Coloring Algorithms for Graphs and Hypergraphs with Forbidden Substructures

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis is based on the following papers. I am a major contributor to all the results included in this thesis.

- The r-coloring and maximum stable set problem in hypergraphs with bounded matching number and edge size, joint work with Sophie Spirkl [31].
- Complexity dichotomy for List-5-Coloring with a forbidden induced subgraph, joint work with Sepehr Hajebi and Sophie Spirkl [24].
- List-3-coloring ordered graphs with a forbidden induced subgraph, joint work with Sepehr Hajebi and Sophie Spirkl [23].

Abstract

This thesis mainly focus on complexity results of the generalized version of the r-Coloring Problem, the r-Pre-Coloring Extension Problem and the List r-Coloring Problem restricted to hypergraphs and ordered graphs with forbidden substructures.

In the context of forbidding non-induced substructure in hypergraphs, we obtain complete complexity dichotomies of the r-Coloring Problem and the r-Pre-Coloring Extension Problem in hypergraphs with bounded edge size and bounded matching number, as well as the r-Pre-Coloring Extension Problem in hypergraphs with uniform edge size and bounded matching number. We also get partial complexity result of the r-Coloring Problem in hypergraphs with uniform edge size and bounded matching number. Additionally, we study the Maximum Stable Set Problem and the Maximum Weight Stable Set Problem in hypergraphs. We obtain complexity dichotomies of these problems in hypergraphs with uniform edge size and bounded matching number.

We then give a polynomial-time algorithm of the 2-Coloring Problem restricted to the class of 3-uniform hypergraphs excluding a fixed one-edge induced subhypergraph. We also consider linear hypergraphs and show that 3-Coloring in linear 3-uniform hypergraphs with either bounded matching size or bounded induced matching size is NP-hard if the bound is a large enough constant.

This thesis also contains a near-dichotomy of complexity results for ordered graphs. We prove that the List-3-Coloring Problem in ordered graphs with a forbidden induced ordered subgraph is polynomial-time solvable if the ordered subgraph contains only one edge, or it is isomorphic to some fixed ordered 3-vertex path plus isolated vertices. On the other hand, it is NP-hard if the ordered subgraph contains at least three edges, or contains a vertex of degree two and does not satisfy the polynomial-time case mentioned before, or contains two non-adjacent edges with a specific ordering. The complexity result when forbidding a few ordered subgraphs with exactly two edges is still unknown.

Acknowledgements

I would like to express my deepest appreciation to my supervisor Sophie Spirkl for her help and guidance. I am extremely grateful for the time we worked together, the valuable feedback and suggestions related to research and writing, and all the support and encouragement I received.

I would like to thank my readers Jane Gao and Joseph Cheriyan for reading my thesis.

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Table of Contents

List of Figures

Chapter 1

Introduction

Graph coloring is one of the fundamental topics in graph theory. A *graph* is a pair (V, E) where V is a finite set and $E \subseteq \binom{V}{2}$. The set V is called the set of *vertices* and E is called the set of *edges*. The r-Coloring Problem is, given a graph, to decide whether there is an assignment of r colors to the vertex set of this graph, such that two vertices receive different colors if there is an edge joining them.

Graph coloring is a useful tool to solve some real-life problems. The idea of graph coloring first came from coloring a map and was discussed by many early graph theorists, see [9] for example. Later it turned to a well-known theorem: the Four Color Theorem [1] [2], which is also known as the first computer-assisted proof.

Though with the assistance of computer, we may be able to find an optimal coloring of some graphs of small size, in general, it is not guaranteed unless we restrict the input graphs. This is because the r-Coloring Problem is a well-known NP-hard problem.

Theorem 1.0.1 (Karp [29]). *For every fixed integer r with $r \geq 3$, the r-Coloring Problem is NP-complete.*

Because of this result, people then ask whether adding some restrictions on the input graphs makes the coloring problem easier. In this thesis, we mainly focus on the complexity results of different variations of the coloring problems restricted to different generalizations of graphs with forbidden structures.

1.1 Definitions

Hypergraphs are a generalization of graphs. A *hypergraph* G is a pair (V, E) where V is a finite set, and $E \subseteq 2^V \backslash \{\emptyset\}$. V is called the set of *vertices* and E is called the set of *edges*. For a hypergraph $G = (V, E)$, we define $V(G) = V$ and $E(G) = E$. For $k \in \mathbb{N}$, we say that G is *k-uniform* if $|e| = k$ for all edges $e \in E$, and G is *k-bounded* if $|e| \leq k$ for all edges $e \in E$. A 2-uniform hypergraph is simply called a graph. An *induced subhypergraph* H of G is a hypergraph with $V(H) \subseteq V(G)$ and $E(H) = \{e \in E(G) : e \subseteq V(H)\}$, and we denote this induced subhypergraph by $G[V(H)]$. Given a hypergraph H, the hypergraph G is called *H-free* if H is not an induced subhypergraph of G.

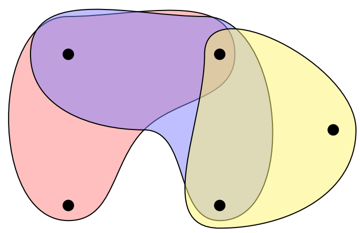

Figure 1.1: A 3-uniform hypergraph.

A *matching* of G is a set of pairwise disjoint edges. A *maximal matching* of G is a matching which is maximal with respect to inclusion. For a hypergraph G, we denote by $\nu(G)$ the maximum integer s such that G contains a matching of size s. A set $S \subseteq V(G)$ of G is *stable* if $e \cap S \neq e$ for every $e \in E(G)$. We say a k-uniform hypergraph is *complete* if its edge set is the set of all k-vertex subsets of its vertex set. A set $S \subseteq V(G)$ is a *clique* if $G[S]$ is complete. The *clique number* of G, denoted $\omega(G)$, is the maximum size of a clique S in G.

We use $[r]$ to denote the set $\{1, \ldots, r\}$. Given a hypergraph G and a positive integer r, a function $c : V(G) \to [r]$ is an *r-coloring* of G if for all $i \in [r]$, $c^{-1}(i)$ is a stable set in G. G is *r-colorable* if there exists an r-coloring of G. The *chromatic number* of G, denoted $\chi(G)$, is the minimum integer r such that G is r-colorable.

A function $c : X \to [r]$ for some $X \subseteq V(G)$ is a *partial r-coloring* of G if c is an r-coloring of $G[X]$. For convenience, we also denote a partial coloring as (X, c). Given a partial r-coloring (X, c) of G, an *r-precoloring extension* of (X, c) is a partial r-coloring (X', c') with $c'(v) = c(v)$ for all $v \in X$, and $X \subset X'$. We say that a partial coloring (X, c) *r-extends* to G if there is an r-precoloring extension $(V(G), c')$ of (X, c).

For a fixed integer r, the HYPERGRAPH r-COLORING PROBLEM is to decide whether a given hypergraph G is r-colorable, and the HYPERGRAPH r-PRECOLORING EXTENSION PROBLEM is to decide given a hypergraph G and a partial r-coloring (X, c), whether (X, c) r-extends to G. When restricting to the class of graphs, it is simply called the r-COLORING PROBLEM and the r-PRECOLORING EXTENSION PROBLEM.

Let G be a graph and let k be a positive integer. A function $c : V(G) \to [k]$ is a *k-coloring* of G if for any $uv \in E(G)$, $c(u) \neq c(v)$. A *k-list-assignment* of G is a function $L : V(G) \to 2^{[k]}$. Given a k-list-assignment L of G, a k-coloring c is an *L-coloring* if $c(v) \in L(v)$ for all $v \in V(G)$. G is *L-colorable* if G has an L-coloring. For a fixed positive integer k, the LIST-k-COLORING PROBLEM is to decide, given an instance (G, L) consisting of a graph G and a k-list-assignment L of G, whether G has an L-coloring or not.

An *ordered graph* G is a triple (V, E, φ) such that (V, E) is a graph with vertex set V and edge set E, and $\varphi : V \to \mathbb{R}$ is an injective function. We say φ is the *ordering* of G. For an ordered graph $G = (V, E, \varphi)$, we define $V(G) = V$, $E(G) = E$, and $\varphi_G = \varphi$. For convenience, we also write the ordered graph (V, E, φ) as (G', φ) where $G' = (V, E)$ is a graph.

Given an ordered graph $G = (V, E, \varphi)$, an ordered graph $G' = (V', E', \varphi')$ is *isomorphic* to G if there exists a bijective function $f : V' \to V$ such that for any two vertices v and w in V', $\varphi'(v) < \varphi'(w)$ if and only if $\varphi(f(v)) < \varphi(f(w))$, and $vw \in E'$ if and only if $f(v)f(w) \in E$. We denote this as $G' \cong G$. An ordered graph H is an *ordered induced subgraph* of G if there exists a set $X \subseteq V(G)$ such that $H \cong G[X]$; otherwise G is called *H-free*.

The ORDERED GRAPH LIST-k-COLORING PROBLEM is the same as coloring the corresponding unordered graph of the input instance.

1.2 Background and Contributions

1.2.1 Graphs

Let P_k denote a path with k vertices. Given graphs G and H, let $G+H$ denote the disjoint union of G and H, and rH denote the disjoint union of r copies of H.

The coloring problem with forbidden induced subgraphs is well studied for graphs. For connected graphs H, the only open case of the complexity of the k-COLORING PROBLEM restricted to H-free graphs is when $k = 3$ and $H = P_t$ for $t \geq 8$. Here are some results about when the k-Coloring Problem and List-k-Coloring Problem with forbidden induced subgraphs are easy.

Theorem 1.2.1. *The k-Coloring Problem restricted to H-free graphs can be solved in polynomial time if:*

- $H = P_6$ *for* $k = 4$ [Chudnovsky, Spirkl and Zhong [13]];
- $H = rP_2$ *for all fixed* $k, r \in \mathbb{N}$ [Golovach, Johnson, Paulusma and Song [21]; Balas and Yu [5]; Tsukiyama, Ide, Ariyoshi and Shirakawa [39]];
- $H = rP_3$ *for* $k = 3$ *and all fixed* $r \in \mathbb{N}$ [Broersma, Golovach, Paulusma and Song [7]];

and the List-k-Coloring Problem restricted to H-free graphs can be solved in polynomial time if:

- $H = P_5$ *for all fixed* $k \in \mathbb{N}$ [Hoàng, Kamiński, Lozin, Sawada and Shu [25]];
- $H = P_7$ *for* $k = 3$ [Bonomo, Chudnovsky, Maceli, Schaudt, Stein and Zhong [6]];
- $H = P_6 + rP_3$ *for* $k = 3$ *and all fixed* $r \in \mathbb{N}$ [Chudnovsky, Huang, Spirkl and Zhong [12]];
- $H = P_5 + rP_1$ *for all fixed* $k, r \in \mathbb{N}$ [Couturier, Golovach, Kratsch and Paulusma [14]];

In particular, we will refer to this result several times later.

Theorem 1.2.2 (Golovach, Johnson, Paulusma and Song [21]; Balas and Yu [5]; Tsukiyama, Ide, Ariyoshi and Shirakawa [39])**.** *For fixed positive integers k and r, the k-Coloring Problem restricted to rP_2-free graphs is polynomial-time solvable.*

On the other hand, the following hardness results are known.

Theorem 1.2.3. *The k-Coloring Problem restricted to H-free graphs is NP-complete if:*

- H *contains a cycle for all* $k \geq 3$ [Kamiński and Lozin [28]];

- H *contains a* $K_{1,3}$ *(a vertex with three non-adjacent neighbors) for all* $k \geq 3$ [Holyer [26]];
- $H = P_6$ *for* $k = 5$*, or* $H = P_7$ *for* $k = 4$ [Huang [27]];
- $H = P_5 + P_2$ *for* $k = 5$ [Chudnovsky, Huang, Spirkl and Zhong [12]];

and the List-k-Coloring Problem *restricted to* H*-free graphs is NP-complete if:*

- $H = P_6$ *for* $k = 4$ [Golovach, Paulusma and Song [20]] *;*
- $H = P_4 + P_2$ *for* $k = 5$ [Couturier, Golovach, Kratsch and Paulusma [14]] .

Our contribution is, in [24], we have proved the following theorems:

Theorem 1.2.4. *For every* $r \in \mathbb{N}$*, the* List-5-Coloring Problem *restricted to* rP_3*-free graphs can be solved in polynomial time.*

With Theorem 1.2.1 and 1.2.3, our result (Theorem 1.2.4) completes the following complexity dichotomy.

Theorem 1.2.5. *Let* H *be a graph. Assuming P≠NP, the* List-5-Coloring Problem *restricted to* H*-free graphs can be solved in polynomial time if and only if* H *is an induced subgraph of* rP_3 *or* $P_5 + rP_1$ *for some* $r \in \mathbb{N}$*.*

In this thesis, we will include the proof of the following theorem from [24]:

Theorem 1.2.6. *The* k-Coloring Problem *restricted to* rP_4*-free graphs is NP-complete for all* $k \geq 5$ *and* $r \geq 2$*.*

1.2.2 Hypergraphs

The hypergraph coloring problem is a natural extension of the graph coloring problem. In the past few decades, it has already attracted many people's attention; see the survey [8] for previous results.

In general, the Hypergraph r-Coloring Problem is harder than the r-Coloring Problem. For example, the Hypergraph 2-Coloring Problem is NP-hard, while the 2-Coloring Problem is polynomial-time solvable. The following result shows the Hypergraph r-Coloring Problem is NP-hard, even restriced to k-uniform hypergraphs with some fixed positive integer k.

Theorem 1.2.7 (Lovász [32]; Phelps and Rödl [36]). *For all $k \geq 3$ and $r \geq 2$, the k-*Uniform Hypergraph r-Coloring Problem *is NP-complete.*

We then focus on bounded or uniform hypergraphs. It is natural to ask whether adding restrictions on the input hypergraphs can still make (some variations of) the coloring problem easier. If so, does a similar condition as used in graph colorings make the hypergraph coloring problem polynomial-time solvable?

We notice that some graphs H have the property that r-coloring H-free graphs can be solved in polynomial time for all r, for example, graphs of the form $H = sP_2$, as shown in Theorem 1.2.2. When turning to hypergraphs, it seems that excluding an induced hyper matching would be one of the potential options. But unfortunately, we will show later that even bounding the maximum size of a matching (a much stronger condition than excluding an induced matching) does not always lead to a polynomial-time algorithm.

In this thesis, we will show the following dichotomies:

Theorem 1.2.8. *Let k, r and s be positive integers with $k, r \geq 2$. The k-*Bounded Hypergraph r-Coloring Problem*, the k-*Bounded Hypergraph r-Precoloring Extension Problem *as well as the k-*Uniform Hypergraph r-Precoloring Extension Problem*, restricted to hypergraphs G with $\nu(G) \leq s$, are polynomial-time solvable if*

- *$s \leq r - 1$, or*
- *$k = 3$ and $r = 2$, or*
- *$k = 2$,*

and NP-complete otherwise.

We will also show the following result:

Theorem 1.2.9. *Let k, r and s be positive integers with $k, r \geq 2$. The k-*Uniform Hypergraph r-Coloring Problem *restricted to hypergraphs G with $\nu(G) \leq s$ is polynomial-time solvable if*

- *$s \leq r - 1$, or*
- *$k = 3$ and $r = 2$, or*

- $k = 2$,

and is NP-complete if

- $s \geq (r-1)k + 1$, *and*
- $k \geq 4$ *or* $r \geq 3$.

Theorem 1.2.2 is based on a result of [5] that sP_2-free graphs have only polynomially many maximal (with respect to inclusion) stable sets. Using this, [39] gave a polynomial-time algorithm for finding a maximum (weight) stable set in an sP_2-free graph. We ask an analogous question in hypergraphs with bounded maximum matching size. We will prove:

Theorem 1.2.10. *For fixed positive integers k and s with $k \geq 3$, the* k-UNIFORM HYPERGRAPH MAXIMUM STABLE SET PROBLEM *restricted to hypergraphs with $\nu(G) \leq s$ is polynomial-time solvable, and the* k-UNIFORM HYPERGRAPH MAXIMUM WEIGHT STABLE SET PROBLEM *restricted to hypergraphs with $\nu(G) \leq s$ is NP-complete.*

We also give a first result for excluding an induced subhypergraph:

Theorem 1.2.11. *Let $t \in \mathbb{N}$ be fixed, and let H be the 3-uniform hypergraph with $t+3$ vertices and one edge. Then there is a polynomial-time algorithm for the* 3-BOUNDED HYPERGRAPH 2-COLORING PROBLEM *restricted to H-free hypergraphs.*

Finally, we will prove results about linear hypergraphs. A hypergraph G is *linear* if $|e \cap e'| \leq 1$ for every two distinct $e, e' \in E(G)$. The restriction to linear hypergraphs does not affect NP-hardness:

Theorem 1.2.12 (Phelps and Rödl [36])**.** *For every $r \geq 2$, the* 3-UNIFORM HYPERGRAPH r-COLORING PROBLEM *restricted to linear hypergraphs is NP-complete.*

The following result gives an algorithm for 2-coloring certain linear hypergraphs:

Theorem 1.2.13 (Chattopadhyay and Reed [10])**.** *There is a polynomial-time algorithm for the* k-UNIFORM HYPERGRAPH 2-COLORING PROBLEM *restricted to linear hypergraphs with maximum degree bounded by a function of k.*

We ask how our results extend to linear 3-uniform hypergraphs. For $s \in \mathbb{N}$, we let M_s denote the 3-uniform hypergraph with $3s$ vertices and s pairwise disjoint edges. We will show that in linear hypergraphs, excluding a fixed induced matching implies bounded matching number, which immediately implies (assuming Theorems 1.2.8 and 1.2.10):

Theorem 1.2.14. *Let $s \in \mathbb{N}$. The* 3-Uniform Hypergraph 2-Coloring Problem*, the* 3-Uniform Hypergraph 2-Precoloring Extension Problem*, and the* 3-Uniform Hypergraph Maximum Stable Set Problem *restricted to linear M_s-free hypergraphs are polynomial-time solvable.*

We will also prove:

Theorem 1.2.15. *The* 3-Uniform Hypergraph 3-Coloring Problem *restricted to linear hypergraphs G with $\nu(G) \leq 532$ is NP-complete.*

1.2.3 Ordered Graphs

Motivated by the fact that excluding some induced subgraphs makes the graph coloring problems easier, another idea comes to our mind is the ordered graph. The idea of ordered graphs first came from Ramsey-type questions (see [38] and [34] for example). In many Ramsey-type questions, it is convenient or necessary to give an ordering to the vertex set of a graph. Later in recent years, some other graph theory questions, such as the Turan-type questions (see [35]) and the chromatic number (see [4]) were also studied for ordered graphs.

Part of the idea of ordered graph coloring comes from tournament coloring. When coloring tournaments, backedge graphs are used to represent tournaments, which are actually ordered graphs. In fact, we are the first group of people to study the complexity of coloring with forbidden ordered induced subgraphs.

We notice that coloring an ordered graph is actually the same as coloring the corresponding unordered graph. But the difference comes when excluding some induced subgraphs. Even though the ordered graph coloring itself is the same as the graph coloring, we still use "ordered graph coloring"to avoid confusion when talking about induced subgraphs. Under the ordered graph setting, we can break the symmetry and only exclude "a part of"the induced subgraph. In fact, the symmetry is often vital to make the graph coloring problem easier. One interesting result we will show later is, there are two different ordered P_3 such that the Ordered Graph List-3-Coloring Problem is NP-hard if forbidding one ordered induced subgraph, while forbidding the other one allows us to construct a polynomial-time coloring algorithm.

In order to state our main results, let us define some ordered graphs first. The remaining ordered graphs are defined later when proving the main results. Let $U' = \{u_1, u_2, u_3, u_4, u_5\}$ and $U = U' \backslash \{u_5\}$, the ordering $\varphi' : U' \to \mathbb{R}$ with $u_i \mapsto i$ for $i \in [5]$. Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $\varphi : V \to \mathbb{R}$ with $v_i \mapsto i$ for $i \in [6]$.

Figure 1.2: Two non-isomorphic orderings of the unordered graph P_3.

- Let $J_9 = (U, \{u_1u_2, u_3u_4\}, \varphi'|_U)$.
- Let $J_{16} = (U \backslash \{u_4\}, \{u_1u_2, u_1u_3\}, \varphi'|_{U\backslash\{u_4\}})$.
- Let $M_1 = (V, \{v_1v_6, v_2v_5\}, \varphi)$.
- Let $M_5 = (V \backslash \{v_6\}, \{v_1v_5, v_2v_3\}, \varphi|_{V\backslash\{v_6\}})$.
- Let $M_6 = (V \backslash \{v_5, v_6\}, \{v_1v_3, v_2v_4\}, \varphi|_{V\backslash\{v_5,v_6\}})$.
- Let $M_7 = (V \backslash \{v_5, v_6\}, \{v_1v_4, v_2v_3\}, \varphi|_{V\backslash\{v_5,v_6\}})$.
- Let $M_8 = (V \backslash \{v_6\}, \{v_1v_5, v_2v_4\}, \varphi|_{V\backslash\{v_6\}})$.

Given an ordered graph $H = (V, E, \varphi)$ and two positive integers k and l, let $H(k, l)$ denote the ordered graph obtained by adding k isolated vertices with ordering $\min \varphi(V) - k, ..., \min \varphi(V) - 1$, and l isolated vertices with ordering $\max \varphi(V) + 1, ..., \max \varphi(V) + l$. We denote $-H = (V, E, v \mapsto -\varphi_H(v))$.

$J_{16}(k, l)$ $\quad$ $-J_{16}(k, l)$

Figure 1.3: The ordered graphs $J_{16}(k, l)$ and $-J_{16}(k, l)$ with $k = 3$ and $l = 2$.

In our paper [23], we proved the following result, which is not included in this thesis.

Theorem 1.2.16. *The* Ordered Graph List-3-Coloring Problem *restricted to (H, φ)-free ordered graphs is polynomial-time solvable if H contains at most one edge.*

In this thesis, we will prove the following theorems from [23]:

Theorem 1.2.17. *For all $k, l \in \mathbb{N}$, the* Ordered Graph List-3-Coloring Problem *restricted to $J_{16}(k, l)$-free ordered graphs is polynomial-time solvable.*

Theorem 1.2.18. *If H is an ordered graph such that at least one of the following holds:*

- *H has at least three edges;*
- *H has a vertex of degree at least 2 and is not isomorphic to $J_{16}(k,l)$ or $-J_{16}(k,l)$ for any $k,l \in \mathbb{N}$;*
- *H contains J_9, M_1 or M_5 as induced ordered subgraph;*

then the Ordered Graph List-3-Coloring Problem *restricted to H-free ordered graphs is NP-hard.*

In summary, Theorem 1.2.18 covers all graphs H except:

- graphs H with at most one edge (polynomial-time by Theorem 1.2.16);
- graphs H isomorphic to $J_{16}(k,l)$ or $-J_{16}(k,l)$ (polynomial-time by Theorem 1.2.17);
- graphs H containing M_1 (NP-hard by Theorem 3.2.2);
- graphs H containing M_5 (NP-hard by Theorem 3.2.2);
- graphs H containing M_6 plus isolated vertices (open);
- graphs H isomorphic to $M_7(k,l)$ or $M_8(k,l)$ (open).

The cases are shown in Figures 1.4 and 1.5, where gray vertices represent an arbitrary number of isolated vertices.

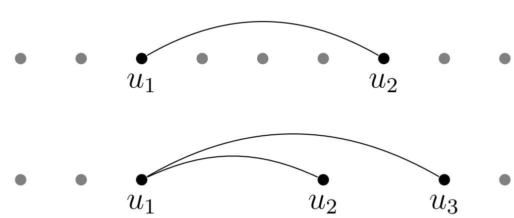

Figure 1.4: The cases of H which are polynomial-time solvable.

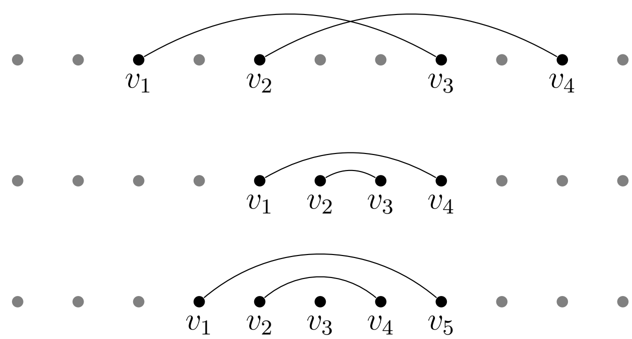

Figure 1.5: The cases of H which are still open.

1.3 Tools

In this section, we will introduce some tools and complexity results that are used later in our proofs.

The *hypergraph Ramsey number*, $R_k(n_1, \ldots, n_t)$, is the smallest integer N such that for every function $f : E(G) \to [t]$ for a complete k-uniform hypergraph G with at least N vertices, there exists $i \in [t]$ and a set $S \subseteq V(G)$ with $|S| \geq n_i$ such that all edges $e \subseteq S$ satisfy $f(e) = i$.

Theorem 1.3.1 (Ramsey [37]). *For all positive integers k, n_1, $\ldots$, n_t, the hypergraph Ramsey number $R_k(n_1, \ldots, n_t)$ exists.*

Given an instance I consisting of n Boolean variables and m clauses, each of which contains 2 literals, the 2-Satisfiability Problem (2-SAT) is to decide whether there exists a truth assignment for every variable such that every clause contains at least one true literal. We say I is *satisfiable* if it admits such an assignment.

Theorem 1.3.2 (Krom [30]; Aspvall, Plass and Tarjan [3]). *The* 2-SAT Problem *can be solved in time $O(n + m)$, where n is the number of variables and m is the number of clauses.*

Given an instance I consisting of n Boolean variables and m clauses, each of which contains 3 literals, the Not-All-Equal-3-Satisfiability Problem (NAE3SAT) is to decide whether there exists a truth assignment for every variable such that every clause contains at least one true literal and one false literal. We say I is *satisfiable* if it admits such an assignment. A *monotone* NAE3SAT is a NAE3SAT restricted to instances with no negated literals.

Theorem 1.3.3 (Garey and Johnson [18]). *Monotone* NAE3SAT *is NP-complete.*

Similar to vertex coloring, an *k-edge-coloring* of a graph G is a function $f : E(G) \to [k]$ such that two edges $e_1, e_2 \in E(G)$ receive different colors if they have a common endpoint.

Theorem 1.3.4 (Vizing [40], Misra and Gries [33]). *There is a $O(mn)$-algorithm for edge-coloring a graph G with $D + 1$ colors, where D is the maximum degree of G, m is the number of edges and n is the number of vertices.*

A graph G is *chordal* if in G, every cycle of length at least 4 has an edge connecting two vertices of the cycle but not in the cycle. Equivalently, every induced cycle in G is a triangle.

There is an old known result derived from [15] that, the treewidth of a chordal graph can be computed in polynomial-time. Indeed, the treewidth of a chordal graph is bounded by its clique number minus 1, and what we compute is the clique number. We also know that the LIST-k-COLORING PROBLEM restricted to graphs with bounded treewidth is polynomial-time solvable with respect to the input size and the treewidth [17]. Thus, we have the following:

Theorem 1.3.5. *The* LIST-3-COLORING PROBLEM *restricted to chordal graphs with bounded clique number is polynomial-time solvable.*

We will also use the following results later.

Theorem 1.3.6 (Edwards [16]). *The* LIST-2-COLORING PROBLEM *can be solved in time $O(n^2)$, where n is the number of vertices of the input graph.*

Theorem 1.3.7 (Garey and Johnson [18]). *The* MAXIMUM STABLE SET PROBLEM *is NP-complete.*

Theorem 1.3.8 (Garey, Johnson and Stockmeyer [19]). *The* 3-COLORING PROBLEM *restricted to graphs with maximum degree at most 4 is NP-complete.*

Theorem 1.3.9 (Chlebík and Chlebíková [11]). *The* LIST-3-COLORING PROBLEM *restricted to bipartite graphs is NP-complete.*

1.4 Outline

Chapter 2 is about hypergraphs. We will give two polynomial-time algorithms for hypergraphs in Section 2.1 and Section 2.2 respectively. In Section 2.3, we will talk about some NP-hard cases and complete the results about hypergraphs with bounded matching size. In Section 2.4, we will talk about the k-UNIFORM HYPERGRAPH MAXIMUM STABLE SET PROBLEM and the k-UNIFORM HYPERGRAPH MAXIMUM WEIGHT STABLE SET PROBLEM. In Section 2.5, we will give a first result for excluding an induced subhypergraph. Finally, in Section 2.6, we will prove results about linear hypergraphs.

Chapter 3 is about ordered graphs. In Section 3.1, we will talk about one of the polynomial-time solvable case as mentioned above. In Section 3.2, we will define some ordered graphs and show that the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to the class of ordered graphs forbidding these ordered graphs as an induced subgraph remains NP-complete.

In Chapter 4, we will cover the hardness result of the k-COLORING PROBLEM restricted to rP_4-free graphs for all $k \geq 5$ and $r \geq 2$.

Chapter 2

Hypergraphs

In this chapter, we will focus on bounded or uniform hypergraphs.

One of the main ideas used in this chapter is "guessing the coloring of a small set". To describe it more carefully, "small"means the size of this set is upper bounded by some constant, and "guessing"means we enumerate and go through all possible choices for such a set. The key point here is the constant bound, which means we can afford going through all choices of the potential set, and for each choice go through all possible colorings of the vertices of this set. This method is used frequently for this kind of problem, for example, in [6] and [24].

Given two partial r-coloring collections $\mathcal{C}$, $\mathcal{C}'$ of a hypergraph G, we say $\mathcal{C}$ and $\mathcal{C}'$ are *r-equivalent* if $\mathcal{C}$ contains a partial r-coloring c_1 which r-extends to G if and only if $\mathcal{C}'$ contains a partial r-coloring c_2 which r-extends to G. We say (X, c) is r-equivalent to $\mathcal{C}$ if the collection $\{(X, c)\}$ is r-equivalent to $\mathcal{C}$. We say that $\mathcal{C}$ is *r-equivalent* to G if G is r-colorable if and only if $\mathcal{C}$ contains a partial r-coloring which r-extends to G.

2.1 Algorithm for the case $k = 3$ and $r = 2$

In this section, we will prove:

Theorem 2.1.1. *For every fixed positive integer s, the* 3-Bounded Hypergraph 2-Coloring Problem *restricted to hypergraphs with $\nu(G) \leq s$ is polynomial-time solvable.*

A common strategy for coloring algorithms is using an algorithm for 2-SAT as a subroutine.

Proof of Theorem 2.1.1. Let G be a 3-bounded hypergraph with $\nu(G) \leq s$. First, we create a collection $\mathcal{C}$ of partial 2-colorings as follows. We fix a maximal matching F of G. We define the set $X^F = \cup_{e \in F} e$. Let $\mathcal{C}$ be the set of all partial 2-colorings $(X^F, c : X^F \to [2])$ of G.

We claim that the collection $\mathcal{C}$ has the following three properties. The theorem follows immediately from these properties.

(1) *$\mathcal{C}$ is 2-equivalent to G.*

It suffices to show that if G has a 2-coloring c, then there is a partial 2-coloring in $\mathcal{C}$ which has a 2-precoloring extension. Let c be a 2-coloring of G. Consider the partial 2-coloring $(X^F, c|_{X^F}) \in \mathcal{C}$. Then c is a 2-precoloring extension of $c|_{X^F}$. Thus we have proved (1).

(2) *$\mathcal{C}$ can be computed in time $O(n^3)$.*

Let $|V(G)| = n$. Since G is 3-bounded, $|E(G)| \leq O(n^3)$. We can go through all edges and construct a maximal matching F in time $O(n^3)$. Checking whether (X, c) is a partial 2-coloring takes time $O(1)$, as the size of X is bounded. Since $|F| \leq \nu(G) \leq s$, we have $|\mathcal{C}| \leq 2^{3s} = O(1)$. Thus, $\mathcal{C}$ can be constructed from F in time $O(n^3)$.

(3) *For every partial 2-coloring c' in $\mathcal{C}$, whether c' has a 2-precoloring extension $(V(G), c)$ can be decided in polynomial time.*

Let $(X^F, c') \in \mathcal{C}$. Since F is a maximal matching and G is 3-bounded, for every edge $e \in E(G) \setminus F$, $|e \setminus X^F| \leq 2$.

We define a 2-precoloring extension (X, c) of c' as follows. We define the sets $X_0, X_1, \ldots$ iteratively. Let $X_0 = X^F$. Let $c(v) = c'(v)$ for all $v \in X^F$. Suppose that we have defined X_i. If there exists an edge $e \in E(G)$ such that $e \subseteq X_i$ and e is monochromatic, then c' does not have a 2-precoloring extension and we return this determination. If there exists an edge $e \in E(G)$ such that $|e \setminus X_i| = 1$ and $c(e \cap X_i) = \{j\}$ for some $j \in [2]$, we define $c(w)$ to be the unique element of $[2] \setminus \{j\}$ for $w \in e \setminus X_i$, and define $X_{i+1} = X_i \cup \{w\}$. Otherwise we stop and let $X = X_i$. This terminates within at most $O(n^3)$ steps. From the construction, clearly $\{(X_i, c)\}$ is equivalent to $\{(X_{i+1}, c)\}$ at every step; and it follows that $\{(X, c)\}$ is equivalent to $\{(X^F, c')\}$, and that if this step returns a determination that (X^F, c') does not 2-extend to G, then this determination is correct.

We define a 2-SAT instance as follows. For every $v \in V(G) \setminus X$, we have a variable x_v. Let $E' \subseteq E(G)$ be the set of edges such that $|e \setminus X| = 2$ for all $e \in E'$. For every edge

$e \in E'$, we create a clause C_e. Let $e = \{v, u, w\}$ with $v \in X$ and $u, w \in V(G) \setminus X$. If $c(v) = 1$, we set $C_e = x_u \vee x_w$. Otherwise, let $C_e = \overline{x_u} \vee \overline{x_w}$.

If the 2-SAT instance has a solution x, where "true"and "false" are represented by 1 and 0 respectively, then we set $c(v) = x_v + 1$ for every $v \in V(G) \setminus X$. Take an edge $e \in E(G)$. If $|e \setminus X| \leq 1$, by the construction of X, e is not monochromatic. If $|e \setminus X| = 2$, the clause C_e of 2-SAT instance and the construction of c guarantees that at least one of the vertices in $e \setminus X$ receives the opposite color from the vertex in $e \cap X$. Since F is maximal, there is no edge e in $E(G)$ with $|e \setminus X| = 3$. Thus, c is a 2-precoloring extension of (X, c').

If there is a 2-precoloring extension d of (X, c), then we set $x_v = d(v) - 1$ for every $v \in V(G) \setminus X$. For every edge $e = \{v, u, w\} \in E'$ with $e \cap X = \{v\}$, if $d(v) = 1$, then $C_e = x_u \vee x_w$. Since e is not monochromatic, without loss of generality we may assume $d(u) = 2$, and so $x_u = d(v) - 1 = 1$. Thus, the clause C_e is satisfied. A similar argument applies for $d(v) = 2$. From the construction of clauses of this 2-SAT instance, we conclude that x is a solution to the 2-SAT instance.

Therefore, deciding whether (X, c) has a 2-coloring extension is equivalent to solving the 2-SAT instance defined above.

It remains to show that this can be done in polynomial-time. Let n be the number of vertices of G. Constructing the set X takes time $O(n^3)$. Constructing the equivalent 2-SAT instance takes time $O(n^3)$. Solving this 2-SAT instance takes time $O(n)$. So the total running time is $O(n^3)$. □

This immediately implies, for fixed r, a polynomial-time algorithm for 2-coloring tournaments with no r vertex-disjoint cyclic triangles, which was first proved by Hajebi [22].

2.2 Algorithm for the case $s \leq r - 1$

In this section, we will prove:

Theorem 2.2.1. *For fixed positive integers r, k, s with $s \leq r-1$, the k-*Bounded Hypergraph r-Precoloring Extension Problem *restricted to hypergraphs G with $\nu(G) \leq s$ is polynomial-time solvable.*

The key idea is to precolor a set of vertices, and in each step, carefully adding vertices to our set such that there will be some color j with the property that edges which only contain vertices precolored j will now contain more precolored vertices than before. Eventually, either all vertices in an edge will be precolored j (and so this precoloring does not lead to a valid coloring) or for some color j, every edge contains a vertex precolored with a color other than j (and so it is safe to color all remaining vertices with color j).

Lemma 2.2.2. *Let $r, k, s \in \mathbb{N}$ with $s \leq r-1$. Let G be a k-bounded hypergraph with $\nu(G) \leq s$. Given a partial r-coloring (X, c) of G, we define $E_i = \{e \in E(G) : e \cap X \subseteq c^{-1}(i)\}$. If $E_i \neq \emptyset$ for all $i \in [r]$, then there is a vertex set $X' \supset X$ and a collection of partial r-colorings $\mathcal{C}$ such that*

- *For every $(X^*, c^*) \in \mathcal{C}$, $X^* = X'$;*
- *$|\mathcal{C}| \leq r^{ks} = O(1)$, and $\mathcal{C}$ can be computed from (X, c) in time $O(n^3)$;*
- *There is a color $j \in [r]$ such that for every edge $e \in E_j$, $|e \cap X'| \geq |e \cap X| + 1$; and*
- *$\mathcal{C}$ is r-equivalent to (X, c).*

Proof. Let S be a matching in G such that $S \subseteq \bigcup_{i \in [r]} E_i$, and S is maximal with respect to this condition. Let $X^S = \cup_{e \in S} e$. Let $X' = X \cup X^S$. Let $\mathcal{C}$ be the set of all partial r-colorings $(X', c' : X' \to [r])$ such that $c'|_X = c$. The first property follows immediately from the construction. Since $|S| \leq s$, $|X^S| \leq ks$, and we have $|\mathcal{C}| \leq r^{ks} = O(1)$. Finding S takes time $O(n^3)$, and thus, $\mathcal{C}$ can be computed from (X, c) in time $O(n^3)$. This proves the second property.

For every $e \in S$, there exists $i \in [r]$ such that $e \in E_i$, and therefore we have that $c(v) = i$ for all $v \in e \cap X$. Since $|S| \leq s \leq r - 1$, there exists a color $j \in [r]$ such that $c(v) \neq j$ for all $v \in X \cap X^S$. Let e be an edge in E_j. We know that $e \cap X \subseteq c^{-1}(j)$, so $e \cap X^S \cap X = \emptyset$. But from the definition of S, we have $e \cap X^S \neq \emptyset$, as otherwise S is not maximal. Thus, $e \cap (X^S \setminus X) \neq \emptyset$. This proves the third property.

Suppose that there is a partial r-coloring $(X', c') \in \mathcal{C}$ which r-extends to G. Then by the construction of $\mathcal{C}$, $c'|_X = c$. Thus, every r-precoloring extension of (X', c') is also an r-precoloring extension of (X, c). Now suppose (X, c) r-extends to $V(G)$, that is, there is a coloring $c' : V(G) \to [r]$ with $c'|_X = c$. Then by the construction of $\mathcal{C}$, $(X', c'|_{X'}) \in \mathcal{C}$. Therefore, the last property holds. $\square$

Theorem 2.2.3. *For fixed positive integers r, k, s with $s \leq r - 1$, there is an algorithm with the following specifications:*

- *Input: A k-bounded hypergraph G with $\nu(G) \leq s$, and an r-precoloring (X, c).*

- *Output: one of*
  - *an r-precoloring extension of (X, c) to $V(G)$;*
  - *a determination that (X, c) does not r-extend to G.*
- *Running time: $O(|V(G)|^3)$.*

Proof. We will define a sequence $\mathcal{C}_0, \dots$ of collections of partial r-colorings iteratively, as follows. Let $\mathcal{C}_0 = \{(X, c)\}$.

Suppose that we have defined $\mathcal{C}_t$. Given a partial r-coloring $(Y, d) \in \mathcal{C}_t$, let $E_{t,i}^{Y,d} = \{e \in E(G) : e \cap Y \subseteq d^{-1}(i)\}$. If $E_{t,i}^{Y,d} = \emptyset$ for some $i \in [r]$ and $(Y, d) \in \mathcal{C}_t$, then we define d' by setting $d'|_Y = d|_Y$ and $d'(v) = i$ for all $v \in V(G) \setminus Y$ and return d'. Note that d' is an r-coloring of G: Since (Y, d) is a partial r-coloring, it follows that no edge of $G[Y]$ is monochromatic. Therefore, if G contains an edge e which is monochromatic with respect to d', then $e \setminus Y \neq \emptyset$. It follows that $e \setminus Y \neq \emptyset$, and since $d'(v) = i$ for all $v \in V(G) \setminus Y$, it follows that every vertex of e is colored i by d'. But then $e \cap Y \subseteq d^{-1}(i)$, a contradiction. This shows that d' is an r-coloring of G.

Otherwise, for every $(Y, d) \in \mathcal{C}_t$, we have that $E_{t,i}^{Y,d} \neq \emptyset$ for all $i \in [r]$, and so there is a collection of partial r-colorings $\mathcal{C}_{t+1}^{Y,d}$ which satisfies the properties in Lemma 2.2.2 applied to G and (Y, d). Let $\mathcal{C}_{t+1} = \cup_{(Y,d) \in \mathcal{C}_t} \mathcal{C}_{t+1}^{Y,d}$. By Lemma 2.2.2, $\mathcal{C}_{t+1}$ is r-equivalent to $\mathcal{C}_t$; and inductively, $\mathcal{C}_{t+1}$ is equivalent to $\mathcal{C}_0 = \{(X, c)\}$. Thus, if $\mathcal{C}_{t+1} = \emptyset$, then (X, c) does not r-extend to G and we return this.

It remains to show that this algorithm terminates in polynomial time. To prove this, we define a potential function $\psi((Y, d)) = \sum_{i \in [r]} \max(\{0\} \cup \{|e \setminus Y| : e \cap Y \subseteq d^{-1}(i)\})$. We have $\psi((X, c)) \leq rk$ since each summand is at most k. We will prove by induction on t that for every $(Y, d) \in \mathcal{C}_t$, we have $\psi((Y, d)) \leq rk - t$.

It suffices to show that if $(Y, d) \in \mathcal{C}_t$ and $(Y', d') \in \mathcal{C}_{t+1}^{Y,d}$ then $\psi((Y', d')) \leq \psi((Y, d)) - 1$. By the third property of Lemma 2.2.2, there is a color $j \in [r]$ such that for every edge $e \in E_{t,j}^{Y,d}$, $|e \cap Y'| \geq |e \cap Y| + 1$, which means that $\max(\{0\} \cup \{|e \setminus Y'| : e \cap Y' \subseteq d'^{-1}(j)\}) \leq \max(\{|e \setminus Y| : e \cap Y \subseteq d^{-1}(j)\}) - 1$. It follows that $\psi((Y', d')) \leq \psi((Y, d)) - 1$, as claimed.

Since $\psi((Y, d)) \geq 0$ for every partial r-coloring (Y, d) of G, it follows that this algorithm terminates in t' steps for some $t' \leq rk$. Since there are $O(1)$ iterations, and by Lemma 2.2.2, we have $|\mathcal{C}_t| = O(1)$ for all $t \leq t'$. Moreover, the set $\mathcal{C}_{t+1}$ can be computed from $\mathcal{C}_t$ in time $|\mathcal{C}_t| \cdot O(n^3) = O(n^3)$. Thus, each step takes time $O(n^3)$. So the total running time is $O(n^3)$. $\square$

2.3 NP-hardness results for bounded matching number

Let G and H be two hypergraphs. We define an operation, $\ltimes$, via $G \ltimes H := (V(G) \cup V(H), E(G) \cup \{e \cup \{x\} : e \in E(H), x \in V(G)\})$.

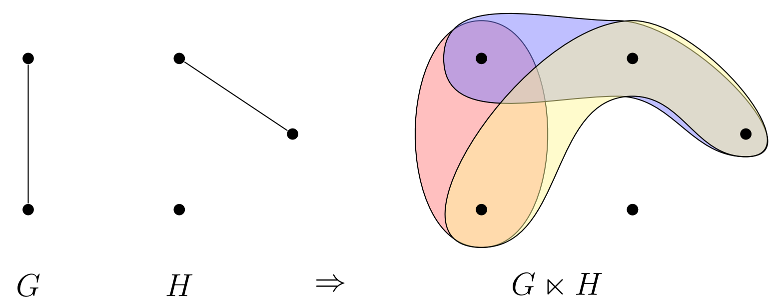

Figure 2.1: An example of $G \ltimes H$.

We have the following properties.

Lemma 2.3.1. *Let G and H be hypergraphs. Then $\nu(G \ltimes H) \leq |V(G)|$.*

Proof. This follows immediately from the fact that every edge in $G \ltimes H$ contains at least one vertex in $V(G)$. $\square$

Lemma 2.3.2. *Let H be a hypergraph. If G is a hypergraph with $\chi(G) = r$, then $G \ltimes H$ is r-colorable if and only if H is r-colorable.*

Proof. Suppose for a contradiction that $G \ltimes H$ has an r-coloring c and H is not r-colorable. Since $c|_{V(H)}$ is not an r-coloring of H, there exists an edge $e \in E(H)$ such that e is monochromatic with respect to $c|_{V(H)}$. Since $\chi(G) = r$ and $(G \ltimes H)[V(G)] = G$, there exist vertices $v_1, \ldots, v_r \in V(G)$ such that $c(v_i) = i$ for all $i \in [r]$. But then one of the edges $e \cup \{v_1\}, \ldots, e \cup \{v_r\}$ is monochromatic, which contradicts the fact that c is an r-coloring of $G \ltimes H$.

Now suppose that H has an r-coloring d. Since $\chi(G) = r$, G has an r-coloring d'. We define a new function $d^* : V(G \ltimes H) \to [r]$ with $d^*(v) = d(v)$ if $v \in V(H)$ and $d^*(v) = d'(v)$ otherwise. For every edge $e \in E(G \ltimes H)$, if $e \in E(G)$, then $d^*|_e = d'|_e$. So e is not monochromatic. Otherwise $e = e' \cup \{v\}$ for some $e' \in E(H)$ and $v \in V(G)$.

Then $d^*|_{e'} = d|_{e'}$, and since the edge e' is not monochromatic, it follows that e is not monochromatic. Thus d^* is an r-coloring of $G \ltimes H$. $\square$

Theorem 2.3.3. *Given fixed integers k and r with $k, r \geq 2$, if the k-*Bounded Hypergraph r-Coloring Problem *is NP-complete, then the $(k+1)$-*Bounded Hypergraph r-Coloring Problem *restricted to hypergraphs with $\nu(G) \leq r$ is NP-complete.*

Proof. Let H be a k-bounded hypergraph. We set the hypergraph $G = K_r$, a complete graph on r vertices. We have $\chi(G) = r$. The hypergraph $G \ltimes H$ can be constructed from G and H in time $O(n^{k+1})$, where $n = |V(G \ltimes H)|$. By the construction, $G \ltimes H$ is $(k+1)$-bounded. The remaining part of the proof follows immediately from Lemmas 2.3.1 and 2.3.2. $\square$

Theorem 2.3.4. *Given fixed integers k and r with $k, r \geq 2$, if the k-*Uniform Hypergraph r-Coloring Problem *is NP-complete, then the $(k+1)$-*Uniform Hypergraph r-Coloring Problem *restricted to hypergraphs with $\nu(G) \leq (r-1)k+1$ is NP-complete.*

Proof. Let H be a k-uniform hypergraph and let G be the complete $(k+1)$-uniform hypergraph with $(r-1)k+1$ vertices. The hypergraph $G \ltimes H$ can be constructed from G and H in time $O(n^{k+1})$, where $n = |V(G \ltimes H)|$. By the construction, $G \ltimes H$ is $(k+1)$-uniform.

We want to show that $\chi(G) = r$. We choose k vertices to color i for every $i \in [r-1]$, and color the remaining vertex r. Since G is $(k+1)$-uniform, every edge of G receives at least two colors. Thus, $\chi(G) \leq r$. Suppose for a contradiction that $\chi(G) \leq r-1$. Then take an $(r-1)$-coloring c of G. There exists one color i with $|c^{-1}(i)| \geq \lceil \frac{(r-1)k+1}{r-1} \rceil \geq \lceil k + \frac{1}{r-1} \rceil = k+1$. This means that there is a monochromatic edge in G, which contradicts the fact that c is an $(r-1)$-coloring of G.

The remaining part of the proof follows immediately from Lemmas 2.3.1 and 2.3.2. $\square$

Theorem 2.3.5. *Given fixed integers k and r with $k, r \geq 2$, if the k-*Uniform Hypergraph r-Coloring Problem *is NP-complete, then the $(k+1)$-*Uniform Hypergraph r-Precoloring Extension Problem *restricted to hypergraphs with $\nu(G) \leq r$ is NP-complete.*

Proof. Let H be a k-uniform hypergraph and let G be a graph with a set of vertices $\{v_1, \ldots, v_r\}$ and no edges. Define the precoloring of $G \ltimes H$ to be $(V(G), c')$ with $c'(v_i) = i$ for all $i \in [r]$. The hypergraph $G \ltimes H$ can be constructed from G and H in time $O(n^{k+1})$, and the precoloring $(V(G), c')$ of $G \ltimes H$ can be constructed in time $O(n)$, where $n = |V(G \ltimes H)|$. The graph H is k-uniform and $E(G) = \emptyset$, so $G \ltimes H$ is $(k+1)$-uniform.

It remains to show that $G \ltimes H$ has an r-precoloring extension with respect to the precoloring $(V(G), c')$ if and only if H is r-colorable.

Suppose $G \ltimes H$ has an r-precoloring extension c. Assume for a contradiction that H is not r-colorable. Since $c|_{V(H)}$ is not an r-coloring of G, there exists an edge $e \in E(H)$ such that e is monochromatic. By the definition of c', one of the vertices $v_1, \ldots, v_r$ receives the same color as e, which contradicts the fact that c is an r-precoloring extension of $G \ltimes H$ and $(V(G), c')$.

Now suppose that H has an r-coloring d. We define a new function $d^* : V(G \ltimes H) \to [r]$ with $d^*(v) = d(v)$ if $v \in V(H)$ and $d^*(v) = c'(v)$ otherwise. For every edge $e \in E(G \ltimes H)$, $e = e' \cup \{v\}$ for some $e' \in E(H)$ and $v \in V(G)$. Then $d^*|_{e'} = d|_{e'}$. The edge e' is not monochromatic, so e is not monochromatic. Thus d^* is an r-coloring of $G \ltimes H$ which r-extends $(V(G), c')$. □

Theorem 2.3.6. *Given fixed integers k and r with $k, r \geq 2$, the k-*Uniform Hypergraph *r-*Coloring Problem *is NP-complete if $k + r \geq 5$.*

Proof. The statement holds for the cases $k = 3$ and $r = 2$ by Theorem 1.2.7, and $k = 2$ and $r \geq 3$ by Theorem 1.0.1. By Theorem 2.3.4, if the k-Uniform Hypergraph r-Coloring Problem is NP-complete, then the $(k+1)$-Uniform Hypergraph r-Coloring Problem is NP-complete. □

Now we are ready to prove our main results.

Theorem 1.2.8. *Let k, r and s be positive integers with $k, r \geq 2$. The k-*Bounded Hypergraph *r-*Coloring Problem*, the k-*Bounded Hypergraph *r-*Precoloring Extension Problem *as well as the k-*Uniform Hypergraph *r-*Precoloring Extension Problem*, restricted to hypergraphs G with $\nu(G) \leq s$, are polynomial-time solvable if*

- *$s \leq r - 1$, or*
- *$k = 3$ and $r = 2$, or*
- *$k = 2$,*

and NP-complete otherwise.

Proof of Theorem 1.2.8. The first and second polynomial-time solvable cases follow from Theorem 2.2.1 and Theorem 2.1.1 respectively. The third polynomial-time solvable case follows from Theorem 1.2.2, as a graph G with $\nu(G) \leq s$ is guaranteed to be $(s+1)P_2$-free. Combining Theorem 2.3.6 with either Theorem 2.3.3 or Theorem 2.3.5, we have completed the dichotomies. □

Theorem 1.2.9. *Let k, r and s be positive integers with $k, r \geq 2$. The k-*Uniform Hypergraph r-Coloring Problem *restricted to hypergraphs G with $\nu(G) \leq s$ is polynomial-time solvable if*

- $s \leq r - 1$, *or*
- $k = 3$ *and* $r = 2$, *or*
- $k = 2$,

and is NP-complete if

- $s \geq (r-1)k + 1$, *and*
- $k \geq 4$ *or* $r \geq 3$.

Proof of Theorem 1.2.9. The first and second polynomial-time solvable cases follow from Theorem 2.2.1 and Theorem 2.1.1 respectively. The third polynomial-time solvable case follows from Theorem 1.2.2, as a graph G with $\nu(G) \leq s$ is guaranteed to be $(s+1)P_2$-free. The NP-completeness result comes from Theorems 2.3.6 and 2.3.4. □

2.4 Stable Set

In this section, we consider the complexity of stable set problems in hypergraphs with bounded matching number. We recall that a set $S \subseteq V(G)$ of G is *stable* if $e \cap S \neq e$ for every $e \in E(G)$. A stable set is *maximal* if it is maximal with respect to inclusion. A stable set is *maximum* if it is a stable set of maximum cardinality. The k-Uniform Hypergraph Maximum Weight Stable Set Problem is the following: Given a k-uniform hypergraph G and a weight function $w : V(G) \to \mathbb{R}_{\geq 0}$, compute a stable set $S \subseteq V(G)$ with $w(S)$ maximized. When all weights are 1, this is called the k-Uniform Hypergraph Maximum Stable Set Problem.

For graphs, the Graph Maximum Weight Stable Set Problem can be solved in polynomial time if the maximum size of an induced matching is bounded:

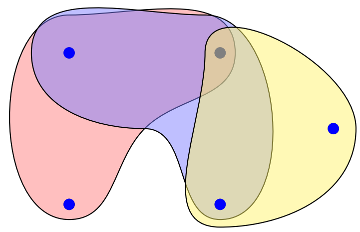

Figure 2.2: A stable set in a 3-uniform hypergraph (blue vertices).

Theorem 2.4.1 (Balas and Yu [5])**.** *For a fixed positive integer s, the* Graph Maximum Weight Stable Set Problem *restricted to sP_2-free graphs can be solved polynomial time.*

For hypergraphs, we notice that:

Theorem 2.4.2. *For fixed positive integers k and s, the k-*Uniform Hypergraph Maximum Stable Set Problem *restricted to hypergraphs with $\nu(G) \leq s$ is polynomial-time solvable.*

Proof. Let G be a k-uniform hypergraph with $\nu(G) \leq s$, and let n be the number of vertices of G. Let $F \subseteq E(G)$ be a maximal matching. We have $|F| \leq s$. The set $V(G) \setminus (\cup_{e \in F} e)$ is stable as F is maximal, and $|V(G) \setminus (\cup_{e \in F} e)| \geq n - ks$. Thus, a maximum stable set of G is of size at least $n - ks$.

Therefore, to find a maximum stable set, we can simply enumerate all choices of a set $U \subseteq V(G)$ with $|U| \leq ks$, and check if the set $V(G) \setminus U$ is stable, and return the largest stable set found this way. There are n^{ks} choices of the set U, and for each U, it takes time $O(n^k)$ to verify stability. Thus, the total running time is $O(n^{ks+k})$. □

In contrast, we will show the following result for the weighted version of the problem:

Theorem 2.4.3. *For a fixed positive integer $k \geq 3$, the k-*Uniform Hypergraph Maximum Weight Stable Set Problem *restricted to hypergraphs with $\nu(G) \leq 1$ is NP-complete.*

In order to prove Theorem 2.4.3, we need the following results. Recall the theorem:

Theorem 1.3.7. *The* Maximum Stable Set Problem *is NP-complete.*

Lemma 2.4.4. *For a fixed positive integers $k \geq 3$, if the $(k-1)$-*Uniform Hypergraph Maximum Weight Stable Set Problem *is NP-complete, then the k-*Uniform Hypergraph Maximum Weight Stable Set Problem *restricted to hypergraphs with $\nu(G) \leq 1$ is NP-complete.*

Proof. Suppose the $(k-1)$-Uniform Hypergraph Maximum Weight Stable Set Problem is NP-complete. Let G be a $(k-1)$-uniform hypergraph with weight function w. We construct a new k-uniform hypergraph H with $V(H) = \{v\} \cup V(G)$ and $E(H) = \{\{v\} \cup e : e \in E(G)\}$. We define the weight function $w' : V(G) \to \mathbb{R}_{\geq 0}$ such that $w'(u) = w(u)$ for each $u \in V(G)$ and $w'(v) = \sum_{u \in V(G)} w(u) + 1$. From the construction, since v is contained in every edge of H, it follows that the hypergraph H satisfies $\nu(H) \leq 1$.

For a set $T \subseteq V(G)$, T is a stable set of G if and only if $T \cup \{v\}$ is a stable set of H. Let S be a maximum weight stable set of H with respect to the weight function w'. By the construction, the vertex v is in S. It follows that $S \setminus \{v\}$ is a maximum weight stable set of G, and thus, to find a maximum weight stable set of G, it suffices to find a maximum weight stable set of H.

Since the construction can be done in polynomial time, we have proved this lemma. □

Proof of Theorem 2.4.3. We prove this by induction on k. When $k = 2$, by Theorem 1.3.7, the Graph Maximum Stable Set Problem is NP-complete. Thus, the Graph Maximum Weight Stable Set Problem is NP-complete.

Suppose that the k-Uniform Hypergraph Maximum Weight Stable Set Problem is NP-complete. By Lemma 2.4.4, the $(k+1)$-Uniform Hypergraph Maximum Weight Stable Set Problem restricted to hypergraphs with $\nu(G) \leq 1$ is NP-complete. Moreover, the $(k+1)$-Uniform Hypergraph Maximum Weight Stable Set Problem is NP-complete. □

2.5 Excluding an induced subhypergraph with one edge

For $t \in \mathbb{N}$ with $t \geq 3$, let H_t be the 3-uniform hypergraph with $t+3$ vertices and one edge. In this section, we will give a polynomial-time algorithm for 2-coloring 3-bounded H_t-free hypergraphs.

Lemma 2.5.1. *Let $t \in \mathbb{N}$, and let G be a 3-bounded H_t-free hypergraph. There is a polynomial-time algorithm to test if G has a 2-coloring with at least t vertices of each color.*

Proof. We may assume that $|e| \geq 2$ for all $e \in E(G)$, since G is not 2-colorable otherwise. Let $\mathcal{C}$ be a partial 2-coloring collection containing all partial 2-colorings $(X \cup Y, c')$ for every pair of disjoint sets $X, Y \subseteq V(G)$ with $|X| = |Y| = t$ and X, Y stable, and $c' : X \cup Y \to [2]$ with $c'(v) = 1$ for all $v \in X$ and $c'(v) = 2$ otherwise. It suffices to show that $\mathcal{C}$ has the following three properties.

(1) *$\mathcal{C}$ is 2-equivalent to the collection of all 2-colorings of G with at least t vertices of each color.*

We only need to show that if G has a 2-coloring c with at least t vertices of each color, then there exists a partial 2-coloring in $\mathcal{C}$ which 2-extends to G. Let c be a 2-coloring of G such that $|c^{-1}(i)| \geq t$ for all $i \in [2]$. Let X and Y be subsets of $c^{-1}(1)$ and $c^{-1}(2)$ respectively, with $|X| = |Y| = t$. We have $(X \cup Y, c|_{X \cup Y}) \in \mathcal{C}$, and c is a 2-precoloring extension of $(X \cup Y, c|_{X \cup Y})$ to $V(G)$. This proves (1).

(2) *$\mathcal{C}$ can be computed in time $O(n^{2t+3})$, where $n = |V(G)|$.*

Since G is 3-bounded, $|E(G)| \leq O(n^3)$. By construction, we have $|\mathcal{C}| \leq O(n^{2t})$. Constructing the sets X, Y and the corresponding partial 2-coloring c takes time $O(n^{2t})$. Checking whether $(X \cup Y, c)$ is a partial 2-coloring takes time $O(n^3)$. Thus, $\mathcal{C}$ can be constructed in time $O(n^{2t+3})$. This proves (2).

(3) *For every partial 2-coloring $(X \cup Y, c)$ in $\mathcal{C}$, whether c 2-extends to G can be decided in polynomial time.*

For convenienve, let us denote $S = X \cup Y$. We define a 2-SAT instance as follows. For every $v \in V(G) \setminus S$, we have a variable x_v. Let $E' \subseteq E(G)$ be the set of edges $e \in E(G)$ with $|c(e \cap S)| = 1$. Note that for every edge $e \in E'$, we have $e \cap S \neq \emptyset$ and $e \setminus S \neq \emptyset$ (since (S, c) is a partial 2-coloring). Thus, $|e \setminus S| \in \{1, 2\}$. For every edge $e \in E'$, we create a clause C_e. Let $u, w \in e \setminus S$ with $u \neq w$ with $|e \setminus S| = 2$. If $c(e \cap S) = \{1\}$, we set $C_e = x_u \vee x_w$. Otherwise, let $C_e = \overline{x_u} \vee \overline{x_w}$. Next, let E'' be the set of edges $e \in E(G)$ with $|e| = 2$ and $e \cap S = \emptyset$. For every $e \in E''$, say $e = \{u, w\}$, we add two clauses $C'_e = x_u \vee x_w$ and $C''_e = \overline{x_u} \vee \overline{x_w}$.

If the 2-SAT instance has a solution $(s_v)_{v \in V(G) \setminus S}$, where "true"and "false"are represented by 1 and 0 respectively, then we set $d(v) = s_v + 1$ for every $v \in V(G) \setminus S$, and

$d(v) = c(v)$ for all $v \in S$. We claim that d is a 2-coloring of G. Consider an edge $e \in E(G)$. If $|c(e \cap S)| > 1$, then e is not monochromatic. If $|c(e \cap S)| = 1$, then $e \in E'$. It follows that the clause C_e of 2-SAT instance and the construction of d guarantees that at least one vertex in $e \setminus S$ receives the opposite color from the vertices in $e \cap S$. Since both sets are non-empty, it follows that e is not monochromatic. It remains to consider the case that $e \cap S = \emptyset$. If $|e| = 2$, then the clauses C'_e and C''_e guarantee that the two vertices of e receive different colors. Therefore, we may assume that $|e| = |e \setminus S| = 3$. Suppose for a contradiction that e is monochromatic. Without loss of generality, assume $d(v) = 1$ for all $v \in e$. Let $X = (S \cap c^{-1}(1))$, and consider the set $X \cup e$. Since all edges with a non-empty intersection with S and all edges of size 2 are non-monochromatic, there is no edge $e' \in E(G)$ with $e' \subseteq X \cup e$ and $e' \neq e$. Thus, $G[X \cup e]$ is an induced copy of H_t in G, which contradicts the fact that G is H_t-free. Therefore, d is a 2-precoloring extension of (S, c).

If there is a 2-precoloring extension d of (S, c), then we set $x_v = d(v) - 1$ for every $v \in V(G) \setminus S$. For every edge $e \in E'$, if $C_e = x_u \vee x_w$, then $e \cap S$ contains only vertices colored 1, and so $d(u) = 2$ or $d(w) = 2$; it follows that C_e is satisfied. If $C_e = \overline{x_u} \vee \overline{x_w}$, then $e \cap S$ contains only vertices colored 2, and so $d(u) = 1$ or $d(w) = 1$; it follows that C_e is satisfied. For every edge $e = \{u, w\} \in E''$, it follows that $d(u) \neq d(w)$, and hence one of x_u, x_w is "true" and the other is "false." It follows that C'_e and C''_e are satisfies. From the construction of clauses of this 2-SAT instance, we conclude that this assignment is a solution to the 2-SAT instance. Therefore, deciding whether (S, c) has a 2-coloring extension is equivalent to solving the 2-SAT instance defined above.

It remains to show that this can be done in polynomial time. Constructing the 2-SAT instance takes time $O(n^3)$. Solving this 2-SAT instance takes time $O(n^3)$. So the total running time is $O(n^3)$. This proves (3) and concludes the proof. □

Theorem 2.5.2. *Let $t \in \mathbb{N}$, and let G be a 3-bounded H_t-free hypergraph. There is a polynomial-time algorithm which takes G as input, and outputs either a 2-coloring of G, or a determination that G is not 2-colorable.*

Proof. If G satisfies the conditions of Lemma 2.5.1, then we are done. Otherwise we can go through every possible coloring such that less than t vertices receive color i for some $i \in [2]$, and check whether it is a 2-coloring, in time $O(n^{t+3})$. □

Note that the proof of Lemma 2.5.1 can be modified to work for the precoloring extension version of the problem, and so can Theorem 2.5.2.

2.6 Linear Hypergraphs

2.6.1 The polynomial-time algorithm

In this subsection, we will use the hypergraph Ramsey number. Recall the theorem:

Theorem 1.3.1. *For all positive integers k, n_1, $\dots$, n_t, the hypergraph Ramsey number $R_k(n_1, \dots, n_t)$ exists.*

Lemma 2.6.1. *For every positive integer s, there exists a positive integer s' such that every 3-uniform linear hypergraph G which contains a matching of size s' contains an induced matching of size s.*

Proof. We may assume that $s \geq 4$. Let $X = \{G_1, \dots, G_t\}$ be the set of all linear 3-uniform hypergraphs with vertex set $\{x_1, \dots, x_9\}$. Since there at most $2^{\binom{9}{3}}$ distinct 3-uniform (labelled) hypergraphs on 9 vertices, it follows that $t \leq 2^{\binom{9}{3}}$. Let $s' = R_3(n_1, \dots, n_t)$ with $n_1 = \dots = n_t = s$.

Let $\{e_1, \dots, e_{s'}\}$ be a matching of size s' in G. For $i \in [s']$, let $e_i = \{u_i, v_i, w_i\}$. Let H be a complete 3-uniform hypergraph $V(H) = \{1, \dots, s'\}$. We define $f : E(H) \to [t]$ as follows. For $e = \{i, j, k\} \subseteq [s']$ with $i < j < k$, we define $f(e)$ =m if $G[e_i \cup e_j \cup e_k]$ is isomorphic to G_m via the isomorphism $u_i \mapsto x_1$, $v_i \mapsto x_2$, $w_i \mapsto x_3$, $u_j \mapsto x_4$, $v_j \mapsto x_5$, $w_j \mapsto x_6$, $u_k \mapsto x_7$, $v_k \mapsto x_8$ and $w_k \mapsto x_9$.

From Theorem 1.3.1, it follows that there is a set $S \subseteq [s']$ with $|S| = s$ and $m \in [t]$ such that $f(e) = m$ for all $e \subseteq S$. We claim that

$$E(G_m) = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8, x_9\}\}.$$

Let $i, j, k, l \in S$ with $i < j < k < l$. Since $G[e_i, e_j, e_k]$ contains the edges e_i, e_j, e_k, it follows that $\{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8, x_9\}\} \subseteq E(G_m)$. Suppose for a contradiction that $E(G_m)$ contains a fourth edge $\{x_a, x_b, x_c\}$. Then, since G_m is linear, we may assume that $a \in \{1, 2, 3\}$, $b \in \{4, 5, 6\}$, and $c \in \{7, 8, 9\}$. The graphs $G[e_i \cup e_j \cup e_k]$ and $G[e_i \cup e_j \cup e_l]$ are isomorphic to G_m via isomorphisms φ, φ', say; and from the definition of f it follows that $\varphi^{-1}(x_a) = \varphi'^{-1}(x_a)$, $\varphi^{-1}(x_b) = \varphi'^{-1}(x_b)$, and $\varphi^{-1}(x_c) \neq \varphi'^{-1}(x_c)$ (since $\varphi^{-1}(x_c) \in e_k$ and $\varphi'^{-1}(x_c) \in e_l$ and $e_k \cap e_l = \emptyset$). But this implies that G contains the edges $\varphi^{-1}(\{x_a, x_b, x_c\})$ and $\varphi'^{-1}(\{x_a, x_b, x_c\})$ which have exactly two vertices in common, contrary to the assumption that G is linear. This proves our claim.

It follows that $G\left[\bigcup_{s \in S} e_s\right]$ is an induced matching of size s in G. $\square$

Theorem 2.6.2. *For all s, the* 2-Precolouring Extension Problem *restricted to 3-uniform linear hypergraphs with no induced matching of size at least s can be solved in polynomial time.*

Proof. By Theorem 2.2.1 and Lemma 2.6.1. □

2.6.2 NP-hardness of 3-coloring with bounded matching number

In this section, we will prove the following result.

Theorem 1.2.15. *The* 3-Uniform Hypergraph 3-Coloring Problem *restricted to linear hypergraphs G with $\nu(G) \leq 532$ is NP-complete.*

We will use the following theorems.

Theorem 1.3.8. *The* 3-Coloring Problem *restricted to graphs with maximum degree at most 4, is NP-complete.*

Theorem 1.3.4. *There is a $O(mn)$-algorithm for edge-coloring a graph G with $D+1$ colors, where D is the maximum degree of G, m is the number of edges and n is the number of vertices.*

Let us introduce a new way to describe 3-uniform hypergraphs. Instead of using edges with three vertices, we use 2-edges labeled with vertices. Given a graph G, we say a function $l : E(G) \to V(G)$ with $l(e) \notin e$ for all $e \in E(G)$ is a *labeling* of G. The vertex $l(e)$ is called the *label* of e, and the edge e is a *labeled edge*.

For a linear 3-uniform hypergraph G, let $l : E(G) \to V(G)$ be a function with $l(e) \in e$ for all $e \in E(G)$. Let G' be the graph with vertex set $V(G)$ and edge set $\{\{e \setminus \{l(e)\} : e \in E(G)\}$, and let $l'(e \setminus \{l(e)\}) = l(e)$. Since G is linear, each edge of G' corresponds to a unique edge of G, and thus l' is well-defined. We call (G', l') a *labeled graph representation* of G. Notice that with a labeled graph representation, we can reconstruct the corresponding linear 3-uniform hypergraph.

In this section, all of the pictures of 3-uniform hypergraphs are drawn using the labeled graph representation.

The following two lemmas give constructions for gadgets we will use in our NP-hardness reduction. The existence of similar gadgets in 3-uniform linear hypergraphs was first proved in [36]. Here we give an explicit construction to obtain a precise bound for the matching number. The construction is shown in Figure 2.3.

Lemma 2.6.3. *There is a linear 3-uniform hypergraph G_1 with three specified vertices a, b, c with the following properties:*

- *For every 3-coloring f of G_1, either $f(a), f(b), f(c)$ are all distinct, or $f(a) = f(b) = f(c)$.*
- *There is a 3-coloring f' of G_1 with $f(a), f(b), f(c)$ all distinct.*
- *There is a set $Z \subseteq V(G_1)$ with $|Z| \leq 19$ such that $G_1 \backslash Z$ has no edges, and $a, b, c \in Z$.*
- *No edge e of G_1 contains more than one of the vertices a, b, c.*

Proof. We want to define G_1 using the labeled graph representation (G_1', l) (see Figure 2.3). First, we create three vertices a, b, c. Then we create 4 copies of K_4, say H_1, H_2, H_3, H_4. For $i \in [4]$, let $V(H_i) = \{s_i, t_i, u_i, v_i\}$. We define the labeling $l(s_i t_i) = l(u_i v_i) = a$, $l(s_i u_i) = l(t_i v_i) = b$, and $l(s_i v_i) = l(t_i u_i) = c$.

Let $S = V(H_1) \times V(H_2) \times V(H_3) \times V(H_4)$. For every 4-tuple $T = (x, y, z, w) \in S$, we create 5 new copies of K_4, say $H_0^T, H_1^T, H_2^T, H_3^T, H_4^T$. Let $V(H_i^T) = \{s_i^T, t_i^T, u_i^T, v_i^T\}$ for $i \in [4]$, and $V(H_0^T) = \{r_1^T, r_2^T, r_3^T, r_4^T\}$. We define the labeling $l(s_i^T t_i^T) = l(u_i^T v_i^T) = a$, $l(s_i^T u_i^T) = l(t_i^T v_i^T) = b$ and $l(s_i^T v_i^T) = l(t_i^T u_i^T) = c$ for $i \in [4]$, and $l(r_1^T r_2^T) = l(r_3^T r_4^T) = a$, $l(r_1^T r_3^T) = l(r_2^T r_4^T) = b$ and $l(r_1^T r_4^T) = l(r_2^T r_3^T) = c$. For each $i \in [4]$, we add edges $s_i^T r_i^T$ with $l(s_i^T r_i^T) = x$, $t_i^T r_i^T$ with $l(t_i^T r_i^T) = y$, $u_i^T r_i^T$ with $l(u_i^T r_i^T) = z$ and $v_i^T r_i^T$ with $l(v_i^T r_i^T) = w$.

Let $V(G_1') = \{a, b, c\} \cup (\cup_{i \in [4]} V(H_i)) \cup (\cup_{T \in S} \cup_{i=0}^{4} V(H_i^T))$, and $E(G_1')$ be the set of all labeled edges defined above. By the construction, the function l defined above is a labeling of G_1'. Notice that there is no edge incident to more than one of the vertices a, b, c, and $l(V(G_1')) = \{a, b, c\} \cup (\cup_{i \in [4]} V(H_i))$. Thus, by taking $Z = l(V(G_1'))$, we have $|Z| \leq 19$ and $a, b, c \in Z$; so Z satisfies the third property of the lemma. We now prove the other properties.

(1) *The 3-uniform hypergraph G_1 is linear.*

Let $X_1 = \{a, b, c\}$, $X_2 = (\cup_{i \in [4]} V(H_i))$ and $X_3 = (\cup_{T \in S} \cup_{i=0}^{4} V(H_i^T))$. From the construction, it follows that for every edge e of G_1, there exist $i, j \in [3]$ with $i < j$ such that e contains one vertex of X_i and two vertices of X_j and with $e \cap X_j \in E(G_1')$ (and therefore, $\{l(e \cap X_j)\} = e \cap X_i$).

Suppose for a contradiction that there exist distinct $e, e' \in E(G_1)$ with $|e \cap e'| = 2$. Let $j, j' \in [3]$ such that $|e \cap X_j| = 2$ and $|e' \cap X_{j'}| = 2$. It follows that $j = j'$. Since G_1' is

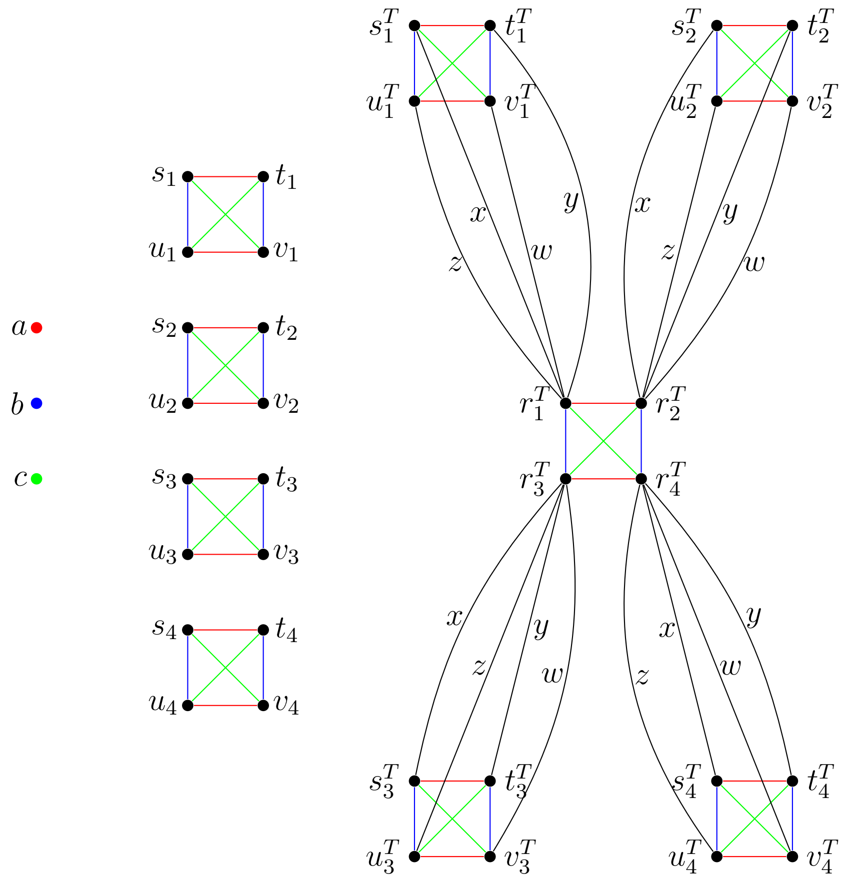

Figure 2.3: The construction from Lemma 2.6.3. The colored edge means the label of this edge is the vertex of the corresponding color. The right-hand side shows $H_0^T, \ldots, H_4^T$ for $T = (x, y, z, w)$.

simple, we have $e \cap X_j \neq e' \cap X_j$, and so $e \setminus X_j = e' \setminus X_j = \{l(e \cap X_j)\} = \{l(e' \cap X_j)\}$. But in G_1', every two edges with the same label are not incident to a common vertex, a contradiction. We conclude that G_1 is linear. This proves (1).

(2) *There is a 3-coloring f' of G_1 with $f'(a), f'(b), f'(c)$ all distinct.*

We define a function $f' : (V(G_1)) \to [3]$ as follows. Let $f'(a) = 1$, $f'(b) = 2$ and $f'(c) = 3$. For each $i \in [4]$, let $f'(s_i) = f'(t_i) = 2$ and $f'(u_i) = f'(v_i) = 3$. Since $l(s_i t_i) = l(u_i v_i) = a$, the edges of G_1 corresponding to labeled edges in $G_1'[V(H_i)]$ are not monochromatic.

For each $T \in S$ and each $i \in [4]$, let $f'(s_i^T) = f'(u_i^T) = 1$, $f'(t_i^T) = f'(v_i^T) = 3$, $f'(r_1^T) = f'(r_4^T) = 1$, and $f'(r_2^T) = f'(r_3^T) = 2$. For $i \in [4]$, no vertex $v \in V(H_i^T)$ has $f'(v) = 2$, and no edge between $V(H_i^T)$ and $V(H_0^T)$ is labeled a. So there is no monochromatic edge e in G_1 with $e \cap \cup_{i=0}^4 V(H_i^T) \neq \emptyset$. Therefore, the function f' is a 3-coloring of G_1. This proves (2).

(3) *For each 3-coloring f of G_1, either $f(a), f(b), f(c)$ are all distinct, or $f(a) = f(b) = f(c)$.*

Assume for a contradiction that, without loss of generality, there is a 3-coloring f of G_1 such that $f(a) = f(b)$. Without loss of generality, we may assume that $f(a) = f(b) = 1$ and $f(c) = 2$.

We claim that there exists $x_0 \in V(H_1)$ such that $f(x_0) = 3$. Assume for a contradiction that every vertex $v \in V(H_1)$ has $f(v) \neq 3$. Since $l(s_1 v_1) = c$ and $f(c) = 2$, without loss of generality let $f(s_1) \neq 2$. So $f(s_1) = 1$. Since $l(s_1 t_1) = a$, $l(s_1 u_1) = b$ and $f(a) = f(b) = 1$, we have $f(t_1) = f(u_1) = 2$. But the edge $t_1 u_1$ is labeled c and $f(c) = 2$, the corresponding edge $\{t_1, u_1, c\}$ of G_1 is monochromatic, which violates the condition that f is a 3-coloring of G_1.

A similar argument holds for every H_i with $i \in \{2, 3, 4\}$, and H_j^T with $T \in S$ and $j \in \{0, 1, \ldots, 4\}$. There exist vertices $y_0 \in V(H_2)$, $z_0 \in V(H_3)$, $w_0 \in V(H_4)$ such that $f(y_0) = f(z_0) = f(w_0) = 3$. Let $T = (x_0, y_0, z_0, w_0)$. By the argument above, there is a $j \in [4]$ such that $f(r_j) = 3$. Since there is a vertex $v \in V(H_j^T)$ with $f(v) = 3$, and $f(l(vr_j)) = 3$ (because $l(vr_j) \in \{x_0, y_0, z_0, w_0\}$), the edge $\{v, r_j, l(vr_j)\}$ of G_1 is monochromatic, which contradicts the condition that f is a 3-coloring of G_1. This proves (3). $\square$

Lemma 2.6.4. *There is a linear 3-uniform hypergraph G_2 with specified vertices a, b, c with the following properties:*

- *For every 3-coloring f of G_2, we have $f(a), f(b), f(c)$ all distinct.*
- *G_2 is 3-colorable.*
- *There is a set $Z \subseteq V(G_2)$ with $|Z| \leq 19$ such that $G_2 \backslash Z$ has no edges, and $a, b, c \in Z$.*
- *At most one edge of G_2 contains more than one of the vertices a, b, c.*

Proof. Let G_2 be obtained from G_1 defined in Lemma 2.6.3 by adding the edge $\{a, b, c\}$. The result follows immediately from Lemma 2.6.3. $\square$

Now we are ready to prove Theorem 1.2.15.

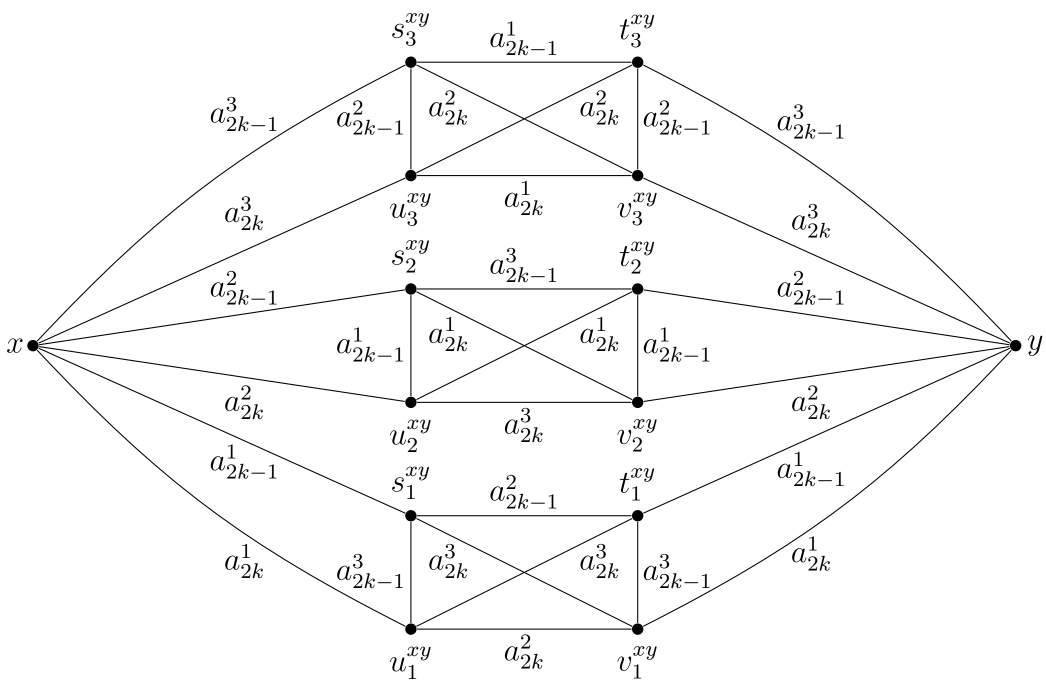

Figure 2.4: The construction of H_3^{xy}, H_2^{xy}, H_1^{xy} (top to bottom) for an edge xy with $f'(xy) = k$.

Proof of Theorem 1.2.15. We give an NP-hardness reduction from the Graph 3-Coloring Problem restricted to graphs with maximum degree at most 4, which is NP-hard by Theorem 1.3.8.

Let G^* be a graph with maximum degree at most 4. Let $f' : E(G^*) \to [5]$ be an edge-coloring of G^*. We construct a labeled graph representation (G', l) of a 3-uniform linear hypergraph G as follows (see Figure 2.4).

We create three sets of vertices $A = \{a_1^1, \ldots, a_{10}^1\}$, $B = \{a_1^2, \ldots, a_{10}^2\}$ and $C = \{a_1^3, \ldots, a_{10}^3\}$. For the vertices a_1^1, a_1^2, a_1^3, we create a new copy of G_2 as defined in Lemma 2.6.4, denoted G^1, with a_1^1, a_1^2, a_1^3 as its specified vertices. For every $i \in \{2, \ldots, 10\}$, we create three new copies of G_1 as defined in Lemma 2.6.3, one with specified vertices a_i^1, a_1^2, a_1^3, one with specified vertices a_1^1, a_i^2, a_1^3 and one with specified vertices a_1^1, a_1^2, a_i^3, respectively. We denote these three hypergraphs $G^{i,1}$, $G^{i,2}$ and $G^{i,3}$ respectively. For convenience, we also define $G^{1,1} = G^{1,2} = G^{1,3} = G^1$.

Next, for all $k \in [5]$ and for each edge $e = xy \in E(G^*)$ with $f'(xy) = k$, we create three copies of K_4, say H_1^e, H_2^e, H_3^e; see Figure 2.4 for a picture of the construction described below. Let $V(H_i^e) = \{s_i^e, t_i^e, u_i^e, v_i^e\}$ for $i \in [3]$. Let $l(s_i^e t_i^e) = a_{2k-1}^{(i+1)}$, $l(s_i^e u_i^e) = l(t_i^e v_i^e) = a_{2k-1}^{(i+2)}$, $l(u_i^e v_i^e) = a_{2k}^{(i+1)}$ and $l(s_i^e v_i^e) = l(t_i^e u_i^e) = a_{2k}^{(i+2)}$ for all $i \in [3]$, where superscripts are read modulo 3, so $a_j^4 = a_j^1$ and $a_j^5 = a_j^2$ for all $j \in [10]$. We also add edges xs_i^e, yt_i^e with $l(xs_i^e) = l(yt_i^e) = a_{2k-1}^i$, and edges xu_i^e, yv_i^e with $l(xu_i^e) = l(yv_i^e) = a_{2k}^i$ for all $i \in [3]$.

Let $\mathcal{G} = \{G^1\} \cup \{G^{i,j} : i \in \{2, \ldots, 10\}, j \in [3]\}$. Let $U = (\cup_{G'' \in \mathcal{G}} V(G'')) \setminus (A \cup B \cup C)$, $W = \cup_{e \in E(G^*)} \cup_{i \in [3]} V(H_i^e)$. Let $V(G') = A \cup B \cup C \cup U \cup W \cup V(G^*)$ and let $E(G')$ be the set of all labeled edges defined above. By the construction, the function l defined above is a labeling of G'. Let G be the corresponding 3-uniform hypergraph of (G', l).

Notice that from the construction, there is no other edge $e \in E(G)$ with $e \cap U \neq \emptyset$ and $e \cap (W \cup V(G^*)) \neq \emptyset$. Furthermore, except for the edge $\{a_1^1, a_1^2, a_1^3\}$, there is no edge $e \in E(G)$ with $e \subseteq A \cup B \cup C \cup V(G^*)$. Moreover, for every edge $e \in E(G) \setminus \{a_1^1, a_1^2, a_1^3\}$, we have $|e \cap (A \cup B \cup C)| \leq 1$. Thus, for each edge $e \in E(G) \setminus \{a_1^1, a_1^2, a_1^3\}$, exactly one of the conditions $|e \cap U| \geq 2$ and $|e \cap (W \cup V(G^*))| = 2$ holds. Moreover, for all $e \in E(G)$, we have that $|e \cap V(G^*)| \leq 1$.

(1) *The 3-uniform hypergraph G is linear.*

We take two edges $e, e' \in E(G)$ with $e \neq e'$. Assume for a contradiction that $|e \cap e'| = 2$. It follows that $e, e' \neq \{a_1^1, a_1^2, a_1^3\}$, since no edge except $\{a_1^1, a_1^2, a_1^3\}$ contains more than one vertex of $A \cup B \cup C$.

If $|e \cap U| \geq 2$, then $e \subseteq G^{a,b}$ for some $a \in [10]$ and $b \in [3]$. Since $|e \cap e'| = 2$, we have that $e' \cap U \neq \emptyset$, and so $|e' \cap U| \geq 2$. It follows that $e' \subseteq V(G^{c,d})$ for some $c \in [\{2, \ldots, 10\}]$ and $d \in [3]$. By Lemma 2.6.3, $(a, b) \neq (c, d)$. But then $V(G^{a,b}) \cap V(G^{c,d}) \subseteq \{a_1^1, a_1^2, a_1^3\}$ and so $e \cap e' \subseteq \{a_1^1, a_1^2, a_1^3\}$. But $|e \cap \{a_1^1, a_1^2, a_1^3\}| \leq 1$, so $|e \cap e'| \leq 1$, which is a contradiction.

If $|e \cap (W \cup V(G^*))| = 2$, then $|e \cap (A \cup B \cup C)| = 1$. Since $|e \cap e'| = 2$ and exactly one of $e' \cap U \neq \emptyset$ and $e' \cap (W \cup V(G^*)) \neq \emptyset$ holds, we have $e' \cap (W \cup V(G^*)) \neq \emptyset$. It follows that $|e' \cap (W \cup V(G^*))| = 2$. Consider the labeled graph G'. Notice that by the construction above, for each $e^* \in E(G^*)$, no two edges of $G'[e^* \cup (\cup_{i\in[3]} V(H_i^{e^*}))]$ with the same label are incident to one common vertex. Thus e and e' are both incident to a common vertex $x \in V(G^*)$. For every $xy_1, xy_2 \in E(G^*)$, since f' is an edge coloring of G^*, $f'(xy_1) \neq f'(xy_2)$. Thus, for every two edges e_1, e_2 of G' incident to x, $|e_1 \cap e_2| = 1$. Hence, we have proved $|e \cap e'| \leq 1$, which leads to a contradiction. This proves (1).

(2) *We have $\nu(G) \leq 532$.*

By Lemmas 2.6.3 and 2.6.4, for every graph $G'' = G^{i,j} \in \mathcal{G}$, there is a set $S_{G''}$ of size at most 19 which contains a_i^j such that $G'' \setminus S_{G''}$ has no edges; for G^1, the set S_{G^1} contains all of a_1^1, a_1^2, a_1^3. Each edge which is not a subset of $A \cup B \cup C \cup U$ contains a vertex in $A \cup B \cup C$. Thus, the set $X = \cup_{G'' \in \mathcal{G}} S_{G''}$ meets all edges of G, and $|X| \leq 19 \cdot 28$. So $\nu(G) \leq 19 \cdot 28 = 532$. This proves (2).

(3) *The graph G^* is 3-colorable if and only if G is 3-colorable.*

Let c' be a 3-coloring of G. By Lemma 2.6.4, $c'(a_1^1)$, $c'(a_1^2)$ and $c'(a_1^3)$ are all distinct. Without loss of generality let $c'(a_1^1) = 1$, $c'(a_1^2) = 2$ and $c'(a_1^3) = 3$. From the construction, by Lemma 2.6.3, $c'(a_i^1) = 1$, $c'(a_i^2) = 2$ and $c'(a_i^3) = 3$ for all $i \in [10]$. We want to prove that $c'|_{V(G^*)}$ is a 3-coloring of G^*.

Suppose for a contradiction that there exists an edge $xy \in E(G^*)$ with $c'(x) = c'(y)$. Let $k = f'(xy)$. Without loss of generality, let $c'(x) = c'(y) = 1$. Then consider the graph H_1^{xy}. Because of the edges $\{x, s_1^{xy}, a_{2k-1}^1\}$, $\{x, u_1^{xy}, a_{2k}^1\}$, $\{y, t_1^{xy}, a_{2k-1}^1\}$ and $\{y, v_1^{xy}, a_{2k}^1\}$, all of the vertices $s_1^{xy}, t_1^{xy}, u_1^{xy}, v_1^{xy}$ are colored 2 or 3. Since $c'(a_{2k-1}^3) = 3$, from the edge $\{s_1^{xy}, u_1^{xy}, a_{2k-1}^3\}$, it follows that one of the vertices s_1^{xy}, u_1^{xy} is not colored 3. Without loss of generality let $c'(s_1^{xy}) = 2$. Because of the edge $\{s_1^{xy}, t_1^{xy}, a_{2k-1}^2\}$, we have $c'(t_1^{xy}) = 3$. Consider the edges $\{t_1^{xy}, u_1^{xy}, a_{2k}^3\}$ and $\{t_1^{xy}, v_1^{xy}, a_{2k-1}^3\}$. Since $c'(a_{2k}^3) = c'(a_{2k-1}^3) = 3$, we have $c'(u_1^{xy}) = c'(v_1^{xy}) = 2$. But then the edge $\{u_1^{xy}, v_1^{xy}, a_{2k}^2\}$ is monochromatic, which contradicts the fact that c' is a 3-coloring of G. This proves that if G is 3-colorable, then so is G^*.

For the converse direction, let c be a 3-coloring of G^*. We want to define a 3-coloring d of G. Let $d(v) = 1$ for all $v \in A$, $d(v) = 2$ for all $v \in B$, and $d(v) = 3$ for all $v \in C$. By Lemmas 2.6.3 and 2.6.4, there is a way to extend d to $G[A \cup B \cup C \cup U]$.

Let $d(v) = c(v)$ for all $v \in V(G^*)$. For each edge $xy \in E(G^*)$ and each $i \in [3]$, since c

is a 3-coloring of G^*, one of the vertices x, y is not colored i. If $c(x) \neq i$, then for the set $V(H_i^{xy})$, we set $d(s_i^{xy}) = d(u_i^{xy}) = i$ and $d(t_i^{xy}) = d(v_i^{xy}) = i + 1$, reading colors modulo 3 (so if this would assign color 4, we assign color 1 instead). If $c(x) = i$, then $c(y) \neq i$, and for the set $V(H_i^{xy})$, we set $d(s_i^{xy}) = d(u_i^{xy}) = i + 1$, again reading colors modulo 3; and $d(t_i^{xy}) = d(v_i^{xy}) = i$. Thus, we have defined the function d for all vertices of G.

We then want to show that d is a 3-coloring of G. From the construction, all edges e with $e \cap U \neq \emptyset$ are contained in $G[A \cup B \cup C \cup U]$ and hence not monochromatic. It remains to consider edges $e \in E(G)$ with $e \cap W \neq \emptyset$. It follows that there is an edge $xy \in E(G^*)$ and $i \in [3]$ such that $\emptyset \neq e \cap V(H_i^{xy}) = e \cap W$. If $x \in e$, then either $s_i^{xy} \in e$ or $t_i^{xy} \in e$ and from the construction of d, we have that $d(e \cap (A \cup B \cup C)) = \{i\}$, and either $d(x) \neq i$ or $d(s_i^{xy}), d(t_i^{xy}) \neq i$. The case $y \in e$ follows analogously. Therefore, we may assume that $|e \cap V(H_i^{xy})| = 2$. Now either the two vertices in $e \cap V(H_i^{xy})$ receive different colors, or they receive the same color in $\{i, i+1\}$ and $d(e \cap (A \cup B \cup C)) = \{i + 2\}$. Thus, the edge e is not monochromatic. This proves (3).

(4) *The 3-hypergraph G can be constructed from G^* in time $O(n^3)$, where $n = |V(G^*)|$.*

Since $|V(G_1)| = O(1)$ and $|E(G_1)| = O(1)$, the 3-uniform hypergraph G_1 can be constructed in time $O(1)$. Similarly, the 3-uniform hypergraph G_2 can be constructed in time $O(1)$. We create $3 \cdot 10 - 2 = 28$ copies of the gadgets G_1 or G_2. This step can be done in time $O(1)$.

Let $n = |V(G^*)|$, and $m = |E(G^*)|$. The edge coloring f' of G^* can be computed in time $O(mn) \leq O(n^3)$ by Theorem 1.3.4. For each edge $e \in E(G^*)$, we create 12 new vertices and 30 edges. Thus, constructing the vertex set W and all edges incident to W takes time $O(n^2)$.

□

Chapter 3

Ordered Graphs

In this chapter, we turn to ordered graphs. We recall that an *ordered graph* G is a triple (V, E, φ) such that (V, E) is a graph with vertex set V and edge set E, and $\varphi : V \to \mathbb{Z}$ is an injective function. For convenience, we also write the ordered graph (V, E, φ) as (G', φ) where $G' = (V, E)$ is a graph.

As promised in Introduction, we now give the full list of all ordered graphs we will use. Let $U' = \{u_1, u_2, u_3, u_4, u_5\}$ and $U = U' \backslash \{u_5\}$, the ordering $\varphi' : U' \to \mathbb{R}$ with $u_i \mapsto i$ for $i \in [5]$.

- Let $J_1 = (U, \{u_1u_2, u_2u_3, u_3u_4\}, \varphi'|_U)$.
- Let $J_2 = (U, \{u_1u_2, u_2u_4, u_3u_4\}, \varphi'|_U)$.
- Let $J_3 = (U, \{u_1u_3, u_2u_3, u_2u_4\}, \varphi'|_U)$.
- Let $J_4 = (U, \{u_1u_3, u_2u_4, u_3u_4\}, \varphi'|_U)$.
- Let $J_5 = (U, \{u_1u_4, u_2u_3, u_2u_4\}, \varphi'|_U)$.
- Let $J_6 = (U, \{u_1u_4, u_2u_3, u_3u_4\}, \varphi'|_U)$.
- Let $J_7 = (U, \{u_1u_2, u_1u_4, u_3u_4\}, \varphi'|_U)$.
- Let $J_8 = (U, \{u_1u_3, u_1u_4, u_2u_4\}, \varphi'|_U)$.
- Let $J_9 = (U, \{u_1u_2, u_3u_4\}, \varphi'|_U)$.

- Let $J_{10} = (U, \{u_1u_2, u_1u_4\}, \varphi'|_U)$.
- Let $J_{11} = (U, \{u_1u_3, u_1u_4\}, \varphi'|_U)$.
- Let $J_{12} = (U, \{u_1u_2, u_2u_4\}, \varphi'|_U)$.
- Let $J_{13} = (U', \{u_1u_5, u_2u_3, u_3u_4\}, \varphi')$.
- Let $J_{14} = (U', \{u_1u_5, u_2u_3, u_2u_4\}, \varphi')$.
- Let $J_{15} = (U\backslash\{u_4\}, \{u_1u_2, u_2u_3\}, \varphi'|_{U\backslash\{u_4\}})$.
- Let $J_{16} = (U\backslash\{u_4\}, \{u_1u_2, u_1u_3\}, \varphi'|_{U\backslash\{u_4\}})$.

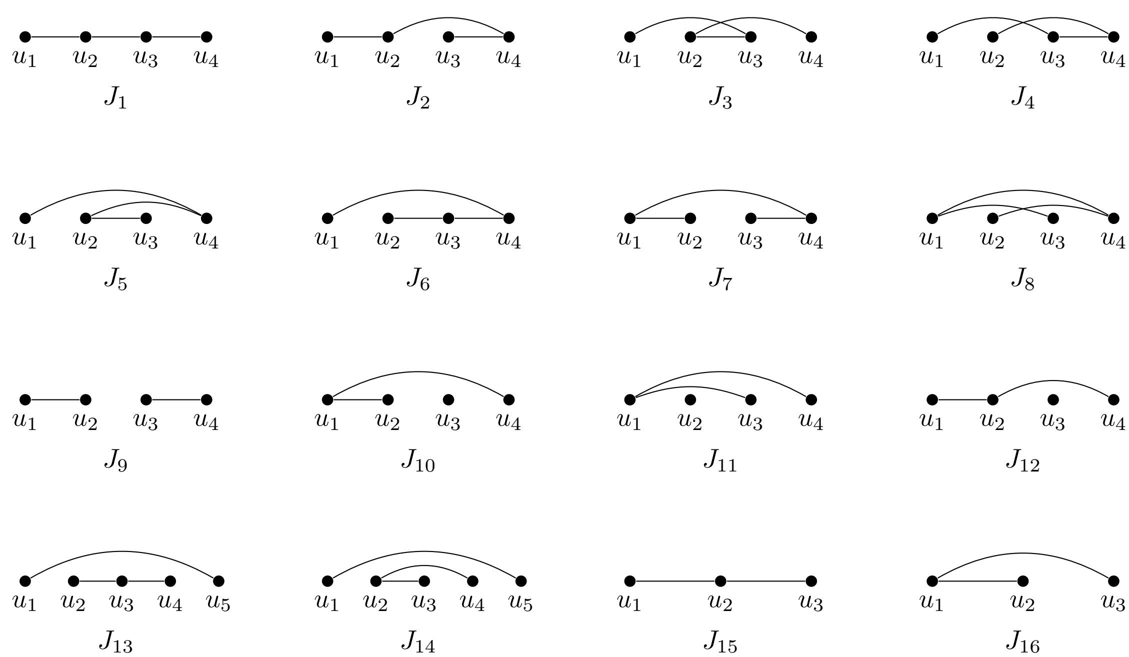

Figure 3.1: The ordered graphs J_i for $i \in [16]$.

Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $\varphi : V \to \mathbb{R}$ with $v_i \mapsto i$ for $i \in [6]$.

- Let $M_1 = (V, \{v_1v_6, v_2v_5\}, \varphi)$.
- Let $M_2 = (V, \{v_1v_6, v_2v_5, v_3v_4\}, \varphi)$.

- Let $M_3 = (V, \{v_1v_4, v_2v_5, v_3v_6\}, \varphi)$.
- Let $M_4 = (V, \{v_1v_5, v_2v_4, v_3v_6\}, \varphi)$.
- Let $M_5 = (V\backslash\{v_6\}, \{v_1v_5, v_2v_3\}, \varphi|_{V\backslash\{v_6\}})$.
- Let $M_6 = (V\backslash\{v_5, v_6\}, \{v_1v_3, v_2v_4\}, \varphi|_{V\backslash\{v_5,v_6\}})$.
- Let $M_7 = (V\backslash\{v_5, v_6\}, \{v_1v_4, v_2v_3\}, \varphi|_{V\backslash\{v_5,v_6\}})$.
- Let $M_8 = (V\backslash\{v_6\}, \{v_1v_5, v_2v_4\}, \varphi|_{V\backslash\{v_6\}})$.

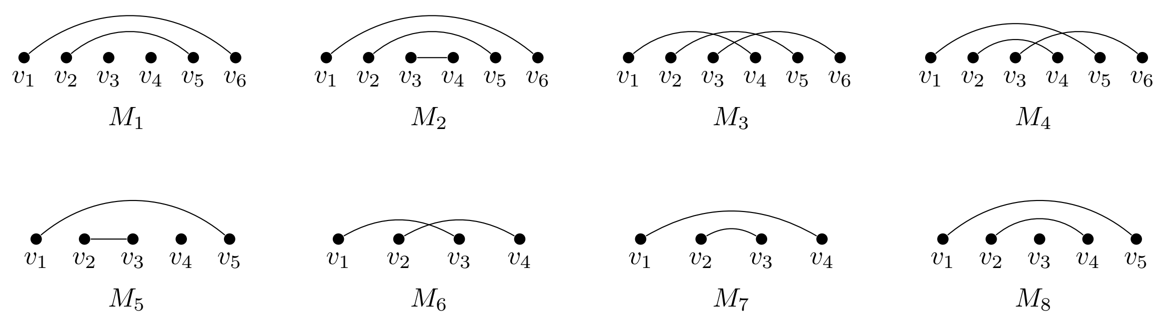

Figure 3.2: The ordered graphs M_i for $i \in [8]$.

Here are some terms and notations we will use in this chapter. Let G be an ordered graph. For $X \subseteq V$, we denote $G[X] = (X, \{e \in E : e \subseteq X\}, \varphi|_X)$. For $x, y \in \mathbb{Z}$ with $x < y$, we denote $G[x : y] = G[\{v \in V : x \leq \varphi_G(v) \leq y\}]$, and $-G = (V, E, v \mapsto -\varphi_G(v))$. For a vertex $v \in V$, the set of *forward neighbors* of v is defined as $N^+(v) = \{u \in N(v) : \varphi(v) < \varphi(u)\}$, and the set of *backward neighbors* of v is $N^-(v) = \{u \in N(v) : \varphi(v) > \varphi(u)\}$. We say two disjoint sets $U, W \subseteq V$ is *anticomplete* if no vertex in U has a neighbor in W.

In section 3.1, we will give a polynomial-time algorithm for list-3-coloring $J_{16}(k, l)$-free graphs. In section 3.2, we will prove except for the open cases, forbidding other ordered graphs from the list is still NP-hard.

3.1 Algorithm for $J_{16}(k, l)$-free ordered graphs

We noticed that a J_{16}-free ordered graph is chordal. Given a $J_{16}(k, l)$-free ordered graph G, if we can find a way to get rid of these $k + l$ isolated vertices and get a J_{16}-free ordered graph G', then we only need to consider coloring a chordal graph. Indeed, this is the main idea of the coloring algorithm in this section. What we do is, first we "guess"the set of first k and last l vertices colored i for each color $i \in [3]$. For those remaining vertices which are not adjacent to these "guessed"vertices, we then use some properties and known results of chordal graphs to finish the coloring.

We start by introducing some terminology. Let (G, L) be an instance of the Ordered Graph List-3-Coloring Problem. An instance (G', L') is a (G, L)-*refinement* if G' is an induced subgraph of G and for all $v \in V(G')$, $L'(v) \subseteq L(v)$. A (G, L)-refinement (G', L') is *spanning* if $G' = G$. A (G, L)-*profile* $\mathcal{L}$ is a set of (G, L)-refinements. A (G, L)-profile is *spanning* if all its elements are spanning. Two list assignments L and L' are *equivalent* for G if every coloring c of G is an L-coloring if and only if it is an L'-coloring.

Let (G, L) be an instance of the Ordered Graph List-3-Coloring Problem.

Lemma 3.1.1. *There exists a spanning (G, L)-refinement (G, L') such that for all $uv \in E(G)$ with $|L(v)| = 1$, $L(u) \cap L(v) = \emptyset$, and L and L' are equivalent for G. Moreover, L' can be computed from L in time $O(n^3)$.*

Proof. We define a sequence of lists recursively. Let $L_0 = L$. Suppose that we have defined L_i. If there is an edge $uv \in E(G)$ with $|L_i(v)| = 1$ and $L_i(u) \cap L_i(v) \neq \emptyset$, let $L_{i+1}(u) = L_i(u) \backslash L_i(v)$, and $L_{i+1}(w) = L_i(w)$ for all $w \in V(G) \backslash \{u\}$. Otherwise stop and let $L' = L_i$.

This terminates within at most $3n$ steps, as $\sum_{w \in V(G)} |L_{i+1}(w)| \leq \sum_{w \in V(G)} |L_i(w)| - 1$, and $\sum_{w \in V(G)} |L_0(w)| \leq 3n$. In each step, finding an edge $uv \in E(G)$ with $|L(v)| = 1$ and $L(u) \cap L(v) \neq \emptyset$ takes time at most $O(n^2)$ and constructing a new list L_{i+1} takes time $O(n)$. Thus L' can be computed from L in time $O(n^3)$.

Since $L_0 = L$, G has an L-coloring if and only if G has an L_0-coloring. For all L_i-colorings c of G and for all edges $uv \in E(G)$, $c(v) \in L_i(v)$ and $c(u) \neq c(v)$. Thus c is an L_{i+1}-coloring of G. For all L_{i+1}-colorings c' of G, since $L_{i+1}(w) \subseteq L_i(w)$ for all $w \in V(G)$, c' is an L_i-coloring of G. Thus, L and L' are equivalent for G. □

Lemma 3.1.2. *Let $k, l \in \mathbb{N}$ be fixed positive integers, and (G, L) be an instance of the* Ordered Graph List-3-Coloring Problem *restricted to $J_{16}(k, l)$-free ordered graphs. There is a spanning (G, L)-profile $\mathcal{L}'_1$ such that:*

- *$|\mathcal{L}'_1| \leq O(n^{3(k+l)})$, and $\mathcal{L}'_1$ can be constructed from L in time $O(n^{3(k+l)+4})$.*
- *For all $(G, L') \in \mathcal{L}'_1$, let $X' = \{v \in V(G) : |L'(v)| \geq 2\}$. Then in the graph $G[X']$, every vertex has at most 2 forward neighbors.*

- *If there is an L-coloring c of G with $|c^{-1}(i)| \geq k+l$ for all $i \in [3]$, then there exists $(G, L') \in \mathcal{L}'_1$ such that c is an L'-coloring.*

Proof. Let $\mathcal{Q}$ be the set of all 6-tuples $Q = (A_1, A_2, A_3, B_1, B_2, B_3)$ of disjoint subsets of $V(G)$ such that for all $i \in [3]$, $|A_i| = k$ and $|B_i| = l$, $i \in L(v)$ for all $v \in A_i \cup B_i$, and $A_i \cup B_i$ are stable. For each $q \in \mathcal{Q}$, we construct a (G, L)-refinement (G, L^Q) as follows.

The list L_0^Q is defined as follows. For each vertex $v \in V(G)$, we let $L_0^{'Q}(v) = \{i\}$ if $v \in A_i \cup B_i$ for some $i \in [3]$, otherwise let $L_0^{'Q}(v) = L(v)$. For each $i \in [3]$, let $m_i = \max\{\varphi_G(a) : a \in A_i\}$, $n_i = \min\{\varphi_G(b) : b \in B_i\}$. Then, for all $i \in [3]$, remove i from $L_0^{'Q}(v)$ for every $v \in V(G[-\infty : m_i])\backslash A_i$ and every $v \in V(G[n_i : \infty])\backslash B_i$. By Lemma 3.1.1, the list L_0^Q such that for all $uv \in E(G)$ with $|L_0^Q(v)| = 1$, $L_0^Q(u) \cap L_0^Q(v) = \emptyset$, can be constructed from $L_0^{'Q}$ in polynomial time.

The list L^Q is constructed recursively. Starting from the list L_0^Q, we construct a sequence of equivalent list assignments $L_1^Q, L_2^Q, \ldots$ until some L_s^Q satisfies the second property of this lemma. For convenience, every time we define L_t^Q for $0 \leq t \leq s$, we also define the following sets. For $\{i, j\} \subseteq [3]$, let $X_t^{ij} = \{v \in V(G) : L_t^Q(v) = \{i, j\}\}$, and let $X_t^{123} = \{v \in V(G) : L_t^Q(v) = \{1, 2, 3\}\}$. Let $X_t = X_t^{12} \cup X_t^{13} \cup X_t^{23} \cup X_t^{123}$.

If in the graph $G[X_t]$, every vertex has at most 2 forward neighbors, then let $L^Q = L_t^Q$. Otherwise, there is a vertex v with at least 3 forward neighbors in the graph $G[X_t]$. Notice that if G contains K_4 as a subgraph, then G is not L'-colorable for any $L' : V(G) \to 2^{[3]}$. We return $\mathcal{L}'_1 = \emptyset$ in this case. Thus, we may assume that G contains no K_4 from now on.

Also, we may assume that

(1) *The vertex v does not have two distinct non-adjacent forward neighbors u, w such that $L_t^Q(v) \cap L_t^Q(u) \cap L_t^Q(w) \neq \emptyset$.*

As otherwise consider a color $i \in L_t^Q(v) \cap L_t^Q(u) \cap L_t^Q(w)$. From the construction of L_t^Q, we know that v, u, w are not adjacent to any vertex from $A_i \cup B_i$. A_i and B_i are disjoint and $A_i \cup B_i$ is a stable set. The vertices u, w are non-adjacent forward neighbors of v. For every $x \in A_i$, $y \in \{u, v, w\}$ and $z \in B_i$, $\varphi(x) < \varphi(y) < \varphi(z)$. From the constructions of L_0^Q, we have $G[A_i \cup B_i \cup \{u, v, w\}] \cong J_{16}(k, l)$, which contradicts to the fact that G is $J_{16}(k, l)$-free. This proves (1).

If $v \in X_t^{123}$, then for every two forward neighbors u, w of v in X_t, $L_t^Q(v) \cap L_t^Q(u) \cap L_t^Q(w) \neq \emptyset$. So by (1), u and w are adjacent. But then, since v has at least 3 forward neighbors, there exists a K_4 as a subgraph of G, which is a contradiction. Thus, this case is impossible.

It remains to consider the case $v \in X_t^{ij}$. Let u, w, x be three distinct forward neighbors of v in X_t. Since G has no K_4, by symmetry, we may assume that $uw \notin E$. By (1), it follows that $L_t^Q(u) \cap L_t^Q(v) \cap L_t^Q(w) = \emptyset$. By symmetry, we may assume that $L_t^Q(u) = \{i, m\}$, $L_t^Q(w) = \{j, m\}$ where $\{i, j, m\} = [3]$. We have the following subcases.

- $L_t^Q(x) \supseteq \{i, j\}$.

  Then $ux, wx \in E(G)$ by (1). Since u and w have two adjacent neighbors in common, it follows that $c(u) = c(w)$ for every 3-coloring of G. We let $L_{t+1}^{\prime Q}(u) = L_{t+1}^{\prime Q}(w) = \{m\}$, $L_{t+1}^{\prime Q}(y) = L_t^Q(y) \backslash \{m\}$ for $y \in N(u) \cup N(w)$, and $L_{t+1}^{\prime Q}(y) = L_t^Q(y)$ for all $y \in V(G) \backslash (N[u] \cup N[w])$.

- $L_t^Q(x) = \{i, m\}$. (The case $\{j, m\}$ follows from symmetry.)

  By (1), we have $ux \in E(G)$. But now v has two adjacent neighbors with list $\{i, m\}$. So in every L_t^Q-coloring c of G, we have $c(v) = j$. We let $L_{t+1}^{\prime Q}(v) = \{j\}$, $L_{t+1}^{\prime Q}(y) = L_t(y) \backslash \{j\}$ for $y \in N(v)$, and $L_{t+1}^{\prime Q}(y) = L_t(y)$ for all $y \in V(G) \backslash N[v]$.

At the end of each step, by applying Lemma 3.1.1, we replace the list $L_{t+1}^{\prime Q}$ by an equivalent list L_{t+1}^Q such that for all $uv \in E(G)$ with $|L_{t+1}^Q(v)| = 1$, we have $L_{t+1}^Q(u) \cap L_{t+1}^Q(v) = \emptyset$, in time $O(n^3)$.

For all t, $|X_{t+1}| \leq |X_t| - 1$, and $|X_0| \leq n$. Thus the algorithm above terminates in at most n steps. In each step t, finding the vertex v with at least 3 forward neighbors in $G[X_t]$ takes time $O(n)$, constructing the list $L_{t+1}^{\prime Q}$ takes time $O(n)$, and constructing the list L_{t+1}^Q takes time $O(n^3)$. So L^Q can be constructed in time $O(n^4)$. We let $\mathcal{L}_1' = \{L^Q : Q \in \mathcal{Q}\}$. There are at most $\binom{n}{k} \cdot \binom{n-k}{k} \cdot \binom{n-2k}{k} = O(n^{3k})$ different choices of the triple (A_1, A_2, A_3), and at most $O(n^{3l})$ different choices of the triple (B_1, B_2, B_3). For each 6-tuple Q we add at most one list to $\mathcal{L}_1'$. Thus, $|\mathcal{L}_1'| \leq O(n^{3(k+l)})$. Therefore, $\mathcal{L}_1$ can be constructed from L in time $O(n^{3(k+l)+4})$.

Finally, let c be an L-coloring of G with $|c^{-1}(i)| \geq k+l$ for all $i \in [3]$. Define $A_i' \subseteq c^{-1}(i)$ to be the set of vertices such that $|A_i'| = k$, and $\varphi(v) > \varphi(u)$ for all $v \in c^{-1}(i) \backslash A_i'$ and $u \in A_i'$, that is, A_i' is the set of first k vertices colored i in c. Similarly, for all $i \in [3]$, define $B_i' \subseteq c^{-1}(i)$ to be the set of vertices such that $|B_i'| = l$, and $\varphi(v) < \varphi(u)$ for any $v \in c^{-1}(i) \backslash B_i'$ and $u \in B_i'$. Let $Q' = (A_1', A_2', A_3', B_1', B_2', B_3')$. It follows that $Q' \in \mathcal{Q}$. Thus, the corresponding (G, L)-refinement $(G, L^{Q'})$ is in $\mathcal{L}_1'$.

We want to show that c is also an $L^{Q'}$-coloring. We will prove this by induction on t. For every vertex $v \in V(G)$, we have $c(v) \in L_0^{Q'}(v)$ from the choice of Q'. Thus, c

is an $L_0^{Q'}$-coloring. Suppose c is an $L_t^{Q'}$-coloring. Then for $t+1$, from our construction, $c(v) \in L_{t+1}^{'Q'}(v)$ for all vertex v. So c is an $L_{t+1}^{'Q'}$-coloring of G. By Lemma 3.1.1, c is also an $L_{t+1}^{Q'}$-coloring. Thus, the L-coloring c is an $L^{Q'}$-coloring of G. □

Lemma 3.1.3. *Let $k, l \in \mathbb{N}$ be fixed positive integers, and (G, L) be an instance of the* Ordered Graph List-3-Coloring Problem *restricted to $J_{16}(k,l)$-free ordered graphs. There is a spanning (G, L)-profile $\mathcal{L}_2'$ such that:*

- *$|\mathcal{L}_2'| \leq 3 \cdot n^{k+l}$, and $\mathcal{L}_2'$ can be constructed from L in time $O(n^{k+l+1})$.*
- *For all $(G, L') \in \mathcal{L}_2'$, let $X = \{v \in V(G) : |L'(v)| \geq 2\}$. Then $|L'(v)| = 2$ and $L'(u) = L'(v)$ for all $u, v \in X$.*
- *If c is an L-coloring of G with $|c^{-1}(i)| < k + l$ for some $i \in [3]$, then there exists $(G, L') \in \mathcal{L}_2'$ such that c is an L'-coloring.*

Proof. Let $\mathcal{P}$ be a set of all pairs $P = (i, A_i)$ such that $i \in [3]$ and $A_i \subseteq V(G)$ with $|A_i| < k + l$, A_i stable and $i \in L(v)$ for all $v \in A_i$. For each $P \in \mathcal{P}$, we construct a (G, L)-refinement (G, L^P) as follows.

Let $L^P(v) = \{i\}$ for all $v \in A_i$, and $L^P(v) = L(v) \backslash \{i\}$ otherwise. It follows that $L^P(v) = [3] \backslash \{i\}$ for all $v \in V(G)$ with $|L^P(v)| \geq 2$.

The set $\mathcal{P}$ is of size at most $3 \cdot n^{k+l}$. For each pair $P \in \mathcal{P}$ we add at most one refinement to $\mathcal{L}_2'$. Thus, $|\mathcal{L}_2'| \leq 3 \cdot n^{k+l}$. Constructing the list L^P takes time $O(n)$. Thus, $\mathcal{L}_2'$ can be constructed from L in time $O(n^{k+l+1})$.

Let c be an L-coloring of G with $|c^{-1}(i)| < k+l$ for some $i \in [3]$. The pair $P' = (i, c^{-1}(i))$ satisfies the property that $|c^{-1}(i)| < k + l$, $c^{-1}(i)$ is stable and $i \in L(v)$ for all $v \in c^{-1}(i)$. Thus, the corresponding (G, L)-refinement $(G, L^{P'})$ is in $\mathcal{L}_2'$. By the construction of $L^{P'}$, c is an $L^{P'}$-coloring. □

A graph G is *chordal* if in G, every cycle of length at least 4 has an edge connecting two vertices of the cycle but not in the cycle. Equivalently, every induced cycle in G is a triangle.

Lemma 3.1.4. *Let $k, l \in \mathbb{N}$ be fixed positive integers, and (G, L) be an instance of the* Ordered Graph List-3-Coloring Problem *restricted to $J_{16}(k,l)$-free ordered graphs. Let $X = \{v \in V(G) : |L(v)| \geq 2\}$ and let us assume that every vertex in X has at most two forward neighbors in $G[X]$ and that $|X| \geq 3k + 3l + 6$. There is a spanning (G, L)-profile $\mathcal{L}_1$ such that:*

- $|\mathcal{L}_1| = O(1)$, *and* $\mathcal{L}_1$ *can be constructed in time* $O(n^3)$.
- *For all* $(G, L^*) \in \mathcal{L}_1$, *let* $X^* = \{v \in V(G) : |L^*(v)| \geq 2\}$. *Then the graph* $G[X^*]$ *is chordal.*
- *If* c *is an* L*-coloring of* G*, then there exists* $(G, L^*) \in \mathcal{L}_1$ *such that* c *is an* L^**-coloring of* G.

Proof. First, we define two sets $C' \subseteq C \subseteq X$ as follows. We start with $C' = C = \emptyset$. In each step, we take the vertex $v \in X \backslash C$ with the smallest $\varphi(v)$. Add v and its forward neighbors in $G[X]$ to C, and add v to C'. We repeat this k times. Since every vertex in X has at most two forward neighbors in $G[X]$, $|C| \leq 3k$. By construction, C' is a stable set of size k. Moreover, no vertex in C' is adjacent to a vertex in $X \backslash C$. Define $D \subseteq X$ to be the set of vertices such that $|D| = 3l + 6$, and $\varphi(v) < \varphi(u)$ for all $v \in X \backslash (C \cup D)$ and $u \in D$, that is, D is the set of last $3l + 6$ vertices in $X \backslash C$. Since $|X| \geq 3k + 3l + 6$, it follows that C, C' and D are well-defined.

Let $\mathcal{F}$ be the set of all functions $f : C \cup D \to [3]$ such that f is an L-coloring of $G[C \cup D]$. For every $f \in \mathcal{F}$, we construct a (G, L)-refinement (G, L'^f) such that $L'^f(v) = \{f(v)\}$ if $v \in C \cup D$, and $L'^f(v) = L(v)$ otherwise. By Lemma 3.1.1, there is an equivalent list L^f of L'^f such that for all $uv \in E(G)$ with $|L^f(v)| = 1$, $L^f(u) \cap L^f(v) = \emptyset$. Let $\mathcal{L}_1 = \{(G, L^f) : f \in \mathcal{F}\}$. There are at most $3^{3k+3l+6} = O(1)$ possible choices of f. Thus, $|\mathcal{L}_1| = O(1)$. Constructing the set C and D takes time $O(1)$. Each L'^f can be constructed in time $O(n)$. Each L^f can be constructed in time $O(n^3)$. So $\mathcal{L}_1$ can be constructed in time $O(n^3)$.

Now let $(G, L^*) \in \mathcal{L}_1$. Every non-chordal ordered graph contains a vertex with two non-adjacent forward neighbors. To be more precise, the vertex with the smallest order in an induced cycle of size at least 4 is a desired vertex. Now we want to show that $G[X^*]$ is chordal using this property. Suppose for a contradiction that in $G[X^*]$, there is a vertex v_1 with two non-adjacent forward neighbors v_2, v_3. There is a stable set $D' \subseteq D$ of size at least l such that D' is anticomplete to $\{v_1, v_2, v_3\}$. That is because $X^* \subseteq X$ and every vertex in X has at most 2 forward neighbors in $G[X]$, so $D \backslash N(\{v_1, v_2, v_3\})$ is of size at least $3l$. Since D has a 3-coloring by construction, there is a stable set $D' \subseteq D \backslash N(\{v_1, v_2, v_3\})$ of size at least l and which is anticomplete to $\{v_1, v_2, v_3\}$. From the construction above, the sets $\{v_1, v_2, v_3\}$, C' and D' are disjoint. Moreover, for every $x \in C'$, $y \in \{v_1, v_2, v_3\}$ and $z \in D'$, $\varphi(x) < \varphi(y) < \varphi(z)$. So $G[C' \cup D' \cup \{v_1, v_2, v_3\}] \cong J_{16}(k, l)$, which is a contradiction. Therefore, $G[X^*]$ is chordal.

Finally, let c be an L-coloring of G. Take the coloring $c' = c|_{C \cup D}$ and consider the corresponding (G, L)-refinement $(G, L'^{c'})$ and $(G, L^{c'})$ defined above. Since we have covered all possible colorings f of $G[C \cup D]$, $(G, L^{c'})$ is in $\mathcal{L}_1$. We can verify that $c(v) \in L'^{c'}(v)$ for all vertices $v \in V(G)$. Thus c is also an $L^{c'}$-coloring. $\square$

Lemma 3.1.5. *Let $k, l \in \mathbb{N}$ be fixed positive integers, and (G, L) be an instance of the* Ordered Graph List-3-Coloring Problem *restricted to $J_{16}(k, l)$-free ordered graphs. Let $X = \{v \in V(G) : |L(v)| \geq 2\}$ and let us assume that $|X| < 3k + 3l + 6$. There is a spanning (G, L)-profile $\mathcal{L}_2$ such that:*

- *$|\mathcal{L}_2| = O(1)$, and $\mathcal{L}_2$ can be constructed in time $O(n^3)$.*
- *For any $(G, L^*) \in \mathcal{L}_2$, $|L^*(v)| \leq 1$ for all $v \in V(G)$.*
- *If c is an L-coloring of G, then there exists $(G, L^*) \in \mathcal{L}_2$ such that c is an L^*-coloring of G.*

Proof. Let $\mathcal{F}$ be the set of all functions $f : X \to [3]$ such that f is an L-coloring of $G[X]$. For every possible function $f \in \mathcal{F}$, we construct a list L^f such that $L^f(v) = \{f(v)\}$ for all $v \in X$, and $L^f(v) = L(v)$ otherwise. Let $\mathcal{L}_2 = \{(G, L^f) : f \in \mathcal{F}\}$.

For every $(G, L^f) \in \mathcal{L}_2$ and for every $v \in V(G)$, if $v \in X$ then $|L^f(v)| \leq 1$; otherwise by the definition of X, we have $|L^f(v)| \leq |L'^f(v)| \leq 1$. Thus $|L^f(v)| \leq 1$ for all $v \in V(G)$.

Since there are at most $3^{3k+3l+6} = O(1)$ possible choices of f, $|\mathcal{L}_2| = O(1)$. Each L'^f can be constructed in time $O(n)$, and L^f can be constructed from L'^f in time $O(n^3)$. So $\mathcal{L}_2$ can be constructed in time $O(n^3)$.

Finally, let c be an L-coloring of G. Let $c' = c|_X$ and consider the corresponding (G, L)-refinements $(G, L'^{c'})$ and $(G, L^{c'})$ defined above. Since we have covered all possible L-colorings $f : X \to [3]$, $(G, L^{c'}) \in \mathcal{L}_2$. By the construction of c' and $L'^{c'}$, c is an $L'^{c'}$-coloring thus is an $L^{c'}$-coloring. $\square$

Recall the theorems:

Theorem 1.3.5. *The* List-3-Coloring Problem *restricted to chordal graphs with bounded clique number is polynomial-time solvable.*

Theorem 1.3.6. *The* List-2-Coloring Problem *can be solved in time $O(n^2)$, where n is the number of vertices of the input graph.*

Theorem 3.1.6. *For fixed $k, l \in \mathbb{N}$, there is an algorithm with the following specifications:*

- *Input:* (G, L)*, which is an instance of the* Ordered Graph List-3-Coloring Problem *and* G *is* $J_{16}(k, l)$*-free.*

- *Output: one of*

  - *an* L*-coloring of* G*;*
  - *a determination that* G *is not* L*-colorable;*
  - *a spanning* (G, L)*-profile* $\mathcal{L}$ *with* $|\mathcal{L}| \leq O(n^{3(k+l)})$ *such that for every* $(G, L^*) \in \mathcal{L}$*, if* $X^{L^*} = \{v \in V(G) : |L^*(v)| \geq 2\}$*, then* $G[X^{L^*}]$ *is chordal.*

- *Running time:* $O(n^{3(k+l+1)})$*.*

Proof. Let $\mathcal{L}'_1$ be as in Lemma 3.1.2. Let $\mathcal{L}'_2$ be as in Lemma 3.1.3. By Theorem 1.3.6, every (G, L)-refinement $(G, L') \in \mathcal{L}'_2$ can be solved in time $O(n^2)$. If this finds an L-coloring of G, we just output the coloring instead of processing with the other things.

For every (G, L)-refinement $(G, L') \in \mathcal{L}'_1$, let $X = \{v \in V(G) : |L'(v)| \geq 2\}$. If $|X| \geq 3k+3l+6$, then there is a spanning (G, L)-profile $\mathcal{L}^{L'}$ which satisfies the properties in Lemma 3.1.4. If $|X| < 3k+3l+6$, then there is a spanning (G, L)-profile $\mathcal{L}^{L'}$ which satisfies the properties in Lemma 3.1.5. Finally, let $\mathcal{L} = \cup_{(G,L')\in\mathcal{L}'_1} \mathcal{L}^{L'}$. From the constructions, for every $(G, L^*) \in \mathcal{L}$, $G[X^{L^*}]$ is chordal. Since $|\mathcal{L}^{L'}| = O(1)$ and $|\mathcal{L}'_1| \leq O(n^{3(k+l)})$, $|\mathcal{L}| \leq O(n^{3(k+l)})$. The collection $\mathcal{L}^{L'}$ can be constructed from L' in time $O(n^3)$, and $|\mathcal{L}'_1| \leq O(n^{3(k+l)})$. Thus, $\mathcal{L}$ can be constructed from L in time $O(n^{3(k+l+1)})$. □

Proof of Theorem 1.2.17. Let G be a $J_{16}(k, l)$-free ordered graph and L be a 3-list-assignment for G. We check in polynomial time if G contains a clique of size 4. If so, then G is not L-colorable and we are done. We apply the algorithm from Theorem 3.1.6 to (G, L). If the output is an L-coloring of G or a determination that G is not L-colorable, then we are done; so we may assume that the output is a (G, L)-profile $\mathcal{L}$. For each $(G, L^*) \in \mathcal{L}$, we let $X^{L^*} = \{v \in V(G) : |L^*(v)| \geq 2\}$ as in Theorem 3.1.6. By Lemma 3.1.1, we may assume that $L^*(u) \cap L^*(v) = \emptyset$ for all $uv \in E(G)$ such that $|L^*(u)| = 1$. If $L^*(u) = \emptyset$ for some $u \in V(G)$, then G has no L^*-coloring and we continue. Otherwise, since Theorem 3.1.6 guarantees that $G[X^{L^*}]$ is chordal, and since G contains no clique of size 4, we can check in polynomial time if $G[X^{L^*}]$ is L^*-colorable. If this returns a coloring f, then by Lemma 3.1.1, we obtain an L-coloring of G as follows:

- for $x \in X^{L^*}$, let $c(x) = f(x)$;
- for all other $x \in V(G)$, let $c(x)$ be the unique color in $L^*(x)$.

If there is no $(G, L^*) \in \mathcal{L}$ such that this returns a coloring of G, then, from the definition of a (G, L)-profile, it follow that G is not L-colorable. This concludes the proof. $\square$

3.2 NP-hardness results

In this section, we will prove the following theorem.

Theorem 1.2.18. *If H is an ordered graph such that at least one of the following holds:*

- *H has at least three edges;*
- *H has a vertex of degree at least 2 and is not isomorphic to $J_{16}(k, l)$ or $-J_{16}(k, l)$ for any k, l;*
- *H contains J_9, M_1 or M_5 as induced ordered subgraph;*

then the Ordered Graph List-3-Coloring Problem *restricted to (H, φ)-free ordered graphs is NP-complete.*

In ordered to show Theorem 1.2.18, we will show the following two theorems.

Theorem 3.2.1. *Let H be a graph and $\varphi : V(H) \to \mathbb{Z}$. The* Ordered Graph List-3-Coloring Problem *restricted to (H, φ)-free ordered graphs is NP-complete if H contains a copy of P_4 or $P_3 + P_2$ as an induced subgraph.*

Theorem 3.2.2. *The* Ordered Graph List-3-Coloring Problem *is NP-complete when restricted to the class of M_j-free ordered graphs, for $j \in [5]$.*

We will use three constructions to show the Ordered Graph List-3-Coloring Problem is NP-complete when restricted to the class of J_i-free ordered graphs, for every $i \in [15]$. Then with these proofs, we will show Theorem 3.2.1.

The first two constructions reduce from NAE3SAT. Notice that we will prove a stronger result: In the following two theorems, we actually prove the NP-hardness of the Ordered Graph 3-Coloring Problem instead of the Ordered Graph List-3-Coloring Problem within specific classes of graphs.

Theorem 3.2.3. *Given a monotone NAE3SAT instance I, there is a graph H_1 such that:*

1. *The graph H_1 can be computed from I in time $O(m+n)$, where m is the number of clauses of I and n is the number of variables of I;*

2. *The graph H_1 is 3-colorable if and only if I is satisfiable;*

3. *There is an injective function $\tau_1 : V(H_1) \to \mathbb{Z}$ such that (H_1, τ_1) is J_3, J_6 and $-J_{11}$-free, and τ_1 can be computed from H_1 in time $O(m+n)$;*

4. *There is an injective function $\tau_2 : V(H_1) \to \mathbb{Z}$ such that (H_1, τ_2) is J_1, J_2, $-J_4$, J_5, J_8 and J_{12}-free, and τ_2 can be computed from H_1 in time $O(m+n)$;*

5. *There is an injective function $\tau_3 : V(H_1) \to \mathbb{Z}$ such that (H_1, τ_3) is J_{10}-free, and τ_3 can be computed from H_1 in time $O(m+n)$.*

Therefore, the Ordered Graph 3-Coloring Problem *is NP-complete when restricted to the class of J_1, J_2, J_3, $-J_4$, J_5, J_6, J_8, J_{10}, $-J_{11}$ or J_{12}-free ordered graphs.*

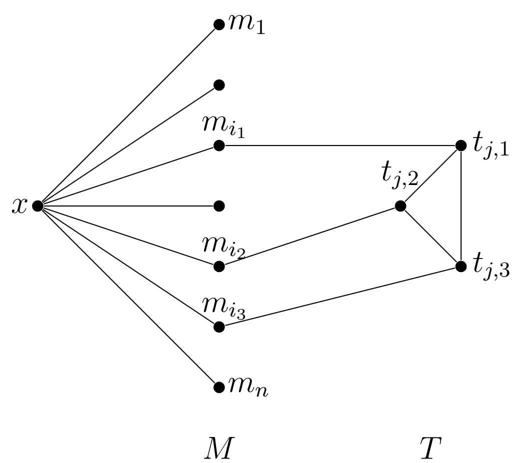

Figure 3.3: The construction of H_1 from Theorem 3.2.3, with M corresponding to variables and T corresponding to clauses.

Proof. The construction of H_1 is shown in Figure 3.3. First we create a vertex x. For every variable x_i of I, we create a vertex m_i, and denote the set of such vertices as M. For every clause C_j of I, we create three vertices $t_{j,k}$ for $k \in [3]$, and denote the set of such vertices as T. Let $V(H_1) = \{x\} \cup M \cup T$.

For every vertex $m_i \in M$, we add an edge xm_i. For every clause C_j of I, if the variables in C_j are $x_{i_1}, x_{i_2}, x_{i_3}$ with $1 \leq i_1 < i_2 < i_3 \leq n$, we add edges $m_{i_k}t_{j,k}$ for $k \in [3]$ and edges $t_{j,1}t_{j,2}$, $t_{j,2}t_{j,3}$, $t_{j,1}t_{j,3}$. Let $E(H_1)$ be the set of all defined edges.

(1) *$H_1[m]$ is stable and $H_1[T]$ is disjoint union of triangles.*

It follows from the construction of H_1.

(2) *The graph H_1 can be computed from I in time $O(m+n)$.*

It takes time $O(m+n)$ to compute both the set $V(H_1)$ and $E(H_1)$.

(3) *The graph H_1 is 3-colorable if and only if I is satisfiable.*

Let $f : V(H_1) \to [3]$ be a 3-coloring of H_1. Without loss of generality, we assume $f(x) = 1$. Since every vertex in M is adjacent to x, we have $f(m_i) \in \{2,3\}$ for every $i \in [n]$. We claim that if the variables in C_j are $x_{i_1}, x_{i_2}, x_{i_3}$ with $1 \leq i_1 < i_2 < i_3 \leq n$, then at least one of $f(m_{i_1})$, $f(m_{i_2})$ and $f(m_{i_3})$ has value 2 and at least one of them has value 3. Suppose for a contradiction, without loss of generality, that $f(m_{i_k}) = 2$ for every $k \in [3]$. Then $f(t_{j,k}) \in \{1,3\}$ for every $k \in [3]$, at least two of the three vertices receive the same color. But by (1), the vertices $t_{j,1}$, $t_{j,2}$ and $t_{j,3}$ form a triangle. which leads to a contradiction as desired. Thus, by assigning true to the variable x_i if $f(m_i) = 2$ and false otherwise, we get a valid truth assignment to the monotone NAE3SAT instance I.

If there is a valid truth assignment to I, we define a 3-coloring $g : V(H_1) \to [3]$ as follows. For every $i \in [n]$, let $g(m_i) = 2$ if the variable v_i is true in this truth assignment, otherwise let $g(m_i) = 3$. Let $g(x) = 1$. For each clause C_j, we denote the variables in C_j as $x_{i_1}, x_{i_2}, x_{i_3}$ with $1 \leq i_1 < i_2 < i_3 \leq n$. Let $g(t_{j,1}) \in \{2,3\} \backslash \{g(m_{i_1})\}$. Since $g|_M$ is constructed from a valid truth assignment of I, at least one of the $g(m_{i_1})$, $g(m_{i_2})$ and $g(m_{i_3})$ has value 2 and at least one of them has value 3. If $g(m_{i_2}) \neq g(m_{i_1})$, then let $g(t_{j,2}) \in \{2,3\} \backslash \{g(m_{i_2})\}$ and $g(t_{j,3}) = 1$, otherwise let $g(t_{j,3}) \in \{2,3\} \backslash \{g(m_{i_3})\}$ and $g(t_{j,2}) = 1$. To verify this is a valid 3-coloring, we simply go through and check every edge in $E(H_1)$.

(4) *There is an injective function $\tau_1 : V(H_1) \to \mathbb{Z}$ such that (H_1, τ_1) is J_3, J_6 and $-J_{11}$-free, and τ_1 can be computed from H_1 in time $O(m+n)$.*

The function $\tau_1 : V(H_1) \to \mathbb{Z}$ is defined as follows. Let $\tau_1(x) = 1$. Let $\tau_1(m_i) = i + 1$

for every $i \in [n]$. For every $j \in [m]$ and $k \in [3]$, let $\tau_1(t_{j,k}) = n + 3j + k - 2$. The function τ_1 can be constructed in time $O(m+n)$ as we go through every vertex once.

From the construction we have $\tau_1|M$ and $\tau_1|T$ are injective, and $\tau_1(x) < \tau_1(m_i) < \tau_1(t_{j,k})$ for every $i \in [n]$, $j \in [m]$ and $k \in [3]$. So the function τ_1 is injective.

Suppose for a contradiction that (H_1, τ_1) contains an induced path $w_1w_3w_2w_4$ with $\tau_1(w_1) < \tau_1(w_2) < \tau_1(w_3) < \tau_1(w_4)$. Since the vertex x does not have any backward neighbor and every $m_i \in M$ has only one backward neighbor x, we have $w_3 \in T$. At most one of w_1 and w_2 is in T as $w_1, w_2 \in N(w_3)$ and $w_1w_2 \notin E(H_1)$. Thus we have $w_1 \in M$, $w_2 \in T \cup M$ and $w_4 \in T$. If $w_2 \in T$, then w_3 is also adjacent to w_4 since $w_3, w_4 \in N(w_2)$, which is a contradiction. If $w_2 \in M$, then w_3 has two neighbors in M, which is a contradiction. Thus, we have proved (H_1, τ_1) is J_3-free.

Suppose for a contradiction that (H_1, τ_1) contains an induced path $w_1w_4w_3w_2$ with $\tau_1(w_1) < \tau_1(w_2) < \tau_1(w_3) < \tau_1(w_4)$. Since w_4 has two backward neighbors, we have $w_4 \in T$. For the two backward neighbors w_1, w_3 of w_4, since $w_1w_3 \notin E(H_1)$ and $\tau_1(w_1) < \tau_1(w_3)$, we have $w_1 \in M$ and $w_3 \in T$. The vertex w_2 is not in the set T as otherwise $w_2w_4 \in E(H_1)$, so $w_2 \in M$. From the construction of τ_1, there exist $i_1, i_2 \in [n]$ with $i_1 < i_2$ and $w_1 = m_{i_1}$, $w_2 = m_{i_2}$. But then we have $\tau_1(w_4) < \tau_1(w_3)$, which is a contradiction. Thus, we have proved (H_1, τ_1) is J_6-free.

Suppose (H_1, τ_1) contains an induced subgraph $(\{w_1, w_2, w_3, w_4\}, \{w_1w_4, w_2w_4\})$ with $\tau_1(w_1) < \tau_1(w_2) < \tau_1(w_3) < \tau_1(w_4)$. Since w_4 has two backward neighbors, we have $w_4 \in T$. For the two backward neighbors w_1, w_2 of w_4, since $w_1w_2 \notin E(H_1)$ and $\tau_1(w_1) < \tau_1(w_2)$, we have $w_1 \in M$ and $w_2 \in T$. From the construction of τ_1, since $w_2w_4 \in E(H_1)$ and $\tau_1(w_2) < \tau_1(w_3) < \tau_1(w_4)$, we have $w_3 \in T$ and $w_2w_3, w_3w_4 \in E(H_1)$, which is a contradiction. Thus, we have proved (H_1, τ_1) is $-J_{11}$-free.

(5) *There is an injective function $\tau_2 : V(H_1) \to \mathbb{Z}$ such that (H_1, τ_2) is J_1, J_2, $-J_4$, J_5, J_8 and J_{12}-free, and τ_2 can be computed from I in time $O(m+n)$.*

The function $\tau_2 : V(H_1) \to \mathbb{Z}$ is defined as follows. Let $\tau_2(m_i) = i$ for every $i \in [n]$. Let $\tau_2(x) = n + 1$. For every $j \in [m]$ and $k \in [3]$, let $\tau_2(t_{j,k}) = n + 3j + k - 2$. The function τ_2 can be constructed in time $O(m+n)$ as we go through every vertex once.

From the construction we have $\tau_2|M$ and $\tau_2|T$ are injective, and $\tau_2(m_i) < \tau_2(x) < \tau_2(t_{j,k})$ for every $i \in [n]$, $j \in [m]$ and $k \in [3]$. So the function τ_2 is injective.

Suppose for a contradiction that (H_1, τ_2) contains an induced path $w_1w_2w_3w_4$ with $\tau_2(w_1) < \tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$. Since the vertex x does not have any forward neighbor, we have $w_1, w_2, w_3 \neq x$. Since every vertex in M has no backward neighbor, we have

$w_2, w_3, w_4 \notin M$. So $w_2, w_3 \in T$. Then we have $w_4 \in T$ as $\tau_2(w_4) > \tau_2(w_3)$. But from the construction of H_1 and τ_2, we also have $w_2w_4 \in E(H_1)$ as $w_2w_3, w_3w_4 \in E(H_1)$, which is a contradiction. Thus, we have proved (H_1, τ_2) is J_1-free.

Suppose for a contradiction that (H_1, τ_2) contains an induced path $w_1w_2w_4w_3$ with $\tau_2(w_1) < \tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$. Since the vertex x does not have any forward neighbor, we have $w_1, w_2, w_3 \neq x$. Since every vertex in M has no backward neighbor, we have $w_2, w_4 \notin M$. So $w_2 \in T$. Then we have $w_3, w_4 \in T$ as $\tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$. But from the construction of H_1 and τ_2, we also have $w_2w_3 \in E(H_1)$ as $w_2w_4 \in E(H_1)$, which is a contradiction. Thus, we have proved (H_1, τ_2) is J_2-free.

Suppose for a contradiction that (H_1, τ_2) contains an induced path $w_3w_1w_2w_4$ with $\tau_2(w_1) < \tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$. Since the vertex x does not have any forward neighbor, we have $w_1, w_2 \neq x$. Since every vertex in M has no backward neighbor, we have $w_2, w_3, w_4 \notin M$. So $w_2 \in T$. Then we have $w_3, w_4 \in T$ as $\tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$. But from the construction of H_1 and τ_2, we also have $w_2w_3 \in E(H_1)$ as $w_2w_4 \in E(H_1)$, which is a contradiction. Thus, we have proved (H_1, τ_2) is $-J_4$-free.

Suppose for a contradiction that (H_1, τ_2) contains an induced path $w_1w_4w_2w_3$ with $\tau_2(w_1) < \tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$. Since the vertex x does not have any forward neighbor, we have $w_1, w_2 \neq x$. Since every vertex in M has no backward neighbor, we have $w_3, w_4 \notin M$. If $w_2 \in T$, then we have $w_3, w_4 \in T$ as $\tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$. But from the construction of H_1 and τ_2, we also have $w_3w_4 \in E(H_1)$ as $w_2w_4 \in E(H_1)$, which is a contradiction. Thus we have $w_2 \in M$, which implies $w_1 \in M$. Also since w_2 has two forward neighbors and $\tau_2(x) < \tau_2(t_{j,k})$ for every $j \in [m]$ and $k \in [3]$, we have $w_4 \in T$. But then w_4 has two neighbors $w_1, w_2 \in M$, which is a contradiction. Thus, we have proved (H_1, τ_2) is J_5-free.

Suppose for a contradiction that (H_1, τ_2) contains an induced path $w_3w_1w_4w_2$ with $\tau_2(w_1) < \tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$. Since the vertex x does not have any forward neighbor, we have $w_1, w_2 \neq x$. Since every vertex in M has no backward neighbor, we have $w_3, w_4 \notin M$. Since w_1 has two non-adjacent forward neighbors w_2, w_4 with $\tau_2(w_2) < \tau_2(w_4)$, we have $w_1 \in M$ and $w_4 \in T$. Thus we have $w_2 \in T$, as w_4 has exactly one neighbor in M. But from the construction of H_1 and τ_2, we also have $w_3 \in T$ and so $w_2w_3, w_3w_4 \in E(H_1)$ as $w_2w_4 \in E(H_1)$ and $\tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$, which is a contradiction. Thus, we have proved (H_1, τ_2) is J_8-free.

Suppose that (H_1, τ_2) contains an induced subgraph $(\{w_1, w_2, w_3, w_4\}, \{w_1w_2, w_2w_4\})$ with $\tau_2(w_1) < \tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$. Since the vertex x does not have any forward neighbor, we have $w_1, w_2 \neq x$. Since every vertex in M has no backward neighbor, we have $w_2, w_4 \notin M$. So $w_2 \in T$. Then we have $w_3, w_4 \in T$ as $\tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$. But from the construction of H_1 and τ_2, we also have $w_2w_3, w_3w_4 \in E(H_1)$ as $w_2w_4 \in E(H_1)$, which is a contradiction. Thus, we have proved (H_1, τ_2) is J_{12}-free.

(6) *There is an injective function $\tau_3 : V(H_1) \to \mathbb{Z}$ such that (H_1, τ_3) is J_{10}-free, and τ_3 can be computed from H_1 in time $O(m+n)$.*

The function $\tau_3 : V(H_1) \to \mathbb{Z}$ is defined as follows. Let $\tau_3(x) = 1$. Let $\tau_1(m_i) = i+1$ for every $i \in [n]$. For every $j \in [m]$ and $k \in [3]$, let $m_i \in M$ be the vertex such that $t_{j,k} \in N(m_i)$, then we set $\tau_3(t_{j,k}) = n+2+\sum_{i'=1}^{i-1}(deg(m_{i'})-1)+|\{t_{j',k'} \in N(m_i) : j' < j\}|$. The function τ_1 can be constructed in time $O(m+n)$ as we go through every vertex once.

From the construction we have $\tau_3|M$ and $\tau_3|T$ are injective, and $\tau_3(x) < \tau_3(m_i) < \tau_3(t_{j,k})$ for every $i \in [n]$, $j \in [m]$ and $k \in [3]$. So the function τ_3 is injective.

Suppose that (H_1, τ_3) contains an induced subgraph $(\{w_1, w_2, w_3, w_4\}, \{w_1w_2, w_1w_4\})$ with $\tau_3(w_1) < \tau_3(w_2) < \tau_3(w_3) < \tau_3(w_4)$. Now we consider the vertex w_1. Since $w_1 = x$ implies $w_1w_3 \in E(H_1)$ as $w_2, w_4 \in N(w_1)$, and $w_1 \in T$ implies $w_2w_4 \in E(H_1)$, we have $w_1 \in M$. But from the construction of τ_3, we also have $w_1w_3 \in E(H_1)$, which is a contradiction. Thus, we have proved (H_1, τ_2) is J_{10}-free. □

Theorem 3.2.4. *Given a monotone NAE3SAT instance I, there is an ordered graph (H_2, τ_4) such that*

1. *The ordered graph (H_2, τ_4) can be computed from I in time $O(m+n)$;*
2. *The graph H_2 is 3-colorable if and only if I is satisfiable;*
3. *The ordered graph (H_2, τ_4) is J_7, J_{13} and J_{14}-free.*

Therefore, the Ordered Graph 3-Coloring Problem *is NP-complete when restricted to the class of J_7, J_{13} or J_{14}-free ordered graphs.*

Proof. The construction of H_2 is shown in Figure 3.4. First we create a vertex x. For every variable x_i of I, we create a vertex m_i and add edge xm_i. We denote the set of such vertices m_i as M. For every clause C_j of I, if the variables in C_j are $x_{i_1}, x_{i_2}, x_{i_3}$ with $1 \le i_1 < i_2 < i_3 \le n$, we create six vertices t_{j,i_k} and $s_{i_k,j}$ for $k \in [3]$. We add edges $t_{j,i_k}t_{j,i_{k'}}$ for $\{k, k'\} \subseteq [3]$, and $xs_{i_k,j}$, $m_{i_k}s_{i_k,j}$ and $s_{i_k,j}t_{j,i_k}$ for $k \in [3]$. We denote the set of vertices t_{j,i_k} as T, and the set of vertices $s_{i_k,j}$ as S. Finally, let $V(H_2) = \{x\} \cup M \cup S \cup T$ and $E(H_2)$ be the set of all edges defined above.

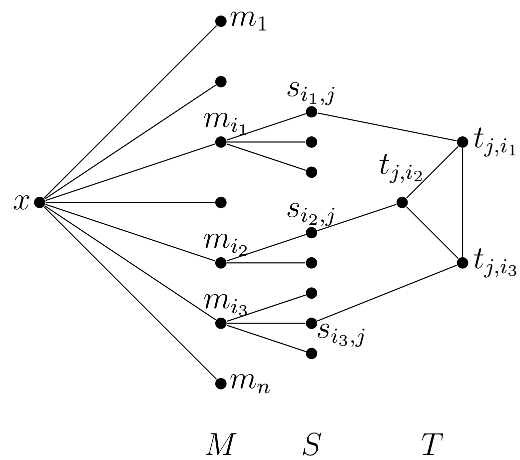

Figure 3.4: The construction of H_2 from Theorem 3.2.4.

The function $\tau_4 : V(H_2) \to \mathbb{Z}$ is defined as follows. Let $\tau_4(x) = 1$ and $\tau_1(m_i) = i + 1$ for every $i \in [n]$. For every $s_{i,j} \in S$, let $\tau_4(s_{i,j}) = n + 2 + \sum_{i'=1}^{i-1}(deg(m_{i'}) - 1) + |\{s_{i,j'} \in N(m_i) : j' < j\}|$. For every $t_{j,i} \in T$, let $\tau_4(t_{j,i}) = \tau_4(s_{i,j}) + 3m$.

(1) *The ordered graph (H_2, τ_4) can be computed from I in time $O(m + n)$.*

It takes time $O(m + n)$ to compute both the set $V(H_2)$ and $E(H_2)$. The function τ_4 can be constructed in time $O(m+n)$, since $|V(H_2)| = n+6m+1$ and we go through every vertex once.

(2) *The graph H_2 is 3-colorable if and only if I is satisfiable.*

Let $f : V(H_2) \to [3]$ be a 3-coloring of H_2. Without loss of generality, we assume $f(x) = 1$. Since every vertex in $M \cup S$ is adjacent to x, we have $f(y) \in \{2, 3\}$ for every $y \in M \cup S$.

We claim that if the variables in C_j are $x_{i_1}, x_{i_2}, x_{i_3}$ with $1 \le i_1 < i_2 < i_3 \le n$, then at least one of $f(m_{i_1})$, $f(m_{i_2})$ and $f(m_{i_3})$ has value 2 and at least one of them has value 3. Suppose for a contradiction, without loss of generality, that $f(m_{i_k}) = 2$ for every $k \in [3]$. We have $f(s_{i_k,j}) = 3$ for every $k \in [3]$. Then $f(t_{j,i_k}) \in \{1, 2\}$ for every $k \in [3]$, at least two of the three vertices receive the same color. But from the construction, the vertices t_{j,i_1}, t_{j,i_2} and t_{j,i_3} form a triangle, which leads to a contradiction as desired. Thus, by assigning true to the variable x_i if $f(m_i) = 2$ and false otherwise, we get a valid truth assignment to the monotone NAE3SAT instance I.

If there is a valid truth assignment to I, we define a 3-coloring $g : V(H_2) \to [3]$ as follows. For every $i \in [n]$, let $g(m_i) = 2$ if the variable v_i is true in this truth assignment, otherwise let $g(m_i) = 3$. Let $g(x) = 1$. For every vertex $s_{i,j} \in S$, we define $g(s_{i,j}) \in \{2, 3\} \backslash \{g(m_i)\}$.

For each clause C_j, we denote the variables in C_j as $x_{i_1}, x_{i_2}, x_{i_3}$ with $1 \leq i_1 < i_2 < i_3 \leq n$. Let $g(t_{j,i_1}) \in \{2, 3\} \backslash \{g(s_{i_1,j})\}$. Since $g|_M$ is constructed from a valid truth assignment of I, at least one of the $g(m_{i_1})$, $g(m_{i_2})$ and $g(m_{i_3})$ has value 2 and at least one of them has value 3. If $g(m_{i_2}) \neq g(m_{i_1})$, then let $g(t_{j,2}) \in \{2, 3\} \backslash \{g(s_{i_2,j})\}$ and $g(t_{j,3}) = 1$, otherwise let $g(t_{j,3}) \in \{2, 3\} \backslash \{g(s_{i_3,j})\}$ and $g(t_{j,2}) = 1$.

To verify this is a valid 3-coloring, we simply go through every edge yz in $E(H_1)$. If without loss of generality $y = x$ and $z \in M \cup S$, then $g(y) = 1$ and $g(z) \in \{2, 3\}$. So $g(y) \neq g(z)$. If $y \in M$ and $z \in S$, then from the construction $g(z) \in \{2, 3\} \backslash \{g(y)\}$, so $g(z) \neq g(y)$. If $y \in S$ and $z \in T$, then from the construction either $g(z) = 1$ or $g(z) \in \{2, 3\} \backslash \{g(y)\}$, so $g(y) \neq g(z)$. If $y, z \in T$, we have $g(y) \neq g(z)$ as we use all three colors to color the vertices whose corresponding variables are in the same clause.

(3) *The ordered graph (H_2, τ_4) is J_7, J_{13} and J_{14}-free.*

Suppose for a contradiction that (H_2, τ_4) contains an induced path $w_2w_1w_4w_3$ with $\tau_4(w_1) < \tau_4(w_2) < \tau_4(w_3) < \tau_4(w_4)$. Now we consider the vertex w_1. Since w_1 has two non-adjacent forward neighbors, we know that $w_1 \notin S \cup T$. If $w_1 = x$, then $w_2, w_4 \in M \cup S$. But from the construction of τ_4, we also have $w_3 \in M \cup S$, which implies $w_3 \in S$ and so $w_1w_3 \in E(H_2)$ as a contradiction. If $w_1 \in M$, then $w_2, w_4 \in S$, which implies $w_1w_3 \in E(H_2)$ as a contradiction. Thus, we have proved (H_2, τ_4) is J_7-free.

Suppose (H_2, τ_4) contains an induced subgraph $(\{w_1, w_2, w_3, w_4, w_5\}, \{w_1w_5, w_2w_3, w_3w_4\})$ with $\tau_4(w_1) < \tau_4(w_2) < \tau_4(w_3) < \tau_4(w_4) < \tau_4(w_5)$. Now we consider the vertex w_1. Since $w_1w_5 \in E(H_2)$ and $w_1w_2 \notin E(H_2)$, we have $w_1 \notin \{x\}$. If $w_1 \in M$, we have $w_5 \in S$, which causes $w_2 \in M$ and $w_3 \in S$. But then w_4 has no place to go, which is a contradiction. If $w_1 \in T$, we know that $w_2, w_3, w_4 \in T$. But then from the construction of H_2 and τ_4, we have $w_2w_4 \in E(H_2)$, which is a contradiction. If $w_1 \in S$, then $w_5 \in T$. Since $w_2w_4 \notin E(H_2)$, at least one of w_2, w_3, w_4 is in S. From the construction of τ_4, we know that $w_2 \in S$. So the forward neighbor w_3 of w_2 is in T. But from the construction of τ_4, the inequality $\tau_4(w_1) < \tau_4(w_2)$ implies $\tau_4(w_5) < \tau_4(w_3)$, which is a contradiction. Thus, we have proved (H_2, τ_4) is J_{13}-free.

A similar argument holds for the case that (H_2, τ_4) is J_{14}-free.

□

In the third construction, we will use the following result.

Theorem 1.3.9. *The* List-3-Coloring Problem *restricted to bipartite graphs is NP-complete.*

Theorem 3.2.5. *The* Ordered Graph List-3-Coloring Problem *restricted to J_9 or J_{15}-free ordered graphs is NP-complete.*

Proof. Given a bipartite graph G with bipartition (X, Y) and its list assignment L, we construct an ordered graph (G, τ_5) as follows. We enumerate the set $X = \{x_1, ..., x_s\}$ and $Y = \{y_1, ..., y_t\}$. Let $\tau_5 : V(G) \to \mathbb{Z}$ be a function with $\tau_5(x_i) = i$ for $i \in [s]$ and $\tau_5(y_j) = s + j$ for $j \in [t]$.

Clearly, the ordered graph (G, τ_5) can be computed in time $O(n)$, and (G, τ_5) is list-3-colorable if and only if G is list-3-colorable. The ordered graph (G, τ_5) is J_9 and J_{15}-free, as for every edge $zw \in E(G)$, without loss of generality, we have $z \in X$ and $w \in Y$. $\square$

Corollary 3.2.6. *If the* Ordered Graph List-3-Coloring Problem *restricted to H-free ordered graphs is NP-complete, then the* Ordered Graph List-3-Coloring Problem *restricted to $-H$-free ordered graphs is NP-complete.*

Now we are ready to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Let (H, φ) be an ordered graph. If H contains a copy of P_4, say Q_1, then since $(Q_1, \varphi|_{Q_1}) \in \{J_i : i \in [8]\} \cup \{-J_i : i \in [8]\}$, by Theorems 3.2.3, 3.2.4, 3.2.5 and Corollary 3.2.6, we have the Ordered Graph List-3-Coloring Problem restricted $(Q_1, \varphi|_{Q_1})$-free ordered graphs is NP-complete.

If H contains a copy of $P_3 + P_2$, we denote it $Q_2 = (\{v_1, v_2, v_3, v_4, v_5\}, \{v_1v_2, v_2v_3, v_4v_5\})$. By symmetry, we assume without loss of generality that $\varphi(v_4) < \varphi(v_5)$. Then we consider $\min\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\}$ and $\max\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\}$. If $\max\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\} < \varphi(v_4)$ or $\min\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\} > \varphi(v_5)$, then Theorem 3.2.5 indicates the Ordered Graph List-3-Coloring Problem restricted $(Q_2, \varphi|_{Q_2})$-free ordered graphs is NP-complete.

If $\varphi(v_4) < \min\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\}$ and $\max\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\} < \varphi(v_5)$, then Q_2 either contains a copy of J_{13} or $-J_{13}$ as induced subgraph, or a copy of J_{14} or $-J_{14}$. By Theorem 3.2.4, the Ordered Graph List-3-Coloring Problem restricted $(Q_2, \varphi|_{Q_2})$-free ordered graphs is NP-complete.

If $\min\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\} < \varphi(v_i) < \max\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\}$ for some $i \in \{4, 5\}$, then Q_2 either contains a copy of J_{10} as induced subgraph, or a copy of J_{11}, or a copy of J_{12}, or

$-J_{10}$, $-J_{11}$, $-J_{12}$. By Theorem 3.2.3 and Corollary 3.2.6, the Ordered Graph List-3-Coloring Problem restricted $(Q_2, \varphi|_{Q_2})$-free ordered graphs is NP-complete. ☐

Before we start proving Theorem 3.2.2, let us prove the following lemma:

Lemma 3.2.7. *Given a graph G, if a graph H satisfies the following conditions:*

1. *$V(G) \subseteq V(H)$.*
2. *For every edge $uv \in E(G)$, we have $uv \notin E(H)$ and there are three vertex disjoint uv-paths P_1^{uv}, P_2^{uv} and P_3^{uv} of length at least 3. Moreover, for all edges $uv, st \in E(G)$ and $i, j \in [3]$, we have $V(P_i^{uv}) \cap V(P_j^{st}) = \{u, v\} \cap \{s, t\}$.*
3. *The correspondence between every edge $uv \in E(G)$ and its paths P_1^{uv}, P_2^{uv} and P_3^{uv} in H is given.*
4. *Every edge in H is contained in some P_i^e, for $i \in [3]$ and $e \in E(G)$.*

Then there is a list assignment $L : V(H) \to 2^{[3]}$ such that:

1. *The list assignment L can be computed from H in time $O(|E(H)|)$.*
2. *For every vertex $u \in V(G)$, we have $L(u) = [3]$.*
3. *For every L-coloring f of H, the function $f|_{V(G)}$ is a 3-coloring of G.*
4. *For every 3-coloring g of G, there is a corresponding L-coloring g' of H with $g'|_{V(G)} = g$.*

Note:

- We say a pair (H, L) as in Lemma 3.2.7 a *realization* of G.
- As the 3-Coloring Problem is NP-hard, to decide whether H is L-colorable is also NP-hard.

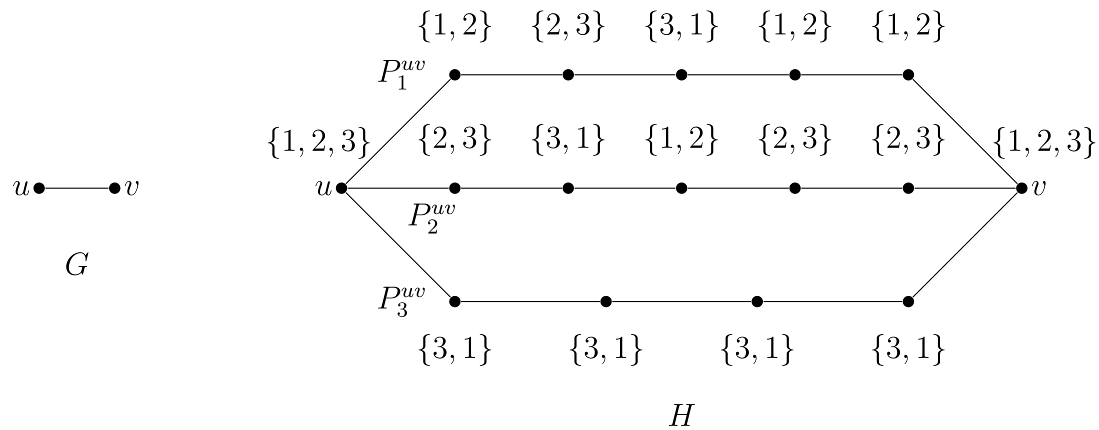

Figure 3.5: An example of a realization (H, L) of G. Each vertex is labeled with its list.

Proof. For every edge $uv \in E(G)$, let the three vertex disjoint uv-paths be P_1^{uv}, P_2^{uv} and P_3^{uv} of length at least 4. For convenience, in this proof, we read every color modulo 3 (so if this would assign color 4, we assign color 1 instead). We define the list assignment $L : V(H) \to 2^{[3]}$ as follows. Let $L(u) = [3]$ for every vertex $u \in V(G)$. Then we take a path P_i^{uv}, $i \in [3]$ and denote $P_i^{uv} = uw_1w_2...w_tv$. If t is even, we set $L(w_j) = \{i, i+1\}$ for all $i \in [t]$. If t is odd, we set $L(w_j) = \{i+j-1, i+j\}$ for $j \in [3]$, and $L(w_j) = \{i, i+1\}$ for $i \in \{4, ..., t\}$.

(1) *The list assignment L can be computed from H in time $O(|E(H)|)$, if given the correspondence between every edge $uv \in E(G)$ and its paths P_1^{uv}, P_2^{uv} and P_3^{uv} in H.*

For a given path P_i^e, defining $L|_{V(P_i^e)}$ takes time $|E(P_i^e)|$. So the running time is $\sum_{e\in E(G)} \sum_{i=1}^{3} |E(P_i^e)| = |E(H)|$.

(2) *For every vertex $u \in V(G)$, we have $L(u) = [3]$.*

This holds immediately from the construction.

(3) *For every L-coloring f of H, the function $f|_{V(G)}$ is a 3-coloring of G.*

Suppose not, then there is an L-coloring f of H such that there is an edge $uv \in E(G)$ with $f(u) = f(v) = i$ for some $i \in [3]$. We consider the path $P_i^{uv} = uw_1w_2...w_tv$ in H. If t is even, then from the construction we have that $f(w_j) = i+1$ if j is odd, and $f(w_j) = i$ if

j is even. But then we have $f(w_t) = f(v)$, which leads to a contradiction. If t is odd, then from the construction we have $f(w_1) = i+1$, $f(w_2) = i+2$, $f(w_3) = i$, and for $j \in \{4, ..., t\}$, $f(w_j) = i+1$ if j is even and $f(w_j) = i$ if j is odd. But then $f(w_t) = f(v) = i$, which is a contradiction. Thus $f|_{V(G)}$ is a 3-coloring of G.

(4) *For every 3-coloring g of G, there is a corresponding L-coloring g' of H with $g'|_{V(G)} = g$.*

For every $u \in V(G)$, let $g'(u) = g(u)$. For every edge $uv \in V(G)$, we denote $g(u) = a$ and $g(v) = b$, $\{a, b\} \subseteq [3]$. Then we consider the path $P_i^{uv} = uw_1w_2...w_tv$. We may assume that $a \in L(w_1)$ and $b \in L(w_t)$, for otherwise P_i^{uv} is L-colorable.

If t is even, let $g'(w_1) \in L(w_1)\backslash\{a\} \neq \emptyset$, $g'(w_2) = a$, and $g'(w_j) = g'(w_1)$ if $j \in \{3, ..., t\}$ is odd, and $g'(w_j) = g'(w_2)$ if $j \in \{3, ..., t\}$ is even. Notice that we have $g'(w_t) = g'(w_2) = a$. Since $a \neq b$, we have $g'(v) \neq g'(w_t)$.

So let us assume t is odd. If $a = i$, or $a = i+2$ and $b = i+1$, then we set $g'(w_1) = i+1$, $g'(w_2) = i+2$, $g'(w_3) = i$, and $g'(w_j) = i$ if $j \in \{4, ..., t\}$ odd, $g'(w_j) = i+1$ if $j \in \{4, ..., t\}$ even. Thus $g'(w_t) = i \neq b = g'(v)$. If $a = i+1$, or $a = i+2$ and $b = i$, then we set $g'(w_1) = i$, $g'(w_2) = i+1$, $g'(w_3) = i+2$, and $g'(w_j) = i+1$ if $j \in \{4, ..., t\}$ is odd, $g'(w_j) = i$ if $j \in \{4, ..., t\}$ is even. Thus $g'(w_t) = i+1 \neq b = g'(v)$.

Therefore, we have defined an L-coloring g' of H with $g'|_{V(G)} = g$. □

The proof of Theorem 3.2.2 is divided into three constructions, all of which use Lemma 3.2.7 as a helper method.

Theorem 3.2.8. *Given a graph G, there is a graph H_3 and two injective functions $\tau_5, \tau_6 : V(G) \to \mathbb{R}$ such that:*

1. *There is a list assignment $L_1 : V(H_3) \to 2^{[3]}$ such that the pair (H_3, L_1) is a realization of G.*

2. *The ordered graphs (H_3, τ_5) and (H_3, τ_6) can be constructed from G in time $O(m^2)$, where $m = |E(G)|$.*

3. *The ordered graph (H_3, τ_5) is M_1 and M_2-free.*

4. *The ordered graph (H_3, τ_6) is M_3-free.*

Therefore, the Ordered Graph List-3-Coloring Problem *is NP-complete when restricted to the class of M_1, M_2 or M_3-free ordered graphs.*

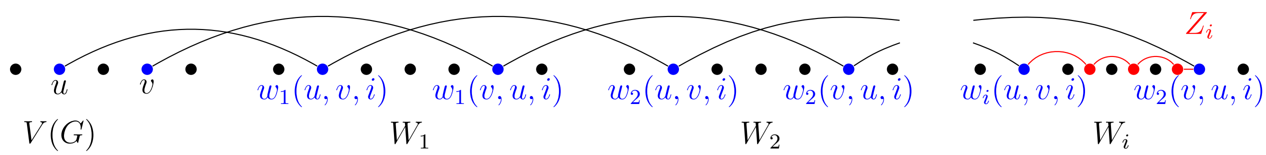

Figure 3.6: The construction of (H_3, τ_5) from Theorem 3.2.8.

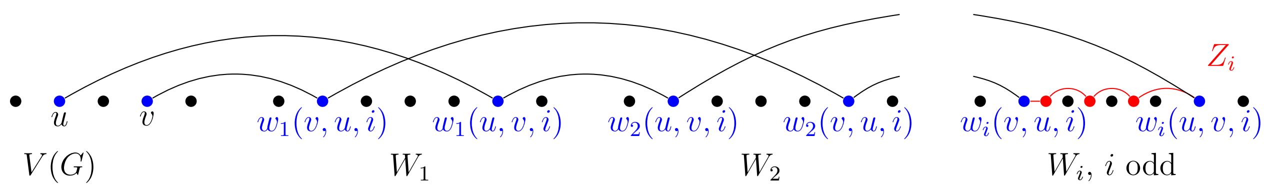

Figure 3.7: The construction of (H_3, τ_6) from Theorem 3.2.8.

Proof. We denote $|V(G)| = n$ and $|E(G)| = m$. Let $f : E(G) \to [m]$ be an ordering of $E(G)$, and $g : V(G) \to [n]$ be an ordering of $V(G)$. We construct the ordered graphs (H_3, τ_5) and (H_3, τ_6) as follows (see Figures 3.6 and 3.7).

For every edge $uv \in E(G)$, we create 6 vertices $w_1(u,v,j)$ and $w_1(v,u,j)$ for $j \in \{3f(uv)-2, 3f(uv)-1, 3f(uv)\}$, and add edges $uw_1(u,v,j)$ and $vw_1(v,u,j)$ for $j \in \{3f(uv)-2, 3f(uv)-1, 3f(uv)\}$. Let W_1 be the set of such vertices $w_1(u,v,j)$. Suppose now W_{i-1} has been defined. We create a new vertex $w_i(u,v,j)$ if $w_{i-1}(u,v,j) \in W_{i-1}$ and $j \geq i$, and add an edge $w_{i-1}(u,v,j)w_i(u,v,j)$. Let W_i be the set of such vertices $w_i(u,v,j)$. For convenience, we also denote $W_0 = V(G)$.

We define $\tau_5(v) = \tau_6(v) = g(v)$ for every $v \in V(G)$. For every $i \in \{1, ..., 3m\}$, we define $\tau_5(w_i(u,v,j)) = n + (\sum_{i'=1}^{i-1} |W_{i'}|) + |\{w_i(x,y,k) \in W_i : g(x) < g(u)\}| + |\{w_i(u,y,k) \in W_i : g(y) < g(v)\}| + j + 3 - 3f(uv)$ for every $w_i(u,v,j) \in W_i$.

Let us consider the vertices $w_i(u',v',i)$ and $w_i(v',u',i)$ in W_i. Without loss of generality we may assume that $\tau_5(w_i(u',v',i)) < \tau_5(w_i(v',u',i))$. For every vertex $w_i(u,v,j) \in W_i$ with $\tau_5(w_i(u',v',i)) < \tau_5(w_i(u,v,j)) < \tau_5(w_i(v',u',i))$, we add one new vertex $z_i(u,v,j)$. Let Z_i be the set of such vertices $z_i(u,v,j)$. Let $\tau_5(z_i(u,v,j)) = \tau_5(w_i(u,v,j)) + \frac{1}{2}$. For every $z_i(u,v,j), z_i(u^*,v^*,j^*) \in Z_i$ with $\tau_5(z_i(u,v,j)) < \tau_5(z_i(u^*,v^*,j^*))$, we add edges $z_i(u,v,j)z_i(u^*,v^*,j^*)$ if $\tau_5(z_i(u,v,j)) = \tau_5(z_i(u^*,v^*,j^*)) - 1$, and edges $w_i(u',v',i)z_i(u,v,j)$ if $\tau_5(z_i(u,v,j)) = \tau_5(w_i(u',v',i)) + \frac{3}{2}$, and $w_i(v',u',i)z_i(u^*,v^*,j^*)$ if $\tau_5(z_i(u^*,v^*,j^*)) = \tau_5(w_i(v',u',i)) - \frac{1}{2}$.

We define $\tau_6(v)$ as follows. Let $\tau_6(v) = \tau_5(v)$ for every $v \in V(G)$. For every $i \in \{1, ..., 3m\}$ and $w_i(u,v,j) \in W_i$, we define $\tau_6(w_i(u,v,j)) = \tau_5(w_i(u,v,j))$ if i is even, and $\tau_6(w_i(u,v,j)) = n + (\sum_{i'=1}^{i-1} |W_{i'}|) + |W_i| + 1 - (|\{w_i(x,y,k) \in W_i : g(x) < g(u)\}| + |\{w_i(u,y,k) \in W_i : g(y) < g(v)\}| + j + 3 - 3f(uv))$ if i is odd. For every $i \in \{1, ..., 3m\}$ and $z_i(u,v,j) \in Z_i$, let $\tau_6(z_i(u,v,j)) = \tau_6(w_i(u,v,j)) + \frac{1}{2}$ if i odd, and $\tau_6(z_i(u,v,j)) = \tau_6(w_i(u,v,j)) - \frac{1}{2}$ if i is even.

Let $V(H_3) = V(G) \cup \bigcup_{i=1}^{3m}(W_i \cup Z_i)$ and $E(H_3)$ be the set of all edges defined above. From the construction, the functions τ_5 and τ_6 are orderings of H_3.

We then let P_i^{uv} consist of vertices u, v, $w_1(u,v,3f(uv)+i-3)$, $w_2(u,v,3f(uv)+i-3)$, ..., $w_{3f(uv)+i-3}(u,v,3f(uv)+i-3)$, $w_1(v,u,3f(uv)+i-3)$, $w_2(v,u,3f(uv)+i-3)$, ..., $w_{3f(uv)+i-3}(v,u,3f(uv)+i-3)$ and all vertices in $Z_{3f(uv)+i-3}$, for $uv \in E(G)$ and $i \in [3]$. The graph H_3 and the paths P_i^{uv} satisfy the condition of Lemma 3.2.7. Thus, letting L_1 be as in Lemma 3.2.7, we have:

(1) *The pair (H_3, L_1) is a realization of G.*

(2) *The ordered graphs (H_3, τ_5) and (H_3, τ_6) can be computed from G in time $O(m^2)$.*

It takes time $O(m)$ and $O(n)$ to get the functions f and g, respectively. The set W_1 can be computed from G and f in time $O(m)$. For $i \in \{2, ..., 3m\}$, the set W_i can be computed from W_{i-1} in time $O(m)$. And the function $\tau_5|_{W_i}$ can be computed in time $O(m)$ for $i \in [3m]$. The set Z_i and the function $\tau_5|_{Z_i}$ can be computed from W_i and $\tau_5|_{W_i}$ in time $O(m)$. Finally, the function τ_6 can be computed in time $O(m^2)$. Thus, the ordered graphs (H_3, τ_5) and (H_3, τ_6) can be computed from G in time $O(m^2)$.

From the construction defined above, we notice that:

(3) *Given $k \in \{5, 6\}$, for each edge $uv \in E(H_3)$ with $\tau_k(u) < \tau_k(v)$ and $u \in Z_i$ or $v \in Z_i$ for some $i \in \{1, ..., 3m\}$, there is at most one vertex w with $\tau_k(u) < \tau_k(w) < \tau_k(v)$.*

With this observation, we are ready to prove the remaining two properties.

(4) *The ordered graph (H_3, τ_5) is M_1 and M_2-free.*

Suppose (H_3, τ_5) contains a copy of M_1 or M_2 as ordered induced subgraph. Let us consider the vertex v_1. Because of the vertices v_3, v_4, by (3) we have $v_1, v_2, v_5, v_6 \notin Z_i$ for every $i \in [3m]$. If $v_1 \in V(G)$, then we have $v_6 \in W_1$. So $v_2 \in V(G)$ and $v_5 \in W_1$. But then either $\tau_5(v_2) < \tau_5(v_1)$ or $\tau_5(v_5) > \tau_5(v_6)$, both of which cause a contradiction. If $v_1 \in W_i$ for some $i \in [3m]$, then $v_6 \in W_{i+1}$, and $v_2 \in W_i$, $v_5 \in W_{i+1}$, which is a contradiction. Thus, we have proved (H_3, τ_5) is M_1-free and M_2-free.

(5) *The ordered graph (H_3, τ_6) is M_3-free.*

Suppose (H_3, τ_6) contains a copy of M_3 as ordered induced subgraph. Let us consider the vertex v_1. Because of the vertices v_2, v_3, by (3) we have $v_1, v_4 \notin Z_i$ for every $i \in [3m]$. Similarly, we have $v_t \notin Z_i$ for every $t \in [6]$ and $i \in [3m]$. Let $v_1 \in W_i$ for some $i \in \{0, 1, ..., 3m\}$, then $v_4 \in W_{i+1}$. Then let us consider the vertex v_2. For every vertex $x \in W_i$ with $\tau_6(x) > \tau_6(v_1)$ and for every vertex $y \in N(x)$ with $\tau_6(y) > \tau_6(x)$ and $y \notin Z_i$, we have $\tau_6(y) < \tau_6(v_4)$. Thus, the vertex v_2 is in W_{i+1}. But then for every vertex $x \in W_{i+1}$ with $\tau_6(v_2) < \tau_6(x) < \tau_6(v_4)$ and for every vertex $y \in N(x)$ with $\tau_6(y) > \tau_6(x)$ and $y \notin Z_i$, we have $\tau_6(y) < \tau_6(v_5)$, which means the vertex v_3 has nowhere to go. Thus, we have proved (H_3, τ_6) is M_3-free. $\square$

Theorem 3.2.9. *Given a graph G, there is an ordered graph (H_4, τ_7) and a list assignment $L_2 : V(H_4) \to 2^{[3]}$ such that:*

1. *The pair (H_4, L_2) is a realization of G.*
2. *The ordered graph (H_4, τ_7) can be constructed from G in time $O(m^2)$.*
3. *The ordered graph (H_4, τ_7) is M_4-free.*

Therefore, the Ordered Graph List-3-Coloring Problem *is NP-complete when restricted to the class of M_4-free ordered graphs.*

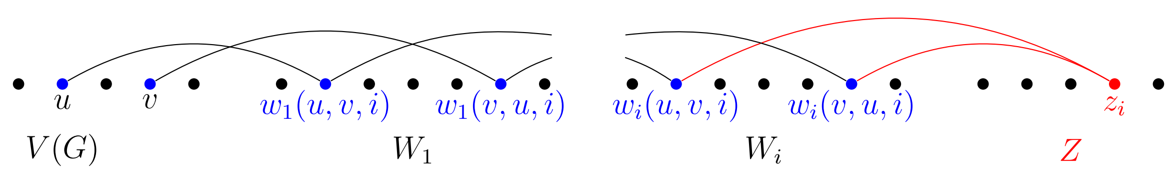

Figure 3.8: The construction of (H_4, τ_7) from Theorem 3.2.9.

Proof. We denote $|V(G)| = n$ and $|E(G)| = m$. Let $f : E(G) \to [m]$ be an ordering of $E(G)$, and $g : V(G) \to [n]$ be an ordering of $V(G)$. We construct the ordered graph (H_4, τ_7) as follows.

For every edge $uv \in E(G)$, we create 6 vertices $w_1(u,v,j)$ and $w_1(v,u,j)$ for $j \in \{3f(uv)-2, 3f(uv)-1, 3f(uv)\}$, and add edges $uw_1(u,v,j)$ and $vw_1(v,u,j)$ for $j \in \{3f(uv)-2, 3f(uv)-1, 3f(uv)\}$. Let W_1 be the set of such vertices $w_1(u,v,j)$. Suppose now W_{i-1} has been defined. We create a new vertex $w_i(u,v,j)$ if $w_{i-1}(u,v,j) \in W_{i-1}$ and $j \geq i$, and add an edge $w_{i-1}(u,v,j)w_i(u,v,j)$. Let W_i be the set of such vertices $w_i(u,v,j)$. For convenience, we also denote $W_0 = V(G)$.

We define $\tau_7(v) = g(v)$ for every $v \in V(G)$. For every $i \in \{1,...,3m\}$, we define $\tau_7(w_i(u,v,j)) = n + \sum_{i'=1}^{i-1} |W_{i'}| + |\{w_i(x,y,k) \in W_i : g(x) < g(u)\}| + |\{w_i(u,y,k) \in W_i : g(y) < g(v)\}| + j + 3 - 3f(uv)$ for every $w_i(u,v,j) \in W_i$.

Let us consider the vertices $w_i(u',v',i)$ and $w_i(v',u',i)$ in W_i. For every $i \in \{1,...,3m\}$, we add one new vertex z_i and edges $w_i(u',v',i)z_i$ and $w_i(v',u',i)z_i$, and let $\tau_7(z_i) = n + 3m(3m+1) + (3m-i+1)$.

Let $V(H_4) = V(G) \cup (\bigcup_{i=1}^{3m} W_i \cup \{z_i\})$ and $E(H_4)$ be the set of all edges defined above. From the construction, the function $\tau_7 : V(H_4) \to \mathbb{R}$ is an ordering of H_4.

We then let P_i^{uv} consist of vertices u, v, $w_1(u,v,3f(uv)+i-3)$, $w_2(u,v,3f(uv)+i-3)$, ..., $w_{3f(uv)+i-3}(u,v,3f(uv)+i-3)$, $w_1(v,u,3f(uv)+i-3)$, $w_2(v,u,3f(uv)+i-3)$, ..., $w_{3f(uv)+i-3}(v,u,3f(uv)+i-3)$ and $z_{3f(uv)+i-3}$, for $uv \in E(G)$ and $i \in [3]$. The graph H_4 and the paths P_i^{uv} satisfy the condition of Lemma 3.2.7. Thus, letting L_2 be as in Lemma 3.2.7, we have:

(1) *The pair (H_4, L_2) is a realization of G.*

(2) *The ordered graph (H_4, τ_7) can be computed from G in time $O(m^2)$.*

It takes time $O(m)$ and $O(n)$ to get the functions f and g, respectively. The set W_1 can be computed from G and f in time $O(m)$. For $i \in \{2,...,3m\}$, the set W_i can be computed from W_{i-1} in time $O(m)$. And the function $\tau_7|_{W_i}$ can be computed in time $O(m)$ for $i \in [3m]$. Thus, the ordered graph (H_4, τ_7) can be computed from G in time $O(m^2)$.

(3) *The ordered graph (H_2, τ_7) is M_4-free.*

Suppose that (H_4, τ_7) contains a copy of M_4 as ordered induced subgraph. Because of the edges of M_4, we have $v_1, v_2, v_3 \notin \{z_1, ..., z_{3m}\}$, and $v_1, v_2 \notin W_{3m}$. Let $v_1 \in W_i$ for some $i \in \{0, ..., 3m-1\}$. If $v_5 \in W_{i+1}$, then we have $v_4 \notin \{z_1, ..., z_{3m}\}$. For every vertex x with $\tau_7(v_1) < \tau_7(x) < \tau_7(v_5)$ and for every vertex $y \in N(x) \backslash \{z_1, ..., z_{3m}\}$ with $\tau_7(y) > \tau_7(x)$, we have $\tau_7(y) > \tau_7(v_5)$. Thus, we conclude that $v_5 \in \{z_1, ..., z_{3m}\}$. But then for every vertex $x \in \{z_1, ..., z_{3m}\}$ with $\tau_7(x) > \tau_7(v_5)$ and for every vertex $y \in N(x)$, we have $\tau_7(y) < \tau_7(v_1)$, which is a contradiction. Thus, we have proved (H_4, τ_7) is M_4-free. $\square$

Theorem 3.2.10. *Given a graph G, there is an ordered graph (H_5, τ_8) and a list assignment $L_3 : V(H_5) \to 2^{[3]}$ such that:*

1. *The pair (H_5, L_3) is a realization of G.*

2. *The ordered graph (H_5, τ_8) can be constructed from G in time $O(m^3)$.*

3. *The ordered graph (H_5, τ_8) is M_5-free.*

Therefore, the Ordered Graph List-3-Coloring Problem *is NP-complete when restricted to the class of M_5-free ordered graphs.*

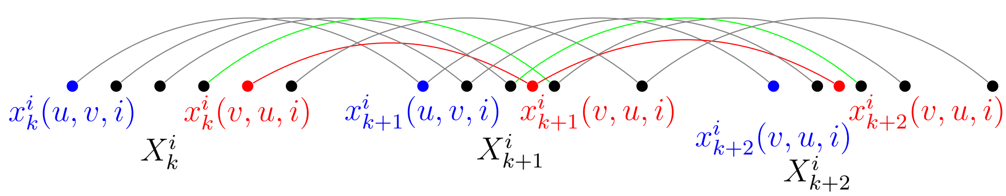

Figure 3.9: The construction of X_k^i, X_{k+1}^i and X_{k+2}^i in (H_5, τ_8) from Theorem 3.2.10.

Proof. We denote $|V(G)| = m$ and $|E(G)| = n$. Let $f : E(G) \to [m]$ be an ordering of $E(G)$, and $g : V(G) \to [n]$ be an ordering of $V(G)$. We construct the ordered graph (H_5, τ_8) as follows.

For every edge $uv \in E(G)$, we create 6 vertices $w_1(u, v, j)$ and $w_1(v, u, j)$ for $j \in \{3f(uv) - 2, 3f(uv) - 1, 3f(uv)\}$, and add edges $uw_1(u, v, j)$ and $vw_1(v, u, j)$ for $j \in \{3f(uv) - 2, 3f(uv) - 1, 3f(uv)\}$. Let W_1 be the set of such vertices $w_1(u, v, j)$. Suppose now W_{i-1} has been defined. We create a new vertex $w_i(u, v, j)$ if $w_{i-1}(u, v, j) \in W_{i-1}$ and $j \geq i$. Let W_i be the set of such vertices $w_i(u, v, j)$. For convenience, we also denote $W_0 = V(G)$.

We define $\tau_8(v) = g(v)$ for every $v \in V(G)$. For every $i \in \{1, ..., 3m\}$, we define $\tau_8(w_i(u, v, j)) = 36m^2(i - 1) + n + \sum_{i'=1}^{i-1} |W_{i'}| + |\{w_i(x, y, k) \in W_i : g(x) < g(u)\}| + |\{w_i(u, y, k) \in W_i : g(y) < g(v)\}| + j + 3 - 3f(uv)$ for every $w_i(u, v, j) \in W_i$.

Let us consider the vertices $w_i(u', v', i)$ and $w_i(v', u', i)$ in W_i. Without loss of generality we may assume that $\tau_8(w_i(u', v', i)) = \tau_8(w_i(v', u', i)) - a_i$ for some $a_i \in \mathbb{N}$. We create a sequence of sets $X_k^i = \{x_k^i(u, v, j) : w_i(u, v, j) \in W_i\}$, for $k \in [a_i - 1]$. For convenience, we also denote $X_{a_i}^i = W_{i+1} = X_0^{i+1}$. For every $i \in [3m]$, we add edges $x_k^i(u, v, j)x_{k+1}^i(u, v, j)$ for every $k \in \{0, ..., a_i - 1\}$ and $x_{k+1}^i(u, v, j) \in X_{k+1}^i$, and edge $x_{a_i-1}^i(u', v', i)x_{a_i-1}^i(v', u', i)$.

For every $k \in [a_i - 1]$, let $\tau_8(x_k^i(v', u', i)) = \tau_8(w_i(v', u', i)) + 6mk - k - \frac{1}{2}$, and $\tau_8(x_k^i(u, v, j)) = \tau_8(w_i(u, v, j)) + 6mk$ otherwise. Notice that this implies $\tau_8(x_{k+1}^i(v', u', i)) - \tau_8(x_k^i(v', u', i)) = \tau_8(x_{k+1}^i(u, v, j)) - \tau_8(x_k^i(u, v, j)) - 1$ for every $k \in \{0, ..., a_i - 1\}$ and $\{u, v\} \neq \{u', v'\}$ and $x_{k+1}^i(u, v, j) \in X_{k+1}^i$.

Let $V(H_5) = V(G) \cup (\bigcup_{i=1}^{3m} W_i \cup (\bigcup_{k=1}^{a_i-1} X_k^i))$ and $E(H_5)$ be the set of all edges defined above. From the construction, the function $\tau_8 : V(H_5) \to \mathbb{R}$ is an ordering of H_5.

For $uv \in E(G)$ and $i \in [3]$, we then let P_i^{uv} consist of vertices u, v, $w_1(u, v, 3f(uv) + i - 3)$, $w_2(u, v, 3f(uv) + i - 3)$, ..., $w_{3f(uv)+i-3}(u, v, 3f(uv) + i - 3)$, $w_1(v, u, 3f(uv) + i - 3)$, $w_2(v, u, 3f(uv) + i - 3)$, ..., $w_{3f(uv)+i-3}(v, u, 3f(uv) + i - 3)$ and all vertices $x_k^j(u, v, 3f(uv) + i - 3)$ and $x_k^j(v, u, 3f(uv) + i - 3)$ for $j \in [3f(uv) + i - 3]$ and $k \in [a_j - 1]$. The graph H_5 and the paths P_i^{uv} satisfy the conditions of Lemma 3.2.7. Thus, letting L_3 be as in Lemma 3.2.7, we have:

(1) *The pair (H_5, L_3) is a realization of G.*

(2) *The ordered graph (H_5, τ_8) can be computed from G in time $O(m^3)$.*

The set W_1 can be computed from G and f in time $O(m)$. For $i \in \{2, ..., 3m\}$, the set W_i can be computed from W_{i-1} in time $O(m)$. And the function $\tau_8|_{W_i}$ can be computed in time $O(m)$ for $i \in [3m]$. The set X_k^i and the function $\tau_8|_{X_k^i}$ can be computed from W_i and $\tau_8|_{W_i}$ in time $O(m)$ for $k \in [a_i - 1]$. And $a_i \leq |W_i| = O(m)$, $|X_k^i| = |W_i|$ for $k \in [a_i - 1]$. Thus, the ordered graph (H_5, τ_8) can be computed from G in time $O(m^3)$.

(3) *The ordered graph (H_5, τ_8) is M_5-free.*

Suppose (H_5, τ_8) contains a copy of M_5 as ordered induced subgraph. Let us consider the edges v_1v_5 and v_2v_3. We claim that $v_2 = x_k^i(v', u', i)$ and $v_3 = x_{k+1}^i(v', u', i)$ for some $i \in [3m]$ and $k \in [a_i - 1]$, and u' and v' being the vertices in $V(H_5)$ such that $g(u') < g(v')$ and $f(u'v') = \lceil \frac{i}{3} \rceil$. This is because otherwise from the construction of τ_8, the condition $\tau_8(v_1) < \tau_8(v_2)$ implies $\tau_8(y) < \tau_8(z)$ for every $y \in N^+(v_1)$ and $z \in N^+(v_2)$.

Since v_1v_5 is an edge, we have $v_1 \in X_k^i$ and $v_5 \in X_{k+1}^i$. Moreover, from the construction of τ_8, for every two distinct $x, x' \in X_{k+1}^i \backslash \{x_k^i(v', u', i)\}$, we have $|\tau_8(x) - \tau_8(x')| \geq 1$. Since $\tau_8(x_{k+1}^i(v', u', i)) - \tau_8(x_k^i(v', u', i)) = \tau_8(x_{k+1}^i(u, v, i)) - \tau_8(x_k^i(u, v, i)) - 1$, there is no vertex in X_{k+1}^i which could be v_4, which gives a contradiction. Thus, we have proved (H_5, τ_8) is M_5-free.

□

Thus, we have proved Theorem 3.2.2. Combining Theorems 3.2.1, 3.2.2, 1.2.17 and 1.2.3, we have proved Theorem 1.2.18.

Chapter 4

An NP-hardness result of k-Coloring

In this chapter, we prove Theorem 1.2.6 by reducing from the monotone NAE3SAT problem. Recall the theorem:

Theorem 1.2.6. *The k-*Colouring Problem *restricted to rP_4-free graphs is NP-complete for all $k \geq 5$ and $r \geq 2$.*

Let I be a monotone NAE3SAT instance with variables $x_1, x_2, ...x_n$ and clauses $C_1, C_2, ..., C_m$. Now let us construct a graph $G = (V, E)$. The set of vertices V is defined as follows:

- There are five vertices $c_1, c_2, ..., c_5$ representing colors.
- For each variable x_i, $i \in [n]$, there is a corresponding vertex x_i in V.
- For each clause C_j, $j \in [m]$, there are two corresponding vertices y_j and z_j in V.
- For each clause C_j and each $k \in [3]$, there are two vertices u_j^k and w_j^k corresponding to each literal.

and the set of edges E is defined as follows:

- For each $i, j \in [5]$ with $i \neq j$, add edges c_ic_j to E.
- For each $i \in [n]$, add edges c_3x_i, c_4x_i and c_5x_i.
- For each $j \in [m]$, add edges c_1y_j, c_2y_j, c_1z_j, c_2z_j.

- For each $j \in [m]$ and $k \in [3]$, add edges $c_1 u_j^k$ and $c_2 w_j^k$.
- For each $j \in [m]$, add edges $c_i u_j^k$ and $c_i w_j^k$ for all pairs (i, k) with $i \in \{3, 4, 5\}$, $k \in \{1, 2, 3\}$, and $i \neq k + 2$.
- For each $i \in [n]$ and $j \in [m]$, add edges $x_i y_j$ and $x_i z_j$.
- For each $j \in [m]$, add edges $y_j u_j^k$ and $z_j w_j^k$ for all $k \in [3]$.
- For each $j \in [m]$, if C_j contains $x_{i_1}, x_{i_2}, x_{i_3}$, then we add edges $x_{i_k} u_j^k$ and $x_{i_k} w_j^k$ for all $k \in [3]$.

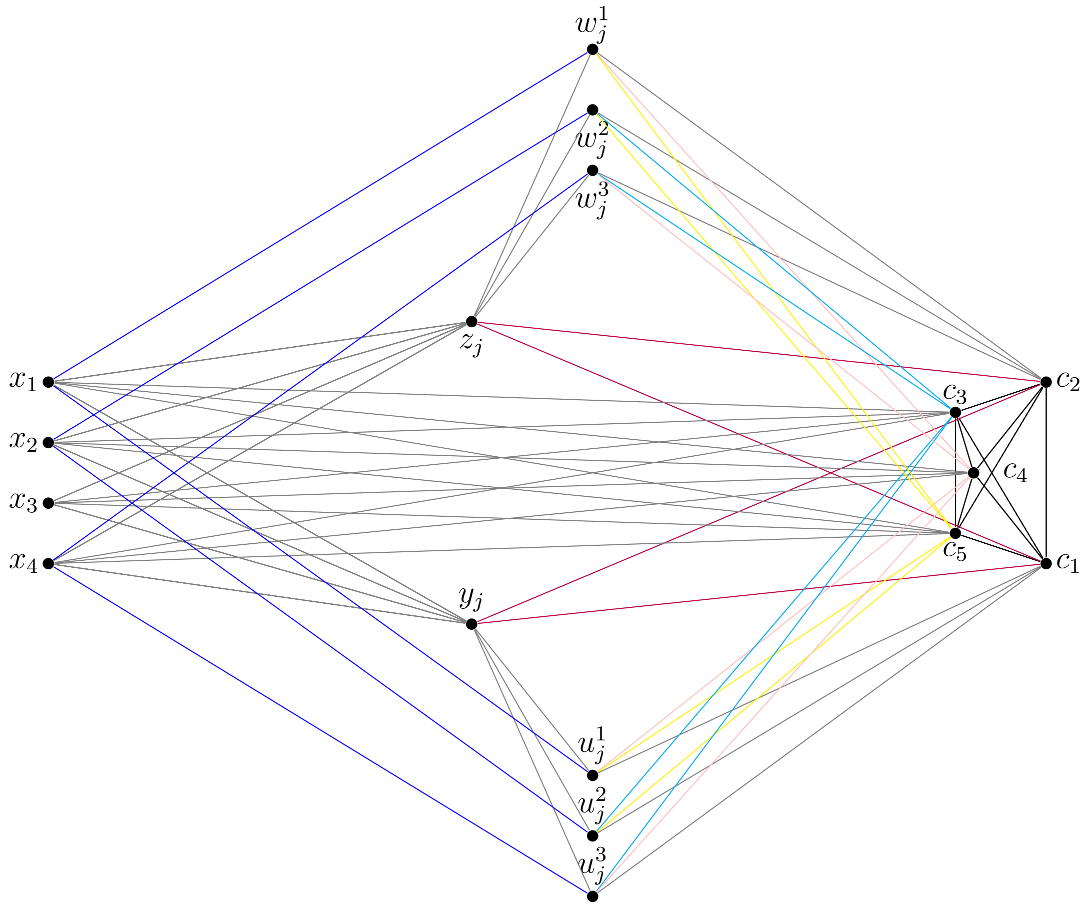

Figure 4.1: The graph G, given a monotone NAE3SAT instance with variables x_1, x_2, x_3, x_4 and a clause C_j containing variables x_1, x_2, x_4.

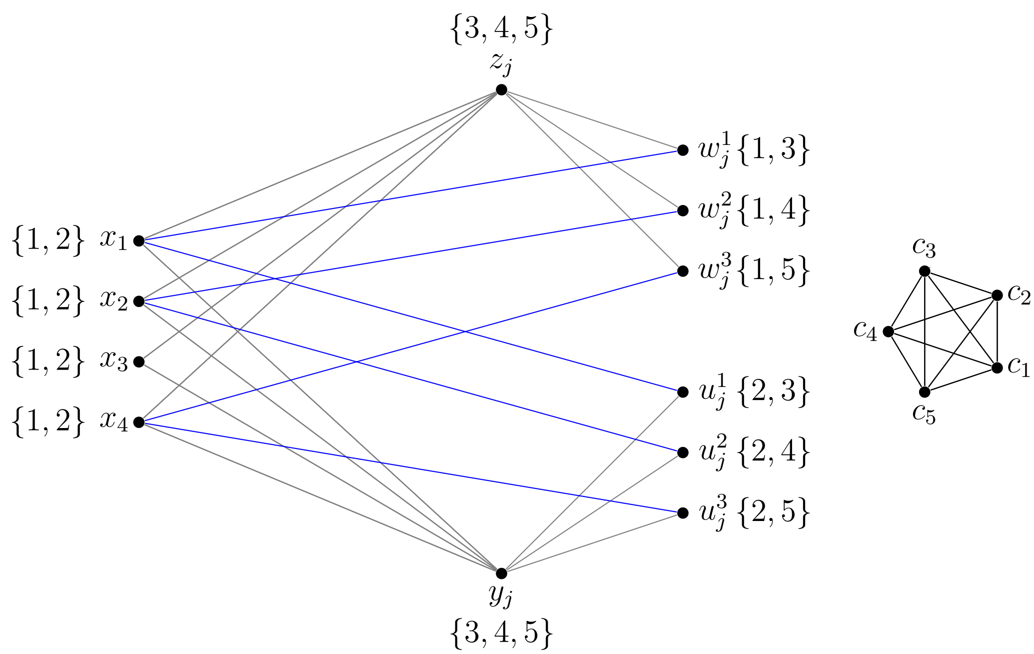

Figure 4.2: The graph G omitting edges incident to c_i for all $i \in [5]$. Each vertex is labeled with its list assuming the vertex c_i receives color i for $i \in [5]$.

From the construction, we have $|V| = n + 8m + 5$ and $|E| = 3n + 34m + 2mn + 10$. Thus the construction has polynomial size and can be done in polynomial time.

For convenience, let us denote $C = \{c_i \in V : i \in [5]\}$, $X = \{x_i \in V : i \in [n]\}$, $U = \{u_j^k \in V : j \in [m], k \in [3]\} \cup \{w_j^k \in V : j \in [m], k \in [3]\}$ and $Y = \{y_j \in V : j \in [m]\} \cup \{z_j \in V : j \in [m]\}$.

Theorem 4.0.1. *A monotone NAE3SAT instance I is satisfiable if and only if G is 5-colorable.*

Proof. Suppose $f : V \to [5]$ is a 5-coloring of G. Since the set $\{c_i : i \in [5]\}$ forms a clique, we may assume that $f(c_i) = i$ for every $i \in [5]$. Thus $f(x_i) \in \{1, 2\}$ for every vertex x_i.

Now take a clause C_j, and let $x_{i_1}, x_{i_2}, x_{i_3}$ be the literals in C. We know that at least one of $f(x_{i_1})$, $f(x_{i_2})$ and $f(x_{i_3})$ is equal to 1. The reason is, if $f(x_{i_1})$, $f(x_{i_2})$ and $f(x_{i_3})$ are all equal to 2, then $f(u_j^1) = 3$, $f(u_j^2) = 4$ and $f(u_j^3) = 5$ hold at the same time. But then the vertex y_j has one neighbour of each color, which is a contradiction. Similarly, by considering the vertices w_j^k for $k \in \{1, 2, 3\}$, we deduce at least one of $f(x_{i_1})$, $f(x_{i_2})$ and $f(x_{i_3})$ is 2. Thus, by setting x_i to be True if $f(x_i) = 1$ and False if $f(x_i) = 2$, we get a solution to I as desired.

Now suppose we have a truth assignment to all variables $x_1, x_2, \ldots, x_n$, and we want to define a coloring $f : V \to [5]$ of G. Let $f(c_i) = i$ for every $i \in [5]$. Let $f(x_i) = 1$ if x_i is assigned True, $f(x_i) = 2$ otherwise. For each clause C_j with literals $x_{i_1}, x_{i_2}, x_{i_3}$, we set $f(u_j^k) = k + 2$ and $f(w_j^k) = 1$ if $f(x_{i_k}) = 2$; we set $f(u_j^k) = 2$ and $f(w_j^k) = k + 2$ if $f(x_{i_k}) = 1$. Since for each clause C_j, at least one literal is assigned True and at least one literal is assigned False, we know that there exists $k_1, k_2 \in [3]$ with $k_1 \neq k_2$ such that $f(u_j^{k_1}) = 2$ and $f(w_j^{k_2}) = 1$. So we can set $f(y_j) = k_1 + 2$ and $f(z_j) = k_2 + 2$. Therefore, we have defined a 5-coloring f of G. □

Next, let us show that G is $2P_4$ free.

Lemma 4.0.2. *Every induced P_4 in G either contains one vertex from C, or one vertex from X and one vertex from Y.*

Proof. Let $P = v_1v_2v_3v_4$ be an induced path in G which contains no vertex from C.

If the vertex v_2 is in U, then without loss of generality, we have $v_1 \in X$ and $v_3 \in Y$. But then v_1v_3 is an edge of G, which contradicts the fact that P is an induced P_4. Similarly, $v_3 \notin U$. So we conclude that v_2 and v_3 can only be in $X \cup Y$.

Since the set X and Y are two independent sets while $v_2v_3 \in E$, v_2 and v_3 cannot be both in X or both in Y. Thus, one of the two vertices is in X and the other is in Y. □

Lemma 4.0.3. *G is $2P_4$-free.*

Proof. Assume G contains two paths P^1 and P^2 of length 4, such that $P^1 \cup P^2 \simeq 2P_4$. From the definition, we have that P^1 and P^2 are vertex disjoint and non-adjacent to each other. From 4.0.2, each of P^1 and P^2 either contains one vertex from C, or one vertex from X and one vertex from Y.

Suppose P^1 has a vertex $c_i \in C$, and P^2 has a vertex $c_j \in C$. Since P^1 and P^2 are vertex disjoint, $c_i \neq c_j$. But then $c_i c_j \in E$, and thus there is an edge between P^1 and P^2, a contradiction.

Suppose P^1 has a vertex $x_{i_1} \in X$ and $y_{i_2} \in Y$, and P^2 has a vertex $x_{j_1} \in X$ and $y_{j_2} \in Y$. Since P^1 and P^2 are vertex disjoint, $x_{i_1} \neq x_{j_1}$, $y_{i_2} \neq y_{j_2}$. Then from the construction of G, we know that $x_{i_1} y_{j_2}$ and $x_{j_1} y_{i_2}$ are edges of G, which contradicts to fact that P^1 and P^2 have no edges between them.

Suppose without loss of generality that P^1 has a vertex $c_i \in C$, and P^2 has a vertex $x_{j_1} \in X$ and $y_{j_2} \in Y$. If $i \in \{1, 2\}$ then $c_i y_{j_2} \in E$, otherwise $c_i x_{j_1} \in E$. Both contradict the fact that there are no edges between P^1 and P^2.

Thus, G is $2P_4$-free. □

Therefore, we have proved Theorem 1.2.6.

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