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# Normal operators with highly incompatible off-diagonal corners

by

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*Dedicated to the memory of
our friend and colleague, Ali-Akbar Jafarian*

**Abstract.** Let $\mathcal{H}$ be a complex, separable Hilbert space, and $\mathcal{B}(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. Given an orthogonal projection $P \in \mathcal{B}(\mathcal{H})$ and an operator $D \in \mathcal{B}(\mathcal{H})$, we may write $D = \left[\begin{smallmatrix} D_1 & D_2 \\ D_3 & D_4 \end{smallmatrix}\right]$ relative to the decomposition $\mathcal{H} = \operatorname{ran} P \oplus \operatorname{ran}(I-P)$. In this paper we study the question: for which non-negative integers $j, k$ can we find a normal operator $D$ and an orthogonal projection $P$ such that $\operatorname{rank} D_2 = j$ and $\operatorname{rank} D_3 = k$? Complete results are obtained in the case where $\dim \mathcal{H} < \infty$, and partial results are obtained in the infinite-dimensional setting.

## 1. Introduction

**1.1.** Let $\mathcal{H}$ denote a complex, separable Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators acting on $\mathcal{H}$, keeping in mind that when $\dim \mathcal{H} = n < \infty$ we may identify $\mathcal{H}$ with $\mathbb{C}^n$, and $\mathcal{B}(\mathcal{H})$ with $\mathbb{M}_n(\mathbb{C})$. We write $\mathcal{P}(\mathcal{H}) := \{P \in \mathcal{B}(\mathcal{H}) : P = P^2 = P^*\}$ to denote the set of orthogonal projections in $\mathcal{B}(\mathcal{H})$. Given $T \in \mathcal{B}(\mathcal{H})$, $T$ admits a natural $2 \times 2$ operator-matrix decomposition

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

with respect to the decomposition $\mathcal{H} = P\mathcal{H} \oplus (I-P)\mathcal{H}$. Of course, $T_j = T_j(P)$, $1 \le j \le 4$.

We are interested in determining to what extent the set $\{(T_2(P), T_3(P)) : P \in \mathcal{P}(\mathcal{H})\}$ determines the structure of the operator $T$. Following [4], we

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say that $T$ has *property* (CR) (the *common rank property*) if $\operatorname{rank} T_2(P) = \operatorname{rank} T_3(P)$ for all $P \in \mathcal{P}(\mathcal{H})$. We recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is said to be *orthogonally reductive* if for every orthogonal projection $P \in \mathcal{P}(\mathcal{H})$ the condition $(I-P)AP = 0$ implies that $PA(I-P) = 0$. That is, every invariant subspace for $A$ is orthogonally reducing for $A$. In the above-cited paper, the following result was obtained:

1.2. Theorem ([4, Theorem 5.8]). *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. If $T$ has property* (CR), *then there exist $\lambda, \mu \in \mathbb{C}$ and $A \in \mathcal{B}(\mathcal{H})$ with $A$ either selfadjoint or an orthogonally reductive unitary operator such that $T = \lambda A + \mu I$.*

**1.3.** In fact, if $\dim \mathcal{H} < \infty$, then the converse is also true (see [4, Theorem 3.15]). We note that every normal operator (and hence every unitary operator) acting on a finite-dimensional Hilbert space is automatically orthogonally reductive; the argument is outlined three paragraphs below. In particular, every operator $T$ that has property (CR) must be normal with spectrum lying either on a line or a circle, and when $\mathcal{H}$ is finite-dimensional, every such normal operator has property (CR).

Property (CR) was termed a "compatibility" condition on the off-diagonal corners of the operator $T$. In this paper, we examine to what extent the off-diagonal corners of a normal operator $D$ may be "incompatible" in the sense of rank. That is, writing $D = \left[\begin{smallmatrix} D_1 & D_2 \\ D_3 & D_4 \end{smallmatrix}\right]$ relative to $\mathcal{H} = P\mathcal{H} \oplus (I-P)\mathcal{H}$, we consider how large

$$|\operatorname{rank} D_2 - \operatorname{rank} D_3|$$

can get.

More generally, our main result (Theorem 2.5 below) shows that if $\dim \mathcal{H} = n < \infty$ and $1 \le j, k \le \lfloor n/2 \rfloor$, then there exist a normal operator $D$ and a projection $P$ such that $\operatorname{rank} D_2(P) = j$ while $\operatorname{rank} D_3(P) = k$. (That $\lfloor n/2 \rfloor$ is the optimal upper bound follows from the argument of Section 2.4.) If $\dim \mathcal{H} = \infty$ and $0 \le j, k \le \infty$, then the same conclusion holds (Theorem 3.2).

The infinite-dimensional setting also allows for certain subtleties which cannot occur in the finite-dimensional setting. For example, if $\dim \mathcal{H} = n < \infty$, $D = \left[\begin{smallmatrix} D_1 & D_2 \\ D_3 & D_4 \end{smallmatrix}\right] \in \mathcal{B}(\mathcal{H})$ is normal and $D_3 = 0$, then $D_2 = 0$. Indeed, this is just a restatement of the fact that every normal matrix is orthogonally reductive. This follows by observing that the normality of $D$ implies that

$$D_1^* D_1 - D_1 D_1^* = D_2 D_2^* - D_3^* D_3.$$

Thus $\operatorname{tr}(D_2 D_2^*) = \operatorname{tr}(D_3^* D_3)$, or equivalently $\|D_2\|_2 = \|D_3\|_2$, where $\|\cdot\|_2$ refers to the Frobenius (or Hilbert–Schmidt) norm. From this, $D_3 = 0$ clearly implies that $D_2 = 0$. We shall show that if $\mathcal{H}$ is infinite-dimensional, then it is possible to have $D_3(P) = 0$ while $D_2(P)$ is a *quasiaffinity* (i.e. $D_2(P)$ has trivial kernel and dense range), although it is not possible for $D_3(P)$ to be compact and $D_2(P)$ to be invertible at the same time (see Proposition 3.3 below).

**1.4.** It is worth mentioning that a related question where ranks are replaced by unitarily invariant norms has been considered by Bhatia and Choi [2]. More specifically, they consider normal matrices $D = \left[\begin{smallmatrix} D_1 & D_2 \\ D_3 & D_4 \end{smallmatrix}\right]$ acting on $\mathcal{H} := \mathbb{C}^n \oplus \mathbb{C}^n$. As noted above, normality of $D$ shows that $\|D_2\|_2 = \|D_3\|_2$. In the case of the operator norm $\|\cdot\|$, it follows that $\|D_3\| \le \sqrt{n}\,\|D_2\|$, and equality can be obtained in this expression if and only if $n \le 3$. (If we denote by $\alpha_n$ the minimum number such that $\|D_3\| \le \alpha_n \|D_2\|$ for all $D \in \mathbb{M}_{2n}(\mathbb{C})$ as above—so that $\alpha_n \le \sqrt{n}$—it is not even known at this time whether or not the sequence $(\alpha_n)_n$ is bounded.)

It is interesting to note that the example given in [2] for the case where $n = 3$ and $\alpha_3 = \sqrt{3}$ is also an example of a normal matrix $D \in \mathbb{M}_6(\mathbb{C})$ for which $\operatorname{rank} D_2 = 1$ and $\operatorname{rank} D_3 = 3$.

## 2. The finite-dimensional setting

**2.1.** In examining the incompatibility of the off-diagonal corners of a normal operator $D \in \mathbb{M}_n(\mathbb{C})$, we first dispense with the trivial cases where $n \in \{2, 3\}$. Indeed, as seen in [4, Proposition 3.7], in this setting, $D$ automatically has property (CR).

For this reason, henceforth we shall assume that $\dim \mathcal{H} \ge 4$.

The key to obtaining the main theorem of this section is Theorem 2.3, which shows that if $\dim \mathcal{H} = 2m$ for some integer $m \ge 2$, then we can find a normal operator $D$ such that $\operatorname{rank} D_3 = 1$ and $\operatorname{rank} D_2 = m$. For $m = 2$, this is an immediate consequence of [4, Theorem 3.15], since in this case, given a normal operator $D \in \mathbb{M}_4(\mathbb{C})$ whose eigenvalues lie neither on a common circle nor on a common line, $D$ fails to have property (CR), and this can only happen if there exists a projection $P \in \mathbb{M}_4(\mathbb{C})$ of rank two such that $\operatorname{rank} D_2(P) = 2$, while $\operatorname{rank} D_3(P) = 1$.

Given $X = [x_{i,j}], Y = [y_{i,j}] \in \mathbb{M}_n(\mathbb{C})$, we shall denote by $X \bullet Y$ the *Hadamard* or *Schur* product of $X$ and $Y$, i.e. $X \bullet Y = [x_{i,j}\, y_{i,j}] \in \mathbb{M}_n(\mathbb{C})$.

2.2. Lemma. *Let $m \ge 3$ be an integer. Let*

$$A = \operatorname{diag}(\alpha_1, \ldots, \alpha_m) \quad \textit{and} \quad B = \operatorname{diag}(\beta_1, \ldots, \beta_m)$$

*be diagonal operators in $\mathbb{M}_m(\mathbb{C})$, and $D := \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. Set $Z := [z_{j,k}] \in \mathbb{M}_m(\mathbb{C})$, where $z_{j,k} := \alpha_j - \beta_k$ for all $1 \le j, k \le m$. Suppose that there exists a positive definite matrix $S \in \mathbb{M}_m(\mathbb{C})$ such that*

$$\operatorname{rank} S \bullet Z = 1 \quad \textit{and} \quad \operatorname{rank} S^t \bullet Z = m,$$

*where $S^t$ denotes the transpose of $S$. Then there exists a projection $P \in$*

$\mathbb{M}_{2m}(\mathbb{C})$ *of rank* $m$ *such that if* $D = \left[\begin{smallmatrix} D_1 & D_2 \\ D_3 & D_4 \end{smallmatrix}\right]$ *relative to* $\mathbb{C}^{2m} = \operatorname{ran} P \oplus \operatorname{ran}(I-P)$, *then* $\operatorname{rank} D_2 = m$ *and* $\operatorname{rank} D_3 = 1$.

*Proof.* We leave it as an exercise for the reader to show that $0 < S \in \mathbb{M}_m(\mathbb{C})$ implies that $S$ can be expressed in the form $S = MN^{-1}$, where $M$ and $N$ are two commuting positive definite matrices satisfying $M^2 + N^2 = I_m$. From this it follows that

$$P := \begin{bmatrix} M^2 & MN \\ MN & N^2 \end{bmatrix}$$

is an orthogonal projection in $\mathbb{M}_{2m}(\mathbb{C})$ whose rank is $m = \operatorname{tr}(P)$. Since

$$P = \begin{bmatrix} M \\ N \end{bmatrix} [M \quad N],$$

we deduce that $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ is an isometry from $\mathbb{C}^m$ into $\mathbb{C}^{2m}$. A straightforward computation shows that

$$I_{2m} - P = \begin{bmatrix} I_m - M^2 & -MN \\ -MN & I_m - N^2 \end{bmatrix} = \begin{bmatrix} N \\ -M \end{bmatrix} [N \quad -M],$$

and that $\left[\begin{smallmatrix} N \\ -M \end{smallmatrix}\right]$ is once again an isometry of $\mathbb{C}^m$ into $\mathbb{C}^{2m}$.

Our goal is to show that $\operatorname{rank}(I-P)DP = 1$, while $\operatorname{rank} PD(I-P) = m$. As both $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} N \\ -M \end{smallmatrix}\right]$ are isometries, this is equivalent to proving that

$$\operatorname{rank}(NAM - MBN) = \operatorname{rank}\left( [N \quad -M] \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} \right) = 1,$$

while

$$\operatorname{rank}(MAN - NBM) = \operatorname{rank}\left( [M \quad N] \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} N \\ -M \end{bmatrix} \right) = m.$$

Now $N$ and $M$ are each invertible in $\mathbb{M}_m(\mathbb{C})$, and $NM = MN$ implies that $N^{-1}$ and $M$ also commute. Thus

$$\begin{aligned} \operatorname{rank}(NAM - MBN) &= \operatorname{rank}(AMN^{-1} - N^{-1}MB) = \operatorname{rank}(AS - SB) \\ &= \operatorname{rank} S \bullet Z = 1, \end{aligned}$$

while

$$\begin{aligned} \operatorname{rank}(MAN - NBM) &= \operatorname{rank}(N^{-1}MA - BMN^{-1}) = \operatorname{rank}(SA - BS) \\ &= \operatorname{rank}(AS^t - S^tB) = \operatorname{rank} S^t \bullet Z = m. \ \blacksquare \end{aligned}$$

2.3. Theorem. *Let* $m \geq 1$ *be an integer. Then there exist a normal operator* $D \in \mathcal{B}(\mathbb{C}^{2m}) \simeq \mathbb{M}_{2m}(\mathbb{C})$ *and an orthogonal projection* $P$ *of rank* $m$ *such that if* $D = \left[\begin{smallmatrix} D_1 & D_2 \\ D_3 & D_4 \end{smallmatrix}\right]$ *relative to* $\mathbb{C}^{2m} = \operatorname{ran} P \oplus \operatorname{ran}(I-P)$, *then* $\operatorname{rank} D_2 = m$ *and* $\operatorname{rank} D_3 = 1$.

*Proof.* The case $m = 1$ is easily satisfied by the operator $D = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and the projection $P = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. The case where $m = 2$ follows from [4, Proposition 3.13].

Suppose, therefore, that $m \geq 3$. By Lemma 2.2, we have reduced our problem to that of finding two diagonal matrices $A = \operatorname{diag}(\alpha_1, \ldots, \alpha_m)$ and $B = \operatorname{diag}(\beta_1, \ldots, \beta_m)$ and a positive definite matrix $0 < S = [s_{j,k}] \in \mathbb{M}_m(\mathbb{C})$ such that

$$\operatorname{rank} S \bullet Z = 1 \quad \text{and} \quad \operatorname{rank} S^t \bullet Z = m.$$

We begin by specifying $A$ and $B$; we first temporarily fix a parameter $1 < \gamma$ whose exact value we shall determine later. For $1 \leq j \leq m$, set $\alpha_j = j\gamma + i$. Set $B = A^*$, so that $\beta_k = \overline{\alpha_k} = k\gamma - i$. Then $Z = [z_{j,k}] = [(j-k)\gamma + 2i]$.

Next, we set $S$ $(= S(\gamma)) = [s_{j,k}]$, where $s_{j,k} = \frac{2i}{(j-k)\gamma+2i}$. Observe first that for $1 \leq j, k \leq m$,

$$\overline{s_{k,j}} = \frac{-2i}{(k-j)\gamma - 2i} = \frac{2i}{(j-k)\gamma + 2i} = s_{j,k},$$

so that $S$ is clearly hermitian, and $s_{j,j} = 1$ for all $1 \leq j \leq m$. It is therefore reasonably straightforward to see that since $m$ is a fixed constant, and since $\lim_{\gamma \to \infty} \frac{2i}{(j-k)\gamma+2i} = 0$ for all $1 \leq j \neq k \leq m$, there exists a constant $\Gamma(m) \geq 1$ such that $\gamma > \Gamma(m)$ ensures that $\|S - I_m\| < 1/4$, and thus $S$ $(= S(\gamma))$ must be positive definite.

For an explicit estimate for $\Gamma(m)$, observe that if $R = [r_{j,k}] \in \mathbb{M}_m(\mathbb{C})$, and if $\|R\|_\infty := \max\{|r_{j,k}| : 1 \leq j, k \leq m\}$, then $\|R\| \leq m\|R\|_\infty$. Indeed, if $x = (x_k)_{k=1}^m \in \mathbb{C}^m$, then (using the Cauchy–Schwarz inequality) we find that

$$\|Rx\|^2 = \sum_{j=1}^m \left| \sum_{k=1}^m r_{j,k} x_k \right|^2 \leq \sum_{j=1}^m m\|R\|_\infty^2 \|x\|^2 = m^2 \|R\|_\infty^2 \, \|x\|^2,$$

from which the result follows. In particular, by choosing $\Gamma(m) = 8m$, we see that $\gamma > \Gamma(m)$ implies that

$$\|S - I_m\| \leq m \max_{1 \leq j,k \leq m} |s_{j,k} - \delta_{j,k}| = m \max_{1 \leq j \neq k \leq m} |s_{j,k}| < m\frac{2}{\gamma} < \frac{1}{4},$$

and so $S$ is a positive invertible operator.

Consider

$$S \bullet Z = [s_{j,k} z_{j,k}] = \left[ \frac{2i}{(j-k)\gamma + 2i} ((j-k)\gamma + 2i) \right] = [2i]_{m \times m}.$$

It is clear that $S \bullet Z \in \mathbb{M}_m(\mathbb{C})$ is a rank-one operator; indeed, $S \bullet Z = 2miQ$, where $Q$ is the rank-one projection whose matrix consists entirely of the entries $1/m$.

We therefore turn our attention to

$$S^t \bullet Z = [s_{k,j} z_{j,k}] = \left[ \frac{2i}{(k-j)\gamma + 2i} \; ((j-k)\gamma + 2i) \right] = [2i\theta_{j,k}],$$

where $\theta_{j,k} = \frac{(j-k)\gamma+2i}{(k-j)\gamma+2i} \in \mathbb{T}$, $1 \le j,k \le m$. Observe that if $1 \le j,k \le m-1$, then $\theta_{j,k} = \theta_{j+1,k+1}$. Thus $T := \frac{1}{2i}(S^t \bullet Z)$ is a Toeplitz matrix, and the diagonal entries of $T$ are all equal to 1.

In fact, for $1 \le j,k \le m$,

$$\overline{\theta_{k,j}} = \frac{(k-j)\gamma - 2i}{(j-k)\gamma - 2i} = \frac{-((j-k)\gamma + 2i)}{-((k-j)\gamma + 2i)} = \theta_{j,k},$$

and therefore $T$ is not only Toeplitz, but hermitian as well.

It only remains to show that the rank of $S^t \bullet Z$ is $m$, or equivalently that $\det T \neq 0$.

Define $\hat{T} = 2I_m - mQ$. Then $\hat{T}$ is invertible and $\hat{T}^{-1} = \frac{1}{2-m}Q + \frac{1}{2}(I_m - Q)$. Note that each diagonal entry of $\hat{T}$ is 1, while each off-diagonal entry is $-1$. From this and the calculations above it follows that

$$\|T - \hat{T}\| \le m\|T - \hat{T}\|_\infty = m\Big(\max_{1\le j\neq k\le m} |\theta_{j,k} + 1|\Big) < m\frac{4}{\gamma} < \frac{1}{2} < \frac{1}{\|\hat{T}^{-1}\|},$$

implying that $T$ is invertible whenever $\gamma > \Gamma(m) = 8m$.

Thus, by choosing $\gamma > \Gamma(m) = 8m$, we see that a positive solution to our problem can be found. ∎

**2.4.** Suppose now that $n \ge 5$ is an integer and that $T \in \mathbb{M}_n(\mathbb{C})$. If $P \in \mathcal{P}(\mathbb{C}^n)$ is any projection, then the minimum of $\operatorname{rank} P$ and $\operatorname{rank}(I-P)$ is at most $\lfloor n/2 \rfloor$. It follows that

$$\max(\operatorname{rank} T_2(P), \operatorname{rank} T_3(P)) \le \lfloor n/2 \rfloor.$$

As already observed, if $D \in \mathbb{M}_n(\mathbb{C})$ is normal, then $D$ is orthogonally reductive, and so if $\operatorname{rank} T_3(P) = 0$, then automatically $\operatorname{rank} T_2(P) = 0$. In light of these observations, we see that the following result is the best possible; it is the main theorem of Section 2.

2.5. Theorem. *Let $n \ge 2$ be a positive integer, $1 \le j,k \le \lfloor n/2 \rfloor$. Then there exist a normal operator $D \in \mathbb{M}_n(\mathbb{C})$ and a projection $P$ such that relative to $\mathbb{C}^n = \operatorname{ran} P \oplus \operatorname{ran}(I-P)$ we can write*

$$D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix},$$

*where $\operatorname{rank} D_2 = k$ and $\operatorname{rank} D_3 = j$.*

*Proof.* Without loss of generality, we can assume that $k \ge j$. First, we set $m := (k-j)+1$. Applying Theorem 2.3 we may choose a normal element

$M \in \mathbb{M}_{2m}(\mathbb{C})$ such that

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix},$$

where $\operatorname{rank} M_2 = (k - j) + 1$ and $\operatorname{rank} M_3 = 1$. Define

$$\hat{D} = \begin{bmatrix} I_{j-1} & & & I_{j-1} \\ & M_1 & M_2 & \\ & M_3 & M_4 & \\ I_{j-1} & & & I_{j-1} \end{bmatrix}.$$

Here, it is understood that if $j = 1$, then $I_0$ acts on a space of dimension zero. Finally, let

$$D = 0_{n-2k} \oplus \hat{D} = \begin{bmatrix} 0_{n-2k} & & & & \\ & I_{j-1} & & & I_{j-1} \\ & & M_1 & M_2 & \\ & & M_3 & M_4 & \\ & I_{j-1} & & & I_{j-1} \end{bmatrix}.$$

(Again, if $n = 2k$, the $0_0$ term is not required.) Set $P = I_{(n-2k)+(j-1)+m} \oplus 0_{m+(j-1)}$, and relabel $D = \left[\begin{smallmatrix} D_1 & D_2 \\ D_3 & D_4 \end{smallmatrix}\right]$ relative to the decomposition $\mathbb{C}^n = \operatorname{ran} P \oplus \operatorname{ran}(I - P)$. It is then routine to verify that $\operatorname{rank} D_2 = k$ and $\operatorname{rank} D_3 = j$. ∎

**2.6.** The operator $D$ constructed in Theorem 2.5 is far from unique. Indeed, we first note that we were free to choose arbitrarily large $\gamma$'s in the definition of $A$ and $B$ earlier. Secondly, it is not hard to show that by choosing $B = A^*$ and $Z$ as we did above, and by defining $S$ such that $S \bullet Z = 2iQ$, $S$ is always hermitian. Thus, given one triple $(A, B, S)$ as above that works, if we slightly perturb the weights $\alpha_j$ of our given $A$ to obtain a diagonal matrix $A_0$ and we set $B_0 = A_0^*$, then the new $S_0$ we require to make $S_0 \bullet Z_0 = 2iQ$ will be sufficiently close to the original $S$ so as to be invertible (since the set of invertible operators is open in $\mathbb{M}_m(\mathbb{C})$).

**2.7.** An interesting, but apparently far more complicated, question is to characterise those normal operators $D \in \mathbb{M}_{2m}(\mathbb{C})$ for which it is possible to find a projection $P$ of rank equal to $m$ such that $\operatorname{rank}(I - P)DP = 1$ and $\operatorname{rank} PD(I - P) = m$. We are not able to resolve this question at this time. We can assert, however, that not only is such a normal operator abstractly "far away" from operators with property (CR); in fact, we are able to quantify this distance, and say a bit more about the structure of $D$.

Let $n \geq 1$ be an integer, and recall that the function

$$\rho : \mathbb{M}_n(\mathbb{C}) \times \mathbb{M}_n(\mathbb{C}) \to \{0, 1, 2, \ldots\}, \qquad (A, B) \mapsto \operatorname{rank}(A - B),$$

defines a metric on $\mathbb{M}_n(\mathbb{C})$.

We also recall that an operator $T \in \mathcal{B}(\mathcal{H})$ (where $\dim \mathcal{H} \in \mathbb{N} \cup \{\infty\}$) is said to be *cyclic* if there exists $x \in \mathcal{H}$ such that $\operatorname{span}\{x, Tx, T^2x, \ldots\}$ is dense in $\mathcal{H}$. Obviously this can only happen if $\mathcal{H}$ is separable, and it is well-known that a normal operator is cyclic if and only if it has *multiplicity one*, that is, its commutant $N' := \{X \in \mathcal{B}(\mathcal{H}) : XN = NX\}$ is a *masa* (i.e. a maximal abelian selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$). If $N$ is a compact, normal operator, then this is equivalent to saying that the eigenspaces corresponding to the eigenvalues of $N$ are all one-dimensional, and together they densely span the Hilbert space.

2.8. Theorem. *Let $m \geq 3$ be an integer, and suppose that $D \in \mathbb{M}_{2m}(\mathbb{C})$ is a normal operator. Suppose that $P \in \mathbb{M}_{2m}(\mathbb{C})$ is an orthogonal projection of rank $m$ such that* $\operatorname{rank}(I - P)DP = 1$ *and* $\operatorname{rank} PD(I - P) = m$. *Then:*

(a) *$D$ has $2m$ distinct eigenvalues (and therefore it is a cyclic operator);*
(b) *$\rho(D, Y) \geq \lfloor (m-1)/2 \rfloor$ for all $Y \in \mathcal{Y}$, where $\mathcal{Y}$ is the set of matrices in $\mathbb{M}_{2m}(\mathbb{C})$ which satisfy property* (CR).

*Proof.* First observe that $\mathcal{Y}$ is closed under perturbations by scalar multiples of the identity operator. Hence, we may assume that $D$ is invertible, since otherwise we simply add a sufficiently large multiple of the identity to $D$, which affects neither the hypotheses nor the conclusion of the theorem.

(a) Next, we set $P_0 := P$, and let $V_0$ be the range of $P_0$. By hypothesis,

$$\dim(V_0 \vee DV_0) = m + 1, \qquad \dim(V_0 \cap D^{-1}V_0) = m - 1.$$

More generally, we claim that the following chain of subspaces has strictly increasing dimensions (from 0 to $n = 2m$):

$$V_{-m} \subset V_{-m+1} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset V_m,$$

where

$$\begin{aligned} V_{k+1} &= V_k \vee DV_k, && \forall\, 0 \leq k \leq m-1, \\ V_{k-1} &= V_k \cap D^{-1}V_k, && \forall\, -m+1 \leq k \leq 0. \end{aligned}$$

Assume to the contrary that this fails. Let $P_k$ be the projection onto the range of $V_k$, $-m \leq k \leq m$.

(i) If $V_{k+1} = V_k$ for some $0 < k < m$, then $DV_k = V_k$. This implies that $D^*V_k = V_k$, i.e., $P_kD(I - P_k) = 0$. Since $P_k \geq P_0$, we deduce that

$$P_0D(I - P_k) = 0$$

($\operatorname{rank} P_k = \dim V_k \leq m + k < 2m$). In other words, $P_0D(I - P_0)$ has non-trivial kernel in $V_0$, a contradiction.

(ii) Similarly, if $V_{k+1} = V_k$ for some $-m \le k < 0$, then once again $DV_k = V_k$ and $P_k D(I - P_k) = 0$. Since $P_k \le P_0$, we deduce that

$$P_k D(I - P_0) = 0$$

($\operatorname{rank} P_k = \dim V_{k+1} = \dim V_k \ge 1$). This implies that the range of $P_0 D(I - P_0)$ is smaller than that of $P_0$; a contradiction.

Thus the claim is proved.

In particular, $V_{-m+1}$ is one-dimensional. Pick a unit vector in $V_{-m+1}$. We next show that $x$ is a cyclic vector for $D$.

Note that $Dx \notin V_{-m+1}$, and hence $x, Dx$ span $V_{-m+2}$. Under the assumption that $\{x, Dx, \ldots, D^j x\}$ spans $V_{-m+j+1}$, we see that $\{x, Dx, \ldots, D^{j+1} x\}$ spans $V_{-m+j+2}$ by construction. This is true for all $0 \le j \le 2m-1$, which proves that $x$ is a cyclic vector for $D$.

(b) With the decomposition $\mathbb{C}^{2m} = \operatorname{ran} P \oplus \operatorname{ran}(I - P)$, we may write

$$D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}.$$

Next, suppose that $Y \in \mathcal{Y}$, so that $Y$ has the common rank property. With respect to the same decomposition of $\mathbb{C}^{2m}$, we have

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}.$$

Define $F := D - Y$ and write

$$F = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}.$$

Clearly $D_2 = Y_2 + F_2$ and $D_3 = Y_3 + F_3$. Denote by $r$ the rank of $F$. Then

$$m = \operatorname{rank} D_2 \le \operatorname{rank} Y_2 + \operatorname{rank} F_2 \le \operatorname{rank} Y_2 + r,$$

and similarly

$$\operatorname{rank} Y_3 \le \operatorname{rank} D_3 + \operatorname{rank} F_3 \le r + 1.$$

But $\operatorname{rank} Y_2 = \operatorname{rank} Y_3$, since $Y$ has property (CR), so it follows that

$$m \le r + r + 1,$$

and thus $r \ge \lfloor (m-1)/2 \rfloor$. Hence, $\rho(D, Y) \ge \lfloor (m-1)/2 \rfloor$. ∎

**2.9.** An inspection of the proof of part (b) above shows that the finite-dimensionality of the underlying Hilbert space did not really play a role. In fact, if $\mathcal{H}$ is infinite-dimensional, $0 \le j, k < \infty$, $D \in \mathcal{B}(\mathcal{H})$ is normal and $P \in \mathcal{B}(\mathcal{H})$ is a projection for which

$$\operatorname{rank}(I - P)DP = j \quad \text{and} \quad \operatorname{rank} PD(I - P) = k,$$

then the same argument shows that $\operatorname{rank}(D - Y) \geq \lfloor |k - j|/2 \rfloor$ for all $Y \in \mathcal{B}(\mathcal{H})$ with property (CR).

## 3. The infinite-dimensional case

**3.1.** Throughout this section, we shall assume that the underlying Hilbert space $\mathcal{H}$ is infinite-dimensional and separable. Our first goal here is to extend Theorem 2.5 to this setting.

3.2. THEOREM. *For all $0 \leq j, k \leq \infty$, there exist a normal operator $D \in \mathcal{B}(\mathcal{H})$ and an orthogonal projection $P \in \mathcal{B}(\mathcal{H})$ for which*

$$\operatorname{rank}(I - P)DP = j \quad \textit{and} \quad \operatorname{rank} PD(I - P) = k.$$

*Proof.* By replacing $P$ by $I - P$ if necessary, it becomes clear that there is no loss of generality in assuming that $j \leq k$.

CASE 1: $j = 0$. If $k = 0$ as well, we may consider $D = I$, the identity operator, and let $P$ be any non-trivial projection.

For $k = 1$, we consider the bilateral shift $U$: that is, let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$, and set $Ue_n = e_{n-1}$ for all $n \in \mathbb{Z}$. Let $P_0$ denote the orthogonal projection of $\mathcal{H}$ onto $\overline{\operatorname{span}}\{e_n\}_{n \leq 0}$. The condition above is satisfied with $D := U$, $P := P_0$.

For $2 \leq k \leq \infty$, we simply consider the tensor product $D := U \otimes I_k$ of $U$ above with $I_k$, the identity operator acting on a Hilbert space $\mathcal{K}$ of dimension $k$, and we set $P := P_0 \otimes I_k$ to obtain the desired rank equalities.

CASE 2: $1 \leq j < \infty$. Let $U$ denote the bilateral shift from Case 1, and $P_0$ denote the orthogonal projection of $\mathcal{H}$ onto $\overline{\operatorname{span}}\{e_n\}_{n \leq 0}$. If $H := (U + U^*) \otimes I_j$, it is relatively straightforward to verify that with $Q_1 := P_0 \otimes I_j$, we have

$$\operatorname{rank}(I - Q_1)HQ_1 = j = \operatorname{rank} Q_1 H(I - Q_1).$$

Next, let $R := U \otimes I_{k-j}$ (where $\infty - j := \infty$) and choose a projection $Q_2 := P_0 \otimes I_{k-j}$ as in Case 1 such that

$$\operatorname{rank}(I - Q_2)RQ_2 = 0 \quad \text{and} \quad \operatorname{rank} Q_2 R(I - Q_2) = k - j.$$

A routine calculation shows that with $D := H \oplus R$ and $P := Q_1 \oplus Q_2$, the desired rank equalities are met.

CASE 3: $j = \infty$. Since we have reduced the problem to the case where $j \leq k$, it follows that $k = \infty$ as well.

Consider the selfadjoint operator $\hat{H} := \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] \in \mathbb{M}_2(\mathbb{C})$. Then $H := \hat{H} \otimes I = \left[\begin{smallmatrix} I & I \\ I & I \end{smallmatrix}\right]$ satisfies the condition relative to the projection $P = I \oplus 0$. ∎

The case where $j = 1$ and $k = \infty$ in the above theorem is only one possible infinite-dimensional analogue of Theorem 2.3. Alternatively, we may view that theorem as requiring that $D_2$ be invertible. Interestingly enough, this is no longer possible in the infinite-dimensional setting. In fact, a stronger (negative) result holds.

3.3. PROPOSITION. *There does not exist a normal operator*

$$D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$$

*in* $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ *such that* $D_2$ *is invertible and* $D_3$ *is compact.*

*Proof.* We argue by contradiction. If such a normal operator $D$ were to exist, it would follow that

$$D_2 D_2^* = (D_1^* D_1 - D_1 D_1^*) + D_3^* D_3.$$

Since $D_2$ is invertible, $D_2 D_2^*$ is positive and invertible, and thus 0 is not in the essential numerical range of $D_2 D_2^*$. On the other hand, by a result of the second author [6, Theorem 8], and keeping in mind that $D_3$ is compact, 0 is indeed in the essential numerical range of $(D_1^* D_1 - D_1 D_1^*) + D_3^* D_3$, a contradiction. ∎

**3.4.** When $1 \leq m < \infty$, it is clear that an operator $D_2 \in \mathbb{M}_m(\mathbb{C})$ is invertible if and only if $D_2$ is a *quasiaffinity*, i.e. it is injective and has dense range. Moreover, in the infinite-dimensional setting, not every normal operator is orthogonally reductive. Despite this, in light of Proposition 3.3, the next example is somewhat surprising.

3.5. THEOREM. *There exists a normal operator*

$$D = \begin{bmatrix} D_1 & D_2 \\ 0 & D_4 \end{bmatrix}$$

*in* $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ *such that* $D_2$ *is a quasiaffinity.*

*Proof.* Let $A = U + 2U^*$ and $B = A^* = U^* + 2U$, where $U$ is the bilateral shift operator (i.e. $Ue_n = e_{n-1}$, $n \in \mathbb{Z}$) from Theorem 3.2. Then $D := A \oplus B$ is easily seen to be a normal operator.

Let $M \in \mathcal{B}(\mathcal{H})$ be a positive contraction, and let $N := (I - M^2)^{1/2}$, so that $MN = NM$ and $M^2 + N^2 = I$. From this it follows that

$$P := \begin{bmatrix} M^2 & MN \\ MN & N^2 \end{bmatrix}$$

is an orthogonal projection in $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Arguing as in Theorem 2.3, we see that $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} N \\ -M \end{smallmatrix}\right]$ are both isometries from $\mathcal{H}$ into $\mathcal{H} \oplus \mathcal{H}$, and that it suffices to find $M$ and $N$ as above such that

$$(NAM - MBN) = \begin{bmatrix} N & -M \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = 0,$$

while

$$(MAN - NBM) = [M \quad N] \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} N \\ -M \end{bmatrix}$$

is injective and has dense range.

We shall choose $M$ (and thus $N$) to be diagonal operators relative to the orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$, $M = \operatorname{diag}((\alpha_n)_{n\in\mathbb{Z}})$, where $\alpha_n := 1/\sqrt{1+4^{-n}}$ for each $n \in \mathbb{Z}$. The condition that $N = (I - M^2)^{1/2}$ implies that $N = \operatorname{diag}((\beta_n)_{n\in\mathbb{Z}})$, where $\beta_n = 2^{-n}/\sqrt{1+4^{-n}}$ for all $n \in \mathbb{Z}$.

It is easy to see that $M$ and $N$ are commutative, positive contractions and $M^2 + N^2 = I$ by construction.

Next,

$$NAMe_n = NA(\alpha_n e_n) = \alpha_n N(e_{n-1} + 2e_{n+1}) = \alpha_n(\beta_{n-1}e_{n-1} + 2\beta_{n+1}e_{n+1}),$$

while

$$MA^*Ne_n = MA^*(\beta_n e_n) = \beta_n M(e_{n+1} + 2e_{n-1}) = \beta_n(\alpha_{n+1}e_{n+1} + 2\alpha_{n-1}e_{n-1}).$$

But

$$\alpha_n\beta_{n-1} = \frac{1}{\sqrt{1+4^{-n}}}\,\frac{2^{-(n-1)}}{\sqrt{1+4^{-(n-1)}}} = \frac{2}{\sqrt{1+4^{-(n-1)}}}\,\frac{2^{-n}}{\sqrt{1+4^{-n}}} = 2\alpha_{n-1}\beta_n,$$

and similarly

$$2\alpha_n\beta_{n+1} = \frac{2}{\sqrt{1+4^{-n}}}\,\frac{2^{-(n+1)}}{\sqrt{1+4^{-(n+1)}}} = \frac{1}{\sqrt{1+4^{-(n+1)}}}\,\frac{2^{-n}}{\sqrt{1+4^{-n}}} = \alpha_{n+1}\beta_n.$$

Since this holds for all $n \in \mathbb{Z}$, $NAM - MA^*N = 0$, as claimed.

As for the second equation we must verify, observe that

$$(MAN - NA^*M)^* = NA^*M - MAN = -(MAN - NA^*M).$$

Hence, we need only show that $MAN - NA^*M$ is injective, since then $(MAN - NA^*M)^*$ is also injective and thus, in particular, both have dense range.

Again, we compute for each $n \in \mathbb{Z}$ that

$$\begin{aligned}
(MAN &- NA^*M)e_n \\
&= MANe_n - NA^*Me_n = MA(\beta_n e_n) - NA^*(\alpha_n e_n) \\
&= \beta_n M(e_{n-1} + 2e_{n+1}) - \alpha_n N(e_{n+1} + 2e_{n-1}) \\
&= \beta_n(\alpha_{n-1}e_{n-1} + 2\alpha_{n+1}e_{n+1}) - \alpha_n(\beta_{n+1}e_{n+1} + 2\beta_{n-1}e_{n-1}) \\
&= (\alpha_{n-1}\beta_n - 2\alpha_n\beta_{n-1})e_{n-1} + (2\alpha_{n+1}\beta_n - \alpha_n\beta_{n+1})e_{n+1}.
\end{aligned}$$

Suppose that $x = \sum_{n\in\mathbb{Z}} x_n e_n \in \ker(MAN - NA^*M)$. Then

$$\begin{aligned} 0 &= (MAN - NA^*M)\sum_{n\in\mathbb{Z}} x_n e_n \\ &= \sum_{n\in\mathbb{Z}} x_n\big((\alpha_{n-1}\beta_n - 2\alpha_n\beta_{n-1})e_{n-1} + (2\alpha_{n+1}\beta_n - \alpha_n\beta_{n+1})e_{n+1}\big). \end{aligned}$$

By equating coefficients, we see that for all $p \in \mathbb{Z}$,

$$x_{p+1}(\alpha_p\beta_{p+1} - 2\alpha_{p+1}\beta_p) + x_{p-1}(2\alpha_p\beta_{p-1} - \alpha_{p-1}\beta_p) = 0,$$

or equivalently

$$x_{p+1} = -\frac{2\alpha_p\beta_{p-1} - \alpha_{p-1}\beta_p}{\alpha_p\beta_{p+1} - 2\alpha_{p+1}\beta_p} x_{p-1} \quad \text{for all } p \in \mathbb{Z}.$$

But a routine calculation shows that

$$\frac{2\alpha_p\beta_{p-1} - \alpha_{p-1}\beta_p}{\alpha_p\beta_{p+1} - 2\alpha_{p+1}\beta_p} = -2\frac{\sqrt{1 + 4^{-(p+1)}}}{\sqrt{1 + 4^{-(p-1)}}},$$

and so the condition $\|x\|^2 = \sum_{p\in\mathbb{Z}} |x_p|^2 < \infty$ clearly implies that

$$x_p = 0 \quad \text{for all } p \in \mathbb{Z}.$$

Thus $\ker(MAN - NA^*M) = 0 = \ker(MAN - NA^*M)^*$, as required to complete the proof. ∎

Using a slightly more subtle "direct sum" device than in Case 2 of Theorem 3.2, we obtain:

3.6. Corollary. *If $j$ is any positive integer, then there exists a normal operator $D \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ and a projection $P \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ of infinite rank and nullity such that*

$$\operatorname{rank}(I - P)DP = j$$

*and $PD(I - P)$ is a quasiaffinity.*

*Proof.* By Theorem 3.5, we can find a normal operator $N = \left[\begin{smallmatrix} N_1 & N_2 \\ 0 & N_4 \end{smallmatrix}\right] \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ such that $N_2$ is a quasiaffinity. Let $Q = \left[\begin{smallmatrix} I_j & I_j \\ I_j & I_j \end{smallmatrix}\right] \in \mathbb{M}_{2j}(\mathbb{C})$, so that $Q$ is (2 times) a projection of rank $j$. Then $D := N \oplus Q$ is clearly normal, and it is unitarily equivalent to

$$\begin{bmatrix} I_j & & & I_j \\ & N_1 & N_2 & \\ & 0 & N_4 & \\ I_j & & & I_j \end{bmatrix}.$$

Set $D_1 = \left[\begin{smallmatrix} I_j & 0 \\ 0 & N_1 \end{smallmatrix}\right]$, $D_2 = \left[\begin{smallmatrix} 0 & I_j \\ N_2 & 0 \end{smallmatrix}\right]$, $D_3 = \left[\begin{smallmatrix} 0 & 0 \\ I_j & 0 \end{smallmatrix}\right]$ and $D_4 = \left[\begin{smallmatrix} N_4 & 0 \\ 0 & I_j \end{smallmatrix}\right]$. Clearly $\operatorname{rank} D_3 = j$ and $D_2$ is a quasiaffinity. ∎

**3.7.** In Theorem 2.8, we saw that if $D \in \mathbb{M}_{2m}(\mathbb{C})$ is a normal matrix, and if $P \in \mathbb{M}_{2m}(\mathbb{C})$ is a projection of rank $m$ such that $\operatorname{rank}(I-P)DP = 1$ and $\operatorname{rank} PD(I-P) = m$, then $D$ is necessarily cyclic. It is reasonable to ask, therefore, whether an analogue of this might hold in the infinite-dimensional setting. In general, the answer is no.

3.8. COROLLARY. *For any integer $j \geq 0$, there exists a non-cyclic normal operator $D \in \mathcal{B}(\mathcal{H})$ and an orthogonal projection $P \in \mathcal{B}(\mathcal{H})$ of infinite rank and nullity such that*

$$\operatorname{rank}(I-P)DP = j$$

*and $PD(I-P)$ is a quasiaffinity.*

*Proof.* By Theorem 3.5, we can choose a normal operator $N \in \mathcal{B}(\mathcal{H})$ with

$$N = \begin{bmatrix} N_1 & N_2 \\ 0 & N_4 \end{bmatrix},$$

where $N_2$ is a quasiaffinity, and by Corollary 3.6 (or by Theorem 3.5 once again if $j = 0$), we may choose a normal operator $M \in \mathcal{B}(\mathcal{H})$ such that

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix},$$

where $\operatorname{rank} M_2 = j$ and $M_2$ is a quasiaffinity.

Define

$$D = \begin{bmatrix} N_1 & & & & & N_2 \\ & N_1 & & & N_2 & \\ & & M_1 & M_2 & & \\ & & M_3 & M_4 & & \\ & 0 & & & N_4 & \\ 0 & & & & & N_4 \end{bmatrix}.$$

Letting $P = I \oplus I \oplus I \oplus 0 \oplus 0 \oplus 0$, we see that $\operatorname{rank}(I-P)DP = \operatorname{rank} M_3 = j$ and $PD(I-P)$ is a quasiaffinity. Moreover, $D$ is unitarily equivalent to $N \oplus N \oplus M$, and thus is not cyclic. ■

## 4. Compact normal operators

**4.1.** Let $D \in \mathcal{B}(\mathcal{H})$ (where $\mathcal{H}$ is either finite- or infinite-dimensional) be a normal operator, and let $P \in \mathcal{B}(\mathcal{H})$ be a non-trivial projection. Write

$$D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$$

relative to the decomposition $\mathcal{H} = \operatorname{ran} P \oplus \operatorname{ran}(I-P)$.

The fact that in the infinite-dimensional setting we can find $D$ and $P$ as above such that $D_3 = 0 \neq D_2$, whereas no such $D$ and $P$ exist when $\dim \mathcal{H} < \infty$, is the statement that not every normal operator acting on an infinite-dimensional Hilbert space is orthogonally reductive, whereas every normal matrix is.

In [1], the concept of an almost invariant subspace for bounded linear operators $T$ acting on infinite-dimensional Banach spaces was introduced. Given a Banach space $\mathfrak{X}$ and an infinite-dimensional (closed) subspace $\mathfrak{M}$ of $\mathfrak{X}$ such that $\mathfrak{X}/\mathfrak{M}$ is again infinite-dimensional ($\mathfrak{M}$ is then called a *half-space* of $\mathfrak{M}$), we say that $\mathfrak{M}$ is *almost invariant* for $T$ if there exists a finite-dimensional subspace $\mathfrak{F}$ of $\mathfrak{X}$ such that $T\mathfrak{M} \subseteq \mathfrak{M}+\mathfrak{F}$. The minimal dimension of such a space $\mathfrak{F}$ is referred to as the *defect* of $T$ relative to $\mathfrak{M}$. In [5] and [7], it was shown that every operator $T$ acting on an infinite-dimensional Banach space admits an almost invariant half-space of defect at most 1. This is a truly remarkable result.

As a possible generalisation of the notion of reductivity for Hilbert space operators, we propose the following definition.

4.2. Definition. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *almost reductive* if for every projection $P \in \mathcal{B}(\mathcal{H})$, the condition $\operatorname{rank}(I-P)TP < \infty$ implies that $\operatorname{rank} PT(I-P) < \infty$.

**4.3.** It is clear that every invariant-half space is automatically almost-invariant for $T$. If the notion of "almost-reductivity" is to make sense, one should expect that every *orthogonally reductive* operator should be "almost reductive".

The relevance of this to the problem we have been examining is as follows: if $K \in \mathcal{B}(\mathcal{H})$ is a *compact* normal operator, then it is well-known [8] that $K$ is orthogonally reductive. This leads to the following question.

4.4. Question. Is every compact normal operator $K$ almost reductive? (More generally, is every *reductive* normal operator $D \in \mathcal{B}(\mathcal{H})$ almost reductive?)

Phrased another way, does there exist a compact normal operator $K$ and a projection $P$ (necessarily of infinite rank and nullity) such that

$$\operatorname{rank}(I-P)KP < \infty \quad \text{and} \quad \operatorname{rank} PK(I-P) = \infty?$$

The normal operators $D$ constructed in Theorems 3.2 and 3.5 for which $\operatorname{rank}(I-P)DP < \infty$ and $\operatorname{rank} PD(I-P) = \infty$ were definitely not compact, nor were they reductive.

So far, we have been unable to resolve this question. Indeed, we propose the following (potentially simpler) one:

4.5. Question. Do there exist a compact normal operator $K \in \mathcal{B}(\mathcal{H})$ and a projection $P \in \mathcal{B}(\mathcal{H})$ such that $\operatorname{rank}(I-P)KP < \infty$ and $PK(I-P)$ is a quasiaffinity?

While we do not have an answer to this question, nevertheless, there are some things that we can say about its structure, should such an operator $K$ exist. First we recall a result of Fan and Fong.

4.6. Theorem ([3, Theorem 1]). *Let $H$ be a compact hermitian operator. Then the following are equivalent:*

(a) $H = [A^*, A]$ *for some compact operator* $A$.
(b) *There exists an orthonormal basis* $\{e_n\}_{n\in\mathbb{N}}$ *such that* $\langle He_n, e_n\rangle = 0$ *for all* $n \in \mathbb{N}$.

Recall that a compact operator $K \in \mathcal{B}(\mathcal{H})$ is said to be a *Hilbert–Schmidt* operator if there exists an orthonormal basis $\{e_n\}_{n=1}^\infty$ for $\mathcal{H}$ such that

$$\|K\|_2 := (\operatorname{tr}(K^*K))^{1/2} = \Big(\sum_{n=1}^\infty \langle K^*Ke_n, e_n\rangle\Big)^{1/2} < \infty.$$

(Equivalently, this holds for *all* orthonormal bases $\{e_n\}_{n=1}^\infty$.) When this is the case, the map $K \mapsto \|K\|_2$ defines a norm on the set $\mathcal{C}_2(\mathcal{H})$ of all Hilbert–Schmidt operators on $\mathcal{H}$. (Although this is not the original definition of $\mathcal{C}_2(\mathcal{H})$, it is equivalent to it.)

4.7. Corollary. *Let*

$$K = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}$$

*be a compact normal operator in* $\mathcal{B}(\mathcal{H}\oplus\mathcal{H})$. *Then* $K_2 \in \mathcal{C}_2(\mathcal{H})$ *if and only if* $K_3 \in \mathcal{C}_2(\mathcal{H})$, *in which case* $\|K_2\|_2 = \|K_3\|_2$.

*In particular, therefore, if* $K_3$ *is a finite-rank operator, then* $K_2$ *must be a Hilbert–Schmidt operator.*

*Proof.* As $K$ is normal, it follows that $K_1^*K_1 + K_3^*K_3 = K_1K_1^* + K_2K_2^*$, and thus $[K_1^*, K_1] = K_2K_2^* - K_3^*K_3$. Now, $K_1$ is compact, and so by the above theorem, there exists an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ such that $\langle (K_2K_2^* - K_3^*K_3)e_n, e_n\rangle = 0$ for all $n \in \mathbb{N}$.

Suppose that $K_3 \in \mathcal{C}_2(\mathcal{H})$. Then

$$\|K_3\|_2^2 = \operatorname{tr}(K_3^*K_3) = \sum_{n=1}^\infty \langle K_3^*K_3e_n, e_n\rangle < \infty.$$

Therefore,

$$\sum_{n=1}^{\infty}\langle K_2K_2^*e_n, e_n\rangle = \sum_{n=1}^{\infty}\langle K_3^*K_3e_n, e_n\rangle < \infty,$$

proving that $K_2 \in \mathcal{C}_2(\mathcal{H})$, and $\|K_2\|_2 = \|K_3\|_2$.

The last statement of the theorem is obvious. ∎

**4.8.** The proof of Theorem 2.8 yields a very specific structure result for normal matrices $D \in \mathbb{M}_{2m}(\mathbb{C})$ for which there exists an orthogonal projection $P$ satisfying $\operatorname{rank}(I-P)DP = 1$ and $\operatorname{rank} PD(I-P) = m$. Since orthogonal reductivity and normality of matrices coincide, Proposition 4.10 below can be seen as an extension of that structure result to the infinite-dimensional setting.

4.9. Definition. By a *simple bilateral chain of subspaces* of a Hilbert space $\mathcal{H}$ we mean a sequence $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$ of closed subspaces with

$$\cdots \subset \mathcal{M}_{-2} \subset \mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots,$$

where $\dim(\mathcal{M}_{j+1} \ominus \mathcal{M}_j) = 1$ for all $j \in \mathbb{Z}$. We say an operator $T \in \mathcal{B}(\mathcal{H})$ *shifts forward* a simple bilateral chain $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$ if

$$T\mathcal{M}_j \subset \mathcal{M}_{j+1}, \quad \forall j \in \mathbb{Z}.$$

4.10. Proposition. *Let $T$ be an orthogonally reductive operator on $\mathcal{H}$ and assume that relative to a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ it has the representation*

$$T = \begin{bmatrix} A & L \\ F & B \end{bmatrix},$$

*where $F$ has rank 1 and $L$ has infinite rank. Then $T$ has an infinite-dimensional invariant subspace $\mathcal{H}_0$ (not necessarily proper) such that the restriction $T_0$ of $T$ to $\mathcal{H}_0$ shifts forward a simple bilateral chain $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$ of subspaces.*

*Proof.* Assume with no loss of generality that $T$ is invertible and let $\mathcal{M}_0 = \mathcal{H}_1$. We will define subspace $\mathcal{M}_j$ inductively: we set

$$\mathcal{M}_{j+1} = \mathcal{M}_j + T\mathcal{M}_j, \qquad \forall j \geq 0,$$
$$\mathcal{M}_{j-1} = \mathcal{M}_j \cap T^{-1}\mathcal{M}_j, \quad \forall j \leq 0.$$

Then

$$\cdots \subset \mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots, \qquad T\mathcal{M}_j \subset \mathcal{M}_{j+1}, \qquad \forall j \in \mathbb{Z}.$$

The assumption that $F$ has rank 1 implies that $\mathcal{M}_1 \ominus \mathcal{M}_0$ has dimension 1. It follows inductively that the dimension of $\mathcal{M}_{j+1} \ominus \mathcal{M}_j$ is at most 1 for all

$j \in \mathbb{Z}$. We shall show that this difference in dimensions is exactly 1 for all $j \in \mathbb{Z}$.

Suppose not. First assume $j > 1$. If $\mathcal{M}_{j+1} = \mathcal{M}_j$, then $\mathcal{M}_j$ is invariant under $T$ and thus reducing. This means that

$$P_j T(I - P_j) = 0,$$

with $P_j$ denoting the orthogonal projection onto $\mathcal{M}_j$. In particular, then,

$$L(I - P_j) = P_0 L(I - P_j) = 0.$$

But this implies that the rank of $L$ is at most $j$, which is a contradiction.

The proof for $j < 1$ is similar. In summary, we conclude that $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$ is a proper bilateral chain of subspaces.

Now $\bigcap_{j=-\infty}^{\infty} \mathcal{M}_j$ and $\bigvee_{j=-\infty}^{\infty} \mathcal{M}_j$ are both invariant, and hence reducing. Let

$$\mathcal{H}_0 = \Big(\bigvee_{j=-\infty}^{\infty} \mathcal{M}_j\Big) \ominus \Big(\bigcap_{j=-\infty}^{\infty} \mathcal{M}_j\Big),$$

and note that if we define

$$\mathcal{M}_j' = \mathcal{M}_j \ominus \Big(\bigcap_{k=-\infty}^{\infty} \mathcal{M}_k\Big),$$

and $T_0 := T|_{\mathcal{H}_0}$, then $\{\mathcal{M}_j'\}_{j=-\infty}^{\infty}$ is the desired bilateral chain in $\mathcal{H}_0$ which $T_0$ shifts forward. ∎

For compact normal operators, we can obtain a stronger result.

4.11. Corollary. *If $K$ is a compact normal operator on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of the form*

$$K = \begin{bmatrix} A & L \\ F & B \end{bmatrix},$$

*where $F$ has rank 1 and $L$ is a quasiaffinity, then $K$ shifts forward a simple bilateral chain $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$ of subspaces.* (*Here it is understood that* $\dim \mathcal{H}_1 = \infty = \dim \mathcal{H}_2$.)

*Proof.* It is well-known that compact normal operators are orthogonally reductive [8]. Thus we must only show that the subspace $\mathcal{H}_0$ of the proposition above coincides with $\mathcal{H}$. In other words,

$$\bigcap_{j=-\infty}^{\infty} \mathcal{M}_j = 0, \qquad \bigvee_{j=-\infty}^{\infty} \mathcal{M}_j = \mathcal{H}.$$

Set

$$\mathcal{N}_1 = \bigcap_{j=-\infty}^{\infty} \mathcal{M}_j, \quad \mathcal{N}_2 = \mathcal{M}_0 \ominus \mathcal{N}_1,$$

$$\mathcal{N}_3 = \Big(\bigvee_{j=-\infty}^{\infty} \mathcal{M}_j\Big) \ominus \mathcal{M}_0, \quad \mathcal{N}_4 = \Big(\bigvee_{j=-\infty}^{\infty} \mathcal{M}_j\Big)^{\perp}.$$

As $\mathcal{N}_1, \bigoplus_{1\le i\le 3} \mathcal{N}_i$ are both invariant and therefore reducing for $K$, with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 = (\mathcal{N}_1 \oplus \mathcal{N}_2) \oplus (\mathcal{N}_3 \oplus \mathcal{N}_4)$, we may write

$$K = \begin{bmatrix} A & L \\ F & B \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix} = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & L' & 0 \\ 0 & F' & B_3 & 0 \\ 0 & 0 & 0 & B_4 \end{bmatrix} \begin{matrix} \mathcal{N}_1 \\ \mathcal{N}_2 \\ \mathcal{N}_3 \\ \mathcal{N}_4 \end{matrix}.$$

Since

$$L = \begin{bmatrix} 0 & 0 \\ L' & 0 \end{bmatrix}$$

is a quasiaffinity, it follows that $\mathcal{N}_1 = 0$, and similarly $\mathcal{N}_4 = 0$. In other words,

$$\bigcap_{j=-\infty}^{\infty} \mathcal{M}_j = \mathcal{N}_1 = 0, \qquad \bigvee_{j=-\infty}^{\infty} \mathcal{M}_j = \mathcal{N}_4^{\perp} = \mathcal{H}. \ \blacksquare$$

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