OFF-DIAGONAL CORNERS OF SUBALGEBRAS OF $\mathcal{L}(\mathbb{C}^n)$

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Abstract. Let $n \in \mathbb{N}$, and consider $\mathbb{C}^n$ equipped with the standard inner product. Let $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^n)$ be a unital algebra and $P \in \mathcal{L}(\mathbb{C}^n)$ be an orthogonal projection. The space $\mathfrak{L} := P^\perp \mathfrak{A}|_{\mathrm{ran}\, P}$ is said to be an off-diagonal corner of $\mathfrak{A}$, and $\mathfrak{L}$ is said to be essential if $\cap\{\ker L : L \in \mathfrak{L}\} = \{0\}$ and $\cap\{\ker L^* : L \in \mathfrak{L}\} = \{0\}$, where L^* denotes the adjoint of L. Our goal in this paper is to determine effective upper bounds on dim $\mathfrak{A}$ in terms of dim $\mathfrak{L}$, where $\mathfrak{L}$ is an essential off-diagonal corner of $\mathfrak{A}$. A detailed structure analysis of $\mathfrak{A}$ based upon the dimension of $\mathfrak{L}$, while seemingly elusive in general, is nevertheless provided in the cases where dim $\mathfrak{L} \in \{1, 2\}$.

1. Introduction and notation

1.1. Let $1 \le n$ be an integer, and consider $\mathbb{C}^n$ equipped with the standard inner product. By $\mathcal{L}(\mathbb{C}^n)$ we denote the self-adjoint algebra of (necessarily continuous) linear maps from the Hilbert space $\mathbb{C}^n$ into itself. We shall often identify elements of $\mathcal{L}(\mathbb{C}^n)$ with their $n \times n$ complex matrices relative to the standard orthonormal basis for $\mathbb{C}^n$. Moreover, to improve the readability of the paper, we shall use the same notation I_n (resp. 0_n) to denote both the $n \times n$ identity matrix (resp. the $n \times n$ zero matrix), as well as the identity map (resp. the zero map) in $\mathcal{L}(\mathbb{C}^n)$. Given a unital subalgebra $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^n)$, and an orthogonal projection $P \in \mathcal{L}(\mathbb{C}^n)$, we shall refer to $P^\perp \mathfrak{A}|_{\mathrm{ran}\, P}$ as an **off-diagonal corner** of $\mathfrak{A}$. [Here, and throughout this paper, projections will always be assumed to be orthogonal, that is – *self-adjoint* idempotents.] We are interested in the question: how much is the structure of $\mathfrak{A}$ determined by the structure of $\mathfrak{L}$? In this generality, very little can be said.

For example, suppose that $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^n)$ (resp. $\mathfrak{B} \subseteq \mathcal{L}(\mathbb{C}^m)$) is a unital algebra and that $P \in \mathcal{L}(\mathbb{C}^n)$ (resp. $Q \in \mathcal{L}(\mathbb{C}^m)$) is a non-trivial projection. Let $\mathcal{M}_1 := \mathrm{ran}\, P$, $\mathcal{M}_2 := \mathrm{ran}\, P^\perp$, $\mathcal{N}_1 := \mathrm{ran}\, Q$ and $\mathcal{N}_2 := \mathrm{ran}\, Q^\perp$. Suppose furthermore that $\mathcal{N}_1$ is invariant for $\mathfrak{B}$ – i.e. $Bx \in \mathcal{N}_1$ for all $B \in \mathfrak{B}$ and $x \in \mathcal{N}_1$.

Let $\mathfrak{C} := \mathfrak{A} \oplus \mathfrak{B}$, and decompose

$$\mathbb{C}^n \oplus \mathbb{C}^m \simeq \mathbb{C}^{n+m} \simeq \mathcal{N}_1 \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{N}_2.$$

Relative to this decomposition, we find that

$$\mathfrak{C} = \left\{ \begin{bmatrix} B_1 & 0 & 0 & B_2 \\ 0 & A_1 & A_2 & 0 \\ 0 & L & A_4 & 0 \\ 0 & 0 & 0 & B_4 \end{bmatrix} : \begin{bmatrix} A_1 & A_2 \\ L & A_4 \end{bmatrix} \in \mathfrak{A},\ \begin{bmatrix} B_1 & B_2 \\ 0 & B_4 \end{bmatrix} \in \mathfrak{B} \right\}.$$

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If R is the projection of $\mathbb{C}^{n+m}$ onto $\mathcal{N}_1 \oplus \mathcal{M}_1$, then the off-diagonal corner $R^\perp \mathfrak{C}|_{\operatorname{ran} R} = \left\{ \begin{bmatrix} 0 & L \\ 0 & 0 \end{bmatrix} : L \in \mathfrak{L} \right\}$ clearly has the same dimension as $\mathfrak{L}$, and yet yields no information whatsoever about $\mathfrak{C}$, since it fails to interact with that component of $\mathfrak{C}$ which stems from the algebra $\mathfrak{B}$. We avoid this obvious pitfall by requiring that $\mathfrak{L}$ be an essential subspace, which we now define.

Given positive integers p and q, a subspace $\mathfrak{L}$ of the set $\mathcal{L}(\mathbb{C}^p, \mathbb{C}^q)$ of linear maps from $\mathbb{C}^p$ to $\mathbb{C}^q$ is said to be an **essential subspace** if $\cap\{\ker L : L \in \mathfrak{L}\} = \{0\}$, and $\operatorname{span}\{\operatorname{ran} L : L \in \mathfrak{L}\} = \mathbb{C}^q$. (The terminology is motivated from the theory of C^*-algebras, where a closed ideal $\mathfrak{J}$ of a C^*-algebra $\mathfrak{A}$ is said to be essential if $0 \neq a \in \mathfrak{A}$ implies that there exist $j_1, j_2 \in \mathfrak{J}$ such that $aj_1, j_2 a \neq 0$. In our setting, the subspace $\mathfrak{L}$ is essential if and only if $0 \neq A \in \mathcal{L}(\mathbb{C}^q)$ and $0 \neq B \in \mathcal{L}(\mathbb{C}^p)$ implies that there exist $L_1, L_2 \in \mathfrak{L}$ such that $AL_1 \neq 0$ and $L_2 B \neq 0$.)

We remark that it can be shown that if $\mathfrak{A} \subseteq \mathbb{C}^n$ is a unital algebra and $\mathfrak{L}$ is a non-essential off-diagonal corner corresponding to a non-trivial projection P with $\mathcal{R} := \operatorname{ran} P$, then there exists a decomposition $\mathcal{R} = \mathcal{N}_1 \oplus \mathcal{M}_1$ and $\mathcal{R}^\perp = \mathcal{M}_2 \oplus \mathcal{N}_2$ such that $\dim \mathcal{N}_1 + \dim \mathcal{N}_2 > 0$, and relative to the decomposition $\mathbb{C}^n = \mathcal{N}_1 \oplus \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{N}_2$, a typical member of $\mathfrak{A}$ is of the form

$$\begin{bmatrix} X_{11} & X_{12} & Y_{11} & Y_{12} \\ 0 & X_{22} & Y_{21} & Y_{22} \\ 0 & L & Z_{11} & Z_{12} \\ 0 & 0 & 0 & Z_{22} \end{bmatrix}.$$

As noted above – the dimension of $\mathfrak{L}$ in general gives no information about (upper bounds) on the dimension of $\mathfrak{A}$.

It is clear that if $p = q$ and if $\mathfrak{L}$ contains an invertible operator, then $\mathfrak{L}$ is essential. The space $\mathfrak{L}$ of linear maps admitting the matrix forms (relative to the standard orthonormal basis for $\mathbb{C}^3$)

$$\mathfrak{L} := \left\{ \begin{bmatrix} \alpha & 0 & \beta \\ 0 & \alpha & 0 \\ 0 & \beta & 0 \end{bmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$

is an example of an essential subspace of $\mathcal{L}(\mathbb{C}^3)$ not containing any invertible elements.

Our goal below is to determine upper bounds on the dimension of $\mathfrak{A}$, given the dimension d of one of its essential corners $\mathfrak{L}$. In general (see Theorem 2.5), the best bound we can find for $\dim \mathfrak{A}$ is on the order of d^3, but under certain conditions on the rank of P and the maximum of the ranks of the elements of $\mathfrak{L}$, we can do much better (see Theorem 2.9).

In Sections 3 and 4 of the paper, we closely examine the cases where $d = 1$ and $d = 2$ respectively. In the first instance, we are in fact able to classify *up to "admissible" similarity* (see Section 3.1 for the definition of an "admissible" similarity) all unital subalgebras of $\mathcal{L}(\mathbb{C}^{2p})$ admitting an essential corner of dimension 1 relative to a projection P of rank p. While the corresponding problem seems at the moment intractable in the case where $\dim \mathfrak{L} = 2$, we are nevertheless able to determine all possibly occurring values for the dimension of $\mathfrak{A}$, and we also demonstrate which values are possible when we further stipulate that $\mathfrak{A}$ should be a *self-adjoint* algebra. Our main emphasis will be on the case where the rank of the projection P is half of the dimension of the space.

1.2. We remind the reader of some notation that will be used throughout the paper. Given a subspace $\mathcal{S} \subseteq \mathcal{L}(\mathbb{C}^p, \mathbb{C}^q)$ and elements $A \in \mathcal{L}(\mathbb{C}^q)$, $B \in \mathcal{L}(\mathbb{C}^p)$, we set

$$A\mathcal{S}B = \{ASB : S \in \mathcal{S}\}.$$

Given vectors $x, y \in \mathbb{C}^n$, we denote by $x \otimes y^*$ the rank-one operator defined by $x \otimes y^*(z) = \langle z, y\rangle x$, $z \in \mathbb{C}^n$. If $\mathcal{K} \subseteq \mathbb{C}^n$ is a subspace, the projection of $\mathbb{C}^n$ onto $\mathcal{K}$ is denoted by $P_{\mathcal{K}}$, and the direct sum of two subspaces $\mathcal{K}$ and $\mathcal{J}$ of $\mathbb{C}^n$ is denoted by $\mathcal{K} \dotplus \mathcal{J}$. For $n \in \mathbb{N}$, we use $\mathfrak{D}_n$ to denote the (self-adjoint) algebra of all linear maps on $\mathbb{C}^n$ admitting a diagonal matrix relative to the standard orthonormal basis. Finally, if $P \in \mathcal{L}(\mathbb{C}^n)$ is a projection and $T_k \in \mathcal{L}(\mathbb{C}^n)$, $k = 1, 2$ admit a decomposition

$$T_k = \begin{bmatrix} A_k & B_k \\ L_k & D_k \end{bmatrix}$$

relative to the decomposition $\mathbb{C}^n = \operatorname{ran} P \oplus \operatorname{ran} P^\perp$, then $P^\perp(T_1T_2)|_{\operatorname{ran} P}$ equals the entry $L_1A_2 + D_1L_2$ of the corresponding matrix product. In the case where $D_1 = 0$, $\operatorname{rank} P = \operatorname{rank} P^\perp$, and we have chosen orthonormal bases for $\operatorname{ran} P$ and $\operatorname{ran} P^\perp$ such that $L_1 = I_p$, we shall often omit the "I_p" from the notation and simply write $A_2 = P^\perp(T_1T_2)|_{\operatorname{ran} P}$. The meaning will be clear from the context.

1.3. The current article can be seen as part of a more general program to study operators and algebras through their compressions. For example, in [5], it was shown that an operator $T \in \mathcal{B}(\mathcal{H})$ (the set of continuous linear operators acting on an infinite-dimensional Hilbert space $\mathcal{H}$) satisfies $\|P^\perp TP\| = \|PTP^\perp\|$ for all projections $P \in \mathcal{B}(\mathcal{H})$ if and only if $T = \alpha I + \beta X$ for some $\alpha, \beta \in \mathbb{C}$, where $X \in \mathcal{B}(\mathcal{H})$ is either a hermitian operator or a unitary operator whose essential spectrum is contained in a half-circle. In [6], those integers j and k for which there exist normal matrices $D \in \mathbb{M}_n(\mathbb{C})$ and a projection P such that $\operatorname{rank} P^\perp DP = j$ while $\operatorname{rank} PDP^\perp = k$ are characterised. Recently, those unital algebras $\mathcal{A} \subseteq \mathbb{M}_n(\mathbb{C})$ for which $P\mathcal{A}P|_{\operatorname{ran} P}$ is an algebra for all projections P were classified in [3] and [2]. In [1] it was shown that $T \in \mathbb{M}_n(\mathbb{C})$ has the property that $T = \alpha I_n + F$, where $F \in \mathbb{M}_n(\mathbb{C})$ has rank at most m if and only if the algebraic degree of $PTP|_{\operatorname{ran} P}$ is less than $m + 2$ whenever $P \in \mathbb{M}_n(\mathbb{C})$ is a projection of rank $m + 2$.

2. General results

2.1. Standard P-decompositions of an algebra. We first establish a decomposition for algebras $\mathfrak{A}$ relative to a projection P that will prove useful below.

Let $2 \leq n$ be an integer and $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^n)$ be a unital algebra. Suppose that $P \in \mathcal{L}(\mathbb{C}^n)$ is a projection of rank p and that the corresponding off-diagonal corner $\mathfrak{L} = P^\perp\mathfrak{A}|_{\operatorname{ran} P}$ of $\mathfrak{A}$ is non-zero. Let $\{L_1, L_2, \ldots, L_d\}$ be a basis for $\mathfrak{L}$, and choose $M_k \in \mathfrak{A}$, $1 \leq k \leq d$ such that $P^\perp M_k P = L_k$. Define $\mathfrak{M} := \operatorname{span}\{M_1, M_2, \ldots, M_d\}$ and set $\mathfrak{T} := \{T \in \mathfrak{A} : P^\perp TP = 0\}$. It is clear that $\dim \mathfrak{M} = \dim \mathfrak{L} = d$, and that

$$\mathfrak{A} = \mathfrak{M} \dotplus \mathfrak{T}.$$

We note that $\mathfrak{T}$ is not only a subspace of $\mathfrak{A}$; it is in fact a unital subalgebra. We further denote by $\mathfrak{N}$ the set $\{N \in \mathfrak{T} : N = PNP^\perp\}$, which we observe to be an ideal of $\mathfrak{T}$. Then $\mathfrak{N}$ admits a (subspace) complement in $\mathfrak{T}$, which we shall denote by $\mathfrak{V}$. That is,

$$\mathfrak{A} = \mathfrak{M} \dotplus \mathfrak{T} = \mathfrak{M} \dotplus \mathfrak{V} \dotplus \mathfrak{N}.$$

We refer to the above decomposition as a **standard P-decomposition** of $\mathfrak{A}$. These are in general far from unique, as the space $\mathfrak{M}$ depends *a priori* upon the choice of $M_1, M_2, \ldots, M_d$, which in turn depend upon our choice of a basis $\{L_1, L_2, \dots, L_d\}$ for $\mathfrak{L}$. However, the spaces $\mathfrak{T}$, $\mathfrak{N}$, and $\mathfrak{V}$, as well as the dimension of $\mathfrak{M}$ – namely $d = \dim\, \mathfrak{L}$ – are independent of the choice of a basis of $\mathfrak{L}$.

Thus, to be precise, a standard P-decomposition refers to the tuple

$$(\mathfrak{A}, P, \{L_1, L_2, \dots, L_d\}, \{M_1, M_2, \dots, M_d\}, \mathfrak{T}, \mathfrak{V}, \mathfrak{N}),$$

where $\mathfrak{T}, \mathfrak{V}$ and $\mathfrak{N}$ depend only upon $\mathfrak{A}$ and P.

Although the following is obvious from the discussion above, we state it as a proposition for ease of referencing.

2.2. Proposition. *Let $2 \le n$ be an integer and $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^n)$ be a unital algebra. Let $P \in \mathcal{L}(\mathbb{C}^n)$ be a projection, and suppose that $\mathfrak{L} := P^\perp \mathfrak{A}|_{\mathrm{ran}\, P}$ is a subspace of dimension d. Let $\mathfrak{A} = \mathfrak{M} \dotplus \mathfrak{T} = \mathfrak{M} \dotplus \mathfrak{V} \dotplus \mathfrak{N}$ be a standard P-decomposition of $\mathfrak{A}$ as defined above. Then*

$$\begin{aligned} \dim\, \mathfrak{A} &= \dim\, \mathfrak{M} + \dim\, \mathfrak{T} = d + \dim\, \mathfrak{T} \\ &= \dim\, \mathfrak{M} + \dim\, \mathfrak{V} + \dim\, \mathfrak{N} = d + \dim\, \mathfrak{V} + \dim\, \mathfrak{N}. \end{aligned}$$

As a consequence, in trying to estimate the dimension of $\mathfrak{A}$ in terms of d, we seek to understand how big the dimensions of $\mathfrak{V}$ and of $\mathfrak{N}$ can be, given the dimension of $\mathfrak{L}$.

The proof of the following Lemma is essentially contained in the proof of Proposition 5.1 of [4].

2.3. Lemma. *Let $1 \le p, q$ be natural numbers and $\mathcal{S} \subseteq \mathcal{L}(\mathbb{C}^p, \mathbb{C}^q)$ be a subspace. Let $\mu := \max\{\mathrm{rank}\, S : S \in \mathcal{S}\}$. There exist invertible linear maps $V \in \mathcal{L}(\mathbb{C}^p)$ and $W \in \mathcal{L}(\mathbb{C}^q)$ such that the linear map whose matrix relative to the standard orthonormal bases for $\mathbb{C}^p$ and $\mathbb{C}^q$ is*

$$\begin{bmatrix} I_\mu & 0 \\ 0 & 0 \end{bmatrix}.$$

lies in $W \mathcal{S} V$. Furthermore, relative to the same block-decomposition, each element $K \in W \mathcal{S} V$ is of the form

$$K = \begin{bmatrix} K_1 & K_2 \\ K_3 & 0 \end{bmatrix},$$

where $K_3 K_1^m K_2 = 0$ for all integers $m \ge 0$.

Of course, if $\mu = p = q$, we conclude that $I_p \in W \mathcal{S} V$, and there is no block-matrix decomposition for K as above.

2.4. Remark. Let $2 \le n$ be an integer and $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^n)$ be a unital algebra. Let $P \in \mathcal{L}(\mathbb{C}^n)$ be a projection of rank p, where $1 \le p < n$, and set $\mathfrak{L} := P^\perp \mathfrak{A}|_{\mathrm{ran}\, P}$. By Lemma 2.3, we can find invertible linear maps $W \in \mathcal{L}(\mathbb{C}^{n-p})$ and $V \in \mathcal{L}(\mathbb{C}^p)$ such that the linear map whose matrix relative to the standard orthonormal bases for $\mathbb{C}^p$ and $\mathbb{C}^q$ is

$$\begin{bmatrix} I_\mu & 0 \\ 0 & 0 \end{bmatrix}$$

lies in $W \mathfrak{L} V$, where $\mu := \max\{\mathrm{rank}\, L : L \in \mathfrak{L}\}$, and relative to this block-matrix decomposition, every element of $W \mathfrak{L} V$ is of the form

$$K = \begin{bmatrix} K_1 & K_2 \\ K_3 & 0 \end{bmatrix}.$$

Set $R := V \oplus W^{-1} \in \mathcal{L}(\mathbb{C}^n)$, so that R is invertible. Then, relative to the decomposition $\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^{n-p}$, we have

$$\mathfrak{B} := R^{-1}\mathfrak{A}R = \{\begin{bmatrix} V^{-1}XV & V^{-1}YW^{-1} \\ WLV & WZW^{-1} \end{bmatrix} : \begin{bmatrix} X & Y \\ L & Z \end{bmatrix} \in \mathfrak{A}\}.$$

If we set $\mathfrak{L}_{\mathfrak{B}} := P^{\perp}\mathfrak{B}|_{\operatorname{ran} P} = W\mathfrak{L}V$, then clearly $\dim \mathfrak{L}_{\mathfrak{B}} = \dim \mathfrak{L}$. Moreover, if $\mathfrak{A} = \mathfrak{M} \dotplus \mathfrak{V} \dotplus \mathfrak{N}$ is a standard P-decomposition for $\mathfrak{A}$, then $\mathfrak{B} = R^{-1}\mathfrak{M}R \dotplus R^{-1}\mathfrak{V}R \dotplus R^{-1}\mathfrak{N}R$ is a standard P-decomposition for $\mathfrak{B}$. Thus, estimating the dimensions of $\mathfrak{M}$, $\mathfrak{V}$ and $\mathfrak{N}$ is equivalent to estimating the sizes of the corresponding subspaces in the P-decomposition of $\mathfrak{B}$. In light of these remarks, when beginning a proof, we shall typically assume (without comment) that we have replaced $\mathfrak{A}$ by $\mathfrak{B}$. We emphasise that if $p = n - p$ and $\mathfrak{L}$ contains an invertible operator, then the action of replacing $\mathfrak{A}$ by $\mathfrak{B}$ means that we are assuming that $I_p \in \mathfrak{L}$; in fact, in this case we assume that $L_1 = I_p$.

2.5. Theorem. *Let $2 \leq n$ be an integer and $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^n)$ be a unital algebra. Let $P \in \mathcal{L}(\mathbb{C}^n)$ be a projection of rank p, where $1 \leq p < n$, and suppose that $\mathfrak{L} := P^{\perp}\mathfrak{A}|_{\operatorname{ran} P}$ is an essential subspace of dimension d. Let $\mathfrak{A} = \mathfrak{M} \dotplus \mathfrak{V} \dotplus \mathfrak{N}$ be a standard P-decomposition of $\mathfrak{A}$ as defined in Section 2.1. Then* $\dim \mathfrak{N} \leq d^3$ *and* $\dim \mathfrak{V} \leq 2d^2$. *Consequently,*

$$\dim \mathfrak{A} \leq d(1 + d)^2.$$

Proof. Let $\mathfrak{X} := P\mathfrak{V}|_{\operatorname{ran} P}$ and $\mathfrak{Z} := P^{\perp}\mathfrak{V}|_{\operatorname{ran} P^{\perp}}$. From the decomposition $\mathfrak{T} = \mathfrak{V} \dotplus \mathfrak{N}$, we see that

$$\dim \mathfrak{T} \leq \dim \mathfrak{X} + \dim \mathfrak{Z} + \dim \mathfrak{N}.$$

Writing $T := \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathfrak{V}$ relative to the decomposition $\mathbb{C}^n = \operatorname{ran} P \oplus \operatorname{ran} P^{\perp}$, it easily follows from the fact that both M_kT and TM_k lie in $\mathfrak{A}$ that $L_kX \in \mathfrak{L}$ and $ZL_k \in \mathfrak{L}$ for all $1 \leq k \leq d$.

For each $1 \leq k \leq d$, define the maps

$$\begin{array}{cccc} \varphi_k : & \mathfrak{X} & \to & \mathfrak{L} \\ & X & \mapsto & L_kX \end{array}$$

and

$$\begin{array}{cccc} \psi_k : & \mathfrak{Z} & \to & \mathfrak{L} \\ & Z & \mapsto & ZL_k \end{array}.$$

Define a linear map $\varphi : \mathfrak{X} \to \mathfrak{L}^d$ by $\varphi(X) = (\varphi_1(X), \varphi_2(X), \ldots, \varphi_d(X))$, and similarly define the map $\psi : \mathfrak{Z} \to \mathfrak{L}^d$ by $\psi(Z) = (\psi_1(Z), \psi_2(Z), \ldots, \psi_d(Z))$. We claim that φ and ψ are injective. Indeed, if $\varphi(X) = 0$, then $L_kX = 0$ for all $1 \leq k \leq d$, whence $\operatorname{ran} X \subseteq \cap_{1 \leq k \leq d} \ker L_k = \{0\}$, and thus $X = 0$. From this we find that $\dim \mathfrak{X} \leq \dim \mathcal{L}^d = d^2$.

In a similar manner, if $\psi(Z) = 0$, then $ZL_k = 0$ for all $1 \leq k \leq d$, and thus $Z|_{\operatorname{span}\{\operatorname{ran} L_k : 1 \leq k \leq d\}} = 0$. But $\operatorname{span}\{\operatorname{ran} L_k : 1 \leq k \leq d\} = \operatorname{ran} P^{\perp}$, as $\mathfrak{L}$ is an essential subspace. Thus $Z = 0$ and so ψ is injective. It follows that $\dim \mathfrak{Z} \leq d^2$.

Set $\mathfrak{Y} := \{Y : \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} \in \mathfrak{N}\}$, again, relative to the decomposition $\mathbb{C}^n = \operatorname{ran} P \oplus \operatorname{ran} P^{\perp}$. For each $1 \leq i, j \leq d$, define the map

$$\begin{array}{cccc} \gamma_{i,j} : & \mathfrak{Y} & \to & \mathfrak{L} \\ & Y & \mapsto & L_jYL_i \end{array},$$

and set $\gamma(Y) = (\gamma_{i,j}(Y))_{i,j=1}^d \in \mathbb{M}_d(\mathfrak{L})$. Once again, we claim that γ is injective. Suppose that $Y \in \ker \gamma$, so that $L_j Y L_i = 0$ for all $1 \le i, j \le d$. Temporarily fix j. Then $(L_j Y) L_i = 0$ for all $1 \le i \le d$, and arguing as above, the fact that $\text{span}\{\text{ran}\, L_i : 1 \le i \le d\} = \text{ran}\, P^\perp$ implies that $L_j Y = 0$. Since this is true for all $1 \le j \le d$, $\text{ran}\, Y \subseteq \cap_{1 \le j \le d} \ker\, L_j = \{0\}$, so that $Y = 0$. It follows that $\dim \mathfrak{N} = \dim \mathfrak{Y} \le \dim \mathbb{M}_d(\mathfrak{L}) = d^3$.

The last statement of the Theorem follows immediately from Proposition 2.2.

□

The estimates from Theorem 2.5 are – up to a constant multiple – optimal, as the following two examples demonstrate.

2.6. Example. Let $2 \le p$ be an integer. Let $\mathcal{E} := \{e_1, e_2, \dots e_p, f_1, f_2, \dots, f_p\}$ be an orthonormal basis for $\mathbb{C}^{2p}$. Let $P \in \mathcal{L}(\mathbb{C}^{2p})$ be the projection onto span $\{e_1, e_2, \dots, e_p\}$, and define $\mathfrak{L} = \text{span}\{f_j \otimes e_1^*, f_1 \otimes e_i^* : 1 \le i, j \le p\}$.

Set $\mathfrak{X} = \text{span}\{e_k \otimes e_1^*, e_j \otimes e_i^* : 1 \le k \le p, 2 \le i, j \le p\}$ and $\mathfrak{Z} = \text{span}\,\{f_1 \otimes f_k^*, f_j \otimes f_i^* : 1 \le k \le p, 2 \le i, j \le p\}$. An elementary calculation shows that the linear space $\mathfrak{A}$ spanned by $\mathfrak{L}$, $\mathfrak{X}$ and $\mathfrak{Z}$ is in fact an algebra. In particular, if $p = 4$, then the algebra $\mathfrak{A}$ generated by $\mathfrak{L}, \mathfrak{X}$ and $\mathfrak{Z}$ consists of those linear maps whose matrices relative to the corresponding orthonormal basis $\mathcal{E}$ for $\mathbb{C}^8$ look like

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * \\ * & 0 & 0 & 0 & 0 & * & * & * \\ * & 0 & 0 & 0 & 0 & * & * & * \\ * & 0 & 0 & 0 & 0 & * & * & * \end{bmatrix}.$$

It is clear that $d := \dim \mathfrak{L} = 2p - 1$, while $\dim \mathfrak{X} = \dim \mathfrak{Z} = p + (p-1)^2$. Thus $\dim \mathfrak{X}$ and $\dim \mathfrak{Z}$ are on the order of $\left(\frac{d}{2}\right)^2$.

We can enlarge the algebra $\mathfrak{A}$ by adding the space $\mathfrak{Y} = \text{span}\,\{e_j \otimes f_i^* : 2 \le i, j \le p\}$ to get an algebra $\mathfrak{B} = \text{span}\{\mathfrak{A}, \mathfrak{Y}\}$. In this case, $\dim \mathfrak{Y} = (p-1)^2$ is on the order of $\frac{d^2}{4}$, and thus $\dim \mathfrak{B}$ is on the order of $\frac{3}{4} d^2$.

2.7. Example. To obtain an algebra $\mathfrak{A}$ whose dimension is on the same order of magnitude as d^3, where d denotes the dimension of an essential corner $\mathfrak{L}$ of $\mathfrak{A}$ requires a bit more effort.

Here we shall begin with a positive integer $1 \le \mu$, and we shall set $p = \mu^3$. Let $P \in \mathcal{L}(\mathbb{C}^{2p})$ be a projection of rank p.

Let $\{e_1, e_2, \dots, e_p\}$ be an orthonormal basis for $\text{ran}\, P$, and $\{f_1, f_2, \dots, f_p\}$ be an orthonormal basis for $\text{ran}\, P^\perp$. For each $1 \le k \le \mu^2$, we define $\mathcal{H}_k = \text{span}\{e_{(k-1)\mu + j} : 1 \le j \le \mu\}$ and $\mathcal{K}_k = \text{span}\{f_{(k-1)\mu + j} : 1 \le j \le \mu\}$, so that the collection $\{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_{\mu^2}, \mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_{\mu^2}\}$ consists of mutually orthogonal spaces, each of dimension μ.

We let $\mathfrak{L}$ be the subspace of operators $L : \text{ran}\, P \to \text{ran}\, P^\perp$ satisfying the following conditions:

- $P_{\mathcal{K}_1} L P_{\mathcal{H}_1}$ is arbitrary.
- For each $2 \le j \le \mu^2$, there exists a scalar $\alpha_{j,1} \in \mathbb{C}$ such that $P_{\mathcal{K}_j} L P_{\mathcal{H}_1}(e_k) = f_{(j-1)\mu + k}$, $1 \le k \le \mu$. (In other words, the operator $P_{\mathcal{K}_j} L P_{\mathcal{H}_1}$ looks like a "scalar" operator with respect to the given bases for those subspaces.)

- For each $2 \le i \le \mu^2$, there exists a scalar $\alpha_{1,i} \in \mathbb{C}$ such that $P_{\mathcal{K}_1} L P_{\mathcal{H}_i}(e_{(i-1)\mu+k}) = f_k$, $1 \le k \le \mu$. (Again, the operator $P_{\mathcal{K}_1} L P_{\mathcal{H}_i}$ looks "scalar" with respect to the given bases for those subspaces.)
- For each $2 \le i, j \le \mu^2$, $P_{\mathcal{K}_j} L P_{\mathcal{H}_i} = 0$.

When $\mu = 4$, $p = 64$, $\mathfrak{L}$ consists of all maps whose matrices relative to the bases $\{e_1, e_2, \ldots, e_{64}\}$ and $\{f_1, f_2, \ldots, f_{64}\}$ are of the form

$$\begin{bmatrix} A & \alpha_{1,2} I_4 & \cdots & \alpha_{1,16} I_4 \\ \alpha_{2,1} I_4 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{16,1} I_4 & 0 & \cdots & 0 \end{bmatrix}.$$

Here, $A \in \mathbb{M}_4(\mathbb{C})$ is arbitrary, while each $\alpha_{i,j} \in \mathbb{C}$.

Next, let $\mathfrak{Y}$ be the subspace of operators $Y : \operatorname{ran} P^\perp \to \operatorname{ran} P$ satisfying the following conditions:

- For each $2 \le i, j \le \mu^2$, $P_{\mathcal{H}_j} Y P_{\mathcal{K}_i}$ is arbitrary.
- If $i = 1$ or $j = 1$, then $P_{\mathcal{H}_j} Y P_{\mathcal{K}_i} = 0$.

Once again, when $\mu = 4$, $p = 64$, $\mathfrak{Y}$ consists of all linear maps whose matrices relative to the bases $\{f_1, f_2, \ldots, f_{64}\}$ and $\{e_1, e_2, \ldots, e_{64}\}$ are of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & Y_{2,2} & \cdots & Y_{2,16} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & Y_{16,2} & \cdots & Y_{16,16} \end{bmatrix}.$$

Here, $Y_{i,j} \in \mathbb{M}_4(\mathbb{C})$ is arbitrary, $2 \le i, j \le 16$.

We then set $\mathfrak{X} := \mathfrak{Y}\,\mathfrak{L} = \{YL : Y \in \mathfrak{Y}, L \in \mathfrak{L}\}$, and $\mathfrak{Z} := \mathfrak{L}\,\mathfrak{Y} = \{LY : L \in \mathfrak{L}, Y \in \mathfrak{Y}\}$. A routine calculation shows that $\mathfrak{X}$ consists of all operators $X : \operatorname{ran} P \to \operatorname{ran} P$ which satisfy $X = P_{\mathcal{H}_1}^\perp X P_{\mathcal{H}_1}$, while $\mathfrak{Z}$ consists of all operators $Z : \operatorname{ran} P^\perp \to \operatorname{ran} P^\perp$ satisfying $Z = P_{\mathcal{K}_1} Z P_{\mathcal{K}_1}^\perp$.

In our example where $\mu = 4$, $p = 64$, we find that $\mathfrak{X}$ is the set of all operators whose matrices relative to the basis $\{e_1, e_2, \ldots, e_{64}\}$ are of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ X_{2,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ X_{16,1} & 0 & \cdots & 0 \end{bmatrix},$$

and that $\mathfrak{Z}$ is the set of all operators whose matrices relative to the basis $\{f_1, f_2, \ldots, f_{64}\}$ are of the form

$$\begin{bmatrix} 0 & Z_{1,2} & \cdots & Z_{1,16} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $X_{i,j}, Z_{i,j}$ are arbitrary (for the respective (i,j)'s for which they appear above).

Clearly $\mathfrak{X}$ and $\mathfrak{Z}$ are algebras (the product of any two elements of $\mathfrak{X}$ (resp. of $\mathfrak{Z}$) is zero). In fact, it is routine – if tedious – to verify that

$$\mathfrak{A} := \left\{ \begin{bmatrix} X & Y \\ L & Z \end{bmatrix} : X \in \mathfrak{X}, Y \in \mathfrak{Y}, L \in \mathfrak{L}, Z \in \mathfrak{Z} \right\}$$

forms an algebra.

Observe that $d := \dim\, \mathfrak{L} = \mu^2 + 2(\mu^2 - 1) = 3\mu^2 - 2$; $\dim\, \mathfrak{X} = \dim\, \mathfrak{Z} = (\mu^2 - 1)(\mu^2) = \mu^4 - \mu^2$, while $\dim\, \mathfrak{Y} = (\mu^2 - 1)^2(\mu^2) = \mu^6 - 2\mu^4 + \mu^2$.

Thus $\dim\, \mathfrak{A} = (3\mu^2 - 2) + 2(\mu^4 - \mu^2) + (\mu^6 - 2\mu^4 + \mu^2) = \mu^6 + 2\mu^2 - 2$, which (when μ and thus d is large) is on the order of $\frac{d^3}{27}$.

2.8. As we have just seen, in general we must expect the dimension of $\mathfrak{A}$ to be on the order of $(\dim\, \mathfrak{L})^3$. However, there are cases where we can do *much* better. Note that in the statement of the following theorem, the condition that $\mu := \max\{\operatorname{rank} L : L \in \mathfrak{L}\} = p$ may be replaced by the condition that $\mathcal{L}$ contain an invertible operator.

2.9. Theorem. *Let $p \geq 1$ be an integer and $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^{2p})$ be a unital algebra. Let $P \in \mathcal{L}(\mathbb{C}^{2p})$ be a projection of rank p, and set $\mathfrak{L} := P^\perp \mathfrak{A}|_{\operatorname{ran} P}$. Suppose that $\mathfrak{L}$ is an essential subspace,* $\dim\, \mathfrak{L} = d$, *and that $\mu := \max\{\operatorname{rank} L : L \in \mathfrak{L}\} = p$. Then*

$$\dim \mathfrak{A} \leq 4d.$$

Proof. Let

$$\mathfrak{A} := \mathfrak{M} \dotplus \mathfrak{V} \dotplus \mathfrak{N}$$

denote a standard P-decomposition of $\mathfrak{A}$ with basis $\{L_k : 1 \leq k \leq d\}$ for $\mathfrak{L}$ and $\{M_k : 1 \leq k \leq d\}$ for $\mathfrak{M}$. We shall show that $\dim\, \mathfrak{V} \leq 2d$ and that $\dim\, \mathfrak{N} \leq d$, from which the result follows. Note that by using Lemma 2.3, we may assume that $L_1 \in \mathfrak{L}$ is an operator whose matrix relative to the the orthonormal bases $\{e_1, e_2, \ldots, e_p\}$ and $\{f_1, f_2, \ldots, f_p\}$ is I_p.

Decompose $\mathbb{C}^{2p} = \operatorname{ran} P \oplus (\operatorname{ran} P)^\perp$. For any $T = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathfrak{V}$, it follows that $X = P^\perp(M_1 T)|_{\operatorname{ran} P} \in \mathfrak{L}$, and similarly, $P^\perp(T M_1)|_{\operatorname{ran} P} = Z \in \mathfrak{L}$. Thus we can find integers $1 \leq d_1, d_2 \leq d$ and $R_1, R_2, \ldots, R_{d_1}, S_1, S_2, \ldots, S_{d_2} \in \mathfrak{T}$ such that

$$\mathfrak{T} = \mathfrak{V} \dotplus \mathfrak{N},$$

where $\mathfrak{V} = \operatorname{span}\{R_1, R_2, \ldots, R_{d_1}, S_1, S_2, \ldots, S_{d_2}\}$ and $\mathfrak{N} = \{N \in \mathfrak{T} : N = PNP^\perp\}$.

For $N = \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} \in \mathfrak{N}$, we also have that $P(NM_1)|_{\operatorname{ran} P} = Y \in P\mathfrak{V}|_{\operatorname{ran} P}$, and so from above, $Y \in \mathfrak{L}$. From this we deduce that $\dim\, \mathfrak{N} \leq d$.

In summary, $\dim \mathfrak{A} \leq d + (d_1 + d_2) + d \leq 4d$.

□

We remark that the above proof includes the fact that $\dim\, \mathfrak{V} \leq 2d = 2 \dim\, \mathfrak{L}$, a fact which will be used later in the paper.

2.10. Example. The upper estimate of Theorem 2.9 is the best we can hope for, even in the case where $\mathfrak{A}$ is a self-adjoint algebra. Indeed, let $1 \le p$ be an integer, and let $C \in \mathcal{L}(\mathbb{C}^p)$ be the unitary operator whose matrix relative to the standard orthonormal basis for $\mathbb{C}^p$ is

$$[C] := \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & & & 0 & 1 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \in \mathbb{M}_p(\mathbb{C}).$$

Thus C is the p-cycle (i.e. the unitary map which permutes the standard orthonormal basis of $\mathbb{C}^p$ cyclically). Clearly $C^p = I_p$ and if $L_j := C^{j-1}$, $1 \le j \le p$, then $\{L_1, L_2, \dots, L_p\}$ is a linearly independent set which forms a group.

Let $\mathfrak{L} := \operatorname{span}\{L_j : 1 \le j \le p\}$, and $\mathfrak{A} := \mathbb{M}_2(\mathbb{C}) \otimes \mathfrak{L} \subseteq \mathcal{L}(\mathbb{C}^{2p})$. Set $P := I_p \oplus 0_p \in \mathcal{L}(\mathbb{C}^{2p})$, and observe that $\mathfrak{L} = P^\perp \mathfrak{A}|_{\operatorname{ran} P}$ is an essential subspace of dimension p. Moreover, $\dim \mathfrak{A} = 4 \dim \mathfrak{L}$.

2.11. In Theorem 2.5, we established bounds on the dimension of a unital algebra $\mathfrak{A}$ admitting an essential off-diagonal corner $\mathfrak{L}$ of dimension d, and showed that it is possible for the dimension of such an algebra $\mathfrak{A}$ to be on the order of d^3 (see Example 2.7). In that Example, one notes that the rank of P is half the dimension of the ambient space, meaning that the corner $\mathfrak{L}$ is "square". When the rank of P differs from that of $P^\perp$ and the corresponding $\mu = p$, a stronger bound is available to us.

2.12. Proposition. *Let p and n be integers with $1 \le p < n$, and $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^n)$ be a unital algebra. Suppose that $P \in \mathcal{L}(\mathbb{C}^n)$ is a projection of rank p and that $p < q := n - p$. If $\mathfrak{L} := P^\perp \mathfrak{A}|_{\operatorname{ran} P}$ is an essential corner of dimension d and $\mathfrak{L}$ contains an element of full rank p, then*

$$\dim \mathfrak{A} \le 2d^2 + d - 1.$$

Proof. Let $\{L_1, L_2, \dots, L_d\}$ be a basis for $\mathfrak{L}$. By Lemma 2.3, we may assume (by applying a similarity if necessary) that the matrix for L_1 relative to the bases $\{e_1, e_2, \dots, e_p\}$ for $\operatorname{ran} P$ and $\{f_1, f_2, \dots, f_q\}$ for $\operatorname{ran} P^\perp$ looks like

$$\begin{bmatrix} I_p \\ 0 \end{bmatrix} \in \mathbb{M}_{q \times p}(\mathbb{C}).$$

Let us decompose $\mathbb{C}^q = \mathbb{C}^p \oplus \mathbb{C}^{q-p}$ and consider $L_1 = \begin{bmatrix} I_p \\ 0 \end{bmatrix} \in \mathfrak{L} \subseteq \mathcal{L}(\mathbb{C}^p, \mathbb{C}^p \oplus \mathbb{C}^q)$. Choose $M_1, M_2, \dots, M_d \in \mathfrak{A}$ such that $P^\perp M_k|_{\operatorname{ran} P} = L_k$, $1 \le k \le d$, and let

$$\mathfrak{A} = \mathfrak{M} \dotplus \mathfrak{T} = \mathfrak{M} \dotplus \mathfrak{V} \dotplus \mathfrak{N}$$

be the corresponding standard P-decomposition of $\mathfrak{A}$. Decompose $\mathbb{C}^n = \operatorname{ran} P \oplus \operatorname{ran} P^\perp$. If $T = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathfrak{V}$ relative to this decomposition, then

$$P^\perp (M_1 T)|_{\operatorname{ran} P} = \begin{bmatrix} X \\ 0 \end{bmatrix} \in \mathfrak{L}.$$

But span$\{\begin{bmatrix} X \\ 0 \end{bmatrix} : T = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathfrak{V}\}$ is obviously not an essential subspace of $\mathfrak{L}$, and thus it follows that if $\mathfrak{X} := P\mathfrak{V}|_{\mathrm{ran}\, P} = P\mathfrak{T}|_{\mathrm{ran}\, P}$, then dim $\mathfrak{X} \le d-1$.

The estimate from the proof of Theorem 2.5 shows that if $\mathfrak{Z} = P^\perp \mathfrak{V}|_{\mathrm{ran}\, P^\perp}$, then dim $\mathfrak{Z} \le d^2$. Together, these imply that dim $\mathfrak{V} \le$ dim $\mathfrak{X}$ + dim $\mathfrak{Z} \le (d-1) + d^2$.

Next, suppose that $N = \begin{bmatrix} 0 & N_2 \\ 0 & 0 \end{bmatrix} \in \mathfrak{N}$, and that $W = \begin{bmatrix} X & Y \\ L & Z \end{bmatrix} \in \mathfrak{A}$. Then $NW = \begin{bmatrix} N_2 L & N_2 Z \\ 0 & 0 \end{bmatrix} \in \mathfrak{T} \subseteq \mathfrak{A}$, whence $N_2 L = P(NW)|_{\mathrm{ran}\, P} \in \mathfrak{X}$.

Consider, for $N \in \mathfrak{N}$ as above, the linear map $\Psi(N) : \mathfrak{L} \to \mathfrak{X}$ defined by $\Psi(N)(L) = N_2 L$. If $\Psi(N) = 0$, then $N_2 L = 0$ for all $L \in \mathfrak{L}$. Since $\mathfrak{L}$ is an essential corner of $\mathfrak{A}$, span $\{\mathrm{ran}\, L : L \in \mathfrak{L}\} = \mathrm{ran}\, P^\perp$, whence $N_2 = 0$. That is, the (clearly) linear map $\Psi : \mathfrak{N} \to \mathcal{L}(\mathfrak{L}, \mathfrak{X}) := \{\Phi : \mathfrak{L} \to \mathfrak{X} : \Phi \text{ is linear}\}$ is injective. Thus dim $\mathfrak{N} \le$ dim $\mathcal{L}(\mathfrak{X}, \mathfrak{N}) \le d(d-1)$.

Finally, dim $\mathfrak{A} \le$ dim $\mathfrak{L}$ + dim $\mathfrak{V}$ + dim $\mathfrak{N} \le d + (d-1) + d^2 + d(d-1) = 2d^2 + d - 1$, as claimed.

□

2.13. Example. Let $1 \le n$. Set $\mathfrak{A} = \mathcal{L}(\mathbb{C}^n)$ and let $P = I_1 \oplus 0_{n-1}$. Then $\mathfrak{L} := P^\perp \mathfrak{A}|_{\mathrm{ran}\, P}$ is clearly an essential corner of dimension $d = n-1$. Meanwhile, dim $\mathfrak{A} = n^2 = (d+1)^2$.

We have seen above (see Theorem 2.5) that in general, given a unital algebra $\mathfrak{A}$ with an essential corner of dimension d, it is possible that the dimension of $\mathfrak{A}$ be on the order of d^3. In the next example, and in the following Theorem, we shall see that depending upon the structure of $\mathfrak{L}$, it is possible that the dimension of $\mathfrak{A}$ be no more than $d+2$.

2.14. Example. Let $3 \le p \in \mathbb{N}$. Then there exists an essential subspace $\mathfrak{L} \subseteq \mathcal{L}(\mathbb{C}^p)$ of dimension $p-1$ with the property that if $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^{2p})$ is a unital algebra, and $P \in \mathcal{L}(\mathbb{C}^{2p})$ is a projection of rank p with $\mathfrak{L} = P^\perp \mathfrak{A}|_{\mathrm{ran}\, P}$, then dim $\mathfrak{A} \le p+1$.

Let $\{e_1, e_2, \ldots, e_p\}$ be the standard orthonormal basis for $\mathbb{C}^p$. Define

$$L_k = e_{k+1} \otimes e_k^* + e_{k+2} \otimes e_{k+1}^*, \quad 1 \le k \le p-2$$
$$L_{p-1} = e_p \otimes e_{p-1}^* + e_1 \otimes e_p^*.$$

Set $\mathfrak{L} :=$ span $\{L_k : 1 \le k \le p-1\}$. Note that for $1 \le k \le p-2$,

$$\mathrm{ran}\, L_k = \mathrm{span}\, \{e_{k+1}, e_{k+2}\}, \quad \mathrm{ran}\, L_k^* = \mathrm{span}\, \{e_k, e_{k+1}\},$$

and for $k = p-1$,

$$\mathrm{ran}\, L_k = \mathrm{span}\, \{e_1, e_p\}, \quad \mathrm{ran}\, L_{p-1}^* = \mathrm{span}\, \{e_{p-1}, e_p\}.$$

Hence, span$\{\mathrm{ran}\, L_k : 1 \le k \le p-1\} =$ span $\{\mathrm{ran}\, L_k^* : 1 \le k \le p-1\} = \mathbb{C}^p$ and so $\mathfrak{L}$ is essential.

Suppose next that $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^{2p})$ is a unital algebra, that $P \in \mathcal{L}(\mathbb{C}^{2p})$ is a projection of rank p, and that $P^\perp \mathfrak{A}|_{\mathrm{ran}\, P} = \mathfrak{L}$. Choose $M_k \in \mathfrak{A}$, $1 \le k \le p-1$ such that $P^\perp M_k|_{\mathrm{ran}\, P} = L_k$, and let $\mathfrak{A} = \mathfrak{M} \dotplus \mathfrak{T} = \mathfrak{M} \dotplus \mathfrak{V} \dotplus \mathfrak{N}$ be a standard P-decomposition of $\mathfrak{A}$.

Recall that if $T = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathfrak{T}$, then $P^\perp (M_k T)|_{\mathrm{ran}\, P} = L_k X \in \mathfrak{L}$, $1 \le k \le p-1$. We shall use this to estimate the size of $\mathfrak{T}$.

(I) Let $X \in \mathcal{L}(\mathbb{C}^p)$, suppose that $\mathfrak{L}X := \{LX : L \in \mathfrak{L}\} \subseteq \mathfrak{L}$.

- Fix $j \in \{2, \cdots, p-2\}$. Since $L_j X = \sum_{k=1}^{p-1} \alpha_k L_k$,

$$L_j X = \alpha_1(e_2 \otimes e_1^*) + \sum_{k=2}^{p-1}(\alpha_{k-1} + \alpha_k)(e_{k+1} \otimes e_k^*) + \alpha_{p-1}(e_1 \otimes e_p^*).$$

Since $\operatorname{ran} L_j \subseteq \operatorname{span}\{e_{j+1}, e_{j+2}\}$, it follows that
(i) $\alpha_1 = 0$;
(ii) $\alpha_{k-1} + \alpha_k = 0, \quad 2 \leq k \leq p-1,\ k \neq j, j+1$;
(iii) $\alpha_{p-1} = 0$.
Thus, $\alpha_k = 0,\ k \neq j$.

- For $j = 1$, suppose that $L_1 X = \sum_{k=1}^{p-1} \alpha_k L_k$. Then

$$L_1 X = \alpha_1(e_2 \otimes e_1^*) + \sum_{k=2}^{p-1}(\alpha_{k-1} + \alpha_k)(e_{k+1} \otimes e_k^*) + \alpha_{p-1}(e_1 \otimes e_p^*).$$

Since $\operatorname{ran} L_1 \subseteq \operatorname{span}\{e_2, e_3\}$, it follows that
(iv) $\alpha_{k-1} + \alpha_k = 0,\ 3 \leq k \leq p-1$;
(v) $\alpha_{p-1} = 0$.
Thus, $\alpha_k = 0$ for all $2 \leq k \leq p-1$.

- For $j = p-1$, suppose that $L_{p-1} X = \sum_{k=1}^{p-1} \alpha_k L_k$. Then

$$L_{p-1} X = \alpha_1(e_2 \otimes e_1^*) + \sum_{k=2}^{p-1}(\alpha_{k-1} + \alpha_k)(e_{k+1} \otimes e_k^*) + \alpha_{p-1}(e_1 \otimes e_p^*).$$

Since $\operatorname{ran} L_{p-1} \subseteq \operatorname{span}\{e_p, e_1\}$, it follows that
(vi) $\alpha_{k-1} + \alpha_k = 0, \quad 2 \leq k \leq p-1$; and
(vii) $\alpha_1 = 0$.
Thus, $\alpha_k = 0,\ 1 \leq k \leq p-2$.

In summary, if $\mathfrak{L} X \subset \mathfrak{L}$, then $L_j X \in \mathbb{C} L_j$, say $L_j X = \beta_j L_j$, $1 \leq j \leq p-1$. Then, for $1 \leq j \leq p-2$, $X^* L_j^* = \overline{\beta_j} L_j^*$, and so

$$X^* L_j^* e_{j+1} = \overline{\beta_j} L_j^* e_{j+1} \text{ and } X^* e_j = \overline{\beta_j} e_j.$$

Moreover,

$$X^* L_j^* e_{j+2} = \overline{\beta_j} L_j^* e_{j+2} \text{ and } X^* e_{j+1} = \overline{\beta_j} e_{j+1}.$$

For $j = p-1$, $X^* L_{p-1}^* = \overline{\beta_{p-1}} L_{p-1}^*$, from which we find that

$$X^* L_{p-1}^* e_p = \overline{\beta_{p-1}} L_{p-1}^* e_p, \quad X^* e_{p-1} = \overline{\beta_{p-1}} e_{p-1},$$

and

$$X^* L_{p-1}^* e_1 = \overline{\beta_j} L_{p-1}^* e_1, \quad X^* e_p = \overline{\beta_{p-1}} e_p.$$

Therefore, $\beta_1 = \cdots = \beta_{p-1}$, whence $X \in \mathbb{C} I_p$.

(II) Let $Z \in \mathcal{L}(\mathbb{C}^p)$ and suppose that $Z\mathfrak{L} := \{ZL : L \in \mathfrak{L}\} \subseteq \mathfrak{L}$. Then $\mathfrak{L}^* Z^* \subseteq \mathfrak{L}^*$.
An argument similar to that above shows that $Z^* \in \mathbb{C} I_p$; that is, $Y \in \mathbb{C}\, I_p$.

(III) Let $W \in \mathcal{L}(\mathbb{C}^p)$, and suppose that $W\mathfrak{L} \subseteq \mathbb{C} I_p$. Given that $\operatorname{rank} L_j = 2$, $1 \leq j \leq p-1$, we see that $W L_j = 0$. Since $\mathfrak{L}$ is essential, we conclude that $W = 0$.

Assembling all of these pieces, we see that if $\mathfrak{A} \subseteq \mathbb{M}_{2p}(\mathbb{C})$ is a unital algebra, $P \in \mathbb{M}_{2p}(\mathbb{C})$ is a projection of rank p, and $\mathfrak{L} = P^\perp \mathfrak{A}|_{\operatorname{ran} P}$, then $\dim\, \mathfrak{A} \leq p + 1 = \dim\, \mathfrak{L} + 2$.

2.15. Theorem. *Suppose that* $2 \le p$ *is an integer and that* $\mathfrak{L} \subseteq \mathcal{L}(\mathbb{C}^p)$ *is an essential subspace of dimension* $d \ge 2$. *Suppose furthermore that there exists an invertible operator* $S \in \mathfrak{L}$ *such that* $S^{-1}\mathfrak{L} := \{S^{-1}L : L \in \mathfrak{L}\}$ *does not contain a non-scalar algebra. If* $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^{2p})$ *is a unital algebra,* $P \in \mathcal{L}(\mathbb{C}^{2p})$ *is a projection of rank* p*, and* $\mathfrak{L} = P^\perp \mathfrak{A}|_{\operatorname{ran} P}$*, then* $\dim \mathfrak{A} \le \dim \mathfrak{L} + 2$.

Proof. By replacing $\mathfrak{A}$ with $R^{-1}\mathfrak{A}R$, where $R = I_p \oplus S \in \mathcal{L}(\mathbb{C}^{2p})$, we can assume without loss of generality that $I_p \in \mathfrak{L}$ and that $\mathfrak{L}$ contains no non-scalar algebra. As always, we let $\{L_1, L_2, \ldots, L_d\}$ be a basis for $\mathfrak{L}$ with $L_1 = I_p$, and choose $M_k \in \mathfrak{A}$, $1 \le k \le d$ such that $P^\perp M_k|_{\operatorname{ran} P} = L_k$. Set $\mathfrak{M} := \operatorname{span}\{M_1, M_2, \ldots, M_d\}$. Let $\mathfrak{A} = \mathfrak{M} \dotplus \mathfrak{T}$ be the corresponding standard P-decomposition of $\mathfrak{A}$, where $\dim \mathfrak{M} = \dim \mathfrak{L} = d$ and $\mathfrak{T} = \{T \in \mathfrak{A} : P^\perp T P = 0\}$.

Given $T = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ relative to $\mathbb{C}^{2p} = \operatorname{ran} P \oplus \operatorname{ran} P^\perp$, we find that $X = P^\perp(M_1 T)|_{\operatorname{ran} P}$ and $Z = P^\perp(T M_1)|_{\operatorname{ran} P} \in \mathfrak{L}$. Since

$$\mathfrak{X} := \left\{ X : \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathfrak{T} \right\} \qquad \text{and} \qquad \mathfrak{Z} := \left\{ Z : \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \right\}$$

are clearly algebras, the hypothesis on $\mathfrak{L}$ implies that $\mathfrak{X} \subseteq \mathbb{C} I_p$ and $\mathfrak{Z} \subseteq \mathbb{C} I_p$.

To finish the proof, it suffices to show that $N = \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} \in \mathfrak{T}$ implies that $N = 0$. Note, however, that

$$Y = P^\perp(M_1 T)|_{\operatorname{ran} P^\perp} \in \mathfrak{Z},$$

whence $Y \in \mathbb{C} I_p$ from above. Since $\dim \mathfrak{L} = d \ge 2$, we can find a non-scalar operator $L \in \mathfrak{L}$. Choose $M \in \mathfrak{A}$ with $P^\perp M|_{\operatorname{ran} P} = L$, say

$$M = \begin{bmatrix} A & B \\ L & D \end{bmatrix}.$$

Then

$$MN = \begin{bmatrix} A & B \\ L & D \end{bmatrix} \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & AY \\ 0 & LY \end{bmatrix} \in \mathfrak{T},$$

so that $LY \in \mathbb{C} I_p$ from above. Thus $Y = 0$.

□

2.16. Example. As an example of a space $\mathfrak{L}$ which satisfies the conditions of Theorem 2.15, consider $p = 3$ and let $\mathfrak{L} \subseteq \mathcal{L}(\mathbb{C}^3)$ be the algebra who elements admit the following matrix structure relative to the standard orthonormal basis for $\mathbb{C}^3$:

$$\left\{ \begin{bmatrix} \alpha - \beta & \gamma & 0 \\ \delta & \alpha & \gamma \\ 0 & \delta & \alpha + \beta \end{bmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{C} \right\}.$$

Clearly $\mathfrak{L}$ is an essential subspace of $\mathcal{L}(\mathbb{C}^3)$, since $I_3 \in \mathfrak{L}$, and $\dim \mathfrak{L} = 4$. Set $S = I_3$ in Theorem 2.15. A routine calculation shows that $\mathfrak{L}$ contains no algebra other than $\mathbb{C} I_3$. Thus, by the above Theorem, if $\mathfrak{L}$ is an essential corner of some unital algebra $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^6)$, then $\dim \mathfrak{A} \le 6$.

Note that by choosing $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^6)$ to be the algebra whose matrix structure relative to the decomposition $\mathbb{C}^6 = \mathbb{C}^3 \oplus \mathbb{C}^3$ looks like

$$\left\{ \begin{bmatrix} \xi I_3 & 0 \\ L & \eta I_3 \end{bmatrix} : \xi, \eta \in \mathbb{C}, L \in \mathfrak{L} \right\},$$

we see that $\mathfrak{A}$ is a unital algebra with dim $\mathfrak{A} = 6$ and that $\mathfrak{L}$ is the essential corner of $\mathfrak{A}$ corresponding to the projection $P = I_3 \oplus 0_3$.

2.17. As we shall see in Section 4.3, part of the difficulty in classifying algebras admitting an essential corner of dimension $d \geq 2$ up to similarity lies in our limited understanding of the structure of essential subspaces of $\mathcal{L}(\mathbb{C}^p, \mathbb{C}^q)$. If one could find a classification scheme for these (say - up to equivalence, where $\mathfrak{L}_1$ is *equivalent* to $\mathfrak{L}_2$ if there exist invertible operators R, S such that $\mathfrak{L}_2 = R\,\mathfrak{L}_1\,S$), this might go a long way to further our understanding of the corresponding algebras.

2.18. We finish by remarking that Theorem 2.5 holds in the infinite-dimensional setting. Theorem 2.9 also holds in the infinite-dimensional setting, provided that we replace the assumption that $\mu = p$ in the statement of that Theorem by the assumption that $\mathfrak{L}$ should contain an invertible element. In the finite-dimensional setting, these two assumptions are of course equivalent.

3. Algebras with essential corners of dimension 1.

3.1. As we shall now see, the dimension of an off-diagonal corner of a unital subalgebra $\mathfrak{A}$ of $\mathcal{L}(\mathbb{C}^n)$ can yield information not only about the dimension of $\mathfrak{A}$, but also about the structure of that algebra. This section is devoted to describing – up to *admissible similarity*, which we now define – all possible algebras $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^{2p})$ for which there exists a projection P of rank p such that dim $P^\perp \mathfrak{A}|_{\operatorname{ran} P} = 1$.

Following a suggestion made by the referee, we introduce the notion of an *admissible* similarity corresponding to P, namely: an invertible element $S \in \mathcal{L}(\mathbb{C}^n)$ will be referred to as **admissible** if ran P is invariant for S. Relative to the decomposition $\mathbb{C}^n = \operatorname{ran} P \oplus \operatorname{ran} P^\perp$, we may write

$$S = \begin{bmatrix} S_1 & S_2 \\ 0 & S_4 \end{bmatrix}.$$

Since $\mathbb{C}^n$ is finite-dimensional, it is not hard to see that S^{-1} is admissible when S is. It will prove important later to note that the invertible operator R from Remark 2.4 is admissible.

3.2. Let $p \geq 1$ be an integer and $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^{2p})$ be a unital algebra. Suppose that $P \in \mathcal{L}(\mathbb{C}^{2p})$ is a projection of rank p and that $\mathfrak{L} := P^\perp \mathfrak{A}|_{\operatorname{ran} P}$ is an essential subspace of dimension 1.

Since we are interested in determining the structure of $\mathfrak{A}$ up to admissible similarity, by using the argument of Remark 2.4 (which, as we have just noted is admissible relative to the projection P) and adopting the notation of our standard P-decomposition of $\mathfrak{A}$, we may assume *a priori* that $\mathfrak{L} = \operatorname{span}\{L_1\}$, where L_1 is an operator whose matrix relative to the standard orthonormal bases $\{e_1, e_2, \ldots, e_p\}$ for ran P and $\{f_1, f_2, \ldots, f_p\}$ for ran $P^\perp$ is I_p. It follows that relative to the decomposition $\mathbb{C}^{2p} = \operatorname{ran} P \oplus \operatorname{ran} P^\perp$, and the above choices of orthonormal bases for ran P and ran $P^\perp$ respectively, $M_1 \in \mathfrak{A}$ is of the form $M_1 = \begin{bmatrix} A_1 & B_1 \\ I_p & D_1 \end{bmatrix}$. We now fix this decomposition of $\mathbb{C}^{2p}$ and these bases for ran P and ran $P^\perp$ the remainder of the argument.

Suppose that $T_1 = \begin{bmatrix} X_1 & Y_1 \\ \lambda I_p & Z_1 \end{bmatrix} \in \mathfrak{A}$ with $\lambda \neq 0$. Then $P^\perp T_1^2|_{\operatorname{ran} P} = \lambda(X_1 + Z_1)$, and thus $X_1 + Z_1 \in \mathfrak{L} = \mathbb{C} I_p$. Similarly, if $T_2 := \begin{bmatrix} X_2 & Y_2 \\ 0 & Z_2 \end{bmatrix} \in \mathfrak{A}$, then $T_1 + T_2 \in \mathfrak{A}$, whence

$$\lambda((X_1 + X_2) + (Z_1 + Z_2)) \in \mathbb{C} I_p.$$

Therefore $X_2 + Z_2 \in \mathbb{C}I_p$ as well. In other words,

$$\text{for any } T = \begin{bmatrix} X & Y \\ \lambda I_p & Z \end{bmatrix} \in \mathfrak{A}, \text{ we have that } X + Z \in \mathbb{C}I_p. \qquad (*)$$

In particular, with M_1 as above, $A_1 + D_1 \in \mathbb{C}I_p$, and so there exists $\theta \in \mathbb{C}$ such that $\theta I_p = A_1 + D_1$. Let

$$M := M_1 - \frac{\theta}{2} I_{2p} = \begin{bmatrix} A & B_1 \\ I_p & -A \end{bmatrix} \in \mathfrak{A},$$

where $A = A_1 - \frac{\theta}{2} I_p$.

Observe that $M^2 = \begin{bmatrix} A^2 + B_1 & AB_1 - B_1 A \\ 0 & B_1 + A^2 \end{bmatrix}$, and thus from the argument above we deduce that there exists $\gamma \in \mathbb{C}$ such that

$$2\gamma I_p = (A^2 + B_1) + (B_1 + A^2),$$

or equivalently that $B_1 = \gamma I_p - A^2$. Note that from this we see that $M = \begin{bmatrix} A & \gamma I_p - A^2 \\ I_p & -A \end{bmatrix}$ and $M^2 = \gamma I_{2p}$.

Now, for an arbitrary $T = \begin{bmatrix} X & Y \\ \lambda I_p & Z \end{bmatrix} \in \mathfrak{A}$, we have that $T = \lambda M + T_0$, where $T_0 := T - \lambda M$ satisfies $P^\perp T_0|_{\operatorname{ran} P} = 0$. From our standard P-decomposition of $\mathfrak{A}$, we have that $\mathfrak{A} = \mathbb{C}M \dotplus \mathfrak{T}$. If $T = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathfrak{T}$, then $TM \in \mathfrak{A}$, and

$$TM = \begin{bmatrix} XA + Y & XB_1 - YA \\ Z & -ZA \end{bmatrix}.$$

It follows that $Z \in \mathbb{C}I_p$, and from condition $(*)$ applied to T we deduce that $X + Z \in \mathbb{C}I_p$. As such, we also have that $X \in \mathbb{C}I_p$. Choose $\alpha, \beta \in \mathbb{C}$ such that

$$T = \begin{bmatrix} \alpha I_p & Y \\ 0 & \beta I_p \end{bmatrix}.$$

Then $(M + T)^2 \in \mathfrak{A}$, and therefore satisfies condition $(*)$. A routine calculation shows that this reduces to the statement that

$$(\alpha I_p + A)^2 + 2(Y + (\gamma I_p - A^2)) + (\beta I_p - A)^2 \in \mathbb{C}I_p,$$

which in turn implies that there exists $\kappa \in \mathbb{C}$ for which

$$Y = (\beta - \alpha)A + \kappa I_p.$$

That is, every element $T \in \mathfrak{T}$ is of the form

$$T = \begin{bmatrix} \alpha I_p & (\beta - \alpha)A + \kappa I_p \\ 0 & \beta I_p \end{bmatrix}$$

for some $\alpha, \beta, \kappa \in \mathbb{C}$. In particular, therefore, $T - \alpha I_{2p} = \begin{bmatrix} 0 & (\beta - \alpha)A + \kappa I_p \\ 0 & (\beta - \alpha)I_p \end{bmatrix} \in \mathfrak{T}$ and

$$\mathfrak{A} \subseteq \mathfrak{C} := \operatorname{span}\{M, I_{2p}, T_{\mu,\nu} : \mu, \nu \in \mathbb{C}\},$$

where each $T_{\mu,\nu} := \begin{bmatrix} 0 & \mu A + \nu I_p \\ 0 & \mu I_p \end{bmatrix}$. It is not hard to check that $\dim \mathfrak{C} = 4$, and it is a routine if somewhat tedious calculation to show that $\mathfrak{C}$ is an algebra.

3.3. Having established a 4-dimensional algebra $\mathfrak{C}$ which contains $\mathfrak{A}$ (up to admissible similarity), we now examine the possibilities for $\mathfrak{A}$ as a subalgebra of $\mathfrak{C}$, based upon its dimension. We recall that $\mathfrak{A}$ is unital and that dim $\mathfrak{L} = 1$. It follows that $2 \leq \dim \mathfrak{A} \leq 4$.

- dim $\mathfrak{A} = 2$. Since $I_{2p}, M \in \mathfrak{A}$ and these are clearly linearly independent, and since $M^2 \in \mathbb{C}I_{2p}$, we see that $\mathfrak{A} = \operatorname{span}\{I_{2p}, M\}$ is indeed a 2-dimensional algebra with an essential off-diagonal corner of dimension one.
- dim $\mathfrak{A} = 3$. Then $\mathfrak{A} = \operatorname{span}\{I_{2p}, M, T_{\mu_0,\nu_0}\}$ for some $\mu_0, \nu_0 \in \mathbb{C}$, and $|\mu_0| + |\nu_0| \neq 0$. Here we have two possibilities (maintaining the notation from the Section 3.2):
  - If $\mu_0 = 0$, then $\nu_0 \neq 0$ and
  $$\nu_0^{-1} T_{0,\nu_0} M = \begin{bmatrix} 0 & I_p \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & \gamma I_p - A^2 \\ I_p & -A \end{bmatrix} = \begin{bmatrix} I_p & -A \\ 0 & 0 \end{bmatrix} \in \mathfrak{A},$$
  a contradiction (since this is clearly not in the span of M, I_{2p} and T_{0,ν_0}).
  - If $\mu_0 \neq 0$, then we may assume without loss of generality that $\mu_0 = 1$. Then
  $$M T_{1,\nu_0} = \begin{bmatrix} A & \gamma I_p - A^2 \\ I_p & -A \end{bmatrix} \begin{bmatrix} 0 & A + \nu_0 I_p \\ 0 & I_p \end{bmatrix} = \begin{bmatrix} 0 & \nu_0 A + \gamma I_p \\ 0 & \nu_0 I_p \end{bmatrix} \in \mathfrak{A}.$$
  Writing
  $$M T_{1,\nu_0} = \xi_1 M + \xi_2 I_{2p} + \xi_3 T_{1,\nu_0}$$
  shows that $\xi_1 = 0 = \xi_2$ and $\xi_3 = \nu_0$, from which we find that we need $\nu_0^2 = \gamma$. On the other hand,
  $$\begin{aligned} T_{1,\nu_0} M &= \begin{bmatrix} A + \nu_0 & -A^2 - \nu_0 A \\ I_p & -A \end{bmatrix} \\ &= \begin{bmatrix} A & \nu_0^2 I_p - A^2 \\ I_p & -A \end{bmatrix} + \nu_0 \begin{bmatrix} I_p & 0 \\ 0 & I_p \end{bmatrix} + (-\nu_0) \begin{bmatrix} 0 & A + \nu_0 \\ 0 & I_p \end{bmatrix} \in \mathfrak{A}. \end{aligned}$$
  Given that $M^2 = \gamma I_{2p} = \nu_0^2 I_{2p}$ and $T_{1,\nu_0}^2 = T_{1,\nu_0}$, it is clear that for either square root of γ, $\mathfrak{A} = \operatorname{span}\{M, I_{2p}, T_{1,\gamma^{1/2}}\}$ is indeed a three-dimensional algebra with an essential off-diagonal corner of dimension one.

  The final conclusion is that
  $$\mathfrak{A} = \operatorname{span}\left\{ \begin{bmatrix} I_p & 0 \\ 0 & I_p \end{bmatrix}, \begin{bmatrix} 0 & A + \gamma^{1/2} I_p \\ 0 & I_p \end{bmatrix}, \begin{bmatrix} A & \gamma I_p - A^2 \\ I_p & -A \end{bmatrix} \right\}.$$
- dim $\mathfrak{A} = 4$. In this case, we clearly have that
$$\mathfrak{A} = \mathfrak{C} = \operatorname{span}\left\{ C_1 := \begin{bmatrix} I_p & 0 \\ 0 & I_p \end{bmatrix}, C_2 := \begin{bmatrix} 0 & A \\ 0 & I_p \end{bmatrix}, C_3 := \begin{bmatrix} 0 & I_p \\ 0 & 0 \end{bmatrix}, C_4 := \begin{bmatrix} A & -A^2 \\ I_p & -A \end{bmatrix} \right\}.$$

3.4. Let $\mathfrak{C}$ be the algebra from paragraph 3.3, and let $R \in \mathcal{L}(\mathbb{C}^{2p})$ be the operator which may be written relative to the above decomposition of $\mathbb{C}^{2p} = \operatorname{ran} P \oplus \operatorname{ran} P^\perp$ and the corresponding orthonormal bases as $R = \begin{bmatrix} I_p & A \\ 0 & I_p \end{bmatrix}$, so that $R^{-1} = \begin{bmatrix} I_p & -A \\ 0 & I_p \end{bmatrix}$. Observe that R is admissible (relative to P) as defined above. Then

$$R^{-1} C_1 R = I_{2p} \qquad R^{-1} C_2 R = \begin{bmatrix} 0 & 0 \\ 0 & I_p \end{bmatrix}$$
$$R^{-1} C_3 R = \begin{bmatrix} 0 & I_p \\ 0 & 0 \end{bmatrix} \qquad R^{-1} C_4 R = \begin{bmatrix} 0 & 0 \\ I_p & 0 \end{bmatrix}.$$

Hence $R^{-1}MR = \begin{bmatrix} 0 & \gamma I_p \\ I_p & 0 \end{bmatrix}$, and $R^{-1}\mathfrak{C}R \simeq \mathbb{M}_2(\mathbb{C}) \otimes I_p$. From this and the characterisations of Section 3.3, we readily obtain the following structure theorem.

Our original statement of Theorem 3.5 below yielded just the "only if" conclusions stated below. We would like to thank the referee for suggesting the use of "admissible" similarities, which allow us to sharpen the results to "if and only if" statements. It can be argued that admissible similarity is perhaps a more natural notion to consider, given that the off-diagonal corners of an algebra $\mathfrak{A}$ refer to $P^\perp \mathfrak{A} P|_{\mathrm{ran}\, P}$, and that these off-diagonal corners rely on an orthogonal decomposition of the underlying space into $\mathrm{ran}\, P \oplus \mathrm{ran}\, P^\perp$.

3.5. Theorem. *Let $p \geq 1$ be an integer and $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^{2p})$ be a unital algebra. Suppose that $P \in \mathcal{L}(\mathbb{C}^{2p})$ is a projection of rank p and that $\mathfrak{L} := \dim P^\perp \mathfrak{A}|_{\mathrm{ran}\, P}$ is an essential subspace of dimension 1. Then $2 \leq \dim \mathfrak{A} \leq 4$, and*

(I) $\dim \mathfrak{A} = 2$ *if and only if either*

(i) $\mathfrak{A}$ *is similar via an admissible similarity to* $\mathrm{span}\left\{ \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}, I_{2p} \right\} \subseteq \mathcal{L}(\mathbb{C}^{2p})$, *or*

(ii) $\mathfrak{A}$ *is similar via an admissible similarity to* $\mathrm{span}\left\{ \begin{bmatrix} 0 & 0 \\ I_p & 0 \end{bmatrix}, I_{2p} \right\} \subseteq \mathcal{L}(\mathbb{C}^{2p})$.

(II) $\dim \mathfrak{A} = 3$ *if and only if* $\mathfrak{A}$ *is similar via an admissible similarity to* $\mathcal{T}_2^*(\mathbb{C}) \otimes I_p$, *where* $\mathcal{T}_2(\mathbb{C})$ *denotes the upper triangular* 2×2 *complex matrices.*

(III) $\dim \mathfrak{A} = 4$ *if and only if* $\mathfrak{A}$ *is similar via an admissible similarity to* $\mathbb{M}_2(\mathbb{C}) \otimes I_p$.

Proof.

(I) Suppose that $\dim \mathfrak{A} = 2$. Note that if we write the original M as

$$M = \begin{bmatrix} A & \gamma I_p - A^2 \\ I_p & -A \end{bmatrix},$$

then $\mathfrak{A} = \mathrm{span}\{I_{2p}, M\}$. After taking the admissible similarity transformation in Section 3.4, we may assume that $\mathfrak{A} = \mathrm{span}\{I_{2p}, \begin{bmatrix} 0 & \gamma I_p \\ I_p & 0 \end{bmatrix}\}$.

If $\gamma \neq 0$, $\mathfrak{A}$ is similar via the admissible similarity $S := \begin{bmatrix} \gamma^{1/2} I_p & 0 \\ 0 & I_p \end{bmatrix}$ to the space indicated in (i).

If $\gamma = 0$, then $\mathfrak{A}$ is equal to the space indicated in (ii).

(II) Suppose that $\dim \mathfrak{A} = 3$. Once again, if we write the original M as

$$M = \begin{bmatrix} A & \gamma I_p - A^2 \\ I_p & -A \end{bmatrix},$$

then $\mathfrak{A} = \mathrm{span}\{I_{2p}, M, T_{1,\gamma^{1/2}}\}$. After taking the admissible similarity transformation in Section 3.4, we may assume that $\mathfrak{A} = \mathrm{span}\{I_{2p}, \begin{bmatrix} 0 & \gamma I_p \\ I_p & 0 \end{bmatrix}, \begin{bmatrix} 0 & \gamma^{1/2} I_p \\ 0 & I_p \end{bmatrix}\}$. Define $S := \begin{bmatrix} I_p & \gamma^{1/2} I_p \\ 0 & I_p \end{bmatrix} \in \mathcal{L}(\mathbb{C}^{2p})$, and note that S is an admissible similarity. A routine computation now shows that

$$S^{-1}\mathfrak{A}S = \mathcal{T}_2^*(\mathbb{C}) \otimes I_p = \left\{ \begin{bmatrix} aI_p & 0 \\ bI_p & cI_p \end{bmatrix} : a, b, c \in \mathbb{C} \right\}.$$

(III) This follows immediately from Sections 3.3 and 3.4.

□

3.6. Remark. It is worth noting that the algebra span $\{\begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}, I_{2p}\}$ appearing in part (I) above is similar (but not admissibly similar) to $\mathfrak{D}_2 \otimes I_p \subseteq \mathcal{L}(\mathbb{C}^{2p})$.

4. Algebras where $d = 2$.

4.1. This section is devoted to an analysis of those algebras $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^{2p})$ for which the space $\mathfrak{L} := P^\perp \mathfrak{A}|_{\operatorname{ran} P}$ is essential and $\dim \mathfrak{L} = 2$, where P is a projection of rank p in $\mathcal{L}(\mathbb{C}^{2p})$.

4.2. Theorem. *Let $2 \le p$ be an integer and $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^{2p})$ be a unital algebra. Let $P \in \mathcal{L}(\mathbb{C}^{2p})$ be a projection of rank p, and set $\mathfrak{L} := P^\perp \mathfrak{A}|_{\operatorname{ran} P}$. Suppose that $\mathfrak{L}$ is essential, $\dim \mathfrak{L} = 2$ and that $\mu := \max\{\operatorname{rank} L : L \in \mathfrak{L}\}$.*

(a) *If $\mu = p$, then* $\dim \mathfrak{A} \le 8$.
(b) *If $\mu < p$, then* $\dim \mathfrak{A} \le 4$.

Proof. (a) This is a direct application of Theorem 2.9.

(b) As per Remark 2.4, we fix a standard P-decomposition of $\mathfrak{A}$ with $\mathfrak{L} = \operatorname{span}\{L_1, L_2\}$, where – decomposing $\mathbb{C}^{2p} = \operatorname{ran} P \oplus \operatorname{ran} P^\perp$, L_1 and L_2 admit a block-operator form $L_1 = \begin{bmatrix} I_\mu & 0 \\ 0 & 0 \end{bmatrix}$ and $L_2 = \begin{bmatrix} K_1 & K_2 \\ K_3 & 0 \end{bmatrix}$ for an appropriate choice of K_1, K_2 and K_3. Recall that $K_3 K_1^j K_2 = 0$ for all integers $j \ge 0$.

Since $\mathfrak{L}$ is essential, it follows that both K_2 and K_3 must have full rank; that is, $\operatorname{rank} K_3 = \operatorname{rank} K_2 = p - \mu > 0$. By definition of μ, we have that $\mu \ge \operatorname{rank} K_2 = p - \mu$, or equivalently, that $p \le 2\mu$. In fact, if it were the case that p were equal to 2μ, then the fact that K_3 has full rank would imply that K_3 is invertible. But the equation $K_3 K_2 = 0$ implies that $\ker K_3 \ne \{0\}$, and thus $p < 2\mu$.

Recall that the standard P-decomposition includes a choice of $M_k \in \mathfrak{A}$, such that $P^\perp M_k|_{\operatorname{ran} P} = L_k$, $k = 1, 2$, and that $\mathfrak{T} = \{T \in \mathfrak{A} : P^\perp T P = 0\}$. Suppose that $T = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathfrak{T}$. Decomposing $\operatorname{ran} P$ as $\operatorname{ran} P = (\ker L_1)^\perp \oplus \ker L_1$ allows us to write X as

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

Thus

$$P^\perp (M_1 T)|_{\operatorname{ran} P} = L_1 X = \begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix} \in \mathfrak{L} = \operatorname{span}\{L_1, L_2\}.$$

But $K_3 \ne 0$ implies that $L_1 X \in \mathbb{C} L_1$, so that $X_1 \in \mathbb{C} I_\mu$ – say $X_1 = \lambda I_\mu$ – and $X_2 = 0$.

Next, consider

$$\begin{bmatrix} K_2 X_3 & K_2 X_4 - \lambda K_2 \\ 0 & 0 \end{bmatrix} = P^\perp (M_2 T)|_{\operatorname{ran} P} - \lambda L_2 \in \mathfrak{L}.$$

As before, it follows that $K_2 X_3 \in \mathbb{C} I_\mu$ and $K_2 (X_4 - \lambda I_{p-\mu}) = 0$. *A fortiori*, $\operatorname{rank} K_2 = p - \mu < \mu$ and $K_2 X_3 \in \mathbb{C} I_\mu$ together imply that $K_2 X_3 = 0$. But $\ker K_2 = \{0\}$ since K_2 has full rank, and thus $X_3 = X_4 - \lambda I_{p-\mu} = 0$. We have shown that $X \in \mathbb{C} I_p$.

A similar argument shows that $Z \in \mathbb{C} I_p$ as well. It follows that $\mathfrak{T} \subseteq \mathfrak{D}_2 \otimes I_p + P \mathfrak{T} P^\perp$.

Thus $T \in \mathfrak{T}$ implies that there exist $\alpha_T, \beta_T \in \mathbb{C}$ such that $T = \begin{bmatrix} \alpha_T I_p & Y_T \\ 0 & \beta_T I_p \end{bmatrix}$. If $\alpha_T = \beta_T$ for all $T \in \mathfrak{T}$, then $T - \alpha_T I_{2p} = \begin{bmatrix} 0 & Y_T \\ 0 & 0 \end{bmatrix} \in \mathfrak{T}$. If there exists $T \in \mathfrak{T}$ such that $\alpha_T \neq \beta_T$, then (by the Riesz functional calculus, for example) there exist operators $E := \begin{bmatrix} I_p & E_2 \\ 0 & 0 \end{bmatrix}$ and $F := \begin{bmatrix} 0 & F_2 \\ 0 & I_p \end{bmatrix} \in \mathfrak{T}$. Then $T - (\alpha E + \beta F)$ is of the form $\begin{bmatrix} 0 & Y_T - (\alpha E_2 + \beta F_2) \\ 0 & 0 \end{bmatrix}$. Either way, we conclude that $\dim \mathfrak{T} \leq 2 + \dim \mathfrak{N}$, where $\mathfrak{N} = \{T \in \mathfrak{T} : T = PTP^\perp\}$.

Suppose that $N = \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} \in \mathfrak{N} \subseteq \mathfrak{T}$. We decompose $\operatorname{ran} P^\perp = \operatorname{ran} L_1 \oplus (\operatorname{ran} L_1)^\perp$, and write

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}$$

relative to this decomposition of $\operatorname{ran} P^\perp$ and that of $\operatorname{ran} P$ given earlier. Clearly NM_1, NM_2 lie in $\mathfrak{T}$, and therefore $YL_1 = \begin{bmatrix} Y_1 & 0 \\ Y_3 & 0 \end{bmatrix}$ and $YL_2 = \begin{bmatrix} Y_1 K_1 + Y_2 K_3 & Y_1 K_2 \\ Y_3 K_1 + Y_4 K_3 & Y_3 K_2 \end{bmatrix}$ both lie in $\mathbb{C} I_p$.

From this we deduce that $Y_1 = Y_3 = 0$ and thus $Y_2 K_3 = Y_4 K_3 = 0$. But K_3 is surjective (by the essentialness of $\mathfrak{L}$), and hence $Y_2 = Y_4 = 0$, i.e. $Y = 0$.

As such, $\dim \mathfrak{N} = 0$ and therefore $\dim \mathfrak{T} \leq 2$. Since $\mathfrak{A} = \operatorname{span}\{M_1, M_2, \mathfrak{T}\}$, we have that $\dim \mathfrak{A} \leq 4$, completing the proof.

□

4.3. Unlike the case where $\dim P^\perp \mathfrak{A}|_{\operatorname{ran} P} = 1$, it does not seem feasible to classify all unital algebras $\mathfrak{A}$ admitting an essential corner $\mathfrak{L}$ as above of dimension 2 up to similarity, admissible or otherwise. Indeed, suppose that $L_1, L_2 \in \mathcal{L}(\mathbb{C}^p)$ are two arbitrary linear maps that generate an essential subspace $\mathfrak{L}$ of $\mathcal{L}(\mathbb{C}^p)$. Write $\mathbb{C}^{2p} = \mathbb{C}^p \oplus \mathbb{C}^p$ and define $M_k := \begin{bmatrix} 0 & 0 \\ L_k & 0 \end{bmatrix}$, $k = 1, 2$ relative to this decomposition. If we set $\mathfrak{A} := \operatorname{span}\{M_1, M_2, I_{2p}\} \subseteq \mathcal{L}(\mathbb{C}^{2p})$, then $\mathfrak{A}$ is a unital algebra of dimension three with essential corner $\mathfrak{L}$ of dimension two.

For example, if $1 < r < \mu < p$ are integers, we may decompose $\mathbb{C}^p = \mathbb{C}^r \oplus \mathbb{C}^{\mu - r} \oplus \mathbb{C}^{p - \mu}$. Relative to this decomposition, define

$$L_1 = \begin{bmatrix} I_r & 0 & 0 \\ 0 & I_{\mu - r} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad L_2 = \begin{bmatrix} K_{11} & K_{12} & K_{21} \\ 0 & K_{14} & 0 \\ 0 & K_{32} & 0 \end{bmatrix}.$$

If K_{21} is injective and K_{32} is surjective, then $\mathfrak{L} = \operatorname{span}\{L_1, L_2\}$ is an essential subspace of $\mathcal{L}(\mathbb{C}^p)$.

It is not at all clear how to classify the corresponding subclass of algebras $\mathfrak{A}$ defined above up to similarity (admissible or not), let alone how to classify all unital, three-dimensional subalgebras of $\mathcal{L}(\mathbb{C}^{2p})$ admitting an essential corner of dimension two.

In light of these facts, our approach will be first to improve the estimates of Theorem 2.5 in this particular setting, after which we shall concentrate on determining (up to *-isomorphism) which unital, *self-adjoint* algebras admit such an essential corner of dimension two. We begin by showing that all dimensions specified by Theorem 4.2 can occur.

Note that if $\mathfrak{A}$, $\mathfrak{L}$ are as in that theorem (and adhering to the notation used therein), if $M_k = \begin{bmatrix} A_k & B_k \\ L_k & D_k \end{bmatrix}$, $k = 1, 2$, then the fact that $\mathfrak{A}$ is unital implies that $\dim \mathfrak{A} \geq 3$.

4.4. Example. Set $p = 2$, and let $\mathcal{E} := \{e_1, e_2\}$, $\mathcal{F} := \{f_1, f_2\}$ denote two copies of the standard orthonormal basis for $\mathbb{C}^2$. Let $J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_2(\mathbb{C})$. Define $P = I_2 \oplus 0_2 \in \mathcal{L}(\mathbb{C}^4)$ and relative to the decomposition $\mathbb{C}^4 = \operatorname{ran} P \oplus \operatorname{ran} P^\perp$, let $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^4)$ denote the algebra of all linear maps whose matrices relative to the orthonormal basis $\{e_1, e_2, f_1, f_2\}$ for $\mathbb{C}^4$ belong to

$$\left\{ \begin{bmatrix} \alpha_1 I_2 + \alpha_2 J & \alpha_3 I_2 + \alpha_4 J \\ \alpha_5 I_2 + \alpha_6 J & \alpha_7 I_2 + \alpha_8 J \end{bmatrix} : \alpha_k \in \mathbb{C}, k = 1, 2, \cdots, 8 \right\}.$$

Set $\mathfrak{L} = P^\perp \mathfrak{A}|_{\operatorname{ran} P}$, so that the elements of $\mathfrak{L}$ are those maps whose matrix forms look like $\{\alpha_5 I_2 + \alpha_6 J : \alpha_5, \alpha_6 \in \mathbb{C}\}$. Then $\mathfrak{L}$ is a two-dimensional space, and the corresponding maximal rank $\mu = 2$. Obviously $\dim \mathfrak{A} = 8$.

Define

$$\begin{aligned}
\mathfrak{A}_7 &:= \{T \in \mathfrak{A} : \alpha_3 = 0\} \\
\mathfrak{A}_6 &:= \{T \in \mathfrak{A} : \alpha_3 = \alpha_4 = 0\} \\
\mathfrak{A}_5 &:= \{T \in \mathfrak{A} : \alpha_2 = \alpha_3 = \alpha_4 = 0\} \\
\mathfrak{A}_4 &:= \{T \in \mathfrak{A} : \alpha_2 = \alpha_3 = \alpha_4 = \alpha_8 = 0\} \\
\mathfrak{A}_3 &:= \{T \in \mathfrak{A} : \alpha_2 = \alpha_3 = \alpha_4 = \alpha_8 = 0, \alpha_1 = \alpha_7\}
\end{aligned}$$

It is routine to verify that each $\mathfrak{A}_k$ is an algebra of dimension k, $k = 3, 4, \ldots, 7$, and that the corresponding $\mathfrak{L}_k := P^\perp \mathfrak{A}_k|_{\operatorname{ran} P}$ is a two-dimensional essential subspace with maximal rank $\mu_k = 2 = p$.

4.5. Example. Let $p = 3$. Define $L_1, L_2 \in \mathcal{L}(\mathbb{C}^3)$ to be those linear maps whose matrices relative to the standard orthonormal basis for $\mathbb{C}^3$ are given by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ respectively. Define $M_k = \begin{bmatrix} 0 & 0 \\ L_k & 0 \end{bmatrix} \in \mathcal{L}(\mathbb{C}^3 \oplus \mathbb{C}^3)$. Set $\mathfrak{A} = \operatorname{span}\{I_6, M_1, M_2\}$, so that $\dim \mathfrak{A} = 3$, and let $P = I_3 \oplus 0_3$. With $\mathfrak{L} = P^\perp \mathfrak{A}|_{\operatorname{ran} P} = \operatorname{span}\{L_1, L_2\}$, we see that $\mathfrak{L}$ is a two-dimensional essential subspace, with corresponding maximal rank $\mu = 2 < p$.

Setting $\mathfrak{B} := \operatorname{span}\{\mathfrak{A}, P\}$ yields a four-dimensional algebra with the same off-diagonal corner $P^\perp \mathfrak{B}|_{\operatorname{ran} P} = \mathfrak{L}$.

4.6. We now turn our attention to classifying (up to *-isomorphism) those C^*-algebras $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^{2p})$ which admit an essential corner $\mathfrak{L}$ with $\dim \mathfrak{L} = 2$, corresponding to a projection P of rank p. By Theorem 4.2, the dimension of such an algebra must lie between 3 and 8. This greatly restricts the class of C^*-algebras we need to consider. In fact, up to *-isomorphism, we need only consider the following:

- the commutative C^*-algebras $\mathfrak{A} := \mathfrak{D}_k$, $k = 3, 4, \ldots, 8$;
- the algebras $\mathfrak{A} := \mathbb{M}_2(\mathbb{C}) \oplus \mathfrak{D}_k$, $k = 0, 1, \ldots, 4$; and
- the algebra $\mathfrak{A} := \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C})$.

Of course, every C^*-subalgebra of a $\mathcal{L}(\mathbb{C}^n)$ is unitarily equivalent to a direct sum of algebras of the form $\mathcal{L}(\mathbb{C}^k)\otimes\mathbb{C}I_j$ for an appropriate choice of k's and j's, and so for each of the above examples, it suffices to describe $\mathfrak{A}$ (up to unitary equivalence) by defining the multiplicities of the components of the matrix algebras. In the hope that some of the techniques we establish below might eventually extend to study off-diagonal corners a wider class of C^*-algebras than $\mathcal{L}(\mathbb{C}^n)$, we shall adopt a more operator-theoretic but equivalent approach, namely: in each case, we shall either describe a *- representation $\rho : \mathfrak{A} \to \mathcal{L}(\mathbb{C}^{2p})$ which admits an essential corner $\mathfrak{L}$ as above, or we shall prove that no such representation exists.

4.7. Example. There exist *-representations $\rho_k : \mathfrak{D}_k \to \mathcal{L}(\mathbb{C}^4)$, $k = 3, 4$ and a projection P of rank 2 such that $\mathfrak{L} := P^\perp \rho_k(\mathfrak{D}_k)|_{\operatorname{ran} P}$ is an essential corner of dimension 2.

- Let $\{e_1, e_2\}$ denote the standard orthonormal basis for $\mathbb{C}^2$ and define $W = e_1 \otimes e_2^* + e_2 \otimes e_1^*$. Note that $W^2 = I_2$. Define $T = \begin{bmatrix} W & I_2 \\ I_2 & W \end{bmatrix} \in \mathcal{L}(\mathbb{C}^2 \oplus \mathbb{C}^2)$, and $\mathfrak{A} = \operatorname{span}\{I_4, T, T^2\}$. Then $\mathfrak{A}$ is a unital commutative C^*-algebra with $\dim \mathfrak{A} = 3$, hence $\mathfrak{A}$ is *-isomorphic to $\mathfrak{D}_3$ (say via ρ_3), and if we set $P = I_2 \oplus 0_2$, then $\mathfrak{L} := P^\perp \rho_3(\mathfrak{D}_3)|_{\operatorname{ran} P}$ is an essential corner of dimension 2.
- Define $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^4)$ to be the algebra whose elements admit the following matrix representation relative to the standard orthonormal basis:
$$\{\begin{bmatrix} \alpha & & \beta & \\ & \gamma & & \delta \\ \beta & & \alpha & \\ & \delta & & \gamma \end{bmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{C}\}.$$
Then $\mathfrak{A}$ is a unital C^*-algebra which is isomorphic to $\mathfrak{D}_4$ (say via ρ_4). Again, if we set $P = I_2 \oplus 0_2$, then $\mathfrak{L} := P^\perp \rho_4(\mathfrak{D}_4)|_{\operatorname{ran} P}$ is an essential corner of dimension 2.

4.8. Proposition. *Let $5 \le k \le 8$. Suppose that $2 \le p \in \mathbb{N}$, and that $\rho : \mathfrak{D}_k \to \mathcal{L}(\mathbb{C}^{2p})$ is an injective, unital *-representation. If $P \in \mathcal{L}(\mathbb{C}^{2p})$ is a projection of rank p and $\mathfrak{L} := P^\perp \rho(\mathfrak{D}_k)|_{\operatorname{ran} P}$, then either $\mathfrak{L}$ is not essential, or* $\dim \mathfrak{L} \neq 2$.

Proof. We argue by contradiction. Suppose that such a representation ρ and projection P exist and fix $\mathfrak{A} = \rho(\mathfrak{D}_k)$. As in Section 2.1, we consider a standard P-decomposition of $\mathfrak{A}$ where $\mathfrak{L}$ has basis $\{L_1, L_2\}$ which is "lifted" to a basis $\{M_1, M_2\}$ for $\mathfrak{M} \subseteq \mathfrak{A}$.

As $5 \le \dim \mathfrak{A} \le 8$, it follows from Theorem 4.2 that $\mu := \max\{\operatorname{rank} L : L \in \mathfrak{L}\} = p$. Since $\dim \mathfrak{L} = 2$, and since $\mathfrak{A} = \mathfrak{M} \dot{+} \mathfrak{T}$, we find that $\dim \mathfrak{T} \ge 3$. From the proof of Theorem 2.9, we see that $\dim P\mathfrak{T}|_{\operatorname{ran} P} \le 2$. This implies that there exists a non-zero operator $T \in \mathfrak{A}$ whose operator matrix relative to the decomposition $\mathbb{C}^{2p} = \operatorname{ran} P \oplus \operatorname{ran} P^\perp$ is given by $\begin{bmatrix} 0 & T_2 \\ 0 & T_4 \end{bmatrix}$.

The fact that every element of $\mathfrak{D}_k$, and hence of $\mathfrak{A}$, is normal implies that $T_2 = 0$. Moreover, $\mathfrak{A}$ is a C^*-algebra, and thus relative to the same decomposition of $\mathbb{C}^{2p}$, $|T| \in \mathfrak{A}$ admits the matrix representation $\begin{bmatrix} 0 & 0 \\ 0 & |T_4| \end{bmatrix}$, and $|T_4| \neq 0$. But $\mathfrak{A}$ is abelian, and so $|T|M_j = M_j|T|$. Then $|T_4||_{\operatorname{ran} L_j} = 0$, $j = 1, 2$, contradicting the fact that $\mathfrak{L}$ is essential.

□

In examining the remaining cases, we find that the argument in the case where $\mathfrak{A}$ is isomorphic to $\mathbb{M}_2(\mathbb{C}) \oplus \mathbb{C}$ is significantly longer and more delicate than the others, and so we isolate this case below.

4.9. Proposition. *Suppose that* $2 \le p \in \mathbb{N}$. *Suppose furthermore that* $\mathfrak{A} \subseteq \mathcal{L}(\mathbb{C}^{2p})$ *is a self-adjoint, unital algebra which is* **-isomorphic to* $\mathbb{M}_2(\mathbb{C}) \oplus \mathcal{D}_1 \simeq \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{C}$. *Let* $P \in \mathcal{L}(\mathbb{C}^{2p})$ *be a projection of rank* p, *and let* $\mathfrak{L} = P^\perp \mathfrak{A}|_{\operatorname{ran} P}$. *It follows that either* $\mathfrak{L}$ *is not essential, or* $\dim \mathfrak{L} \neq 2$.

Proof. We shall argue by contradiction. Suppose that $\mathfrak{L}$ is essential and that $\dim \mathfrak{L} = 2$.

Below, we shall decompose $\mathbb{C}^{2p} = \operatorname{ran} P \oplus \operatorname{ran} P^\perp$, and we shall decompose all elements of $\mathcal{L}(\mathbb{C}^{2p})$ relative to this decomposition of $\mathbb{C}^{2p}$. Clearly $P = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}$ relative to this decomposition.

Now $\mathfrak{A}$ is *-isomorphic to $\mathfrak{A}_0 := \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{C}$, and thus there exists $1 < \gamma < p$ such that $\mathfrak{A}$ is *unitarily equivalent* to

$$\mathbb{M}_2(\mathbb{C})^{(\gamma)} \oplus \mathbb{C} I_{2p-2\gamma}.$$

More specifically, let us consider $\mathfrak{A}$ to be an injective, unital *-representation ρ of $\mathfrak{A}_0$ on $\mathbb{C}^{2p}$. Consider next $q := I_2 \oplus 0 \in \mathfrak{A}_0$, and denote by Q the projection $\rho(q)$.

Recall that the algebra $\mathfrak{T}$ in any standard P-decomposition of $\mathfrak{A}$ is entirely determined by P, and as such its dimension is independent of the particular basis $\{L_1, L_2\}$ for $\mathfrak{L}$ we may choose (and thus for whichever complement $\mathfrak{M}$ to $\mathfrak{T}$ in $\mathfrak{A}$ we may choose). In our case, $\dim \mathfrak{A} = 5$ and $\dim \mathfrak{L} = 2$, and so by the definition of $\mathfrak{T}$, $\dim \mathfrak{T} = 3$.

Step One.

Suppose that $Q \in \mathfrak{T}$, say $Q = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix}$ relative to $\mathbb{C}^{2p} = \operatorname{ran} P \oplus \operatorname{ran} P^\perp$. Since Q is a projection, we see that $Q_2 = 0$ and that Q_1, Q_4 are projections as well. Without loss of generality, we may write $Q_1 = I_r \oplus 0_{p-r}$ and $Q_4 = I_s \oplus 0_{p-s}$. Since $Q \neq I_{2p}$, either $r < p$ or $s < p$. Thus we have

$$Q = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0_{p-r} & 0 & 0 \\ 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & 0_{p-s} \end{bmatrix}.$$

(Of course, if $r = p$, the second row and column are absent. The main point here is that at least one of the second row (and column) and the fourth row (and column) is present.)

Since ρ is a unital *-representation, $I_{2p} - Q = \rho((I_2 \oplus 1) - q) = \rho(0_2 \oplus 1)$. Let $a = a_0 \oplus \alpha \in \mathfrak{A}_0$, where $a_0 \in \mathbb{M}_2(\mathbb{C})$ and $\alpha \in \mathbb{C}$.

Now $a = qaq + \alpha((I_2 \oplus 1) - q)$, and thus $\rho(a) = Q\rho(a)Q + \alpha(I - Q)$. That is,

$$\rho(a) = \left[\begin{array}{cc|cc} * & 0 & * & 0 \\ 0 & \alpha I_{p-r} & 0 & 0 \\ \hline * & 0 & * & 0 \\ 0 & 0 & 0 & \alpha I_{p-s} \end{array}\right].$$

Recall from Theorem 4.2 that $\dim \mathfrak{A} = 5$ implies that $\mu = p$, where $\mu = \max\{\operatorname{rank} L : L \in \mathfrak{L}\}$. Then, since $\mathfrak{L}$ is essential, $r = p = s$.

Thus $Q \notin \mathfrak{T}$.

Step Two.

Let $\mathfrak{T}_0 := \{t \in \mathfrak{A}_0 : \rho(t) \in \mathfrak{T}\}$ be the pre-image of $\mathfrak{T}$ in $\mathfrak{A}_0$, and note that $\mathfrak{T}_0$ is a three-dimensional, unital algebra. Choose $x, y \in \mathfrak{A}_0$ such that $\mathfrak{T}_0 = \operatorname{span}\{1, x, y\}$, where $1 := I_2 \oplus 1$ is the identity of $\mathfrak{A}_0$. Without loss of generality, we may assume that $x := \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \oplus 0$ and that $y = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} \oplus 0$ for an appropriate choice of $x_k, y_k \in \mathbb{C}$, $1 \le k \le 4$. Let

$x_e := \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, $y_e := \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}$. Then $\mathfrak{T}_e := \text{span}\,\{x_e, y_e\}$ is a two-dimensional subalgebra of $\mathbb{M}_2(\mathbb{C})$. (That it is closed under multiplication is an easy consequence of the fact that the powers and products of x and y are in $\mathfrak{T}_0$ and as such can be written as a linear combination of $1, x$ and y, but the components arising from the second summand are always zero, and therefore the coefficient of 1 is always zero.)

We first claim that $\mathfrak{T}$ is non-abelian. Indeed, otherwise $\mathfrak{T}_e \subseteq \mathbb{M}_2(\mathbb{C})$ is abelian. It is a relatively straightforward consequence of Burnside's Theorem that any two-dimensional, abelian subalgebra of $\mathbb{M}_2(\mathbb{C})$ is either similar to $\mathfrak{D}_2$ or to the algebra generated by the identity I_2 and the 2×2 nilpotent Jordan cell J. Both of these algebras are unital, however, which contradicts the fact established in Step One that $Q \not\in \mathfrak{T}$.

Thus $\mathfrak{T}_e$ is a non-unital, two-dimensional subalgebra of $\mathbb{M}_2(\mathbb{C})$, and hence it is unitarily equivalent to one of

- $\mathfrak{R} := \left\{ \begin{bmatrix} z_1 & z_2 \\ 0 & 0 \end{bmatrix} : z_1, z_2 \in \mathbb{C} \right\}$,
- $\mathfrak{C} := \left\{ \begin{bmatrix} z_1 & 0 \\ z_3 & 0 \end{bmatrix} : z_1, z_3 \in \mathbb{C} \right\}$.

We shall argue the case where $\mathfrak{T}_e = \mathfrak{R}$ (i.e. in particular we embed the unitary equivalence mentioned above into the definition of the map ρ). The case where $\mathfrak{T}_e$ is unitarily equivalent to $\mathfrak{C}$ is handled similarly.

Let $e_{i,j}$, $1 \le i, j \le 2$ denote the canonical (i, j)-matrix unit of $\mathbb{M}_2(\mathbb{C})$. Set $U := \rho(e_{11} \oplus 0)$ and $V := \rho(e_{12} \oplus 0)$, and note that $\mathfrak{T} := \text{span}\,\{I_{2p}, U, V\}$.

Define $M_1 := \rho(e_{21} \oplus 0)$ and $M_2 := \rho(e_{22} \oplus 0)$. Since $\{e_{21} \oplus 0, e_{22} \oplus 0\}$ is linearly independent from $\{I_1 \oplus 1, e_{11} \oplus 0, e_{12} \oplus 0\}$, we find that $\{M_1, M_2\}$ is linearly independent from $\{I_{2p}, U, V\}$, and as such, if we set $\mathfrak{M} := \text{span}\,\{M_1, M_2\}$, then $\mathfrak{M}$ is a complement to $\mathfrak{T}$ in $\mathfrak{A}$. It follows that $\mathfrak{L} = P^\perp \mathfrak{M}|_{\text{ran}\, P}$.

The equations $e_{11}e_{21} = 0$ and $e_{11}e_{22} = 0$ imply that $UM_1 = 0 = UM_2$. Writing

- $U = \begin{bmatrix} U_1 & 0 \\ 0 & U_4 \end{bmatrix}$ (note that e_{11} is a projection and thus so is U, whence $U = U^*$) and
- $M_k = \begin{bmatrix} A_k & B_k \\ L_k & D_k \end{bmatrix}$, $k = 1, 2$,

we find that $U_4 L_1 = 0 = U_4 L_2$.

The hypothesis that $\mathfrak{L}$ is essential implies that $\text{ran}\, P^\perp = \text{span}\,\{\text{ran}\, L_1, \text{ran}\, L_2\}$. From this we see that $U_4 = 0$, and thus

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

In fact, let us refine this decomposition further. Since U is a projection, so is U_1, and so we may decompose $\text{ran}\, P = \text{ran}\, U_1 \oplus (\text{ran}\, P \ominus \text{ran}\, U_1)$. With respect to the decomposition $\mathbb{C}^{2p} = \text{ran}\, U_1 \oplus (\text{ran}\, P \ominus \text{ran}\, U_1) \oplus \text{ran}\, P^\perp$, we may then write

$$U = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} V_{11} & V_{12} & V_{21} \\ V_{13} & V_{14} & V_{22} \\ 0 & 0 & V_4 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} A_{11} & A_{12} & B_{11} \\ A_{13} & A_{14} & B_{12} \\ L_{11} & L_{12} & D_1 \end{bmatrix} \quad M_2 = \begin{bmatrix} A_{21} & A_{22} & B_{21} \\ A_{23} & A_{24} & B_{22} \\ L_{21} & L_{22} & D_2 \end{bmatrix}.$$

Next, the equations $e_{11}e_{12} = e_{12}$ and $e_{12}e_{11} = 0$ imply that $UV = V$ and $VU = 0$. Thus

$$V = \begin{bmatrix} 0 & V_{12} & V_{21} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From the equations $e_{21}e_{11} = e_{21}$ and $e_{11}e_{21} = 0$ we find that $M_1U = M_1$ and $UM_1 = 0$, while the equation $e_{11}e_{22} = 0 = e_{22}e_{11}$ implies that $UM_2 = 0 = M_2U$. Thus

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ A_{13} & 0 & 0 \\ L_{11} & 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{24} & B_{22} \\ 0 & L_{22} & D_2 \end{bmatrix}.$$

Finally, since $e_{21}e_{12} = e_{22}$, we have that $M_1V = M_2$. Hence

$$M_2 = M_1V = \begin{bmatrix} 0 & 0 & 0 \\ A_{13} & 0 & 0 \\ L_{11} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & V_{12} & V_{21} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{13}V_{12} & A_{13}V_{21} \\ 0 & L_{11}V_{12} & L_{11}V_{21} \end{bmatrix}.$$

It follows that $L_{22} = L_{11}V_{12}$. Note that

$$\mathfrak{L} = \text{span}\{[L_{11} \quad 0], [0 \quad L_{22}]\} = \text{span}\{[L_{11} \quad 0], [0 \quad L_{11}V_{12}]\}.$$

Since $\mathfrak{L}$ is essential, we deduce that $\operatorname{ran} L_{11} = \operatorname{ran} P^\perp$. Keeping in mind that $\operatorname{rank} P = \operatorname{rank} P^\perp = p$, $L_{11} \in \mathcal{L}(\mathbb{C}^p)$. This means that, under the decomposition $\mathbb{C}^{2p} = \operatorname{ran} U_1 \oplus (\operatorname{ran} P \ominus \operatorname{ran} U_1) \oplus \operatorname{ran} P^\perp$, the second summand is absent. Therefore, $U = P$.

Recall that $UM_1 = 0$, $M_1U = M_1$, and $UM_2 = M_2U = 0$. With respect to the decomposition $\mathbb{C}^{2p} = \operatorname{ran} P \oplus \operatorname{ran} P^\perp$, we get

$$M_1 = \begin{bmatrix} 0 & 0 \\ L_1 & 0 \end{bmatrix}, \qquad \text{and} \qquad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & D_2 \end{bmatrix}.$$

Thus $\mathfrak{L} = P^\perp \mathfrak{M}|_{\operatorname{ran} P} = \text{span}\{L_1, 0\}$ has dimension 1, a contradiction.

This completes the proof.

□

4.10. Example. There exist injective, unital *-representations $\rho_k : \mathbb{M}_2(\mathbb{C}) \oplus \mathfrak{D}_k \to \mathcal{L}(\mathbb{C}^4)$, $k = 0, 2$ and a projection P of rank 2 such that $\mathfrak{L} := P^\perp \rho_k(\mathbb{M}_2(\mathbb{C}) \oplus \mathfrak{D}_k)|_{\operatorname{ran} P}$ is an essential corner of dimension 2.

- Consider the case where $k = 0$. Define $\rho_0 : \mathbb{M}_2(\mathbb{C}) \to \mathcal{L}(\mathbb{C}^4)$ via

$$\rho_0(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}) = \begin{bmatrix} \alpha & 0 & \beta & 0 \\ 0 & \delta & 0 & \gamma \\ \gamma & 0 & \delta & 0 \\ 0 & \beta & 0 & \alpha \end{bmatrix}.$$

  With $P = I_2 \oplus 0_2 \in \mathcal{L}(\mathbb{C}^4)$ and $\mathfrak{A} = \rho_0(\mathbb{M}_2(\mathbb{C}))$, $\mathfrak{L} := P^\perp \mathfrak{A}|_{\operatorname{ran} P}$ is an essential corner of dimension 2.
- Consider the case where $k = 2$. Define $\rho_2 : \mathbb{M}_2(\mathbb{C}) \oplus \mathfrak{D}_2 \to \mathcal{L}(\mathbb{C}^4) \simeq \mathcal{L}(\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C})$ via

$$\rho_2(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \oplus (\omega \oplus \theta)) = \begin{bmatrix} \alpha & 0 & \beta & 0 \\ 0 & \frac{\omega+\theta}{2} & 0 & \frac{\omega-\theta}{2} \\ \gamma & 0 & \delta & 0 \\ 0 & \frac{\omega-\theta}{2} & 0 & \frac{\omega+\theta}{2} \end{bmatrix}.$$

With $P = I_2 \oplus 0_2 \in \mathcal{L}(\mathbb{C}^4)$ and $\mathfrak{A} := \rho_2(\mathbb{M}_2(\mathbb{C}) \oplus \mathfrak{D}_2)$, $\mathfrak{L} := P^\perp \mathfrak{A}|_{\operatorname{ran} P}$ is an essential corner of dimension 2.

4.11. Proposition. *Suppose that $2 \le p \in \mathbb{N}$ and that $k \in \{1, 3, 4\}$. Suppose furthermore that $\rho : \mathbb{M}_2(\mathbb{C}) \oplus \mathfrak{D}_k \to \mathcal{L}(\mathbb{C}^{2p})$ is an injective, unital *-representation. Let $\mathfrak{A} := \rho(\mathbb{M}_2(\mathbb{C}) \oplus \mathfrak{D}_k)$, and let $P \in \mathcal{L}(\mathbb{C}^{2p})$ be a projection of rank p. If $\mathfrak{L} := P^\perp \mathfrak{A}|_{\operatorname{ran} P}$, then either $\mathfrak{L}$ is not essential, or* $\dim \mathfrak{L} \neq 2$.

Proof. In each case, we shall argue by contradiction. Suppose that $\dim \mathfrak{L} = 2$ and that $\mathfrak{L}$ is essential.

- The case where $k = 1$; i.e. where $\mathfrak{A}$ is a representation of $\mathbb{M}_2(\mathbb{C}) \oplus \mathfrak{D}_1 \simeq \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{C}$ is handled by Proposition 4.9.
- Consider the case where $k \in \{3, 4\}$ and $\rho : \mathbb{M}_2(\mathbb{C}) \oplus \mathfrak{D}_k \to \mathcal{L}(\mathbb{C}^{2p})$ is an injective, unital *-representation with $\mathfrak{A} = \rho(\mathbb{M}_2(\mathbb{C}) \oplus \mathfrak{D}_k)$.

  Let $P \in \mathcal{L}(\mathbb{C}^{2p})$ be a projection of rank p, and let $\mathfrak{A} = \mathfrak{M} \dotplus \mathfrak{V} \dotplus \mathfrak{N}$ be a standard P-decomposition of $\mathfrak{A}$. As per the comment following Theorem 2.9, we see that $\dim \mathfrak{V} \le 2 \dim \mathfrak{L} = 4$. Hence $\dim \mathfrak{N} \ge 1$. As always, we decompose $\mathbb{C}^{2p} = \operatorname{ran} P \oplus \operatorname{ran} P^\perp$.

  From above, we see that there exists $0 \neq N = \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} \in \mathfrak{A}$. Since N is a nilpotent of order two, it follows that there exists a nilpotent y of order two in $\mathbb{M}_2(\mathbb{C}) \oplus \mathfrak{D}_k$ such that $N = \rho(y)$. But then (without loss of generality),

$$y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus 0_k.$$

  Note that y^*y and yy^* are projections which add to $q := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus 0_k$. Thus NN^* and N^*N are projections which add to $Q = \rho(q)$. That is,

$$Q = \begin{bmatrix} YY^* & 0 \\ 0 & Y^*Y \end{bmatrix}$$

  is a *central* projection in $\mathfrak{A}$. Noting that $\operatorname{rank} YY^* = \operatorname{rank} Y^*Y$, we see that without loss of generality, we can find $1 \le r < p$ such that

$$Q = I_r \oplus 0_{p-r} \oplus I_r \oplus 0_{p-r}.$$

  The fact that Q is a central projection implies that (relative to the corresponding decomposition of $\mathbb{C}^{2p}$) for all $A \in \mathfrak{A}$,

$$A = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ 0 & A_{22} & 0 & A_{24} \\ A_{31} & 0 & A_{33} & 0 \\ 0 & A_{42} & 0 & A_{44} \end{bmatrix}.$$

  Furthermore, if $w \in \mathbb{M}_2(\mathbb{C}) \oplus 0_k$, then $qw = w = wq$ and writing $W = \rho(w)$,

$$QW = W = WQ,$$

  while if $X \in 0_2 \oplus \mathfrak{D}_k$, then $qx = 0 = xq$, so that with $X = \rho(x)$,

$$QX = 0 = XQ.$$

In matrix form, this yields

$$W = \rho(w) = \begin{bmatrix} W_{11} & 0 & W_{13} & 0 \\ 0 & 0 & 0 & 0 \\ W_{31} & 0 & W_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$X = \rho(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & X_{22} & 0 & X_{24} \\ 0 & 0 & 0 & 0 \\ 0 & X_{42} & 0 & X_{44} \end{bmatrix}.$$

The fact that $\mathfrak{L}$ is essential implies that there exist W, X such that $W_{31} \neq 0 \neq X_{42}$. Note that dim $\mathfrak{L} = 2$. But then the map

$$\varphi(x) = \begin{bmatrix} X_{22} & X_{24} \\ X_{42} & X_{44} \end{bmatrix}$$

yields a representation of $\mathfrak{D}_k$ such that the corresponding $\mathfrak{L}_{\mathfrak{D}_k} = P^\perp \varphi(\mathfrak{D}_k)|_{\operatorname{ran} P}$ is essential and has dimension 1. This contradicts Theorem 3.5.

□

The final case is handled by the following example.

4.12. Example. There exists an injective, unital *-representation $\rho : \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C}) \to \mathcal{L}(\mathbb{C}^4)$ and a projection P of rank 2 such that $\mathfrak{L} := P^\perp \rho(\mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C}))|_{\operatorname{ran} P}$ is an essential corner of dimension 2.

Proof. Define $\rho : \mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C}) \to \mathcal{L}(\mathbb{C}^4) = \mathcal{L}(\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C})$ via

$$\rho(\begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} \oplus \begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{bmatrix}) = \begin{bmatrix} \alpha_1 & 0 & \beta_1 & 0 \\ 0 & \alpha_2 & 0 & \beta_2 \\ \gamma_1 & 0 & \delta_1 & 0 \\ 0 & \gamma_2 & 0 & \delta_2 \end{bmatrix}.$$

If $\mathfrak{A} := \rho(\mathbb{M}_2(\mathbb{C}) \oplus \mathbb{M}_2(\mathbb{C}))$ and $P = I_2 \oplus 0_2$, then $\mathfrak{L} := P^\perp \mathfrak{A}|_{\operatorname{ran}\ P}$ is an essential corner of dimension 2.

□

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