# Non-perturbative aspects of gauge theories

by

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## Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

## Statement of Contributions

Chapters 1 and 5 correspond to the paper [1], written in collaboration with Davide Gaiotto and Jaume Gomis.

Chapter 2 corresponds to the paper [2], written in collaboration with Jaume Gomis and Matthew Yu.

Chapter 3 corresponds to the paper [3], written in collaboration with Changha Choi, Jaume Gomis, and Zohar Komargodski.

Chapter 4 corresponds to the paper [4], written in collaboration with Jaume Gomis.

Chapter 6 corresponds to the paper [5], written in collaboration with Jaume Gomis.

## Abstract

The purpose of this thesis is to establish some non-perturbative results in gauge theories in $d \leq 4$ spacetime dimensions. We generically refer to such theories as *Quantum Chromodynamics* (QCD). Gauge theories are asymptotically free, which means that at short distances interactions become weak, and one can reliably use perturbation theory in order to solve the theory. Most of our current understanding of QCD stems from studying the theory at short distances; for example, this is the regime where the bulk of experimental tests of the Standard Model of particle physics has been performed. By contrast, the long distance regime is characterized by strong interactions, and here perturbation theory generically stops working. Hence, the dynamics of QCD at macroscopic scales is to a large extent still a mystery. For example, our *ab initio* understanding of the spectrum of hadrons is still very limited, most data coming from numerics or phenomenological models.

In order to study gauge theories in their strongly-interacting regime, the perturbative approach is mostly useless, and one must develop new tools. In this work we study some quantitative properties of QCD in its strongly-coupled phase. Some of our main tools are supersymmetry, 't Hooft anomalies, dualities, topological fields theories, and two-dimensional chiral algebras.

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I also want to thank my collaborators Changha Choi, Davide Gaiotto, Zohar Komargodski, and Matthew Yu, from whom I learned a lot during our many discussions. This thesis consists of articles we wrote together; any chapter written in collaboration is as much theirs as it is mine, after all.

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Last but not least, I want to thank the members of my committee, Davide and Jaume, together with Niayesh Afshordi, Ruxandra Moraru, and Erich Poppitz, for their valuable input, and for accommodating last-minute meetings caused by my frequent absent-mindedness in regard to bureaucracy and deadlines.

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# Chapter 0

# Introduction and summary.

Say we are given a quantum field theory. An extremely useful tool to understand its dynamics is *perturbation theory*: we expand observables in powers of a coupling constant $g(\mu)$, where $\mu$ is the energy scale of interest. In general, this expansion is reliable if and only if $g(\mu)$ is sufficiently small, a situation that is controlled by the beta function $\beta(g) := \mu\partial_\mu g(\mu)$. (See e.g. [6] for the textbook analysis.) If $\beta < 0$ one has $g \to 0$ as $\mu \to \infty$, which means that the theory is ultraviolet free, and perturbation theory works at large energies. Conversely, if $\beta > 0$, one has $g \to 0$ as $\mu \to 0$, which means that the theory is infrared free, and perturbation theory works at low energies. In either case, perturbation theory does *not* work in the opposite limit, and the theory becomes strongly interacting, in which case new tools are required.

The leading behaviour of the beta function is controlled by dimensional analysis. For example, let us look at a gauge theory. Let $A$ be the gauge field and $F = \mathrm{d}A + [A, A]$. The Lagrangian reads

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{tr}(F \wedge \star F) + \cdots \tag{0.0.1}$$

By dimensional analysis it is easy to see that $g^2$ has units of $\mu^{4-d}$ where $d$ is the number of spacetime dimensions, which means that

$$\beta(g^2) = (d-4)g^2 + \mathcal{O}(g^3) \tag{0.0.2}$$

Therefore, for $d > 4$ the theory is infrared free, and the ultraviolet is ill-behaved. Similarly, for $d < 4$, the theory is ultraviolet free and the infrared is non-perturbative. The edge case $d = 4$ is of special interest because it is the physical dimension, but also because $g$ is classically marginal. A one-loop computation is necessary, and one learns that the sign of $\beta$ depends on the details of the matter fields. For realistic field content, such as the Standard Model of particle physics, the beta function is negative, and the theory behaves as in the $d < 4$ case. So, to summarize, in $d \leq 4$ the coupling constant of gauge theories can be as small as we want, just by looking at shorter and shorter distances, which means that the high energy regime can be studied reliably using perturbation theory.

Understanding the behaviour of the theory at low energies is, thus, a very non-trivial yet important task: first, it is a non-perturbative problem, thus requiring new tools; and second, it is of practical interest, because it describes the real world at real (macroscopic) energy scales. For example, the phenomenological spectrum of hadrons looks nothing like what one would expect from weakly-coupled quarks, a reflection of the fact that the dynamics are strongly interacting. While lattice QCD has been able to shed much light on the problem, understanding it analytically is still one of the long-standing open problems in QFT.

Our goal in this thesis is to say a few things about some non-perturbative properties of gauge theories in $d \leq 4$. In the reminder of this chapter we summarize some of our main findings. For the sake of readability, our arguments here are more heuristic that in the main text, and we include fewer references to the literature. We generically organize the the document from low dimensions to high dimensions.

## 0.1 Chapter 1.

In chapter 1 we discuss some aspects of 't Hooft anomalies [7] in QFT. Anomalies are arguably the most systematic and powerful tool that we have in order to analyze non-perturbative dynamics (other than numerics). They are one of the few properties of strongly interacting theories that can in fact be determined explicitly and reliably, although sometimes the computation can require substantial mathematical skill.

Consider a generic QFT with some algebra of operators. The symmetry group of the theory is, by definition, the group of transformations on this algebra that leaves the dynamics invariant. Importantly, the group of symmetries of the theory is typically realized on the Hilbert space $\mathcal{H}$ *projectively* [6]. The physical reason is that only transition amplitudes are observable, so the physical states are rays on $\mathcal{H}$: their phase is unobservable and therefore symmetries need only compose properly modulo a phase. Mathematically, operators live in $\mathrm{End}(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}^*$ and an action on $\mathcal{H} \otimes \mathcal{H}^*$ fixes an action on $\mathcal{H}$ only up to a phase.

What determines the projective representation of the symmetry group on the Hilbert space? How can we compute this phase in practice? And what consequences does it have for the properties of the theory? The main idea of chapter 1 is that, as it turns out, the projective phase is entirely determined by the anomalies of the symmetry group. In this way, if one understands the anomalies of the system, one can determine the projective action. Often enough, the converse is also true, in the sense that if one is able to compute the projective action independently, one can use that information to recover (at least some partial) information about the anomalies of the theory. Therefore, this insight not only simplifies the calculations of some anomalies, but it also gives a somewhat transparent picture of what the anomaly represents and what its consequences are.

The simplest illustration of this idea is in a theory with a bunch of fermions and time-reversal symmetry. Being fermionic, the system has a $\mathbb{Z}_2$ symmetry denoted by $(-1)^F$, which measures whether a given operator is bosonic (if even) or fermionic (if odd). Time-reversal, denoted by $\mathsf{T}$, will be chosen to satisfy the simplest symmetry group there is, namely $\mathbb{Z}_2$, i.e., $\mathsf{T}^2 = 1$. All in all, in this example we take the symmetry group to be $\mathbb{Z}_2 \times \mathbb{Z}_2$, represented as

$$((-1)^F)^2 = 1, \qquad \mathsf{T}^2 = 1, \qquad (-1)^F \mathsf{T} = \mathsf{T}(-1)^F \tag{0.1.1}$$

An important property to keep in mind is that time-reversal is an anti-unitary operator, meaning that it conjugates complex numbers.

Given the symmetry group above, how will it be represented on the Hilbert space? In full generality, the group structure need only be realized up to a phase, which means that, as a matter of principle, we can have the following:

$$((-1)^F)^2 = \alpha, \qquad \mathsf{T}^2 = \beta, \qquad (-1)^F \mathsf{T} = \gamma \mathsf{T}(-1)^F \tag{0.1.2}$$

where $\alpha, \beta, \gamma \in \mathrm{U}(1)$ are three phases. We can always redefine $(-1)^F \mapsto \alpha^{-1/2}(-1)^F$ such that $\alpha$ disappears from the first equation, hence we can always fix fermion parity to square to 1. Importantly, we cannot do this for time-reversal, since $\mathsf{T}$ is anti-unitary and therefore $\mathsf{T}^2$ is invariant under rephasing. Explicitly, if we let $\mathsf{T} \to \xi \mathsf{T}$ for some phase $\xi$, then $\mathsf{T}^2 \to \xi \mathsf{T} \xi \mathsf{T} = |\xi|^2 \mathsf{T}^2$, which is still equal to $\mathsf{T}^2$. In this sense, it seems that the phase $\beta$ cannot be redefined away.

An important condition on $\beta$ can be obtained by looking at $\mathsf{T}^3$. This can be parenthesized in two ways,

$$\mathsf{T}^2\mathsf{T} = \mathsf{T}\mathsf{T}^2 \tag{0.1.3}$$

which means that $\beta$ commutes with $\mathsf{T}$ and therefore it has to be real, i.e., $\beta = \pm 1$. Finally, a similar constraint on $\gamma$ can be obtained by multiplying both sides of $(-1)^F \mathsf{T} = \gamma \mathsf{T}(-1)^F$ by $(-1)^F$, once from the left and once from the right, and comparing the result. This yields $\gamma^2 \mathsf{T} = \mathsf{T}$, i.e., $\gamma = \pm 1$. So, to summarize, the most general projective action consistent with this symmetry group is

$$((-1)^F)^2 = 1, \qquad \mathsf{T}^2 = \pm 1, \qquad (-1)^F \mathsf{T} = \pm \mathsf{T}(-1)^F \tag{0.1.4}$$

where the two signs are independent. (Note that if we considered a unitary symmetry instead of an anti-unitary one, then the second sign $\beta$ could be redefined away too, and only the third sign $\gamma$ would remain.)

A key property of this result is that the space of possible projective phases is *disconnected*. Consequently, if we deform the system continuously, while preserving the symmetry, we cannot change the two signs above – they are *invariants* of the system. Any continuous map from $\beta = +1$ to $\beta = -1$ would have to go through a point where $\beta$ is a complex phase, which is inconsistent with the symmetry group, as we just showed. In this way, the two signs $\pm$ above are a property of the theory that is protected under smooth deformations. This gives a convenient way to compute them: if we are able to find a deformation from whatever theory we are interested in, into a theory we already understand, we can use that information to determine the two signs. We just deform into the simpler theory, which will preserve the signs, and compute them there. We can then deform back into the original theory, with the guarantee that the signs will stay the same. For example, one can show that a system of $d = 1$ dimensional fermions, with arbitrary self-interactions, has $\gamma = -1$ if the number of fermions is odd; this can be shown by turning off the interactions, which leaves us with a system of free fermions, where the result $\gamma = -1$ can be established in a few lines of straightforward algebra.

The exact same phenomenon occurs for many other symmetry groups beyond time-reversal, and with any group structure beyond $\mathbb{Z}_2$. Generically, the space of possible projective phases is disconnected, and therefore the specific phase chosen by a system becomes an invariant of the same. As such, the phase can be computed by deforming the theory into a simpler one and determining the phase there. For example, many theories we are interested in can be deformed into free theories, such as QCD, where $g \to 0$ takes us to a system of free quarks and free gluons. Free systems can be solved exactly and therefore any projective phase can be calculated explicitly. The projective phase of the original system is therefore determined too.

That being said, and in order to not give the impression that these features are specific to systems of free fields, in chapter 1 we also illustrate them by looking at topological quantum field theories. These are not continuously connected to a free phase (that is, not connected via a weakly-coupled path), inasmuch as their coupling constants are discrete and hence cannot be smoothly turned off. The study of fermionic TQFTs requires some machinery that we shall develop more systematically in chapter 5, although some basic notions are reviewed in chapter 1. In any event, the final conclusion is that these theories too acquire projective phases, exactly like the theories that can be deformed into free phases, which is of course unsurprising since, as we argue, these projective phases are a universal features of QFT.

These projective phases are not a mere curiosity, but they have important implications. For example, if a given system has time-reversal as above, but the group is realized as $\mathsf{T}^2 = -1$, then one can immediately conclude that the spectrum of the Hamiltonian is necessarily degenerate, with at least two-fold degeneracy. This is the well-known Kramers degeneracy [8].[1] The same conclusion can be reached if the anomalous sign appears in $(-1)^F \mathsf{T} = -\mathsf{T}(-1)^F$, with the added property that the degeneracy is now of the supersymmetric type: the degenerate states have opposite fermion parity.[2]

[1] The proof is straightforward: given an energy eigenstate $|E\rangle$, we can construct another state $\mathsf{T}|E\rangle$; as $\mathsf{T}$ is by definition a symmetry, it commutes with the Hamiltonian, and therefore the new state has the same energy as the old one. And these two states are clearly linearly independent, because a putative relation of the form $\mathsf{T}|E\rangle = c|E\rangle$ for some constant $c$ is incompatible with $\mathsf{T}^2 = -1$, inasmuch as $-|E\rangle = \mathsf{T}^2|E\rangle = |c|^2|E\rangle$ is inconsistent.

[2] The proof is analogous. The two states $|E\rangle$ and $\mathsf{T}|E\rangle$ have opposite signs under $(-1)^F$ as $\mathsf{T}$ by assumption anti-commutes with $(-1)^F$. This automatically precludes a linear dependence of the form $\mathsf{T}|E\rangle \propto |E\rangle$ as both sides are eigenvectors of $(-1)^F$ with different eigenvalue.

Other similar conclusions can also be established for other symmetry groups. For example, in chapter 1 we argue that *any theory* in $d = 1 + 1$ that is invariant under (chiral) parity and an odd number of fermions necessarily has a two-fold degenerate supersymmetric spectrum. Similarly, *any theory* in $d = 2 + 1$ invariant under time-reversal and an odd number of fermions also has a supersymmetric spectrum. The method we use to prove these claims is to deform the theory into a system of free fermions, and analyzing the realization of the symmetry group in the free case. We also argue that this result is consistent with some dynamical scenarios about the strongly-coupled regime of these theories that had previously been suggested in the literature. For example, it had been conjectured that some gauge theories flow to certain topological theories at large distances; we check that these topological theories are supersymmetric precisely when the original gauge theory has an odd number of fermions, as predicted by the general argument above.

To summarize the discussion so far,

We have argued that the symmetry group of a given quantum theory can and often is realized projectively, and that this projective action is protected under small deformations. The invariance under deformations allows us to explicitly compute these projective phases, by deforming into a simpler theory. And these projective phases, when present, imply non-trivial predictions about the properties of the theory, such as degeneracy in the spectrum of states. These predictions can then be tested against any potential dynamical scenario about the non-perturbative regime of the theory.

The second main result of chapter 1 is a discussion of how this relates to 't Hooft anomalies. The claim is that the projective phases are entirely determined by the anomalies of the symmetry group, and therefore they can be fixed in terms of the anomaly data, if the data is known. In practice, we can turn this around and use the projective phases, if they are known, to figure out the anomaly itself, which is often a more subtle task. Let us summarize the main points here.

In general, given a theory in $d$ dimensions and symmetry group $G$, the anomalies of the theory take values in $\mathrm{Tor}(\Omega_{d+1}(G)) \oplus \mathrm{Free}(\Omega_{d+2}(G))$ (see e.g. [9] for a nice review). Here Tor and Free denote the torsion and free parts of $\Omega$, respectively, and where $\Omega_n(G)$ denotes the $n$-dimensional bordism group of $G$, namely the space of $n$-dimensional manifolds that carry a $G$-structure, modulo bordism, where two manifolds are considered *bordant* if they can be obtained as the boundary of an $n + 1$-dimensional manifold that extends the $G$-structure. The bordism group $\Omega_n(G)$ is a generalized cohomology theory, similar in spirit to the classical group cohomology $H^n(G)$ but that requires a more refined mathematical treatment.

The factor $\mathrm{Free}(\Omega_{d+2}(G))$ is generically well-understood by now: it classifies the *local* anomalies, namely those that follow from triangle diagrams. In other words, the local anomalies are those that are captured by perturbation theory. As such, their computation and consequences are textbook material. By contrast, the factor $\mathrm{Tor}(\Omega_{d+1}(G))$, which classifies *global* anomalies, is much harder to get a grip on. On the one hand, computing this bordism group is a rather non-trivial task by itself, requiring advanced mathematical tools; and on the other hand, even if we understood this group, establishing which element is chosen by a specific quantum theory is also very hard, since these anomalies are not captured by perturbation theory.

At the formal level, a possible path is the following. Consider a certain quantum theory with symmetry $G$ and anomaly $\alpha \in \Omega_{d+1}(G)$. If we put this theory on a compact spatial manifold $X_{d-1}$ (equipped with whatever structures are required to properly define the system, such as boundary conditions), we can associate a Hilbert space $\mathcal{H}(X_{d-1})$ to it via canonical quantization. This compactification of the theory yields an effective two-dimensional class $\tilde{\alpha}$ obtained by integrating the original class $\alpha$ over $X_{d-1}$,

$$\tilde{\alpha} = \int_{X_{d-1}} \alpha \tag{0.1.5}$$

The two-dimensional class $\tilde{\alpha}$ has a simple interpretation: it describes the anomaly of the effective $d = 1$ theory obtained after compactification. But in $d = 1$, i.e., point-particle quantum mechanics, anomalies are essentially a synonym for a projective representation. In this sense, the projective realization of $G$ on $X_{d-1}$ is determined by the image $\tilde{\alpha}$ of $\alpha$ under the reduction along $X_{d-1}$. This does not fix $\alpha$ uniquely, since some information is lost after integration, but it does contain some partial information. We may be able to recover more information if we perform the same procedure for several different manifolds $X_{d-1}$ and, more optimistically, it is conceivable that the full class $\alpha$ can be determined if we understand $\tilde{\alpha}$ for sufficiently many distinct spatial manifolds. At the very least, if $\tilde{\alpha}$ is non-trivial for at least one $X_{d-1}$ (i.e., if the symmetry is actually realized projectively), then we can automatically conclude that $\alpha$ has to be non-trivial too, and thus the symmetry must have an anomaly.

Another interesting piece of information is contained in the isometries of $X_{d-1}$. In general, when we compactify a theory on a given spatial manifold, the isometries of the same descend to internal symmetries of the compact system. Therefore, the Hilbert space $\mathcal{H}(X_{d-1})$ actually realizes a representation of the extended group $G \times \mathrm{Iso}(X_{d-1})$. This extended group can be realized projectively too, and this contains more information that just the projective realization of $G$ alone. These extra projective phases typically give us more information about the original class $\alpha$ than the pure $G$ part.

There is more to the story. There are several techniques to calculate the group $\Omega_{d+1}(G)$ in the mathematics literature. One that seems to be particularly useful in physics is the so-called *Atiyah-Hirzebruch spectral sequence* [10]. This spectral sequence has a nice physical interpretation in terms of decorating defects and junctions with invertible theories that keep track of the different obstructions that go into a proper definition of how the symmetry acts on the theory [11].

In very broad terms, the end result of the spectral sequence is not quite $\Omega_{d+1}(G)$ but rather the associated graded of a filtration thereof. Specifically, the sequence converges to quotients of the form $F_i\Omega(G)/F_{i-1}\Omega(G)$, where $0 \subset F_0\Omega(G) \subset F_1\Omega(G) \subset \cdots \subset F_d\Omega(G) \subset \Omega(G)$ is a filtration of $\Omega(G)$. The different successive quotients of this filtration turn out to have specific implications for the projective realization of the symmetry. In other words, the different projective phases that appear on the Hilbert space have a natural layer structure, and each layer has a specific image in the filtration of $\Omega_{d+1}(G)$.

To give a very concrete example, we can look at the time-reversal symmetry discussed at the beginning of this section, whose projective realization was argued to be

$$((-1)^F)^2 = 1, \qquad \mathsf{T}^2 = \pm 1, \qquad (-1)^F\mathsf{T} = \pm\mathsf{T}(-1)^F \tag{0.1.6}$$

The bordism group predicts anomalies classified by $\Omega_2^{\mathrm{pin}^-} = \mathbb{Z}_8$. The Atiyah-Hirzebruch spectral sequence yields three quotients isomorphic to $\mathbb{Z}_2$, coming from the filtration $0 \subset \mathbb{Z}_2 \subset \mathbb{Z}_4 \subset \mathbb{Z}_8$ (i.e., $\mathbb{Z}_8/\mathbb{Z}_4 = \mathbb{Z}_4/\mathbb{Z}_2 = \mathbb{Z}_2/0 = \mathbb{Z}_2$). The three $\mathbb{Z}_2$ layers have specific effects on the projective realization of the time-reversal symmetry: two of them are just the two signs in (0.1.6), while the third $\mathbb{Z}_2$ is a more subtle phenomenon where the operator $(-1)^F$ happens to become ill-defined (in the sense that we cannot even realize it on the Hilbert space. This third sign does not tell us whether the representation is projective, but rather whether a representation is possible at all. We discuss this in more detail in chapter 1 but also give a very down to earth description in appendix A.)

For another example with a new twist, we can look at the (chiral) parity symmetry in $d = 1+1$ dimensions that we mentioned in passing earlier. The bordism group predicts that this symmetry has anomalies classified by $\Omega_3^{\mathrm{spin}}(\mathbb{Z}_2) = \mathbb{Z}_8$. Interestingly, the projective realization of the parity symmetry on the Hilbert space only admits a single sign $\pm 1$ in its group structure (as discussed right below (0.1.4)), and therefore it can only detect a single $\mathbb{Z}_2$ inside the full $\mathbb{Z}_8$ group. In this case, the isometry group of the spatial manifold turns out to be non-trivial, and when one looks at the projective realization of the extended symmetry group, the full $\mathbb{Z}_8$ anomaly emerges. This example highlights that in order to reconstruct the full anomaly given the projective representation of a symmetry, it is not always enough to look at the symmetry alone, but rather at how it interacts with the emergent symmetry of the compactified theory.

To summarize this second part of the discussion,

> We have argued that the projective realization of a given symmetry group is entirely encoded in its 't Hooft anomaly. Anomalies are classified by a certain generalized cohomology theory, namely group bordism. This group has a natural layer structure, and this structure is inherited by the projective representation. If one already knows the anomaly of a system, one can use that knowledge to determine the projective realization. Conversely, if one understands the projective realization instead, one can use that information to reconstruct the anomaly.

This gives a convenient method to analyze some anomalies that could be hard to compute otherwise. A projective phase in a symmetry group is a very concrete object, which can be fixed using very elementary tools, while an anomaly can be rather abstract and hard to approach given more traditional definitions. It also gives a somewhat explicit picture of what the anomalies imply, namely the failure for symmetries to compose as expected in the presence of a given background. This has the advantage that some consequences of the anomaly are easy to extract, since for example an extra sign in a group relation can immediately imply degeneracy in the spectrum, as argued above. Finally, this story also gives a window towards the intricacies of bordism theory, since some of its aspects sometimes have a simple physical interpretation.

We would like to make one final comment here. The fact that the class $\tilde{\alpha}$ that encodes the projective representation is obtained by integrating over $X_{d-1}$ makes it clear that, if $X_{d-1}$ is a boundary, then $\tilde{\alpha}$ is trivial. This means that if whatever structures are required to define the theory (such as a $G$-connection or spin structures) extend to one higher dimension, the symmetry group will not be realized projectively. This does not mean that the symmetry is non-anomalous, since $\tilde{\alpha} = 0$ does not imply that $\alpha = 0$, but rather that the chosen background was not able to detect the anomaly. In this sense, in order to be able to say anything about anomalies, we must always choose structures that do not extend. This is also clear from the physics point of view, since the non-bounding structures are precisely those that allow for zero-modes. In a compactified theory only these matter, and therefore in the absence of zero-modes the vacuum is trivial. Indeed, there are no non-trivial unitary one-dimensional projective representations, so a unique ground state always realizes symmetries non-projectively.

This explains why we only give examples up to dimension $d = 3$. In $d = 4$ any orientable spatial manifold is bounding, hence unless we turn on non-trivial holonomies we will not observe any projective phases. One could try to find simple examples in $d = 4$ by looking at non-orientable spatial manifolds (as $\Omega_3^{\mathrm{pin}_+} = \mathbb{Z}_2$ is non-trivial), which would detect anomalies for time-reversal symmetry in four dimensions. We do not explore this possibility in this thesis.

## 0.2 Chapter 2.

In chapter 2 we discuss some aspects of gauge theories in $d = 1 + 1$ dimensions. As $d \leq 4$, these theories are asymptotically free, which means that the perturbative loop expansion in terms of Feynman diagrams only gives useful information at high energies (i.e., short distances). By contrast, the behaviour of the theory at low energies (i.e., long distances) is much harder to explore, since the coupling constant grows and the perturbative expansion breaks down. The goal of chapter 2 is to try and say a few things about the low energy regime of QCD using non-perturbative tools.

The most basic question one may ask about a non-perturbative QFT is whether it is gapped or not. This question controls the leading macroscopic behaviour of the system, since a gapless theory contains non-trivial phenomena at any energy scale, while a gapped theory has rather boring dynamics at energies below the gap. More specifically, a gapless system always flows to a conformal field theory at large distances, while a gapped one to a (possibly trivial) topological field theory. In this sense, the first question we would like to ask is whether QCD is gapless or not; and, in either case, perhaps more ambitiously, which specific CFT or TQFT describes its macroscopic dynamics.

A situation where the infrared can be studied reliably is when the mass of the quarks $m$ is very large. Integrating out a massive field generates a local effective action for the remaining fields. In particular, what is left in QCD is an effective action for the gauge field; by gauge invariance and dimensional analysis, the only local term that survives the limit $m \to \infty$ is the gauge kinetic term.[3] In other words, if the mass of the fermion is sufficiently large, the dynamics are given by those of pure Yang-Mills, up to a renormalization of the coupling constant and irrelevant corrections suppressed by powers of $1/m$. Importantly, two-dimensional Yang-Mills is exactly solvable [12, 13] and, in particular, it is a gapped theory, with a unique vacuum.[4] In conclusion, QCD with very massive fermions is a gapped theory with unique vacuum, and the infrared TQFT is trivial.

Given this observation, we learn that the really hard case is that of massless quarks.[5] Are such theories gapless or not? And what is their infrared behaviour?

The main idea of chapter 2, somewhat oversimplified, is the following. First, we compactify the spatial direction, i.e., we put the theory on a cylinder of finite radius $L$. In this situation, and after some careful considerations regarding composite operators and gauge invariance, one can show that the Hamiltonian of QCD takes the form

$$H = g^2 L^3 \sum_{n>0} \frac{1}{n^2} |J_n^a|^2 \tag{0.2.1}$$

where $J_n$ are the Fourier modes of the gauge current $J^a = \bar{\psi} t^a \psi$,

$$J^a(x) = \sum_{n\geq 0} J_n^a e^{inx/L} \tag{0.2.2}$$

The conclusion is that the space of massless states, defined by $H = 0$, translates to the condition $J_n^a = 0$, as $H$ is expressed as a sum of squares. Consequently, the massless part of the spectrum is obtained by setting $J^a(x) \equiv 0$:

[3] There is also theta terms for abelian factors, if any. One can also generate terms for background fields if they couple to the fermions, such as the Arf invariant for gravity.

[4] Abelian factors at $\theta = \pi$ lead to two-fold vacuum degeneracy.

[5] Note that the massless point is well-defined since there are symmetries that only exist at that point, such as time-reversal and (chiral) fermion parity. This shall not be the case for $d = 2 + 1$ that we will discuss later on.

$$\begin{array}{ccc} \text{Space of massless} & \equiv & \text{Free algebra generated by } \psi \\ \text{operators in QCD} & & \text{subject to the relation } J^a \equiv 0. \end{array} \tag{0.2.3}$$

Note that this is a purely algebraic problem. If we are given some gauge-invariant operator $\mathcal{O}$ constructed out of the quark fields $\psi$, and such that this operator can be factorized as $\mathcal{O} = J^a \mathcal{O}^a$ for some other operator $\mathcal{O}^a$, then the observable $\mathcal{O}$ creates massive excitations only, and it decouples at large distances. Conversely, if $\mathcal{O}$ *cannot* be factorized in terms of gauge currents $J^a$, then that operator creates massless excitations and thus survives the infrared limit. With this in mind, a given theory is gapped if and only if there are finitely-many massless operators $\mathcal{O}$. Indeed, a non-trivial CFT always has an infinite number of operators, while a TQFT has a finite number of operators.

Importantly, this problem can be solved. One can show that a QCD theory defined by some gauge group $G$ and quarks in some representation $R$ has finitely-many massless operators (that is, finitely-many operators that do not contain the gauge current $J^a$) if and only if it is one of these:

- Any gauge group $G$ and quarks in the adjoint representation $R = \text{adj}$.
- $G = S(\mathrm{U}(N) \times \mathrm{U}(M))$, $G = \mathrm{SO}(N) \times \mathrm{SO}(M)$, $G = \mathrm{Sp}(N) \times \mathrm{Sp}(M)$, and quarks in the bifundamental representation $R = (\square, \square)$.
- $G = \mathrm{U}(N)$, $G = \mathrm{SO}(N)$, $G = \mathrm{Sp}(N)$, and quarks in the rank-2 representations $R = \square\!\square, \begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$.
- A finite list of isolated theories (e.g., $G = \mathrm{Spin}(9)$ and quarks in the spinor representation).
- Combinations of the above.

Any other choice of $(G, R)$ will lead to a gapless theory (i.e., one which contains an infinite number of operators that do not contain gauge currents $J^a$).

The idea behind this classification is as follows. One can show that the space of operators described by (0.2.3) coincides with the space of operators of a well-known rational CFT [14–18], namely the theory whose chiral algebra is the Goddard-Kent-Olive coset[6]

$$\frac{\mathrm{SO}(\dim R)_1}{G_{T(R)}} \tag{0.2.4}$$

Here $\mathrm{SO}(\dim R)_1$ denotes the free chiral algebra generated by the fermion field $\psi$, while $G_{T(R)}$ the Wess-Zumino-Witten subalgebra generated by the currents $J^a = \bar{\psi} t^a \psi$. The quotient of

[6] $T(R)$ is the Dynkin index of $R$, a certain group theory factor; we define it properly and give its value for several important representations in appendix B.

these two algebras precisely enumerates operators of the numerator (i.e., combinations of quark fields $\psi$) modulo operators of the denominator (i.e., combinations of gauge currents $J^a$). Let us stress that, at this point, we are not claiming that the coset (0.2.4) is the low-energy effective theory of QCD. We are only claiming that the infrared of QCD, and the coset (0.2.4), have the same spectrum of operators, as sets (i.e., without assuming that they describe the same dynamics). As the two theories have the same number of operators, they are either both gapped (if finite) or both gapless (if infinite).

With this in mind, a QCD theory is gapped if and only if its associated GKO coset $\mathrm{SO}(\dim R)_1/G_{T(R)}$ has a finite spectrum. Luckily, the classification of finite GKO models is known, and it gives rise to the list above. Therefore, this completes the problem of figuring out whether a given gauge theory in two-dimensions is gapless or gapped.

Needless to say, this is not the end of the story. In order to understand the large distance dynamics of QCD, it is not enough to known whether it is gapless or not, but we would also like to know which degrees of freedom describe the infrared physics. At this point we know the space of massless operators, but not their properties at the infrared fixed-point. In other words, we know whether a given operator defined in the ultraviolet survives the infrared limit, but we do not known what are its dynamical properties in the infrared. For example, we do not know the scaling dimensions of operators at large distances.

A very natural conjecture for the infrared CFT is precisely the GKO coset (0.2.4). In other words, this chiral algebra not only classifies massless operators, but it also contains the CFT data thereof. We are thus lead to consider the conjecture [19]

$$\text{Infrared of QCD} \quad \equiv \quad \frac{\mathrm{SO}(\dim R)_1}{G_{T(R)}} \tag{0.2.5}$$

Note that this conjecture contains our previous result (0.2.3), and extends it. The claim in (0.2.3) is reliable – it follows from the explicit analysis of the Hamiltonian of QCD – while the conjecture (0.2.5) is not entirely justified from first principles. That being said, the conjecture is the simplest CFT whose spectrum coincides with the reliable result (0.2.3) and, as we argue in chapter 2, it also matches all the 't Hooft anomalies of QCD, which makes the conjecture particularly attractive.

The conjecture (0.2.5), if correct, completely solves the problem of figuring out the macroscopic behaviour of QCD in two dimensions. If a given theory is in the list above, the infrared is gapped, while if it is not, it is gapless. And, moreover, in either case, the long distance degrees of freedom are those contained in the GKO coset $\mathrm{SO}(\dim R)_1/G_{T(R)}$. If the theory is gapped, this coset is a TQFT, with a finite-dimensional Hilbert space that describes the vacua of the theory. Instead, if the theory is gapless, the coset describes a CFT, whose chiral algebra takes the form of a specific gauged-WZW model.

To give a few examples of what this looks like, consider the following two prototypical QCD theories:

- Adjoint QCD, namely one quark in the adjoint representation. This is in the list above, and therefore the theory is gapped. The coset that describes its space of vacua is

$$\frac{\mathrm{SO}(\dim G)_1}{G_h} \tag{0.2.6}$$

where $h = T(\mathrm{adj})$. This coset describes a TQFT; by working out its properties one learns that its Hilbert space has dimension $2^{\mathrm{rank}(G)}$, so adjoint QCD has these many discrete vacuum states. The properties of this TQFT, such as the line operators and their correlators, can be extracted from the coset $\mathrm{SO}(\dim G)_1/G_h$.

- Fundamental QCD, namely quarks in the fundamental representation. This is (generically) not in the list above, and therefore the theory is gapless. The coset that describes its massless degrees of freedom is

$$\frac{\mathrm{SO}(\nu N N_F)_1}{G_{N_F}} \tag{0.2.7}$$

where $N$ is the number of colors, $N_F$ the number of flavors, and $\nu = 1, 2, 4$ depending on whether the representation is real, complex, or pseudo-real, respectively. This coset describes a CFT; by level-rank duality, it turns out that it is in fact isomorphic to an ungauged WZW model for the flavor symmetry, at level $N$. (For example, QCD with $G = \mathrm{SU}(N)$ and fundamental quarks has $\mathrm{U}(N_F)$ flavor symmetry, and therefore the infrared CFT is an $\mathrm{U}(N_F)_N$ WZW model.)

This also partially solves the problem of light quarks, since there is a well-defined method to identify the image of the mass operator $m\bar{\psi}\psi$ in the coset (0.2.5). The mass operator is mapped into a certain operator in this coset, and therefore turning on small masses is equivalent to deforming the coset by that certain operator. The fate of the theory under such deformation (or any other deformation one may be interested in, such as four-fermi interactions), depends on the specific situation but is at least a well-defined problem, which can be worked out explicitly in a case-by-case analysis.

There are several consistency checks on this picture. For example, we can see that the gapped theories all have quarks in dimensions that grow, at most, quadratically with the number of colors. To some extent this was to be expected, since if the number of quarks grows faster than the number of colors, then for sufficiently large $N$ the gluon dynamics are subleading, and the gap of the theory is controlled by the mass of the fermions. In particular, for massless fermions the theory is gapless, and therefore we should only find gapped theories for finitely many choices of $N$; these are precisely the isolated cases in the list of gapped theories.

Another simple consistency check regards the fact that none of the theories in the list above have continuous flavor symmetries. This was also to be expected since in two dimensions, any continuous (chiral) symmetry necessarily has an 't Hooft anomaly, a situation that is incompatible with a mass gap. More generally, one can show that the coset (0.2.5) has in fact the exact same symmetries (both discrete and continuous, zero-form and one-form) and anomalies (both local and global) as the original QCD system, which is a necessary condition for the latter to describe the macroscopic limit of the former.

## 0.3 Chapter 3.

Our next object of study is gauge theories in $d = 2 + 1$ dimensions. As before, in three dimensions the theory is still asymptotically free, so the strongly-coupled regime corresponds to low energies, i.e., large distances. Our goal is the same as before, namely to try and say a few things about the non-perturbative regime of QCD. Some aspects are shared with the previous section, but some are new.

One of the most important elements of three-dimensional gauge theories is the existence of Chern-Simons terms [20–22]. This means that, on top of the standard gluon kinetic term $\mathrm{tr}(F^2)$, one can also add terms of the form $\mathrm{tr}(FA)$. Three key properties of this last term are that: First, it has one fewer derivatives than the Yang-Mills kinetic term, hence it is more relevant at large distances. Second, it depends on $A$ rather than $F$, so gauge invariance is not entirely manifest; in fact, checking invariance under large transformations yields the condition that the coefficient $k$ of the Chern-Simons term must be integrally quantized. And, third, it turns out that the pure Chern-Simons theory, where the only term in the Lagrangian is $\mathrm{tr}(FA)$, is actually exactly solvable; the resulting theory is topological, in the sense that it has a finite-dimensional Hilbert space and observables only depend on the topology of spacetime. All of these properties will play an essential role below.

With this in mind, we are set to study the low energy dynamics of three dimensional QCD. The theory is specified by some gauge group $G$, which carries a coupling constant and a Chern-Simons term with coefficient $k$, plus some fermions in some representation $R$ and some mass $m$. We shall denote the corresponding theory as $G_k + \psi_R$.

As in two-dimensions, a regime where one can study the theory reliably is the case of very large $m$. Integrating out massive matter fields yields an effective action for the gauge fields. By gauge invariance and dimensional analysis, the only local terms one can generate are the gauge kinetic term and a Chern-Simons term. Therefore, up to corrections of order $1/m$, massive fermions only generate a renormalization of the coupling constant and the Chern-Simons coefficient. Importantly, the Chern-Simons coefficient is quantized, and therefore it cannot depend on the gauge coupling constant. This means that the renormalization, if any, can be computed in perturbation theory. By performing the corresponding one-loop computation, one learns [23, 24] that the Chern-Simons coefficient gets renormalized as $k \mapsto k + \mathrm{sign}(m)T(R)$, where $\mathrm{sign}(m) = m/|m|$ is the sign of the quark mass, and $T(R)$ is the Dynkin index of the representation $R$ (see appendix B for the definition of $T(R)$ and its value for several important representations). Once the fermion has been integrated out, we can take the low-energy limit explicitly: since the gluon kinetic term is less relevant than the Chern-Simons one, we can drop the former,[7] and we are left with a pure Chern-Simons theory, which is topological (so it does not flow). This is the end result: the macroscopic limit of the theory is a pure Chern-Simons theory at level $k \pm T(R)$, which is now exactly solvable.

So, to summarize,

A QCD theory with gauge group $G$, Chern-Simons level $k$, and quarks in a representation $R$ with sufficiently large mass $m$, flows at long distances to a Chern-Simons TQFT:

$$\lim_{m\to\pm\infty} G_k + \psi_R = G_{k\pm T(R)} \tag{0.3.1}$$

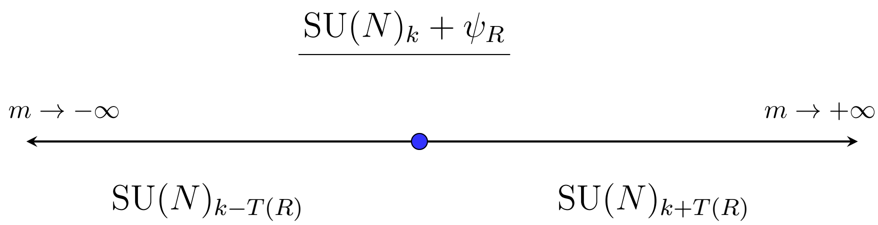
Unlike in two-dimensions, here the large-mass regime is not entirely trivial, but rather it is given by a certain TQFT. In any case, we are still faced with the same situation, where the really hard case is that of light quarks.[8] We cannot really follow the same strategy as before (for example, the $3d$ Hamiltonian is not as simple as the $2d$ one; and, moreover, $3d$ CFTs are not nearly as well-understood and constrained as $2d$ ones, so the gauged Wess-Zumino-Witten formulation would not be particularly useful anyway). So we must try something new.

The approach we take in chapter 3 is the following. To be concrete we choose the gauge group $G = \mathrm{SU}(N)$, since this is the group we study in that chapter, but the story for other groups is qualitatively identical. We begin by looking at the large mass case, where the infrared is given by $\mathrm{SU}(N)_{k\pm T(R)}$, and ask ourselves what happens as we decrease the value of the mass. Clearly, there must be at least one phase transition, since the two large-mass phases $\mathrm{SU}(N)_{k-T(R)}$ and $\mathrm{SU}(N)_{k+T(R)}$ are distinct. One can in fact show that, for sufficiently large $k$, or large $N_F$, this is all there is [25, 26] (see also [27]): the two large-mass phases continue

[7]Needless to say, we can drop the gluon kinetic term in favor of the Chern-Simons term only when the latter has a non-zero coefficient. If for some reason we are faced with a theory with zero Chern-Simons level, then the gluon kinetic term is the most relevant interaction. This is a rare situation but it does play a role below, when we look at a $\mathrm{U}(1)_0$ theory.

[8]Unlike in $d = 2$, here the massless point is in general not well-defined, since there are no special symmetries at that point (so the choice $m = 0$ depends on the renormalization scheme). The reason time-reversal does not exist is that the Chern-Simons term, being proportional to $\epsilon^{\mu\nu\rho}$, is not invariant. This does have an exception: given that time-reversal maps $k \to -k$, the case $k = 0$ does have a well-defined massless point $m = 0$, since at that point the theory becomes invariant under time-reversal, and hence $m = 0$ is protected from radiative corrections.

all the way down to some finite value of $m$, at which there is a unique phase transition:

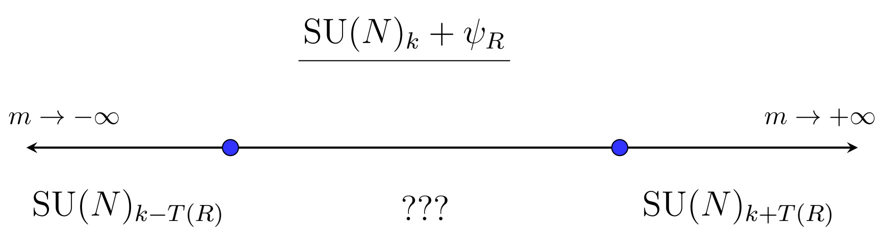



(0.3.2)

The degrees of freedom at the transition point can be studied reliably using $1/k$ or $1/N_F$ expansions.

What happens for small $k$ and small $N_F$? The most immediate possibility is that the diagram above continues to hold in this regime. In this case, the degrees of freedom at the transition point must still be non-trivial (e.g., in order to match several 't Hooft anomalies), but we can no longer study them in any controlled fashion. This seems like a roadblock. A more attractive possibility is that, for small $k, N_F$, a new phase opens up, and there are two transition points. The advantage of this scenario is that, if correct, it gives a consistent method to reliably understand the degrees of freedom at small mass. When doing so, one learns that these degrees of freedom pass many non-trivial consistency checks, which a posteriori justifies the assumptions that allowed us to obtain them in the first place.

With this in mind, let us assume that for small $k$ and small $N_F$ the phase diagram of the theory looks something like this:




(0.3.3)

where the intermediate phase is to be determined.

A hint of what the dynamics at the transition point are can be obtained thanks to the so-called level-rank dualities of Chern-Simons (see e.g. [28]). The two asymptotic phases $\mathrm{SU}(N)_{k\pm T(R)}$ have in fact an alternative description: these TQFTs can also be expressed in terms of a unitary gauge group:

$$\mathrm{SU}(N)_{k\pm T(R)} \quad \longleftrightarrow \quad \mathrm{U}(T(R)\pm k)_{\mp N} \tag{0.3.4}$$

Under level-rank duality of two groups $G \leftrightarrow G'$, representations are exchanged as $R \leftrightarrow R^t \times f^r$, where $t$ denotes the transpose of the Young diagram, $f$ the transparent fermion (an invisible anyon that carries spin 1/2), and $r$ is the rank of $R$ (i.e., the number of boxes in its Young diagram). This invites us to guess that, perhaps, *the full gauge theory* at the transition point may also enjoy a non-trivial duality of the form

$$\mathrm{SU}(N)_k + \psi_R \quad \longleftrightarrow \quad \mathrm{U}(T(R)\pm k)_{k_1,k_2} + \mathcal{O}_{R^t} \tag{0.3.5}$$

where $\mathcal{O}$ is a boson if $r$ is odd, and a fermion if even. This is clearly a very optimistic guess, and with little motivation so far, but it turns out to be surprisingly self-consistent as we shall see. At this point, the dual levels $k_1, k_2$ are unknown. The first level $k_1$ denotes the coefficient of the $\mathrm{SU}(T(R) \pm k)$ subgroup, while the second level $k_2$ denotes the coefficient of the U(1) part. For very large mass, U(1) levels shift as $k_2 \mapsto k_2 \pm \frac{2}{T(R)\pm k}\dim(R^t)$.

The case of rank $r = 1$ fermions was analyzed in quite detail in [29], where the dual field $\mathcal{O}$ is a boson (since $r$ is odd). The analysis of fermions in rank $r = 2$ representations was initiated in [30], where the dual field $\mathcal{O}$ is a fermion (since $r$ is even). That paper studied the case of $N_F = 1$ copies of the adjoint representation, and also the symmetric and anti-symmetric representations for $G = \mathrm{SO}(N)$ and $G = \mathrm{SU}(N)$. The remaining rank-2 representations, namely the symmetric and anti-symmetric of $\mathrm{SU}(N)$, is the content of chapter 3. Higher rank representations are not expected to follow a similar pattern, since the number of fermionic degrees of freedom grows faster than the number of gluons, and therefore the gauge interactions are subdominant, so to leading order the system behaves more like weakly coupled fermions (and in particular it is unlikely that the system has a quantum phase similar to that of $r \leq 2$).

Let us, then, focus on the symmetric and anti-symmetric representations of $\mathrm{SU}(N)$. These are denoted as $R = \square\!\square$ and $R = \begin{smallmatrix}\square\\\square\end{smallmatrix}$, respectively. The working hypothesis is that, close to the two transition points in (0.3.3), the system can also be described using dual variables

$$\mathrm{SU}(N)_k + \psi_R \quad \longleftrightarrow \quad \mathrm{U}(T(R) \pm k)_{k_1,k_2} + \psi_{R^t} \tag{0.3.6}$$

where the dual field is a fermion since $r$ is even. Here, the transposition $R^t$ interchanges the symmetric and anti-symmetric representations. The dual levels $k_1, k_2$ can be fixed as follows. If we send $m \to \pm\infty$, then the original gauge theory flows to $\mathrm{SU}(N)_{k\pm T(R)}$. If we assume that the dual mass is $m' = -m$, then sending $m \to \pm\infty$ is equivalent to $m' \to \mp\infty$, and therefore the dual theory flows to $\mathrm{U}(T(R) \pm k)_{k_1 \mp T(R^t), k_2 \mp \frac{2}{T(R)\pm k}\dim(R^t)}$. As these two theories are supposed to describe the same phase, and given the level-rank duality in (0.3.4), we conclude that the dual level must be

$$\begin{aligned} k_1 &= \pm(T(R^t) - N) \\ k_2 &= \mp N \pm \frac{2}{T(R) \pm k}\dim(R^t) \end{aligned} \tag{0.3.7}$$

And now comes the key point: by sending the mass $m'$ to infinity in the opposite direction, namely $m' \to \pm\infty$, we must land on the intermediate phase denoted by "???" in (0.3.3). The intermediate phase predicted by the two duals is

$$\mathrm{U}(T(R) \pm k)_{\pm(2T(R^t)-N), \mp N \pm \frac{4}{T(R)\pm k}\dim(R^t)} \tag{0.3.8}$$

Apparently, the two duals predict a different phase, which would be an inconsistent situation. Luckily, there is another non-trivial equivalence of Chern-Simons theories, this time taking the form

$$U(A)_{B,B\pm A} \quad\longleftrightarrow\quad U(|B|)_{-A\,\mathrm{sign}(B),(-A\mp B)\,\mathrm{sign}(B)} \tag{0.3.9}$$

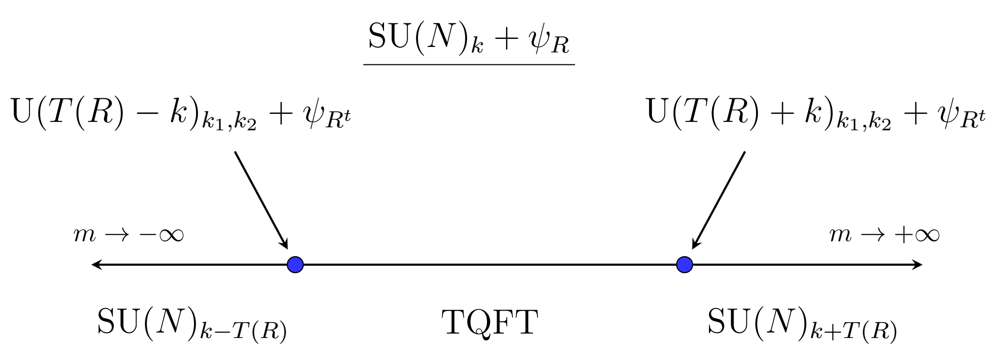
In order for the two theories in (0.3.8) to be dual to each other we must have

$$\begin{aligned} T(R)+k &= -2T(R^t)+N \\ \frac{4}{T(R)+k}\dim(R^t) &= T(R)+k+2T(R^t) \end{aligned} \tag{0.3.10}$$

Using $T(\yng(1,1))=\frac{1}{2}(M-2)$, $\dim(\yng(1,1))=\frac{1}{2}M(M-1)$, together with $T(\yng(2))=\frac{1}{2}(M+2)$, $\dim(\yng(2))=\frac{1}{2}M(M+1)$, where $M$ is the number of colors (either $N$ or $T(R)\pm k$, depending on whether we are looking at the original fermion $R$ or its dual $R^t$), one can check that these conditions are indeed satisfied! This is a very non-trivial coincidence of group theory factors, and it works out thanks to an independent level-rank duality of TQFTs, which is a very good sign that the conjecture is on the right track.

One thing that should be noticed is that the dual theories are $\mathrm{U}(T(R)\pm k)$; of course, this only makes sense for $|k|\le T(R)$, so this picture can hold at most in the range $k\in(-T(R),T(R))$. We conjecture that it does in fact hold in the whole range, while for any $|k|\ge T(R)$, the theory is in the semi-classical regime with only two phases (0.3.2). This is consistent with our general expectations, since we know that for sufficiently large $k$ the theory definitely is in the two-phase regime. Note also that the dual theory is also given by a unitary group with a rank 2 representation, and in constructing the full phase diagram we assumed that it was in its two-phase regime. This is only self-consistent if $|k|<T(R)$ holds if and only if $|k_1|\ge T(R^t)$ which, using the expressions above, is indeed correct. So this constitutes another non-trivial consistency check of the whole picture.[9]

To summarize, the proposed phase diagram looks like this:




(0.3.11)

where at each phase transition we write a weakly-coupled dual gauge theory, and in the intermediate phase the TQFT that this dual theory predicts, namely (0.3.8). The two duals

[9]Unfortunately, this also means that the dual theories are never both weakly-coupled simultaneously: one is in the semi-classical phase when and only when the other is in the quantum phase. So we cannot really test the duality using perturbative tools.

could in principle predict distinct intermediate phases, but they in fact predict the same, thanks to a level-rank duality. Otherwise the picture would not be self-consistent. Here $k_1, k_2$ are given by (0.3.7).

An interesting fact about the conjecture just described is the following. If we plug in the values of $T(R^t)$ and $\dim(R^t)$ into (0.3.8), we get the following intermediate quantum phase for the theory:

$$\mathrm{U}(T(R) \pm k)_{\mp T(R)+k,2k} \tag{0.3.12}$$

Recall that the original gauge theory is not in general invariant under time-reversal. Such operation takes $k \to -k$, so this is not a symmetry for non-zero $k$. Of course, we could formally restore time-reversal if, on top of reversing the orientation of spacetime, we also manually take $k \to -k$; in other words, we think of $k$ as something like a spurion. We now look at the intermediate phase in (0.3.12). If we reverse the orientation of spacetime, this flips the sign of the two levels. If we also manually change $k \to -k$, we end up with $\mathrm{U}(T(R) \mp k)_{\pm T(R)+k,2k}$, which is the exact same TQFT we began with. So the intermediate phase is consistent with time-reversal. In particular, for $k = 0$, the operation of time-reversal becomes an actual symmetry of the system, and the intermediate TQFT becomes time-reversal invariant as well, which is a non-trivial fact since most TQFTs are not invariant under time reversal.[10]

Let us look at the time-reversal invariant point $k = 0$ in more detail. For this value of the level the TQFT in (0.3.12) becomes $\mathrm{U}(T(R))_{\mp T(R),0} \equiv \mathrm{PSU}(T(R))_{\mp T(R)} \times U(1)_0$. An interesting aspect of this theory is the factor of $\mathrm{U}(1)_0$. This just describes a free, massless photon, with gauge group U(1) and no Chern-Simons coefficient. In three dimensions, a free photon is equivalent to a compact scalar (by Hodge duality, where a two-form is dual to a zero-form, i.e., a scalar). This massless scalar that appears at large distances is nothing but the Goldstone boson associated to the U(1) flavor symmetry of the original system, which acts as $\psi \mapsto e^{i\alpha}\psi$. So this conjecture predicts that this global symmetry is spontaneously broken, a phenomenon that is very hard to see semi-classically. So this constitutes a very concrete, non-trivial prediction about this gauge theory.

Needless to say, this whole story hinges on several educated guesses, so it is by no means a rigorous derivation. In order to gain confidence in the conjecture we must perform as many consistency checks as possible. We already mentioned some simple ones above. Another set of consistency checks come from embedding the $d = 3$ theories above into $d = 4$ gauge theories, whose large distance behaviour must be compatible with the dynamics predicted by the conjecture above. We will perform several quantitative tests of the four-dimensional picture in chapter 6, and also some more qualitative ones in chapter 3.

Going back to the pure $d = 3$ dynamics, another large set of tests is obtained by turning on background fields for the continuous symmetries. Given a continuous symmetry with

[10] In general $G_k$ is not dual to $G_{-k}$, with $\mathrm{U}(k)_{k,0}$ being an exception thanks to the level-rank duality (0.3.9), as can be seen by taking $A = B = k$ and the bottom sign therein.

group $K$, one may add to the original Lagrangian any term constructed out of the background $K$ connection. Non-trivial tests come from looking a Chern-Simons terms for the group $K$. As the fermions transform non-trivially under $K$, integrating them out will in general shift the the coefficient of the Chern-Simons terms for $K$. Furthermore, such coefficient also generically shifts under level-rank duality, and also under integrating out the fermions in the dual theory. All in all, if we follow the Chern-Simons term for $K$ as we go around the conjectured phase diagram, its coefficient will change in the different phases. In particular, if we reach the intermediate phase from either the negative mass (left) or the positive mass (right) side of the diagram, we go through different shifts, and the accumulated shift may be different. Of course, the intermediate phase is unique, so consistency of the picture requires that the shift computed through the left part of the diagram must be identical to the shift computed through the right part of the diagram, which is a non-trivial consistency condition. Furthermore, given the spurion argument above, this accumulated shift must be in fact an odd function of $k$, since the coefficient of a Chern-Simons term is odd under time-reversal, i.e., under $k \to -k$. These two conditions – that the left and right shifts are equal, and that they are odd functions of $k$ – are rather non-trivial tests of the phase diagram, and they can be checked for any continuous symmetry of the system. In chapter 3 we show that these conditions check out for the proposed phase diagram, and we also include the analysis of other phase diagrams that appeared in the literature, for completeness.

A final consistency check that we discuss is the matching of 't Hooft anomalies for discrete symmetries. Such anomalies are global in the sense of chapter 1, so they are somewhat subtle to analyze. We focus on the case of time-reversal for the $k = 0$ theory, which can be argued to have anomalies classified by $\Omega_4^{\mathrm{pin}^+} = \mathbb{Z}_{16}$. The anomaly computed in the original ultraviolet variables must be equal to the anomaly of the intermediate TQFT, which is a very constraining condition for the phase diagram. In the theory studied above, the time-reversal invariant TQFT also includes a free scalar, which in a sense trivializes the test; but similar theories, where we look at other rank 2 representations, do not have a compact scalar and here the test is much more stringent. Therefore, in chapter 3 we also include an analysis of the $\mathbb{Z}_{16}$ time-reversal anomaly and check for its matching.

## 0.4 Chapter 4.

As discussed in the previous section, three-dimensional gapped QFTs flow at large distances – in their strongly-coupled regime – to topological field theories. It is therefore worthwhile to study TQFTs in detail, since they describe the macroscopic dynamics of realistic systems. Conversely, an a priori understanding of the properties of TQFTs (such as dualities, and symmetries) is of much help in figuring out which TQFT appears in the low-energy limit of a given QFT.

There are other reasons one may be interested in TQFTs in their own right. For example, it is possible to realize them in actual experiments, such as the Hall effect [31]. Moreover, they are an example of an exactly solvable QFT [22], and therefore they are a great playground to analyze generic features of QFTs, such as symmetries, anomalies, etc.

With these motivations in mind, the goal of chapter 4 is to determine the symmetries of abelian TQFTs. A TQFT is said to be *abelian* if it can be described by a Chern-Simons theory over an abelian gauge group. Equivalently, it is abelian if and only if every Wilson line admits an inverse, i.e., another line such that their combination is the identity [32]. Abelian theories thus form an abelian group, as opposed to a fusion category for non-abelian theories. Groups are more constrained than general fusion categories, and therefore the space of symmetries is much more manageable and, in a sense, rich. Generic TQFTs have very few symmetries, most often they are "trivial" symmetries associated to permutations of a Dynkin diagram, and the non-trivial symmetries are exceptional and hard to determine. By contrast, the symmetries of abelian theories are more accessible since they are (a subset of) the automorphisms of the abelian group above, which is a classic problem in group theory. The result is a very rich set of symmetries, often with fun connections to other areas such as number theory.

Let us summarize the main points of chapter 4. We begin by quickly reviewing some background material. One way to characterize an abelian TQFT is by its Chern-Simons presentation. The most general Lagrangian over an abelian gauge group is

$$\mathcal{L} = \frac{1}{4\pi} K_{ij} a^i \wedge \mathrm{d}a^j \tag{0.4.1}$$

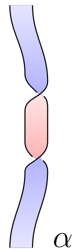
where $a^i$ is a tuple of $\mathrm{U}(1)$ gauge fields and $K_{ij}$ is an integral symmetric matrix. This matrix specifies the TQFT. While in chapter 4 we deal with the general case, in this section we will simplify the discussion by assuming that all diagonal components of $K$ are even; this ensures that the TQFT is bosonic, i.e., it does not depend on the spin structure of spacetime. The case of spin theories is almost identical, with some extra factors of 2 here and there.

The observables of a TQFT are the Wilson lines. In the abelian case, such a line is specified by a tuple of integers $\alpha_i \in \mathbb{Z}$, namely

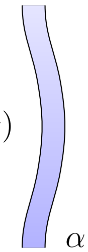
$$W_\alpha(\mathsf{c}) = e^{i\alpha_i \oint_{\mathsf{c}} a^i} \tag{0.4.2}$$

where $\mathsf{c}$ is some cycle in spacetime. There are two crucial properties of $W_\alpha(\mathsf{c})$ that we need to describe:

- The operator $W$ does not only depend on $\mathsf{c}$, but there is also a dependence on the choice of trivalization of the normal bundle thereof [22]. The idea is that short-distance divergences force us to introduce a regulator. A choice that preserves topological invariance is to specify a vector normal to the line, which dictates the direction we are to deform it in case of coinciding singularities. The line thus becomes a ribbon. One can show that the dependence on this choice of normal vector is rather mild: it merely introduces a (computable) phase:



$$=\quad \theta(\alpha) \tag{0.4.3}$$

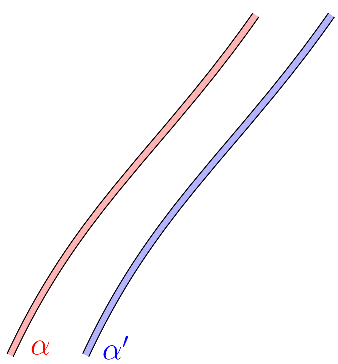



where

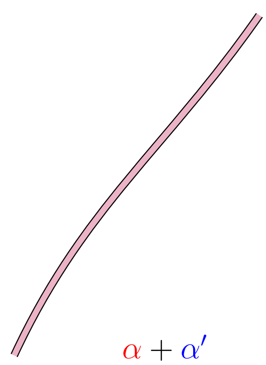
$$\theta(\alpha) := e^{i\pi\alpha^t K^{-1}\alpha} \tag{0.4.4}$$

- Not every choice of $\alpha_i \in \mathbb{Z}$ leads to a distinct operator. Two Wilson lines $W_\alpha$ and $W_{\alpha'}$ have identical correlation functions if and only if $\alpha \equiv \alpha' \mod K$. The idea is that the propagator of two gauge fields is $2\pi K^{-1}$ (this being the inverse of the gauge kinetic term in (0.4.1)). Therefore, correlation functions of $W_\alpha$ can be expressed as a certain function of $e^{2\pi i\alpha K^{-1}\beta}$, where $\beta$ is the charge of any other Wilson line. If $\alpha \equiv \alpha'$ mod $K$, then $e^{2\pi i\alpha K^{-1}\beta} \equiv e^{2\pi i\alpha' K^{-1}\beta}$ for all $\beta$, and $W_\alpha, W_{\alpha'}$ have the same correlators with respect to any other line. Thus, there are finitely-many distinct Wilson lines, which take values in the coset $\mathbb{Z}/K\mathbb{Z}$. (This coset is finite and has $|\det(K)|$ elements)

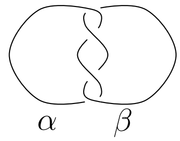
If we have two lines $W_\alpha(\mathsf{c})$ and $W_{\alpha'}(\mathsf{c}')$ such that $\mathsf{c}, \mathsf{c}'$ are homologous (and the normal bundles are in the same homotopy class), then we can bring the cycles $\mathsf{c}, \mathsf{c}'$ together, since the theory is topological. The end result is a single Wilson line, with charge $\alpha + \alpha'$:




$$= \tag{0.4.5}$$




Abelian theories are fully specified by these pieces of data we just described: the coset $\mathcal{A} := \mathbb{Z}/K\mathbb{Z}$, the operation $\mathcal{A}\times\mathcal{A} \to \mathcal{A}$ given by entry-wise addition $\alpha+\alpha'$, and the quadratic function $\theta\colon \mathcal{A} \to \mathrm{U}(1)$. Indeed, the space of all such structures is isomorphic to the space of integral symmetric matrices $K$ (up to stable equivalence [33]). Any correlation function of a configuration of Wilson lines can be computed if we know the abelian group $(\mathcal{A}, +)$ and the function $\theta$. For example, using the two moves just described (0.4.3), (0.4.5), one can show that




$$\Big\langle \qquad \Big\rangle = \left( \frac{1}{\sqrt{|\mathcal{A}|}} \frac{\theta(\alpha+\beta)}{\theta(\alpha)\theta(\beta)} \right)^{\mathrm{link}(\mathsf{c}_1,\mathsf{c}_2)} \tag{0.4.6}$$

where $\mathrm{link}(\mathsf{c}_1, \mathsf{c}_2)$ is the linking number of the cycles $\mathsf{c}_1, \mathsf{c}_2$ (in this example, $\mathrm{link}(\mathsf{c}_1, \mathsf{c}_2) = 2$).

We can summarize this review of abelian theories as follows:

> A general abelian TQFT is characterized by an integral symmetric matrix $K$. From this matrix one can extract a finite abelian group $(\mathcal{A}, +)$, with $\mathcal{A} = \mathbb{Z}/K\mathbb{Z}$, and a quadratic, homogeneous function on this group $\theta(\alpha) = \exp(i\pi\alpha^t K^{-1}\alpha)$, with $\alpha \in \mathcal{A}$. Any correlation function in the theory can be calculated in terms of these basic pieces of data, namely undoing self-twists (0.4.3) and fusing lines (0.4.5).

We can now move on to the main point of chapter 4, to wit, classifying all symmetries of such TQFTs. Given the discussion above, we can characterize symmetries as follows. A symmetry is, by definition, an operation on the observables of the theory that leaves correlation functions invariant. In the case at hand, the observables are the Wilson lines, and therefore a symmetry is a permutation $\rho\colon \mathcal{A} \to \mathcal{A}$ such that

$$\langle W_{\rho(\alpha)}(\mathsf{c}) \cdots \rangle \equiv \langle W_\alpha(\mathsf{c}) \cdots \rangle \tag{0.4.7}$$

where $\cdots$ denotes any insertion of other Wilson lines. As we just reviewed, this correlation function can be determined entirely in terms of fusion and $\theta$, and therefore $\rho$ will leave correlation functions invariant if and only if

$$\rho(\alpha + \alpha') = \rho(\alpha) + \rho(\alpha') \mod K, \qquad \theta(\rho(\alpha)) = \theta(\alpha) \tag{0.4.8}$$

for all $\alpha, \alpha' \in \mathcal{A}$.

Similarly, anti-unitary symmetries by definition reverse the orientation of spacetime, i.e., they are permutations that satisfy

$$\langle W_{\rho(\alpha)}(\bar{\mathsf{c}}) \cdots \rangle \equiv \langle W_\alpha(\mathsf{c}) \cdots \rangle^* \tag{0.4.9}$$

where $\bar{\mathsf{c}}$ is the cycle $\mathsf{c}$ with the opposite orientation, and the complex conjugation $*$ is due to the symmetry being anti-unitary. Thus, we learn that $\rho$ is an anti-unitary symmetry if and only if

$$\rho(\alpha + \alpha') = \rho(\alpha) + \rho(\alpha') \mod K, \qquad \theta(\rho(\alpha)) = \theta(\alpha)^* \tag{0.4.10}$$

for all $\alpha, \alpha' \in \mathcal{A}$.

In more abstract terms, an abelian TQFT is specified by its data $(\mathcal{A}, +, \theta)$, and a symmetry is an automorphism of the group $(\mathcal{A}, +)$ that leaves the quadratic function $\theta$ invariant, up to complex conjugation for anti-unitary symmetries.

This formulation makes the problem straightforward: for a given abelian group, we first identify all the automorphisms, and then we check which of these leave the quadratic form invariant. The abelian group is finite, and so in the worst-case-scenario we brute-force through finitely many permutations. As usual, it is enough to check the generators, on which an action of a homomorphism determines an action on the whole group. The recipe is now clear: if we are given some abelian TQFT, we must first determine its abelian group and quadratic form data, and then we list all automorphisms thereof. We explain how to do this in practice for arbitrary theories, paying special attention to the case of spin (a.k.a. fermionic) TQFTs, that is, theories that depend on the spin structure of the underlying manifold. QCD with quarks is fermionic, and therefore the infrared TQFT most often is, too.

Let us illustrate this with the most basic abelian theory, $\mathrm{U}(1)_k$, namely a Chern-Simons theory with gauge group $\mathrm{U}(1)$ and level $k$. This theory admits both a bosonic and a fermionic version. In order to simplify the presentation, we do not make a distinction between these two options here, and also drop some factors of 2. The abelian group is $\mathcal{A} = \mathbb{Z}_k$, and the quadratic form is $\theta(\alpha) = \exp(\pi i\, \alpha^2/k)$ for $\alpha \in \mathbb{Z}_k$, where we use additive notation for the group law. A general automorphism of $\mathbb{Z}_k$ is a map of the form $\alpha \mapsto q\alpha$ where $q$ is an integer coprime to $k$, and insisting that this map leaves the quadratic form invariant yields the condition $q^2 = \pm 1 \mod k$, where the upper sign corresponds to unitary symmetries and the lower one to anti-unitary ones. Solving the congruence $q^2 = \pm 1 \mod k$ yields all the symmetries of the theory.

Unitary symmetries are given by the solutions to $q^2 = +1 \mod k$. There is always at least one non-trivial solution, namely $q = -1$. This corresponds to the usual charge-conjugation symmetry of gauge theories, which more traditionally acts as $a^i \mapsto -a^i$. The equation $q^2 = +1 \mod k$ may or may not admit solutions other than $q = \pm 1$; when it does, the permutation $\alpha \mapsto q\alpha$ will not be a symmetry of the Lagrangian $\mathcal{L}$, since the only transformations that leave $\frac{k}{4\pi} a \wedge \mathrm{d}a$ invariant are $a \mapsto \pm a$. Therefore, charge-conjugation is the only symmetry of the classical theory, and other symmetries, if any, will only be symmetries of the quantum theory.

Anti-unitary symmetries, on the other hand, are given by the solutions to $q^2 = -1 \mod k$. This equation does not always admit solutions. One can show that this congruence equation admits solutions if and only if all prime factors of $k$ are congruent to 1 modulo 4, i.e., only for those values of the level is the theory time-reversal invariant. Furthermore, one can argue that the number of unitary – and anti-unitary, if any – symmetries is $2^{\omega(k)}$, where $\omega(k)$ is the number of distinct prime factors of $k$.

The first few values of $k$ for which $\mathrm{U}(1)_k$ is time-reversal invariant are $k = 1, 2, 5, 10, 13, \ldots$. Indeed, these are the first few integers for which the congruence $q^2 = -1 \mod k$ admits solutions. As before, this symmetry is somewhat unusual since it again does not leave the Lagrangian invariant. This means that, for those special values of $k$ above, the symmetry is a quantum symmetry, which is not present in the classical theory. It is a non-trivial self-duality of the theory. Another exotic aspect of this symmetry is that, except for $k = 2$, it is an order-four operation, i.e., it satisfies $\mathsf{T}^4 = 1$ as opposed to the more standard algebra $\mathsf{T}^2 = 1$.

We thus learn that $\mathrm{U}(1)_k$ generically has lots of symmetries, and their type and number depends on the arithmetic properties of the level $k$. Most of these symmetries are quantum symmetries, i.e., they are transformations that leave correlation functions invariant, even if they do not leave the Lagrangian invariant in any meaningful sense. That being said, it is interesting to note that TQFTs often have lots of equivalent descriptions: theories described by different matrices $K_1, K_2$ may in fact give rise to the same theory, via a non-trivial duality. This is so if the matrices $K_1, K_2$ are associated to the same topological data $(\mathcal{A}, +, \theta)$. A symmetry that is non-Lagrangian with respect to the matrix $K_1$ may become Lagrangian with respect to the matrix $K_2$. In other words, whether a symmetry is classical or quantum depends on the duality frame we choose to describe the theory, and is not intrinsic to the theory itself. To give a very concrete example, consider the theory $\mathrm{U}(1)_k$. This theory is in fact equivalent to another Chern-Simons theory, described by three gauge fields coupled via

$$K = \begin{pmatrix} k & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{0.4.11}$$

In the duality frame of $K$, the symmetry $\alpha \to q\alpha$ becomes a Lagrangian symmetry, since it can be effected by the change of variables

$$\begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix} \mapsto Q \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix}, \qquad Q := \begin{pmatrix} q & -p & 1 \\ pk & \frac{1}{2}p(-q \pm 1) & \frac{1}{2}(q \pm 1) \\ -k & \frac{1}{2}(q \pm 1) & \frac{1}{2p}(-q \pm 1) \end{pmatrix} \tag{0.4.12}$$

where $p$ is an integer such that $q^2 = 1 + 2pk$. Although it may not be obvious at first glance, the matrix $Q$ is integral and unimodular, which means that it is a valid change of variables (it respects the periodicity conditions on the $\mathrm{U}(1)$ gauge fields $a^i$). It is easily checked that the matrix $K$ satisfies $Q^t K Q \equiv K$, and therefore the change of variables (0.4.12) leaves the Lagrangian $a^t K \mathrm{d}a$ invariant. On the first gauge field $a^1$, the change of variables acts as $a^1 \to qa^1 + \cdots$, and therefore this redefinition implements the symmetry $\alpha \to q\alpha$, as required. The conclusion is that, while this unitary symmetry is non-Lagrangian with respect to $\mathrm{U}(1)_k$, it becomes Lagrangian with respect to a dual description $K = k \oplus \sigma_x$.

The general case is conceptually very similar. The abelian group for arbitrary abelian TQFTs is always a product of cyclic group, and symmetries are defined by polynomial congruences on these groups, roughly of the form $Q^2 = \pm 1 \mod K$, with matrices $Q, K$. Solving this system of congruences classifies all symmetries; the end result depends sensitively on the number-theoretic properties of the levels of the Chern-Simons terms. Generically, these symmetries are quantum in the sense of the previous paragraph: they are symmetries of the theory but not of the classical Lagrangian. In a suitable dual description, some symmetries may become Lagrangian. And, as above, the symmetries often satisfy exotic algebras, such as $\mathsf{T}^4 = 1$, or even non-abelian algebras. See chapter 4 for a long list of examples.

This completes the problem of determining the symmetries of abelian theories. While interesting in its own right, it is also useful to try and apply the conclusions to more general QCD theories. We can list a couple of examples here. Two abelian theories that appear in the low-energy regime of QCD are $\mathrm{U}(1)_2$ and $\mathrm{U}(1)_5$, which describe the macroscopic dynamics of SU(2) plus an adjoint fermion, and PSp(5) plus an anti-symmetric fermion, both at the $m = k = 0$ point [30] (see also section 3.6 for a review). As the gauge theory is time-reversal invariant, so must the low-energy TQFT. And, using the results of this chapter, this is indeed seen to be the case, since $k = 2, 5$ both satisfy the condition for time-reversal invariance of $\mathrm{U}(1)_k$. Moreover, $\mathrm{U}(1)_2$ has order-two time-reversal invariance while $\mathrm{U}(1)_5$ has order-four (i.e., $\mathsf{T}^2 = 1$ and $\mathsf{T}^4 = 1$, respectively). This is precisely what one expects from the gauge theory point of view, since PSp(5) is not simply-connected, and the corresponding magnetic symmetry modifies the time-reversal algebra from $\mathsf{T}^2 = 1$ to $\mathsf{T}^2 = \mathcal{M}$, where $\mathcal{M}^2 = 1$ [34]. On the other hand, SU(2) is simply-connected and thus there is no magnetic symmetry, and the time-reversal algebra is the standard order-two algebra. We also discuss other possible applications of our results to other gauge theories in chapter 4.

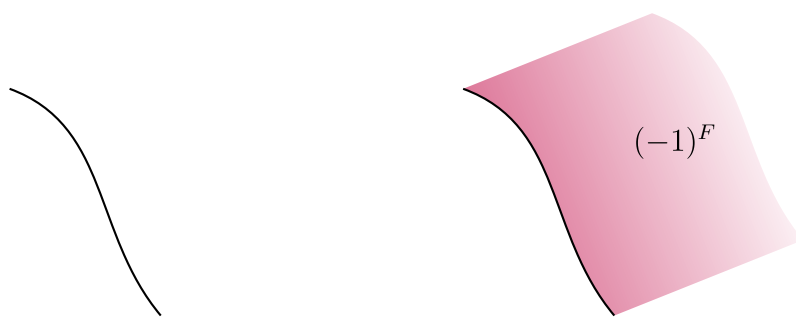
## 0.5 Chapter 5.

In chapter 5 we continue our study of topological theories in three dimensions. The motivation is the same as in the previous section: these theories describe the macroscopic regime of gapped QFTs, so any knowledge we can extract from the former will potentially teach us something about the latter. Furthermore, these theories share many of the features of more complex QFTs while being exactly solvable, which makes them a great arena to explore general properties of quantum systems.

To be concrete, the topic of chapter 5 is the dependence of TQFTs on the *spin structure* of spacetime [35]. Gauge theories that contain quarks require spin structures to be defined (i.e., a choice of boundary conditions for fermions), and therefore their macroscopic effective description often require a spin structure too. Topological theories that depend on the spin structure – known as *spin* TQFTs or *fermionic* TQFTs – go beyond the scope of traditional TQFTs – known as *bosonic* TQFTs – which depend only on the topology of spacetime. In other words, spin TQFTs are more sensitive to the underlying manifold than bosonic ones. This means that their construction requires a more refined treatment than the traditional approach.

In chapter 5 we describe how spin TQFTs are constructed, with special emphasis to their Hilbert space. This space is a super-vector space, i.e., it is $\mathbb{Z}_2$-graded, owing to the fact that the states in fermionic theories can be labelled by their fermion parity, i.e., by whether they are a boson or a fermion. This super-vector space only depends on the topology of the spatial manifold and the associated spin structure, up to isomorphism. Large diffeomorphisms that leave the topological space invariant induce isomorphisms of the associated Hilbert spaces, which must be compatible with the $\mathbb{Z}_2$ grading. The specific form of these isomorphisms – known as the *modular data* of the theory – is crucial for solving the theory on arbitrary manifolds (which can always be reconstructed by cutting it into simpler manifolds, and pasting them back together via a suitable diffeomorphism [22]).

The degrees of freedom of three-dimensional TQFTs are line operators, also known as *anyons*. The distinguishing feature of a fermionic TQFT, as opposed to a bosonic one, is that the line operators come in two different types. The "regular" anyons, which we refer to as *Neveu-Schwarz* lines, behave quite similarly to the anyons of a bosonic theory. They are topological lines, in the sense that correlation functions only depend on the global structure of the configuration of operators. By contrast, fermionic theories also contain a different type of anyons, which we call *Ramond* lines, which are not genuine line operators but rather they exist at the end of the topological surface that implements fermion parity $(-1)^F$:




(0.5.1)

Neveu-Schwarz anyon Ramond anyon

We denote the set of Neveu-Schwarz anyons as $\mathcal{A}_-$ and the set of Ramond anyons as $\mathcal{A}_+$. Similarly to bosonic TQFTs, these anyons have associated topological data, such as $S$ and $T$ matrices, which measure the response of the theory to large diffeomorphisms.

A nice example of a spin three-manifold whose partition function can be evaluated explicitly is the lens space $L(a,1) \equiv S^3/\mathbb{Z}_a$. For $a$ even this space has two spin structures, and the corresponding partition functions read

$$Z[L(a,1);\pm] \equiv \sum_{\alpha\in\mathcal{A}_\pm} (T_\alpha)^{-a}(S_{0,\alpha})^2 \tag{0.5.2}$$

This shows that, if we understand the sets $\mathcal{A}_\pm$ and the associated topological data $S, T$, we can compute the partition function on the lens space with spin structure $\pm$. More generally, the same is true for any three-dimensional spin manifold: any observable can be determined in terms of the anyons $\mathcal{A}_\pm$ and the topological data of the theory (i.e., the action of large diffeomorphisms).

How can we determine the topological data of a fermionic theory, in practice? The strategy we follow in order to construct fermionic theories is the following [36, 37]. If we have one such theory to begin with, one can always sum over all possible spin structures, which yields a TQFT that no longer depends on the spin structure. In other words, summing over spin structures turns a fermionic TQFT into a bosonic TQFT. Importantly, this process can be reversed: one can also turn the bosonic TQFT into the fermionic one back. The way one does this is by gauging a suitable one-form symmetry, an operation that is dual to summing over spin structures. Specifically, this gauging means summing over all possible two-form gauge fields for the symmetry. In conclusion, any fermionic TQFT can be obtained by a suitable sum of bosonic TQFTs. As the latter are very well-understood, we can use this approach to give an explicit construction of the former. For example, we show how all the modular data of the fermionic theory can be written entirely in terms of the modular data of the dual bosonic theory. Similarly, the fermionic Hilbert space is a direct sum of suitable bosonic Hilbert spaces.

From this point of view, the Neveu-Schwarz lines of the fermionic theory are the anyons of the bosonic theory that are uncharged under the one-form symmetry, and the Ramond line those that are charged. This explains why, after gauging, the latter appear at the end of the $(-1)^F$-surface.

To summarize,

Fermionic TQFTs always have an associated bosonic TQFT, and one can go back and forth between these two theories by suitable gaugings. Any object on either side of the duality can be expressed as a linear combination of similar objects on the other side of the duality. As bosonic theories are better understood than fermionic ones, one can use this correspondence to solve the latter using known facts about the former.

To given an explicit example of what the duality looks like, consider the following. Let $\mathcal{H}(\Sigma^{\alpha_1\cdots\alpha_n})$ denote the Hilbert space of the bosonic theory, quantized on an arbitrary (orientable) Riemann surface $\Sigma$ with external punctures $\alpha_1,\cdots,\alpha_n$ (i.e., labelled boundary components). Let also $\hat{\mathcal{H}}(\Sigma_s^{\alpha_1\cdots\alpha_n})$ denote the super-Hilbert space, on the same surface, with spin structure $s$, as defined by the fermionic theory. Then these two spaces are related by

$$\begin{aligned}\bigoplus_{\mathsf{c}\in H_1(\Sigma,\mathbb{Z}_2)} P_{\mathsf{c},s}\big[\mathcal{H}(\Sigma^{\alpha_1\cdots\alpha_n})\oplus\mathcal{H}(\Sigma^{\alpha_1\cdots\alpha_n\psi})\big]&=\hat{\mathcal{H}}(\Sigma_s^{\alpha_1\cdots\alpha_n})\\ \bigoplus_{s\in H^1(\Sigma,\mathbb{Z}_2)}\hat{\mathcal{H}}(\Sigma_s^{\alpha_1\cdots\alpha_n})&=\mathcal{H}(\Sigma^{\alpha_1\cdots\alpha_n})\oplus\mathcal{H}(\Sigma^{\alpha_1\cdots\alpha_n\psi})\end{aligned}\tag{0.5.3}$$

where $P_{\mathsf{c},s}$ are projectors of the form $P_{\mathsf{c},s}=s^{\mathsf{c}}W_\psi^{\mathsf{c}}$, with $s^{\mathsf{c}}=\pm1$ the spin structure around the cycle $\mathsf{c}$ and $W_\psi^{\mathsf{c}}$ the Wilson line around $\mathsf{c}$ associated to the transparent anyon $\psi$ (the generator of the one-form symmetry). The projectors are defined in such a way that $W_\psi^{\mathsf{c}}\equiv s^{\mathsf{c}}$ on $\hat{\mathcal{H}}$, i.e., such that the transparent fermion around a cycle measures precisely the boundary condition around the same. In any case, these expressions (0.5.3) demonstrate the idea that summing over spin structures $H^1(\Sigma,\mathbb{Z}_2)$ "undoes" the sum over $\mathbb{Z}_2$ gauge fields $H_1(\Sigma,\mathbb{Z}_2)$, and vice versa.

The modular data of the fermionic theory can be obtained by acting with the projectors $P_{\mathsf{c},s}$ on the data of the bosonic theory. This data is very important since it allows us to compute observables on arbitrary spin 3-manifolds, via surgery. In order to illustrate the procedure we explicitly calculate this data for several examples of common TQFTs, and then use it to compute the path integral on lens spaces, a non-trivial family of spin 3-manifolds.

While the construction of the Hilbert space as above is complete and self-contained, it might sometimes be time-consuming for high-genus surfaces. If the explicit basis is not needed, but just the total number of states, then there is a simple formula that computes that number directly for us. In the bosonic case it is the so-called Verlinde formula [38, 39]. In chapter 5 we also discuss the fermionic version of this formula, which gives us the number of bosons and fermions of an arbitrary spin TQFT, on an arbitrary surface, in terms of the torus modular data of the bosonic parent, which is much more accessible than the data for higher genus. (In particular, we only need the vacuum $S$-matrix, so the $F$-symbols are not required.)

As in the previous section, these results are definitely useful in their own right, since spin TQFTs are non-trivial examples of QFTs and therefore any progress towards their understanding is welcome. But, of course, these considerations become much more interesting when applied to other problems. We can in fact make contact with other chapters in this thesis. For example, by using our explicit construction of the Hilbert space of fermionic TQFTs, in chapter 1 we show that these spaces often realize projective representations of the symmetry group, signaling an 't Hooft anomaly for the symmetry, as reviewed in section 0.1. Similarly, in chapter 4 we determine the symmetries of abelian TQFTs, including fermionic theories. One aspect that was left unanswered in that chapter is how the symmetry group is extended by fermion parity, if at all, since this symmetry does not act on the anyons and hence cannot be detected by the methods in that chapter (cf. footnote 92 on p. 209). Using our refined understanding of fermionic TQFTs from this chapter, one could in principle answer this question, since the Hilbert space does in fact detect fermion parity (although we do not do this in this thesis). Finally, the strongest connection to other topics in this thesis is that with chapter 6. As we shall review momentarily, that chapter deals with four dimensional gauge theories and their domain walls. The Witten index of these walls is computed by counting states in the Hilbert space of certain three-dimensional fermionic TQFTs, a calculation that crucially utilizes the results of this present chapter.

## 0.6 Chapter 6.

The final chapter of this thesis concerns the concept of domain walls in $d = 3 + 1$ gauge theories. These walls are non-perturbative codimension-1 dynamical objects in a QFT, which generically support degrees of freedom in one lower dimension. Understanding the dynamics of these walls, and the localized degrees of freedom they host, is a strong-coupling problem which is in general very hard to address. The goal of chapter 6 is to try and say a few things about them in $d = 3+1$ dimensions, and make contact with dynamics of gauge theories in $d = 2+1$ dimensions as discussed above.

Consider a four-dimensional theory that has some discrete symmetry $\Gamma$ that is spontaneously broken. As such, the system has $|\Gamma|$ vacuum sectors, labeled by elements $g \in \Gamma$; assume further that all these vacua are trivial, i.e., that the system is gapped and the vacua do not carry topological degrees of freedom. In this situation, one can define boundary conditions such that the fields approach different vacua in opposite asymptotic directions, say at $z \to \pm\infty$. This configuration is topologically protected since there is no finite-energy process through which it could decay everywhere to the same vacuum. From very far away, i.e., at low energies, all the energy density of the configuration is confined to a codimension-1 region of spacetime, orthogonal to the $z$ direction. This defines a domain wall, that is, a dynamical surface that separates two vacuum states.

What are the dynamics of this surface? To leading order the surface will tend to minimize its tension, i.e., its area. In other words, the overall configuration of the surface is determined by the Nambu-Goto action, $S = -TA$. This center-of-mass mode can be interpreted as the Goldstone boson associated to the spontaneously broken translation symmetry along the $z$ direction caused by the wall. The hard question is, what is there beyond this mode. Specifically, what are the degrees of freedom that are localized on the wall? Note that, at low energies, these degrees of freedom define a purely $d = 2+1$ dimensional theory, since they do not have enough energy to leave the wall, and the vacua on both sides of the wall do not have topological degrees of freedom which could interact with the $d = 2+1$ dimensional dynamics. We would like to identify this $d = 2+1$ theory explicitly.

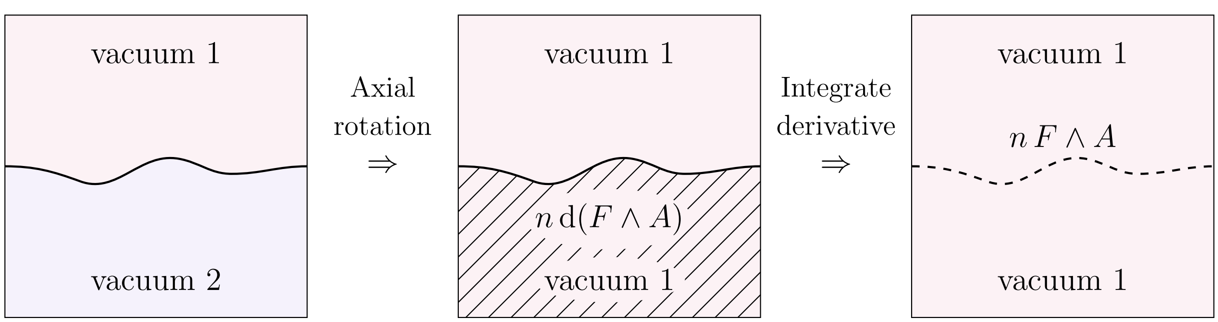
We can begin by doing some kinematics [40]. A general wall is specified by the two vacua it separates, say $\gamma_1, \gamma_2 \in \Gamma$. In fact, the wall only depends on the combination $\gamma_1\gamma_2^{-1}$, since we can use the affinely-realized symmetry $\Gamma$ to relabel the vacua. So the wall really only depends on a single group element, and we can denote the wall as $\mathrm{W}_\gamma$. Another simple fact is that the parity-reversal of the wall, denoted by $\overline{\mathrm{W}_\gamma}$, can be obtained by $z \to -z$, i.e., by interchanging the elements $\gamma_1, \gamma_2$. In other words, $\overline{\mathrm{W}_\gamma} \equiv \mathrm{W}_{\gamma^{-1}}$. So whatever the $3d$ theory on $\mathrm{W}_\gamma$ is, it must be such that, under time-reversal, it is mapped into $\mathrm{W}_{\gamma^{-1}}$. This is as much as we can say without invoking actual dynamical arguments.

In order to make progress, let us be more concrete. Consider a gauge theory with gauge group $G$ and massless quarks in some representation $R$. The case of fundamental quarks, $R = \square$, was analyzed in [41]; there, it is assumed that the time-reversal symmetry was spontaneously broken, leading two two vacua and one domain wall. The $3d$ degrees of freedom were identified. Here we consider, instead, the case of rank-2 quarks, i.e., the adjoint representation or the symmetric and anti-symmetric representations $R = \square\!\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}$. Generally speaking, we do not expect non-trivial infrared dynamics for higher-rank representations, since the beta function is only negative for finitely many values of $N$.

We assume that the gauge group $G$ is simply-connected, for otherwise the vacua will generically have topological degrees of freedom and domain walls are not well-defined as $3d$ theories. The $4d$ theory has an axial symmetry $\Gamma = \mathbb{Z}_{4T(R)}$, where $T(R)$ is the Dynkin index of $R$ (see appendix B for the definition of $T(R)$ and its value for several important representations); this symmetry is what remains of the classical axial U(1) symmetry $\psi \mapsto e^{i\alpha\gamma_\star}\psi$, which is explicitly broken – down to a discrete subgroup – in the quantum theory by a mixed anomaly with the gauge group.

We now make the following key assumption: the fermion bilinear $\psi^t\psi$ condenses. This is definitely true for the adjoint case [42–46], and this also implies the condensation for the other rank-2 representations for sufficiently large $N$, since all rank-2 theories are roughly the same to leading order in $1/N$ [47]. We assume that the condensation continues to hold for all finite $N$. The condensation of the fermion bilinear spontaneously breaks $\mathbb{Z}_{4T(R)} \to \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as $\psi \to -\psi$; this leads to $2T(R)$ discrete, trivial vacua, labelled by elements of $\mathbb{Z}_{4T(R)}/\mathbb{Z}_2 = \mathbb{Z}_{2T(R)}$. We thus have $2T(R)$ domain walls, labelled by this cyclic group.

What is the $d = 2+1$ theory that lives on these walls? We propose that the three-dimensional theory supported on the worldvolume of $\mathrm{W}_n$, where $n \in \mathbb{Z}_{2T(R)}$, is QCD with gauge group $G$ and fermions in the representation $R$, together with a Chern-Simons term at (bare) level $k_0 = n$. This value of the Chern-Simons level can be motivated as follows (see diagram below). The domain wall $\mathrm{W}_n$ is defined by the condition that the fermions have $n$ more units of phase on one side than the other; we can therefore perform an axial rotation on half space, with parameter $n$, to bring both sides to the same phase. Due to the axial anomaly, the phase rotation induces a theta term, which is a total derivative. On half space, the total derivative localizes to the boundary; but the theta term is in fact a derivative of the Chern-Simons interaction, $F \wedge F = \mathrm{d}(F \wedge A)$, and therefore the theta term with phase $n$ induces a Chern-Simons term with coefficient $n$ on the boundary, that is, on the domain wall. In other words, the configuration that defines $\mathrm{W}_n$ is equivalent to having the same vacuum everywhere, plus a Chern-Simons term localized on the wall.




The first step defines the domain wall as the interface between vacua 1 and 2. On this configuration (second step), we perform an axial rotation on the bottom region only, with $n$ units of phase, equal to the difference between vacua 2 and 1. This axial rotation generates a theta term, which is a total derivative and hence integrates to the boundary, namely the interface. This yields a configuration where we have the same vacuum everywhere, but with a Chern-Simons term on a surface (third step).

To summarize the discussion so far,

> A $d = 3+1$ gauge theory with simply-connected gauge group $G$ and fermions in the rank-2 representation $R$ has $2T(R)$ vacua labelled by $\mathbb{Z}_{2T(R)}$. The domain wall separating vacua differing by $n \in \mathbb{Z}_{2T(R)}$ units of phase supports a $d = 2+1$ dimensional gauge theory with gauge group $G$, fermions in the representation $R$, and a Chern-Simons term with bare coefficient $n$.

We often label the $3d$ gauge theory by the renormalized level, defined as the bare level minus $T(R)$. In this convention, the claim is that $\mathrm{W}_n \equiv G_{n-T(R)} + \psi_R$.

A simple consistency check of this proposal is the following. Recall that the kinematics of domain walls require that time-reversal acting on $\mathrm{W}_\gamma$ is equivalent to $\mathrm{W}_{\gamma^{-1}}$; here $\gamma = n$ and $\gamma^{-1} = 2T(R) - n$, as $n \in \mathbb{Z}_{2T(R)}$. So we better have $\overline{\mathrm{W}}_n \equiv \mathrm{W}_{2T(R)-n}$. This is indeed satisfied by the conjecture $\mathrm{W}_n \equiv G_{n-T(R)} + \psi_R$, inasmuch as time-reversal flips the sign of the (renormalized) Chern-Simons level, i.e., $\overline{G_k} \equiv G_{-k}$. So at this level of the analysis, the overall picture is seen to be self-consistent.

In chapter 3 we give some qualitative arguments that support the conjecture for the symmetric and anti-symmetric representations of the gauge group. In the case of the adjoint representation we can be much more explicit and give some quantitative evidence. The property that makes the adjoint case more tractable is that the theory is, in fact, supersymmetric – a gauge field plus an adjoint fermion define an $\mathcal{N} = 1$ vector multiplet. Being supersymmetric, there are some observables that can be computed exactly, which can then be used to test the proposal above.

The observable we shall use is the Witten index [42, 46]. This is defined as $I := \mathrm{tr}(-1)^F e^{-\beta H}$, and it has the nice property that it is invariant under smooth (supersymmetry preserving) deformations. Indeed, in a supersymmetric theory all the excited states are paired-up in bose-fermi doublets, and therefore they cancel out in $I$, as $(-1)^F$ weights them with opposite sign. Therefore, the index is sensitive to the vacua only. Furthermore, any reasonable deformation can only add or remove vacuum states in bose-fermi pairs, so the index does not change under it. All in all, the Witten index is an observable of supersymmetric theories that does not change under small deformations, and this makes the index computable, since we can often deform the theory into one that we know how to solve. One can also study twisted Witten indices, where one turns on fugacities for flavor symmetries of the theory.

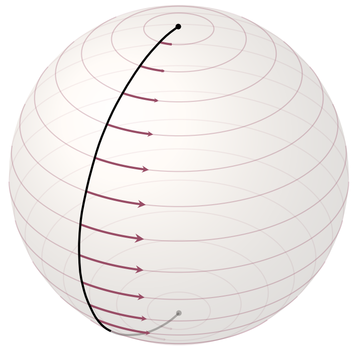
In chapter 6 we compute the (regular and twisted) Witten indices of the domain walls of adjoint QCD using both the $4d$ variables and the (conjectured) $3d$ variables. Consistency of the proposed $3d$ theory requires these two computations to agree. This matching also lends some evidence to the pure $3d$ dynamics discussed in the previous sections, since the computation of the Witten index requires knowledge of the infrared (the vacua) of these theories, which we previously conjectured to be certain TQFTs via alternative methods. So the agreement of the Witten index for the domain walls not only tests the proposal that the domain wall supports specific $3d$ QFTs, but it also tests the proposal that these QFTs flow, at large distances, to specific TQFTs.

The matching of Witten indices is our quantitative check on the domain wall proposal for adjoint quarks. In chapter 3 we also describe the domain walls in the case of symmetric and anti-symmetric quarks. These theories are not supersymmetric, and therefore we cannot compute any partition function reliably and check for matching. That being said, in some cases we can give some milder evidence that the proposal is correct. We close this introductory chapter with a rough description of the argument presented in chapter 3.

Consider a $4d$ gauge theory with group $\mathrm{SU}(N)$ and a quark in the (anti-)symmetric representation. The domain walls are labelled by $n \in \mathbb{Z}_{N\pm 2}$, with the upper sign referring to the symmetric representation and the lower sign to the anti-symmetric. The TQFT that we derive in chapter 3 for this gauge theory is $\mathrm{U}(n)_{N\pm 2-n, N\pm 2-2n}$. The most interesting wall is the time-reversal invariant one, $n = (N \pm 2)/2$, since for this value of $n$ the TQFT becomes $\mathrm{U}(n)_{n,0} \equiv \mathrm{PSU}(n)_n \times \mathrm{U}(1)_0$. The factor of $\mathrm{U}(1)_0$ gives a $\mathrm{U}(1)$ gauge field with zero topological mass, which in three-dimensions is equivalent (by Hodge duality) to a compact scalar, namely a free, massless, circle-valued scalar field. This massless field is to be thought of as the Goldstone mode of the $\mathrm{U}(1)$ baryon symmetry. Hence, in this case the TQFT predicts that, while the baryon symmetry is unbroken in the bulk, it breaks on the wall!

As a matter of fact, there is a situation where this spontaneous breaking can be seen explicitly, directly in the $4d$ theory. In general, a semi-classical computation will not see a topological sector such as $\mathrm{PSU}(n)_n$, which means that we can only hope to obtain a reliable semi-classical description if this topological sector is trivial. This TQFT is only trivial for $n = 2$, since $\mathrm{PSU}(2)_2 \equiv \mathrm{SO}(3)_1$ is an SPT. Recalling that $n = (N \pm 2)/2$, we are invited to look more carefully at the theories $\mathrm{SU}(2) + \square\!\square$ and $\mathrm{SU}(4) + \begin{smallmatrix}\square\\\square\end{smallmatrix}$. We claim that in these two cases, the compact scalar $\mathrm{U}(1)_0$ is visible on the wall in a reliable effective analysis.

What makes the two theories $\mathrm{SU}(2) + \square\!\square$ and $\mathrm{SU}(4) + \begin{smallmatrix}\square\\\square\end{smallmatrix}$ special? The answer is that, in these two cases, the flavor symmetry is enhanced, since $\mathrm{SU}(2) = \mathrm{Spin}(3)$ and $\mathrm{SU}(4) = \mathrm{Spin}(6)$, and $\square\!\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}$ are both equivalent to *two* copies of the fundamental representation. Hence, for these two values of $N$, the flavor symmetry is $\mathrm{SU}(2)$ instead of $\mathrm{U}(1)$. Importantly, a mass term preserves $\mathrm{U}(1)$ but not $\mathrm{SU}(2)$, and therefore the former cannot break spontaneously while the latter may. We assume that, indeed, $\mathrm{SU}(2)$ breaks to $\mathrm{U}(1)$, which leads to a vacuum manifold isomorphic to a sphere, $\mathbb{S}^2 = \mathrm{SU}(2)/\mathrm{U}(1)$. In this description, the two vacua separated by the $n = 2$ domain wall are just two antipodal points on the sphere, and the domain wall itself is a geodesic connecting them. Importantly, this geodesic is not unique: all half circles connecting antipodal points have the same length, and thus there is an $\mathbb{S}^1$-worth of possible geodesics. This means that $\mathrm{U}(1)$ is spontaneously broken, the compact scalar corresponding to motion along the azimuthal direction:



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# Chapter 1

# Global Anomalies on the Hilbert Space.

**Authorship.** The content of this chapter is taken almost verbatim from the paper [1], written in collaboration with Davide Gaiotto and Jaume Gomis.

**Abstract.** We show that certain global anomalies can be detected in an elementary fashion by analyzing the way the symmetry algebra is realized on the torus Hilbert space of the anomalous theory. Distinct anomalous behaviours imprinted in the Hilbert space are identified with the distinct cohomology "layers" that appear in the classification of anomalies in terms of cobordism groups. We illustrate the manifestation of the layers in the Hilbert for a variety of anomalous symmetries and spacetime dimensions, including time-reversal symmetry, and both in systems of fermions and in anomalous topological quantum field theories (TQFTs) in $2+1d$. We argue that anomalies can imply an exact bose-fermi degeneracy in the Hilbert space, thus revealing a supersymmetric spectrum of states; we provide a sharp characterization of when this phenomenon occurs and give nontrivial examples in various dimensions, including in strongly coupled QFTs.

**Acknowledgments.** We would like to thank Theo Johnson-Freyd, Nathan Seiberg, Ryan Thorngren, and Alex Turzillo for useful discussions.

## 1.1 Introduction and Summary

Consider a system with a classical global symmetry group $G$. Powerful constraints on the dynamics can be derived by coupling the system to a background connection $A$ for the symmetry $G$. The system has an 't Hooft anomaly [7] if the non-invariance of the partition function under background gauge transformations generated by $g \in G$

$$Z[A] \mapsto e^{i\alpha(g,A)} Z[A] \tag{1.1.1}$$

cannot be cancelled by a local counterterm constructed out of the background fields. This is the physics that endows anomalies with a cohomological formulation [6, 48, 49].

The anomaly $\alpha(g, A)$ is a local functional of the background connection and of the transformation $g \in G$. An 't Hooft anomaly is captured via anomaly inflow [50] from a topological term in one dimension higher. Each such topological term can be thought of as the effective action characterizing a symmetry protected topological (SPT) phase [51–53] with symmetry $G$ in one higher dimension. The topological term is gauge invariant on a closed manifold and reproduces the anomaly on a manifold with a boundary. Being topological, an 't Hooft anomaly is robust under deformations that preserve the symmetry, including renormalization group transformations. 't Hooft anomalies give physicists some of the very few clues into the nonpertubative dynamics of a quantum system.

A combination of insights from condensed matter physics, particle physics, quantum information and mathematics has culminated in a conjecturally complete answer to the problem of classifying the possible anomalies in various dimensions [52, 54–60]. This includes anomalies in bosonic as well as fermionic systems, for discrete and continuous internal symmetry groups as well as discrete spacetime symmetries such as time-reversal and parity.[11] This has led to the topological classification of anomalies in terms of cobordism theory and generalized cohomology theories [11, 36, 54, 56, 58–60, 65–70].

Consider first a bosonic system, one which can be defined without a choice of spin structure of the underlying manifold. By Wigner's theorem, symmetries come in two flavours: linear and unitary, or antilinear and antiunitary, with time-reversal being the prototypical example of an antiunitary symmetry. Thus, the symmetry data of a bosonic system is specified by the pair

$$(G, w_1)\,, \tag{1.1.2}$$

where $G$ is a group and $w_1 \in H^1(G, \mathbb{Z}_2)$ a certain cohomology class $w_1 \colon G \to \mathbb{Z}_2$ that encodes the unitarity/antiunitarity of the group elements in $G$. The anomalies of a bosonic system with symmetry data $(G, w_1)$ in $D$ spacetime dimensions are classified by the twisted cobordism group [58]

$$\Omega_{\mathrm{so}}^{D+1}(G; w_1)\,. \tag{1.1.3}$$

[11] Coupling to a time-reversal background requires defining the system on unoriented manifolds [58, 61–64].

In low spacetime dimensions, for $D \leq 2$, the anomaly classification reduces to group cohomology: $\Omega_{\text{so}}^{D+1}(G; w_1) = H^{D+1}(G, \mathrm{U}(1))$, extending the classic result that anomalies in quantum mechanics (i.e. $D = 1$) are classified by $H^2(G, \mathrm{U}(1))$, that is, by the projective representations of $G$.[12] In higher dimensions, $\Omega_{\text{so}}^{D+1}(G; w_1)$ can be reconstructed (losing some information about the addition law) from the Atiyah-Hirzebruch spectral sequence [10], that combines $H^{D+1}(G, \mathrm{U}(1))$ with other cohomology groups of lower degrees.

Now recall the characterization of symmetries and classification anomalies of a fermionic system, which requires the choice of a ($G$-twisted) spin structure to be defined. A fermionic system has a universal and unbreakable $\mathbb{Z}_2^F$ unitary symmetry generated by fermion parity, denoted by $(-1)^F$. This symmetry induces a $\mathbb{Z}_2$-grading in the Hilbert space $\mathcal{H}$ of fermionic systems, which become super-vector spaces. Since (classically) symmetries cannot change the fermion parity, that is $[g, (-1)^F] = 0$, the symmetry group $G_f$ acting on the local operators of a fermionic system is necessarily a $\mathbb{Z}_2^F$ central extension of a group $G$, such that $G = G_f/\mathbb{Z}_2^F$. Also, by virtue of Wigner's theorem, a symmetry can be either unitary or antiunitary. Therefore, the symmetries of a fermionic system are characterized by a cocycle $w_2 \in H^2(G, \mathbb{Z}_2^F)$ specifying the $\mathbb{Z}_2^F$ central extension and by a cocycle $w_1 \in H^1(G, \mathbb{Z}_2)$ encoding the unitarity/antiunitarity of group elements. The anomalies of a fermionic system with symmetry data[13]

$$(G; w_1, w_2) \tag{1.1.4}$$

in $D$ spacetime dimensions are classified by the twisted cobordism group [11, 59]

$$\Omega_{\text{spin}}^{D+1}(G; w_1, w_2)\,. \tag{1.1.5}$$

State-of-the-art mathematical techniques allow for the computation of these twisted cobordism groups; see [9, 71–73] for many relevant examples together with reviews aimed at physicists. A particularly convenient computational tool is again the Atiyah-Hirzebruch spectral sequence. The different ingredients that go into the computation of (1.1.5) in this spectral sequence can be given a nice physical interpretation in terms of layers in various dimensions (see below).

While the topological classification of anomalies is rather well understood, *detecting* whether a physical system is anomalous can be a difficult task. Intuitively, one has to keep track of all the arbitrary choices required for a sharp definition of the system on a general background and then quantify the topological obstruction to the trivialization of these choices. A concrete calculation may involve hard-to-determine data characterizing the system.[14] While

[12]When $w_1$ is nontrivial the cocycle condition defining $H^*(G, K)$ is twisted by the action of $w_1$, which acts as an involution on $K$. This action is nontrivial for $K = U(1)$ and $K = \mathbb{Z}$ but trivial for $K = \mathbb{Z}_2$. In order to avoid clutter we do not write the twisting by $w_1$ explicitly.

[13]For example, for $G = \mathbb{Z}_2$, and taking $w_1$ and $w_2$ to be the nontrivial $\mathbb{Z}_2 = \{0, 1\}$ element in $H^1(\mathbb{Z}_2, \mathbb{Z}_2)$ and $H^2(\mathbb{Z}_2, \mathbb{Z}_2^F)$ yields the symmetry group generated by time-reversal $\mathsf{T}$ obeying $\mathsf{T}^2 = (-1)^F$, sometimes denoted by $\mathbb{Z}_4^{\mathsf{T}}$. This is the relevant symmetry group of the celebrated topological superconductors.

[14]For example, detecting anomalies in bosonic topological quantum field theories (TQFTs) requires knowing the $F$-symbols [74].

detecting the anomalies induced by transformations connected to the identity of a Lie group $G$ is textbook material, the detection of *global* anomalies, which includes anomalies for all discrete symmetries, is more subtle [75, 76].[15] The approach is often indirect, for example by embedding some global anomalies into perturbative ones (see for example the recent work [85] and references therein).

In this paper we exhibit an elementary method for detecting *some* anomalies, based on constructing the Hilbert space of the theory on a flat (spatial) torus $T^{D-1}$ as well as determining how the algebra of symmetries is realized on the Hilbert space. This can be given the following physical interpretation. The anomaly of a $D$-dimensional theory can be represented by the class

$$\alpha_{D+1} \in \Omega^{D+1}_{\text{spin}}(G; w_1, w_2)\,. \tag{1.1.6}$$

Studying the Hilbert space of the $D$-dimensional anomalous theory on a spatial torus[16] produces upon integration a class

$$\tilde{\alpha}_2 = \int_{T^{D-1}} \alpha_{D+1}\,. \tag{1.1.7}$$

The class $\tilde{\alpha}_2$ can be viewed as the effective anomaly class of a quantum mechanical theory in $0+1d$, which we recognize from the properties of the Hilbert space. As a result, we expect to be able to detect this way all anomalies whose cobordism class can be recognized from the values on manifolds of the form $T^{D-1} \times \Sigma_2$, equipped with generic flat connections, spin structures, etc.

This perspective also shows that a torus compactification can provide useful anomaly information only if the relevant structures – either the background $G$ connection or spin structure – do not extend to one higher dimension (i.e. if they are not the boundary of a manifold in one higher dimension). Indeed, if these structures were all bounding such that $T^{D-1} = \partial M^D$, then $\int_{T^{D-1}} \alpha_{D+1} = \int_{M^D} d\alpha_{D+1} = 0$, and the effective anomaly in $0+1d$ vanishes. This means that in order to detect the anomaly in the torus Hilbert space we must either turn on non-trivial holonomies for the symmetry $G$ or we must consider periodic boundary conditions on the torus for fermionic theories – or both.[17]

In practice, we find that this method captures a surprisingly large amount of anomaly information. This is especially true for fermionic systems.

In order to illustrate how various anomalies are manifested in the Hilbert space, it is useful to recall some ingredients of the (partial) reconstruction of $\Omega^{D+1}_{\text{spin}}(G; w_1, w_2)$ via the

[15] In the case of antiunitary symmetries it requires, for example, learning how to define spin TQFTs on unoriented manifolds, which is an open problem. Numerous interesting partial results have been obtained, however [63, 77–84].

[16] One could also study the reduction of the anomaly class on more general manifolds, potentially detecting more anomalies.

[17] Turning on non-trivial holonomies for the symmetry $G$ defines a $G$-twisted Hilbert spaces, which are the Hilbert spaces where to detect anomalies if the spin structures are bounding.

Atiyah-Hirzebruch spectral sequence. The starting point is a collection of layers in various degrees[18] (see e.g. [11, 66, 67])

$$\begin{array}{rll} & \vdots & \\ \nu_{D-2} \in H^{D-2}(G,\mathbb{Z}) & & p_x + ip_y \text{ layer} \\ \nu_{D-1} \in H^{D-1}(G,\mathbb{Z}_2) & & \text{Arf layer} \\ \nu_{D} \in H^{D}(G,\mathbb{Z}_2) & & \psi \text{ layer} \\ \nu_{D+1} \in H^{D+1}(G,\mathrm{U}(1)) & & \text{Bosonic layer} \end{array} \tag{1.1.8}$$

with nontrivial differentials connecting the various classes. Each layer has a physical and geometric interpretation (see section 1.2 for more details). In particular, the groups which appear in the second slot of $H^{D-k}(G,\,\cdot\,)$ are the groups of $k$-dimensional SPT phases with no symmetries. We summarize them in table 1.1.

| | $0+1d$ | $1+1d$ | $2+1d$ |
|---|---|---|---|
| SPTs | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| generator | $\psi$ | Arf | $p_x + ip_y$ aka $\mathrm{SO}(1)_1$ |
| $Z$ | $S_\pm \mapsto \pm 1$ | $\Sigma \mapsto (-1)^{\mathrm{Arf}(\Sigma)}$ | $M_3 \mapsto e^{i\mathrm{CS}_{\mathrm{grav}}[M_3]}$ |

Table 1.1: The first row gives the classification of SPT phases with no symmetries, the second the generators of the SPT classes, and the last the partition functions of the generators. $S_\pm$ denotes a circle with periodic/antiperiodic (R/NS) boundary conditions; $\Sigma$ is a compact Riemann surface, and $\mathrm{Arf}(\Sigma)$ is the Arf-invariant of the surface with spin structure, which evaluates to 0 on even and to 1 on odd spin structures; and $M_3$ is a three-manifold, with $\mathrm{CS}_{\mathrm{grav}} = \frac{1}{4\pi}\frac{1}{48}\int_{M_3} \mathrm{tr}(\omega\, \mathrm{d}\omega + \frac{2}{3}\omega^3)$.

The endpoint of the Atiyah-Hirzebruch spectral sequence calculation is the *associated graded* of a filtration of $\Omega^{D+1}_{\mathrm{spin}}(G; w_1, w_2)$: the addition law on the $k$-th layer is modified by unknown carry-overs from lower layers, which are somewhat tricky to compute. Physically, that means that even if the non-trivial differentials vanish, we can only really assign a specific value to $\nu_{D-k}$ if all $\nu_{D-k'}$ with $k' > k$ vanish, or we can only discuss the difference in the $\nu_{D-k}$ anomaly of two theories for which all $\nu_{D-k'}$ with $k' > k$ are the same.

We now demonstrate the Hilbert space manifestation of the layers in the anomalies of $0+1d$ fermionic systems with an antiunitary time-reversal symmetry $\mathsf{T}$ with $\mathsf{T}^2 = 1$, so that $G_f = \mathbb{Z}_2^{\mathsf{T}} \times \mathbb{Z}_2^F$. The anomalies of such a system are classified by $\Omega^2_{\mathrm{spin}}(\mathbb{Z}_2; 1, 0) = \Omega^2_{\mathrm{pin}^-} = \mathbb{Z}_8$

[18] Recall that the action of $w_1$ is trivial on $\mathbb{Z}_2$ coefficients.

[86]. The anomaly arises from three layers

$$\begin{aligned} &\nu_0 \in H^0(\mathbb{Z}_2^{\mathsf{T}}, \mathbb{Z}_2) \simeq \mathbb{Z}_2 && \text{Arf layer} \\ &\nu_1 \in H^1(\mathbb{Z}_2^{\mathsf{T}}, \mathbb{Z}_2) \simeq \mathbb{Z}_2 && \psi \text{ layer} \\ &\nu_2 \in H^2(\mathbb{Z}_2^{\mathsf{T}}, \mathrm{U}(1)) \simeq \mathbb{Z}_2 && \text{Bosonic layer} \end{aligned} \tag{1.1.9}$$

We can think about these three groups as compiling into $\mathbb{Z}_8$, corresponding to the binary expansion

$$\nu = \nu_0 + 2\nu_1 + 4\nu_2 \quad \mod 8\,. \tag{1.1.10}$$

The simplest $0+1d$ system with anomaly $\nu \in \mathbb{Z}_8$ is a set of $\nu$ free massless Majorana fermions, with time-reversal acting as $\mathsf{T}(\psi(t)) = \psi(-t)$ on all $\nu$ fermions. At the level of operators, this system has symmetries generated by $\mathsf{T}$ and $(-1)^F$, these two operations commuting and being both of order two, i.e., $G_f = \mathbb{Z}_2^{\mathsf{T}} \times \mathbb{Z}_2^F$. At the level of the Hilbert space, the anomaly $\nu$ is manifested through the following anomalous pattern:

- $\underline{\nu = 1 \mod 2}$: There is no graded Hilbert space $\mathcal{H}$. This arises from the Arf layer $\nu_0$.
- $\underline{\nu = 2 \mod 4}$: There is a graded Hilbert space $\mathcal{H}$ but the symmetry generators on $\mathcal{H}$ do not commute. Instead, they anti-commute:

  $$\{\mathsf{T}, (-1)^F\} = 0\,. \tag{1.1.11}$$

  This arises from the fermion $\psi$ layer $\nu_1$.
- $\underline{\nu = 4 \mod 8}$: There is a graded Hilbert space $\mathcal{H}$ with $[\mathsf{T}, (-1)^F] = 0$ on it, but the symmetry algebra $\mathsf{T}^2 = 1$ is realized projectively on $\mathcal{H}$, that is

  $$\mathsf{T}^2 = -1 \qquad \text{on} \qquad \mathcal{H}\,. \tag{1.1.12}$$

  This arises from the bosonic layer $\nu_2$.

As we compactify higher-dimensional systems on tori, we will use this characterization to recognize the image of various anomalies.[19]

Let us explain our approach to detecting anomalies in the celebrated example of topological superconductors. Consider the anomalies of $2+1d$ fermionic systems with time-reversal symmetry $\mathsf{T}$ obeying $\mathsf{T}^2 = (-1)^F$. The symmetry group is $G_f = \mathbb{Z}_4^{\mathsf{T}}$, with $G = G_f/\mathbb{Z}_2^F = \mathbb{Z}_2^{\mathsf{T}}$ and the symmetry is twisted by the nontrivial $\mathbb{Z}_2$ classes $w_1$ and $w_2$ in $H^1(\mathbb{Z}_2^{\mathsf{T}}, \mathbb{Z}_2)$ and

[19] An elegant instance of this general idea is Witten's SU(2) anomaly [75], which is described by a cobordism class $\eta \wedge c_2(F)$ [73], that when integrated over a four sphere with background gauge fields with minimal instanton number, yields the SPT class $\eta$ in $0+1d$ with no symmetries, which describes the $\psi$-phase (see table 1.1). Therefore the SU(2) global anomaly is detected as an anomaly in $(-1)^F$ due to a fermion zero mode in the instanton background and arises from the $\psi$-layer.

$H^2(\mathbb{Z}_2^{\mathsf{T}}, \mathbb{Z}_2^F)$. The anomalies are classified by $\Omega^4_{\text{spin}}(\mathbb{Z}_2, 1, 1) = \Omega^4_{\text{pin}^+} = \mathbb{Z}_{16}$ [59, 87–89]. By anomaly inflow, this is the same as the classification of topological superconductors in $3+1d$. The anomalies are constructed from the following layers

$$\begin{aligned}
&\nu_1 \in H^1(\mathbb{Z}_2^{\mathsf{T}}, \mathbb{Z}) \simeq \mathbb{Z}_2 && p_x + ip_y \text{ layer} \\
&\nu_2 \in H^2(\mathbb{Z}_2^{\mathsf{T}}, \mathbb{Z}_2) \simeq \mathbb{Z}_2 && \text{Arf layer} \\
&\nu_3 \in H^3(\mathbb{Z}_2^{\mathsf{T}}, \mathbb{Z}_2)) \simeq \mathbb{Z}_2 && \psi \text{ layer} \\
&\nu_4 \in H^4(\mathbb{Z}_2^{\mathsf{T}}, \mathrm{U}(1)) \simeq \mathbb{Z}_2 && \text{Bosonic layer}
\end{aligned} \tag{1.1.13}$$

These four groups compile into $\mathbb{Z}_{16}$, corresponding to the binary expansion

$$\nu = \nu_1 + 2\nu_2 + 4\nu_3 + 8\nu_4 \mod 16\,. \tag{1.1.14}$$

Anomalies $\nu \in \mathbb{Z}_{16}$ can be detected by studying the Hilbert spaces $\mathcal{H}_{XY}$ of the theory on the two-torus $T^2$, which depend on the choice of spin structure on $T^2$, where $X, Y \in \{\text{NS}, \text{R}\}$. This gives rise to the Hilbert spaces associated to even spin structures $\mathcal{H}_{\text{NS-NS}}, \mathcal{H}_{\text{NS-R}}, \mathcal{H}_{\text{R-NS}}$, and to the odd spin structure $\mathcal{H}_{\text{R-R}}$. As explained above, anomalies can only appear in $\mathcal{H}_{\text{R-R}}$, as the other three spin structures are bounding.

The following anomalies can be detected on the Hilbert space, as we show in both the study of spin TQFTs and fermions in $2+1d$:

- $\underline{\nu = 1 \mod 2}$: In $\mathcal{H}_{\text{R-R}}$ the classically $(-1)^F$-even time-reversal symmetry generator $\mathsf{T}$ becomes $(-1)^F$-odd, thus changing the parity of the states in $\mathcal{H}_{\text{R-R}}$. This corresponds to $\mathsf{T}$ anticommuting with $(-1)^F$ instead of commuting in $\mathcal{H}_{\text{R-R}}$:

  $$\{\mathsf{T}, (-1)^F\} = 0\,. \tag{1.1.15}$$

  This anomalous behaviour is associated with the $p_x + ip_y$ layer in (1.1.13). For $\nu$ even $[\mathsf{T}, (-1)^F] = 0$ on $\mathcal{H}_{XY}$.

- $\underline{\nu = 2 \mod 4}$: The $\mathbb{Z}_4^{\mathsf{T}}$ symmetry algebra on $\mathcal{H}_{XY}$ is

  $$\mathsf{T}^2 = (-1)^F \times (-1)^{\text{Arf}(T^2)} \qquad \text{on} \qquad \mathcal{H}_{XY}\,. \tag{1.1.16}$$

  The symmetry algebra is undeformed on the Hilbert spaces with even spin structure and deformed in the Hilbert space with odd spin structure. This anomalous behaviour is associated with the Arf layer in (1.1.13). For $\nu = 0 \mod 4$, $\mathsf{T}^2 = (-1)^F$ on $\mathcal{H}_{XY}$.

- The next two layers $\nu_3$ and $\nu_4$, corresponding to $\nu = 4 \mod 8$ and $\nu = 8 \mod 16$, are not visible on the torus Hilbert space and require other observables to detect them.

The analysis of anomalies for time-reversal symmetry $\mathsf{T}^2 = (-1)^F$ in the Hilbert space of spin TQFTs [35] requires constructing $\mathcal{H}_{XY}$ in the first place, and also learning how to compute the action of the operators (Wilson lines) on $\mathcal{H}_{XY}$. A systematic construction of the Hilbert space of fermionic TQFTs is presented in chapter 5, although we also quickly summarize the essential points in this chapter in section 1.4.

Another interesting example where the anomaly layers can be detected on the Hilbert space is in $1+1d$ fermionic systems with a unitary $\mathbb{Z}_2$ symmetry[20]. The overall symmetry is $G_f = \mathbb{Z}_2 \times \mathbb{Z}_2^F$ and the anomalies are classified by $\Omega^3_{\text{spin}}(\mathbb{Z}_2, 0, 0) = \mathbb{Z}_8$ [59, 66, 92, 93], constructed from the layers

$$\begin{aligned} &\nu_1 \in H^1(\mathbb{Z}_2, \mathbb{Z}_2) \simeq \mathbb{Z}_2 && \text{Arf layer} \\ &\nu_2 \in H^2(\mathbb{Z}_2, \mathbb{Z}_2) \simeq \mathbb{Z}_2 && \psi \text{ layer} \\ &\nu_3 \in H^3(\mathbb{Z}_2, \mathrm{U}(1)) \simeq \mathbb{Z}_2 && \text{Bosonic layer} \end{aligned} \tag{1.1.17}$$

These three groups compile into $\mathbb{Z}_8$, corresponding to the binary expansion

$$\nu = \nu_1 + 2\nu_2 + 4\nu_3 \qquad \text{mod } 8\,. \tag{1.1.18}$$

The simplest example of a $1+1d$ theory with symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2^F$ that realizes the $\nu \in \mathbb{Z}_8$ anomaly is a system of $\nu$ Majorana fermions. The generator of $\mathbb{Z}_2$ is the chiral symmetry $g = (-1)^{F_L}$, which acts trivially on the right-moving fermions and negates the left-moving fermions.

The anomaly $\nu \in \mathbb{Z}_8$ can be detected by studying the untwisted Hilbert space $\mathcal{H}_X$ and the $\mathbb{Z}_2$-twisted Hilbert space $\mathcal{H}_X^g$ of the $\nu$ Majorana fermions, where $X \in \{\text{NS}, \text{R}\}$ labels the spin structure on the (spatial) circle. We observe the following pattern:

- $\underline{\nu = 1 \mod 2:}$ The theory does not have proper graded twisted Hilbert spaces $\mathcal{H}^g_{\text{NS}}$ and $\mathcal{H}^g_{\text{R}}$.

  Also, while the untwisted Hilbert spaces $\mathcal{H}_X$ are well-defined, $(-1)^{F_L}$ and $(-1)^F$ do not commute on $\mathcal{H}_{\text{R}}$
  $$\{(-1)^{F_L}, (-1)^F\} = 0 \qquad \text{on } \mathcal{H}_R\,. \tag{1.1.19}$$
  For $\nu$ even $\mathcal{H}_X$ and $\mathcal{H}_X^g$ are properly graded and $[(-1)^{F_L}, (-1)^F] = 0$.

- The $\nu = 2 \mod 4$ and $\nu = 4 \mod 8$ layers are not visible on the Hilbert space as an anomalous realization of symmetry or a projective representation. Indeed reducing the $H^3$ class in (1.1.17) on the circle produces a trivial class in $H^2(\mathbb{Z}_2, \mathrm{U}(1))$, signaling that there are no nontrivial projective representations of $\mathbb{Z}_2$ in $\mathcal{H}_X$ or $\mathcal{H}_X^g$. We note, however, that the anomaly can be detected by measuring the spin of states in the twisted Hilbert spaces $\mathcal{H}_X^g$ (see e.g. [94] for a similar discussion for bosonic systems). In an anomalous theory this spin has a fractional part, which means that the rotation symmetry is realized projectively.

[20]This symmetry is related to the two previous examples via the Smith map [90, 91]. The results below can also be derived using this perspective.

An interesting application of these results is the following. As explained above, some anomalies imply that the symmetry generator is fermion-odd in the Hilbert space $\mathcal{H}$ with the appropriate (non-bounding) structure: the operator that implements the symmetry anticommutes with $(-1)^F$ instead of commuting. This immediately implies that the spectrum of the theory is supersymmetric, namely for any state in $\mathcal{H}$ there is a partner with the same energy and with opposite fermion parity. This property of the theory is rather surprising: the bose-fermi degeneracy is a consequence of an anomaly instead of a conventional supersymmetry. This provides a unified perspective on several observations in the literature, and it leads to generalizations and new predictions:

- Any $0+1d$ theory with an antiunitary $\mathbb{Z}_2^{\mathsf{T}}$ symmetry and an odd number of Dirac fermions has exact bose-fermi degeneracy. The supersymmetric spectrum of an odd number of free Dirac fermions was described in [95], and has been studied more recently in [96–98]. From our perspective, any theory with a $\mathbb{Z}_2^{\mathsf{T}}$ anomaly $\nu = 2 \mod 4$ has a supersymmetric spectrum.

- Any $1+1d$ theory with a unitary chiral $\mathbb{Z}_2$ symmetry and an odd number of Majorana fermions has exact bose-fermi degeneracy in the untwisted Ramond Hilbert space $\mathcal{H}_{\mathrm{R}}$. This includes the supersymmetric spectrum of $\mathrm{SU}(N)$ adjoint QCD with $N$ even in $1+1d$ recently discussed in [99] (the spectrum is supersymmetric in spite of the fact that Lagrangian of adjoint QCD is not supersymmetric). The fact that a $\mathbb{Z}_2$-symmetric theory with an odd number of Majorana fermions has $\{(-1)^{F_L}, (-1)^F\} = 0$ in $\mathcal{H}_{\mathrm{R}}$ implies that any such a theory will have a supersymmetric spectrum. This includes examples in Yang-Mills with $\mathrm{Spin}(N)$ gauge group, e.g. in the fundamental representation for $N$ odd and in the traceless symmetric representation for $N = 0, 3 \mod 4$. It would be interesting to exhibit this bose-fermi degeneracy explicitly.

- Any $2+1d$ theory with antiunitary $\mathbb{Z}_4^{\mathsf{T}}$ symmetry and an odd number of Majorana fermions has exact bose-fermi degeneracy in the odd spin structure Hilbert space $\mathcal{H}_{\text{R-R}}$. This is a nontrivial prediction for the spectrum of gauge theories in $2+1d$ theories, which are strongly coupled in the infrared. An instance of a theory that should have a supersymmetric spectrum is $\mathrm{SO}(N)$ gauge theory (with vanishing Chern-Simons coupling) with a fermion in the traceless symmetric representation. Time-reversal invariance requires that $N$ is even, and $\nu$ odd further requires that $N = 0 \mod 4$. While the Lagrangian of this theory is not supersymmetric, the anomaly implies that the spectrum is nonetheless supersymmetric. We can provide nontrivial evidence for this claim. In [30] the infrared dynamics of this theory was proposed to be captured by $\mathrm{SO}(\frac{N+2}{2})_{\frac{N+2}{2}}$ Chern-Simons theory. Using the formulae in [5] for the number of bosonic and fermionic states in $\mathcal{H}_{\text{R-R}}$ of this Chern-Simons theory[21] we find that the spectrum

[21] In [5] it was shown that $\mathrm{SO}(2n+1)_{2k+1}$ Chern-Simons theory has $\binom{n+k-1}{k-1}$ bosons and $\binom{n+k-1}{k}$ fermions

is indeed supersymmetric! Our argument applies to other gauge theories with higher rank real representations and is a nontrivial prediction of their spectrum.

The next layer, measuring the projectivity of the symmetry algebra on $\mathcal{H}$, also has nontrivial implications, the most famous being Kramers theorem. From our analysis one can conclude that any theory in $2+1d$ with $\mathbb{Z}_4^{\mathsf{T}}$ symmetry and anomaly $\nu = 2 \mod 4$ has (at least) two-fold degeneracy in the fermionic part of the even-spin-structure Hilbert spaces, and in the bosonic part of the odd-spin-structure Hilbert space. When $\nu = 0 \mod 4$ there is (at least) two-fold degeneracy for all the fermionic states, in any of the spin structures.

Finally, we should stress that our analysis may not yet capture all the information about anomalies which is encoded in the torus Hilbert spaces. Isometries of the internal space will act on the Hilbert space of a compactified theory. As a result, one could study anomalies for the combination of the original symmetries and the new internal symmetries of the compactified system. We leave this to future work.

The plan for the rest of the chapter is as follows. In section 1.3 we study free fermions in various dimensions and illustrate how anomalies manifest themselves at the level of their Hilbert space. We consider antiunitary time-reversal symmetry in $0+1$ and $2+1$ dimensions, where the algebra is $\mathsf{T}^2 = 1$ and $\mathsf{T}^2 = (-1)^F$, respectively; and we also consider unitary chiral symmetry in $1+1$ dimensions with algebra $g^2 = 1$. After that, in section 1.4 we consider the same problem in $2+1d$ spin TQFTs. We study how their anomalies are seen by constructing their Hilbert spaces. Here we revisit the algebra $\mathsf{T}^2 = (-1)^F$ and find the same behaviour as in the case of free fermions.

## 1.2 Anomalies from Layers

The classification of SPT phases in terms of generalized cohomology/cobordism and associated "layers" is somewhat forbidding, but has a rather transparent physical meaning. Ultimately, we want to have a procedure to associate a partition function to a manifold equipped with appropriate structures. First, we can triangulate the manifold, equipping it with a discretization of the various structures we want to endow it with: a flat connection along the edges of the triangulation, some discrete version of the spin structure and orientation, etc. Next, we can take the cell decomposition $C$ dual to the triangulation, and place on the facets of $C$ some collection of invertible TFTs (meaning here SPTs with no symmetries) of appropriate dimension, following some rules which take into account the discrete data we put on the manifold. The partition function is then defined as the partition function of the collection of invertible TFTs.

in the Hilbert space $\mathcal{H}_{\text{R-R}}$. The spectrum is supersymmetric for $n = k$, when the theory is time-reversal invariant.

The "layers" of the cohomology theory are simply a way to encode which rules we use to place invertible TFTs on facets. The differential in the generalized cohomology theory imposes the constraint that the final answer should be independent of the choice of triangulation as well as any other choices made at intermediate steps of the construction. It also identifies pairs of rules which give the same final answer.

As an example, consider orientable, spin SPTs for unitary symmetries. A discretized $G$ flat connection is given as a collection of $G$ elements on the edges of the triangulation.

1. If we were to include only the bottom layer, we would leave all facets bare and only focus on vertices of $C$. At each vertex we place some complex phase (aka elements of $\mathrm{U}(1)$) determined by the group elements along the edges of the dual simplex. This is literally the cochain $\nu_{D+1}$ representing an element in the group cohomology $H^{D+1}(G, \mathrm{U}(1))$. The cocycle condition ensures that the partition function defined as the product of all the phases is independent of the choice of triangulation and gauge. Coboundaries give partition functions which evaluate to 1 in a trivial manner.

2. Following [54], the next refinement of the story involves placing a fermionic one-dimensional Hilbert space along some of the edges of $C$. The choice is the cochain $\nu_D$ representing an element in the group cohomology $H^D(G, \mathbb{Z}_2)$. The cocycle condition ensures that each vertex is connected to an even number of fermionic edges. At each vertex we now get to pick a vector in the (one-dimensional, Grassmann even) tensor product of these vector spaces. This is roughly the same as a choice of $\nu_{D+1}$, but not canonically, because of sign ambiguities in the tensor product. The Grassmann combinatorics needed to rearrange the tensor products when contracting states at the endpoints of fermionic edges, as well as the (spin structure dependent) signs arising from fermion loops contribute to the overall sign of the partition function.

3. At the next level of refinement, we can place Arf theories on some two-dimensional facets according to some $\nu_{D-1}$. The cocycle condition ensures that we have an even number of Arf facets impinging on an edge, but the edge must now carry a specific choice of how to gap the corresponding Majorana modes. The two possible choices have opposite Grassmann parity, so this choice is similar but not canonically equivalent to a choice of $\nu_D$, etc. The evaluation of the partition function will now require a careful manipulation of the Majorana modes.

4. Next, we can place $\mathrm{SO}(n)_{\pm 1}$ Chern-Simon theories on three-dimensional facets according to some $\nu_{D-2}$. The cocycle condition insures that we have the same number of chiral and anti-chiral fermions at two-dimensional facets, but facets must now carry a specific choice of how to gap these $2d$ fermions. Two inequivalent choices differ by a factor of Arf. The evaluation of the partition function must now cope with this extra level of complication.

5. In principle, we could continue, selecting some invertible fermionic theories to place on four-dimensional facets, etc. In practice, no non-trivial invertible theories are expected to exist up to dimension 7, so we can safely stop here for most physical systems.

On general grounds, the differential in the generalized cohomology theory takes a triangular form, with the diagonal being the standard differential for $H^{D+1-k}(G, T_k)$, where $T_k$ is the group of invertible theories we can place on the $k$-th dimensional facets. The off-diagonal components of the differential are non-trivial and somewhat tricky to compute far from the diagonal. Furthermore, the "stacking" operation on generalized cohomology classes, i.e. the sum of anomalies, is also defined in a triangular manner, with the diagonal being the usual operation of stacking invertible theories.

As one compactifies an SPT on, say, a circle, one can take a triangulation of the $D$-dimensional manifold $M$ and refine it to a triangulation of $M \times S^1$ in a systematic way. Applying the rules above to $M \times S^1$ and reducing them to some evaluation on the triangulation of $M$ one can figure out the resulting SPT theory in one dimension lower. This was done for the Gu-Wen layer in [100], but has not been done in full generality.

## 1.3 Anomalies in free fermion Hilbert space

In this section we demonstrate how the Hilbert space on the torus detects a variety of anomalies in systems of free fermions in various dimensions.

### 1.3.1 Anomalous $\mathbb{Z}_2^{\mathsf{T}}$ in 0+1 dimensions

The anomalies of a fermionic system in 0+1 dimensions with an antiunitary time-reversal symmetry $\mathsf{T}$ with $\mathsf{T}^2 = 1$, so that $G_f = \mathbb{Z}_2^{\mathsf{T}} \times \mathbb{Z}_2^F$, are classified by $\Omega^2_{\text{spin}}(\mathbb{Z}_2; 1, 0) = \Omega^2_{\text{pin}^-} = \mathbb{Z}_8$. These anomalies arise from three layers

$$\begin{aligned} &\nu_0 \in H^0(\mathbb{Z}_2^{\mathsf{T}}, \mathbb{Z}_2) \simeq \mathbb{Z}_2 \\ &\nu_1 \in H^1(\mathbb{Z}_2^{\mathsf{T}}, \mathbb{Z}_2) \simeq \mathbb{Z}_2 \\ &\nu_2 \in H^2(\mathbb{Z}_2^{\mathsf{T}}, \mathrm{U}(1)) \simeq \mathbb{Z}_2 \,, \end{aligned} \tag{1.3.1}$$

which generate the $\mathbb{Z}_8$ anomaly.

We shall study the $\mathbb{Z}_8$ anomaly in a system of free fermions. Related considerations can be found in [101–103].

Consider $\nu$ Majorana fermions in $0 + 1$ dimensions

$$\mathcal{L} = \sum_{a=1}^{\nu} \frac{i}{4} \psi^a \partial_t \psi^a \,. \tag{1.3.2}$$

The theory has a $\mathbb{Z}_2^{\mathsf{T}}$ time-reversal symmetry which acts as[22]

$$\mathsf{T}\psi^a(t) = \psi^a(-t)\mathsf{T} \tag{1.3.3}$$

and $\mathbb{Z}_2^F$ fermion parity

$$\{(-1)^F, \psi^a(t)\} = 0\,. \tag{1.3.4}$$

It is known that a $\mathbb{Z}_2^{\mathsf{T}}$-symmetric quartic interaction that gaps out the fermions can be written for $\nu = 8$ [86]. This realizes in the fermion system the $\mathbb{Z}_8$ anomaly expected from the cobordism classification.

Canonical quantization of (1.3.2) leads to a Clifford algebra of rank $\nu$

$$\{\psi^a, \psi^b\} = 2\delta^{ab} \qquad a, b = 1, 2, \ldots, \nu\,. \tag{1.3.5}$$

We now proceed to identifying the anomaly layers (1.3.1). Each layer is implemented in a characteristic way in the fashion that symmetries are realized on the Hilbert space $\mathcal{H}$.

- $\nu = 1 \mod 2$. There is a rather severe anomaly for $\nu$ odd as the operator $(-1)^F$ generating the $\mathbb{Z}_2^F$ symmetry does not exist. The theory does not admit a proper graded Hilbert space of states. Equivalently stated, the Clifford algebra of odd rank has two irreducible representations, and $(-1)^F$ exchanges them, instead of acting within an irreducible representation. This anomaly is associated with the $H^0(\mathbb{Z}_2^{\mathsf{T}}, \mathbb{Z}_2) = \mathbb{Z}_2$ layer, the Arf layer.

  This anomaly due to the lack of proper Hilbert space can also be detected by studying partition functions. Consider the partition function on the circle with antiperiodic (NS) and periodic (R) boundary conditions. The partition function with NS boundary conditions is[23]

  $$Z_{\text{NS}} = 2^{\nu/2}\,. \tag{1.3.6}$$

  Nominally, this partition function should count the number of states in $\mathcal{H}$, that is $Z_{\text{NS}} = \text{tr}_{\mathcal{H}}(\mathbf{1}) = \dim(\mathcal{H})$. The answer (1.3.6) mirrors the statement that there is no proper Hilbert space for $\nu$ odd, as $2^{\nu/2}$ is not an integer. Likewise, while the partition

[22] For the purposes of studying anomalies it suffices to take all fermions to transform with the same sign under $\mathsf{T}$. If a fermion is assigned the transformation $\mathsf{T}\psi_-(t) = -\psi_-(-t)\mathsf{T}$, we can then write a $\mathbb{Z}_2^{\mathsf{T}}$-invariant mass term $i\psi_+\psi_-$ that couples a pair of fermions which transform with opposite signs under $\mathsf{T}$. This lifts both fermions and therefore without loss of generality we can focus on a collection of fermions that transform with the same sign under $\mathsf{T}$.

[23] This partition function can be evaluated by taking the square root of partition function of $2\nu$ Majorana fermions, which has a $2^\nu$-dimensional Hilbert space. It can also be computed by zeta-regularizing $Z \equiv \text{Pf}(i\partial_t)^\nu = \prod_{n\in\mathbb{Z}} \lambda_n^\nu$, where the eigenvalues of the $0+1d$ Dirac operator are $\lambda_n = n + 1/2$ in the NS sector, and $\lambda_n = n$ in the R sector.

function with R boundary conditions vanishes due to the presence of zero-modes, i.e. $Z_{\mathrm{R}} = \mathrm{tr}_{\mathcal{H}}(-1)^F = 0$, the correlator

$$\langle \psi^1 \psi^2 \cdots \psi^\nu \rangle_{\mathrm{R}} \tag{1.3.7}$$

is non-vanishing, as the insertions compensate the zero-modes. This observable (1.3.7) changes sign under the action of $(-1)^F$, signaling that $(-1)^F$ is anomalous as the $\mathbb{Z}_2^F$ Ward identities are violated.

- $\nu = 2 \mod 4$. For $\nu$ even the theory has a well-defined Hilbert space and operator $(-1)^F$ acting on it. The Clifford algebra of even rank $\nu$ has a unique irreducible representation of dimension $2^{\nu/2}$, thus all representations are unitarily equivalent, and we can study the implementation of symmetries in any choice of basis. We can construct $\mathcal{H}$ by defining the creation and annihilation operators

$$\psi_\pm^A = \frac{1}{2}\big(\psi^{2A-1} \pm i\psi^{2A}\big) \qquad A = 1, \ldots, \nu/2\,, \tag{1.3.8}$$

which obey

$$\{\psi_+^A, \psi_-^B\} = \delta^{AB}\,,\ \{\psi_+^A, \psi_+^B\} = \{\psi_-^A, \psi_-^B\} = 0 \qquad A, B = 1, \ldots, \nu/2\,, \tag{1.3.9}$$

We define the vacuum by

$$\psi_-^A|0\rangle = 0 \qquad A = 1, \ldots, \nu/2\,, \tag{1.3.10}$$

and create the whole module by acting with the different $\psi_+^A$ on it. Time-reversal acts by exchanging the creation and annihilation operators (see (1.3.3) and recall that $\mathsf{T}$ is antilinear)

$$\mathsf{T}\psi_\pm^A = \psi_\mp^A \mathsf{T}\,. \tag{1.3.11}$$

This allows us to determine the action of $\mathsf{T}$ on the vacuum $|0\rangle$ by considering the most general state

$$\mathsf{T}|0\rangle = \alpha|0\rangle + \alpha_A \psi_+^A|0\rangle + \cdots + \alpha_{12\ldots\nu/2}\psi_+^1\psi_+^2\cdots\psi_+^{\nu/2}|0\rangle\,, \tag{1.3.12}$$

for some yet-to-be-fixed coefficients $\{\alpha\}$. Acting on both sides with $\psi_-^A$ and using (1.3.9), (1.3.10) and (1.3.11) we conclude that all but the last coefficient vanish, namely[24]

$$\mathsf{T}|0\rangle = \alpha_{12\ldots\nu/2}\psi_+^1\psi_+^2\cdots\psi_+^{\nu/2}|0\rangle\,, \tag{1.3.13}$$

[24] This corresponds to what is usually referred to as *particle-hole symmetry*: time-reversal exchanges $\psi_+$ and $\psi_-$, so a state full of $\psi_+$ is mapped to a state full of $\psi_-$, and vice-versa.

with $|\alpha_{12\ldots\nu/2}| = 1$ so that state is normalized. Note that $\mathsf{T}$ adds $\nu/2$ fermionic modes, so it changes the fermion parity of the state if $\nu/2$ is odd. This implies that the $\mathbb{Z}_2^{\mathsf{T}} \times \mathbb{Z}_2^F$ symmetry generators on $\mathcal{H}$ obey

$$\{\mathsf{T}, (-1)^F\} = 0 \qquad \text{for } \nu = 2 \mod 4\,. \tag{1.3.14}$$

and

$$[\mathsf{T}, (-1)^F] = 0 \qquad \text{for } \nu = 0 \mod 4\,, \tag{1.3.15}$$

Therefore, the anomaly corresponding to the $\nu = 2 \mod 4$ layer is detected by virtue of the operators $\mathsf{T}$ and $(-1)^F$ anticommuting in $\mathcal{H}$. This anomaly is associated with the $H^1(\mathbb{Z}_2^{\mathsf{T}}, \mathbb{Z}_2) = \mathbb{Z}_2$ layer, the $\psi$-layer.

- $\nu = 4 \mod 8$. For $\nu = 4 \mod 8$ the theory has a proper Hilbert space and $[\mathsf{T}, (-1)^F] = 0$. We now proceed to study how the $\mathbb{Z}_2^{\mathsf{T}}$ symmetry is realized on the Hilbert space. Acting with $\mathsf{T}$ again in (1.3.12) yields

$$\mathsf{T}^2|0\rangle = |\alpha_{12\ldots\nu/2}|^2\psi_-^1\psi_-^2\cdots\psi_-^{\nu/2}\psi_+^1\psi_+^2\cdots\psi_+^{\nu/2}|0\rangle = (-1)^{\nu/4(\nu/2-1)}|0\rangle\,. \tag{1.3.16}$$

Therefore, for $\nu = 4 \mod 8$ the $\mathbb{Z}_2^{\mathsf{T}}$ symmetry is realized projectively on the Hilbert space, that is

$$\mathsf{T}^2 = -1\,. \tag{1.3.17}$$

Therefore, the anomaly corresponding to the $\nu = 4 \mod 8$ layer is detected by virtue of the $\mathbb{Z}_2^{\mathsf{T}}$ symmetry being realized projectively on the Hilbert space.[25] This anomaly is associated with the $H^2(\mathbb{Z}_2^{\mathsf{T}}, \mathrm{U}(1)) = \mathbb{Z}_2$ layer, the bosonic layer.

**SYK model.** To close this section we can consider a very simple application of these results. One of the most well-known systems in $0+1d$ is the celebrated SYK model [104, 105], which consists of a system of $N$ Majorana fermions interacting via four-fermi terms

$$\mathcal{L} = \sum_a \frac{i}{4}\psi^a\partial_t\psi^a - \sum_{abcd} J_{abcd}\psi^a\psi^b\psi^c\psi^d\,, \tag{1.3.18}$$

where the coupling constants $J$ are real. This Lagrangian is invariant under $\mathsf{T}(\psi^i(t)) = \psi^i(-t)$, and therefore all the conclusions from our previous discussion hold. The time-reversal anomaly of the system is $\nu = N \mod 8$. We immediately conclude that,

[25] We note that $\mathsf{T}^2 = -1$ for $\nu = 4, 6 \mod 8$ and $\mathsf{T}^2 = 1$ for $\nu = 0, 2 \mod 8$. This follows from the properties of the antilinear involution acting on the Clifford algebra as $\mathsf{T}^{-1}\gamma^a\mathsf{T} = \gamma^a$, or equivalently $U\gamma^aU^{-1} = (\gamma^a)^*$, where we have written $\mathsf{T} = UK$, with $K$ denoting complex conjugation and $U$ a unitary. Therefore, $\mathsf{T}^2 = U^*U = \pm 1$ with $\mathsf{T}^2 = 1$ corresponding to a real involution and $\mathsf{T}^2 = -1$ to a pseudoreal/quaternionic involution. Using the explicit form of the gamma matrices we constructed we find the signs as discussed, implying that for $\nu = 0, 2 \mod 8$ and $\nu = 4, 6 \mod 8$ the Clifford algebra admits a real and quaternionic involution respectively.

- If $N$ is odd, the SYK model does not admit a satisfactory ($\mathbb{Z}_2$-graded) Hilbert space.
- If $N$ is even, $N = 2 \mod 4$, then $\mathsf{T}$ is fermion-odd, and therefore the spectrum of the Hamiltonian is (at least) two-fold degenerate, with energy doublets having opposite fermion parity (the Hilbert space is supersymmetric).
- If $N$ is even, $N = 4 \mod 8$, then $\mathsf{T}$ squares to $-1$, and therefore the spectrum of the Hamiltonian is (at least) two-fold degenerate, with energy doublets having the same fermion parity (they are Kramers doublets).
- If $N$ is even, $N = 0 \mod 8$, the symmetry is non-anomalous and we cannot conclude anything about the spectrum of the Hamiltonian. Unless we tune the coefficients $J$ to have some special symmetry, we do not expect any degeneracy in the Hilbert space.

It is a rather entertaining exercise to explicitly check these claims by numerically diagonalizing the SYK Hamiltonian for small values of $N$. We also note that similar ideas can be found in e.g. [106, 107].

### 1.3.2 Anomalous $\mathbb{Z}_4^{\mathsf{T}}$ in 2+1 dimensions

The anomalies of a fermionic system in $2+1d$ with an antiunitary time-reversal symmetry $\mathsf{T}^2 = (-1)^F$ are classified by $\Omega^4(\mathbb{Z}_2; 1, 1) = \Omega^4_{\text{pin}^+} = \mathbb{Z}_{16}$. These anomalies arise from four layers (1.1.13)

$$\begin{aligned}
\nu_1 &\in H^1(\mathbb{Z}_2^{\mathsf{T}}, \mathbb{Z}) \simeq \mathbb{Z}_2 \\
\nu_2 &\in H^2(\mathbb{Z}_2^{\mathsf{T}}, \mathbb{Z}_2) \simeq \mathbb{Z}_2 \\
\nu_3 &\in H^3(\mathbb{Z}_2^{\mathsf{T}}, \mathbb{Z}_2)) \simeq \mathbb{Z}_2 \\
\nu_4 &\in H^4(\mathbb{Z}_2^{\mathsf{T}}, \mathrm{U}(1)) \simeq \mathbb{Z}_2 \,,
\end{aligned} \tag{1.3.19}$$

which compile into $\mathbb{Z}_{16}$.

In this section we study this anomaly in a system of free Majorana fermions, and in section 1.4 we will do the same in time-reversal symmetric spin TQFTs.

Consider a system of $\nu$ Majorana fermions $\psi$. We shall work in the Majorana basis where the gamma matrices are real,

$$\gamma^0 = i\sigma^2, \qquad \gamma^1 = \sigma^1, \qquad \gamma^2 = \sigma^3 \,. \tag{1.3.20}$$

In this basis the Majorana condition is simply $\psi^* = \psi$ so $\psi$ is a real two-component Grassmann-odd spinor. We can without loss of generality take time-reversal to act as

$$\mathsf{T}(\psi(t)) = \pm\gamma^0\psi(-t) \,. \tag{1.3.21}$$

Given a pair of fermions transforming with opposite signs, we can write down a $\mathsf{T}$-invariant mass term, which means that such a pair does not contribute to anomalies. Therefore, as far as anomalies is concerned, we can take all fermions to transform with the same sign, say $+1$. It is known that a $\mathsf{T}$-invariant interaction exists with 16 fermions that lifts all of them [56, 87–89, 108–110].

We now construct the torus Hilbert space of the system and study how the time-reversal anomaly manifests itself on it. A subtle but important difference in $2+1d$ as opposed to the examples in $0+1d$ and $1+1d$ is that $\Omega_4^{\text{spin}} = \mathbb{Z}$ contains a free part: in $2+1d$ there exists a purely gravitational SPT. This invertible theory is intertwined with time-reversal in an interesting way, which we review next. The generator of SPTs with no symmetry in $2+1d$ is given by the spin TQFT denoted by $\mathrm{SO}(1)_1$, corresponding to the super Ising category [63, 111–113]. The partition function of this theory is $e^{-i\mathrm{CS}_{\text{grav}}}$, where locally

$$\mathrm{CS}_{\text{grav}} = \frac{1}{4\pi}\int_{M_3} \mathrm{tr}\left(\omega\,\mathrm{d}\omega + \frac{2}{3}\omega^3\right)\,, \tag{1.3.22}$$

where $\omega$ is the spin connection for the gravitational background of $M_3$. An arbitrary SPT with no symmetry is given by a number $n \in \mathbb{Z}$ of copies of the generator, namely $\mathrm{SO}(n)_1 := \mathrm{SO}(1)_1^n$, whose partition function is $e^{-in\mathrm{CS}_{\text{grav}}}$. As a spin TQFT, $\mathrm{SO}(n)_1$ can be obtained by condensing a certain fermion in the bosonic TQFT $\mathrm{Spin}(n)_1$, that is by gauging a certain $\mathbb{Z}_2$ one-form symmetry (see section 1.4). Note that the Chern-Simons form $\mathrm{CS}_{\text{grav}}$ is a volume form, so it is odd under time-reversal.

If the manifold is non-trivial, the fermions automatically couple to the Chern-Simons term for the background gravitational field, because the Dirac operator contains a piece proportional to the spin connection. In $2+1d$ time-reversal acts both on the fermions and on the Chern-Simons interactions, and the combined system is only time-reversal invariant if the coefficient of the latter is adjusted appropriately. This behaviour should be thought of as a mixed time-reversal-gravitational anomaly, and it can be ascribed to a controlled non-invariance of the fermion path-integral measure $D\psi$. This non-invariance is a topological phase, the eta invariant $\eta$, and we can summarize the anomaly as the statement that each massless Majorana fermion $\psi$ transforms as

$$\mathsf{T}\colon\ D\psi \mapsto e^{-i\pi\eta/2} D\psi\,. \tag{1.3.23}$$

In absence of other background fields, the eta invariant is precisely the gravitational Chern-Simons term,

$$\frac{1}{2}\pi\eta = \mathrm{CS}_{\text{grav}} \mod 2\pi\mathbb{Z}\,. \tag{1.3.24}$$

In this sense, time-reversal does not map the QFT of a single massless fermion into itself, but rather into itself tensored with a copy of the SPT $\mathrm{SO}(1)_1$; schematically

$$\mathsf{T}\big(\text{massless } \psi\big) = \text{massless } \psi \times \mathrm{SO}(1)_1\,. \tag{1.3.25}$$

In order to compensate for the anomalous phase $e^{-i\pi\eta/2}$, we formally need to attach to each massless Majorana fermion a copy of $\frac{1}{2}\mathrm{CS}_{\text{grav}}$, i.e., to a copy of a "square root" of $\mathrm{SO}(1)_1$.

The combined object $e^{\frac{1}{2}\mathrm{CS}_{\mathrm{grav}}} D\psi$ is now time-reversal invariant. In the notation of (1.3.25), we formally need to move "half" of $\mathrm{SO}(1)_1$ to the left, so as to have $\mathsf{T}$ mapping a QFT into itself instead of into a second QFT.

The discussion above is equivalent to the statement that a massless Majorana fermion carries chiral central charge $c = 1/4$ (this is also known as the *framing anomaly* [22]; recall that $c$ measures the coupling of the theory to $\mathrm{CS}_{\mathrm{grav}}$). As $c$ is odd under time-reversal, a system with $c \neq 0$ is not invariant by itself, but must be coupled to a suitable SPT, whose central charge is $-c$, in order to make the total central charge zero. The generator of SPTs $\mathrm{SO}(1)_1$ has $c = 1/2$, so in order to compensate for the $c = 1/4$ of the fermion we formally need to couple it to a square root of $\mathrm{SO}(1)_1$. More generally, given an arbitrary number $\nu$ of massless Majorana fermions, the system $\psi^\nu$ is not actually time-reversal invariant, but the combined system $\psi^\nu \times \mathrm{SO}(\nu/2)_{-1}$ is. Naturally, if the number of fermions $\nu$ is odd, the coefficient of $\mathrm{CS}_{\mathrm{grav}}$ is not properly normalized, and the system does not make sense as a purely $2+1d$ object: we either give up time-reversal invariance and drop the gravitational counterterm, or we keep the symmetry and regard the system as the boundary of a $3+1d$ theory. For $\nu$ even, we can maintain time-reversal invariance and still have a conventional $2+1d$ theory, but only after coupling the fermions to $\mathrm{SO}(\nu/2)_{-1}$. For now, we will consider the $\nu$ fermions alone, and later on we will study the effect of turning on $\mathrm{SO}(\nu/2)_{-1}$ for $\nu$ even.

With this in mind, let us go back to studying the system of $2+1d$ $\nu$ massless Majorana fermions on the torus $T^2$. Anomalies, being renormalization-group invariant, always arise in the realization of the symmetry on the low energy states; therefore, in order to detect the anomalies, it suffices to look at the vacuum sector. For even spin structure on $T^2$ there are no zero modes and no anomalies; this agrees with the general discussion in section 1.1 where we argued that anomalies can only be detected on manifolds that do not bound.

For odd spin structure there are zero modes and potential anomalies. Roughly, the system with odd spin structure on $T^2$ behaves as $2\nu$ copies of the $0+1d$ system of Majoranas we analyzed earlier, the factor of 2 being due to the fact that each $\psi$ has two real components instead of one. In this sense, the analogous to the first layer in $0+1d$ is never activated in $2+1d$, because the number of Majorana components is always even. In other words, the Hilbert space $\mathcal{H}_{XY}$ of $2+1d$ Majorana fermions is always well-defined, regardless of the parity of the number of fermions. But the other two layers, those measured by the fermion parity of $\mathsf{T}$ and the sign in $\mathsf{T}^2 = \pm 1$, are potentially activated. The first one is measured by the parity of $\nu$, and the second one by the parity of $\nu/2$. We will exhibit the following anomalous behavior in the Hilbert space:

- $\nu = 1 \mod 2$: In $\mathcal{H}_{\text{R-R}}$ time-reversal is fermion-odd, it anticommutes with $(-1)^F$

$$\{\mathsf{T}, (-1)^F\} = 0\,. \tag{1.3.26}$$

  This anomaly is associated to the $p_x + ip_y$ layer $\nu_1$. For $\nu$ even $[\mathsf{T}, (-1)^F] = 0$.

- $\nu = 2 \mod 4$: In the even spin structure Hilbert spaces $\mathcal{H}_{\text{NS-NS}}, \mathcal{H}_{\text{NS-R}}, \mathcal{H}_{\text{R-NS}}$ time-reversal satisfies the standard algebra $\mathsf{T} = (-1)^F$, but in the odd spin structure Hilbert space $\mathcal{H}_{\text{R-R}}$ this algebra is realized projectively, namely $\mathsf{T}^2 = -(-1)^F$. In other words, the time-reversal symmetry on $\mathcal{H}_{XY}$ satisfies

$$\mathsf{T}^2 = (-1)^F \times (-1)^{\text{Arf}(T^2)} . \tag{1.3.27}$$

 This anomaly is associated to the Arf layer $\nu_2$.

The next two layers, $\nu_3, \nu_4$, which measure $\nu \mod 8$ and $\nu \mod 16$, respectively, are invisible on the torus Hilbert spaces.

The discussion regarding the first two layers is essentially identical to the $0+1d$ case, so we only highlight the differences. The fermions now depend on both time $t$ and the spatial coordinate $\boldsymbol{x}$, which we take to coordinatize a torus $T^2$. The Hilbert space associated to this spatial slice is built by acting with the spatial modes on the vacuum sector. If $\text{Arf}(T^2) = 0$, then there are no zero-modes, and the vacuum Hilbert space is trivial: there is a unique vacuum state $|0\rangle$. Therefore, here time-reversal acts quite trivially: $\mathsf{T}$ is fermion-even and satisfies $\mathsf{T}^2 = (-1)^F$ on the nose: neither layer is activated. In order to detect the anomaly we have to look at the non-bounding torus, i.e., where both boundary conditions are periodic such that $\text{Arf}(T^2) = 1$. Here there is a single zero-mode for each Majorana fermion, which is spatially constant. In what follows we shall study this vacuum module generated by these zero-modes in $\mathcal{H}_{\text{R-R}}$.

First of all, since $\mathsf{T}(\psi) = \gamma^0\psi$ where $\gamma^0 = i\sigma^2$, time-reversal acts on the two components of the Majorana fermion as

$$\mathsf{T}(\psi^1) = +\psi^2, \qquad \mathsf{T}(\psi^2) = -\psi^1 . \tag{1.3.28}$$

In terms of the complex spinor $\Psi = \frac{1}{\sqrt{2}}(\psi^1 + i\psi^2)$ this becomes

$$\mathsf{T}(\Psi) = i\Psi^* . \tag{1.3.29}$$

The Hilbert space is built by declaring that $\Psi_i|0\rangle = 0$ for all $i = 1, 2, \ldots, \nu$, and by repeatedly acting with $\Psi_i^*$ on $|0\rangle$. The action of time-reversal on the whole vacuum Hilbert space is uniquely fixed in terms of its action on $|0\rangle$, which again reads

$$\mathsf{T}|0\rangle = \Psi_1^*\Psi_2^* \cdots \Psi_\nu^*|0\rangle \tag{1.3.30}$$

up to an inconsequential phase. We thus see that, indeed, if $\nu$ is odd $\mathsf{T}$ anticommutes with $(-1)^F$. Now, if we act with $\mathsf{T}$ twice we get

$$\begin{aligned} \mathsf{T}^2|0\rangle &= \mathsf{T}\Psi_1^*\Psi_2^* \cdots \Psi_\nu^*|0\rangle \\ &= (-i)^\nu \Psi_1\Psi_2 \cdots \Psi_\nu \mathsf{T}|0\rangle \\ &= (-i)^\nu \Psi_1\Psi_2 \cdots \Psi_\nu \Psi_1^*\Psi_2^* \cdots \Psi_\nu^*|0\rangle \\ &= i^{-\nu^2}|0\rangle . \end{aligned} \tag{1.3.31}$$

When $\nu$ is even we get $\mathsf{T}^2|0\rangle = +|0\rangle$. More generally, as $\mathsf{T}^2 = (-1)^F$ when acting on the creation operators, the relation $\mathsf{T}^2|0\rangle = +|0\rangle$ lifts to $\mathsf{T}^2 = (-1)^F$ on the whole Hilbert space.[26]

We now return to the effect of the gravitational SPT $\mathrm{SO}(\nu/2)_{-1}$ for $\nu$ even that is needed in order to have a time-reversal symmetric theory. This SPT has a unique state on any spin structure Hilbert space $\mathcal{H}_{XY}$. The fermion parity of this state is known to be $(-1)^F = (-1)^{\mathrm{Arf}(T^2)\nu/2}$ (see [5, 36] and section 5.2.2). Therefore, the $\mathsf{T}$-invariant combined system $\psi^\nu \times \mathrm{SO}(\nu/2)_{-1}$ has a time-reversal algebra

$$\mathsf{T}^2 = (-1)^F \times (-1)^{\mathrm{Arf}(T^2)\nu/2} . \tag{1.3.32}$$

This means that the operator algebra $\mathsf{T}^2 = (-1)^F$ is undeformed for $\nu = 0 \mod 4$, while it gets deformed by the Arf theory for $\nu = 2 \mod 4$ in $\mathcal{H}_{\text{R-R}}$, as claimed.

### 1.3.3 Anomalous $\mathbb{Z}_2$ in 1+1 dimensions

The anomalies of a fermionic system in $1+1d$ with a unitary $\mathbb{Z}_2$ symmetry such that $G_f = \mathbb{Z}_2 \times \mathbb{Z}_2^F$ are classified by $\Omega^2_{\text{spin}}(\mathbb{Z}_2; 0, 0) = \mathbb{Z}_8$. These anomalies arise from three layers

$$\begin{aligned} &\nu_1 \in H^1(\mathbb{Z}_2, \mathbb{Z}_2) \simeq \mathbb{Z}_2 \\ &\nu_2 \in H^2(\mathbb{Z}_2, \mathbb{Z}_2) \simeq \mathbb{Z}_2 \\ &\nu_3 \in H^3(\mathbb{Z}_2, \mathrm{U}(1)) \simeq \mathbb{Z}_2 \, , \end{aligned} \tag{1.3.33}$$

which generate the $\mathbb{Z}_8$ anomaly.

Consider $\nu$ Majorana fermions in $1+1d$[27]

$$\mathcal{L} = \sum_{a=1}^{\nu} i\psi_L^a \partial_+ \psi_L^a + i\psi_R^a \partial_- \psi_R^a \, , \tag{1.3.34}$$

where $\partial_\pm = \partial_t \pm \partial_x$. This system has a chiral $\mathbb{Z}_2$ unitary symmetry generated by $g = (-1)^{F_L}$ which combines with the nonchiral $\mathbb{Z}_2^F$ symmetry generated by $(-1)^F$ to yield the symmetry group $G_f = \mathbb{Z}_2 \times \mathbb{Z}_2^F$. These symmetries act on the fermions as

$$\begin{aligned} \{(-1)^{F_L}, \psi_L^a\} = [(-1)^{F_L}, \psi_R^a] = 0 \\ \{(-1)^F, \psi_L^a\} = \{(-1)^F, \psi_R^a\} = 0 \, . \end{aligned} \tag{1.3.35}$$

[26] Here we are fixing the fermion parity of the odd-spin-structure ground state to be $+1$, the same as in the even-spin-structures case. We could declare instead that the ground state has fermion parity $-1$. This would eliminate the anomalous sign in the time-reversal algebra below. The anomaly would manifest itself, instead, in the presence of an Arf-dependent anomalous sign in the fermion parity of the ground state, $(-1)^F|0\rangle = (-1)^{\mathrm{Arf}(T^2)\nu/2}|0\rangle$. This is simply a redefinition $(-1)^F \mapsto (-1)^{\mathrm{Arf}(T^2)\nu/2}(-1)^F$.

[27] The anomalies in $1+1d$ are actually $\mathbb{Z}_8 \times \mathbb{Z}$, the second factor being the gravitational anomaly. We take $\nu_L = \nu_R = \nu$ to cancel this gravitational anomaly and focus directly on the $\mathbb{Z}_8$ factor.

It is known that a $\mathbb{Z}_2$-symmetric interaction that gaps out the fermions can be written for $\nu = 8$ [92, 114–116]. This realizes in the fermion system the $\mathbb{Z}_8$ anomaly expected from the cobordism classification.

We now analyze the anomaly layers that can be detected in the Hilbert space. We discuss in turn the Hilbert space $\mathcal{H}_X$ and the $\mathbb{Z}_2$-twisted Hilbert space $\mathcal{H}_X^g$, where $X \in \{\mathrm{NS}, \mathrm{R}\}$ denotes the spin structure on the spatial circle. The twisted Hilbert space $\mathcal{H}_X^g$ is defined by quantizing in the presence of a nontrivial $\mathbb{Z}_2$ (flat) connection around the circle for the $\mathbb{Z}_2$ symmetry.

In order to detect the anomalies we proceed to study the implementation of symmetries on the zero-mode operators in $\mathcal{H}_X$ and $\mathcal{H}_X^g$ in turn.

<u>Anomalies in $\mathcal{H}_X$</u>

Since in the NS sector there are no fermion zero-modes, there is a unique, trivial vacuum and symmetries are realized on $\mathcal{H}_{\mathrm{NS}}$ in a non-anomalous fashion. In the R sector there are fermion zero-modes which upon quantization furnish a Clifford algebra of rank $2\nu$

$$\{\psi_L^a, \psi_L^b\} = \{\psi_R^a, \psi_R^b\} = 2\delta^{ab}\,, \qquad \{\psi_L^a, \psi_R^b\} = 0 \qquad a, b = 1, 2, \ldots, \nu\,. \tag{1.3.36}$$

This Clifford algebra has a unique irreducible representation of dimension $2^\nu$, thus all representations are unitarily equivalent, and we can study the implementation of symmetries in any choice of basis. We can construct $\mathcal{H}_{\mathrm{R}}$ by defining the creation and annihilation operators $\psi_+^a = \frac{1}{2}(\psi_R^a + i\psi_L^a)$ and $\psi_-^a = \frac{1}{2}(\psi_R^a - i\psi_L^a)$, such that $\psi_-^a|0\rangle = 0$. It follows from (1.3.35) that

$$(-1)^{F_L}\psi_+^a = \psi_-^a(-1)^{F_L}\,. \tag{1.3.37}$$

The $\mathbb{Z}_2$ symmetry generator thus maps the empty vacuum to the completely filled state

$$(-1)^{F_L}|0\rangle = \alpha\psi_+^1\psi_+^2\cdots\psi_+^\nu|0\rangle\,. \tag{1.3.38}$$

for some phase $\alpha$. This implies that the $\mathbb{Z}_2 \times \mathbb{Z}_2^F$ symmetry generators on $\mathcal{H}_R$ obey

$$\{(-1)^{F_L}, (-1)^F\} = 0 \qquad \text{for } \nu \text{ odd}\,, \tag{1.3.39}$$

and

$$[(-1)^{F_L}, (-1)^F] = 0 \qquad \text{for } \nu \text{ even}\,. \tag{1.3.40}$$

Therefore, the anomaly corresponding to the $\nu$ odd layer is detected by virtue of the operators $(-1)^{F_L}$ and $(-1)^F$ anticommuting in $\mathcal{H}_R$. This has also been noticed in [11, 117].

The anomaly associated to the $\nu$ odd layer can also be detected in the torus partition function with periodic boundary conditions around both the spatial circle and temporal circle, that is with $(\mathrm{R}, \mathrm{R})$ boundary conditions along the two cycles of the torus. The zero-modes in $\mathcal{H}_{\rm R}$ imply that the partition function vanishes, but the partition function with fermion zero-modes saturated is nonvanishing:

$$\langle \psi_L^1 \psi_L^2 \cdots \psi_L^\nu \psi_R^1 \psi_R^2 \cdots \psi_R^\nu \rangle \neq 0\,. \tag{1.3.41}$$

This implies that the $(-1)^{F_L}$ Ward identities are violated for $\nu$ odd, that is, there is an anomaly for the chiral $\mathbb{Z}_2$ symmetry.

Anomalies in $\mathcal{H}_X^g$

This Hilbert space is constructed by imposing boundary conditions twisted by $(-1)^{F_L}$ when fermions are transported around the spatial circle. This yields different boundary conditions for the left-moving and right-moving fermions, which we will denote by $[X_L, X_R]$, where $X_{L/R} \in \{\rm NS, R\}$.[28]

Let us consider $\mathcal{H}_X^g$ in turn:

$\boldsymbol{\mathcal{H}^g_{\rm NS}}$. This corresponds to [R, NS] boundary conditions on the fermions. There are $\nu$ zero modes from the left movers and none from the right movers. The zero mode algebra is thus a Clifford algebra of rank $\nu$

$$\{\psi_L^a, \psi_L^b\} = 2\delta^{ab} \qquad a, b = 1, 2, \ldots, \nu\,. \tag{1.3.42}$$

- $\nu$ odd. There is a rather severe anomaly for $\nu$ odd as the operator $(-1)^F$ generating $\mathbb{Z}_2^F$ obeying $\{(-1)^F, \psi_L^a\} = 0$ does not exist. The theory does not admit a proper graded Hilbert space of states. Equivalently stated, the Clifford algebra of odd rank has two irreducible representations, and $(-1)^F$ exchanges them, instead of acting within an irreducible representation. This anomaly is associated with the $H^1(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ layer, the Arf layer.

  The $\mathbb{Z}_2$ anomaly associated to the Arf layer and the corresponding lack of a Hilbert space can also be detected in the torus partition function with antiperiodic boundary conditions around both the spatial circle and the temporal circle, that is, with (NS, NS) boundary conditions along the two cycles of the torus. This partition function is given by

  $$Z^{(g,1)}_{\rm (NS,NS)} = (\sqrt{2}\chi_\sigma)^\nu (\overline{\chi}_1 + \overline{\chi}_\epsilon)^\nu\,, \tag{1.3.43}$$

  where $\chi_1, \chi_\sigma$ and $\chi_\epsilon$ are the Virasoro characters with weight $0, 1/16$ and $1/2$. For $\nu$ odd, the partition function indeed does not have an integral expansion, and thus there is no suitable Hilbert space. The dimensions of the modules are integers times a factor of $\sqrt{2}$, consistent with the fact that the anomaly is due to a dangling $\mathrm{C}\ell(1)$, whose dimension is formally $\sqrt{2}$.

[28]This is to be contrasted with the boundary condition in the untwisted Hilbert space, where $X_L = X_R$.

- $\nu$ even. For $\nu$ even the theory has a well-defined Hilbert space $\mathcal{H}^g_{\mathrm{NS}}$ and well defined $(-1)^F$ and $(-1)^{F_L}$ operators obeying $[(-1)^{F_L}, (-1)^F] = 0$ in the Hilbert space. The $\mathbb{Z}_2$ symmetry generator maps the empty vacuum in $\mathcal{H}^g_X$ to itself up to phase $(-1)^{F_L}|0\rangle = \alpha|0\rangle$.

$\boldsymbol{\mathcal{H}^g_{\mathrm{R}}}$. This corresponds to $[\mathrm{NS}, \mathrm{R}]$ boundary conditions on the fermions. There are $\nu$ zero modes from the right-movers and none from the left-movers. The zero mode algebra is thus a Clifford algebra of rank $\nu$

$$\{\psi^a_R, \psi^b_R\} = 2\delta^{ab} \qquad a, b = 1, 2, \ldots, \nu\,. \tag{1.3.44}$$

- $\nu$ odd. Verbatim our discussion above: this system has a $\mathbb{Z}_2$ anomaly associated to the Arf layer, diagnosed by the lack of a proper graded Hilbert space of states. This can also be seen from the lack of integral expansion of the torus partition function with $(\mathrm{R}, \mathrm{NS})$ boundary conditions along the two cycles of the torus

$$Z^{(g,1)}_{(\mathrm{R,NS})} = (\chi_1 + \chi_\epsilon)^\nu (\sqrt{2}\overline{\chi}_\sigma)^\nu \tag{1.3.45}$$

  which again does not have a properly quantized expansion.

- $\nu$ even. For $\nu$ even the theory has a well-defined Hilbert space $\mathcal{H}^g_{\mathrm{R}}$ and well defined $(-1)^F$ and $(-1)^{F_L}$ operators obeying $[(-1)^{F_L}, (-1)^F] = 0$ in the Hilbert space.

Twisting by the symmetry eliminates the zero modes of one chirality. So effectively the twisting halves the number of zero-modes, and the system moves one step up the ladder of layers. In the untwisted Hilbert spaces $\mathcal{H}_X$, $\nu = 1 \mod 2$ means that $g$ anticommutes with $(-1)^F$. For twisted Hilbert spaces $\mathcal{H}^g_X$, $\nu = 1 \mod 2$ means lack of graded Hilbert spaces.

**Projective rotations.** In the discussion so far, we argued that the $\nu = 1 \mod 2$ layer of the anomaly can be detected by the symmetry algebra of $g$ and $(-1)^F$. It is noteworthy that, if we look at a more general group of symmetries, the *full* anomaly $\nu \mod 8$ can be detected.

The detection of the $\nu \in \mathbb{Z}_8$ anomaly is well-understood by now. This anomaly measures the anomalous spin of operators on the twisted Hilbert spaces $\mathcal{H}^g_X$. From our point of view, this statement is understood as follows. When we compactify the theory on a circle, the model acquires an extra flavor symmetry: translations along the compact direction become internal symmetries of the effective $0+1$ dimensional quantum mechanical model. Therefore, the Hilbert spaces on the circle realize a representation of a larger symmetry group: the initial $G_f = \mathbb{Z}_2 \times \mathbb{Z}_2^F$ symmetry is enhanced to $G_f \times \mathrm{U}(1)$, with $\mathrm{U}(1)$ the group of rotations around the spatial circle. While the bottom layer $\nu \mod 2$ only measures a projective action of $G_f$, the full anomaly $\nu \mod 8$ measures a projective realization of the extended group $G_f \times \mathrm{U}(1)$.

In a non-anomalous theory, the Hilbert space realizes a double-cover of the symmetry group U(1), because of the presence of fermions — the system is invariant under $4\pi$ rotations. In an anomalous theory, the Hilbert space realizes a higher cover of U(1), because of the presence of operators with fractional spin. We claim that in a system with anomaly $\nu \in \mathbb{Z}_8$, the twisted Hilbert spaces $\mathcal{H}_X^g$ realize a $2 \times (8/\gcd(\nu, 8))$-cover of U(1).

This claim can be established by looking again at a free fermion system. In this system, the anomalous spin comes from the "zero-point momentum" of the fermions, namely 1/16 units for each Ramond zero-mode. In the untwisted Hilbert space there is no fractional momentum: if both chiralities are NS, there are no zero-modes, and if they are both R, the zero-point momenta cancel out. In the twisted Hilbert space, one of the two chiralities is NS and the other is R, so there is no cancellation, and there are $\nu/16$ units of fractional momentum. Therefore, the eigenvalues of the momentum operator are half-integers (from the oscillator modes), plus $\nu/16$ (from the zero-modes). In other words, $2P \pm \nu/8 \in \mathbb{Z}$. The momentum operator $P$ is precisely the generator of U(1) rotations; therefore, instead of two circles, we need to perform $2 \times (8/\gcd(\nu, 8))$ full turns instead in order to compensate for this fractional momentum. This proves the claim.

This also admits a path-integral interpretation. In a system with $\nu \in \mathbb{Z}_8$ units of anomaly, the partition function twisted by the $g$ symmetry picks up a phase $e^{\pm 2\pi i \nu/8}$ under a $4\pi$ rotation, which signals the presence of operators with fractional momentum. We show this by looking at the free fermion system, whose partition function over the twisted Hilbert space $\mathcal{H}_X^g$ is (1.3.43), (1.3.45)

$$\begin{aligned} \mathcal{H}_{\mathrm{NS}}^g: \quad & Z_{(\mathrm{NS,NS})}^{(g,1)} = (\sqrt{2}\chi_\sigma)^\nu (\overline{\chi}_1 + \overline{\chi}_\epsilon)^\nu \\ \mathcal{H}_{\mathrm{R}}^g: \quad & Z_{(\mathrm{R,NS})}^{(g,1)} = (\chi_1 + \chi_\epsilon)^\nu (\sqrt{2}\overline{\chi}_\sigma)^\nu \end{aligned} \tag{1.3.46}$$

A $4\pi$ rotation corresponds to a $T^2$ modular transformation, which induces the phase $e^{\pm 4\pi i \nu h_\sigma} \equiv e^{\pm 2\pi i \nu/8}$, with $h_\sigma = 1/16$ the conformal weight of $\chi_\sigma$.

To summarize, the effect of the anomaly $\nu \in \mathbb{Z}_8$ is that the twisted Hilbert spaces realize a projective representation of the rotation symmetry group U(1), and from this projective action one can read off the anomaly. For a given value of $\nu$, the Hilbert space realizes a $2 \times (8/\gcd(\nu, 8))$ cover of U(1), as opposed to a double-cover, and the generator of rotations has fractional momentum $\nu/16$. In a conformal theory, this can also be detected by performing a $T^2$ modular transformation on the twisted partition function. A very similar philosophy was used in [63] to derive the time-reversal anomaly in a $2+1d$ dimensional system.

## 1.4 Anomalies in spin TQFT Hilbert space

In this section we demonstrate the existence of anomalies in fermionic TQFTs by looking directly at their Hilbert space. We follow the construction of the Hilbert space of a fermionic

TQFT in [5]; see [36, 37, 118–121] for related work. Here we summarize the main ingredients, leaving most details to chapter 5.

Given an arbitrary spin TQFT, one may sum over all spin structures in order to yield a bosonic TQFT. We refer to this theory as the *bosonic parent/shadow* of the original spin TQFT. This process of summing over spin structures corresponds to gauging the zero-form symmetry generated by fermion parity $\mathbb{Z}_2^F = \langle(-1)^F\rangle$. Given such bosonic theory, one may undo the gauging, i.e., we can recover the original spin TQFT by gauging a dual $\mathbb{Z}_2$ symmetry, this time a one-form symmetry [32]. This symmetry is generated by a certain fermionic line operator $\psi$, i.e., $\mathbb{Z}_2^{(1)} = \langle\psi\rangle$. Gauging this one-form symmetry is also known as *condensing* the anyon $\psi$.

With this in mind, the Hilbert space of the spin TQFT is easily obtained in terms of the Hilbert space of the bosonic parent, by means of the standard procedure of gauging a symmetry. The Hilbert space of the bosonic parent, being a standard TQFT, is well-understood. Specifically, the torus Hilbert space has a basis of states labelled by the anyons [22]:

$$\mathcal{H}(T^2) = \operatorname*{Span}_{\beta\in\mathcal{A}}\left[\ \bigcirc\!\!\!\!\!\times\ \ \beta\ \right] \tag{1.4.1}$$

Here $\mathcal{A}$ denotes the set of all anyons in the bosonic parent – a finite set – and the loop denotes a Wilson line labeled by $\beta$ wrapped along the $\mathsf{b}$-cycle of the torus (see figure 1.1).

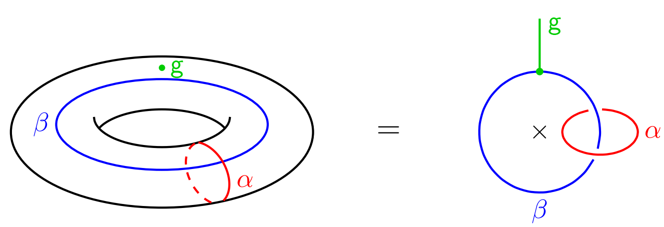
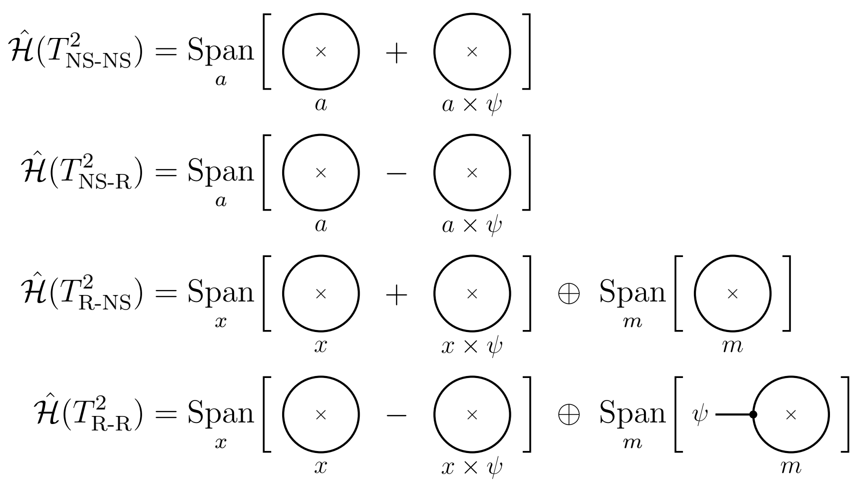



Figure 1.1: Schematic notation for an arbitrary configuration of anyons on the torus, in the presence of a puncture $\mathsf{g}$. The green line represents the vertical (time) direction, orthogonal to the torus. We insert an anyon $\mathsf{g}$ along this direction, which from the point of view of the torus becomes a puncture (a marked point). The red line represents the $\mathsf{a}$-cycle, and the blue one the $\mathsf{b}$-cycle. We insert Wilson lines with anyons $\alpha, \beta$ along these cycles, respectively. The cross $\times$ represents the hole. The states in the Hilbert space $\mathcal{H}$ are created by wrapping anyons around the $\mathsf{b}$-cycle. The states in the *twisted* Hilbert space $\mathcal{H}^{\mathsf{g}}$ are created by wrapping anyons around the $\mathsf{b}$-cycle, in presence of a vertical anyon $\mathsf{g}$.

The set $\mathcal{A}$ comes equipped with extra structure; for example, we have the modular matrix $S\colon \mathcal{A}\times\mathcal{A}\to\mathbb{C}$ that implements the large diffeomorphism $(\mathsf{a},\mathsf{b})\mapsto(\mathsf{b},-\mathsf{a})$. Similarly, we also have the modular matrix $T\colon \mathcal{A}\times\mathcal{A}\to\mathbb{C}$ that implements the large diffeomorphism $(\mathsf{a},\mathsf{b})\mapsto(\mathsf{a},\mathsf{a}+\mathsf{b})$; these two transformations $S, T$ generate the set of all large diffeomorphisms, the modular group $SL_2(\mathbb{Z})$. By our choice of basis (1.4.1) where the states are wrapped around the $\mathsf{b}$-cycle, the $T$-matrix is diagonal, with $T = \mathrm{diag}(e^{2\pi(h-c/24)})$, where $c$ is the central charge of the system and $h\colon \mathcal{A} \to \mathbb{Q}/\mathbb{Z}$ denotes the topological spin of the lines.

The Hilbert space of the fermionic theory, let us call it $\hat{\mathcal{H}}$, depends on the spin structure of the torus. We denote these tori as $T^2_{s^{\mathsf{a}},s^{\mathsf{b}}}$, where $s^{\mathsf{a}}, s^{\mathsf{b}} = \pm 1$ refers to the boundary condition around the cycle $\mathsf{a}$- and $\mathsf{b}$-cycles, respectively. We also denote the $s = -1$ boundary condition by NS and the $s = +1$ boundary condition by R. We claim that the corresponding Hilbert spaces are spanned by the following bases:

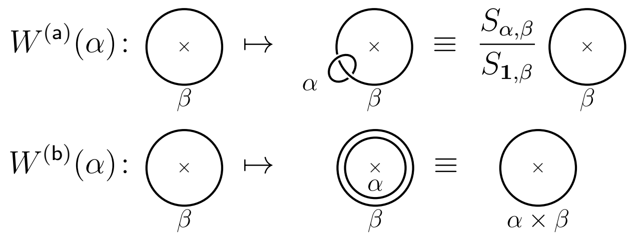


(1.4.2)

Here, $a$ takes values in the subset of lines of $\mathcal{A}$ with the property that $h_\alpha \equiv h_{\alpha\times\psi} \mod 1$. On the other hand, both $x$ and $m$ denote the lines in $\mathcal{A}$ such that $h_\alpha \equiv h_{\alpha\times\psi} + 1/2 \mod 1$; the difference between $x$ lines and $m$ lines is that the former satisfy $x \times \psi \neq x$ while the latter satisfy $m \times \psi = m$. Finally, the state with an open line denotes the anyon $\beta$ around the spatial torus and the anyon $\psi$ running along the time direction, cf. figure 1.1. (From the point of view of the spatial torus, the line operator $\psi$ looks like a puncture, i.e., a local operator; it is essentially a constant spinor, a zero-mode, which explains why it only exists in the odd spin structure).

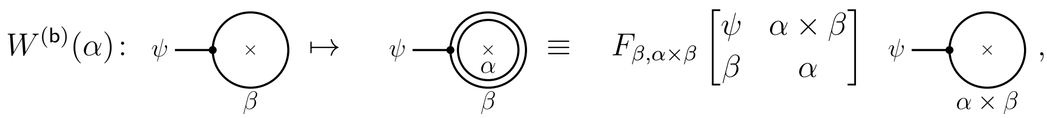
The rationale behind the structure above is the following. Given the bosonic theory, gauging the one-form symmetry generated by $\psi$ means inserting this operator in all possible ways; summing over insertions along the spatial cycles projects the spectrum into the invariant states, and summing over insertions along the time circle introduces twisted sectors. One can check that the states in (1.4.2) are indeed invariant under insertion of $\psi$ along any of the spatial cycles. The twisted sectors are precisely the states with a puncture, which do not live in $\mathcal{H}$ but in the defect Hilbert space $\mathcal{H}^\psi$ instead. The details of this construction are elaborated upon in section 5.1.

The explicit geometric structure of states in (1.4.2) also allows us find how the operators of the spin TQFT act on the different states. For example, the Wilson lines act by inserting an anyon around the cycle they are supported on, and therefore they act as



(1.4.3)

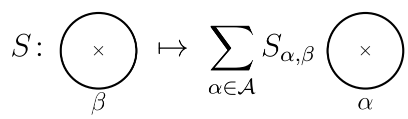
on states without puncture, and as [122]



(1.4.4)

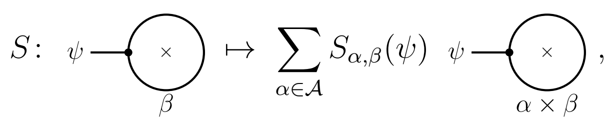
on states with a puncture. Here $S\colon \mathcal{A}\times\mathcal{A}$ denotes the modular matrix of the bosonic parent, and $F$ its $F$-symbols. From these expressions it is trivial to write down the Wilson operators as matrices acting on $\hat{\mathcal{H}}$ (with respect to the basis (1.4.2)).

Similarly, one can also write down how modular transformations map the different Hilbert spaces $\hat{\mathcal{H}}(T^2_{XY})$ into each other. For example, an $S$-transformation acts as



(1.4.5)

on states without puncture, and as



(1.4.6)

on states with a puncture. Here $S(\psi)$ is the $S$-matrix of the bosonic parent in the once-punctured torus (cf. (5.1.18)). Given these two expressions one can easily check that $S$-transformations reshuffle the different spin structures as expected, namely $(X,Y)\mapsto(Y,X)$. Identical considerations hold for $T$-transformations (these are actually simpler because they do not see the puncture, so the formula $T=\mathrm{diag}(e^{2\pi(h-c/24)})$ is still valid for states in $\mathcal{H}^\psi$).

The final important remark concerning the fermionic Hilbert space $\hat{\mathcal{H}}$ is that it is a super-vector space, i.e., its states are either bosons or fermions. Given that $(-1)^F$ is, by definition, dual to the gauged one-form symmetry $\mathbb{Z}_2^{(1)}$, it is clear that the states charged under the former correspond to the states coming from the twisted sector, i.e., the fermions in $\hat{\mathcal{H}}$ are precisely those that include a $\psi$-puncture. In this sense, $(-1)^F$ is trivial in the even spin structure Hilbert spaces, and in the odd spin structure Hilbert space it equals $+1$ on $x$-lines and $-1$ on $m$-lines. In a fermionic theory the modular group is no longer $SL_2(\mathbb{Z})$, but rather a $\mathbb{Z}_2^F$ extension, known as the metaplectic group $Mp_1(\mathbb{Z})$, defined by the relations

$$S^2=(ST)^3,\qquad S^4=(-1)^F\,. \tag{1.4.7}$$

The torus Hilbert spaces of spin TQFTs realize a unitary representation of this group.

We next illustrate all these considerations by explicitly working out several specific examples of spin TQFTs. We show that time-reversal invariant theories with $\mathsf{T}^2 = (-1)^F$ with $\mathbb{Z}_{16}$ anomalies $\nu = 2 \mod 4$ have time-reversal in the Hilbert space realized as

$$\mathsf{T}^2 = (-1)^F \times (-1)^{\mathrm{Arf}(T^2)}\,, \tag{1.4.8}$$

thus exhibiting the anomalies associated with the Arf layer. We then show that spin TQFTs with $\nu$ odd have a time-reversal symmetry that anticommutes with $(-1)^F$ on $\mathcal{H}_{\text{R-R}}$

$$\{\mathsf{T}, (-1)^F\} = 0\,, \tag{1.4.9}$$

thus exhibiting the anomalies associated with the $\psi$ layer

### 1.4.1 $\nu = 2 \mod 4$: Arf layer

In this section we consider time-reversal invariant theories with $\nu = 2 \mod 4$ and show that they realize the expected behavior associated with the Arf layer. We work out in detail here the example of the semion-fermion theory which has $\nu = 2$, although the same behaviour is observed in any theory with $\nu = 2 \mod 4$. Other common examples of time-reversal invariant spin TQFTs are $\mathrm{Sp}(n)_n$ and $\mathrm{SO}(n)_n$, which have $\nu = 2n$ and $\nu = n$, respectively. One can check that e.g. $\mathrm{Sp}(3)_3$ and $\mathrm{SO}(6)_6$, which have $\nu = 6$, exhibit the same behaviour. We do not reproduce the explicit computation here because the matrices are very large and the details do not contain any new ingredients.

The semion-fermion theory is a fermionic TQFT with four anyons: the vacuum $\mathbf{1}$, a semion $s$, a transparent fermion $\psi$, and the composite $s \times \psi$. A Chern-Simons realization of this theory is $\mathrm{U}(1)_2 \times \{\mathbf{1}, \psi\}$, where $\mathrm{U}(1)_2 = \{\mathbf{1}, s\}$ is the TQFT of a single semion, and $\psi$ denotes a transparent fermion. The invertible factor can be written as $\{\mathbf{1}, \psi\} = \mathrm{SO}(n)_1$ for any $n \in \mathbb{Z}$; the most convenient choice is $n = -2$, so that the theory has vanishing central charge (as required by time-reversal). In other words, we shall consider $\mathrm{U}(1)_2 \times \mathrm{U}(1)_{-1}$, whose Lagrangian reads

$$\mathcal{L} = \frac{1}{4\pi}(2a\,\mathrm{d}a - b\,\mathrm{d}b)\,, \tag{1.4.10}$$

where $a, b$ are $\mathrm{U}(1)$ connections. The time-reversal symmetry of this Lagrangian acts as follows [4, 111]:

$$\mathsf{T}(a) = a - b, \qquad \mathsf{T}(b) = 2a - b\,. \tag{1.4.11}$$

One way of constructing the Hilbert space of this theory is to take $\mathrm{U}(1)_2 \times \mathrm{U}(1)_{-4}$, which is bosonic, and condense the line $\psi = (0, 2)$; this is a fermionic quotient, and so the result is a spin TQFT, where $\psi$ becomes transparent.

We begin by constructing the Hilbert space of the bosonic parent, $\mathrm{U}(1)_2 \times \mathrm{U}(1)_{-4}$. This is a $2 \times 4 = 8$-dimensional space, whose basis can be taken to be

$$|\alpha, \beta\rangle = W^{(\mathsf{b})}(\alpha, \beta)|0\rangle, \qquad (\alpha, \beta) \in \mathbb{Z}_2 \times \mathbb{Z}_4\,, \tag{1.4.12}$$

where $|0\rangle$ denotes the vacuum state – the empty torus – and $W$ denotes a Wilson line:

$$W^{(\mathsf{c})}(\alpha, \beta) := \exp i \oint_{\mathsf{c}} (\alpha a + \beta b)\,, \tag{1.4.13}$$

for any $\mathsf{c} \in H_1(T^2, \mathbb{Z}) = \mathbb{Z}[\mathsf{a}] \oplus \mathbb{Z}[\mathsf{b}]$.

The Wilson lines along the $\mathsf{b}$-cycle act on a generic state as follows:

$$W^{(\mathsf{b})}(\alpha, \beta)|\alpha', \beta'\rangle = |\alpha + \alpha' \mod 2,\ \beta + \beta' \mod 4\rangle\,. \tag{1.4.14}$$

The action of the Wilson lines associated to other cycles can be obtained using the modular operations. For example, on the $\mathsf{a}$-cycle, one has

$$W^{(\mathsf{a})}(\alpha)|\alpha'\rangle = \frac{S_{\alpha\alpha'}}{S_{\mathbf{1}\alpha'}}|\alpha'\rangle\,, \tag{1.4.15}$$

where $S$ denotes the $S$-matrix of the system. In the theory $\mathrm{U}(1)_2 \times \mathrm{U}(1)_{-4}$, this matrix reads $S_{(\alpha,\beta),(\alpha',\beta')} = e^{\pi i(\beta\beta'/2 - \alpha\alpha')}/2\sqrt{2}$.

We now condense the fermion $\psi = (0, 2)$. The braiding phase of a generic line $(\alpha, \beta)$ with respect to the fermion is $B((\alpha, \beta), \psi) = e^{-i\pi\beta}$, and so the NS- and R-lines are as follows:

$$\begin{aligned} \text{NS}\colon \quad & \mathcal{A}_{\text{NS}} = \{(\alpha, \beta) \colon \beta = 0, 2\} \\ \text{R}\colon \quad & \mathcal{A}_{\text{R}} \ = \{(\alpha, \beta) \colon \beta = 1, 3\} \end{aligned} \tag{1.4.16}$$

Furthermore, there are no fixed-points under fusion with $\psi$. Indeed, the lines are all in two-dimensional orbits, paired as follows:

$$\begin{aligned} \text{NS}\colon \quad & (\alpha, 0) \xleftrightarrow{\times\psi} (\alpha, 2) \\ \text{R}\colon \quad & (\alpha, 1) \xleftrightarrow{\times\psi} (\alpha, 3)\,. \end{aligned} \tag{1.4.17}$$

The lack of fixed-points indicates that there are no Majorana lines in the theory, i.e., all states are bosonic. In the terminology of section 5.1.2, all lines $(\alpha, \beta)$ are $a$-type or $x$-type, depending on whether $\beta$ is even or odd; and there are no $m$-lines.

**Hilbert space.** With these preliminaries in mind, we now construct the torus Hilbert space(s) of the fermionic theory. As in section 5.1.2, the states of the condensed phase are expressed as linear combinations of those of the bosonic parent, and the specific combinations are determined by the spin structure (cf. (5.1.33)):

- If we take **NS-NS** boundary conditions, the two states are

$$\begin{aligned}|0;\text{NS-NS}\rangle &= \frac{1}{\sqrt{2}}(|0,0\rangle + |0,2\rangle)\\ |1;\text{NS-NS}\rangle &= \frac{1}{\sqrt{2}}(|1,0\rangle + |1,2\rangle)\,.\end{aligned} \tag{1.4.18}$$

- If we take **NS-R** boundary conditions, the two states are

$$\begin{aligned}|0;\text{NS-R}\rangle &= \frac{1}{\sqrt{2}}(|0,0\rangle - |0,2\rangle)\\ |1;\text{NS-R}\rangle &= \frac{1}{\sqrt{2}}(|1,0\rangle - |1,2\rangle)\,.\end{aligned} \tag{1.4.19}$$

- If we take **R-NS** boundary conditions, the two states are

$$\begin{aligned}|0;\text{R-NS}\rangle &= \frac{1}{\sqrt{2}}(|0,1\rangle + |0,3\rangle)\\ |1;\text{R-NS}\rangle &= \frac{1}{\sqrt{2}}(|1,1\rangle + |1,3\rangle)\,.\end{aligned} \tag{1.4.20}$$

- If we take **R-R** boundary conditions, the two states are

$$\begin{aligned}|0;\text{R-R}\rangle &= \frac{1}{\sqrt{2}}(|0,1\rangle - |0,3\rangle)\\ |1;\text{R-R}\rangle &= \frac{1}{\sqrt{2}}(|1,1\rangle - |1,3\rangle)\,.\end{aligned} \tag{1.4.21}$$

**Modularity.** As a consistency check, we can study how modular transformations move us around these four Hilbert spaces. Take for example the $S$-transformation. In the bosonic parent, this operation acts as

$$S|\alpha,\beta\rangle = \sum_{\substack{\alpha'\in\mathbb{Z}_2\\ \beta'\in\mathbb{Z}_4}} S_{(\alpha,\beta),(\alpha',\beta')}|\alpha',\beta'\rangle \tag{1.4.22}$$

with $S_{(\alpha,\beta),(\alpha',\beta')} = e^{i\pi(\beta\beta'/2-\alpha\alpha')}/2\sqrt{2}$. Using this we obtain the action of $S$ on the fermionic theory. For example, it acts on the NS-NS states as follows:

$$\begin{aligned}S|0;\text{NS-NS}\rangle &= \frac{1}{\sqrt{2}}\Big[S|0,0\rangle + S|0,2\rangle\Big]\\ &= \frac{1}{4}\Big[|0,0\rangle + |0,1\rangle + |0,2\rangle + |0,3\rangle + |1,0\rangle + |1,1\rangle + |1,2\rangle + |1,3\rangle +\\ &\qquad |0,0\rangle - |0,1\rangle + |0,2\rangle - |0,3\rangle + |1,0\rangle - |1,1\rangle + |1,2\rangle - |1,3\rangle\Big]\\ &= \frac{1}{2}(|0,0\rangle + |0,2\rangle + |1,0\rangle + |1,2\rangle)\,.\end{aligned} \tag{1.4.23}$$

We recognize this state as $\frac{1}{\sqrt{2}}(|0;\text{NS-NS}\rangle+|1;\text{NS-NS}\rangle)$. Through an identical computation one can show that $S$ maps $|1;\text{NS-NS}\rangle$ into $\frac{1}{\sqrt{2}}(|0;\text{NS-NS}\rangle-|1;\text{NS-NS}\rangle)$. In both cases we see that an $S$-transformation maps states in $\hat{\mathcal{H}}_{\text{NS-NS}}$ into $\hat{\mathcal{H}}_{\text{NS-NS}}$, precisely as expected (cf. (5.1.26)); and, moreover, the specific matrix that realizes this transformation is

$$\hat{S}_{\text{NS-NS}\to\text{NS-NS}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\,. \tag{1.4.24}$$

By performing $S$-transformations on the other three Hilbert spaces we see that they are permuted exactly as they should, namely $S\colon \hat{\mathcal{H}}_{s^{\mathsf{a}},s^{\mathsf{b}}}\to\hat{\mathcal{H}}_{s^{\mathsf{b}},s^{\mathsf{a}}}$; and that they act as the following matrices:

$$\hat{S}_{\text{NS-R}\to\text{R-NS}} = \hat{S}_{\text{R-NS}\to\text{NS-R}} = -i\hat{S}_{\text{R-R}\to\text{R-R}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\,. \tag{1.4.25}$$

A $T$-transformation, on the other hand, acts in the bosonic parent as

$$T|\alpha,\beta\rangle = e^{i\pi(\alpha^2/2-\beta^2/4)}|\alpha,\beta\rangle\,, \tag{1.4.26}$$

which induces the following transformation in the fermionic quotient: $T\colon \hat{\mathcal{H}}_{s^{\mathsf{a}},s^{\mathsf{b}}}\to\hat{\mathcal{H}}_{s^{\mathsf{a}},s^{\mathsf{a}}s^{\mathsf{b}}}$, with matrices

$$\hat{T}_{\text{NS-NS}\to\text{NS-R}} = \hat{T}_{\text{NS-R}\to\text{NS-NS}} = \hat{T}_{\text{R-NS}\to\text{R-R}} = \hat{T}_{\text{R-R}\to\text{NS-R}} = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\,. \tag{1.4.27}$$

(The semion-fermion theory is rather degenerate, not least due to the fact that it factorizes into a bosonic TQFT and a trivial spin TQFT; in the general case, the matrices $\hat{S}_{s^{\mathsf{a}},s^{\mathsf{b}}},\hat{T}_{s^{\mathsf{a}},s^{\mathsf{b}}}$ are all typically distinct.)

A final ingredient as regards modularity is the charge-conjugation operation, which acts in homology as $\mathsf{C}\colon(\mathsf{a},\mathsf{b})\mapsto(-\mathsf{a},-\mathsf{b})$. Unlike the $S$- and $T$-operations, charge-conjugation fixes all spin structures: $\mathsf{C}\colon\hat{\mathcal{H}}_{s^{\mathsf{a}},s^{\mathsf{b}}}\to\hat{\mathcal{H}}_{s^{\mathsf{a}},s^{\mathsf{b}}}$. This operation acts on the U(1) connection as $a\mapsto -a$ or, equivalently, on the anyons as $\alpha\mapsto\bar\alpha=-\alpha$. The semion and the fermion are both self-conjugate (cf. $-1=1 \mod 2$), which means that $\mathsf{C}$ acts trivially on all the anyons of the theory. That being said, this operator need not act trivially on the Hilbert space. Its action is easily computed given the expression of the fermionic Hilbert space in terms of the bosonic parent, namely

$$\mathsf{C}|\alpha,\beta\rangle = |-\alpha \mod 2,\, -\beta \mod 4\rangle\,. \tag{1.4.28}$$

For example, this action induces the following action on the quotient theory:

$$\begin{aligned}\mathsf{C}|0;\text{NS-NS}\rangle &= \frac{1}{\sqrt{2}}(\mathsf{C}|0,0\rangle+\mathsf{C}|0,2\rangle)\\ &= \frac{1}{\sqrt{2}}(|0,0\rangle+|0,2\rangle)\\ &= |0;\text{NS-NS}\rangle\,.\end{aligned} \tag{1.4.29}$$

Repeating this operation for the rest of basis vectors, we arrive at

$$\hat{\mathsf{C}}_{\text{NS-NS}\to\text{NS-NS}} = \hat{\mathsf{C}}_{\text{NS-R}\to\text{NS-R}} = \hat{\mathsf{C}}_{\text{R-NS}\to\text{R-NS}} = -\hat{\mathsf{C}}_{\text{R-R}\to\text{R-R}} = \mathbf{1}_2 \tag{1.4.30}$$

or, more succinctly, $\hat{\mathsf{C}} = (-1)^{\text{Arf}(s)}$.

Given the explicit expressions for the $\hat{S}, \hat{T}, \hat{C}$ matrices, we can check that they realize a unitary representation of the modular group. In this case, the lack of Majorana lines means that $(-1)^F$ is trivial, which means that the modular algebra is just that of the regular torus with no punctures. In other words, the modular transformations satisfy $S^2 = (ST)^3 = \mathsf{C}$, with $\mathsf{C}^2 = 1$. The matrices $\hat{S}, \hat{T}, \hat{C}$ calculated above indeed satisfy this algebra, as expected. In checking this one must keep in mind that $\hat{S}, \hat{T}$ do not live in $\text{End}(\hat{\mathcal{H}}_s)$ (unlike in the bosonic case) but rather in $\text{Hom}(\hat{\mathcal{H}}_s, \hat{\mathcal{H}}_{s'})$ (cf. (5.1.41)).

**Wilson lines.** An arbitrary element $|v\rangle \in \hat{\mathcal{H}}_{s^{\mathsf{a}},s^{\mathsf{b}}} \cong \mathbb{C}^{2|0}$ can be written as $|v\rangle = c_0|0; s^{\mathsf{a}}s^{\mathsf{b}}\rangle + c_1|1; s^{\mathsf{a}}s^{\mathsf{b}}\rangle$ for some coefficients $c_0, c_1 \in \mathbb{C}$. Furthermore, all operators $\mathcal{O} \in \text{End}(\mathcal{H}_{s^{\mathsf{a}},s^{\mathsf{b}}})$ can be represented as $2\times 2$ complex matrices. The Wilson lines themselves, in particular, are represented as $2\times 2$ matrices. Their explicit form can be inferred from the expression for our basis vectors in terms of those of the bosonic parent, and the fact that the Wilson lines in the bosonic theory act as in (1.4.14). For example,

$$\begin{aligned} W^{(\mathsf{a})}(1,0)|1;\text{NS-NS}\rangle &= \frac{1}{\sqrt{2}} W^{(\mathsf{a})}(1,0)(|1,0\rangle + |1,2\rangle) \\ &= \frac{1}{\sqrt{2}} e^{-i\pi}(|1,0\rangle + |1,2\rangle) \\ &= -|1;\text{NS-NS}\rangle\,, \end{aligned} \tag{1.4.31}$$

and

$$\begin{aligned} W^{(\mathsf{b})}(1,0)|1;\text{NS-NS}\rangle &= \frac{1}{\sqrt{2}} W^{(\mathsf{b})}(1,0)(|1,0\rangle + |1,2\rangle) \\ &= \frac{1}{\sqrt{2}} (|2,0\rangle + |2,2\rangle) \\ &= +|0;\text{NS-NS}\rangle\,. \end{aligned} \tag{1.4.32}$$

The rest of matrix elements are computed using the same idea. Denoting the semion by $\varsigma = (1,0)$, and the fermion by $\psi = (0,2)$, the end result is the Pauli matrices

$$W^{(\mathsf{a})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(\varsigma) = \sigma^3, \qquad W^{(\mathsf{b})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(\varsigma) = \sigma^1, \qquad W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(\psi) = -s^{\mathsf{c}}\mathbf{1}_2\,. \tag{1.4.33}$$

(As before, the fact that $W^{(\mathsf{c})}(\varsigma)$ is independent of $s^{\mathsf{a}}, s^{\mathsf{b}}$ is rather particular to this simple system; in generic spin TQFTs these matrices depend non-trivially on the spin structure.)

One can easily check that the Wilson lines satisfy the fusion rules of the theory, namely $\varsigma^2 = \psi^2 = \mathbf{1}$.

**Time-reversal.** Finally, we implement time-reversal as an explicit operator in $\hat{\mathcal{H}}_{s^{\mathsf{a}},s^{\mathsf{b}}}$. We write $\mathsf{T}=\tau K$, where $K$ denotes complex conjugation and $\tau\in\mathbb{C}^2\times\mathbb{C}^2$; this factorisation is not canonical, in the sense that $\tau$ and $K$ are separately convention-dependent – only their product is meaningful. We shall fix them by declaring that our basis is real, $K|\alpha;s^{\mathsf{a}}s^{\mathsf{b}}\rangle=|\alpha;s^{\mathsf{a}}s^{\mathsf{b}}\rangle$, where $\alpha=0,1$. In other words, $K$ acts by complex-conjugating the coefficients:

$$K(c_0|0;s^{\mathsf{a}}s^{\mathsf{b}}\rangle+c_1|1;s^{\mathsf{a}}s^{\mathsf{b}}\rangle)\equiv c_0^*|0;s^{\mathsf{a}}s^{\mathsf{b}}\rangle+c_1^*|1;s^{\mathsf{a}}s^{\mathsf{b}}\rangle\,. \tag{1.4.34}$$

Naturally, this definition is not basis-independent. But $\mathsf{T}$, which is the object we care about, is, so this is enough for our purposes.

We recall that $\mathsf{T}$ acts on the U(1) fields as $\mathsf{T}(a)=a-b,\ \mathsf{T}(b)=2a-b$ (cf. (1.4.11)). This induces the following transformation on the Wilson lines:

$$\begin{aligned}W^{(\mathsf{c})}(\varsigma)&=\exp i\oint_{\mathsf{c}}a\mapsto\exp i\oint_{\mathsf{c}}(a-b)\equiv W^{(\mathsf{c})}(\varsigma)W^{(\mathsf{c})}(\psi)\\W^{(\mathsf{c})}(\psi)&=\exp i\oint_{\mathsf{c}}b\mapsto\exp i\oint_{\mathsf{c}}(2a-b)\equiv W^{(\mathsf{c})}(\psi)\end{aligned} \tag{1.4.35}$$

where we have used $W(\varsigma)^2=W(\varsigma^2)=\mathbf{1}_2$. More to the point, time-reversal acts on the anyons by fixing the vacuum and the fermion, and by exchanging $\varsigma\leftrightarrow\varsigma\times\psi$.

With this, time-reversal acts on the Hilbert space as follows:

$$\mathsf{T}\,W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(\alpha)\mathsf{T}^{-1}=W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(\mathsf{T}(\alpha))\qquad\mathsf{c}\in\{\mathsf{a},\mathsf{b}\}, \tag{1.4.36}$$

where $\alpha\in\{\mathbf{1},\varsigma,\psi,\psi\times\varsigma\}$. As $W(\psi)\propto\mathbf{1}_2$, the only non-trivial equation corresponds to the semion, $W(\varsigma)$, which in our basis reads

$$\tau_{s^{\mathsf{a}},s^{\mathsf{b}}}(W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(\varsigma))^*\tau^{-1}_{s^{\mathsf{a}},s^{\mathsf{b}}}=-s^{\mathsf{c}}W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(\varsigma)\qquad\mathsf{c}\in\{\mathsf{a},\mathsf{b}\}. \tag{1.4.37}$$

This is nothing but a set of linear equations in the components of $\tau$, with solution

$$(\tau_{s^{\mathsf{a}},s^{\mathsf{b}}})_{\alpha,\beta}=(-s^{\mathsf{b}})^{\alpha}\delta_{\alpha+\beta+\frac{1}{2}(s^{\mathsf{a}}+1)} \tag{1.4.38}$$

up to an inconsequential global phase, and where $\alpha,\beta=0,1$ and $\delta_x=1$ if $x$ is even, and $\delta_x=0$ if odd.

We next note that

$$\begin{aligned}(\tau\tau^*)_{\alpha,\beta}&=\sum_\gamma\tau_{\alpha,\gamma}\tau^*_{\gamma,\beta}\\&=\sum_\gamma(-s^{\mathsf{b}})^{\alpha+\gamma}\delta_{\alpha+\gamma+\frac{1}{2}(s^{\mathsf{a}}+1)}\delta_{\beta+\gamma+\frac{1}{2}(s^{\mathsf{a}}+1)}\\&=(-s^{\mathsf{b}})^{\frac{1}{2}(s^{\mathsf{a}}+1)}\delta_{\alpha+\beta}\,.\end{aligned} \tag{1.4.39}$$

Finally, observe that the expression above can also be written as $\tau\tau^* = (-1)^{\mathrm{Arf}(s)}$, which means that time-reversal satisfies

$$\mathsf{T}^2 = \tau\tau^* \equiv (-1)^{\mathrm{Arf}(s)} . \tag{1.4.40}$$

In this theory, fermion parity is trivial, $(-1)^F \equiv 1$. Therefore, the equation above means that the time-reversal algebra $\mathsf{T}^2 = (-1)^F$ is deformed when acting on the Hilbert space, in the form

$$\mathsf{T}^2 = (-1)^F \times (-1)^{\mathrm{Arf}(s)} \tag{1.4.41}$$

which is precisely what we expected, given that the theory has $\nu = 2 \mod 4$.

### 1.4.2 $\nu = 1 \mod 2$: fermion layer

In this section we consider time-reversal invariant theories with $\nu$ odd and show that they realize the expected behavior associated with the fermion layer. We work out in detail here the example of $\mathrm{SO}(3)_3$ Chern-Simons theory that has $\nu = 3$. One can repeat the exercise for other theories with $\nu$ odd, such as $\mathrm{SO}(5)_5$, which has $\nu = 5$; the main conclusions are identical.

With this in mind, we next study the spin TQFT $\mathrm{SO}(3)_3$. This is the smallest intrinsically fermionic topological theory, in the sense that it supports both bosonic and fermionic states, unlike the previous section, where all states were bosonic. The presence of fermionic states is directly related to the presence of Majorana lines, i.e., of fixed-points of the condensing fermion. These will introduce new ingredients into the picture.

The bosonic parent of the theory is $\mathrm{Spin}(3)_3 = \mathrm{SU}(2)_6$, which becomes $\mathrm{SO}(3)_3$ upon gauging the $\mathbb{Z}_2$ center. So we first construct the bosonic theory. This theory has seven states, labelled by their isospin: $|j\rangle$, where $j = 0, \frac{1}{2}, 1, \dots, 3$. Under modular transformations, these states transform as

$$\begin{aligned} S|j\rangle &= \sum_{j'} S_{j,j'} |j'\rangle, \qquad S_{j,j'} = \frac{1}{2} \sin \frac{\pi}{8}(2j+1)(2j'+1) \\ T|j\rangle &= e^{\pi i (2j(2j+2)-3)/16} |j\rangle . \end{aligned} \tag{1.4.42}$$

The condensing fermion $\psi$ corresponds to the line $j = 3$. The braiding phase of a generic line $j$ with respect to $\psi$ is $B(j, \psi) = (-1)^{2j}$, which means that the NS-lines are those with integral isospin, and the R-lines are those with half-integral isospin:

$$\begin{aligned} \text{NS}: \quad & \mathcal{A}_{\text{NS}} = \{j = 0, 1, 2, 3\} \\ \text{R}: \quad & \mathcal{A}_{\text{R}} \ = \{j = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}\} . \end{aligned} \tag{1.4.43}$$

The NS-lines are all in two-dimensional orbits, paired up as follows:

$$0 \xleftrightarrow{\times\psi} 3, \qquad 1 \xleftrightarrow{\times\psi} 2 . \tag{1.4.44}$$

On the other hand, in the R sector there is one two-dimensional orbit, and one fixed-point:

$$\frac{1}{2} \overset{\times\psi}{\longleftrightarrow} \frac{5}{2}, \qquad \frac{3}{2} \circlearrowleft \times\psi \ . \tag{1.4.45}$$

In other words, $0, 1, 2, 3$ are all $a$-type; $\frac{1}{2}, \frac{5}{2}$ are both $x$-type; and $\frac{3}{2}$ is $m$-type.

**Hilbert space.** With the information above we have all we need in order to construct the Hilbert space of the fermionic theory. As usual, the states of the condensed phase $SO(3)_3$ are expressed in terms of those of its parent, the specific expression being determined by the choice of spin structure:

- If we take **NS-NS** boundary conditions, the two states are

$$\begin{aligned} |0;\text{NS-NS}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |3\rangle) \\ |1;\text{NS-NS}\rangle &= \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\,. \end{aligned} \tag{1.4.46}$$

- If we take **NS-R** boundary conditions, the two states are

$$\begin{aligned} |0;\text{NS-R}\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |3\rangle) \\ |1;\text{NS-R}\rangle &= \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)\,. \end{aligned} \tag{1.4.47}$$

- If we take **R-NS** boundary conditions, the two states are

$$\begin{aligned} |0;\text{R-NS}\rangle &= \frac{1}{\sqrt{2}}(|1/2\rangle + |5/2\rangle) \\ |1;\text{R-NS}\rangle &= |3/2\rangle\,. \end{aligned} \tag{1.4.48}$$

- If we take **R-R** boundary conditions, the two states are

$$\begin{aligned} |0;\text{R-R}\rangle &= \frac{1}{\sqrt{2}}(|1/2\rangle - |5/2\rangle) \\ |1;\text{R-R}\rangle &= |3/2;\psi\rangle\,, \end{aligned} \tag{1.4.49}$$

  where, we remind the reader, $|\alpha;\beta\rangle$ denotes the anyon $\alpha$ in presence of a $\beta$ puncture (cf. (5.1.6)).

**Modularity.** As a check of the formalism so far, let us construct the modular data associated to these states, and check that it behaves as expected from general considerations. The even-spin-structure Hilbert spaces do not contain punctures, which means their modular data is computed in exactly the same way as in the previous section. For example, performing an $S$-transformation on an NS-R state we expect to obtain an R-NS state, which is easily confirmed:

$$\begin{aligned}
&S|0;\text{NS-R}\rangle \\
&\quad=\frac{1}{\sqrt{2}}(S|0\rangle-S|3\rangle) \\
&\quad=\frac{1}{4}\Big[ \\
&\quad+\sqrt{1-\xi}|0\rangle+|1/2\rangle+\sqrt{1+\xi}|1\rangle+\sqrt{2}|3/2\rangle+\sqrt{1+\xi}|2\rangle+|5/2\rangle+\sqrt{1-\xi}|3\rangle \\
&\quad-\sqrt{1-\xi}|0\rangle+|1/2\rangle-\sqrt{1+\xi}|1\rangle+\sqrt{2}|3/2\rangle-\sqrt{1+\xi}|2\rangle+|5/2\rangle-\sqrt{1-\xi}|3\rangle \\
&\qquad\Big] \\
&\quad=\frac{1}{2}(|1/2\rangle+\sqrt{2}|3/2\rangle+|5/2\rangle) \\
&\quad=\frac{1}{\sqrt{2}}(|0;\text{R-NS}\rangle+|1;\text{R-NS}\rangle)\,,
\end{aligned}\tag{1.4.50}$$

where in the second line we have denoted $\xi=\sin\pi/4=1/\sqrt{2}$.

Acting with $S$ on the rest of even-spin-structure Hilbert spaces we see that they are indeed permuted as $S\colon \hat{\mathcal{H}}_{s^{\mathsf{a}},s^{\mathsf{b}}}\to\hat{\mathcal{H}}_{s^{\mathsf{b}},s^{\mathsf{a}}}$; and, moreover, this action is effected by the following matrices:

$$\begin{aligned}
\hat{S}_{\text{NS-NS}\to\text{NS-NS}}&=\frac{1}{2}\begin{pmatrix}+\sqrt{2-\sqrt{2}} & +\sqrt{2+\sqrt{2}}\\ +\sqrt{2+\sqrt{2}} & -\sqrt{2-\sqrt{2}}\end{pmatrix} \\
\hat{S}_{\text{NS-R}\to\text{R-NS}}=\hat{S}_{\text{R-NS}\to\text{NS-R}}&=\frac{1}{\sqrt{2}}\begin{pmatrix}+1 & +1\\ +1 & -1\end{pmatrix}\,.
\end{aligned}\tag{1.4.51}$$

$T$-transformations work similarly. For example, they should map states in the NS-NS sector into the NS-R sector, which is indeed what happens:

$$\begin{aligned}
T|0;\text{NS-NS}\rangle&=\frac{1}{\sqrt{2}}(T|0\rangle+T|3\rangle) \\
&=e^{-3\pi i/16}\frac{1}{\sqrt{2}}(|0\rangle-|3\rangle) \\
&=e^{-3\pi i/16}|0;\text{NS-R}\rangle\,.
\end{aligned}\tag{1.4.52}$$

Acting with $T$ on the rest of basis vectors, one confirms that $T$-transformations map

$T\colon \hat{\mathcal{H}}_{s^{\mathsf{a}},s^{\mathsf{b}}} \to \hat{\mathcal{H}}_{s^{\mathsf{a}},s^{\mathsf{a}}s^{\mathsf{b}}}$, through the following matrices:

$$\begin{aligned}\hat{T}_{\text{NS-NS}\to\text{NS-R}} = \hat{T}_{\text{NS-R}\to\text{NS-NS}} &= e^{-3\pi i/16}\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}\\ \hat{T}_{\text{R-NS}\to\text{R-NS}} &= \begin{pmatrix}1 & 0\\ 0 & e^{3\pi i/4}\end{pmatrix}\,.\end{aligned} \tag{1.4.53}$$

One can easily check that all the expected properties of the modular group (i.e., (5.1.41)) are satisfied, where $(-1)^F = +\mathbf{1}_2$, as all states are bosonic.

The odd-spin-structure Hilbert space $\hat{\mathcal{H}}_{\text{R-R}}$ is much more interesting. The state $|0;\text{R-R}\rangle \sim |1/2\rangle - |5/2\rangle$ contains no punctures, so it is a boson, whereas the state $|1;\text{R-R}\rangle = |3/2;\psi\rangle$ has a $\psi$-puncture, so it is a fermion. In other words, in the R-R sector the fermion parity operator is non-trivial, $(-1)^F = \sigma_z$. This makes the analysis of modular transformations more involved. In particular, these transformations should not mix these two states, and they should not take us outside $\hat{\mathcal{H}}_{\text{R-R}}$ (as the $s = (+1,+1)$ spin structure is fixed by all of the mapping class group). These expectations are confirmed by direct computation. For example, acting on $|0;\text{R-R}\rangle$ with an $S$-transformation we get

$$\begin{aligned}S|0;\text{R-R}\rangle &= \frac{1}{\sqrt{2}}(S|1/2\rangle - S|5/2\rangle)\\ &= \frac{1}{\sqrt{2}}(|1/2\rangle - |5/2\rangle)\\ &= |0;\text{R-R}\rangle\end{aligned} \tag{1.4.54}$$

which is indeed in $\hat{\mathcal{H}}_{\text{R-R}}$ (and has not mixed with the fermion, as it never could: modular transformations do not mix configurations with different punctures).

The action of $S$ on $|1;\text{R-R}\rangle = |3/2;\psi\rangle$ is more subtle, because the state contains a puncture, so we need the $S$-matrix in the once-punctured torus. This matrix is given by (5.1.18)

$$\begin{aligned}S_{3/2,3/2}(\psi) &= \sum_{j=0}^{3}\frac{\theta_j}{\theta_{3/2}^2}S_{0,j}F_{3/2,3/2}\begin{bmatrix}\psi & 3/2\\ 3/2 & j\end{bmatrix}\\ &= \sum_{j=0}^{3} e^{\pi i(2j(j+1)-15)/8}\times\frac{1}{2}\sin\frac{\pi}{8}(2j+1)\times(-1)^j\\ &= (-1)^{7/4}\,,\end{aligned} \tag{1.4.55}$$

which means that, altogether,

$$\hat{S}_{\text{R-R}\to\text{R-R}} = \begin{pmatrix}1 & 0\\ 0 & (-1)^{7/4}\end{pmatrix}\,. \tag{1.4.56}$$

$T$-transformations, on the other hand, do not care about the puncture, so they are just given by the spin of the states:

$$\hat{T}_{\text{R-R}\to\text{R-R}} = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{3/4} \end{pmatrix} . \tag{1.4.57}$$

These two matrices are also easily seen to satisfy the expected modular properties, namely they are unitary and obey the algebra of $Mp_1(\mathbb{Z})$, to wit, $\hat{S}^2 = (\hat{S}\hat{T})^3$, $\hat{S}^4 = (-1)^F$.

**Wilson lines.** Given the choice of basis for the different Hilbert spaces as above, one can express the operators of the theory – the Wilson lines – as $2 \times 2$ complex matrices. The even-spin-structure Hilbert spaces contain no punctures, so the basic idea is identical to the previous sections, namely we induce the action on the quotient theory from that of its bosonic parent. In the bosonic parent the Wilson lines act as

$$W^{(\mathsf{b})}(j)|j'\rangle = |j \times j'\rangle \equiv \sum_{j''=|j-j'|}^{\min(j+j',6-j-j')} |j''\rangle , \tag{1.4.58}$$

which means that, for example,

$$\begin{aligned} W^{(\mathsf{b})}(1)|0;\text{NS-NS}\rangle &= \frac{1}{\sqrt{2}}(W^{(\mathsf{b})}(1)|0\rangle + W^{(\mathsf{b})}(1)|3\rangle) \\ &= \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) , \end{aligned} \tag{1.4.59}$$

which we identify as $|1;\text{NS-NS}\rangle$. Similarly, $W^{(\mathsf{b})}(1)|1;\text{NS-NS}\rangle = |0;\text{NS-NS}\rangle + 2|1;\text{NS-NS}\rangle$. The $\mathsf{a}$-cycle computation is analogous: in the bosonic parent Wilson lines act as

$$W^{(\mathsf{a})}(j)|j'\rangle = \frac{S_{j,j'}}{S_{0,j'}}|j'\rangle , \tag{1.4.60}$$

which implies that

$$\begin{aligned} W^{(\mathsf{a})}(1)|0;\text{NS-NS}\rangle &= \frac{1}{\sqrt{2}}(W^{(\mathsf{a})}(1)|0\rangle + W^{(\mathsf{a})}(1)|3\rangle) \\ &= \frac{1}{\sqrt{2}}(1+\sqrt{2})(|0\rangle + |3\rangle) , \end{aligned} \tag{1.4.61}$$

which equals $(1+\sqrt{2})|0;\text{NS-NS}\rangle$. Repeating this calculation on all even-spin-structure Hilbert spaces, we obtain the following collection of matrices:

$$\begin{aligned} W^{(\mathsf{a})}_{\text{NS-NS}}(1) &= \begin{pmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{pmatrix} & W^{(\mathsf{b})}_{\text{NS-NS}}(1) &= \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \\ W^{(\mathsf{a})}_{\text{NS-R}}(1) &= \begin{pmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{pmatrix} & W^{(\mathsf{b})}_{\text{NS-R}}(1) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ W^{(\mathsf{a})}_{\text{R-NS}}(1) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & W^{(\mathsf{b})}_{\text{R-NS}}(1) &= \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} , \end{aligned} \tag{1.4.62}$$

The same computation for the rest of NS-lines yields $W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(3)=-s^{\mathsf{c}}\mathbf{1}_2$ for the transparent fermion, and $W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(2)=W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(1)W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(3)$ (which is the expected relation given the fusion rule $2=1\times 3$, i.e., that the lines $j=1,2$ are paired up through fusion with $j=3$). One can also check that these matrices satisfy the algebra required by the fusion rule $1\times 1=0+1+2$, namely $W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(1)^2=\mathbf{1}_2+(1-s^{\mathsf{c}})W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(1)$. Finally, it is also checked that, under modular transformations, these matrices are permuted as they should, e.g. $S_{s^{\mathsf{a}},s^{\mathsf{b}}}W^{(\mathsf{a})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(\alpha)(S_{s^{\mathsf{a}},s^{\mathsf{b}}})^\dagger=W^{(\mathsf{b})}_{s^{\mathsf{b}},s^{\mathsf{a}}}(\bar{\alpha})$.

We now move on to the odd-spin-structure sector, the R-R Hilbert space. The bosonic state $|0;\text{R-R}\rangle\sim|1/2\rangle-|5/2\rangle$ contains no punctures, so it behaves in the same manner as the states in the even-spin-structure sector, for example

$$\begin{aligned}W^{(\mathsf{b})}(1)|0;\text{R-R}\rangle&=\frac{1}{\sqrt{2}}(W^{(\mathsf{b})}(1)|1/2\rangle-W^{(\mathsf{b})}(1)|5/2\rangle)\\&=\frac{1}{\sqrt{2}}(|1/2\rangle+|3/2\rangle-|3/2\rangle-|5/2\rangle)\\&=|0;\text{R-R}\rangle\,.\end{aligned}\tag{1.4.63}$$

The fermionic state $|1;\text{R-R}\rangle=|3/2;\psi\rangle$, on the other hand, requires using the data of the once-punctured torus, cf. (5.1.11):

$$\begin{aligned}W^{(\mathsf{b})}(1)|1;\text{R-R}\rangle&=W^{(\mathsf{b})}(1)|3/2;\psi\rangle\\&=F_{3/2,3/2}\begin{bmatrix}3&3/2\\3/2&1\end{bmatrix}|3/2;\psi\rangle\end{aligned}\tag{1.4.64}$$

which evaluates to $-|3/2;\psi\rangle$.

The $\mathsf{a}$-cycle does not see the puncture (cf. (5.1.10)), and so its evaluation is straightforward. All in all, the Wilson lines in the R-R sector read

$$W^{(\mathsf{a})}_{\text{R-R}}(1)=W^{(\mathsf{b})}_{\text{R-R}}(1)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\,,\tag{1.4.65}$$

together with $W^{(\mathsf{c})}_{\text{R-R}}(3)=-\mathbf{1}_2$ for the transparent fermion, and $W^{(\mathsf{c})}_{\text{R-R}}(2)=-W^{(\mathsf{c})}_{\text{R-R}}(1)$ (as expected from the fusion rule $2=1\times 3$). It is easily checked that these matrices behave properly under modular transformations, e.g. $S_{\text{R-R}}W^{(\mathsf{a})}_{\text{R-R}}(\alpha)(S_{\text{R-R}})^\dagger=W^{(\mathsf{b})}_{\text{R-R}}(\bar{\alpha})$.

**Time-reversal.** Finally, we discuss the behaviour of the theory under time-reversal. Recall that $\mathrm{SO}(3)_3$ is time-reversal symmetric thanks to the level-rank duality [112]

$$\mathrm{SO}(3)_3\;\longleftrightarrow\;\mathrm{SO}(3)_{-3}\times\mathrm{SO}(9)_1\tag{1.4.66}$$

A key aspect of this duality is that time-reversal is not really a symmetry of $\mathrm{SO}(3)_3$, but rather a map $\mathsf{T}\colon\mathrm{SO}(3)_3\mapsto\mathrm{SO}(3)_3\times\mathrm{SO}(9)_{-1}$. The factor $\mathrm{SO}(9)_1$ is invertible, so we should think of $\mathrm{SO}(3)_3$ being time-reversal invariant only if we mod out by SPTs. In the strict sense, it is not.

In the $\mathrm{U}(1)_k$ case this obstruction was easily circumvented: the duality $\mathrm{U}(1)_k \leftrightarrow \mathrm{U}(1)_{-k} \times \mathrm{SO}(4)_1$ could be rewritten as $\mathrm{U}(1)_k \times \mathrm{U}(1)_{-1} \leftrightarrow \mathrm{U}(1)_{-k} \times \mathrm{U}(1)_{+1}$, i.e., we could break up the invertible factor into two, and split them symmetrically into the two theories. In this situation, we would say time-reversal is not a symmetry of $\mathrm{U}(1)_k$, but rather of $\mathrm{U}(1)_k \times \mathrm{U}(1)_{-1}$: this theory is identical to its conjugate, even taking into account SPTs.

In the $\mathrm{SO}(3)_3$ case, no such solution is possible: the SPT $\mathrm{SO}(9)_1$ cannot be split into two equal factors; such splitting would require fractional levels $\mathrm{SO}(9/2)_1^2$, which is not well-defined as a $3d$ theory.

An equivalent way to phrase this discussion is by thinking of the central charge – indeed, this number is what classifies $3d$-SPTs with no symmetry. Time-reversal always maps $c$ into $-c$, which means a theory can only be time-reversal invariant, in the strict sense, if $c = 0$. If $c \neq 0$, we may be able to correct this by multiplying by a suitable SPT, but this is not always possible. Indeed, the SPT $\mathrm{SO}(n)_1$ has $c = n/2$, which means we can only correct the central charge in multiples of $1/2$. In other words, the minimal SPT has $c = 1/2$, corresponding to a single edge Majorana fermion. Any other SPT will consist of an integral number of copies of this system.

In the $\mathrm{U}(1)_k$ case, the central charge takes value $c = 1$, so this obstruction is avoidable: we just have to tensor the theory with two copies of the Majorana fermion, i.e., $\mathrm{SO}(1)_1^2 = \mathrm{U}(1)_1$. This makes the central charge of the product theory, $\mathrm{U}(1)_k \times \mathrm{U}(1)_{-1}$, vanish, making it a valid candidate for a time-reversal invariant theory. In the $\mathrm{SO}(3)_3$ case, $c = 9/4$, which is not a multiple of $1/2$, which means no redefinition can correct the central charge. The theory is not, and cannot be made, time-reversal invariant in the strict sense. Only in the relative sense when we think of QFTs as absolute theories, modulo invertible ones.

More generally, if a given theory $A$ is known to be time-reversal invariant in the relative sense, then necessarily $c \propto 1/4$. Indeed, time-reversal maps $A$ into $\bar{A}$, modulo some SPT, and so $A \leftrightarrow \bar{A} \times \mathrm{SO}(n)_1$ for some $n$. The central charge of $A$ therefore satisfies $c(A) = -c(A) + n/2$, which means that $c(A) = n/4$, as claimed. If $c = 0 \mod 1/4$, i.e., if $c \propto 1/2$, then the theory can be made time-reversal invariant in the strict sense, by considering $A \times \mathrm{SO}(2c(A))_{-1}$, whose central charge vanishes. If $c \neq 0 \mod 1/4$, this is not possible. In other words, $c \mod 1/4$ measures the most primitive obstruction to being time-reversal invariant as a pure $3d$ theory. This is the $\nu_1$ layer.[29]

The discussion above is set up in the framework of spin TQFTs, but an analogous situation happens in other families of theories. For example, the minimal *bosonic* SPT is $(E_8)_1$, which has $c = 8$, which means that bosonic time-reversal invariant theories always have $c \propto 4$, and that $c \mod 4$ is the first layer in the anomaly of time-reversal invariance.

Going back to our example of $\mathrm{SO}(3)_3$, let us see what we can say about time-reversal,

[29] See [123] for a very similar recent discussion.

neglecting the fact that it is not an operator acting on $SO(3)_3$, but rather a map from this theory into $SO(3)_3 \times SO(9)_{-1}$. Time-reversal acts on the lines of $SO(3)_3$ as $1 \leftrightarrow 2 \equiv 1 \times \psi$. If we ignore the SPT, this descends to the Hilbert space action

$$\begin{aligned} \tau\,(W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(1))^*\,\tau^{-1} &= W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(2) \\ &= -s^{\mathsf{c}} W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(1) \end{aligned} \tag{1.4.67}$$

where $\mathsf{T} = \tau K$. The solution to this matrix equation is

$$\begin{aligned} \tau_{\text{NS-NS}} &\propto \mathbf{1}_2 \\ \tau_{\text{NS-R}} &\propto \sigma^z \\ \tau_{\text{R-NS}} &\propto \sigma^x \end{aligned} \tag{1.4.68}$$

for the even spin structures, and

$$\tau_{\text{R-R}} = \begin{pmatrix} 0 & z_1 \\ z_2 & 0 \end{pmatrix} \tag{1.4.69}$$

for the odd spin structure, where $z_{1,2} \in \mathbb{C}$ are some arbitrary coefficients.

Finally, recall that $(-1)^F = \mathbf{1}_2$ for even spin structure, and $(-1)^F = \sigma^z$ for the odd spin structure. It is clear from these expressions that $\tau$ commutes with $(-1)^F$ for even spin structures, and anti-commutes for the odd spin structure

$$\{\mathsf{T}, (-1)^F\} = 0\,, \tag{1.4.70}$$

precisely as expected from a system with $\nu$ odd and associated with the $\psi$ layer. We have also established that this behavior is present in other time-reversal invariant theories with $\nu$ odd.

# Chapter 2

# Infrared phases of 2d QCD.

**Authorship.** The content of this chapter is taken almost verbatim from the paper [2] written in collaboration with Jaume Gomis and Matthew Yu.

**Abstract.** We derive the necessary and sufficient conditions for a $2d$ gauge theory to develop a mass gap, by studying the QCD Hamiltonian. The conditions can be explicitly solved, and we provide the complete list of all $2d$ QCD theories that have a quantum mechanical gap in their spectrum. The list of gapped theories includes QCD models with quarks in vector-like as well as chiral representations. The gapped theories consist of several infinite families of classical gauge groups with quarks in rank 1 and 2 representations, plus a finite number of isolated cases. We also put forward and analyze the effective infrared description of QCD — TQFTs for gapped theories and CFTs for gapless theories — and exhibit several interesting features in the infrared, such as the existence of non-trivial global 't Hooft anomalies and emergent supersymmetry. We identify $2d$ QCD theories that flow in the infrared to celebrated CFTs such as minimal models, Wess-Zumino-Witten and Kazama-Suzuki models.

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## 2.1 Introduction

A central theme in physics is unraveling the low energy phenomena that emerges from a physical system described by a collection of microscopic degrees of freedom and interactions. The long distance behavior of the system crucially depends on whether the spectrum of the Hamiltonian is gapped or gapless, but determining which phase is realized is often a nonperturbative problem.

In broad terms, a gapped system is described at low energies by a topological quantum field theory (TQFT), while the asymptotic low energy dynamics of a gapless one is captured by a conformal field theory (CFT).[30] Ascertaining whether a system flows to a TQFT or a CFT, and to which one, can be out of reach because of large quantum fluctuations, which are responsible for a wealth of low energy phenomena.

QCD theories are an important class of strongly coupled systems in which it is nontrivial to postulate the infrared dynamics. Determining whether Yang-Mills theory with gauge group $G$ coupled to massless quarks in a representation $R$ of $G$ in $d \leq 4$ spacetime dimensions is gapped or gapless remains an open problem. We henceforth refer to such theories of massless quarks and gluons as *QCD theories*. The following qualitative picture is expected:

- QCD theories without quarks, that is, pure Yang-Mills theory, are believed to be gapped. For simply connected gauge group $G$, the infrared is described by the trivial TQFT.[31]

- QCD theories with a large number of quarks – more precisely, with a large Dynkin index[32] – are gapless. In $4d$, this is by virtue of the beta function [124, 125] being positive for a sufficiently large number of quarks, which implies that the infrared is described by a CFT of free massless particles. In $3d$, the fact that QCD theories flow to a weakly coupled CFT can be established in the limit of large Dynkin index [25, 126].

While gapped QCD theories in $4d$ have been known for some time [42, 43], it is only recently that examples of gapped QCD theories in $3d$, together with their infrared TQFTs, have been put forward [30] (see also [3, 113, 127–130]). Little is otherwise known about whether

[30]The CFT can be either a symmetry preserving nontrivial fixed point of the renormalization group, the extreme infrared limit of the nonlinear theory of Goldstone bosons when the vacuum spontaneously breaks a continuous symmetry, or free massless particles in a symmetric vacuum (e.g. infrared free gauge theories).

[31]In $4d$ the theory has a unique vacuum for $\theta \neq \pi$ while for $\theta = \pi$ the time-reversal symmetry is spontaneously broken and there are two trivially gapped vacua [40]. Yang-Mills theory in $3d$ can be enriched by a Chern-Simons term and then the theory in the infrared is gapped and described by a nontrivial TQFT.

[32]$N_F$ fermions in a representation $R$ of $G$ has Dynkin index $N_F \times I(R)$, where $\mathrm{tr}(t^a_R t^b_R) = I(R)\delta^{ab}$.

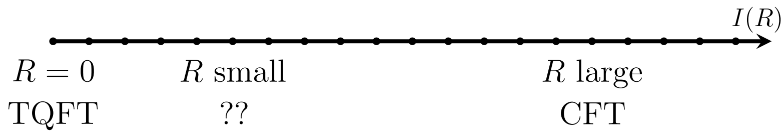


Figure 2.1: Diagram describing infrared dynamics of QCD with massless quarks in a representation $R$ of the gauge group $G$. The theory without quarks is expected to be gapped, and is gapless for large enough $R$. The intermediate regime where the representation $R$ is small remains an open problem.

a given QCD theory is gapped or not, and which TQFT/CFT describes its infrared limit (see fig. 2.1).

In this paper we determine all the QCD theories in $2d$ that are gapped, and therefore those that are gapless. The full classification of gapped QCD theories is summarized in tables 2.1 and 2.2.

In $2d$ QCD, the quark content is specified by a pair of representations $(R_\ell, R_r)$ of the gauge group $G$ acting on the left and right chiral quarks. We denote such a QCD theory by $(G; R_\ell, R_r)$.[33] We derive the necessary and sufficient conditions for a QCD theory $(G; R_\ell, R_r)$ to be gapped by analyzing the explicit lighcone and temporal Hamiltonians of QCD. Lightcone quantization, where $x^+$ *and* $x^-$ are time, and the canonical Hamiltonian formalism, where $x^0$ is time, yield exactly the same conditions.

From our Hamiltonian analysis, the following criterion is derived: a QCD theory is gapless if and only if there exists a canonical, chiral, dimension 2 primary operator of the quark current algebra constructed from either the left chiral quarks or right chiral quarks. A QCD theory is gapped if and only if both these left and right chiral operators vanish identically. These operator equations, derived by studying the Hamiltonian(s) in the ultraviolet, can be completely solved, yielding the classification of gapped theories.[34] The exhaustive list of gauge groups $G$ and quark contents $(R_\ell, R_r)$ of all the QCD theories that are gapped appears in tables 2.1 and 2.2, corresponding to vector-like and chiral QCD theories respectively. Any

[33]See section 2.2 for the role of the topology of the gauge group $G$ in defining topological sectors, discrete theta angles, gauge anomalies, etc.

[34]The operators equations that are the necessary and sufficient conditions for a QCD theory to be gapped

$$T_{\mathfrak{so}(\dim R_\ell)_1} - T_{G_{I(R_\ell)}} = 0$$
$$\overline{T}_{\mathfrak{so}(\dim R_r)_1} - \overline{T}_{G_{I(R_r)}} = 0$$

correspond to *all* the conformal embeddings into the $\mathfrak{so}(\dim R_\ell)_1$ *and* $\mathfrak{so}(\dim R_r)_1$ current algebras, and are in one-to-one correspondence with Cartan's classification of symmetric spaces. $T_{\mathfrak{so}(\dim R_\ell)_1}/\overline{T}_{\mathfrak{so}(\dim R_r)_1}$ is the canonical energy-momentum tensor of the left/right chiral quarks in the ultraviolet, and $T_{G_{I(R_\ell)}}/\overline{T}_{G_{I(R_r)}}$ is the left/right moving Sugawara energy-momentum tensor of the current algebra $G_{I(R)}$ at level $I(R)$. See section 2.4 for details.

other QCD theory not in the tables is gapless.

| $\mathfrak{g}$ | $R$ | $\mathfrak{g}$ | $R$ |
|---|---|---|---|
| $\forall \mathfrak{g}$ | adjoint | $\mathfrak{su}(2)$ | $\mathbf{5}$ |
| $\mathfrak{so}(N)$ | $\square$ | $\mathfrak{so}(9)$ | $\mathbf{16}$ |
| $\mathfrak{u}(N)$ | $\square_q$ | $F_4$ | $\mathbf{26}$ |
| $\mathfrak{so}(N)$ | $\square\!\square$ | $\mathfrak{sp}(4)$ | $\mathbf{42}$ |
| $\mathfrak{sp}(N)$ | $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ | $\mathfrak{su}(8)$ | $\mathbf{70}$ |
| $\mathfrak{u}(N)$ | $\begin{smallmatrix}\square\\\square\end{smallmatrix}_q$ | $\mathfrak{so}(16)$ | $\mathbf{128}$ |
| $\mathfrak{u}(N)$ | $\square\!\square_q$ | $\mathfrak{so}(10)+\mathfrak{u}(1)$ | $\mathbf{16}_q$ |
| $\mathfrak{su}(M)+\mathfrak{su}(N)+\mathfrak{u}(1)$ | $(\square,\square)_q$ | $E_6+\mathfrak{u}(1)$ | $\mathbf{27}_q$ |
| $\mathfrak{so}(M)+\mathfrak{so}(N)$ | $(\square,\square)$ | $\mathfrak{su}(2)+\mathfrak{su}(2)$ | $(\mathbf{2},\mathbf{4})$ |
| $\mathfrak{sp}(M)+\mathfrak{sp}(N)$ | $(\square,\square)$ | $\mathfrak{su}(2)+\mathfrak{sp}(3)$ | $(\mathbf{2},\mathbf{14})$ |
| | | $\mathfrak{su}(2)+\mathfrak{su}(6)$ | $(\mathbf{2},\mathbf{20})$ |
| | | $\mathfrak{su}(2)+\mathfrak{so}(12)$ | $(\mathbf{2},\mathbf{32})$ |
| | | $\mathfrak{su}(2)+E_7$ | $(\mathbf{2},\mathbf{56})$ |
| $\bigoplus_i \mathfrak{g}_i$ | $\bigoplus_i(\mathbf{1},\dots,R_i,\dots,\mathbf{1})_{\vec{q}_i}$ | | |

Table 2.1: Classification of vector-like gapped QCD theories $(G;R,R)$. $\mathfrak{g}$ denotes the Lie algebra of the gauge group and $R$ the representation of the quarks (given in terms of a Young diagram or dimension of the representation). The global form of the gauge group $G$ is arbitrary, as long as it admits $R$ as a representation. $q\in\mathbb{Z}$ is the charge under the $\mathfrak{u}(1)$ gauge group factor. The left columns include adjoint QCD for arbitrary gauge group and families of theories for the classical groups, while the right columns contain isolated theories. Gapped QCD theories with classical gauge groups must have quarks transforming in rank-one or rank-two representations, any other representation leading to a gapless theory. The bottom entry indicates an arbitrary tensor product of gapped theories $(\mathfrak{g}_i,R_i)$ constructed from the entries in the table that have a U(1) gauge group factor. These theories are coupled together via the $\mathfrak{u}(1)$ matrix of charges $\{\vec{q}_i\}$, which must be non-singular (see section 2.4.3 for details).

Remarkably, there exist chiral QCD theories that are gapped. The complete classification of chiral gapped QCD theories is given in table 2.2. From the classification of vector-like gapped theories in table 2.1, we can construct chiral gapped theories in two ways:

- A chiral QCD theory $(G;R_\ell,R_r)$ is gapped if and only if $(R_\ell,R_r)=(\sigma_\ell\cdot R,\sigma_r\cdot R)$ and the vector-like theory $(G;R,R)$ appears in table 2.1, where $\sigma_\ell,\sigma_r$ are outer automorphisms of $\mathfrak{g}$. Here $\sigma\cdot R$ denotes the action of $\sigma$ on $R$.[35] An example of such a chiral gapped

[35]See table 2.3 for a list of the automorphisms of simple Lie algebras. If $G$ contains a U(1) factor then $\sigma$

| $\mathfrak{g}$ | $(R_\ell, R_r)$ |
|---|---|
| $\mathfrak{g}$ | $(\sigma_\ell \cdot R, \sigma_r \cdot R)$ |
| $\bigoplus_i \mathfrak{g}_i$ | $\bigoplus_i (\mathbf{1}, \ldots, (\sigma_{\ell,i} \cdot R_i, \sigma_{r,i} \cdot R_i), \ldots, \mathbf{1})_{\vec{q}_{\ell,i}, \vec{q}_{r,i}}$ |

Table 2.2: Classification of chiral gapped QCD theories. Here $\mathfrak{g}$ and $R$ label the vector-like gapped theories from table 2.1. $\sigma_{\ell,i}, \sigma_{r,i}$ denote outer automorphisms of $\mathfrak{g}_i$ (such as complex conjugation for simply-laced groups, or triality for $\mathfrak{so}(8)$). $\vec{q}_{\ell,i}, \vec{q}_{r,i}$ are tuples of charges for $\mathfrak{u}(1)$ factors. These charge matrices must have trivial kernel and cancel gauge anomalies, but are otherwise arbitrary (see section 2.4.3 for details).

theory is

$$(\mathrm{Spin}(8); \mathbf{8}_v, \mathbf{8}_c)\,, \tag{2.1.1}$$

corresponding to the triality automorphism acting on the vector-like gapped theory $(\mathrm{Spin}(8); \mathbf{8}_v, \mathbf{8}_v)$ that appears in table 2.1.

- A chiral gapped QCD theory can be constructed by taking arbitrary tensor products of the basic gapped theories with quarks in a complex representation (there are seven such entries in table 2.1). The theories are coupled via the integral matrices $q_\ell$ and $q_r$ that specify the charges under the $\mathfrak{u}(1)$ gauge group factors for the left and right chiral quarks. In order for theory to be gapped these matrices must be non-singular. A concrete example of such a chiral gapped theory is

$$\prod_i \mathrm{U}(n_i) \text{ with quarks } R = \bigoplus_i (\mathbf{1},\ldots,\Box_i,\ldots,\mathbf{1})_{\vec{q}_{\ell,i},\vec{q}_{r,i}}\,, \tag{2.1.2}$$

corresponding to the tensor product of the vector-like gapped theories $(\mathrm{U}(n_i); \Box_q, \Box_q)$, coupled via their $\mathrm{U}(1)$ gauge subgroups.

Having established which QCD theories are gapped and which are gapless, our next goal is to put forward the explicit low energy description of all QCD theories. In the gapped case this means finding the specific topological degrees of freedom carried by the vacua, the infrared TQFT, and in the gapless case finding the specific massless degrees of freedom of the infrared CFT.

Determining the long distance description of a given QCD theory is nontrivial. Unlike the question of whether a QCD theory is gapped or gapless, which can be answered rigorously, the task of finding the specific infrared degrees of freedom requires some guesswork. The most natural and straightforward conjecture is that the infrared description of $(G; R_\ell, R_r)$ is given by the $g^2 \to \infty$ limit of the QCD Lagrangian, since $g$ has mass dimension. This limit

can also be chosen to act on the U(1) charge by reversing its sign.

yields the gauged WZW description of a CFT with the following left and right chiral algebras

$$\frac{\mathrm{SO}(\dim(R_\ell))_1}{G_{I(R_\ell)}} \times \frac{\mathrm{SO}(\dim(R_r))_1}{G_{I(R_r)}} \,. \tag{2.1.3}$$

This coset can be shown to be a TQFT – and hence to correspond to a gapped QCD theory – if and only if $(G; R_\ell, R_r)$ is in table 2.1 or 2.2. In this sense, our results derived from the Hamiltonian analysis are perfectly consistent with the conjectured infrared description. One can study many interesting aspects of QCD beyond the existence of a gap using the infrared coset description (2.1.3). For example, many well-known CFTs, such as minimal models, emerge in the infrared of QCD, which allows us to map non-trivial questions about the dynamics of QCD theories into questions about these CFTs, which can then be answered explicitly.

If a QCD theory has continuous chiral symmetries, then the infrared CFT necessarily contains a Wess–Zumino–Witten (WZW) factor for these symmetries; this sector carries the corresponding perturbative 't Hooft anomalies for the continuous symmetries. Even in the absence of continuous chiral symmetries, the infrared CFT is nontrivial when the QCD theory is gapless, and is based on a chiral algebra without spin one currents.[36] Interestingly, 't Hooft anomaly matching predicts the existence of some hitherto unknown 't Hooft anomalies for discrete global symmetries in these CFTs, such as nonperturbative anomalies for time-reversal symmetry and discrete chiral symmetries.

As an example, the proposal implies that QCD with a classical gauge group and with fundamental quarks flows in the infrared to a WZW CFT:

$$\begin{aligned} \mathrm{SU}(N) + N_F\,\square \quad &\xrightarrow{\text{infrared}} \quad \mathrm{U}(N_F)_N \text{ WZW} \\ \mathrm{SO}(N) + N_F\,\square \quad &\xrightarrow{\text{infrared}} \quad \mathrm{SO}(N_F)_N \text{ WZW} \\ \mathrm{Sp}(N) + N_F\,\square \quad &\xrightarrow{\text{infrared}} \quad \mathrm{Sp}(N_F)_N \text{ WZW}\,. \end{aligned} \tag{2.1.4}$$

These theories indeed carry the 't Hooft anomalies for the continuous flavor symmetries. Moreover, using that the 't Hooft anomalies must match predicts that these WZW models are endowed with several non-trivial global anomalies, which can be exhibited by general arguments or by brute-force computation in specific examples. For instance, the renormalization group flow predicts that the $\mathrm{SO}(N_F)_N$ WZW model has a global anomaly associated with time-reversal symmetry, with $\mathsf{T}^2 = (-1)^F$, which takes the value $NN_F \mod 2$. Many other such examples can be constructed, leading to a wealth of CFTs in the infrared and 't Hooft anomalies thereof.

[36] For example, the QCD theories with $G = \mathrm{SU}(2)$ with a single quark in a spin $j \in \mathbb{Z}$ representation, that is $(\mathrm{SU}(2), j, j)$, has an infrared chiral algebra given by the W-algebra $\mathcal{W}(2, 4, \ldots, 2j)$. For $j = 1, 2$ the theory is gapped and flows to a TQFT, while for $j = 3$ the spin 4 and 6 currents become null and the chiral algebra is the Virasoro algebra, and the theory flows to the fermionic tricritical Ising model.

The plan for the rest of the chapter is as follows. In section 2.2 we set up the stage by carefully analyzing the microscopic description of QCD, its free parameters, topological sectors, and gauge anomalies. In section 2.3 we begin our investigations of the mass gap problem; in particular, here we exploit the symmetries and 't Hooft anomalies of $2d$ theories to constrain as much as possible the theories that can potentially be gapped. We find several simple criteria that automatically force the theory to be gapless, thus reducing considerably the landscape of gapped theories. These criteria alone are not enough to actually prove that a given theory is gapped, so in section 2.4 we turn our attention to an explicit analysis of the Hamiltonian of QCD. We recover the necessary conditions laid out in the previous section, and also find sufficient conditions as well, culminating in a concrete list of gapped theories. In section 2.5 we reconsider our results, this time in light of the conjecture (2.1.3) which proposes a concrete description of QCD at low energies. We give further evidence for the correctness of this conjecture, and subsequently apply it to many explicit examples.

We also include several extra sections (which were initially sub-appendices, but this was not allowed by University formatting code). Section 2.6 can be used as a reference for our conventions, and it contains some technical computations that supplement the main text. Section 2.7 reviews some relevant facts about $2d$ CFTs and, in particular, cosets of the form (2.1.3) that conjecturally encapsulate the low-energy degrees of freedom of QCD. We also work out some examples in some detail. Finally, section 2.8 contains a separate discussion of QCD theories where the gauge group is abelian, i.e., where $G$ consists of factors of U(1) only. While this type of theories is covered by our general discussion from other sections, when studied in isolation one can be more explicit in some of our claims. Also, they illustrate some general features of other non-abelian QCD theories that contain U(1) factors, such as the breaking of some U(1) flavor symmetries as the result of flavor-gauge mixed anomalies.

## 2.2 $2d$ QCD theories

The field content of a $2d$ QCD theory is specified by a choice of gauge group $G$ and a pair of representations $R_\ell$ and $R_r$ of $G$ acting on left and right chiral quarks.[37] We label such a QCD theory by the triple $(G; R_\ell, R_r)$. $G$ is an arbitrary compact, connected Lie group with Lie algebra $\mathfrak{g} = \oplus_I \mathfrak{g}_I \oplus_m \mathfrak{u}(1)_m$, a direct sum of simple Lie algebras $\mathfrak{g}_I$ and abelian Lie algebras $\mathfrak{u}(1)_m$. The Lagrangian of a QCD theory with massless quarks is

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2}\operatorname{tr}(g^{-2}F_{\mu\nu}F^{\mu\nu}) + i\psi_\ell^\dagger D_-\psi_\ell + i\psi_r^\dagger D_+\psi_r\,, \tag{2.2.1}$$

where

$$D_-\psi_\ell = (\partial_- - iA_-^a t_\ell^a)\psi_\ell\,, \qquad D_+\psi_r = (\partial_+ - iA_+^a t_r^a)\psi_r\,, \tag{2.2.2}$$

[37]In $2d$, complex conjugation does not reverse the chirality of a fermion since the chirality matrix $\gamma_3 = \gamma^0\gamma^1$ does not include an $i$, unlike in $4d$ where $\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and conjugation does flip chirality. This implies that the most general $2d$ QCD theory cannot be written using just left chiral fermions, in contrast to $4d$.

$t_\ell^a$ ($t_r^a$) are the generators of the Lie algebra $\mathfrak{g}$ in representation $R_\ell$ ($R_r$) and we have introduced lightcone coordinates $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$, and $A_\pm^a = \frac{1}{2}(A_0^a \pm A_1^a)$. Each gauge group factor has a gauge coupling, which is captured by $g^{-2}$ inside the trace. See section 2.6 for details and conventions. See also table 2.3 for a summary of simple Lie groups and relevant properties.

| $G_{\text{sc}}$ | $\mathrm{SU}(N)$ | $\mathrm{Sp}(N)$ | $\mathrm{Spin}(2N+1)$ | $\mathrm{Spin}(4N)$ | $\mathrm{Spin}(4N+2)$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $h$ | $N$ | $N+1$ | $2N-1$ | $4N-2$ | $4N$ | 12 | 18 | 30 | 9 | 4 |
| $\mathrm{Out}(\mathfrak{g})$ | $\mathbb{Z}_2$ | $\cdot$ | $\cdot$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $Z(G_{\text{sc}})$ | $\mathbb{Z}_N$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_4$ | $\mathbb{Z}_3$ | $\mathbb{Z}_2$ | $\cdot$ | $\cdot$ | $\cdot$ |

Table 2.3: Lie data for the simply-connected simple Lie groups $G_{\text{sc}}$. Here $h$ denotes the dual Coxeter number (defined as the Dynkin index of the adjoint representation). $\mathrm{Out}(\mathfrak{g})$ is the group of outer automorphisms of $\mathfrak{g}$, which corresponds to charge conjugation symmetry of QCD. $Z(G_{\text{sc}})$ is the center of the gauge group, which contains the one-form center symmetry of QCD. For $\mathrm{SU}(2)$, $\mathrm{Out}(\mathfrak{g})$ is trivial, and for $\mathrm{Spin}(8)$, it is enhanced to $\mathrm{Out}(\mathfrak{g}) = \mathbb{S}_3$ (triality).

**Global issues, flux tubes and theta terms.**

A QCD theory requires specifying a global choice of a gauge group $G$ with Lie algebra $\mathfrak{g}$. We consider first QCD with the simply-connected form of the gauge group $G_{\text{sc}}$, which we denote by $(G_{\text{sc}}; R_\ell, R_r)$. Such a QCD theory may have a one-form symmetry $\Gamma$ [32], where $\Gamma \subset Z(G_{\text{sc}})$ is a subgroup of the center (cf. table 2.3).

$2d$ QFTs with a one-form symmetry $\Gamma$ have topological sectors labeled by a representation $\rho \in \Gamma^\vee$ of $\Gamma$, where $\Gamma^\vee$ is the Pontryagin dual group. Physically, a topological sector labeled by $\rho \in \Gamma^\vee$ describes the theory in the presence of a flux tube created by a quark-antiquark pair of charge $\rho$ at $\pm$-infinity [131, 132], a background that preserves Poincaré invariance in $2d$.

We now consider the theory with gauge group $G = G_{\text{sc}}/\Gamma$.[38] Since $G$-bundles are classified by $H^2(M, \pi_1(G)) \cong \Gamma$, the sum over gauge fields in the functional integral can be weighted by a discrete theta term labeled by $\rho \in \Gamma^\vee$, which takes the form of a generalized Stiefel-Whitney class

$$i \int_M w_\rho(G)\,. \tag{2.2.3}$$

We label such a QCD theory by $(G; R_\ell, R_r)_\rho$.

We proceed to prove that:

[38] The discussion can be easily extended to the case $G_{\text{sc}}/K$, where $K \subset \Gamma$.

- $(G_{\rm sc}; R_\ell, R_r)$ with a $\rho$-flux tube is the same as $(G; R_\ell, R_r)_\rho$

The one-form global symmetry $\Gamma$ of $(G_{\rm sc}; R_\ell, R_r)$ implies that there is a topological local operator $U_g$, with $g \in \Gamma$, which acts on line operators.[39] Diagonalizing the topological local operators $U_g$ on the Hilbert space leads to the decomposition

$$\mathcal{H} = \bigoplus_{\rho\in\Gamma^\vee} \mathcal{H}_\rho \,, \tag{2.2.6}$$

where $\rho$ is an irreducible representation of $\Gamma$ and

$$|\psi\rangle \in \mathcal{H}_\rho \quad \Longleftrightarrow \quad U_g|\psi\rangle = \chi_\rho(g)|\psi\rangle \,. \tag{2.2.7}$$

The Hilbert space $\mathcal{H}_\rho$ corresponds to $(G_{\rm sc}; R_\ell, R_r)$ in the presence of a $\rho$-flux tube.

The QCD theory $(G; R_\ell, R_r)_\rho$ can be constructed by gauging the one-form symmetry of $(G_{\rm sc}; R_\ell, R_r)$ tensored with an SPT phase for the one-form symmetry $\Gamma$; such SPTs are labeled by an element $\rho$ of the reduced cobordism group $\tilde{\Omega}^2_{\rm Spin}(B^2\Gamma) \cong \Gamma^\vee$. A nontrivial $\mathrm{SPT}_\rho$ weights the sum over $g \in \Gamma$ that defines $(G; R_\ell, R_r)_\rho$ by gauging $\Gamma$ with the phase

$$\chi^*_\rho(g) \,, \tag{2.2.8}$$

where $\rho \in \Gamma^\vee$ is a representation of $\Gamma$ and $\chi_\rho(g)$ is a character of $\Gamma$ in the representation $\rho$. This is an alternative way to think about the discrete theta term (2.2.3).

Consider a theory $T$ in the presence of a fixed two-form gauge field $B_2$ for the one-form symmetry $\Gamma$, which takes values in $H^2(M, \Gamma) = \Gamma$. The partition function of the theory in such a background is given by

$$Z_T[g] = \sum_{\rho\in\Gamma^\vee} Z_T(\rho)\chi_\rho(g) \,, \tag{2.2.9}$$

where the sum over $\rho$ is due to the Hilbert space structure (2.2.6) and $g \in \Gamma$ labels the choice of background gauge field $B_2$. $Z_T(\rho)$ is the partition function of the theory in the presence of a $\rho$-flux tube.

[39]The charge of a line operator $\mathcal{L}$ under $\Gamma$ is measured by $U_g$ as

$$U_g\, \mathcal{L}\, U_g^{-1} = \chi_\rho(g)\mathcal{L} \,, \tag{2.2.4}$$

where $\rho \in \Gamma^\vee$ is an irreducible representation of $\Gamma$ and $\chi_\rho(g)$ is a character of $\Gamma$ in the representation $\rho$. This means that the spectrum of line operators in the theory can be organized according to their charges under $\Gamma$ as

$$[\mathcal{L}] = \bigoplus_{\rho\in\Gamma^\vee} [\mathcal{L}]_\rho \,, \tag{2.2.5}$$

where line operators in $[\mathcal{L}]_\rho$ carry charge $\rho$.

The theory $T/\Gamma$ obtained by gauging $\Gamma$ has a dual $(-1)$-form symmetry $\Gamma^\vee$ [133–135], and $T/\Gamma$ can be coupled to a background zero-form gauge field for this symmetry, which corresponds to an element $\hat{\rho}$ of $\Gamma^\vee$. The partition of the gauged theory in the presence of this background gauge field is

$$Z_{T/\Gamma}[\hat{\rho}] = \frac{1}{|\Gamma|}\sum_{g\in\Gamma} Z_T[g]\chi^*_{\hat{\rho}}(g)\,, \tag{2.2.10}$$

where $\chi^*_{\hat{\rho}}(\hat{g})$ encodes the coupling of the background two-form gauge field for $\Gamma$ with the zero-form gauge field for $\Gamma^\vee$. This describes the theory $T/\Gamma$ with a discrete theta term labeled by $\hat{\rho}\in\Gamma^\vee$. Using equation (2.2.9) we arrive at

$$Z_{T/\Gamma}[\hat{\rho}] = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\sum_{\rho\in\Gamma^\vee} Z_T(\rho)\chi_\rho(g)\chi^*_{\hat{\rho}}(g) = Z_T(\hat{\rho})\,, \tag{2.2.11}$$

where we have used that $\sum_{g\in\Gamma}\chi_\rho(g)\chi^*_{\hat{\rho}}(g) = |\Gamma|\delta_{\rho,\hat{\rho}}$. Therefore, the partition function of the theory $T/\Gamma$ with a theta term $\rho\in\Gamma^\vee$ is the same as the partition function of the original theory $T$ in the sector with a with a $\rho$-flux tube, thus completing the proof that $(G_{\rm sc}; R_\ell, R_r)$ with a $\rho$-flux tube is the same as $(G; R_\ell, R_r)_\rho$. This implies that it is sufficient to study QCD theories with a simply connected gauge group, which we will do henceforth.

We now turn to the next result

- $(G_{\rm sc}; R_\ell, R_r)$ is gapless if and only if it is gapless in the $\rho = 0$ flux tube sector.

This implies that for the purposes of classifying gapped QCD theories it suffices to consider QCD theories with simply connected gauge group and in the trivial flux tube sector.

This conclusion is a consequence of the fact that the (massless) QCD theory $(G_{\rm sc}; R_\ell, R_r)$ admits topological line operators $\mathcal{L}$ that carry any charge under the one-form symmetry $\Gamma$ [136].[40] Since $\mathcal{L}$ carries one-form symmetry charge $\rho$, it defines a map between the Hilbert spaces $\mathcal{H}_{\rho=0}$ and $\mathcal{H}_\rho$: acting with $\mathcal{L}$ on $\mathcal{H}_{\rho=0}$ creates states in $\mathcal{H}_\rho$. Physically, acting with a topological line $\mathcal{L}$ inserts static probe charges $\rho$ at $\pm$-infinity. Such a topological line operator $\mathcal{L}$ interpolates between the Hamiltonian of the theory in distinct flux tube sectors

$$\mathcal{L}H_{\rho=0} = H_\rho\mathcal{L}\,, \tag{2.2.12}$$

where $H_{\rho=0}$ and $H_\rho$ are the Hamiltonians of the theory in the trivial and $\rho$-flux tube sector respectively. Note that in general $\mathcal{L}$ is a non-invertible topological operator and therefore (2.2.12) cannot be written as a similarity transformation. Since $\mathcal{L}$ is topological it carries vanishing energy density (zero tension). Therefore it cannot lower the energy and the sector with a $\rho$-flux tube is gapless if and only it is gapless in the sector with no string (that is with $\rho = 0$).[41] We note that this conclusion relies on the existence of topological line

[40] The global symmetries of the QCD Lagrangian (2.2.1) together with these topological lines make it technically natural to study the theory without four-fermi terms, cf. [99, 136].

[41] If $|\Omega\rangle$ is the ground state of $H_{\rho=0}$, the ground state of $H_\rho$ is $\mathcal{L}|\Omega\rangle$.

operators, and these are not present generically in the theory with massive quarks, where indeed a massless particle can appear in the theory with a flux tube (see e.g. [137]).

In summary, for the purposes of classifying all gapped QCD theories we can, without loss of generality, consider the theory with simply connected gauge group in the trivial topological sector, without a string. Henceforth, we will use $G$ to refer to the simply connected gauge group or use instead the Lie algebra $\mathfrak{g}$.

**Gauge anomaly cancellation.**

In order to define a consistent QCD theory, the global symmetry $G$ acting on the free fermions in the deep ultraviolet must have no obstructions to being gauged. Therefore all anomalies for $G$ gauge transformations, perturbative and nonperturbative, must cancel. Perturbative anomalies, that is, anomalies associated to $G$ gauge transformations connected to the identity, are classified by the first summand in the free part of the spin cobordism group

$$\mathrm{Free}\big(\Omega^4_{\mathrm{spin}}(BG)\big) = \mathbb{Z}^{|I|+\frac{1}{2}|m|(|m|+1)} \oplus \mathbb{Z}\,, \tag{2.2.13}$$

where $|I|$ and $|m|$ is the number of simple and abelian factors in $\mathfrak{g}$ respectively. These anomalies are determined by a one-loop diagram and encoded in the first line of the anomaly polynomial

$$\begin{aligned}\big(\mathrm{tr}_{R_\ell}\, e^{F/2\pi} - \mathrm{tr}_{R_r}\, e^{F/2\pi}\big)\hat{A}(\mathcal{R})\big|_4 = &\sum_{a,b}\big[\mathrm{tr}(t^a_\ell t^b_\ell) - \mathrm{tr}(t^a_r t^b_r)\big]\frac{F^a\wedge F^b}{8\pi^2}\\ &- \frac{p_1(\mathcal{R})}{24}\big[\dim(R_\ell)-\dim(R_r)\big]\,,\end{aligned} \tag{2.2.14}$$

where $F^a$ is the two-form field strength and $p_1(\mathcal{R})$ the first Pontryagin class for the background metric. Gauge anomaly cancelation requires that the representations $R_\ell$ and $R_r$ of the left and right chiral quarks obey

$$\mathrm{tr}(t^a_\ell t^b_\ell) = \mathrm{tr}(t^a_r t^b_r) \qquad \forall\ a,b\,. \tag{2.2.15}$$

The nontrivial anomaly constraints in (2.2.15) are:[42]

1. $\mathfrak{g}_I$-$\mathfrak{g}_I$ anomaly: $t^a_{\ell,r}$ are generators of the simple Lie algebra $\mathfrak{g}_I$. The anomaly cancelation condition requires that

$$I(R_\ell) - I(R_r) = 0\,, \tag{2.2.16}$$

[42]This is to be contrasted with the $4d$ anomaly cancelation equation $\mathrm{tr}(t^a_\ell\{t^b_\ell, t^c_\ell\}) = 0$ when the theory is written using left chiral fermions, which is nontrivial for the $\mathfrak{g}_I$-$\mathfrak{g}_I$-$\mathfrak{g}_I$, $\mathfrak{g}_I$-$\mathfrak{g}_I$-$\mathfrak{u}(1)_m$ and $\mathfrak{u}(1)_m$-$\mathfrak{u}(1)_n$-$\mathfrak{u}(1)_p$ anomalies. In $2d$ a chiral fermion in any irreducible representation of any $\mathfrak{g}$ contributes to the $\mathfrak{g}_I$-$\mathfrak{g}_I$ anomaly, while in $4d$ only chiral fermions transforming in a complex representation of $\mathrm{SU}(N)$ contribute to the pure $\mathfrak{g}_I$-$\mathfrak{g}_I$-$\mathfrak{g}_I$ anomaly, because the rest of the simple Lie algebras have no cubic Casimir. Since a chiral fermion cannot be given a mass in $2d$, unlike for a $4d$ chiral fermion in a real representation, any $2d$ chiral fermion can potentially contribute to the anomaly and, indeed, it does.

where $I(R)$ is the Dynkin index of the representation $R$, defined by $\mathrm{tr}(t^a t^b) = I(R)\delta^{ab}$. The index of a reducible representation follows from $I(R_1 \oplus R_2) = I(R_1) + I(R_2)$.

2. $\mathfrak{u}(1)_m$-$\mathfrak{u}(1)_n$ anomaly: $t^a_{\ell,r}$ are generators of an abelian Lie algebra. The anomaly cancelation condition is

$$\sum_\ell Q_{\ell,m} Q_{\ell,n} - \sum_r Q_{r,m} Q_{r,n} = 0\,, \tag{2.2.17}$$

where $Q_{\ell,m}$ and $Q_{r,m}$ are the left and right $\mathrm{U}(1)_m$ charges of the quarks.

A global symmetry $G$ may have a more subtle obstruction to being gauged associated to a background $G$ gauge transformation not connected to the identity, like the celebrated SU(2) global anomaly in $4d$ [75]. If the symmetry group $G$ is gauged, like in QCD, global anomalies for $G$ must also cancel for the gauge theory to be consistent. Topologically nontrivial gauge transformations in (compactified) $2d$ flat spacetime are classified by $\pi_2(G)$, which vanishes for any continuous Lie group $G$, and $2d$ gauge theories do not have this type of global anomalies. From the cobordism point of view of anomalies, the vanishing of the anomalies is seen through the fact that $\Omega^3_{\text{spin}}(BG) = 0$ (see e.g. [72]).[43] Therefore the anomaly cancelation conditions (2.2.15) are necessary and sufficient for a QCD theory to be consistent.

Since gravity couples to QCD as a nondynamical background field, it can be afflicted by gravitational anomalies without rendering the theory inconsistent. These anomalies are captured by the second $\mathbb{Z}$ summand in (2.2.13) and by the second line of the anomaly polynomial in (2.2.14).[44] We discuss in the next section the implications that 't Hooft anomalies, including gravitational anomalies, have for the infrared dynamics of QCD theories.

Of course, vector-like theories $(G; R, R)$, with $R_\ell = R_r$, are manifestly free of gauge anomalies. But in $2d$, gauge-anomaly-free chiral QCD theories are abundant. Most of these chiral theories, however, have gravitational anomalies. There are, nonetheless, chiral gauge theories with neither gauge nor gravitational anomalies, i.e., simultaneous solutions to[45]

$$\begin{aligned} \mathrm{tr}(t^a_\ell t^b_\ell) &= \mathrm{tr}(t^a_r t^b_r) \qquad \forall\, a, b \\ \dim(R_\ell) &= \dim(R_r)\,. \end{aligned} \tag{2.2.18}$$

Unlike in $4d$, where the beta-function for the gauge coupling constrains the quark content of $4d$ QCD theories that are strongly coupled in the infrared, any $2d$ QCD flows to strong coupling at low energies. Our first goal is to determine which $2d$ QCD theories are gapped, and which are gapless.

[43] Global anomalies for a discrete symmetry group can be nontrivial. For example $\Omega^3_{\text{spin}}(B\mathbb{Z}_2) = \mathbb{Z}_8$.

[44] A mixed $\mathfrak{u}(1)$-gravity anomaly governed by $\nabla_\mu J^\mu = \alpha R$ can be written down, where $J^\mu$ is the $\mathfrak{u}(1)$ current. But $\alpha = 0$ in a unitary theory. It can be nonvanishing in a nonunitary theory, like in the string theory $bc$ ghost system. Thus there are are no mixed gauge-gravity anomalies in $2d$ QCD.

[45] A simple example of a chiral theory with no gravitational anomalies is $(\mathrm{Spin}(5); \mathbf{35}, \mathbf{5} + \mathbf{30})$. As a matter of fact, this theory has no continuous flavor symmetries, so it does not have any perturbative 't Hooft anomalies whatsoever.

## 2.3 Symmetries, 't Hooft Anomalies and Gaplessness

In this section we use symmetry and 't Hooft anomaly considerations to derive necessary conditions for a $2d$ QFT theory to be gapped. We start with a discussion of symmetries and 't Hooft anomalies and then use them to constraint the phases of $2d$ QFTs.

Symmetries provide a powerful organizing principle parametrizing the most general solution of a QFT consistent with the symmetries. But without further input, either perturbative or nonperturbative, symmetries do not inform the actual dynamics of a physical system.

An 't Hooft anomaly for a global symmetry, diagnosed by violations of Ward identities in the presence of nondynamical background gauge fields for global symmetries, instead, does inform the dynamics of the system. Since 't Hooft anomalies are quantized, they are invariant under symmetric deformations, and define invariants in the space of symmetric QFTs. In particular they are invariant under renormalization group transformations. While 't Hooft anomalies alone cannot determine the dynamics of a system, they rule out any dynamical scenario that does not match the microscopic 't Hooft anomalies. As such, 't Hooft anomalies provide nonperturbative guidance about the dynamics of QFTs.

A system defined at short distances with an 't Hooft anomaly for a symmetry cannot flow in the deep infrared to a trivially gapped theory, as this has vanishing 't Hooft anomalies. A system with an 't Hooft anomaly can flow either to a symmetry-preserving gapless phase or a symmetry breaking phase, which is gapless if the broken symmetry is continuous[46] and a TQFT if the broken symmetry is discrete.[47] If the anomalous symmetry transformation is discrete, the system may also flow to a symmetry preserving gapped phase described by a TQFT with topological order, which can saturate anomalies that are torsion classes.

A system with an 't Hooft anomaly for a continuous symmetry transformation cannot flow to a TQFT because an 't Hooft anomaly for a continuous symmetry implies a nonvanishing correlation function for conserved currents at separated points, and a TQFT, being topological, does not have such correlation functions. This implies that a system with perturbative anomalies, corresponding to anomalies for continuous symmetry transformations connected to the identity, can only flow to a symmetry preserving gapless phase or a symmetry breaking gapless phase.

In $2d$, the fate of a system with an 't Hooft anomaly is further constrained by important theorems. These theorems, once combined with the discussion above, leads to the following implications:

1. Coleman-Mermin-Wagner theorem [138, 139]: a continuous global symmetry cannot be spontaneously broken in $2d$.

[46] More precisely, the anomaly is not torsion.

[47] More precisely, the anomaly is torsion.

A $2d$ system with an 't Hooft anomaly for a continuous symmetry must flow to a symmetry preserving gapless phase.

2. A $2d$ TQFT does not have intrinsic topological order [140]: in $2d$ a symmetry preserving gapped phase cannot saturate 't Hooft anomalies

   A $2d$ system with an 't Hooft anomaly for a discrete symmetry must flow to a symmetry preserving gapless phase or a symmetry breaking gapped phase described by a TQFT.

We are now ready to state the following far-reaching result for the dynamics of $2d$ QFTs:

**Proposition 2.3.1** *A 2d QFT with a continuous chiral global symmetry is symmetry preserving and gapless.*

Consider a QFT with a U(1) global symmetry. The one-form current for the U(1) global symmetry is $J = J_\mu \, \mathrm{d}x^\mu \equiv J_+ \mathrm{d}x^+ + J_- \mathrm{d}x^-$, with $J_\pm = \frac{1}{\sqrt{2}}(J_0 \pm J_1)$. This obeys the conservation equation (see section 2.6 for conventions)

$$\partial_- J_+ + \partial_+ J_- = 0 \,. \tag{2.3.1}$$

This is an operator equation that holds inside any correlation function as long as the location of the current $J$ does not coincide with any operator insertions (conservation may fail at coincident points). Poincaré's lemma implies that locally the current takes the following form

$$J_\mu = \epsilon_{\mu\nu}\partial^\nu \phi \Longleftrightarrow J_\pm = \pm \partial_\pm \phi(x^+, x^-) \,, \tag{2.3.2}$$

where $\phi(x^+, x^-)$ is a scalar operator in the theory.

Consider now a chiral symmetry. A right-moving $\mathrm{U}(1)_r$ symmetry is implemented by a conserved current that is antiselfdual

$$U(1)_r : \; J = -\star J \Longleftrightarrow J_+ = 0 \Longleftrightarrow \partial_+ J_- = 0 \,. \tag{2.3.3}$$

By virtue of (2.3.2), a theory with a $\mathrm{U}(1)_r$ symmetry contains a scalar operator $\phi$ that is right-moving

$$J_+ = 0 \Longrightarrow \phi = \phi(x^-) \,. \tag{2.3.4}$$

Likewise, a left-moving $\mathrm{U}(1)_\ell$ symmetry is generated by a conserved current that is selfdual

$$U(1)_\ell : \; J = +\star J \Longleftrightarrow J_- = 0 \Longleftrightarrow \partial_- J_+ = 0 \,, \tag{2.3.5}$$

and a $\mathrm{U}(1)_\ell$ symmetry implies the existence of a left-moving scalar operator

$$J_- = 0 \Longrightarrow \phi = \phi(x^+) \,. \tag{2.3.6}$$

This implies that a QFT with either a left or a right moving U(1) symmetry is necessarily gapless: the theory has a chiral scalar operator that creates chiral massless states when acting on the vacuum.

This proposition may seem at odds with our discussion above since, typically, symmetries alone cannot determine the infrared phase of a system. The reason that it does in this case is that a chiral U(1) symmetry automatically leads to an 't Hooft anomaly for that symmetry, as we show below. And as we mentioned above, an 't Hooft anomaly for a continuous global symmetry in $2d$ necessarily leads to a symmetry preserving gapless phase.

Consider the renormalization group flow out of a CFT with a $\mathrm{U}(1)_\ell$ symmetry that is triggered by a $\mathrm{U}(1)_\ell$-invariant relevant operator.[48] Since the flow preserves the $\mathrm{U}(1)_\ell$ symmetry, the most general two-point function for the $\mathrm{U}(1)_\ell$ current consistent with dimensional analysis and Poincaré invariance is

$$\langle J_+(x)J_+(0)\rangle = \frac{K_\ell(x^+x^-/\mu^2)}{x^+x^+}\,, \tag{2.3.7}$$

where $\mu$ is a scale generated along the renormalization group flow. In a unitary theory $K_\ell \geq 0$, with $K_\ell = 0$ if and only if $J_+ = 0$. Demanding conservation law of $\mathrm{U}(1)_\ell$ symmetry current at separated points $\partial_- J_+ = 0$ implies that $K_\ell$ is a renormalization group invariant

$$\frac{\partial}{\partial \rho^2} K_\ell = 0\,, \tag{2.3.8}$$

where we have introduced Rindler coordinates $x^\pm = \rho\, e^{\pm\sigma}$, so that

$$\langle J_+(x)J_+(0)\rangle = \frac{k_\ell}{x^+x^+}\,. \tag{2.3.9}$$

In the deep ultraviolet, $K_\ell^{\mathrm{UV}} = k_\ell \in \mathbb{Z}$ is the level of the $\mathrm{U}(1)_\ell$ current algebra of the ultraviolet CFT. Therefore, $k_\ell$ is the 't Hooft anomaly coefficient for the $\mathrm{U}(1)_\ell$ symmetry.[49] Since $k_\ell \neq 0$ implies that correlators have support at separated points (2.3.9), and $\mathrm{U}(1)_\ell$ cannot be spontaneously broken, the infrared of a system with a $\mathrm{U}(1)_\ell$ symmetry must be symmetry preserving and gapless.

This argument admits an interesting generalization. Consider now the renormalization group flow of a U(1)-symmetric CFT triggered by a U(1)-invariant relevant operator. In a unitary CFT with a normalizable vacuum, the conservation law for the U(1) current $J = J_+ \mathrm{d}x^+ + J_- \mathrm{d}x^-$ implies a separate conservation law for chiral $\mathrm{U}(1)_\ell$ and $\mathrm{U}(1)_r$ symmetries,

[48]In QCD, the CFT in the ultraviolet is the CFT of free fermions, and the renormalization group flow is triggered by the gauge coupling.

[49]The contact term implied by (2.3.9) leads to a violation of the conservation equation $\partial_- j_+ = \frac{k_\ell}{2\pi}\partial_+ A_-$ upon coupling system to a background gauge field for $\mathrm{U}(1)_\ell$ via $\int d^2x A_- J_+$. In QCD, $k_\ell = \sum_i q_{i,\ell}^2$, where $q_{i,\ell}$ are the $\mathrm{U}(1)_\ell$ charges of chiral fermions.

generated by $J_-$ and $J_+$ respectively [141]. The most general current two-point functions consistent with dimensional analysis and Poincaré invariance are

$$\begin{aligned}
\langle J_+(x)J_+(0)\rangle &= \frac{K_\ell(x^+x^-/\mu^2)}{x^+x^+}\,,\\
\langle J_-(x)J_-(0)\rangle &= \frac{K_r(x^+x^-/\mu^2)}{x^-x^-}\,,\\
\langle J_+(x)J_-(0)\rangle &= \frac{G(x^+x^-/\mu^2)}{x^+x^-}\,.
\end{aligned} \tag{2.3.10}$$

In a parity invariant system $K_\ell = K_r$. Conservation of the U(1) current $\partial_- J_+ + \partial_+ J_- = 0$ at separated points implies that

$$\rho^2 \frac{\partial}{\partial\rho^2}(K_\ell + G) = G\,, \qquad \rho^2 \frac{\partial}{\partial\rho^2}(K_r + G) = G\,. \tag{2.3.11}$$

Therefore, the quantity $K_\ell - K_r$ is a renormalization group invariant

$$\rho^2 \frac{\partial}{\partial\rho^2}(K_\ell - K_r) = 0\,. \tag{2.3.12}$$

In the deep ultraviolet, $K_\ell^{\rm UV} = k_\ell \in \mathbb{Z}$ and $K_r^{\rm UV} = k_r \in \mathbb{Z}$ are the levels of the $\mathrm{U}(1)_\ell$ and $\mathrm{U}(1)_r$ current algebras of the ultraviolet CFT. $K_\ell - K_r = k_\ell - k_r$ is the 't Hooft anomaly coefficient for the U(1) symmetry and is constant everywhere in the flow.[50] This must be reproduced by be infrared phase, and it can only be realized by a symmetry preserving gapless phase.

**Proposition 2.3.2** *A 2d QFT with a gravitational anomaly is gapless.*

An almost identical reasoning applies to the conservation law of the energy-momentum tensor $T_{\mu\nu}$ along a renormalization group flow out of a CFT, for which we have

$$\partial_- T_{+\pm} + \partial_+ T_{-\pm} = 0\,. \tag{2.3.13}$$

In a unitary CFT with a normalizable vacuum, $T_{+-} = 0$ and $T_{++}$ and $T_{--}$ are chiral, that is $\partial_- T_{++} = \partial_+ T_{--} = 0$, so that in the ultraviolet CFT, the correlators with support at separated points are

$$\text{UV CFT:}\quad \langle T_{++}(x)T_{++}(0)\rangle = \frac{c_\ell^{\rm UV}}{2(x^+)^4}\,, \qquad \langle T_{--}(x)T_{--}(0)\rangle = \frac{c_r^{\rm UV}}{2(x^-)^4}\,, \tag{2.3.14}$$

where $c_\ell^{\rm UV}$ and $c_r^{\rm UV}$ are the central chargers of the left and right-moving Virasoro algebras. In a parity invariant theory $c_\ell^{\rm UV} = c_r^{\rm UV}$. The quantity $c_\ell^{\rm UV} - c_r^{\rm UV}$ detects a gravitational 't Hooft

[50] In QCD $k_\ell - k_r = \mathrm{tr}_{\rm fermions}(\gamma^3 Q^t Q)$, where $Q$ is the U(1) charge of the fermions.

anomaly, and must be matched by the infrared phase.[51] While the $c$-theorem [142] says that $c_\ell$ and $c_r$ decrease along a renormalization group flow, the difference $c_\ell - c_r$ must remain constant. In a theory with a gravitational 't Hooft anomaly the energy-momentum tensor is a nontrivial operator, with separated point correlation functions. Since such correlation functions cannot be realized by a TQFT, a $2d$ theory with a gravitational 't Hooft anomaly is necessarily gapless.

Comparing with the anomaly polynomial (2.2.14), we have that the gravitational 't Hooft anomaly of the QCD theory $(G; R_\ell, R_r)$ is

$$c_\ell^{\mathrm{UV}} - c_r^{\mathrm{UV}} = \frac{1}{2}\left(\dim(R_\ell) - \dim(R_r)\right) . \tag{2.3.15}$$

$(G; R_\ell, R_r)$ with a gravitational 't Hooft anomaly in the ultraviolet is gapless.

### 2.3.1 Towards Gapped QCD Theories

In the previous section we established that a $2d$ theory with a continuous chiral symmetry is automatically gapless. Therefore, if we wish to classify QCD theories that are gapped, the first step is to determine which QCD theories have no such symmetries. These can be expressed as conditions on the quark content of the QCD theory as follows:[52]

A $2d$ QFT with a gravitational anomaly is gapless.

**Proposition 2.3.3** *A necessary condition for $(G; R_\ell, R_r)$ with semisimple $G$ to be gapped is that the representations $R_\ell$ and $R_r$ of $G$ are the direct sum of distinct, real irreducible representations of $G$. A QCD theory with a quark content that is not of this type is necessarily gapless.*

$(G; R_\ell, R_r)$ is obtained by gauging a diagonal subgroup $G$ of the global symmetry acting on the left and right chiral quarks, and giving the gauge fields a kinetic term. For the purpose of identifying the continuous global symmetries of a QCD theory it suffices to discuss the Lie algebra of symmetries. The continuous global symmetry algebra acting on the quarks in the ultraviolet is

$$\mathfrak{so}(\dim(R_\ell)) \oplus \mathfrak{so}(\dim(R_r)) , \tag{2.3.16}$$

where $\dim(R_{\ell/r})$ is the real dimension of the representation $R_{\ell/r}$ of $\mathfrak{g}$. Our immediate task is to answer for what choices of $R_\ell$ and $R_r$ does a QCD theory admit a continuous chiral global symmetry, and is therefore gapless (cf. proposition 2.3.1).

In order to answer this question it suffices to consider the left chiral fermions, as an identical discussion holds for the right chiral ones. Consider left chiral fermions transforming

[51] This can be derived by imposing energy-momentum conservation law (2.3.13) on the most general two-point functions of $T_{++}$, $T_{--}$ and $T_{+-}$ at separated points.

[52] We discuss QCD with a reductive gauge group, that is with abelian gauge group factors, below.

in an irreducible representation $R$ of a semisimple Lie algebra $\mathfrak{g}$. A QCD theory with this quark content has a left chiral flavor symmetry if and only if the embedding

$$\mathfrak{so}(\dim(R)) \supset \mathfrak{g} \tag{2.3.17}$$

has a nontrivial commutant, that is, there exists an algebra $\mathfrak{h} \subset \mathfrak{so}(\dim(R))$ such that $[\mathfrak{g}, \mathfrak{h}] = 0$. This depends on the nature of the representation $R$, which for now we take to be irreducible:

• A chiral quark in a <u>complex representation</u> has a chiral U(1) global symmetry.

A complex representation $R$ of a semisimple Lie algebra $\mathfrak{g}$ is described by traceless, antihermitian $(\dim(R)/2) \times (\dim(R)/2)$ matrices $t$. Therefore the pair $(\mathfrak{g}, R)$ defines the following Lie algebra embedding and branching

$$\begin{gathered} \mathfrak{su}(\dim(R)/2) \supset \mathfrak{g} \\ \text{fundamental} \mapsto R\,. \end{gathered} \tag{2.3.18}$$

Since $R$ is irreducible, the commutant of $\mathfrak{g}$ in $\mathfrak{su}(\dim(R)/2)$ is trivial by Schur's lemma.

Let us now determine whether there is a commutant of $\mathfrak{g}$ in $\mathfrak{so}(\dim(R))$, the symmetry algebra acting on the quarks. The Lie algebra $\mathfrak{su}(\dim(R)/2)$ embeds into the $\mathfrak{so}(\dim(R))$ symmetry algebra of the quarks as

$$\hat{t} = \begin{pmatrix} \mathrm{re}(t) & \mathrm{im}(t) \\ -\mathrm{im}(t) & \mathrm{re}(t) \end{pmatrix} \subset \mathfrak{so}(\dim(R))\,, \tag{2.3.19}$$

where $\mathrm{re}(t)^T = -\mathrm{re}(t)$, $\mathrm{im}(t)^T = \mathrm{im}(t)$, with $T$ denoting the transpose, and $\mathrm{tr}(t) = 0$. Since

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \subset \mathfrak{so}(\dim(R)) \tag{2.3.20}$$

commutes with $\hat{t}$ and $U \not\subset \mathfrak{su}(\dim(R)/2)$, a chiral quark in a complex representation has a chiral U(1) global symmetry. This also follows from the following sequence of embeddings

$$\mathfrak{so}(\dim(R)) \supset \mathfrak{su}(\dim(R)/2) \oplus \mathfrak{u}(1) \supset \mathfrak{g} \oplus \mathfrak{u}(1)\,. \tag{2.3.21}$$

• A chiral quark in a <u>pseudoreal representation</u> has a chiral $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$ global symmetry.

A pseudoreal representation $R$ of a Lie algebra $\mathfrak{g}$ is described by traceless, antihermitian $(\dim(R)/2) \times (\dim(R)/2)$ matrices $t$ obeying

$$-t^T = JtJ^{-1}\,, \tag{2.3.22}$$

where $J$ is the canonical antisymmetric matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,. \tag{2.3.23}$$

These are precisely the generators of the $\mathfrak{sp}(\dim(R)/4)$ Lie algebra in the fundamental representation. Therefore the pair $(\mathfrak{g}, R)$ defines the following Lie algebra embedding and branching

$$\begin{aligned} \mathfrak{sp}(\dim(R)/4) &\supset \mathfrak{g} \\ \text{fundamental} &\mapsto R\,. \end{aligned} \tag{2.3.24}$$

Since $R$ is irreducible, the commutant of $\mathfrak{g}$ in $\mathfrak{sp}(\dim(R)/4)$ is trivial by Schur's lemma.

Let us now determine whether there is a commutant of $\mathfrak{g}$ in $\mathfrak{so}(\dim(R))$, the symmetry algebra acting on the quarks. Since $J = i\sigma_2 \otimes 1$, the $\mathfrak{sp}(\dim(R)/4)$ matrices $t$, which obey (2.3.22), can be written as

$$t = \sum_{M=1}^{4} t_M \otimes q_M\,, \tag{2.3.25}$$

where $t_M$ are real matrices obeying $t_a^T = t_a$ for $a = 1, 2, 3$ and $t_4^T = -t_4$.[53] Here we denote $q_M = (i\vec{\sigma}, 1)$, with $\vec{\sigma}$ the Pauli matrices, a two-dimensional complex-valued representation of the quaternions.

The embedding of $\mathfrak{sp}(\dim(R)/4)$ into the $\mathfrak{so}(\dim(R))$ symmetry algebra of the quarks is

$$\hat{t} = \sum_{M=1}^{4} t_M \otimes \hat{\sigma}_M \subset \mathfrak{so}(\dim(R))\,, \tag{2.3.26}$$

where $\hat{\sigma}_M = (\sigma_1 \otimes i\sigma_2, i\sigma_2 \otimes 1, \sigma_3 \otimes i\sigma_2, 1 \otimes 1)$ is a four-dimensional real-valued representation of the quaternions. Since the matrices $U_a \subset \mathfrak{so}(\dim(R))$

$$\begin{aligned} U_1 &= 1 \otimes i\sigma_2 \otimes \sigma_1 \\ U_2 &= 1 \otimes 1 \otimes i\sigma_2 \\ U_3 &= 1 \otimes i\sigma_2 \otimes \sigma_3 \end{aligned} \tag{2.3.27}$$

commute with $\hat{t}$, generate an $\mathfrak{sp}(1)$ algebra (namely, $[\tau_a, \tau_b] = i\epsilon_{abc}\tau_c$ with $\tau_a = \frac{1}{2i}U_a$) and $U_a \not\subset \mathfrak{sp}(\dim(R)/4)$, a chiral quark in a pseudoreal representation has a chiral $\mathfrak{sp}(1)$ global symmetry. This also follows from the following sequence of embeddings

$$\mathfrak{so}(\dim(R)) \supset \mathfrak{sp}(\dim(R)/4) \oplus \mathfrak{sp}(1) \supset \mathfrak{g} \oplus \mathfrak{sp}(1)\,. \tag{2.3.28}$$

• A chiral quark in a <u>real representation</u> has no continuous global symmetry.

A real irreducible representation $R$ of a Lie algebra $\mathfrak{g}$ is described by traceless, antihermitian $(\dim(R)) \times (\dim(R))$ matrices $t$ obeying

$$-t^T = t\,. \tag{2.3.29}$$

[53] There are three symmetric matrices and one antisymmetric, each being $n \times n$ with $n = \dim(R)/4$. Thus, there are $\frac{3}{2}n(n+1) + \frac{1}{2}n(n-1) = n(2n+1)$ degrees of freedom, which is precisely the dimension of the algebra $\mathfrak{sp}(n) \equiv \mathfrak{sp}(\dim(R)/4)$.

These are precisely the generators of the $\mathfrak{so}(\dim(R))$ Lie algebra in the fundamental representation. Therefore the pair $(\mathfrak{g}, R)$ defines the following Lie algebra embedding and branching

$$\begin{aligned} \mathfrak{so}(\dim(R)) &\supset \mathfrak{g} \\ \text{fundamental} &\mapsto R\,. \end{aligned} \tag{2.3.30}$$

Since $R$ is irreducible, the commutant of $\mathfrak{g}$ in $\mathfrak{so}(\dim(R))$ is trivial by Schur's lemma. Therefore, a chiral quark in a real representation has no continuous global symmetry.

Let us now consider the case where the representation $R$ is reducible. Since we are seeking QCD theories that are gapped, which means that they cannot have any continuous flavor symmetries, we take $R$ to be the direct sum of irreducible, real representations $R_\alpha$ with multiplicity $M_\alpha$

$$R = \bigoplus_\alpha M_\alpha \cdot R_\alpha\,, \qquad \text{with} \qquad M_\alpha \in \{0, 1, 2, \dots\}\,. \tag{2.3.31}$$

If $M_\alpha > 1$, then there is an $\mathfrak{so}(M_\alpha)$ chiral flavor symmetry acting on the quarks. Indeed, the representation matrix for the reducible representation $M_\alpha \cdot R_\alpha$ can be written as

$$t = 1 \otimes t_\alpha\,, \tag{2.3.32}$$

where $t_\alpha$ is a representation of $R_\alpha$. The matrix $U \subset \mathfrak{so}(M_\alpha \cdot \dim(R_\alpha))$

$$U = O \otimes 1 \qquad \text{with} \qquad O^T = -O \tag{2.3.33}$$

is a representation of $\mathfrak{so}(M_\alpha)$ and commutes with $t$. It therefore generates an $\mathfrak{so}(M_\alpha)$ flavor symmetry. This is also a consequence of the sequence of embeddings

$$\mathfrak{so}(M_\alpha \cdot \dim(R_\alpha)) \supset \mathfrak{so}(\dim(R_\alpha)) \oplus \mathfrak{so}(M_\alpha) \supset \mathfrak{g} \oplus \mathfrak{so}(M_\alpha)\,. \tag{2.3.34}$$

Finally, by virtue of Schur's lemma, the direct sum of distinct irreducible real representations $\oplus_\alpha R_\alpha$ does not have a continuous flavor symmetry, as the commutant of $\mathfrak{so}(\sum_\alpha \dim(R_\alpha)) \supset \mathfrak{g}$ is trivial.

Let us now consider QCD with a reductive gauge group $G = K \times U(1)^n$, where $K$ is semisimple. It suffices to consider the left chiral fermions, as an identical discussion holds for the right chiral ones. When the gauge group has U(1) factors, a classical $\mathrm{U}(1)_F$ chiral flavor symmetry may be broken by the Adler-Bell-Jackiw (ABJ) anomaly, that is, by a mixed a $\mathrm{U}(1)$-$\mathrm{U}(1)_F$ anomaly. A classical semisimple symmetry always remains unbroken. Therefore, for the purposes of classifying QCD theories with $K \times \mathrm{U}(1)^n$ gauge group and no flavor symmetries, consider QCD with chiral quarks transforming under $K \times \mathrm{U}(1)^n$ as

$$\bigoplus_{I=1}^{N_F} (R_I, \vec{q}_I)\,, \tag{2.3.35}$$

where $R_I$ is a complex irreducible representation of $K$, $q_I$ is an integral $n$-component charge vector under $\mathrm{U}(1)^n$ and all pairs $(R_I, \vec{q}_I)$ are distinct (any other quark content leads to a flavor symmetry). The classical chiral flavor symmetry is

$$\mathfrak{h}_{\text{classical}} = \bigoplus_{I=1}^{N_F} \mathfrak{u}(1)_I \,. \tag{2.3.36}$$

We want to determine under what conditions this symmetry is completely broken by the ABJ anomaly. Upon defining the integral matrix $Q$ whose columns are

$$Q = (\dim(R_1)\vec{q}_1, \dim(R_2)\vec{q}_2, \ldots, \dim(R_{N_F})\vec{q}_{N_F}) \tag{2.3.37}$$

we arrive at the following result

**Proposition 2.3.4** *A necessary condition for $(G; R_\ell, R_r)$ with $G = K \times \mathrm{U}(1)^n$ to be gapped is that the representations $R_\ell$ and $R_r$ of $G$ are of the irreducible form (2.3.35) and the charge matrices $Q_\ell, Q_r$ have trivial kernel. A QCD theory with a quark content that is not of this type is necessarily gapless.*

The proof is straightforward. An arbitrary chiral $\mathfrak{u}(1) \subseteq \mathfrak{h}_{\text{classical}}$ flavor symmetry is specified by an integer vector $\vec{n} = (n_1, n_2, \ldots, n_{N_F})$ such that, given an angle $\alpha \in \mathfrak{u}(1)$, the $I$-th quark is rotated by an angle $\alpha n_I$. The mixed ABJ anomaly between this $\mathfrak{u}(1)$ flavor symmetry and the $\mathrm{U}(1)^n$ gauge group is $Q\vec{n}$. Therefore, no flavor symmetries remain if and only if $Q$ has empty kernel (so that there are no nontrivial solutions to $Q\vec{n} = 0$). See also section 2.8 for a more in-depth discussion of QCD theories with $\mathrm{U}(1)^n$ gauge group.

Our findings thus far are summarized in the two propositions of this subsection. This is as far as one can get using 't Hooft anomaly considerations. Answering whether a QCD theory $(G; R_\ell, R_r)$ obeying the conditions in the propositions is gapped or gapless requires studying the dynamics. The analysis thus far does not say, for example, whether the vector-like QCD theory with $G = \mathrm{SU}(2)$ and quarks in the isospin $j \in \mathbb{Z}$ representation (which is real) is gapped or gapless. We will provide a complete answer to these questions in the rest of the paper.

Before closing this section let us make one final remark. In this section we have capitalized on the symmetries of $2d$ QFTs as much as we could. In a nutshell, we showed that, if $\mathfrak{h}$ denotes the chiral symmetry algebra of a $2d$ system, then the system is automatically gapless as soon as $\mathfrak{h}$ is non-trivial. There is a nice physical interpretation of this result. In a unitary CFT, a chiral symmetry $\mathfrak{h}$ is always enhanced to $\mathfrak{h}_k$ affine algebra, for a suitable level $k$. Therefore, if $\mathfrak{h}$ is non-zero, the system contains $\mathfrak{h}_k$ massless currents which automatically make the system gapless: the infrared contains, at the very least, an $\mathfrak{h}_k$ WZW CFT subsector. Therefore, a necessary condition for being gapped is that the chiral symmetry is trivial, $\mathfrak{h} \equiv 0$. We will see the $\mathfrak{h}_k$ currents reappear explicitly in the Hamiltonian of QCD in the following section, and we will study them in more detail when we look at the infrared of QCD in section 2.5.

## 2.4 Mass spectrum and QCD Hamiltonians

In this section we analyze the mass spectrum of QCD by studying the quantization of the QCD Hamiltonians. The main result is a derivation of the necessary and sufficient conditions for a QCD theory to be gapped. Along the way, we prove that a QCD theory with a continuous global symmetry has massless particles in the spectrum, reproducing the result derived in section 2.3 by symmetry and 't Hooft anomaly arguments. We analyze the lightcone and temporal Hamiltonians, both yielding the same conditions for QCD to be gapped.

### 2.4.1 Lightcone Hamiltonian

Our aim in this section is to study the mass spectrum of QCD by quantizing the theory in the lightcone frame [143–145]. We review here the most salient features and formulas (see [146, 147] for reviews and recent work). The basic idea is to use the a lightcone coordinate, say $x^+ = \frac{1}{\sqrt{2}}(x^0 + x^1)$, as the time variable. This quantization defines the Hilbert space and the Cauchy data of the theory on a constant $x^+$ surface, and the conjugate lightcone Hamiltonian $P^-$ evolves states in $x^+$. When we choose $x^+$ to play the role of time, the lightcone coordinate $x^-$ plays the role of a spatial coordinate. The momentum $P^+$ conjugate to $x^-$ commutes with the lightcone Hamiltonian $P^-$, i.e., $[P^+, P^-] = 0$. Therefore, the mass spectrum of QCD can be obtained by simultaneously diagonalizing the operators $P^+$ and $P^-$, since

$$M^2 = 2P^+P^- \,. \tag{2.4.1}$$

Positive semidefiniteness of $M^2$ and of the lightcone Hamiltonian $P^-$ implies that all states in the Hilbert space have $P^+ \geq 0$. This combined with the fact that interactions preserve $P^+$ implies that the vacuum state in lightcone quantization is trivial, that is, the vacuum has no particles in it, and the nonperturbative vacuum coincides with the Fock vacuum. This makes lightcone quantization well adapted to study the meson and hadron spectrum of QCD.

Since $P^-$ evolves states along $x^+$, left-moving massless particles are not visible in the $P^-$ Hamiltonian. This implies that the spectrum of the Hamiltonian $P^-$ correctly accounts for all massive and right-moving massless particles, but does not detect left-moving massless particles. This shortcoming can be overcome by quantizing QCD using instead $x^- = \frac{1}{\sqrt{2}}(x^0 - x^1)$ as the lightcone time. In this frame, the lightcone Hamiltonian is instead $\hat{P}^+$ and its spectrum contains all the massive and left-moving massless particles. Therefore, by diagonalizing the QCD lightcone Hamiltonians $P^-$ and $\hat{P}^+$ in the quantizations where $x^+$ and $x^-$ is time respectively, all the massless particles of QCD are accounted for.

Let us proceed with the lightcone quantization of QCD with left and right chiral quarks in representations $R_\ell$ and $R_r$ of the gauge group $G$. We start with the QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = \frac{1}{g^2} F^a_{+-} F^a_{+-} + i\psi^\dagger_\ell(\partial_- - iA^a_- t^a_\ell)\psi_\ell + i\psi^\dagger_r(\partial_+ - iA^a_+ t^a_r)\psi_r \,. \tag{2.4.2}$$

We fix the gauge $A_-^a = 0$, in which all states have positive norm and there are no ghosts. In the lightcone gauge with $x^+$ being time, the left-chiral quarks $\psi_\ell$ and $A_+^a$ are not dynamical. They can be integrated out to yield

$$\mathcal{L}_{\text{QCD}} = i\psi_r^\dagger \partial_+ \psi_r - g^2 J_r^a \frac{1}{\partial_-^2} J_r^a \,, \tag{2.4.3}$$

where

$$J_r^a = \frac{1}{2} {:}\psi_r^\dagger t_r^a \psi_r{:} \tag{2.4.4}$$

generates the right-moving quark current for $\mathfrak{g}$, the Lie algebra of $G$, which obeys $\partial_- J_r^a = 0$. The Noether charges for the lightcone Hamiltonian and momentum are

$$\begin{aligned} P^- &= -g^2 \int \mathrm{d}x^- \, {:}J_r^a \frac{1}{\partial_-^2} J_r^a{:} \\ P^+ &= i \int \mathrm{d}x^- \, {:}\psi_r^\dagger \partial_- \psi_r{:} \,. \end{aligned} \tag{2.4.5}$$

Choosing $x^-$ as lightcone time instead results in the associated lightcone Hamiltonian and momentum

$$\begin{aligned} \hat{P}^+ &= -g^2 \int \mathrm{d}x^+ \, {:}J_\ell^a \frac{1}{\partial_+^2} J_\ell^a{:} \\ \hat{P}^- &= i \int \mathrm{d}x^+ \, {:}\psi_\ell^\dagger \partial_+ \psi_\ell{:} \,, \end{aligned} \tag{2.4.6}$$

where now

$$J_\ell^a = \frac{1}{2} {:}\psi_\ell^\dagger t_\ell^a \psi_\ell{:} \tag{2.4.7}$$

generates the left-moving quark current for $G$, with $\partial_+ J_\ell^a = 0$.

We are interested in determining when a QCD theory is gapped, and when it is gapless. This requires determining when the lightcone Hamiltonians (2.4.5) and (2.4.6) have zero eigenvalues. We will answer this question by studying the eigenvalues of the operators

$$\begin{aligned} H &= -g^2 \int \mathrm{d}x \, {:}J^a \frac{1}{\partial_x^2} J^a{:} \\ \hat{P} &= i \int \mathrm{d}x \, {:}\psi^\dagger \partial_x \psi{:} \,, \end{aligned} \tag{2.4.8}$$

where $x = x^-$ or $x = x^+$ and $J^a = J_r^a$ or $J^a = J_\ell^a$ depending on whether $x^+$ or $x^-$ is the lightcone time. Canonical quantization yields the following equal-time commutation relations

$$\{\psi_i^\dagger(x), \psi^j(y)\} = \delta_i^j \delta(x-y) \,, \tag{2.4.9}$$

where the Latin indices are the representation labels for the representation $R$ of $\mathfrak{g}$ of the relevant chiral quarks. The quark field expansion in Fourier modes is

$$\psi^i(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}k \left( a^i(k) e^{-ikx} + b_i^\dagger(k) e^{ikx} \right) \,, \tag{2.4.10}$$

where $k = k_-$ or $k = k_+$ depending on whether $x^+$ or $x^-$ is taken as the lightcone time. Note that in the lightcone frame the Fourier modes carry nonnegative longitudinal momentum. Using (2.4.9) we get the anticommutation relations

$$\{a^i(k), a_j^\dagger(k')\} = \delta^i_j \delta(k-k')\,, \qquad \{b^i(k), b_j^\dagger(k')\} = \delta^i_j \delta(k-k')\,, \tag{2.4.11}$$

with $a_i^\dagger(k) = b_i^\dagger(k)$ if the quark field is a Majorana fermion. These operators define the Fock vacuum, and in fact the nonperturbative vacuum $|0\rangle$ of lightcone QCD

$$a^i(k)|0\rangle = 0\,, \qquad b^i(k)|0\rangle = 0 \qquad \forall\ k\,. \tag{2.4.12}$$

Normal ordering in (2.4.8) implies that the vacuum has zero lightcone energy and momentum

$$H|0\rangle = 0\,, \qquad P|0\rangle = 0\,. \tag{2.4.13}$$

The goal is to diagonalize the lightcone Hamiltonian(s) $H$ on the Hilbert space $\mathcal{H}$ created by the quarks

$$|\Psi^{i_1 i_2 \ldots i_L}\rangle \equiv a_{i_1}^\dagger(k_1) a_{i_2}^\dagger(k_2) \ldots a_{i_L}^\dagger(k_L)|0\rangle\,. \tag{2.4.14}$$

The physical states of QCD must be gauge invariant, which implies that all physical states must be invariant under the action of $\mathfrak{g}$

$$\int \mathrm{d}x\, J^a(x)|\Psi^{i_1 i_2 \ldots i_L}\rangle = 0\,. \tag{2.4.15}$$

While $H$ mixes states in the Hilbert space $\mathcal{H}$, the longitudinal momentum operator $P$ is diagonal, it is the sum of the longitudinal momentum of each parton.

QCD has massless particles if and only if there exist states $|\Psi^{i_1 i_2 \ldots i_L}\rangle \in \mathcal{H}$, other than the vacuum state, that are gauge invariant and have zero lightcone energy,

$$\int \mathrm{d}x :J^a \frac{1}{\partial^2} J^a: |\Psi^{i_1 i_2 \ldots i_L}\rangle = 0\,. \tag{2.4.16}$$

The currents $J^a = \frac{1}{2}{:}\psi^\dagger t^a \psi{:}$ constructed out of the chiral quarks $\psi$ transforming in a representation $R$ of $\mathfrak{g}$ generate an affine chiral current algebra $\mathfrak{g}_{I(R)}$

$$J^a(x) J^b(0) \sim \frac{I(R)\delta^{ab}}{x^2} + \frac{i f^{ab}{}_c J^c(0)}{x}\,, \tag{2.4.17}$$

where the level is the Dynkin index $I(R)$ of the representation $R$. The OPE (2.4.17) implies, upon putting the longitudinal coordinate on the circle, that the Fourier modes obey

$$[J^a_n, J^b_m] = i f^{ab}{}_c J^c_{n+m} + I(R)\delta^{ab} n \delta_{m+n,0}\,. \tag{2.4.18}$$

The zero energy state condition (2.4.16) takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n^2} J^a_{-n} J^a_n |\Psi^{i_1 i_2 \ldots i_L}\rangle = 0\,. \tag{2.4.19}$$

Since the Hamiltonian $H$ in (2.4.8) is a positive semidefinite operator, the necessary and sufficient conditions for a state $|\Psi^{i_1 i_2 \ldots i_L}\rangle$ to have zero energy are:

1. $|\Psi^{i_1 i_2 \cdots i_L}\rangle$ is a primary state of the current algebra $\mathfrak{g}_{I(R)}$, that is $J^a_n|\Psi^{i_1 i_2 \cdots i_L}\rangle = 0$ $\forall\, n \geq 1$.
2. $|\Psi^{i_1 i_2 \cdots i_L}\rangle$ transforms in the trivial representation of $\mathfrak{g}$.

We now proceed to study under what conditions these states exist.

The quark Hilbert space $\mathcal{H}$ decomposes into modules of the $\mathfrak{so}(\dim(R))_1$ current algebra at level one [14] (see also [148]), labeled by representations of $\mathfrak{so}(\dim(R))$. These are labeled by $(\mathbf{0}, \mathbf{v}, \mathbf{s}, \mathbf{c})$ when $\dim(R)$ is even and by $(\mathbf{0}, \mathbf{v}, \mathbf{s})$ when it is odd, where $\mathbf{0}, \mathbf{v}, \mathbf{s}, \mathbf{c}$ are the trivial, the vector and spinor representation(s) of $\mathfrak{so}(\dim(R))$. The precise relation between the modules of $\mathfrak{so}(\dim(R))_1$ current algebra and fermion Hilbert space is (e.g. for $\dim(R)$ even)

$$\begin{aligned} \mathcal{H}_{\mathrm{NS}} &= \mathcal{H}_{\mathbf{0}} \oplus \mathcal{H}_{\mathbf{v}} \\ \mathcal{H}_{\mathrm{R}} &= \mathcal{H}_{\mathbf{s}} \oplus \mathcal{H}_{\mathbf{c}}\,, \end{aligned} \tag{2.4.20}$$

where $\mathcal{H}_{\mathrm{X}}$ denotes the fermion Hilbert space with fermions obeying $\mathrm{X} \in \{\mathrm{NS}, \mathrm{R}\}$ (Neveu-Schwarz and Ramond) boundary conditions on the circle.

Since QCD is obtained by gauging the subalgebra $\mathfrak{g} \subset \mathfrak{so}(\dim(R))$, the current algebra embeds as $\mathfrak{g}_{I(R)} \subset \mathfrak{so}(\dim(R))_1$ into the fermion current algebra. That means that any state in $\mathcal{H}$ fits inside a module of the $\mathfrak{g}_{I(R)}$ current algebra. An elegant way to describe how states embed is through the branching functions $b_{\Lambda\lambda}$, which encode how the characters of the $\mathfrak{so}(\dim(R))_1$ current algebra decompose into $\mathfrak{g}_{I(R)}$ characters:

$$\chi_\Lambda(q) = \sum_\lambda b_{\Lambda\lambda}(q)\chi_\lambda(q)\,. \tag{2.4.21}$$

Here $\Lambda \in \{\mathbf{0}, \mathbf{v}, \mathbf{s}, \mathbf{c}\}$ or $\{\mathbf{0}, \mathbf{v}, \mathbf{s}\}$ and $\lambda$ is a highest weight vector labeling the integrable representations of $\mathfrak{g}_{I(R)}$, which obeys

$$\sum_{i=1}^{\mathrm{rank}(\mathfrak{g})} a_i^\vee \lambda_i \leq I(R)\,, \tag{2.4.22}$$

where $a_i^\vee$ are the comarks of $\mathfrak{g}$. The function $b_{\Lambda\lambda}(q)$ counts how many primary states of $\mathfrak{g}_{I(R)}$ with highest weight $\lambda$ appear in the decomposition of the module of $\mathfrak{so}(\dim(R))_1$ with highest weight $\Lambda$. They also capture at which level the primaries of $\mathfrak{g}_{I(R)}$ appear in the $\mathfrak{so}(\dim(R))_1$ modules.

The branching function $b_{\Lambda\lambda}(q)$ has a module interpretation. $b_{\Lambda\lambda}(q)$ is a character of the chiral algebra $\mathcal{A}$, where $\mathcal{A}$ is the commutant chiral algebra of $\mathfrak{g}_{I(R)}$ inside $\mathfrak{so}(\dim(R))_1$. $\mathcal{A}$ necessarily contains the Virasoro algebra with central charge the difference of the central charges of the two current algebras $c_{\mathcal{A}} = c(\mathfrak{so}(\dim R)_1) - c(\mathfrak{g}_{I(R)}) = \frac{1}{2}\dim R - \frac{I(R)\dim(\mathfrak{g})}{I(R)+h}$, where $h$ is the dual Coxeter number of $\mathfrak{g}$ (see table 2.3). Also, if $\mathfrak{h}$ is the commutant of $\mathfrak{g}$ inside

$\mathfrak{so}(\dim R)$, then $\mathcal{A}$ contains a current algebra $\mathfrak{h}_k$. This current algebra has the interpretation as the flavor symmetry current algebra in QCD.

We have established that a QCD theory is gapless if and only if the fermion Hilbert space contains a nontrivial primary state of $\mathfrak{g}_{I(R)}$ labeled by the trivial representation of $\mathfrak{g}$. We are therefore interested in the functions $b_{\Lambda\hat{\mathbf{0}}}(q)$, where we denote the trivial representation of $\mathfrak{g}$ by $\hat{\mathbf{0}}$. In order to determine whether a QCD theory has a massless state it will suffice to look at the branching function for the integrable representation of $\mathfrak{so}(\dim(R))_1$ labeled by the trivial representation of $\mathfrak{so}(\dim(R))$ into the trivial representation of $\mathfrak{g}_{I(R)}$. Whether the theory has massless states or not is encoded in the properties of the function $b_{\mathbf{0}\hat{\mathbf{0}}}(q)$.

Given that $b_{\mathbf{0}\hat{\mathbf{0}}}(q)$ is the vacuum character of the chiral algebra $\mathcal{A}$, this branching function takes the following general form[54]

$$b_{\mathbf{0}\hat{\mathbf{0}}}(q) = q^{-c_{\mathcal{A}}/24}\big(1 + a_1 q + a_2 q^2 + \ldots\big)\,. \tag{2.4.23}$$

The theory has massless particles if $a_l \neq 0$ for any $l$.

We explain now the physical meaning of the coefficients $a_l$. The coefficient of the $q^0$ term being $a_0 = 1$ corresponds to the vacuum state $|0\rangle$, which is unique. The coefficient $a_1 = \dim(\mathfrak{h})$ is the dimension of the commutant of $\mathfrak{g}$ in $\mathfrak{so}(\dim(R))$. If $\mathfrak{h}$ is nontrivial, we can build the primary, singlet states of $\mathfrak{g}_{I(R)}$ at level one by acting on the vacuum with the flavor symmetry currents

$$\tilde{J}^\alpha = \frac{1}{2}\psi^\dagger \tilde{t}^\alpha \psi, \qquad \alpha = 1, \ldots, \dim(\mathfrak{h})\,. \tag{2.4.24}$$

The currents $\tilde{J}^\alpha$ generate an $\mathfrak{h}_k$ current algebra, the level $k$ being determined by the embedding $\mathfrak{so}(\dim R) \supset \mathfrak{h} \oplus \mathfrak{g}$. Indeed, these states are annihilated by $J^a$ since $[\tilde{J}^\alpha, J^a] = 0$, by virtue of $\mathfrak{h}$ commuting with $\mathfrak{g}$. If such an operator $\tilde{J}^\alpha$ exists, then the theory is gapless. This reproduces the result we proved in section 2.3 stating that any theory with a continuous, chiral global symmetry — which means $\mathfrak{h}$ is nontrivial — is necessarily gapless. Therefore, a necessary condition for the QCD theory to be gapped is that the theory has no continuous, chiral flavor symmetries and therefore that $a_1 = 0$.

We turn our attention to the physics of the coefficient $a_2$. Recall that given a current algebra $\mathfrak{g}_{I(R)}$, one can construct the chiral energy momentum tensor [149, 150]

$$T_{\mathfrak{g}_{I(R)}} = \frac{1}{8(I(R)+h)} {:}J^a J^a{:}\,. \tag{2.4.25}$$

The operator $T_{\mathfrak{g}_{I(R)}}$ generates a Virasoro algebra of central charge $c(\mathfrak{g}_{I(R)}) = \frac{I(R)\dim(G)}{I(R)+h}$. The canonical level 2 state in the Hilbert space $\mathcal{H}$

$$(T_{\mathfrak{so}(\dim(R))_1} - T_{\mathfrak{g}_{I(R)}})|0\rangle \tag{2.4.26}$$

[54] There are no $q^{n+1/2}$ terms in the expansion since $\mathcal{H}_{\mathbf{0}}$ contains states created with an even number of fermions.

is a singlet, primary state of the $\mathfrak{g}_{I(R)}$ current algebra, where $T_{\mathfrak{so}(\dim(R))_1}$ is the energy momentum tensor of the $\mathfrak{so}(\dim(R))_1$ current algebra generated by the quarks in the ultraviolet. This is a consequence of the OPEs

$$\begin{aligned} T_{\mathfrak{g}_{I(R)}}(x) J^a(0) &\sim \frac{J^a(0)}{x^2} + \frac{\partial J^a(0)}{x} \\ T_{\mathfrak{so}(\dim(R))_1}(x) J^a(0) &\sim \frac{J^a(0)}{x^2} + \frac{\partial J^a(0)}{x}\,, \end{aligned} \tag{2.4.27}$$

so that

$$[J^a, T_{\mathfrak{so}(\dim(R))_1} - T_{\mathfrak{g}_{I(R)}}] = 0\,. \tag{2.4.28}$$

This proves that the state (2.4.26) is a gauge invariant, primary state of $\mathfrak{g}_{I(R)}$. Therefore as long as $T_{\mathfrak{so}(\dim(R))_1} - T_{\mathfrak{g}_{I(R)}} \neq 0$, so that $a_2 \neq 0$, the theory has massless states and the spectrum is gapless.[55]

This implies that a necessary (and as we will show also sufficient) condition for a QCD theory to be gapped is that the following operator equation holds

$$T_{\mathfrak{so}(\dim(R))_1} - T_{\mathfrak{g}_{I(R)}} = 0\,. \tag{2.4.30}$$

This equation is very constraining, it implies that $a_l = 0$ for all $l$. Indeed, when (2.4.30) is obeyed then $b_{\Lambda\lambda}(q)$ is as a character of the Virasoro algebra with $c_{\mathcal{A}} = 0$, which has a unique, trivial unitary representation. Therefore, when equation (2.4.30) holds then[56]

$$b_{\Lambda\hat{\mathbf{0}}}(q) = \delta_{\Lambda\mathbf{0}}\,, \tag{2.4.31}$$

and the only singlet primary state of $\mathfrak{g}_{I(R)}$ is the vacuum state: the theory has no massless particles.

By demanding that there are no left-moving or right-moving massless particles, we arrive at the following proposition:

**Proposition 2.4.1** *QCD theory $(G; R_\ell, R_r)$ is gapped if and only if both operator equations hold*

$$\begin{aligned} T_{\mathfrak{so}(\dim R_\ell)_1} - T_{\mathfrak{g}_{I(R_\ell)}} &= 0\,, \\ \overline{T}_{\mathfrak{so}(\dim R_r)_1} - \overline{T}_{\mathfrak{g}_{I(R_r)}} &= 0\,. \end{aligned} \tag{2.4.32}$$

[55] In the presence of a global symmetry $\mathfrak{h}$ there are additional level 2 primary, singlet states of $\mathfrak{g}_{I(R)}$

$$\tilde{J}^\alpha_{-2}|0\rangle\,, \qquad \tilde{J}^\alpha_{-1}\tilde{J}^\beta_{-1}|0\rangle\,. \tag{2.4.29}$$

Generically, there are $\dim(\mathfrak{h})(\dim(\mathfrak{h})+3)/2$ such states constructed using the flavor affine algebra $\mathfrak{h}_k$ currents (for very small values of $k$ one may need to subtract some null states).

[56] This follows by comparing the scaling dimensions of operators.

This was derived by looking at the lightcone Hamiltonians with $x^+$ and $x^-$ being time.

We will come back to these equations momentarily. Before we do that we shall also derive these equations in a different quantization scheme in order to gain more insight into the mechanism behind the gap.

### 2.4.2 Temporal gauge Hamiltonian

In this section we rederive the necessary and sufficient conditions (2.4.32) for a QCD theory $(G; R_\ell, R_r)$ to be gapped by studying the canonical Hamiltonian where time is the time-like coordinate $x^0 = t$. The lightcone coordinates are well-suited to algebraic considerations because the two chiralities are mostly decoupled. By contrast, a time-like coordinate requires more work but it also leads to a more transparent understanding of the spectrum, because the Hamiltonian takes the traditional form, which evolves states in physical time. For previous work on the temporal Hamiltonian of QCD see [19, 151, 152].

We follow the same conventions as in §2.2, which we present here for convenience

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1), \quad A^a_\pm = \frac{1}{\sqrt{2}}(A^a_0 \pm A^a_1)\,. \tag{2.4.33}$$

We start with the Lagrangian

$$\mathcal{L} = i\psi^\dagger_{i\,\ell}\partial_-\psi_{i\,\ell} + i\psi^\dagger_{i\,r}\partial_+\psi_{i\,r} + A^a_- J^a_\ell + A^a_+ J^a_r - \frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\,, \tag{2.4.34}$$

where $J^a_{\ell,r} = \frac{1}{2}\psi^\dagger_{i\,\ell,r}t^a_{ij}\psi_{j\,\ell,r}$. Canonical quantization leads to the following equal-time commutation relations

$$\begin{aligned}
\{\psi^\dagger_{i\,\ell}(x), \psi_{j\,\ell}(y)\} &= \delta_{ij}\delta(x-y)\,,\\
\{\psi^\dagger_{i\,r}(x), \psi_{j\,r}(y)\} &= \delta_{ij}\delta(x-y)\,,\\
[\Pi^a(x), A^b_1(y)] &= i\delta^{ab}\delta(x-y)\,,
\end{aligned} \tag{2.4.35}$$

where $\Pi^a = \frac{\delta\mathcal{L}}{\delta\dot{A}^a_0} = \frac{1}{g^2}F^a_{10} = \frac{1}{g^2}E^a(x)$. Consequently, classically the currents obey the commutation relations

$$[J^a_{\ell,r}(x), J^b_{\ell,r}(y)] = if^{abc}J^c_{\ell,r}(x)\delta(x-y)\,. \tag{2.4.36}$$

The Hamiltonian density is

$$\begin{aligned}
\mathscr{H} &= g^2E^a(x)^2 + A^a_0G^a + \frac{1}{\sqrt{2}}A^a_1(x)(J^a_\ell - J^a_r)(x)\\
&\quad + \frac{1}{\sqrt{2}}i\psi^\dagger_{i\,\ell}\partial_x\psi_{i\,\ell}(x) - \frac{1}{\sqrt{2}}i\psi^\dagger_{i\,r}\partial_x\psi_{i\,r}(x)\,,
\end{aligned} \tag{2.4.37}$$

and

$$G^a(x) = J^a_\ell(x) + J^a_r(x) - (\partial_x\Pi^a + if^{abc}A^b_1\Pi^c)(x) \tag{2.4.38}$$

is the Gauss' law operator, which obeys the following commutation relation with the currents

$$[G^a(x), J^b_{\ell,r}(y)] = if^{abc}J^c_{\ell,r}(x)\delta(x-y)\,. \tag{2.4.39}$$

The Gauss law operator commutes with the Hamiltonian, that is $[G^a(x), H] = 0$, where $H = \int dx\, \mathscr{H}$.

The phase space of QCD has a primary constraint, namely the momentum conjugate to $A_0$ vanishes

$$\Pi^a = \frac{\delta \mathcal{L}}{\delta \dot{A}^a_0} = 0\,. \tag{2.4.40}$$

Demanding stability of this constraint leads to the secondary constraint

$$\frac{d\Pi^a}{dt} = [H, \Pi^a] = G^a = 0\,, \tag{2.4.41}$$

Since $\frac{dG^a}{dt} = [H, G^a] = 0$ on the constraint surface, there are no further constraints. Hamilton's equations derived from (2.4.37) reproduce the equations of motion obtained by varying the Lagrangian (2.4.34). We work in the gauge with $A^a_0 = 0$ so that $A^a_+ = A^a_1 \equiv A^a$.

In the quantum theory, fields are promoted to operators and composites need to be renormalized due to quantum fluctuations at arbitrary short distances. The quark currents in the quantum theory must be normal ordered $\widehat{J}^a_{\ell,r} = \frac{1}{2}{:}\psi^\dagger_{i\,\ell,r}t^a_{ij}\psi_{j\,\ell,r}{:}$. In the quantum theory, the commutation relations become

$$\begin{aligned}
[\widehat{J}^a_{\ell,r}(x), \widehat{J}^b_{\ell,r}(y)] &= if^{abc}J^c_{\ell,r}(x)\delta(x-y) \pm iI(R)\delta^{ab}\partial_x\delta(x-y)\,,\\
[\widehat{\Pi}^a(x), \widehat{A}^b(y)] &= i\delta^{ab}\delta(x-y)\,,
\end{aligned} \tag{2.4.42}$$

where $I(R) \equiv I(R_\ell) = I(R_r)$ by virtue of gauge anomaly cancellation, and $\pm$ corresponds to $\ell$ and $r$ respectively. The operators $\widehat{J}^a_{\ell,r}$ generate the current algebra $\mathfrak{g}_{I(R_{\ell,r})}$. Quantization leads to the Schwinger term in the current commutators (2.4.42) (cf. with (2.4.36)), which will have important implications.

In order to determine the conditions for the spectrum of Hamiltonian to be gapped, we first define the fermion operators in (2.4.37) in normal ordered form

$$\begin{aligned}
\frac{1}{\sqrt{2}}i\psi^\dagger_{i\,\ell}\partial_x\psi_{i\,\ell}(x) &= \frac{1}{\sqrt{2}}{:}i\psi^\dagger_{i\,\ell}\partial_x\psi_{i\,\ell}(x){:} - \frac{1}{\sqrt{2}}i\lim_{\epsilon\to 0}\langle\psi^\dagger_{i\,\ell}(x+\epsilon)\partial_x\psi_{i\,\ell}(x-\epsilon)\rangle\\
-\frac{1}{\sqrt{2}}i\psi^\dagger_{i\,r}\partial_x\psi_{i\,r}(x) &= -\frac{1}{\sqrt{2}}{:}i\psi^\dagger_{i\,r}\partial_x\psi_{i\,r}(x){:} + \frac{1}{\sqrt{2}}i\lim_{\epsilon\to 0}\langle\psi^\dagger_{i\,r}(x+\epsilon)\partial_x\psi_{i\,r}(x-\epsilon)\rangle\,.
\end{aligned} \tag{2.4.43}$$

where $\langle\psi^\dagger_{i\ell,\,r}(x)\psi_{j\ell,\,r}(y)\rangle \sim \frac{i\delta_{ij}}{x-y}$. We then express $\pm\frac{1}{\sqrt{2}}{:}i\psi^\dagger_{\ell,r}\partial_x\psi_{\ell,r}{:}$ in terms of the Sugawara tensor for $\mathfrak{so}(\dim R_{\ell,r})_1$ via

$$\begin{aligned}
\frac{1}{2}T_{\mathfrak{so}(\dim R_\ell)_1} &= \frac{i}{8(I(R)+h)}{:}\widehat{J}^a_\ell\widehat{J}^a_\ell(x){:} = \frac{1}{\sqrt{2}}{:}i\psi^\dagger_{i\,\ell}\partial_x\psi_{i\,\ell}(x){:}\\
\frac{1}{2}\overline{T}_{\mathfrak{so}(\dim R_r)_1} &= \frac{i}{8(I(R)+h)}{:}\widehat{J}^a_r\widehat{J}^a_r(x){:} = -\frac{1}{\sqrt{2}}{:}i\psi^\dagger_{i\,r}\partial_x\psi_{i\,r}(x){:}\,.
\end{aligned} \tag{2.4.44}$$

The crucial insight (see also [19, 152]) is that we can split the energy momentum tensor into a piece that couples to the gauge fields and a piece that is decoupled

$$
\begin{aligned}
T_{\mathfrak{so}(\dim R_\ell)_1} &= T_{\mathfrak{g}_{I(R_\ell)}} + \big(T_{\mathfrak{so}(\dim R_\ell)_1} - T_{\mathfrak{g}_{I(R_\ell)}}\big)\\
\overline{T}_{\mathfrak{so}(\dim R_r)_1} &= \overline{T}_{\mathfrak{g}_{I(R_r)}} + \big(\overline{T}_{\mathfrak{so}(\dim R_r)_1} - \overline{T}_{\mathfrak{g}_{I(R_r)}}\big)\,.
\end{aligned}
\tag{2.4.45}
$$

The quantized Hamiltonian must commute with the quantum Gauss' law operator

$$
\widehat{G}^a(x) = \widehat{J}^a_\ell(x) + \widehat{J}^a_r(x) - (\partial_x \widehat{\Pi}^a + i f^{abc}\widehat{A}^b\,\widehat{\Pi}^c)(x)\,. \tag{2.4.46}
$$

It is given by (see section 2.6.2 for details)

$$
\begin{aligned}
\widehat{\mathscr{H}} &= \frac{1}{2}\big(T_{\mathfrak{g}_{I(R_\ell)}} + T_{\mathfrak{g}_{I(R_r)}}\big) + \frac{1}{\sqrt{2}}\widehat{A}^a(x)(\widehat{J}^a_\ell - \widehat{J}^a_r)(x) + \frac{1}{\sqrt{2}} I(R)\widehat{A}^a(x)^2 + g^2\widehat{E}^a(x)^2\\
&+ \frac{1}{\sqrt{2}} i \lim_{\epsilon\to 0}\big(\langle \psi^\dagger_{i\,\ell}(x+\epsilon)\partial_x\psi_{i\,\ell}(x-\epsilon)\rangle - \langle \psi^\dagger_{i\,r}(x+\epsilon)\partial_x\psi_{i\,r}(x-\epsilon)\rangle\big)\\
&+ \frac{1}{2}\big(T_{\mathfrak{so}(\dim R_\ell)_1} - T_{\mathfrak{g}_{I(R_\ell)}}\big) + \frac{1}{2}\big(\overline{T}_{\mathfrak{so}(\dim R_r)_1} - \overline{T}_{\mathfrak{g}_{I(R_r)}}\big)\,.
\end{aligned}
\tag{2.4.47}
$$

Let us discuss some of the most salient features of this Hamiltonian. Expressing the Hamiltonian in terms of the energy momentum tensor and splitting it as in (2.4.45) shows that there is a decoupled CFT with energy momentum tensors $T_{\mathfrak{so}(\dim R_\ell)_1} - T_{\mathfrak{g}_{I(R_\ell)}}$ and $\overline{T}_{\mathfrak{so}(\dim R_r)_1} - \overline{T}_{\mathfrak{g}_{I(R_r)}}$. The last line in (2.4.47) describes a gapless sector. The term $I(R)\widehat{A}^a(x)^2$ in (2.4.47), which is present due to the Schwinger term in (2.4.42), gaps out the gauge fields, and strongly suggests that the first two lines in (2.4.47) describe a gapped Hamiltonian.[57] This analysis makes manifest that the massless degrees of freedom decouple in the ultraviolet and go along for the ride during the renormalization group flow (see [19]). In conclusion, the QCD theory $(G; R_\ell, R_r)$ is gapped if and only if the decoupled CFT is trivial, that is if

$$
\begin{aligned}
T_{\mathfrak{so}(\dim R_\ell)_1} - T_{\mathfrak{g}_{I(R_\ell)}} &= 0\\
\overline{T}_{\mathfrak{so}(\dim R_r)_1} - \overline{T}_{\mathfrak{g}_{I(R_r)}} &= 0
\end{aligned}
\tag{2.4.48}
$$

We have thus recovered the conditions (2.4.32) we had derived in the previous section using the lightcone Hamiltonians.

### 2.4.3 Classification of gapped theories

We now return to our main task: deducing whether a given QCD theory is gapped or not. We showed, both looking at lightcone quantization and standard canonical quantization, that a theory labelled by $(G; R_\ell, R_r)$ is gapped if and only if the two operator equations hold:

$$
\begin{aligned}
T_{\mathfrak{so}(\dim R_\ell)_1} - T_{\mathfrak{g}_{I(R_\ell)}} &= 0\,,\\
\overline{T}_{\mathfrak{so}(\dim R_r)_1} - \overline{T}_{\mathfrak{g}_{I(R_r)}} &= 0\,.
\end{aligned}
\tag{2.4.49}
$$

[57]It would be interesting to give a rigorous proof that it is gapped.

The key point is that the equations (2.4.49) can in fact be solved. It suffices to consider one chirality first, and then combine solutions that merge left and right chiral sectors. In [153, 154] it was shown that the energy-momentum tensor of the affine algebra $\mathfrak{g}_{I(R)}$ coincides with that of a free fermion theory $\mathfrak{so}(\dim(R))_1$ if and only if the matrices $t^a_{ij}$ that generate the representation $R$ satisfy the Jacobi-like identity (see section 2.6.1 for derivation)

$$t^a_{ij}t^a_{k\ell} + t^a_{ik}t^a_{\ell j} + t^a_{i\ell}t^a_{jk} = 0\,. \tag{2.4.50}$$

In turn, this identity is satisfied if and only if there exists some Lie algebra $\hat{\mathfrak{g}}$ that contains $\mathfrak{g}$ such that the homogeneous space[58] $\hat{G}/G$ is symmetric, and the fermions transform with respect to $\mathfrak{g}$ in the same way as the generators of $\hat{G}/G$. The condition $t^a_{ij}t^a_{k\ell} + t^a_{ik}t^a_{\ell j} + t^a_{i\ell}t^a_{jk} = 0$ is nothing but the Bianchi identity for the Riemann tensor of $\hat{G}/G$. The symmetric spaces have been fully classified [155], and we can read off the list of gapped QCD theories from this classification: for each symmetric space of the form $\hat{G}/G$ where the symmetric space generators transform according to a representation $R$ of $G$, there is a gapped QCD theory with gauge group $G$ and quarks in the representation $R$, and vice versa.

A different perspective yields the same answer. The equality of the energy-momentum tensors of $\mathfrak{so}(\dim(R))_1$ and $\mathfrak{g}_{I(R)}$ implies, by definition, that the affine algebra $\mathfrak{g}_{I(R)}$ embeds conformally into the affine algebra $\mathfrak{so}(\dim(R))_1$. The conformal embeddings have been fully classified [156–158], and we can read off the list of gapped QCD theories from this classification: for each conformal embedding of an algebra $\mathfrak{g}_k$ into $\mathfrak{so}(n)_1$ via a representation $R$ of $\mathfrak{g}$, there is a gapped QCD theory with gauge group $G$ and quarks in the representation $R$, and vice versa.

Either point of view yields table 2.4. This table contains the list of "minimal" gapped QCD theories. Naturally, one can also take the tensor product of two gapped theories to obtain another gapped theory. This operation corresponds to reducible symmetric spaces, or non-maximal conformal embeddings; the most general symmetric space is a product of irreducible ones, and the most general conformal embedding is a sequence of maximal ones.

In QCD, this stacking operation gives rise to theories with decoupled gapped sectors so they are also gapped in a trivial way – and so they are of little interest by themselves. The exception is when the different theories contain abelian $\mathfrak{u}(1)$ factors in their gauge group, in which case we can couple the minimal theories through these factors, which generates another gapped theory which is *not* just the product of decoupled theories. With this in mind, the most general gapped QCD theory is either a theory in table 2.4, or a product of such theories provided they contain a $\mathfrak{u}(1)$ gauge group, in which case the matrix of charges for these $\mathfrak{u}(1)$ factors must be non-singular. Any other gapped theory is a trivial product of these two options.

[58]The global structure of $\hat{G}, G$ is arbitrary, the condition $t^a_{ij}t^a_{k\ell} + t^a_{ik}t^a_{\ell j} + t^a_{i\ell}t^a_{jk} = 0$ is purely algebraic and insensitive to the choice of $\hat{G}, G$ for a given $\hat{\mathfrak{g}}, \mathfrak{g}$. At the algebraic level, the condition can be recast as the existence of an algebra $\hat{\mathfrak{g}} = \mathfrak{g} + \mathfrak{p}$ such that $[p^i, p^j] \in \mathfrak{g}$ for $p^i, p^j \in \mathfrak{p}$, and $[p^i, g^a] = t^a_{ij}p^j$ for $g^a \in \mathfrak{g}$.

| $\mathfrak{g}$ | $R$ | IR TQFT | $\hat{\mathfrak{g}}$ |
|---|---|---|---|
| $\mathfrak{g}$ | Adj | $\mathrm{SO}(\dim G)_1/G_h$ | $\mathfrak{g}+\mathfrak{g}$ |
| $\mathfrak{so}(N)$ | $\square\!\square$ | $\mathrm{SO}(\frac{1}{2}(N+2)(N-1))_1/\mathrm{Spin}(N)_{N+2}$ | $\mathfrak{su}(N)$ |
| $\mathfrak{sp}(N)$ | $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ | $\mathrm{SO}((2N+1)(N-1))_1/\mathrm{Sp}(N)_{N-1}$ | $\mathfrak{su}(2N)$ |
| $\mathfrak{s}(\mathfrak{u}(M)+\mathfrak{u}(N))$ | $(\square,\square)_q$ | $\mathrm{U}(MN)_1/S(\mathrm{U}(M)_N\times\mathrm{U}(N)_M)$ | $\mathfrak{su}(M+N)$ |
| $\mathfrak{so}(M)+\mathfrak{so}(N)$ | $(\square,\square)$ | $\mathrm{SO}(MN)_1/(\mathrm{Spin}(M)_N\times\mathrm{Spin}(N)_M)$ | $\mathfrak{so}(M+N)$ |
| $\mathfrak{u}(N)$ | $\begin{smallmatrix}\square\\\square\end{smallmatrix}_q$ | $\mathrm{U}(\frac{1}{2}N(N-1))_1/\mathrm{U}(N)_{N-2,2(N-1)q^2}$ | $\mathfrak{so}(2N)$ |
| $\mathfrak{u}(N)$ | $\square\!\square_q$ | $\mathrm{U}(\frac{1}{2}N(N+1))_1/\mathrm{U}(N)_{N+2,2(N+1)q^2}$ | $\mathfrak{sp}(N)$ |
| $\mathfrak{sp}(M)+\mathfrak{sp}(N)$ | $(\square,\square)$ | $\mathrm{SO}(4MN)_1/(\mathrm{Sp}(M)_N\times\mathrm{Sp}(N)_M)$ | $\mathfrak{sp}(M+N)$ |
| $\mathfrak{sp}(4)$ | $\mathbf{42}$ | $\mathrm{SO}(42)_1/\mathrm{Sp}(4)_7$ | $E_6$ |
| $\mathfrak{su}(2)+\mathfrak{su}(6)$ | $(\mathbf{2},\mathbf{20})$ | $\mathrm{SO}(40)_1/(\mathrm{SU}(2)_{10}\times\mathrm{SU}(6)_6)$ | $E_6$ |
| $\mathfrak{so}(10)+\mathfrak{u}(1)$ | $\mathbf{16}_q$ | $\mathrm{U}(16)_1/(\mathrm{Spin}(10)_4\times\mathrm{U}(1)_{16q^2})$ | $E_6$ |
| $F_4$ | $\mathbf{26}$ | $\mathrm{SO}(26)_1/F_{4,3}$ | $E_6$ |
| $\mathfrak{su}(8)$ | $\mathbf{70}$ | $\mathrm{SO}(70)_1/\mathrm{SU}(8)_{10}$ | $E_7$ |
| $\mathfrak{su}(2)+\mathfrak{so}(12)$ | $(\mathbf{2},\mathbf{32})$ | $\mathrm{SO}(64)_1/(\mathrm{SU}(2)_{16}\times\mathrm{Spin}(12)_8)$ | $E_7$ |
| $E_6+\mathfrak{u}(1)$ | $\mathbf{27}_q$ | $\mathrm{U}(27)_1/(E_{6,6}\times\mathrm{U}(1)_{27q^2})$ | $E_7$ |
| $\mathfrak{so}(16)$ | $\mathbf{128}$ | $\mathrm{SO}(128)_1/\mathrm{Spin}(16)_{16}$ | $E_8$ |
| $\mathfrak{su}(2)+E_7$ | $(\mathbf{2},\mathbf{56})$ | $\mathrm{SO}(112)_1/(\mathrm{SU}(2)_{28}\times E_{7,12})$ | $E_8$ |
| $\mathfrak{su}(2)+\mathfrak{sp}(3)$ | $(\mathbf{2},\mathbf{14})$ | $\mathrm{SO}(28)_1/(\mathrm{SU}(2)_7\times\mathrm{Sp}(3)_5)$ | $F_4$ |
| $\mathfrak{so}(9)$ | $\mathbf{16}$ | $\mathrm{SO}(16)_1/\mathrm{Spin}(9)_2$ | $F_4$ |
| $\mathfrak{so}(4)$ | $(\mathbf{2},\mathbf{4})$ | $\mathrm{SO}(8)_1/\mathrm{Spin}(4)_{10,2}$ | $G_2$ |

Table 2.4: List of irreducible gapped theories. The **first column** denotes the gauge algebra. Any global choice of $G$ for a given $\mathfrak{g}$ leads to a gapped theory. The **second column** denotes the representation of the quarks, either in the Young diagram notation or directly in terms of its dimension. $\square, \square\!\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}$ denote the fundamental, symmetric, and anti-symmetric representations, respectively (with appropriate reality conditions, e.g. Majorana if real, and with traces removed, if possible). $q$ denotes the charge of the fermions under $\mathfrak{u}(1)$ factors, if any; this charge can be chosen arbitrarily. The **third column** denotes the TQFT that describes the space of vacua of these gapped theories (see section 2.5); here we choose the simply-connected form $G_{\mathrm{sc}}$ for concreteness. The **fourth column** denotes the Lie algebra $\hat{\mathfrak{g}}$ that makes $\hat{G}/G$ a symmetric spaces.

For illustration purposes, consider the gapped theory $\mathfrak{u}(N)+\square_q$, where $\square_q$ denotes the fundamental representation and $q\in\mathbb{Z}$ is an arbitrary integer that specifies the charge of the quark under the trace part $\mathfrak{u}(1)\subset\mathfrak{u}(N)$. Stacking a family of these gapped theories, and coupling the abelian factors via an arbitrary matrix of charges, one obtains the gapped theory

$$\prod_{i=1}^{m} \mathfrak{u}(N_i), \qquad R = \bigoplus_{i=1}^{m} (\mathbf{1}, \ldots, \mathbf{1}, \square, \mathbf{1}, \ldots, \mathbf{1})_{\vec{q}_i}\,, \tag{2.4.51}$$

where $\vec{q}_i$ is a vector of charges that specifies how the $i$-th fermion couples to $\mathfrak{u}(1)^m$. The special case where $N_i = 1$ for all $i$ corresponds to abelian QCD, i.e., QED with $m$ photons and $m$ fermions, which we analyze in more detail in section 2.8.

The theories in table 2.4 are written in terms of non-chiral data. As a matter of fact, one can also modify these theories to obtain gapped theories that are chiral. The idea is that, if the theory with vector-like matter $(R_\ell, R_r) = (R, R)$ is gapped, then the theory with chiral matter $(R_\ell, R_r) = (\sigma_\ell \cdot R, \sigma_r \cdot R)$ is also gapped, where $\sigma_\ell, \sigma_r$ denote outer automorphisms of $\mathfrak{g}$ (see table 2.3). For example, for simply-laced groups $\sigma \cdot R$ may denote the conjugate representation $\bar{R}$, while for $\mathfrak{g} = \mathfrak{so}(8)$, $\sigma \cdot R$ may denote any representation related by triality. As $\dim(\sigma \cdot R) = \dim(R)$ and $I(\sigma \cdot R) = I(R)$, the chiral theory with $(R_\ell, R_r) = (\sigma_\ell \cdot R, \sigma_r \cdot R)$ is also gapped. In the case of theories that contain abelian gauge groups, the statement becomes that one can use different charges for the two chiralities, $(q_\ell, q_r)$, provided they satisfy the gauge anomaly cancellation condition (2.2.17).

For example, given the gapped theory in (2.4.51), one can generate other gapped theories by replacing some of the fundamentals by anti-fundamentals (for one chirality only, or for both), and also by assigning generically different U(1) charges to the two chiralities $\vec{q}_i \mapsto (\vec{q}_{\ell,i}, \vec{q}_{r,i})$.

Table 2.4, together with the two operations we just described (stacking gapped theories and coupling them together via their abelian factors, and acting with outer automorphisms on the representations), give the extensive list of gapped QCD theories. See tables 2.1 and 2.2 for summary of gapped vector-like and chiral theories respectively. Any other theory is either a trivial product of gapped theories, or is gapless.

## 2.5 Infrared dynamics of 2$d$ QCD

Having classified all QCD theories that are gapped, and consequently, those that are gapless, it remains an interesting open question to determine the effective field theory describing the low energy dynamics. The most natural proposal is that the low energy theory is a gauged WZW coset model with chiral algebra (see [19] and more recently [136, 159])

$$\frac{\mathrm{SO}(\dim R_\ell)_1}{G_{I(R_\ell)}} \times \frac{\mathrm{SO}(\dim R_r)_1}{G_{I(R_r)}}\,. \tag{2.5.1}$$

In order to simplify notation we focus on the chiral half, with the understanding that the full theory is constructed by putting together the left and right sectors.

The idea behind (2.5.1) is that QCD can be thought of as $\dim(R)$ free fermions, which can be described as the fermionic WZW theory $\mathrm{SO}(\dim(R))_1$, where a symmetry $G \subset \mathrm{SO}(\dim(R))$ has been gauged, and one has added a kinetic term for the gluons. The coupling constant $g$ is dimensionful so it grows in the infrared and it is self-consistent to assume that $g \to \infty$ as $E \to 0$, which means that we can drop the gluon kinetic term for very low energies. All in all, it is expected that the deep infrared of QCD theories is described by the CFT coset (2.5.1), namely an $\mathrm{SO}(\dim(R))_1$ WZW model with gauged $G_{I(R)}$ symmetry. The level of the gauge current algebra is determined by the Dynkin embedding index of $\mathrm{SO}(\dim(R)) \supset G$; this embedding is defined by the branching rule $\square \mapsto R$, and hence the embedding index is $I(R)$.

This proposal is also suggested by our canonical analysis of section 2.4, where we highlighted the presence of $\mathfrak{so}(\dim(R))_1, \mathfrak{g}_{I(R)}$ current algebras in the Hamiltonian of QCD, and the fact that the operator $T_{\mathfrak{so}(\dim(R))_1} - T_{\mathfrak{g}_{I(R)}}$ naturally appears in this Hamiltonian, playing the role of the energy-momentum of a low-energy CFT that is decoupled from massive modes, which disappear in the deep infrared.

**Gapped spectrum.** One nice aspect of (2.5.1) is that it is perfectly consistent with our classification from the previous section, because the coset (2.5.1) is a full-fledged CFT if its central charge is non-zero, but describes a TQFT when its central charge vanishes. In other words, the chiral energy-momentum tensor of the coset is $T_{\mathfrak{so}(\dim(R))_1/\mathfrak{g}_{I(R)}} \equiv T_{\mathfrak{so}(\dim(R))_1} - T_{\mathfrak{g}_{I(R)}}$, and this is a non-trivial operator if and only if the theory is gapless. In any case, it is important to stress that the criterion for masslessness $T_{\mathfrak{so}(\dim(R))_1/\mathfrak{g}_{I(R)}} \neq 0$ was obtained in previous sections independently of the conjecture (2.5.1), but the two are perfectly consistent with each other, a fact that gives more evidence for the latter.

**Continuous symmetries.** In $2d$, continuous chiral symmetries cannot appear nor disappear along a symmetric renormalization group flow. Therefore, the effective low energy description of QCD must have the exact same such symmetries as the original ultraviolet theory. This is nicely reproduced by the coset, because the symmetries of both theories have the same definition: the flavor symmetry group is the commutant of $G$ inside $\mathrm{SO}(\dim(R))$, i.e., the rotations of the chiral quarks that commute with gauge transformations.

**'t Hooft anomalies** Another nice property of the conjecture (2.5.1) is that it automatically matches all the 't Hooft anomalies of the original QCD theory. Indeed, while the argument above does not strictly speaking prove that this coset is the low energy limit of the ultraviolet theory, it does prove that they are in the same deformation class. In other words, even though in principle the limits $E \to 0$ and $g^2 \to \infty$ need not be equivalent, it is still true that they are connected by a path in parameter space. Therefore, these two theories will carry the same 't Hooft anomalies for all the symmetries that are preserved along the path. This provides a strong consistency check on the proposal that the coset really is the low-energy limit of QCD.

The case of perturbative anomalies can be exhibited explicitly. The chiral flavor symmetry $H$ in the ultraviolet is generated by the free fermion currents that commute with the gauge group, that is, commutant of $G$ inside $\mathrm{SO}(\dim(R))$. If $R$ is given by $N_F$ copies of a given irreducible representation $R_0$, that is $R = N_F \cdot R_0$ the flavor symmetry is $H(N_F)$, with $H = \mathrm{O}, \mathrm{Sp}, \mathrm{U}$ for real, pseudo-real, and complex representations, respectively (in the complex case, the symmetry may be either $\mathrm{U}(N_F)$ or $\mathrm{SU}(N_F)$, depending on whether the diagonal $\mathrm{U}(1)$ is broken by the ABJ anomaly or not, see section 2.3.1).

The 't Hooft anomaly for $H$ is the Dynkin index of the representation under the flavor group (2.2.14), in this case the fundamental representation. This means that the flavor symmetry carries $\dim(R_0)$ units of anomaly. This is reproduced by the coset in a straightforward manner, because one can write

$$\frac{\mathrm{SO}(\dim(R))_1}{G_{I(R)}} \equiv H_{\dim(R_0)} \times \frac{\mathrm{SO}(\dim(R))_1}{G_{I(R)} \times H_{\dim(R_0)}} \,. \tag{2.5.2}$$

The factor $H_{\dim(R_0)}$ matches the ultraviolet 't Hooft anomaly, and the factor $\frac{\mathrm{SO}(\dim(R))_1}{G_{I(R)} \times H_{\dim(R_0)}}$ has no continuous global symmetries (no commutant). We point out that this latter coset is actually well-defined, which might not be entirely obvious. One way to see this is that one could imagine gauging the diagonal symmetry $H$ in the ultraviolet (which is anomaly-free), to yield the gauge theory $G \times H + (R_0, \square)$. The infrared coset for this theory is precisely $\frac{\mathrm{SO}(\dim(R))_1}{G_{I(R)} \times H_{\dim(R_0)}}$.

The case of nonperturbative global anomalies is more subtle, and requires a case-by-case analysis. That being said, the argument above proves that the coset CFT will automatically match all the anomalies, perturbative and global. This has a nice bonus consequence, namely that it predicts that many well-known CFTs actually carry nonperturbative anomalies, a fact that may not have been fully appreciated in the past. For example, below we will describe many gauge theories that flow in the infrared to common CFTs such as minimal models or WZW models. These theories necessarily carry the same nonperturbative anomalies of the ultraviolet theory, and the latter are often easy to determine (because one can flow to the deep ultraviolet, where the fermions and gluons are essentially free and semiclassical considerations often suffice). Among others, this predicts global 't Hooft anomalies for discrete symmetries such time-reversal, whose presence is seldom discussed in the CFT literature.

While there is not much one can say about global anomalies in full generality, there is one feature that is actually rather universal. There are several discrete symmetries, such as discrete chiral symmetries or antiunitary time-reversal symmetry, whose anomalies have the following effect on the Hilbert space: when the number fermions in ultraviolet is odd, the Ramond Hilbert space is automatically supersymmetric [1]. This is a nonperturbative statement that affects the whole spectrum of the theory and, in particular, the low-energy spectrum. Therefore, the effective infrared description must satisfy this property as well. This is indeed reproduced by the coset (2.5.1), because the states in the Ramond sector come from branchings from the spinor representation(s) of $\mathrm{SO}(\dim(R))_1$; and, famously, when $\dim(R)$ is odd there is a single spinor whose Ramond-Ramond character is identically zero, a property that is inherited to the full coset. (Another diagnosis of this anomaly is that the twisted Hilbert space becomes ill-defined, which is also reproduced by the coset because the spinor character of $\mathrm{SO}(\dim(R))_1$ carries a factor of $\sqrt{2}$ for odd $\dim(R)$, and hence the twisted partition function does not have an integral expansion; see (2.7.20) for the characters of $\mathrm{SO}(n)_1$).

**One-form symmetry.** QCD theories can have one-form symmetry associated to a subgroup center of the gauge group (see table 2.3). This symmetry is discrete,[59] and hence by the generalized Coleman-Mermin-Wagner theorem [32], it cannot break spontaneously. Therefore, the infrared effective description must realize all the one-form symmetries of the ultraviolet theory.

In two dimensions, the effect of a one-form symmetry is to break up the theory into distinct sectors, or universes. The full Hilbert space of the theory is the direct sum of the Hilbert spaces of the different universes (see section 2.2). The total theory suffers from a mild violation of cluster decomposition, but the theory projected to a given universe is perfectly well-defined by itself, and satisfies decomposition.

Given a QFT with one-form symmetry, the emergent infrared CFT inherits it. Hence, in these CFTs the vacuum is not unique (the coefficient of the vacuum character in the torus partition function is an integer larger than 1). Instead, the infrared CFT is a direct sum (not a direct product) of "conventional" CFTs with a unique vacuum each.

In QCD, the one-form symmetry is the subgroup of the center that is not screened by the fermions, namely the kernel of the representation $R$ under which the quarks transform, $\Gamma = \ker(R) \subseteq Z(G)$. This is a symmetry for all $g^2$ and in particular it remains a symmetry in the $g^2 \to \infty$ limit, and therefore the coset CFT also has a $\Gamma$ one-form symmetry. As $\Gamma$ does not act on the fermions, it does not embed into $\mathrm{SO}(\dim(R))$, and hence in the quotient $\mathrm{SO}(\dim(R))_1/G_{I(R)}$ we are trying to gauge a group $\Gamma$ that does not act on anything – this is an orbifold by a symmetry that does not act faithfully (cf. with [135, 160, 161]). This indeed leads to $|\Gamma|$ different universes, labelled by elements $\rho \in \Gamma^\vee$ (see section 2.2).

The CFT on a given universe labelled by $\rho$ corresponds to the coset $\mathrm{SO}(\dim(R))_1/(G/\Gamma)_{I(R)}$ with a theta term labelled by $\rho \in \Gamma^\vee$. The functional integral of the coset sums over $G/\Gamma$ bundles, which are labeled by $\Gamma$. The sum over bundles is weighted by the theta term. It is interesting to compare this perspective with the algebraic approach to cosets in the literature

[59] If the gauge group is reductive, then the one-form symmetry may include $\mathrm{U}(1)$ factors associated to the photons. These $\mathrm{U}(1)$ groups exist only when the photons are free; otherwise the screening by quarks explicitly breaks $\mathrm{U}(1)$ down to a discrete subgroup.

[150, 162, 163]. In the algebraic approach to cosets one organizes representations of the coset into long and short(er) orbits under the action of $\Gamma$, as $\Gamma$ permutes the coset representations. When the action of $\Gamma$ has only long orbits, the algebraic prescription is to divide the partition function by $\Gamma$, so that the vacuum character appears with multiplicity one, and only the trivial bundles contribute. This yields the partition function in one universe, which when there are no fixed points, is the same in all universes. When the model has shorter orbits, one has to deal with "fixed point resolution", and correct by a series of prescriptions and ansätze for the fact that characters enter with fractional multiplicity. These prescriptions have a rather clear interpretation from our perspective. When the coset has no fixed points, the CFT in each universe is the same and only trivial bundles contribute. Instead, when the coset has fixed points, the CFT in each universe is generically different. In order to identify the partition function in a given universe when there are fixed points, one must sum over nontrivial bundles, weighted by a discrete theta term, which gives a non-vanishing constant partition function [164]. These contributions combine with those of the long orbits to produce a partition function that is modular invariant in each universe. In a sense, the algebraic approach to cosets in the literature constructs the partition function in one universe, while from our perspective one can construct more modular invariant partition functions by weighing the sum over nontrivial bundles (which are constant) by distinct discrete theta terms.

**Central charge.** The central charge of the CFT in the deep ultraviolet is $\frac{1}{2}\dim(R_{\ell/r})$, and in the deep infrared is $\frac{1}{2}\dim(R_{\ell/r}) - c(G_{I(R_{\ell/r})})$. Note that $c$ decreases and dynamics is compatible with the $c$-theorem. Note also that both $c_\ell$ and $c_r$ decrease by the same amount (because $I(R_\ell) \equiv I(R_r)$, by gauge anomaly cancellation, cf. (2.2.16)), which is a consequence of the conservation of the gravitational anomaly $c_\ell - c_r$. It might be interesting to note that gapped theories "erase information maximally" in the sense that they decrease the $c$ function as much as possible.

It should be pointed out that the infrared theory described by the coset (2.5.1) is not expected to be robust under deformations in the ultraviolet. If we add mass terms or four-fermi terms, in general one would find that the infrared theory is deformed as well, and the coset (2.5.1) flows to a different theory. This new theory has smaller (or equal) central charge. In the case of TQFTs, the central charge is already zero so deformations in the ultraviolet will map the infrared theory to a different TQFT, with generically fewer vacua. This is to be contrasted with the similar situation in $3d$: here, infrared TQFTs are actually robust under small ultraviolet deformations. The reason is that $2d$ TQFTs have local operators, while $3d$ TQFTs do not; therefore, local deformations in the ultraviolet map to non-trivial infrared operators in $2d$, but to the trivial operator in $3d$.

**Some simple examples.** While we will work out plenty of examples in the next few subsections, we can list a couple of simple examples here, which will hopefully illustrate some of the main features.

Take the QCD theory $(\mathrm{SU}(2); \mathbf{7}, \mathbf{7})$. The infrared dynamics is conjecturally described by the coset $\mathrm{SO}(7)_1/\mathrm{SU}(2)_{28}$. This CFT has central charge $c = 7/10$, which agrees with the central charge of the tricritical Ising model. There are only two fermionic CFTs with this central charge: the tricritical Ising model itself (thought of as a fermionic CFT that does not in fact depend on the spin structure), or its fermionization. In other words, it is either a bosonic minimal model, promoted to fermionic in a trivial way, or it is a fermionic minimal model [165–167].

Here it is easy to determine which of these options is correct. In the deep ultraviolet there are 7 free fermions, so the system carries $-1 \mod 8$ units of 't Hooft anomaly under the chiral $\mathbb{Z}_2$ symmetry. A bosonic theory cannot match this, so the second option is correct: this QCD system flows in the infrared to the fermionized tricritical Ising model.[60] Note that this theory precisely matches the 't Hooft anomaly for the discrete chiral symmetry [168].

Once the correct low-energy degrees of freedom have been identified, one can ask several interesting questions. For example, one could try to determine the mapping between relevant operators in the ultraviolet to operators in the infrared. The spectrum of infrared operators, together with their quantum numbers, is well understood. The most relevant operator in the ultraviolet is the mass term, and the most relevant operator in the infrared is the $(1/10, 1/10)$ operator, so it is a very natural guess that these operators are identified. Moreover, both are odd under the chiral $\mathbb{Z}_2$ symmetry. A similar analysis can be performed for the rest of operators. When the mapping is complete, one can study the deformed theory, where one adds suitable scalar operators to the Lagrangian; this gives us a window to the infrared of the *massive* QCD theory, by turning on the deformation $(1/10, 1/10)$ to the infrared CFT.

Finally, this scenario predicts that the fermionic tricritical Ising model is invariant under time-reversal, with $\mathsf{T}^2 = (-1)^F$, and that this symmetry has a nonperturbative 't Hooft anomaly. This symmetry, and anomaly, are manifest in the ultraviolet, where it acts as $\psi(t) \mapsto \gamma^0 \psi(-t)$, with 't Hooft anomaly measured by the number of fermions mod 2, in this case $7 \equiv 1 \mod 2$. It would be interesting to understand how this symmetry acts on the infrared CFT, and to determine its anomaly directly.

A very similar story holds for the QCD theory $(\mathrm{Spin}(7); \mathbf{8}, \mathbf{8})$. The infrared dynamics is conjecturally described by the coset $\mathrm{SO}(8)_1/\mathrm{Spin}(7)_1$. This CFT has central charge $c = 1/2$, so it is either the bosonic Ising model (promoted to a fermionic theory in a trivial way), or the fermionized Ising model, i.e., a free Majorana fermion. As before, it is easy to determine which of these options is actually realized: there are 8 fermions in the deep ultraviolet, so the $\mathbb{Z}_2$ chiral symmetry has no 't Hooft anomalies. This is only matched by the first option,

[60] We also show this directly in section 2.7.2.

namely the bosonic Ising model; hence, this is what QCD flows to in the infrared.[61]

Much like above, one can try to determine how the ultraviolet operators are mapped to the infrared ones, and what happens when we deform the theory by these operators.

In these two examples we extracted the physics of the coset directly from its central charge. This was possible thanks to the fact that they are both smaller than unity: $c < 1$. For generic QCD theories, the central charge is $c > 1$ and its knowledge alone does not uniquely determine the CFT. In this situation, the properties of the infrared are to be extracted from the CFT $\mathrm{SO}(n)/G_k$ by the standard coset construction. We review this construction in section 2.7. Here we also revisit the $c = 7/10$ and $c = 1/2$ examples again, and confirm that they correspond to the fermionic tricritical Ising model and the bosonic Ising mode, respectively, by explicitly working out the branching functions of the coset.

From now on we will assume that the conjecture (2.5.1) is correct. We can use it to propose explicit descriptions of the strongly coupled infrared dynamics of interesting QCD theories.

### 2.5.1 Gapped theories

Let us make a few remarks about QCD theories on table 2.4; these theories are gapped, so their infrared involves a certain TQFT that describes their vacua.

**Adjoint QCD.** The first interesting example is adjoint QCD, namely the gauge theory with gauge group $G$ and a fermion in the adjoint representation. This theory has received a lot of attention in the past, see [99, 136, 137, 169–173] for a sample of papers.

The vacua of these theories are described by the topological coset

$$\frac{\mathrm{SO}(\dim(\mathfrak{g}))_1}{G_h}\,, \tag{2.5.3}$$

where $h$ is the dual Coxeter number of $\mathfrak{g}$ (cf. table 2.3). The branching functions of this coset are well understood [174]:

$$\begin{aligned} d_{\text{NS-NS}} &= \sum_{\lambda\in\mathcal{R}} \chi_\lambda \\ d_{\text{NS-R}} &= \sum_{\lambda\in\mathcal{R}} (-1)^{h_\lambda} \chi_\lambda \\ d_{\text{R-NS}} &= 2^{r/2} \chi_\rho \\ d_{\text{R-R}} &= 0\,, \end{aligned} \tag{2.5.4}$$

where $r = \mathrm{rank}(\mathfrak{g})$, $\rho$ denotes the Weyl vector and

$$\mathcal{R} = \{\lambda \mid \exists \hat{w} \in \hat{W} \text{ such that } \lambda = h\hat{\omega}_0 + (\hat{w} - 1)\hat{\rho}\}\,. \tag{2.5.5}$$

[61] We also show this directly in section 2.7.2.

From these equations it immediately follows that $\mathcal{H}_{\mathrm{R}}$ is always supersymmetric and has $2^r$ states. The space $\mathcal{H}_{\mathrm{NS}}$ also has these many states and it is purely bosonic. In other words,

$$\text{infrared of adjoint QCD:}\qquad \begin{cases}\mathcal{H}_{\mathrm{NS}} &= \mathbb{C}^{2N|0}\\ \mathcal{H}_{\mathrm{R}} &= \mathbb{C}^{N|N}\end{cases}, \qquad N := 2^{r-1}\,. \tag{2.5.6}$$

Furthermore, by explicitly constructing $\mathcal{R}$ for the different simple algebras, one observes that half the states in $\mathcal{H}_{\mathrm{NS}}$ are charged under $(-1)^{F_L}$, and the other half is not. In other words, half the representations in $\mathcal{R}$ have integral spin $h_\lambda \in \mathbb{Z}$, and the other half have half-integral spin $h_\lambda \in \mathbb{Z}+\frac{1}{2}$. The only exception is $\mathrm{SU}(2n+1)$, which has $2^{n-1}(2^n+1)$ states with integral spin and $2^{n-1}(2^n-1)$ states with half-integral spin.

**QCD with bifundamentals.** The next few interesting examples correspond to theories with gauge group $G\times G$ and fermions in the bifundamental representation, namely

$$\begin{aligned} &S(\mathrm{U}(N)\times\mathrm{U}(M)) + (\square,\square)\\ &\mathrm{SO}(N)\times\mathrm{SO}(M) + (\square,\square)\\ &\mathrm{Sp}(N)\times\mathrm{Sp}(M) + (\square,\square)\,, \end{aligned} \tag{2.5.7}$$

whose space of vacua are described by the following cosets:

$$\frac{\mathrm{U}(NM)_1}{S(\mathrm{U}(N)_M\times\mathrm{U}(M)_N)},\qquad \frac{\mathrm{SO}(NM)_1}{\mathrm{SO}(N)_M\times\mathrm{SO}(M)_N},\qquad \frac{\mathrm{SO}(4NM)_1}{\mathrm{Sp}(N)_M\times\mathrm{Sp}(M)_N}\,. \tag{2.5.8}$$

The branching rules of these cosets are well-known [175–178]: they describe the level-rank dualities $\mathfrak{g}(N)_k \leftrightarrow \mathfrak{g}(k)_N$. The decomposition of numerator characters $d_{\pm,\pm}$ into denominator characters $\chi$ takes the following general form:

$$\begin{aligned} d_{\text{NS-X}}(q,g_1,g_2) &= \sum_\lambda (\pm 1)^{2(h_\lambda+h_{\lambda^t})}\chi_\lambda(q,g_1)\chi_{\lambda^t}(q,g_2)\\ d_{\text{R-X}}(q,g_1,g_2) &= \sum_\lambda (\pm 1)^{2(h_\lambda+h_{\lambda^t})}\chi_{\gamma\cdot\lambda}(q,g_1)\chi_{\lambda^t}(q,g_2)\,, \end{aligned} \tag{2.5.9}$$

where $\lambda$ denotes a primary of $\mathfrak{g}(N)_k$, and $\lambda^t$ the primary of $\mathfrak{g}(k)_N$ obtained from $\lambda$ by transposing the Young diagram. Moreover, $h_\lambda$ denotes the conformal dimension of $\lambda$, and $\gamma\in Z(G_{\mathrm{sc}})$ a suitable simple current. Finally, $g_1, g_2$ denote flavor symmetry elements of $\mathfrak{g}(N), \mathfrak{g}(k)$, respectively.

These branching functions imply that there are as many vacua as primaries in $\mathfrak{g}$, i.e., the number of vacua is (see e.g. [5])

$$
\begin{aligned}
&S(\mathrm{U}(N)\times\mathrm{U}(M))\colon\quad \binom{N+M-1}{M}\\
&\mathrm{SO}(N)\times\mathrm{SO}(M)\colon\quad \begin{cases}
\binom{n+m}{m}+\binom{n+m-1}{m-1} & (N,M)=(2n,2m+1)\\
\frac{1}{2}\binom{n+m}{m}+\binom{n+m-1}{m-1}+\frac{1}{2}\binom{n+m-2}{m-2}+\frac{3}{2}\binom{n+m-2}{m} & (N,M)=(2n,2m)\\
\frac{1}{2}\binom{n+m}{m}+\frac{1}{2}\binom{n+m-1}{m-1}+\frac{3}{2}\binom{n+m-1}{m} & (N,M)=(2n+1,2m)\\
\binom{n+m}{m} & (N,M)=(2n+1,2m+1)
\end{cases}\\
&\mathrm{Sp}(N)\times\mathrm{Sp}(M)\colon\quad \binom{N+M}{M}.
\end{aligned}
\tag{2.5.10}
$$

**QCD with rank-2.** Another interesting example is the theories with rank-2 quarks. Here we discuss the two theories

$$
\begin{aligned}
\mathrm{Spin}(N)&+\square\!\square\\
\mathrm{Sp}(N)&+\begin{smallmatrix}\square\\\square\end{smallmatrix}
\end{aligned}
\tag{2.5.11}
$$

whose vacua are described by the cosets

$$
\frac{\mathrm{SO}((N+2)(N-1)/2)_1}{\mathrm{Spin}(N)_{N+2}},\qquad \frac{\mathrm{SO}((2N+1)(N-1))_1}{\mathrm{Sp}(N)_{N-1}}\,. \tag{2.5.12}
$$

We are not aware of an explicit discussion of the branchings of these cosets in the literature. That being said, the Ramond sector turns out to be particularly simple, and is reminiscent of the adjoint case (2.5.4):

$$
\begin{aligned}
&& d_{\text{R-R}} &= 0\,,\\
\mathrm{Spin}(N),\ N\text{ odd:} && d_{\text{R-NS}} &= 2^{(N-1)/4}\chi_{[3,1,1,\dots,1,1,3]}+2^{(N-1)/4}\chi_{[1,3,1,1,\dots,1,1,3]}\\
\mathrm{Spin}(N),\ N\text{ even:} && d_{\text{R-NS}} &= 2^{(N-2)/4}\chi_{[3,1,1,\dots,1,1,3]}+2^{(N-2)/4}\chi_{[1,3,1,1,\dots,1,1,3]}+\text{c.c.}\\
\mathrm{Sp}(N)\colon && d_{\text{R-NS}} &= 2^{(N-1)/2}\chi_{[0,1,\dots,1,0]}\,.
\end{aligned}
\tag{2.5.13}
$$

From this we automatically conclude that there are $2^{\lfloor N/2\rfloor+1}$ and $2^{N-1}$ vacua, respectively. In the Ramond sector these vacua are split half-and-half into bosons and fermions, while in the Neveu-Schwartz sector they are all bosonic. In this latter sector, half the states are charged under $(-1)^{F_L}$ and the other half is neutral (i.e., half the primaries have integral spin and the other half have half-integral spin).

The branching rules for the other gapped theories with rank-2 quarks, namely $\mathrm{U}(N)$ plus a symmetric or anti-symmetric quark, are analyzed in [179].

**Exceptionals.** We close this section with an example involving an exceptional Lie group, to wit

$$F_4 + \mathbf{26}\,. \tag{2.5.14}$$

The vacua of this theory are described by the coset

$$\frac{\mathrm{SO}(26)_1}{F_{4,3}}\,, \tag{2.5.15}$$

whose branching functions are

$$\begin{aligned} d_{\text{NS-X}} &= \chi_{\mathbf{1}} \pm \chi_{\mathbf{26}} + \chi_{\mathbf{273}} \pm \chi_{\mathbf{1274}} \\ d_{\text{R-NS}} &= 2\chi_{\mathbf{4096}} \\ d_{\text{R-R}} &= 0\,. \end{aligned} \tag{2.5.16}$$

Hence, this theory has $\mathcal{H}_{\mathrm{NS}} = \mathbb{C}^{4|0}$ and $\mathcal{H}_{\mathrm{R}} = \mathbb{C}^{2|2}$.

### 2.5.2 QCD with fundamental matter

Here we describe the infrared dynamics of QCD with quarks in the fundamental representation. More precisely, we shall discuss the following theories

- $\mathrm{SU}(N) + N_F\,\square$.
- $\mathrm{SO}(N) + N_F\,\square$.
- $\mathrm{Sp}(N) + N_F\,\square$.

These describe the celebrated 't Hooft model [143].

The coset CFTs that describe the low energy limit of these theories are

$$\frac{\mathrm{U}(NN_F)_1}{\mathrm{SU}(N)_{N_F}}, \qquad \frac{\mathrm{SO}(NN_F)_1}{\mathrm{SO}(N)_{N_F}}, \qquad \frac{\mathrm{SO}(4NN_F)_1}{\mathrm{Sp}(N)_{N_F}}\,. \tag{2.5.17}$$

We now claim that these CFTs are in fact the well-known WZW theories

$$\mathrm{U}(N_F)_N, \qquad \mathrm{SO}(N_F)_N, \qquad \mathrm{Sp}(N_F)_N\,. \tag{2.5.18}$$

Indeed, the characters of the coset $\mathrm{SO}(\cdots)_1/\mathfrak{g}(N)_{N_F}$ are given by the coefficients of the characters of $\mathfrak{g}(N)_{N_F}$ in the decomposition of $\mathrm{SO}(\cdots)_1$; but, as in (2.5.9), these coefficients are precisely the characters of $\mathfrak{g}(N_F)_N$. In other words, the equality $\mathrm{SO}(\cdots)_1/\mathfrak{g}(N)_{N_F} \equiv \mathfrak{g}(N_F)_N$ is tantamount to the level-rank duality $\mathfrak{g}(N)_{N_F} \leftrightarrow \mathfrak{g}(N_F)_N$.

Let us make a few remarks:

- Note that the infrared CFT is just the WZW model for the flavor symmetry. This CFT manifestly matches the perturbative 't Hooft anomalies for the flavor symmetry in the ultraviolet. So this is the simplest scenario for the infrared dynamics, and could have been guessed independently of the general conjecture (2.5.1). These WZW models also match the nonperturbative anomalies, although in a less obvious way (see below for an explicit example).

- The equality $\mathrm{SO}(\cdots)_1/\mathfrak{g}(N)_{N_F} \equiv \mathfrak{g}(N_F)_N$ can also be understood as the consequence of the triviality of the coset $\mathrm{SO}(\cdots)_1/(\mathfrak{g}(N)_{N_F} \times \mathfrak{g}(N_F)_N)$, i.e., of the fact that the gauge theory obtained from $G + N_F\,\square$ by gauging the flavor symmetry is gapped.

- Similar considerations hold for other gauge groups; for example, if we use $\mathrm{Spin}(N)$ instead of $\mathrm{SO}(N)$, the flavor symmetry is $\mathrm{O}(N_F)$ instead of $\mathrm{SO}(N_F)$, and the infrared CFT is a WZW model with target space $\mathrm{O}(N_F)$. This is again a consequence of the level-rank duality $\mathrm{Spin}(N)_{N_F} \leftrightarrow \mathrm{O}(N_F)_N$ [113]. Similarly, one could use $\mathrm{U}(N)$ instead of $\mathrm{SU}(N)$, in which case the infrared CFT is $\mathrm{SU}(N_F)_N$, again by level-rank duality.

- This predicts for example that $\mathrm{SO}(N_F)_N$ has an 't Hooft anomaly for time-reversal, measured by $NN_F \mod 2$.

An interesting special case is $\mathrm{SU}(N) + \square$, i.e., a single copy of the fundamental representation $N_F = 1$. The infrared coset in this case is $\mathrm{U}(1)_N$. For $N = 3$ this coset is actually a supersymmetric minimal model, $\mathrm{U}(1)_3 = \mathcal{M}_1^{\mathcal{N}=2}$, and therefore $\mathrm{SU}(3) + \mathbf{3}$ has emergent supersymmetry in the infrared. This is a consequence of the fact that $\wedge^3\square$ contains a gauge singlet, if and only if $N = 3$ [153].

In the ultraviolet of $\mathrm{SU}(N) + \square$ there is a manifest $\mathbb{Z}_2$ chiral symmetry that acts as $(-1)^{F_L} : \psi \mapsto \gamma^3\psi$, and whose 't Hooft anomaly is the number of Majorana fermions, $2N$ mod 8. This anomaly must be reproduced by the infrared degrees of freedom, i.e., by $\mathrm{U}(1)_N$. We check this as follows.

The equality $\mathrm{U}(N)_1/\mathrm{SU}(N)_1 = \mathrm{U}(1)_N$ is due to the character decomposition

$$\begin{aligned} d_{\text{NS-X}}(q,g,\theta) &= \sum_{\ell=0}^{N-1} (\pm 1)^{\ell} \chi_{\ell}^{\text{NS-X}}(q,\theta)\chi_{\gamma^{\ell}\cdot\mathbf{0}}(q,g) \\ d_{\text{R-X}}(q,g,\theta) &= \sum_{\ell=0}^{N-1} (\pm 1)^{\ell+1} \chi_{\ell+\lfloor N/2\rfloor}^{\text{R-X}}(q,\theta)\chi_{\gamma^{\ell}\cdot\mathbf{0}}(q,g)\,, \end{aligned} \tag{2.5.19}$$

where $\gamma$ is the generator of the $Z(\mathrm{SU}(N)) = \mathbb{Z}_N$ center symmetry and $\mathbf{0}$ is the vacuum character. Also, $\chi_\ell(q,\theta)$ denote the regular (bosonic) characters of $\mathrm{U}(1)_N$ if $N$ is even, and the super-characters if $N$ is odd; and $\theta$ denotes a $\mathrm{U}(1)$ flavor fugacity. Finally, $\chi_\lambda(q,g)$ denotes an $\mathrm{SU}(N)_1$ character with $g \in \mathrm{SU}(N)$ flavor fugacity.

These branching relations imply that the characters of the coset CFT are

$$\begin{aligned}
b_\ell^{\text{NS-NS}} &= \chi_\ell^{\text{NS-NS}} \\
b_\ell^{\text{NS-R}} &= (-1)^\ell \chi_\ell^{\text{NS-R}} \\
b_\ell^{\text{R-NS}} &= \chi_{\ell+\lfloor N/2\rfloor}^{\text{R-NS}} \\
b_\ell^{\text{R-R}} &= (-1)^\ell \chi_{\ell+\lfloor N/2\rfloor}^{\text{R-R}} \,.
\end{aligned} \tag{2.5.20}$$

Hence, the partition function twisted by the $(-1)^{F_L}$ symmetry is

$$\begin{aligned}
\operatorname{tr}_{\mathcal{H}_{\text{NS}}}\left((-1)^{F_L} q^{L_0-1/24}\bar{q}^{\bar{L}_0-1/24}\right) &= \sum_{\ell=0}^{N-1} \bar{b}_\ell^{\text{NS-R}} b_\ell^{\text{NS-NS}} \\
&= \sum_{\ell=0}^{N-1} (-1)^\ell \bar{\chi}_\ell^{\text{NS-R}} \chi_\ell^{\text{NS-NS}} \,.
\end{aligned} \tag{2.5.21}$$

The 't Hooft anomaly under $(-1)^{F_L}$ is measured by the phase acquired by this partition function under an $ST^2S^{-1}$ modular transformation. The idea is that $S$ moves the operator $(-1)^{F_L}$ from the spatial cycle into the temporal cycle, so it allows us to access the spin of the operators in the twisted sector. In a non-anomalous fermionic theory, the spin should be half-integral; hence, $T^2$ measures precisely the extent to which this condition fails. If we use the modular matrices of $\mathrm{U}(1)_N$ (see e.g. [1, 180]), we obtain $ST^2S^{-1} = e^{2\pi i \frac{2N}{8}}$, precisely as in the ultraviolet.

### 2.5.3 WZW models

We noticed in the previous section that the infrared CFT that describes $G + N_F \square$ is *just* the WZW model for the chiral flavor symmetry. An interesting question one could ask is how general this situation is, i.e., for which QCD theories are the infrared degrees of freedom just the affinization of the ultraviolet currents. As stated in (2.5.2)

$$\frac{\mathrm{SO}(\dim(R))_1}{G_{I(R)}} \equiv H_{\dim(R_0)} \times \frac{\mathrm{SO}(\dim(R))_1}{G_{I(R)} \times H_{\dim(R_0)}}\,, \tag{2.5.22}$$

one can always factor out these currents from the infrared coset, and the question becomes for which theories is the remaining sector a trivial CFT. Note that this extra part is precisely the infrared coset of the QCD theory where one gauges the (diagonal) flavor symmetry, $G \times H + (R_0, \square)$. From this perspective, the answer is straightforward: the theory $G + R$ flows in the infrared to $H_{\dim(R_0)}$ (plus possibly a trivial CFT, i.e., a TQFT) if and only if the theory $G \times H + (R_0, \square)$ is gapped. But now we can utilize our classification of gapped theories (cf. table 2.4) to give the list we are after. This way one obtains table 2.5.

| $G$ | $R$ | IR WZW |
|---|---|---|
| $G \in$ table 2.4 | $R$ | $\varnothing$ |
| $\mathrm{SU}(N)$ | $N_F\,\square$ | $\mathrm{U}(N_F)_N$ |
| $\mathrm{U}(N)$ | $N_F\,\square$ | $\mathrm{SU}(N_F)_N$ |
| $\mathrm{SO}(N)$ | $N_F\,\square$ | $\mathrm{SO}(N_F)_N$ |
| $\mathrm{Sp}(N)$ | $N_F\,\square$ | $\mathrm{Sp}(N_F)_N$ |
| $\mathrm{SU}(N)$ | $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ | $\mathrm{U}(1)_{\frac{1}{2}N(N-1)}$ |
| $\mathrm{SU}(N)$ | $\square\square$ | $\mathrm{U}(1)_{\frac{1}{2}N(N+1)}$ |
| $\mathrm{Spin}(10)$ | $\mathbf{16}$ | $\mathrm{U}(1)_{16}$ |
| $E_6$ | $\mathbf{27}$ | $\mathrm{U}(1)_{27}$ |
| $\mathrm{SU}(2)$ | $\mathbf{2}$ | $\mathrm{SU}(2)_1$ |
| $\mathrm{SU}(2)$ | $\mathbf{4}$ | $\mathrm{SU}(2)_2$ |
| $\mathrm{Sp}(3)$ | $\mathbf{14}$ | $\mathrm{SU}(2)_7$ |
| $\mathrm{SU}(6)$ | $\mathbf{20}$ | $\mathrm{SU}(2)_{10}$ |
| $\mathrm{Spin}(12)$ | $\mathbf{32}$ | $\mathrm{SU}(2)_{16}$ |
| $E_7$ | $\mathbf{56}$ | $\mathrm{SU}(2)_{28}$ |

Table 2.5: Classification of QCD theories that realize in the infrared a pure WZW model for the ultraviolet flavor symmetry (modulo a TQFT). Any theory not on this table will flow in the infrared to $H$ current algebra plus a non-trivial CFT (which has no continuous flavor symmetry). The first line "$G \in$ table 2.4" refers to the fact that gapped theories themselves satisfy this criterion, in the sense that both their flavor symmetry and their infrared CFT are trivial.

### 2.5.4 Minimal models

It is interesting to note that the (regular and supersymmetric) minimal models [181–187], a celebrated family of $2d$ CFTs, also appear in the infrared of QCD gauge theories. We already noticed three instances of this phenomenon so far, where we found QCD theories that flow to the $c = 1/2$, $c = 7/10$ and $c = 1$ theories in the infrared. Here we describe some families of QCD theories that realize all the minimal models. The examples are by no means exhaustive: there are many other QCD theories that also flow to minimal models in the infrared. In order to simplify the discussion, in this and subsequent examples we shall make no distinction between the bosonized/fermionized versions of a given CFT, and we will not be careful with certain discrete quotients of the gauge group (so for example $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$).

**Virasoro minimal models.** Consider the following QCD theories:

$$\begin{aligned} &\mathrm{SU}(2)^3 \times \mathrm{SO}(k) + (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{2},\mathbf{1},\square) \\ &\mathrm{SU}(2)^2 \times \mathrm{Sp}(k) + (\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2},\square)\,. \end{aligned} \tag{2.5.23}$$

Their infrared is

$$\frac{\mathrm{SO}(4k+4)_1}{\mathrm{SU}(2)_{k+1}\times\mathrm{SU}(2)_k\times\mathrm{SU}(2)_1\times\mathrm{SO}(k)_4}\,,\qquad\frac{\mathrm{SO}(4k+4)_1}{\mathrm{SU}(2)_1\times\mathrm{SU}(2)_{k+1}\times\mathrm{Sp}(k)_1}\,,\tag{2.5.24}$$

and we claim that these are both coset realizations of $\mathcal{M}_k$. Indeed, they can both be written as

$$\mathcal{M}_k\equiv\frac{\mathrm{SU}(2)_k\times\mathrm{SU}(2)_1}{\mathrm{SU}(2)_{k+1}}\tag{2.5.25}$$

thanks to the level-rank dualities $\mathrm{SO}(nm)_1/\mathrm{SO}(n)_m=\mathrm{SO}(m)_n$ and $\mathrm{SO}(4nm)_1/\mathrm{Sp}(n)_m=\mathrm{Sp}(m)_n$.

**$\mathcal{N}=1$ minimal model.** Consider the QCD theories

$$\begin{aligned}&\mathrm{SU}(2)^2\times\mathrm{Sp}(2)\times\mathrm{SO}(k)+(\mathbf{2},\mathbf{1},\mathbf{4},\mathbf{1})+(\mathbf{2},\mathbf{2},\mathbf{1},\square)\\&\mathrm{SU}(2)^2\times\mathrm{SO}(2)\times\mathrm{Sp}(k)+(\mathbf{2},\mathbf{1},0,\square)+(\mathbf{2},\mathbf{2},1,\mathbf{1})\,.\end{aligned}\tag{2.5.26}$$

Their infrared is

$$\frac{\mathrm{SO}(4k+8)_1}{\mathrm{SU}(2)_{k+2}\times\mathrm{SU}(2)_k\times\mathrm{Sp}(2)_1\times\mathrm{SO}(k)_4}\,,\quad\frac{\mathrm{SO}(4k+8)_1}{\mathrm{SU}(2)_{k+2}\times\mathrm{SU}(2)_2\times\mathrm{SO}(2)_4\times\mathrm{Sp}(k)_1}\tag{2.5.27}$$

and we claim that these are both coset realizations of $\mathcal{M}_k^{\mathcal{N}=1}$. Indeed, they can both be written as

$$\mathcal{M}_k^{\mathcal{N}=1}=\frac{\mathrm{SU}(2)_k\times\mathrm{SU}(2)_2}{\mathrm{SU}(2)_{k+2}}\tag{2.5.28}$$

for the same reason as for $\mathcal{M}_k$.

### 2.5.5 Diagonal coset

Here we discuss a class of QCD theories whose infrared leads to the so-called diagonal cosets $(\mathfrak{g}_k\times\mathfrak{g}_{k'})/\mathfrak{g}_{k+k'}$, whose structure is better understood than that of generic cosets [188, 189]. In particular, consider the following linear quivers:

$$\begin{aligned}S(\mathrm{U}(N)\times\mathrm{U}(M)\times\mathrm{U}(L))+(\square,\square,\mathbf{1})+(\mathbf{1},\square,\square)\\\mathrm{SO}(N)\times\mathrm{SO}(M)\times\mathrm{SO}(L)+(\square,\square,\mathbf{1})+(\mathbf{1},\square,\square)\\\mathrm{Sp}(N)\times\mathrm{Sp}(M)\times\mathrm{Sp}(L)+(\square,\square,\mathbf{1})+(\mathbf{1},\square,\square)\,.\end{aligned}\tag{2.5.29}$$

Their infrared theories are

$$\begin{aligned}&\frac{\mathrm{U}(NM+LM)_1}{S(\mathrm{U}(N)_M\times\mathrm{U}(M)_{N+L}\times\mathrm{U}(L)_M)}\,,\\&\qquad\qquad\frac{\mathrm{SO}(NM+LM)_1}{\mathrm{SO}(N)_M\times\mathrm{SO}(M)_{N+L}\times\mathrm{SO}(L)_M}\,,\\&\qquad\qquad\qquad\qquad\frac{\mathrm{SO}(4NM+4LM)_1}{\mathrm{Sp}(N)_M\times\mathrm{Sp}(M)_{N+L}\times\mathrm{Sp}(L)_M}\,,\end{aligned}\tag{2.5.30}$$

which, thanks to level-rank duality, become the following diagonal cosets

$$\frac{\mathrm{SU}(M)_N \times \mathrm{SU}(M)_L}{\mathrm{SU}(M)_{N+L}}, \qquad \frac{\mathrm{SO}(M)_N \times \mathrm{SO}(M)_L}{\mathrm{SO}(M)_{N+L}}, \qquad \frac{\mathrm{Sp}(M)_N \times \mathrm{Sp}(M)_L}{\mathrm{Sp}(M)_{N+L}}\,. \tag{2.5.31}$$

### 2.5.6 Kazama-Suzuki

Here we describe QCD theories that acquire an emergent $\mathcal{N}=2$ supersymmetry in the infrared. In particular, they become Kazama-Suzuki models [190].

Consider the QCD quiver associated to an arbitrary complete graph $K_n$, where each node represents a gauge group $G_i$, and each edge a bifundamental quark (see figure 2.2). We take the gauge groups to be any of

$$\begin{aligned} G &= S(\mathrm{U}(N_1)\times \mathrm{U}(N_2)\times\cdots\times \mathrm{U}(N_n)) \\ G &= \mathrm{SO}(N_1)\times \mathrm{SO}(N_2)\times\cdots\times \mathrm{SO}(N_n) \\ G &= \mathrm{Sp}(N_1)\times \mathrm{Sp}(N_2)\times\cdots\times \mathrm{Sp}(N_n)\,. \end{aligned} \tag{2.5.32}$$

Note that for $n=1,2$ these are pure Yang-Mills and $G\times G+(\square,\square)$, i.e., they are both gapped theories (the latter are entries on table 2.4). We claim that for $n\geq 3$, the theory has emergent $\mathcal{N}=2$ supersymmetry in the infrared, they are Kazama-Suzuki models. Indeed, their infrared cosets are

$$\frac{\mathrm{U}\big(\sum_{i>j} N_i N_j\big)_1}{S\big(\prod_i \mathrm{U}(N_i)_{\sum_{j\neq i} N_j}\big)}, \qquad \frac{\mathrm{SO}\big(\sum_{i>j} N_i N_j\big)_1}{\prod_i \mathrm{SO}(N_i)_{\sum_{j\neq i} N_j}}, \qquad \frac{\mathrm{SO}\big(4\sum_{i>j} N_i N_j\big)_1}{\prod_i \mathrm{Sp}(N_i)_{\sum_{j\neq i} N_j}}\,, \tag{2.5.33}$$

which are the cosets that describe Kazama-Suzuki models associated to the embeddings

$$\begin{aligned} &\mathrm{SU}\big(\sum_{i\neq\star} N_i\big) \supset S\big(\prod_{i\neq\star}\mathrm{U}(N_i)\big), \\ &\qquad\qquad \mathrm{SO}\big(\sum_{i\neq\star} N_i\big) \supset \prod_{i\neq\star}\mathrm{SO}(N_i), \\ &\qquad\qquad\qquad\qquad \mathrm{Sp}\big(\sum_{i\neq\star} N_i\big) \supset \prod_{i\neq\star}\mathrm{Sp}(N_i) \end{aligned} \tag{2.5.34}$$

at level $N_\star$, where $\star$ is an arbitrary node of $K_n$.

## 2.6 Conventions and Background

We work in $1+1$ dimensional Minkowski spacetime with metric $\eta=\mathrm{diag}(-1,+1)$ and $2\times 2$ gamma matrices $\gamma^\mu$, $\mu=0,1$. The Hodge dual of a one-form is $(\star j)_\mu=\epsilon_{\mu\nu}j^\nu$ where we take $\epsilon^{tx}=-\epsilon_{tx}=+1$. In particular, $\star\mathrm{d}t=\mathrm{d}x$, $\star\mathrm{d}x=\mathrm{d}t$. In null coordinates $x^\pm$ the metric is $\eta_{++}=\eta_{--}=0$, $\eta_{+-}=\eta_{-+}=1$, and the star is $\star\mathrm{d}x^\pm=\pm\mathrm{d}x^\pm$.

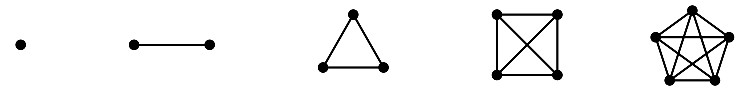

Figure 2.2: The first few complete graphs $K_1, K_2, \ldots, K_5$. If we associate to each node a gauge group $\mathrm{U}(n_i), \mathrm{SO}(n_i), \mathrm{Sp}(n_i)$ (with the global U(1) modded out in the unitary case), and to each edge a bifundamental quark, then the quiver gauge theory associated to $K_n$ has a Kazama-Suzuki model as its effective low energy description.

The minimal spinor is Majorana-Weyl, namely one can impose the simultaneous conditions $\gamma^3\psi = \pm\psi$ and $\psi^* = C\psi$, where $\gamma^3 = \gamma^0\gamma^1$ is the chirality matrix and $C$ is the charge-conjugation matrix, defined by $(\gamma^\mu)^* = C\gamma^\mu C^{-1}$. It is convenient to choose the Majorana basis $\gamma^\mu = (i\sigma^2, \sigma^1)$ where $\gamma^3 = \sigma^3$ and $C = 1$. In this basis, Majorana fermions are real $\psi^* = \psi$, and chiral fermions are either $\psi \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\psi \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We take $\psi = 2^{-1/4}\begin{pmatrix} \psi_\ell \\ \psi_r \end{pmatrix}$ and $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$. The fermion kinetic term is

$$-i\bar{\psi}\gamma^\mu\partial_\mu\psi = i\psi_\ell^*\partial_-\psi_\ell + i\psi_r^*\partial_+\psi_r\,. \tag{2.6.1}$$

For Grassmann odd $a, b$ we use $(ab)^* = b^*a^*$.

**Mass terms.** For massless fermions the two chiralities are decoupled. These couple through mass terms

$$\begin{aligned} i\bar{\psi}\psi &= \frac{i}{\sqrt{2}}(\psi_\ell^*\psi_r + \psi_\ell\psi_r^*) \\ i\bar{\psi}\gamma^3\psi &= \frac{-i}{\sqrt{2}}(\psi_\ell^*\psi_r - \psi_\ell\psi_r^*)\,. \end{aligned} \tag{2.6.2}$$

both of which are hermitian. Although less obvious, one can also use bilinears of the form

$$\begin{aligned} \mathrm{re}(\bar{\psi}^*\psi) &= \frac{1}{\sqrt{2}}(\psi_\ell\psi_r - \psi_\ell^*\psi_r^*) \\ \mathrm{im}(\bar{\psi}^*\psi) &= \frac{i}{\sqrt{2}}(\psi_\ell\psi_r + \psi_\ell^*\psi_r^*)\,. \end{aligned} \tag{2.6.3}$$

both of which are hermitian and Lorentz scalars (recall that $\psi_\ell \mapsto e^{\eta/2}\psi_\ell$ and $\psi_r \mapsto e^{-\eta/2}\psi_r$ under a boost with rapidity $\eta \in \mathbb{R}$).

If the fermion is real, $\psi_{\ell,r}^* = \psi_{\ell,r}$, then the bilinears $i\bar{\psi}\gamma^3\psi$ and $\mathrm{re}(\bar{\psi}^*\psi)$ both become zero, due to fermi statistics, and the other two bilinears $i\bar{\psi}\psi$ and $\mathrm{im}(\bar{\psi}^*\psi)$ become identical.

**Symmetries.** Let us list some of the manifest symmetries of QCD theories. The center one-form symmetry is straightforward: it is given by the subgroup of the center that is not screened by the fermions, namely $\ker(R) \subseteq Z(G)$, the kernel of the representation $R$.

As a matter of principle, it is possible that there are other one-form symmetries that are not associated to the center of the gauge group, although exhibiting these is a much more complicated task.[62]

Zero-form symmetries are abundant too. For example, if we have $N_F$ massless chiral fermions in a given representation $R$, one has the following continuous flavor symmetries acting on them:

$$\begin{array}{ll} R \text{ complex:} & \mathrm{U}(N_F) \\ R \text{ real:} & \mathrm{O}(N_F) \\ R \text{ pseudoreal:} & \mathrm{Sp}(N_F)\,. \end{array} \tag{2.6.4}$$

When $G$ has no U(1) factors, so that the generators are traceless, the classical axial symmetries are unbroken (unlike in $3+1d$). If there are abelian factors then U(1) flavor symmetries are generically broken into discrete subgroups. An important discrete chiral subgroup, which is very often present even if U(1) is broken, is the chiral fermion parity $\mathbb{Z}_2^L$, which negates left movers and fixes right movers, to wit, $\mathbb{Z}_2^L\colon \psi \mapsto \gamma^3\psi$. This subgroup has a very well understood group of 't Hooft anomalies, valued in $\mathbb{Z}_8$ and responsible for many interesting properties of QCD theories.

In presence of mass terms, the continuous chiral symmetries (2.6.4) descend to their diagonal vector-like subgroups, or even smaller subgroups if the different flavors have different masses.

Finally, there are two other discrete symmetries that are quite useful: charge conjugation $\mathbb{Z}_2^{\mathsf{C}}$, which sends a representation $R$ to its conjugate $\bar{R}$, and time-reversal $\mathbb{Z}_2^{\mathsf{T}}$, which is anti-linear and satisfies $\mathsf{T}^2 = (-1)^F$. Note that charge conjugation exists only if the group admits complex representations, for otherwise the operation is a gauge transformation and thus not a symmetry. On the other hand, time-reversal is a symmetry only if the theory is vector-like $R_\ell = R_r$, because $\gamma^0$ interchanges the two chiralities. The combination $\mathsf{CT}$ is a symmetry only if $R_\ell = \bar{R}_r$. Charge conjugation and time-reversal act on the gauge fields as

$$\begin{aligned} &\mathsf{C}\colon \begin{cases} \psi(x,t) \mapsto \psi^*(x,t) \\ A_\mu(x,t) \mapsto -A_\mu^t(x,t) \end{cases} \\ &\mathsf{T}\colon \begin{cases} \psi(x,t) \mapsto \gamma^0\psi(x,-t) \\ A_0(x,t) \mapsto +A_0^t(x,-t) \\ A_1(x,t) \mapsto -A_1^t(x,-t)\,, \end{cases} \end{aligned} \tag{2.6.5}$$

where $t$ denotes transposition.

[62]For example it has recently been appreciated [191] that pure Yang-Mills has a very large space of non-invertible one-form symmetries, valued in a maximal torus of $G$. It is not clear which if these, if any, survive the introduction of matter, or whether there are other one-form symmetries – invertible or otherwise – beyond these, although at face value it seems unlikely.

The three discrete $\mathbb{Z}_2$ transformations, chiral fermion parity, charge conjugation, and time-reversal, act as follows on the fermion mass terms:

$$\begin{array}{ccccc} & i\bar{\psi}\psi & i\bar{\psi}\gamma^3\psi & \mathrm{re}(\bar{\psi}^*\psi) & \mathrm{im}(\bar{\psi}^*\psi) \\ \mathbb{Z}_2^L & - & - & - & - \\ \mathbb{Z}_2^{\mathsf{C}} & + & - & - & + \\ \mathbb{Z}_2^{\mathsf{T}} & - & + & - & - \end{array} \tag{2.6.6}$$

Note that all the mass terms are odd under the chiral symmetry, which suggests that this symmetry might be anomalous. On the other hand, there is at least one mass term that is even under the other two symmetries, so they are non-anomalous. The exception is when the fermions are real, because in that case the mass terms $i\bar{\psi}\gamma^3\psi$ and $\mathrm{re}(\bar{\psi}^*\psi)$ vanish due to fermi statistics. In this situation, the remaining mass terms $i\bar{\psi}\psi$ and $\mathrm{im}(\bar{\psi}^*\psi)$ (which are in fact equal) are odd under time-reversal, which suggests that such symmetry might be anomalous too (and, if so, the anomaly will be at most a mod 2 effect; this is confirmed by $\Omega_3^{\mathrm{pin}^+} = \mathbb{Z}_2$).

### 2.6.1 Matching Energy Momentum Tensors

We have the canonical energy momentum tensor built from the fermion fields which takes the form

$$T(z) = -\frac{1}{2}\sum_i :\psi^i\partial\psi^i:(z)\,. \tag{2.6.7}$$

The normal ordering is defined as the constant part of the OPE of AB. The notation above for two operators is defined by

$$:AB:(z) = \frac{1}{2\pi i}\oint_z \frac{dx}{x-z}A(x)B(z)\,, \tag{2.6.8}$$

and extracts the constant part of the OPE of $A$ and $B$ via a contour integral. We can define the current $J^a(z) = \frac{1}{2}:\psi^i t^a_{ij}\psi^j:(z)$, for $t^a_{ij}$ the generators of the $\mathfrak{g}$ Lie algebra. We write the energy momentum tensor given by the Sugawara construction as

$$\begin{aligned} \mathcal{T}(z) &= \gamma\sum_a ::\psi^i t^a_{ij}\psi^j::\psi^k t^a_{kl}\psi^l::(z) \\ &= \gamma\sum_a t^a_{ij}t^a_{kl}\,::\psi^i\psi^j::\psi^k\psi^l::(z)\,. \end{aligned} \tag{2.6.9}$$

We first consider the term $::\psi^i\psi^j::\psi^k\psi^l::$. By the rearrangement lemma [150, Appendix 6.C] we have

$$\begin{aligned} ::\psi^i\psi^j::\psi^k\psi^l::(z) = {}&:\psi^i\,:\psi^j\,:\psi^k\psi^l:::(z) + :[:\psi^i\psi^j:,:\psi^k\psi^l:]:(z) \\ &+ ::[:\psi^k\psi^l:,\psi^i]:\psi^j:(z) + :\psi^i\,:[:\psi^k\psi^l:,\psi^j]::(z)\,, \end{aligned} \tag{2.6.10}$$

When we substitute these results into (2.6.9) we get

$$\begin{aligned}\gamma\sum_a t^a_{ij}t^a_{kl} \;{:}{:}\psi^i\psi^j{:}{:}\psi^k\psi^l{:}{:}\;(z) &= \gamma\sum_a t^a_{ij}t^a_{kl}\;{:}\psi^i\;{:}\psi^j\;{:}\psi^k\psi^l{:}{:}{:}\;(z)\\ &\quad+\gamma\sum_a t^a_{ij}t^a_{kl}[-\delta^{jk}\;{:}\psi^l\partial_z\psi^i{:}\;+\delta^{jl}\;{:}\psi^k\partial_z\psi^i{:}\;-\delta^{il}\;{:}\psi^k\partial_z\psi^j{:}\;+\delta^{ik}\;{:}\psi^l\partial_z\psi^j{:}]\\ &= \gamma\sum_a t^a_{ij}t^a_{kl}\;{:}\psi^i\;{:}\psi^j\;{:}\psi^k\psi^l{:}{:}{:}\;(z)+\gamma\sum_a[-4t^a_{ij}t^a_{jl}\;\;{:}\psi^l\partial_z\psi^i{:}]\,.\end{aligned} \tag{2.6.11}$$

The second term above has the form of (2.6.7), where we use the fact that

$$\sum_a t^a_{ij}t^a_{jl} = 2I(R)\frac{\dim(G)}{\dim(R)}\,\delta^{il}\,,\quad \gamma=\frac{1}{8(I(R)+h)}\,, \tag{2.6.12}$$

and take $G$ and $R$ from our list of gapped theories. A necessary condition for equality of $T(z)$ and $\mathcal{T}(z)$ is if $\gamma\sum_a t^a_{ij}t^a_{kl}\;{:}\psi^i\;{:}\psi^j\;{:}\psi^k\psi^l{:}{:}{:}\;(z)=0$. By definition, this term is

$$\frac{\gamma}{(2\pi i)^4}\sum_a\oint\frac{dw}{w-z}\,\frac{dx}{x-z}\,\frac{dy}{y-z}t^a_{ij}t^a_{kl}\,\psi^i(w)\psi^j(x)\psi^k(y)\psi^l(z), \tag{2.6.13}$$

and since $\psi^i$ is Grassmann, this vanishes if the total antisymmetrization of $\sum_a t^a_{ij}t^a_{kl}$ vanishes, and this gives us the condition

$$\sum_a t^a_{ij}t^a_{kl}+t^a_{ik}t^a_{lj}+t^a_{il}t^a_{jk}=0\,. \tag{2.6.14}$$

If the group $G=\tilde{G}\times\mathrm{U}(1)$ which has a U(1) factor, we can write $t^a_{ij}$ as a decomposition under $\tilde{G}$ and U(1), where it is charge $q$ under the U(1) i.e. the current is $J(z)=\frac{1}{2}\;{:}\psi^i q\,\psi^i{:}$. As an example we take U($N$) with a fermion that is in the antisymmetric representation of SU($N$) and charge $q$ under U(1). Then the SU($N$) part of the Sugawara tensor reads

$$\begin{aligned}\mathcal{T}_{\mathrm{SU}(N)}(z) &= \frac{1}{8(2N-2)}\Big[-4\frac{2(N^2-N-2)}{N}\Big]\;{:}\psi^i\partial_z\psi^i{:}\\ &= -\frac{1}{2}\;{:}\psi^i\partial_z\psi^i{:}\;+\frac{1}{N(N-1)}\;{:}\psi^i\partial_z\psi^i{:}\;\;.\end{aligned} \tag{2.6.15}$$

The U(1) part of the Sugawara tensor reads

$$\mathcal{T}_{\mathrm{U}(1)}(z)=\frac{1}{4N(N-1)}\;{:}{:}\psi^i\psi^i{:}{:}\psi^j\psi^j{:}{:}=\frac{-1}{N(N-1)}\;{:}\psi^i\partial_z\psi^i{:}\,, \tag{2.6.16}$$

and by summing (2.6.15) and (2.6.16) we reproduce (2.6.7).

Now suppose we are working with an Abelian theory and we consider $n$ complex fermions, with $q_{Ia}$ charge matrix, i.e. $\mathrm{U}(1)^n_{q^tq}$. The canonical energy momentum is given by

$$T(z)=\frac{1}{2}({:}\partial_z\psi^{I\dagger}\psi^I{:}\;-\;{:}\psi^{I\dagger}\partial_z\psi^I{:})(z). \tag{2.6.17}$$

With the charge matrix we can define the current $J_a = \psi^{I\dagger} q_{Ia} \psi^I$, which satisfies the OPE

$$J_a(z)J_b(0) \sim \frac{k_{ab}}{z^2}. \tag{2.6.18}$$

From this we build the Sugawara tensor

$$\mathcal{T}(z) = \frac{1}{2}\sum_{a,b}(k_{ab})^{-1} ::\psi^{I\dagger} q_{Ia}\psi^I::\psi^{J\dagger} q_{Jb}\,\psi^J:: (z)\,, \tag{2.6.19}$$

where $k_{ab} = (q^t q)_{ab} = q_{Ia}\, q_{Ib}$, so $(k_{ab})^{-1} = (q_{Ia}\, q_{Ib})^{-1}$.and we can define the current as $J_a =:\psi^{I\dagger} q_{Ia} \psi^I:$. Again by the rearrangement lemma we have

$$\begin{aligned} q_{Ia}\, q_{Jb}\ ::\psi^{I\dagger}\psi^I::\psi^{J\dagger}\,\psi^J:: (z) &= q_{Ia}\, q_{Jb}\ :\psi^{I\dagger}\ :\psi^I\ :\psi^{J\dagger}\psi^J:::\ (z) \\ &\quad + q_{Ia}\, q_{Jb}(-\delta^{IJ}\psi^J\ :\partial_z\psi^{I\dagger}:\ -\delta^{IJ}\ :\psi^{J\dagger}\partial_z\psi^I:)(z) \\ &= q_{Ia}\, q_{Ib}\,(-\ :\psi^I\partial_z\psi^{I\dagger}:\ -\ :\psi^{I\dagger}\partial_z\psi^I:)(z)\,. \end{aligned} \tag{2.6.20}$$

After substituting into (2.6.19) we get

$$\begin{aligned} \mathcal{T}(z) &= \frac{1}{2}\sum_{a,b}(q_{Ib}^{-1}q_{Ia}^{-1})[q_{Ia}\, q_{Jb}\ :\psi^{I\dagger}\ :\psi^I\ :\psi^{J\dagger}\psi^J:::\ (z) \\ &\quad + q_{Ia}\, q_{Jb}(-\delta^{IJ}\ :\psi^J\partial_z\psi^{I\dagger}:\ -\delta^{IJ}\ :\psi^{J\dagger}\partial_z\psi^I:)(z)] \\ &= \frac{1}{2}[:\psi^{I\dagger}\ :\psi^I\ :\psi^{J\dagger}\psi^J:::\ (z) + (-\delta^{IJ}\ :\psi^J\partial_z\psi^{I\dagger}:\ -\delta^{IJ}\ :\psi^{J\dagger}\partial_z\psi^I:)(z)] \\ &= \frac{1}{2}(:\partial_z\psi^{I\dagger}\psi^I:\ -\ :\psi^{I\dagger}\partial_z\psi^I:)(z) \\ &= T(z)\,, \end{aligned} \tag{2.6.21}$$

which is the canonical energy momentum tensor. We have used the fact that $\sum_b (q_{Ib})^{-1}\, q_{Jb} = \delta_{IJ}$, and the first term in the second equality vanishes by applying (2.6.13) and evaluating the contour integrals.

### 2.6.2 Temporal Gauge Hamiltonian commutation

The quantized Hamiltonian is given by integrating the Hamiltonian action in (2.4.47), where we take the left and right handed fermions $\psi_\ell(x)$ and $\psi_r(x)$ to be operators on a circle where $0 \le x \le 2\pi$:

$$\begin{aligned} \widehat{H} = \int_0^{2\pi} \Big[&\frac{i}{8(I(R)+h)}\left(:\widehat{J}^a_\ell \widehat{J}^a_\ell(x): + :\widehat{J}^a_r \widehat{J}^a_r(x):\right) \\ &+ \frac{1}{\sqrt{2}} i \lim_{\epsilon\to 0}\left(\langle\psi^\dagger_{i\,\ell}(x+\epsilon)\partial_x\psi_{i\,\ell}(x-\epsilon)\rangle - \langle\psi^\dagger_{i\,r}(x+\epsilon)\partial_x\psi_{i\,r}(x-\epsilon)\rangle\right) \\ &+ \frac{1}{\sqrt{2}}\widehat{A}^a(x)(\widehat{J}^a_\ell - \widehat{J}^a_r)(x) + \frac{1}{\sqrt{2}} I(R)\widehat{A}^a(x)^2\Big]\,dx \\ &+ g^2\int_0^{2\pi}(\widehat{E}^a(x))^2 dx. \end{aligned} \tag{2.6.22}$$

We use the expression for $\widehat{J}^a_{\ell,r}$ as specifically given in section 2.4.2.

For the commutator $[\widehat{G}^a, \widehat{H}] = 0$, the pieces proportional to $g^0$ and $g^2$ vanish separately. The commutator of $\widehat{G}^a$ with $:\widehat{J}^a_{\ell,r}\widehat{J}^a_{\ell,r}(x):$ vanishes [192], and we proceed to use the commutation relations in (2.4.42) to show that the other terms in (2.6.22) vanish when commuted with $\widehat{G}^a$. Consider first the terms:

$$[\widehat{J}^a_\ell(x), \int_0^{2\pi} \widehat{A}^b \widehat{J}^b_\ell(y)dy] = \int_0^{2\pi} \widehat{A}^b(y) \left(if^{abc}J^c_\ell(x)\delta(x-y) + iI(R)\delta^{ab}\partial_x\delta(x-y)\right) dy \quad (2.6.23)$$

$$[\widehat{J}^a_r(x), \int_0^{2\pi} \widehat{A}^b \widehat{J}^b_r(y)dy] = \int_0^{2\pi} \widehat{A}^b(y) \left(if^{abc}J^c_r(x)\delta(x-y) - iI(R)\delta^{ab}\partial_x\delta(x-y)\right) dy\,. \quad (2.6.24)$$

We then look at the terms:

$$\begin{aligned} -[D_x\widehat{\Pi}^a(x), \int_0^{2\pi} \widehat{A}^d(\widehat{J}^d_\ell - \widehat{J}^d_r)(y)dy] = \\ &- \int_0^{2\pi} i\delta^{ab}\partial_x\delta(x-y)\left(\widehat{J}^b_\ell - \widehat{J}^b_r\right)(y)dy \quad (2.6.25)\\ &- \int_0^{2\pi} if^{abc}\widehat{A}^b(x)\delta^{cd}\left(\widehat{J}^d_\ell - \widehat{J}^d_r\right)(y)\delta(x-y)dy\,, \\ -I(R)[D_x\widehat{\Pi}^a(x), \int_0^{2\pi} \widehat{A}^d(y)^2dy] = \\ &- 2iI(R)\int_0^{2\pi} \widehat{A}^d(y)\delta^{ad}\partial_x\delta(x-y)dy \quad (2.6.26)\\ &- 2iI(R)\int_0^{2\pi} if^{abc}\widehat{A}^d(y)\widehat{A}^b(x)\delta^{cd}\delta(x-y)dy\,. \end{aligned}$$

The second term in (2.6.23) and (2.6.24) cancel with the first term in (2.6.26). The first term in (2.6.23) and (2.6.24) cancel with the second term in (2.6.25); the last term in (2.6.26) vanishes by antisymmetry. We are thus left to negotiate the term

$$-\int_0^{2\pi} i\delta^{ab}\partial_x\delta(x-y)\left(\widehat{J}^b_\ell - \widehat{J}^b_r\right)(y)dy \quad (2.6.27)$$

in (2.6.25). For this we consider

$$[\widehat{J}^a_{\ell,r}(x), \pm\int_0^{2\pi} i\lim_{\epsilon\to 0}\langle\psi^\dagger_{k\,\ell,r}(y+\epsilon)\partial_y\psi_{k\,\ell,r}(y-\epsilon)\rangle dy] \quad (2.6.28)$$

where we first compute the commutator by treating the second term as an operator, and then using the propagator while taking the $\epsilon\to 0$ limit. This is the same procedure used to prove that $[\widehat{J}^a_{\ell,r}(x), \widehat{J}^b_{\ell,r}(y)]$ contains a Schwinger term. Working with just the left handed part of 2.6.28 we get

$$
\begin{aligned}
&[\widehat{J}^a_\ell(x), \int_0^{2\pi} i \lim_{\epsilon\to 0}\langle \psi^\dagger_{k\,\ell}(y+\epsilon)\partial_y\psi_{k\,\ell}(y-\epsilon)\rangle dy] \\
&= [\tfrac{1}{2}\psi^\dagger_{i\,\ell}t^a_{ij}\psi_{j\,\ell}(x), \int_0^{2\pi} i \lim_{\epsilon\to 0}\langle \psi^\dagger_{k\,\ell}(y+\epsilon)\partial_y\psi_{k\,\ell}(y-\epsilon)\rangle dy] \\
&= \frac{i}{2}\lim_{\epsilon\to 0}\Big\{\int_0^{2\pi}\delta(x-y-\epsilon)\partial_y(\psi^\dagger_{i\,\ell}(y+\epsilon)t^a_{ij}\psi_{j\,\ell}(y-\epsilon))dy \\
&\qquad\qquad + \int_0^{2\pi}\partial_y\delta(x-y+\epsilon)\psi_{j\,\ell}(y+\epsilon)t^a_{ij}\psi^\dagger_{i\,\ell}(y-\epsilon)dy\Big\} \\
&= \frac{i}{2}\lim_{\epsilon\to 0}\Big\{-\int_0^{2\pi}\partial_y\delta(x-y-\epsilon)\psi^\dagger_{i\,\ell}(y+\epsilon)t^a_{ij}\psi_{j\,\ell}(y-\epsilon)dy \\
&\qquad\qquad + \int_0^{2\pi}\partial_y\delta(x-y+\epsilon)\psi_{j\,\ell}(y+\epsilon)t^a_{ij}\psi^\dagger_{i\,\ell}(y-\epsilon)dy\Big\} \\
&= i\Big\{-\int_0^{2\pi}\partial_y\delta(x-y){:}\psi^\dagger_{i\,\ell}(y)t^a_{ij}\psi_{j\,\ell}(y){:}dy\Big\} \\
&= i\int_0^{2\pi}\partial_x\delta(x-y)\widehat{J}^a(y)dy\,,
\end{aligned}
\tag{2.6.29}
$$

which cancels the first term in (2.6.27), and we can do an analogous computation for $\widehat{J}^a_r$. The first equality we use the fact that $\widehat{J}^a(x) = \frac{1}{2}\psi^\dagger_{i\,\ell}t^a_{ij}\psi_{j\,\ell}(x) + (\text{singular term})$, where the singular term vanishes in the commutator. To go from the third equal sign to the fourth equal sign we replace the fermions by the normal ordered version where [193]

$$
\psi^\dagger_{i\,\ell}(y+\epsilon)t^a_{ij}\psi_{j\,\ell}(y-\epsilon) = {:}\psi^\dagger_{i\,\ell}(y+\epsilon)t^a_{ij}\psi_{j\,\ell}(y-\epsilon){:} + \lim_{\tilde{\epsilon}\to 0}\langle\psi^\dagger_{i\,\ell}(x+\epsilon+\tilde{\epsilon})t^a_{ij}\psi_{j\,\ell}(x-\epsilon-\tilde{\epsilon})\rangle\,, \tag{2.6.30}
$$

and the second term vanishes under the derivative. For the term proportional to $g^2$ we consider

$$
[D_x\widehat{\Pi}^a(x), \int_0^{2\pi}\widehat{E}^d(y)^2dy] = -2i\int_0^{2\pi} i f^{abc}\widehat{E}^b(x)\widehat{E}^c(y)\delta(x-y)dy\,, \tag{2.6.31}
$$

which vanishes due to the antisymmetry of $f^{abc}$.

## 2.7 Infrared coset CFTs

In this section we review the formalism of coset CFTs [150], our primary goal being to understand the CFTs that appear in the deep infrared of QCD (2.5.1).

**Chiral characters.** One of the most important concepts in RCFT is that of a chiral character. These consist of a finite family of functions $\{\chi_\lambda(q)\}$ of the complex structure

$q = e^{2\pi i\tau}$, holomorphic in the upper half plane, and labelled by the primaries of the theory $\lambda$ (the representations of the chiral algebra). Given these characters, the torus partition function of the theory takes the form

$$Z(q) = \sum_{\lambda,\bar{\lambda}} M_{\bar{\lambda},\lambda}\, \bar{\chi}_{\bar{\lambda}}(\bar{q})\chi_\lambda(q) \equiv \bar{\chi}^\dagger M \chi\,. \tag{2.7.1}$$

Here $M$, the so-called mass matrix, specifies how the left-moving sectors are paired up with right-moving ones. The possible choices for $M$ are constrained by the requirement of $Z$ being a modular-invariant function of $q$. This is archived by a key property of the characters, to wit, their covariance under modular transformations. Under a generic such transformation, the characters mix with each other in a well-defined fashion, and the role of $M$ is to ensure that the sesquilinear form $Z = \chi^\dagger M \chi$ is a scalar under these transformations. As a result, while $Z(q)$ is a well-defined function on $\mathbb{H}/SL(2,\mathbb{Z})$, the tuple $\chi$ is best thought of as a non-trivial section thereon.

When $\chi_\lambda(q)$ admits a Hilbert space interpretation, it is defined as

$$\chi_\lambda(q) := \mathrm{tr}_{\mathcal{H}_\lambda}(q^{L_0 - c/24})\,, \tag{2.7.2}$$

such that

$$Z(q) = \mathrm{tr}_{\mathcal{H}}(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}), \qquad \mathcal{H} = \bigoplus_{\lambda,\bar{\lambda}} M_{\bar{\lambda},\lambda}\mathcal{H}_{\bar{\lambda}} \otimes \mathcal{H}_\lambda\,. \tag{2.7.3}$$

Here $\mathcal{H}$ is the full Hilbert space of the theory, and $\mathcal{H}_\lambda$ is the representation space (module) for $\lambda$. The mass matrix $M$ dictates how these chiral modules combine into $\mathcal{H}$. From now on, and in order to simplify the notation and presentation, we always have in mind the diagonal theory $M_{\bar{\lambda},\lambda} = \delta_{\bar{\lambda},\lambda}$. Non-diagonal theories can often be thought of as the diagonal theory of a larger algebra via (potentially non-abelian) anyon condensation.

**Fermionic CFTs.** A CFT is said to be *fermionic* if, on top of the dependence on the conformal structure of spacetime, it also depends on the choice of spin structure thereof. In other words, a fermionic CFT depends on the boundary conditions for fermionic fields. The CFTs that appear at RG fixed points of QCD theories are naturally fermionic, because the microscopic theory contains quarks. Hence our main interest is in fermionic CFTs.

In the case of the torus there are four spin structures, corresponding to either periodic or anti-periodic boundary conditions around the two non-trivial cycles. We also refer to these boundary conditions as Ramond and Neveu-Schwartz, respectively, and we use the notation $+ = \mathrm{R}$, $- = \mathrm{NS}$ interchangeably.

In a fermionic CFT, the characters acquire a dependence on the spin structure: they become super-characters. Consequently, we denote them as $\chi_\lambda^{\pm,\pm}$ where

$$
\begin{aligned}
\chi_\lambda^{\text{NS-NS}}(q) &:= \mathrm{tr}_{\mathcal{H}_{\text{NS};\lambda}}(q^{L_0-c/24}) \\
\chi_\lambda^{\text{NS-R}}(q) &:= \mathrm{tr}_{\mathcal{H}_{\text{NS};\lambda}}((-1)^{F_L} q^{L_0-c/24}) \\
\chi_\lambda^{\text{R-NS}}(q) &:= \mathrm{tr}_{\mathcal{H}_{\text{R};\lambda}}(q^{L_0-c/24}) \\
\chi_\lambda^{\text{R-R}}(q) &:= \mathrm{tr}_{\mathcal{H}_{\text{R};\lambda}}((-1)^{F_L} q^{L_0-c/24})\,.
\end{aligned}
\tag{2.7.4}
$$

Here, $\mathcal{H}_{\pm;\lambda}$ denotes the module of $\lambda$ with fermion boundary conditions $\pm$, while $(-1)^{F_L}$ denotes the chiral fermion parity operator, which assigns $+1$ to bosonic left-movers and $-1$ to fermionic left-movers, while it acts trivially on the right-moving modes.

The partition function of a fermionic CFT is obtained by combining the two chiral halves in a modular covariant way:

$$
Z_{\pm,\pm}(q) = \sum_{\bar\lambda,\lambda} M^{\pm}_{\bar\lambda,\lambda} \bar\chi^{\pm,\pm}_{\bar\lambda}(\bar q)\chi^{\pm,\pm}_{\lambda}(q)\,, \tag{2.7.5}
$$

which computes

$$
Z_{\pm,\pm}(q) = \mathrm{tr}_{\mathcal{H}_\pm}((\mp 1)^F q^{L_0-c/24}\bar q^{\bar L_0-c/24}), \qquad \mathcal{H}_\pm = \bigoplus_{\lambda,\bar\lambda} M^{\pm}_{\bar\lambda,\lambda}\mathcal{H}_{\pm;\bar\lambda}\otimes\mathcal{H}_{\pm;\lambda}\,, \tag{2.7.6}
$$

where $(-1)^F \equiv (-1)^{F_L}(-1)^{F_R}$ is the total fermion parity.

Here the mass matrices $M^\pm$, which dictate how the two chiral halves $\mathcal{H}_{\pm;\bar\lambda}, \mathcal{H}_{\pm;\lambda}$ combine into the full Hilbert space $\mathcal{H}_\pm$, are chosen so as to ensure that $Z_{\pm,\pm}$ transforms appropriately under modular transformations. Unlike in the case of bosonic CFTs, $Z_{\pm,\pm}$ is not in general invariant under $SL(2,\mathbb{Z})$. Indeed, modular transformations generically map the different spin structures into each other, which induces a reshuffling of the partition functions $Z_{\pm,\pm}$. Specifically, under the standard generators of $SL(2,\mathbb{Z}) = \langle S, T\rangle$, the partition functions transform as

$$
\begin{tikzcd}
Z_{\text{NS-NS}} \arrow[loop left, "S"] \arrow[r, leftrightarrow, "T"] & Z_{\text{NS-R}} \arrow[r, leftrightarrow, "S"] & Z_{\text{R-NS}} \arrow[loop right, "T"] \\
& Z_{\text{R-R}} \arrow[loop left, "S"] \arrow[loop right, "T"] &
\end{tikzcd}
\tag{2.7.7}
$$

The choices for the mass matrices $M^\pm$ are constrained by the requirement of $Z_{\pm,\pm}$ being a modular-covariant function of $q$. As usual, we will always have in mind the diagonal theory $M^{\pm}_{\bar\lambda,\lambda} = \delta_{\bar\lambda,\lambda}$.

In order to simplify the notation, we shall frequently leave the dependence on the spin structure $\pm,\pm$ implicit.

**Flavor-twisted characters.** If the chiral algebra has some flavor symmetry $U$, then it is often useful to introduce flavor-twisted characters (i.e., we turn on fugacities for the Cartan generators; these are roots of $\mathfrak{u}$). This allows us to organize the modules $\mathcal{H}_\lambda$ into irreducible representations of $U$ so as to have a more transparent understanding of the structure of states therein. To this end, we can define extended characters as

$$\chi_\lambda(q,g) := \mathrm{tr}_{\mathcal{H}_\lambda}(q^{L_0-c/24}\rho(g))\,, \tag{2.7.8}$$

where $g \in U$ is a symmetry group element and $\rho\colon g \to \mathcal{H}_\lambda$ is its representation on the Hilbert space. The character is a class function, so its dependence on $g$ is only through its conjugacy class.

Regular characters $\chi_\lambda(q)$ are obtained from the extended ones $\chi_\lambda(q,g)$ by setting $g=1$. The former only keep track of the conformal weights of the states in $\mathcal{H}_\lambda$, while the latter also keeps track of their quantum numbers under $U$.

**Coset CFTs.** Whenever the chiral algebra has a subalgebra, one can expand the characters of the former in terms of those of the latter,

$$\chi_\lambda(q) = \sum_\Lambda b_\lambda^\Lambda(q)\chi_\Lambda(q)\,, \tag{2.7.9}$$

where $\chi_\lambda$ are the characters of the original chiral algebra, and $\chi_\Lambda$ those of the subalgebra.

The key point of this construction is that, if $\chi_\lambda$ and $\chi_\Lambda$ are both modular covariant, then so are the coefficients $b_\lambda^\Lambda$. This means that one can think of these coefficients as the characters of a new theory, which we call the *coset CFT*; this is the celebrated GKO coset construction [194]. If at least one of $\chi_\lambda, \chi_\Lambda$ is a super-character, then so is $b_\lambda^\Lambda$, and hence the coset is a fermionic CFT.

This new theory, the coset CFT, has characters $b_\lambda^\Lambda(q)$, and therefore its partition function takes the form

$$Z^{\text{coset}}(q) = \sum_b \bar{b}(\bar{q})b(q)\,, \tag{2.7.10}$$

where we restrict to diagonal partition functions for simplicity. If $b$ is a super-character, the expression above defines the partition function of a fermionic CFT, while if it is a regular character, it defines the partition function of a bosonic CFT.

It should be remarked that, in order to actually calculate the coefficients $b_\lambda^\Lambda$, it is often unavoidable to turn on flavor fugacities, for otherwise the computation becomes impracticable. One is therefore forced to look at the extended characters $\chi_\lambda(q,g)$, $\chi_\Lambda(q,g')$, where $g$ is a symmetry group element of the original algebra, and $g'$ its restriction to the subalgebra. Specifically, if the original algebra has flavor symmetry $U$ and its subalgebra has flavor symmetry $U' \subseteq U$, then the character decomposition can be extended to

$$\chi_\lambda(q,g) = \sum_\Lambda b_\lambda^\Lambda(q,g'')\chi_\Lambda(q,g')\,, \tag{2.7.11}$$

where $g \in U$, $g' \in U'$, and $g''$ is a group element of the flavor symmetry of the coset, namely the commutant of $U'$ inside $U$.

**WZW CFTs.** The discussion so far has been rather abstract and general. To be concrete, let us discuss Wess–Zumino–Witten (WZW) theories, which we review next. WZW theories are labelled by a compact Lie group $G$, which we take to be simple and connected, and a "level" $k$, an integer that specifies the central extension for the loop algebra of $G$. We denote the corresponding model by $G_k$. When $G$ is simply-connected (which we henceforth assume, unless specified otherwise) the chiral algebra is a Kač-Moody algebra:

$$[J_n^a, J_m^b] = f^{ab}{}_c \, J_{n+m}^c + kn\delta_{ab}\delta_{n+m} \,, \tag{2.7.12}$$

while if $G$ is not simply-connected, then the chiral algebra is Kač-Moody extended by the simple currents that generate $\pi_1(G)$. WZW theories with $\pi_1(G) = 0$ are always bosonic, while those with $\pi_1(G) \neq 0$ are fermionic if any of the currents that generate $\pi_1(G)$ is a fermion (it has half-integral conformal weight).

The representations of the chiral algebra are required to be unitary with respect to $(J_n^a)^\dagger := J_{-n}^a$.

The enveloping algebra of (2.7.12) contains the Virasoro algebra via the Sugawara construction:

$$L_n = \frac{1}{2(k+h)} \sum_{a,m} {:}J_m^a J_{n-m}^a{:} \,, \tag{2.7.13}$$

such that

$$\begin{aligned} &[L_n, L_m] = (m-n)L_{n+m} + \frac{c(G_k)}{12} n(n^2-1)\delta_{n+m}, \qquad c(G_k) := \frac{k}{k+h}\dim(\mathfrak{g}) \\ &[L_n, J_m^a] = -mJ_{n+m}^a \,. \end{aligned} \tag{2.7.14}$$

Note that this last expression indicates that $J_{-n}$ carries $n$ units of $L_0$ eigenvalue.

The primaries of the theory are labelled by the integrable representations of $\mathfrak{g}_k$, to wit, the highest-weight representations $\lambda \in \mathrm{Rep}(\mathfrak{g})$ that satisfy

$$(\theta, \lambda) \leq k \,, \tag{2.7.15}$$

with $\theta$ the highest root of $\mathfrak{g}$ (the highest weight of the adjoint representation) and $(\cdot,\cdot)$ its Killing form, normalized to $(\theta,\theta) = 2$. The module $\mathcal{H}_\lambda$ is constructed as follows: at lowest grade, one begins with the vacuum states $|\lambda\rangle$, which live in the finite dimensional representation $\lambda$ generated by $J_0^a$. On top of these vacua one constructs the excited states. For example, at grade-one one has the states $J_{-1}^a|\lambda\rangle$, which live inside the representation $\theta\otimes\lambda$. At grade-two one has the states $J_{-2}^a|\lambda\rangle$ and $J_{-1}^a J_{-1}^b|\lambda\rangle$, so the states live inside $\theta\otimes\lambda + \theta^2_{\text{sym}}\otimes\lambda$.

Etc. Importantly, not all of these states are physical: we must project out the null states, i.e., the states whose norm vanishes. Continuing this way one obtains the character

$$\chi_\lambda(q,g) = q^{h_\lambda - c/24}(\chi_\lambda(g) + \chi_{R_1}(g)q + \chi_{R_2}(g)q^2 + \chi_{R_3}(g)q^3 + \cdots)\,, \tag{2.7.16}$$

where $R_n$ is the space of physical states at grade $n$, and $\chi_R(g) := \mathrm{tr}_R(g)$ the finite character of $R$.[63] A more convenient way to obtain the characters is the so-called Weyl-Kač formula [195],

$$\chi_\lambda(q,g) = \frac{\hat\chi_{\lambda+\rho}(q,g)}{\hat\chi_\rho(q,g)}, \qquad \hat\chi_\lambda(q,g) := \sum_{\substack{w\in W\\ \alpha^\vee\in Q^\vee}} \det(w)\, z^{k\alpha^\vee + w\lambda} q^{|k\alpha^\vee + w\lambda|^2/2k}\,, \tag{2.7.17}$$

where $W$ denotes the Weyl group of $\mathfrak{g}$, $Q^\vee$ its coroot lattice, $\rho$ its Weyl vector, and $z$ is the value of $g \in G$ when conjugated to any maximal torus.

**Spin($n$)$_1$ CFT.** A particularly important family of WZW models is Spin$(n)_1$. For $n$ odd this theory has three primaries $0, v, \sigma$ (the scalar, vector, and spinor representation, respectively), and for $n$ even it has four, $0, v, s, c$ (where $s, c$ denote the two spinors). The corresponding characters read

$$\begin{aligned}
\chi_0(q,g) &= q^{-n/48}\Big[\bullet + \ydiagram{1,1}\,q + \Big(\bullet + \ydiagram{1,1} + \ydiagram{2} + \ydiagram{1,1,1,1}\Big)q^2 \\
&\qquad + \Big(\bullet + 3\,\ydiagram{1,1} + \ydiagram{2} + \ydiagram{1,1,1,1} + \ydiagram{2,1,1} + \ydiagram{1,1,1,1,1,1}\Big)q^3 + \cdots\Big] \\
\chi_v(q,g) &= q^{-n/48+1/2}\,\ydiagram{1}\Big[\bullet + \bullet\, q + \Big(\bullet + \ydiagram{1,1}\Big)q^2 + \Big(\bullet + 2\,\ydiagram{1,1} + \ydiagram{1,1,1,1}\Big)q^3 + \cdots\Big] \\
&\qquad + q^{-n/48+3/2}\Big[\ydiagram{1,1,1} + \ydiagram{1,1,1,1,1}\,q + \ydiagram{1,1,1,1,1,1,1}\,q^2 + \cdots\Big] \\
&\equiv q^{-n/48+1/2}\Big[\ydiagram{1} + \Big(\ydiagram{1} + \ydiagram{1,1,1}\Big)q + \Big(2\,\ydiagram{1} + \ydiagram{2,1} + \ydiagram{1,1,1} + \ydiagram{1,1,1,1,1}\Big)q^2 \\
&\qquad + \Big(3\,\ydiagram{1} + 2\,\ydiagram{2,1} + 3\,\ydiagram{1,1,1} + \ydiagram{2,1,1,1} + \ydiagram{1,1,1,1,1} + \ydiagram{1,1,1,1,1,1,1}\Big)q^3 + \cdots\Big] \\
\chi_\sigma(q,g) &= q^{n/24}\,\sigma\,\Big[\bullet + \ydiagram{1,1}\,q^2 + \Big(\bullet + \ydiagram{1,1} + \ydiagram{2}\Big)q^3 + \Big(\bullet + 2\,\ydiagram{1,1} + \ydiagram{2} + \ydiagram{1,1,1,1}\Big)q^4 + \cdots\Big] \\
&\qquad + q^{n/24+1}\,\bar\sigma\,\Big[\ydiagram{1} + \ydiagram{1}\,q + \Big(\ydiagram{1} + \ydiagram{1,1,1}\Big)q^2 + \Big(2\,\ydiagram{1} + \ydiagram{1,1,1} + \ydiagram{2,1}\Big)q^3 + \cdots\Big]
\end{aligned} \tag{2.7.18}$$

[63] By definition, $\chi_R(g) := \sum_{\lambda\in\Omega(R)} z^\lambda$, where $\Omega(R)$ is the space of weights of the representation $R$, with multiplicities.

$$
\begin{aligned}
&\equiv q^{n/24}\Big[\sigma + \big(\sigma + \overline{\dot{\ydiagram{1}}}\big)q + \big(2\sigma + \dot{\ydiagram{1,1}} + 2\,\overline{\dot{\ydiagram{1}}}\big)q^2 \\
&\qquad\qquad + \big(4\sigma + \overline{\dot{\ydiagram{1,1,1}}} + 2\,\dot{\ydiagram{1,1}} + 4\,\overline{\dot{\ydiagram{1}}} + \dot{\ydiagram{2}}\big)q^3 + \cdots\Big]\,,
\end{aligned}
$$

where $\sigma$ denotes the spinor of $\mathrm{Spin}(n)$ when $n$ is odd, and any of the two spinors when $n$ is even (in which case $\bar{\sigma}$ denotes the conjugate spinor, obtained by permuting the last two Dynkin labels). Here, a Young diagram stands for the finite character $\chi_R(g)$ associated to the representation $R$ of $\mathfrak{so}(n)$, and a dot $\dot{R}$ is a short-hand notation for the representation whose highest weight is given by $\lambda_{\dot{R}} := \lambda_R + \omega_{\lfloor n/2 \rfloor}$ (so for example $\dot{\ydiagram{1}} = (1, 0, \ldots, 0, 1)$).

These expressions make it manifest how the different states of $\mathrm{Spin}(n)_1$ appear at each level. For example, in $\chi_0$ at grade $n = 2$ the states live inside

$$
J^a_{-2}|0\rangle + J^a_{-1}J^b_{-1}|0\rangle \subseteq \ydiagram{1,1} + (\ydiagram{1,1} \otimes \ydiagram{1,1})_{\mathrm{sym}} = \left(\bullet + \ydiagram{1,1} + \ydiagram{2} + \ydiagram{1,1,1,1}\right) + \ydiagram{2,2}, \tag{2.7.19}
$$

and one can check that the representations in parentheses are physical and the isolated one is null (its norm vanishes).

**$\mathbf{SO}(\boldsymbol{n})_{\mathbf{1}}$ CFT.** A related – and also very important – WZW model is $\mathrm{SO}(n)_1$. This theory is obtained from $\mathrm{Spin}(n)_1$ through fermionization. There are two key properties of $\mathrm{SO}(n)_1$ that make it special. First, it is a *fermionic* theory, meaning that its characters and partition functions depend on the choice of spin structure. Second, it is a *holomorphic* theory, meaning that its partition function factorizes as $Z_{\pm,\pm} = |d_{\pm,\pm}|^2$ (as opposed to non-holomorphic theories whose partition function is a sum of such terms). In other words, $\mathrm{SO}(n)_1$ has a *unique* primary (for fixed spin structure). Following the convention of [150] we denote this unique character as $d_{\pm,\pm}$.

The characters of $\mathrm{SO}(n)_1$ can be obtained from those of $\mathrm{Spin}(n)_1$ (cf. equation (2.7.18)) via the standard bosonization/fermionization dictionary:

$$
\begin{aligned}
&& d_{\text{NS-X}}(q,g) &= \chi_0(q,g) \pm \chi_v(q,g) \\
&n \text{ odd:} & d_{\text{R-NS}}(q,g) &= \sqrt{2}\chi_\sigma(q,g) \\
&& d_{\text{R-R}}(q,g) &= 0 \\
&n \text{ even:} & d_{\text{R-X}}(q,g) &= \chi_s(q,g) \pm \chi_c(q,g)\,.
\end{aligned} \tag{2.7.20}
$$

**$\mathbf{U(1)}_{\boldsymbol{k}}$ CFT.** So far we have described WZW models for simple groups. The case of $\mathrm{U}(1)$ requires a separate discussion. By $\mathrm{U}(1)_k$, with $k$ even, we mean a free compact boson at radius $R^2 = k$. This has a chiral $\mathrm{U}(1)$ flavor symmetry; if we turn on a fugacity $z \in \mathrm{U}(1)$ for this symmetry, the characters of this CFT are

$$
\chi_\ell(q,z) = \eta(q)^{-1} \sum_{u \in \mathbb{Z}} q^{\frac{1}{2}k(u+\ell/k)^2} z^{\sqrt{k}u + \ell/\sqrt{k}}\,, \tag{2.7.21}
$$

with $\ell = 0, 1, \ldots, k-1$, and where $\eta$ is the Dedekind function.

When $k$ is odd, by $\mathrm{U}(1)_k$ we mean the theory $\mathrm{U}(1)_{4k}$ extended by the vertex operator of weight $k/2$, i.e., by $\ell = 2k$. As this weight is half-integral, the operator is a fermion and the extension results in a fermionic CFT. Its super-characters can be obtained for example by following the rules of [1]:

$$\begin{aligned} \chi^{(k)}_{\text{NS-X};\ell}(q,z) &= \chi^{(4k)}_{2\ell}(q,z) \pm \chi^{(4k)}_{2\ell+2k}(q,z) \\ \chi^{(k)}_{\text{R-X};\ell}(q,z) &= \chi^{(4k)}_{2\ell+1}(q,z) \pm \chi^{(4k)}_{2\ell+1+2k}(q,z)\,. \end{aligned} \tag{2.7.22}$$

These characters have also been discussed in e.g. [180]. The special case $k = 1$ is equivalent to a free fermion theory, $\mathrm{U}(1)_1 = \mathrm{SO}(2)_1$, as can be checked by comparing the corresponding super-characters; this is nothing but the trivial statement that one complex fermion equals two real fermions. More generally, $\mathrm{U}(n)_1$ denotes the CFT of $n$ complex fermions, and we shall use the notation $\mathrm{U}(n)_1 = \mathrm{SO}(2n)_1$ interchangeably. (The former is more natural when the free fermions are associated to a complex representation of the gauge group $G$).

**WZW coset models.** We are now ready to discuss the class of models of interest, namely cosets of the form $\mathrm{SO}(n)_1/G_k$, which appear at the deep infrared of QCD theories with gauge group $G$.

The CFT $\mathrm{SO}(n)_1/G_k$ is obtained by embedding $G_k$ into $\mathrm{SO}(n)_1$. As above, this embedding gives rise to a character decomposition of the form

$$d_{\pm,\pm} = \sum_\lambda b^{\pm,\pm}_\lambda \chi_\lambda\,, \tag{2.7.23}$$

where $d_{\pm,\pm}$ are the characters of $\mathrm{SO}(n)_1$, and $\chi_\lambda$ those of $G_k$. The denominator theory $G_k$ is allowed to be fermionic, in which case it is understood that $\chi_\lambda = \chi^{\pm,\pm}_\lambda$ is a super-character at spin structure $\pm,\pm$. In any case, whether $G_k$ is fermionic or not, the coset is a fermionic theory, because the numerator $\mathrm{SO}(n)_1$ is fermionic. Consequently, the coefficients $b^{\pm,\pm}_\lambda$ depend on the spin structure, as indicated by the superscript. These coefficients are the super-characters of the coset CFT $\mathrm{SO}(n)_1/G_k$, and they determine the dynamics of QCD in the infrared. In particular, at low energies the partition function of QCD becomes

$$Z_{\pm,\pm} = \sum_\lambda |b^{\pm,\pm}_\lambda|^2\,. \tag{2.7.24}$$

In this sense, the whole problem of describing the strongly coupled dynamics of QCD has been reduced to the task of finding the coefficients $b_\lambda$ in (2.7.23). While for generic cosets this is a computationally demanding task, for cosets of the form $\mathrm{SO}(n)_1/G_k$ there is a substantial simplification: the numerator $\mathrm{SO}(n)_1$ is in fact equivalent to $n$ free Majorana fermions.

This free fermion representation can be exploited as follows. Consider the coset $\mathrm{SO}(n)_1/G_k$, where $n = \dim(R)$ and $k = I(R)$ with embedding $G \subseteq \mathrm{SO}(n)$ via the representation $R$.

In the Neveu-Schwartz sector the free fermions have half-integral modding $\psi_{r+1/2}$, while in the Ramond sector they have integral modding $\psi_r$. These modes are independent, so the $\mathrm{SO}(\dim(R))_1$ partition function is just the product of the individual partition functions over all $r \in \mathbb{N}$. The fermions $\psi_{r+1/2}, \psi_r$ all generate $R$-modules except for the Ramond zero modes $\psi_0$, which generate a spinor module. With this, the different partition functions of $\mathrm{SO}(\dim(R))_1$ read

$$\begin{aligned} & d_{\text{NS-X}}(q,g;R) = q^{-\dim(R)/48} \prod_{r=0}^{\infty} \prod_{\lambda \in \Omega(R)} 1 \pm z^\lambda q^{r+1/2} \\ \dim(R) \text{ odd}: \quad & d_{\text{R-NS}}(q,g;R) = \sqrt{2}\, q^{\dim(R)/24} \chi_\sigma(g) \prod_{r=1}^{\infty} \prod_{\lambda \in \Omega(R)} 1 + z^\lambda q^r \\ & d_{\text{R-R}}(q,g;R) = 0 \\ \dim(R) \text{ even}: \quad & d_{\text{R-X}}(q,g;R) = q^{\dim(R)/24} (\chi_s(g) \pm \chi_c(g)) \prod_{r=1}^{\infty} \prod_{\lambda \in \Omega(R)} 1 \pm z^\lambda q^r \,, \end{aligned} \tag{2.7.25}$$

where $g \in G$ is the restriction of any flavor $\mathrm{SO}(\dim(R))$ symmetry to the subgroup $G$, and $z$ its value on any maximal torus.

One can use these expressions to compute the first few terms of the $q$-expansion of $d_{\pm,\pm}$. These terms are then reorganized into $G_{I(R)}$ characters, whose $q$-expansion can be obtained with e.g. the Weyl-Kač formula (2.7.17). The characters of the coset $\mathrm{SO}(\dim(R))_1/G_{I(R)}$ are identified with the coefficients of this reorganized series. Computer software is often instrumental in these computations, for example the LieART Mathematica package [196]. The extensive tables of Lie algebras, representations, and branchings in [197] can also come in handy.

**Topological cosets and conformal embeddings.** A special role is played by cosets $\mathrm{SO}(n)_1/G_k$ where $G_k$ embeds into $\mathrm{SO}(n)_1$ *conformally*, i.e., when the central charge of $\mathrm{SO}(n)_1/G_k$ vanishes. We argued in the main text that this happens if and only if the QCD theory with group $G$ is gapped. When this happens, the infrared theory becomes a trivial CFT. That being said, the coset is not an empty theory, even though it has no local degrees of freedom; in other words, it is a *topological* QFT. The low energy dynamics of gapped theories is entirely contained in the topological degrees of freedom carried by the topological coset $\mathrm{SO}(n)_1/G_k$.

By topological invariance, all observables of such cosets become $q$-independent, so the branching functions $b_\lambda$ are just numbers instead of functions of $q$. Note that topological invariance is just a special case of conformal invariance: TQFTs are invariant under all diffeomorphisms instead of just the conformal ones. This means, in particular, that the formula (2.7.24) is still valid for TQFTs. In this case, as $L_0 \equiv 0$, the partition function actually computes the total number of states in the theory, which has a finite-dimensional Hilbert space:

$$\begin{aligned} Z_{\pm,-} &= \mathrm{tr}_{\mathcal{H}_\pm}(1) \equiv \text{bosons plus fermions in } \mathcal{H}_\pm \\ Z_{\pm,+} &= \mathrm{tr}_{\mathcal{H}_\pm}(-1)^F \equiv \text{bosons minus fermions in } \mathcal{H}_\pm \end{aligned} \tag{2.7.26}$$

or, equivalently,

$$\begin{aligned} \text{number of bosons in } \mathcal{H}_\pm &= \frac{1}{2}(Z_{\pm,-} + Z_{\pm,+}) \equiv \sum_\lambda \frac{1}{2}(|b_\lambda^{\pm,-}|^2 + |b_\lambda^{\pm,+}|^2) \\ \text{number of fermions in } \mathcal{H}_\pm &= \frac{1}{2}(Z_{\pm,-} - Z_{\pm,+}) \equiv \sum_\lambda \frac{1}{2}(|b_\lambda^{\pm,-}|^2 - |b_\lambda^{\pm,+}|^2)\,. \end{aligned} \tag{2.7.27}$$

In what follows we shall work out several examples in some detail in order to illustrate some of the previous considerations.

### 2.7.1 Examples of topological cosets

Here we demonstrate the coset construction for the conformal embedding $\mathrm{SO}(8)_1 \supset \mathrm{SU}(3)_3$. As $c = 8/2 - 3\cdot 8/6 = 0$, the resulting theory is topological, i.e., it has a finite-dimensional Hilbert space. In order to find this Hilbert space we need to decompose the SO(8) characters into SU(3) characters. The former are given by (2.7.25):

$$\begin{aligned} \chi_0(q,g) &= q^{-1/6}\Big[\mathbf{1} + (\mathbf{8} + \mathbf{10} + \overline{\mathbf{10}})\,q + (\mathbf{1} + 4\,\mathbf{8} + \mathbf{10} + \overline{\mathbf{10}} + 3\,\mathbf{27})q^2 \\ &\qquad + (2\,\mathbf{1} + 10\,\mathbf{8} + 6\,\mathbf{10} + 6\,\overline{\mathbf{10}} + 6\,\mathbf{27} + 2\,\mathbf{35} + 2\,\overline{\mathbf{35}} + \mathbf{64})q^3 + \cdots\Big] \\ \chi_v(q,g) &= q^{1/3}\Big[\mathbf{8} + (\mathbf{1} + 2\,\mathbf{8} + \mathbf{10} + \overline{\mathbf{10}} + \mathbf{27})q \\ &\qquad + (2\,\mathbf{1} + 6\,\mathbf{8} + 3\,\mathbf{10} + 3\,\overline{\mathbf{10}} + 4\,\mathbf{27} + \mathbf{35} + \overline{\mathbf{35}})q^2 + \cdots\Big]\,, \end{aligned} \tag{2.7.28}$$

and, by triality,

$$\chi_s(q,g) = \chi_c(q,g) = \chi_v(q,g)\,. \tag{2.7.29}$$

We next reorganize these characters in terms of $\mathrm{SU}(3)_3$ characters. The characters of $\mathrm{SU}(3)_3$ are given by

$$\begin{aligned} \chi_{\mathbf{1}}(q,g) &= q^{-1/6}\Big[\mathbf{1} + \mathbf{8}q + \big(\mathbf{1} + 2\,\mathbf{8} + \mathbf{27}\big)q^2 \\ &\qquad + (2\,\mathbf{1} + 4\,\mathbf{8} + 2\,\mathbf{10} + 2\,\overline{\mathbf{10}} + 2\,\mathbf{27} + \mathbf{64})q^3 + \cdots\Big] \\ \chi_{\mathbf{3}}(q,g) &= q^{1/18}\Big[\mathbf{3} + \big(\mathbf{3} + \mathbf{6} + \mathbf{15}\big)q + \big(3\,\mathbf{3} + 2\,\mathbf{6} + 3\,\mathbf{15} + \mathbf{24} + \mathbf{42}\big)q^2 + \cdots\Big] \end{aligned}$$

$$
\begin{aligned}
\chi_{\mathbf{6}}(q,g) &= q^{7/18}\Big[\mathbf{6} + \big(\mathbf{3}+\mathbf{6}+\mathbf{15}+\mathbf{24}\big)q \\
&\qquad\qquad + \big(2\,\mathbf{3}+4\,\mathbf{6}+3\,\mathbf{15}+\mathbf{15}'+3\,\mathbf{24}+\mathbf{42}\big)q^2+\cdots\Big] \\
\chi_{\mathbf{8}}(q,g) &= q^{1/3}\Big[\mathbf{8} + \big(\mathbf{1}+2\,\mathbf{8}+\mathbf{10}+\overline{\mathbf{10}}+\mathbf{27}\big)q \\
&\qquad\qquad + \big(2\,\mathbf{1}+6\,\mathbf{8}+3\,\mathbf{10}+3\,\overline{\mathbf{10}}+4\,\mathbf{27}+\mathbf{35}+\overline{\mathbf{35}}\big)q^2+\cdots\Big] \\
\chi_{\mathbf{10}}(q,g) &= q^{5/6}\Big[\mathbf{10} + \big(\mathbf{8}+\mathbf{10}+\mathbf{27}\big)q \\
&\qquad\qquad + \big(3\,\mathbf{8}+3\,\mathbf{10}+\overline{\mathbf{10}}+2\,\mathbf{27}+\mathbf{35}+\overline{\mathbf{35}}\big)q^2+\cdots\Big] \\
\chi_{\mathbf{15}}(q,g) &= q^{13/18}\Big[\mathbf{15} + \big(\mathbf{3}+\mathbf{6}+2\,\mathbf{15}+\mathbf{15}'+\mathbf{24}\big)q \\
&\qquad\qquad + \big(3\,\mathbf{3}+3\,\mathbf{6}+6\,\mathbf{15}+2\,\mathbf{15}'+\mathbf{21}+3\,\mathbf{24}+2\,\mathbf{42}\big)q^2+\cdots\Big]\,.
\end{aligned}
\tag{2.7.30}
$$

By comparing (2.7.28) to (2.7.30) it is easily checked that

$$
\begin{aligned}
d_{\text{NS-NS}} &= \chi_{\mathbf{1}}+\chi_{\mathbf{8}}+\chi_{\mathbf{10}}+\chi_{\overline{\mathbf{10}}} \\
d_{\text{NS-R}} &= \chi_{\mathbf{1}}-\chi_{\mathbf{8}}+\chi_{\mathbf{10}}+\chi_{\overline{\mathbf{10}}} \\
d_{\text{R-NS}} &= 2\chi_{\mathbf{8}} \\
d_{\text{R-R}} &= 0\,,
\end{aligned}
\tag{2.7.31}
$$

which implies that the NS sector of $\mathrm{SO}(8)_1/\mathrm{SU}(3)_3$ has four bosons and no fermions, and the R sector has two and two, i.e., $\mathcal{H}_{\mathrm{NS}}=\mathbb{C}^{4|0}$ and $\mathcal{H}_{\mathrm{R}}=\mathbb{C}^{2|2}$.

It is interesting to note that, out of the four bosons in $\mathcal{H}_{\mathrm{NS}}$, one of them (the one corresponding to $b_{\mathbf{8}}$) is charged under $(-1)^{F_R}$, while the other three are neutral. In the full non-chiral theory, this state is a boson because it comes from $\bar{b}_{\mathbf{8}}b_{\mathbf{8}}$, which is charged under both $(-1)^{F_L}$ and $(-1)^{F_R}$ (and is therefore neutral under $(-1)^F=(-1)^{F_L}(-1)^{F_R}$).

In the previous example we found that $\mathcal{H}_{\mathrm{R}}$ was supersymmetric (it contains the same number of bosons as fermions), which was a consequence of $d_{\text{R-R}}$ vanishing. In order to show that this is not always the case, we next describe an example where $\mathcal{H}_{\mathrm{R}}$ is *not* supersymmetric.

Consider the coset $\mathrm{SO}(16)_1/\mathrm{Spin}(9)_2$. Using (2.7.25), the characters of the numerator are

$$
\begin{aligned}
\chi_0(q,g) &= q^{-1/3}\big(\mathbf{1}+(\mathbf{36}+\mathbf{84})q \\
&\quad + (\mathbf{9}+\mathbf{16}+\mathbf{36}+\mathbf{44}+\mathbf{84}+2\,\mathbf{126}+\mathbf{231}+\mathbf{495}+\mathbf{924})q^2 \\
&\quad + (2\,\mathbf{9}+\mathbf{16}+5\,\mathbf{36}+\mathbf{44}+5\,\mathbf{84}+3\,\mathbf{126}+3\,\mathbf{231}+\mathbf{495} \\
&\quad + 3\,\mathbf{594}+\mathbf{910}+3\,\mathbf{924}+\mathbf{1650}+\mathbf{2457}+2\,\mathbf{2772})q^3+\cdots\big) \\
\chi_v(q,g) &= q^{1/6}\big(\mathbf{16}+(\mathbf{16}+\mathbf{128}+\mathbf{432})q
\end{aligned}
$$

$$
\begin{aligned}
&+ (3\,\mathbf{16} + 3\,\mathbf{128} + 3\,\mathbf{432} + \mathbf{576} + \mathbf{672} + \mathbf{768} + \mathbf{2560})q^2 \\
&+ (7\,\mathbf{16} + 9\,\mathbf{128} + 9\,\mathbf{432} + 4\,\mathbf{576} + 2\,\mathbf{672} + 5\,\mathbf{768} \\
&+ \mathbf{1920} + 4\,\mathbf{2560} + \mathbf{4608} + \mathbf{4928} + 2\,\mathbf{5040})q^3 + \cdots\big)
\end{aligned}
$$

$$
\begin{aligned}
\chi_s(q,g) = q^{2/3}\big(&\mathbf{44} + \mathbf{84} + (\mathbf{9} + \mathbf{36} + \mathbf{44} + \mathbf{84} + \mathbf{126} + \mathbf{231} + \mathbf{594} + \mathbf{924})q \\
&+ (\mathbf{1} + 2\,\mathbf{9} + 3\,\mathbf{36} + 3\,\mathbf{44} + 4\,\mathbf{84} + 3\,\mathbf{126} + 3\,\mathbf{231} + \mathbf{495} \\
&+ 3\,\mathbf{594} + \mathbf{910} + 4\,\mathbf{924} + \mathbf{1650} + \mathbf{1980} + \mathbf{2457} + \mathbf{2772})q^2 \\
&+ (2\,\mathbf{1} + 6\,\mathbf{9} + 8\,\mathbf{36} + 6\,\mathbf{44} + 10\,\mathbf{84} + 10\,\mathbf{126} + 2\,\mathbf{156} + 9\,\mathbf{231} \\
&+ 3\,\mathbf{495} + 11\,\mathbf{594} + 3\,\mathbf{910} + 11\,\mathbf{924} + 5\,\mathbf{1650} + 2\,\mathbf{1980} \\
&+ 3\,\mathbf{2457} + \mathbf{2574} + 5\,\mathbf{2772} + 3\,\mathbf{3900} + 2\,\mathbf{4158} + \mathbf{9009} + \mathbf{15444})q^3 + \cdots\big) \\
\chi_c(q,g) = q^{2/3}\big(&\mathbf{128} + (\mathbf{16} + 2\,\mathbf{128} + \mathbf{432} + \mathbf{576} + \mathbf{768})q \\
&+ (3\,\mathbf{16} + 6\,\mathbf{128} + 4\,\mathbf{432} + 3\,\mathbf{576} + \mathbf{672} + 3\,\mathbf{768} + 2\,\mathbf{2560} + \mathbf{5040})q^2 \\
&+ (8\,\mathbf{16} + 16\,\mathbf{128} + 13\,\mathbf{432} + 9\,\mathbf{576} + 4\,\mathbf{672} + 9\,\mathbf{768} + \mathbf{1920} \\
&+ 8\,\mathbf{2560} + 2\,\mathbf{4608} + \mathbf{4928} + 5\,\mathbf{5040} + \mathbf{9504} + \mathbf{12672})q^3 + \cdots\big)\,.
\end{aligned}
\tag{2.7.32}
$$

In order to express these in terms of the characters of the denominator, we need the $\mathrm{Spin}(9)_2$ characters:

$$
\begin{aligned}
\chi_{\mathbf{1}}(q,g) = q^{-1/3}\big(&\mathbf{1} + \mathbf{36}\,q + (\mathbf{1} + \mathbf{36} + \mathbf{44} + \mathbf{126} + \mathbf{495})q^2 \\
&+ (\mathbf{1} + 4\,\mathbf{36} + \mathbf{44} + \mathbf{84} + \mathbf{126} + \mathbf{495} + 2\,\mathbf{594} + \mathbf{910} + \mathbf{2772})q^3 + \cdots\big) \\
\chi_{\mathbf{16}}(q,g) = q^{1/6}\big(&\mathbf{16} + (\mathbf{16} + \mathbf{128} + \mathbf{432})q \\
&+ (3\,\mathbf{16} + 3\,\mathbf{128} + 3\,\mathbf{432} + \mathbf{576} + \mathbf{672} + \mathbf{768} + \mathbf{2560})q^2 \\
&+ (7\,\mathbf{16} + 9\,\mathbf{128} + 9\,\mathbf{432} + 4\,\mathbf{576} + 2\,\mathbf{672} + 5\,\mathbf{768} \\
&+ \mathbf{1920} + 4\,\mathbf{2560} + \mathbf{4608} + \mathbf{4928} + 2\,\mathbf{5040})q^3 + \cdots\big) \\
\chi_{\mathbf{44}}(q,g) = q^{2/3}\big(&\mathbf{44} + (\mathbf{36} + \mathbf{44} + \mathbf{594})q \\
&+ (\mathbf{1} + 2\,\mathbf{36} + 3\,\mathbf{44} + \mathbf{126} + \mathbf{495} + 2\,\mathbf{594} + \mathbf{910} + \mathbf{924} + \mathbf{1980})q^2 \\
&+ (\mathbf{1} + 6\,\mathbf{36} + 5\,\mathbf{44} + \mathbf{84} + 3\,\mathbf{126} + \mathbf{231} + 2\,\mathbf{495} + 7\,\mathbf{594} + 3\,\mathbf{910} \\
&+ 2\,\mathbf{924} + \mathbf{1980} + 2\,\mathbf{2772} + \mathbf{3900} + \mathbf{4158} + \mathbf{9009})q^3 + \cdots\big) \\
\chi_{\mathbf{84}}(q,g) = q^{2/3}\big(&\mathbf{84} + (\mathbf{9} + \mathbf{84} + \mathbf{126} + \mathbf{231} + \mathbf{924})q \\
&+ (2\,\mathbf{9} + \mathbf{36} + 4\,\mathbf{84} + 2\,\mathbf{126} + 3\,\mathbf{231} + \mathbf{594} + 3\,\mathbf{924} + \mathbf{1650} + \mathbf{2457} + \mathbf{2772})q^2 \\
&+ (\mathbf{1} + 6\,\mathbf{9} + 2\,\mathbf{36} + \mathbf{44} + 9\,\mathbf{84} + 7\,\mathbf{126} + 2\,\mathbf{156} + 8\,\mathbf{231} + \mathbf{495} + 4\,\mathbf{594} + 9\,\mathbf{924} \\
&+ 5\,\mathbf{1650} + \mathbf{1980} + 3\,\mathbf{2457} + \mathbf{2574} + 3\,\mathbf{2772} + 2\,\mathbf{3900} + \mathbf{4158} + \mathbf{15444})q^3 + \cdots\big) \\
\chi_{\mathbf{128}}(q,g) = q^{2/3}\big(&\mathbf{128} + (\mathbf{16} + 2\,\mathbf{128} + \mathbf{432} + \mathbf{576} + \mathbf{768})q \\
&+ (3\,\mathbf{16} + 6\,\mathbf{128} + 4\,\mathbf{432} + 3\,\mathbf{576} + \mathbf{672} + 3\,\mathbf{768} + 2\,\mathbf{2560} + \mathbf{5040})q^2 \\
&+ (8\,\mathbf{16} + 16\,\mathbf{128} + 13\,\mathbf{432} + 9\,\mathbf{576} + 4\,\mathbf{672} + 9\,\mathbf{768} + \mathbf{1920} \\
&+ 8\,\mathbf{2560} + 2\,\mathbf{4608} + \mathbf{4928} + 5\,\mathbf{5040} + \mathbf{9504} + \mathbf{12672})q^3 + \cdots\big)\,,
\end{aligned}
\tag{2.7.33}
$$

in terms of which one can write

$$
\begin{aligned}
d_{\text{NS-NS}}(q,g) &= \chi_{\mathbf{1}}(q,g)+\chi_{\mathbf{16}}(q,g)+\chi_{\mathbf{84}}(q,g)\\
d_{\text{NS-R}}(q,g) &= \chi_{\mathbf{1}}(q,g)-\chi_{\mathbf{16}}(q,g)+\chi_{\mathbf{84}}(q,g)\\
d_{\text{R-NS}}(q,g) &= \chi_{\mathbf{44}}(q,g)+\chi_{\mathbf{84}}(q,g)+\chi_{\mathbf{128}}(q,g)\\
d_{\text{R-R}}(q,g) &= \chi_{\mathbf{44}}(q,g)+\chi_{\mathbf{84}}(q,g)-\chi_{\mathbf{128}}(q,g)\,.
\end{aligned}
\tag{2.7.34}
$$

These affine branching rules imply that the Hilbert spaces of $\mathrm{SO}(16)_1/\mathrm{Spin}(9)_2$ are $\mathcal{H}_{\text{NS}}=\mathcal{H}_{\text{R}}=\mathbb{C}^{3|0}$. As promised, $\mathcal{H}_{\text{R}}$ is not supersymmetric. While the unextended character $d_{\text{R-R}}(q)=0$ vanishes, the extended one $d_{\text{R-R}}(q,g)=\chi_{\mathbf{84}}(q,g)+\chi_{\mathbf{44}}(q,g)-\chi_{\mathbf{128}}(q,g)$ is non-zero, and hence the coset does not have the same number of bosons and fermions.

In section 2.5 we attributed the supersymmetry of the Ramond sector to certain 't Hooft anomalies. These anomalies are present when the number of quarks is odd. In the theory $\mathrm{SO}(16)_1/\mathrm{Spin}(9)_2$ the number of fermions is even, 16, so there is no reason to expect that the Ramond sector is supersymmetric – and indeed it is not.

### 2.7.2 Example of a non-topological coset

Here we study the coset $\mathrm{Spin}(7)_1/\mathrm{SU}(2)_{28}$, which has $c=7/2-3\times 28/30\equiv 7/10$. This is non-zero so the coset is non-topological, i.e., it is a traditional CFT. We begin by writing the characters of the numerator $\mathrm{Spin}(7)_1$:

$$
\begin{aligned}
\chi_0(q,g) &= q^{-7/48}\Big[\mathbf{1}+(\mathbf{3}+\mathbf{7}+\mathbf{11})q+(2\,\mathbf{1}+\mathbf{3}+2\,\mathbf{5}+2\,\mathbf{7}+2\,\mathbf{9}+\mathbf{11}+2\,\mathbf{13})q^2\\
&\qquad +(2\,\mathbf{1}+5\,\mathbf{3}+4\,\mathbf{5}+8\,\mathbf{7}+5\,\mathbf{9}+6\,\mathbf{11}+4\,\mathbf{13}+2\,\mathbf{15}+\mathbf{17}+\mathbf{19})q^3+\cdots\Big]\\
\chi_v(q,g) &= q^{17/48}\Big[\mathbf{7}+(\mathbf{1}+\mathbf{5}+2\,\mathbf{7}+\mathbf{9}+\mathbf{13})q\\
&\qquad +(\mathbf{1}+2\,\mathbf{3}+3\,\mathbf{7}+\mathbf{11}+3\,\mathbf{5}+2\,\mathbf{7}+3\,\mathbf{9}+2\,\mathbf{11}+2\,\mathbf{13}+\mathbf{15}+\mathbf{17})q^2\\
&\qquad +(4\,\mathbf{1}+5\,\mathbf{3}+9\,\mathbf{5}+12\,\mathbf{7}+10\,\mathbf{9}+8\,\mathbf{11}+7\,\mathbf{13}+2\,\mathbf{15}+3\,\mathbf{17}+\mathbf{19})q^3+\cdots\Big]\\
\chi_\sigma(q,g) &= q^{7/24}\Big[\mathbf{1}+\mathbf{7}+(\mathbf{1}+\mathbf{3}+\mathbf{5}+2\,\mathbf{7}+\mathbf{9}+\mathbf{11}+\mathbf{13})q\\
&\qquad +(2\,\mathbf{1}+3\,\mathbf{3}+4\,\mathbf{5}+6\,\mathbf{7}+4\,\mathbf{9}+4\,\mathbf{11}+3\,\mathbf{13}+\mathbf{15}+\mathbf{17})q^2+\\
&\qquad +(2\,\mathbf{1}+8\,\mathbf{3}+11\,\mathbf{5}+16\,\mathbf{7}+13\,\mathbf{9}+11\,\mathbf{11}\\
&\qquad\quad +10\,\mathbf{13}+5\,\mathbf{15}+3\,\mathbf{17}+2\,\mathbf{19})q^3+\cdots\Big]\,.
\end{aligned}
\tag{2.7.35}
$$

Similarly, the characters of the denominator $\mathrm{SU}(2)_{28}$ are

$$\begin{aligned}
\chi_{\mathbf{1}}(q,g) &= q^{-7/60}(\mathbf{1}+\mathbf{3}\,q+(\mathbf{1}+\mathbf{3}+\mathbf{5})q^2+(\mathbf{1}+3\,\mathbf{3}+\mathbf{5}+\mathbf{7})q^3+\cdots) \\
\chi_{\mathbf{7}}(q,g) &= q^{17/60}(\mathbf{7}+(\mathbf{5}+\mathbf{7}+\mathbf{9})q+(\mathbf{3}+2\,\mathbf{5}+3\,\mathbf{7}+2\,\mathbf{9}+\mathbf{11})q^2 \\
&\quad+(\mathbf{1}+2\,\mathbf{3}+5\,\mathbf{5}+6\,\mathbf{7}+5\,\mathbf{9}+2\,\mathbf{11}+\mathbf{13})q^3+\cdots) \\
\chi_{\mathbf{11}}(q,g) &= q^{53/60}(\mathbf{11}+(\mathbf{9}+\mathbf{11}+\mathbf{13})q+(\mathbf{7}+2\,\mathbf{9}+3\,\mathbf{11}+2\,\mathbf{13}+\mathbf{15})q^2 \\
&\quad+(\mathbf{5}+2\,\mathbf{7}+5\,\mathbf{9}+6\,\mathbf{11}+5\,\mathbf{13}+2\,\mathbf{15}+\mathbf{17})q^3+\cdots) \\
\chi_{\mathbf{13}}(q,g) &= q^{77/60}(\mathbf{13}+(\mathbf{11}+\mathbf{13}+\mathbf{15})q+(\mathbf{9}+2\,\mathbf{11}+3\,\mathbf{13}+2\,\mathbf{15}+\mathbf{17})q^2 \\
&\quad+(\mathbf{7}+2\,\mathbf{9}+5\,\mathbf{11}+6\,\mathbf{13}+5\,\mathbf{15}+2\,\mathbf{17}+\mathbf{19})q^3+\cdots) \\
\chi_{\mathbf{17}}(q,g) &= q^{137/60}(\mathbf{17}+(\mathbf{15}+\mathbf{17}+\mathbf{19})q+(\mathbf{13}+2\,\mathbf{15}+3\,\mathbf{17}+2\,\mathbf{19}+\mathbf{21})q^2 \\
&\quad+(\mathbf{11}+2\,\mathbf{13}+5\,\mathbf{15}+6\,\mathbf{17}+5\,\mathbf{19}+2\,\mathbf{21}+\mathbf{23})q^3+\cdots) \\
\chi_{\mathbf{19}}(q,g) &= q^{173/60}(\mathbf{19}+(\mathbf{17}+\mathbf{19}+\mathbf{21})q+(\mathbf{15}+2\,\mathbf{17}+3\,\mathbf{19}+2\,\mathbf{21}+\mathbf{23})q^2 \\
&\quad+(\mathbf{13}+2\,\mathbf{15}+5\,\mathbf{17}+6\,\mathbf{19}+5\,\mathbf{21}+2\,\mathbf{23}+\mathbf{25})q^3+\cdots) \\
\chi_{\mathbf{23}}(q,g) &= q^{257/60}(\mathbf{23}+(\mathbf{21}+\mathbf{23}+\mathbf{25})q+(\mathbf{19}+2\,\mathbf{21}+3\,\mathbf{23}+2\,\mathbf{25}+\mathbf{27})q^2 \\
&\quad+(\mathbf{17}+2\,\mathbf{19}+5\,\mathbf{21}+6\,\mathbf{23}+5\,\mathbf{25}+2\,\mathbf{27}+\mathbf{29})q^3+\cdots) \\
\chi_{\mathbf{29}}(q,g) &= q^{413/60}(\mathbf{29}+(\mathbf{27}+\mathbf{29})q+(\mathbf{25}+2\,\mathbf{27}+2\,\mathbf{29}+\mathbf{31})q^2 \\
&\quad+(\mathbf{23}+2\,\mathbf{25}+4\,\mathbf{27}+4\,\mathbf{29}+2\,\mathbf{31})q^3+\cdots) \\
&\quad\cdots
\end{aligned} \tag{2.7.36}$$

Given these expressions one can check that the $\mathrm{Spin}(7)_1$ characters decompose into $\mathrm{SU}(2)_{28}$ characters as

$$\begin{aligned}
\chi_0(q,g) &= q^{-7/240}(\chi_{\mathbf{1}}(q,g)+\chi_{\mathbf{11}}(q,g)+\chi_{\mathbf{19}}(q,g)+\chi_{\mathbf{29}}(q,g))(1+q^2+q^3+2q^4+2q^5+4q^6+\cdots) \\
&\quad+q^{137/240}(\chi_{\mathbf{7}}(q,g)+\chi_{\mathbf{13}}(q,g)+\chi_{\mathbf{17}}(q,g)+\chi_{\mathbf{23}}(q,g))(1+q+2q^2+2q^3+4q^4+5q^5+\cdots) \\
\chi_v(q,g) &= q^{353/240}(\chi_{\mathbf{1}}(q,g)+\chi_{\mathbf{11}}(q,g)+\chi_{\mathbf{19}}(q,g)+\chi_{\mathbf{29}}(q,g))(1+q+2q^2+2q^3+3q^4+4q^5+\cdots) \\
&\quad+q^{17/240}(\chi_{\mathbf{7}}(q,g)+\chi_{\mathbf{13}}(q,g)+\chi_{\mathbf{17}}(q,g)+\chi_{\mathbf{23}}(q,g))(1+q+q^2+2q^3+3q^4+4q^5+\cdots) \\
\chi_\sigma(q,g) &= q^{49/120}(\chi_{\mathbf{1}}(q,g)+\chi_{\mathbf{11}}(q,g)+\chi_{\mathbf{19}}(q,g)+\chi_{\mathbf{29}}(q,g))(1+q+q^2+2q^3+3q^4+4q^5+\cdots) \\
&\quad+q^{1/120}(\chi_{\mathbf{7}}(q,g)+\chi_{\mathbf{13}}(q,g)+\chi_{\mathbf{17}}(q,g)+\chi_{\mathbf{23}}(q,g))(1+q+2q^2+3q^3+4q^4+6q^5+\cdots)\,.
\end{aligned} \tag{2.7.37}$$

According to the coset prescription we are instructed to regard the $q$-dependent coefficients as the characters of a new theory, which we denote as

$$\begin{aligned}
\chi_{1,1}(q) &= q^{-7/240}(1+q^2+q^3+2q^4+2q^5+4q^6+4q^7+7q^8+\cdots) \\
\chi_{1,3}(q) &= q^{137/240}(1+q+2q^2+2q^3+4q^4+5q^5+7q^6+9q^7+\cdots) \\
\chi_{1,4}(q) &= q^{353/240}(1+q+2q^2+2q^3+3q^4+4q^5+6q^6+7q^7+\cdots) \\
\chi_{1,2}(q) &= q^{17/240}(1+q+q^2+2q^3+3q^4+4q^5+6q^6+8q^7+\cdots) \\
\chi_{2,1}(q) &= q^{49/120}(1+q+q^2+2q^3+3q^4+4q^5+6q^6+8q^7+\cdots) \\
\chi_{2,2}(q) &= q^{1/120}(1+q+2q^2+3q^3+4q^4+6q^5+8q^6+11q^7+\cdots)\,,
\end{aligned} \tag{2.7.38}$$

which we recognize as the Virasoro characters of the minimal model $M(4,5)$ with central charge $c = 7/10$:

$$\chi_{r,s}(q) = k_s(q) - k_{-s}(q), \qquad k_s(q) := \eta(q)^{-1} \sum_{n\in\mathbb{Z}} q^{((2np(p+1)+r(p+1)-sp)^2)/4p(p+1)} . \tag{2.7.39}$$

Fermionizing, we get the super-characters

$$\begin{aligned} \chi^{\text{NS-X}}(q) &= \chi_{1,1} \pm \chi_{1,4} \quad \& \quad \chi_{1,3} \pm \chi_{1,2} \\ \chi^{\text{R-NS}}(q) &= \chi_{2,1} \quad \& \quad \chi_{2,2} \,, \end{aligned} \tag{2.7.40}$$

which, nonsurprisingly, are the characters of the fermionic $c = 7/10$ minimal model. In other words, the infrared theory of $\mathrm{SU}(2) + \mathbf{7}$ is the fermionic $M(4,5)$ minimal model, with coset realization $\mathrm{SO}(7)_1/\mathrm{SU}(2)_{28}$.

Another interesting example is $\mathrm{SO}(8)_1/\mathrm{Spin}(7)_1$. The numerator and denominator algebras are both of the type $\mathfrak{so}(n)_1$, so the characters are straightforward. By working out the decomposition one obtains

$$\begin{aligned} d_{\text{NS-X}} &= \chi_0^{(7)}\chi_0^{(1)} + \chi_v^{(7)}\chi_v^{(1)} \pm \chi_\sigma^{(7)}\chi_\sigma^{(1)} \\ d_{\text{R-X}} &= \chi_0^{(7)}\chi_v^{(1)} + \chi_v^{(7)}\chi_0^{(1)} \pm \chi_\sigma^{(7)}\chi_\sigma^{(1)} \,, \end{aligned} \tag{2.7.41}$$

where $\chi^{(7)}$ are the characters of $\mathrm{Spin}(7)_1$ and $\chi^{(1)}$ are the Ising characters. Therefore, the infrared chiral algebra of $\mathrm{Spin}(8) + \mathbf{8}$ is the (bosonic) Ising CFT.

## 2.8 Abelian theories

Consider a QED theory with $N_c$ photons and $N_F$ non-chiral Dirac fermions:

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{N_c} g_{ij}^{-2} \mathrm{d}a_i \wedge \star \mathrm{d}a_j + \frac{i}{2\pi}\sum_{i=1}^{N_c} \theta_i \mathrm{d}a_i + \sum_{I=1}^{N_F} \bar{\psi}_I \not{D}_I \psi^I \,, \tag{2.8.1}$$

where the gauge fields are normalized to integral periods:

$$\int \frac{\mathrm{d}a_i}{2\pi} \in \mathbb{Z} \,, \tag{2.8.2}$$

and $D$ denotes the covariant derivative

$$D_I = \partial + i\sum_{i=1}^{N_c} Q_{Ii} a_i \,, \tag{2.8.3}$$

with $Q_{Ii} \in \mathbb{Z}$ the charge of the field $\psi_I$ under $\mathrm{U}(1)_i$.

In order to simplify the notation we will often think of indexed objects $g_{ij}, \theta_i, a_i, Q_{Ii}$ as arrays of suitable shape, e.g., $g$ is a square matrix of dimension $N_c \times N_c$, $\theta$ a row vector of dimension $1 \times N_c$, $Q$ a rectangular matrix of dimension $N_F \times N_c$, etc.

In (2.8.1), $g_{ij}, \theta_i$ denote the coupling constants of the model. These are not all independent: for example, one can always perform linear changes of basis in photon space $a \mapsto Aa$, under which $g^{-2} \mapsto A^t g^{-2} A$, $\theta \mapsto \theta A$. Here $A \in GL(N_c, \mathbb{Z})$ is an integral unimodular matrix so as to preserve the quantization condition (2.8.2). Under this redefinition, the matrix of charges $Q$ transforms as $Q \mapsto QA$. We will come back to this momentarily.

Classically, the model has flavor symmetry $\mathrm{U}(1)^{N_F}$ corresponding to the axial rotations of the fermions $\psi_I \mapsto e^{i\alpha_I \gamma^3}\psi_I$. This flavor symmetry may be enhanced to a non-abelian group if some of the rows of $Q$ are equal. These classical symmetries often have a mixed anomaly with the gauge group $\mathrm{U}(1)^{N_c}$. Specifically, consider the flavor subgroup $\mathrm{U}(1)_F \subset \mathrm{U}(1)^{N_F}$ defined by a certain row vector of integers $n = (n_1, n_2, \dots, n_{N_F})$ such that, under $\alpha \in \mathrm{U}(1)_F$, the fermions rotate with angle $\alpha_I = \alpha n_I$. Under $\mathrm{U}(1)_F$ the left-movers have charge $+n_I$, while the right-movers have charge $-n_I$. Under $\mathrm{U}(1)_i$, both chiralities have charge $Q_{Ii}$. Therefore, the mixed flavor-gauge anomaly is

$$\mathrm{U}(1)_F \text{ - } \mathrm{U}(1)_i\colon \qquad \sum_{I=1}^{N_F} 2n_I Q_{Ii} \equiv 2(nQ)_i \,. \tag{2.8.4}$$

This mixed anomaly has two (dual) interpretations: first, it stems from the fact that, under $\mathrm{U}(1)_F$ rotations, the theta terms in (2.8.1) shift, so these terms are rendered unphysical; and second, it corresponds to the fact that the current that generates $\mathrm{U}(1)_F$ is not conserved, and hence $\mathrm{U}(1)_F$ is not an actual symmetry. Let us analyze both these points in turn.

**Theta terms.** The mixed anomaly between $\mathrm{U}(1)^{N_F}$ and $\mathrm{U}(1)^{N_c}$ can be understood as the statement that, under $\mathrm{U}(1)^{N_F}$ rotations, the theta terms in (2.8.1) shift. In particular, under the $\mathrm{U}(1)_F$ subgroup specified by the integer vector $n$, one has

$$\mathrm{U}(1)_F\colon \theta \mapsto \theta + 2\alpha n Q \,. \tag{2.8.5}$$

This means that $\theta$ are unphysical parameters, as they can generically be rotated away. More precisely, the linear combination $\theta v$, where $v$ is a $N_c \times 1$ column vector, shifts as $\Delta(\theta v) = 2\alpha n Q v$, and this is zero for all $\alpha \in \mathrm{U}(1)_F$ if and only if $Qv \equiv 0$. In other words, the physical theta parameters of the system are in correspondence with the vectors that are annihilated by $Q$ on the right:

- In the non-chiral QED system (2.8.1), the space of physical theta parameters is given by the kernel (right-null-space) of $Q$. The linear combination $\theta v$ is physical if and only if $v \in \ker Q$.

**Axial currents.** The mixed anomaly between $\mathrm{U}(1)^{N_F}$ and $\mathrm{U}(1)^{N_c}$ can also be understood as the statement that, in the quantum theory, some of the currents that generate $\mathrm{U}(1)^{N_F}$ are not conserved. In particular, the subgroup $\mathrm{U}(1)_F$ specified by the integer vector $n$ is generated by the current

$$J_n := \sum_{I=1}^{N_F} n_I J_I \,, \tag{2.8.6}$$

where $J_I^\mu = \bar{\psi}_I \gamma^3 \gamma^\mu \psi_I$ is the current that generates axial rotations of the $I$-th fermion. The mixed $\mathrm{U}(1)_F$-$\mathrm{U}(1)_i$ anomaly violates the conservation law for $J_n$ if and only if $2nQ$ is non-zero. In other words, there are as many conserved axial currents as there are row vectors that are annihilated by $Q$ on the left:

• In the non-chiral QED system (2.8.1), the algebra of axial flavor symmetries is given by the cokernel (left-null-space) of $Q$. The linear combination $n \cdot J$ is conserved if and only if $n \in \operatorname{coker} Q$.

**Hermite Normal form.** We noticed above that one can always perform changes of basis in photon space according to $a \mapsto Aa$, where $A$ is a matrix in $GL(N_c, \mathbb{Z})$. This change of basis redefines the matrix of charges according to $Q \mapsto QA$. One can always use this freedom to put $Q$ in (column) Hermite normal form, namely $Q = \tilde{Q}A$ where $\tilde{Q}$ is lower triangular and columns of zeros, if any, are to the far right (see figure 2.3).

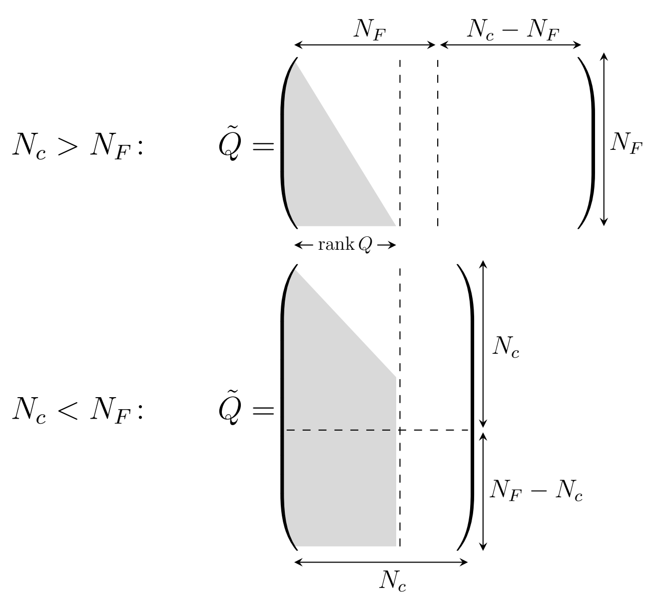


Figure 2.3: Hermite normal form of an $N_F \times N_c$ integral matrix $Q$. The gray region represents the non-zero entries. This decomposition is unique if we impose some further restrictions, such as positivity of pivots; this shall play no role in this work.

In this basis the matrix of charges has $\ker Q$ columns that are identically zero. This means that the corresponding photons are essentially decoupled. (They still couple topologically, via the kinetic term. This does not affect local properties like the existence of a mass gap). This explains why the physical theta terms come from $\ker Q$: a theta term is physical if and only if it multiplies a free photon. In other words, the parameter $\theta_i v_i$ is physical if and only if the photon $a_i v_i$ is decoupled from the fermions.

Putting together the last two observations we learn that, as far as classifying gapped theories is concerned, we can assume without loss of generality that there is the same number of photons than fermions, $N_F \equiv N_c$. This follows from the rank-nullity theorem, $|\ker Q| - |\operatorname{coker} Q| \equiv N_c - N_F$, which implies that if $N_c \neq N_F$, then at least one of $\ker Q, \operatorname{coker} Q$ will be non-empty. If $\ker Q$ is non-empty the system contains decoupled photons, which are gapped and hence do not affect the classification. Conversely, if $\operatorname{coker} Q$ is non-empty, the system contains continuous chiral symmetries and hence it is automatically gapless. The interesting question is, therefore, what happens if $Q$ is square and non-singular.

If $N_F \equiv N_c$ and $Q$ is full-rank, there are no rows nor columns that are zero, so there are no decoupled sectors and no continuous axial symmetries. There are no $\mathrm{U}(1)^{N_F}$ axial symmetries because of the mixed anomaly, and no non-abelian chiral symmetries because there are no repeated rows in $Q$ (for otherwise the matrix would not be full-rank). We now claim that these conditions are not only necessary for being gapped, but also sufficient:

**Proposition 2.8.1** *The non-chiral QED system* (2.8.1) *defined by a square matrix of charges $Q$ is gapped if and only if $Q$ is full rank.*

This claim follows from the analysis of section 2.4, and the fact that free fermions have chiral algebra $\mathrm{U}(N_F)_1$ and the photons a chiral algebra $\mathrm{U}(1)_{Q^tQ}$. The lattice generated by the compact scalars in $\mathrm{U}(1)_{Q^tQ}$ is non-degenerate if and only if $Q$ is full-rank. Equality of the energy-momentum tensors of $\mathrm{U}(N_F)_1$ and $\mathrm{U}(1)_{Q^tQ}$ is nothing but the standard boson-fermion correspondence in $2d$.

One can reach the this conclusion by looking directly at the central charges. The central charge of the free fermions is $N_F$ and, and that of the compact bosons is[64] $\operatorname{sign}(Q^tQ) \equiv \operatorname{rank}(Q)$, and these match if and only if $Q$ is full-rank, as required.

As an interesting remark, note that if $Q^tQ$ is not full rank, then in the gauge chiral algebra $\mathrm{U}(1)_{Q^tQ}$ there is a factor of $\mathrm{U}(1)_0$ for each zero eigenvalue of $Q^tQ$. This factor of $\mathrm{U}(1)_0$ should be thought of as a free photon (a similar phenomenon was observed in [3] in a $3d$ QCD system). This is consistent with the discussion so far, in the sense that if the rank is not maximal there will be columns of zeros in $Q$, signaling decoupled photons.

[64]Here $\operatorname{sign}(K)$ denotes the signature of the matrix $K$, defined as $+1$ for each positive eigenvalue, $-1$ for each negative eigenvalue, and 0 for each zero eigenvalue. As $K = Q^tQ$ is positive semi-definite, $\operatorname{sign}(Q^tQ) \equiv \operatorname{rank}(Q^tQ)$.

**Discrete symmetries.** If $\operatorname{rank} Q = N_c$, we saw earlier that $Q$ defines a QED theory with no continuous chiral symmetries. That being said, the system in general enjoys several discrete symmetries. Let us look at purely left-handed transformations. If we consider a $\mathrm{U}(1)_\ell$ transformation defined by an integer vector $n$, then the theta term shifts as

$$\mathrm{U}(1)_\ell \colon \theta \mapsto \theta + \alpha n Q\,, \tag{2.8.7}$$

as per the flavor-gauge mixed anomaly. This shift means that $\mathrm{U}(1)_\ell$ is not a true symmetry of the quantum system. On the other hand, if we choose $\alpha$ in such a way that $\theta$ stays invariant modulo $2\pi$, then the corresponding transformation does constitute a true symmetry of the quantum theory. This is simplest to ensure in the Hermite basis 2.3. In this basis it becomes clear that there is a $\mathbb{Z}_{\tilde{q}}$ discrete symmetry for each diagonal component of $\tilde{Q}$, obtained by choosing $\alpha = 2\pi k/\tilde{q}$ with $k = 0, 1, \dots, \tilde{q} - 1$. Note that there is another factor of $\mathbb{Z}_{\tilde{q}}$ that acts on the right-handed fermions alone, but the two factors of $\mathbb{Z}_{\tilde{q}}$ are not two distinct symmetries, inasmuch as their simultaneous action is nothing but a gauge transformation. Hence, all in all, the flavor symmetry group of QED is

$$\prod_{\tilde{q} \in \operatorname{diag}(\tilde{Q})} \mathbb{Z}_{\tilde{q}}\,. \tag{2.8.8}$$

Note that if $\operatorname{rank} Q < N_c$, then some of the diagonal components of $\tilde{Q}$ will be zero. If for $\tilde{q} = 0$ we agree to denote $\mathbb{Z}_0 \equiv \mathrm{U}(1)$, then the group above also contains the case where the system has non-trivial continuous symmetries.

For future reference we mention the fact that the order of the symmetry group is $\prod \tilde{q} \equiv \det(Q)$.

**Chiral theories.** We finally make a few remarks concerning chiral theories. These are labelled by pairs of integral matrices $Q_\ell, Q_r$ which specify the charges of the left-movers and right-movers, respectively. Gauge anomaly cancellation requires (2.2.17)

$$Q_\ell^t Q_\ell \equiv Q_r^t Q_r\,. \tag{2.8.9}$$

The reader might find it useful to have at their disposal examples of chiral theories. A trivial class of examples is $Q_\ell = -Q_r$. A more interesting class of examples is provided by choosing any non-symmetric normal matrix $Q$, and taking $Q_\ell = Q$, $Q_r = Q^t$. More generally, it is easy to show that if $Q_\ell, Q_r$ satisfy the gauge anomaly cancellation condition (2.8.9), then there exists some orthogonal matrix $\mathcal{O}$ such that $Q_\ell \equiv \mathcal{O} Q_r$. Therefore, we can generate other families of examples by fixing $Q_r$ and looking for orthogonal matrices $\mathcal{O}$ that make $\mathcal{O} Q_r$ integral.

In any case, many of the previous claims for non-chiral theories can be easily generalized to chiral theories. For example, if $N_c > N_F$, the extra photons are still decoupled. Indeed, if the matrices $Q_\ell, Q_r$ are fat (more columns than rows, see figure 2.3) then they necessarily have a non-trivial kernel. The anomaly cancellation condition says that they in fact *share* the kernel: the equality $Q_\ell^t Q_\ell \equiv Q_r^t Q_r$ implies that $||Q_\ell v||^2 = ||Q_r v||^2$ for any vector $v$, so $v$ is either annihilated by both $Q_\ell, Q_r$ or by neither. If $v$ is in their kernel, then the linear combination $v_i a_i$ is indeed a decoupled photon.

Similarly, if $N_c < N_F$, then there will necessarily be some anomalous continuous symmetry, because the matrices $Q_\ell, Q_r$ will have a non-trivial cokernel, so the associated currents will be conserved — the mixed anomaly with the gauged U(1) will vanish.

All in all, in classifying gapped theories we can assume without loss of generality that $N_c \equiv N_F$, and that $Q_\ell, Q_r$ are full rank. In this situation, the exact same argument from before proves that these are not only necessary conditions for being gapped, but also sufficient:

**Proposition 2.8.2** *A chiral QED system defined by a pair of square matrices of charges $(Q_\ell, Q_r)$, subject to the gauge anomaly cancellation condition (2.8.9), is gapped if and only if $(Q_\ell, Q_r)$ are full rank. (Both matrices necessarily have the same rank, due to (2.8.9)).*

As a consistency check, note that a gapped theory cannot have continuous chiral symmetries, and it is not entirely obvious from the discussion above that a model with full rank matrices $(Q_\ell, Q_r)$ has no such symmetries. It is clear that, being full rank, there are no purely left handed (nor purely right handed) symmetries; but there is no immediate reason that excludes symmetries where both chiralities transform at the same time. It is not hard to show that, as a matter of fact, no such symmetries exist either: any would-be flavor symmetry where both chiralities transform simultaneously is either broken by a mixed flavor-gauge anomaly, or a pure gauge transformation itself. Hence, chiral models with full rank $(Q_\ell, Q_r)$ have no continuous chiral symmetries, as required for a supposedly gapped theory.

The conjectural infrared TQFT has left chiral algebra $\mathrm{U}(N_F)_1/\mathrm{U}(1)_{Q_\ell^t Q_\ell}$, and right chiral algebra $\mathrm{U}(N_F)_1/\mathrm{U}(1)_{Q_r^t Q_r}$. These two are in fact identical, by the anomaly cancellation condition. The sectors of the two chiral halves are combined via the orthogonal matrix $\mathcal{O}$ discussed above, namely $\mathcal{O} := Q_\ell Q_r^{-1}$.

### 2.8.1 U(1) with $N$ charge-$q$ Dirac fermions

Here we analyse the infrared dynamics of U(1) plus $N$ copies of a charge-$q$ Dirac fermion. The claim is that the low energy theory of this system corresponds to a copy of the $\mathrm{SU}(N)_1$ WZW model on each of the $q$ universes. (These universes are the result of the $\mathbb{Z}_q$ one-form symmetry). This nicely reproduces the analysis of [198].

According to the general conjecture (2.5.1), the infrared dynamics of the model are described by the coset

$$\frac{\mathrm{U}(N)_1}{\mathrm{U}(1)_{Nq^2}}\,. \tag{2.8.10}$$

Note that if $N > 1$, the central charge is non-zero, so the coset describes a gapless theory.

In order to project the theory into a specific universe we gauge the one-form symmetry $\mathbb{Z}_q$, to wit

$$\left(\frac{\mathrm{U}(N)_1}{\mathrm{U}(1)_{Nq^2}}\right)/\mathbb{Z}_q \equiv \frac{\mathrm{U}(N)_1}{\mathrm{U}(1)_N} \equiv \mathrm{SU}(N)_1\,. \tag{2.8.11}$$

The second equality involves the character decomposition

$$\begin{aligned} d_{\text{NS-X}}(q,g,\theta) &= \sum_{n=0}^{N-1}(-1)^n\chi_n(q,\theta)\chi_{\gamma^n\cdot\mathbf{0}}(q,g) \\ d_{\text{R-X}}(q,g,\theta) &= \sum_{n=0}^{N-1}(-1)^{n+1}\chi_{n+\lfloor N/2\rfloor}(q,\theta)\chi_{\gamma^n\cdot\mathbf{0}}(q,g)\,, \end{aligned} \tag{2.8.12}$$

where $\theta$ is a flavor U(1) parameter and $g$ an SU($N$) flavor parameter. The characters of $\mathrm{U}(1)_N$ are denoted by $\chi_n(q,\theta)$ and those of $\mathrm{SU}(N)_1$ by $\chi_\lambda(q,g)$. When $N$ is odd, $\chi_n$ denotes a super-character and when even, a regular character.

Note that when $N=1$ the CFT $\mathrm{SU}(1)_1$ becomes trivial, which means that the charge-$q$ Schwinger model has a unique, trivial vacuum in each universe. In this case, the infrared coset $\mathrm{U}(1)_1/\mathrm{U}(1)_{q^2}$ describes a gapped theory, and the character decomposition is

$$\begin{aligned} d_{\text{NS-X}}(q,\theta) &= \sum_{\ell=0}^{q-1}(\pm 1)^\ell\chi^{\text{NS-X}}_{\ell q}(q,\theta) \\ d_{\text{R-X}}(q,\theta) &= \sum_{\ell=0}^{q-1}(\pm 1)^\ell\chi^{\text{R-X}}_{\ell q+\lfloor q/2\rfloor}(q,\theta)\,, \end{aligned} \tag{2.8.13}$$

where $\chi_\ell$ are the characters of $\mathrm{U}(1)_{q^2}$; these are regular (bosonic) characters when $q$ is even (cf. (2.7.21)), and super-characters when odd (cf. (2.7.22)). From this character decomposition we learn that the theory has $q$ vacua, all bosonic, in both sectors $\mathcal{H}_{\mathrm{NS}}=\mathcal{H}_{\mathrm{R}}=\mathbb{C}^{q|0}$. These $q$ vacua live in the $q$ universes, one in each[65]. Out of these, $\lceil q/2\rceil$ are neutral under $(-1)^{F_L}$, and the rest $\lfloor q/2\rfloor$ are charged.

### 2.8.2 Vacua of gapped theories

In the previous section we described the infrared dynamics of a gapless theory. Here we study the gapped case, namely those theories where the matrix of charges $Q$ is square and full rank. Conjecturally, the vacua of such theories are described by the coset (2.5.1)

$$\frac{\mathrm{U}(n)_1}{\mathrm{U}(1)_{Q^tQ}}\,. \tag{2.8.14}$$

[65]More precisely, the states that live in a specific universe are linear combination of these $q$ states. $(-1)^{F_L}$ does not commute with the one-form symmetry, and therefore the states in a given universe do not have well-defined charge under $(-1)^{F_L}$.

We focus on the case where $Q$ describes an even lattice, i.e., where all the diagonal components of $Q^tQ$ are even. In this situation the CFT $\mathrm{U}(1)_{Q^tQ}$ is bosonic, that is, its characters do not depend on the spin structure.

The characters of the numerator $\mathrm{U}(n)_1$ are given by

$$\begin{aligned} d_{\text{NS-X}}(q,\theta) &= q^{-n/24}\prod_{r=0}^{\infty}\prod_{i=1}^{n}(1\pm z_i q^{r+1/2})(1\pm z_i^{-1}q^{r+1/2}) \\ d_{\text{R-X}}(q,\theta) &= q^{n/12}\left[\prod_{i=1}^{n} z_i^{1/2}\pm z_i^{-1/2}\right]\prod_{r=1}^{\infty}\prod_{i=1}^{n}(1\pm z_i q^{r})(1\pm z_i^{-1}q^{r})\,, \end{aligned} \tag{2.8.15}$$

where $\theta\in\mathrm{U}(n)$ is a flavor parameter conjugate to $\theta\sim\mathrm{diag}(z_1,\dots,z_n)$.

On the other hand, the characters of the denominator are

$$\chi_\ell(q,\theta)=\frac{1}{\eta(q)^n}\sum_{u\in\mathbb{Z}^n+K^{-1}\ell} q^{\frac{1}{2}u^tKu}z^{Qu},\qquad K:=Q^tQ\,, \tag{2.8.16}$$

where $\eta$ is the Dedekind eta function.

The vacua of the QED theory labelled by a matrix $Q$ are determined by the branching functions of $\mathrm{U}(n)_1$ into $\mathrm{U}(1)_{Q^tQ}$, i.e., by the decomposition of $d_{\pm,\pm}$ characters into $\chi_\ell$ characters. We propose that the branching functions of the coset $\mathrm{U}(n)_1/\mathrm{U}(1)_{Q^tQ}$ are given by the following:

$$\begin{aligned} d_{\text{NS-X}}(q,\theta) &= \sum_{\ell\in\Gamma(Q)}(\pm1)^{||\ell||^2}\chi_{Q^t\ell}(q,\theta) \\ d_{\text{R-X}}(q,\theta) &= \sum_{\ell\in\Gamma(Q)}(\pm1)^{||\ell||^2}\chi_{Q^t(\ell+1/2)}(q,\theta)\,, \end{aligned} \tag{2.8.17}$$

where $\Gamma(Q)\cong\mathbb{Z}^n/\sim$ is the set of all integral vectors modulo the identification through rows of $Q$: two vectors are declared to be equivalent if they differ by some integral linear combination of the rows of $Q$. As a consistency check, note that $2h_{Q^t\ell}=||\ell||^2$ so the chiral fermion parity in the NS-sector corresponds precisely to the spin of the characters. Similarly, in the R-sector the spin is $2h_{Q^t(\ell+1/2)}=n/4\mod 1$ so the spin is in $\mathbb{Z}+\frac{1}{8}n$, as expected from the zero-point energy of the fermions.

The branchings above predict that the QED theory with matrix of charges $Q$ has $|\Gamma(Q)|\equiv|\det(Q)|$ vacuum states. As in the Schwinger model, these can be thought of as the result of the spontaneous symmetry breaking of the axial symmetry, which also has $|\det(Q)|$ elements. Note also that $|\det(Q)|$ is the order of the one-form symmetry, which suggests that each universe has a single vacuum state.

# Chapter 3

# Three-Dimensional Gauge Theories with Two-Index Matter Fields.

**Authorship.** The content of this chapter is reproduced almost verbatim from the paper [3] written in collaboration with Changha Choi, Jaume Gomis, and Zohar Komargodski.

**Abstract.** We study the nonperturbative dynamics of $3d$ SU($N$) gauge theories coupled to a fermion in a rank-two representation. We argue that when the Chern-Simons level is sufficiently small the theory develops a quantum phase with an emergent topological field theory. When the level vanishes, we further argue that a baryon condenses and hence baryon symmetry is spontaneously broken. The infrared theory then consists of a Nambu-Goldstone boson coupled to a TQFT. Our proposals lead to new fermion-fermion dualities involving fermions in two-index representations. We make contact with some recently discussed aspects of four-dimensional gauge theories and their (non supersymmetric) domain walls. Finally, we discuss some aspects of the time-reversal anomaly in theories with a one-form symmetry.

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## 3.1 Introduction

Strongly coupled Quantum Field Theories (QFTs) can develop interesting infrared "quantum" (nonperturbative) phases, distinct from those that can be inferred by semiclassical considerations. Recently the study of the infrared dynamics of $2+1$ dimensional gauge theories has resulted in the discovery of novel nonperturbative quantum phases [29, 30, 113, 127–129]. In addition, some related aspects have been already studied on the lattice [199, 200]. The subject involves several conceptual differences from the (perhaps) more familiar setting of $3+1$ dimensional gauge theories coupled to matter:

- Since the gauge coupling has positive mass dimension, $2+1$ dimensional gauge theories are always asymptotically free. In particular, these theories are interesting even when the gauge group is U(1).

- $2+1$ dimensional gauge theories are labeled by a gauge group, matter fields, and the Chern-Simons couplings, which are quantized.

- Since in $2+1$ dimensions there is no notion of spinor chirality, one can typically add a mass term for the matter fields preserving all the continuous symmetries of the massless theory. In some theories, however, the massless point is distinguished by the presence of (a discrete) antilinear time-reversal symmetry. The phases of these theories can be studied as a function of the continuous mass deformations for the matter fields.

- There are no continuous 't Hooft anomalies for the symmetries of $2+1$ dimensional theories. However, there are many discrete anomalies and they have to be consistent with the infrared phases of these theories. Even though these anomalies are discrete they nevertheless provide highly nontrivial constraints on the infrared dynamics. In addition, such quantum phases are constrained by the matching of some counterterms. This matching of counterterms physically corresponds to consistency conditions on conductivity coefficients. We will encounter some examples in this paper.

The subject of $2+1$ dimensional gauge theories connects in an obvious way to condensed matter physics (where the gauge symmetry is typically emergent) and in somewhat less obvious ways to particle physics. Many $3+1$ dimensional gauge theories have degenerate trivial vacua. As a result, a domain wall connecting two vacua supports at low energies a $2+1$ dimensional theory, and one can often make this connection very natural (as we will see in this paper). In addition, one can study $3+1$ dimensional gauge theories compactified on a circle (e.g. in the context of finite temperature physics), a problem that similarly reduces to the study of $2+1$ dimensional systems.

In spite of the fact that $2+1$ dimensional gauge theories are asymptotically free, these theories admit regimes in parameter space where they are weakly coupled and the infrared dynamics can be inferred by a careful semiclassical analysis:

- When all the matter fields have a large mass (in units of the Yang-Mills coupling constant) they can be integrated out at energy scales above the strong coupling scale. This only leads to a shift in the infrared Chern-Simons level (this shift is one-loop exact for fermions and trivial for bosons) and hence the deep infrared theory is given by the infrared dynamics of the pure Yang-Mills-Chern-Simons gauge theory, which is quite well understood. This typically leads in the infrared to a Chern-Simons Topological Quantum Field Theory (TQFT).[66] Since the shift of the Chern-Simons level depends on the sign of the mass of the fermion [24, 201, 202], the large mass limit defines an asymptotically large-positive mass phase and an asymptotically large-negative mass phase, described by two distinct TQFTs.

- When the number of matter fields is very large[67] [25, 126] (i.e. many species or large representations) one can demonstrate that there exists a weakly coupled conformal field theory (CFT) which interpolates between the TQFTs describing the two asymptotically large mass phases. Such a theory does not develop interesting new quantum phases.

- Likewise, when the Chern-Simons level is very large [26] one can show that there exists a weakly coupled CFT interpolating between the TQFTs describing the two asymptotically large mass phases. For large $k$ the theory does not develop interesting new quantum phases.

  While no new quantum phases emerge for "large representations" or large Chern-Simons level, it is a wide-open nonperturbative problem to determine for which representations and which levels new quantum phases develop. For some recent work on such questions in the context of quiver gauge theories see [203, 204] and references therein.

It follows from our discussion above that the dynamics of $2+1$ dimensional gauge theories is especially interesting when neither the Chern-Simons level, the dimension of the representation, nor the mass of the matter fields are too large. In this regime there is no semiclassical approximation to the dynamics of the theory. This is when we may expect

[66] An exception to this is when the Chern-Simons level vanishes upon integrating out massive matter fields. In that case, for simply connected groups, the TQFT is trivial. For non-simply connected groups the situation is more complicated, but we will not need it here except in the case of U(1), where the low energy theory is the gapless theory of a compact scalar. This fact will play an important role in this paper.

[67] The relevant group theory factor is the index of the (possibly reducible) representation of the matter fields.

quantum effects to dominate the dynamics and new interesting phenomena may emerge, including new nonperturbative phases.

In this paper we study a class of $2+1$ dimensional gauge theories for which we provide a large body of evidence that they indeed develop new nonperturbative phases along with new phase transitions, for which we propose novel dual descriptions. (We do not know, in general, if these phase transitions are 1st or 2nd order.) The theories we analyze are SU($N$) Yang-Mills gauge theories coupled to a fermion in the rank-two symmetric or antisymmetric representation of SU($N$) and a Chern-Simons term at level $k$. (The fermion is a Dirac fermion, i.e. a complex fermion with two components.)

For generic $k$, these models have a global baryon number symmetry, $\mathrm{U}(1)_B$, acting on the fermion as $\psi \mapsto e^{i\alpha}\psi$. In addition, there is charge-conjugation symmetry. Both of these symmetries are unbroken by the mass term $im\bar{\psi}\psi$. For $k=0$ (which is only allowed for even $N$) the model also admits a time-reversal symmetry. The mass term breaks the time-reversal symmetry. Finally, since there are no dynamical degrees of freedom in the fundamental representation, these theories have a one-form $\mathbb{Z}_2$ symmetry when $N$ is even. (In the context of condensed matter physics, the one-form symmetry is expected to be accidental.)

Let us now summarize the main results:

1. These theories have a critical value of the level $k_{\text{crit}}$ below which a new intermediate quantum phase appears between the semiclassically accessible asymptotic large-positive and large-negative mass phases. The critical value is[68]

$$\text{symmetric: } k_{\text{crit}} = \frac{N+2}{2}\,, \qquad \text{antisymmetric: } k_{\text{crit}} = \frac{N-2}{2}\,. \tag{3.1.1}$$

2. For $0 \neq k < k_{\text{crit}}$ there is an intermediate quantum phase described by the following "emergent" TQFTs[69]

$$\text{symmetric: } \mathrm{U}\left(\frac{N+2}{2}-k\right)_{\frac{N+2}{2}+k,2k}\,, \quad \text{antisymmetric: } \mathrm{U}\left(\frac{N-2}{2}-k\right)_{\frac{N-2}{2}+k,2k}\,. \tag{3.1.2}$$

3. For $k=0$ there is an intermediate quantum phase that includes a Nambu-Goldstone boson (NGB) corresponding to the spontaneous breaking of $\mathrm{U}(1)_B$, along with a TQFT:

$$\text{symmetric: } \frac{\mathrm{SU}\left(\frac{N+2}{2}\right)_{\frac{N+2}{2}} \times S^1}{\mathbb{Z}_{\frac{N+2}{2}}}\,, \quad \text{antisymmetric: } \frac{\mathrm{SU}\left(\frac{N-2}{2}\right)_{\frac{N-2}{2}} \times S^1}{\mathbb{Z}_{\frac{N-2}{2}}}\,. \tag{3.1.3}$$

[68] The quantized level $k$ must be integer for $N$ even and half-integer for $N$ odd. See section 3.2.

[69] We recall that $\mathrm{U}(N)_{P,Q} := \frac{SU(N)_P \times U(1)_{NQ}}{\mathbb{Z}_N}$ with $P \equiv Q \bmod N$. The quotient by $\mathbb{Z}_N$ gauges an anomaly-free one-form symmetry [32, 205].

The notation $S^1$ stands for the linear sigma model of a compact real scalar field $\{\phi \sim \phi + 2\pi\}$, which is dual to pure $\mathrm{U}(1)_0$ gauge theory in $2+1$ dimensions.[70] The scalar couples to the TQFT $\mathrm{SU}\big(\frac{N\pm 2}{2}\big)_{\frac{N\pm 2}{2}}$ by gauging a diagonal, anomaly-free $\mathbb{Z}_{\frac{N\pm 2}{2}}$ one-form symmetry.

The microscopic (ultraviolet) theory for $k = m = 0$ is time-reversal invariant. It may then seem odd that we are proposing that this model flows to a TQFT coupled to a Nambu-Goldstone boson, as TQFTs are typically non-time-reversal invariant. It is encouraging to observe that the $\mathrm{SU}(n)_n/\mathbb{Z}_n$ Chern-Simons theory is in fact also a time-reversal invariant (spin) TQFT [112]. This is a nontrivial consistency check of our proposal.

4. For $k < k_{\text{crit}}$ these theories undergo two phase transitions as a function of the mass of the fermion. The phase transitions connect the intermediate quantum phase with the asymptotic large-positive mass phase and with the asymptotic large-negative mass phase, respectively. We propose that these transitions have a dual description in terms of another $2+1$ dimensional gauge theory. This leads us to propose new (fermion-fermion) dualities in $2+1$ dimensions.

   Dualities for $\mathrm{SU}(N)_k +$ symmetric $\psi$ for $k < \frac{N+2}{2}$:

$$\begin{aligned}
\mathrm{SU}(N)_k + \text{ symmetric } \psi &\longleftrightarrow \mathrm{U}\left(\frac{N+2}{2}+k\right)_{\frac{1}{2}(-1+k-3N/2),k-N/2} + \text{ antisymmetric } \tilde{\psi} \\
\mathrm{SU}(N)_k + \text{ symmetric } \psi &\longleftrightarrow \mathrm{U}\left(\frac{N+2}{2}-k\right)_{\frac{1}{2}(+1+k+3N/2),k+N/2} + \text{ antisymmetric } \hat{\psi}\,.
\end{aligned} \tag{3.1.4}$$

   Dualities for $\mathrm{SU}(N)_k +$ antisymmetric $\psi$ for $k < \frac{N-2}{2}$:

$$\begin{aligned}
\mathrm{SU}(N)_k + \text{ antisymmetric } \psi &\longleftrightarrow \mathrm{U}\left(\frac{N-2}{2}+k\right)_{\frac{1}{2}(+1+k-3N/2),k-N/2} + \text{ symmetric } \tilde{\psi} \\
\mathrm{SU}(N)_k + \text{ antisymmetric } \psi &\longleftrightarrow \mathrm{U}\left(\frac{N-2}{2}-k\right)_{\frac{1}{2}(-1+k+3N/2),k+N/2} + \text{ symmetric } \hat{\psi}\,.
\end{aligned} \tag{3.1.5}$$

   We note that the fermion in the dual gauge theory transforms in the other rank-two representation compared to the fermion in the original gauge theory.

5. For $k \geq k_{\text{crit}}$ the phase diagram has just two phases: the asymptotic large-positive mass and large-negative mass semiclassical phases, separated by a phase transition. For very

[70] For $k = 0$, the $\mathrm{U}(1)_0$ factor in (3.1.2) should not be interpreted as a TQFT, but rather as a gapless $\mathrm{U}(1)_0$ gauge theory, which can be dualized to the compact scalar.

large $k$ the phase transition is controlled by a weakly coupled CFT. The asymptotic large mass phases are the TQFTs

$$\text{symmetric: } \mathrm{SU}(N)_{k\pm\frac{N+2}{2}} \,, \qquad \text{antisymmetric: } \mathrm{SU}(N)_{k\pm\frac{N-2}{2}} \,, \tag{3.1.6}$$

where the upper/lower sign is for the large positive/negative mass asymptotic phase. These phases are present in these theories for any $k$, but for $k < k_{\text{crit}}$ they are separated by the intermediate quantum phases discussed above while for $k \geq k_{\text{crit}}$ they are separated by a single transition (which for sufficiently large $k$ must be given by a CFT).

The plan for the rest of the chapter is as follows. In section 3.2 we present our conjectures for the phases of $\mathrm{SU}(N)$ Yang-Mills gauge theory coupled to a fermion in the symmetric and antisymmetric representation and with a Chern-Simons term. In section 3.3 we present a list of nontrivial consistency checks, involving a comparison to known special cases, and the highly nontrivial matching of some contact terms. In section 3.4 we study some domain wall solutions in four-dimensional gauge theories and show that at least in one case one can explicitly demonstrate baryon symmetry breaking on the domain wall, in agreement with the predictions for the infrared behavior of the corresponding three-dimensional gauge theory. In section 3.5 we make a few forward-looking comments about the time-reversal anomaly and about baryons.

## 3.2 Phase Diagrams

Consider Yang-Mills theory with gauge group $\mathrm{SU}(N)$, a Chern-Simons term and a Dirac fermion $\psi$ in the rank-two symmetric or antisymmetric representation $R$ of $\mathrm{SU}(N)$:

$$\mathcal{L} = \operatorname{tr}\left(-\frac{1}{2g^2}F^2 + \frac{ik_{\text{bare}}}{4\pi}\left(A\mathrm{d}A + \frac{2}{3}A^3\right) + i\bar{\psi}\not{D}\psi + im\bar{\psi}\psi\right) . \tag{3.2.1}$$

In this section we make a proposal for the phase diagram of these theories as a function of the effective Chern-Simons level $k := k_{\text{bare}} - T(R)$ and of the mass $m \in \mathbb{R}$ of the fermion.[71] $T(R)$ is the Dynkin index of the $\mathrm{SU}(N)$ representation under which the Dirac fermion transforms. Since under time-reversal $k \to -k$ (along with reversing the sign of the mass), we restrict our discussion to $k \geq 0$. Since $k_{\text{bare}} \in \mathbb{Z}$, it follows from table 3.1 that $k \in \mathbb{Z}$ for $N$ even and $k \in \mathbb{Z} + \frac{1}{2}$ for $N$ odd.

[71] The shift of the Chern-Simons level in (3.2.1) by $T(R)$ arises from the determinant of the massless fermion. It is convenient to use $k_{\text{bare}}$ when writing Lagrangians since $k_{\text{bare}}$ is always an integer. However, the infrared phases of the theory are more conveniently labeled by $k := k_{\text{bare}} - T(R)$ because time-reversal symmetry acts on $k$ by simply reversing it, along with reversing the mass of the fermion.

| | symmetric | antisymmetric | adjoint |
|---|---|---|---|
| $T(R)$ | $\frac{1}{2}(N+2)$ | $\frac{1}{2}(N-2)$ | $N$ |

Table 3.1: Index for SU($N$) rank-two and adjoint representations. (Since the adjoint representation is real, one could also take the corresponding fermion to be a Majorana fermion – that model was discussed in detail in [30].)

We now discuss the global symmetries of these theories. There is a $\mathrm{U}(1)_B$ flavor symmetry and a $\mathbb{Z}_2^{\mathsf{C}}$ charge-conjugation symmetry $\mathsf{C}$ acting as[72]

$$\mathrm{U}(1)_B: \ \psi \mapsto e^{i\alpha}\psi\,, \qquad\qquad \mathsf{C}: \ \begin{cases} A_\mu \mapsto -A_\mu^T \\ \psi \ \mapsto +\psi^*\,. \end{cases} \tag{3.2.2}$$

These transformations do not commute and generate the group $\mathrm{O}(2)_B = \mathrm{U}(1)_B \rtimes \mathbb{Z}_2^{\mathsf{C}}$. Since the center of the gauge group $\mathbb{Z}_N \subset \mathrm{SU}(N)$ acts as $\psi \mapsto e^{\frac{4\pi i}{N}}\psi$, the global symmetry group is $\mathrm{O}(2)_B/\mathbb{Z}_{N/2}$ for $N$ even and $\mathrm{O}(2)_B/\mathbb{Z}_N$ for $N$ odd. The operators charged under this symmetry are baryons, which will be discussed briefly at the end of this paper.

Since the gauge group is simply-connected, the magnetic symmetry group is trivial. For $N$ even a $\mathbb{Z}_2 \subset \mathbb{Z}_N$ subgroup of the center acts trivially on $\psi$ and the theory has a $\mathbb{Z}_2$ one-form global symmetry. For $N$ odd the one-form symmetry is trivial. Finally, for $k = m = 0$ the theory is time-reversal invariant. Time-reversal symmetry acts on the fermion by

$$\mathsf{T}: \ \psi \mapsto \gamma_0\psi^* \ . \tag{3.2.3}$$

Therefore, in these theories, $\mathsf{T}^2 = (-1)^F$. It is easy to verify that the mass term is odd under time-reversal symmetry but it preserves all other symmetries.

We proceed now to analyzing the phase diagram. We start with the phases that can be established by a semiclassical analysis. When $|m| \gg g^2$ we can reliably integrate out the fermion before the interactions become strong. Integrating out a massive fermion shifts the Chern-Simons level to $k + \mathrm{sign}(m)T(R)$, and the resulting effective theory is pure SU($N$) Yang-Mills with an integer-quantized Chern-Simons term at level $k + \mathrm{sign}(m)T(R)$. This theory, which now has no matter fields, flows at low energies to the topological SU($N$) Chern-Simons theory at level $k + \mathrm{sign}(m)T(R)$, which we denote by $\mathrm{SU}(N)_{k+\mathrm{sign}(m)T(R)}$. Therefore the infrared dynamics is captured by the TQFTs $\mathrm{SU}(N)_{k\pm T(R)}$ for large positive and negative mass respectively. These asymptotic large mass phases are present for all $k$ and all $N$. In the above discussion of the asymptotic phases, $k = T(R)$ is an exception since the infrared theory (after integrating the fermions out) is pure Yang-Mills theory (without a Chern-Simons term) with gauge group SU($N$). In this case the infrared theory is trivial and gapped due to confinement and due to the fact that the gauge group is simply-connected.

[72]For $N = 2$ the action of $\mathsf{C}$ on the gauge field is a gauge transformation.

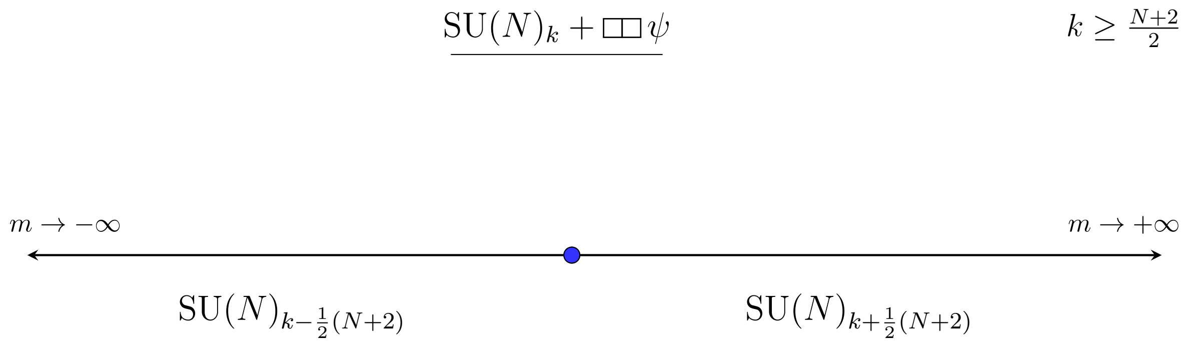



Figure 3.1: Phase diagram of $\mathrm{SU}(N)$ gauge theory with a symmetric fermion for $k \geq \frac{N+2}{2}$. The solid circle represents a phase transition between the asymptotic phases. For sufficiently large $k$ we know for certain that the phase transition is associated with a CFT.

As long as $k$ is sufficiently large, the above two topological phases $\mathrm{SU}(N)_{k+\mathrm{sign}(m)T(R)}$ are separated by a single transition. The question is below which value of $k$ additional phases appear. Our proposal is that as long as $k \geq T(R)$ the above picture holds true, namely, the two semiclassically accessible phases are separated by a phase transition at some value of the mass. The phase diagrams for $k \geq T(R)$ are summarized in figures 3.1 and 3.2. Note that the boundary of the above region, $k = T(R)$, is exactly where one of the phases becomes a trivial infrared theory, with no Chern-Simons TQFT.

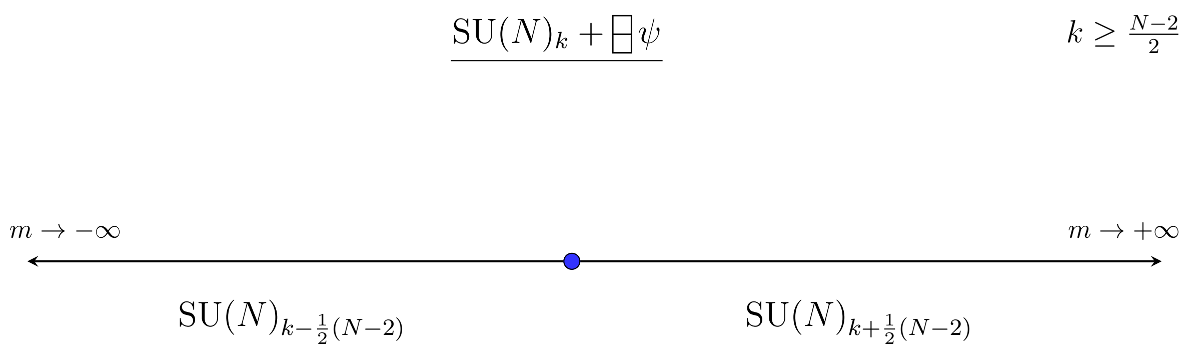



Figure 3.2: Phase diagram of $\mathrm{SU}(N)$ with an antisymmetric fermion for $k \geq \frac{N-2}{2}$. The solid circle represents a phase transition between the asymptotic phases. For sufficiently large $k$ we know for certain that the phase transition is associated with a CFT.

For $0 \leq k < T(R)$ we propose that there is a new intermediate "quantum phase" in between the asymptotic large mass phases.[73] This phase is inherently quantum mechanical, and is not visible semiclassically. This new quantum phase connects to each of the asymptotic phases through a phase transition. The phase diagrams for $0 < k < T(R)$ are summarized in

[73] In the case of symmetric fermion the new phase appears at $k = T(R) - 2 = \frac{N}{2} - 1$. See below.

figures 3.3 and 3.4. The reason that $k = 0$ is excluded from the figures is that it requires a separate discussion, as we shall see below.

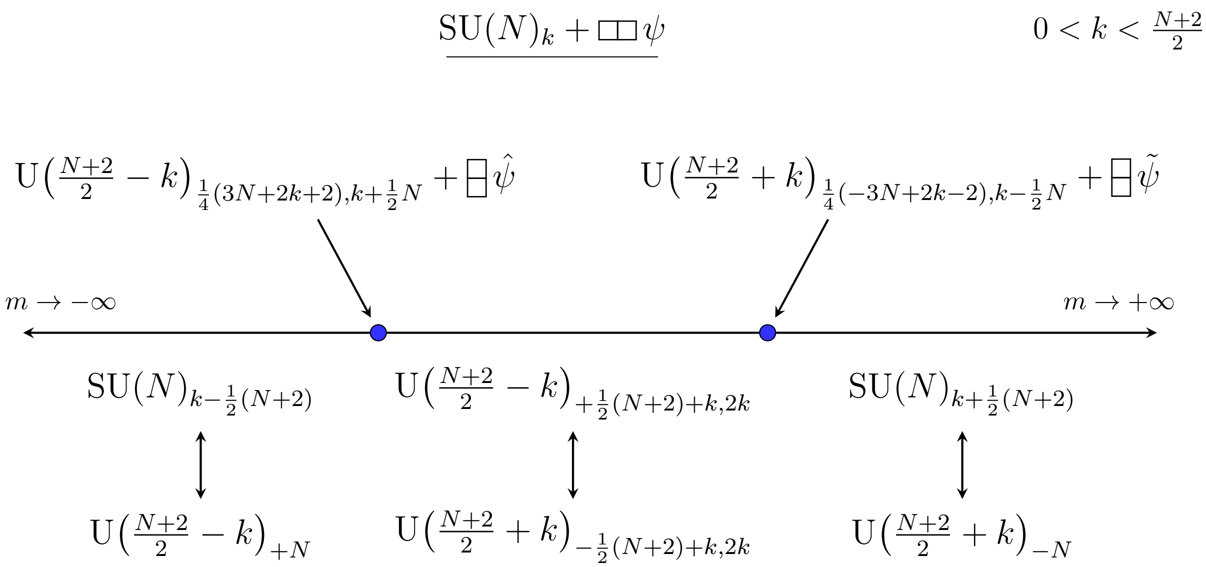



Figure 3.3: Phase diagram of $\mathrm{SU}(N)$ with a symmetric fermion for $0 < k < \frac{N+2}{2}$. The solid circles represent a phase transition between the asymptotic phases and the intermediate quantum phase. Each phase transition has a dual gauge theory description, which appears with an arrow pointing to the phase transition. The mass deformations are related by $m_\psi = -m_{\hat{\psi}}$ and $m_\psi = -m_{\tilde{\psi}}$.

The way we arrive at the phase diagrams in figures 3.3 and 3.4 is as follows. As mentioned above the asymptotic positive and negative mass phases are described by the TQFTs $\mathrm{SU}(N)_{k\pm T(R)}$. These TQFTs admit a level/rank $SU/U$ dual description [28, 206][74]

$$\mathrm{SU}(N)_{k\pm T(R)} \;\longleftrightarrow\; \mathrm{U}(T(R) \pm k)_{\mp N,\mp N}\,. \tag{3.2.4}$$

We start with the level/rank dual description of the asymptotic positive mass phase and search for a dual description that would allow us to understand the quantum phase semiclassically in the dual variables. Similarly, we consider the asymptotic negative mass phase and search for an ultraviolet gauge theory that could describe the phase transition between the semiclassical asymptotic negative mass phase and the quantum phase. These steps involve some guesswork. The fact that this can be done at all is already a highly nontrivial consistency check. Indeed, the two required dual descriptions are mutually non-local[75] but

[74]Level/rank dualities are generically valid only as spin TQFTs, and therefore, whenever the theory on one side of the duality is not spin (i.e. it does not have a transparent spin 1/2 line) we must tensor that theory with a trivial spin TQFT. $\mathrm{SU}(N)_k$ is never spin and $\mathrm{U}(N)_{k,k}$ is spin for $k$ odd.

[75]By "mutually non-local" we mean that there exists no local map between the fields in the two dual

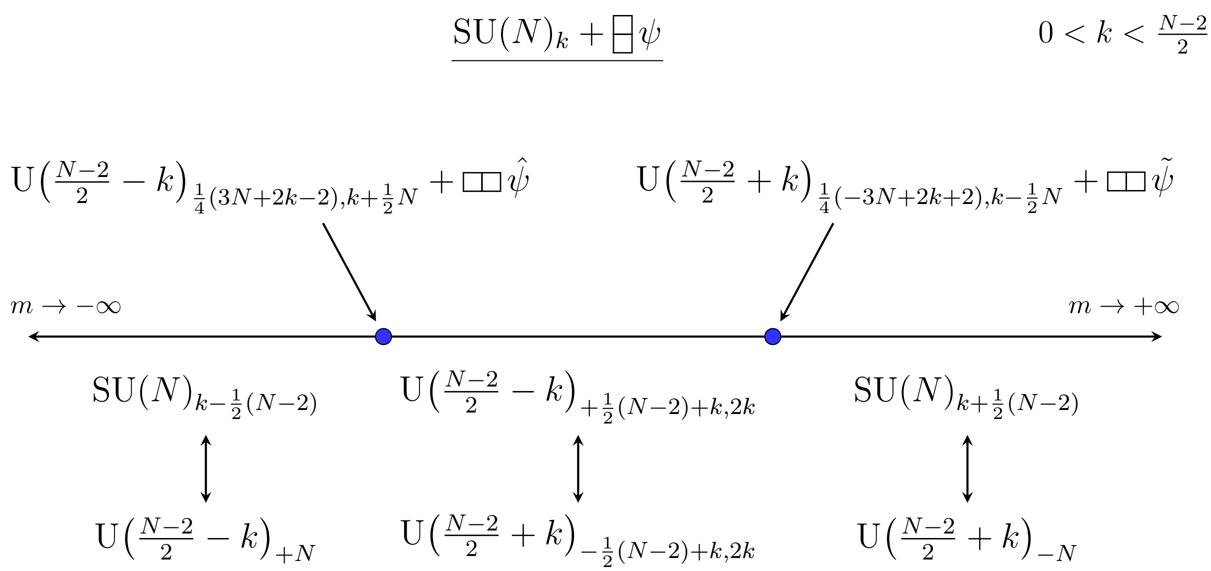



Figure 3.4: Phase diagram of $\mathrm{SU}(N)$ with an antisymmetric fermion for $0 < k < \frac{N-2}{2}$. The solid circles represent a phase transition between the asymptotic phases and the intermediate quantum phase. Each phase transition has a dual gauge theory description, which appears with an arrow pointing to the phase transition. The mass deformations are related by $m_\psi = -m_{\hat{\psi}}$ and $m_\psi = -m_{\tilde{\psi}}$.

there is only one quantum phase, which both of them have to describe simultaneously (in our case this happens thanks to the new level/rank duality in [28]). In the present context, luckily, we were able to find consistent dual descriptions describing the same quantum phase. Furthermore, this guess satisfies very nontrivial additional consistency checks, as we shall see. One of the dual theories is based on the gauge group $\mathrm{U}(T(R) - k)$ and the other on the gauge group $\mathrm{U}(T(R) + k)$ with appropriate Chern-Simons levels and matter representations.

We are thus led to propose the following new fermion-fermion dualities for $0 \leq k < T(R)$:

Dualities for $\mathrm{SU}(N)_k +$ symmetric $\psi$ for $k < \frac{N+2}{2}$:

$$\mathrm{SU}(N)_k + \text{ symmetric } \psi \longleftrightarrow \mathrm{U}\left(\frac{N+2}{2} + k\right)_{\frac{1}{2}(-1+k-3N/2),k-N/2} + \text{ antisymmetric } \tilde{\psi}$$
$$\mathrm{SU}(N)_k + \text{ symmetric } \psi \longleftrightarrow \mathrm{U}\left(\frac{N+2}{2} - k\right)_{\frac{1}{2}(+1+k+3N/2),k+N/2} + \text{ antisymmetric } \hat{\psi}\,. \tag{3.2.5}$$

descriptions, yet, they have some region of overlap in the physics they describe. This is reminiscent of the Seiberg-Witten solution, which has two mutually non-local theories describing two different massless theories, with a region of overlap.

Dualities for $\mathrm{SU}(N)_k +$ antisymmetric $\psi$ for $k < \frac{N-2}{2}$:

$$\mathrm{SU}(N)_k + \text{ antisymmetric } \psi \longleftrightarrow \mathrm{U}\left(\frac{N-2}{2}+k\right)_{\frac{1}{2}(+1+k-3N/2),k-N/2} + \text{ symmetric } \tilde{\psi}$$
$$\mathrm{SU}(N)_k + \text{ antisymmetric } \psi \longleftrightarrow \mathrm{U}\left(\frac{N-2}{2}-k\right)_{\frac{1}{2}(-1+k+3N/2),k+N/2} + \text{ symmetric } \hat{\psi}\,. \tag{3.2.6}$$

For $k = T(R) - 1 = N/2$ in the theory of the fermion in the symmetric representation the intermediate phase coincides with the asymptotic large negative mass phase and the first phase transition is therefore unnecessary. Indeed, the associated duality in the second line of (3.2.5) trivializes since the antisymmetric representation of U(1) is trivial.

The case of $k = 0$ is particularly interesting and requires a separate discussion. The quantum phase that has appeared in the figure 3.3 and 3.4 would seem to make sense also for $k = 0$. However, while for $k > 0$ it is a pure TQFT, for $k = 0$ it is not. Indeed, after integrating the fermion in the dual theory with gauge group $\mathrm{U}(T(R))$, we are left with pure $\mathrm{U}(T(R))_{T(R),0}$ Yang-Mills-Chern-Simons theory, with $T(R) = \frac{N}{2} \pm 1$ in the symmetric/antisymmetric case. The crucial point is that the $\mathrm{U}(1)_0$ factor is not topological. This latter theory can be dualized to the theory of a compact, real scalar field $\phi$

$$\mathcal{L}_{\mathrm{U}(1)_0} = \frac{f_\pi^2}{2}(\partial\phi)^2\,. \tag{3.2.7}$$

with $\phi \sim \phi + 2\pi$ and $f_\pi^2$ the "decay constant".

This theory is combined with the Chern-Simons theory in the following way: the NGB theory (3.2.7) has a non-anomalous U(1) one-form symmetry, corresponding to the conserved two-form current $\epsilon_{\mu\nu\rho}\partial^\rho\phi$. Likewise, $\mathrm{SU}(T(R))_{T(R)}$ Chern-Simons theory has a non-anomalous $\mathbb{Z}_{T(R)}$ one-form symmetry (it is non-anomalous when one views $\mathrm{SU}(T(R))_{T(R)}$ as a spin TQFT). We gauge the diagonal $\mathbb{Z}_{T(R)}$ symmetry, and denote this by

$$\frac{\mathrm{SU}(T(R))_{T(R)} \times S^1}{\mathbb{Z}_{T(R)}}\,, \tag{3.2.8}$$

where $\mathbb{Z}_{T(R)}$ is the diagonal one-form symmetry. The phase diagrams for $k = 0$ are summarized in figures 3.5 and 3.6.

An immediate consistency check of this scenario – which we have already discussed in the introduction – is that the ultraviolet theory is time-reversal symmetric, so it is reassuring to realize that $\mathrm{SU}\big(\frac{N}{2} \pm 1\big)_{\frac{N}{2}\pm 1}/\mathbb{Z}_{\frac{N}{2}\pm 1}$ Chern-Simons theory and the NGB theory are time-reversal invariant. The time-reversal invariance of the quotient $\mathrm{SU}\big(\frac{N}{2} \pm 1\big)_{\frac{N}{2}\pm 1}/\mathbb{Z}_{\frac{N}{2}\pm 1}$ can be shown from level/rank duality as in [112].

The main feature of the $k = 0$ model is, of course, the nonperturbative spontaneous breaking of the $\mathrm{U}(1)_B$ baryon number symmetry. The $\mathrm{U}(1)_B$ symmetry breaking occurs due

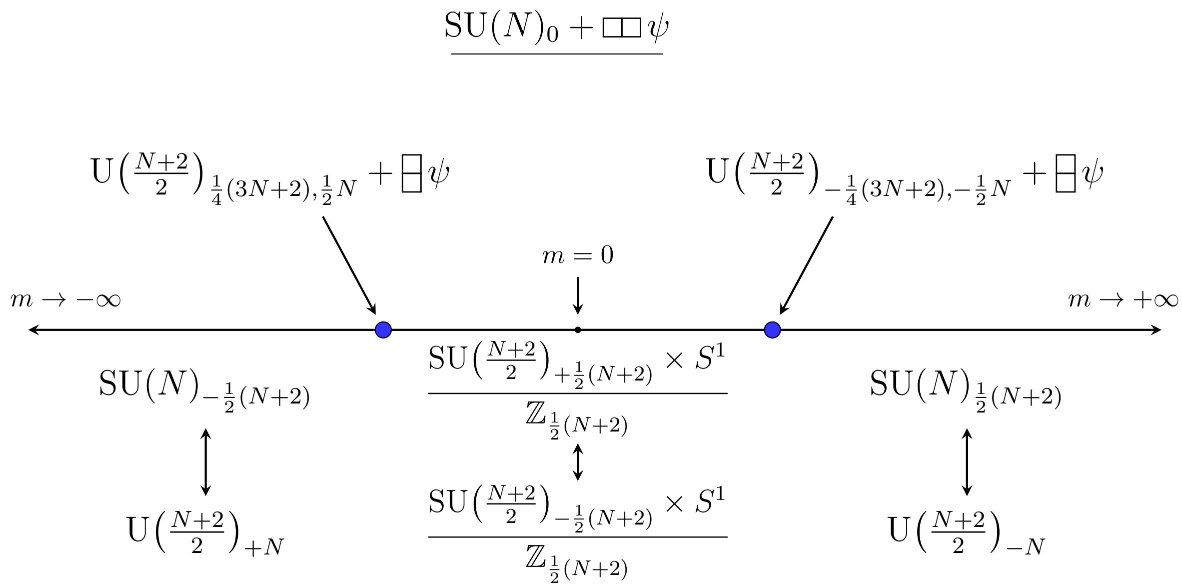



Figure 3.5: Phase diagram of $\mathrm{SU}(N)$ gauge theory with a symmetric fermion for $k = 0$. The circle $S^1$ represents the corresponding sigma model. Each phase transition has a dual gauge theory description, which appears with an arrow pointing to the phase transition. The mass deformations are related by $m_\psi = -m_{\hat{\psi}}$ and $m_\psi = -m_{\tilde{\psi}}$.

to the condensation of a baryon. We will discuss the baryon operators in this theory very briefly in the last section.

This spontaneous symmetry breaking is not in contradiction with the Vafa-Witten type theorems [207]. In essence, if a symmetry cannot be preserved by a time-reversal invariant mass term then there is no obstruction for the spontaneous breaking of that symmetry.

Indeed, in our class of theories, it is not possible to deform by a mass term while preserving both $\mathrm{U}(1)_B$ baryon number symmetry and time-reversal symmetry. This is obvious for $N > 2$ since there is no mass term whatsoever that preserves time-reversal symmetry. However, the case of $N = 2$ requires special attention. In the case of $N = 2$ with an antisymmetric fermion the theory is always in the large $k$ two-phase regime, and there is no quantum phase and no spontaneous breaking occurs, of course, since the fermion is completely decoupled. For $N = 2$ with a symmetric fermion, the situation is more interesting. A Dirac fermion in the rank-two symmetric representation is equivalent to *two* Majorana fermions in the adjoint representation. Let us denote the two Majorana fermions by $\Psi_1, \Psi_2$, such that $\psi = \Psi_1 + i\Psi_2$. The $\mathrm{U}(1)_B = \mathrm{SO}(2)_B$ baryon symmetry simply rotates these two real fermions

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} . \tag{3.2.9}$$

Time-reversal symmetry can be taken to act as $\Psi_1 \mapsto \gamma_0 \Psi_1$, and $\Psi_2 \mapsto \gamma_0 \Psi_2$. Finally, we

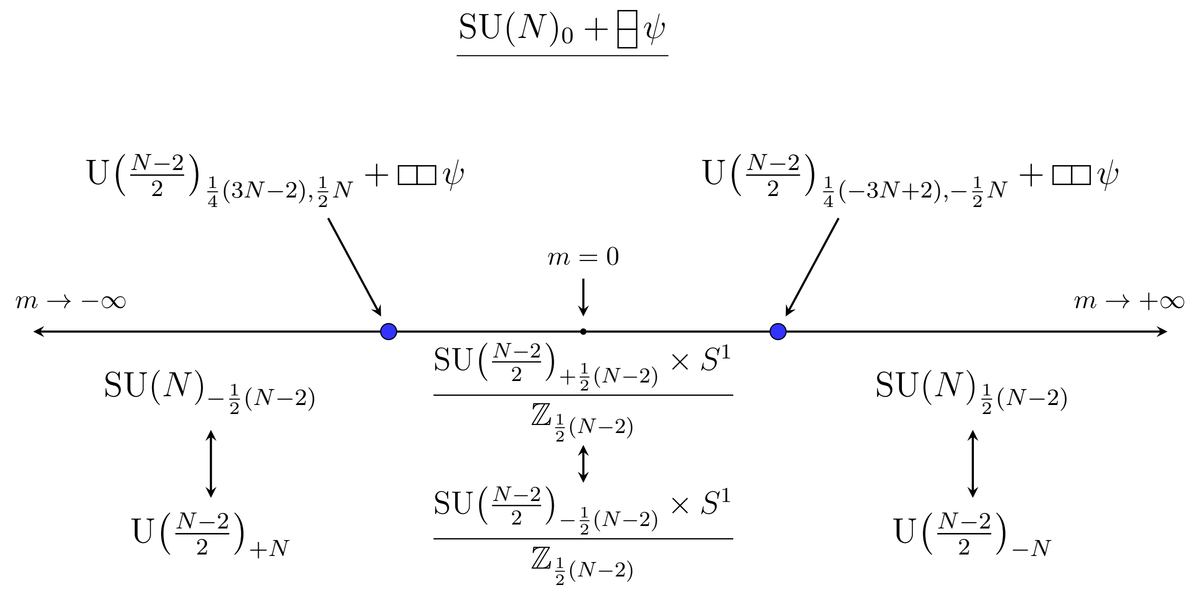



Figure 3.6: Phase diagram of $\mathrm{SU}(N)$ gauge theory with a antisymmetric fermion for $k = 0$. The circle $S^1$ represents the gapless sigma model with $S^1$ target space. Each phase transition has a dual gauge theory description, which appears with an arrow pointing to the phase transition. The mass deformations are related by $m_\psi = -m_{\hat{\psi}}$ and $m_\psi = -m_{\tilde{\psi}}$.

have a charge-conjugation symmetry $\mathsf{C}$ that acts as $\Psi_2 \mapsto -\Psi_2$ while keeping $\Psi_1$ intact. The minimal scalar baryon operator transforms with charge 2 under $\mathrm{U}(1)_B$. The hermitian combinations $i(\bar{\Psi}_1\Psi_1 - \bar{\Psi}_2\Psi_2)$, and $i\bar{\Psi}_1\Psi_2$ are the components of this baryon operator.[76] The hermitian combination $i(\bar{\Psi}_1\Psi_1 + \bar{\Psi}_2\Psi_2)$ is instead invariant under baryon symmetry, but obviously breaks time-reversal symmetry when added to the Lagrangian.

Therefore, clearly, if we want to preserve a time-reversal symmetry, we must use the hermitian baryon operators above. Indeed, for example, adding $i\bar{\Psi}_1\Psi_2$ to the Lagrangian would preserve $\mathsf{CT}$. However, there is no way to add a time-reversal invariant mass term that also preserves the $\mathrm{U}(1)_B$ baryon symmetry. Therefore, there is no obstruction for baryon symmetry to be spontaneously broken. Interestingly, in the case of $N = 2$, the TQFT trivializes (see figure 3.5) because $\mathrm{SU}(2)_2/\mathbb{Z}_2 = \mathrm{SO}(3)_1$, which is a trivial spin TQFT. The fact that in the particular case $N = 2$ the NGB is not accompanied by a TQFT will be crucial later, when we make contact with $3+1$ dimensional physics. Note also that in the case of $N = 2$ it is quite clear that the operator which condenses and leads to the NGB is (without loss of generality) $i(\bar{\Psi}_1\Psi_1 - \bar{\Psi}_2\Psi_2)$.

[76]They are the real and imaginary parts of the baryon constructed with a Dirac spinor in the symmetric representation $\epsilon_{\alpha_1\alpha_2}\epsilon_{\beta_1\beta_2}\psi^{\alpha_1\beta_1}\psi^{\alpha_2\beta_2}$.

## 3.3 Additional Consistency Checks

### 3.3.1 Special Cases

Here we discuss special values of $N$ where we can compare our proposed dynamics with previously conjectured phase diagrams for other families of theories. We also embed SU(2) with a rank-two symmetric fermion in a renormalization group flow of $\mathcal{N}=2$ supersymmetric SU(2) pure gauge theory, and contrast our proposed phase diagram with the expected infrared dynamics of the supersymmetric theory.

• $\mathrm{SU}(4)_k$ *with antisymmetric fermion*

A consistency check on our proposed dynamics follows from the isomorphism $\mathrm{SU}(4) \simeq \mathrm{Spin}(6)$ and the fact that the antisymmetric representation of SU(4) is the six-dimensional, vector, real representation of Spin(6). Since $\psi$ is a Dirac fermion we have the following equivalence of theories

$$\mathrm{SU}(4)_k + \text{ antisymmetric } \psi \equiv \mathrm{Spin}(6)_k + \ (N_f = 2)\ \Psi\,, \tag{3.3.1}$$

where by $N_f = 2$ we mean two Majorana fermions in the vector representation of Spin(6).

For $k \geq 1$ the phase diagram of $\mathrm{SU}(4)_k +$ antisymmetric has two asymptotic phases (see figure 3.2). The phase diagram of $\mathrm{Spin}(6)_k + N_f\ \Psi$ was derived in [113]. For $N_f = 2$ and $k \geq 1$ the phase diagrams agree trivially by virtue of the identity of the TQFTs $\mathrm{SU}(4)_n = \mathrm{Spin}(6)_n$.

We now proceed to the nontrivial matching for $k = 0$ where both theories have an intermediate phase, which we want to compare. Plugging $N = 4$ in figure 3.4 we find that the intermediate phase of $\mathrm{SU}(4)_k +$ antisymmetric $\psi$ is described by

$$\mathrm{U}(1)_0\,, \tag{3.3.2}$$

which, as we explained in detail, is simply a free compact scalar $\phi$ with periodicity $2\pi$. By contrast, the intermediate phase of $\mathrm{Spin}(6)_k + N_f\ \Psi$ is described by the following coset [113]

$$\frac{\mathrm{SO}(N_F)}{S\big(\mathrm{O}(\frac{N_F}{2}) \times \mathrm{O}(\frac{N_F}{2})\big)} + 6\Gamma_{WZ}\,, \tag{3.3.3}$$

where $\Gamma_{WZ}$ is a Wess-Zumino term. The coset in (3.3.3) can be described more explicitly by the equivalence relation of $\mathrm{SO}(N_F)$ matrices $O$

$$O \sim P \cdot O\,, \tag{3.3.4}$$

where $P$ is a block-diagonal matrix with two $\frac{N_F}{2} \times \frac{N_F}{2}$ blocks $A$ and $B$

$$P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{3.3.5}$$

such that $\det(A \cdot B) = 1$.

For $N_f = 2$ the Wess-Zumino term vanishes (because $\pi_3(S^1) = 0$) and we are left with a sigma model on the space of SO(2) matrices $O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ subject to the equivalence relation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \sim \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{3.3.6}$$

with implies that $\theta \sim \theta + \pi$. Therefore, after the quotient the space is still isomorphic to a circle (albeit with a radius smaller by a factor of 2). Therefore this precisely coincides with the result that we obtained for SU(4) with an antisymmetric fermion.

In summary, our phase diagram for $\mathrm{SU}(N)_k$ with a fermion in the antisymmetric representation for $N = 4$ precisely matches the proposed phase diagram of $\mathrm{Spin}(M)$ with $N_f$ Majorana fermions in the vector representation for $M = 6$ and $N_f = 2$. This supports the validity of both phase diagrams.

• $\mathrm{SU}(3)_k$ *with an antisymmetric fermion*

A somewhat more trivial consistency check can be made for $\mathrm{SU}(3)_k$ with an antisymmetric fermion by noting that the rank-two antisymmetric representation of SU(3) is the same as the complex conjugate of the fundamental, three dimensional representation of SU(3). Thus we have the equivalence of theories

$$\mathrm{SU}(3)_k + \text{ antisymmetric } \psi \equiv \mathrm{SU}(3)_k + \ (N_f = 1)\ \Upsilon\,, \tag{3.3.7}$$

where $\Upsilon$ is a Dirac fermion. These theories, which always have two phases (i.e. there is no intermediate phase regime), can be seen to have the same phase diagram by comparing figure 3.4 for $N = 3$ with the phase diagram of $\mathrm{SU}(N)_k$ with $N_f$ fermions in the fundamental representation for $N = 3$ and $N_f = 1$ in [29].

• $\mathrm{SU}(2)_k$ *with an antisymmetric fermion*

A very degenerate special case is $\mathrm{SU}(2)_k$ with a fermion in the antisymmetric representation. In this case the fermion is decoupled from the gauge dynamics and there is no intermediate phase. The infrared is captured by $\mathrm{SU}(2)_k$ Chern-Simons theory except at one point in the phase diagram, which coincides with the phase transition in figure 3.2. The phase transition simply corresponds to a neutral fermion becoming massless.

• $\mathrm{SU}(2)_k$ *with a symmetric fermion*

The rank-two symmetric representation of SU(2) is the adjoint representation. Therefore, figures 3.1 and 3.3 for $N = 2$ describe the infrared dynamics of SU(2) QCD with $N_f = 2$ adjoint Majorana fermions.[77] This extends the phase diagram of $SU/SO/Sp$ adjoint QCD with $N_f = 1$ put forward in [30].

[77]We recall that our theory is based on a Dirac fermion and therefore $N_f = 2$ Majorana adjoint fermions of SU(2).

The particular case $N = 2$ with a symmetric fermion admits an embedding in the $\mathcal{N} = 2$ supersymmetric theory of one SU(2) vector multiplet. In addition to the fields we have in our theory, this model also has a real scalar field $\varphi$ in the adjoint representation, with couples to the fermions via Yukawa terms. What we called $\mathrm{U}(1)_B$ is naturally referred to as the $R$-symmetry in the supersymmetric context. We can flow from the supersymmetric theory to our theory by simply adding a (supersymmetry-breaking) mass term for the real scalar field $\varphi$.[78] Below we analyze what happens if that mass term is very small compared to the scale set by the gauge coupling.

The infrared of the supersymmetric model consists of [208] a complex scalar field $Y$, whose imaginary part transforms inhomogeneously under the U(1) $R$-symmetry, thus signaling spontaneous symmetry breaking of this symmetry. The kinetic term for $Y$ is approximately canonical for large $\mathrm{re}(Y)$ and the potential is the runaway potential $V \sim e^{-\frac{1}{g}\mathrm{re}(Y)}$. Adding a small (supersymmetry-breaking) mass term $m^2\varphi^2$ in the ultraviolet translates to adding a small (supersymmetry-breaking) mass term $m^2\,\mathrm{re}(Y)^2$ in the infrared. (The map between the deformations in the UV and IR is rather simple for large $\mathrm{re}(Y)$ because the theory is weakly coupled there.) For small enough mass of $\varphi$ the minimum of

$$V = e^{-\frac{1}{g}\mathrm{re}(Y)} + m^2(\mathrm{re}(Y))^2 \tag{3.3.8}$$

is therefore at large $\mathrm{re}(Y)$ and we can analyze the physics semiclassically. The fermions are all lifted due to the Yukawa couplings and $\mathrm{re}(Y)$ is likewise massive at the minimum of the potential. The deep infrared theory therefore consists of just the (compact) Nambu-Goldstone boson $\mathrm{im}(Y)$ without an additional TQFT, exactly as in the scenario we proposed above for SU(2) with a Dirac fermion in the rank-two symmetric representation.

We shall return to this theory later when we discuss domain walls in $3+1$ dimensional SU(2) Yang-Mills with $N_f = 2$ adjoint Majorana fermions.

### 3.3.2 Gravitational Counterterm Matching

Another nontrivial check of our proposed phase diagrams in figures 3.3 and 3.4 can be devised by coupling the theories to background gravity. A well-defined (scheme independent) observable is the difference in the gravitational counterterm $2\Delta c\,\mathrm{CS}_{\mathrm{grav}}$ between the asymptotic negative and asymptotic positive mass phases. This difference is closely related to the difference in the thermal conductance in the two phases. This is an interesting quantity to study because it can be easily computed in the original "electric variables" where it is one loop exact. But it can also be computed in the dual variables, followed by traversing the quantum phase, and using the dual variables again. Therefore, one can devise a concrete nontrivial consistency check. Such computations were done in the context of supersymmetric

[78] A closely related $\mathcal{N} = 1$ preserving mass deformation was analyzed in [127].

dualities in [209, 210], where the connection to the physically observable thermal conductance (or the analogous charge conductance which we will study soon) is explained.

The jump in the gravitational counterterm in the electric variables is given by the number of fermions in the ultraviolet gauge theory (this is related to the parity anomaly [24, 201, 202]). Therefore, in our theories

$$\Delta c = -\dim(R) \ , \tag{3.3.9}$$

where $R$ is either the rank-two symmetric or antisymmetric representation, respectively.

Our phase diagrams for $k < T(R)$ in figures 3.3 and 3.4 provide us with another way to compute this difference. The two computations must agree for consistency. We start with the TQFT in the asymptotic negative mass phase $\mathrm{SU}(N)_{k-T(R)}$ and work our way towards the asymptotic positive mass phase $\mathrm{SU}(N)_{k+T(R)}$. This requires tracking the jump of the gravitational counterterm across level/rank dualities, where a gravitational counterterm is generated, and across the phase transitions:

- $SU/U$ TQFT level/rank duality in the asymptotic negative phase [28]:

$$\Delta c_1 = N(k - T(R)) \tag{3.3.10}$$

- Jump induced from positive to negative mass of the leftmost dual gauge theory:

$$\Delta c_2 = \dim(\hat{R}) \tag{3.3.11}$$

- $U/U$ TQFT level/rank duality in the intermediate phase [28]:

$$\Delta c_3 = (T(R) - k)(T(R) + k) - 1 \tag{3.3.12}$$

- Jump induced from positive to negative mass of the rightmost dual gauge theory:

$$\Delta c_4 = \dim(\tilde{R}) \tag{3.3.13}$$

- $U/SU$ TQFT level/rank duality in the asymptotic positive phase [28]:

$$\Delta c_5 = -(k + T(R))N \tag{3.3.14}$$

Here $\hat{R}$ and $\tilde{R}$ denote the representation of the fermion in the leftmost and rightmost dual descriptions in figures 3.3 and 3.4.

Adding up the contributions to the jump following this path we find that it precisely matches that in (3.3.9)

$$\sum_{I=1}^{5} \Delta c_I \equiv \Delta c \,, \tag{3.3.15}$$

both for the theory with a symmetric and antisymmetric fermion!

### 3.3.3 Baryon Counterterm Matching

An entirely analogous exercise to (3.3.15) is to match the difference in the baryon number conductivity coefficient. This coefficient is simply the difference between the two asymptotic phases in the Chern-Simons term for the baryon background gauge field $B$, i.e., we are after the difference

$$\frac{\Delta\kappa}{4\pi}\int B\wedge \mathrm{d}B\,. \tag{3.3.16}$$

Here we think about $B$ as an ordinary U(1) connection and the space-time is assumed to be a spin manifold.[79]

We need to carefully normalize the baryon charge of the fermion. The most convenient choice is to imagine that the fermion is in the (anti-)symmetric representation of $\mathrm{U}(N)$ rather than $\mathrm{SU}(N)$ and the diagonal of $\mathrm{U}(N)$ is the baryon number. This would lead to the fermion carrying charge 2. However, the corresponding baryon gauge field would then have possible fractional fluxes and in order to fix that we need to take the charge to be $2/N$.

We can therefore compute $\Delta\kappa$ straightforwardly in the electric variables as

$$\Delta\kappa=\left(\frac{2}{N}\right)^2\dim(R)=\frac{4}{N^2}\frac{1}{2}N(N\pm 1)=\frac{2}{N}(N\pm 1)\,. \tag{3.3.17}$$

As before, we can also compute $\Delta\kappa$ using the dual description:

- First, in the phase with large negative mass we need to perform level/rank duality between $\mathrm{SU}(N)_{k-T(R)}$ and $\mathrm{U}(T(R)-k)_N$, which leads to a jump[80] [28]

$$\Delta\kappa_1=-\frac{k-T(R)}{N}=-\frac{k-\frac{1}{2}(N\pm 2)}{N}\,. \tag{3.3.20}$$

- Next, there is a crucial difference with our computation of the thermal conductivity. Since the baryon symmetry maps to the magnetic symmetry in the dual variables, and

[79] As all our theories are fermionic and the baryon number clearly satisfies a spin-charge relation, we can in principle also study our theories on spin$_c$ manifolds. This leads to some additional nontrivial consistency checks which we do not present here.

[80] Let us explain briefly how to derive this shift from [28]. Using the notation of [28], the Lagrangian for $\mathrm{SU}(N)_K$ is

$$\mathcal{L}_{SU(N)_K}=\frac{K}{4\pi}\operatorname{tr}\left[b\mathrm{d}b-\frac{2}{3}ib^3\right]+\frac{\epsilon_K}{4\pi}(\operatorname{tr}b)\mathrm{d}(\operatorname{tr}b)+\frac{1}{2\pi}c\mathrm{d}(\operatorname{tr}b+B) \tag{3.3.18}$$

where $b$ us a $\mathfrak{u}(N)$ gauge field. If we integrate out $c$ and remove the trace $b:=\tilde{b}-\frac{1}{N}B$ (with $\operatorname{tr}\tilde{b}\equiv 0$), we get

$$\mathcal{L}_{SU(N)_K}\to\frac{K}{4\pi}\operatorname{tr}\left[\tilde{b}\mathrm{d}\tilde{b}-\frac{2}{3}i\tilde{b}^3\right]+\frac{K}{4\pi N}B\mathrm{d}B+\frac{\epsilon_K}{4\pi}B\mathrm{d}B\,. \tag{3.3.19}$$

The level/rank dual $\mathrm{U}(K)_{-N}$ also has a term $\frac{\epsilon_K}{4\pi}B\mathrm{d}B$, so the relative shift by the contact term is only given by the term $\frac{K/N}{4\pi}B\mathrm{d}B$.

since the dual fermions are not charged under the magnetic symmetry, we get that $\Delta\kappa_2 \equiv \Delta\kappa_4 \equiv 0$. In other words, the phase transitions do not lead to a jump in the conductivity.

- Next we need to address $\Delta\kappa_3$, which arises from the $U/U$ level/rank duality in the quantum phase. We find again from [28] that

$$\Delta\kappa_3 = 1\,. \tag{3.3.21}$$

- Finally, the positive-mass $SU/U$ level/rank duality leads to

$$\Delta\kappa_5 = \frac{k + \frac{1}{2}(N \pm 2)}{N}\,. \tag{3.3.22}$$

We see that if we add all the partial jumps $\Delta\kappa_I$ we get precisely the same shift in the baryon counterterm (3.3.17) computed in the electric theory:

$$\sum_{I=1}^{5} \Delta\kappa_I \equiv \Delta\kappa \tag{3.3.23}$$

for both the symmetric and antisymmetric representation!

The matching of the gravitational contact term guarantees that the phase diagram remains consistent in curved space and the thermal conductivities are single-valued, as they should be in physical theories. The matching of the baryon contact term further guarantees that we can consistently gauge the ultraviolet baryon symmetry in all the phases. Therefore, one can derive from our phase diagrams and dualities also the phase diagrams and corresponding dualities for $\mathrm{U}(N)$ gauge theories coupled to two-index matter fields.

### 3.3.4 Self-consistency check for the dualities

The fact that the conductivity coefficient $\kappa$ is continuous across the phase diagram means that $\mathrm{U}(1)_B$ can be consistently gauged. Moreover, given our choice of normalization for the baryon charge of the fermion, one can also gauge the diagonal $\mathbb{Z}_N$ one-form symmetry, and this process conserves the phase diagram as well [29]. The end result is the phase diagram of $\mathrm{U}(N)_{k,k'} + \psi$, where $\psi$ is a fermion in the symmetric or antisymmetric representation; the details of these phase diagrams will be discussed elsewhere.

For our purposes, it suffices to note that $\mathrm{SU}(N)_k + \psi$ and $\mathrm{U}(N)_{k,k'} + \psi$ have a qualitatively identical phase diagram and, in particular, they have the same number of phases. More specifically, either they are both in the two-phase regime, or they both develop an intermediate nonperturbative phase, depending on whether $k$ is larger or smaller than $T(R)$. This implies yet another consistency check of our phase diagram, because the dualities (3.2.5) and (3.2.6)

$$\begin{aligned}
\mathrm{SU}(N)_k + \text{ symmetric } \psi &\longleftrightarrow \mathrm{U}\left(\frac{N+2}{2}+k\right)_{\frac{1}{2}(-1+k-3N/2),k-N/2} + \text{ antisymmetric } \tilde{\psi} \\
\mathrm{SU}(N)_k + \text{ symmetric } \psi &\longleftrightarrow \mathrm{U}\left(\frac{N+2}{2}-k\right)_{\frac{1}{2}(+1+k+3N/2),k+N/2} + \text{ antisymmetric } \hat{\psi} \\
\mathrm{SU}(N)_k + \text{ antisymmetric } \psi &\longleftrightarrow \mathrm{U}\left(\frac{N-2}{2}+k\right)_{\frac{1}{2}(+1+k-3N/2),k-N/2} + \text{ symmetric } \tilde{\psi} \\
\mathrm{SU}(N)_k + \text{ antisymmetric } \psi &\longleftrightarrow \mathrm{U}\left(\frac{N-2}{2}-k\right)_{\frac{1}{2}(-1+k+3N/2),k+N/2} + \text{ symmetric } \hat{\psi}
\end{aligned} \tag{3.3.24}$$

are only consistent if one side is in the two-phase regime and the other one is in the three-phase regime, and vice versa[81]. It is straightforward to check that this is indeed the case: assuming that $0 < k < T(R)$, one has

$$\begin{aligned}
-\frac{1}{2}(-1+k-3N/2) &> \frac{1}{2}\left[\frac{N+2}{2}+k-2\right] \\
+\frac{1}{2}(+1+k+3N/2) &> \frac{1}{2}\left[\frac{N+2}{2}-k-2\right]
\end{aligned} \tag{3.3.25}$$

for the antisymmetric case, and

$$\begin{aligned}
-\frac{1}{2}(+1+k-3N/2) &> \frac{1}{2}\left[\frac{N-2}{2}+k+2\right] \\
+\frac{1}{2}(-1+k+3N/2) &> \frac{1}{2}\left[\frac{N-2}{2}-k+2\right]
\end{aligned} \tag{3.3.26}$$

for the symmetric case, as required.

Notice the interplay between the symmetric and antisymmetric representations in the original UV description and in the dual description, which are manifested non-trivially in the above checks. This makes the criteria for quantum phases in (3.1.2) natural.

## 3.4 Domain Walls in Four Dimensional Gauge Theories

Let us consider the four-dimensional theory of a Dirac fermion in the symmetric/antisymmetric representation coupled to $\mathrm{SU}(N)$ gauge fields. We can equivalently think about it as $\mathrm{SU}(N)$ gauge theory coupled to a Weyl fermion in the symmetric/antisymmetric representation and another Weyl fermion in the conjugate representation. Let us begin with the massless theory.

[81] We would like to thank A. Baumgartner for an interesting discussion regarding this point.

This theory has a ($\mathbb{Z}_{2(N-2)}$) $\mathbb{Z}_{2(N+2)}$ discrete chiral symmetry that acts by re-phasing the two Weyl fermions together,

$$\mathbb{Z}_{2(N\pm2)}: \ \psi, \tilde{\psi} \mapsto e^{i\alpha}\psi, e^{i\alpha}\tilde{\psi} , \qquad \alpha = \frac{2\pi k}{2(N\pm 2)} \ , \ k \in \mathbb{Z} \ . \tag{3.4.1}$$

In addition the theory enjoys baryon number symmetry $\mathrm{U}(1)_B$ which acts by re-phasing the two fermions in an opposite fashion

$$\mathrm{U}(1)_B: \ \psi, \tilde{\psi} \mapsto e^{i\beta}\psi, e^{-i\beta}\tilde{\psi} \ . \tag{3.4.2}$$

The special case of $N = 2$ with a symmetric representation Dirac fermion is equivalent to SU(2) gauge theory with two Weyl fermions in the adjoint representation. In this case the $\mathrm{U}(1)_B$ symmetry is in fact enhanced to $\mathrm{SU}(2)_F$ flavor symmetry (and in addition, there is the discrete $\mathbb{Z}_8$ axial symmetry, where the order-two generator in $\mathbb{Z}_8$ is identified with the center of $\mathrm{SU}(2)_F$ flavor symmetry. This order-two generator coincides with the fermion number symmetry, and it is hence unbreakable as long as the vacuum is Poincaré invariant.)

These theories admit a mass deformation, $M\psi\tilde{\psi}$, and we can take $M$ to be non-negative without loss of generality, at the expense of having to keep track of the $\theta$ parameter of the gauge theory. The mass perturbation breaks the $\mathbb{Z}_{2(N\pm2)}$ symmetry down to $\mathbb{Z}_2$. However, the mass term preserves $\mathrm{U}(1)_B$. For $\theta = 0, \pi$ also the time-reversal symmetry is preserved.

In the special case of $N = 2$, the mass perturbation breaks $\mathrm{SU}(2)_F$ symmetry, but it preserves baryon number, which can be identified with the Cartan subgroup of $\mathrm{SU}(2)_F$. The Vafa-Witten-like theorems would imply that the massless theory cannot break $\mathrm{U}(1)_B$. We will assume that $\mathrm{SU}(2)_F$ is broken to $\mathrm{U}(1)_B$.

It is reasonable to assume that the massless theory breaks the chiral symmetry $\mathbb{Z}_{2(N\pm2)}$ as[82]

$$\mathbb{Z}_{2(N\pm2)} \to \mathbb{Z}_2 \ . \tag{3.4.3}$$

According to these assumptions, the vacuum structure of the theory is therefore:

- $N > 2$: The massless theory has $N \pm 2$ vacua, each of which is trivial and gapped. The order parameter distinguishing these vacua is the fermion bilinear $\langle\psi\tilde{\psi}\rangle$, which is charged under $\mathbb{Z}_{2(N\pm2)}/\mathbb{Z}_2$.

- $N = 2$: Here $\mathrm{SU}(2)_F$ breaks spontaneously to $\mathrm{U}(1)_B$ but also the axial symmetry is spontaneously broken. The fermion bilinear is in the adjoint representation and it is

[82]This can be proven in the planar limit [47], where the theory in the meson sector is equivalent to $\mathcal{N} = 1$ Supersymmetric Yang Mills theory, which is known to develop a condensate. Therefore our statement about the symmetry breaking pattern certainly holds for large enough finite $N$.

the order parameter leading to this symmetry breaking pattern.[83] Let us parameterize it without loss of generality by $\langle\psi\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This leads to the breaking pattern $\mathrm{SU}(2)_F \to \mathrm{U}(1)_B$, and the corresponding coset manifold is $S^2$. Acting with the generator of $\mathbb{Z}_8$ we get a new vacuum, and therefore we have at least two copies of the coset $S^2$. However, acting with the square of the generator of $\mathbb{Z}_8$ we get the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, but this is in fact on the same coset as the original condensate (more precisely, the Weyl group of $\mathrm{SU}(2)_F$ relates these two configurations). Hence we have exactly two copies of $S^2$, and the broken axial symmetry allows us to move from one copy to the other copy.[84]

Let us now turn on a small positive mass $M$. This corresponds to a small potential on the above space of vacua which is a function of $M, \theta$. For $N > 2$, for any $\theta \neq 0$ this lifts the degeneracy and picks up one of the $N \pm 2$ vacua. At $\theta = \pi$ there is a first order transition and two (adjacent) vacua are exactly degenerate. While this analysis is done at small $M$, it is natural to assume that this is true for any positive $M$. In particular, at asymptotically large $M$ this agrees with the expectations from pure Yang-Mills theory, which is supposed to have a trivial ground state for any $\theta \neq \pi$ and two degenerate vacua at $\theta = \pi$.

For $N = 2$ we can again turn on some mass $M$ and fix the theta angle. But now the mass $M$ is an adjoint SU(2) matrix. Without loss of generality we take this matrix to be in the Cartan and hence the eigenvalues are real (and we keep track of $\theta$). Therefore we choose $M = M_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with positive $M_0$. The physics depends only on the combination

$$M_0 e^{i\theta/4} \tag{3.4.4}$$

and, furthermore, all the physical observables must be periodic under $\theta \to \theta + 2\pi$.

We can parameterize the vacuum configurations by the adjoint $\mathrm{SU}(2)_F$ matrix of fermion condensates

$$\mathcal{M}_{\mathrm{VAC}} = \left\{ U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U^{-1} \right\} \cup \left\{ V \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} V^{-1} \right\} \tag{3.4.5}$$

where $U, V$ are SU(2) matrices. This is just the union of two $S^2$'s, as we explained above. The potential (up to an unimportant proportionality factor) induced by the deformation by

[83] Indeed, since the order parameter must be a scalar in space-time the Lorentz indices are contracted antisymmetrically, and since it must be gauge invariant, the gauge indices are contracted symmetrically and hence the flavor indices must be contracted symmetrically as well, leading to the symmetric product of the fundamental representation of $\mathrm{SU}(2)_F$ with itself, namely, the adjoint representation.

[84] This scenario of the SU(2) gauge theory with two adjoint fermions flowing in the infrared to two copies of $S^2$ has been recently connected to the Seiberg-Witten solution of the $\mathcal{N} = 2$ vector multiplet theory [211]. Other possibilities for the infrared dynamics were recently discussed also in [212, 213].

$M, \theta$ on the space $\mathcal{M}_{\text{VAC}}$ is

$$E_1 = M_0 e^{i\theta/4}\, \mathrm{tr}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} U \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} U^{-1}\right) + c.c. \tag{3.4.6}$$

on the first $S^2$ and

$$E_2 = iM_0 e^{i\theta/4}\, \mathrm{tr}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} V \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} V^{-1}\right) + c.c. \tag{3.4.7}$$

on the second $S^2$. We need to minimize over $U$ and $V$ and then find the global minimum by comparing these two sectors. First we simplify the expressions for $E_{1,2}$ and write (again, up to unimportant proportionality factors)

$$\begin{aligned} E_1 &= M_0 \cos(\theta/4)\, \mathrm{tr}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} U \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} U^{-1}\right) , \\ E_2 &= M_0 \sin(\theta/4)\, \mathrm{tr}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} V \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} V^{-1}\right) . \end{aligned} \tag{3.4.8}$$

Using the Cauchy-Schwarz inequality we see that the expression in the trace is minimized when the gaugino condensate is $\langle\psi\psi\rangle = -\sigma_z$ and it is maximized when the gaugino condensate is $\langle\psi\psi\rangle = \sigma_z$. We refer to these two points on $S^2$ as the south and north pole of $S^2$. We therefore have the following phases for small enough $M_0$, as a function of $\theta$:

- $-\pi < \theta < \pi$: The true vacuum is at the south pole of the first $S^2$.
- $\pi < \theta < 3\pi$: The true vacuum is at the south pole of the second $S^2$
- $3\pi < \theta < 5\pi$: The true vacuum is at the north pole of the first $S^2$
- $5\pi < \theta < 7\pi$: The true vacuum is at the north pole of the second $S^2$
- $\theta = \pi, 3\pi, 5\pi, 7\pi$: The two vacua on the two sides of the transition are exactly degenerate.

While the periodicity of the above list of vacua is $8\pi$, of course the physical observables are $2\pi$ periodic. In addition, as before, while our analysis is reliable for small $M_0$, it is consistent to assume that the bulk vacua behave as above for any $M_0$.

We now turn to analyzing domain walls in this theory. We can start from the massless case, which is the most difficult (and the richest) case. We consider first $N > 2$. Since we have $N \pm 2$ vacua, we can study the domain wall between any pair of vacua. However, using the spontaneously broken axial symmetry we see that the result only depends on the difference of the phases of the gaugino condensates on the two sides. Let us label the vacua by $J = 1, \ldots, N \pm 2$ according to the phase of the gaugino condensate $\langle\psi\tilde{\psi}\rangle = e^{\frac{2\pi i J}{N\pm 2}}$. A natural conjecture for the domain wall theory is to identify it with the quantum phase in the corresponding $2+1$ dimensional gauge theory of $\mathrm{SU}(N)$ with a symmetric (antisymmetric) fermion. This leads to the proposal that the theory on the domain wall connecting the vacuum $I$ and the vacuum $J$. We can choose the orientation such that without loss of generality, say, $I > J$ and $J$ is on the right hand side of the wall. The other cases are obtained by simply reversing the orientation. We thus identify the rank of the gauge group of the quantum phase in (3.1.2) with the jump $I-J$, that is, $\frac{N\pm 2}{2} - k \leftrightarrow I - J$. Equivalently, the $2+1$ level maps to $k \leftrightarrow \frac{N\pm 2}{2} - (I-J)$, whence the domain wall theory is

$$\mathrm{U}\left(I-J\right)_{N\pm 2-I+J,N\pm 2-2I+2J} \ . \tag{3.4.9}$$

Of course, this TQFT has to be accompanied by the decoupled center of mass degree of freedom (which is described to leading order by the Nambu-Goto action).

The proposal (3.4.9) is a generalization of the Acharya-Vafa theory for SYM [46]. Note a few interesting facts that follow from (3.4.9).

- If we take $I = J+1$ we obtain the TQFT $\mathrm{U}(1)_N$ in the case of the symmetric fermion, and $\mathrm{U}(1)_{N-4}$ in the case of the antisymmetric fermion. Note that for finite (positive) $M$, and $\theta = \pi$ precisely these two adjacent vacua are degenerate and hence this domain wall continues to exist also at finite $M$. At very large $M$ we can integrate out the fermion and remain with the pure Yang-Mills theory, where the domain wall is given by $\mathrm{U}(1)_N$ (more precisely, the domain wall of pure Yang-Mills theory is level/rank dual to the $\mathrm{U}(1)_N$ TQFT). We see that in the symmetric fermion case no phase transition occurs. This is similar to the theory with the adjoint fermion. But in the case of the antisymmetric fermion we see that a phase transition does occur as we crank up the mass. If the phase transition is second order, it would be natural to assume that it is in the same universality class as the corresponding 3d phase transition, namely, it is given by $\mathrm{U}(1)_{N-2}$ plus a charge-2 fermion. This discussion (beautifully) makes sense also for the degenerate case $N = 2$, where the corresponding theory with the antisymmetric tensor is equivalent to the pure Yang-Mills theory with a neutral Dirac fermion. The domain wall theory is always $\mathrm{U}(1)_2$ (here we use that $\mathrm{U}(1)_2 \simeq \mathrm{U}(1)_{-2}$) and the $\mathrm{U}(1)_0$ plus a charge-2 fermion leads to a massless fermion on the wall [214–217], which can be thought of as arising precisely from the massless Dirac fermion in the bulk!

- In general, combining the four-dimensional spontaneously broken axial symmetry and time-reversal symmetry, we can derive that the domain wall theory connecting $I$ and $J$ should be isomorphic to the one connecting $N \pm 2 - I + J$ and 0. Indeed, this is merely the statement that

$$\mathrm{U}\left(I-J\right)_{N\pm 2-I+J,N\pm 2-2I+2J} \simeq \mathrm{U}\left(N\pm 2-I+J\right)_{J-I,N\pm 2-2I+2J} \tag{3.4.10}$$

which is nothing but level/rank duality in three dimensions.

- If $N$ is even then there exists a time-reversal invariant domain wall, given by $I - J = \frac{N\pm2}{2}$. The corresponding theory on the wall is $\mathrm{U}\left(\frac{N\pm2}{2}\right)_{\frac{N\pm2}{2},0}$. We discussed in detail that this theory should be interpreted as a massless NGB coupled to some TQFT. While baryon symmetry is not broken in the bulk, we see that it is broken on the wall! In principle, this domain wall theory is a nonperturbative object and it is hard to understand where this massless NGB comes from and why (recall that in the case of the Acharya-Vafa domain walls, there is no such massless field). We can think of this domain wall theory as a bound state of $\frac{N\pm2}{2}$ elementary domain walls (the elementary domain walls are the ones described in the first bullet point). There is however one case, $N = 2$ with a symmetric fermion, where this massless NGB can be seen explicitly. We describe this mechanism below.

Now let us discuss in detail the case of $N = 2$ with a Dirac fermion in the adjoint representation. Recall that, as explained above, instead of having $N + 2 = 4$ isolated trivial vacua we have two copies of $S^2$. The "elementary" domain wall corresponds to connecting the south pole of the first $S^2$ and the south pole of the second $S^2$ (or any isomorphic configuration thereof). This wall is hard to understand since it passes in regions of field space which are not within our effective theory. However, our prediction above for the physics of this domain wall is the $\mathrm{U}(1)_2$ TQFT, which makes a lot of sense (in the limit of softly deformed pure $\mathcal{N} = 2$ SYM theory, this result can be derived directly from the Seiberg-Witten solution along the lines of [218]).

The bound state of two such elementary walls corresponds to a jump of $\theta$ by $4\pi$ and what it does is to connect the north pole and the south pole of the same given $S^2$. Now, the wall can be analyzed entirely within classical field theory, since the wall is merely a geodesic running from the north to the south pole of the $\mathrm{O}(3)$ NLSM. The massless bosonic mode that our 3d model predicts is simply the azimuthal degree of freedom of that geodesic trajectory, see figure 3.7. Therefore, the domain wall has indeed a compact NGB which corresponds to the spontaneous breaking of baryon symmetry. In figure 3.8 we depict the four vacua at the north and south poles of the two $S^2$'s as the vertices of the square and we draw the two $S^2$'s that stretch along the diagonals, parameterizing the vacua of the theory.

For this story to hold up it is essential that the TQFT that accompanies the NGB is trivial. Indeed, it is $\mathrm{PSU}(2)_2 = \mathrm{SO}(3)_1$ which is a trivial theory. In the four dimensional theory there is nowhere for the domain wall to obtain a Chern-Simons term from since the sphere has the standard non-singular round metric and the domain wall can be understood entirely within effective field theory. Therefore the NGB in the infrared is not accompanied by a TQFT, in accordance with our prediction from the 3d analysis.

Note that the emergence of the NGB on the wall here is quite analogous to the way the symmetry breaking phases of $\mathrm{QCD}_3$ emerge from the corresponding four-dimensional construction [41]. For similar constructions see also [40, 219–225]

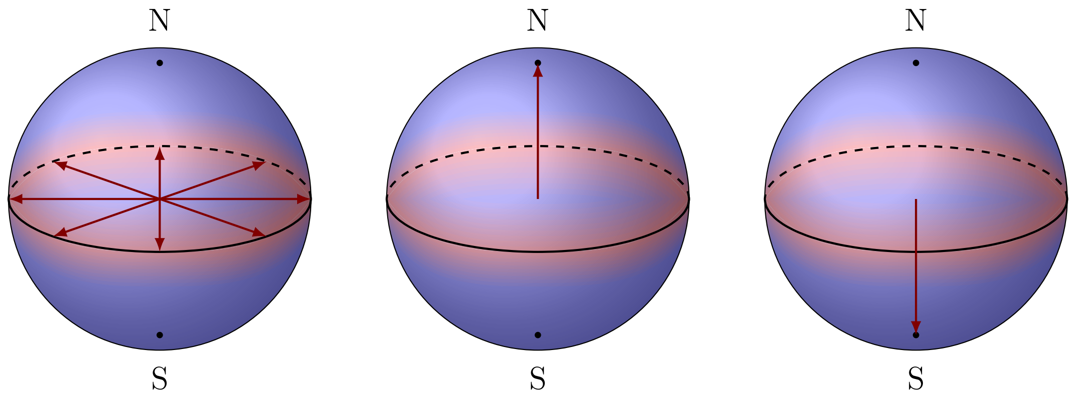



Figure 3.7: These figures represent the NG boson that arises on the domain wall connecting the vacuum at the north pole with the vacuum at the south pole. The first figure is the NG boson on the domain wall and the second and third figures represent the bulk vacua.

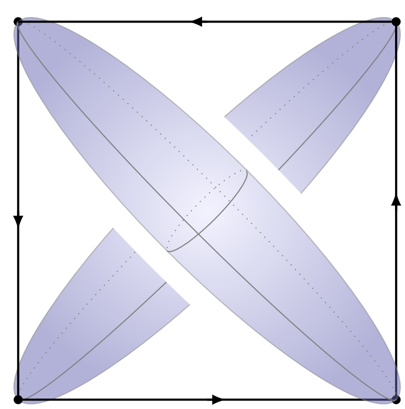


Figure 3.8: Moduli space of SU(2) plus a symmetric fermion, consisting of two copies of $S^2$ (the $S^2$'s in the figure are stretched only for the convenience of the picture).

## 3.5 Comments on Future Directions

Our theories with $m = k = 0$ all have a time-reversal symmetry, satisfying

$$\mathsf{T}^2 = (-1)^F\,. \tag{3.5.1}$$

It is therefore possible to study these theories on $\text{pin}_+$ manifolds. It is well known that there is a purely gravitational $\mathbb{Z}_{16}$ time-reversal anomaly in such cases [56, 87–89, 108]. We denote the anomaly by $\nu \in \mathbb{Z}_{16}$. Here we would like to briefly mention some new aspects of this anomaly, which will be explained in more detail elsewhere.

A naive approach to computing the anomaly $\nu$ in the ultraviolet is to disregard the gauge interactions and compute the total (weighted) number of Majorana fermions. This seems physically justified because the gauge interactions are arbitrarily weak in the deep ultraviolet. However, in [30] it was noted that this procedure is often ambiguous; it is not invariant under gauge transformations, in the sense that if we compose $\mathsf{T}$ with a gauge transformation then $\nu$ may change by an integer which is not a multiple of 16.

In [34] it was pointed out that in theories where the gauge group is not simply connected, the time-reversal symmetry algebra could be deformed by the magnetic symmetry and as a result $\nu$ would not be well defined. Here we will demonstrate that $\nu \in \mathbb{Z}_{16}$ cannot be canonically defined (in general) in theories with a one-form symmetry. This even happens in theories without a magnetic symmetry (i.e. based on simply connected gauge groups). The lack of a canonical choice of $\nu$ has to do with the fact that the full symmetry group of the theory has more generators than just $\mathsf{T}$, and there is no canonical way to set to zero the background fields for the one-form symmetry. See below.

Let us consider a simple example, which was already studied in [30]. Consider SU(2) gauge theory coupled to a single Majorana adjoint fermion (and $k = m = 0$). Let the time-reversal symmetry act in the standard way, which leads to $\nu = 3$, simply because there are three Majorana fermions in total. However, let us now compose $\mathsf{T}$ with the gauge transformation $U = i\sigma_y$. The (traceless) Hermitian matrix of Majorana fermions transforms under this gauge transformation as

$$\begin{pmatrix} \Psi & \chi \\ \chi^\dagger & -\Psi \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Psi & \chi \\ \chi^\dagger & -\Psi \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\Psi & -\chi^\dagger \\ -\chi & \Psi \end{pmatrix} . \tag{3.5.2}$$

If our original time-reversal symmetry acts as $\mathsf{T}\Psi = \gamma^0\Psi$ and $\mathsf{T}\chi = \gamma^0\chi^\dagger$ as usual, then combining this with the gauge transformation we see that $\mathsf{T}\circ U\Psi = -\gamma^0\Psi$ and $\mathsf{T}\circ U\chi = -\gamma^0\chi$. Remembering that $\mathsf{T}\circ U$ is antilinear (squaring to $(-1)^F$) we thus see that now it would appear that the time-reversal anomaly is $\nu = -1$. We therefore clearly see that in theories with a one-form symmetry the $\nu \in \mathbb{Z}_{16}$ anomaly is not uniquely defined in spite of the fact that there is no magnetic symmetry. The same phenomenon takes place in many of the examples studied in [30, 34]. We will now give a physical as well as a mathematical interpretation of this phenomenon.

The SU(2) gauge theory coupled to the adjoint Majorana fermion was claimed to flow in the infrared to a free massless Majorana fermion alongside with the $\mathrm{U}(1)_2$ TQFT [30]. The time-reversal anomaly of the $\mathrm{U}(1)_2$ TQFT could be either $\nu = +2$ or $\nu = -2$ – this depends on the time-reversal transformation of the semion in the TQFT. The contribution of the decoupled Majorana fermion is always $\nu_{\mathrm{Majorana}} = +1$ in our conventions (i.e. with our choice of orientation). Therefore the total time-reversal anomaly of the infrared theory is either $-1$

or 3, which is in exact agreement with the values found above in the ultraviolet gauge theory. (Analogous matching of the multiple possible values in the ultraviolet of $\nu$ can be carried out in the examples appearing in [30, 34].) Indeed, the ultraviolet theory has an unscreened Wilson line in the fundamental representation and the properties of the particle defining the worldline are not part of the definition of the ultraviolet theory.

A closely related point of view [212] is that we can break the one-form symmetry by adding heavy particles that transform faithfully under the center of the gauge group. We need to assign time-reversal transformations to these new particles. The fundamental Wilson line is now screened by these particles. In particular, $\nu \in \mathbb{Z}_{16}$ is well defined and the transformation properties of the infrared semions are completely determined in the presence of these heavy particles.[85]

The mathematical interpretation[86] is that one can show that in the presence of a $\mathbb{Z}_2$ one-form symmetry the value of $\nu$ can be shifted by a change of variables involving the two-form gauge field and $w_1^2$. This can lead to $\Delta\nu = 4$ which is precisely what we have found above. Ideally in such theories we should compute the full anomaly polynomial involving the two-form gauge field $B$ and the time-reversal gauge field $w_1$, and this should be compared across renormalization group flows and dualities. We leave this for the future.

An additional subtlety that arises in our theories (but not in the theories with an adjoint Majorana) at $m = k = 0$ is the presence of a massless scalar field. In the presence of such a gapless mode, it is nontrivial to evaluate the infrared contribution to $\nu$ (and to various other discrete anomalies).

To avoid the complications of having the Nambu-Goldstone boson when we study the theory on pin$_+$ manifolds, we could try to add a baryon to the Lagrangian, in such a way that some time-reversal symmetry remains and at the same time the Nambu-Goldstone boson would be lifted due to the explicit breaking of baryon symmetry.

This brings us back to the discussion of which baryon, in fact, condenses (see e.g. [226] for a discussion of some baryons in such theories). It turns out that the baryons in these theories are not as simple as one may initially expect. For instance, the naive baryons for the rank-two representation that were discussed in the literature

$$\epsilon_{i_1\cdots i_N}\epsilon_{j_1\cdots j_N}\psi^{i_1 j_1}\cdots\psi^{i_N j_N} \tag{3.5.3}$$

vanish identically because of Fermi statistics:

$$\epsilon_{i_1\ldots i_N}\epsilon_{j_1\cdots j_N}\psi^{i_1 j_1}\psi^{i_2 j_2}\psi^{i_3 j_3}\cdots \equiv 0 \tag{3.5.4}$$

for arbitrary values of the spinor indices and regardless of which other insertions are used. To construct a baryon with minimal baryon charge one therefore has to add various derivatives, insertions of the field strength or mesons. It would be nice to return to this in the future.

[85]We thank T. Senthil for discussions on this.

[86]We thank R. Thorngren for collaborating with us on this result.

## 3.6 Review of $\mathrm{QCD}_3$ dynamics.

In this section we collect some known phase diagrams of QCD in $d = 2+1$ dimensions. We mostly focus on [3, 29, 30, 113]; we make no attempt at reviewing the subject comprehensively since the existing literature is huge. For concreteness we restrict ourselves to either $N_F$ fundamental quarks, or $N_F = 1$ quarks in rank-2 representations; and to simple groups. In the literature one can also find $N_F = 2$ rank-2 quarks [130], as well as mixtures of quarks and scalars [227–231], including theories with supersymmetry [127–129, 232, 233]; and also quiver gauge theories [203, 234–236] and exceptional Lie groups [237]. The general picture is qualitatively similar although the finer points require a more specific analysis.

The following basic level-rank dualities of Chern-Simons theories play an important role (we take $k > 0$) [28, 112, 113]:

$$\begin{array}{rcl} \mathrm{SU}(N)_k & \longleftrightarrow & \mathrm{U}(k)_{-N,-N} \\ \mathrm{SO}(N)_k & \longleftrightarrow & \mathrm{SO}(k)_{-N} \\ \mathrm{Spin}(N)_k & \longleftrightarrow & \mathrm{O}(k)^0_{-N,-N} \\ \mathrm{Sp}(N)_k & \longleftrightarrow & \mathrm{Sp}(k)_{-N} \end{array} \tag{3.6.1}$$

We schematically write these dualities as $G(N)_k \leftrightarrow \hat{G}(k)_{-N}$, where $\hat{G}$ is the "dual" group. Clearly, $\hat{\hat{G}} = G$. Note that $\hat{G}(N_F)$ can be thought of as the flavor symmetry of the gauge theory with gauge group $G(N)$.

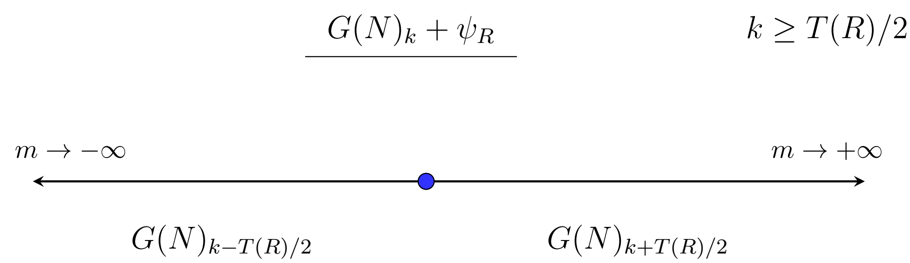
Other important dualities are

$$\begin{array}{rcl} \mathrm{U}(N)_{k,k\pm N} & \longleftrightarrow & \mathrm{U}(k)_{-N,\mp(k\pm N)} \\ \mathrm{O}(N)^1_{k,k-1+L} & \longleftrightarrow & \mathrm{O}(k)^1_{-N,-N+1+L} \end{array} \tag{3.6.2}$$

Note that, under level-rank duality $G(N)_k \leftrightarrow \hat{G}(k)_{-N}$, integrable representations are exchanged as $R \leftrightarrow R^t \times f^r$, where $R^t$ the representation whose Young diagram is the transpose of that of $R$, $f$ is the transparent fermion, and $r$ is the rank of $R$ (i.e., the number of boxes in its Young diagram). Therefore, at a very heuristic level, it is reasonable to expect that if we deform $G(N)$ by a field in the representation $R$, then the duality remains true if we also deform $\hat{G}(k)$ by a field in the representation $R^t$, with the same fermion parity as $R$ if $r$ is even, and with the opposite parity if odd. For example, adding fundamental fermions to $G$ is equivalent to adding fundamental bosons to $\hat{G}$; and adding symmetric fermions is equivalent to adding ant-symmetric fermions to the dual.

With this in mind, consider now QCD. We denote the gauge group as $G(N)$, the renormalized Chern-Simons level as $k$, and the quark representation as $R$. The level is defined as transforming as $k \to -k$ under time-reversal. The gauge theory is denoted as $G(N)_k + \psi_R$. The Dynkin index of the representation is denoted as $T(R)$, and it is scaled up by a factor of 2 when the representation is complex (so, for example, the fundamental representation always has $T(\square) = 1$, whether the group is $\mathrm{SU}(N)$ or $\mathrm{SO}(N)$).

The general structure of the phase diagram is the following. When the level is large enough, presumably $k \geq T(R)/2$, the theory is in the two-phase regime, namely the infrared phases are the two semiclassical phases that follow from integrating out massive fermions:




(3.6.3)

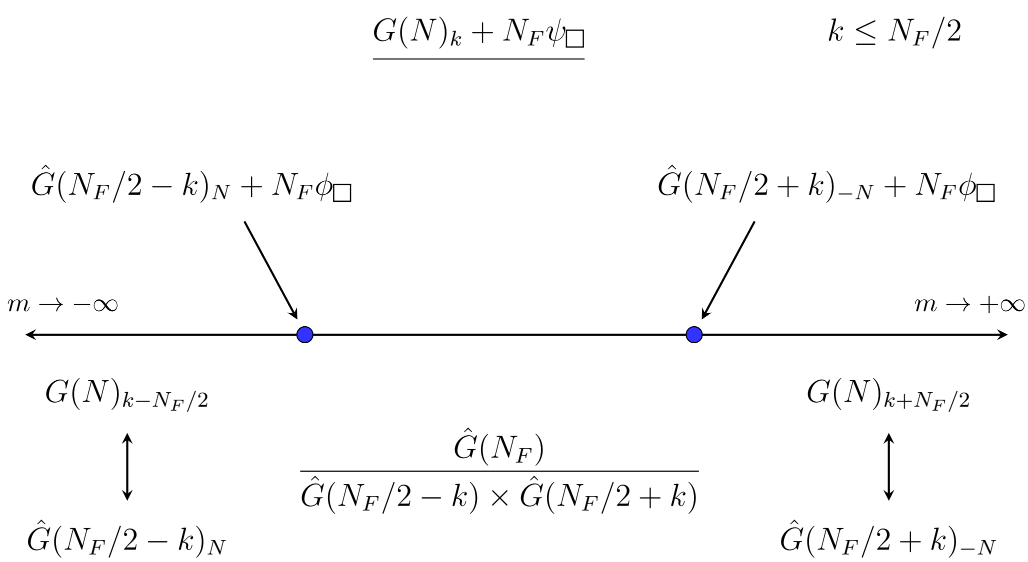
In the case of fundamental quarks $R = N_F \square$, the transition point admits a weakly coupled dual theory, which takes the form [238]

$$\hat{G}(k + N_F/2)_{-N} + N_F \phi_\square \tag{3.6.4}$$

where $\hat{G}$ is the dual group (cf. (3.6.1)). The masses are mapped as $\mathrm{sign}(m) = \mathrm{sign}(m'^2)$. Integrating out the fields, the two dual theories reproduce the same phase thanks to level-rank duality. For higher-rank fermions, no weakly coupled dual theory is known.

When the level is small, presumably $k < T(R)/2$, an intrinsically quantum phase opens up, and the transition points admit weakly coupled dual theories. The dual theory has matter fields in the representation $R^t$, and these are bosons if $R$ is a rank-1 representation, and fermions if a rank-2 representation.

The diagram in the quantum regime is as follows. When $R$ is the fundamental representation, the dual theory contains scalars:

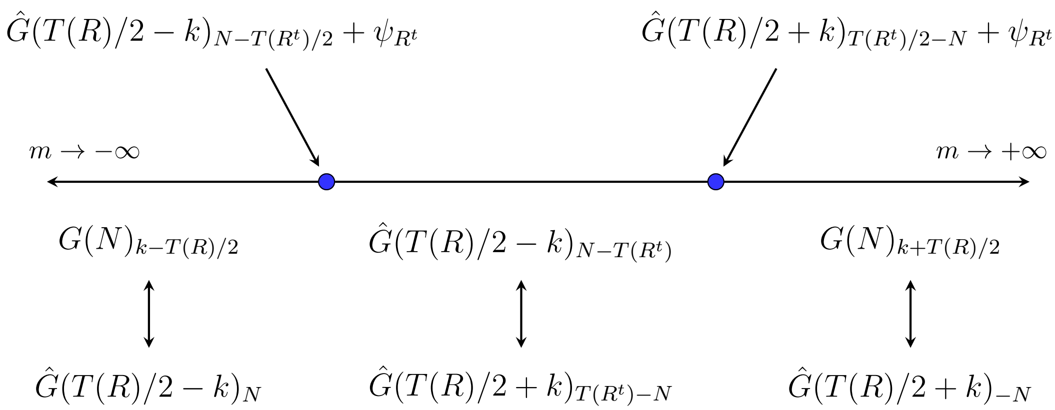



(3.6.5)

Here the intermediate phase represents a sigma model over the displayed manifold, and it is the consequence of the spontaneous breaking of the flavor symmetry by a fermion bilinear condensate; this breaking is manifest in the dual variables. The sigma model also includes a Wess-Zumino term at level $N$. The mass deformations are mapped as $\mathrm{sign}(m) = -\,\mathrm{sign}(m'^2)$ for the left transition, and as $\mathrm{sign}(m) = +\,\mathrm{sign}(m'^2)$ for the right transition.

When $R$ is a rank-2 representation, the dual theory contains fermions:

$$\underline{G(N)_k + \psi_R} \qquad\qquad k \leq T(R)/2$$



(3.6.6)

Here the intermediate phase is a topological phase; it is a non-perturbative phase from the point of view of the original fermions, but it is semi-classical with respect to the dual fermions. The mass deformations are mapped as $\mathrm{sign}(m) = -\,\mathrm{sign}(m')$. The two descriptions of the intermediate phase are apparently inequivalent, but they are in fact identical thanks to the level-rank dualities above.

Disconnected and non-simple groups have more than one level. The phase diagrams above require the following additions:

- In the case of fundamental quarks, the dual to $\mathrm{Spin}(N)$ is $\mathrm{O}(k)$, and the $\mathbb{Z}_2$ level is zero throughout, i.e., $\hat{G} \to \mathrm{O}(\cdots)^0$.

- In the case of rank-2 quarks, the dual to $\mathrm{Spin}(N)$ is $\mathrm{O}(k)$, and the $\mathbb{Z}_2$ level is $\mp 1/2$ on the dual gauge theories, and 1 in the intermediate phase, i.e., the theories are $\mathrm{O}(\cdots)^{\mp 1/2} + \psi_{R^t}$ and $\mathrm{O}(\cdots)^1$ respectively.

- In the case of rank-2 quarks, the dual to $\mathrm{SU}(N)$ is $\mathrm{U}(k)$, and the $\mathrm{U}(1)$ level is $\mp N \pm \frac{q}{T(R)\pm k}\dim(R^t)$ on the dual theories, and $\mp N \pm \frac{2q}{T(R)\pm k}\dim(R^t)$ in the intermediate phase, where $q$ is the $\mathrm{U}(1)$ charge of the fermions, i.e., $q = 0$ for adjoint quarks and $q = 2$ for symmetric and anti-symmetric quarks.

- It should also be noted that at small $m$, the theory with adjoint fermions is supersymmetric, and this symmetry is spontaneously broken, hence the infrared contains a gapless point where, on top of the TQFT above, there is a decoupled Majorana fermion (i.e., a Goldstino).

Note that the picture above is only consistent if the dual theories are in the two-phase regime when the original theory is in the three-phase regime – indeed, the intermediate phase was derived by simply integrating out the fields in the dual theory. An immediate consistency check for the rank-2 phase diagrams comes from noticing that the dual theories are rank-2 gauge theories too, and therefore $k \leq T(R)/2$ must hold precisely when $N > T(R^t)$. Using the actual values of $T(R)$ and $T(R^t)$ this is indeed seen to hold.

A simple test in the case of fundamental matter comes from looking at the $k = 0$ case, which is time-reversal invariant. The Vafa-Witten analysis [239] strongly suggests that here, the flavor symmetry is broken as $\hat{G}(N_F) \to \hat{G}(N_F/2) \times \hat{G}(N_F/2)$, consistently with the general arguments above. This, if correct, then implies the case of general $k$. Indeed, if the picture is correct for a given $(N_F, k)$, then it is also correct for $(N_F - 1, k \pm 1/2)$, as can be seen by turning on a mass for a single flavor in the conjectured dualities. This mass gaps out that single fermion, which decreases $N_F \to N_F - 1$ and it also shifts $k \to k \pm 1/2$, depending on the sign of the deformation.

## 3.7 Time-reversal and flavor symmetries.

A non-trivial consistency check comes from looking at how time-reversal acts on the phase diagram. Generically, this is not a symmetry, as it is explicitly broken by $m$ and $k$. We can formally restore this symmetry by letting it act on these parameters, i.e., we take $(m, k) \to (-m, -k)$. In a loose sense, we think of $m, k$ as some sort of spurions. In practice, this means that, if we reverse the orientation of spacetime (i.e., reverse the signs of all Chern-Simons levels), and also manually reverse the sign of $m$ and $k$, the diagram should stay invariant. This is manifestly true for the diagram as written,[87] but becomes a much more stringent condition once we turn on background fields.

We check this as follows. First, we turn on background fields for all symmetries – in our case, gravity and the baryon symmetry – and adjust their Chern-Simons coefficients in the ultraviolet to their time-reversal invariant point (i.e., the renormalized levels for background fields are chosen to vanish). Then, we track how this coefficient changes as we move around the diagram. The coefficient found in the infrared quantum phase must be the same whether we compute it via the left part of the diagram (the negative mass phase) or the right part of the diagram (the positive mass phase). Furthermore, this coefficient must be an odd function

[87] This explains why the mass deformation was chosen as $\operatorname{sign}(m) = -\operatorname{sign}(m')$ for the fermion-fermion dualities (see below (3.6.5)), but as $\operatorname{sign}(m) = \pm\operatorname{sign}(m'^2)$ for the fermion-boson dualities (see below (3.6.6)): the mass of a scalar is even under time-reversal, while the mass of the fermion is odd. Therefore, the two transitions must have the same sign in the case of fermion-fermion, but opposite signs for fermion-boson.

of $k$; this last condition is what extends the discussion of sections 3.3.2 and 3.3.3 (there we only checked that the coefficient matched across the diagram, but not whether it was odd under $k \to -k$).

We next perform this test for the phase diagrams above. In all cases we checked, the test works out correctly and thus the phase diagrams are consistent as far as time-reversal and background symmetries is concerned. For a given Lie algebra, it is enough to only check one form of the gauge group (e.g., for $\mathfrak{so}(N)$ we only do $\mathrm{SO}(N)$ instead of also doing $\mathrm{Spin}(N)$). The computation is insensitive to the global form of the group.

The Chern-Simons term for the gravitational term is denoted as $\mathrm{CS}[g]$ and that for the baryon symmetry as $\mathrm{CS}[B]$. In the adjoint case, the infrared includes a Goldstino, which we shall denote by $\chi$.

### 3.7.1 Unitary group.

Consider $\mathrm{SU}(N)_k + \psi_R$. The asymptotic phases are

$$\begin{array}{c} \mathrm{SU}(N)_{k\pm T(R)/2} \\ \pm\,\frac{1}{2}\dim(R)\mathrm{CS}[g] \\ \pm\,\frac{1}{2}\frac{Q-T(R)}{N}\mathrm{CS}[B] \end{array} \quad\longleftrightarrow\quad \begin{array}{c} \mathrm{U}(T(R)/2\pm k)_{\mp N} \\ \pm\big(\frac{1}{2}\dim(R) - N(T(R)\pm 2k)\big)\mathrm{CS}[g] \\ \pm\,\frac{1}{2}\frac{Q-T(R)}{N}\mathrm{CS}[B] \end{array} \tag{3.7.1}$$

Here $Q := \frac{1}{2}q^2\dim(R)/N$ is the charge under $\mathrm{U}(1)$, with $q$ the number of boxes of $R$. In particular, $Q(\square) = N_F$, $Q(\mathrm{adj}) = 0$, $Q(\substack{\square\\\square}) = 2(N-1)$, and $Q(\square\!\square) = 2(N+1)$.

The dual theories are

- $R = N_F\square$:
$$\mathrm{U}(N_F/2\pm k)_{\mp N} + N_F\phi_{\square} - 2Nk\,\mathrm{CS}[g] \pm 0\ \mathrm{CS}[B] \tag{3.7.2}$$

- $R = \mathrm{adj}$:
$$\begin{aligned} &\mathrm{U}(N/2\pm k)_{\mp\frac{3}{4}N+\frac{1}{2}k,\mp N} + \psi_{\mathrm{adj}} \\ &\qquad \mp\big(\tfrac{3}{8}N^2 - \tfrac{3}{2}Nk \pm \tfrac{1}{2}k^2 \mp 1\big)\mathrm{CS}[g] \mp \tfrac{1}{2}\ \mathrm{CS}[B] \end{aligned} \tag{3.7.3}$$

- $R = \substack{\square\\\square}$
$$\begin{aligned} &\mathrm{U}((N-2)/2\pm k)_{\mp\frac{3}{4}N+\frac{1}{2}k\pm\frac{1}{2},k\mp\frac{1}{2}N} + \psi_{\square\!\square} \\ &\qquad \mp\big(\tfrac{3}{8}N^2 - \tfrac{3}{2}Nk \pm \tfrac{5}{4}N \pm \tfrac{1}{2}k^2 - \tfrac{1}{2}k\big)\mathrm{CS}[g] \pm \tfrac{1}{2}\ \mathrm{CS}[B] \end{aligned} \tag{3.7.4}$$

- $R = \square\!\square$:
$$\begin{aligned} &\mathrm{U}((N+2)/2\pm k)_{\mp\frac{3}{4}N+\frac{1}{2}k\mp\frac{1}{2},k\mp\frac{1}{2}N} + \psi_{\substack{\square\\\square}} \\ &\qquad \mp\big(\tfrac{3}{8}N^2 - \tfrac{3}{2}Nk \mp \frac{5}{4}N \pm \tfrac{1}{2}k^2 + \tfrac{1}{2}k\big)\mathrm{CS}[g] \pm \tfrac{1}{2}\ \mathrm{CS}[B] \end{aligned} \tag{3.7.5}$$

Integrating out the fermions in the dual theories, the intermediate phase is found to be

- $R = N_F\square$:

$$\frac{\mathrm{U}(N_F)}{\mathrm{U}(N_F/2+k)\times\mathrm{U}(N_F/2-k)} + N\Gamma - 2Nk\,\mathrm{CS}[g] + 0\ \mathrm{CS}[B] \tag{3.7.6}$$

- $R = \mathrm{adj}$:

$$\mathrm{U}(N/2\pm k)_{\mp\frac{1}{2}N+k,\mp N} + \chi \mp \left(\tfrac{1}{4}N^2 - Nk \pm k^2 \mp \tfrac{3}{2}\right)\mathrm{CS}[g] \mp \tfrac{1}{2}\ \mathrm{CS}[B] \tag{3.7.7}$$

- $R = \begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$

$$\mathrm{U}((N-2)/2\pm k)_{\mp\frac{1}{2}(N-2)+k,2k} \mp \left(\tfrac{1}{4}N^2 - Nk \pm N \pm k^2 - k\right)\mathrm{CS}[g] \pm \tfrac{1}{2}\ \mathrm{CS}[B] \tag{3.7.8}$$

- $R = \square\square$:

$$\mathrm{U}((N+2)/2\pm k)_{\mp\frac{1}{2}(N+2)+k,2k} \mp \left(\tfrac{1}{4}N^2 - Nk \mp N \pm k^2 + k\right)\mathrm{CS}[g] \pm \tfrac{1}{2}\ \mathrm{CS}[B] \tag{3.7.9}$$

The central charge of the infrared phase is

- $R = N_F\square$:

$$c = Nk \tag{3.7.10}$$

- $R = \mathrm{adj}$:

$$c = k\left[\frac{k^2-1}{N} + \frac{1}{4}N\right] \tag{3.7.11}$$

- $R = \begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$

$$c = k\left[\frac{k^2-1}{N-2} + \frac{1}{4}N + 1\right] + \mathrm{sign}(k) \tag{3.7.12}$$

- $R = \square\square$:

$$c = k\left[\frac{k^2-1}{N+2} + \frac{1}{4}N - 1\right] + \mathrm{sign}(k) \tag{3.7.13}$$

In all cases we see that $c$ is an odd function of $k$, as expected.

Furthermore, the infrared coefficient of $\mathrm{CS}[B]$ matches when coming from both directions, as it should; indeed, recall that level-rank duality reads

$$U(N)_{+N,+(N\pm k)}[B] \quad\longleftrightarrow\quad U(k)_{-N,-(N\pm k)}[B] \mp \mathrm{CS}[B] \tag{3.7.14}$$

Not only does the counterterm match, but its value is precisely the one required by time-reversal.

### 3.7.2 Orthogonal group.

Consider $\mathrm{SO}(N)_k + \psi_R$ with $R$ a rank-1 or rank-2 representation. The asymptotic phases are

$$\begin{array}{c} \mathrm{SO}(N)_{k \pm T(R)/2} \\ \pm\, \frac{1}{2}\dim(R)\mathrm{CS}[g] \end{array} \quad \longleftrightarrow \quad \begin{array}{c} \mathrm{SO}(T(R)/2 \pm k)_{\mp N} \\ \pm\big(\frac{1}{2}\dim(R) - N(\frac{1}{2}T(R) \pm k)\big)\mathrm{CS}[g] \end{array} \tag{3.7.15}$$

The dual theories are

- $R = N_F \square$:

$$\mathrm{SO}(N_F/2 \pm k)_{\mp N} + N_F \phi_{\square} - Nk\,\mathrm{CS}[g] \tag{3.7.16}$$

- $R = \mathrm{adj}$:

$$\mathrm{SO}((N-2)/2 \pm k)_{\mp\frac{3}{4}N + \frac{1}{2}k \pm \frac{1}{2}} + \psi_{\square\square} \mp \big(\tfrac{3}{16}N^2 - \tfrac{3}{4}Nk \pm \tfrac{5}{8}N \pm \tfrac{1}{4}k^2 - \tfrac{1}{4}k \mp \tfrac{1}{2}\big)\mathrm{CS}[g] \tag{3.7.17}$$

- $R = \square\square$:

$$\mathrm{SO}((N+2)/2 \pm k)_{\mp\frac{3}{4}N + \frac{1}{2}k \mp \frac{1}{2}} + \psi_{\mathrm{adj}} \mp \big(\tfrac{3}{16}N^2 - \tfrac{3}{4}Nk \mp \tfrac{5}{8}N \pm \tfrac{1}{4}k^2 + \tfrac{1}{4}k \mp \tfrac{1}{2}\big)\mathrm{CS}[g] \tag{3.7.18}$$

Integrating out the fermions in the dual theories, the intermediate phase is found to be

- $R = N_F \square$:

$$\frac{\mathrm{SO}(N_F)}{\mathrm{SO}(N_F/2 + k) \times \mathrm{SO}(N_F/2 - k)} + N\Gamma - Nk\,\mathrm{CS}[g] \tag{3.7.19}$$

- $R = \mathrm{adj}$:

$$\mathrm{SO}((N-2)/2 \pm k)_{\mp\frac{1}{2}(N-2)+k} + \chi \mp \big(\tfrac{1}{8}N^2 - \tfrac{1}{2}Nk \pm \tfrac{1}{2}N \pm \tfrac{1}{2}k^2 - \tfrac{1}{2}k \mp 1\big)\mathrm{CS}[g] \tag{3.7.20}$$

- $R = \square\square$:

$$\mathrm{SO}((N+2)/2 \pm k)_{\mp\frac{1}{2}(N+2)+k} \mp \big(\tfrac{1}{8}N^2 - \tfrac{1}{2}Nk \mp \tfrac{1}{2}N \pm \tfrac{1}{2}k^2 + \tfrac{1}{2}k \mp \tfrac{1}{2}\big)\mathrm{CS}[g] \tag{3.7.21}$$

The central charge of the infrared phase is

- $R = N_F \square$:

$$c = \frac{1}{2}Nk \tag{3.7.22}$$

- $R = \mathrm{adj}$:

$$c = \frac{k}{2(N-4)}\left[\frac{1}{4}N(N-2) + k^2 - 3\right] \tag{3.7.23}$$

- $R = \square\square$:

$$c = \frac{k}{2N}\left[\frac{1}{2}N(N-6) + k^2 - 1\right] \tag{3.7.24}$$

In all cases we see that $c$ is an odd function of $k$, as expected.

### 3.7.3 Symplectic group.

Consider $\mathrm{Sp}(N)_k + \psi_R$ with $R$ a rank-1 or rank-2 representation. The asymptotic phases are

$$\begin{array}{c} \mathrm{Sp}(N)_{k\pm T(R)/2} \\ \pm\,\frac{1}{2}\dim(R)\mathrm{CS}[g] \end{array} \quad \longleftrightarrow \quad \begin{array}{c} \mathrm{Sp}(T(R)/2\pm k)_{\mp N} \\ \pm\big(\frac{1}{2}\dim(R) - 2N(T(R)\pm 2k)\big)\mathrm{CS}[g] \end{array} \tag{3.7.25}$$

The dual theories are

- $R = N_F\square$:

$$\mathrm{Sp}(N_F/2 \pm k)_{\mp N} + N_F\phi_{\square} - 4Nk\,\mathrm{CS}[g] \tag{3.7.26}$$

- $R = \mathrm{adj}$:

$$\mathrm{Sp}((N+1)/2 \pm k)_{\mp\frac{3}{4}N + \frac{1}{2}k \mp \frac{1}{4}} + \psi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \mp \big(\tfrac{3}{4}N^2 - 3Nk \mp \tfrac{5}{4}N \pm k^2 + \tfrac{1}{2}k \mp \tfrac{1}{2}\big)\mathrm{CS}[g] \tag{3.7.27}$$

- $R = \begin{smallmatrix}\square\\\square\end{smallmatrix}$:

$$\mathrm{Sp}((N-1)/2 \pm k)_{\mp\frac{3}{4}N + \frac{1}{2}k \pm \frac{1}{4}} + \psi_{\mathrm{adj}} \mp \big(\tfrac{3}{4}N^2 - 3Nk \pm \tfrac{5}{4}N \pm k^2 - \tfrac{1}{2}k \mp \tfrac{1}{2}\big)\mathrm{CS}[g] \tag{3.7.28}$$

Integrating out the fermions in the dual theories, the intermediate phase is found to be

- $R = N_F\square$:

$$\frac{\mathrm{Sp}(N_F)}{\mathrm{Sp}(N_F/2+k)\times \mathrm{Sp}(N_F/2-k)} + N\Gamma - 4Nk\,\mathrm{CS}[g] \tag{3.7.29}$$

- $R = \mathrm{adj}$:

$$\mathrm{Sp}((N+1)/2 \pm k)_{\mp\frac{1}{2}(N+1)+k} + \chi \mp \big(\tfrac{1}{2}N^2 - 2Nk \mp N \pm 2k^2 + k \mp 1\big)\mathrm{CS}[g] \tag{3.7.30}$$

- $R = \square\square$:

$$\mathrm{Sp}((N-1)/2 \pm k)_{\mp\frac{1}{2}(N-1)+k} \mp \big(\tfrac{1}{2}N^2 - 2Nk \pm N \pm 2k^2 - k \mp \tfrac{1}{2}\big)\mathrm{CS}[g] \tag{3.7.31}$$

The central charge of the infrared phase is

- $R = N_F\square$:

$$c = 2Nk \tag{3.7.32}$$

- $R = \mathrm{adj}$:

$$c = \frac{k}{2(N+2)}\Big[N(N+1) + 4k^2 - 3\Big] \tag{3.7.33}$$

- $R = \square\square$:

$$c = \frac{k}{2N}\Big[N(N+3) + 4k^2 - 1\Big] \tag{3.7.34}$$

In all cases we see that $c$ is an odd function of $k$, as expected.

## 3.8 Time-reversal anomaly.

Another non-trivial consistency check comes from looking at the pure time-reversal anomaly. In $d = 2+1$, an anti-unitary symmetry $\mathsf{T}$ with algebra $\mathsf{T}^2 = (-1)^F$ has a $\mathbb{Z}_{16}$ valued 't Hooft anomaly, often denoted as $\nu$. For systems continuously connected to a free fermion phase, this anomaly is easily computed as follows.

Consider a system of free Majorana fermions (see A.2 for a quick review). The most general time-reversal operation is $\psi \mapsto V\gamma^0\psi$, where $V$ is a real matrix (that does not act in spinor space). This operation satisfies $\mathsf{T}^2 = (-1)^F$ if $V^2 = 1$. Note that such a matrix is diagonalizable, and has eigenvalues $\pm 1$; the anomaly associated to this transformation is the number of positive eigenvalues, minus the number of negative ones, modulo 16 [56]. In other words,

$$\nu = \operatorname{tr} V \mod 16 \tag{3.8.1}$$

As this is invariant under smooth deformation, this formula remains true for interacting theories, as long as they can be continuously connected to the free phase. Such is the case, for example, in QCD, where $g \to 0$ turns off the interactions and leaves us with free fermions (and free gluons, which do not contribute to $\nu$). The value of $\nu$ is also invariant under RG flow, and therefore it must be reproduced by whatever phase emerges at large distances, in particular, by the conjectured quantum phase in the phase diagrams reviewed above.

In QCD, the matrix $V$ can be chosen to act on both the color indices and the flavor indices. For simplicity we shall restrict ourselves to color indices only, but the idea for the general case is the same. In this situation, for any matrix $V$ in color space that squares to 1, we will have a value of $\nu$, which must be matched by the infrared phase. If we denote the gauge group by $G$, and the quark representation as $R$, then the matrix $V$ takes the form $V = R(g)$ for any $g$ such that $R(g^2) = 1$. In other words, for any $g \in G$ such that $g^2 \in \ker R$, the value of $\nu \equiv \chi_R(g) := \operatorname{tr}_R g$ modulo 16 is an invariant of the system, which is constant along any $\mathsf{T}$-preserving deformation. Here $\chi_R$ denotes the group character of $R$, multiplied by 2 if $R$ is complex (as we count Majorana fermions). Note that $\nu$ depends only on the conjugacy class of $g$, so it is enough to look at a maximal torus of $G$. Even simpler, note that for non-trivial $R$ one has $\ker R \subseteq Z(G)$.

We will next compute the value of $\nu$ for all the phase diagrams reviewed above, and check for 't Hooft anomaly matching whenever it is possible. Note that $G_k + \psi_R$ is time-reversal invariant only when $k = 0$. The bare level $k - T(R)/2$ has to be integral, which means that $k = 0$ only exists when $T(R)$ is even.

### 3.8.1 Unitary group.

The special unitary group $\mathrm{SU}(N)$ is defined as the set of $N \times N$ complex matrices such that $gg^\dagger = \mathbf{1}_N$ and $\det(g) = 1$. It corresponds to the compact form of the Cartan series $A_{N+1}$.

**Fundamental quarks.** Consider $\mathrm{SU}(N)$ plus $N_F$ fundamental quarks. The index is $T(R)=N_F$ and therefore $\nu$ is defined only when $N_F$ is even. The most general matrix with $R(g^2)=1$ is $g=-\mathbf{1}_{2p}\oplus\mathbf{1}_{N-2p}$ for some integer $p$. Evaluating the character,

$$\nu=2N_F\operatorname{tr}(g)=2N_F(N-4p) \tag{3.8.2}$$

Note that $N_F$ is even and therefore $\nu\equiv 2N_FN \mod 16$, the $p$-dependence drops out.

The low energy effective description is postulated to be a non-linear sigma model with a Wess-Zumino term [29]; the 't Hooft anomaly of such a system is not known, and therefore we cannot check for matching.

**Adjoint quarks.** Consider $\mathrm{SU}(N)$ plus one adjoint quark. The index is $T(R)=N$ and therefore $\nu$ is defined only when $N$ is even. The most general matrix with $R(g^2)=1$ is $g=(-\mathbf{1}_p\oplus\mathbf{1}_{N-p})\theta$ where $p$ is an integer and $\theta$ a phase subject to $\theta^N=(-1)^a$ (such that $\det(g)=1$). Evaluating the character (cf. (B.2.3)),

$$\nu=\operatorname{tr}(g)\operatorname{tr}(g^{-1})-1\equiv(N-2p)^2-1 \tag{3.8.3}$$

Reducing modulo 16, the anomaly is $\nu=-1$ and $\nu=3$.

The low energy effective description is postulated to be the TQFT $\mathrm{U}(N/2)_{N/2,N}$ plus a decoupled Majorana fermion [30]. The former has anomaly $\nu=\pm2$ while the latter has anomaly $\nu=+1$. Therefore, the infrared phase has anomaly $\nu=-1$ and $\nu=3$, precisely as in the ultraviolet.

**Symmetric quarks.** Consider $\mathrm{SU}(N)$ plus one symmetric quark. The index is $T(R)=N+2$ and therefore $\nu$ is defined only when $N$ is even. The most general matrix with $R(g^2)=1$ is either $g=-\mathbf{1}_{2p}\oplus\mathbf{1}_{N-2p}$ where $p$ is an integer, or $g=(-\mathbf{1}_p\oplus\mathbf{1}_{N-p})i$ with $p\equiv N/2 \mod 2$ (such that $\det(g)=1$). Evaluating the character (cf. (B.2.2)),

$$\nu=\operatorname{tr}(g^2)+\operatorname{tr}(g)^2=\begin{cases}N+(N-4p)^2\\ -N-(N-2p)^2\end{cases} \tag{3.8.4}$$

for the two families of matrices above. Note that $N$ is even and therefore the first option is in fact independent of $p$, and equal to $N(N+1)$ mod 16.

Simplifying the expression above, we get the following anomalies:

$$\nu=\begin{cases}0 & N\equiv 0 \mod 16 & 8 & N\equiv 8 \mod 16\\ -2,6 & N\equiv 2 \mod 16 & -2,6 & N\equiv 10 \mod 16\\ \pm4 & N\equiv 4 \mod 16 & \pm4 & N\equiv 12 \mod 16\\ -6 & N\equiv 6 \mod 16 & 2 & N\equiv 14 \mod 16\end{cases} \tag{3.8.5}$$

The low energy effective description is postulated to be the TQFT $\mathrm{PSU}((N+2)/2)_{(N+2)/2}$ plus a decoupled compact scalar [3]; the 't Hooft anomaly of such a system is not known, and therefore we cannot check for matching.

**Anti-symmetric quarks.** Consider $\mathrm{SU}(N)$ plus one anti-symmetric quark. The index is $T(R) = N - 2$ and therefore $\nu$ is defined only when $N$ is even. The most general matrix with $R(g^2) = 1$ is either $g = -\mathbf{1}_{2p} \oplus \mathbf{1}_{N-2p}$ where $p$ is an integer, or $g = (-\mathbf{1}_p \oplus \mathbf{1}_{N-p})i$ with $p \equiv N/2 \mod 2$ (such that $\det(g) = 1$). Evaluating the character (cf. (B.2.2)),

$$\nu = -\operatorname{tr}(g^2) + \operatorname{tr}(g)^2 = \begin{cases} -N + (N-4p)^2 \\ N - (N-2p)^2 \end{cases} \tag{3.8.6}$$

for the two families of matrices above. Note that $N$ is even and therefore the first option is in fact independent of $p$, and equal to $N(N-1)$ mod 16.

Simplifying the expression above, we get the following anomalies:

$$\nu = \begin{cases} 0 & N \equiv 0 \mod 16 & 8 & N \equiv \ 8 \mod 16 \\ 2 & N \equiv 2 \mod 16 & -6 & N \equiv 10 \mod 16 \\ \pm 4 & N \equiv 4 \mod 16 & \pm 4 & N \equiv 12 \mod 16 \\ -2, 6 & N \equiv 6 \mod 16 & -2, 6 & N \equiv 14 \mod 16 \end{cases} \tag{3.8.7}$$

The low energy effective description is postulated to be the TQFT $\mathrm{PSU}((N-2)/2)_{(N-2)/2}$ plus a decoupled compact scalar [3]; the 't Hooft anomaly of such a system is not known, and therefore we cannot check for matching.

### 3.8.2 Orthogonal group.

The special orthogonal group $\mathrm{SO}(N)$ is defined as the set of $N \times N$ real matrices such that $gg^t = \mathbf{1}_N$ and $\det(g) = 1$. It corresponds to the compact form of the Cartan series $B_{(N-1)/2}, D_{N/2}$ after quotienting out a central $\mathbb{Z}_2$.

**Fundamental quarks.** Consider $\mathrm{SO}(N)$ plus $N_F$ fundamental quarks. The index is $T(R) = N_F$ and therefore $\nu$ is defined only when $N_F$ is even. The most general matrix with $R(g^2) = 1$ is $g = -\mathbf{1}_{2p} \oplus \mathbf{1}_{N-2p}$ for some integer $p$. Evaluating the character,

$$\nu = N_F \operatorname{tr}(g) = N_F(N - 4p) \tag{3.8.8}$$

When either $N_F/2$ or $p$ is even, then $\nu = NN_F \mod 16$; on the other hand, if they are both odd, then $\nu = NN_F + 8 \mod 16$. Note that $\nu$ is only defined when $\mathsf{T}^2 = (-1)^F$, i.e., when the time-reversal algebra is not modified by the magnetic symmetry [34]; this happens if and only if $N_F/2$ is even, whence $\nu \equiv NN_F$, the $p$-dependence drops out.

The low energy effective description is postulated to be a non-linear sigma model with a Wess-Zumino term [29]; the 't Hooft anomaly of such a system is not known, and therefore we cannot check for matching.

**Adjoint quarks.** Consider $\mathrm{SO}(N)$ plus one adjoint quark. The index is $T(R) = N-2$ and therefore $\nu$ is defined only when $N$ is even. The most general matrix with $R(g^2) = 1$ is either $g = -\mathbf{1}_{2p} \oplus \mathbf{1}_{N-2p}$ for some integer $p$, or $g = \mathrm{diag}(i\sigma_y, \cdots, i\sigma_y)$. Evaluating the character (cf. (B.2.7)),

$$\nu = \frac{1}{2}(-\operatorname{tr}(g^2) + \operatorname{tr}(g)^2) \equiv \begin{cases} \frac{1}{2}(-N + (N-4p)^2) \\ \frac{1}{2}N \end{cases} \tag{3.8.9}$$

for the two families of matrices above.

Simplifying the expressions above, we get the following anomalies:

$$\nu = \begin{cases} 0,8 & N \equiv 0 \mod 32 & 0,8 & N \equiv 16 \mod 32 \\ 1 & N \equiv 2 \mod 32 & -7 & N \equiv 18 \mod 32 \\ \pm 2, 6 & N \equiv 4 \mod 32 & -2, \pm 6 & N \equiv 20 \mod 32 \\ -1, 3 & N \equiv 6 \mod 32 & -5, 7 & N \equiv 22 \mod 32 \\ \pm 4 & N \equiv 8 \mod 32 & \pm 4 & N \equiv 24 \mod 32 \\ -3, 5 & N \equiv 10 \mod 32 & -3, 5 & N \equiv 26 \mod 32 \\ 2, \pm 6 & N \equiv 12 \mod 32 & \pm 2, -6 & N \equiv 28 \mod 32 \\ -5, 7 & N \equiv 14 \mod 32 & -1, 3 & N \equiv 30 \mod 32 \end{cases} \tag{3.8.10}$$

The low energy effective description is postulated to be the TQFT $\mathrm{SO}((N-2)/2)_{(N-2)/2}$ plus a decoupled Majorana fermion [30]. The former has anomaly $\nu = \pm(N-2)/2$ while the latter has anomaly $\nu = +1$. One can see that this matches the computation in the ultraviolet if and only if $N/2$ is odd. For $N/2$ even, the time-reversal algebra is modified by the monopole symmetry $\pi_1(\mathrm{SO}(N)) = \mathbb{Z}_2$ [34], and hence the 't Hooft anomaly is no longer given by $\mathbb{Z}_{16}$, i.e., there is no reason to expect matching (as we are computing the anomaly for the wrong symmetry group).

A way around this is to consider, instead, the cover of $\mathrm{SO}(N)$, which has trivial fundamental group and hence no magnetic symmetry. Thus, we look at $\mathrm{Spin}(N)$ plus an adjoint quark. The computation of $\nu$ in the ultraviolet is the same, but now the infrared TQFT is $\mathrm{O}((N-2)/2)^1_{(N-2)/2,N/2+2}$ [113]. For $N/2$ even this can also be written as $\mathrm{SO}((N-2)/2)_{(N-2)/2} \times (\mathbb{Z}_2)_N$; the second factor has anomaly $\nu = 0, 8$ for $N = 0$ and $\nu = 0, \pm 4$ for $N = 4$ (with mod 8 periodicity in $N$). Adding the contribution of $(\mathbb{Z}_2)_N$ to the anomaly of $\mathrm{SO}((N-2)/2)_{(N-2)/2}$ nicely matches the ultraviolet computation for $N/2$ even.

Note that another way around the monopole symmetry is to conjugate by a flavor symmetry. Reference [34] shows that time-reversal, composed with a reflection in color space, has no monopole deformation when $N/2$ is even. Following the same analysis as before, but adding the contribution of the flavor symmetry, the ultraviolet anomaly can be shown to be $\nu = \frac{1}{2}(N^2 - 8Np - 5N + 4)$ for some integer $p$. Again, this nicely matches the infrared anomaly $\nu = \pm(N-2)/2 + 1$ for $N/2$ even.

**Symmetric quarks.** Consider $\mathrm{SO}(N)$ plus one symmetric quark. The index is $T(R) = N+2$ and therefore $\nu$ is defined only when $N$ is even. The most general matrix with $R(g^2) = 1$ is either $g = -\mathbf{1}_{2p} \oplus \mathbf{1}_{N-2p}$ for some integer $p$, or $g = \mathrm{diag}(i\sigma_y, \cdots, i\sigma_y)$. Evaluating the character (cf. (B.2.7)),

$$\nu = \frac{1}{2}(\mathrm{tr}(g^2) + \mathrm{tr}(g)^2) - 1 \equiv \begin{cases} \frac{1}{2}(N + (N-4p)^2) - 1 \\ -\frac{1}{2}N - 1 \end{cases} \tag{3.8.11}$$

for the two families of matrices above.

Simplifying the expressions above, we get the following anomalies:

$$\nu = \begin{cases} -1, 7 & N \equiv \ 0 \mod 32 & -1, 7 & N \equiv 16 \mod 32 \\ \pm 2 & N \equiv \ 2 \mod 32 & \pm 6 & N \equiv 18 \mod 32 \\ 1, -3, -7 & N \equiv \ 4 \mod 32 & 1, 5, -7 & N \equiv 20 \mod 32 \\ \pm 4 & N \equiv \ 6 \mod 32 & \pm 4 & N \equiv 22 \mod 32 \\ 3, -5 & N \equiv \ 8 \mod 32 & 3, -5 & N \equiv 24 \mod 32 \\ \pm 6 & N \equiv 10 \mod 32 & \pm 2 & N \equiv 26 \mod 32 \\ -3, 5, -7 & N \equiv 12 \mod 32 & 1, -3, 5 & N \equiv 28 \mod 32 \\ 8 & N \equiv 14 \mod 32 & 0 & N \equiv 30 \mod 32 \end{cases} \tag{3.8.12}$$

The low energy effective description is postulated to be the TQFT $\mathrm{SO}((N+2)/2)_{(N+2)/2}$ [30]. This has anomaly $\nu = \pm(N+2)/2$. One can see that this matches the computation in the ultraviolet if and only if $N/2$ is odd. For $N/2$ even, the time-reversal algebra is modified by the monopole symmetry $\pi_1(\mathrm{SO}(N)) = \mathbb{Z}_2$ [34], and hence the 't Hooft anomaly is no longer given by $\mathbb{Z}_{16}$, i.e., there is no reason to expect matching (as we are computing the anomaly for the wrong symmetry group).

A way around this is to consider, instead, the cover of $\mathrm{SO}(N)$, which has trivial fundamental group and hence no magnetic symmetry. Thus, we look at $\mathrm{Spin}(N)$ plus a symmetric quark. The computation of $\nu$ in the ultraviolet is the same, but now the infrared TQFT is $\mathrm{O}((N+2)/2)^1_{(N+2)/2,N/2}$ [113]. For $N/2$ even this can also be written as $\mathrm{SO}((N+2)/2)_{(N+2)/2} \times (\mathbb{Z}_2)_N$; the second factor has anomaly $\nu = 0, 8$ for $N = 0$ and $\nu = 0, \pm 4$ for $N = 4$ (with mod 8 periodicity in $N$). Adding the contribution of $(\mathbb{Z}_2)_N$ to the anomaly of $\mathrm{SO}((N+2)/2)_{(N+2)/2}$ nicely matches the ultraviolet computation for $N/2$ even.

Note that another way around the monopole symmetry is to conjugate by a flavor symmetry. Reference [34] shows that time-reversal, composed with a reflection in color space, has no monopole deformation when $N/2$ is even. Following the same analysis as before, but adding the contribution of the flavor symmetry, the ultraviolet anomaly can be shown to be $\nu = \frac{1}{2}(N^2 - 8Np - 3N + 2)$ for some integer $p$. Again, this nicely matches the infrared anomaly $\nu = \pm(N+2)/2$ for $N/2$ even.

### 3.8.3 Symplectic group.

The symplectic group $\mathrm{Sp}(N)$ is defined as the set of $\mathrm{SU}(2N)$ matrices such that $g\Omega g^t = \Omega$, where $\Omega$ is the canonical anti-symmetric matrix $\Omega = \mathrm{diag}(i\sigma_y, \cdots, i\sigma_y)$. It corresponds to the compact form of the Cartan series $C_N$. The alternative notation $\mathrm{USp}(2N)$ is also sometimes used for this group.

**Fundamental quarks.** Consider $\mathrm{Sp}(N)$ plus $N_F$ fundamental quarks. The index is $T(R) = N_F$ and therefore $\nu$ is defined only when $N_F$ is even. The most general matrix with $R(g^2) = 1$ is $g = -\mathbf{1}_{2p} \oplus \mathbf{1}_{2N-2p}$ for some integer $p$. Evaluating the character,

$$\nu = 2N_F \operatorname{tr}(g) = 4N_F(N - 2p) \tag{3.8.13}$$

Note that $N_F$ is even and therefore $\nu \equiv 4N_F N \mod 16$, the $p$-dependence drops out.

The low energy effective description is postulated to be a non-linear sigma model with a Wess-Zumino term [29]; the ’t Hooft anomaly of such a system is not known, and therefore we cannot check for matching.

**Adjoint quarks.** Consider $\mathrm{Sp}(N)$ plus one adjoint quark. The index is $T(R) = N + 1$ and therefore $\nu$ is defined only when $N$ is odd. The most general matrix with $R(g^2) = 1$ is either $g = -\mathbf{1}_{2p} \oplus \mathbf{1}_{2N-2p}$ for some integer $p$, or $g = (-\mathbf{1}_N \oplus \mathbf{1}_N)i$. Evaluating the character (cf. (B.2.8)),

$$\nu = \frac{1}{2}(\operatorname{tr}(g^2) + \operatorname{tr}(g)^2) \equiv \begin{cases} \frac{1}{2}(2N + (2N - 4p)^2) \\ -N \end{cases} \tag{3.8.14}$$

for the two families of matrices above.

Simplifying the expressions above, we get the following anomalies:

$$\nu = \begin{cases} -1, 3 & N \equiv 1 \mod 16 & -5, 7 & N \equiv 9 \mod 16 \\ -3, 5 & N \equiv 3 \mod 16 & -3, 5 & N \equiv 11 \mod 16 \\ -5, 7 & N \equiv 5 \mod 16 & -1, 3 & N \equiv 13 \mod 16 \\ -7 & N \equiv 7 \mod 16 & 1 & N \equiv 15 \mod 16 \end{cases} \tag{3.8.15}$$

The low energy effective description is postulated to be the TQFT $\mathrm{Sp}((N + 1)/2)_{(N+1)/2}$ plus a decoupled Majorana fermion [30]. The former has anomaly $\nu = \pm(N + 1)$ while the latter has anomaly $\nu = +1$. Adding up these contributions, the infrared phase has anomaly $\nu = \pm(N + 1) + 1$, which nicely matches the ultraviolet computation above.

**Anti-symmetric quarks.** Consider $\mathrm{Sp}(N)$ plus one anti-symmetric quark. The index is $T(R) = N - 1$ and therefore $\nu$ is defined only when $N$ is odd. The most general matrix with

$R(g^2)=1$ is either $g=-\mathbf{1}_{2p}\oplus\mathbf{1}_{2N-2p}$ for some integer $p$, or $g=(-\mathbf{1}_N\oplus\mathbf{1}_N)i$. Evaluating the character (cf. (B.2.8)),

$$\nu=\frac{1}{2}(-\operatorname{tr}(g^2)+\operatorname{tr}(g)^2)-1\equiv\begin{cases}\frac{1}{2}(-2N+(2N-4p)^2)-1\\ N-1\end{cases}\tag{3.8.16}$$

for the two families of matrices above.

Simplifying the expressions above, we get the following anomalies:

$$\nu=\begin{cases}0 & N\equiv 1 \mod 16 & 8 & N\equiv\ 9 \mod 16\\ \pm 2 & N\equiv 3 \mod 16 & \pm 6 & N\equiv 11 \mod 16\\ \pm 4 & N\equiv 5 \mod 16 & \pm 4 & N\equiv 13 \mod 16\\ \pm 6 & N\equiv 7 \mod 16 & \pm 2 & N\equiv 15 \mod 16\end{cases}\tag{3.8.17}$$

The low energy effective description is postulated to be the TQFT $\mathrm{Sp}((N-1)/2)_{(N-1)/2}$ [30]. This theory has anomaly $\nu=\pm(N-1)$, which nicely matches the ultraviolet computation above.

# Chapter 4

# Symmetries of Abelian Chern-Simons Theories and Arithmetic.

**Authorship.** The content of this chapter is reproduced almost verbatim from the paper [4] written in collaboration with Jaume Gomis.

**Abstract.** We determine the unitary and anti-unitary symmetries of arbitrary abelian Chern-Simons theories. The symmetries depend sensitively on the arithmetic properties (e.g. prime factorization) of the matrix of Chern-Simons levels, revealing interesting connections with number theory. We give a complete characterization of the symmetries of abelian topological field theories and along the way find many theories that are non-trivially time-reversal invariant, including $\mathrm{U}(1)_k$ Chern-Simons theory and $(\mathbb{Z}_k)_\ell$ gauge theories. For example, we prove that $\mathrm{U}(1)_k$ Chern-Simons theory is time-reversal invariant if and only if $-1$ is a quadratic residue modulo $k$, which happens if and only if all the prime factors of $k$ are Pythagorean (i.e., of the form $4n+1$), or Pythagorean with a single additional factor of 2. Many distinct non-abelian finite symmetry groups are found.

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## 4.1 Introduction and Summary

Symmetries play a pivotal role in our description of nature. In classical physics symmetries generate solutions of the equations of motion and in quantum mechanics symmetries imply selection rules and constrain physical observables. 't Hooft anomalies for global symmetries, being renormalization-group invariant, provide powerful nonperturbative constraints on the dynamics. By a classic result of Wigner, symmetries in quantum mechanics are implemented in the Hilbert space either by unitary or anti-unitary operators, and the corresponding transformations are linear and anti-linear, respectively.

Invariance of the classical action under a transformation $g$ imposes nontrivial constraints on the correlation functions of the theory. These are encapsulated in Ward identities. Invariance of the action under a transformation $g$ is a sufficient condition for $g$ to be a symmetry. However, this is not necessary. A transformation $g$ that does not leave the action $S$ invariant

$$g \cdot S \neq S \tag{4.1.1}$$

is nevertheless a symmetry of the quantum theory if it obeys the Ward identities

$$\langle g \cdot \mathcal{O}_1 \cdots g \cdot \mathcal{O}_m \rangle = \begin{cases} \langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle & g \text{ unitary} \\ \langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle^* & g \text{ anti-unitary} \end{cases}, \tag{4.1.2}$$

where $*$ implements complex conjugation. We shall refer to such non-Lagrangian symmetries as *quantum symmetries*. Naturally, determining whether a theory has a quantum symmetry is nontrivial. In this work we characterize all the symmetries, quantum or otherwise, of abelian Chern-Simons theories.

Chern-Simons theories are ubiquitous in physics and mathematics. They arise as the emergent infrared description of gapped, quantum phases of matter such as the integer and fractional quantum Hall effect, quantum spin liquids and analogs of topological insulators and superconductors (see e.g [240, 241]). Chern-Simons theories capture the nonperturbative infrared dynamics of $2+1$ dimensional gauge theories with massless fermions [3, 30, 113, 127–129, 237], and describe the low-energy dynamics of domain walls connecting vacua of $3+1$ dimensional gauge theories [3, 40, 41, 46]. Chern-Simons theory, a topological quantum field theory (TQFT), has also found beautiful and profound applications in mathematics, starting with Witten's work [22] on the topological invariants of knots and three-manifolds.

In this paper we give a complete description of all the unitary and anti-unitary symmetries of abelian Chern-Simons theories, the simplest incarnation being $\mathrm{U}(1)_k$ Chern-Simons theory, described by the Lagrangian

$$\mathcal{L} = \frac{k}{4\pi} a\, \mathrm{d}a\,, \tag{4.1.3}$$

where $a$ is a U(1) gauge field and the coupling constant is quantized, $k \in \mathbb{Z}$. More generally, an arbitrary abelian TQFT can be described by a collection of such fields coupled via an integral symmetric matrix $K$ with Lagrangian

$$\mathcal{L} = \frac{1}{4\pi} a^t K \mathrm{d}a\,, \tag{4.1.4}$$

where $a^t = (a_1, \ldots, a_n)$. These theories have been studied intensely and enjoy a myriad of applications. In spite of this, we unearth a rich structure of symmetries in these theories, which depends on the arithmetic properties of the Chern-Simons levels $K$, revealing interesting connections with number theory.

Symmetries in topological phases of matter have been at the forefront of recent developments at the intersection of condensed matter, particle physics, and mathematics. These gapped phases are encoded by emergent TQFTs. Gapped phases with no topological order (no nontrivial anyons) and protected by symmetries describe SPT phases (see e.g. [52, 242–245]) while phases with topological order (with nontrivial anyons) and enriched by symmetries give rise to the so-called SET phases (see e.g. [246–250]). Symmetries and 't Hooft anomalies of TQFTs have recently played a key role in understanding the nonperturbative infrared dynamics of gauge theories [3, 30, 113, 127–129, 237]. Despite a lot of work, little is concretely known about the symmetries of TQFTs. Here we tackle this problem for abelian TQFTs.

For the reader's convenience we summarize here a sample of our main results:

- $\mathrm{U}(1)_k$ is a time-reversal invariant spin TQFT,[88] that is, it admits an anti-unitary symmetry, if and only if $-1$ is a quadratic residue modulo $k$ (cf. proposition 4.3.2). Equivalently:

  $$U(1)_{+k} \longleftrightarrow U(1)_{-k} \qquad \Longleftrightarrow \qquad q^2 = -1 \mod k \quad \text{for some } q \in \mathbb{Z}. \tag{4.1.5}$$

  Therefore, $\mathrm{U}(1)_k$ Chern-Simons theory is time-reversal invariant if and only if

  $$k \in \mathbb{T} := \{k \in \mathbb{Z} \,|\, kp - q^2 = 1 \quad \text{for some } p, q \in \mathbb{Z}\}\,. \tag{4.1.6}$$

  This result can also be stated as $\mathrm{U}(1)_k$ being dual to $\mathrm{U}(1)_{-k}$ when $k \in \mathbb{T}$, which we denote by $\mathrm{U}(1)_{+k} \longleftrightarrow \mathrm{U}(1)_{-k}$. The integer $k$ is in $\mathbb{T}$ if and only if all its prime factors are Pythagorean (i.e., congruent to 1 modulo 4), or Pythagorean with a single factor of 2. Any time-reversal symmetry is of order 4, except for $k = 1, 2$, when it is of order 2 (cf. proposition 4.3.3).

[88] If $k$ is odd, $\mathrm{U}(1)_k$ is a spin TQFT. For $k$ even it is bosonic but can be turned into a spin TQFT by tensoring with a transparent fermion $\{\mathbf{1}, \psi\}$. See section 4.2 for details.

The set of time-reversal invariant $\mathrm{U}(1)_k$ Chern-Simons theories includes the subset $k \in \mathbb{P} := \{k \in \mathbb{Z} \,|\, kp^2 - q^2 = 1 \quad \text{for some } p, q \in \mathbb{Z}\} \subset \mathbb{T}$. The set $\mathbb{P}$ corresponds to those values of the level for which the (negative) Pell equation is solvable, which was shown by Witten [251, 252] to lead to time-reversal invariance.

We prove that the time-reversal symmetry is a quantum symmetry if and only if $k \in \mathbb{T}\backslash\mathbb{P}$ (cf. proposition 4.3.6). By studying the time-reversal invariance of $\mathrm{U}(1)_k \times \mathrm{U}(1)_{k'}$ we obtain an interesting number-theoretic conjecture, to wit, $k \in \mathbb{T}$ if and only if there exist some $k' \in \mathbb{P}$ such that $kk' \in \mathbb{P}$. We argue that this conjecture follows from a well-known conjecture by Hardy-Littlewood (cf. conjecture 4.7.1).

- All the unitary symmetries of $\mathrm{U}(1)_k$ are of order 2, and the number of such symmetries depends on the number of distinct prime factors of $k$, usually denoted by $\omega(k)$. More precisely, the group of unitary symmetries of $\mathrm{U}(1)_k$ is (cf. proposition 4.3.10)

$$(\mathbb{Z}_2)^{\varpi(k)}\,, \qquad \varpi(k) := \begin{cases} \omega(k) & k \text{ odd} \\ \omega(k/2) & k \text{ even.} \end{cases} \tag{4.1.7}$$

  When $\mathrm{U}(1)_k$ with $k$ even is upgraded to an spin TQFT by considering $\mathrm{U}(1)_k \times \{\mathbf{1}, \psi\}$, an additional factor of $\mathbb{Z}_2$ appears when $k$ is a multiple of 8. All but one factor of $\mathbb{Z}_2$ in (4.1.7), which corresponds to charge conjugation, are quantum symmetries. When $k \in \mathbb{T}$, the total group of symmetries is the central product of its unitary subgroup and $\mathbb{Z}_4$.

- The unitary and anti-unitary symmetries of $\mathrm{U}(1)^n$ Chern-Simons theory with matrix of levels $K$ correspond to the integral-valued matrices $Q$, invertible modulo $K$, that solve (cf. proposition 4.4.4)

$$\begin{aligned} \text{unitary:} \qquad & Q^t K^{-1} Q - K^{-1} = P \\ \text{anti-unitary:} \qquad & Q^t K^{-1} Q + K^{-1} = P \end{aligned} \tag{4.1.8}$$

  for some integral-valued matrix $P$. While the first equation always admits solutions, the second one need not, and only when there is a solution is the theory time-reversal invariant. The group of symmetries is finite and generically non-abelian. A given symmetry is quantum if and only if $P \neq 0$ for all the $Q$'s that implement it.

- Two abelian Chern-Simons theories described by matrices $K_1, K_2$ (not necessarily of the same dimension) are dual if and only if there exist suitable matrices $Q, P$ such that

$$Q^t K_1^{-1} Q - K_2^{-1} = P \tag{4.1.9}$$

  (see section 4.4.2 for the precise formulation and the conditions on $Q, P$). In this sense, the unitary symmetries of $K$ correspond to the self-dualities $K \leftrightarrow K$, and the anti-unitary symmetries to dualities $K \leftrightarrow -K$.

- The twisted gauge theory $(\mathbb{Z}_{k_1})_{k_2}$ (also known as $\mathbb{Z}_{k_1}$ Dijkgraaf-Witten theory [35] when $k_2$ is even, and which can be realized by the $\mathrm{U}(1)^2$ Chern-Simons theory with $K = \begin{pmatrix} 0 & k_1 \\ k_1 & k_2 \end{pmatrix}$ with $k_2 \in [0, 2k_1))$ is conjectured to be time-reversal invariant if and only if $k_2$ is proportional to $\mu(k_1)$ (cf. conjecture 4.4.2)

$$k_2 \propto \mu(k_1) \tag{4.1.10}$$

where $\mu(n)$ equals $n$ divided by all its Pythagorean prime factors (e.g. $\mu(10) = \frac{2\times 5}{5} = 2$). The conjecture has been verified for $k_1 \in [0, 200]$ and all $k_2$. We compute the explicit group of unitary and anti-unitary symmetries of $(\mathbb{Z}_{k_1})_{k_2}$ for small values of the levels; see table 4.1 for a sample. The time-reversal symmetry of $(\mathbb{Z}_{k_1})_{k_2}$ implies in particular a duality between abelian TQFTs

$$(\mathbb{Z}_{k_1})_{+k_2} \longleftrightarrow (\mathbb{Z}_{k_1})_{-k_2} \qquad \Longleftrightarrow \qquad k_2 \propto \mu(k_1)\,. \tag{4.1.11}$$

The theory $(\mathbb{Z}_k)_0$ has conjecturally $2^{\omega(k)}\phi(k)$ unitary transformations and as many anti-unitary ones (where $\phi(k)$ is the Euler totient function, which counts the number of integers $q \in [1, k)$ relatively prime to $k$). Among these symmetries, there is a unitary $\mathbb{Z}_2$ subgroup which is Lagrangian, and four anti-unitary Lagrangian symmetries (except for $k = 2$, which only has two). For $k > 2$ the group of symmetries is non-abelian (see 4.4.5 for the explicit conjecture), while for $k = 2$, the group of symmetries is $\mathbb{Z}_2^2$, with a $\mathbb{Z}_2$ unitary subgroup.

- The so-called "minimal abelian TQFT" $\mathcal{A}^{N,t}$ is proven to be time-reversal invariant invariant if and only if $t$ is proportional to $\mu(N)$ (cf. subsection 4.3.2)

$$t \propto \mu(N)\,. \tag{4.1.12}$$

These minimal theories have $N$ anyons with a $\mathbb{Z}_N$ fusion algebra, and their spin depends on the integer $t$.

TQFTs can also have a one-form symmetry group [32, 205] on top of the usual (zero-form) symmetry group that we study in this paper. The Wilson lines describing the worldline of anyons transform in representations of this group. The one-form symmetries of abelian Chern-Simons theories are well understood (see e.g. [253]). Given an abelian TQFT with an abelian Chern-Simons representation, the one-form symmetry group is $\mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \ldots \times \mathbb{Z}_{k_n}$, where $\{k_i\}$ are the Smith invariants of $K$ (cf. section 4.4). Interestingly, given a QFT with a zero-form symmetry group and a one-form symmetry group, these can combine into a nontrivial extension known as a 2-group (see e.g. [252, 254]). When a theory has a 2-group symmetry, the zero-form and one-form symmetries do not factorize; rather, they are mixed non-trivially. However, it is known that abelian TQFTs have a trivial 2-group of

| $k$ | $\mathrm{Aut}((\mathbb{Z}_k)_0)$ | $\mathrm{Aut}_U((\mathbb{Z}_k)_0)$ | $\mathrm{Aut}((\mathbb{Z}_k)_{\mu(k)})$ | $\mathrm{Aut}_U((\mathbb{Z}_k)_{\mu(k)})$ |
|---|---|---|---|---|
| 2 | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 |
| 3 | $D_8$ | $\mathbb{Z}_2^2$ | $D_8$ | $\mathbb{Z}_2^2$ |
| 4 | $D_8$ | $\mathbb{Z}_2^2$ | $D_8$ | $\mathbb{Z}_2^2$ |
| 5 | $\mathbb{Z}_4 \circ D_8$ | $D_8$ | $\mathbb{Z}_4$ | $\mathbb{Z}_2$ |
| 6 | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_2^3$ | $D_8$ | $\mathbb{Z}_2^2$ |
| 7 | $\mathbb{Z}_3 \rtimes D_8$ | $D_{12}$ | $\mathbb{Z}_3 \rtimes D_8$ | $D_{12}$ |
| 8 | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_2^3$ |
| 9 | $\mathbb{Z}_3 \rtimes D_8$ | $D_{12}$ | $\mathbb{Z}_3 \rtimes D_8$ | $D_{12}$ |
| 10 | $\mathbb{Z}_2 \times \mathbb{Z}_4 \circ D_8$ | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_4$ | $\mathbb{Z}_2$ |
| 11 | $\mathbb{Z}_5 \rtimes D_8$ | $D_{20}$ | $\mathbb{Z}_5 \rtimes D_8$ | $D_{20}$ |
| 12 | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\mathbb{Z}_2^4$ | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\mathbb{Z}_2^4$ |

Table 4.1: The group of symmetries of $(\mathbb{Z}_{k_1})_{k_2}$, denoted by $\mathrm{Aut}(\star)$, and its unitary subgroup $\mathrm{Aut}_U(\star)$, for $k_1 \in [0, 12]$ and $k_2 = 0, \mu(k_1)$. For $k_2 \not\propto \mu(k_1)$ there are no anti-unitary symmetries. See table 4.2 for the group of symmetries up to $k_1 = 27$. (See section 4.6 for basic definitions).

symmetries [252, 255, 256]: the zero-form and one-form symmetries factorize, and since the one-form symmetries are completely understood, what remains are the zero-form symmetries, which is the problem we address in this paper. Furthermore, since the 2-group in an abelian TQFT is trivial, the zero-form and one-form 't Hooft anomalies are well defined and can be classified using cohomology and cobordism groups [55, 58, 59, 69, 71, 257], and "anomaly indicators" detecting the 't Hooft anomalies (see e.g. [63, 78]) can be investigated. These anomaly indicators – which are the partition function evaluated on the generators of the corresponding cobordism groups, and expressed in terms of the modular data of the TQFT (see below) – are only known for a handful of symmetry groups.

The plan for the remainder of the chapter is as follows. In section 4.2 we describe the general paradigm of symmetries in topological quantum field theories, and the simplifications that occur for abelian TQFTs. In section 4.3 we completely describe all the symmetries for the most characteristic abelian system: $\mathrm{U}(1)_k$ Chern-Simons theory. In section 4.4 we generalize the analysis to arbitrary abelian TQFTs, by realizing them as $\mathrm{U}(1)^n$ Chern-Simons theories. We prove several results, and make a number of conjectures. In section 4.5 we work out a couple dozen examples in some detail, so as to illustrate the general formalism. Finally, we summarize definitions and notations in section 4.6 and leave some proofs and further results to section 4.7.

## 4.2 TQFTs and Symmetry

Before delving into the study of the symmetries of abelian Chern-Simons theories we describe how symmetries are realized in a TQFT in $2+1$ dimensions. We informally review the data defining a TQFT and how, in an abelian TQFT, it is completely fixed in terms of most elementary data, to wit, the anyon fusion algebra and the anyon spins. We then proceed with the physical and mathematical characterization of a symmetry in a TQFT. More details and mathematical elaborations can be found in the literature [38, 249, 258–261].

A TQFT can be understood as a finite collection of anyons – particles with fractional statistics – belonging to an anyon set $\mathcal{A}$ endowed with the following additional data:

- **Fusion:** A commutative, associative product $\times \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ describing the fusion of anyons $a, b \in \mathcal{A}$ (see figure 4.1):

$$a \times b = \sum_{c \in \mathcal{A}} N_{ab}{}^{c}\, c\,, \tag{4.2.1}$$

  where $N_{ab}{}^{c} \in \mathbb{Z}_{\geq 0}$ are the so-called *fusion coefficients*. We denote the trivial anyon by $\mathbf{1}$.

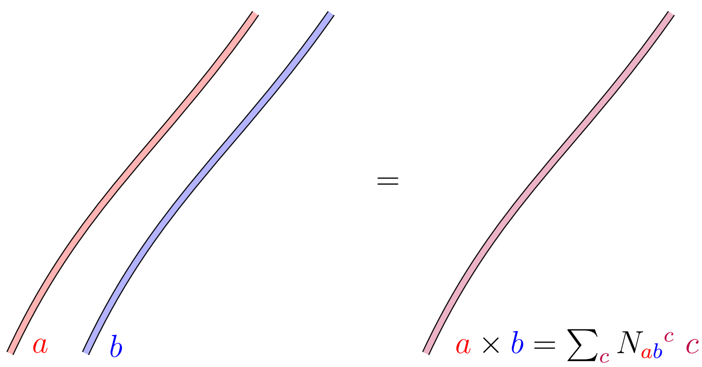



Figure 4.1: Fusion of anyons: two unbraided lines with labels $a, b$ can be replaced by one with label $\sum_c N_{ab}{}^{c}\, c$.

- **Topological spin:** A map $\theta \colon \mathcal{A} \to \mathrm{U}(1)$. The topological spin determines the anyonic character of an anyon. One usually writes $\theta(a) =: \exp(2\pi i h_a)$, where $h_a \colon \mathcal{A} \to \mathbb{Q}/\mathbb{Z}$ is the spin of $a$. The topological spin controls the framing anomaly of a knot (the dependence of observables on the choice of the homotopy class of a normal vector field, see figure 4.2).

- ***S*- and *T*-matrices:** A representation of the modular group. The $S$-matrix determines the braiding phase $B \colon \mathcal{A} \times \mathcal{A} \to \mathrm{U}(1)$ between anyons (see figure 4.3)

$$B(a, b) = \frac{S_{ab}}{S_{\mathbf{1}b}}\,, \tag{4.2.2}$$

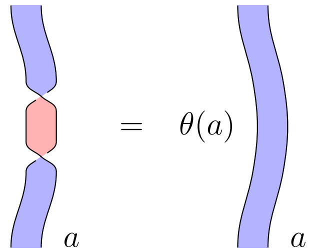



Figure 4.2: Topological spin: anyons are to be thought of as ribbons rather than knots. Observables depend on the twisting thereof, through their spin.

while $T_{ab} = \theta_a e^{-2\pi i c/24}\delta_{ab}$, where $c$ is the *chiral central charge* of the TQFT, which controls the framing anomaly (the dependence of observables on the 2-framing of the manifold).

- ***F*- and *R*-symbols:** The associator and braiding isomorphism, encoding the fusion of multiple anyons and their half-braiding. This data is defined modulo local, redundant isomorphisms (gauge transformations) $U$ defined on fusion vector spaces. The gauge-transformed data, which we denote by $UF$ and $UR$, is physically equivalent to $F$ and $R$, and define the same TQFT.


Figure 4.3: Braiding of anyons: if at least one of $a, b$ is abelian, then the two lines may be unbraided, a process that generates a phase $B(a,b) \in \mathrm{U}(1)$.

This data is subject to nontrivial consistency conditions, known as the Moore-Seiberg relations, which include the hexagon and pentagon relations involving the $F$- and $R$-symbols. These relations imply that some of the data above is actually redundant; for example, the topological spin $\theta$ is a gauge invariant combination of the $F$- and $R$-symbols. The TQFT data defines a modular tensor category. This data can be used to compute an arbitrary correlation function of the TQFT (cf. (4.1.2)).

An anyon $a$ is said to be *abelian* if the fusion of $a$ with an arbitrary anyon $b$ contains a single anyon $c = c(a, b)$, i.e.

$$a \times b = c \qquad\qquad \forall b \in \mathcal{A}\,. \tag{4.2.3}$$

In terms of the fusion coefficients (4.2.1), $a$ is abelian if for any $b$ the sum $\sum_{c\in\mathcal{A}} N_{ab}{}^{c}$ equals 1. An abelian anyon $a \in \mathcal{A}$ has a unique inverse $\bar{a} \in \mathcal{A}$ such that $a \times \bar{a} = \mathbf{1}$, and therefore abelian anyons form a finite abelian group, the one-form symmetry group of the TQFT [32].

An abelian TQFT is a TQFT in which all anyons in $\mathcal{A}$ are abelian. Therefore, in an abelian TQFT the anyon fusion algebra defines a finite abelian group, which we also denote by $\mathcal{A}$. Remarkably, an abelian TQFT is completely determined by the group $\mathcal{A}$ encoding the fusion of anyons, and by the topological spin $\theta\colon \mathcal{A} \to \mathrm{U}(1)$ of the anyons, which is a quadratic, homogeneous function on $\mathcal{A}$ [256, 262–264].[89] The entire TQFT data can be reconstructed from $\mathcal{A}$ and such a $\theta$.[90] The braiding phase of the abelian TQFT with fusion $\mathcal{A}$ and spin $\theta$ takes the form

$$B(a, b) = \frac{\theta(a \times b)}{\theta(a)\theta(b)}\,, \qquad a, b \in \mathcal{A}\,, \tag{4.2.4}$$

while the corresponding $S$-matrix is

$$S(a, b) = \frac{B(a, b)}{\sqrt{|\mathcal{A}|}}\,. \tag{4.2.5}$$

Importantly, given $(\mathcal{A}, \theta)$ there is a unique equivalence class of $F$ and $R$ symbols, and therefore a unique TQFT with that $(\mathcal{A}, \theta)$. Summarizing, in an abelian TQFT the entire theory is completely fixed in terms of $(\mathcal{A}, \theta)$. This statement is *not* true in a generic non-abelian TQFT, which is what makes the abelian case more tractable.

The discussion above applies as stated for a bosonic TQFT, a theory that does not require specifying a spin structure on the three-manifold where it is defined. Many interesting TQFTs, including abelian Chern-Simons theories, do require a choice of a spin structure to be defined. Such TQFTs are known as *spin* TQFTs. In a spin TQFT there is a distinguished abelian anyon $\psi$ with topological spin $\theta(\psi) = -1$ and trivial braiding with all other anyons. This implies that $\psi$ squares to the trivial anyon, i.e. $\psi \times \psi = \mathbf{1}$, and that $\theta(a \times \psi) = -\theta(a)$ for all $a \in \mathcal{A}$. In other words, a spin TQFT has a local (spin 1/2) fermion, which endows the data above with a $\mathbb{Z}_2$-grading (i.e., anyons come in pairs $\{a, a \times \psi\}$).

[89] $\theta$ is a quadratic function if the symmetric form in (4.2.4) is bilinear, i.e. $B(a \times b, c) = B(a, c)B(b, c)$. Homogeneity means that $\theta(a^n) = \theta(a)^{n^2}$ for any $n \in \mathbb{Z}$, which implies that $\theta(\mathbf{1}) = 1$.

[90] The central charge is determined by $(\mathcal{A}, \theta)$ only modulo 8. This indeterminacy can be understood as coming from the fact that one may always tensor by an even unimodular lattice, which has no lines, but may add central charge; the minimal such lattice is $E_8$, which has signature 8. Some more refined observables (see e.g. [262, 264]) are sensitive to the actual value of $c$, and not only to it modulo 8. If we are interested in such observables, the TQFT data should be taken as $(\mathcal{A}, \theta, c)$ rather than just $(\mathcal{A}, \theta)$. This will not play a major role in this work.

Any abelian TQFT, bosonic or spin, admits a representation as an abelian Chern-Simons theory [33, 256, 262, 264, 265], and is completely determined by $(\mathcal{A}, \theta)$. Therefore, in spite that a complete and universally accepted axiomatization of a spin TQFT from a categorical point of view is lacking, the abelian Chern-Simons realization of the TQFT and its datum $(\mathcal{A}, \theta)$ suffice to determine the symmetries of spin abelian TQFTs (we also provide path integral arguments to exhibit the symmetries of abelian Chern-Simons theories that do not rely on the precise categorical characterization of spin TQFTs).

The symmetries of a TQFT are, by definition, the automorphisms of its data [249]. An automorphism $g$ of a TQFT is a permutation of the anyons $g\colon \mathcal{A} \to \mathcal{A}$

$$a \mapsto g(a) \tag{4.2.6}$$

that preserves the fusion algebra $\mathcal{A}$

$$g(a \times b) = g(a) \times g(b) \qquad \Longleftrightarrow \qquad {N_{g(a)g(b)}}^{g(c)} = {N_{ab}}^{c}\,. \tag{4.2.7}$$

If the symmetry of the TQFT is unitary it must preserve the data modulo gauge transformations

$$\theta(g(a)) = \theta(a)\,, \qquad S_{g(a)g(b)} = S_{ab}\,, \qquad g \cdot F = UF \qquad g \cdot R = UR\,, \tag{4.2.8}$$

while if the symmetry is anti-unitary it preserves the data modulo gauge transformations, up to complex conjugation

$$\theta(g(a)) = \theta(a)^* \qquad S_{g(a)g(b)} = S^*_{ab} \qquad g \cdot F = UF^*\,, \qquad g \cdot R = UR^*\,. \tag{4.2.9}$$

Despite this explicit characterization, little is known about the actual symmetries of TQFTs. By contrast, the one-form symmetries of a TQFT are completely understood; they are determined by the abelian anyons and their fusion. Henceforth, when we discuss symmetries we refer to usual (zero-form) symmetries.

As reviewed above, in an abelian TQFT the entire data is completely determined by the abelian group $\mathcal{A}$ encoding the fusion algebra and the topological spin $\theta$. A necessary condition for the transformation $g$ to a symmetry of an abelian TQFT is that $g\colon \mathcal{A} \to \mathcal{A}$ is an automorphism of the finite group $\mathcal{A}$

$$g(a \times b) = g(a) \times g(b)\,. \tag{4.2.10}$$

The set of automorphisms of $\mathcal{A}$, denoted by $\mathrm{Aut}(\mathcal{A})$, is a finite, generically nonabelian group. An automorphism $g$ of $\mathcal{A}$ lifts to a unitary symmetry of the abelian TQFT if and only if

$$\theta(g(a)) = \theta(a)\,, \tag{4.2.11}$$

and to an anti-unitary symmetry if and only if

$$\theta(g(a)) = \theta(a)^*\,. \tag{4.2.12}$$

If such an automorphism $g$ exists, it is guaranteed that the entire data of the abelian TQFT is preserved and $g$ is a symmetry. In other words, the group of symmetries of an abelian TQFT is the subgroup of $\mathrm{Aut}(\mathcal{A})$ that preserves the topological spins (up to complex conjugation for anti-unitary symmetries). We introduce the following notation for this group:

**Definition 4.2.1** Given an abelian TQFT, we let $\mathrm{Aut}(\mathcal{A},\theta) \subseteq \mathrm{Aut}(\mathcal{A})$ denote the group of all symmetries, and $\mathrm{Aut}_U(\mathcal{A},\theta) \subseteq \mathrm{Aut}(\mathcal{A},\theta)$ the subgroup of unitary symmetries.

The main goal of this work is to study the object $\mathrm{Aut}(\mathcal{A},\theta)$. We determine it explicitly in the case of $\mathrm{U}(1)_k$, and give a complete characterization thereof for arbitrary abelian theories. We will also work out a few illustrative examples in some detail.

## 4.3 $\mathrm{U}(1)_k$ Chern-Simons

We begin by reviewing Chern-Simons theory with gauge group $\mathrm{U}(1)$. The generalization to the gauge group $\mathrm{U}(1)^n$ is the content of section 4.4.

The Lagrangian of $\mathrm{U}(1)_k$ Chern-Simons theory is

$$\mathcal{L} = \frac{k}{4\pi} a\,\mathrm{d}a\,, \tag{4.3.1}$$

where $a$ is a $\mathrm{U}(1)$ gauge gauge field and the coupling $k \in \mathbb{Z}$ is quantized. Being topological, the theory can be defined on an arbitrary (oriented, framed) three-manifold, perhaps with a choice of spin structure depending on the parity of $k$. The equations of motion are

$$\mathrm{d}a = 0 \tag{4.3.2}$$

and the classical field configurations are flat connections.

The gauge invariant operators in this theory are the Wilson lines

$$W_\alpha(\gamma) := \exp\left[i\alpha\int_\gamma a\right], \qquad \alpha \in \mathbb{Z}\,. \tag{4.3.3}$$

Physically, $W_\alpha$ describes the worldline of an anyon $\alpha$ with topological spin

$$\theta(\alpha) = e^{2\pi i h_\alpha}\,, \qquad h_\alpha = \frac{\alpha^2}{2k}\,. \tag{4.3.4}$$

The spin of an anyon $h_\alpha$ is only well-defined modulo an integer, because it cannot be distinguished from an anyon enriched with a soft $a$-photon, which has spin $h = 1$. If we introduce a background electromagnetic field, the anyon $\alpha$ is seen to carry a fractional charge given by $\alpha/k$, as follows from the coupling $\frac{1}{2\pi} A\,\mathrm{d}a$.

The anyon fusion algebra is determined by the OPE of the corresponding Wilson lines: $\alpha \times \beta = \alpha + \beta$. The braiding phase acquired by an anyon $\alpha$ circumnavigating around an anyon $\beta$ is

$$B(\alpha,\beta) = \frac{\theta(\alpha \times \beta)}{\theta(\alpha)\theta(\beta)} = \mathrm{e}^{2\pi i \, \frac{\alpha\beta}{k}} \,. \tag{4.3.5}$$

It follows from (4.3.5) and (4.3.4) that the anyon $\alpha = k$ has trivial braiding with respect to all other anyons, and has spin $h = 0 \mod 1$ for $k$ even and spin $h = 1/2 \mod 1$ for $k$ odd. Therefore $\mathrm{U}(1)_k$ is a spin TQFT for odd $k$, and a bosonic TQFT for even $k$. The former describes, for example, the fractional quantum Hall fluid at filling fraction $\nu = 1/k$, where the anyon $\alpha = k$ represents the microscopic electron.

Since the anyons $\alpha$ and $\alpha + k$ have indistinguishable braiding properties, and identical spins for $k$ even, and spins that differ by $1/2$ for $k$ odd, the lines of $\mathrm{U}(1)_k$ are subject to an equivalence relation: anyons related by a transparent bosonic anyon are to be identified. A bosonic theory can be made into a spin theory by tensoring with the trivial spin TQFT of a transparent fermion $\{\mathbf{1}, \psi\}$. We will often follow the convention of leaving this factor implicit when discussing spin TQFTs.

Summarizing, the anyon set and the fusion algebra of $\mathrm{U}(1)_k$ is:

- $\mathrm{U}(1)_k$, $k$ even: the theory has $k$ anyons labeled by $\alpha \in \{0, 1, \ldots, k-1\}$ and a $\mathcal{A} \cong \mathbb{Z}_k$ fusion algebra
$$\alpha \times \beta = \alpha + \beta \mod k \,. \tag{4.3.6}$$
The theory is bosonic and can be defined on an arbitrary three-manifold.

- $\mathrm{U}(1)_k$, $k$ odd: the theory has $2k$ anyons labeled by $\alpha \in \{0, 1, \ldots, 2k-1\}$ and a $\mathcal{A} \cong \mathbb{Z}_{2k}$ fusion algebra
$$\alpha \times \beta = \alpha + \beta \mod 2k \,. \tag{4.3.7}$$
It is a spin TQFT, as signalled by the presence of the transparent fermion $\alpha = k$.

- $\mathrm{U}(1)_k \times \{\mathbf{1}, \psi\}$, $k$ even: the theory has $2k$ anyons labeled by the pair $(\alpha, \beta)$, where $\alpha \in \{0, 1, \ldots, k-1\}$ and $\beta \in \{0, 1\}$, and the fusion algebra is $\mathcal{A} \cong \mathbb{Z}_k \times \mathbb{Z}_2$
$$(\alpha, \beta) \times (\alpha', \beta') = (\alpha + \alpha' \mod k, \ \beta + \beta' \mod 2) \,. \tag{4.3.8}$$
It is a spin TQFT by virtue of the tensoring with $\{\mathbf{1}, \psi\}$, where $\psi$ is represented by the Wilson line with charges $(0, 1)$.

We now proceed to determine the full set of symmetries of $\mathrm{U}(1)_k$ Chern-Simons theory.

### 4.3.1 Symmetries of $\mathrm{U}(1)_k$

We start with the manifest Lagrangian symmetries. $\mathrm{U}(1)_k$ with $k > 2$ has a $\mathbb{Z}_2$ unitary Lagrangian symmetry $\mathsf{C} : a \mapsto -a$, charge conjugation, under which $\mathcal{L} \mapsto \mathcal{L}$, and that acts on the anyons as

$$\mathsf{C}\colon \alpha \mapsto -\alpha\,. \tag{4.3.9}$$

The operation $\mathsf{C}$ is not a symmetry of $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$ because charge conjugation acts trivially on all the lines, since $1 = -1 \mod 2$.

Time-reversal is an anti-unitary transformation

$$\mathsf{T}\colon \quad \begin{cases} a_0(x^0) \mapsto +a_0(-x^0) \\ a_i(x^0) \mapsto -a_i(-x^0) \end{cases} \tag{4.3.10}$$

which acts on the Wilson lines as $\mathsf{T}\colon W_\alpha(\gamma) \mapsto W_\alpha(\mathsf{T}\gamma)$, where $\mathsf{T}\gamma$ denotes the time-reflected image of the curve $\gamma$. While $\mathsf{T}$ is a symmetry of the equations of motion (4.3.2), it does not leave the action invariant, i.e. $\mathcal{L} \mapsto -\mathcal{L}$. This transformation is not a quantum symmetry either since it does not obey the corresponding Ward identity (4.1.2). Therefore, if $\mathsf{T}$ is to be a symmetry of $\mathrm{U}(1)_k$, it must act non-trivially on the anyon labels:

$$\mathsf{T}\colon W_\alpha(\gamma) \mapsto W_{\mathsf{T}(\alpha)}(\mathsf{T}\gamma) \tag{4.3.11}$$

for some $\mathsf{T}\colon \mathcal{A} \to \mathcal{A}$.

In order to study the quantum symmetries of $\mathrm{U}(1)_k$ Chern-Simons theory we first need to understand the automorphisms of its fusion algebra $\mathcal{A}$. Indeed, as explained in section 4.2, a transformation $g$ is a symmetry of a TQFT if it is an automorphism of its data $(\mathcal{A}, \theta)$ which requires, first and foremost, that $g \in \mathrm{Aut}(\mathcal{A})$. As usual, any element of $\mathrm{Aut}(\mathcal{A})$ is completely determined by its action on the generators of $\mathcal{A}$. With this in mind, the automorphisms of the fusion algebra $\mathcal{A}$ of $\mathrm{U}(1)_k$ Chern-Simons theory are as follows:

- $\mathrm{U}(1)_k$, $k$ even. The most general endomorphism of $\mathcal{A} \cong \mathbb{Z}_k$ acts as $g : \alpha \mapsto q\alpha \mod k$, where $q := g(1) \in \mathcal{A}$ and $\alpha \in \{0, 1\, \ldots, k-1\}$. This lifts to an automorphism of $\mathbb{Z}_k$ if and only if $g$ maps a generator of $\mathbb{Z}_k$ into a generator of $\mathbb{Z}_k$. This requires $q$ to be relatively prime to $k$, i.e. $\gcd(q, k) = 1$:

  $$g\colon \alpha \mapsto q\alpha \mod k\,, \qquad \gcd(q,k) = 1\,. \tag{4.3.12}$$

  The number of automorphisms (and of generators) of $\mathbb{Z}_k$ is the number of totatives of $k$: the number of integers $1 \le q \le k$ such that $\gcd(q, k) = 1$. This number is counted by the Euler totient function $\phi(k)$. The automorphism group $\mathrm{Aut}(\mathbb{Z}_k)$ is the multiplicative group of integers modulo $k$, an abelian group often denoted as $\mathbb{Z}_k^\times$.

- $\mathrm{U}(1)_k$, $k$ odd. The most general endomorphism of $\mathcal{A} \cong \mathbb{Z}_{2k}$ acts as $g\colon \alpha \mapsto q\alpha \mod 2k$, where $q := g(1) \in \mathcal{A}$ and $\alpha \in \{0, 1, \ldots, 2k-1\}$. It is an automorphism if and only if $q$ is coprime to $2k$:
$$g\colon \alpha \mapsto q\alpha \mod 2k\,, \qquad \gcd(q, 2k) = 1\,. \tag{4.3.13}$$
The automorphisms automatically preserve the transparent fermion ($\alpha = k$) since $qk = k \mod 2k$ for $q$ odd. The number of automorphisms of $\mathbb{Z}_{2k}$ is the Euler totient function $\phi(2k) = \phi(k)$, the last equality by virtue of $k$ being odd. The automorphism group is $\mathrm{Aut}(\mathbb{Z}_{2k}) = \mathbb{Z}_{2k}^{\times}$.

- $\mathrm{U}(1)_k \times \{\mathbf{1}, \psi\}$, $k$ even. The most general endomorphism of $\mathcal{A} \cong \mathbb{Z}_k \times \mathbb{Z}_2$ acts as
$$g\colon \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \mapsto \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \quad \begin{matrix}\mod k\\ \mod 2\end{matrix}\,, \qquad \begin{matrix}a, b \in \mathbb{Z}_k\\ c, d \in \mathbb{Z}_2\end{matrix} \tag{4.3.14}$$
where $\alpha \in \{0, 1, \ldots, k-1\}$ and $\beta \in \{0, 1\}$. Such a map is an automorphism if and only if it is invertible ($\mathrm{mod}\ k, \mathrm{mod}\ 2$). The automorphism group of $\mathbb{Z}_k \times \mathbb{Z}_2$ does not admit as straightforward a description as in the previous cases, but its order is known: $4\phi(k)$ if $k/2$ is even, and $6\phi(k)$ if $k/2$ is odd [266, 267]. The automorphism group is generically non-abelian.

  Locality of the TQFT requires that the automorphism $g$ preserves the transparent fermion, $g(\psi) = \psi$, that is, it fixes the anyon $(0, 1)$. This implies that the candidate symmetries of $\mathrm{U}(1)_k \times \{\mathbf{1}, \psi\}$ with $k$ even are the automorphisms of $\mathbb{Z}_k \times \mathbb{Z}_2$ with $b = 0$ and $d = 1$. In order for the transformation to be invertible, one must have $\gcd(a, k) = 1$ or, if $k/2$ is odd, $\gcd(a, k/2) = 1$. The number of such transformations is $2\phi(k)$ and $3\phi(k)$ for $k/2$ even and odd, respectively.

This immediately shows that $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$ have no symmetries since $\mathrm{Aut}(\mathbb{Z}_2)$ is trivial, and indeed charge conjugation $\mathsf{C}$ acts trivially in these theories.

We have thus characterised all the automorphisms of $\mathcal{A}$. These are the candidate transformations to be a symmetry of the TQFT. They uplift to symmetries if they respect the topological spin of the lines (up to complex conjugation for anti-unitary symmetries). We turn to this question next.

**Anti-unitary Symmetries.**

We start by studying the anti-unitary symmetries of $\mathrm{U}(1)_k$ Chern-Simons theory. We already established that the canonical time-reversal transformation (4.3.10) is not a symmetry of $\mathrm{U}(1)_k$. Since the TQFT data of $\mathrm{U}(1)_k$ Chern-Simons theory is determined by the fusion algebra $\mathcal{A}$ and the topological spin $\theta$, an automorphism $\mathsf{T} \in \mathrm{Aut}(\mathcal{A})$ will lead to an anti-unitary symmetry if and only if
$$\theta(\mathsf{T}(\alpha)) = \theta(\mathsf{T}(\alpha))^* \quad \Longleftrightarrow \quad h_{\mathsf{T}(\alpha)} = -h_\alpha \mod 1\,. \tag{4.3.15}$$

This condition is not satisfied by every automorphism of $\mathcal{A}$. More importantly, depending on the value of $k$, there will be cases where there are no automorphisms at all that satisfy (4.3.15). This is precisely what happens for even $k$, when we regard $\mathrm{U}(1)_k$ as a bosonic theory[91]:

**Proposition 4.3.1** *The bosonic theory* $\mathrm{U}(1)_k$*, with* $k$ *even, is never time-reversal invariant.*

*Proof.* Consider the permutation $\mathsf{T}\colon \alpha \mapsto q\alpha$ for some $q \in [0, k)$. This operation satisfies $h_{\mathsf{T}(\alpha)} = -h_\alpha \mod 1$ if and only if

$$\frac{q^2\alpha^2}{2k} + \frac{\alpha^2}{2k} \in \mathbb{Z}\,. \tag{4.3.16}$$

If we take, for example, the fundamental line $\alpha = 1$, this requires $\frac{1+q^2}{2k}$ to be an integer. But $q$ must odd for $\mathsf{T}$ to be an automorphism, and so $1 + q^2 = 2 \mod 4$, which means that $\frac{1+q^2}{2k}$ cannot be an integer. □

We therefore see that the theory $\mathrm{U}(1)_k$ can only possibly be time-reversal invariant if we regard it as a spin TQFT. And even if we do so, there will still be some values of $k$ for which $\mathrm{U}(1)_k$ admits no time-reversal permutation at all. To see this, define the following:

**Definition 4.3.1** We let $\mathbb{T} \subset \mathbb{Z}$ be the set of integers $k$ such that $-1$ is a quadratic residue modulo $k$, i.e. $q^2 = -1 \mod k$ for some $q \in \mathbb{Z}$. In other words,

$$\mathbb{T} := \{k \in \mathbb{Z} \,|\, kp - q^2 = 1 \quad \text{for some } p, q \in \mathbb{Z}\}\,. \tag{4.3.17}$$

With this, we prove that

**Proposition 4.3.2** *The spin theory* $\mathrm{U}(1)_k$ *is time-reversal invariant if and only if* $k \in \mathbb{T}$*.*

*Proof.* We begin with the case of odd $k$, that is, $\mathrm{U}(1)_k$, where $\mathcal{A} \cong \mathbb{Z}_{2k}$. We shall look for the most general automorphism $\mathsf{T} \in \mathrm{Aut}(\mathcal{A})$ that satisfies (4.3.15). Any such operation is of the form

$$\mathsf{T}(\alpha) = q\alpha, \qquad q := \mathsf{T}(1) \in [0, 2k)\,. \tag{4.3.18}$$

If we impose that $h_{\mathsf{T}(1)} = -h_1 \mod 1$, we get $1 + q^2 = 2pk$ for some integer $p$. It is easy to show that this equation is solvable if and only if $k \in \mathbb{T}$. One direction is obvious; for the opposite direction, assume that $1 + \tilde{q}^2 = \tilde{p}k$. If $\tilde{p}$ is even, we are done; if it is odd, then we can set

$$q := \tilde{q} + k, \qquad p := \tilde{q} + \frac{\tilde{p} + k}{2} \tag{4.3.19}$$

which satisfy $1 + q^2 = 2pk$, as required (note that $\tilde{p} + k$ is even, and so $p \in \mathbb{Z}$).

[91]This result also follows from the fact that the central charge of $\mathrm{U}(1)_k$ is not proportional to 4.

Once we ensure the spin of the generator transforms properly under $\mathsf{T}$, it is easy to show that so do the rest of lines. Indeed,

$$h_{\mathsf{T}(\alpha)} = \frac{q^2\alpha^2}{2k} = \frac{(2pk-1)\alpha^2}{2k} = -\frac{\alpha^2}{2k} \mod 1\,, \tag{4.3.20}$$

where we have used that $1+q^2 = 2pk$.

Finally, it is also easy to show that *any* integer $q$ that solves $1+q^2 = 2pk$ will be a time-reversal operation. Indeed, $1+q^2 = 2pk$ implies that any common factor to $k$ and $q$ must divide 1, and so $\gcd(q,k) = 1$, which means that $\alpha \mapsto q\alpha$ is invertible.

We now move on to the even $k$ case, that is, $\mathrm{U}(1)_k \times \{\mathbf{1}, \psi\}$, where $\mathcal{A} \cong \mathbb{Z}_k \times \mathbb{Z}_2$, where the first factor is generated by the fundamental line $(1,0)$, and the second one by the transparent fermion $\psi = (0,1)$.

Any fusion endomorphism is fixed once we choose its action on the generators. In fact, the transparent fermion is the only spin $h = 1/2$ line that braids trivially to all other lines (because $\mathrm{U}(1)_k$ is bosonic), and thus the action of time-reversal on it is fixed to $\mathsf{T}(\psi) \equiv \psi$. Therefore, we only have freedom to choose how time-reversal acts on $(1,0)$. We write $\mathsf{T}(1,0) := (q_1, q_2)$ for a pair of integers $q_1, q_2$, where $q_1 \in \{0, 1, \dots, k-1\}$ and $q_2 \in \{0,1\}$.

Proposition 4.3.1 implies that $q_2 = 0$ is not possible. Therefore, the candidate anti-unitary transformation is $\mathsf{T}(1,0) \equiv (q,1)$ for some integer $q \in [0,k)$, and so the most general endomorphism is of the form

$$\mathsf{T}(\alpha,\beta) = (q\alpha, \alpha+\beta)\,. \tag{4.3.21}$$

We now insist that the spin of $(1,0)$ is mapped into its negative under time-reversal. Imposing that $h_1 = -h_{\mathsf{T}(1)\otimes\psi} \mod 1$ we get $1+q^2 = (2p-1)k$ for some integer $p$. Once again, it is easy to show that this equation is solvable if and only if $k \in \mathbb{T}$. One direction is obvious; for the opposite direction, assume that $1+\tilde{q}^2 = \tilde{p}k$. Then, upon reducing the equation modulo 4, it becomes clear that $\tilde{p}$ has to be odd, and so we can write $\tilde{p} = 2p-1$, as we wanted to show.

Once we ensure the spin of the generator transforms properly under $\mathsf{T}$, it is easy to show that so do the rest of lines. Indeed,

$$h_{\mathsf{T}(\alpha,\beta)} = \frac{q^2\alpha^2}{2k} + \frac{1}{2}(\alpha+\beta)^2 = \frac{(-1+k(2p-1))\alpha^2}{2k} + \frac{1}{2}(\alpha+\beta)^2\,, \tag{4.3.22}$$

where we have used $q^2 = -1 + k(2p-1)$. This is clearly equal to

$$h_{\mathsf{T}(\alpha,\beta)} = -\frac{\alpha^2}{2k} + \frac{1}{2}\beta^2 = -h_{(\alpha,\beta)} \mod 1 \tag{4.3.23}$$

as required.

Finally, it is also easy to show that *any* integer $q$ that solves $1+q^2 = (2p-1)k$ will be a time-reversal operation. Indeed, and as before, this equation can only be satisfied if

$\gcd(q,k) = \gcd(q,k/2) = 1$, and so $(\alpha,\beta) \mapsto (q\alpha, \alpha+\beta)$ is invertible (i.e. an automorphism of $\mathbb{Z}_k \times \mathbb{Z}_2$). $\square$

As we can see, the set $\mathbb{T} \subset \mathbb{Z}$ plays a key role in the study of the time-reversal properties of $\mathrm{U}(1)_k$ (and, as we shall see, of $\mathrm{U}(1)^n$). We therefore make a few remarks about this set:

- The first few solutions are $k = 1, 2, 5, 10, 13, 17, 25, 26, 29, 34, 37, 41, 50, 53, 58, 61, 65, 73, 74, 82, 85, 89, 97, \dots$.
- A given $k$ is in $\mathbb{T}$ if and only if it can be written as $k = a^2 + b^2$ for relatively prime $a, b \in \mathbb{Z}$ (see e.g. [268], theorem 3.21).
- Given the prime decomposition of $k$
$$k = 2^e \left[ \prod_{\pi = 1 \bmod 4} \pi^\alpha \right] \left[ \prod_{\pi = 3 \bmod 4} \pi^\beta \right], \tag{4.3.24}$$
$k \in \mathbb{T}$ if and only if $e \in \{0,1\}$ and $\beta_i \equiv 0$ (see e.g. [268], theorem 3.20). In other words, $k \in \mathbb{T}$ if and only if all its prime factors are Pythagorean, or Pythagorean with a single factor of 2. This implies, for example, that $\mathbb{T}^{\text{even}} = 2\mathbb{T}^{\text{odd}}$.
- The set $\mathbb{T}$ contains a special subset $\mathbb{P}$, defined as those integers $k$ for which the (negative) Pell equation is solvable:
$$\mathbb{P} := \{k \in \mathbb{Z} \,|\, kp^2 - q^2 = 1 \quad \text{for some } p, q \in \mathbb{Z}\}. \tag{4.3.25}$$
Unlike $\mathbb{T}$, the set $\mathbb{P}$ has no simple characterization in terms of the prime decomposition of $k$. See section 4.7 for some mode details about Pell numbers.
- The density of $\mathbb{T}$ is $\#\{k \in \mathbb{T} \mid k \leq x\} \sim x/\sqrt{\log x}$. It is conjectured that around 57% of the numbers in $\mathbb{T}$ are in $\mathbb{P}$ [269, 270].

If $k \in \mathbb{T}$, there exists an integer $q \in [0,k)$ such that $q^2 = -1 \mod k$. We explain in the section 4.7 how to construct $q$ explicitly.

We now go back to the theory $\mathrm{U}(1)_k$. We have the following:

**Proposition 4.3.3** *The time-reversal symmetry of* $\mathrm{U}(1)_k$ *is an order-four operation (except for* $k = 1, 2$*, where it is of order two).*

*Proof.* We shall prove that $\mathsf{T}^2 = \mathsf{C}$, where $\mathsf{C}\colon \alpha \mapsto -\alpha$ is the unitary $\mathbb{Z}_2$ charge conjugation symmetry (4.3.9). From this it follows that $\mathsf{T}^4 = 1$, and therefore $\mathsf{T}$ is an order-four operation (except for $k = 1, 2$, where $\mathsf{C}$ is trivial).[92]

[92] The symmetry algebra $\mathsf{T}^2 = \mathsf{C}$ can in principle be extended by fermion parity $(-1)^F$, which does not act on the Wilson lines. The full symmetry algebra is, therefore, either $\mathbb{Z}_4 \times \mathbb{Z}_2$ (corresponding to $\mathsf{T}^4 = 1$) or $\mathbb{Z}_8$ (corresponding to $\mathsf{T}^4 = (-1)^F$). Figuring out which of these options is realized requires determining how $(-1)^F$ acts on these theories, a subject that is beyond the scope of this chapter.

Showing that $\mathsf{T}^2 = \mathsf{C}$ is straightforward. If $k$ is odd, then

$$-\mathsf{T}^2(\alpha) = -q^2\alpha = (1-2pk)\alpha = \alpha \mod 2k\,. \tag{4.3.26}$$

Similarly, if $k$ is even, then

$$\begin{aligned} -\mathsf{T}^2(\alpha,\beta) &= (-q^2\alpha, (q+1)\alpha+\beta) \\ &= ((1-(2p-1)k)\alpha, (q+1)\alpha+\beta) \\ &= (\alpha,\beta) \mod (k,2) \end{aligned} \tag{4.3.27}$$

where we have used that $q$ is odd. ☐

We see that if $k \in \mathbb{T}$, then there exists some anti-unitary operation $\mathsf{T}$ which satisfies a $\mathbb{Z}_4$ algebra. That being said, there will be, in general, more than one such permutations, and therefore the time-reversal transformation is not unique. We have the following result:

**Proposition 4.3.4** *If* $\mathrm{U}(1)_k$ *is time-reversal invariant, there are* $2^{\varpi(k)}$ *different anti-unitary permutations, where* $\varpi(k)$ *denotes the number of distinct prime factors of* $k$ *for* $k$ *odd and of* $k/2$ *for* $k$ *even* (*cf.* (4.1.7)).

*Proof.* Indeed, there are as many permutations as there are solutions to $q^2 = -1+(2p-1)k$ with $q \in [0,k)$ for $k$ even, and to $q^2 = -1+2pk$ with $q \in [0,2k)$ for $k$ odd. We shall first show that this problem is equivalent to counting the solutions to $\tilde{q}^2 = -1 \mod k$:

- Consider the case with $k$ even. Then any solution to $\tilde{q}^2 = -1+\tilde{p}k$ must necessarily have $\tilde{p}$ odd (for otherwise we reach a contradiction upon reducing the equation modulo 4), and so we can write $(\tilde{q},\tilde{p}) = (q, 2p-1)$, which yields $q^2 = -1+(2p-1)k$, as required.

- We now consider the case with $k$ odd. We claim that the solutions to $q^2 = -1+2pk$ with $q \in [0,2k)$ can be put in a bijection with solutions to $\tilde{q}^2 = -1+\tilde{p}k$ with $\tilde{q} \in [0,k)$. First, assume we are given the set $\{\tilde{q} \in [0,k)\}$; we construct the set $\{q \in [0,2k)\}$ as follows: if $\tilde{q}$ is odd, then $\tilde{p}$ must be even, and so $(q,2p) = (\tilde{q},\tilde{p})$; on the other hand, if $\tilde{q}$ is even, then $\tilde{p}$ must be odd, and so $(q,2p) = (\tilde{q}+k, \tilde{p}+2\tilde{q}+k)$. Conversely, if we are given the set $\{q \in [0,2k)\}$, we write $(\tilde{q},\tilde{p}) = (q,p)$ if $q \in [0,k)$, and $(\tilde{q},\tilde{p}) = (q-k, p-2q+k)$ if $\tilde{q} \in [k,2k)$.

We thus see that we may reduce our problem to counting solutions to $q^2 = -1 \mod k$, both for $k$ even and odd. It is a well-known result that the number of solutions is precisely $2^{\varpi(k)}$, see for example theorem 6.3 in [271] (together with remark 6.2 therein). The intuition behind this result (and which can be generalised to any polynomial congruence) is the following. Any solution to $q^2 = -1 \mod k$ can be reconstructed uniquely from the solutions to $q_i^2 = -1 \mod \pi_i$, where $\pi_i$ are the prime factors of $k$. Each congruence $q_i^2 = -1 \mod \pi_i$ is solvable (because $\pi_i$ is Pythagorean), and it has two solutions $\pm q_i$ (and only two, as per

Lagrange's theorem, except for $\pi = 2$, where only solution is $q_i = 1$, inasmuch as $1 = -1 \mod 2$). As there are $\varpi(k)$ congruences, each having two solutions, the total number of solutions is $2^{\varpi(k)}$, as claimed. $\square$

For completeness, we mention that one can prove that $k \in \mathbb{T}$ is sufficient for time-reversal invariance using a path integral argument, which is quite similar to one in [251, 252] where it was used to show time-reversal invariance for $k \in \mathbb{P} \subset \mathbb{T}$. The argument is straightforward but it does not prove that the condition $k \in \mathbb{T}$ is also necessary.

**Proposition 4.3.5** *It follows from a path integral argument that when $k \in \mathbb{T}$ the theory $\mathrm{U}(1)_k$ is time-reversal invariant as a spin TQFT.*

*Proof.* Take two arbitrary integers $m, n$ with $m$ is odd and $n$ even, and such that

$$mn - q^2 = 1 \tag{4.3.28}$$

for some integer $q$ (which can easily seen to be odd). We shall prove that $\mathrm{U}(1)_m$ and $\mathrm{U}(1)_n \times \{\mathbf{1}, \psi\}$ are both time-reversal invariant.

Take the Lagrangian of $\mathrm{U}(1)_m \times \mathrm{U}(1)_{-n}$

$$4\pi\mathcal{L} = m\, a\, \mathrm{d}a - n\, b\, \mathrm{d}b \tag{4.3.29}$$

whose Wilson lines are of the form

$$\exp\left[i\alpha \int a + i\beta \int b\right], \qquad (\alpha, \beta) \in \mathbb{Z}_{2m} \times \mathbb{Z}_n\,. \tag{4.3.30}$$

Under the $\mathrm{GL}_2(\mathbb{Z})$ transformation

$$\mathsf{T}\colon \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} q & -n \\ m & -q \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \tag{4.3.31}$$

the Lagrangian becomes

$$\mathsf{T}\colon 4\pi\mathcal{L} \mapsto -m\, a\, \mathrm{d}a + n\, b\, \mathrm{d}b \equiv -4\pi\mathcal{L} \tag{4.3.32}$$

and the lines map according to

$$\mathsf{T}\colon (\alpha, \beta) \mapsto (q\alpha + m\beta, -n\alpha - q\beta)\,. \tag{4.3.33}$$

We therefore see that $\mathrm{U}(1)_m \times \mathrm{U}(1)_{-n} \longleftrightarrow \mathrm{U}(1)_{-m} \times \mathrm{U}(1)_n$, i.e., the product is time-reversal invariant. The explicit duality map is given by (4.3.33).

We now prove that $\mathrm{U}(1)_m$ is time-reversal invariant. To this end, we note that the theory above contains a sub-group of lines of the form $(\alpha, 0)$, which is isomorphic to $\mathrm{U}(1)_m$, with isomorphism $\alpha \leftrightarrow (\alpha, 0)$. Time-reversal restricts to a well-defined action on $\mathrm{U}(1)_m$, because

$$\mathsf{T}\colon (\alpha, 0) \mapsto (q\alpha, -n\alpha) \sim (q\alpha, 0) \tag{4.3.34}$$

where we have used the fact that $n$ is even.

We next prove that $\mathrm{U}(1)_n \times \{\mathbf{1},\psi\}$ is time-reversal invariant. To this end, we note that the theory above contains a sub-group of lines of the form $(0,\beta)$ and $(m,\beta)$, which is isomorphic to $\mathrm{U}(1)_n \times \{\mathbf{1},\psi\}$, with isomorphism $\beta\otimes\mathbf{1} \leftrightarrow (0,\beta)$ and $\beta\otimes\psi \leftrightarrow (m,\beta)$. Time-reversal restricts to a well-defined action on $\mathrm{U}(1)_n \times \{\mathbf{1},\psi\}$, because

$$\begin{aligned}
&\mathsf{T}\colon (\,0\,,\beta) \mapsto (m\beta,-q\beta)\\
&\mathsf{T}\colon (m,\beta) \mapsto (qm+m\beta,-nm-q\beta) \sim (m(1+\beta),-q\beta)
\end{aligned} \tag{4.3.35}$$

where we have used the fact that $n$ is even and $q$ is odd. This completes the proof. □

As a consistency check, we note that the action of time-reversal on the lines of $\mathrm{U}(1)_m$ is $\mathsf{T}(\alpha)=q\alpha$, and that on $\mathrm{U}(1)_n \times \{\mathbf{1},\psi\}$ is $\mathsf{T}(\beta\otimes\psi^\gamma)=q\beta\otimes\psi^{\beta+\gamma}$, with $\gamma=0,1$. This is precisely the same map we found in proposition 4.3.2.

One can couple the theory $\mathrm{U}(1)_k$ to electromagnetism by turning on a background $\mathrm{U}(1)_B$ connection. If $k\in\mathbb{T}$, then time-reversal remains a symmetry in the presence of this background field, but at the cost of introducing a Chern-Simons counterterm for the electromagnetic field, with fractional coefficient. This means that there is a mixed $\mathsf{T}-\mathrm{U}(1)_B$ 't Hooft anomaly,[93] and so the system can only be defined on the boundary of a $3+1$ manifold. Using the Lagrangian argument above, and following the same reasoning as in [111, 252], it is easy to prove that the anomaly is given by a $3+1$ dimensional topological term $\theta=2\pi/k$ for $\mathrm{U}(1)_B$.

**Remark 4.3.1** It is common that in theories that are symmetric under both time-reversal and charge conjugation, the operators $\mathsf{T}$ and $\mathsf{CT}$ constitute two separate $\mathbb{Z}_2$ symmetries, both of which represent suitable time-reversal operations. These two symmetries are independent: they have different anomalies, they may be affected by magnetic symmetries (if any), and may be interchanged under duality (see e.g. [34]). In our case, these two symmetries in fact combine into a single $\mathbb{Z}_4$ algebra, $\mathsf{T}^3=\mathsf{CT}$, and so they do not correspond to independent symmetries.

**Remark 4.3.2** It is interesting to note that we obtained $k\in\mathbb{T}$ as a *necessary* condition just by insisting that the fundamental line has a partner with opposite spin. In turns, this condition was also seen to be *sufficient*, so one may wonder if a similar phenomenon may occur in other topological systems. In other words, given an arbitrary TQFT, does the matching of the spin of a single line guarantee that the theory is time-reversal invariant? Generically speaking, the answer is *no*, as there are many examples where a specific pair of lines match but others do not. A much stronger test is the matching of *all* the lines, that is, the condition that $\{h\}=\{-h\} \mod 1$ (with equality as multisets, that is, taking into account multiplicities). For example, one may we check that the set of spins matches for the

[93]We thank N. Seiberg for this comment.

theory $\mathrm{SU}(N)_N$, for $N = 1, 5, 13, 17, \ldots$ (both as a bosonic and a spin TQFTs), all of which happen to be Pythagorean primes. As suggestive as this may seem, the pairs of lines that have opposite spin do not in general have the same quantum dimension, so these theories are not time-reversal invariant. ($\mathrm{SU}(N)_N/\mathbb{Z}_N$ is, however, time-reversal invariant for all $N$ [112])

Upon turning on a background metric, the duality $\mathrm{U}(1)_k \longleftrightarrow \mathrm{U}(1)_{-k}$ no longer holds as written, because the two theories have a different framing anomaly, and so they couple to the background gravitational field differently. This can be interpreted as a mixed anomaly between time-reversal and gravity. To maintain the duality one must adjust gravitational Chern-Simons counterterms on both sides so that their central charges agree. In particular, one may use $\mathrm{U}(1)_{\pm 1}$ to add/subtract one unit of central charge, without otherwise changing the topological content of the theory. With this in mind, the precise duality reads

$$\mathrm{U}(1)_{+k} \times \mathrm{U}(1)_{-1} \longleftrightarrow \mathrm{U}(1)_{-k} \times \mathrm{U}(1)_{+1} \,. \tag{4.3.36}$$

These theories can be represented by the matrices $K = \mathrm{diag}(\pm k, \mp 1)$. In the bosonic case, we already included a factor $\{\mathbf{1}, \psi\}$ to make the theory into a spin theory; here we see that this factor also fixes the central charge, provided we identify $\{\mathbf{1}, \psi\} \equiv \mathrm{U}(1)_{-\,\mathrm{sign}(k)}$. In the spin case, this factor also fixes the central charge, but leaves the spectrum of lines unaffected.

It is clear that without the factor of $\mathrm{U}(1)_{\pm 1}$, time-reversal cannot possibly be a Lagrangian symmetry of the $\mathrm{U}(1)_k$ theory, because the only $\mathrm{GL}_1(\mathbb{Z})$ transformations are $a \to \pm a$, neither of which maps $k \to -k$. More generally, the signature of the $K$-matrix is invariant under congruence ($\mathrm{sign}(K) \equiv \mathrm{sign}(G^t K G)$ for any $G \in \mathrm{GL}_n(\mathbb{Z})$, as per the Sylvester law of inertia) and so time-reversal can only be a Lagrangian symmetry if the signature vanishes (inasmuch as the chiral central charge is odd under time-reversal). Once we fix the central charge, time-reversal may (but need not) become a Lagrangian symmetry. It is interesting to note that, in the case at hand, this happens only for a subset of $\mathbb{T}$: only for a specific set of values of $k$ is the Lagrangian time-reversal invariant. One can show that this is so if and only if $k \in \mathbb{P}$:

**Proposition 4.3.6** *The Lagrangian of the theory* $\mathrm{U}(1)_k \times \mathrm{U}(1)_{-1}$ *is time-reversal invariant if and only if $k$ satisfies the negative Pell equation.*

*Proof.* The fact that this condition is necessary can be obtained by looking at the bottom-right component of the equation $K = -G^t K G$, where $K = \mathrm{diag}(k, -1)$. That this is also sufficient was originally shown in [251, 252], and follows from the explicit change of variables

$$\mathsf{T}\colon \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} q & p \\ -kp & -q \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \qquad kp^2 - q^2 = 1 \,. \tag{4.3.37}$$

Proposition 4.7.4 in section 4.7 generalizes the construction to $K = \mathrm{diag}(k, k')$. □

This means that if $k \in \mathbb{T}$ but it is not in $\mathbb{P}$, then $\mathrm{U}(1)_k$ will be time-reversal invariant, but the invariance will not be a symmetry of the Lagrangian, not even if we include the factor of $\mathrm{U}(1)_{\pm 1}$. It is a quantum symmetry of $\mathrm{U}(1)_k \times \mathrm{U}(1)_{-1}$. However, it is possible that in a different abelian Chern-Simons realization of the same TQFT data that the symmetry becomes Lagrangian.

**Remark 4.3.3** As a physical application of proposition 4.3.2, note that given an integer $\ell$ such that both $k$ and $k+\ell^2$ are in $\mathbb{T}$, the theory

$$\mathrm{U}(1)_{k+\ell^2/2} + \Psi \tag{4.3.38}$$

with $\Psi$ a Dirac fermions of charge $\ell$ is infrared time-reversal invariant for $m \neq 0$. Indeed, integrating the fermions out we get

$$\mathrm{U}(1)_k \;\longleftrightarrow\; \mathrm{U}(1)_{-k} \tag{4.3.39}$$

for $m \to -\infty$, and

$$\mathrm{U}(1)_{k+\ell^2} \;\longleftrightarrow\; \mathrm{U}(1)_{-(k+\ell^2)} \tag{4.3.40}$$

for $m \to +\infty$. This suggests that the CFT at the massless point $m = 0$ may be time-reversal invariant as well. These gauge theories, in spite of not being time-reversal invariant in the ultraviolet, have an emergent time-reversal symmetry across the entire infrared phase diagram. The first few solutions of $(k, k+\ell^2) \in \mathbb{T} \times \mathbb{T}$ are

$$\begin{aligned}
&\ell = 0\colon && k = 1, 2, 5, 10, 13, 17, 26, 25, 29, \ldots \\
&\ell = 1\colon && k = 1, 25, 73, 145, 169, 193, 289, \ldots \\
&\ell = 2\colon && k = 1, 13, 37, 61, 85, 97, 109, 181, \ldots \\
&\ell = 3\colon && k = 1, 17, 25, 41, 65, 73, 97, 113, \ldots \\
&\text{etc.}
\end{aligned} \tag{4.3.41}$$

(For $\ell = 0$ one gets an infrared emergent time-reversal symmetry in Maxwell-Chern-Simons theories). A similar phenomenon occurs in non-abelian theories. For example, using the Chern-Simons dualities $\mathrm{U}(1)_k \leftrightarrow \mathrm{SU}(k)_{-1} \leftrightarrow \mathrm{SO}(k)_{-2}$ we observe that the theories $\mathrm{SU}(k)_0$ and $\mathrm{SO}(k)_0$ with two fundamental Dirac fermions, and $k \in \mathbb{T}$, are time-reversal invariant in their massive phases (necessarily also in their massless phase, because the UV theory is time-reversal invariant).

It is an interesting number-theoretic problem whether there exists, for a given $\ell \in \mathbb{Z}$, an infinite number of pairs with $(k, k+\ell^2) \in \mathbb{T}^2$. This is similar in spirit to the so-called Polignac conjecture, which states that there exists an infinite number of pairs of primes of the form $(\pi, \pi+n)$ for any $n \in 2\mathbb{N}$ (recall that primes $\pi > 2$ are in $\mathbb{T}$ iff they are Pythagorean). Assuming this conjecture with $\ell^2 = n$ (which requires $\ell$ to be even), and noting that $\pi$ and $\pi + \ell^2$ are either both Pythagorean or neither is, suggests that indeed there exists an infinite number of pairs $(k, k+\ell^2) \in \mathbb{T}^2$, at least for $\ell$ even.

**Unitary Symmetries.**

We now move on to the unitary symmetries of $\mathrm{U}(1)_k$. The principle is identical to the anti-unitary case, the only difference being a sign flip. By definition, an automorphism $\mathsf{U} \in \mathrm{Aut}(\mathcal{A})$ is a unitary symmetry of $(\mathcal{A}, \theta)$ if and only if

$$\theta(\mathsf{U}(\alpha)) = \theta(\alpha) \quad \Longleftrightarrow \quad h_{\mathsf{U}(\alpha)} = h_\alpha \mod 1\,. \tag{4.3.42}$$

As in the anti-unitary case, any permutation is fixed once we choose how the generators transform. The corresponding permutation will be a symmetry if it satisfies (4.3.42). But, unlike the case of anti-unitary symmetries, here the equation $h_{\mathsf{U}(\alpha)} = h_\alpha \mod 1$ always admits solutions: at least, the trivial permutation and charge conjugation $\mathsf{C}$ exist. These are transformations that leave the action of the theory invariant. We thus solve a more refined problem: the interesting automorphisms will be those that are neither trivial nor $\mathsf{C}$. Another difference with the anti-unitary case is that, in general, we will find non-trivial symmetries also in the bosonic case.

We begin with the following observation:

**Proposition 4.3.7** *All the unitary symmetries of* $\mathrm{U}(1)_k$ *(as a bosonic TQFT if* $k$ *is even) are transformations of the form*

$$\mathsf{U}\colon \alpha \mapsto q\alpha \tag{4.3.43}$$

*for some integer* $q$ *that satisfies*

$$q^2 = 1 + 2pk\,. \tag{4.3.44}$$

*Similarly, the unitary symmetries of* $\mathrm{U}(1)_k \times \{\mathbf{1}, \psi\}$ *for* $k$ *even are of the form*

$$\mathsf{U}\colon (\alpha, \beta) \mapsto (q\alpha, p\alpha + \beta) \tag{4.3.45}$$

*for some integer* $q$ *that satisfies*

$$q^2 = 1 + pk\,. \tag{4.3.46}$$

*The solutions* $q = \pm 1$ *(with* $p = 0$*) always exist and corresponds to the trivial permutation, and charge conjugation* $\mathsf{C}$ *(4.3.9), respectively. All other solutions correspond to quantum symmetries.*

*Proof.* The case of $\mathrm{U}(1)_k$ (as a bosonic TQFT if $k$ is even) is essentially identical to the anti-unitary case. Let us therefore consider $\mathrm{U}(1)_k \times \{\mathbf{1}, \psi\}$ with $k$ even. Any fusion endomorphism that fixes the transparent fermion is of the form

$$\mathsf{U}\colon (\alpha, \beta) \mapsto (q\alpha, c\alpha + \beta) \tag{4.3.47}$$

for a pair of integers $c, q$. If $c$ is even, $\mathsf{U}$ does not mix the lines of $\mathrm{U}(1)_k$ with the transparent fermion, and so this is a symmetry that was also present in the bosonic case. If $c$ is odd, the permutation does mix the lines, and so it is only a symmetry of the fermionic theory. In any case, requiring that the spin of the fundamental line is equal to the spin of its image under $\mathsf{U}$, we get

$$q^2 = 1 - (c^2 + 2\tilde{p})k \tag{4.3.48}$$

for some integer $\tilde{p}$. Letting $-p := c^2 + 2\tilde{p}$ we get the expression in the proposition (note that $p$ and $c$ have the same parity, and therefore we can replace the latter by the former in the transformation $\mathsf{U}$). It is straightforward to check that if the spin of the fundamental line is invariant under $\mathsf{U}$, so is the spin of the rest of lines. Finally, it is easy to show that *any* solution of (4.3.48) corresponds to a permutation (i.e. $q$ automatically has the appropriate coprimality with $k$ to define an automorphism). $\square$

As in the anti-unitary case, all the unitary permutations have the same order:

**Proposition 4.3.8** *All the unitary symmetries of* $\mathrm{U}(1)_k$ *(either as a bosonic or as an spin TQFT) are of order-two.*

*Proof.* For $\mathrm{U}(1)_k$ we have

$$\mathsf{U}^2\colon \alpha \mapsto q^2\alpha = \alpha + 2pk\alpha \tag{4.3.49}$$

which indeed equals $\alpha$. In the case of $\mathrm{U}(1)_k \times \{\mathbf{1}, \psi\}$, the argument is identical:

$$\mathsf{U}^2\colon (\alpha, \beta) \mapsto (q^2\alpha, p\alpha(q+1) + \beta) = (\alpha + pk\alpha, p\alpha(q+1) + \beta) \tag{4.3.50}$$

which, using the fact that $q$ is odd, yields $(\alpha, \beta)$, as claimed. $\square$

Take the theory $\mathrm{U}(1)_k$, without the factor of $\{\mathbf{1}, \psi\}$ for $k$ even. A slight modification of the argument in proposition 4.3.4 proves that the number of solutions in the range $q \in [0, 2k)$ for $k$ odd, and in the range $q \in [0, k)$ for $k$ even, is $2^{\varpi(k)}$, as in the anti-unitary case. Therefore, in order to have solutions other than $\mathsf{U} \in \{1, \mathsf{C}\}$, the level $k$ must not be a prime power or twice a prime power. Such non-trivial solutions will not be a symmetry of the classical Lagrangian, because $p \neq 0$. They correspond to quantum symmetries.

For $k$ even, one may also study the unitary symmetries of the theory as a spin TQFT, that is, of $\mathrm{U}(1)_k \times \{\mathbf{1}, \psi\}$. The symmetries of the bosonic theory are inherited in the fermionic theory, but new symmetries may appear – those under which the transparent fermion mixes non-trivially. The automorphisms are given by the integers $q$ that satisfy $q^2 = 1 + pk$, and whether the transparent fermion mixes is controlled by the parity of $p$. It is easy to show that the number of solutions is $2^{\varpi(k)}$ for $k = 2 \mod 4$, and $2^{\varpi(k/2)+1}$ for $k = 0 \mod 4$. Therefore, there is an enhancement of symmetry when going from the bosonic theory to the spin theory if and only if $k$ is a multiple of 8: only in that case may the fermion mix. The additional transformation that appears when the theory is uplifted from bosonic to spin is generated by $q = k/2 - 1$ (with $p = k/4 - 1$). We summarise these claims as follows:

**Proposition 4.3.9** *All the unitary symmetries of* $\mathrm{U}(1)_k$ *(both as a spin theory and as a bosonic theory in the case of* $k$ *even) are* $\mathbb{Z}_2$*-valued. There are* $2^{\varpi(k)}$ *permutations if* $k$ *is not a multiple of* 8*. If* $k = 0 \mod 8$*, then there are* $2^{\varpi(k)}$ *permutations in the bosonic theory, and twice as many in the spin theory.*

Needless to say, one may compose any non-trivial unitary symmetry with a given $\mathsf{T}$ to yield a different notion of time-reversal. Similarly, composing any two time-reversal operations results in a unitary symmetry, and composing two unitary symmetries leads to another unitary symmetry. In fact, a stronger result is true. Let $\{\mathsf{T}_i\}$ be all time-reversal symmetries, and $\{\mathsf{U}_i\}$ be unitary ones. Let $\mathsf{U}_0 := 1$, pick some element of $\{\mathsf{T}_i\}$, and denote it by $\mathsf{T}_0$. Then any $\mathsf{T}_i$ can be obtained by acting with some $\mathsf{U}_i$ on $\mathsf{T}_0$. Indeed, it is easy to see that the sets

$$\{\mathsf{T}_i\} \qquad \text{and} \qquad \{\mathsf{T}_0\mathsf{U}_i\} \tag{4.3.51}$$

contain the same number of elements (because $\mathsf{T}_0$ is invertible, so $\mathsf{T}_0\mathsf{U}_i \neq \mathsf{T}_0\mathsf{U}_j$ for $i \neq j$), and so they must be identical. Thus, perhaps after relabelling its elements, we have

$$\mathsf{T}_i \equiv \mathsf{U}_i\mathsf{T}_0 \tag{4.3.52}$$

and so one time-reversal permutation suffices to generate them all.

Recalling definition 4.2.1, all these considerations can be put together to obtain the following:

**Proposition 4.3.10** *The group of symmetries of* $\mathrm{U}(1)_k$ *as a spin TQFT is*

$$\begin{aligned}\mathrm{Aut}(\mathrm{U}(1)_k^{\mathrm{spin}}) &= \mathbb{Z}_4 \times (\mathbb{Z}_2)^{\varpi(k)-1}\\ \mathrm{Aut}_U(\mathrm{U}(1)_k^{\mathrm{spin}}) &= (\mathbb{Z}_2)^{\varpi(k)}\end{aligned} \tag{4.3.53}$$

*if* $k \in \mathbb{T}$*, and*

$$\mathrm{Aut}(\mathrm{U}(1)_k^{\mathrm{spin}}) = \mathrm{Aut}_U(\mathrm{U}(1)_k^{\mathrm{spin}}) = \begin{cases}(\mathbb{Z}_2)^{\varpi(k)+1} & k = 0 \mod 8\\ (\mathbb{Z}_2)^{\varpi(k)} & \text{otherwise}\end{cases} \tag{4.3.54}$$

*otherwise. On the other hand, as a bosonic theory (with* $k$ *even), the group reads*

$$\mathrm{Aut}(\mathrm{U}(1)_k^{\mathrm{bosonic}}) = \mathrm{Aut}_U(\mathrm{U}(1)_k^{\mathrm{bosonic}}) = (\mathbb{Z}_2)^{\varpi(k)}\,. \tag{4.3.55}$$

### 4.3.2 Minimal abelian TQFT

An important abelian theory that appears in the study of the one-form symmetries of three-dimensional TQFTs is the so-called "minimal abelian TQFT" [38, 249, 253, 272]. This theory is denoted by $\mathcal{A}^{N,t}$ (also by $\mathbb{Z}_N^{(t)}$), with $N, t$ two integers, which must be coprime if we require the theory to be modular. The number of lines is $N$, which can be labelled as $s = 1, 2, \ldots, N$. Fusion corresponds to addition modulo $N$, $s \times s' = (s + s' \mod N)$, i.e. the fusion algebra is $\mathbb{Z}_N$. The spin of the line $s$ is $h_s = t\frac{s^2}{2N}$. For example, if $k$ is even, then $\mathrm{U}(1)_k = \mathcal{A}^{k,1}$; if $k$ is odd, then $\mathrm{U}(1)_k = \mathcal{A}^{2k,2}$ (which, indeed, is not modular, because the braiding matrix has a non-trivial kernel). All these theories admit an abelian Chern-Simons representation (e.g. for $t = N - 1$ the $K$-matrix is the Cartan matrix of $\mathrm{SU}(N)$).

The analysis of the symmetries of $\mathcal{A}^{N,t}$ is essentially identical to that of $\mathrm{U}(1)_k$ because the fusion algebra is also cyclic. For example, following the same reasoning as in the $1 \times 1$ case, this theory is seen to be time-reversal invariant if and only if

$$2Np = t(1 + q^2) \tag{4.3.56}$$

is solvable for some integers $p, q$. It is easy to prove that this equation is solvable if and only if

$$t \in \mu(N)\mathbb{Z}\,. \tag{4.3.57}$$

Indeed, by reducing (4.3.56) modulo $\mu(N)$ we get $t(1 + q^2) = 0 \mod \mu(N)$; but $(1 + q^2)$ is never divisible by a prime of the form $4n + 3$, and so $t$ itself mush vanish modulo $\mu(N)$, showing that $t \propto \mu(N)$ is necessary. Conversely, noting that $N/\mu(N)$ is always in $\mathbb{T}^{\text{odd}}$, we know that there exists a pair of integers $\tilde{p}, q$ such that $2\tilde{p}N = \mu(N)(1 + q^2)$; multiplying this equation by $t/\mu(N)$ and letting $p = \tilde{p}t/\mu(N)$ we find that $t \propto \mu(N)$ is also sufficient.

Alternatively, one may rewrite (4.3.57) as a condition on $N$ instead of $t$, as follows:

$$N \in \bigcup_{d|t} d\,\mathbb{T}^{\text{odd}}\,. \tag{4.3.58}$$

Indeed, if $N \in d\,\mathbb{T}^{\text{odd}}$ for some $d|t$, then there exists some $\tilde{p}, q$ such that $2(N/d)\tilde{p} = 1 + q^2$; multiplying this equation by $t/d$ and letting $p = \tilde{p}t/d$ shows that (4.3.56) is solvable. Conversely, if $N \notin d\,\mathbb{T}^{\text{odd}}$ for any $d|t$ then, in particular, $N \notin t\,\mathbb{T}^{\text{odd}}$ (and, if $t \in 2\mathbb{Z}$, then $N \notin (t/2)\,\mathbb{T}^{\text{odd}}$ either), and so equation (4.3.56) is not solvable (note that if $t$ is odd then $N$ must be odd as well).

If we further assume that $(N, t) = 1$, the expression (4.3.58) can be simplified into

$$N \in \mathbb{T}^{\text{odd}} \setminus \left( \bigcup_{\pi|p} \pi\,\mathbb{Z} \right) \tag{4.3.59}$$

where $\pi$ are the Pythagorean prime factors of $p$.

As $\mathcal{A}^{N,t}$ has a single generator, its group of symmetries is abelian, and can be studied along the same lines as in the $\mathrm{U}(1)_k$ case.

## 4.4 $\mathrm{U}(1)^n$ Chern-Simons theory

We now move on to Chern-Simons theories that contain an arbitrary number of factors of U(1). As a Lagrangian theory, the system is described by

$$\mathcal{L} = \frac{1}{4\pi} a^t K \mathrm{d}a \tag{4.4.1}$$

for a $\mathrm{U}(1)^n$ gauge field $a^t = (a_1, a_2, \ldots, a_n)$. The Lagrangian is metric independent and, although not manifestly so, gauge invariant provided the coefficient matrix $K \in \mathbb{Z}^{n\times n}$ is symmetric and integral-valued. Generically speaking, the theory depends on the orientation of spacetime and, if at least diagonal component of $K$ is odd, on the spin structure. The theory has central charge $c = \operatorname{sign}(K)$ (the signature of $K$), which controls the coupling to the Chern-Simons form for the background metric, via the framing anomaly. To keep matters simple, we shall often turn off this metric, and any other background field one may ultimately want to couple $a$ to.

The observables of the theory are the Wilson lines, modulo local bosonic operators. These lines are of the form

$$W_{\vec{\alpha}}(\gamma) := \exp\left[i\vec{\alpha}^t \int_\gamma a\right] , \tag{4.4.2}$$

where $\vec{\alpha} \in \mathbb{Z}^n$ is the representation $\mathrm{U}(1)^n \ni \theta \mapsto \mathrm{e}^{i\vec{\alpha}\cdot\theta}$. We shall call $\vec{\alpha}$ the *charge* of $W_{\vec{\alpha}}$, and we will often denote the line itself by $\vec{\alpha}$.

These lines can be thought of as the worldlines of anyons, i.e., particles with fractional statistics. In particular, they have spin and may braid non-trivially. If a line $\vec{\alpha}$ braids around a line $\vec{\beta}$, their product picks up a phase $B(\vec{\alpha},\vec{\beta}) \in \mathrm{U}(1)$, where

$$B(\vec{\alpha},\vec{\beta}) := \exp\left[2\pi i\, \vec{\alpha}^t K^{-1} \vec{\beta}\right] . \tag{4.4.3}$$

Similarly, the topological spin of the line corresponds to half self-braiding,

$$\theta(\vec{\alpha}) := \exp\left[2\pi i\, h_{\vec{\alpha}}\right] , \qquad h_{\vec{\alpha}} := \frac{1}{2}\vec{\alpha}^t K^{-1}\vec{\alpha} \,. \tag{4.4.4}$$

The function $\theta$ is said to be a *quadratic refinement* of the bilinear form $B$, because one has

$$B(\vec{\alpha},\vec{\beta}) \equiv \frac{\theta(\vec{\alpha}+\vec{\beta})}{\theta(\vec{\alpha})\theta(\vec{\beta})} \,. \tag{4.4.5}$$

This implies that the spin of the lines determines their braiding unambiguously; one need not keep track of the latter.

An operator is said to be *local* if it braids trivially with any other line. In particular, any line with $\vec{\alpha}$ proportional to a column of $K$ satisfies $B(\vec{\alpha},\vec{\beta}) \equiv 1$ for any $\vec{\beta}$, and so it will be local. If, furthermore, the corresponding column has even diagonal element, then $h_{\vec{\alpha}} = 0$

mod 1, and so the local line will be bosonic. As before, lines differing by such a local operator are identified, and so the degrees of freedom of the theory are in fact finite. More explicitly, we have the following:

- If all the diagonal components of $K$ are even, then all the local operators are bosonic, and we need not specify a spin structure to define the theory. It is a bosonic TQFT. Any two lines that are congruent modulo some linear combination (with integer coefficients) of the columns of $K$ are identified, which means that the lines live in the lattice $\mathbb{Z}^n/K\mathbb{Z}^n$. There are $|\det K|$ independent lines, which can be taken to be all the lattice points in the $n$-dimensional parallelepiped spanned by the columns of $K$.

- If at least one diagonal component of $K$ is odd, the theory contains a local fermionic operator, which requires a choice of spin structure. The theory is a spin TQFT. Any two lines that are congruent modulo some linear combination (with integer coefficients) of the columns of $K$ are identified, except if they differ by a local fermion. This means that the lines live in the lattice $(\mathbb{Z}^n/K\mathbb{Z}^n) \times \mathbb{Z}_2$. There are $2|\det K|$ independent lines, which can be taken to be all the lattice points in the $n$-dimensional parallelepiped spanned by the columns of $K$, together with a $\mathbb{Z}_2$ label that specifies if the line carries a local fermion or not. Alternatively, a basis of lines can be taken to be all the lattice points in the $n$-dimensional parallelepiped spanned by the columns of $\tilde{K}$, where $\tilde{K}$ is the matrix given by doubling any one column of $K$ with odd diagonal component.

The spectrum of lines is given by the set $\mathcal{A} := \mathbb{Z}^n/\sim$, where

$$\vec{\alpha} \sim \vec{\beta} \quad \Longleftrightarrow \quad \vec{\alpha} = \vec{\beta} + K\vec{\gamma}\,, \tag{4.4.6}$$

where $\vec{\gamma}$ is any tuple of integers with

$$\sum_{K_{ii}\ \text{odd}} \gamma_i = \text{even}\,. \tag{4.4.7}$$

Reducing $\mathbb{Z}^n$ modulo $K$, instead of modulo $\sim$, would be tantamount to identifying the local fermion, if any, with the vacuum. In other words, we would forget about the information carried by such a line. This would not be correct: we need the $\mathbb{Z}_2$ label to signal the presence of $\psi$. This extra piece of information resolves the ambiguity in lifting the symmetric form $B$ into the quadratic form $\theta$. We shall nevertheless often refer to the equivalence $\sim$ as "reduction modulo $K$", in order to keep the notation as simple as possible.

Due to the abelian nature of the gauge fields, any pair of unbraided lines $\vec{\alpha}, \vec{\beta}$ can be brought together to form a line of charge $\vec{\alpha}+\vec{\beta}$. In other words, the fusion rules of the theory are

$$\vec{\alpha} \times \vec{\beta} := (\vec{\alpha} + \vec{\beta} \mod K)\,. \tag{4.4.8}$$

The theory described by a given matrix $K$ may have several symmetries. The main focus of this paper is to study the zero-form symmetries, but for completeness we mention that the one-form symmetry group can be obtained by bringing $K$ into its Smith normal form $K \to \mathrm{diag}(k_1, k_2, \dots, k_n)$, where $k_i$ is the greatest common divisor of all $i \times i$ minors of $K$. Given this canonical decomposition, the one-form symmetry group is

$$\bigoplus_{i=1}^{n} \mathbb{Z}_{k_i} \,. \tag{4.4.9}$$

We now move on to the zero-form symmetries of the system. These are, by definition, the permutations of the lines that respect their topological properties. A unitary zero-form symmetry of the corresponding system is an automorphism $\mathsf{U}\colon \mathcal{A} \to \mathcal{A}$ that satisfies

$$\begin{aligned} \mathsf{U}(a \times b) &= \mathsf{U}(a) \times \mathsf{U}(b) \\ \theta(\mathsf{U}(a)) &= \theta(a) \\ B(\mathsf{U}(a), \mathsf{U}(b)) &= B(a, b) \end{aligned} \tag{4.4.10}$$

for all $a, b \in \mathcal{A}$. Similarly, an anti-unitary zero-form symmetry is an automorphism $\mathsf{T}\colon \mathcal{A} \to \mathcal{A}$ that satisfies

$$\begin{aligned} \mathsf{T}(a \times b) &= \mathsf{T}(a) \times \mathsf{T}(b) \\ \theta(\mathsf{T}(a)) &= \theta(a)^* \\ B(\mathsf{T}(a), \mathsf{T}(b)) &= B(a, b)^* \,. \end{aligned} \tag{4.4.11}$$

Thanks to (4.4.5), the braiding is determined by the spin, and so the third condition is automatically guaranteed to hold if the first two do; we nevertheless find it convenient to keep track of the braiding matrix explicitly.

We have denoted the anti-unitary symmetries by $\mathsf{T}$ because we will think of them as a time-reversal operation (or a reflection in the Euclidean setting). These symmetries do not always exist: only for some special matrices $K$ is the system independent of the orientation of spacetime. In particular, as the Lagrangian is odd under the reversal of orientation, we require $K$ and $-K$ to describe equivalent theories: the theories with matrices $K$ and $-K$ must be dual.

A sufficient condition for the theories described by two matrices $K_1, K_2$ to be equivalent is that they are congruent, i.e., $\mathrm{GL}_n(\mathbb{Z})$-equivalent: that there exists a unimodular matrix $G$ such that $K_1 \equiv G^t K_2 G$, as follows from the redefinition $a_2 := G a_1$. The matrix $G$ is required to be unimodular because the change of variables has to be invertible and respect the normalisation of the gauge fields. We shall refer to these equivalences of theories as *Lagrangian* (or classical) *symmetries*, because they are manifest symmetries of the Lagrangian. As we shall show, one may have matrices $K_1, K_2$ that are not $\mathrm{GL}_n(\mathbb{Z})$-equivalent, and yet the theories described by them are nevertheless equivalent. This latter notion of equivalence we refer to as a *quantum symmetry*, or as a *duality*.

Dualities of TQFTs are often valid only when the theory is regarded as a spin TQFT. In order to turn a bosonic theory into a spin TQFT, it suffices to tensor the theory by the trivial spin TQFT $\mathrm{U}(1)_{\pm 1} = \{\mathbf{1}, \psi\}$, where $\mathbf{1}$ is a local boson and $\psi$ a local fermion. Tensoring a theory that is already spin by this trivial factor leaves the TQFT unaffected, inasmuch as we identify local fermions anyway (because they differ by a local boson: $\psi_1 = (\psi_1\psi_2)\psi_2$).

If we turn on some background field that couples to a given TQFT, then one may need to adjust appropriate counterterms for it on both sides of the duality. The canonical example is the coupling to background gravity, which is controlled by the central charge of the theory (through the framing anomaly). In particular, the central charge – being the signature of the $K$-matrix – is odd under time-reversal, which means that a theory can only be time-reversal invariant in the presence of gravity if the central charge vanishes. In this sense, a theory being invariant in flat spacetime may require a gravitational counterterm to remain invariant when the metric is nontrivial. Noting that $\mathrm{U}(1)_{\pm 1}$ is essentially trivial (it is an SPT) but has central charge $\pm 1$, one may add as many factors of this theory as necessary so that the theory under consideration has vanishing central charge, as required to maintain the time-reversal symmetry when turning on a background metric. If the theory is already spin, tensoring by $\mathrm{U}(1)_{\pm 1} = \{\mathbf{1}, \psi\}$ has no effect other than changing the central charge; but for a bosonic system, this factor turns the theory into a spin TQFT.

### 4.4.1 Symmetries of $\mathrm{U}(1)^n$

The analysis of the symmetries of a system described by a matrix $K$ is essentially identical to that of $\mathrm{U}(1)_k$: the symmetries are those automorphisms of the fusion algebra that respect the spin of the lines. The most general endomorphism of $\mathcal{A} \cong \mathbb{Z}^n/\sim$ is

$$g\colon \vec{\alpha} \mapsto Q\vec{\alpha} \tag{4.4.12}$$

for some matrix $Q$, its $i$-th column being $g(\vec{e}_i)$, with $\vec{e}_i$ the $i$-th unit vector. This map is an automorphism if the action of $Q$ is invertible modulo $\sim$, i.e., if it is a permutation of $\mathcal{A}$. Finally, this permutation shall be a symmetry if it conserves the spin of all the lines, up to complex conjugation in the anti-unitary case. We discuss this in some more detail below.

**Anti-unitary symmetries.**

A natural generalisation of proposition 4.3.2 reads

**Proposition 4.4.1** *A necessary condition for the Chern-Simons theory with matrix $K$ to admit an anti-unitary symmetry is that there exists a pair of matrices $(Q, P) \in \mathbb{Z}^{n\times n} \times \mathbb{Z}^{n\times n}$ where $P$ has even diagonal elements, and such that*

$$PK - Q^t K^{-1} Q K = 1_n\,. \tag{4.4.13}$$

*Proof.* We shall look for the most general permutation that satisfies the conditions (4.4.11).

As in the case of a single U(1) factor, any putative time-reversal operation is fixed once we know how the generators transform. The most general fusion endomorphism reads

$$\mathsf{T}(\vec{\alpha}) = Q\vec{\alpha} \tag{4.4.14}$$

for some matrix $Q$, the $i$-th column of which represents the action of $\mathsf{T}$ on the unit vector in the $i$-th direction $\vec{e}_i$.

Imposing that the spin of $\vec{e}_i$ is the opposite of that of $\mathsf{T}(\vec{e}_i)$, we get

$$\frac{1}{2}\vec{e}_i^{\,t} K^{-1}\vec{e}_i = -\frac{1}{2}\vec{e}_i^{\,t} Q^t K^{-1} Q\vec{e}_i + P_{ii} \tag{4.4.15}$$

for some integer $P_{ii}$. Similarly, imposing that $\mathsf{T}$ commutes with braiding, $B(\vec{e}_i,\vec{e}_j) = B(\mathsf{T}(\vec{e}_i),\mathsf{T}(\vec{e}_j))^*$, we get

$$\vec{e}_i^{\,t} K^{-1}\vec{e}_j = -\vec{e}_i^{\,t} Q^t K^{-1} Q\vec{e}_j + P_{ij} \tag{4.4.16}$$

for some integer $P_{ij}$. These two equations, in matrix form, take the form quoted in the proposition, as claimed. Note that if this equation is satisfied, then the spin of all the lines behaves as expected, and not only that of the generators:

$$\begin{aligned} h_{\mathsf{T}(\vec{\alpha})} &= \frac{1}{2}\vec{\alpha}^t \left(Q^t K^{-1} Q\right)\vec{\alpha} \\ &= \frac{1}{2}\vec{\alpha}^t \left(-K^{-1} + P\right)\vec{\alpha}, \end{aligned} \tag{4.4.17}$$

which indeed equals $-h_{\vec{\alpha}}$ modulo 1. □

We stress that, unlike in the case of a single U(1) factor, the argument in proposition 4.4.1 does not prove that any map $\vec{\alpha} \mapsto Q\vec{\alpha}$ with $PK - Q^t K^{-1} QK = 1_n$ represents a time-reversal operation, even though the conditions (4.4.11) are satisfied. One must also require $Q$ to be a permutation, that is, invertible modulo $K$ over the integers. This is a non-trivial condition that is not satisfied for every solution of $PK - Q^t K^{-1} QK = 1_n$. (In the $1\times 1$ case, the equation $pk - q^2 = 1$ implies that $\gcd(k,q) = 1$, and so any solution is invertible; this is no longer necessarily true in the $n \times n$ case: some solutions may fail to be invertible).

As in proposition 4.3.5, one can also examine the time-reversal invariance of $\mathrm{U}(1)^n$ through a Lagrangian argument:

**Proposition 4.4.2** *A sufficient condition for the Chern-Simons theory described by the matrix $K$ to admit an anti-unitary symmetry is that there exists a pair of matrices $(Q,P) \in \mathbb{Z}^{n\times n} \times \mathbb{Z}^{n\times n}$ where $P$ has even diagonal elements, and such that*

$$PK - Q^t K^{-1} QK = 1_n \tag{4.4.18}$$

*subject to the conditions*

$$[K^{-1}QK, Q^t] = 0, \qquad [KP, Q] = 0\,. \tag{4.4.19}$$

(Note that if $Q$ is normal and commutes with $K$, then these equations are automatically satisfied).

*Proof.* By solving for $P$ in (4.4.18), and taking the transpose, it becomes clear that $P$ is symmetric, and so it defines a (bosonic) abelian Chern-Simons theory. Take the Lagrangian with matrix $K \oplus -P$

$$4\pi\mathcal{L} = a^t K \mathrm{d}a - b^t P \mathrm{d}b \tag{4.4.20}$$

and perform the $\mathrm{GL}_{2n}(\mathbb{Z})$ transformation

$$\mathsf{T}\colon \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} Q^t & -P \\ K & -Q \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{4.4.21}$$

under which

$$\begin{aligned} \begin{pmatrix} K & 0 \\ 0 & -P \end{pmatrix} &\mapsto \begin{pmatrix} -KPK + QKQ^t & [KP, Q] \\ [Q^t, PK] & PKP - Q^tPQ \end{pmatrix} \\ &\equiv \begin{pmatrix} -K & 0 \\ 0 & P \end{pmatrix}. \end{aligned} \tag{4.4.22}$$

The off-diagonal entries vanish by virtue of $[KP, Q] = 0$ and $P$ being symmetric, and the equality for the diagonal entries follows from the assumptions in the proposition.

This proves that the theory is time-reversal invariant. The mapping of lines reads

$$(\vec{\alpha}, \vec{\beta}) \mapsto (Q\vec{\alpha} + K\vec{\beta}, -P\vec{\alpha} - Q^t\vec{\beta})\,. \tag{4.4.23}$$

Finally, and thanks to the evenness of $P$, the action of $\mathsf{T}$ descends to a well-defined operation on the lines of $\mathrm{U}(1)_k$:

$$(\vec{\alpha}, 0) \mapsto (Q\vec{\alpha}, -P\vec{\alpha}) \sim (Q\vec{\alpha}, 0) \tag{4.4.24}$$

as required. □

**Remark 4.4.1** It is easy to argue that the conditions in proposition 4.4.2 are $\mathrm{GL}_n(\mathbb{Z})$-invariant. Indeed, if we redefine our gauge fields according to $a := Ga'$ for some $G \in \mathrm{GL}_n(\mathbb{Z})$, then the lines transform as $\vec{\alpha} = (G^{-1})^t\vec{\alpha}'$, and

$$\begin{aligned} K &= (G^{-1})^t K' G^{-1} \\ Q &= (G^{-1})^t Q' G^t \\ P &= G P' G^t \end{aligned} \tag{4.4.25}$$

which leaves the equations (4.4.18), (4.4.19) invariant. This was to be expected, inasmuch as a Chern-Simons theory depends on $K$ modulo congruences. (Two $K$-matrices in the same congruence class have the same determinant; however, the converse is not true: there can multiple congruence classes with a given determinant. The number of congruence classes depends nontrivially on the value of the determinant.)

Deciding whether the equation $PK - Q^t K^{-1} Q K = 1_n$ is solvable for a given $K$ is a rather non-trivial problem, unlike in the case of $\mathrm{U}(1)_k$ (where it suffices to scan $q \in [0,k)$ for solutions; moreover, and thanks to proposition 4.7.1, deciding whether $k \in \mathbb{T}$ requires at most $\omega(k) \le \frac{2\log k}{\log\log k}$ operations if given the prime divisors of $k$). We shall make no attempt at finding an efficient characterisation of the set of $K$-matrices that solve this equation. We will content ourselves with focusing specifically to the case where $K$ is a $2\times 2$ matrix. In particular, we will consider the following two families of $K$-matrices:

- Diagonal $\mathrm{U}(1)_{k_1} \times \mathrm{U}(1)_{k_2}$, with matrix $K = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$, and

- $\mathbb{Z}_{k_1}$ twisted gauge theory at level $k_2$, denoted by $(\mathbb{Z}_{k_1})_{k_2}$, with matrix $K = \begin{pmatrix} 0 & k_1 \\ k_1 & k_2 \end{pmatrix}$.

**Remark 4.4.2** The theory $(\mathbb{Z}_{k_1})_{k_2}$ is also known as Dijkgraaf-Witten theory when $k_2$ is even [35]. It admits a Chern-Simons gauge theory realization [205, 273]. One can show that any $2\times 2$ matrix $K$ with $\det(K) = -n^2$ for some integer $n$ can be brought into this form by a $\mathrm{GL}_2(\mathbb{Z})$ congruence transformation $G^t K G$ (see e.g. [274]). Furthermore, it is easy to show that $(\mathbb{Z}_{k_1})_{k_2} \sim (\mathbb{Z}_{k_1})_{k_2+2k_1}$, because the corresponding matrices are congruent[94].

We conjecture the following:

**Conjecture 4.4.1** *The diagonal theory* $\mathrm{U}(1)_{k_1} \times \mathrm{U}(1)_{k_2}$*:*

- *If* $k_1 k_2 > 0$*,*
  - *Never time-reversal invariant if* $k_1 k_2 = 0 \mod 4$*,*
  - *If* $k_1 k_2 = 2 \mod 4$*, say,* $k_1 = 2\tilde{k}_1$*, then the theory is* $\mathsf{T}$*-invariant if and only if* $k_2 \in \mu(\tilde{k}_1)\mathbb{T}$*, i.e., if* $\mu(\tilde{k}_1) = \mu(k_2)$*,*
  - *If* $k_1 k_2$ *is odd, then the theory is* $\mathsf{T}$*-invariant if and only if* $k_2 \in \mu(k_1)\mathbb{T}$*, i.e., if* $\mu(k_1) = \mu(k_2)$

[94] Given $(\mathbb{Z}_a)_b$, not all the theories in $b \in [0, 2a)$ are independent. For example, if $a$ is odd, one has the duality of spin TQFTs $(\mathbb{Z}_a)_b \longleftrightarrow (\mathbb{Z}_a)_{b+a}$. This follows from the more general duality

$$(\mathbb{Z}_a)_b \times \mathrm{U}(1)_k \longleftrightarrow (\mathbb{Z}_a)_{b+a} \times \mathrm{U}(1)_k \tag{4.4.26}$$

which holds if and only if $a = 2^\alpha(2m+1)$ and $k = 2^\alpha(2n+1)$ for some integers $\alpha, m, n$. The explicit change of variables is $G^t K_{a,b} G \equiv K_{a,a+b}$, where

$$G := \begin{pmatrix} -1 & m+n+2mn & 2n+1 \\ 0 & -1 & 0 \\ 0 & 2m+1 & 1 \end{pmatrix}, \quad K_{a,b} := \begin{pmatrix} 0 & a & 0 \\ a & b & 0 \\ 0 & 0 & k \end{pmatrix}. \tag{4.4.27}$$

- *If* $k_1k_2 < 0$,
  - *If* $k_1$ *is odd, the theory is* $\mathsf{T}$*-invariant if and only if* $k_2 \in \mu(k_1)\mathbb{T}$,
  - *If* $k_1 = 2\tilde{k}_1$ *is even, the theory is* $\mathsf{T}$*-invariant if and only if* $k_2 \in \mu(\tilde{k}_1)(\mathbb{T} \cup 2\mathbb{T})$.

**Conjecture 4.4.2** *The theory* $(\mathbb{Z}_{k_1})_{k_2}$ *is time-reversal invariant if and only if* $k_2 \in \mu(k_1)\mathbb{Z}$.

Some of these claims are easy to prove. For example, if $k_1$ and $k_2$ are both even and positive, then the theory $\mathrm{U}(1)_{k_1} \times \mathrm{U}(1)_{k_2}$ is bosonic and has central charge $+2$, and so it cannot be time-reversal invariant. More generally, the conditions above can be seen to be necessary just by insisting that the generating lines $\vec{e}_1, \vec{e}_2$ have a line with opposite spin. Proving that they are also sufficient requires more work, but in principle does not seem out of reach: an approach similar to the one-dimensional case $\mathrm{U}(1)_k$ should work. In any case, we checked that the conjecture is correct up to $|k_i| \leq 200$ in the diagonal case, and $|k_1| \leq 200$ and $k_2 \in [0, 2k_1)$ in the $(\mathbb{Z}_{k_1})_{k_2}$ gauge theory case. We stress that the diagonal theory can be be time-reversal invariant even when neither of the factors by itself is; naturally, this also holds for more general theories: a product may have more symmetries than its individual factors.

Note that if the conjecture above is true, then any odd non-Pythagorean prime factor of $\det(K)$ must appear an even number of times. In fact, it seems that this is true for any $2 \times 2$ matrix, whether it is of the forms above or not:

**Conjecture 4.4.3** *A necessary condition for the matrix* $K \in \mathbb{Z}^{2\times 2}$ *to describe a time-reversal invariant theory is that* $\lambda(\det(K)) \in \mathbb{T}$*, where* $\lambda(n)$ *denotes the squarefree part of* $n$ .

We recall that a number is said to be squarefree if its prime decomposition contains no repeated factors. We have checked that this conjecture is true for all matrices with $|\det(K)| \leq 500$. (For completeness, we remark that $\lambda(n) \in \mathbb{T}$ if and only if $n$ can be expressed as the sum of two perfect squares, not necessarily coprime).

It also appears that all primitive matrices with $\det(K) > 0$, if time-reversal invariant, have $\mathsf{T}^2 = \mathsf{C}$, as in the $1 \times 1$ case:

**Conjecture 4.4.4** *If* $K \in \mathbb{Z}^{2\times 2}$ *is positive definite and primitive (i.e. with* $\gcd(K_{ij}) = 1$ *for all* $i, j$*), then* $\mathsf{T}^2 = \mathsf{C}$.

We checked that this is true for all matrices with $\det(K) \leq 400$.

**Unitary Symmetries.**

An essentially identical philosophy allows us to study unitary symmetries rather than anti-unitary ones. Following an argument equivalent to that of proposition 4.4.1 it is easy to prove that

**Proposition 4.4.3** *Given some $K \in \mathbb{Z}^{n\times n}$, the most general unitary symmetry (i.e., a permutation subject to (4.4.10)) is of the form*

$$\mathsf{U}\colon \vec{\alpha} \mapsto Q\vec{\alpha} \tag{4.4.28}$$

*for some $Q \in \mathbb{Z}^{n\times n}$, invertible over $\mathcal{A}$, the $i$-th column of which represents $\mathsf{U}(\vec{e}_i)$, the action of the unitary symmetry on the unit vector in the $i$-th direction. Invariance of spin and braiding requires*

$$PK + Q^t K^{-1} QK = 1_n \tag{4.4.29}$$

*for some integral matrix $P$ with even diagonal components.*

There is always the trivial solution $Q = 1_n$, which leaves all the lines invariant, and its negative $Q = -1_n$, which corresponds to charge-conjugation $\mathsf{C}\colon \vec{\alpha} \mapsto -\vec{\alpha}$. Any other solution $Q$ (invertible modulo $K$) will correspond to some non-trivial unitary zero-form symmetry of the system.

We can finally write down the general expression for the group of symmetries of a given theory:

**Proposition 4.4.4** *Given an arbitrary abelian TQFT realized as a $\mathrm{U}(1)^n$ Chern-Simons theory with matrix of levels $K$, the group of (unitary and anti-unitary) zero-form symmetries can be expressed as*

$$\mathrm{Aut}(K) \cong \{Q \in \mathbb{Z}^{n\times n} \,|\, PK \pm Q^t K^{-1} QK = 1_n \text{ for some } P \in \mathbb{Z}^{n\times n}\}/\sim \tag{4.4.30}$$

*where $P$ is required to have even diagonal components, $Q$ is required to be invertible modulo $K$, and $\sim$ denotes the equivalence*

$$Q \sim Q' \qquad \Longleftrightarrow \qquad Q\vec{e}_i \sim Q'\vec{e}_i, \qquad i = 1, 2, \dots, n \tag{4.4.31}$$

*where the last $\sim$ denotes equivalence in $\mathcal{A}$ (cf. (4.4.6)). The subgroup of unitary symmetries is given by*

$$\mathrm{Aut}_U(K) \cong \{Q \in \mathbb{Z}^{n\times n} \,|\, PK + Q^t K^{-1} QK = 1_n \text{ for some } P \in \mathbb{Z}^{n\times n}\}/\sim \tag{4.4.32}$$

*with the same restrictions as before. A given symmetry $[Q]$ is quantum if and only if $P \neq 0$ for all $Q \in [Q]$.*

**Remark 4.4.3** Here we are making a slight abuse of notation in order to simplify the presentation: strictly speaking, if a given matrix $Q$ satisfies both $PK + Q^t K^{-1} QK = 1_n$ and $PK - Q^t K^{-1} QK = 1_n$ (possibly with different $P$'s), they are *different* symmetries, and so distinct elements of $\mathrm{Aut}(K)$. The same permutation on the anyons constitutes both a unitary, and an anti-unitary symmetry of the system. In other words, the group of symmetries is the *disjoint union* of the set of anti-unitary symmetries, and the set of unitary symmetries. In order to implement this, one should think of $\mathrm{Aut}(K)$ as pairs $(Q,\sigma)$, where $\sigma=\pm1$ keeps track of whether a given permutation is unitary or anti-unitary, and one must add the condition $\sigma(Q)=\sigma(Q')$ to the equivalence relation $\sim$.

We propose the following conjecture:

**Conjecture 4.4.5** *The group of unitary symmetries of $(\mathbb{Z}_k)_0$ is multiplicative in $k$:*

$$\mathrm{Aut}_U((\mathbb{Z}_{ab})_0)=\mathrm{Aut}_U((\mathbb{Z}_a)_0)\times\mathrm{Aut}_U((\mathbb{Z}_b)_0),\qquad \gcd(a,b)=1\,. \tag{4.4.33}$$

*Furthermore, for prime powers, it is given by*

$$\mathrm{Aut}_U((\mathbb{Z}_{\pi^n})_0)=D_{2\phi(\pi^n)},\qquad \mathrm{Aut}_U((\mathbb{Z}_{2^n})_0)=\mathbb{Z}_2\times D_{\phi(2^n)}\,, \tag{4.4.34}$$

*where $D_{2n}$ denotes the dihedral group of order $2n$. The full group of symmetries, including anti-unitary transformations, is a $\mathbb{Z}_2$ extension of the unitary sub-group:*

$$\mathrm{Aut}((\mathbb{Z}_k)_0)=\mathbb{Z}_2\ltimes\mathrm{Aut}_U((\mathbb{Z}_k)_0)\,. \tag{4.4.35}$$

**Remark 4.4.4** Note the similarity of this group and $\mathbb{Z}_k^\times:=\mathrm{Aut}(\mathbb{Z}_k)$, the multiplicative group of integers modulo $k$. As per a classic result of Gauss, this latter group is also multiplicative, and given by $\mathrm{Aut}(\mathbb{Z}_{\pi^n})=\mathbb{Z}_{\phi(\pi^n)}$ and $\mathrm{Aut}(\mathbb{Z}_{2^n})=\mathbb{Z}_2\times\mathbb{Z}_{\phi(2^n)/2}$. For $k=\pi$ a prime, the group $\mathrm{Aut}_U((\mathbb{Z}_\pi)_0)=D_{2(\pi-1)}$ has been computed in [275].

We next illustrate how to compute $\mathrm{Aut}(\star)$ step by step, through a couple of examples. More examples are worked out, to a lesser degree of detail, in section 4.5.

Consider the theory $(\mathbb{Z}_{k_1})_{k_2}$, whose matrix is

$$K=\begin{pmatrix}0 & k_1\\ k_1 & k_2\end{pmatrix} \tag{4.4.36}$$

where we can take without loss of generality $k_1>0$ and $0\le k_2<2k_1$. The theory is bosonic if $k_2$ is even, and spin otherwise. In the first case, the lines are of the form $(\alpha,\beta)\in\mathbb{Z}_{k_1}\times\mathbb{Z}_{k_1}$, and in the second case $(\alpha,\beta)\in\mathbb{Z}_{2k_1}\times\mathbb{Z}_{k_1}$. The spin of an arbitrary line is

$$h_{\alpha,\beta}=\frac{\alpha\beta}{k_1}-\frac{k_2\alpha^2}{2k_1^2} \tag{4.4.37}$$

A common notation for the lines of $(\mathbb{Z}_{k_1})_{k_2}$ is $\mathsf{e}_i=(i,0)$, called the *electric* lines, and $\mathsf{m}_j=(0,j)$, called the *magnetic* lines. Their product is $\mathsf{e}_i\mathsf{m}_j=(i,j)$. There are $i\in[0,k_1)$ electric lines if $k_2$ is even, and $i\in[0,2k_1)$ lines of odd; and $j\in[0,k_1)$ magnetic lines. (The electric line $\mathsf{e}_i = (i, 0)$ should not be confused with the unit vector $\vec{e}_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ in the $i$-th direction). The line $\mathsf{e}_{k_1} \equiv \psi$ is the transparent fermion, and so $\mathsf{e}_{i+k_1} \equiv \mathsf{e}_i \times \psi$.

Take for example $(\mathbb{Z}_3)_0$. A basis of lines is

$$\begin{aligned}\mathcal{A} &= \{(0,0),\ \ (1,0),\ \ (2,0),\ \ (0,1),\ \ (1,1),\ \ (2,1),\ \ (0,2),\ \ (1,2),\ \ (2,2)\}\\ &= \{\,\mathbf{1},\ \ \mathsf{e}_1,\ \ \mathsf{e}_2,\ \ \mathsf{m}_1,\ \ \mathsf{e}_1\mathsf{m}_1,\ \ \mathsf{e}_2\mathsf{m}_1,\ \ \mathsf{m}_2,\ \ \mathsf{e}_1\mathsf{m}_2,\ \ \mathsf{e}_2\mathsf{m}_2\}\end{aligned} \tag{4.4.38}$$

with spins

$$h = \{0,\ 0,\ 0,\ 0,\ 1/3,\ 2/3,\ 0,\ 2/3,\ 1/3\}\,. \tag{4.4.39}$$

Any endomorphism of the fusion algebra is of the form

$$g\colon \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \mapsto Q\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \qquad Q = \begin{pmatrix}g(\vec{e}_1) & g(\vec{e}_2)\end{pmatrix}\,. \tag{4.4.40}$$

As $\vec{e}_i$ both have vanishing spin, the condition $g\colon h \mapsto \pm h$ requires

$$g(\vec{e}_i) \in \{(1,0),\ (2,0),\ (0,1),\ (0,2)\} \tag{4.4.41}$$

and so there are $4^2 - 4 = 12$ candidates for the matrices $Q$:

$$\begin{array}{c|cccc} g(\vec{e}_i) & (1,0) & (2,0) & (0,1) & (0,2)\\ \hline (1,0) & \cdot & \begin{pmatrix}1&2\\0&0\end{pmatrix} & \begin{pmatrix}1&0\\0&1\end{pmatrix} & \begin{pmatrix}1&0\\0&2\end{pmatrix}\\ (2,0) & \begin{pmatrix}2&1\\0&0\end{pmatrix} & \cdot & \begin{pmatrix}2&0\\0&1\end{pmatrix} & \begin{pmatrix}2&0\\0&2\end{pmatrix}\\ (0,1) & \begin{pmatrix}0&1\\1&0\end{pmatrix} & \begin{pmatrix}0&2\\1&0\end{pmatrix} & \cdot & \begin{pmatrix}0&0\\1&2\end{pmatrix}\\ (0,2) & \begin{pmatrix}0&1\\2&0\end{pmatrix} & \begin{pmatrix}0&2\\2&0\end{pmatrix} & \begin{pmatrix}0&0\\2&1\end{pmatrix} & \cdot \end{array} \tag{4.4.42}$$

By explicit computation, one may check that the only endomorphisms that are actually automorphisms (i.e., the only matrices $Q$ that are invertible modulo $K$) are

$$\begin{gathered}\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}2&0\\0&2\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&2\\2&0\end{pmatrix}\\ \begin{pmatrix}1&0\\0&2\end{pmatrix}, \begin{pmatrix}2&0\\0&1\end{pmatrix}, \begin{pmatrix}0&2\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\2&0\end{pmatrix}\end{gathered} \tag{4.4.43}$$

and that the first line satisfies $K^{-1} - Q^t K^{-1} Q = P$, and the second one $K^{-1} + Q^t K^{-1} Q = P$, for some integral-valued matrix $P$. Therefore, the former generate unitary symmetries, and the latter anti-unitary symmetries.

One may check that the two matrices

$$\mathsf{T}\colon \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \qquad \mathsf{U}\colon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.4.44}$$

generate the whole group of symmetries, and they satisfy

$$\mathsf{T}^4 = \mathsf{U}^2 = (\mathsf{TU})^2 = 1 \tag{4.4.45}$$

and so the group of symmetries is dihedral:

$$\mathrm{Aut}((\mathbb{Z}_3)_0) = D_8 = \langle \mathsf{T}, \mathsf{U} \rangle\,. \tag{4.4.46}$$

Similarly, the pair of matrices $\mathsf{C} := \mathsf{T}^2$ and $\mathsf{U}$ generate the subgroup of unitary symmetries, and they satisfy

$$\mathsf{C}^2 = \mathsf{U}^2 = 1 \tag{4.4.47}$$

and so the latter is cyclic:

$$\mathrm{Aut}_U((\mathbb{Z}_3)_0) = \mathbb{Z}_2^2 = \langle \mathsf{C}, \mathsf{U} \rangle\,. \tag{4.4.48}$$

Consider now what happens when we turn on a non-trivial twisting, say, $(\mathbb{Z}_3)_2$. The spin of the lines is modified into

$$h = \{0,\, 8/9,\, 5/9,\, 0,\, 2/9,\, 2/9,\, 0,\, 5/9,\, 8/9\}\,. \tag{4.4.49}$$

As we can see, there is no line with spin $-8/9 = 1/9 \mod 1$, and so $\vec{e}_1$ has no partner under time-reversal: the theory does not admit anti-unitary symmetries. Therefore the symmetries, if any, must be unitary, and so they must fix the spin; thus, the condition $h \mapsto +h$ requires

$$\begin{aligned} \mathsf{U}(\vec{e}_1) &\in \{(1,0),(2,2)\} \\ \mathsf{U}(\vec{e}_2) &\in \{(0,1),(0,2)\} \end{aligned} \tag{4.4.50}$$

from where it follows that all the candidates for $Q$ are

$$\begin{array}{c|cc} \mathsf{U}(\vec{e}_i) & (1,0) & (2,2) \\ \hline (0,1) & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \\ (0,2) & \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix} \end{array} \tag{4.4.51}$$

One may check that all these matrices are invertible, but the only two that satisfy $K^{-1} - Q^t K^{-1} Q = P$ for some integral-valued matrix $P$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}\,. \tag{4.4.52}$$

Finally, the second matrix is easily seen to implement charge conjugation $\mathsf{C}$, and so it squares to the identity. In other words, the group of symmetries of the system is

$$\mathrm{Aut}((\mathbb{Z}_3)_2) = \mathrm{Aut}_U((\mathbb{Z}_3)_2) = \mathbb{Z}_2 = \langle \mathsf{C} \rangle \,. \tag{4.4.53}$$

By an identical argument one may calculate the group of symmetries of an arbitrary abelian theory. In table 4.2 we include the group of symmetries of $(\mathbb{Z}_{k_1})_{k_2}$ for small values of the levels.

| $k$ | $\mathrm{Aut}((\mathbb{Z}_k)_0)$ | $\mathrm{Aut}_U((\mathbb{Z}_k)_0)$ | $\mathrm{Aut}((\mathbb{Z}_k)_{\mu(k)})$ | $\mathrm{Aut}_U((\mathbb{Z}_k)_{\mu(k)})$ |
|---|---|---|---|---|
| 2 | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 |
| 3 | $D_8$ | $\mathbb{Z}_2^2$ | $D_8$ | $\mathbb{Z}_2^2$ |
| 4 | $D_8$ | $\mathbb{Z}_2^2$ | $D_8$ | $\mathbb{Z}_2^2$ |
| 5 | $\mathbb{Z}_4 \circ D_8$ | $D_8$ | $\mathbb{Z}_4$ | $\mathbb{Z}_2$ |
| 6 | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_2^3$ | $D_8$ | $\mathbb{Z}_2^2$ |
| 7 | $\mathbb{Z}_3 \rtimes D_8$ | $D_{12}$ | $\mathbb{Z}_3 \rtimes D_8$ | $D_{12}$ |
| 8 | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_2^3$ |
| 9 | $\mathbb{Z}_3 \rtimes D_8$ | $D_{12}$ | $\mathbb{Z}_3 \rtimes D_8$ | $D_{12}$ |
| 10 | $\mathbb{Z}_2 \times \mathbb{Z}_4 \circ D_8$ | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_4$ | $\mathbb{Z}_2$ |
| 11 | $\mathbb{Z}_5 \rtimes D_8$ | $D_{20}$ | $\mathbb{Z}_5 \rtimes D_8$ | $D_{20}$ |
| 12 | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\mathbb{Z}_2^4$ | $\mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\mathbb{Z}_2^4$ |
| 13 | $\mathbb{Z}_4 \circ D_{24}$ | $D_{24}$ | $\mathbb{Z}_4$ | $\mathbb{Z}_2$ |
| 14 | $\mathbb{Z}_2 \times \mathbb{Z}_3 \rtimes D_8$ | $\mathbb{Z}_2^2 \times S_3$ | $\mathbb{Z}_3 \rtimes D_8$ | $D_{12}$ |
| 15 | $D_8 \rtimes_5 D_8$ | $\mathbb{Z}_2^2 \times D_8$ | $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$ | $\mathbb{Z}_2^3$ |
| 16 | $\mathbb{Z}_4 \rtimes D_8$ | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_4 \rtimes D_8$ | $\mathbb{Z}_2 \times D_8$ |
| 17 | $\mathbb{Z}_4 \circ D_{32}$ | $D_{32}$ | $\mathbb{Z}_4$ | $\mathbb{Z}_2$ |
| 18 | $\mathbb{Z}_2 \times \mathbb{Z}_3 \rtimes D_8$ | $\mathbb{Z}_2^2 \times S_3$ | $\mathbb{Z}_3 \rtimes D_8$ | $D_{12}$ |
| 19 | $\mathbb{Z}_9 \rtimes D_8$ | $D_{36}$ | $\mathbb{Z}_9 \rtimes D_8$ | $D_{36}$ |
| 20 | $D_8 \rtimes_5 D_8$ | $\mathbb{Z}_2^2 \times D_8$ | $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_4$ | $\mathbb{Z}_2^3$ |
| 21 | $\mathbb{Z}_2^3 \rtimes_2 D_{12}$ | $\mathbb{Z}_2^3 \times S_3$ | $\mathbb{Z}_2^3 \rtimes_2 D_{12}$ | $\mathbb{Z}_2^3 \times S_3$ |
| 22 | $\mathbb{Z}_2 \times \mathbb{Z}_5 \rtimes D_8$ | $\mathbb{Z}_2^2 \times D_{10}$ | $\mathbb{Z}_5 \rtimes D_8$ | $D_{20}$ |
| 23 | $\mathbb{Z}_{11} \rtimes D_8$ | $D_{44}$ | $\mathbb{Z}_{11} \rtimes D_8$ | $D_{44}$ |
| 24 | $\mathbb{Z}_2 \times \mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\mathbb{Z}_2^5$ | $\mathbb{Z}_2 \times \mathbb{Z}_2^2 \wr \mathbb{Z}_2$ | $\mathbb{Z}_2^5$ |
| 25 | $\mathbb{Z}_4 \circ D_{40}$ | $D_{40}$ | $\mathbb{Z}_4$ | $\mathbb{Z}_2$ |
| 26 | $\mathbb{Z}_2 \times \mathbb{Z}_4 \circ D_{24}$ | $\mathbb{Z}_2 \times D_{24}$ | $\mathbb{Z}_4$ | $\mathbb{Z}_2$ |
| 27 | $\mathbb{Z}_9 \rtimes D_8$ | $D_{36}$ | $\mathbb{Z}_9 \rtimes D_8$ | $D_{36}$ |

Table 4.2: The group of symmetries of $(\mathbb{Z}_{k_1})_{k_2}$, denoted by $\mathrm{Aut}(\star)$, and its unitary subgroup $\mathrm{Aut}_U(\star)$, for $k_1 \in [0,27]$ and $k_2 = 0, \mu(k_1)$. For $k_2 \not\propto \mu(k_1)$ there are no anti-unitary symmetries. (See section 4.6 for basic definitions).

Similarly, in tables 4.3 and 4.4 we include the group of symmetries of the diagonal theory $\mathrm{U}(1)_{k_1} \times \mathrm{U}(1)_{k_2}$.

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_4$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_4$ |
| 3 | $\cdot$ | $SD_{16}$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $D_8$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $D_{12}$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_2^2$ |
| 4 | $\cdot$ | $\cdot$ | $D_8$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ |
| 5 | $\cdot$ | $\cdot$ | $\cdot$ | $\mathbb{Z}_4 \circ D_8$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_4 \times S_3$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_4 \times \mathbb{Z}_2$ |
| 6 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $D_{12}$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ |
| 7 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $SD_{32}$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ |
| 8 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^4$ | $\mathbb{Z}_2^3$ |
| 9 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\mathbb{Z}_{24} \rtimes \mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2 \times S_3$ | $\mathbb{Z}_2^2$ |
| 10 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_4 \times \mathbb{Z}_2$ |
| 11 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\mathbb{Z}_{24} \rtimes \mathbb{Z}_2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ |
| 12 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $D_8 \times D_8$ | $\mathbb{Z}_2^3$ |
| 13 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\mathbb{Z}_4 \circ D_{24}$ |

Table 4.3: The group of symmetries of the diagonal theory $K = \mathrm{diag}(k_1, k_2)$, to wit, $\mathrm{Aut}(\mathrm{U}(1)_{k_1} \times \mathrm{U}(1)_{k_2})$, for small values of $k_i$.

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2$ |
| 3 | $\cdot$ | $D_8$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $D_{12}$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_2^2$ |
| 4 | $\cdot$ | $\cdot$ | $D_8$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ |
| 5 | $\cdot$ | $\cdot$ | $\cdot$ | $D_8$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ | $D_{12}$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ |
| 6 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $D_{12}$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ |
| 7 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $D_{16}$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ |
| 8 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^4$ | $\mathbb{Z}_2^3$ |
| 9 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $D_{24}$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2 \times S_3$ | $\mathbb{Z}_2^2$ |
| 10 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\mathbb{Z}_2 \times D_8$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ |
| 11 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $D_{24}$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ |
| 12 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $D_8 \times D_8$ | $\mathbb{Z}_2^3$ |
| 13 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ | $D_{24}$ |

Table 4.4: The group of unitary symmetries of the diagonal theory $K = \mathrm{diag}(k_1, k_2)$, to wit, $\mathrm{Aut}_U(\mathrm{U}(1)_{k_1} \times \mathrm{U}(1)_{k_2})$, for small values of $k_i$.

### 4.4.2 Dualities of $\mathrm{U}(1)^n$

A straightforward extension of the formalism so far can be used to diagnose dualities between different abelian Chern-Simons theories. Given two systems described by matrices $K_1, K_2$, $n_1 \times n_1$ and $n_2 \times n_2$, respectively, the theories shall describe the same TQFT if they give rise to isomorphic anyon data $\mathcal{A}_i, \theta_i$. This corresponds to a bijection $f\colon \mathcal{A}_1 \leftrightarrow \mathcal{A}_2$ that preserves fusion and spin. If we write the anyons $\mathcal{A}_i$ as $n_i$-tuples of integral charges $\vec{\alpha}_i$, then preservation of fusion requires $f$ to act linearly, say, $\vec{\alpha}_1 = Q\vec{\alpha}_2$, with $Q$ an $n_1 \times n_2$ integral matrix that has a left inverse modulo $K_2$ (equivalently, $\vec{\alpha}_2 = \tilde{Q}\vec{\alpha}_1$ with $\tilde{Q}$ an $n_2 \times n_1$ integral matrix with left inverse modulo $K_1$). Preservation of spin requires the existence of an $n_2 \times n_2$ integral matrix $P$, with even diagonal components, such that

$$Q^t K_1^{-1} Q - K_2^{-1} = P\,. \tag{4.4.54}$$

The theories described by $K_1, K_2$ are dual if and only if such matrices $Q, P$ exist:

$$K_1 \quad \longleftrightarrow \quad K_2\,. \tag{4.4.55}$$

In light of this discussion, we can summarize the content of the previous sections as follows: a theory with matrix $K$ has a unitary symmetry if and only if there is a self-duality $K \leftrightarrow K$, and an anti-unitary symmetry if and only if there is a duality $K \leftrightarrow -K$.

For example, it is a well-known fact that $\mathrm{U}(1)_{-8}$ is level-rank dual (as bosonic TQFTs) to $\mathrm{SU}(8)_1$. This latter theory can be represented as an abelian Chern-Simons theory with $K$-matrix equal to the Cartan matrix of $\mathrm{SU}(8)$. In our terminology, this duality is implemented, for example, via the matrix $Q = (0, 0, 1, 0, 0, 0, 0)$, which indeed satisfies (4.4.54) with $P \equiv (2)$.

Other interesting examples of dual abelian theories can be found in twisted gauge theories $(\mathbb{Z}_a)_b$. Recall that if $a$ is odd, then $(\mathbb{Z}_a)_b \leftrightarrow (\mathbb{Z}_a)_{a+b}$, which is already clear at the Lagrangian level (see footnote 94). There are extra dualities that go beyond this trivial one, for example $(\mathbb{Z}_7)_2 \leftrightarrow (\mathbb{Z}_7)_4$ and $(\mathbb{Z}_7)_3 \leftrightarrow (\mathbb{Z}_7)_5$. One can easily check these dualities by finding a suitable $Q, P$ in (4.4.54).

Many more examples of (trivial and non-trivial) dualities between abelian theories can be exhibited. In contrast to the non-abelian case, abelian TQFTs enjoy infinitely-many dualities. For a given finite abelian group $\mathcal{A}$ and a quadratic form $\theta$ on it, there are infinitely many integral matrices, of varying dimension, that generate the pair $(\mathcal{A}, \theta)$, so all these matrices are dual. Trivially dual theories can be obtained by looking directly at the Lagrangian: matrices related by $\mathrm{GL}(\mathbb{Z})$-conjugation give rise to the same dynamics, $GK_1G^t = K_2$. Non-trivially dual theories, which are not dual at the Lagrangian level, require the generalized condition $Q^tK_1^{-1}Q - K_2^{-1} = P$, which allows for matrices of different dimension. In any case, fixing $K_1$, one can find infinitely-many matrices $K_2$ that are dual to it, just by varying $G$ or $P, Q$ in these equations.

## 4.5 Examples

Finally, we discuss some illustrative examples. To avoid repetition, we typically include a theory only if it incorporates a new feature that was not present in the previous examples. We begin by the case of a single abelian factor, $\mathrm{U}(1)_k$.

**Example 4.5.1** **($k=2$)** We have $\varpi(2)=0$, and so the system has no unitary symmetries. As the system is bosonic, there are no anti-unitary symmetries either.

One may regard the system as a spin TQFT, in which case it is usually known as the *semion-fermion theory* [108, 111]. The system now admits one anti-unitary symmetry, which can be found by solving $2p-q^2=1$, whose only solution in the range $q\in[0,2)$ is $q=1$. This means that the permutation is $s\leftrightarrow s\times\psi$, as is well-known.

We thus have

$$\begin{aligned}\mathrm{Aut}(\mathrm{U}(1)_2)=\mathrm{Aut}_U(\mathrm{U}(1)_2)=0\\ \mathrm{Aut}(\mathrm{U}(1)_2\times\{\mathbf{1},\psi\})=\mathbb{Z}_2=\langle\mathsf{T}\rangle\\ \mathrm{Aut}_U(\mathrm{U}(1)_2\times\{\mathbf{1},\psi\})=0\,.\end{aligned}\tag{4.5.1}$$

The integer $k=2$ is Pell, and so the time-reversal permutation above is a symmetry of the Lagrangian (provided by $\{\mathbf{1},\psi\}$ we mean $\mathrm{U}(1)_{-1}$ rather than $\mathrm{U}(1)_{+1}$).

**Example 4.5.2** **($k=3$)** We have $\varpi(3)=1$, and so the system only has one unitary symmetry: charge conjugation. This is a Lagrangian symmetry.

Similarly, $3\neq1\mod 4$, and so the system is not time-reversal invariant.

We thus have

$$\mathrm{Aut}(\mathrm{U}(1)_3)=\mathrm{Aut}_U(\mathrm{U}(1)_3)=\mathbb{Z}_2=\langle\mathsf{C}\rangle\,.\tag{4.5.2}$$

**Example 4.5.3** **($k=5$)** We have $\varpi(5)=1$, and so the system only has one unitary symmetry: charge conjugation. This is a Lagrangian symmetry.

The level satisfies $5=1\mod 4$, and so the system is time-reversal invariant. The permutation can be found using equation (4.7.2): $q=\frac{5-1}{2}!+5=7$ (there is a second solution, which differs by a sign: $q=-7=3\mod 10$). The explicit map of lines is

$$\begin{array}{c|c|c|c|c|c|c|c|c|c|c|}\alpha & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ \mathsf{T}(\alpha) & 0 & 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3\end{array}\tag{4.5.3}$$

We thus have

$$\begin{aligned}\mathrm{Aut}(\mathrm{U}(1)_5)=\mathbb{Z}_4=\langle\mathsf{T}\rangle\\ \mathrm{Aut}_U(\mathrm{U}(1)_5)=\mathbb{Z}_2=\langle\mathsf{C}\rangle\,.\end{aligned}\tag{4.5.4}$$

The integer $k=5$ is Pell, and so the time-reversal permutation above is a symmetry of the Lagrangian once we include the gravitational counterterm (but not without it).

**Example 4.5.4** **(**$k = 8$**)** We have $\varpi(8) = 1$, and so the system only has one unitary symmetry: charge conjugation. This is a Lagrangian symmetry.

The system is bosonic, and so it is not time-reversal invariant. One may regard the system as a spin TQFT, but $8 = 0 \mod 4$, and so it is not time-reversal invariant either.

As a spin TQFT, one has $\varpi(4) + 1 = 2$, and so the system has three unitary symmetries: charge-conjugation and multiplication by $\pm 3$. The latter are not Lagrangian symmetries.

We thus have

$$\begin{aligned} \mathrm{Aut}(\mathrm{U}(1)_8) = \mathrm{Aut}_U(\mathrm{U}(1)_8) = \mathbb{Z}_2 = \langle \mathsf{C} \rangle \\ \mathrm{Aut}(\mathrm{U}(1)_8 \times \{\mathbf{1}, \psi\}) = \mathrm{Aut}_U(\mathrm{U}(1)_8 \times \{\mathbf{1}, \psi\}) = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \mathsf{C}, \mathsf{U} \rangle \,. \end{aligned} \tag{4.5.5}$$

where $\mathsf{U}$ is either of $(\alpha, \beta) \mapsto (\pm 3\alpha, \alpha + \beta)$ (the other sign being $\mathsf{CU}$).

**Example 4.5.5** **(**$k = 12$**)** We have $\varpi(12) = 2$, and so the system has three unitary symmetries: charge conjugation and multiplication by $\pm 5$. The latter are not Lagrangian symmetries.

The system is bosonic, and so it is not time-reversal invariant. One may regard the system as a spin TQFT, but $12 = 0 \mod 4$, and so it is not time-reversal invariant either.

As a spin TQFT, one has $\varpi(6) + 1 = 2$, and so the unitary symmetries are the same as in the bosonic case. They are not Lagrangian symmetries either.

We thus have

$$\begin{aligned} \mathrm{Aut}(\mathrm{U}(1)_{12}) = \mathrm{Aut}_U(\mathrm{U}(1)_{12}) = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \mathsf{C}, \mathsf{U} \rangle \\ \mathrm{Aut}(\mathrm{U}(1)_{12} \times \{\mathbf{1}, \psi\}) = \mathrm{Aut}_U(\mathrm{U}(1)_{12} \times \{\mathbf{1}, \psi\}) = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \mathsf{C}, \mathsf{U} \rangle \,. \end{aligned} \tag{4.5.6}$$

where $\mathsf{U}$ denotes multiplication by either of $\pm 5$ (the other sign being $\mathsf{CU}$), while fixing the local fermion, if any.

**Example 4.5.6** **(**$k = 15$**)** We have $\varpi(15) = 2$, and so the system has three unitary symmetries: charge conjugation and multiplication by $\pm 11$. The latter are not Lagrangian symmetries.

The level can be factored as $15 = 3 \cdot 5$, and $3 \neq 1 \mod 4$, and so the system is not time-reversal invariant.

We thus have

$$\mathrm{Aut}(\mathrm{U}(1)_{15}) = \mathrm{Aut}_U(\mathrm{U}(1)_{15}) = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \mathsf{C}, \mathsf{U} \rangle \,. \tag{4.5.7}$$

where $\mathsf{U}$ denotes multiplication by either of $\pm 11$ (the other sign being $\mathsf{CU}$).

**Example 4.5.7** **(**$k = 24$**)** We have $\varpi(24) = 2$, and so the system has three unitary symmetries: charge conjugation and multiplication by $\pm 7$. The latter are not Lagrangian symmetries.

The system is bosonic, and so it is not time-reversal invariant. One may regard the system as a spin TQFT, but $24 = 0 \mod 4$, and so it is not time-reversal invariant either.

As a spin TQFT, one has $\varpi(12) + 1 = 3$, and so the number of unitary symmetries is doubled. The new symmetries, those that mix the bosonic lines with the transparent fermions, are generated by multiplication by 13.

We thus have

$$\begin{aligned} \mathrm{Aut}(\mathrm{U}(1)_{24}) &= \mathrm{Aut}_U(\mathrm{U}(1)_{24}) = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \mathsf{C}, \mathsf{U} \rangle \\ \mathrm{Aut}(\mathrm{U}(1)_{24} \times \{\mathbf{1}, \psi\}) &= \mathrm{Aut}_U(\mathrm{U}(1)_{24} \times \{\mathbf{1}, \psi\}) = \mathbb{Z}_2^3 = \langle \mathsf{C}, \mathsf{U}, \mathsf{U}' \rangle \,. \end{aligned} \tag{4.5.8}$$

where $\mathsf{U}$ denotes multiplication by either of $\pm 7$ (the other sign being $\mathsf{CU}$) while fixing the local fermion, if any, and $\mathsf{U}'$ denotes multiplication by 13, while mixing the local fermion.

**Example 4.5.8** **(**$k = 25$**)** We have $\varpi(25) = 1$, and so the system only has one unitary symmetry: charge conjugation. This is a Lagrangian symmetry.

The level can be factored as $25 = 5^2$, and $5 = 1 \mod 4$, which means that the system is time-reversal invariant. In order to find the solution to $q^2 = -1 \mod 25$ one may use Hensel lifting (4.7.4): the solutions modulo 5 are $\pm 3$, and so the solutions modulo $5^2$ are $\pm 3 \mp (3^2 + 1) = \pm 7$. The explicit map of lines is

$$\begin{array}{c|c|c|c|c|c|c|c|c|c|c|} \alpha & 0 & 1 & 2 & 3 & 4 & \cdots & 46 & 47 & 48 & 49 \\ \mathsf{T}(\alpha) & 0 & 7 & 14 & 21 & 28 & \cdots & 22 & 29 & 36 & 43 \end{array} \tag{4.5.9}$$

We thus have

$$\begin{aligned} \mathrm{Aut}(\mathrm{U}(1)_{25}) &= \mathbb{Z}_4 = \langle \mathsf{T} \rangle \\ \mathrm{Aut}_U(\mathrm{U}(1)_{25}) &= \mathbb{Z}_2 = \langle \mathsf{C} \rangle \,. \end{aligned} \tag{4.5.10}$$

The integer $k = 25$ is not Pell, and so the time-reversal permutation above is not a symmetry of the Lagrangian, not even if we include the gravitational counterterm.

**Example 4.5.9** **(**$k = 65$**)** We have $\varpi(65) = 2$, and so the system has three unitary symmetries: charge conjugation and multiplication by $\pm 51$. The latter are not Lagrangian symmetries.

The level can be factored as $65 = 5 \cdot 13$, and $5 = 13 = 1 \mod 4$, which means that the system is time-reversal invariant. In order to find the solution to $q^2 = -1 \mod 65$ one may use the Chinese Remainder Theorem (cf. the discussion below (4.7.4)). The solutions modulo 5 are $q = \pm 3$, and the solutions modulo 13 are $q = \pm 5$. Take for example the solution with $q = 3 \mod 5$ and $q = 5 \mod 13$; then, using the Euclidean algorithm, we find $5 \cdot 8 + 13 \cdot (-3) = 1$, which means that $q = 3 \cdot 13 \cdot (-3) + 5 \cdot 5 \cdot 8 = 83 \mod 65$. Similarly, taking the solution with $q = -3 \mod 5$ and $q = 5 \mod 13$ leads to $q = 57 \mod 65$. All in all, the solutions of $q^2 = -1 \mod 65$ are $q = \pm 47, \pm 57$. The explicit map of lines is

$$\begin{array}{c|c|c|c|c|c|c|c|c|c|c|} \alpha & 0 & 1 & 2 & 3 & 4 & \cdots & 126 & 127 & 128 & 129 \\ \mathsf{T}_1(\alpha) & 0 & 47 & 94 & 11 & 58 & \cdots & 72 & 119 & 36 & 83 \\ \mathsf{T}_2(\alpha) & 0 & 57 & 114 & 41 & 98 & \cdots & 32 & 89 & 16 & 73 \end{array} \tag{4.5.11}$$

We thus have

$$\begin{aligned}\operatorname{Aut}(\mathrm{U}(1)_{65}) &= \mathbb{Z}_4\times\mathbb{Z}_2 = \langle \mathsf{T},\mathsf{U}\rangle\\ \operatorname{Aut}_U(\mathrm{U}(1)_{65}) &= \mathbb{Z}_2\times\mathbb{Z}_2 = \langle \mathsf{C},\mathsf{U}\rangle\,.\end{aligned}\tag{4.5.12}$$

where $\mathsf{T}$ denotes either of $\mathsf{T}_1,\mathsf{T}_2$ (the other one being $\mathsf{UT}$), and $\mathsf{U}$ denotes multiplication by either of $\pm 51$ (the other sign being $\mathsf{T}^2\mathsf{U}$).

The integer $k=65$ is Pell, and so the permutation above is a symmetry of the Lagrangian once we include the gravitational counterterm (but not without it).

We now move on to $2\times 2$ matrices. We denote by $[a,b,c]$ the equivalence class (with respect to congruence) of all matrices of which $\begin{pmatrix}a&b\\b&c\end{pmatrix}$ is a representative. We begin with positive-definite $K$, and order them by $\det(K)$. (We recall that there can be more than one congruence class with a given value of $\det(K)$).

**Example 4.5.10** $(\det(K)=2)$ The first non-trivial positive-definite time-reversal invariant theory is $K=[2,0,1]$, where the permutation is $Q=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, which is of order 2. The system has no unitary symmetries. Therefore,

$$\begin{aligned}\operatorname{Aut}([2,0,1]) &= \mathbb{Z}_2 = \langle \mathsf{T}\rangle\\ \operatorname{Aut}_U([2,0,1]) &= 0\,.\end{aligned}\tag{4.5.13}$$

The transformation $\mathsf{T}$ is not a symmetry of the Lagrangian (because the central charge is 2), but it becomes one once we subtract two units of central charge (i.e., we consider the theory $\operatorname{diag}(K,-1,-1)$, which is dual to $\mathrm{U}(1)_2\times\mathrm{U}(1)_{-1}$).

There are no other $2\times 2$ congruence classes with determinant equal to 2.

**Example 4.5.11** $(\det(K)=5)$ The next positive-definite time-reversal invariant theories are $K=[5,0,1]$ and $K=[2,1,3]$, where the permutations are

$$\pm Q=\begin{pmatrix}3&0\\0&1\end{pmatrix}\tag{4.5.14}$$

and

$$\pm Q=\begin{pmatrix}1&2\\2&3\end{pmatrix}\tag{4.5.15}$$

respectively. They all satisfy $\mathsf{T}^2=\mathsf{C}$ (and thus are of order 4). The system has no non-trivial unitary symmetries. Therefore,

$$\begin{aligned}\operatorname{Aut}([5,0,1]) = \operatorname{Aut}([2,1,3]) &= \mathbb{Z}_4 = \langle \mathsf{T}\rangle\\ \operatorname{Aut}_U([5,0,1]) = \operatorname{Aut}_U([2,1,3]) &= \mathbb{Z}_2 = \langle \mathsf{C}\rangle\,.\end{aligned}\tag{4.5.16}$$

These are the only $2\times 2$ congruence classes with determinant equal to 5.

**Example 4.5.12** ($\det(K)=9$) The next positive-definite time-reversal invariant theory is $K=[3,0,3]$, where the permutations are

$$\pm Q=\begin{pmatrix}1&4\\2&1\end{pmatrix},\begin{pmatrix}4&2\\1&1\end{pmatrix}\tag{4.5.17}$$

all of which satisfy $\mathsf{T}^4=\mathsf{C}$ (and are thus of order 8), and

$$\pm Q=\begin{pmatrix}2&4\\1&1\end{pmatrix},\begin{pmatrix}4&1\\1&2\end{pmatrix}\tag{4.5.18}$$

all of which satisfy $\mathsf{T}^2=\mathsf{C}$ (and are thus of order 4).

Similarly, the non-trivial unitary symmetries are

$$\pm Q=\begin{pmatrix}0&5\\1&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}5&0\\0&1\end{pmatrix}\tag{4.5.19}$$

the first one of which squares to $\mathsf{C}$ (and is thus of order 4), and the other two are of order 2.

As it turns out, all these symmetries can be generated from just the two matrices

$$\mathsf{T}:\ \begin{pmatrix}1&4\\2&1\end{pmatrix},\qquad \mathsf{U}_1:\ \begin{pmatrix}0&1\\1&0\end{pmatrix}\tag{4.5.20}$$

which satisfy $\mathsf{T}^8=\mathsf{U}_1^2=1$ and $\mathsf{U}_1\mathsf{T}=\mathsf{T}^3\mathsf{U}_1$, and so the group of symmetries is semidihedral, $SD_{16}\cong\mathbb{Z}_8\rtimes\mathbb{Z}_2$. Similarly, the matrices $\mathsf{U}_1$ and $\mathsf{U}_2:=\mathsf{T}^2$ satisfy $\mathsf{U}_2^4=\mathsf{U}_1^2=1$ and $\mathsf{U}_2\mathsf{U}_1\mathsf{U}_2=\mathsf{U}_1$, and generate the whole group of unitary symmetries, and so the latter is dihedral, $D_8\cong\mathbb{Z}_4\rtimes\mathbb{Z}_2$. All in all, the group of symmetries is

$$\begin{aligned}\mathrm{Aut}([3,0,3])&=SD_{16}=\langle\mathsf{T},\mathsf{U}_1\rangle\\ \mathrm{Aut}_U([3,0,3])&=D_8=\langle\mathsf{U}_1,\mathsf{U}_2\rangle\,.\end{aligned}\tag{4.5.21}$$

The rest of binary forms with $\det(K)=9$ are $K=[1,0,9]$ and $K=[2,1,5]$, neither of which is time-reversal invariant. They have no non-trivial unitary symmetries either.

**Example 4.5.13** ($\det(K)=12$) The matrix $K=[4,2,4]$ has a unitary symmetry with $\mathsf{U}^6=1$ (and no time-reversal).

The non-trivial permutations are

$$\pm Q=\begin{pmatrix}1&3\\1&2\end{pmatrix},\begin{pmatrix}2&1\\3&1\end{pmatrix},\begin{pmatrix}1&2\\1&3\end{pmatrix},\begin{pmatrix}2&3\\3&2\end{pmatrix},\begin{pmatrix}3&1\\2&1\end{pmatrix}\tag{4.5.22}$$

which are of order $6,6,2,2,2$, respectively.

The whole group can be generated from one of the order 6 permutations, and one of the order 2 ones. They satisfy $\mathsf{U}_1^6 = \mathsf{U}_2^2 = 1$, together with $\mathsf{U}_1\mathsf{U}_2\mathsf{U}_1 = \mathsf{U}_2$, and so the group structure is dihedral:

$$\mathrm{Aut}([4,2,4]) = \mathrm{Aut}_U([4,2,4]) = D_{12} = \langle \mathsf{U}_1, \mathsf{U}_2 \rangle \,. \tag{4.5.23}$$

The rest of binary forms of the same determinant are $K = [1,0,12]$, $K = [2,0,6]$, and $K = [3,0,4]$, none of which is time-reversal invariant. One has

$$\begin{aligned} \mathrm{Aut}([1,0,12]) &= \mathrm{Aut}_U([1,0,12]) = \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \mathrm{Aut}([2,0,6]) &= \mathrm{Aut}_U([2,0,6]) = \mathbb{Z}_2 \\ \mathrm{Aut}([3,0,4]) &= \mathrm{Aut}_U([3,0,4]) = \mathbb{Z}_2 \times \mathbb{Z}_2 \,. \end{aligned} \tag{4.5.24}$$

**Example 4.5.14** ($\det(K) = 18$) The first positive-definite time-reversal invariant theory with a $\mathsf{T}$ such that $\mathsf{T}^2 \neq \mathsf{C}$ and $\det(K)$ not a perfect square is $K = [3,0,6]$, where the permutations are

$$\pm Q = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix} \tag{4.5.25}$$

both of which satisfy $\mathsf{T}^2 = \mathsf{C}$ (and thus are of order 4), and

$$\pm Q = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \tag{4.5.26}$$

both of which are of order 2, i.e. $\mathsf{T}^2 = 1$. If one chooses $\mathsf{T}^2 = (-1)^F$, the latter admit a well-defined $\mathbb{Z}_{16}$ anomaly, which is easily evaluated to be $\pm 2$.

The only non-trivial unitary symmetry is

$$\pm Q = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \tag{4.5.27}$$

which is of order 2.

If we denote by $\mathsf{T}$ one of the order 4 time-reversal symmetries, and by $\mathsf{U}$ one of the unitary ones, then one may check that these two operations generate the whole group of symmetries. One has $\mathsf{T}^4 = \mathsf{U}^2 = 1$ and $\mathsf{TUT} = \mathsf{U}$, and so

$$\begin{aligned} \mathrm{Aut}([3,0,6]) &= D_8 = \langle \mathsf{T}, \mathsf{U} \rangle \\ \mathrm{Aut}_U([3,0,6]) &= \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \mathsf{C}, \mathsf{U} \rangle \,. \end{aligned} \tag{4.5.28}$$

The rest of binary forms with $\det(K) = 18$ are $K = [1,0,18]$ and $K = [2,0,9]$, neither of which is time-reversal invariant. They have no non-trivial unitary symmetries either.

**Example 4.5.15** **(**$\det(K) = 49$**)** The first example of a time-reversal invariant theory where the order of the symmetry is greater than 8 is $K = [7, 0, 7]$, where the permutations are

$$\pm Q = \begin{pmatrix} 2 & 11 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 12 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 9 & 3 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 10 & 2 \\ 5 & 3 \end{pmatrix}, \tag{4.5.29}$$

all of which satisfy $\mathsf{T}^8 = \mathsf{C}$ (and thus are of order 16), and

$$\pm Q = \begin{pmatrix} 2 & 10 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 9 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 9 & 4 \\ 4 & 5 \end{pmatrix}, \begin{pmatrix} 10 & 5 \\ 5 & 4 \end{pmatrix}, \tag{4.5.30}$$

all of which satisfy $\mathsf{T}^2 = \mathsf{C}$ (and thus are of order 4).

The non-trivial unitary symmetries are

$$\pm Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 13 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 9 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 9 & 2 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 0 & 13 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 9 & 5 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 9 \\ 5 & 2 \end{pmatrix} \tag{4.5.31}$$

which are of order $2, 2, 2, 2, 4, 8, 8$, respectively.

If we let $\mathsf{T}$ denote one of the order 16 time-reversal permutations, and $\mathsf{U}_1$ one of the order 2 unitary permutations, then one may check that these two operations generate the whole group. Furthermore, one has $\mathsf{T}^{16} = \mathsf{U}_1^2 = 1$ and $\mathsf{U}_1\mathsf{T}\mathsf{U}_1 = \mathsf{T}^7$, and so the group is the semidihedral group of order 32. On the other hand, if we let $\mathsf{U}_2$ be one of the order 8 unitary symmetries, then one may check that these two operations generate the whole unitary group. One has $\mathsf{U}_2^8 = \mathsf{U}_1^2 = 1$ and $\mathsf{U}_2\mathsf{U}_1\mathsf{U}_2 = \mathsf{U}_1$, which is the dihedral group of order 16. All in all, the group of symmetries is

$$\begin{aligned} \mathrm{Aut}([7,0,7]) &= SD_{32} = \langle \mathsf{T}, \mathsf{U}_1 \rangle \\ \mathrm{Aut}_U([7,0,7]) &= D_{16} = \langle \mathsf{U}_1, \mathsf{U}_2 \rangle \,. \end{aligned} \tag{4.5.32}$$

The rest of binary forms with $\det(K) = 49$ are $K = [1, 0, 49]$, $K = [2, 1, 25]$, and $K = [5, \pm 1, 10]$, neither of which is time-reversal invariant. They have no non-trivial unitary symmetries either.

**Example 4.5.16** **(**$\det(K) = 50$**)** Take for example $K = [5, 0, 10]$. The anti-unitary symmetries are

$$\pm Q = \begin{pmatrix} 1 & 3 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 7 \\ 6 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 7 \\ 4 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 6 & 9 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 0 & 7 \end{pmatrix} \tag{4.5.33}$$

the first two of which satisfy $\mathsf{T}^6 = \mathsf{C}$ (and are thus of order 12), and the rest of which satisfy $\mathsf{T}^2 = \mathsf{C}$ (and are thus of order 4).

The non-trivial unitary symmetries are

$$\pm Q = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 6 \\ 8 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 6 \\ 2 & 7 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 8 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix} \tag{4.5.34}$$

the first two of which satisfy $\mathsf{U}^3 = \mathsf{C}$ (and are thus of order 6), and the rest of which are of order 2.

One may check that the three matrices

$$\mathsf{T}\colon \begin{pmatrix} 3 & 5 \\ 0 & 3 \end{pmatrix}, \quad \mathsf{U}_1\colon \begin{pmatrix} 7 & 4 \\ 2 & 7 \end{pmatrix}, \quad \mathsf{U}_2\colon \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix} \tag{4.5.35}$$

generate the whole group, and satisfy $\mathsf{T}^4 = \mathsf{U}_1^3 = \mathsf{U}_2^2 = [\mathsf{T}, \mathsf{U}_i] = (\mathsf{U}_1\mathsf{U}_2)^2 = 1$, and therefore

$$\begin{aligned} \mathrm{Aut}([5,0,10]) &= \mathbb{Z}_4 \times D_6 = \langle \mathsf{T}, \mathsf{U}_1, \mathsf{U}_2 \rangle \\ \mathrm{Aut}_U([5,0,10]) &= D_{12} = \langle \mathsf{T}^2\mathsf{U}_1, \mathsf{U}_2 \rangle\,. \end{aligned} \tag{4.5.36}$$

The rest of binary forms with $\det(K) = 50$ are $[6, \pm 2, 9]$, $[3, \pm 1, 17]$, $[1, 0, 50]$, and $[2, 0, 25]$, and they are all time-reversal invariant with symmetry group $\mathrm{Aut}(\star) = \mathbb{Z}_4 = \langle \mathsf{T} \rangle$ and $\mathrm{Aut}_U(\star) = \mathbb{Z}_2 = \langle \mathsf{C} \rangle$.

We now move on to $2 \times 2$ indefinite matrices.

**Example 4.5.17** ($\det(K) = -2$) The only binary form with $\det(K) = -2$ is $[1, 1, -1]$, which contains four lines. The theory has no non-trivial unitary permutations, and one anti-unitary one, effected by

$$Q = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.5.37}$$

which squares to the identity. Therefore,

$$\begin{aligned} \mathrm{Aut}([1,1,-1]) &= \mathbb{Z}_2 = \langle \mathsf{T} \rangle \\ \mathrm{Aut}_U([1,1,-1]) &= 0\,. \end{aligned} \tag{4.5.38}$$

When $\mathsf{T}^2 = (-1)^F$, this symmetry admits a well-defined $\mathbb{Z}_{16}$ anomaly, which is easily evaluated to be $\nu = \pm 2$.

**Example 4.5.18** ($\det(K) = -3$) The two binary forms are $K = [1, 1, -2]$ and $K = [2, 1, -1]$, neither of which admits an anti-unitary permutation. The unitary permutations are the trivial one, i.e.,

$$\begin{aligned} \mathrm{Aut}([1,1,-2]) = \mathrm{Aut}_U([1,1,-2]) = \mathbb{Z}_2 = \langle \mathsf{C} \rangle \\ \mathrm{Aut}([2,1,-1]) = \mathrm{Aut}_U([2,1,-1]) = \mathbb{Z}_2 = \langle \mathsf{C} \rangle\,. \end{aligned} \tag{4.5.39}$$

**Example 4.5.19** ($\det(K) = -4$) All the matrices are of the twisted gauge theory type, $K = [0, 2, k]$, with $k = 0, 1, 2, 3$. For $k$ odd there are no anti-unitary symmetries, while the unitary ones are trivial:

$$\mathrm{Aut}([0,2,k]) = \mathrm{Aut}_U([0,2,k]) = \mathbb{Z}_2 = \langle \mathsf{C} \rangle, \qquad k = 1, 3\,. \tag{4.5.40}$$

For $k$ even, there are anti-unitary symmetries. In particular, for $k = 0$ we have the trivial permutation and the electric-magnetic duality $\mathsf{e} \leftrightarrow \mathsf{m}$, as is well known. There is also the unitary symmetry $\mathsf{e} \leftrightarrow \mathsf{m}$, which can be obtained from composing the two anti-unitary symmetries. Similarly, for $k = 2$, the anti-unitary permutation is $\mathsf{m} \leftrightarrow \mathsf{em}$, and there are no unitary symmetries. In short,

$$\begin{aligned} \mathrm{Aut}([0,2,0]) &= \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \mathsf{T}, \mathsf{T}' \rangle \\ \mathrm{Aut}_U([0,2,0]) &= \mathbb{Z}_2 = \langle \mathsf{T}\mathsf{T}' \rangle \\ \mathrm{Aut}([0,2,2]) &= \mathbb{Z}_2 = \langle \mathsf{T} \rangle \\ \mathrm{Aut}_U([0,2,2]) &= 0\,. \end{aligned} \tag{4.5.41}$$

**Example 4.5.20** ($\det(K) = -5$) The representatives are $K = [2, 1, -2]$ and $K = [1, 2, -1]$. They both have a $\mathsf{T}^2 = \mathsf{C}$ permutation, and no non-trivial unitary symmetries. In other words,

$$\begin{aligned} \mathrm{Aut}([2,1,-2]) = \mathrm{Aut}([1,2,-1]) &= \mathbb{Z}_4 = \langle \mathsf{T} \rangle \\ \mathrm{Aut}_U([2,1,-2]) = \mathrm{Aut}_U([1,2,-1]) &= \mathbb{Z}_2 = \langle \mathsf{C} \rangle\,. \end{aligned} \tag{4.5.42}$$

**Example 4.5.21** ($\det(K) = -9$) All the matrices are of the twisted gauge theory type, $K = [0, 3, k]$, with $k = 0, \ldots, 5$. There are anti-unitary symmetries only for $k = 0, 3$:

$$\begin{aligned} \mathrm{Aut}([0,3,k]) &= D_8 \\ \mathrm{Aut}_U([0,3,k]) &= \mathbb{Z}_2 \times \mathbb{Z}_2 \end{aligned}, \qquad k = 0, 3 \tag{4.5.43}$$

while for the rest of levels the only symmetry is charge conjugation:

$$\mathrm{Aut}([0,3,k]) = \mathrm{Aut}_U([0,3,k]) = \mathbb{Z}_2, \qquad k = 1, 2, 4, 5\,. \tag{4.5.44}$$

**Example 4.5.22** ($\det(K) = -18$) The first example with time-reversal with order greater than 4 is $K = [3, 3, -3]$, whose anti-unitary permutations read

$$\pm Q = \begin{pmatrix} 2 & 5 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 6 & 7 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} 6 & 5 \\ 1 & 0 \end{pmatrix} \tag{4.5.45}$$

(which are of order $8, 8, 4, 4$), and whose non-trivial unitary permutations read

$$\pm Q = \begin{pmatrix} 5 & 4 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 5 & 6 \\ 4 & 1 \end{pmatrix} \tag{4.5.46}$$

(which are of order $4, 2, 2$). It is a simple exercise to check that

$$\begin{aligned} \mathrm{Aut}([3,3,-3]) &= SD_{16} \\ \mathrm{Aut}_U([3,3,-3]) &= D_8\,. \end{aligned} \tag{4.5.47}$$

The rest of the binary forms with the same determinant are $[1, 4, -2]$ and $[2, 4, -1]$, which have $\mathrm{Aut}(\star) = \mathrm{Aut}_U(\star) = \mathbb{Z}_2 = \langle \mathsf{C} \rangle$.

**Example 4.5.23** ($\det(K) = -20$) The next interesting example is $K = [4, 2, -4]$, which has

$$\begin{aligned} \mathrm{Aut}([4,2,-4]) &= \mathbb{Z}_4 \times D_6 \\ \mathrm{Aut}_U([4,2,-4]) &= D_{12}\,. \end{aligned} \tag{4.5.48}$$

The rest of binary forms with the same determinant are $[2, 4, -2]$, which has $\mathrm{Aut}(\star) = \mathbb{Z}_4$ and $\mathrm{Aut}_U(\star) = \mathbb{Z}_2$, and $[1, 4, -4]$ and $[4, 4, -1]$, which have $\mathrm{Aut}(\star) = \mathrm{Aut}_U(\star) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

**Example 4.5.24** ($\det(K) = -27$) Another interesting example is the pair $K = [3, 3, -6]$, $K = [6, 3, -3]$, which has

$$\mathrm{Aut}(\star) = \mathrm{Aut}_U(\star) = D_{12}\,. \tag{4.5.49}$$

The rest of binary forms with the same determinant are $[1, 5, -2]$ and $[2, 5, -1]$, which have $\mathrm{Aut}(\star) = \mathrm{Aut}_U(\star) = \mathbb{Z}_2$.

**Example 4.5.25** ($\det(K) = -49$) As 49 is a perfect square, these matrices are of the twisted gauge theory type. One has

$$\begin{aligned} \mathrm{Aut}([0,7,k]) &= \mathbb{Z}_3 \rtimes D_8 \\ \mathrm{Aut}_U([0,7,k]) &= D_{12} \end{aligned} \tag{4.5.50}$$

if $k \propto 7$, and

$$\mathrm{Aut}([0,7,k]) = \mathrm{Aut}_U([0,7,k]) = \mathbb{Z}_2 \tag{4.5.51}$$

otherwise.

**Example 4.5.26** ($\det(K) = -121$) The next interesting example is, again, of the twisted gauge theory type. One has

$$\begin{aligned} \mathrm{Aut}([0,11,k]) &= \mathbb{Z}_5 \rtimes D_8 \\ \mathrm{Aut}_U([0,11,k]) &= D_{20} \end{aligned} \tag{4.5.52}$$

if $k \propto 11$, and

$$\mathrm{Aut}([0,11,k]) = \mathrm{Aut}_U([0,11,k]) = \mathbb{Z}_2 \tag{4.5.53}$$

otherwise.

Finally, we consider a few higher-dimensional examples, chosen at random:

**Example 4.5.27** ($\det(K) = 16$) The theory with matrix

$$K = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \tag{4.5.54}$$

has

$$\begin{aligned} \mathrm{Aut}(\star) &= A_4 \rtimes D_8 \\ \mathrm{Aut}_U(\star) &= \mathbb{Z}_2 \times S_4\,. \end{aligned} \tag{4.5.55}$$

**Example 4.5.28** ($\det(K) = 36$) The theory with matrix

$$K = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 4 \end{pmatrix} \tag{4.5.56}$$

has

$$\begin{aligned} \mathrm{Aut}(\star) &= S_3 \times SD_{32} \\ \mathrm{Aut}_U(\star) &= S_3 \times D_8 \,. \end{aligned} \tag{4.5.57}$$

**Example 4.5.29** ($\det(K) = 48$) The theory with matrix

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 8 \end{pmatrix} \tag{4.5.58}$$

has

$$\mathrm{Aut}(\star) = \mathrm{Aut}_U(\star) = \mathbb{Z}_2^2 \times S_4 \,. \tag{4.5.59}$$

## 4.6 Notation and definitions.

For the convenience of the reader, we gather here some common definitions we use throughout the text.

We denote by $\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}$ the set of all integers, and by $\mathbb{T}, \mathbb{P}$ the two subsets

$$\begin{aligned} \mathbb{T} &:= \{k \in \mathbb{Z} \mid kp - q^2 = 1 \text{ for some } p, q \in \mathbb{Z}\} \\ \mathbb{P} &:= \{k \in \mathbb{Z} \mid kp^2 - q^2 = 1 \text{ for some } p, q \in \mathbb{Z}\} \,. \end{aligned} \tag{4.6.1}$$

One has $\mathbb{P} \subset \mathbb{T} \subset \mathbb{Z}$.

All primes greater than 2 are odd, and so they can be written as $4n \pm 1$ for some integer $n$. Those of the form $4n + 1$ are called *Pythagorean* (because they can be written as the sum of two squares, unlike those of the form $4n - 1$, as per Fermat's theorem).

The function $\phi \colon \mathbb{Z} \to \mathbb{Z}$ denotes the Euler totient function: $\phi(k)$ is the number of integers $q$ such that $0 < q < k$ and $\gcd(q, k) = 1$, where gcd denotes the greatest common divisor. In other words, there are $\phi(k)$ integers smaller than $k$ that are coprime to it. This function is multiplicative, $\phi(ab) = \phi(a)\phi(b)$ for any $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$, and is given by $\phi(\pi^n) = \pi^{n-1}(\pi - 1)$ for prime $\pi$ and integer $n$.

The function $\omega \colon \mathbb{Z} \to \mathbb{Z}$ counts the number of distinct prime factors, i.e. the prime decomposition of a given $k \in \mathbb{Z}$ reads

$$k \equiv \prod_{i=1}^{\omega(k)} \pi_i^{n_i} \,. \tag{4.6.2}$$

We also denote $\varpi(k) := \omega(k)$ if $k$ is odd, and $\varpi(k) := \omega(k/2)$ if even. For example,

$$\begin{aligned} &\omega(1) = 0, \quad \omega(2) = \omega(3) = \omega(4) = \omega(5) = 1, \quad \omega(6) = 2, \ldots \\ &\varpi(1) = \varpi(2) = 0, \quad \varpi(3) = \varpi(4) = \cdots = \varpi(11) = 1, \quad \varpi(12) = 2, \ldots \end{aligned} \tag{4.6.3}$$

The function $\mu\colon \mathbb{Z} \to \mathbb{Z}$ denotes the operation of removing the Pythagorean prime factors:

$$\mu(1) = 1, \quad \mu(2) = 2, \quad \mu(3) = 3, \quad \mu(4) = 4, \quad \mu(5) = 1, \quad \mu(6) = 6, \ldots \tag{4.6.4}$$

One has $k \in \mathbb{T}$ if and only if $\mu(k) = 1$ or $\mu(k) = 2$. The function $\lambda\colon \mathbb{Z} \to \mathbb{Z}$ denotes the squarefree part (i.e., $\lambda(k)$ is the smallest divisor of $k$ such that $k/\lambda(k)$ is a perfect square):

$$\lambda(1) = 1, \quad \lambda(2) = 2, \quad \lambda(3) = 3, \quad \lambda(4) = 1, \quad \lambda(5) = 5, \ldots, \lambda(8) = 2, \ldots \tag{4.6.5}$$

We denote by $\mathbb{Z}^{n\times n}$ the set of all integral $n \times n$ matrices, and by $\mathrm{GL}_n(\mathbb{Z}) \subset \mathbb{Z}^{n\times n}$ the subset of invertible matrices over $\mathbb{Z}$. A given matrix is invertible over $\mathbb{Z}$ if and only if its determinant is $\pm 1$, and so the elements of $\mathrm{GL}_n(\mathbb{Z})$ are known as *unimodular matrices.*

Given some set $A$ with some extra structure $\sigma$, we denote by $\mathrm{Aut}(A, \sigma) \subseteq S_A$ the set of all permutations of $A$ that "respect" the structure $\sigma$, and whose group operation is that inherited from $S_A$ (i.e., composition). For example, if $\times\colon A \times A \to A$ is a binary product such that $(A, \times)$ is a group, then $\mathrm{Aut}(A, \times)$ is the set of permutations that are group homomorphisms. Similarly, if $A$ is a group and $\theta\colon A \to U(1)$ is a quadratic form on it, $\mathrm{Aut}(A, \theta)$ denotes the set of automorphisms of $A$ that leave $\theta$ invariant, perhaps up to complex conjugation: $\theta(\pi(a)) = \theta(a)^{\pm 1}$ for all $a \in A$ and $\pi \in \mathrm{Aut}(A)$. If the data $(A, \theta)$ comes from a Chern-Simons theory with matrix $K$, we also use the notation $\mathrm{Aut}(K) \equiv \mathrm{Aut}(A, \theta)$, or even $\mathrm{Aut}(\mathrm{U}(1)_k)$ in the $1 \times 1$ case.

Given some unital ring $A$, we denote by $A^\times$ the group of units of $A$ – the set of its invertible elements. For example, one has $\mathrm{GL}_n(\mathbb{Z}) \equiv (\mathbb{Z}^{n\times n})^\times$.

The group $\mathbb{Z}_k$ denotes the cyclic group of order $k$, which consists of the set $\{0, 1, \ldots, k-1\}$, where the product operation is just addition, followed by reduction modulo $k$. One can also endow $\mathbb{Z}_k$ with integer product, which makes it into a ring (integer product is not usually invertible); the group of units is denoted by $\mathbb{Z}_k^\times$, and its order is $\phi(k)$.

We also recall some basic definitions from group theory, following [276].

**Definition 2.1.3** Let $N$ and $G$ be groups. Then an *action* of $G$ on $N$ is a homomorphism $\theta\colon G \to \mathrm{Aut}(N)$. This is described by saying that $G$ *acts* on $N$ or that $N$ is a *$G$-group.*

**Definition 2.1.4** Let $G$ and $N$ be groups such that $G$ acts on $N$ with action given by $\theta$. Then the *semi-direct product $N \rtimes_\theta G$ of $N$ by $G$ with this action* is defined as follows. The underlying set of $N \rtimes_\theta G$ is $G \times N$ and the multiplication is defined by $(g_1, n_2)(g_2, n_2) = (g_1 g_2, (n_1^{g_2\theta}) n_2)$.

**Definition 2.2.6** [...] A group $G$ is an *external central product* $H \circ K$ of two groups $H$ and $K$ if there exists an isomorphism $\theta\colon Z(H) \to Z(G)$ such that $G$ is $(H \times K)/N$ where $N = \{(h, h^{-1}\theta) \mid h \in Z(H)\}$.

**Definition 2.3.1** Let $G$ be a group and $\Omega$ a non-empty finite set. Then $G$ *acts on* $\Omega$ if, to each $\omega \in \Omega$ and $g \in G$, there corresponds a unique element $\omega^g \in \Omega$ such that, if $g_1$ and $g_2 \in G$ then $(\omega^{g_1})^{g_2} = \omega^{g_1 g_2}$; and $\omega^1 = \omega$. If $G$ acts on $\Omega$ then the *permutation representation* of $G$ corresponding to the action is the homomorphism $\rho\colon G \to \Sigma_\Omega$, the symmetric group on $\Omega$, defined by $\omega(g\rho) = \omega^g$ for all $\omega \in \Omega$ and all $g \in G$.

**Definition 2.3.2** Let $H$ be a group and $\Omega$ a non-empty finite set. Then $H^\Omega$ denotes the set of all maps from $\Omega$ to $H$. For $f_1, f_2 \in H$, define $f_1 f_2 \in H^\Omega$ by $\omega(f_1 f_2) = (\omega f_1)(\omega f_2)$ for all $\omega \in \Omega$.

**Definition 2.3.3** Let $H$ be a group, and $G$ be a finite group acting on a non-empty finite set $\Omega$. Then an action of $G$ on the group $H^\Omega$ is defined as follows. For each $g \in G$ and $f \in H^\Omega$, define $f^g \in H^\Omega$ by $\omega f^g = \omega^{g^{-1}} f$ for all $\omega \in \Omega$. The *(permutational) wreath product* $H \wr G$ of $H$ with $G$ corresponding to this action of $G$ on $\Omega$ is the split extension $H^\Omega \rtimes G$ with this action of $G$ on $H^\Omega$.

Finally, we define a few important finite groups (see e.g. Definition 2.1.11 in [276]):

- The *dihedral group* $D_{2n}$ of order $2n$ is defined by

$$D_{2n} = \langle x, y \mid y^n = x^2 = (xy)^2 = 1 \rangle \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2 \tag{4.6.6}$$

- The *semidihedral group* $SD_{2^{n+1}}$ of order $2^{n+1}$ is defined by

$$SD_{2^{n+1}} = \langle x, y \mid y^{2^n} = x^2 = (xy)^2 y^{2^{n-1}} = 1 \rangle \tag{4.6.7}$$

- The *symmetric group* $S_n$ of order $n!$, corresponding to all the permutations of $n$ objects, and its commutator subgroup $A_n$, of order $n!/2$, known as the *alternating group* and given by the even permutations of $S_n$. One has $S_n = A_n \rtimes \mathbb{Z}_2$ for $n \geq 5$.

## 4.7 Further results.

In this section we collect some further results concerning the theory $\mathrm{U}(1)_k$ which may prove useful in subsequent studies of this system. We begin by making some remarks concerning the set $\mathbb{T}$, defined as those integers $k$ such that $-1$ is a quadratic residue modulo $k$, i.e., those integers for which the equation $q^2 = -1 + pk$ is solvable for some integers $p, q$.

It is straightforward to show that any solution $(p, q)$ is such that $q$ is congruent to $q_0$ modulo $k$, where $(p_0, q_0)$ is a solution with $q_0 \in [0, k)$. More precisely, if $(p_0, q_0)$ is a solution, then so is $(P(n), Q(n))$ for any $n \in \mathbb{Z}$, where

$$\begin{aligned} P(n) &:= p_0 + 2q_0 n + kn^2 \\ Q(n) &:= q_0 + kn \end{aligned} \tag{4.7.1}$$

as is easily checked. This is not particular to our problem; the solutions to congruences of the form $f(q) = 0 \mod k$, for some polynomial $f\colon \mathbb{Z} \to \mathbb{Z}$, are always defined modulo $k$.

Generically speaking, this type of congruences are solved by first solving them modulo the prime divisors of $k$. Indeed, if $k$ is to divide $f(q)$, then so must its divisors. This means that the prime divisors of $k$ are essential in deciding whether $q^2 + 1 = 0 \mod k$ is solvable or not. To be precise, one of the key results concerning the set $\mathbb{T}$ is the following:

**Proposition 4.7.1** *A given $k$ is in $\mathbb{T}$ if and only if all its prime factors are Pythagorean (that is, congruent to 1 modulo 4), perhaps up to a single factor of 2.*

*Proof.* By reducing $kp = 1 + q^2$ modulo 4, and considering the odd $q$ and even $q$ cases separately, it becomes clear that $k$ cannot be a multiple of 4. Similarly, by Gaussian reciprocity, $-1$ is a quadratic residue modulo a prime $\pi$ if and only if $\pi$ is Pythagorean, and so $k$ cannot be a multiple of a non-Pythagorean prime either. This proves that the conditions above are necessary; proving that they are also sufficient can be done by explicitly constructing a solution $q$. We now sketch how this can be done.

First off, if $k$ is a Pythagorean prime, we can use Wilson's theorem to obtain an explicit expression for $q$. Indeed,

$$q = \left(\frac{k-1}{2}\right)! \tag{4.7.2}$$

satisfies $q^2 = -1 \mod k$. One can also take

$$q = (k-a)!! \tag{4.7.3}$$

where $a$ is any of $\{\pm 1, 2, 3\}$.

Lifting the solution to a prime power $k = \pi^n$ can be done using the Hensel lemma. If we let $q_1$ be the solution for $n = 1$, then the general solution can be obtained via the quadratic map

$$q_n = q_{n-1} - a(q_{n-1}^2 + 1) \tag{4.7.4}$$

where $a$ is a solution to $2q_1 a = 1 \mod \pi$ (e.g., $a = (2q_1)^{\pi-2}$, as per Fermat's little theorem).

Finally, finding a solution for arbitrary $k$ requires the use of the Chinese Remainder Theorem. For example, let $k = a_1 a_2$ with $a_1, a_2$ two prime powers. Then $q^2 = -1 \mod k$ requires $q^2 = -1 \mod a_i$, which by the previous paragraph has a solution $q_i$. With this, the solution of $q^2 = -1 \mod k$ is $q = q_1\alpha_1 a_2 + q_2\alpha_2 a_1 \mod k$, where $\alpha_1, \alpha_2$ are the Bézout coefficients for $a_1, a_2$ (i.e., a pair of integers such that $a_1\alpha_1 + a_2\alpha_2 = 1$, which can be computed using the Euclidean algorithm). By iteration we can easily find the solutions for an arbitrary integer $k = a_1 a_2 \ldots a_n$, and so the conditions in proposition 4.7.1 are also sufficient. $\square$

The integers $q$ that solve $q^2 = -1 \mod k$ implement the time-reversal permutations on the anyons of $\mathrm{U}(1)_k$. The lines $a \in \mathcal{A}$ that are fixed under time-reversal (modulo local operators) play a special role in analysing the time-reversal symmetry of a system (and its anomalies), see e.g. [64, 77]. We have the following:

**Proposition 4.7.2** *The only lines that satisfy $\mathsf{T}(a) \equiv a$ are the identity and the transparent fermion. If $k$ is odd, no line satisfies $\mathsf{T}(a) = a \times \psi$, while if $k$ is even, the only lines satisfying $\mathsf{T}(a) = a \times \psi$ are $a = k/2 \times \mathbf{1}$ and $a = k/2 \times \psi$.*

*Proof.* Any line fixed by $\mathsf{T}$ (perhaps up to $\psi$) has $a = \mathsf{T}^2(a) = \mathsf{C}(a)$. Let $k$ be odd; then lines fixed by $\mathsf{C}$ satisfy $2\alpha = 0 \mod 2k$, that is, $\alpha \propto k$. Both lines $\alpha = 0, k$ have $\mathsf{T}(\alpha) = \alpha$, and so there are no lines with $\mathsf{T}(a) = a \times \psi$.

Now let $k$ be even; then lines fixed by $\mathsf{C}$ satisfy $2\alpha = 0 \mod k$, that is, $\alpha \propto k/2$. One may check that $a = (0, \beta)$ satisfies $\mathsf{T}(a) = a$, and $a = (k/2, \beta)$ satisfies $\mathsf{T}(a) = a \times \psi$. $\square$

We thus see that the property $\mathsf{T}^2 = \mathsf{C}$ implies that the set of lines that are fixed by time-reversal is very small. More generally, it is possible to argue that, due to $\theta(\mathsf{T}(a)) = \theta(a)^*$, an anyon can only be fixed by $\mathsf{T}$ (perhaps up to $\psi$) if its spin is either $\theta(a) = \pm 1$ or $\theta(a) = \pm i$, i.e., if $h \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$. These are the bosons, fermions, semions, and anti-semions of the theory. For some purposes, it may be useful to know how many of these lines the theory supports. We have the following:

**Proposition 4.7.3** *Let $k \in \mathbb{Z}$, and denote by $N_h$ the number of lines of spin $h$ in $\mathrm{U}(1)_k$ (as a spin TQFT), and by $\lambda(k)$ the squarefree part of $k$. Then we have $N_0 = N_{1/2} = \sqrt{k/\lambda(k)}$. Furthermore, if we write $k = 2^e \tilde{k}$, with $\tilde{k}$ odd, then $N_{1/4} = N_{3/4} = 0$ if $e$ is even, and $N_{1/4} = N_{3/4} = \sqrt{k/\lambda(k)}$ if odd.*

*Proof.* We shall need the following trivial fact: given some integer $k \in \mathbb{Z}$, all solutions to the equation

$$\alpha^2 = k\beta, \qquad \alpha, \beta \in \mathbb{Z} \tag{4.7.5}$$

are of the form $(\alpha_n, \beta_n) = (n\sqrt{k\lambda(k)}, n^2\lambda(k))$ for some integer $n$. Indeed, if $k\beta$ is to be a perfect square, then $\beta$ must be proportional to $\lambda(k)$; and the constant of proportionality must itself be a perfect square.

We next count the bosons and fermions of $\mathrm{U}(1)_k$.

We begin with the $k$ odd case. An anyon $\alpha \in [0, 2k)$ has vanishing spin iff $\alpha^2 = 2k\beta$ for some integer $\beta$. All the solutions to this equation are of the form $\alpha = n\sqrt{2k\lambda(2k)}$ for some integer $n = 0, 1, \ldots, \lfloor \frac{2k-1}{\sqrt{2k\lambda(2k)}} \rfloor$. Therefore, there are

$$\left\lfloor \frac{2k-1}{\sqrt{2k\lambda(2k)}} \right\rfloor + 1 \equiv \sqrt{\frac{k}{\lambda(k)}} \tag{4.7.6}$$

bosons. Similarly, the fermions are given by the solutions to $\alpha^2 = k(2\beta+1)$, that is, $\alpha = n\sqrt{k\lambda(k)}$ with $n = 1, 3, \ldots, \lfloor \frac{2k-1}{\sqrt{k\lambda(k)}} \rfloor$. Therefore, there are

$$\frac{1}{2}\left(\left\lfloor \frac{2k-1}{\sqrt{k\lambda(k)}} \right\rfloor + 1\right) \equiv \sqrt{\frac{k}{\lambda(k)}} \tag{4.7.7}$$

fermions.

We now move on the the $k$ even case. The bosons in the spin theory come from the bosons and fermions in the non-spin theory. The former solve $\alpha^2 = 2k\beta$ and the latter solve $\alpha^2 = k(2\beta+1)$. Together, they solve $\alpha^2 = k\beta$, that is, $\alpha = n\sqrt{k\lambda(k)}$, with $n = 0, 1, \ldots, \lfloor \frac{k-1}{\sqrt{k\lambda(k)}} \rfloor$. Therefore, there are

$$\left\lfloor \frac{k-1}{\sqrt{k\lambda(k)}} \right\rfloor + 1 \equiv \sqrt{\frac{k}{\lambda(k)}} \tag{4.7.8}$$

bosons. The counting of the fermions is identical.

A very similar argument proves the claim for the semions. For $k$ odd, the counting is straightforward. For $k$ even, one is to count the spin 1/4 and 3/4 lines in the bosonic theory, which solve $2\alpha^2 = k(2p+1)$. Writing $k = 2^e\tilde{k}$, with $\tilde{k}$ odd, it is clear that no solutions exist for $e$ even (because $\sqrt{2}$ is not integral). For $e$ odd, the solution is $\alpha = 2^{(e-1)/2}n\sqrt{\tilde{k}\lambda(\tilde{k})}$, with $n = 1, 3, \ldots, \lfloor \frac{2^e\tilde{k}-1}{2^{(e-1)/2}\sqrt{\tilde{k}\lambda(\tilde{k})}} \rfloor$. Thus, there are

$$\frac{1}{2}\left(\left\lfloor \frac{2^e\tilde{k}-1}{2^{(e-1)/2}\sqrt{\tilde{k}\lambda(\tilde{k})}} \right\rfloor + 1\right) \equiv \sqrt{\frac{k}{\lambda(k)}} \tag{4.7.9}$$

spin $h = 1/4$ lines in the spin theory, and as many spin 3/4 lines. □

A similar technique can be applied to counting other lines $N_h$.

We now move on to the so-called Pell numbers:

**Definition 4.7.1** An integer $k$ is said to be *Pell* if there exists a pair of integers $p, q$ such that $kp^2 - q^2 = 1$. The set of Pell numbers is denoted by $\mathbb{P}$.

We include here some known facts about Pell numbers, the first few of which are $k = 1, 2, 5, 10, 13, 17, 26, 29, \ldots$:

- No perfect square other than 1 is ever Pell. (Indeed, $n^2 - m^2 > 2m$ for $n > m > 0$, and so this expression cannot equal 1).

- All Pell numbers are in $\mathbb{T}$ (but the converse is not true; the first few exceptions are $\mathbb{T} \setminus \mathbb{P} = \{25, 34, 146, 169, 178, 194, \ldots\}$).

- A squarefree integer $k$ is Pell iff the fundamental unit $\sigma$ of $\mathbb{Q}(\sqrt{k})$ has norm $-1$. The rest of units are of the form $\pm\sigma^n$ for some integer $n$ (see e.g. [277], theorem 11.4.1).

- $k$ is Pell iff the convergents of $\sqrt{k}$ have odd period. If $(p_0, q_0)$ denotes the fundamental solution, then the rest of solutions are $q_n + p_n\sqrt{k} = (q_0 + p_0\sqrt{k})^{2n+1}$ (see e.g. [271], theorems 5.15 and 5.16). Equivalently,

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} q_0 & p_0 \\ kp_0 & q_0 \end{pmatrix}^{2n} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \tag{4.7.10}$$

  (Note that the determinant of this matrix is $-1$, and so its odd powers generate *positive* norm units).

- $k$ is Pell iff it can be written as $k = a^2 + b^2$ for relatively prime $a, b \in \mathbb{Z}$, with $b$ odd, and such that the Gauss-type Diophantine equation $b(V^2 - W^2) - 2aVW = 1$ is solvable with $V, W \in \mathbb{Z}$ [278].

- Let $\pi$ denote a prime not congruent to 3 mod 4. Then any integer of the form $k = \pi$, or $k = \pi_1\pi_2$ with $(\pi_1, \pi_2) = -1$, is Pell (where $(\cdot, \cdot)$ is the Legendre symbol; see e.g. [277], theorem 11.5.7). Furthermore, any odd integer of the form $k = \pi_1\pi_2\cdots\pi_{2n+1}$ such that there is no triplet $(a, b, c)$ with $(\pi_a, \pi_b) = (\pi_b, \pi_c) = +1$, is Pell [279].

Pell numbers appear naturally in the study of the time-reversal properties of $\mathrm{U}(1)_k$. For example, one has the following:

**Proposition 4.7.4** *If $kk'$ satisfies the Pell equation the theory $\mathrm{U}(1)_k \times \mathrm{U}(1)_{-k'}$ is time-reversal invariant.*

*Proof.* Assume that

$$kk'p^2 - q^2 = 1, \qquad p, q \in \mathbb{Z} \tag{4.7.11}$$

Let

$$4\pi\mathcal{L} = ka\,\mathrm{d}a - k'b\,\mathrm{d}b \tag{4.7.12}$$

and introduce the $\mathrm{GL}_2(\mathbb{Z})$ transformation

$$\mathsf{T}\colon \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} q & k'p \\ kp & q \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{4.7.13}$$

The Lagrangian becomes

$$\mathsf{T}\colon 4\pi\mathcal{L} \mapsto -k\, a\, \mathrm{d}a + k'\, b\, \mathrm{d}b \tag{4.7.14}$$

as required. □

Taking $k' = 1$ leads to the invariance of $\mathrm{U}(1)_k \times \mathrm{U}(1)_{-1}$ (cf. proposition 4.3.6). Moreover, this result, together with conjecture 4.4.1, leads to the following interesting purely number-theoretic conjecture:

**Conjecture 4.7.1** *An integer $k$ satisfies $q^2 = -1 \mod k$ for some $q \in \mathbb{Z}$ if and only if there exists some Pell integer $k'$ such that $kk'$ is also Pell.*

Recall that any solution of $q^2 = -1 + pk$ is of the form $p = p_0 + 2q_0 n + kn^2$ (cf. (4.7.1)). If $p$ is Pell for some $n$, then it suffices to take $k' = p$, from where the conjecture would follow (because $kp = q^2 + 1$ is automatically Pell). Noting that whenever this polynomial is prime, it is also Pell, our conjecture actually follows from the so-called Hardy-Littlewood "conjecture F" [280], which states that $ax^2 + bx + c$ is prime infinitely often unless $b^2 - 4ac$ is a perfect square or $a + b$ and $c$ are both even (neither condition being satisfied by our polynomials). It is widely believed that the Hardy-Littlewood conjecture is true, which implies that our conjecture – being much weaker – should be true as well.

There is a more specific result due to Lemke Oliver and Iwaniec [281, 282] that states that a polynomial of the type above represent primes or semiprimes infinitely often. But any prime, or any semiprime $\pi_1\pi_2$ with $(\pi_1, \pi_2) = -1$ is Pell. Having no reason to expect otherwise, one is lead to conjecture that both options $(\pi_1, \pi_2) = \pm 1$ appear with the same probability – which is confirmed by numerical analysis – from where it would follow that $p_0 + 2q_0 n + kn^2$ generates infinitely many Pell numbers. In fact, the only possibility for a failure of our conjecture is that this polynomial never represents a prime (disproving the Hardy-Littlewood conjecture), and that all the semiprimes it represents have $(\pi_1, \pi_2) = +1$. This is extremely unlikely, but we have no proof that it cannot happen.

In any event, we checked that the conjecture is true for $k$ up to $10^9$. For now it remains an interesting open question.

If the conjecture is true, we can in fact invert the logic and use the time-reversal invariance of $\mathrm{U}(1)_k \times \mathrm{U}(1)_{-k'}$ to argue that of $\mathrm{U}(1)_k$, for any $k \in \mathbb{T}$, by mimicking the argument of proposition 4.3.5.

**Added note:** An unconditional proof of conjecture 4.7.1 has been discussed in MathOverflow.

# Chapter 5

# Fermionic TQFTs in three dimensions.

**Authorship.** The first two sections of this chapter are taken almost verbatim from the paper [1], written in collaboration with Davide Gaiotto and Jaume Gomis. We also append here a few extra sections that were developed around the same time but not included in the original version of the paper.

**Abstract.** We systematically study the construction of three-dimensional topological theories that depend on the spin structure of spacetime. We explain how the Hilbert space is obtained, and how large diffeomorphisms act on it, which in turn is enough to calculate the partition function on arbitrary three-manifolds via surgery. The action of line operators (Wilson lines, a.k.a. anyons) on the fermionic Hilbert space is described. We also discuss the fermionic version of the Verlinde formula on arbitrary genus and with arbitrary punctures. The general picture involves some novel ingredients, for example, the use of some $F$-symbols of the bosonic parent of the TQFT.

## 5.1 Spin TQFTs and anyon condensation

In this section we outline the construction of TQFTs that depend on the spin structure of the underlying manifold. The strategy we will pursue is the following. Given one such theory, one may sum over all spin structures to yield a bosonic TQFT. This corresponds to gauging the zero-form symmetry generated by fermion parity $\mathbb{Z}_2 = \langle(-1)^F\rangle$. This gauging generates a dual $\mathbb{Z}_2$ $(d-2)$-form symmetry, whose gauging takes us back to the original spin TQFT. Therefore, any spin TQFT can be constructed by gauging a certain $(d-2)$-form symmetry in a bosonic TQFT,[95] and we reduce the problem of constructing spin TQFTs to the more familiar problem of gauging a higher-form symmetry in regular (bosonic) TQFTs. We shall follow this strategy in $d = 3$ spacetime dimensions, where one can be quite explicit, thanks to the powerful formalism of modular tensor categories and two-dimensional chiral algebras.

With this in mind, we begin by reviewing known facts about $3d$ TQFTs, and the gauging of one-form symmetries. From the $2d$ point of view this corresponds to extending the chiral algebra by a simple current, and in the condensed-matter language to (abelian) anyon condensation.

Consider a $3d$ bosonic TQFT. The most basic observable of the theory is the partition function $Z(M)$, where $M$ is a compact 3-manifold. For example, if the manifold takes the form $M = \mathbb{S}^1 \times \Sigma$, with $\mathbb{S}^1$ a circle representing the time direction, and $\Sigma$ a compact surface, then the partition function computes the dimension of the Hilbert space assigned, by canonical quantization, to the spatial slice:

$$Z(\mathbb{S}^1 \times \Sigma) = \dim(\mathcal{H}(\Sigma)) \,. \tag{5.1.1}$$

The observables of the TQFT depend only on the topology of $M$, and therefore diffeomorphisms of $\Sigma$ must act unitarily in $\mathcal{H}(\Sigma)$. Transformations that are continuously connected to the identity act trivially, so effectively we get a unitary representation of the mapping class group, the group of (equivalence classes of) large diffeomorphisms. If one understands the Hilbert space $\mathcal{H}(\Sigma)$, and the action of the MCG on it, one can compute – via surgery – the partition function on an arbitrary 3-manifold $M$.

With this in mind, our main task is to understand the Hilbert space assigned by a TQFT to a compact Riemann surface $\Sigma$, and how Dehn twists act on it. The basic data of the TQFT that determines this information is the following:

- The set of anyons $\mathcal{A}$, a finite set. This set contains a distinguished anyon, the vacuum $\mathbf{1}$.

[95] For example, in $d = 2$ this corresponds to a zero-form symmetry. This has been studied recently, see e.g. [117, 166, 167]. In $d = 4$ one gauges a two-form symmetry, cf. e.g. [283]. One should keep in mind that, potentially, an anomaly could make these gaugings ill-defined, e.g. if summing over spin structures leads to an identically vanishing partition function. This subtlety shall play no role in this work.

- The modular matrix $S\colon \mathcal{A}\times\mathcal{A}\to\mathbb{C}$.
- The topological spin $\theta = e^{2\pi i h}\colon \mathcal{A}\to \mathrm{U}(1)$.

The full data of the TQFT involves other, more subtle objects, the so-called $F$- and $R$-symbols. These will play a role later on; for now, the $S$-matrix is enough. By a key result of Verlinde, the dimension of $\mathcal{H}(\Sigma)$ is determined by $S$ as follows [38, 39]:

$$\dim(\mathcal{H}(\Sigma)) = \sum_{\alpha\in\mathcal{A}} S_{\mathbf{1},\alpha}^{\chi(\Sigma)}\,, \tag{5.1.2}$$

where $\chi$ denotes the Euler characteristic ($\chi(\Sigma_g) = 2-2g$ for a genus $g$ surface $\Sigma_g$). In particular, the torus has $\chi(\Sigma_1)=0$, which means that $\mathcal{H}(\Sigma_1)\cong\mathbb{C}[\mathcal{A}]$, i.e., a basis of states of the torus Hilbert space is labelled by the anyons of the TQFT. The MCG of the torus, $SL_2(\mathbb{Z}) = \langle S,T\rangle$, is generated by $S$ and $T := e^{-2\pi i c/24}\,\mathrm{diag}(\theta)$.

The theory also admits line defect operators, also labelled by $\mathcal{A}$. Namely, we can wrap an anyon $\alpha\in\mathcal{A}$ around the time circle $\mathbb{S}^1$, which produces a defect Hilbert space $\mathcal{H}(\Sigma^\alpha)$. From the point of view of the spatial surface, the anyon $\alpha$ looks like a marked point. Given a family of such punctures $\alpha_1,\dots,\alpha_n$, the generalization of the Verlinde formula is [38]

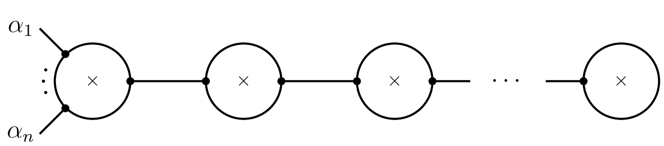
$$\dim(\mathcal{H}(\Sigma^{\alpha_1\cdots\alpha_n})) = \sum_{\alpha\in\mathcal{A}} S_{\mathbf{1},\alpha}^{\chi(\Sigma)}\prod_{i=1}^{n} S_{\alpha_i,\alpha}\,, \tag{5.1.3}$$

where $\chi(\Sigma_g^{\alpha_1\cdots\alpha_n}) = 2-2g-n$ for a surface with $g$ handles and $n$ boundary components. The most fundamental surface is the so-called *trinion*, i.e., a sphere with three punctures. This surface defines the *fusion coefficients*:

$$\mathcal{N}_{\alpha_1,\alpha_2,\alpha_3} := \dim(\mathcal{H}(\Sigma_0^{\alpha_1\alpha_2\alpha_3})) \equiv \sum_{\beta\in\mathcal{A}} \frac{S_{\alpha_1,\beta}S_{\alpha_2,\beta}S_{\alpha_3,\beta}}{S_{\mathbf{1},\beta}}\,, \tag{5.1.4}$$

which endows $\mathcal{A}$ with a product structure, leading to the *fusion ring* of the TQFT. The partition function on an arbitrary surface can be computed by gluing trinions (cf. the "pants decomposition"). Using unitarity of $S$ one can recover the general case (5.1.3) from the trinion (5.1.4).

An explicit basis of states on $\mathcal{H}(\Sigma_g^{\alpha_1\cdots\alpha_n})$ can be written as follows:




, (5.1.5)

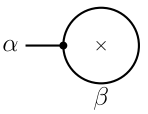
where each cross $\times$ represents a handle of $\Sigma_g$. Each segment carries an orientation and an anyon label (which we omit to simplify the notation). Each trivalent vertex with incoming anyons $\alpha, \beta, \gamma$ carries an internal vector index, taking values in $1, 2, \ldots, \mathcal{N}_{\alpha,\beta,\gamma}$, which we always leave implicit.

In short, the states of $\mathcal{H}(\Sigma_g^{\alpha_1 \cdots \alpha_n})$ can be represented as labelled oriented graphs with $g$ cycles; leaves labelled by the punctures $\alpha_1, \ldots, \alpha_n$; edges labelled by anyons $\alpha \in \mathcal{A}$; and trivalent vertices labelled by internal vector indices taking $\mathcal{N}_{\alpha,\beta,\gamma}$ values, as determined by the incident edges $\alpha, \beta, \gamma$.

Different bases of the Hilbert space are related to the one above by the $F$- and $R$-moves, effected by the aforementioned $F$- and $R$-symbols. These are in correspondence with the different pants decompositions of the surface.

Of particular relevance is the case of the torus with a single puncture, whose states we label as $|\beta; \alpha\rangle \in \mathcal{H}(\Sigma_1^\alpha)$, corresponding to the configuration

$$|\beta; \alpha\rangle \;=\;$$

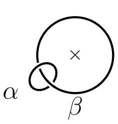



$$. \qquad (5.1.6)$$

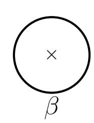
There is one such state for each possible vertex, i.e., the degeneracy of $|\beta; \alpha\rangle$ is given by the fusion coefficient $\mathcal{N}_{\alpha,\beta,\bar{\beta}}$. In particular, the diagram is allowed only if $\alpha \times \beta \propto \beta + \cdots$, i.e., if $\beta$ may "absorb" the puncture $\alpha$. The case of no punctures corresponds to the vacuum anyon $\alpha = \mathbf{1}$, so that all $\beta \in \mathcal{A}$ are allowed, and they all carry degeneracy $\mathcal{N}_{\mathbf{1},\beta,\bar{\beta}} = 1$. For non-trivial $\alpha$, some $\beta \in \mathcal{A}$ may not contribute to $\mathcal{H}(\Sigma_1^\alpha)$, and some other $\beta \in \mathcal{A}$ may contribute more than one state.

Given a basis of states for $\mathcal{H}(\Sigma_1)$ one can write down the operators acting on this space as matrices. In particular, the Wilson loop operators admit such a representation. Let $W^{(\mathsf{c})}(\alpha)$ denote the Wilson loop labelled by the anyon $\alpha \in \mathcal{A}$ running through the cycle $\mathsf{c} \in H_1(\Sigma_1, \mathbb{Z}) = \mathbb{Z}[\mathsf{a}] \oplus \mathbb{Z}[\mathsf{b}]$, where $\mathsf{a}, \mathsf{b}$ are the standard homology cycles. These operators act by inserting $\alpha$ along the given cycle, e.g.

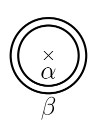
$$W^{(\mathsf{a})}(\alpha)|\beta\rangle \;=\;$$



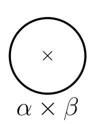

$$=\; \frac{S_{\alpha,\beta}}{S_{\mathbf{1},\beta}}$$




$$W^{(\mathsf{b})}(\alpha)|\beta\rangle \;=\;$$




$$=$$




$$(5.1.7)$$

whence

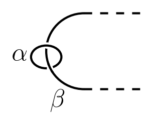
$$\begin{aligned} \langle\beta'|W^{(\mathsf{a})}(\alpha)|\beta\rangle &= \delta_{\beta\beta'} \frac{S_{\alpha\beta}}{S_{\mathbf{1},\beta}} \\ \langle\beta'|W^{(\mathsf{b})}(\alpha)|\beta\rangle &= \mathcal{N}^{\beta'}{}_{\alpha\beta} \end{aligned} \qquad (5.1.8)$$

Naturally, given that $S$ interchanges the cycles $\mathsf{a}$ and $\mathsf{b}$ (up to a sign), one has

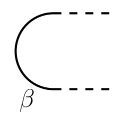
$$W^{(\mathsf{a})}(\alpha) = S W^{(\mathsf{b})}(\alpha) S^\dagger, \qquad W^{(\mathsf{b})}(\alpha) = S W^{(\mathsf{a})}(\bar{\alpha}) S^\dagger \qquad (5.1.9)$$

which is just the statement that the characters $S_{\alpha\beta}/S_{\mathbf{1},\beta}$ diagonalize the fusion rules.

The higher-genus case is analogous. Given a basis of $\mathcal{H}(\Sigma_g^{\alpha_1\cdots\alpha_n})$ one can express the Wilson lines as matrices. As above, a Wilson loop $W^{(\mathsf{c})}(\alpha)$ inserts the anyon $\alpha$ along the cycle $\mathsf{c}\in H_1(\Sigma_g^{\alpha_1\cdots\alpha_n},\mathbb{Z})$. For example, the $\mathsf{a}$-cycles are identical to the torus, inasmuch as wrapping an anyon in the orthogonal cycle is a local operation: one can shrink it to a point. The value of $W^{(\mathsf{a}^i)}$ acting on a given state only depends on the line running through the segment orthogonal to $\mathsf{a}^i$, irrespective of what the rest of the state is doing:

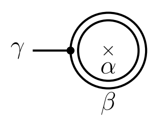



$$= \frac{S_{\alpha,\beta}}{S_{\mathbf{1},\beta}} \tag{5.1.10}$$

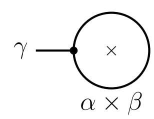



The $\mathsf{b}$-cycles, on the other hand, cannot be shrunk, and so depend on the entire configuration around such cycle: the lines running therein, and the punctures going in and out. For example, the once-punctured torus has [122]

$$W^{(\mathsf{b})}(\alpha)|\beta;\gamma\rangle \;=\;$$




$$= \; F_{\beta,\alpha\times\beta}\begin{bmatrix}\gamma & \alpha\times\beta\\ \beta & \alpha\end{bmatrix} \tag{5.1.11}$$




Configurations with more punctures carry more factors of $F$. The matrix elements of arbitrary configurations of Wilson lines, on surfaces with arbitrary genus and arbitrary punctures, is entirely determined in terms of the TQFT data of the theory. Generalizing (5.1.9), the lines around the different cycles are unitarily related through the MCG of the surface.

The invertible defects – the abelian punctures – correspond to group symmetries of the theory. These are line operators, so the symmetry is a higher-form symmetry [32], in this case a one-form symmetry. Gauging the symmetry corresponds to summing over all possible insertions of the defect. This produces a new TQFT, whose set of anyons $\hat{\mathcal{A}}$ and modular data $\hat{S}$ are fixed in terms of the data of the ungauged theory. Making this procedure precise is the goal of the rest of this section.

The one-form symmetry group is always a finite abelian group, i.e., a product of cyclic groups. Each abelian anyon in $\mathcal{A}$ generates a cyclic subgroup; condensing this anyon means gauging this subgroup. If we are interested in gauging a product of cyclic groups, we can always condense a single generator at a time, iteratively. We can therefore assume without loss of generality that the group to be gauged is cyclic, say, $\mathbb{Z}_n^{(1)}=\langle\mathsf{g}\rangle$, with $\mathsf{g}\in\mathcal{A}$ a certain abelian anyon. The $\mathbb{Z}_n^{(1)}$ symmetry partitions the spectrum $\mathcal{A}$ into $n$ equivalence classes, according to their braiding with respect to $\mathsf{g}$:

$$\mathcal{A}=\bigsqcup_{q=0}^{n-1}\mathcal{A}_q,\qquad \mathcal{A}_q:=\{\alpha\in\mathcal{A}\mid B(\mathsf{g},\alpha)=e^{2\pi i q/n}\}\,, \tag{5.1.12}$$

where $B(\mathsf{g},\alpha) := S_{\mathsf{g},\alpha}/S_{\mathbf{1},\alpha} \equiv \theta(\mathsf{g}\times\alpha)/\theta(\mathsf{g})\theta(\alpha)$ is the braiding phase with respect to $\mathsf{g}$. The modular data of the theory behaves nicely with respect to this grading, e.g. [284]

$$S_{\mathsf{g}^i\times\alpha,\mathsf{g}^j\times\beta} = B(\mathsf{g},\mathsf{g})^{ij} B(\mathsf{g},\alpha)^j B(\mathsf{g},\beta)^i S_{\alpha,\beta} \tag{5.1.13}$$

The 't Hooft anomaly of $\mathbb{Z}_n^{(1)}$ is given by $B(\mathsf{g},\mathsf{g})$, which must equal 1 if the symmetry is to be gauged. In this situation, one can prove that $\theta(\mathsf{g}) = \pm 1$, i.e., the generator is either a boson or a fermion. In the former case, the gauging yields another bosonic TQFT, while in the latter case the theory acquires a dependence on the spin structure, i.e., it becomes a spin TQFT. For now, we assume that $\mathsf{g}$ is a boson, and return to the more interesting case of fermionic quotients later on.

### 5.1.1 Boson anyon condensation

We begin with some bosonic TQFT with anyons $\mathcal{A}$ and modular matrix $S$, and wish to condense some boson $\mathsf{g} \in \mathcal{A}$, to produce some other bosonic TQFT, with anyons $\hat{\mathcal{A}}$ and new modular matrix $\hat{S}$. The standard lore of this procedure is as follows. First, in order to construct $\hat{\mathcal{A}}$, one performs the following three steps [285]:

1. Select the set of neutral lines, $\mathcal{A}_0$ (cf. (5.1.12)), i.e., those with trivial braiding with respect to $\mathsf{g}$.
2. Identify any two lines that are in the same $\mathbb{Z}_n^{(1)}$-orbit, i.e., if they differ by the action of $\mathsf{g}^j$ for some $j \in \mathbb{Z}_n$.
3. If a given $\mathbb{Z}_n^{(1)}$-orbit has less than $n$ elements, it splits into several different anyons in $\hat{\mathcal{A}}$. Specifically, if the length is $\ell$, then the orbit descends to $n/\ell$ copies in the condensed theory.

In what follows we shall describe the geometric interpretation of these rules, which will allow us to compute the modular data of the condensed theory from first principles, with no need to introduce any ansätze. It also admits a natural extension to spin TQFTs which shines a new light – and goes beyond – what is currently understood about such theories.

The main idea to obtain $\hat{\mathcal{A}}$ is to find the torus Hilbert space of the condensed theory, from which one can read off the set of anyons by writing down a basis of vectors (recall that $\mathcal{H}(\Sigma_1) \cong \mathbb{C}[\mathcal{A}]$). The condensed theory is obtained by gauging the $\mathbb{Z}_n$ one-form symmetry, which means we are to insert the generator $\mathsf{g}$ in all possible ways. Summing over all insertions $\mathsf{g}^j$, $j = 0, 1, \dots, n-1$, along the spatial cycles project the Hilbert space into the invariant states. Insertions along the temporal cycles introduce twisted sectors. We will next see that insertions along the three cycles in $M = \mathbb{S}^1 \times \Sigma_1$ indeed reproduce the three steps above.

Let us begin with the time circle. Inserting $\mathsf{g}^j$ along the time direction means taking the torus with a puncture labelled by $\mathsf{g}^j$. Therefore, the states of the condensed theory are generically of the form $|\alpha; \mathsf{g}^j\rangle$ for some $\alpha \in \mathcal{A}$. In other words, the Hilbert space of the condensed theory must be a subspace of the Hilbert space of the original theory, in the presence of an arbitrary puncture:

$$\hat{\mathcal{H}}(\Sigma_1) \subseteq \bigoplus_{j=0}^{n-1} \mathcal{H}(\Sigma_1^{\mathsf{g}^j}) \,. \tag{5.1.14}$$

The states of $\mathcal{H}(\Sigma_1^{\mathsf{g}^j})$ are labelled by anyons $\alpha \in \mathcal{A}$ with the property $\mathsf{g}^j \times \alpha = \alpha$. In particular, $j$ must be proportional to the length of the $\mathbb{Z}_n^{(1)}$-orbit of $\alpha$. We shall denote this orbit by $[\alpha]$, and its length $\ell_\alpha := |[\alpha]|$ equals the minimal integer such that $\mathsf{g}^{\ell_\alpha} \times \alpha = \alpha$. This integer divides $n$, and any other integer $j$ with $\mathsf{g}^j \times \alpha = \alpha$ is of the form $j = \ell_\alpha k$, for some integer $k = 0, 1, \ldots, n/\ell_\alpha - 1$. This reproduces the third condition above, namely if a given orbit is shorter than $\ell_\alpha = n$, it descends to $n/\ell_\alpha$ copies in the condensed theory. The copies just label the number of insertions of the symmetry defect we use to create the state.

Let us now move on to the spatial circles; insertions of the symmetry elements along these circles shall project into the invariant subspace. In this case, the meaning of *invariant* depends on which cycle we insert the symmetry element on; a symmetry along the $\mathsf{a}$-cycle acts via braiding, and along the $\mathsf{b}$-cycle via fusion. In the end, we must have states that are invariant under $\mathsf{g}$, both with respect to braiding and to fusion. Let us discuss these two cases in turn.

- Take first the $\mathsf{a}$-cycle, which is the circle that is orthogonal to the one we use to create states. Given a state created by a line $\alpha \in \mathcal{A}$ running along the $\mathsf{b}$-cycle, the configuration we obtain by inserting $\mathsf{g}$ is $B(\alpha, \mathsf{g})|\alpha; \mathsf{g}^{\ell_\alpha k}\rangle$ (cf. (5.1.10)). The phase $B(\alpha, \mathsf{g})$ equals $e^{2\pi i q/n}$ for $\alpha \in \mathcal{A}_q$ (cf. (5.1.12)). Summing over all insertions $\mathsf{g}^j$ produces the phase

$$\begin{aligned} \sum_{j=0}^{n-1} B(\alpha, \mathsf{g}^j) &= \sum_{j=1}^{n-1} e^{2\pi i q j/n} \\ &= n\delta_{q,0} \end{aligned} \tag{5.1.15}$$

  which indeed projects to the states with $q = 0$, i.e., to $\alpha \in \mathcal{A}_0$. We thus reproduce the first condition, namely the states in the condensed theory must be neutral under $\mathbb{Z}_n^{(1)}$, i.e., taken from $\mathcal{A}_0$.

- Finally, if we now consider the second spatial circle, the $\mathsf{b}$-cycle, and insert $\mathsf{g}^j$, we obtain $|\mathsf{g}^j \times \alpha; \mathsf{g}^{\ell_\alpha k}\rangle$. Summing over all $j$ (and normalizing to have unit norm) leads to

$$|[\alpha], k\rangle := \frac{1}{\sqrt{\ell_\alpha}} \sum_{j=0}^{\ell_\alpha - 1} |g^j \times \alpha; g^{k\ell_\alpha}\rangle, \qquad [\alpha] \in \mathcal{A}_0/\!\sim,\ k \in \mathbb{Z}_{n/\ell_\alpha} \,. \tag{5.1.16}$$

  which is indeed invariant under $\mathsf{g}$, where now this symmetry acts via fusion (i.e., $\alpha \mapsto \mathsf{g} \times \alpha$). We thus reproduce the second condition, namely the fact that the anyons of the condensed theory $\hat{\mathcal{A}}$ are labelled by $\mathbb{Z}_n^{(1)}$ orbits.

We thus see that, as expected, the insertions along the three circles indeed reproduce the three conditions we are used to. All in all, a basis of states is labelled by a pair of indices: $[\alpha]$, denoting a $\mathbb{Z}_n^{(1)}$ orbit of neutral lines $\alpha \in \mathcal{A}_0$, plus a degeneracy label taking values in $k = 0, 1, \ldots, n/\ell_\alpha - 1$. This degeneracy label, arguably the most subtle ingredient so far, is in fact quite natural from the point of view of gauging $\mathbb{Z}_n^{(1)}$: $k\ell_\alpha$ just denotes how many copies of the $\mathsf{g}$-puncture we insert in order to create the state, i.e., from which twisted Hilbert space the state comes from.

The presentation of $\hat{\mathcal{H}}(\Sigma_1)$ above also gives us a natural way to compute the modular data of the condensed theory, in particular, the modular matrix $\hat{S}$. Specifically, a modular transformation acting on a state $|[\alpha]; k\rangle$ is nothing but the $S$-matrix of the uncondensed theory, in the presence of a puncture $g^{k\ell_\alpha}$:

$$\langle[\alpha]; k|\hat{S}|[\alpha']; k'\rangle = \delta_{k\ell_\alpha, k'\ell_{\alpha'}} \sqrt{\ell_\alpha \ell_{\alpha'}}\, S_{\alpha,\alpha'}(\mathsf{g}^{k\ell_\alpha})\,. \tag{5.1.17}$$

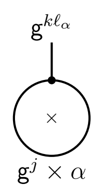
The modular matrix in the once-punctured torus can be expressed in terms of the $F$-symbols of the parent theory, namely [122]

$$S_{\alpha,\alpha'}(\mathsf{g}^j) = \sum_{\beta \in \alpha\times\alpha'} \frac{\theta(\beta)}{\theta(\alpha)\theta(\alpha')} S_{\mathbf{1},\beta} F_{\alpha,\alpha'} \begin{bmatrix} \mathsf{g}^j & \alpha' \\ \alpha & \beta \end{bmatrix}\,. \tag{5.1.18}$$

The basis (5.1.16) of $\hat{\mathcal{H}}(\Sigma_1)$ is natural because it makes $\hat{S}$ block-diagonal, but it does not correspond to the anyon basis. The most obvious way to see this is that the would-be quantum dimension $d_{[\alpha];k} = \hat{S}_{[\alpha];k,[\mathbf{1}];0}/\hat{S}_{[\mathbf{1}];0,[\mathbf{1}];0}$ vanishes for $k \neq 0$.

In order to identify the anyon basis we can look at the dual $\mathbb{Z}_n^{(0)}$ symmetry. The charged states are those with the puncture. More precisely, in the diagonal basis the states transform as $\mathbb{Z}_n^{(0)}$: $|[\alpha]; k\rangle \mapsto e^{2\pi i k/\ell_\alpha}|[\alpha]; k\rangle$. In the anyon basis, this symmetry should act as a cyclic permutation of the anyons, that is, as $\mathbb{Z}_n^{(0)}$: $|[\alpha]; \hat{k}\rangle \mapsto |[\alpha]; \hat{k}+1\rangle$ for some label $\hat{k}$. We conclude that the anyon basis is in fact nothing but the Fourier transform (Pontryagin dual) of the diagonal basis (5.1.16):

$$\begin{aligned} |[\alpha], \hat{k}\rangle :&= \frac{1}{\sqrt{n/\ell_\alpha}} \sum_{k=0}^{n/\ell_\alpha - 1} e^{2\pi i \hat{k} k \ell_\alpha/n} |[\alpha]; k\rangle \\ &= \frac{1}{\sqrt{n}} \sum_{k=0}^{n/\ell_\alpha-1} \sum_{j=0}^{\ell_\alpha - 1} e^{2\pi i \hat{k} k \ell_\alpha/n} \end{aligned}$$




, (5.1.19)

where $\hat{k} \in \mathbb{Z}^*_{n/\ell_\alpha}$. In this basis, the quantum dimension takes the expected value $d_{[\alpha];\hat{k}} = (\ell_\alpha/n) S_{\alpha,\mathbf{1}}/S_{\mathbf{1},\mathbf{1}} = (\ell_\alpha/n) d_\alpha$. The anyons of the quotient $\hat{\mathcal{A}}$ create the states $|[\alpha], \hat{k}\rangle$ by acting on the vacuum.

Given the matrix $\hat{S}$ in the fusion basis, one can use the Verlinde formula to compute the dimension of the Hilbert space of the condensed theory, for an arbitrary Riemann surface, with an arbitrary number of punctures. In the particular case of no external punctures, the formula only involves matrix elements with the vacuum, in which case the matrix with punctures $S(\mathsf{g}^j)$ does not contribute except for the vacuum insertion, that is, the regular $S$ matrix of the uncondensed theory. In other words, the dimension of the Hilbert space of the condensed theory, in the case of no punctures, can conveniently be computed using only the $S$ matrix of the parent theory, without the need to know the $F$-symbols:

$$\begin{aligned}\dim(\hat{\mathcal{H}}(\Sigma_g)) &= \sum_{\alpha\in\hat{\mathcal{A}}} \hat{S}_{\mathbf{1},\alpha}^{\chi(\Sigma)} \\ &\equiv \sum_{\alpha\in\mathcal{A}_0} \frac{n}{\ell_\alpha^{2g}} S_{\mathbf{1},\alpha}^{\chi(\Sigma)}\,.\end{aligned} \tag{5.1.20}$$

It is possible to generalize the expressions above to the case where there is a non-trivial background flux for the $\mathbb{Z}_n^{(0)}$ magnetic symmetry dual to the gauged $\mathbb{Z}_n^{(1)}$ symmetry. For example, if the flux of such background is $q \in \mathbb{Z}_n$, then the states are created from the subset $\mathcal{A}_q$ instead of $\mathcal{A}_0$. Summing over all such backgrounds, i.e., gauging the $\mathbb{Z}_n^{(0)}$ symmetry, takes us back to the original ungauged theory $\mathcal{A}$. We shall not need this generalization here.

### 5.1.2 Fermion anyon condensation

We now move on to the more interesting case of fermion condensation: we have some bosonic TQFT, and we wish to condense a certain abelian fermion, which we denote by $\psi \in \mathcal{A}$. We can assume without loss of generality that this line generates a $\mathbb{Z}_2^{(1)}$ symmetry, i.e. $\psi^2 = \mathbf{1}$, for otherwise we can first condense the boson $\mathsf{g} = \psi^2$ (as in the previous section), and then condense the resulting fermion, which will satisfy $\psi^2 = \mathbf{1}$.

The philosophy underlying fermion condensation is essentially the same as in boson condensation: we construct the Hilbert space of the condensed theory from the states in the parent theory, perhaps in presence of $\psi$-punctures. Roughly speaking, the configurations with non-trivial background flux can be thought of as the different spin structures on $\Sigma$.

Before actually constructing the spin TQFT by condensing a fermion in a bosonic TQFT, let us discuss what we are to expect from this condensation in the first place. A spin TQFT should assign to manifolds of the form $\mathbb{S}^1 \times \Sigma$ a super-vector space $\hat{\mathcal{H}}(\Sigma)$, which depends only on the topology of $\Sigma$, together with its spin structure $s$. (We use a hat to denote the Hilbert space of the condensed theory, and reserve the notation $\mathcal{H}(\Sigma)$ for that of the bosonic parent). Depending on the spin structure on the time circle $\mathbb{S}^1$, the partition function computes either the regular trace over $\hat{\mathcal{H}}(\Sigma)$, or the super-trace (i.e., the trace weighted by fermion parity).

Specifically, the spin generalization of (5.1.1) is

$$
\begin{aligned}
Z(\mathbb{S}^1_{\mathrm{NS}}\times\Sigma)&=\operatorname{tr}_{\hat{\mathcal{H}}(\Sigma)}(\mathrm{id})\\
Z(\mathbb{S}^1_{\mathrm{R}}\times\Sigma)&=\operatorname{tr}_{\hat{\mathcal{H}}(\Sigma)}(-1)^F
\end{aligned}
\tag{5.1.21}
$$

where $\mathbb{S}^1_{\mathrm{NS}}$ denotes the circle with anti-periodic boundary conditions, and $\mathbb{S}^1_{\mathrm{R}}$ the circle with periodic boundary conditions. Therefore, if the super-vector space $\hat{\mathcal{H}}(\Sigma)$ is $\mathbb{C}^{b|f}$, then Neveu-Schwarz boundary conditions compute $b+f$, and Ramond boundary conditions compute $b-f$.

We shall denote a compact surface with genus $g$ and spin structure $s$ by $\Sigma_{g;s}$. As in the bosonic case, large diffeomorphisms act unitarily in $\hat{\mathcal{H}}(\Sigma_{g;s})$. The MCG as a spin surface is a subgroup of the MCG as a surface, $\mathrm{MCG}(\Sigma_{g;s})\subseteq\mathrm{MCG}(\Sigma_g)$. The reason for this is that some diffeomorphisms that leave $\Sigma$ invariant as a topological space, actually change the spin structure $s\mapsto s'$, and so do not constitute elements of the MCG as a spin surface. The canonical example is the $T$-transformation on the torus, which performs a Dehn twist around the $\mathsf{a}$-cycle. As such, it maps $(s^{\mathsf{a}},s^{\mathsf{b}})\mapsto(s^{\mathsf{a}},s^{\mathsf{a}}s^{\mathsf{b}})$. This is an element of the spin MCG if $s^{\mathsf{a}}=+1$, but it is not if $s^{\mathsf{a}}=-1$. On the other hand, $T^2$ is in the spin MCG for any spin structure.

Elements of the spin MCG act unitarily in the Hilbert space, namely,

$$
\mathrm{MCG}(\Sigma_{g;s})\colon\hat{\mathcal{H}}(\Sigma_{g;s})\to\hat{\mathcal{H}}(\Sigma_{g;s})\,.
\tag{5.1.22}
$$

On the other hand, elements of the regular MCG induce isomorphisms of (generically distinct) super-vector spaces,

$$
\mathrm{MCG}(\Sigma_g)\colon\hat{\mathcal{H}}(\Sigma_{g;s})\to\hat{\mathcal{H}}(\Sigma_{g;s'})\,.
\tag{5.1.23}
$$

This means, for example, that the partition function $Z(\mathbb{S}^1\times\Sigma_{g;s})$ is invariant under $\mathrm{MCG}(\Sigma_g)$; and, more generally, observables only depend on the equivalence class of $s$ under the regular MCG. It is known that there are only two equivalence classes of spin structures modulo MCG, the so-called *even* and *odd* spin structures. These are distinguished by the *Arf invariant* [286]. If two spin structures have the same Arf parity, then there exists some MCG element that maps one into the other. If they have different Arf parity, no such MCG element exists. In conclusion, observables of spin TQFTs depend on $s$ only through $\mathrm{Arf}(s)$.

For fixed spin structure, $\mathrm{MCG}(\Sigma_{g;s})$ is represented by a unitary operator in $\hat{\mathcal{H}}(\Sigma_{g;s})$. That being said, due to the $\mathbb{Z}_2$ grading of this vector space, this action typically gets extended. Namely, the Hilbert space of spin TQFTs realize a unitary representation of a certain non-trivial $\mathbb{Z}_2$ extension of the spin MCG. To be explicit, (modding out by Torelli, i.e., working in homology) the MCG of $\Sigma_g$ is the integral symplectic group $Sp_g(\mathbb{Z})$, and the spin MCG is some subgroup thereof. The Hilbert space of the theory realizes a unitary representation of the so-called *metaplectic group* $Mp_g(\mathbb{Z})$, which is defined as the (essentially unique) $\mathbb{Z}_2$ extension of the symplectic group

$$
\mathbb{Z}_2\hookrightarrow Mp_g(\mathbb{Z})\twoheadrightarrow Sp_g(\mathbb{Z})\,.
\tag{5.1.24}
$$

This extension corresponds to the fact that a $2\pi$ rotation is represented by the trivial element in $\mathrm{MCG}(\Sigma_{g;s})$, while it lifts to $(-1)^F$ in $\hat{\mathcal{H}}(\Sigma_{g;s})$.

In order to illustrate these ideas it proves useful to focus on the torus, $\Sigma_1$. There are four spin tori, depending on the boundary conditions on the two spatial circles $\Sigma_{1;s^{\mathsf{a}},s^{\mathsf{b}}} = \mathbb{S}^1_{s^{\mathsf{a}}} \times \mathbb{S}^1_{s^{\mathsf{b}}}$:

$$\begin{aligned} (-1)^{\mathrm{Arf}(s^{\mathsf{a}},s^{\mathsf{b}})} = +1\colon &\quad \begin{cases} \mathbb{S}^1_{\mathrm{NS}} \times \mathbb{S}^1_{\mathrm{NS}} \\ \mathbb{S}^1_{\mathrm{NS}} \times \mathbb{S}^1_{\mathrm{R}} \\ \mathbb{S}^1_{\mathrm{R}} \times \mathbb{S}^1_{\mathrm{NS}} \end{cases} \\ (-1)^{\mathrm{Arf}(s^{\mathsf{a}},s^{\mathsf{b}})} = -1\colon &\quad \mathbb{S}^1_{\mathrm{R}} \times \mathbb{S}^1_{\mathrm{R}} \end{aligned} \tag{5.1.25}$$

The MCG of the torus is the modular group $SL_2(\mathbb{Z}) = \langle \hat{S}, \hat{T} \rangle$, acting as

$$\begin{aligned} \hat{S}\colon \Sigma_{1;s^{\mathsf{a}},s^{\mathsf{b}}} &\to \Sigma_{1;s^{\mathsf{b}},s^{\mathsf{a}}} \\ \hat{T}\colon \Sigma_{1;s^{\mathsf{a}},s^{\mathsf{b}}} &\to \Sigma_{1;s^{\mathsf{a}},s^{\mathsf{a}}s^{\mathsf{b}}} \end{aligned} \tag{5.1.26}$$

Therefore, the subgroup that fixes each spin structure is

$$\begin{aligned} \mathrm{MCG}(\Sigma_{1;--}) &= \langle \hat{S}, \hat{T}^2 \rangle \\ \mathrm{MCG}(\Sigma_{1;-+}) &= \langle \hat{S}\hat{T}\hat{S}, \hat{T}^2 \rangle \\ \mathrm{MCG}(\Sigma_{1;+-}) &= \langle \hat{S}\hat{T}^2\hat{S}, \hat{T} \rangle \\ \mathrm{MCG}(\Sigma_{1;++}) &= \langle \hat{S}, \hat{T} \rangle \end{aligned} \tag{5.1.27}$$

Needless to say, the first three groups are all isomorphic, as they are related through $SL_2(\mathbb{Z})$ conjugation:

$$\langle \hat{S}\hat{T}\hat{S}, \hat{T}^2 \rangle = \hat{T}\langle \hat{S}, \hat{T}^2 \rangle \hat{T}^{-1}, \qquad \langle \hat{S}\hat{T}^2\hat{S}, \hat{T} \rangle = (\hat{S}\hat{T})\langle \hat{S}, \hat{T}^2 \rangle (\hat{S}\hat{T})^{-1}\,. \tag{5.1.28}$$

This group is a congruence subgroup of $SL_2(\mathbb{Z})$ of index 3, usually denoted by $\Gamma_0(2)$. The fourth group, on the other hand, is $SL_2(\mathbb{Z})$ itself.

The diffeomorphism $\hat{S}^4$ corresponds to a $2\pi$ rotation and, as such, acts trivially in a bosonic theory and so is represented by the identity element in $SL_2(\mathbb{Z})$; conversely, in a fermionic theory it is represented by $(-1)^F$. Thus, modular transformations in spin theories satisfy

$$\hat{S}^2 = (\hat{S}\hat{T})^3, \qquad \hat{S}^4 = (-1)^F\,, \tag{5.1.29}$$

with $(-1)^F$ an order 2 central element. These relations define the group $Mp_1(\mathbb{Z})$.[96]

[96] One should keep in mind that $Mp_1(\mathbb{Z})$ does not act faithfully in $\hat{\mathcal{H}}(\Sigma_{1;s})$ (in fact, the metaplectic group is not a matrix group; it does not admit faithful finite-dimensional representations). This fact is most drastic when the theory, for whatever reason, has no fermionic states at all: in such cases, the actions of $Sp_1(\mathbb{Z})$ and $Mp_1(\mathbb{Z})$ are indistinguishable, inasmuch as $(-1)^F$ is trivial. For example, the theory lacks fermionic states if $s$ is an even spin structure, or if the theory is secretly bosonic (through some non-trivial duality). In such cases, one can think of the modular group as being $SL_2(\mathbb{Z})$ instead of $Mp_1(\mathbb{Z})$: their difference is invisible in the Hilbert space anyway. Similar considerations hold in higher genus.

The Hilbert spaces $\hat{\mathcal{H}}(\Sigma_{g;s})$ are best understood by giving an explicit basis for them. As in the previous section, the Hilbert space of the fermionic theory can be constructed by condensing a fermion in the bosonic parent. Indeed, the Hilbert space of the spin theory is a subspace of the Hilbert space of the bosonic parent, together with the space with a $\psi$-puncture,

$$\hat{\mathcal{H}}(\Sigma_{g;s}) \subseteq \mathcal{H}(\Sigma_g) \oplus \mathcal{H}(\Sigma_g^{\psi})\,. \tag{5.1.30}$$

Let us write down a basis for $\hat{\mathcal{H}}(\Sigma_{1;s})$ in terms of the states of the bosonic parent. Recall that the states on the torus in the bosonic theory are labelled by the anyons $\mathcal{A}$. The $\mathbb{Z}_2^{(1)}$ symmetry generated by $\psi$ partitions the spectrum $\mathcal{A}$ into two equivalence classes, distinguished by the braiding phase $B(\psi,\,\cdot\,) = \pm 1$. In the case of boson condensation we denoted these two equivalence classes by $\mathcal{A}_0$ and $\mathcal{A}_1$; in the present context it is more natural to denote them by $\mathcal{A}_{\mathrm{NS}}$ and $\mathcal{A}_{\mathrm{R}}$:

$$\mathcal{A} = \mathcal{A}_{\mathrm{NS}} \sqcup \mathcal{A}_{\mathrm{R}}, \qquad \begin{cases} \mathcal{A}_{\mathrm{NS}} := \{\alpha \in \mathcal{A} \mid B(\alpha,\psi) = +1\} \\ \mathcal{A}_{\mathrm{R}} \;\,:= \{\alpha \in \mathcal{A} \mid B(\alpha,\psi) = -1\} \end{cases} \tag{5.1.31}$$

The two equivalence classes are further partitioned according to the length of the orbits. For a generic $\mathbb{Z}_n^{(1)}$ symmetry, the orbits come in lengths that divide $n$; for $n = 2$, we have two-dimensional orbits and one-dimensional ones. We refer to the latter as *Majorana lines*. It is easy to convince oneself that these can only appear in $\mathcal{A}_{\mathrm{R}}$. We shall use the label $\alpha$ to denote generic lines of $\mathcal{A}$; on the other hand, lines of $\mathcal{A}_{\mathrm{NS}}$ will be denoted by the more specific label $a$, while two-dimensional orbits of $\mathcal{A}_{\mathrm{R}}$ by $x$ and one-dimensional ones by $m$. We will say that $\alpha$ is $a$-type, $x$-type, or $m$-type, according to this classification:

$$\begin{aligned} &a \in \mathcal{A}_{\mathrm{NS}} \\ &x \in \mathcal{A}_{\mathrm{R}} \quad \& \quad |\,x\,| = 2 \\ &m \in \mathcal{A}_{\mathrm{R}} \quad \& \quad |m| = 1\,. \end{aligned} \tag{5.1.32}$$

In other words, $x$-lines satisfy $x \times \psi \neq x$, while $m$-lines satisfy $m \times \psi = m$.

As in the previous section, we take the bosonic theory, and condense a fermion $\psi$. In the condensed theory, the Wilson line of $\psi$ becomes almost trivial: it should be represented by the identity operator, up to a sign, depending on the spin structure around the cycle it is supported on. In other words, the anyon $\psi$ represents the wordline of a local fermion. This determines how the states in the condensed phase are obtained in terms of those of the unconcondensed one. We claim that the basis of $\hat{\mathcal{H}}(\Sigma_{1;s})$ can be taken as

$$
\begin{aligned}
\hat{\mathcal{H}}(\Sigma_{1;--}) &= \operatorname*{Span}_{a\in\mathcal{A}_{\mathrm{NS}}}\left[\tfrac{1}{\sqrt{2}}\left(\Big|\underset{a}{\bigcirc}\Big\rangle + \Big|\underset{a\times\psi}{\bigcirc}\Big\rangle\right)\right] \\
\hat{\mathcal{H}}(\Sigma_{1;-+}) &= \operatorname*{Span}_{a\in\mathcal{A}_{\mathrm{NS}}}\left[\tfrac{1}{\sqrt{2}}\left(\Big|\underset{a}{\bigcirc}\Big\rangle - \Big|\underset{a\times\psi}{\bigcirc}\Big\rangle\right)\right] \\
\hat{\mathcal{H}}(\Sigma_{1;+-}) &= \operatorname*{Span}_{x\in\mathcal{A}_{\mathrm{R}}}\left[\tfrac{1}{\sqrt{2}}\left(\Big|\underset{x}{\bigcirc}\Big\rangle + \Big|\underset{x\times\psi}{\bigcirc}\Big\rangle\right)\right] \oplus \operatorname*{Span}_{m\in\mathcal{A}_{\mathrm{R}}}\left[\Big|\underset{m}{\bigcirc}\Big\rangle\right] \\
\hat{\mathcal{H}}(\Sigma_{1;++}) &= \operatorname*{Span}_{x\in\mathcal{A}_{\mathrm{R}}}\left[\tfrac{1}{\sqrt{2}}\left(\Big|\underset{x}{\bigcirc}\Big\rangle - \Big|\underset{x\times\psi}{\bigcirc}\Big\rangle\right)\right] \oplus \operatorname*{Span}_{m\in\mathcal{A}_{\mathrm{R}}}\left[\Big|\psi\!\!-\!\!\bullet\!\underset{m}{\bigcirc}\Big\rangle\right]
\end{aligned}
\tag{5.1.33}
$$

The reasoning behind the construction of this basis is the same as in the case of boson condensation. Namely, the gauged theory is obtained by inserting $\psi^j$ in all possible ways. Here $\psi$ is of order two, so there are only two possible blocks: $j = 0$ or $j = 1$, i.e., no insertion, or a single $\psi$-insertion. Furthermore, the specific linear combination of states is decided by the spin structure. For example, inserting $\psi$ along the $\mathsf{a}$-cycle inserts the phase $|\alpha;\psi^j\rangle \mapsto B(\alpha,\psi)|\alpha;\psi^j\rangle$. This should reproduce the sign $s^{\mathsf{a}}$, which means that $\mathcal{A}_{\mathrm{NS}}$ lines create states in $s^{\mathsf{a}} = -1$ boundary conditions, and $\mathcal{A}_{\mathrm{R}}$ lines create states in $s^{\mathsf{a}} = +1$ boundary conditions. This explains why the basis is constructed using $a$-type lines in $\hat{\mathcal{H}}(\Sigma_{1;-,\bullet})$, and $x$- and $m$-type lines in $\hat{\mathcal{H}}(\Sigma_{1;+,\bullet})$.

Similarly, inserting $\psi$ along the $\mathsf{b}$-cycle fuses the state into $|\alpha;\psi^j\rangle \mapsto (-1)^j|\psi\times\alpha;\psi^j\rangle$. If this is to reproduce the boundary condition $s^{\mathsf{b}}$, we are required to consider the linear combination $|\alpha\rangle - s^{\mathsf{b}}|\psi\times\alpha\rangle$ for two-dimensional orbits; and, for Majorana lines, the puncture should be present if and only if $s^{\mathsf{b}} = +1$.

Note that, unlike in the case of bosonic condensation, here the multiple copies associated to the short orbits live in different spaces. Moreover, there are no short orbits in the NS sector, so the observables of the theories (the Wilson lines associated to the NS anyons) do not require fixed-point resolution. In this sense, the fusion rules of the condensed theory are inherited from those of the parent in a straightforward manner, without the need of knowing the once-punctured $S$-matrix. In the bosonic case, the fusion rules of the short orbits do require this extra structure.

As a consistency check for the basis above, we can easily show that modular transformations map the different Hilbert spaces as expected. For example, take $\hat{\mathcal{H}}(\Sigma_{1;--})$, and apply an $S$-transformation:

$$\frac{1}{\sqrt{2}}(|a\rangle + |\psi\times a\rangle) \overset{S}{\mapsto} \sum_{\alpha'\in\mathcal{A}} \frac{1}{\sqrt{2}}(S_{a,\alpha'} + S_{\psi\times a,\alpha'})|\alpha'\rangle \tag{5.1.34}$$

As in equation (5.1.13), we have $S_{\psi\times a,\alpha'} = B(\psi,\alpha')S_{a,\alpha'}$, which means that we may restrict the sum over $\alpha'$ to $a$-type lines, for the Ramond ones do not contribute – they cancel out pairwise. With this,

$$\frac{1}{\sqrt{2}}(|a\rangle + |\psi\times a\rangle) \overset{S}{\mapsto} \sum_{a'\in\mathcal{A}_{\mathrm{NS}}/\sim} 2S_{a,a'}\frac{1}{\sqrt{2}}(|a'\rangle + |\psi\times a'\rangle) \tag{5.1.35}$$

which shows that $S$ maps $\hat{\mathcal{H}}(\Sigma_{1;--})$ to itself, as expected (cf. (5.1.26)). Similarly, $T$-transformations map

$$\frac{1}{\sqrt{2}}(|a\rangle + |\psi\times a\rangle) \overset{T}{\mapsto} e^{-2\pi ic/24}\frac{1}{\sqrt{2}}(\theta(a)|a\rangle + \theta(\psi\times a)|\psi\times a\rangle) \tag{5.1.36}$$

Noting that $\theta(\psi\times a) = -\theta(a)$, this becomes

$$\frac{1}{\sqrt{2}}(|a\rangle + |\psi\times a\rangle) \overset{T}{\mapsto} e^{-2\pi ic/24}\theta(a)\frac{1}{\sqrt{2}}(|a\rangle - |\psi\times a\rangle) \tag{5.1.37}$$

which shows that $T$ maps $\hat{\mathcal{H}}(\Sigma_{1;--})$ into $\hat{\mathcal{H}}(\Sigma_{1;-+})$, again as expected (cf. (5.1.26)).

The other three Hilbert spaces can also be seen to transform into each other in the expected manner. Not only that, but the exercise gives us the explicit expression for the $\hat{S}$ and $\hat{T}$ matrices of the condensed theory:

$$\begin{aligned}
&\hat{S}\colon \hat{\mathcal{H}}(\Sigma_{1;--}) \to \hat{\mathcal{H}}(\Sigma_{1;--}) \implies \begin{cases}\hat{S}_{a,a'} = 2S_{a,a'}\end{cases}\\
&\hat{S}\colon \hat{\mathcal{H}}(\Sigma_{1;-+}) \to \hat{\mathcal{H}}(\Sigma_{1;+-}) \implies \begin{cases}\hat{S}_{a,x} = 2S_{a,x}\\ \hat{S}_{a,m} = \sqrt{2}S_{a,m}\end{cases}\\
&\hat{S}\colon \hat{\mathcal{H}}(\Sigma_{1;+-}) \to \hat{\mathcal{H}}(\Sigma_{1;-+}) \implies \begin{cases}\hat{S}_{x,a} = 2S_{x,a}\\ \hat{S}_{m,a} = \sqrt{2}S_{m,a}\end{cases}\\
&\hat{S}\colon \hat{\mathcal{H}}(\Sigma_{1;++}) \to \hat{\mathcal{H}}(\Sigma_{1;++}) \implies \begin{cases}\hat{S}_{x,x'} = 2S_{x,x'}\\ \hat{S}_{x,m} = 0\\ \hat{S}_{m,m'} = S_{m,m'}(\psi)\end{cases}
\end{aligned} \tag{5.1.38}$$

where $S_{m,m'}(\psi)$ denotes the $S$-matrix of the bosonic parent, in the once-punctured torus (cf. (5.1.18)). The $\hat{T}$-matrix is given by a similar expression:

$$\begin{aligned}
&\hat{T}\colon \hat{\mathcal{H}}(\Sigma_{1;--}) \to \hat{\mathcal{H}}(\Sigma_{1;-+}) \implies \Big\{ \hat{T}_{a,a'} = e^{-2\pi i c/24}\theta(a)(\delta_{a,a'} - \delta_{a,\psi\times a'}) \\
&\hat{T}\colon \hat{\mathcal{H}}(\Sigma_{1;-+}) \to \hat{\mathcal{H}}(\Sigma_{1;--}) \implies \Big\{ \hat{T}_{a,a'} = e^{-2\pi i c/24}\theta(a)(\delta_{a,a'} + \delta_{a,\psi\times a'}) \\
&\hat{T}\colon \hat{\mathcal{H}}(\Sigma_{1;+-}) \to \hat{\mathcal{H}}(\Sigma_{1;+-}) \implies \begin{cases} \hat{T}_{x,x'} = e^{-2\pi i c/24}\theta(a)(\delta_{x,x'} + \delta_{x,\psi\times x'}) \\ \hat{T}_{x,m} = 0 \\ \hat{T}_{m,m'} = e^{-2\pi i c/24}\theta(m)\delta_{m,m'} \end{cases} \\
&\hat{T}\colon \hat{\mathcal{H}}(\Sigma_{1;++}) \to \hat{\mathcal{H}}(\Sigma_{1;++}) \implies \begin{cases} \hat{T}_{x,x'} = e^{-2\pi i c/24}\theta(a)(\delta_{x,x'} - \delta_{x,\psi\times x'}) \\ \hat{T}_{x,m} = 0 \\ \hat{T}_{m,m'} = e^{-2\pi i c/24}\theta(m)\delta_{m,m'} \end{cases}
\end{aligned} \tag{5.1.39}$$

Finally, we discuss the third generator of the spin modular group, fermion parity. This zero-form symmetry is the dual symmetry to the gauged $\mathbb{Z}_2^{(1)}$, which means that the states with odd fermion parity are those that carry the puncture. This means that $(-1)^F = 1$ in all even spin structures, while in the R-R sector one has

$$\begin{aligned}
((-1)^F)_{x,x'} &= +\delta_{x,x'} \\
((-1)^F)_{m,m'} &= -\delta_{m,m'}
\end{aligned} \tag{5.1.40}$$

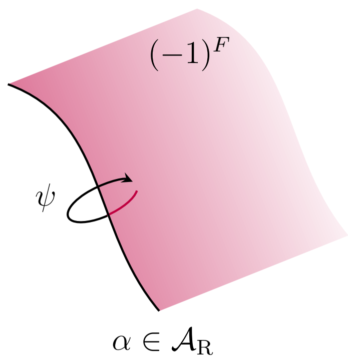
These matrices are unitary, symmetric, and satisfy $\hat{S}^4 = (-1)^F$ and $\hat{S}^2 = (\hat{S}\hat{T})^3$. It is important to remark that these properties are understood in the $\mathbb{Z}_2$-graded sense, i.e., taking into account (5.1.26). In other words, the precise relations are

$$\begin{aligned}
\hat{S}^\dagger_{s^{\mathsf{a}},s^{\mathsf{b}}} &= \hat{S}^{-1}_{s^{\mathsf{a}},s^{\mathsf{b}}} \\
\hat{T}^\dagger_{s^{\mathsf{a}},s^{\mathsf{b}}} &= \hat{T}^{-1}_{s^{\mathsf{a}},s^{\mathsf{b}}} \\
\hat{S}^t_{s^{\mathsf{b}},s^{\mathsf{a}}} &= \hat{S}_{s^{\mathsf{a}},s^{\mathsf{b}}} \\
(\hat{S}_{s^{\mathsf{b}},s^{\mathsf{a}}}\hat{S}_{s^{\mathsf{a}},s^{\mathsf{b}}})^2 &= (-1)^F_{s^{\mathsf{a}},s^{\mathsf{b}}} \\
\hat{S}_{s^{\mathsf{b}},s^{\mathsf{a}}}\hat{S}_{s^{\mathsf{a}},s^{\mathsf{b}}} &= \hat{S}_{s^{\mathsf{b}},s^{\mathsf{a}}}\hat{T}_{s^{\mathsf{b}},s^{\mathsf{a}}s^{\mathsf{b}}}\hat{S}_{s^{\mathsf{a}}s^{\mathsf{b}},s^{\mathsf{b}}}\hat{T}_{s^{\mathsf{a}}s^{\mathsf{b}},s^{\mathsf{a}}}\hat{S}_{s^{\mathsf{a}},s^{\mathsf{a}}s^{\mathsf{b}}}\hat{T}_{s^{\mathsf{a}},s^{\mathsf{b}}} ,
\end{aligned} \tag{5.1.41}$$

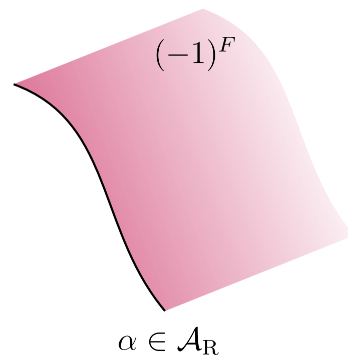
where $\mathcal{O}_{s^{\mathsf{a}},s^{\mathsf{b}}}$ denotes the operator $\mathcal{O} \in \{\hat{S}, \hat{T}, (-1)^F\}$ when acting on $\hat{\mathcal{H}}(\Sigma_{1;s^{\mathsf{a}},s^{\mathsf{b}}})$.

In section 5.2 we construct several examples of quotient TQFTs, namely $\mathrm{SO}(N)_k$ with $k = 1, 2, 3$. Some of these illustrate bosonic anyon condensation, and some others fermionic anyon condensation.

One final comment is in order: it is important to stress that the line operators in the fermionic theory – the anyons – are the NS-lines. The R-lines, on the other hand, are not genuine line operators: they live at the end of the topological surface that implements the $(-1)^F$ symmetry. Indeed, moving a local fermion around a Ramond line generates a minus sign:




$= -$




(5.1.42)

This is analogous to the situation in two-dimensional fermionic CFTs, where Ramond operators are not genuine point operators, but rather exist at the end of the $(-1)^F$-line (see e.g. [287]).

## 5.2 Examples of anyon condensation

Here we collect some extra examples of boson and fermion condensation, using theories of the form $\mathrm{SO}(n)_k$ for small values of $k$. In particular, $k = 1$ and $k = 3$ exemplify fermion condensation, and $k = 2$ boson condensation. The case $k = 1$, i.e., $\mathrm{SO}(n)_1$, is the generator of fermionic SPTs with no symmetry, and so is a key theory in the study of fermionic TQFTs. The case $k = 2$ will be related to U(1) theories, through the level-rank duality $\mathrm{SO}(n)_2 \leftrightarrow \mathrm{SO}(2)_{-n} \equiv \mathrm{U}(1)_{-n}$. Finally, the case $k = 3$ will be constructed through the SU(2) theory, thanks to the level-rank duality $\mathrm{SO}(n)_3 \leftrightarrow \mathrm{SO}(3)_{-n} \equiv \mathrm{SU}(2)_{-2n}/\mathbb{Z}_2$. We also include the case of $\mathrm{U}(1)_k$ separately, this time focusing on its time-reversal invariance.

### 5.2.1 $\mathrm{SO}(n)_1$

This is the minimal spin TQFT, and it has central charge $n/2$, so corresponds to $n$ boundary Majorana fermions. A single fermion, $\mathrm{SO}(1)_1$, is the generator of the group of fermionic SPTs with no extra symmetries, $\Omega^4_{\mathrm{spin}} = \mathbb{Z}$. In other words, any invertible fermionic phase is equivalent to $\mathrm{SO}(n)_1$ for some $n$. The theory can also be written as $n$ copies of (the inverse of) the gravitational Chern-Simons theory.

The bosonic parent of this theory is $\mathrm{Spin}(n)_1$. The details of this theory depend on the parity of $n$.

$\boldsymbol{n = 2m + 1}$**.** One can construct $\mathrm{SO}(n)_1$ by condensing the fermion in the Ising category. The modular data for the parent theory is that of $\mathrm{Ising}_m = \mathrm{Spin}(2m + 1)_1$, which has three anyons:

$$\begin{aligned}\mathbf{1} &= [1,0,0,\dots,0,0] \\ \sigma &= [0,0,0,\dots,0,1] \\ \psi &= [0,1,0,\dots,0,0]\end{aligned} \tag{5.2.1}$$

where $[\lambda_0, \lambda_1, \dots, \lambda_m]$ denote the extended Dynkin labels of the representation. These lines fuse according to $\psi \times \sigma = \sigma$ and $\sigma^2 = \mathbf{1} + \psi$, and transform under modular transformations as follows:

$$\begin{aligned} S|\mathbf{1}\rangle &= \frac{1}{2}|\mathbf{1}\rangle + \frac{1}{\sqrt{2}}|\sigma\rangle + \frac{1}{2}|\psi\rangle & T|\mathbf{1}\rangle &= \mathrm{e}^{2\pi i\left(0-\frac{n}{48}\right)}|\mathbf{1}\rangle \\ S|\sigma\rangle &= \frac{1}{\sqrt{2}}|\mathbf{1}\rangle - \frac{1}{\sqrt{2}}|\psi\rangle & T|\sigma\rangle &= \mathrm{e}^{2\pi i\left(\frac{n}{16}-\frac{n}{48}\right)}|\sigma\rangle \\ S|\psi\rangle &= \frac{1}{2}|\mathbf{1}\rangle - \frac{1}{\sqrt{2}}|\sigma\rangle + \frac{1}{2}|\psi\rangle & T|\psi\rangle &= \mathrm{e}^{2\pi i\left(\frac{1}{2}-\frac{n}{48}\right)}|\psi\rangle \end{aligned} \tag{5.2.2}$$

The lines are partitioned according to their braiding with respect to $\psi$ as

$$\begin{aligned} \mathrm{NS:}\quad & \mathcal{A}_{\mathrm{NS}} = \{\mathbf{1},\ \psi\} \\ \mathrm{R:}\quad & \mathcal{A}_{\mathrm{R}} \ = \{\sigma\} \end{aligned} \tag{5.2.3}$$

and they are paired-up under fusion as

$$\mathbf{1} \xleftrightarrow{\times\psi} \psi, \qquad \sigma \circlearrowleft {\scriptstyle\times\psi} \ . \tag{5.2.4}$$

Therefore, the four Hilbert spaces of the theory are

- If we take **NS-NS** boundary conditions, the state is

$$|0;\mathrm{NS\text{-}NS}\rangle = \frac{1}{\sqrt{2}}(|\mathbf{1}\rangle + |\psi\rangle)\,. \tag{5.2.5}$$

- If we take **NS-R** boundary conditions, the state is

$$|0;\mathrm{NS\text{-}R}\rangle = \frac{1}{\sqrt{2}}(|\mathbf{1}\rangle - |\psi\rangle)\,. \tag{5.2.6}$$

- If we take **R-NS** boundary conditions, the state is

$$|0;\mathrm{R\text{-}NS}\rangle = |\sigma\rangle\,. \tag{5.2.7}$$

- If we take **R-R** boundary conditions, the state is

$$|0;\mathrm{R\text{-}R}\rangle = |\sigma;\psi\rangle\,, \tag{5.2.8}$$

  where, we remind the reader, $|\alpha;\beta\rangle$ denotes the anyon $\alpha$ in presence of a $\beta$ puncture (cf. (5.1.6)).

We see that these spaces are all one-dimensional, as expected from an invertible theory. Furthermore, all states are bosonic, except for the one with a puncture, $|\sigma;\psi\rangle$, which means that $(-1)^F = (-1)^{\mathrm{Arf}(s)}$.

The modular data of the quotient can be computed in a straightforward manner. The only non-trivial case is the $S$ matrix in the R-R sector, which has a puncture. We can compute this matrix element using the general formula (5.1.18), namely

$$S_{\sigma,\sigma}(\psi) = \sum_{\alpha=\mathbf{1},\psi} \frac{\theta(\alpha)}{\theta(\sigma)^2} S_{\mathbf{1},\alpha} F_{\sigma,\sigma} \begin{bmatrix} \psi & \sigma \\ \sigma & \alpha \end{bmatrix} \tag{5.2.9}$$

The $F$-symbols of the Ising category are well-known, cf. $F(\alpha = \mathbf{1}) \equiv +1$ and $F(\alpha = \psi) \equiv -1$. With this,

$$\hat{S}_{\text{R-R}} = \frac{1}{2}(-1)^{3/4}(-i)^m (F(\psi) - F(\mathbf{1})) \equiv e^{i\pi(6m+7)/4} \tag{5.2.10}$$

This result, together with $\hat{T}_{\text{R-R}} = \mathrm{e}^{\pi i(2m+1)/12}$, confirms that the theory satisfies the expected modularity relations, $S^2 = (ST)^3$ and $S^4 = (-1)^F$.

A different perspective yields the same answer. The CFT $\mathrm{SO}(n)_1$ is identical to $n$ free Majorana fermions, and the once-puctured conformal block $|\sigma;\psi\rangle$ is nothing but the torus one-point function of $\psi$. The insertion of $\psi$ removes the Ramond zero-mode, and hence this one-point function is $\langle\psi\rangle = q^{1/24}\prod_{r=1}^{\infty}(1-q^r) \equiv \eta(\tau)$, the Dedekind eta function. The punctured $S$-matrix is nothing but the phase acquired by $\langle\psi\rangle$ under an $S$-transformation, namely $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$. The factor of $\tau^{1/2}$ is the weight associated to a primary of spin $h = 1/2$, while the factor of $\sqrt{-i}$ is the sought-after $S$-matrix. For a system of $n = 2m+1$ fermions, the $S$-matrix is $(\sqrt{-i})^{2m+1} \equiv e^{i\pi(6m+7)/4}$, in agreement with (5.2.10).

$\boldsymbol{n = 2m.}$ Here the bosonic parent $\mathrm{Spin}(2m)_1$ has four lines, the trivial representation, the vector representation, and the two spinor representations. The quotient is obtained by condensing the vector. The Lie algebra is simply-laced, which automatically implies that the fusion rules are abelian, and so there are no fixed-points under fusion. Therefore, all states are bosonic. The four lines are split into two NS-lines (the trivial and the vector) and two R-lines (the two spinors), and each pair belongs to a two-dimensional orbit. This means that each Hilbert space is one-dimensional, as expected from an invertible theory, and moreover all states have $(-1)^F = +1$. The modular data is trivially computed, given that there are no short orbits.

### 5.2.2 $\mathrm{SO}(n)_2$

Here we illustrate the construction of the bosonic theory $\mathrm{SO}(n)_2$, by condensing an abelian boson in $\mathrm{Spin}(n)_2$. We focus in particular on the odd-$n$ case, where all the modular data – especially the $F$-symbols – is fully known [288]. We follow the notation therein.

Consider the algebra $\mathfrak{so}_{2n+1} = B_n$. Its comarks are $a^\vee = 1, 2, 2, \ldots, 2, 1$, which means that the theory $\mathrm{Spin}(2n+1)_2$ has $n+4$ lines. We denote them as $\mathbf{1}, \epsilon, \phi_i, \psi_\pm$, with $i = 1, 2, \ldots, n$. The corresponding affine Dynkin labels are as follows:

$$\begin{aligned}
\mathbf{1} &= [2, 0, 0, \ldots, 0] \\
\epsilon &= [0, 2, 0, \ldots, 0] \\
\phi_1 &= [1, 1, 0, \ldots, 0] \\
\phi_i &= [0, \ldots, 0, 1, 0, \ldots, 0] \qquad \text{at position } i+1 \\
\phi_n &= [0, 0, \ldots, 0, 2] \\
\psi_+ &= [1, 0, 0, \ldots, 0, 1] \\
\psi_- &= [0, 1, 0, \ldots, 0, 1]
\end{aligned} \tag{5.2.11}$$

The $S$-matrix reads

$$\begin{aligned}
S_{\mathbf{1},\mathbf{1}} = S_{\mathbf{1},\epsilon} = S_{\epsilon,\epsilon} &= \frac{1}{2\sqrt{2n+1}} \\
S_{\mathbf{1},\psi_\pm} &= +\frac{1}{2} \\
S_{\epsilon,\psi_\pm} &= -\frac{1}{2} \\
S_{\psi_s,\psi_{s'}} &= \frac{1}{2} s s' \\
S_{\mathbf{1},\phi_i} = S_{\epsilon,\phi_i} &= \frac{1}{\sqrt{2n+1}} \\
S_{\psi_\pm,\phi_i} &= 0 \\
S_{\phi_i,\phi_j} &= \frac{2}{\sqrt{2n+1}} \cos \frac{2\pi i j}{2n+1}
\end{aligned} \tag{5.2.12}$$

and the spins are

$$\begin{aligned}
h_{\mathbf{1}} &= 0 \\
h_\epsilon &= 1 \\
h_{\phi_i} &= \frac{1}{2} \frac{i(2n+1-i)}{2n+1} \\
h_{\psi_+} &= \frac{1}{8} n \\
h_{\psi_-} &= \frac{1}{8} n + \frac{1}{2}
\end{aligned} \tag{5.2.13}$$

From this one derives the quantum dimensions

$$\begin{aligned}
d_{\mathbf{1}} &= 1 \\
d_\epsilon &= 1 \\
d_{\phi_i} &= 2 \\
d_{\psi_\pm} &= \sqrt{2n+1}
\end{aligned} \tag{5.2.14}$$

and fusion rules,

$$\begin{aligned}\epsilon \times \epsilon = \mathbf{1}, \quad \psi_\pm \times \psi_\pm = \mathbf{1} + \sum_{j=1}^{n} \phi_j, \quad \psi_\pm \times \psi_\mp = \epsilon + \sum_{j=1}^{n} \phi_j \\ \epsilon \times \phi_i = \phi_i, \quad \epsilon \times \psi_\pm = \psi_\mp, \quad \phi_i \times \psi_\pm = \psi_\pm + \psi_\mp \\ \phi_i \times \phi_i = \mathbf{1} + \epsilon + \phi_{g(2i)}, \quad \phi_i \times \phi_j = \phi_{g(i-j)} \times \phi_{g(i+j)}, \quad i > j\end{aligned} \tag{5.2.15}$$

where $g(i) = i$ if $1 \leq i \leq n$ and $g(i) = 2n + 1 - i$ otherwise.

We see that there are no multiplicities, and all anyons are self-conjugate. We also note that $\epsilon$ is condensable, which leads to the bosonic theory $\mathrm{SO}(2n+1)_2$. Let us analyse the quotient explicitly.

By looking at the braiding phase $B(\alpha, \epsilon)$ one learns that the unscreened anyons are $\mathbf{1}, \epsilon, \phi_i$, while the screened anyons are $\psi_\pm$. Moreover, $\mathbf{1}$ and $\epsilon$ are in the same orbit, while all the $\phi_i$ are fixed points. Thus, a basis for the condensed Hilbert space is as follows:

$$\begin{aligned}|0\rangle &= \frac{1}{\sqrt{2}}(|\mathbf{1}\rangle + |\epsilon\rangle) \\ |i\rangle &= |\phi_i\rangle \\ |n+i\rangle &= |\phi_i; \epsilon\rangle\end{aligned} \tag{5.2.16}$$

for $i = 1, 2, \ldots, n$, and where $|\cdot\,; \epsilon\rangle$ denotes a state in the once-punctured torus. The condensed $S$-matrix is

$$\begin{aligned}\hat{S}_{0,0} &= 2S_{\mathbf{1},\mathbf{1}} = \frac{1}{\sqrt{2n+1}} \\ \hat{S}_{0,i} &= \sqrt{2}S_{\mathbf{1},\phi_i} = \sqrt{\frac{2}{2n+1}} \\ \hat{S}_{0,n+i} &= 0 \\ \hat{S}_{i,j} &= S_{\phi_i,\phi_j} = \frac{2}{\sqrt{2n+1}} \cos\frac{2\pi ij}{2n+1} \\ \hat{S}_{i,j+n} &= 0 \\ \hat{S}_{i+n,j+n} &= S_{\phi_i,\phi_j}(\epsilon)\end{aligned} \tag{5.2.17}$$

where $S_{\phi_i,\phi_j}(\epsilon)$ is the $S$-matrix of $\mathrm{Spin}(2n+1)_2$ in the presence of a puncture. This matrix element can be obtain as in (5.1.18). For example, we compute

$$
\begin{aligned}
S_{\phi_i,\phi_i}(\epsilon) &= \sum_{\beta\in\phi_i\times\phi_i} \frac{\theta(\beta)}{\theta(\phi_i)^2} S_{\mathbf{1},\beta} F_{\phi_i,\phi_i}\begin{bmatrix}\epsilon & \phi_i\\ \phi_i & \beta\end{bmatrix}\\
&= e^{-2\pi i\frac{i(2n+1-i)}{2n+1}} \frac{1}{\sqrt{2n+1}}\times\\
&\left(\frac{1}{2}\underbrace{F_{\phi_i,\phi_i}\begin{bmatrix}\epsilon & \phi_i\\ \phi_i & \mathbf{1}\end{bmatrix}}_{+1} + \frac{1}{2}\underbrace{F_{\phi_i,\phi_i}\begin{bmatrix}\epsilon & \phi_i\\ \phi_i & \epsilon\end{bmatrix}}_{+1} + e^{2\pi i(\frac{1}{2}\frac{2i(2n+1-2i)}{2n+1})}\underbrace{F_{\phi_i,\phi_i}\begin{bmatrix}\epsilon & \phi_i\\ \phi_i & \phi_{g(2i)}\end{bmatrix}}_{-1}\right)\\
&= \frac{2i}{\sqrt{2n+1}}\sin\frac{2\pi i^2}{2n+1}
\end{aligned}
\tag{5.2.18}
$$

while for $i \neq j$,

$$
\begin{aligned}
S_{\phi_i,\phi_j}(\epsilon) &= \sum_{\beta\in\phi_i\times\phi_j} \frac{\theta(\beta)}{\theta(\phi_i)\theta(\phi_j)} S_{\mathbf{1},\beta} F_{\phi_i,\phi_j}\begin{bmatrix}\epsilon & \phi_j\\ \phi_i & \beta\end{bmatrix}\\
&= \frac{1}{\sqrt{2n+1}}\left(e^{\frac{2i\pi ij}{2n+1}}\underbrace{F_{\phi_i,\phi_j}\begin{bmatrix}\epsilon & \phi_j\\ \phi_i & \phi_{g(i-j)}\end{bmatrix}}_{+1} + e^{-\frac{2i\pi ij}{2n+1}}\underbrace{F_{\phi_i,\phi_j}\begin{bmatrix}\epsilon & \phi_j\\ \phi_i & \phi_{g(i+j)}\end{bmatrix}}_{-1}\right)\\
&= \frac{2i}{\sqrt{2n+1}}\sin\frac{2\pi ij}{2n+1}
\end{aligned}
\tag{5.2.19}
$$

All in all, the $S$-matrix of the quotient takes the form

$$
\hat{S} = \frac{1}{\sqrt{2n+1}}
\begin{array}{c}
\left(\begin{array}{c|c|c}
1 & \sqrt{2} & 0\\ \hline
\sqrt{2} & 2\cos\frac{2\pi ij}{2n+1} & 0\\ \hline
0 & 0 & 2i\sin\frac{2\pi ij}{2n+1}
\end{array}\right)
\end{array}
\begin{array}{l}
\leftarrow |0\rangle\\
\leftarrow |i\rangle\\
\leftarrow |i;\epsilon\rangle
\end{array}
\tag{5.2.20}
$$

One can easily check that this matrix is unitary, and satisfies the algebra of the (bosonic) modular group, $S^2 = (ST)^3$, $S^4 = 1$.

In order to write down the fusion rules of the quotient we have to switch into the fusion basis, namely

$$
|\phi_i, \pm\rangle = \frac{1}{\sqrt{2}}(|\phi_i\rangle \pm |\phi_i;\epsilon\rangle)\,. \tag{5.2.21}
$$

In this basis, the $S$-matrix becomes $\hat{S}_{ij} \sim \frac{1}{\sqrt{2n+1}}e^{2\pi i\frac{ij}{2n+1}}$. This is in agreement with the level-rank duality $\mathrm{SO}(n)_2 \sim \mathrm{SO}(2)_{-n} = \mathrm{U}(1)_{-n}$, where the $S$-matrix of $\mathrm{U}(1)_k$ is $e^{-2\pi i\alpha\beta/k}/\sqrt{k}$. (Here $\sim$ denotes duality modulo $\{\mathbf{1},\psi\}$, since $\mathrm{U}(1)_k$ is spin for odd $k$.)

### 5.2.3 $\mathrm{SO}(n)_3$

We construct the theory using level-rank duality $\mathrm{SO}(n)_3 = \mathrm{SO}(3)_{-n} = \mathrm{SU}(2)_{-2n}/\mathbb{Z}_2$. So we consider $\mathrm{SU}(2)_k$ first.

There are $k+1$ lines, which we label as $j = 0, \frac{1}{2}, 1, \ldots, \frac{1}{2}k$. The $S$-matrix reads

$$S_{ij} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi(2i+1)(2j+1)}{k+2} \tag{5.2.22}$$

and the spins are $h_j = \frac{j(j+1)}{k+2}$. The fusion rules read

$$j_1 \times j_2 = \sum_{j=|j_1-j_2|}^{\min(J,k-J)} j, \qquad J = j_1 + j_2 \tag{5.2.23}$$

The quantum dimensions are $d_j = \frac{\sin \frac{\pi(2j+1)}{k+2}}{\sin \frac{\pi}{k+2}}$, and so $j = k/2$ is abelian. The corresponding $\mathbb{Z}_2$ symmetry acts as $j \mapsto \frac{1}{2}k - j$. The spin of this line is $k/4$, and so the symmetry is condensable if and only if $k$ is even, $k = 2n$. The quotient theory is $\mathrm{PSU}(2)_{2n} = \mathrm{SO}(3)_n$; it is spin if $n$ is odd.

The only fixed-point is $j = n/2$, whose $S$-matrix element is given by (5.1.18)

$$\begin{aligned} S_{n/2,n/2}(n) &= \sum_{\beta \in n/2 \times n/2} \frac{\theta(\beta)}{\theta(n/2)^2} S_{\mathbf{1},\beta} F_{n/2,n/2} \begin{bmatrix} n & n/2 \\ n/2 & \beta \end{bmatrix} \\ &= \frac{\theta(n/2)^{-2}}{\sqrt{n+1}} \sum_{j=0}^{n} \theta(j) \sin \frac{\pi}{2} \frac{2j+1}{n+1} F_{n/2,n/2} \begin{bmatrix} n & n/2 \\ n/2 & j \end{bmatrix} \end{aligned} \tag{5.2.24}$$

which, using $F = (-1)^j$, becomes $S_{n/2,n/2}(n) = e^{-3i\pi n/4}$. One may check that modularity is satisfied, $S^2 = (ST)^3$ and $S^4 = (-1)^F \equiv (-1)^n$. This is indeed consistent with the quotient being bosonic if $n$ is even, and fermionic if odd.

Let us consider the case of even $n$, where the quotient is bosonic. The unscreened lines are those with integer $j$, and the only fixed point is $j = n/2$. Thus, a basis for the Hilbert space is

$$\begin{aligned} |j\rangle &= \frac{1}{\sqrt{2}}(|j\rangle + |n-j\rangle), \qquad j = 0, 1, \ldots, n/2 - 1 \\ |a_1\rangle &= |n/2\rangle \\ |a_2\rangle &= |n/2; n\rangle \end{aligned} \tag{5.2.25}$$

where $|\cdot\,; \alpha\rangle$ denotes the corresponding state with an $\alpha$-puncture.

The $S$-matrix of the quotient is

$$
\begin{aligned}
\hat{S}_{i,j} &= 2S_{i,j} = 2\sqrt{\frac{1}{n+1}}\sin\frac{\pi(2i+1)(2j+1)}{2n+2} \\
\hat{S}_{i,a_1} &= \sqrt{2}S_{i,n/2} = (-1)^i\sqrt{\frac{2}{n+1}} \\
\hat{S}_{a_1,a_1} &= S_{n/2,n/2} = (-1)^{n/2}\sqrt{\frac{1}{n+1}} \\
\hat{S}_{a_1,a_2} &= 0 \\
\hat{S}_{a_2,a_2} &= S_{n/2,n/2}(n) = i^{n/2}\,.
\end{aligned}
\tag{5.2.26}
$$

The fusion basis is defined by

$$
|a_\pm\rangle = \frac{1}{\sqrt{2}}(|a_1\rangle \pm |a_2\rangle)\,. \tag{5.2.27}
$$

One can easily compute the $S$-matrix in this basis, from where one can compute, for example, the fusion rules of the theory.

Consider now the case of odd $n$, where the quotient is spin. The NS lines are those with integral isospin, and the R lines with half-integral isospin. A basis of the quotient Hilbert space is

- If we take **NS-NS** boundary conditions, the states are

$$
|j;\text{NS-NS}\rangle = \frac{1}{\sqrt{2}}(|j\rangle + |2n-j\rangle), \quad j = 0,1,\ldots,n\,. \tag{5.2.28}
$$

- If we take **NS-R** boundary conditions, the states are

$$
|j;\text{NS-R}\rangle = \frac{1}{\sqrt{2}}(|j\rangle - |2n-j\rangle), \quad j = 0,1,\ldots,n\,. \tag{5.2.29}
$$

- If we take **R-NS** boundary conditions, the states are

$$
\begin{aligned}
|j;\text{R-NS}\rangle &= \frac{1}{\sqrt{2}}(|j\rangle + |2n-j\rangle), \quad j = \frac{1}{2},\frac{3}{2},\ldots,\cancel{\frac{n}{2}},\ldots,n \\
|n/2;\text{R-NS}\rangle &= |n/2\rangle\,.
\end{aligned}
\tag{5.2.30}
$$

- If we take **R-R** boundary conditions, the states are

$$
\begin{aligned}
|j;\text{R-R}\rangle &= \frac{1}{\sqrt{2}}(|j\rangle - |2n-j\rangle), \quad j = \frac{1}{2},\frac{3}{2},\ldots,\cancel{\frac{n}{2}},\ldots,n \\
|n/2;\text{R-R}\rangle &= |n/2;n\rangle\,.
\end{aligned}
\tag{5.2.31}
$$

The modular data easily follows from this decomposition, and the once-punctured torus matrix element (5.2.24).

### 5.2.4 $\mathrm{U}(1)_k$

In this section we construct the Hilbert space of the spin TQFT $\mathrm{U}(1)_k$. This generalizes the construction of the semion-fermion theory of section 1.4.1. For some special values of $k$ this theory is time-reversal invariant. The semion-fermion theory is recovered by taking $k = 2$. For $k > 2$, the time-reversal symmetry (when present) satisfies a more exotic algebra [4], namely $\mathsf{T}^2 = \mathsf{C}$, where $\mathsf{C}$ denotes an order-2 unitary symmetry (charge conjugation).

The construction of $\mathrm{U}(1)_k$ is slightly different depending on whether $k$ is even or odd. Indeed, for $k$ odd the theory is naturally spin; but, for $k$ even, it is bosonic, and so it has to be multiplied by the trivial factor $\{\mathbf{1}, \psi\}$ if we are interested in its spin version. The latter case is rather similar to the semion-fermion theory, so here we will focus on the $k$ odd case here, and sketch the main differences for $k$ even at the end.

Consider the theory $\mathrm{U}(1)_k$ with $k$ odd. Its bosonic parent is $\mathrm{U}(1)_{4k}$, whose anyons are labelled as $\alpha \in \mathbb{Z}_{4k}$. The spin theory is obtained by condensing the fermion $\psi = 2k$. The braiding phase of an arbitrary line $\alpha$ with respect to the fermion is $B(\alpha, \psi) = e^{\pi i \alpha}$, which means that the anyons are split as

$$\begin{aligned} \text{NS}\colon & \ 2\alpha, && \alpha = 0, 1, \ldots, 2k-1 \\ \text{R}: & \ 2\alpha + 1, && \alpha = 0, 1, \ldots, 2k-1\,. \end{aligned} \tag{5.2.32}$$

These are all in two-dimensional orbits, paired up as

$$\alpha \overset{\times\psi}{\longleftrightarrow} \alpha + 2k\,. \tag{5.2.33}$$

As there are no fixed-points, all states are bosonic.

**Hilbert space and modularity.** Given the knowledge of the Hilbert space of the bosonic parent, and the action of the modular group on it, we easily construct the same objects in the quotient theory. In particular, the Hilbert space is $\mathcal{H} \cong \mathbb{C}^{4k}$, with states $|\alpha\rangle$, and modular transformations act as

$$\begin{aligned} S|\alpha\rangle &= \sum_{\alpha' \in \mathbb{Z}_{4k}} S_{\alpha,\alpha'} |\alpha'\rangle \\ T|\alpha\rangle &= e^{2\pi i(\alpha^2/4k - 1/24)} |\alpha\rangle \\ \mathsf{C}|\alpha\rangle &= |-\alpha \mod 4k\rangle\,, \end{aligned} \tag{5.2.34}$$

where $S_{\alpha,\alpha'} = e^{-2\pi i \alpha\alpha'/4k}/2\sqrt{k}$, and the term $-1/24$ in the $T$-transformation refers to the central charge of the theory.

The quotient space is as follows:

- If we take **NS-NS** boundary conditions, the states are

$$|\alpha; \text{NS-NS}\rangle = \frac{1}{\sqrt{2}} \left[ |2\alpha\rangle + |2\alpha + 2k\rangle \right], \quad \alpha = 0, \ldots, k-1 \tag{5.2.35}$$

and one has

$$\begin{aligned}
\hat{S}|\alpha;\text{NS-NS}\rangle &= \frac{1}{\sqrt{k}}\sum_{\alpha'=0}^{k-1} \mathrm{e}^{2\pi i\alpha\alpha'/k}|\alpha';\text{NS-NS}\rangle \\
\hat{T}|\alpha;\text{NS-NS}\rangle &= \mathrm{e}^{2\pi i\left(\frac{\alpha^2}{2k}-\frac{1}{24}\right)}|\alpha;\text{NS-R}\rangle \\
\hat{\mathsf{C}}|\alpha;\text{NS-NS}\rangle &= \sum_{\alpha'=0}^{k-1}(\delta_{\alpha+\alpha'}+\delta_{\alpha+\alpha'-k})|\alpha';\text{NS-NS}\rangle
\end{aligned} \tag{5.2.36}$$

where $\delta_x = 1$ if $x \equiv 0 \mod 2k$, and $\delta_x = 0$ otherwise.

- If we take **NS-R** boundary conditions, the states are

$$|\alpha;\text{NS-R}\rangle = \frac{1}{\sqrt{2}}\left[|2\alpha\rangle - |2\alpha+2k\rangle\right], \quad \alpha = 0,\ldots,k-1 \tag{5.2.37}$$

and one has

$$\begin{aligned}
\hat{S}|\alpha;\text{NS-R}\rangle &= \frac{1}{\sqrt{k}}\sum_{\alpha'=0}^{k-1} \mathrm{e}^{2\pi i(2\alpha+1)\alpha'/2k}|\alpha';\text{R-NS}\rangle \\
\hat{T}|\alpha;\text{NS-R}\rangle &= \mathrm{e}^{2\pi i\left(\frac{\alpha^2}{2k}-\frac{1}{24}\right)}|\alpha;\text{NS-NS}\rangle \\
\hat{\mathsf{C}}|\alpha;\text{NS-R}\rangle &= \sum_{\alpha'=0}^{k-1}(\delta_{\alpha+\alpha'}-\delta_{\alpha+\alpha'-k})|\alpha';\text{NS-R}\rangle
\end{aligned} \tag{5.2.38}$$

- If we take **R-NS** boundary conditions, the states are

$$|\alpha;\text{R-NS}\rangle = \frac{1}{\sqrt{2}}\left[|2\alpha+1\rangle + |2\alpha+1+2k\rangle\right], \quad \alpha = 0,\ldots,k-1 \tag{5.2.39}$$

and one has

$$\begin{aligned}
\hat{S}|\alpha;\text{R-NS}\rangle &= \frac{1}{\sqrt{k}}\sum_{\alpha'=0}^{k-1} \mathrm{e}^{2\pi i\alpha(2\alpha'+1)/2k}|\alpha';\text{NS-R}\rangle \\
\hat{T}|\alpha;\text{R-NS}\rangle &= \mathrm{e}^{2\pi i\left(\frac{(2\alpha+1)^2}{8k}-\frac{1}{24}\right)}|\alpha;\text{R-NS}\rangle \\
\hat{\mathsf{C}}|\alpha;\text{R-NS}\rangle &= \sum_{\alpha'=0}^{k-1}(\delta_{\alpha+\alpha'+1}+\delta_{\alpha+\alpha'+1-k})|\alpha';\text{R-NS}\rangle
\end{aligned} \tag{5.2.40}$$

- If we take **R-R** boundary conditions, the states are

$$|\alpha;\text{R-R}\rangle = \frac{1}{\sqrt{2}}\left[|2\alpha+1\rangle - |2\alpha+1+2k\rangle\right], \quad \alpha = 0,\ldots,k-1 \tag{5.2.41}$$

and one has

$$\begin{aligned}\hat{S}|\alpha;\text{R-R}\rangle &= \frac{1}{\sqrt{k}}\sum_{\alpha'=0}^{k-1}\mathrm{e}^{2\pi i(2\alpha+1)(2\alpha'+1)/4k}|\alpha';\text{R-R}\rangle \\ \hat{T}|\alpha;\text{R-R}\rangle &= \mathrm{e}^{2\pi i\left(\frac{(2\alpha+1)^2}{8k}-\frac{1}{24}\right)}|\alpha;\text{R-R}\rangle \\ \hat{\mathsf{C}}|\alpha;\text{R-R}\rangle &= \sum_{\alpha'=0}^{k-1}(\delta_{\alpha+\alpha'+1}-\delta_{\alpha+\alpha'+1-k})|\alpha';\text{R-R}\rangle\end{aligned} \tag{5.2.42}$$

It is reassuring to see that these modular transformations map the different Hilbert spaces precisely as they should (cf. (5.1.26)). Moreover, these matrices are unitary, $\hat{S}$ is symmetric ($\hat{S}^t_{s^{\mathsf{a}},s^{\mathsf{b}}}=\hat{S}_{s^{\mathsf{b}},s^{\mathsf{a}}}$), and they satisfy the modular algebra $(\hat{S}\hat{T})^3=\hat{S}^2=\hat{\mathsf{C}}$ with $\hat{\mathsf{C}}^2=1$.

**Wilson lines.** The Wilson lines are given by

$$\begin{aligned}W^{(\mathsf{a})}(\alpha)|\gamma;s^{\mathsf{a}}s^{\mathsf{b}}\rangle &= \mathrm{e}^{-2\pi i\alpha(\gamma+(1+s^{\mathsf{a}})/4)/k}|\gamma;s^{\mathsf{a}}s^{\mathsf{b}}\rangle \\ W^{(\mathsf{b})}(\alpha)|\gamma;s^{\mathsf{a}}s^{\mathsf{b}}\rangle &= |\alpha+\gamma;s^{\mathsf{a}}s^{\mathsf{b}}\rangle - s^{\mathsf{b}}|\alpha+\gamma+k;s^{\mathsf{a}}s^{\mathsf{b}}\rangle\end{aligned} \tag{5.2.43}$$

where $\alpha\in\mathbb{Z}_{2k}$. They satisfy the expected properties, e.g.,

$$\begin{aligned}W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(\psi) &= W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(k) = -s^{\mathsf{c}}\mathbf{1}_k \\ W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(\alpha\times\alpha') &= W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(\alpha)W^{(\mathsf{c})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(\alpha') \\ S_{s^{\mathsf{a}},s^{\mathsf{b}}}W^{(\mathsf{a})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(\alpha)(S_{s^{\mathsf{a}},s^{\mathsf{b}}})^\dagger &= W^{(\mathsf{b})}_{s^{\mathsf{b}},s^{\mathsf{a}}}(\bar{\alpha}) \\ S_{s^{\mathsf{a}},s^{\mathsf{b}}}W^{(\mathsf{b})}_{s^{\mathsf{a}},s^{\mathsf{b}}}(\alpha)(S_{s^{\mathsf{a}},s^{\mathsf{b}}})^\dagger &= W^{(\mathsf{a})}_{s^{\mathsf{b}},s^{\mathsf{a}}}(\alpha)\end{aligned} \tag{5.2.44}$$

**Time-reversal.** We now implement time-reversal invariance. Recall that $\mathrm{U}(1)_k$ is time-reversal invariant if and only if $q^2=-1 \mod k$ is solvable for some $q\in\mathbb{Z}$, in which case time-reversal acts as $\alpha\mapsto q\alpha$ [4]. This means that, given $\mathsf{T}=\tau K$, we require

$$\begin{aligned}\tau(W_2^{(\mathsf{c})})^*\tau^{-1} &= W_{2q}^{(\mathsf{c})} \\ &= (W_2^{(\mathsf{c})})^q\end{aligned} \tag{5.2.45}$$

with solution

$$\tau_{\alpha,\beta}=(-s^{\mathsf{b}})^{\alpha+\beta}\delta_{2\alpha q+2\beta+\frac{1}{2}(s^{\mathsf{a}}+1)(q+1)} \tag{5.2.46}$$

up to a global phase. One can check that

$$\tau\tau^*=(-1)^{\mathrm{Arf}(s)}\left(\delta_{\frac{1}{2}(s^{\mathsf{a}}+1)+\alpha+\beta}-s^{\mathsf{b}}\delta_{\frac{1}{2}(s^{\mathsf{a}}+1)+\alpha+\beta-k}\right) \tag{5.2.47}$$

and so $\mathsf{T}^2=(-1)^{\mathrm{Arf}(s)}\hat{\mathsf{C}}$.

We see that the time-reversal algebra is deformed by Arf, signaling an anomaly. In this case, the source of the anomaly is clear: the theory has non-vanishing central charge, $c=1$, so it is not time-reversal invariant in the strict sense. We have to multiply by a suitable SPT in order to subtract off the central charge. In this case, $\mathrm{U}(1)_{-1}$ does the trick, as this SPT has $c = -1$.

The Hilbert space of $\mathrm{U}(1)_{-1}$ is straightforward: it suffices to take $k = 1$ in the discussion above. Looking at the action of $\hat{\mathsf{C}}$ on the (one-dimensional) Hilbert space of $\mathrm{U}(1)_{-1}$ we learn that $\hat{\mathsf{C}} = (-1)^{\mathrm{Arf}(s)}$. Therefore, multiplying a given theory by $\mathrm{U}(1)_{\pm 1}$ has the effect of redefining $\hat{\mathsf{C}} \to \hat{\mathsf{C}} \times (-1)^{\mathrm{Arf}(s)}$, which means that the theory $\mathrm{U}(1)_k \times \mathrm{U}(1)_{-1}$ has undeformed algebra, namely $\mathsf{T}^2 = \hat{\mathsf{C}}$. The Arf deformation in the case of $\mathrm{U}(1)_k$ was just signaling that we had not corrected the central charge down to zero; after doing so, the deformation disappears from the time-reversal algebra.

Finally, we make a few remarks concerning the $k$ even case. Now the theory is naturally bosonic, and can be made spin by tensoring with an invertible spin TQFT. If we are interested in time-reversal invariance, the natural choice is $\mathrm{U}(1)_k \times \mathrm{U}(1)_{-1}$, so as to have vanishing central charge. As the theory is a tensor product, one factor being bosonic, the total Hilbert space is straightforward:

$$\hat{\mathcal{H}}_s(\mathrm{U}(1)_k \times \mathrm{U}(1)_{-1}) = \mathcal{H}(\mathrm{U}(1)_k) \otimes \hat{\mathcal{H}}_s(\mathrm{U}(1)_{-1}) \tag{5.2.48}$$

where $\mathcal{H}(\mathrm{U}(1)_k)$ is the space of the bosonic theory $\mathrm{U}(1)_k$, and $\hat{\mathcal{H}}_s(\mathrm{U}(1)_{-1})$ is the space of the fermionic theory $\mathrm{U}(1)_{-1}$. The Hilbert space of $\mathrm{U}(1)_{-1}$ was discussed above, and that of $\mathrm{U}(1)_k$ is well-known, being bosonic. In this sense, no new computation is required in the case of $\mathrm{U}(1)_k$ with $k$ even. One can easily check through straightforward computation that the main conclusions are identical to those of the $k$ odd case, in particular, time-reversal satisfies $\mathsf{T}^2 = \hat{\mathsf{C}}$, with no deformation. (If we reintroduce a non-zero value of the central charge, by multiplying by an extra factor of $\mathrm{U}(1)_{\pm 1}$, the deformation reappears, and we get $\mathsf{T}^2 = (-1)^{\mathrm{Arf}(s)}\hat{\mathsf{C}}$, again signalling the anomaly due to $c$).

## 5.3 Fermionic surgery.

The partition function of a topological theory on an arbitrary three-manifold can be computed explicitly via surgery [22] (see also [289] for a nice review). The idea is that any manifold $M$ can be written as $M_1 \cup M_2$, where $M_1, M_2$ are two handlebodies glued via some homeomorphism $D$ acting on their shared boundary $\Sigma$. The path integral of the theory, perhaps in the presence of operator insertions, can be expressed as suitable matrix elements of $D$, i.e., $\langle v_1|D|v_2\rangle$, where $v_1, v_2$ are the states prepared by $M_1, M_2$. In this sense, if we understand the action of homeomorphisms on $\Sigma$, we can compute any observable on any manifold $M$. The novelty in the spin case is that the topological spaces depend on a choice of spin structure, and this dependence is reflected in the choice of splitting and the action of $D$.

The simplest case is that of $M = L(a, b)$, a lens space, since for these spaces (and these only) the surface $\Sigma$ has genus $g = 1$, i.e., it is a torus. Here we will content ourselves with illustrating the general idea through a few simple examples, mainly following [180].

A lens space $L(a,b)$ is defined as the quotient of a sphere $S^3$ by a $\mathbb{Z}_a$ group, whose action on $S^3$ is specified by $b$, with $a,b$ two coprime integers. This space can be expressed as the gluing of two solid tori via the action of some $D \in SL_2(\mathbb{Z})$ matrix that depends on $a,b$; for example, one can take

$$D = \begin{pmatrix} b & a' \\ a & b' \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{5.3.1}$$

where $a', b'$ are any two integers that make $D$ unimodular, i.e., such that $bb' - aa' = 1$ (these exist because $a,b$ are coprime). We shall mostly be interested in the case where $a$ is even, where $L(a,b)$ has two spin structures, while for $a$ odd it has a unique spin structure.

For $a$ even, the spin structures of $L(a,b)$ are described as follows. There are two such structures, which we denote as $\pm$. Given the surgical presentation of the lens space as the sum of two tori, these two structures are induced from the spin structures of the tori. The $\mathsf{a}$-cycle of these tori are filled-up so their spin structures take the form $(\mathrm{NS}, s_i^{\mathsf{b}})$, where $i = 1,2$ refers to the sign around the $\mathsf{b}$-cycle of these two tori. We can choose one of the signs $s_i^{\mathsf{b}}$ at will, but then the other will be determined too. For example, if we choose $s_1^{\mathsf{b}} = \pm 1$, then (5.3.1) fixes $s_2^{\mathsf{b}} = (-1)^{a'}(\pm 1)^{b'}$. In other words, the $3d$ spin structures $\pm$ of $L(a,b)$ correspond to tori with $2d$ spin structures $(-1, \pm 1)$ and $(-1, (-1)^{a'}(\pm 1)^{b'})$, respectively. Therefore, the spin-dependent partition function of the lens space is

$$Z[L(a,b);\pm] = \langle 0; \mathrm{NS}\text{-}\pm | D | 0; \mathrm{NS}\text{-}(-1)^{a'}(\pm 1)^{b'} \rangle \tag{5.3.2}$$

where $|0; \pm\text{-}\pm\rangle$ is the vacuum state in the torus Hilbert space with spin structure $\pm, \pm$.

Let us do a few simple examples. If we take $b = 1$, then we can take $a' = 0$, $b' = 1$, and so the modular transformation reads

$$D = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = ST^{-a}S \tag{5.3.3}$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The spin-dependent partition functions of the lens space are, then,

$$Z[L(a,1);\pm] = \langle 0; \mathrm{NS}\text{-}\pm | ST^{-a}S | 0; \mathrm{NS}\text{-}\pm \rangle \tag{5.3.4}$$

or, more explicitly,

$$\begin{aligned} Z[L(a,1);-1] &= (ST^{-a}S)_{\text{NS-NS}\to\text{NS-NS}} \\ &= S_{\text{NS-NS}\to\text{NS-NS}}(T_{\text{NS-R}\to\text{NS-NS}}T_{\text{NS-NS}\to\text{NS-R}})^{-a/2}S_{\text{NS-NS}\to\text{NS-NS}} \\ Z[L(a,1);+1] &= (ST^{-a}S)_{\text{NS-R}\to\text{NS-R}} \\ &= S_{\text{R-NS}\to\text{NS-R}}T^{-a}_{\text{R-NS}\to\text{R-NS}}S_{\text{NS-R}\to\text{R-NS}} \end{aligned} \tag{5.3.5}$$

respectively. Using the known expressions for the fermionic modular data in terms of the data of the bosonic parent (cf. (5.1.38), (5.1.39)), this is easily evaluated as

$$Z[L(a,1);\pm] \equiv \frac{2e^{2\pi iac/24}}{\mathscr{D}^2} \sum_{\alpha\in\mathcal{A}_\pm} \theta(\alpha)^{-a} d_\alpha^2 \tag{5.3.6}$$

where: $c$ denotes the chiral central charge; $\mathscr{D}^2 \equiv \sum_{\alpha\in\mathcal{A}} d_\alpha^2$ the total quantum dimension of the bosonic parent; $\theta(\alpha) = e^{2\pi i h_\alpha}$ the topological spin; and $d_\alpha := S_{\alpha,0}/S_{0,0}$ the quantum dimension of $\alpha \in \mathcal{A}$.

For example, the bosonic parent of $\mathrm{U}(1)_k$ (with $k$ odd) is given by $\mathrm{U}(1)_{4k}$. Being abelian, one has $d_\alpha = 1$ and $\mathscr{D}^2 = 4k$. Also, the central charge is $c = 1$ and the NS lines are the even lines of $\mathcal{A} \cong \mathbb{Z}_{4k}$ and the R-lines are the odd lines. Plugging these facts into the general formula (5.3.6) one finds

$$\begin{aligned} Z[L(a,1);-1] &= \frac{1}{k}\sum_{\alpha=0}^{k-1} \mathrm{e}^{-2\pi ia\left(\frac{\alpha^2}{2k}-\frac{1}{24}\right)} \\ Z[L(a,1);+1] &= \frac{1}{k}\sum_{\alpha=0}^{k-1} \mathrm{e}^{-2\pi ia\left(\frac{(2\alpha+1)^2}{8k}-\frac{1}{24}\right)} \end{aligned} \tag{5.3.7}$$

which agree with the expressions in [180]. Of course, one also obtains the same result by plugging the fermionic modular matrices of $\mathrm{U}(1)_k$ (cf. (5.2.36), (5.2.38), (5.2.40)) directly into (5.3.5).

One can also look at other theories, for example $\mathrm{SO}(n)_1$, which is invertible and for which the partition function should be just a phase. Say, for $n$ odd, one has three lines with quantum dimensions $d_0 = d_\psi = 1$ and $d_\sigma = \sqrt{2}$ and spins $\theta(0) = 1, \theta(\psi) = -1, \theta(\sigma) = e^{2\pi in/16}$. Then, using $\mathscr{D}^2 = 4$ and $c = n/2$ in (5.3.6), one gets

$$\begin{aligned} Z[L(a,1);-1] &= e^{2\pi ina/48} \\ Z[L(a,1);+1] &= e^{-2\pi ina/24} \end{aligned} \tag{5.3.8}$$

As before, one can also derive this by plugging the fermionic data (cf. (5.2.2)) directly into (5.3.5). Using the modular data of $\mathrm{SO}(n)_1$ for $n$ even, it is not hard to show that (5.3.8) is also correct for $n$ even, i.e., these formulas are valid for any $n$.

As a simple consistency check of (5.3.7) and (5.3.8), note that

$$\left.\frac{Z[L(a,1);-1]}{Z[L(a,1);+1]}\right|_{\mathrm{U}(1)_k} \equiv \left.\frac{Z[L(a,1);-1]}{Z[L(a,1);+1]}\right|_{\mathrm{SO}(2k)_1} \tag{5.3.9}$$

which is the expected relation, given the level-rank duality $\mathrm{U}(1)_k \leftrightarrow \mathrm{SU}(k)_{-1} \times \mathrm{SO}(2k)_1$, and the fact that $\mathrm{SU}(k)_{-1}$ is bosonic (i.e., independent of the spin structure).

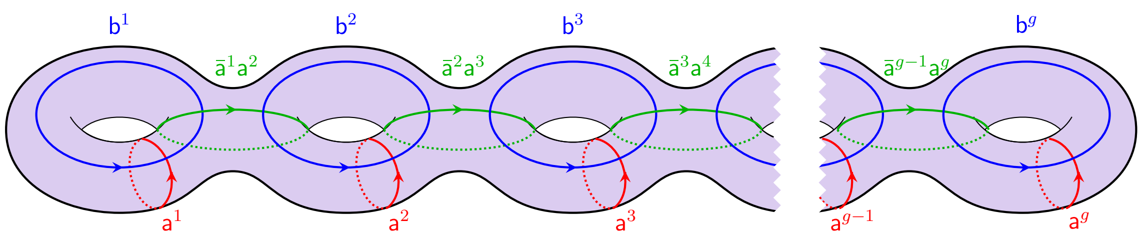
One can use similar techniques to obtain the partition function of other lens spaces. For example, for $b = 3$ and $a = 3n \pm 1$, one can use $a' = \mp 1$ and $b' = \mp n$, whence

$$D = \begin{pmatrix} 3 & \mp 1 \\ a & (1 \mp a)/3 \end{pmatrix} = \mp S T^{-n} S T^{\pm 3} S \tag{5.3.10}$$

Using the known values of $S$ and $T$ one can compute $Z[L(a, 3); \pm]$ for any spin structure. The general case is analogous: for a given $a, b$ one writes down the corresponding $SL_2(\mathbb{Z})$ gluing matrix as a word in $S$ and $T$, and evaluates the partition function using the values of these matrices as computed in the previous sections, e.g. cf. (5.1.38), (5.1.39).

## 5.4 Higher genus. Fermionic Verlinde formula.

We saw in the previous section how the knowledge of the modular data on the torus allowed us to compute the partition function of fermionic theories on more general spin 3-manifolds. The torus data was only good for lens spaces, while more general 3-manifolds usually split into higher genus Riemann surfaces. The purpose of this section is to explore fermionic theories on such surfaces. We let $\Sigma$ denote the Riemann surface such that the manifold is $M = \mathbb{S}^1 \times \Sigma$, with the circle denoting the time direction. We choose to display the surface like so:




where $\mathsf{a}^i, \mathsf{b}^i$ is a basis of homology. Punctures, if any, are taken to be placed to the very left of the surface.

**Spin structures.** In order to define fermionic theories on $\Sigma$ we must specify a spin structure, i.e., a consistent choice of signs a spinor picks up as we move it around. A contractible cycle is necessarily anti-periodic, but non-contractible ones are allowed to be periodic. A basis of non-contractible cycles is $\{\mathsf{a}^i, \mathsf{b}^i\}_{i=1,2,\dots,g}$, so a spin structure is specified by $2^{2g}$ signs, which we denote as $s^1, s^2, \dots, s^g$, where each $s^i = (s^{\mathsf{a}^i}, s^{\mathsf{b}^i})$ is a pair of signs. We denote by $\Sigma_{g;s}$ the surface $\Sigma_g$ with choice $(s^1, \dots, s^g)$. We say a choice signs is *even* if $(-1)^{\mathrm{Arf}(s)} = +1$, and *odd* if $(-1)^{\mathrm{Arf}(s)} = -1$, where

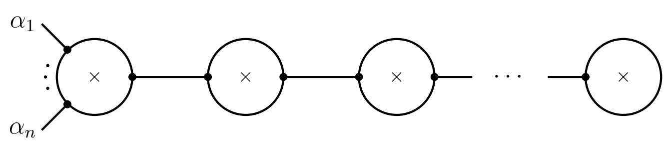
$$\mathrm{Arf}(s_1, \dots, s_g) := \sum_{i=1}^{g} (s^{\mathsf{a}^i} + 1)(s^{\mathsf{b}^i} + 1)/4 \mod 2 \tag{5.4.1}$$

By construction, $\mathrm{Arf}(s^1 \cup s^2) = \mathrm{Arf}(s^1) + \mathrm{Arf}(s^2) \mod 2$, and so $s^1 \cup s^2$ is even if both $s^1, s^2$ are even, or both odd, and odd if only one of them is odd. Thus, the number $\#$ of even and odd spin structures satisfies

$$\begin{aligned} \#_{\text{even}}(g_1 + g_2) &= \#_{\text{even}}(g_1)\#_{\text{even}}(g_2) + \#_{\text{odd}}(g_1)\#_{\text{odd}}(g_2) \\ \#_{\text{odd}}(g_1 + g_2) &= \#_{\text{even}}(g_1)\#_{\text{odd}}(g_2) + \#_{\text{odd}}(g_1)\#_{\text{even}}(g_2) \end{aligned} \tag{5.4.2}$$

Solving this recurrence with the obvious initial conditions $\#_{\text{even}}(1) = 3$ and $\#_{\text{odd}}(1) = 1$ we find the well-known results $\#_{\text{even}}(g) = 2^{g-1}(2^g + 1)$, $\#_{\text{odd}}(g) = 2^{g-1}(2^g - 1)$. Naturally, $\#_{\text{even}}(g) + \#_{\text{odd}}(g) = 2^{2g}$, the total number of spin structures.

**Hilbert space.** Recall that the Hilbert space of the fermionic theory can be obtained by gauging a suitable one-form symmetry in the bosonic parent theory. On a general surface $\Sigma$, the states of the bosonic theory are labelled as in (5.1.5), namely they consist of anyon configurations of the form

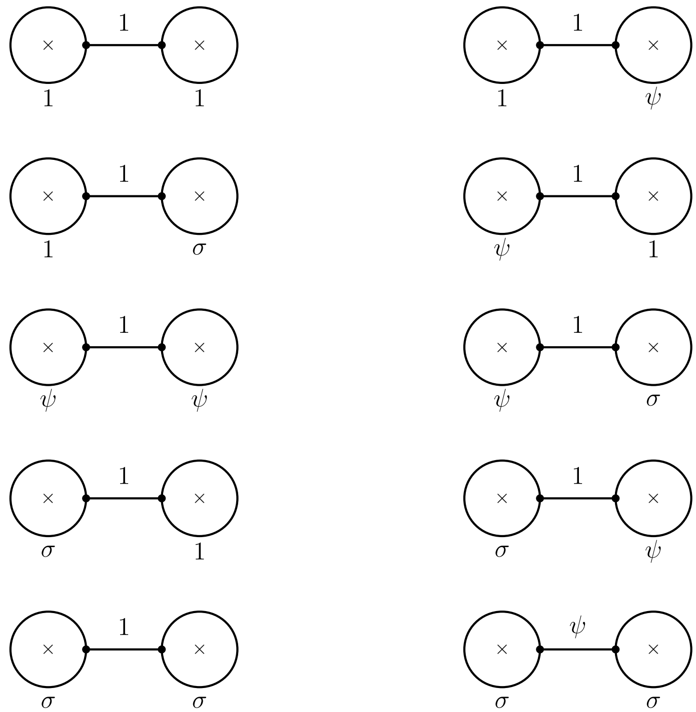



(5.4.3)

where $\alpha_i$ denote external punctures (i.e., Wilson lines running in the time direction), and the internal lines denote anyons running along the interior of $\Sigma$ (with the $\mathsf{a}$-cycles being contractible), the crosses $\times$ representing the holes. Gauging the one-form symmetry means inserting the condensing line $\psi$ in all possible ways, i.e., once in the time-direction (which introduces an extra $\psi$-puncture), and then also in a homology basis of $\Sigma$ (which projects into the invariant states). For example, wrapping $\psi$ around a given $\mathsf{a}^i$-cycle must reproduce the spin structure around the same, $s^{\mathsf{a}^i}$, which means that the anyon around the orthogonal cycle $\mathsf{b}^i$ must be in $\mathcal{A}_{s^{\mathsf{a}^i}}$, i.e., an NS line if $s^{\mathsf{a}^i} = -1$ and a R-line if $s^{\mathsf{a}^i} = +1$. Similarly, wrapping $\psi$ around a $\mathsf{b}^i$-cycle must reproduce the spin structure around the same, $s^{\mathsf{b}^i}$, which instructs us to take suitable linear combinations such that fusion with $\psi$ produces the correct sign.

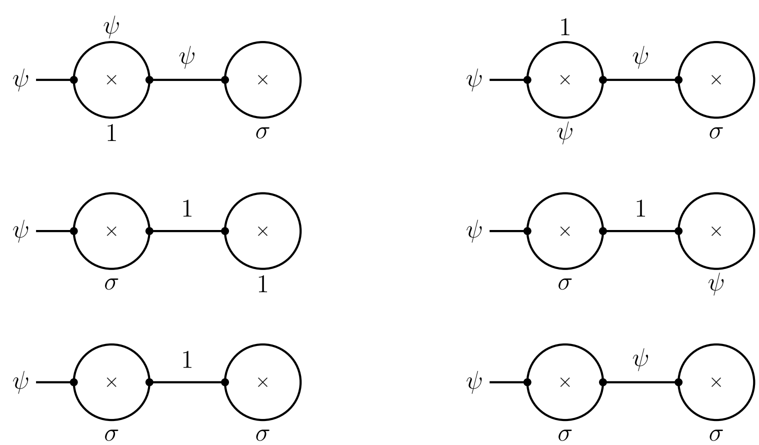
Let us do a few explicit examples. We begin with the bosonic theory $\mathrm{Spin}(n)_1$ (with $n$

odd); all its genus-2 untwisted states are



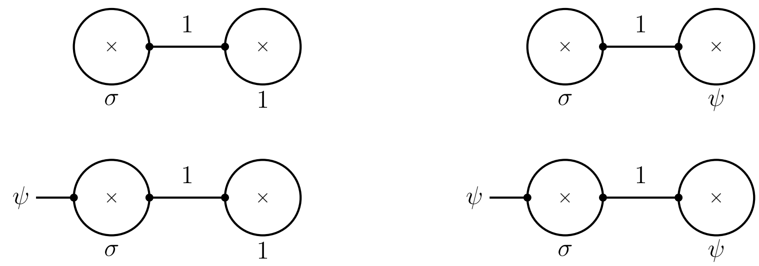
(5.4.4)

while the twisted states are



(5.4.5)

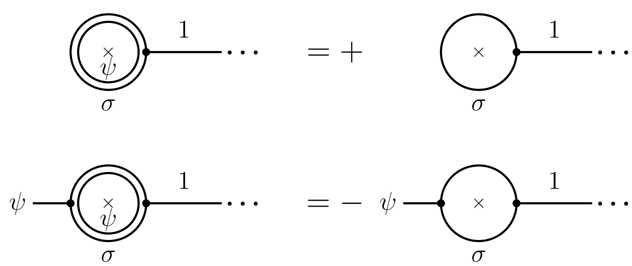
Condensing $\psi$ yields the fermionc theory $\mathrm{SO}(n)_1$, on where these $10+6$ states descend to a unique state on each of the $10+6$ spin structures of $\Sigma_2$. For example, say we are interested in the Hilbert space with spin structure $s^{\mathsf{a}^1} = +1$ and $s^{\mathsf{a}^2} = -1$. Then the states in the condensed theory are linear combinations of the following bosonic states:




(5.4.6)

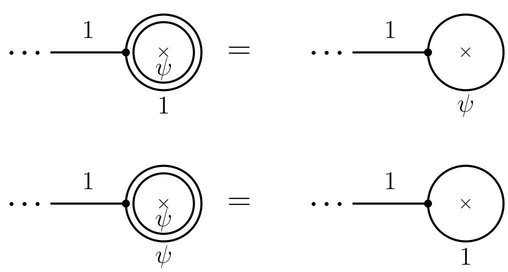
The reason is that wrapping $\psi$ around the left anyon should yield $s^{\mathsf{a}^1} = +1$, and hence the left anyon must be Ramond type (in this example, $\mathcal{A}_{\mathrm{R}} = \{\sigma\}$), while wrapping $\psi$ around the right anyon should yield $s^{\mathsf{a}^2} = -1$, and hence the right anyon must be Neveu-Schwarz type (in this example, $\mathcal{A}_{\mathrm{NS}} = \{1, \psi\}$).

The specific linear combinations of states in (5.4.6) is decided by the boundary conditions around the $\mathsf{b}$-cycles. For example, given that




(5.4.7)

we learn that the first two lines in (5.4.6) must correspond to $s^{\mathsf{b}^1} = -1$ boundary conditions, while the second two lines must correspond to $s^{\mathsf{b}^1} = +1$ boundary conditions. Finally, given that

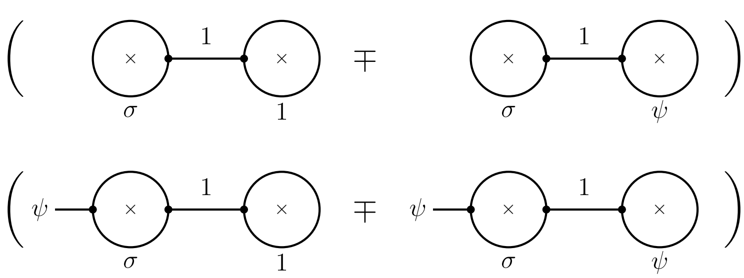



(5.4.8)

we learn that we must sum the two columns in (5.4.6) if $s^{\mathsf{b}^2} = -1$, and subtract them if $s^{\mathsf{b}^2} = +1$. In other words, the final result is the following four Hilbert spaces:

$$\hat{\mathcal{H}}(\Sigma_{2;+-,-\pm}) = \frac{1}{\sqrt{2}}$$

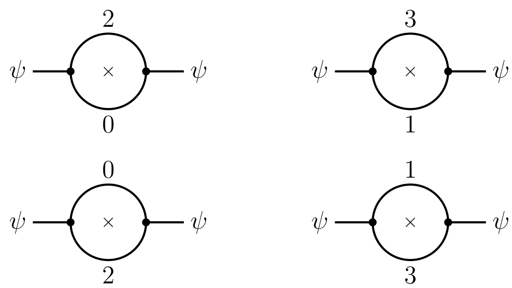
$$\hat{\mathcal{H}}(\Sigma_{2;++,-\pm}) = \frac{1}{\sqrt{2}}$$




(5.4.9)

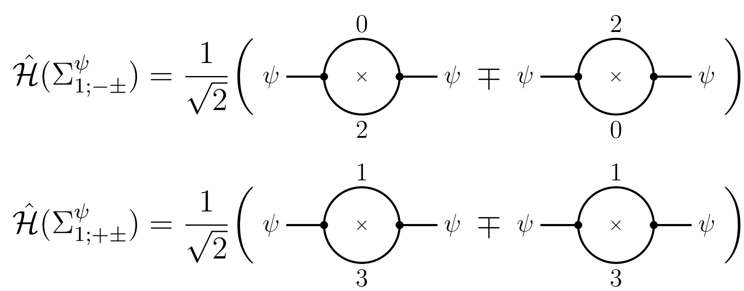
The other twelve Hilbert spaces of the fermionic phase are analogous. They are all one-dimensional, as expected from an invertible theory $\mathrm{SO}(n)_1$. Furthermore, the fermionic punctures only appear for the odd spin structures of $\Sigma_2$, i.e., the even-spin-structure Hilbert spaces have a single bosonic vacuum while the odd-spin-structure Hilbert spaces have a single fermionic vacuum. In general, the partition function of $\mathrm{SO}(n)_1$ on an arbitrary surface $\Sigma$ is $(-1)^{n\,\mathrm{Arf}(s)}$, where $s$ is the spin structure of $\Sigma$, as we shall show below.

Let us now do an example that includes external punctures (on top of the condensing line). We can look at, for example, the torus Hilbert space of $\mathrm{U}(1)_1$ in the presence of a $\psi$ puncture. This means that we must look at the states of the bosonic parent $\mathrm{U}(1)_4$, both in the presence of one $\psi$ puncture and two $\psi$ punctures. Being abelian, it is clear that there are, in fact, no states with a single $\psi$ puncture, so all states in this example actually come from the twisted Hilbert space (the untwisted one is empty):




(5.4.10)

where $0,1,2,3 \in \mathbb{Z}_4$ denote the four lines of $\mathrm{U}(1)_4$, and $\psi = 2$ is the condensing line. Following the same reasoning as before, the four twisted Hilbert spaces are




(5.4.11)

Note that all of the states come from the twisted Hilbert space of the bosonic parent; this means that they are all fermionic states. In particular, the path integral with Ramond boundary conditions on the time-circle, which computes $\mathrm{tr}_{\hat{\mathcal{H}}(\Sigma^{\psi}_{1;s})}(-1)^F$, yields $-1$ for all spin structures $s$, while the same path integral with Neveu-Schwarz boundary conditions, which computes $\mathrm{tr}_{\hat{\mathcal{H}}(\Sigma^{\psi}_{1;s})}(\mathrm{id})$, yields $+1$. This reproduces the geometric computation in appendix A of [216].

The construction of the Hilbert space for an arbitrary surface $\Sigma^{\alpha}_{g;s}$ (with arbitrary genus $g$, punctures $\alpha$, and spin structure $s$), for an arbitrary fermionic TQFT, is a straightforward generalization. We begin by writing down all the states in the bosonic parent, both untwisted and twisted. The former correspond to the states in the presence of external $\alpha$ punctures, while the latter we include $\alpha$ punctures and an extra $\psi$ puncture. Then, the choice of spin structure $s$ decides how these bosonic states are distributed among the different Hilbert spaces $\hat{\mathcal{H}}(\Sigma_{g;s}^{\alpha})$, namely, the boundary condition $s^{\mathsf{a}^i}$ decide the anyon type on the $\mathsf{b}^i$ cycle, while the boundary condition $s^{\mathsf{b}^i}$ decides which specific linear combinations of states to take. The states coming from the untwisted Hilbert space descend to bosons in the condensed phase, while those coming from the twisted Hilbert space descend to fermions.

**Modular data.** Given the explicit basis of states, the modular data of the fermionic theory is directly read off the data of the bosonic theory: as the fermionc states are written as linear combinations of bosonic states, the action of a given Dehn twist on the former can be written in terms of its action on the latter, which is assumed known if the bosonic parent is sufficiently understood. Needless to say, this last part can be quite tricky since this requires the $F$-symbols of the bosonic parent, which are calculable in principle but time-consuming in practice. In any case, as a matter of principle, the problem is solved: all the modular data of the fermionic theory is uniquely specified as a function of the modular data of the bosonic parent. Given this, and a suitable surgery presentation of an arbitrary three-manifold $M$, one can compute any path integral on any fermionic TQFT, which completely solves the theory, at least formally.

**Verlinde formula.** Writing down an explicit basis of states of $\Sigma_{g;s}^{\alpha}$ is useful if we intend to calculate the modular data, as a means to compute partition functions on general 3-manifolds. If one is just interested in the path integral over $\mathbb{S}^1\times\Sigma_{g;s}^{\alpha}$, then it is enough to know the number of states in $\hat{\mathcal{H}}(\Sigma_{g;s}^{\alpha})$, and not the form of an explicit basis. To this end, the Verlinde formula [38, 39] offers a significant simplification, since it gives us directly the number of states.

The fermionic generalization of the Verlinde formula is straightforward: if we denote $\hat{\mathcal{H}}(\Sigma_{g;s}^{\alpha_1\cdots\alpha_n}) := \mathbb{C}^{b|f}$, with $b$ the number of bosons and $f$ the number of fermions, then

$$\begin{aligned} b &= \sum_{\alpha\in\mathcal{A}} c_\alpha S_{\mathbf{1},\alpha}^{\chi(\Sigma)}\prod_{i=1}^n S_{\alpha_i,\alpha} \\ f &= \sum_{\alpha\in\mathcal{A}} B(\psi,\alpha) c_\alpha S_{\mathbf{1},\alpha}^{\chi(\Sigma)}\prod_{i=1}^n S_{\alpha_i,\alpha} \end{aligned} \tag{5.4.12}$$

where: $\mathcal{A}$ denotes the set of anyons in the bosonic parent; $c_\alpha$ is defined as $c_\alpha = 2^{-2g}$ if $\alpha$ is non-Majorana (i.e., either $a$-type or $x$-type) and as $c_\alpha = 2^{-g}(-1)^{\mathrm{Arf}(s)}$ if $\alpha$ is Majorana (i.e., $m$-type); $S_{\alpha,\beta}$ is the $S$-matrix of the bosonic parent; and $B(\psi,\alpha) := S_{\psi,\alpha}/S_{\mathbf{1},\alpha} \equiv -\theta(\psi\times\alpha)/\theta(\alpha)$ is the braiding phase with respect to $\psi$ (such that $B(\psi,\alpha)=+1$ if $\alpha$ is an NS line and $B(\psi,\alpha)=-1$ if $\alpha$ is an R line).

This is just the regular Verlinde formula, but the fermionic states are weighted by the braiding phase $B(\psi,\alpha)$ (because these are the twisted sectors with respect to the one-form gauging), and there is a normalization factor $c_\alpha$ for Majorana lines (since non-Majorana lines are paired up as $\alpha, \alpha\times\psi$ while Majorana lines are not, as they are their own fermionic partner $m\times\psi = m$, cf. e.g. (5.1.33)).

As a note, we mention that one can isolate the contribution of Majorana lines and write the formula in the equivalent form

$$
\begin{aligned}
b &= 2^{-2g}\dim(\Sigma_g^{\alpha_1\cdots\alpha_n}) + 2^{-g}((-1)^{\mathrm{Arf}(s)} - 2^{-g})\left(\sum_m S_{\mathbf{1},m}^{\chi(\Sigma)}\prod_{i=1}^n S_{\alpha_i,m}\right)\\
f &= 2^{-2g}\dim(\Sigma_g^{\alpha_1\cdots\alpha_n\psi}) - 2^{-g}((-1)^{\mathrm{Arf}(s)} - 2^{-g})\left(\sum_m S_{\mathbf{1},m}^{\chi(\Sigma)}\prod_{i=1}^n S_{\alpha_i,m}\right)
\end{aligned}
\tag{5.4.13}
$$

where $\dim(\Sigma_g^{\alpha_1\cdots\alpha_n})$ is the dimension of the bosonic Hilbert space (5.1.3). If we write $d_\alpha = S_{\alpha,\mathbf{1}}/S_{\mathbf{1},\mathbf{1}}$ for the quantum dimension of $\alpha$, we can also write the formula as

$$
\begin{aligned}
b &= \left(\sum_{\alpha\in\mathcal{A}} d_\alpha^2\right)^{g-1}\left(\sum_{\alpha\in\mathcal{A}} c_\alpha d_\alpha^{2-2g}\prod_{i=1}^n \frac{S_{\alpha_i,\alpha}}{S_{\mathbf{1},\alpha}}\right)\\
f &= \left(\sum_{\alpha\in\mathcal{A}} d_\alpha^2\right)^{g-1}\left(\sum_{\alpha\in\mathcal{A}} B(\psi,\alpha) c_\alpha d_\alpha^{2-2g}\prod_{i=1}^n \frac{S_{\alpha_i,\alpha}}{S_{\mathbf{1},\alpha}}\right)
\end{aligned}
\tag{5.4.14}
$$

**Examples.** In order to illustrate the formula we can look at a few examples. The first one we shall look at is an arbitrary fermionic abelian order (with integral central charge), i.e., one where all lines have unit quantum dimension $d_\alpha = 1$. If we let $2k$ denote the number of lines in the theory (where $k$ is the absolute value of the determinant of the $K$-matrix), then the bosonic parent is another abelian theory, this time bosonic, and which has $|\mathcal{A}| = 4k$ lines, half of which are NS lines (and descend to the $2k$ lines in the fermionic theory) and the other half R lines. As the bosonic parent is abelian, the theory has no Majorana lines, since $m\times\psi = m$ is incompatible with abelian fusion rules (which are group-like). Using this information, the Verlinde formula predicts that the number of fermions is $f = 0$, since half the lines have $B(\psi,\alpha) = +1$ and the other half $B(\psi,\alpha) = -1$, which cancel out pairwise. On the other hand, the number of bosons is $b = 2^{-2g}|\mathcal{A}|^g \equiv k^g$. So, to summarize, the fermionic Verlinde formula predicts no fermionic states, and $k^g$ bosonic ones, which is what one expects in abelian theories.

We can do another simple example, namely $\mathrm{SO}(n)_1$. For $n$ odd, the bosonic parent $\mathrm{Spin}(n)_1$ has three lines $\mathcal{A} = \{\mathbf{1},\psi,\sigma\}$, the first two being NS lines and the last one a Majorana line. The quantum dimensions are $d_{\mathbf{1}} = d_\psi = 1$ and $d_\sigma = \sqrt{2}$. Plugging this information into the Verlinde formula one finds $b = (1+(-1)^{\mathrm{Arf}(s)})/2$ and $f = (1-(-1)^{\mathrm{Arf}(s)})/2$. In other words, there is a single state for any $s$, and it is a boson for even spin structures, and a fermion for odd spin structures, which we already saw in an explicit example above (cf. the discussion below (5.4.9)). For $n$ even the theory is abelian, and therefore follows the analysis in the previous paragraph, namely $b=1$ and $f=0$. All in all, the Ramond partition function of $\mathrm{SO}(n)_1$ is $(-1)^{n\,\mathrm{Arf}(s)}$.

As a final example, consider $\mathrm{SO}(3)_k$ for $k$ odd, in which case the theory is fermionic with bosonic parent is $\mathrm{SU}(2)_{2k}$. This theory was discussed in section 5.2.3, where we showed that the NS lines are those with integer isospin $j$, while the R lines are those with half-integer isospin $j$; and, furthermore, there is a single Majorana line with $j=k/2$. Finally, the quantum dimension of a generic line is $d_j=\frac{\sin\frac{\pi(2j+1)}{2k+2}}{\sin\frac{\pi}{2k+2}}$. Plugging this information into the Verlinde formula, we find that

$$\begin{aligned}
b&=\left(\sum_{j=0}^{k}\frac{\sin\left[\frac{\pi(2j+1)}{2k+2}\right]^2}{\sin\left[\frac{\pi}{2k+2}\right]^2}\right)^{g-1}\left(\sum_{j=0}^{k}c_j\frac{\sin\left[\frac{\pi(2j+1)}{2k+2}\right]^{2-2g}}{\sin\left[\frac{\pi}{2k+2}\right]^{2-2g}}\right)\\
&=\left(\frac{k+1}{2}\right)^{g-1}\left(\tfrac{1}{2}(-1)^{\mathrm{Arf}(s)}+2^{-g}\sum_{\lambda=1}^{k}\sin\left[\frac{\pi\lambda}{2k+2}\right]^{2-2g}\right)\\
f&=\left(\sum_{j=0}^{k}\frac{\sin\left[\frac{\pi(2j+1)}{2k+2}\right]^2}{\sin\left[\frac{\pi}{2k+2}\right]^2}\right)^{g-1}\left(\sum_{j=0}^{k}(-1)^{2j}c_j\frac{\sin\left[\frac{\pi(2j+1)}{2k+2}\right]^{2-2g}}{\sin\left[\frac{\pi}{2k+2}\right]^{2-2g}}\right)\\
&=\left(\frac{k+1}{2}\right)^{g-1}\left(-\tfrac{1}{2}(-1)^{\mathrm{Arf}(s)}+2^{-g}\sum_{\lambda=1}^{k}(-1)^{\lambda+1}\sin\left[\frac{\pi\lambda}{2k+2}\right]^{2-2g}\right)
\end{aligned}\tag{5.4.15}$$

where we relabelled $\lambda=2j+1$. For $k=1$ we recover $b=(1+(-1)^{\mathrm{Arf}(s)})/2$ and $f=(1-(-1)^{\mathrm{Arf}(s)})/2$, as expected. These formulas agree with older derivations in the literature, see e.g. [290].

**Generalities.** After these simple examples, and to close this section, we shall next make a few interesting general remarks concerning the spin Verlinde formula. First off, let us note that, by definition, $S_{\alpha\times\psi,\beta}=B(\psi,\beta)S_{\alpha,\beta}$ and $S_{\alpha,\beta\times\psi}=B(\psi,\alpha)S_{\alpha,\beta}$. This means that, if we redefine one of the external punctures as $\alpha_i\to\alpha_i\times\psi$, this just inserts a factor of $B(\psi,\alpha)$ into the sums above, which has the effect of interchanging $b$ and $f$. In formulas,

$$\hat{\mathcal{H}}(\Sigma_{g;s}^{\alpha_1\cdots\alpha_i\times\psi\cdots\alpha_n})=\mathbb{C}^{0|1}\otimes\hat{\mathcal{H}}(\Sigma_{g;s}^{\alpha_1\cdots\alpha_i\cdots\alpha_n})\tag{5.4.16}$$

In other words, adding a fermion $\psi$ along the time direction has the sole effect of reversing the fermion parity of all states in the Hilbert space.

It also follows from the previous equation that, if one of the punctures $\alpha_i$ is Majorana, then the Hilbert space satisfies $\hat{\mathcal{H}}=\mathbb{C}^{0|1}\otimes\hat{\mathcal{H}}$, i.e., it has the same number of bosons and fermions, $b=f$. This means that the trace $\mathrm{tr}_{\hat{\mathcal{H}}}(-1)^F=b-f$ vanishes or, in other words, the path integral with Ramond boundary conditions around the time circle is zero. This is as expected since the Majorana line carries a zero-mode which makes the path integral vanish.

Another simple fact that follows from Verlinde is that if we have an odd number of Ramond lines, the path integral is identically zero. Indeed, in the sum over $\alpha$ the terms with $\alpha$ and $\alpha \times \psi$ carry the opposite sign and thus cancel out pairwise. The terms with $\alpha$ Majorana do not appear pairwise but they do not contribute anyway, since $m$-lines have no $S$-matrix elements with Ramond lines, $S_{m,\alpha} \equiv 0$ for $\alpha \in \mathcal{A}_{\mathrm{R}}$. This vanishing of the path integral is precisely what one would expect, since a surface with an odd number of Ramond lines is inconsistent (the boundary condition around the punctures is $+1$ if contracted towards the punctures, but $-1$ if contracted away from the punctures).

Furthermore, it is easy to see that the Hilbert space $\hat{\mathcal{H}}(\Sigma_{g;s}^{\alpha})$ depends on $s$ only through $\mathrm{Arf}(s)$, which is as expected since large diffeomorphisms induce isomorphisms $\hat{\mathcal{H}}(\Sigma_{g;s}^{\alpha}) \to \hat{\mathcal{H}}(\Sigma_{g;s'}^{\alpha})$ if and only if $\mathrm{Arf}(s) \equiv \mathrm{Arf}(s')$. In other words, Hilbert spaces with the same Arf parity are isomorphic. Along the same lines, it is easily checked that the total dimension $b+f$ is in fact independent of $s$, as the Arf-dependent terms cancel out (since these only appear with $m$-lines, which have $B(\psi,\alpha) = -1$). This is a generic feature of spin TQFTs, where the distribution of bosons and fermions depends on the spin structure but the total number of states does not.

Moreover, if we sum over all spin structures, we recover the Hilbert spaces of the bosonic parent, namely

$$\bigoplus_{s \in H^1(\Sigma_g,\mathbb{Z}_2)} \hat{\mathcal{H}}(\Sigma_{g;s}^{\alpha}) = \mathcal{H}(\Sigma_g^{\alpha}) \oplus \mathcal{H}(\Sigma_g^{\alpha,\psi}) \tag{5.4.17}$$

as one would expect. This can be shown by noting that the number of even spin structures is $\#_{\mathrm{even}} = 2^g(2^{g-1}+1)$ and the number of odd spin structures is $\#_{\mathrm{odd}} = 2^g(2^{g-1}-1)$, and hence

$$\begin{aligned}
\#_{\mathrm{even}} b_{\mathrm{even}} + \#_{\mathrm{odd}} b_{\mathrm{odd}} &= \left(\sum_{\alpha\in\mathcal{A}} d_\alpha^2\right)^{g-1} \sum_{\alpha\in\mathcal{A}} d_\alpha^{2-2g} \prod_{i=1}^{n} \frac{S_{\alpha_i,\alpha}}{S_{\mathbf{1},\alpha}} \\
\#_{\mathrm{even}} f_{\mathrm{even}} + \#_{\mathrm{odd}} f_{\mathrm{odd}} &= \left(\sum_{\alpha\in\mathcal{A}} d_\alpha^2\right)^{g-1} \sum_{\alpha\in\mathcal{A}} B(\psi,\alpha) d_\alpha^{2-2g} \prod_{i=1}^{n} \frac{S_{\alpha_i,\alpha}}{S_{\mathbf{1},\alpha}}
\end{aligned} \tag{5.4.18}$$

which are precisely the dimension of $\mathcal{H}(\Sigma_g^{\alpha})$ and $\mathcal{H}(\Sigma_g^{\alpha,\psi})$, respectively, as computed by the bosonic Verlinde formula (cf. (5.1.3)).

Finally, we mention that the factorization property of TQFTs holds, in the appropriate (fermionic) sense. Namely, if we take two surfaces $\Sigma_{g_1;s_1}^{\alpha_1\cdots\alpha_m\beta}$ and $\Sigma_{g_2;s_2}^{\alpha_{m+1}\cdots\alpha_n\beta'}$, then we can glue them together to yield a surface $\Sigma_{g_1+g_2;s_1\cup s_2}^{\alpha_1\cdots\alpha_n} \sqcup \Sigma_0^{\beta\beta'}$, where $\Sigma_0$ is a two-sphere used to glue the Riemann surfaces along the boundaries $\beta$, $\beta'$. Noting that $\hat{\mathcal{H}}(\Sigma_0^{\beta,\beta'}) = \delta_{\beta,\bar{\beta}'}\mathbb{C}^{1|0} \oplus \delta_{\beta,\bar{\beta}'\times\psi}\mathbb{C}^{0|1}$, the factorization property of spin TQFTs requires that, for non-Majorana $\beta$, the fermionic

Hilbert spaces must satisfy

$$\begin{aligned}\mathbb{C}^{1|0} \otimes \hat{\mathcal{H}}(\Sigma_{g;s}^{\alpha_1\cdots\alpha_n}) &\cong \bigoplus_{\beta\in\mathcal{A}/\sim} \hat{\mathcal{H}}(\Sigma_{g_1;s_1}^{\alpha_1\cdots\alpha_m\beta}) \otimes \hat{\mathcal{H}}(\Sigma_{g_2;s_2}^{\bar{\beta}\alpha_{m+1}\cdots\alpha_n}) \\ \mathbb{C}^{0|1} \otimes \hat{\mathcal{H}}(\Sigma_{g;s}^{\alpha_1\cdots\alpha_n}) &\cong \bigoplus_{\beta\in\mathcal{A}/\sim} \hat{\mathcal{H}}(\Sigma_{g_1;s_1}^{\alpha_1\cdots\alpha_m\beta}) \otimes \hat{\mathcal{H}}(\Sigma_{g_2;s_2}^{(\psi\times\bar{\beta})\alpha_{m+1}\cdots\alpha_n})\end{aligned} \tag{5.4.19}$$

Here, $\beta \in \mathcal{A}/\sim$ means that we only consider one representative for each two-dimensional orbit $\beta \leftrightarrow \psi \times \beta$ (it does not matter which one we choose, due to (5.4.16)). By a straightforward algebraic manipulation, it is not hard to show that the spin Verlinde formula indeed satisfies this property, as required.

# Chapter 6

# Domain Walls in $4d$ $\mathcal{N} = 1$ SYM.

**Authorship.** The content of this chapter is reproduced almost verbatim from the paper [5] written in collaboration with Jaume Gomis.

**Abstract.** $4d$ $\mathcal{N} = 1$ super Yang-Mills (SYM) with simply connected gauge group $G$ has $h$ gapped vacua arising from the spontaneously broken discrete $R$-symmetry, where $h$ is the dual Coxeter number of $G$. Therefore, the theory admits stable domain walls interpolating between any two vacua, but it is a nonperturbative problem to determine the low energy theory on the domain wall. We put forward an explicit answer to this question for all the domain walls for $G = \mathrm{SU}(N), \mathrm{Sp}(N), \mathrm{Spin}(N)$ and $G_2$, and for the minimal domain wall connecting neighboring vacua for arbitrary $G$. We propose that the domain wall theories support specific nontrivial topological quantum field theories (TQFTs), which include the Chern-Simons theory proposed long ago by Acharya-Vafa for $\mathrm{SU}(N)$. We provide nontrivial evidence for our proposals by exactly matching renormalization group invariant partition functions twisted by global symmetries of SYM computed in the ultraviolet with those computed in our proposed infrared TQFTs.

**Acknowledgments.** We would like to thank Davide Gaiotto, Zohar Komargodski, Bruno Le Floch and Nathan Seiberg for useful discussions.

## 6.1 Domain Walls in $4d$ $\mathcal{N}=1$ SYM

$4d$ $\mathcal{N}=1$ super Yang-Mills (SYM) – Yang-Mills theory with a massless adjoint fermion – is believed to share with QCD nonperturbative phenomena such as confinement, existence of a mass gap, and chiral symmetry breaking. $4d$ $\mathcal{N}=1$ SYM with simple and simply-connected gauge group $G$ has $h$ trivial vacua arising from the spontaneously broken $\mathbb{Z}_{2h}$ chiral $R$-symmetry down to $\mathbb{Z}_2$, where $h$ is the dual Coxeter number of $G$ (see table 6.1). The vacua are distinguished by the value of the gluino condensate [43–45]

$$\langle \operatorname{tr} \lambda\lambda \rangle = \Lambda^3 e^{2\pi i a/h}\,, \qquad a = 0, 1, \ldots, h-1\,. \tag{6.1.1}$$

A supersymmetric domain wall that interpolates between two arbitrary vacua $a$ and $b$ at $x_3 \to \pm\infty$ can be defined. The $\mathbb{Z}_{2h}$ symmetry implies that the domain wall theory depends only on the difference between vacua, on $n \equiv a - b \mod h$. We denote the resulting $3d$ low energy theory on the wall by $\mathrm{W}_n$ (see figure 6.1).

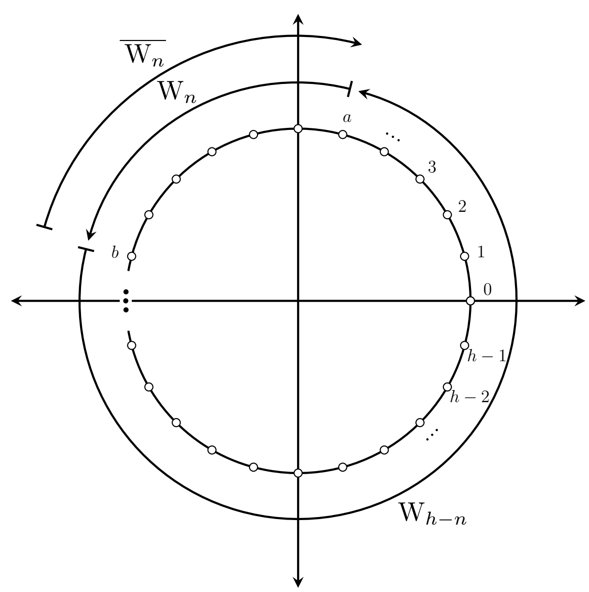



Figure 6.1: Vacua of $4d$ $\mathcal{N}=1$ SYM realized as the $h$ roots of unity in the $\langle \operatorname{tr} \lambda\lambda \rangle$-plane, where $\mathbb{Z}_{2h}$ acts by a $2\pi/h$ rotation. $\mathrm{W}_n$ denotes the domain wall interpolating between vacua separated by $n$ steps counterclockwise, and $\overline{\mathrm{W}_n}$ the domain wall connecting the same vacua but with its orientation reversed. Clearly, $\overline{\mathrm{W}_n} = \mathrm{W}_{h-n}$.

While the $n$-wall tension is fixed by the supersymmetry algebra [291], it is a nonperturbative problem to determine the low energy (i.e. $E \ll \Lambda$) effective theory on the domain wall. A supersymmetric domain wall preserves $3d$ $\mathcal{N}=1$ supersymmetry and therefore a universal $3d$ $\mathcal{N}=1$ Goldstone multiplet describes the spontaneously broken translation and supersymmetry. The nontrivial dynamical question is whether anything else remains in the infrared, a topological quantum field theory (TQFT) or gapless modes and, if so, which one(s).

In this paper we put forward a detailed answer to this question for all the $n$-domain walls for $G=\mathrm{SU}(N)$, $\mathrm{Sp}(N)$, $\mathrm{Spin}(N)$ and $G_2$, and for $n=1$ for arbitrary gauge group $G$.[97] The proposal for $G=\mathrm{SU}(N)$ was put forward long ago by Acharya-Vafa [46] motivated by brane constructions.[98] We provide nontrivial new evidence for the $\mathrm{SU}(N)$ proposal and for all the new proposals in this paper. The case $G=\mathrm{Spin}(N)$ is particularly subtle and rich.

We conjecture that the infrared of the $n$-domain wall theory in $4d$ $\mathcal{N}=1$ SYM with gauge group $G$ is the infrared phase of $3d$ $\mathcal{N}=1$ SYM with gauge group $G$ and Chern-Simons level

$$k=\tfrac{1}{2}h-n\,. \tag{6.1.2}$$

In other words, we propose the infrared description

$$\mathrm{W}_n \text{ in } 4d\ \mathcal{N}=1 \text{ SYM with } G \ \longleftrightarrow\ 3d\ \mathcal{N}=1 \text{ SYM with } G_{h/2-n}\,. \tag{6.1.3}$$

Since the $n$ and $h-n$ domain walls are related by time-reversal (see figure 6.1), consistency of this proposal requires that the corresponding infrared phases must also be related by time-reversal, that is, by sending $k\to -k$ in the $3d$ theory. This requirement is indeed fulfilled by the identification between $n$ and $k$ in (6.1.2).

Determining the infrared phase of $3d$ $\mathcal{N}=1$ SYM is also a nonperturbative problem. In [30] it was proposed that this theory flows in the infrared to a nontrivial TQFT. The domain wall theories, we conjecture, are the "quantum phases" put forward in [30, 113, 237]. This predicts the following domain wall theories:[99]

- $G=\mathrm{SU}(N)$. The $n$-domain wall theory is $\mathrm{W}_n=\mathrm{U}(n)_{N-n,N}$ Chern-Simons theory. This reproduces the proposal in [46].
- $G=\mathrm{Sp}(N)$. The $n$-domain wall theory is $\mathrm{W}_n=\mathrm{Sp}(n)_{N+1-n}$ Chern-Simons theory.
- $G=\mathrm{Spin}(N)$. The $n$-domain wall theory is $\mathrm{W}_n=\mathrm{O}(n)^1_{N-2-n,N-n+1}$ Chern-Simons theory. We review the construction of this TQFT in section 6.4.3.

[97]Vacua of $4d$ $\mathcal{N}=1$ SYM when $G$ is not simply connected can be nontrivial [292, 293]. The presence of a $4d$ TQFT means that there is no purely $3d$ wall theory.

[98]The $\mathrm{Sp}(N)$ case was mentioned in [294, 295]. For a partial list of references on domain walls in $4d$ gauge theories see e.g. [3, 40, 41, 223, 253, 296–302].

[99]The notation $G_k$ for Chern-Simons theories refers to Chern-Simons theory with gauge group $G$ at level $k\in\mathbb{Z}$. The Chern-Simons theory $\mathrm{U}(n)_{k,k'}\equiv\frac{\mathrm{SU}(n)_k\times\mathrm{U}(1)_{nk'}}{\mathbb{Z}_n}$ has two levels, and the theory based on $\mathrm{O}(n)$ has three levels (see section 6.4.3).

- $G = G_2$. The theory has $h = 4$ vacua and two independent walls: $n = 1, 2$. The 2-domain wall theory is $\mathrm{W}_2 = \mathrm{SO}(3)_3 \times S^1$, with $\mathrm{SO}(3)_3$ Chern-Simons theory and $S^1$ the $3d$ sigma model on the circle. For the $n = 1$ wall see below.

The expectation is that the $n$ and $(h-n)$-domain walls are related by time-reversal, that is $\mathrm{W}_{h-n} = \overline{\mathrm{W}_n}$ (see figure 6.1). This is realized by virtue of the level-rank dualities of Chern-Simons theories [28, 112, 113] and time-reversal flipping the sign of the Chern-Simons levels:

$$\begin{aligned} \mathrm{U}(N-n)_{n,N} &\longleftrightarrow \mathrm{U}(n)_{-(N-n),-N} \\ \mathrm{Sp}(N+1-n)_n &\longleftrightarrow \mathrm{Sp}(n)_{-(N+1-n)} \\ \mathrm{O}(N-2-n)^1_{n,n+3} &\longleftrightarrow \mathrm{O}(n)^1_{-(N-2-n),-(N-n+1)} \,. \end{aligned} \tag{6.1.4}$$

The domain walls with $n = h/2$ are nontrivially time-reversal invariant. These TQFTs emerge in the infrared of $3d$ $\mathcal{N} = 1$ SYM $G_0$, with vanishing Chern-Simons level, which is time-reversal invariant.

We also conjecture that:

- Arbitrary group $G$. The $n = 1$ domain wall theory connecting neighboring vacua is $\mathrm{W}_1 = G_{-1}$ Chern-Simons theory. This is consistent with the proposals put forward above due to the level-rank dualities $\mathrm{U}(1)_N \leftrightarrow \mathrm{SU}(N)_{-1}$, $\mathrm{Sp}(1)_N \leftrightarrow \mathrm{Sp}(N)_{-1}$ and $\mathrm{O}(1)^1_{N-3,N} = (\mathbb{Z}_2)_N \leftrightarrow \mathrm{Spin}(N)_{-1}$.

We subject the conjecture (6.1.3) to a number of nontrivial quantitative tests. We exactly match renormalization-group invariant partition functions computed in the $4d$ $\mathcal{N} = 1$ domain walls with the corresponding partition functions computed in the proposed infrared $3d$ TQFTs. This lends nontrivial support for our domain wall proposals in $4d$ $\mathcal{N} = 1$ SYM. We stress that one computation is performed using the $4d$ degrees of freedom, and the other using the proposed $3d$ TQFT degrees of freedom.

The most basic partition function of the $n$-domain wall is the Witten index [42, 303]

$$I_n = \mathrm{tr}_{\mathrm{W}_n}(-1)^F \,, \tag{6.1.5}$$

where $\mathrm{tr}_{\mathrm{W}_n}$ denotes the trace over the torus Hilbert space of $\mathrm{W}_n$ with periodic boundary conditions, and $(-1)^F$ fermion parity. This partition function was first computed by Acharya and Vafa in [46] using the $4d$ $\mathcal{N} = 1$ SYM fields.

We introduce and compute additional partition functions on the domain wall theory where the Witten index is twisted by a global symmetry of SYM. $4d$ $\mathcal{N} = 1$ SYM with gauge group $G$ can have charge conjugation zero-form symmetry $\mathsf{C}$ and one-form symmetry $\Gamma$ [32].[100] $\Gamma$ is the center of $G$, since the fermion in $4d$ $\mathcal{N} = 1$ SYM is in the adjoint representation of the

[100] $\mathsf{C}$ is the outer automorphism group of the Dynkin diagram $\mathfrak{g}$ of $G$ while $\mathsf{S}$ is the outer automorphism group of the extended Dynkin diagram $\mathfrak{g}^{(1)}$ of the affine Lie algebra associated to $G$. The group $\Gamma$ is defined as the quotient $\mathsf{S}/\mathsf{C}$, i.e., the symmetries of $\mathfrak{g}^{(1)}$ that are *not* symmetries of $\mathfrak{g}$.

gauge group. The symmetries $\mathsf{C}$ and $\Gamma$ do not commute when acting on Wilson lines, and combine into $\mathsf{S} = \Gamma \rtimes \mathsf{C}$ (see table 6.1). $\mathsf{C}$ acts on local operators and Wilson lines, and $\Gamma$ on the Wilson lines of the theory. These symmetries are unbroken in each of the $h$ vacua of $4d$ $\mathcal{N} = 1$ SYM. $\mathsf{S}$ is the unbroken symmetry at each vacuum, while $\mathbb{Z}_{2h}$ is spontaneously broken to $\mathbb{Z}_2$. This allows us to define the following twisted Witten indices on the $n$-domain wall theory[101]

$$I_n^{\mathsf{c}} = \mathrm{tr}_{\mathrm{W}_n}(-1)^F \mathsf{c}\,, \tag{6.1.6}$$

where $\mathsf{c} \in \mathsf{C}$, and

$$I_n^{\mathsf{g}} = \mathrm{tr}_{\mathrm{W}_n}(-1)^F \mathsf{g}\,, \tag{6.1.7}$$

where $\mathsf{g} \in \Gamma$. Consistency of our conjecture requires that these partition functions, computed on either side of (6.1.3), match. We compute the Witten indices in terms of the $4d$ degrees of freedom in section 6.2, and in the $3d$ TQFTs in section 6.4.

Computing the domain wall Witten indices on the $3d$ side of the proposal requires understanding the Hilbert space of spin TQFTs, and not merely counting the number of states on the torus, as has been often stated in the literature. We delve into the details of constructing the Hilbert space of spin TQFTs and determining the fermionic parity of the states in section 6.3. The (twisted) Witten indices (6.1.5), (6.1.6), (6.1.7) map to twisted partition functions in the infrared spin TQFT. Importantly, the dimension of the Hilbert space and the index differ in general, as we shall see. In particular, the index sometimes vanishes in theories of interest. While the index can vanish, the twisted indices are non-vanishing, and supersymmetry on the domain wall is unbroken.

| $G$ | $\mathrm{SU}(N)$ | $\mathrm{Sp}(N)$ | $\mathrm{Spin}(2N+1)$ | $\mathrm{Spin}(4N)$ | $\mathrm{Spin}(4N+2)$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $h$ | $N$ | $N+1$ | $2N-1$ | $4N-2$ | $4N$ | 12 | 18 | 30 | 9 | 4 |
| $\mathsf{C}$ | $\mathbb{Z}_2$ | $\cdot$ | $\cdot$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $\Gamma$ | $\mathbb{Z}_N$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_4$ | $\mathbb{Z}_3$ | $\mathbb{Z}_2$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $\mathsf{S}$ | $\mathbb{D}_N$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{D}_4$ | $\mathbb{D}_4$ | $\mathbb{S}_3$ | $\mathbb{Z}_2$ | $\cdot$ | $\cdot$ | $\cdot$ |

Table 6.1: Lie data for the simple Lie groups $G$. Here $h$ denotes the dual Coxeter number (defined as $\mathrm{tr}(t_{\mathrm{adj}} t'_{\mathrm{adj}}) \equiv 2h(t,t')$, where $(\cdot,\cdot)$ denotes the Killing form on $\mathfrak{g}$, normalized so that the highest root has $(\theta,\theta) = 2$). $\mathsf{C}$, $\Gamma$ are the zero-form and one-form symmetry groups of $4d$ $\mathcal{N} = 1$ SYM with gauge group $G$, and $\mathsf{S} = \Gamma \rtimes \mathsf{C}$. $\mathbb{D}_N$ denotes the dihedral group with $2N$ elements, and $\mathbb{S}_N$ the symmetric group with $N!$ elements. For SU(2) the zero-form symmetry group is trivial, and for Spin(8) the zero-form symmetry group is enhanced to $\mathbb{S}_3$ and the total symmetry group to $\mathbb{S}_4$. The $\mathbb{D}_N$ symmetry of pure $\mathrm{SU}(N)$ YM was considered in [304].

[101] One could also twist by any element $\mathsf{cg} \in \mathsf{S}$.

We summarize here the results of our computations, performed both in terms of the $4d$ fields and the conjectured $3d$ topological degrees of freedom, for which we find perfect agreement. We find it convenient to organize the results into master partition functions, which are defined as the generating functions for the twisted Witten indices. In other words, we sum the (twisted) partition functions over all $n$-walls:

$$Z^{\mathsf{s}}(q) := \sum_{n=0}^{h} I_n^{\mathsf{s}} q^n \,, \tag{6.1.8}$$

where $q$ is a fugacity parameter, and where $\mathsf{s} \in \mathsf{S}$ is an element of the unbroken symmetry group. These partition functions have an elegant interpretation as twisted partition functions of a collection of free fermions in $0+1$ dimensions with energies determined by the Lie data of $G$ (see section 6.2). Interestingly, the twisted partition function can be expressed as the untwisted partition function of an associated affine Lie algebra, whose extended Dynkin diagram is obtained by the "folding procedure" introduced in [305].

The master partition functions take a rather simple form:

- SU($N$):

$$Z(q) = (1-q)^N \tag{6.1.9}$$

$$Z^{\mathsf{c}}(q) = \begin{cases} (1-q)(1-q^2)^{(N-1)/2} & N \text{ odd,} \\ (1-q)^2(1-q^2)^{(N-2)/2} & N \text{ even,} \end{cases} \tag{6.1.10}$$

$$Z^{\mathsf{g}}(q) = \prod_{i=0}^{N-1} (1-\mathsf{g}^i q)\,, \tag{6.1.11}$$

  where $\mathsf{c}$ denotes the non-trivial element of $\mathsf{C} = \mathbb{Z}_2$, and $\mathsf{g}$ is any element of $\Gamma = \mathbb{Z}_N$, thought of as an $N$-th root of unity.

- Sp($N$):

$$Z(q) = (1-q)^{N+1} \tag{6.1.12}$$

$$Z^{\mathsf{g}}(q) = \begin{cases} (1-q)(1-q^2)^{N/2} & N \text{ even,} \\ (1-q^2)^{(N+1)/2} & N \text{ odd,} \end{cases} \tag{6.1.13}$$

  where $\mathsf{g}$ denotes the non-trivial element of $\Gamma = \mathbb{Z}_2$.

- Spin($N$), $N$ odd:

$$Z(q) = (1-q)^3(1-q^2)^{(N-1)/2-2} \tag{6.1.14}$$

$$Z^{\mathsf{g}}(q) = (1+q)(1-q)^2(1-q^2)^{(N-1)/2-2}\,, \tag{6.1.15}$$

  where $\mathsf{g}$ denotes the non-trivial element of $\Gamma = \mathbb{Z}_2$.

- Spin($N$), $N$ even:

$$Z(q) = (1-q)^4(1-q^2)^{N/2-3} \tag{6.1.16}$$

$$Z^{\mathsf{c}}(q) = (1-q)^2(1-q^2)^{N/2-2}\,, \tag{6.1.17}$$

where $\mathsf{c}$ denotes the non-trivial element of $\mathsf{C} = \mathbb{Z}_2$.

  - $N = 0 \mod 4$: $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\mathbf{1}, \mathsf{g}_1, \mathsf{g}_2, \mathsf{g}_1\mathsf{g}_2\}$

$$\begin{aligned} Z^{\mathsf{g}_1}(q) &= (1-q^2)^{N/2-1} \\ Z^{\mathsf{g}_2}(q) &= Z^{\mathsf{g}_1\mathsf{g}_2}(q) = (1-q^2)^3(1-q^4)^{N/4-2}\,. \end{aligned} \tag{6.1.18}$$

  - $N = 2 \mod 4$: $\Gamma = \mathbb{Z}_4 = \{\mathbf{1}, \mathsf{g}, \mathsf{g}^2, \mathsf{g}^3\}$

$$\begin{aligned} Z^{\mathsf{g}}(q) &= Z^{\mathsf{g}^3}(q) = (1-q^4)^{(N/2-1)/2} \\ Z^{\mathsf{g}^2}(q) &= (1-q^2)^{N/2-1}\,. \end{aligned} \tag{6.1.19}$$

- $G_2$:

$$Z(q) = (1-q)^2(1-q^2)\,. \tag{6.1.20}$$

Expanding these formulas in a series in $q$ yields $I_n^{\mathsf{s}}$ (see section 6.2). See also section 6.2.6 for the $n = 1$ domain wall twisted Witten indices for arbitrary simply-connected $G$.

The plan of the rest of the chapter is as follows. In section 6.2 we review the calculation of the untwisted Witten index for general domain walls in $4d$ $\mathcal{N} = 1$ SYM, develop the necessary tools to study the twisted indices, and present a detailed calculation thereof, for all the classical Lie groups. In section 6.3 we explain how the Hilbert space of a spin Chern-Simons theory is constructed and, in particular, how to determine the fermion parity $(-1)^F$ of the different states. In section 6.4 we use this refined understanding of spin Chern-Simons theories to compute the twisted partition functions of the $3d$ TQFTs that, conjecturally, describe the infrared dynamics of the domain walls, and show exact agreement. We end with some forward-looking comments in section 6.5. We delegate to extra sections some technical details that are needed in the computation of the twisted partition functions in section 6.4 and some additional material.

## 6.2 Twisted Witten Indices

In this section we study the twisted Witten indices on the $3d$ $\mathcal{N} = 1$ domain walls, as computed in terms of the ultraviolet $4d$ degrees of freedom, namely the gluons and gluinos. This requires considering $4d$ $\mathcal{N} = 1$ SYM on a two-torus and quantizing the space of zero energy states. This leads to a $2d$ $\mathcal{N} = (2,2)$ sigma model on the moduli space of flat

$G$-connections on a two-torus, which is the weighted projective space $\mathbf{WCP}^r_{a_0^\vee,a_1^\vee,\ldots,a_r^\vee}$, where $a_i^\vee$ is the comark for the $i$-th node in the extended Dynkin diagram $\mathfrak{g}^{(1)}$ of the affine Lie algebra associated to $G$ and $r = \mathrm{rank}(G)$ [306, 307]. Just as $4d$ $\mathcal{N}=1$ SYM, this $2d$ theory also has $h$ quantum vacua. A supersymmetric domain wall in $4d$ $\mathcal{N}=1$ SYM corresponds to a supersymmetric soliton in the $2d$ $\mathcal{N}=(2,2)$ sigma model [46].

Using the $2d$ $\mathcal{N}=(2,2)$ sigma model, Acharya and Vafa argued that the Witten index of the domain wall is encoded in the Hilbert space of $r+1$ free fermions in $0+1$ dimensions. Each fermion $\psi_i$ is associated to the $i$-th node of the extended Dynkin diagram $\mathfrak{g}^{(1)}$ of $G$ and the energy of each fermion is $a_i^\vee$. The fermion Hilbert space is graded by the energy of the states

$$\mathcal{H}_{\mathrm{F}} = \bigoplus_{n=0}^{h} \mathcal{H}_{\mathrm{F}}^n \,, \tag{6.2.1}$$

where the maximal energy is $h$ since $h = \sum_{i=0}^r a_i^\vee$. $\mathcal{H}_{\mathrm{F}}^n$ denotes the subspace of energy $n$, that is, the configurations such that

$$\sum_{i=0}^{r} \lambda_i a_i^\vee = n \,, \tag{6.2.2}$$

where $\lambda_i \in \{0,1\}$ is the occupation number of the $i$-th fermion. The Witten index for the $n$-domain wall (6.1.5), with the Goldstino multiplet contribution removed, is the trace over the fermion Hilbert space $\mathcal{H}_{\mathrm{F}}^n$ [46]

$$I_n = \mathrm{tr}_{\mathrm{W}_n}(-1)^F \equiv \mathrm{tr}_{\mathcal{H}_{\mathrm{F}}^n}(-1)^F \,. \tag{6.2.3}$$

The Witten index of all $n$-domain walls is encoded in the partition function of the fermions with periodic boundary conditions on a circle, corresponding to a sum over all states weighted by the energy:

$$Z(q) = \mathrm{tr}_{\mathcal{H}_{\mathrm{F}}}(-1)^F q^H = \sum_{n=0}^{h} I_n q^n \,. \tag{6.2.4}$$

This partition function is readily evaluated

$$Z(q) = \prod_{i=0}^{r} (1 - q^{a_i^\vee}) \,, \tag{6.2.5}$$

which implies, in particular, that the Witten index for the $n$ and $h-n$ wall are the same ($I_n = (-1)^{r+1} I_{h-n}$) since the fermionic Hilbert space for the $n$ and $h-n$ walls are related by particle-hole symmetry. This beautifully reproduces the expectation that the $n$ domain wall and the $h-n$ domain wall (cf. figure 6.1) are related to each other by time-reversal!

A symmetry $\mathsf{s} \in \mathsf{S}$ of $4d$ $\mathcal{N}=1$ SYM acts in a simple way on the Wilson lines of the gauge theory. A Wilson line is labeled by a representation of $G$ with highest weight

$\lambda \equiv \lambda_1\omega_1 + \lambda_2\omega_2 + \cdots + \lambda_r\omega_r$, where $\omega_i$ is the fundamental weight associated to the $i$-th node of the Dynkin diagram $\mathfrak{g}$. The Wilson line $W_i$ labeled by the fundamental weight $\omega_i$ transforms under $\mathsf{c} \in \mathsf{C}$ as

$$\mathsf{c}\colon W_i \mapsto W_{\mathsf{c}(i)}\,, \tag{6.2.6}$$

where $\omega_{\mathsf{c}(i)}$ is the fundamental weight which is charge conjugate to $\omega_i$. An element $\mathsf{g} \in \Gamma$ acts by

$$\mathsf{g}\colon W_i \mapsto \alpha_{\mathsf{g}}(\omega_i)\, W_i\,, \tag{6.2.7}$$

where $\alpha_{\mathsf{g}}(\omega_i) \in \Gamma^*$ is the charge of $\omega_i$ under the center $\Gamma$ of $G$. The action of a symmetry on the fundamental Wilson lines $W_i$ induces an action on the fermions $\psi_i$, which are labeled by a node in the extended Dynkin diagram $\mathfrak{g}^{(1)}$. We recall that $\mathsf{C}$ acts as an outer automorphism of $\mathfrak{g}$, $\mathsf{S}$ acts as an outer automorphism of $\mathfrak{g}^{(1)}$ and $\Gamma = \mathsf{S}/\mathsf{C}$.

We now proceed to compute the Witten index on the domain wall twisted by the symmetries of the system, $\mathsf{S}$. This group is identified with the group of symmetries of the extended Dynkin diagram $\mathfrak{g}^{(1)}$, i.e., a given $\mathsf{s} \in \mathsf{S}$ can be thought of as a permutation of the nodes $i \mapsto \mathsf{s}(i)$ that leaves the diagram $\mathfrak{g}^{(1)}$ invariant. The induced action on the effective $0+1$ system of fermions is

$$\mathsf{s}\colon \psi_i \mapsto \psi_{\mathsf{s}(i)}\,, \tag{6.2.8}$$

where $\psi_i$ is the fermion associated to the $i$-th node of $\mathfrak{g}^{(1)}$. This means that the symmetry $\mathsf{s}$ lifts to a map $\mathcal{H}_{\mathrm{F}} \to \mathcal{H}_{\mathrm{F}}$ which, by definition, commutes with the Hamiltonian,

$$[H, \mathsf{s}] = 0\,, \tag{6.2.9}$$

inasmuch as $a_i^\vee \equiv a_{\mathsf{s}(i)}^\vee$. Thus, $\mathsf{s}$ restricts to a well-defined action on $\mathcal{H}_{\mathsf{F}}^n$, i.e., it preserves the grading (6.2.1). The twisted Witten index is

$$I_n^{\mathsf{s}} = \mathrm{tr}_{\mathrm{W}_n}(-1)^F \mathsf{s} \equiv \mathrm{tr}_{\mathcal{H}_{\mathrm{F}}^n}(-1)^F \mathsf{s}\,. \tag{6.2.10}$$

Similarly, the twisted partition function computes the generating function of twisted indices:

$$Z^{\mathsf{s}}(q) = \mathrm{tr}_{\mathcal{H}_{\mathrm{F}}}(-1)^F \mathsf{s}\, q^H = \sum_{n=0}^{h} I_n^{\mathsf{s}} q^n\,. \tag{6.2.11}$$

An efficient way to compute this partition function is as follows. Take the $i$-th fermion, and consider its orbit under $\mathsf{s}$:

$$\psi_i \mapsto \psi_{\mathsf{s}(i)} \mapsto \psi_{\mathsf{s}^2(i)} \mapsto \cdots \mapsto \psi_{\mathsf{s}^{N_i}(i)} \equiv \psi_i\,, \tag{6.2.12}$$

where $N_i$ denotes the length of the orbit of the $i$-th node under the symmetry $\mathsf{s}$, i.e., the minimal integer such that $\mathsf{s}^{N_i}(i) \equiv i$. In the trace (6.2.10), the only configurations that contribute are those where the occupation number $\lambda_i$ in (6.2.2) is constant along the orbit:

$$\lambda_i = \lambda_{\mathsf{s}(i)} = \lambda_{\mathsf{s}^2(i)} = \cdots = \lambda_{\mathsf{s}^{N_i-1}(i)}\,. \tag{6.2.13}$$

This means that we may restrict the sum over $\mathcal{H}_{\mathrm{F}}^n$ in (6.2.2) to those configurations where this identity is satisfied. We enforce this by dropping all but one of these labels, and multiplying its energy by $N_i$, i.e., we replace (6.2.2) by

$$\sum_{i=0}^{r'} \lambda_i a_i'^\vee = n\,, \tag{6.2.14}$$

where the sum is over one representative for each orbit, $r'$ is the number of orbits of $\mathsf{s}$, and $a_i'^\vee = N_i a_i^\vee$ is the combined energy of all the elements of the orbit of $\lambda_i$. With this, the twisted Witten index (6.1.6) on the domain wall can be computed as the untwisted partition function of $r'+1$ free fermions with energies $a_i'^\vee$:

$$Z^{\mathsf{s}}(q) = \prod_{i=0}^{r'} (1 - q^{a_i'^\vee})\,. \tag{6.2.15}$$

Since $h = \sum_{i=0}^{r'} a_i'^\vee$ we see that $I_n^{\mathsf{s}} = (-1)^{r'+1} I_{h-n}^{\mathsf{s}}$, as required by time-reversal.

Diagrammatically, twisting by a symmetry folds the Dynkin diagram $\mathfrak{g}^{(1)}$ according to the action of $\mathsf{s}$ on the nodes [305]. This yields a new affine Dynkin diagram, which has $r'+1 < r+1$ nodes, and comarks $a_i'^\vee = N_i a_i^\vee$. The twisted Witten index is identical to the untwisted Witten index of the folded diagram.

A quick remark is in order. Let $\lambda_i'$ be a node in the folded diagram, and let $N_i$ be the number of nodes in the original diagram that folded into $\lambda_i'$. The node $\lambda_i'$ is therefore a bound state of $N_i$ fermions, and thus has fermion parity $(-1)^F = (-1)^{N_i}$. Moreover, the symmetry $\mathsf{s}$ permutes these fermions, which generates an extra sign corresponding to the signature of the permutation. When the permutation is cyclic, which is the case relevant to this paper, the signature is just $N_i - 1$. All in all, the contribution of $\lambda_i'$ to the twisted trace is $(-1)^F \mathsf{s} = (-1)^{(N_i + N_i - 1)} = -1$. Therefore, in the folded diagram the node behaves as a regular fermion, just with more energy, and so (6.2.15) is correct as written: the fermionic signs are all taken care of automatically by the folding.

The twisted Witten index can also be computed by diagonalizing the action of $\mathsf{s}$ (6.2.12) by a direct sum of unitary transformations, one for each orbit, which is a symmetry of the collection of fermions. In this basis, $\mathsf{s}$ acts with eigenvalue $\mathsf{s}_i$ on the $i$-th fermion, where $\mathsf{s}_i$ is an $N_i$-th root of unity. The twisted partition function can, therefore, also be expressed as

$$Z^{\mathsf{s}}(q) = \prod_{i=0}^{r} (1 - \mathsf{s}_i\, q^{a_i^\vee})\,. \tag{6.2.16}$$

This also makes the action of time-reversal symmetry on the domain walls manifest, cf. $I_n^{\mathsf{s}} = (-1)^{r+1} \det(\mathsf{s}) I_{h-n}^{\mathsf{s}}$, where $\det(\mathsf{s}) = \mathsf{s}_0 \mathsf{s}_1 \cdots \mathsf{s}_r = (-1)^{r+r'}$ is the parity of the permutation induced by $\mathsf{s}$. We now discuss zero-form symmetries and one-form symmetries in turn.

**Zero-form symmetries.** $4d$ $\mathcal{N}=1$ SYM has charge conjugation symmetry $\mathsf{C}$ if and only if the (unextended) Dynkin diagram $\mathfrak{g}$ of the Lie algebra of $G$ has a symmetry. This corresponds to an outer automorphism of the Lie algebra of $G$ (see table 6.1). Such symmetry is present for the $A_r, D_r, E_6$ algebras, where $\mathsf{C}$ acts as a transposition (order-two permutation) on the nodes of the Dynkin diagram (with low-rank exceptions $A_1, D_4$). In this case, folding the diagram by $\mathsf{C}$ gives rise to what is usually called the twisted affine Dynkin diagram $\mathfrak{g}^{(2)}$ [308, 309], which is constructed by identifying the nodes of $\mathfrak{g}$ that are permuted by $\mathsf{C}$ (and adding the extending node).

As $\mathsf{C}=\mathbb{Z}_2$, the eigenvalues in (6.2.16) are trivial to determine: if a node $i$ is fixed by $\mathsf{c}\in\mathsf{C}$, then its eigenvalue is $\mathsf{c}_i=+1$. On the other hand, if the pair of nodes $i,j$ are swapped, then the eigenvalues are $\pm1$, which can be assigned as $\mathsf{c}_i=+1$ and $\mathsf{c}_j=-1$ (or vice-versa).

**One-form symmetries.** As discussed in section 6.1, $4d$ $\mathcal{N}=1$ has a one-form symmetry group $\Gamma$ given by the center of the gauge group $G$. Here, the eigenvalues $\mathsf{s}_i$ in (6.2.16) have a very natural interpretation. An element $\mathsf{g}\in\Gamma$ acts as an outer automorphism of $\mathfrak{g}^{(1)}$, a permutation of the $r+1$ nodes. Diagonalizing this permutation results on an eigenvalue $\mathsf{g}_i$ on the fermion $\psi_i$ associated to the $i$-th node, which is the charge of the element of the center $\mathsf{g}\in\Gamma$ on the $i$-th fundamental weight of $\mathfrak{g}$ [15, 284, 310–312], that is

$$\mathsf{g}_i=\alpha_{\mathsf{g}}(\omega_i)\,. \tag{6.2.17}$$

Note that this is precisely how $\mathsf{g}\in\Gamma$ acts on the ultraviolet Wilson loops $W_i$ (cf. (6.2.7)).

We now proceed to compute the twisted Witten indices for $G=\mathrm{SU}(N)$, $\mathrm{Sp}(N)$, $\mathrm{Spin}(N)$, and $G_2$ respectively.

### 6.2.1 $G=\mathrm{SU}(N)$

Consider the algebra $A_{N-1}=\mathfrak{su}_N$. The symmetries of this algebra are as follows:

- The group $\mathrm{SU}(N)$ has a $\mathbb{Z}_2$ zero-form symmetry, which corresponds to complex conjugation. It acts by interchanging the $i$-th node with the $(N-i)$-th node in $\mathfrak{g}$. The associated diagonal action can be chosen as follows: take $\mathsf{c}_0=+1$ for the extended node, and $\mathsf{c}_i=+1$ for the first half of the unextended nodes, and $\mathsf{c}_i=-1$ for the second half.

- The group $\mathrm{SU}(N)$ has a $\mathbb{Z}_N$ one-form symmetry, whose associated charge is the $N$-ality (the number of boxes in the Young diagram modulo $N$). If $\mathsf{g}$ denotes a primitive root of unity, then a generic element $\mathsf{g}^t\in\mathbb{Z}_N$ acts on the extended diagram $\mathfrak{g}^{(1)}$ as a cyclic permutation by $t$ units, $\lambda_i\mapsto\lambda_{i+t \mod N}$. The center acts on a representation with highest weight $\lambda$ as follows:

$$\alpha_{\mathsf{g}}(\lambda)\equiv\prod_{i=0}^{N-1}\mathsf{g}^{i\lambda_i}\,, \tag{6.2.18}$$

which means that the eigenvalues in (6.2.16) are $\mathsf{g}_i = \mathsf{g}^i$.

Let us begin by computing the untwisted partition function. The comarks are all $a_i^\vee = 1$. Plugging this into equation (6.2.5) we obtain the untwisted partition function

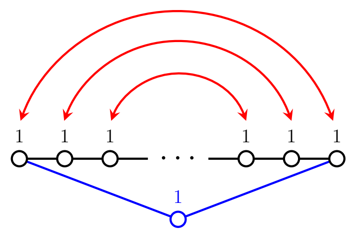
$$Z(q) = (1-q)^N\,, \tag{6.2.19}$$

and, expanding, the Witten index

$$I_n = (-1)^n \binom{N}{n}\,. \tag{6.2.20}$$

This result was also obtained, by an entirely different method, in [313].

We now move on to the twisted indices. Charge conjugation acts on the extended Dynkin diagram $A_{N-1}^{(1)}$ as follows:




(6.2.21)

where the blue node denotes the affine root, and the integers denote the comarks $a_i^\vee$. The automorphism folds the diagram in half.[102] The result is

$A_{2m-1}^{(1)} \mapsto A_{2m-1}^{(2)}$ :


(6.2.22)

$A_{2m}^{(1)} \mapsto A_{2m}^{(2)}$ :


where the folded diagrams both have $m+1 = \lfloor N/2 \rfloor + 1$ nodes. From this we conclude that the folded diagram has $r'+1 = (N+1)/2$ and $r' = (N+2)/2$ nodes for $N$ odd and $N$ even, respectively. In the first case, one node has comark equal to 1, and the rest equal to 2; while in the second case, there are two nodes with comark 1, and the rest equal to 2. Using (6.2.15), the $\mathsf{c}$-twisted partition function is[103]

$$Z^{\mathsf{c}}(q) = \begin{cases} (1-q)(1-q^2)^{(N-1)/2} & N \text{ odd,} \\ (1-q)^2(1-q^2)^{(N-2)/2} & N \text{ even,} \end{cases} \tag{6.2.23}$$

[102]For $N$ odd, the $(N\pm1)/2$-th nodes would naively fold into a loop, which does not yield a valid Dynkin diagram. The correct folding is given by the theory of twisted Kač-Moody algebras [308, 309]. We henceforth fold the diagrams following [308, 309]. The only information we need from the diagram are the comarks.

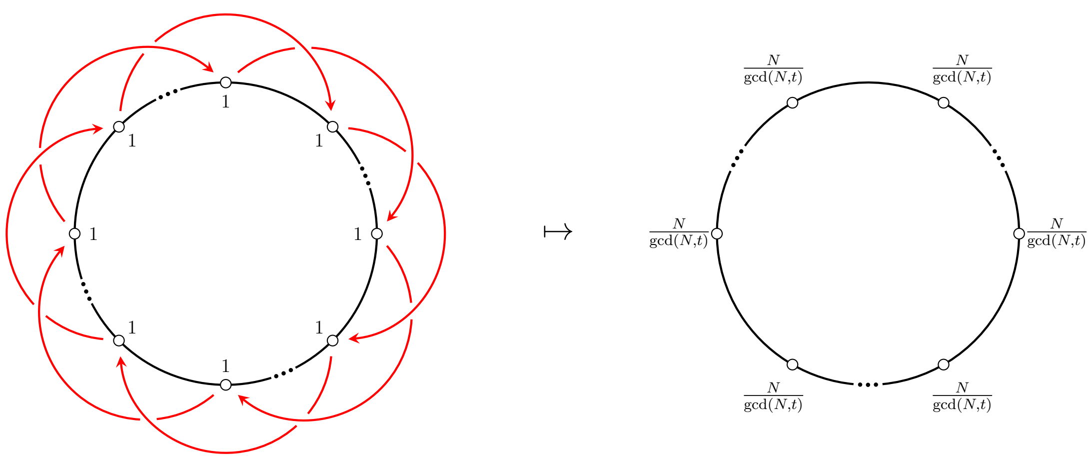
[103]In SU(2) the action of $\mathsf{c}$ is a gauge transformation and $\mathsf{c}$ is not a symmetry; indeed, $Z^{\mathsf{c}}(q) = Z(q)$.

and, expanding, the $\mathsf{c}$-twisted Witten indices are

$$I_n^{\mathsf{c}} = \begin{cases} (-1)^{n/2}\dbinom{(N-1)/2}{n/2} & N \text{ odd, } n \text{ even,} \\ (-1)^{(n+1)/2}\dbinom{(N-1)/2}{(n-1)/2} & N \text{ odd, } n \text{ odd,} \\ (-1)^{n/2}\left[\dbinom{(N-2)/2}{n/2} - \dbinom{(N-2)/2}{n/2-1}\right] & N \text{ even, } n \text{ even,} \\ 2(-1)^{(n+1)/2}\dbinom{(N-2)/2}{(n-1)/2} & N \text{ even, } n \text{ odd.} \end{cases} \tag{6.2.24}$$

One can also compute the partition function in the diagonal basis, where $\mathsf{c}_i = +1$ for the first half of the nodes, and $\mathsf{c}_i = -1$ for the second half. Plugging this into (6.2.16) yields the same expression for the twisted partition function.

Let us now consider the partition function twisted by the $\Gamma = \mathbb{Z}_N$ one-form symmetry. If $\mathsf{g}$ denotes a primitive $N$-th root of unity, then a generic element $\mathsf{g}^t \in \mathbb{Z}_N$ acts on the extended diagram as follows:



(6.2.25)

where the folded diagram has $\gcd(N,t)$ nodes, each with energy $N/\gcd(N,t)$. In other words, $\mathsf{g}^t$ folds the diagram into the affine diagram of $\mathrm{SU}(\gcd(N,t))$, with comarks $N/\gcd(N,t)$. This immediately yields the twisted partition function as (6.2.15)

$$Z^{\mathsf{g}^t}(q) = (1 - q^{N/\gcd(N,t)})^{\gcd(N,t)}\,. \tag{6.2.26}$$

The twisted index reads

$$I_n^{\mathsf{g}^t} = \begin{cases} (-1)^{n\gcd(N,t)/N}\dbinom{\gcd(N,t)}{n\gcd(N,t)/N} & N | n\gcd(N,t), \\ 0 & \text{otherwise.} \end{cases} \tag{6.2.27}$$

Naturally, for $t = 0$ this reduces to the untwisted result.

Alternatively, we may compute the same partition function in the diagonal basis. Using equations (6.2.16) and (6.2.18), the twisted partition function is given by

$$Z^{\mathsf{g}}(q) = \prod_{i=0}^{N-1} (1 - \mathsf{g}^i q) \equiv (q; \mathsf{g})_N \,, \tag{6.2.28}$$

the so-called $q$-Pochhammer symbol, essentially defined by this product. One may prove that this is in fact identical to (6.2.26). Expanding the product, the twisted index becomes

$$I_n^{\mathsf{g}} = (-1)^n \mathsf{g}^{\frac{1}{2}n(n-1)} \binom{N}{n}_{\mathsf{g}}, \qquad \binom{N}{n}_{\mathsf{g}} := \frac{(\mathsf{g}; \mathsf{g})_N}{(\mathsf{g}; \mathsf{g})_n (\mathsf{g}; \mathsf{g})_{N-n}} \,, \tag{6.2.29}$$

where the term in parentheses denotes the so-called $q$-binomial coefficient. This is again identical to (6.2.27).

### 6.2.2 $G = \mathrm{Sp}(N)$

Consider the algebra $C_N = \mathfrak{sp}_N$. The symmetries of this algebra are as follows:

- The group $\mathrm{Sp}(N)$ has no zero-form symmetry.
- The group $\mathrm{Sp}(N)$ has a $\mathbb{Z}_2$ one-form symmetry, whose charged representations are the pseudo-real ones. The non-trivial element $\mathsf{g} \in \mathbb{Z}_2$ acts on the extended diagram by reversing the nodes $\lambda_i \mapsto \lambda_{N-i}$. The center acts on a representation $\lambda$ as follows:

$$\alpha_{\mathsf{g}}(\lambda) \equiv (-1)^{\sum_{i=0}^{\lfloor (N-1)/2 \rfloor} \lambda_{2i+1}} \,, \tag{6.2.30}$$

  which means that the eigenvalues in (6.2.16) are $\mathsf{g}_i = (-1)^i$.

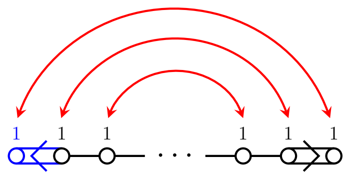
Let us begin by computing the untwisted partition function. The comarks for $\mathrm{Sp}(N)$ are all equal to one, i.e. $a_i^\vee = 1$ for $i = 0, 1, \ldots, N$. Plugging this into equation (6.2.5) we obtain the untwisted partition function

$$Z(q) = (1 - q)^{N+1} \,, \tag{6.2.31}$$

and, expanding, the Witten index

$$I_n = (-1)^n \binom{N+1}{n} . \tag{6.2.32}$$

$4d$ $\mathrm{Sp}(N)$ $\mathcal{N} = 1$ SYM has no charge conjugation symmetry. We can consider instead the index twisted by the $\Gamma = \mathbb{Z}_2$ one-form center symmetry, which acts on the extended Dynkin diagram as follows:



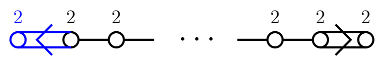

(6.2.33)

where the blue node denotes the affine root, and the integers denote the comarks $a_i^\vee$. The automorphism folds the diagram in half (see footnote 102). The result is

$C^{(1)}_{2m} \mapsto A^{(2)}_{2m}$ :


(6.2.34)

$C^{(1)}_{2m-1} \mapsto C^{(1)}_{m-1}$:




where the folded diagrams have $m + 1 = \lfloor N/2 \rfloor + 1$ nodes.

From this we learn that the folded diagram has $r' + 1 = (N + 2)/2$ and $r' + 1 = (N + 1)/2$ nodes, for $N$ even and $N$ odd, respectively. In the first case, one of these nodes has energy equal to 1, and the rest equal to 2; while in the second case, they are all of energy 2. Plugging this into (6.2.15) the one-form twisted partition function is

$$Z^{\mathfrak{g}}(q) = \begin{cases} (1-q)(1-q^2)^{N/2} & N \text{ even}, \\ (1-q^2)^{(N+1)/2} & N \text{ odd}, \end{cases} \tag{6.2.35}$$

and, expanding, the twisted Witten index

$$I^{\mathfrak{g}}_n = \begin{cases} (-1)^{(n+1)/2}\binom{N/2}{(n-1)/2} & N \text{ even}, n \text{ odd}, \\ (-1)^{n/2}\binom{N/2}{n/2} & N \text{ even}, n \text{ even}, \\ 0 & N \text{ odd}, n \text{ odd}, \\ (-1)^{n/2}\binom{(N+1)/2}{n/2} & N \text{ odd}, n \text{ even}. \end{cases} \tag{6.2.36}$$

One can also compute the partition function in the diagonal basis, where $\mathfrak{g}_i = +1$ for the even nodes, and $\mathfrak{g}_i = -1$ for the odd ones. Plugging this into (6.2.16) yields the same expression for the twisted partition function.

### 6.2.3 $G = \mathrm{Spin}(2N + 1)$

Consider the algebra $B_N = \mathfrak{so}_{2N+1}$. The symmetries of this algebra are as follows:

- The group $\mathrm{Spin}(2N+1)$ has no zero-form symmetry.
- The group $\mathrm{Spin}(2N+1)$ has a $\mathbb{Z}_2$ one-form symmetry, whose charged representations are the spinors. The non-trivial element $\mathsf{g} \in \mathbb{Z}_2$ acts on the extended diagram by permuting the zeroth and first nodes, $\lambda_0 \leftrightarrow \lambda_1$. The center acts on a representation $\lambda$ as follows:
$$\alpha_{\mathsf{g}}(\lambda) \equiv (-1)^{\lambda_N} \tag{6.2.37}$$
which means that the eigenvalues in (6.2.16) are $\mathsf{g}_i = (-1)^{\delta_{i,N}}$.

Let us begin by computing the untwisted partition function. The comarks for $\mathrm{Spin}(2N+1)$ are $a_i^\vee = 1$ for $i = 0, 1, N$, and $a_i^\vee = 2$ for $i = 2, 3, \ldots, N-1$. Plugging this into equation (6.2.5) we obtain the untwisted partition function

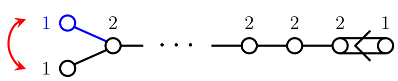
$$Z(q) = (1-q)^3(1-q^2)^{N-2}\,, \tag{6.2.38}$$

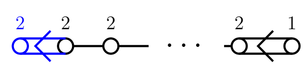
and, expanding, the Witten index

$$I_n = \begin{cases} (-1)^{n/2}\left[\binom{N-2}{n/2} - 3\binom{N-2}{n/2-1}\right] & n \text{ even,} \\ (-1)^{(n-1)/2}\left[\binom{N-2}{(n-1)/2-1} - 3\binom{N-2}{(n-1)/2}\right] & n \text{ odd.} \end{cases} \tag{6.2.39}$$

Note that the index vanishes for $N = 1 \mod 4$ and $n = (N-1)/2$ and by time-reversal for $n' = h - n = (3N-1)/2$. This clearly illustrates the crucial difference between the dimension of the Hilbert space and the index.

$4d$ $\mathrm{Spin}(2N+1)$ $\mathcal{N}=1$ SYM has no charge conjugation symmetry. We can consider instead the index twisted by the $\Gamma = \mathbb{Z}_2$ one-form center symmetry. The non-trivial element $\mathsf{g} \in \mathbb{Z}_2$ acts on the extended Dynkin diagram as follows:

$B_N^{(1)}:$ 

$\mapsto$ $A_{2N}^{(2)}:$ 

(6.2.40)

where the blue node denotes the affine root, and the integers denote the comarks $a_i^\vee$.

From this we learn that the folded diagram has $r'+1 = N$ nodes, one of which has energy equal to 1, and the rest all energy equal to 2. Plugging this into (6.2.15) the one-form twisted partition function is

$$Z^{\mathsf{g}}(q) = (1-q)(1-q^2)^{N-1}\,, \tag{6.2.41}$$

and, expanding, the Witten index

$$I_n^{\mathsf{g}} = \begin{cases} (-1)^{n/2}\binom{N-1}{n/2} & n \text{ even,} \\ (-1)^{(n+1)/2}\binom{N-1}{(n-1)/2} & n \text{ odd.} \end{cases} \tag{6.2.42}$$

One can also compute the partition function in the diagonal basis, where $\mathsf{g}_i = +1$ for all the nodes except for the last one, which has $\mathsf{g}_N = -1$. Plugging this into (6.2.16) yields the same expression for the twisted partition function.

### 6.2.4 $G = \mathrm{Spin}(2N)$

Consider the algebra $D_N = \mathfrak{so}_{2N}$. The symmetries of this algebra are as follows:

- The group $\mathrm{Spin}(2N)$ has a $\mathbb{Z}_2$ zero-form symmetry. The corresponding charge is the chirality of the representation. This symmetry acts by permuting the last two nodes in the unextended Dynkin diagram. The associated diagonal action can be chosen as follows: take $\mathsf{c}_i = +1$ for all but the last two nodes, and $\mathsf{c}_{N-1} = +1$ and $\mathsf{c}_N = -1$.

- The group $\mathrm{Spin}(2N)$ has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ one-form symmetry if $N$ is even, and $\mathbb{Z}_4$ if odd. They act on the extended Dynkin diagram as follows: one of the $\mathbb{Z}_2$'s for $N$ even, and the $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_4$ for $N$ odd, acts as the permutation $\lambda_0 \leftrightarrow \lambda_1$ and $\lambda_{N-1} \leftrightarrow \lambda_N$, while fixing the rest of Dynkin labels in the extended diagram. The other $\mathbb{Z}_2$ factor reverses the order of the extended Dynkin labels, while $\mathbb{Z}_4$ acts as $\lambda_0 \mapsto \lambda_N \mapsto \lambda_1 \mapsto \lambda_{N-1} \mapsto \lambda_0$, and it reverses the order of the rest of Dynkin labels.

  For $N$ even, let $(\mathsf{g}_1, \mathsf{g}_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$; and, for $N$ odd, let $\mathsf{g} \in \mathbb{Z}_4$; all thought of as roots of unity. The center acts on a representation $\lambda$ as follows:

$$\begin{aligned} \alpha_{\mathsf{g}_1,\mathsf{g}_2}(\lambda) &\equiv \mathsf{g}_1^{\lambda_{N-1}+\lambda_N} \mathsf{g}_2^{(N/2-1)\lambda_{N-1}+(N/2)\lambda_N+\sum_{i=0}^{N/2-2}\lambda_{2i+1}} \\ \alpha_{\mathsf{g}}(\lambda) &\equiv \mathsf{g}^{-(N-2)\lambda_{N-1}-N\lambda_N+2\sum_{i=0}^{(N-3)/2}\lambda_{2i+1}}\,. \end{aligned} \tag{6.2.43}$$

  Therefore, the eigenvalues in (6.2.16) are

$$\begin{aligned} (\mathsf{g}_1, \mathsf{g}_2)_{2i} &= 1, && i \in [0, N/2-1) \\ (\mathsf{g}_1, \mathsf{g}_2)_{2i+1} &= \mathsf{g}_2, && i \in [0, N/2-1) \\ (\mathsf{g}_1, \mathsf{g}_2)_{N-1} &= \mathsf{g}_1 \mathsf{g}_2^{N/2-1} \\ (\mathsf{g}_1, \mathsf{g}_2)_{N} &= \mathsf{g}_1 \mathsf{g}_2^{N/2} \\ \mathsf{g}_{2i+1} &= \mathsf{g}^2, && i \in [0, (N-1)/2) \\ \mathsf{g}_{2i} &= 1, && i \in [1, (N-1)/2) \\ \mathsf{g}_{N-1} &= \mathsf{g}^{N-2} \\ \mathsf{g}_N &= \mathsf{g}^N\,. \end{aligned} \tag{6.2.44}$$

Let us begin by computing the untwisted partition function. The comarks of $\mathrm{Spin}(2N)$ are $a_i^\vee = 1$ for $i = 0, 1, N-1, N$, and $a_i^\vee = 2$ for $i = 2, 3, \ldots, N-2$. Plugging this into equation (6.2.5) we obtain the untwisted partition function

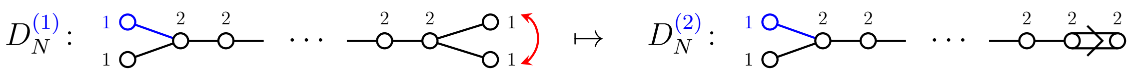
$$Z(q) = (1-q)^4 (1-q^2)^{N-3}\,, \tag{6.2.45}$$

and, expanding, the Witten index

$$I_n = \begin{cases} (-1)^{n/2}\left[\binom{N-3}{n/2} - 6\binom{N-3}{n/2-1} + \binom{N-3}{n/2-2}\right] & n \text{ even}, \\ 4(-1)^{(n-1)/2}\left[\binom{N-3}{(n-1)/2-1} - \binom{N-3}{(n-1)/2}\right] & n \text{ odd}. \end{cases} \tag{6.2.46}$$

Note that the index vanishes when $N$ is even and $n$ corresponds to the time-reversal symmetric wall $n = h/2 = N-1$. It also vanishes for the exceptional pairs $(N, n)$ such that $2 + 4n + 2n^2 - 3N - 4nN + N^2 = 0$.

Let us now consider the index twisted by charge conjugation. Its action on the extended Dynkin diagram, and the resulting folded diagram, are as follows:




(6.2.47)

where the folded diagram has $N$ nodes. From this we learn that the folded diagram has $r' + 1 = N$ nodes, two of which have energy equal to 1, and the rest all energy equal to 2. Plugging this into (6.2.15) the zero-form twisted partition function is

$$Z^{\mathsf{c}}(q) = (1-q)^2(1-q^2)^{N-2} \tag{6.2.48}$$

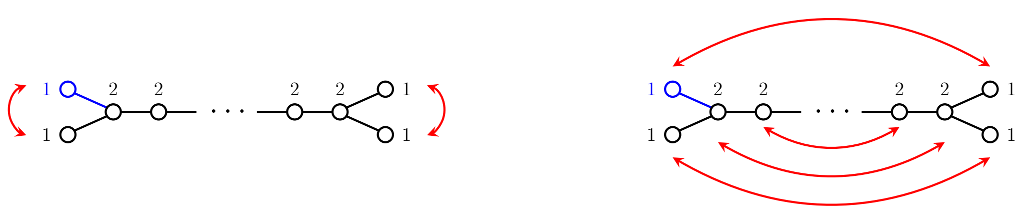
and, expanding, the twisted Witten index

$$I_n^{\mathsf{c}} = \begin{cases} (-1)^{n/2}\left[\binom{N-2}{n/2} - \binom{N-2}{n/2-1}\right] & n \text{ even}, \\ 2(-1)^{(n+1)/2}\binom{N-2}{(n-1)/2} & n \text{ odd}. \end{cases} \tag{6.2.49}$$

One can also compute the partition function in the diagonal basis, where $\mathsf{c}_i = +1$ for all the nodes except for the last one, which has $\mathsf{c}_N = -1$. Plugging this into (6.2.16) yields the same expression for the twisted partition function.

Let us now consider the one-form-twisted partition functions. The symmetry depends on whether $N$ is even or odd, which we consider in turn.

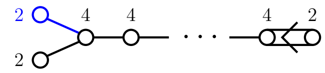
***N* even.** Here the symmetry is $\mathbb{Z}_2 \times \mathbb{Z}_2$. $\mathsf{g}_1$ and $\mathsf{g}_2$ act as follows:




(6.2.50)

The folded diagrams are

$C^{(1)}_{N-2}$ : 

$B^{(1)}_{N/2}$ : 

(6.2.51)

which have $N-1$ and $N/2+1$ nodes, respectively. The folding by $\mathsf{g}_1\mathsf{g}_2$ is in fact identical to that of $\mathsf{g}_2$, i.e., the second diagram.

The twisted partition functions read

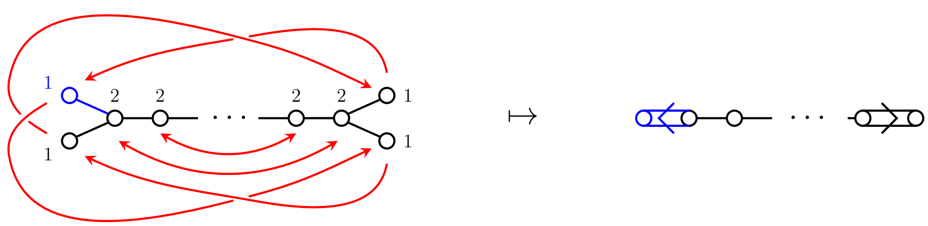
$$\begin{aligned} Z^{\mathsf{g}_1}(q) &= (1-q^2)^{N-1} \\ Z^{\mathsf{g}_2}(q) &= (1-q^2)^3(1-q^4)^{N/2-2}\,, \end{aligned} \tag{6.2.52}$$

and, expanding, the twisted Witten indices

$$\begin{aligned} I^{\mathsf{g}_1}_n &= \begin{cases} (-1)^{n/2}\binom{N-1}{n/2} & n \text{ even,} \\ 0 & n \text{ odd.} \end{cases} \\ I^{\mathsf{g}_2}_n &= \begin{cases} (-1)^{n/4}\left[\binom{N/2-2}{n/4} - 3\binom{N/2-2}{n/4-1}\right] & n\equiv 0 \mod 4, \\ (-1)^{(n-2)/4}\left[\binom{N/2-2}{(n-2)/4-1} - 3\binom{N/2-2}{(n-2)/4}\right] & n\equiv 2 \mod 4, \\ 0 & n \text{ odd,} \end{cases} \end{aligned} \tag{6.2.53}$$

while $I^{\mathsf{g}_1\mathsf{g}_2}_n = I^{\mathsf{g}_2}_n$.

***N* odd.** Here the one-form symmetry is $\mathbb{Z}_4$, whose action on the extended Dynkin diagram, and the corresponding folded diagram, are as follows:



(6.2.54)

where the comarks are all 4 if we fold by a generator of $\mathbb{Z}_4$, and all 2 if we fold by a generator squared. The number of nodes is $(N-1)/2$ in the first case, and $N-1$ in the second case. The folded diagram corresponds to $C^{(1)}_{(N-1)/2}$ and $C^{(1)}_{N-1}$, respectively.

If we let $\mathsf{g}$ denote a generator of $\mathbb{Z}_4$, the twisted partition function is

$$\begin{aligned} Z^{\mathsf{g}}(q) = Z^{\mathsf{g}^3}(q) &= (1-q^4)^{(N-1)/2}\,, \\ Z^{\mathsf{g}^2}(q) &= (1-q^2)^{N-1}\,, \end{aligned} \tag{6.2.55}$$

and, expanding, the twisted Witten indices

$$\begin{aligned} I_n^{\mathsf{g}} = I_n^{\mathsf{g}^3} &= \begin{cases} (-1)^{n/4}\binom{(N-1)/2}{n/4} & n \equiv 0 \mod 4\,, \\ 0 & \text{otherwise}\,, \end{cases} \\ I_n^{\mathsf{g}^2} &= \begin{cases} (-1)^{n/2}\binom{N-1}{n/2} & n \text{ even}\,, \\ 0 & n \text{ odd}\,. \end{cases} \end{aligned} \tag{6.2.56}$$

As usual, one may also compute these partition functions in the diagonal basis. Using the phases (6.2.44) in (6.2.16) yields the same expressions for the twisted partition functions, as expected.

### 6.2.5 $G = G_2$

$G_2$ has no zero-form or one-form symmetry. The comarks for $G_2$ are $a_0^\vee = a_2^\vee = 1$ and $a_1^\vee = 2$. Plugging this into equation (6.2.5) we obtain the untwisted partition function

$$Z(q) = (1-q)^2(1-q^2)\,, \tag{6.2.57}$$

and, expanding, the Witten indices

$$\begin{aligned} I_1 &= -2 \\ I_2 &= 0 \\ I_3 &= 2\,. \end{aligned} \tag{6.2.58}$$

Note that $I_3 = -I_1$, as expected from the action of time-reversal on domain walls.

### 6.2.6 Minimal Wall for Arbitrary Gauge Group

The domain wall theory for $n = 1$ admits a uniform description for all simply-connected groups, including the exceptional ones. Indeed, the only fermion configurations with total energy equal to 1, that is the solutions to (6.2.2)

$$\sum_{i=0}^{r} \lambda_i a_i^\vee = 1\,, \tag{6.2.59}$$

are clearly of the form $\lambda_i = 1$ for one $i$ such that $a_i^\vee = 1$, and $\lambda_j = 0$ for all $j \neq i$. In other words, in each configuration there is only one excited fermion, which moreover necessarily has energy $a_i^\vee = 1$. All these configurations have the same fermion number, namely $(-1)^F = -1$, which means that the index is

$$I_1 \equiv -m_1 \tag{6.2.60}$$

where $m_1$ denotes the number of nodes in the extended Dynkin diagram $\mathfrak{g}^{(1)}$ with comark equal to 1. The values of $m_1$ are given in the following table:

| $G$ | SU($N$) | Sp($N$) | Spin($2N+1$) | Spin($2N$) | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|---|---|---|---|---|---|---|---|---|---|
| $m_1$ | $N$ | $N+1$ | 3 | 4 | 3 | 2 | 1 | 2 | 2 |

(6.2.61)

Note that, for simply-laced $G$, $m_1$ is the order of $\Gamma$.

The index twisted by a symmetry $\mathsf{s} \in \mathsf{S}$ is

$$I_1^{\mathsf{s}} \equiv -m_1^{\mathsf{s}}\,, \tag{6.2.62}$$

where $m_1^{\mathsf{s}}$ denotes the number of nodes in the extended Dynkin diagram $\mathfrak{g}^{(1)}$ with comark equal to 1 that are fixed by $\mathsf{s}$. $m_1^{\mathsf{s}}$ has already been computed for the classical groups $\mathrm{SU}(N), \mathrm{Sp}(N), \mathrm{Spin}(2N+1)$ and $\mathrm{Spin}(2N)$. For the exceptional groups, only $E_6$ and $E_7$ have non-trivial symmetry group $\mathsf{S}$ (see table 6.1). In $E_6$, the zero-form charge-conjugation symmetry leaves invariant the extended node, which has comark 1, and permutes the other two nodes with comark 1. In $E_6$ and $E_7$, the one-form center symmetry permutes all the nodes with comark 1. Therefore, letting $\mathsf{c}$ denote the non-trivial element of $\mathsf{C}$, and $\mathsf{g}$ any non-trivial element of $\Gamma$, the indices are

$$\begin{array}{lllll} E_6: & I_1^{\mathsf{c}} = -1, & I_1^{\mathsf{g}} = 0 & & \mathsf{g} \in \Gamma = \mathbb{Z}_3 \\ E_7: & I_1^{\mathsf{g}} = 0 & & & \mathsf{g} \in \Gamma = \mathbb{Z}_2\,. \end{array} \tag{6.2.63}$$

The (twisted) indices for the exceptional groups $E_6, E_7, E_8, F_4$ and arbitrary $n$ have been included in section 6.7 for completeness.

This concludes our discussion of the twisted Witten indices of the domain walls, as computed in terms of the the $4d$ ultraviolet degrees of freedom. A rather nontrivial consistency test of our proposal is that the Witten indices on the domain walls we just computed are reproduced by the corresponding partition functions of our conjectured $3d$ TQFTs. Computing the image of the (twisted) Witten indices in these TQFTs is nontrivial and requires understanding in detail the Hilbert space of spin TQFTs and the action of $(-1)^F$ on it, a subject to which we now turn.

## 6.3 Hilbert Space of Spin TQFTs

The domain wall theory preserves $3d$ $\mathcal{N}=1$ supersymmetry and observables depend on the choice of a spin structure. Therefore, the TQFT that emerges in the deep infrared of the domain wall must also depend on a choice of a spin structure, that is, it must be a spin TQFT [35].

The data of a TQFT in $3d$ includes the set of anyons $\mathcal{A}$ (or Wilson lines) and the braiding matrix $B\colon \mathcal{A} \times \mathcal{A} \to U(1)$ encoding their braiding. A spin TQFT is a TQFT that has an abelian[104] line $\psi$ that braids trivially with all lines in $\mathcal{A}$ and has half-integral spin.[105] Transparency of $\psi$ implies that it fuses with itself into the vacuum, that is $\psi \times \psi = \mathbf{1}$. Since $\psi$ is transparent and has half-integral spin, the observables of a spin TQFT depend on the choice of a spin structure.

A spin TQFT can be constructed from a parent bosonic TQFT which has an abelian, non-transparent fermion $\psi$ with $\psi \times \psi = \mathbf{1}$, that is, a bosonic TQFT that has a $\mathbb{Z}_2^\psi$ one-form symmetry generated by a fermion [36, 37, 120, 314, 315]. The bosonic parent theory defines a spin TQFT upon gauging its $\mathbb{Z}_2^\psi$ one-form symmetry generated by $\psi$

$$\text{spin TQFT} = \frac{\text{bosonic TQFT}}{\mathbb{Z}_2^\psi}\,. \tag{6.3.1}$$

This procedure is an extension of the notion of bosonic "anyon condensation" [285, 316, 317].[106] Upon gauging, the fermion $\psi$ in the parent bosonic theory becomes the transparent fermion $\psi$ in the spin TQFT. The gauged one-form symmetry $\mathbb{Z}_2^\psi$ of the parent bosonic theory gives rise to an emergent zero-form symmetry $\mathbb{Z}_2$ in the spin TQFT that is generated by the fermion parity operator $(-1)^F$, and which acts on the "twisted sector". We will discuss the action of $(-1)^F$ on the Hilbert space of spin TQFTs shortly.

The lines of the parent bosonic theory $\mathcal{A}$ can be arranged as the disjoint union of two sets $\mathcal{A} = \mathcal{A}_{\rm NS} \cup \mathcal{A}_{\rm R}$ according to their braiding with $\psi$. Lines in $\mathcal{A}_{\rm NS}$, by definition, braid trivially with $\psi$ while lines in $\mathcal{A}_{\rm R}$ have braiding $-1$ with $\psi$. This partitions the lines of the bosonic TQFT according to their $\mathbb{Z}_2^\psi$ quantum number. The lines in each set can be organized into orbits of $\mathbb{Z}_2^\psi$, generated by fusion with $\psi$. The orbits can be either two- or one-dimensional. The lines in one-dimensional orbits are referred to as "Majorana lines" in that they can freely absorb the fermion $\psi$:

$$\psi \times m = m\,. \tag{6.3.2}$$

The Majorana lines, if any, are necessarily in $\mathcal{A}_{\rm R}$.[107] The lines of the bosonic parent theory thus split as

$$\begin{aligned} \mathcal{A}_{\rm NS} &= \{\{a, a\times\psi\} \mid B(\psi,a)=+1\} \\ \mathcal{A}_{\rm R} &= \{\{x, x\times\psi\},\ \{m\} \mid B(\psi,x)=B(\psi,m)=-1\}\,. \end{aligned} \tag{6.3.4}$$

[104] An abelian line is one that yields a single line in its fusion with any line in $\mathcal{A}$.

[105] A TQFT that has a line with half-integral spin which braids nontrivially with at least one line in the theory is not spin. There is no unambiguous way to assign a sign to the fermion as we move it around a circle since the phase it acquires depends on which lines link with the circle, unlike when the fermion is transparent.

[106] More precisely, the parent bosonic TQFT must be attached to a suitable $4d$ SPT phase so that the combined system is non-anomalous, and the symmetry can be gauged.

[107] To prove this we compute the braiding of $\psi$ with a Majorana line $m$ and show that it necessarily has braiding $-1$ with $\psi$

$$B(\psi,m) = e^{2\pi i(h_\psi + h_m - h_{\psi\times m})} = e^{2\pi h_\psi} = -1\,, \quad \Longrightarrow \quad m \in \mathcal{A}_{\rm R}\,, \tag{6.3.3}$$

where $h$ denotes spin of lines and in the second equality we have used the defining relation for a Majorana line (6.3.2). See also [120].

The first set, referred to as the Neveu-Schwarz (NS) lines, is what is usually regarded as the set of Wilson line operators in the spin TQFT. The second set, the Ramond (R) lines, change the spin structure background. This decomposition will be useful shortly in the construction of the Hilbert space of the spin TQFT.

The Hilbert space of the spin TQFT on the spatial torus depends on the choice of spin structure. There are two equivalence classes of spin structures on the torus (or, more generally, on any Riemann surface): even and odd spin structures. Consider the even and odd spin structure Hilbert spaces $\mathcal{H}_{\text{NS-NS}}$ and $\mathcal{H}_{\text{R-R}}$. $\mathcal{H}_{\text{NS-NS}}$ correspond to choosing antiperiodic boundary conditions on the two circles while $\mathcal{H}_{\text{R-R}}$ corresponds to periodic boundary conditions. The other two even spin-structure Hilbert spaces $\mathcal{H}_{\text{NS-R}}$ and $\mathcal{H}_{\text{R-NS}}$ can be obtained from $\mathcal{H}_{\text{NS-NS}}$ by the action of the mapping class group. This group is a non-trivial extension of the modular group $\mathrm{SL}_2(\mathbb{Z})$ by the $\mathbb{Z}_2$ fermion parity symmetry. It is known as the *metaplectic group* $\mathrm{Mp}_1(\mathbb{Z})$. It does not preserve the individual spin structures but it does preserve their equivalence class. The Hilbert spaces of spin TQFTs realize a unitary representation of this group.

The states in the Hilbert space $\mathcal{H}_B$ of a bosonic TQFT are constructed from the path integral on a solid torus by inserting lines $M \in \mathcal{A}$ along the non-contractible cycle [22]. This defines conformal blocks on the torus. We represent this pictorially by

$$|M\rangle = \Big|\, \underset{M}{\bigcirc} \,\Big\rangle \in \mathcal{H}_B\,. \tag{6.3.5}$$

The Hilbert space of the spin TQFT can be constructed from its definition as a quotient of the bosonic parent TQFT (6.3.1).[108] The states in $\mathcal{H}_{\text{NS-NS}}$ are labeled by $a \in \mathcal{A}_{\text{NS}}$, and are represented as

$$|a\rangle_{\text{spin}} = \tfrac{1}{\sqrt{2}} \left( \Big|\, \underset{a}{\bigcirc} \,\Big\rangle + \Big|\, \underset{a\times\psi}{\bigcirc} \,\Big\rangle \right) \in \mathcal{H}_{\text{NS-NS}}\,. \tag{6.3.6}$$

The states in $\mathcal{H}_{\text{R-R}}$ are constructed from conformal blocks of the bosonic parent TQFT on the torus and, in the presence of Majorana lines (6.3.2), from the once-punctured torus conformal blocks of the bosonic parent TQFT. By virtue of $m$ being a Majorana line obeying the fusion rule $\psi \times m = m$, the one-point conformal block on the torus with $m$ along the cycle and $\psi$ at the puncture is nontrivial, as it is allowed by the fusion rules. The states in

[108] More details of the explicit construction of the Hilbert space of spin TQFTs will appear elsewhere [1].

$\mathcal{H}_{\text{R-R}}$ are labeled by $x, m \in \mathcal{A}_{\text{R}}$, and are represented as[109]

$$
\begin{aligned}
|\, x \,\rangle_{\text{spin}} &= \tfrac{1}{\sqrt{2}} \left( \Big|\, \underset{x}{\bigcirc} \,\Big\rangle \; - \; \Big|\, \underset{x \times \psi}{\bigcirc} \,\Big\rangle \right) \in \mathcal{H}_{\text{R-R}}\,, \\
|m\rangle_{\text{spin}} &= \Big|\, \overset{\psi}{\underset{m}{\bigcirc}} \,\Big\rangle \in \mathcal{H}_{\text{R-R}}\,.
\end{aligned}
\tag{6.3.7}
$$

Modular transformations preserve the odd spin structure, i.e., they map $\mathcal{H}_{\text{R-R}}$ into itself. The negative sign in (6.3.7) guarantees that under under modular transformations the states in $\mathcal{H}_{\text{R-R}}$ are mapped into themselves.

Note that the pair of lines $a$ and $a \times \psi$ in the bosonic parent descend to a pair of lines in the spin TQFT, because these are distinct anyons, being distinguishable by their spin. On the other hand, the pair of states $|a\rangle$ and $|a \times \psi\rangle$ descend to a *single* state in the spin TQFT. Thus, while in a bosonic TQFT the number of states is the same as the number of lines, in a spin TQFT there are twice as many lines as there are states.

Our next task is to compute the action of fermion parity, i.e. $(-1)^F$, on the Hilbert space of the spin TQFT. The $\mathbb{Z}_2$ symmetry generated by $(-1)^F$ is the emergent zero-form symmetry that appears upon quotienting the parent bosonic theory by $\mathbb{Z}_2^{\psi}$ in (6.3.1). The charged states are therefore those constructed from the once-punctured torus in the bosonic theory

$$
\begin{aligned}
(-1)^F |m\rangle_{\text{spin}} &= -|m\rangle_{\text{spin}}\,, \\
(-1)^F |\, a \,\rangle_{\text{spin}} &= +|\, a \,\rangle_{\text{spin}}\,, \\
(-1)^F |\, x \,\rangle_{\text{spin}} &= +|\, x \,\rangle_{\text{spin}}\,.
\end{aligned}
\tag{6.3.8}
$$

$(-1)^F$ acts nontrivially on the $\psi$ puncture in the once-punctured torus.

Depending on the choice of spin structure on the "time" circle we can define the following $2^3 = 8$ partition functions for spin TQFTs:[110]

$$
\begin{aligned}
\text{tr}_{-,-}(\mathcal{O}) &\equiv \text{tr}_{\mathcal{H}_{\text{NS-NS}}}(\mathcal{O})\,, \\
\text{tr}_{-,+}(\mathcal{O}) &\equiv \text{tr}_{\mathcal{H}_{\text{NS-R}}}(\mathcal{O})\,, \\
\text{tr}_{+,-}(\mathcal{O}) &\equiv \text{tr}_{\mathcal{H}_{\text{R-NS}}}(\mathcal{O})\,, \\
\text{tr}_{+,+}(\mathcal{O}) &\equiv \text{tr}_{\mathcal{H}_{\text{R-R}}}(\mathcal{O})\,,
\end{aligned}
\tag{6.3.9}
$$

[109] Unlike in bosonic anyon condensation, where a fixed line in the parent theory yields multiples states in the quotient theory, a Majorana line is in an irreducible representation of Cliff$(1|1)$ and yields a unique state in the quotient (spin) TQFT.

[110] $-$ represents antiperiodic boundary condition while $+$ period boundary conditions.

and

$$\begin{aligned}
\operatorname{tr}_{-,-}((-1)^F\mathcal{O}) &\equiv \operatorname{tr}_{\mathcal{H}_{\text{NS-NS}}}((-1)^F\mathcal{O})\,,\\
\operatorname{tr}_{-,+}((-1)^F\mathcal{O}) &\equiv \operatorname{tr}_{\mathcal{H}_{\text{NS-R}}}((-1)^F\mathcal{O})\,,\\
\operatorname{tr}_{+,-}((-1)^F\mathcal{O}) &\equiv \operatorname{tr}_{\mathcal{H}_{\text{R-NS}}}((-1)^F\mathcal{O})\,,\\
\operatorname{tr}_{+,+}((-1)^F\mathcal{O}) &\equiv \operatorname{tr}_{\mathcal{H}_{\text{R-R}}}((-1)^F\mathcal{O})\,,
\end{aligned} \tag{6.3.10}$$

where $\mathcal{O}$ is an operator in the theory. We will be interested in the case when $\mathcal{O}$ is a symmetry of the TQFT. We note that $(-1)^F$ is only non-trivial in the R-R sector, because this is the only Hilbert space that may contain Majorana states. This is the most subtle and rich sector, and the one of interest as far as the twisted Witten indices is concerned.

### 6.3.1 Partition Function of Spin TQFTs

The twisted Witten index of the domain wall theory is computed by considering the odd spin structure on the spatial torus and periodic boundary condition on the time circle. This implies that the twisted Witten indices, computed via the $4d$ ultraviolet fields, must be reproduced by appropriate odd-spin-structure partition functions of our conjectured infrared $3d$ spin TQFTs. In other words, a nontrivial check that our proposed infrared spin TQFTs describe the $n$-domain wall theories is proving that

$$I_n^{\mathsf{s}} \equiv \operatorname{tr}_{\mathcal{H}_{\text{R-R}}}(-1)^F\mathsf{s} \tag{6.3.11}$$

for symmetries $\mathsf{s}\in\mathsf{S}$. This requires, in particular, identifying the image of the symmetries $\mathsf{s}\in\mathsf{S}$ in the infrared TQFT.

Let us begin by considering the untwisted partition function. Given the construction of the Hilbert space $\mathcal{H}_{\text{R-R}}$ in (6.3.7) and the action of $(-1)^F$ in (6.3.8) we can compute the desired partition function as follows

$$\operatorname{tr}_{\mathcal{H}_{\text{R-R}}}(-1)^F = N_x - N_m\,. \tag{6.3.12}$$

This requires determining in the bosonic parent theory the number $N_x$ of two-dimensional orbits and the number $N_m$ of one-dimensional orbits (the number of Majorana lines) in $\mathcal{A}_{\text{R}}$ (see (6.3.4)) under fusion with $\psi$.

Let us illustrate this in a simple example. The simplest spin TQFT is $\mathrm{SO}(N)_1$ Chern-Simons theory, which is is a trivial, invertible spin TQFT with lines $\{\mathbf{1},\psi\}$. The bosonic parent theory is $\mathrm{Spin}(N)_1$ Chern-Simons theory:

- For $N$ odd, $\mathrm{Spin}(N)_1$ is the Ising category, which has three lines $\{\mathbf{1},\sigma,\psi\}$: the vacuum $\mathbf{1}$, the spin operator $\sigma$, and the energy operator $\psi$. These primaries have spins $h=0,\frac{N}{16},\frac{1}{2}$, and fusion rules $\sigma^2=\mathbf{1}+\psi$, $\psi^2=\mathbf{1}$, $\psi\times\sigma=\sigma$.

- For $N$ even, $\mathrm{Spin}(N)_1$ has four lines, with spins $h = 0, \frac{N}{16}, \frac{N}{16}, \frac{1}{2}$, and which we denote by $\{\mathbf{1}, \mathsf{e}, \mathsf{m}, \psi\}$, which correspond to the trivial representation, the two fundamental spinor representations, and the vector representation, respectively. The theory for $N \equiv 0 \mod 4$ has $\mathbb{Z}_2^2$ fusion rules, with $\mathsf{e}^2 = \mathsf{m}^2 = \psi^2 = \mathbf{1}$ and $\mathsf{e} \times \mathsf{m} = \psi$. For $N \equiv 2 \mod 4$ the fusion ring is $\mathbb{Z}_4$, with $\mathsf{e}^2 = \mathsf{m}^2 = \psi$, and $\psi^2 = \mathsf{e} \times \mathsf{m} = \mathbf{1}$.

The theory $\mathrm{SO}(N)_1$ is obtained by condensing $\psi$, that is $\mathrm{SO}(N)_1 = \mathrm{Spin}(N)_1/\mathbb{Z}_2^\psi$. Using the fusion rules and the spins we see that $\mathbf{1}, \psi$ are neutral under $\mathbb{Z}_2^\psi$ and $\mathsf{e}, \mathsf{m}$ and $\sigma$ are charged. In other words, the Neveu-Schwarz sector is

$$\mathcal{A}_{\mathrm{NS}} = \{\mathbf{1}, \psi\}\,, \tag{6.3.13}$$

while the Ramond sector is

$$\mathcal{A}_{\mathrm{R}} = \begin{cases} \{\sigma\} & N \text{ odd,} \\ \{\mathsf{e}, \mathsf{m}\} & N \text{ even.} \end{cases} \tag{6.3.14}$$

This implies that $N_a = 1$ in the NS sector. On the other hand, in the R sector, $(N_x, N_m) = (0, 1)$ for odd $N$ and $(N_x, N_m) = (1, 0)$ for even $N$. Thus there is a unique state in each spin structure, and all states are bosonic except in $\mathcal{H}_{\text{R-R}}$, where $(-1)^F = (-1)^N$, since there is a Majorana line for odd $N$. The $(-1)^F$ odd state is created by the well-known once-punctured torus conformal block in the Ising category with the insertion of $\sigma$, and $\psi$ at the puncture.

For a second example let us now consider the spin TQFT $\mathrm{SO}(3)_3$ Chern-Simons theory, which is the simplest non-trivial spin TQFT. The bosonic parent theory is $\mathrm{SU}(2)_6$ since

$$\mathrm{SO}(3)_3 = \frac{\mathrm{SU}(2)_6}{\mathbb{Z}_2^\psi}\,, \tag{6.3.15}$$

where the abelian line $\psi$ is the line in $\mathrm{SU}(2)_6$ with $j = 3$ and spin $h = 3/2$. The lines in $\mathcal{A} = \{j = 0, \frac{1}{2}, 1, \ldots, 3\}$ which have braiding $-1$ with $\psi$ are those with half-integral isospin: $\mathcal{A}_{\mathrm{R}} = \{j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}\}$. Under fusion with $\psi$ we have the following $\mathbb{Z}_2^\psi$ orbits

$$\begin{aligned} 3 \times \tfrac{1}{2} &= \tfrac{5}{2} \\ 3 \times \tfrac{3}{2} &= \tfrac{3}{2}\,. \end{aligned} \tag{6.3.16}$$

Therefore, in the R-R sector of $\mathrm{SO}(3)_3$ there is a length-2 orbit with $(j = \frac{1}{2}, \frac{5}{2})$ and a Majorana line with $j = \frac{3}{2}$. Thus, $N_x = N_m = 1$. There are $N_x + N_m = 2$ states, but one of them is a boson and the other is a fermion, which means that the partition function with periodic boundary conditions vanishes

$$\mathrm{tr}_{\mathcal{H}_{\text{R-R}}}(-1)^F = N_x - N_m \equiv 0\,. \tag{6.3.17}$$

The vanishing of this trace will be important when discussing the 2-domain wall theory in $4d$ $\mathcal{N} = 1$ SYM with gauge group $G_2$ (cf. section 6.4.4). This example clearly illustrates the importance of looking at the appropriate partition function and not merely at the dimension of the Hilbert space.

We now discuss a different way to compute the partition function that does not rely on computing $N_x$ and $N_m$ directly. The basic idea is to gauge the emergent zero-form $\mathbb{Z}_2$ symmetry generated by $(-1)^F$ in the spin TQFT to obtain the bosonic parent theory back [11, 36, 37, 66, 318]

$$\frac{\text{spin TQFT}}{\mathbb{Z}_2} = \text{bosonic TQFT}\,. \tag{6.3.18}$$

Gauging this $\mathbb{Z}_2$ amounts to summing the spin TQFT over all spin structures of the three-manifold $M$. Taking $M$ to the three-torus, and summing over the $2^3 = 8$ spin structures, corresponding to either periodic or antiperiodic boundary conditions around each of the three circles, we find that

$$\frac{1}{2}\sum_{\pm,\pm}(\mathrm{tr}_{\pm,\pm}(\mathbf{1}) + \mathrm{tr}_{\pm,\pm}(-1)^F) = \mathrm{tr}_{\mathcal{H}_B}(\mathbf{1})\,. \tag{6.3.19}$$

$\mathrm{tr}_{\pm,\pm}$ (see (6.3.9)–(6.3.10)) denotes the trace over the Hilbert space on the spatial torus with boundary conditions $\pm,\pm$, and $\mathrm{tr}_{\mathcal{H}_B}$ the trace over the torus Hilbert space of the bosonic parent theory.

Using the fact that the dimension of the torus Hilbert space is the same in all spin structures and that $(-1)^F$ acts nontrivially only in $\mathcal{H}_{\text{R-R}}$ (see (6.3.8)), we find the formula

$$\mathrm{tr}_{\mathcal{H}_{\text{R-R}}}(-1)^F = 2\dim(\mathcal{H}_B) - 7\dim(\mathcal{H}_F)\,, \tag{6.3.20}$$

where $\dim(\mathcal{H}_B)$ is the dimension of the torus Hilbert space of the bosonic parent TQFT and $\dim(\mathcal{H}_F)$ the dimension of the torus Hilbert space in any one spin structure of the spin TQFT. This formula offers a significant advantage in that it requires computing the total number of states $\dim(\mathcal{H}_F) = N_x + N_m$, and not separately $N_x$ and $N_m$, as in formula (6.3.12). Even simpler, one may compute $\dim(\mathcal{H}_F) = N_a$ in the NS-NS sector directly, where all orbits are of length-2: the number of states is just half the number of lines of the spin TQFT.[111]

As a consistency check, consider the case where $G_F$ is the product of a bosonic theory $\tilde{G}$ times a trivial/invertible spin TQFT

$$G_F = \tilde{G} \times \mathrm{SO}(N)_1\,, \tag{6.3.21}$$

whose bosonic parent is $G_B = \tilde{G} \times \mathrm{Spin}(N)_1$. As $\mathrm{SO}(N)_1$ is a trivial spin TQFT, we get $\dim(\mathcal{H}_F) = \dim(\mathcal{H}_{\tilde{G}})$. Similarly, using that $\mathrm{Spin}(N)_1$ has a four-dimensional Hilbert space if $N$ is even, and a three dimensional Hilbert space if $N$ is odd, we get $\dim(\mathcal{H}_B) = \frac{1}{2}(7 + (-1)^N)\dim(\mathcal{H}_{\tilde{G}})$. Plugging this into (6.3.20), we get

$$\mathrm{tr}_{\mathcal{H}^{G_F}_{\text{R-R}}}(-1)^F = (-1)^N \dim(\mathcal{H}_{\tilde{G}})\,, \tag{6.3.22}$$

[111] In the $\mathrm{SO}(3)_3$ example $\dim(\mathcal{H}_B) = 7$ and $\dim(\mathcal{H}_F) = 2$, and using (6.3.20) the partition function indeed vanishes.

which is precisely what one would expect, given the tensor product structure of $G_F$ and the fact that the trace over $\mathrm{SO}(N)_1$ is $(-1)^N$. Put differently, in the Hilbert space of $G_F = \tilde{G} \times \mathrm{SO}(N)_1$ we have $N_a = N_x$ and $N_m = 0$ for $N$ even, and $N_a = N_m$ and $N_x = 0$ for $N$ odd. That is, in the R-R sector, either no states are Majorana or all are, depending on the parity of $N$. This implies that

$$\mathrm{tr}_{\mathcal{H}^{G_F}_{\text{R-R}}}(-1)^F = \begin{cases} +N_a & N \text{ even} \\ -N_a & N \text{ odd}\,, \end{cases} \tag{6.3.23}$$

which indeed equals (6.3.22).

There are spin TQFTs which factorize in a nontrivial fashion into the product of a bosonic TQFT and a trivial spin TQFT by virtue of a level-rank duality, as for example $\mathrm{U}(1)_k \leftrightarrow \mathrm{SU}(k)_{-1} \times \{\mathbf{1}, \psi\}$. In these theories $\mathrm{tr}_{\mathcal{H}_{\text{R-R}}}(-1)^F$ also just measures the dimension of the Hilbert space (up to possibly a global sign).

More generally, stacking a TQFT, spin or bosonic, with a trivial spin TQFT defines

$$\mathrm{TQFT} \times \mathrm{SO}(N)_1\,. \tag{6.3.24}$$

This theory has the same number of states as the original TQFT but $\mathrm{SO}(N)_1$ can change the global sign of the action of $(-1)^F$ on all states in $\mathcal{H}_{\text{R-R}}$. The partition functions are the same up to possibly a sign

$$\mathrm{tr}_{\mathcal{H}^{\mathrm{TQFT}\times\mathrm{SO}(N)_1}_{\text{R-R}}}(-1)^F = (-1)^N\, \mathrm{tr}_{\mathcal{H}^{\mathrm{TQFT}}_{\text{R-R}}}(-1)^F\,. \tag{6.3.25}$$

Indeed, the single state of $\mathrm{SO}(N)_1$ in $\mathcal{H}_{\text{R-R}}$ is a Majorana state and thus has odd fermion parity for $N$ odd only. Therefore, when comparing the spin TQFT partition function with the Witten index of the domain wall, we will match their absolute values, as if those match, the signs can be also be matched by stacking a suitable trivial spin TQFT, which can be thought of as a purely gravitational counterterm [111].[112]

The generalization to twisted indices is straightforward. Given a symmetry $\mathsf{s} \in \mathsf{S}_{\mathrm{TQFT}}$ of the TQFT, which acts $\mathsf{s}\colon \mathcal{H}_{\text{R-R}} \to \mathcal{H}_{\text{R-R}}$, the partition function

$$\mathrm{tr}_{\mathcal{H}_{\text{R-R}}}((-1)^F \mathsf{s}) \tag{6.3.26}$$

counts the number of bosons fixed by $\mathsf{s}$, minus the number of fermions fixed by $\mathsf{s}$. That being said, there are some subtleties that must be kept in mind. A state fixed by $\mathsf{s}$ does not necessarily contribute with $\mathsf{s} = +1$ to the trace – it might contribute with $\mathsf{s} = -1$ instead, the reason being that the symmetry $\mathsf{s}$ might be realized projectively in the Hilbert space.

The most common example where this may happen is charge-conjugation $\mathsf{c}$. We can illustrate this in $\mathrm{U}(1)_1$ Chern-Simons theory, the simplest theory where this phenomenon

[112]Staking $\mathrm{SO}(N)_1$ for odd $N$ to a $3d$ theory has the same effect as stacking to a $2d$ theory the trivial spin TQFT known as the Arf-invariant, which changes the sign of the partition with odd spin structure.

occurs. This is an invertible spin TQFT, which means that it has a unique state on any spin structure. This state is clearly fixed by $\mathsf{c}$ but, interestingly, it has $\mathsf{c} = -1$ in the odd-spin-structure Hilbert space $\mathcal{H}_{\text{R-R}}$. We can show this as follows. The bosonic parent theory is $\mathrm{U}(1)_4$ Chern-Simons theory, which has four states, labeled by $q = 0, 1, 2, 3$. The $\mathrm{U}(1)_1$ theory is obtained by condensing the fermion $\psi$, which has $q = 2$. The Ramond lines are easily checked to be $q = 1, 3$, and they are paired by fusion with $\psi$ into a single two-dimensional orbit, since $1 \times 2 = 3$. Thus, the unique state in the R-R sector is (cf. (6.3.7))

$$|1\rangle_{\text{spin}} = \frac{1}{\sqrt{2}}(|1\rangle - |3\rangle)\,. \tag{6.3.27}$$

This indeed satisfies $\mathsf{c}|1\rangle_{\text{spin}} = -|1\rangle_{\text{spin}}$, inasmuch as $\mathsf{c}\colon q \mapsto -q \mod 4$ in the bosonic parent, which exchanges $|1\rangle$ and $|3\rangle$.

## 6.4 Domain Wall TQFT Partition Functions

In this section we calculate partition functions twisted by a symmetry $\mathsf{s} \in \mathsf{S}_{\text{TQFT}}$

$$\mathrm{tr}_{\mathcal{H}_{\text{R-R}}}(-1)^F \mathsf{s} \tag{6.4.1}$$

of the Chern-Simons TQFTs we proposed emerge in the infrared of the domain wall theories (see section 6.1). Our calculations beautifully reproduce the results obtained in section 6.2. Namely, we will now demonstrate that the trace (6.4.1) agrees with the twisted Witten index on the $n$-domain wall $\mathrm{W}_n$ (cf. (6.2.10)) as computed in terms of the original $4d$ fields

$$\mathrm{tr}_{\mathcal{H}_{\text{R-R}}}(-1)^F \mathsf{s} \equiv I_n^{\mathsf{s}}\,. \tag{6.4.2}$$

We identify each symmetry $\mathsf{s} \in \mathsf{S}$ in $4d$ $\mathcal{N} = 1$ SYM with a symmetry $\mathsf{s} \in \mathsf{S}_{\text{TQFT}}$ in the infrared TQFT.

In $4d$ $\mathcal{N} = 1$ SYM with gauge group $\mathrm{Sp}(N)$ our proposed domain wall theory corresponds to a Chern-Simons theory based on a group that is simple, connected, and simply-connected, whereas for SYM with $\mathrm{SU}(N)$, $\mathrm{Spin}(N)$ and $G_2$ gauge groups, the proposed infrared Chern-Simons theories are based on a group that is neither. We discuss both cases in turn.

Chern-Simons theory $G_k$, with $G$ simple, connected and simply-connected is always a bosonic TQFT. These theories are made spin by tensoring with the trivial spin TQFT $\mathrm{SO}(N)_1$. It follows from our discussion in section 6.3 that

$$\mathrm{tr}_{\mathcal{H}_{\text{R-R}}^{G_k \times \mathrm{SO}(N)_1}}(-1)^F = (-1)^N \, \mathrm{tr}_{G_k}(\mathbf{1})\,, \tag{6.4.3}$$

since all states have the same fermion parity – all bosonic, or all fermionic, depending on the parity of $N$. Therefore the partition function of $G_k \times \mathrm{SO}(N)_1$ in (6.4.3) is, up to possibly a sign, the dimension of the Hilbert space of $G_k$ Chern-Simons theory on the two-torus.

The states in the torus Hilbert space of $G_k$ Chern-Simons theory are conformal blocks on the torus, which are labeled by the integrable representations of the corresponding affine lie algebra $\mathfrak{g}^{(1)}$ at level $k$ [22, 319]. By definition, the representations of $G$ that are integrable are those whose highest weight $\lambda$ satisfies $(\lambda, \theta) \leq k$, with $\theta$ the highest root of $G$. Expanding the latter in a basis of simple coroots, and introducing an extended label $\lambda_0 := k - (\lambda, \theta)$, integrability can be expressed as

$$\sum_{i=0}^{r} \lambda_i a_i^\vee = k, \qquad \lambda_i \in \mathbb{Z}_{\geq 0} \,. \tag{6.4.4}$$

The dimension of the Hilbert space $\mathrm{tr}_{G_k}(\mathbf{1})$ is equal to the number of solutions to this equation.

Much like the discussion in section 6.2, where the Witten index on the domain wall was computed through an auxiliary system of free fermions, $\mathrm{tr}_{G_k}(\mathbf{1})$ has a nice combinatorial interpretation in terms of a system of free bosons in $0+1$ dimensions. Indeed, the number of integrable representations $\mathrm{tr}_{G_k}(\mathbf{1})$ is the number of ways of creating a state of energy $k$ from $r+1$ free bosons, each with energy $a_i^\vee$. Each boson is associated with a node in the extended Dynkin diagram $\mathfrak{g}^{(1)}$, and $\lambda_i \in \{0, 1, 2, \dots\}$ in (6.4.4) corresponds to the occupation number of the $i$-th boson. Introducing a fugacity parameter $q$ defines a generating function, which is the partition function of the bosons on the circle

$$Z(G, q) \equiv \sum_{k \geq 0} \mathrm{tr}_{G_k}(\mathbf{1})\, q^k \,. \tag{6.4.5}$$

The partition function is thus

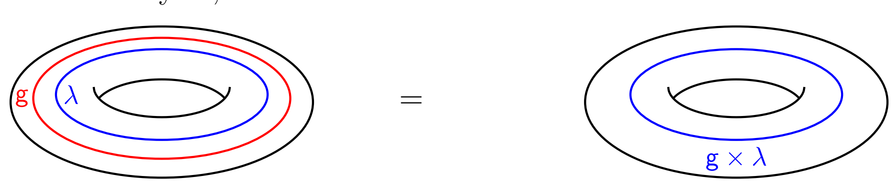
$$Z(G, q) = \prod_{i=0}^{r} (1 - q^{a_i^\vee})^{-1} \,. \tag{6.4.6}$$

The Chern-Simons trace $\mathrm{tr}_{G_k}(\mathbf{1})$ is the coefficient of $q^k$ in (6.4.6).

In a similar fashion, we define the trace twisted by a symmetry $\mathsf{s} \in \mathsf{S}_{\mathrm{TQFT}}$ of $G_k$ Chern-Simons theory:

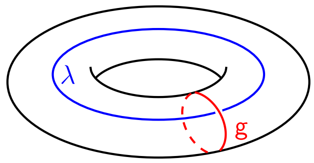
$$\mathrm{tr}_{G_k}(\mathsf{s}) \,. \tag{6.4.7}$$

When $\mathsf{s} = \mathsf{c}$ is a zero-form symmetry, this corresponds to inserting a surface operator, i.e., the symmetry defect is supported on the whole spatial torus. On the other hand, if $\mathsf{s} = \mathsf{g}$ denotes a one-form symmetry, the symmetry defect is a line operator, and one must specify a homology cycle on the torus on which it is is supported. The states of $G_k$ Chern-Simons are created by wrapping on a cycle Wilson lines labeled by integrable representations $\lambda$; if $\mathsf{g}$ is supported on the same cycle, it acts on the states via fusion:

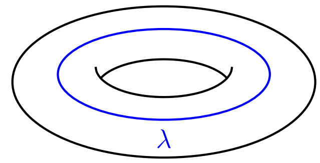



(6.4.8)

Conversely, if $\mathsf{g}$ is supported on the dual cycle, it acts on the states via braiding:




$$= \quad \alpha_{\mathsf{g}}(\lambda) \tag{6.4.9}$$




where $\alpha_{\mathsf{g}}(\lambda)$ is the charge of $\lambda$ under the center of $G$ (cf. (6.2.7)). More generally, one can wrap a pair of symmetry defects on both cycles, but one can always conjugate such configuration via a modular transformation to either of the two options above. This operation, being a similarity transformation, does not affect the value of the trace. In other words, the value of $\mathrm{tr}_{G_k}(\mathsf{g})$ is independent of which cycle we define $\mathsf{g}$ on.

When $\mathsf{s}$ is a symmetry of the classical action of $G_k$ Chern-Simons theory, it is induced by an outer automorphism of the extended Dynkin diagram $\mathfrak{g}^{(1)}$, and it acts as a permutation of the nodes thereof. In that case, $\mathsf{s}$ induces an action on the system of bosons, which permutes them in the same way it permutes the nodes of the Dynkin diagram. As in the system of free fermions, the trace above can be obtained from the partition function of these bosons

$$Z^{\mathsf{s}}(G,q) \equiv \sum_{k\geq 0} \mathrm{tr}_{G_k}(\mathsf{s}) q^k \,, \tag{6.4.10}$$

where $Z^{\mathsf{s}}(G,q)$ denotes the bosonic partition function twisted by the permutation $\mathsf{s}$. One can evaluate this partition function by the same methods as in section 6.2, i.e., by diagram folding or directly in the diagonal basis

$$Z^{\mathsf{s}}(G,q) = \prod_{i=0}^{r} (1 - \mathsf{s}_i q^{a_i^\vee})^{-1} \,, \tag{6.4.11}$$

where $\mathsf{s}_i$ are the eigenvalues of the permutation. Note that, for $\mathsf{s} = \mathsf{g}$ a one-form symmetry, diagram folding naturally corresponds to $\mathsf{g}$ acting as a permutation, i.e. (6.4.8), while the diagonal action corresponds to $\mathsf{g}$ acting via braiding, i.e. (6.4.9). Indeed, it is a well-known fact that an $S$ modular transformation – which interchanges the two cycles – diagonalizes the fusion rules.

It should be noted that Chern-Simons theories can have "quantum symmetries". These are symmetries of the entire TQFT data that are not symmetries of the Lagrangian. Many explicit examples of these symmetries have been found in [4]. These symmetries permute the Wilson lines of the theory, in a way that does not necessarily correspond to a permutation of their Dynkin labels. As such, the free boson representation cannot be used to evaluate the twisted trace, but it must be computed from the action of the symmetry on the Hilbert space of the TQFT. That being said, we find that the symmetries $\mathsf{S}$ in the ultraviolet domain wall map to classical symmetries of the infrared Chern-Simons theories, and we can compute the twisted index using (6.4.11).

Some of our proposed domain wall TQFTs are Chern-Simons theory with a group $G$ that is not connected and/or simply-connected, in which case the theory $G_k$ (with $k$ a set of integers that defines the Chern-Simons action) may depend on the spin structure of the underlying manifold. There are four distinct Hilbert spaces corresponding to the four spin structures on the spatial torus (see section 6.3), but our interest here is in the Hilbert space $\mathcal{H}_{\text{R-R}}$. Now $G_k$ can have fermionic states, which correspond to once-punctured conformal blocks of the parent bosonic theory, and $(-1)^F$ is in general a non-trivial operator. Our goal is to compute

$$\mathrm{tr}_{\mathcal{H}^{G_k}_{\text{R-R}}}((-1)^F \mathsf{s}) \equiv \mathrm{tr}_{G_k}((-1)^F \mathsf{s}) \,, \tag{6.4.12}$$

where we use the latter to simplify notation.

We shall next compute the twisted traces for all the Chern-Simons theories of interest. We begin by considering the simply-connected group $\mathrm{Sp}(n)$, and then we move on to the more subtle and interesting cases $\mathrm{U}(n), \mathrm{O}(n)$. We finally make a few remarks concerning the exceptional groups. The remaining simply-connected groups $\mathrm{SU}(n), \mathrm{Spin}(n)$ as well as $\mathrm{SO}(n)$ are studied in section 6.6.

### 6.4.1 $G = \mathrm{Sp}(n)$

The $n$-domain wall theory for $4d$ $\mathcal{N}=1$ SYM with $G = \mathrm{Sp}(N)$ is proposed to be $\mathrm{Sp}(n)_{N+1-n}$ Chern-Simons theory. Let us proceed to study the partition functions of $\mathrm{Sp}(n)_k$.

Consider the algebra $C_n = \mathfrak{sp}_n$. The comarks are all $a_i^\vee = 1$. Plugging this into (6.4.11) we obtain the generating function as

$$Z(\mathrm{Sp}(n), q) = (1-q)^{-(n+1)} \,, \tag{6.4.13}$$

and, by expanding, the untwisted trace

$$\mathrm{tr}_{\mathrm{Sp}(n)_k}(\mathbf{1}) = \binom{n+k}{k} \,. \tag{6.4.14}$$

This is the number of integrable representations of $\mathrm{Sp}(n)_k$, that is, the dimension of the torus Hilbert space of this Chern-Simons theory.

Let us also compute the partition function twisted by the one-form symmetry $\Gamma = \mathbb{Z}_2$. This symmetry reverses the order of the extended labels, and the charged representations are the pseudo-real ones. Denoting by $\mathsf{g}$ the non-trivial element of $\mathbb{Z}_2$, and using (6.4.11) and (6.2.30), we get the twisted partition function

$$Z^{\mathsf{g}}(\mathrm{Sp}(n), q) = (1-q)^{-\lfloor n/2 \rfloor - 1}(1+q)^{-\lceil n/2 \rceil} \,, \tag{6.4.15}$$

and by expanding

$$\mathrm{tr}_{\mathrm{Sp}(n)_k}(\mathsf{g}) = \begin{cases} 0 & n \text{ odd, } k \text{ odd,} \\[2ex] \binom{(n+k-1)/2}{k/2} & n \text{ odd, } k \text{ even,} \\[2ex] \binom{(n+k-1)/2}{(k-1)/2} & n \text{ even, } k \text{ odd,} \\[2ex] \binom{(n+k)/2}{k/2} & n \text{ even, } k \text{ even.} \end{cases} \tag{6.4.16}$$

Note that $Z^{\mathsf{g}}$ is nothing but the untwisted partition function associated to the Dynkin diagram given by folding the original diagram by the one-form symmetry (6.2.33).

We are now ready to test our proposal. Recall that the conjectured infrared theory corresponding to the $n$-domain wall of $\mathrm{Sp}(N)$ SYM was $\mathrm{W}_n = \mathrm{Sp}(n)_k$, with $k = N + 1 - n$. Using this value of the level in (6.4.14) and (6.4.16) indeed reproduces the (twisted) Witten indices computed in the ultraviolet, cf. (6.2.32) and (6.2.36).

### 6.4.2 $G = \mathrm{U}(n)$

The $n$-domain wall theory for $4d$ $\mathcal{N} = 1$ SYM with $G = \mathrm{SU}(N)$ is proposed to be $\mathrm{U}(n)_{N-n,N}$ Chern-Simons theory. Let us proceed to study the partition functions of $\mathrm{U}(n)_{k,n+k}$.

The Chern-Simons gauge group is not simply connected. The theory is defined as

$$\mathrm{U}(n)_{k,n+k} := \frac{\mathrm{SU}(n)_k \times \mathrm{U}(1)_{n(n+k)}}{\mathbb{Z}_n}\,, \tag{6.4.17}$$

where $\mathbb{Z}_n$ is the one-form symmetry generated by the line $\psi = [0, k, 0, \dots, 0] \otimes (n + k)$. Here and in what follows, $[\lambda_0, \lambda_1, \dots, \lambda_{n-1}]$ denotes the Dynkin labels of an $\mathrm{SU}(n)_k$ representation, and $(q) \in \mathbb{Z}$ the charge of a $\mathrm{U}(1)_{n(n+k)}$ representation. The spin of the generator is easily computed to be $h_\psi = \frac{k(n-1)}{2n} + \frac{n+k}{2n} = \frac{k+1}{2}$. This theory is spin if and only if $k$ is even.

We now proceed to compute the relevant traces. As the theory can be a spin TQFT, the theory may contain fermionic states, and $(-1)^F$ will in general be a non-trivial operator, which we need to understand to compute $\mathrm{tr}((-1)^F \mathsf{s})$. In other words, we have to identify which of the states of this Chern-Simons theory are bosons, and which are fermions.

Note that, unlike the general discussion of section 6.3, this theory is more conveniently presented as a $\mathbb{Z}_n$ quotient rather than a $\mathbb{Z}_2$ quotient, so let us slightly generalize the discussion in that section to such quotients. In section 6.3 we argued that the bosonic states after a $\mathbb{Z}_2$ fermionic quotient are the length-2 orbits, while the fermions are the fixed-points. We now claim that the general statement for $\mathbb{Z}_n$ fermionic quotients is that the bosonic states are the orbits of even length, while the fermions are the orbits of odd length.

To prove this claim, consider a general spin TQFT that can be written as

$$G_F = \frac{G_B}{\mathbb{Z}_n}\,, \tag{6.4.18}$$

where $G_B$ is some bosonic TQFT, and where $\mathbb{Z}_n$ is a one-form symmetry generated by a fermion $\psi$. Since $h_{\psi^p} = ph_\psi$, $\psi^p$ is a fermion if $p$ is odd, and a boson if $p$ is even. This means that $n$ is necessarily even, because $\psi^n = \mathbf{1}$ is a boson.

The braiding phase with respect to $\psi$ is always an $n$-th root of unity, it is the charge with respect to the $\mathbb{Z}_n$ symmetry. This fact allows us to partition the lines of $G_B$ into $n$ equivalence classes according to their $n$-ality $j$, i.e., the value of braiding $B(\alpha,\psi) = e^{2\pi i j/n}$, $j = 0, 1, \dots, n-1$. The lines with $j = 0$ are the NS lines (so that $B(\alpha,\psi) = +1$), and those with $j = n/2$ are the R lines (so that $B(\alpha,\psi) = -1$). The rest of lines are projected out by the $\mathbb{Z}_n$ quotient (unless we turn on a suitable background for the dual $\mathbb{Z}_n$ zero-form symmetry). Furthermore, in each sector the lines are organized into $\mathbb{Z}_n$ orbits,

$$\{\alpha, \psi\alpha, \psi^2\alpha, \dots, \psi^{|\alpha|-1}\alpha\}, \tag{6.4.19}$$

where $|\alpha| \in [1, n]$ denotes the length of the orbit – the minimal integer such that $\psi^{|\alpha|} \times \alpha = \alpha$. An orbit is Majorana if and only if its length is odd, for then and only then it may absorb a fermion. Indeed, the conformal block with puncture $\psi^{|\alpha|}$ is non-vanishing only if $\psi^{|\alpha|} \times \alpha = \alpha$. In conclusion, the fermionic states in the R-R sector of the quotient theory $G_F$ correspond to the orbits of $G_B$ R-lines with an odd number of elements, as claimed.

We are now in position to study the theory $\mathrm{U}(n)_{k,n+k}$. The discussion above has taught us how to identify fermionic states in the Hilbert space of the theory. Rather anticlimactically, we shall now argue that this theory has, in fact, no fermionic states at all! This means that the trace $\mathrm{tr}(-1)^F$ actually just counts the number of states of the theory, much like in a bosonic theory. This explains why the counting of states in [46] matched the domain wall index – because all states are bosonic. This, importantly, is not always the case for other spin TQFTs, such as $\mathrm{O}(n)$ (see below, section 6.4.3).

Let us prove that the theory has no fermionic states. $\mathrm{U}(n)_{k,n+k}$ is level-rank dual to $\mathrm{U}(k)_{-n,-(n+k)}$ as a spin TQFT. Therefore, if either $k$ or $n$ is odd, the theory factorizes as a bosonic theory times an invertible spin TQFT, and so the theory clearly has no Majorana states. The only non-trivial case is, therefore, that of $n, k$ both even, which we assume in what follows.

The theory in the numerator of the quotient description of $\mathrm{U}(n)_{k,n+k}$ in (6.4.17) is bosonic (recall that $\mathrm{U}(1)_K$ is spin for $K$ odd and bosonic for $K$ even; here $K = n(n+k)$, which is even). The states of $\mathrm{U}(n)_{k,n+k}$ are $\mathbb{Z}_n$ orbits of $\mathrm{SU}(n)_k \times \mathrm{U}(1)_{n(n+k)}$ representations. If we manage to prove that there are no orbits of odd length, we succeed in proving that the theory has no Majorana states. In fact, we show that, more generally, all orbits have length-$n$, i.e., all orbits are long. This implies that the states correspond to conformal blocks with no punctures, i.e., all states are bosonic, $(-1)^F \equiv +1$.

Write $\alpha = (R, q)$, where $R$ is an $\mathrm{SU}(n)_k$ representation, and $q \in [0, n(n+k))$ labels a $\mathrm{U}(1)_{n(n+k)}$ representation. The abelian part of the condition $\psi^{|\alpha|} \times \alpha = \alpha$ reads

$$q + |\alpha|(n+k) = q \mod n(n+k)\,, \tag{6.4.20}$$

which can be written as $|\alpha| = 0 \mod n$, i.e., $|\alpha| = n$, as claimed. This proves that all orbits are long, which indeed implies the absence of Majorana lines.

Let us now use this information to compute the different $\mathrm{U}(n)_{k,n+k}$ partition functions. The untwisted trace is the number of conformal blocks (in any of the $2^3$ spin structures). Counting this is a straightforward exercise in combinatorics: we have a factor of $n(n+k)$ due to $\mathrm{U}(1)_{n(n+k)}$, times a factor of $\binom{n+k-1}{k}$ due to $\mathrm{SU}(n)_k$ (cf. (6.6.2)), and a factor of $1/n^2$ due to the quotient $\mathbb{Z}_n$ (one factor of $n$ is due to the projecting out of lines, and the other one because the neutral lines are organized into length-$n$ orbits). All in all, the number of states – the untwisted trace – is

$$\mathrm{tr}_{\mathrm{U}(n)_{k,n+k}}(\mathbf{1}) = \frac{n(n+k)}{n^2}\binom{n+k-1}{k} \equiv \binom{n+k}{k}. \tag{6.4.21}$$

This standard argument was already used in [46]. An important aspect of this computation, much overlooked in the literature, is that this equals $\mathrm{tr}(-1)^F$ only because all the states have trivial fermion parity, which is nontrivially true in this theory. This shall not be the case in the orthogonal group $\mathrm{O}(n)$, where $\mathrm{tr}(-1)^F$ does not just count the total number of states, but rather the bosons minus the fermions, both sets being typically nonempty.

Recall that the conjectured infrared TQFT corresponding to the $n$-domain wall of $\mathrm{SU}(N)$ is $\mathrm{W}_n = \mathrm{U}(n)_{k,n+k}$, with $k = N-n$. Using this value of the level in (6.4.21) indeed reproduces the Witten index computed in the ultraviolet, cf. (6.2.20).

We now proceed to computing the trace twisted by the charge conjugation symmetry $\mathsf{c}$ of $\mathrm{U}(n)_{k,n+k}$. Consider first the case of odd $k$, where the theory is naturally bosonic. In this case, computing the trace amounts to counting the real representations of $\mathrm{U}(n)_{k,n+k}$. A representation of $\mathrm{U}(n)_{k,n+k}$ can be labeled by the pair $(R,q)$, where $R$ is an $\mathrm{SU}(n)_k$ representation, and $q \in [0, n(n+k))$, subject to $|R| = q \mod n$, where $|R|$ denotes the number of boxes in the Young diagram of $R$. Representations $(R,q)$ and $(\sigma^\ell \cdot R, q+\ell(n+k))$, with $\sigma^\ell \cdot R$ the $\mathrm{SU}(n)$ representation with Dynkin labels $(\sigma^\ell \cdot \lambda)_i = \lambda_{i-\ell \mod n}$, are identified by $\mathbb{Z}_n$ spectral flow.

The abelian charge $q$ is correlated with the $\mathrm{SU}(n)$ representation. Indeed, if $n$ is even and $R$ is real modulo $\sigma^\ell$, there is a single charge $q \in [0, n(n+k))$ that makes $(R,q)$ real; if $n$ is odd, there are two such charges.[113] Therefore, the number of real representations in $\mathrm{U}(n)_{k,n+k}$ is the number of representations of $\mathrm{SU}(n)_k$ that are real up to the action of $\sigma$,

[113] Namely, we are looking for solutions to $2q = \ell(n+k) \mod n(n+k)$. These are $q = \frac{\ell}{2}(n+k)$ and $q = \frac{\ell+n}{2}(n+k)$ (except for $(n,\ell) = (\text{even}, \text{odd})$, where there is no solution; see also (6.4.23)). For $n$ even, only one of these two solutions is valid, depending on the parity of $|R|$ (recall that we require $q = |R| \mod n$).

divided by $n$ (the length of the orbits), and multiplied by 2 if $n$ is odd. Let us now count the $\mathrm{SU}(n)_k$ representations.

An $\mathrm{SU}(n)_k$ representation $[\lambda_0, \lambda_1, \ldots, \lambda_{n-1}]$ is real up to the action of $\sigma$ if $\lambda_i = \lambda_{\ell-i \mod n}$ for some $\ell$. The number of such representations is the number of integer solutions to

$$\begin{aligned} 2\lambda_0 + 2\lambda_1 + \cdots + 2\lambda_{\lfloor(\ell-1)/2\rfloor} + \left\{ \begin{array}{cc} 0 & \ell \text{ odd} \\ \lambda_{\ell/2} & \ell \text{ even} \end{array} \right\} + \\ + 2\lambda_{\ell+1} + \cdots + 2\lambda_{\lfloor(n+\ell-1)/2\rfloor} + \left\{ \begin{array}{cc} 0 & n+\ell \text{ odd} \\ \lambda_{(n+\ell)/2} & n+\ell \text{ even} \end{array} \right\} = k\,. \end{aligned} \tag{6.4.22}$$

The number of solutions to this equation is

$$N_\ell = \begin{cases} \binom{(n+k)/2-1}{(k-1)/2} & n \text{ odd, } k \text{ odd,} \\ 0 & n \text{ even, } \ell \text{ odd, } k \text{ odd,} \\ 2\binom{(n+k-1)/2}{(k-1)/2} & n \text{ even, } \ell \text{ even, } k \text{ odd,} \\ \binom{(n+k)/2}{k/2} + \binom{(n+k)/2-1}{k/2-1} & n \text{ even, } \ell \text{ even, } k \text{ even,} \\ \binom{(n+k)/2-1}{k/2} & n \text{ even, } \ell \text{ odd, } k \text{ even,} \end{cases} \tag{6.4.23}$$

where, for future reference, we have also included the case of $k$ even.

We now sum over all $\ell = 0, 1, \ldots, n-1$. For $n$ odd, this just multiplies $\binom{(n+k)/2-1}{(k-1)/2}$ by $n$. If $n$ is even, it multiplies $2\binom{(n+k-1)/2}{(k-1)/2}$ by $n/2$, because half the cases yield no solutions. Next, we divide by $n$ (due to the quotient), and multiply by 2 if $n$ is odd. This yields the number of real representations of $\mathrm{U}(n)_{k,n+k}$ with $k$ odd as[114]

$$\mathrm{tr}_{\mathrm{U}(n)_{k,n+k}}(\mathsf{c}) = \begin{cases} 2\binom{(n+k)/2-1}{(k-1)/2} & n \text{ odd, } k \text{ odd,} \\ \binom{(n+k-1)/2}{(k-1)/2} & n \text{ even, } k \text{ odd.} \end{cases} \tag{6.4.24}$$

By plugging $k = N - n$ in (6.4.24), the partition function reproduces the Witten index twisted by charge conjugation computed in the ultraviolet, cf. (6.2.24).

The case of $k$ even is slightly more complicated because the theory is naturally spin. For $n$ odd we can obtain the twisted trace from the $k$ odd case by using level-rank duality

[114]Note that the expression for $n$ odd is invariant under $n \leftrightarrow k$, as required by level-rank duality.

$\mathrm{U}(n)_{k,n+k} \leftrightarrow \mathrm{U}(k)_{-n,-(n+k)}$. But for $n, k$ both even, the theory is spin, and cannot be written as a bosonic theory times a trivial spin theory – at least not using the standard level-rank duality. Thus, we have to explicitly compute the trace of $\mathsf{c}$ in the R-R sector. This is non-trivial because, among other things, $\mathsf{c}$ may act as $-1$ on some states (see the discussion around (6.3.27)), and thus it is not enough to just count real representations.

A shortcut to compute the trace of $\mathsf{c}$ over the odd spin structure, for $k, n$ both even, is to sum over all spin structures:

$$\frac{1}{2}\left(\sum_{\sigma} \mathrm{tr}_{\sigma}\,\mathsf{c} + \mathrm{tr}_{\sigma}(-1)^F \mathsf{c}\right) = \mathrm{tr}_B\,\mathsf{c}\,, \tag{6.4.25}$$

where $\mathrm{tr}_B$ denotes the trace over the bosonic parent. From this expression, and noting that $(-1)^F$ is trivial in $\mathrm{U}(n)_{k,n+k}$ theories (due to the lack of Majorana lines), we can solve for the trace we are after:

$$\mathrm{tr}_{\text{R-R}}\,\mathsf{c} = \mathrm{tr}_B\,\mathsf{c} - 3\,\mathrm{tr}_{\text{NS-NS}}\,\mathsf{c}\,. \tag{6.4.26}$$

Let us begin with the first term. As this is a trace over a bosonic Hilbert space, we are just to count real representations of $\mathrm{SU}(n)_k \times U(1)_{n(n+k)}$. The first factor corresponds to $\ell = 0$ in (6.4.23), while the second factor has two real representations (namely, $q = 0$ and $q = n(n+k)/2$). The end result is

$$\mathrm{tr}_B\,\mathsf{c} = 2\left[\binom{(n+k)/2}{k/2} + \binom{(n+k)/2-1}{k/2-1}\right]. \tag{6.4.27}$$

Let us now compute the second term in (6.4.26). This is a trace over a fermionic Hilbert space, but over the NS-NS sector, and so we only have to count fixed-points, as they all contribute with $\mathsf{c} = +1$. In other words, the trace is just the number of real representations of $\mathrm{U}(n)_{k,n+k}$, that is, the number of solutions to (6.4.22), summed over $\ell = 0, 1, \ldots, n-1$, and divided by $n$ due to the quotient. Using (6.4.23), we get

$$\begin{aligned} \mathrm{tr}_{\text{NS-NS}}\,\mathsf{c} &\equiv \frac{1}{n}\left(\frac{n}{2}\left[\binom{(n+k)/2}{k/2} + \binom{(n+k)/2-1}{k/2-1}\right] + \frac{n}{2}\binom{(n+k)/2-1}{k/2}\right) \\ &= \frac{1}{2}\binom{(n+k)/2}{k/2} + \frac{1}{2}\binom{(n+k)/2-1}{k/2-1} + \frac{1}{2}\binom{(n+k)/2-1}{k/2}\,. \end{aligned} \tag{6.4.28}$$

Plugging these two traces into (6.4.26), the twisted index, for $n, k$ even, becomes

$$\mathrm{tr}_{\mathrm{U}(n)_{k,n+k}}(\mathsf{c}) = \binom{(n+k)/2-1}{n/2} - \binom{(n+k)/2-1}{k/2}\,. \tag{6.4.29}$$

Note that this is invariant under $n \leftrightarrow k$, as required by level-rank duality.[115] This expression for the twisted partition function of $\mathrm{U}(n)_{k,n+k}$ with $k = N - n$ matches the twisted Witten index computed in the ultraviolet, cf. (6.2.24).

[115] That is, invariant up to a sign. This is due to the fact that $\mathsf{c} = -1$ in the R-R sector of $\mathrm{U}(1)_1$ (cf. (6.3.27)), and the level-rank pair has a difference in their framing anomaly equal to $nk + 1 \equiv 1 \mod 2$, cf. [28].

Finally, we briefly sketch the computation of the trace twisted by the one-form symmetry $\mathsf{g}^t \in \mathbb{Z}_{n+k}$ of $\mathrm{U}(n)_{k,n+k}$, where $\mathsf{g}$ denotes a primitive root of unity, and $t \in [0, n+k)$. The states of $\mathrm{U}(n)_{k,n+k}$ are orbits of the form

$$\{(\sigma^\ell \cdot R, q + \ell(n+k))\}\,, \tag{6.4.30}$$

where $\ell$ ranges from 0 to $n-1$. All the orbits are of length-$n$. The theory has a $\mathbb{Z}_{n+k}$ one-form symmetry that acts as $q \mapsto q + tn$, where $t \in [0, n+k)$. A state is invariant if and only if this transformation cyclically permutes the elements of the orbit, i.e., if a representative $(R, q)$ is mapped into itself up to spectral flow,

$$(R, q + tn) \equiv (\sigma^\ell \cdot R, q + \ell(n+k))\,. \tag{6.4.31}$$

It is clear that if $tn$ is not of the form $\ell(n+k)$ for some $\ell \in \mathbb{Z}$, then no state is invariant, and the twisted trace vanishes. So let us assume that such an $\ell$ exists; it is clear that it is unique, so counting invariant orbits reduces to counting appropriate $\mathrm{SU}(n)_k$ representations. More specifically, the number of invariant states is

$$\mathrm{tr}_{\mathrm{U}(n)_{k,n+k}}(\mathsf{g}^t) = \frac{n(n+k)}{n^2} \hat{N}_\ell\,, \tag{6.4.32}$$

where $n(n+k)$ denotes the number of states in $\mathrm{U}(1)_{n(n+k)}$, and the factor of $n^2$ is due to the $\mathbb{Z}_n$ quotient. $\hat{N}_\ell$ denotes the number of $\mathrm{SU}(n)_k$ representations that satisfy $R = \sigma^\ell \cdot R$, with $\ell := tn/(n+k) \in \mathbb{Z}$.

Counting such $\mathrm{SU}(n)_k$ representations is easy, because this is a simply-connected group, so the states are labelled by the Dynkin labels, $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$, which can be thought of a collection of independent bosons (cf. (6.4.11)). The most efficient way to count the representations that are invariant under $\sigma^\ell$ is to recall that the associated diagonal phase is just the charge under the center (6.2.18), which is a multiplicative phase, so the partition function factorizes:

$$\prod_{j=0}^{n-1} (1 - e^{2\pi i j\ell/n} q)^{-1} \equiv \sum_{k\geq 0} \hat{N}_\ell q^k\,, \tag{6.4.33}$$

and therefore

$$\hat{N}_\ell = \binom{n+k-1}{k}_{e^{2\pi i\ell/n}}\,. \tag{6.4.34}$$

Recall that the $q$-binomial coefficient at a root of unity can be expressed as a regular binomial coefficient, cf. (6.2.29) and (6.2.27).

Putting everything together, the one-form symmetry twisted trace reads

$$\begin{aligned}\mathrm{tr}_{\mathrm{U}(n)_{k,n+k}}(\mathsf{g}^t) &= \binom{n+k}{n}_{\mathsf{g}^t} \\ &\equiv \begin{cases} \begin{pmatrix} \gcd(n+k,t) \\ n\gcd(n+k,t)/(n+k)\end{pmatrix} & tn\equiv 0 \mod n+k, \\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{6.4.35}$$

It is easily checked that, if we plug $k = N - n$ in (6.4.35), the twisted trace for $\mathrm{U}(n)_{k,n+k}$ exactly reproduces the twisted Witten index of the $n$-domain wall of $\mathrm{SU}(N)$ computed in the ultraviolet, cf. (6.2.27).

### 6.4.3 $G = \mathrm{O}(n)$

The $n$-domain wall theory for $4d$ $\mathcal{N} = 1$ SYM with $G = \mathrm{Spin}(N)$ is proposed to be $\mathrm{O}(n)^1_{N-2-n,N-n+1}$ Chern-Simons theory. Let us proceed to study the partition functions of $\mathrm{O}(n)^1_{k,L}$.

The $\mathrm{O}(n)^1_{k,L}$ Chern-Simons theory is defined as [113]

$$\mathrm{O}(n)^1_{k,L} := \frac{\mathrm{O}(n)^1_{k,0}\times(\mathbb{Z}_2)_L}{\mathbb{Z}_2}\,, \tag{6.4.36}$$

where $(\mathbb{Z}_2)_L \leftrightarrow \mathrm{Spin}(L)_{-1}$ denotes a $\mathbb{Z}_2$ gauge theory with twist $L$, and the quotient denotes the gauging of a diagonal $\mathbb{Z}_2$ one-form symmetry. The value of the level we shall be interested in is $L = k + 3$. On the other hand, the first factor is given by the following:

- If $n$ is even, the theory $\mathrm{O}(n)^1_{k,0}$ is defined as the $\mathsf{CM}$-orbifold of $\mathrm{SO}(n)_k$. Here $\mathsf{C}$ denotes the charge-conjugation $\mathbb{Z}_2$ zero-form symmetry that acts by permuting the last two Dynkin labels in $\mathrm{SO}(n)$, and $\mathsf{M}$ is the magnetic $\mathbb{Z}_2$ zero-form symmetry that is dual to the gauged one-form $\mathbb{Z}_2$ symmetry in the denominator of $\mathrm{SO}(n)_k \equiv \mathrm{Spin}(n)_k/\mathbb{Z}_2$. As such, it permutes the lines that split in the quotient, i.e., the lines of $\mathrm{Spin}(n)_k$ that are fixed by fusion with the extending simple current.

- If $n$ is odd, the group $\mathrm{O}(n)$ is a direct product of $\mathrm{SO}(n)$ and $\mathbb{Z}_2$. The Chern-Simons theory $\mathrm{O}(n)^1_{k,0}$ itself does not necessarily factorize, because of the convention of which $\mathbb{Z}_2$ subgroup the reflection represents. The choice in [113] was

$$\mathrm{O}(n)^1_{k,0} := \begin{cases} \dfrac{\mathrm{Spin}(n)_k\times(\mathbb{Z}_2)_{(k-2)(n-1)}}{\mathbb{Z}_2} & n \text{ odd, } k \text{ even,} \\ \mathrm{SO}(n)_k\times(\mathbb{Z}_2)_{(k-2)(n-1)} & n \text{ odd, } k \text{ odd.}\end{cases} \tag{6.4.37}$$

Let us compute the different traces in this theory. As above, the details depend sensitively on the parity of $n$ and $k$, so we consider each case separately.

**Even/Even.** We begin with the theory $\mathrm{Spin}(2n)_{2k}$. Its integrable representations satisfy

$$\lambda_0+\lambda_1+2(\lambda_2+\cdots+\lambda_{n-2})+\lambda_{n-1}+\lambda_n=2k\,, \tag{6.4.38}$$

which has $\mathrm{tr}_{\mathrm{Spin}(2n)_{2k}}(\mathbf{1})$ solutions (cf. (6.6.12)).

We now construct $\mathrm{SO}(2n)_{2k}$, i.e., we gauge a $\mathbb{Z}_2$ one-form symmetry, which acts as $\lambda_0\leftrightarrow\lambda_1$ and $\lambda_{n-1}\leftrightarrow\lambda_n$. This is a bosonic quotient. The neutral representations satisfy $\lambda_{n-1}+\lambda_n=\text{even}$, which has

$$\mathbb{N}:=\binom{n+k}{k}+2\binom{n+k-1}{k-1}+\binom{n+k-2}{k-2}\,, \tag{6.4.39}$$

solutions. These are divided into length-2 orbits, and fixed points. The former satisfy $\lambda_0\neq\lambda_1\vee\lambda_{n-1}\neq\lambda_n$, and the latter $\lambda_0=\lambda_1\wedge\lambda_{n-1}=\lambda_n$. The number of fixed points is

$$\mathbb{F}:=\binom{n+k-2}{k}\,, \tag{6.4.40}$$

and the number of length-2 orbits is $\frac{1}{2}(\mathbb{N}-\mathbb{F})$. Finally, the number of representations of $\mathrm{SO}(2n)_{2k}$ is

$$\mathrm{tr}_{\mathrm{SO}(2n)_{2k}}(\mathbf{1})=2\mathbb{F}+\frac{1}{2}(\mathbb{N}-\mathbb{F})\equiv 2\binom{n+k-2}{k}+\binom{n+k-2}{k-1}+(n\leftrightarrow k)\,, \tag{6.4.41}$$

which is invariant under $n\leftrightarrow k$, as expected by level-rank duality. This also agrees with expression (6.6.15).

We now orbifold by $\mathsf{CM}$, which acts by swapping the lines in $2\mathbb{F}$ pairwise, and as $\lambda_{n-1}\leftrightarrow\lambda_n$. The representations that are fixed under $\mathsf{CM}$ are the subset of the length-2 orbits that satisfy either $\lambda_0\neq\lambda_1\wedge\lambda_{n-1}=\lambda_n$ or $\lambda_0=\lambda_1\wedge\lambda_{n-1}\neq\lambda_n$. In other words, the lines that satisfy either of

$$\begin{aligned}\lambda_0+\lambda_1+2(\lambda_2+\cdots+\lambda_{n-2}+\lambda_{n-1})&=2k, &\qquad \lambda_0&\neq\lambda_1\,,\\ 2(\lambda_1+\lambda_2+\cdots+\lambda_{n-2})+\lambda_{n-1}+\lambda_n&=2k, &\qquad \lambda_{n-1}&\neq\lambda_n\,.\end{aligned} \tag{6.4.42}$$

By symmetry, both conditions have the same number of solutions. In total,

$$\mathbb{A}:=2\left[\binom{n+k-1}{k}+\binom{n+k-2}{k-1}-\binom{n+k-2}{k}\right] \tag{6.4.43}$$

solutions. Note that these are length-2 orbits of $\mathrm{Spin}(2n)_{2k}$, so the number of lines is $\mathbb{A}/2$.

The representations that are interchanged under $\mathsf{CM}$ are all of $\mathbb{F}$, plus the subset of the length-2 orbits that satisfy $\lambda_{n-1}+\lambda_n=\text{even}$ and $\lambda_{n-1}\neq\lambda_n$, minus the solutions to the second line in (6.4.42). The latter are

$$\begin{aligned}\mathbb{B}:=&\binom{n+k}{k}+2\binom{n+k-1}{k-1}+\binom{n+k-2}{k-2}\\ &-\binom{n+k-1}{k}-\binom{n+k-2}{k-1}-\frac{1}{2}\mathbb{A}\\ =&\,4\binom{n+k-2}{k-2}\,.\end{aligned} \tag{6.4.44}$$

Note that $\mathbb{F}$ are fixed points of $\mathbb{Z}_2$, while $\mathbb{B}$ are length-2 orbits, so the number of lines is $2\mathbb{F}+\mathbb{B}/2$. Adding the lines in $\mathbb{A}$ we get $\frac{1}{2}(\mathbb{A}+\mathbb{B})+2\mathbb{F}\equiv \mathrm{tr}_{\mathrm{SO}(2n)_{2k}}(\mathbf{1})$, as one would expect.

Putting all these results together, we see that the number of twisted and untwisted lines in the orbifold is [320]

$$\begin{aligned} N_{\text{twisted}} &= \mathbb{A}\,, \\ N_{\text{untwisted}} &= \mathbb{A}+\frac{1}{4}\mathbb{B}+\mathbb{F}\,, \end{aligned} \tag{6.4.45}$$

and so the theory has

$$\begin{aligned} \mathrm{tr}_{\mathrm{O}(2n)^1_{2k,0}}(\mathbf{1}) &= N_{\text{twisted}}+N_{\text{untwisted}} \\ &= -\frac{9}{8}\binom{n+k-2}{k}+4\binom{n+k-2}{k-1}+\frac{17}{8}\binom{n+k-2}{k-2}+(n\leftrightarrow k) \end{aligned} \tag{6.4.46}$$

lines. This expression agrees with [113].

We now move on to $\mathrm{O}(2n)^1_{2k,2k+3}$. This is obtained by taking the theory we just constructed, $\mathrm{O}(2n)^1_{2k,0}$, tensoring with $\mathrm{Spin}(2k+3)_{-1}$, and gauging a diagonal $\mathbb{Z}_2$ one-form symmetry:

$$\mathrm{O}(2n)^1_{2k,2k+3}=\frac{\mathrm{O}(2n)^1_{2k,0}\times \mathrm{Spin}(2k+3)_{-1}}{\mathbb{Z}_2}\,, \tag{6.4.47}$$

where the quotient is fermionic.

Take the states of $\mathrm{O}(2n)^1_{2k,0}$ as above, i.e., $N_{\text{twisted}}$ and $N_{\text{untwisted}}$, and tensor by $\mathrm{Spin}(2k+3)_{-1}=\{\mathbf{1},\sigma,\chi\}$. The NS and R lines are as follows:

$$\begin{aligned} \text{NS}: &\qquad N_{\text{untwisted}}\otimes\mathbf{1}, \quad N_{\text{twisted}}\otimes\sigma, \quad N_{\text{untwisted}}\otimes\chi \\ \text{R}: &\qquad N_{\text{twisted}}\otimes\mathbf{1}, \quad N_{\text{untwisted}}\otimes\sigma, \quad N_{\text{twisted}}\otimes\chi\,. \end{aligned} \tag{6.4.48}$$

We now quotient by the $\mathbb{Z}_2$ one-form symmetry. This symmetry maps $\mathbf{1}\leftrightarrow\chi$, and it fixes $\sigma$; and, also, it permutes lines in $\mathbb{A}$ pairwise, $a\leftrightarrow a'$, and it fixes those in $\frac{1}{4}\mathbb{B}+\mathbb{F}$. Therefore, in the NS sector it acts as

$$\begin{aligned} \mathbb{A}\otimes\mathbf{1} &\leftrightarrow \mathbb{A}'\otimes\chi \\ (\tfrac{1}{4}\mathbb{B}+\mathbb{F})\otimes\mathbf{1} &\leftrightarrow (\tfrac{1}{4}\mathbb{B}+\mathbb{F})\otimes\chi \\ \mathbb{A}\otimes\sigma &\leftrightarrow \mathbb{A}'\otimes\sigma \end{aligned} \tag{6.4.49}$$

which are all length-two orbits (recall that there are never fixed-points in the NS sector). Thus, the dimension of the Hilbert space is

$$\begin{aligned} \dim(\mathrm{O}(2n)^1_{2k,2k+3}) &= \mathbb{A}+(\tfrac{1}{4}\mathbb{B}+\mathbb{F})+\tfrac{1}{2}\mathbb{A} \\ &\equiv \tfrac{1}{2}N_{\text{twisted}}+N_{\text{untwisted}}\,. \end{aligned} \tag{6.4.50}$$

This corresponds to the trace of $\mathbf{1}$ over the Hilbert space on any of the spatial spin structures.

Consider now the R sector. The one-form symmetry acts as

$$\begin{aligned} \mathbb{A}\otimes\mathbf{1} &\leftrightarrow \mathbb{A}'\otimes\chi \\ (\tfrac{1}{4}\mathbb{B}+\mathbb{F})\otimes\sigma &\leftrightarrow (\tfrac{1}{4}\mathbb{B}+\mathbb{F})\otimes\sigma \\ \mathbb{A}\otimes\sigma &\leftrightarrow \mathbb{A}'\otimes\sigma\,, \end{aligned} \tag{6.4.51}$$

and so all of $\mathbb{A}$ are in length-two orbits, while all of $\frac{1}{4}\mathbb{B}+\mathbb{F}$ are fixed-points. Thus, the number of fermions and bosons is

$$\begin{aligned} N_{\text{boson}} &= \mathbb{A}+\tfrac{1}{2}\mathbb{A}\equiv\tfrac{3}{2}N_{\text{twisted}}\,, \\ N_{\text{fermion}} &= \tfrac{1}{4}\mathbb{B}+\mathbb{F}\equiv N_{\text{untwisted}}-N_{\text{twisted}}\,. \end{aligned} \tag{6.4.52}$$

Note that $N_{\text{boson}}+N_{\text{fermion}}$ agrees with the dimension of the Hilbert space as computed in the NS sector (cf. (6.4.50)). On the other hand, the trace in the odd spin structure, weighted by fermion parity, is $N_{\text{boson}}-N_{\text{fermion}}$:

$$\operatorname{tr}_{\mathrm{O}(2n)^1_{2k,2k+3}}(-1)^F\equiv\tfrac{5}{2}N_{\text{twisted}}-N_{\text{untwisted}}\,. \tag{6.4.53}$$

As a consistency check, recall that one can also express the fermionic trace as $\operatorname{tr}_{\mathcal{H}_{\text{R-R}}}(-1)^F=2\dim(\mathcal{H}_B)-7\dim(\mathcal{H}_F)$ (cf. (6.3.20)). The dimension of the bosonic Hilbert space is

$$\dim(\mathrm{O}(2n)^1_{2k,0}\times\mathrm{Spin}(2k+3)_{-1})\equiv 3(N_{\text{untwisted}}+N_{\text{twisted}})\,, \tag{6.4.54}$$

while the dimension of the fermionic Hilbert space is half the number of lines, i.e., $\frac{1}{2}(2N_{\text{untwisted}}+N_{\text{twisted}})$. Thus,

$$\operatorname{tr}_{\mathrm{O}(2n)^1_{2k,2k+3}}(-1)^F=6N_{\text{twisted}}+6N_{\text{untwisted}}-\tfrac{7}{2}N_{\text{twisted}}-7N_{\text{untwisted}}\,, \tag{6.4.55}$$

which indeed matches the expression above.

Recall that the conjectured infrared theory corresponding to the $n$-domain wall of $\mathrm{Spin}(N)$ was $\mathrm{W}_n=\mathrm{O}(n)^1_{k,k+3}$, with $k=N-2-n$. Using this value of the level in (6.4.53) indeed reproduces the Witten indices computed in the ultraviolet, cf. (6.2.46).

**Odd/Odd.** We consider

$$\mathrm{O}(2n+1)^1_{2k+1,2k+4}=\mathrm{SO}(2n+1)_{2k+1}\times(\mathbb{Z}_2)_{2(n+k)}\,. \tag{6.4.56}$$

As the theory is a tensor product, the traces factorize:

$$\operatorname{tr}_{\mathrm{O}(2n+1)^1_{2k+1,2k+4}}(\mathcal{O}_1\otimes\mathcal{O}_2)\equiv\operatorname{tr}_{\mathrm{SO}(2n+1)_{2k+1}}(\mathcal{O}_1)\cdot\operatorname{tr}_{(\mathbb{Z}_2)_{2(n+k)}}(\mathcal{O}_2)\,. \tag{6.4.57}$$

For example, the $\mathbb{Z}_2$ gauge theory has four states, all bosonic, $\operatorname{tr}_{(\mathbb{Z}_2)_{2(n+k)}}(-1)^F\equiv 4$, which means that the untwisted index is

$$\operatorname{tr}_{\mathrm{O}(2n+1)^1_{2k+1,2k+4}}(-1)^F=4\left[\binom{n+k}{k}-2\binom{n+k-1}{k}\right], \tag{6.4.58}$$

where we have used the trace of $\mathrm{SO}(2n+1)_{2k+1}$ as given in (6.6.10).

Similarly, the index twisted by the zero-form symmetry $\mathsf{c}$ has $\operatorname{tr}_{(\mathbb{Z}_2)_{2(n+k)}}(\mathsf{c}) \equiv 2$, where $\mathsf{c}$ acts by permuting the two spinors (this is the only zero-form symmetry of this $\mathbb{Z}_2$ gauge theory, cf. [4, 321]; it fixes both the identity and the vector). On the other hand, the only zero-form symmetry of $\mathrm{SO}(2n+1)_{2k+1}$ is fermion parity,[116] and there is in fact a natural identification $\mathsf{c} = (-1)^F$ (cf. [113]). Thus, the $\mathsf{c}$-twisted trace weighted by fermion parity actually computes the untwisted trace, with antiperiodic (NS) boundary conditions on the time circle:

$$\operatorname{tr}_{\mathrm{SO}(2n+1)_{2k+1}}((-1)^F\mathsf{c}) \equiv \operatorname{tr}_{\mathrm{SO}(2n+1)_{2k+1}}(\mathbf{1}) \equiv \binom{n+k}{k}, \tag{6.4.59}$$

where we have used (6.6.9). All in all, the twisted trace of $\mathrm{O}(2n+1)^1_{2k+1,2k+4}$ is

$$\operatorname{tr}_{\mathrm{O}(2n+1)^1_{2k+1,2k+4}}((-1)^F\mathsf{c}) = 2\binom{n+k}{k}. \tag{6.4.60}$$

The index twisted by the one-form symmetry is also straightforward. This symmetry is $\mathbb{Z}_2^2$ for $\mathrm{O}(4n+1)_{4k+1}$ and $\mathrm{O}(4n+3)_{4k+3}$, and $\mathbb{Z}_4$ for $\mathrm{O}(4n+1)_{4k+3}$ and $\mathrm{O}(4n+3)_{4k+1}$. These correspond to fusion with the abelian anyons of $\mathrm{Spin}(L)_{-1}$, with $L = 0 \mod 4$ and $L = 2 \mod 4$ respectively, which indeed have a $\mathbb{Z}_2^2/\mathbb{Z}_4$ fusion algebra. As abelian fusion has no fixed-points, all the twisted traces vanish:

$$\begin{aligned} \operatorname{tr}_{\mathrm{O}(2n+1)^1_{2k+1,2k+4}}((-1)^F\mathsf{g}_1\mathsf{g}_2) &\equiv 0\,, \\ \operatorname{tr}_{\mathrm{O}(2n+1)^1_{2k+1,2k+4}}((-1)^F\mathsf{g}) &\equiv 0\,, \end{aligned} \tag{6.4.61}$$

where $(\mathsf{g}_1,\mathsf{g}_2) \in \mathbb{Z}_2^2$ and $\mathsf{g} \in \mathbb{Z}_4$, respectively.

Recall that the conjectured infrared theory corresponding to the $n$-domain wall of $\mathrm{Spin}(N)$ was $\mathrm{W}_n = \mathrm{O}(n)^1_{k,k+3}$, with $k = N-2-n$. Using this value of the level in (6.4.58), (6.4.60), (6.4.61) indeed reproduces the Witten indices computed in the ultraviolet, cf. (6.2.46), (6.2.49), (6.2.53), (6.2.56).

**Odd/Even & Even/Odd.** We only need to consider one; the other follows by the level-rank duality. Take

$$\mathrm{O}(2n+1)^1_{2k,0} = \frac{\mathrm{Spin}(2n+1)_{2k} \times (\mathbb{Z}_2)_{4n(k-1)}}{\mathbb{Z}_2}, \tag{6.4.62}$$

where the gauged one-form symmetry is generated by $a \otimes \mathsf{e}$, where $a = [0, 2k, 0, \ldots, 0]$ and $\mathsf{e}$ is the electric line of the toric code. This is a bosonic quotient.

[116]The Dynkin diagram of $\mathrm{SO}(N)$ for $N$ odd has no reflection symmetries, i.e., its outer automorphism group is trivial. Thus, the zero-form symmetries of $\mathrm{SO}(N)$, if any, must be due to the global structure of the group, as its algebra has no symmetries. Indeed, the zero-form symmetry comes from $\pi_1(\mathrm{SO}(N)) = \mathbb{Z}_2$, but this is just the magnetic dual to the gauged $\mathbb{Z}_2$ one-form symmetry, which means that the magnetic symmetry is formally just $(-1)^F$. If we were to gauge this symmetry, we would recover $\mathrm{Spin}(N)$.

The one-form symmetry acts as $\lambda_0 \leftrightarrow \lambda_1$ and $\mathsf{e}\colon \mathsf{m} \leftrightarrow \mathsf{em}$. The neutral lines are of the form

$$\begin{aligned} &\lambda\otimes\mathbf{1}, \quad \lambda\otimes\mathsf{e}, \qquad \lambda_n = \text{even} \\ &\lambda\otimes\mathsf{m}, \quad \lambda\otimes\mathsf{em}, \qquad \lambda_n = \text{odd}\,. \end{aligned} \tag{6.4.63}$$

Note that there are no fixed points, and all orbits are of length 2:

$$\{\lambda\otimes\mathbf{1}, \quad (a\times\lambda)\otimes\mathsf{e}\}, \qquad \{\lambda\otimes\mathsf{m}, \quad (a\times\lambda)\otimes\mathsf{em}\}\,. \tag{6.4.64}$$

Therefore, a set of representatives can be taken as $\lambda_{\text{tensor}}\otimes\mathbf{1}$ and $\lambda_{\text{spinor}}\otimes\mathsf{m}$. In what follows we drop the second label, as it is correlated with $\lambda$ in a unique way. The number of tensors and spinors is (cf. (6.6.5))

$$N_{\text{tensor}} = \binom{n+k-1}{k-1} + \binom{n+k}{k}, \qquad N_{\text{spinor}} = 2\binom{n+k-1}{k-1}. \tag{6.4.65}$$

We now tensor the theory by a factor of $\mathrm{Spin}(2k+3)_{-1} = \{\mathbf{1},\sigma,\chi\}$, and gauge the fermionic one-form symmetry generated by $f = a\otimes\chi$. The Ramond sector requires $h_{\alpha\times f} = h_\alpha \mod 1$, which means that the lines are

$$(\lambda_{\text{tensor}},\sigma), \qquad (\lambda_{\text{spinor}},\mathbf{1}\text{ or }\chi)\,. \tag{6.4.66}$$

Note that only the former can be a fixed-point under the fermionic quotient, inasmuch as $\chi\times\sigma=\sigma$ while $\chi\colon \mathbf{1}\leftrightarrow\chi$. In particular, the fixed-points are

$$\lambda_{\text{tensor}}, \qquad \lambda_0=\lambda_1\,, \tag{6.4.67}$$

while the rest of lines are all in length-2 orbits. The fixed-points satisfy $\lambda_1+\lambda_2+\cdots+\lambda_{n-1}+\lambda_n/2 = k$, which has

$$\mathbb{F} := \binom{n+k-1}{k}, \tag{6.4.68}$$

solutions. Thus, finally

$$\begin{aligned} \operatorname{tr}_{\mathrm{O}(2n+1)^1_{2k,2k+3}}(-1)^F &= N_{\text{spinor}} + \tfrac12(N_{\text{tensor}}-\mathbb{F}) - \mathbb{F} \\ &= \tfrac52\binom{n+k-1}{k-1} + \tfrac12\binom{n+k}{k} - \tfrac32\binom{n+k-1}{k}. \end{aligned} \tag{6.4.69}$$

The trace over $\mathrm{O}(2n)^1_{2k+1,2k+4}$ can be obtained by using the orthogonal level-rank duality $\mathrm{O}(2n)^1_{2k+1,2k+4} \leftrightarrow \mathrm{O}(2k+1)^1_{-2n,-(2n+3)}$.

One can similarly compute the index twisted by the $\mathbb{Z}_2$ one-form symmetry, which acts via fusion with the electric line $\mathsf{e}$. The charged states are those that include the magnetic line $\mathsf{m}$, to wit, the spinors. In other words, the one-form symmetry correlates (gauge) spin and (spacetime) statistics, so that the states with $(-1)^F\mathsf{e}=+1$ are the tensor bosons and spinor fermions, and states with $(-1)^F \mathsf{e} = -1$ are the spinor bosons and the tensor fermions. With this,

$$\begin{aligned}\operatorname{tr}_{\mathrm{O}(2n+1)^1_{2k,2k+3}}((-1)^F \mathsf{e}) &= -N_{\text{spinor}} + \tfrac{1}{2}(N_{\text{tensor}} - \mathbb{F}) - \mathbb{F} \\ &= -\binom{n+k}{k}.\end{aligned} \tag{6.4.70}$$

As above, the trace for $\mathrm{O}(2n)^1_{2k+1,2k+4}$ is obtained by level-rank duality.

Recall that the conjectured infrared theory corresponding to the $n$-domain wall of $\mathrm{Spin}(N)$ was $\mathrm{W}_n = \mathrm{O}(n)^1_{k,k+3}$, with $k = N - 2 - n$. Using this value of the level in (6.4.69), (6.4.70) indeed reproduces the Witten indices computed in the ultraviolet, cf. (6.2.39), (6.2.42).

### 6.4.4 $G = G_2$

The 2-domain wall theory for $4d$ $\mathcal{N} = 1$ SYM with $G = G_2$ is $\mathrm{SO}(3)_3 \times S^1$, where $S^1$ denotes then nonlinear sigma model with $S^1$ target space. We already proved in section 6.3.1 that the theory $\mathrm{SO}(3)_3$ has vanishing Witten index, and since there is a unique vacuum of the $S^1$ sigma model on the torus, the infrared index vanishes. This matches the Witten index computed in the ultraviolet, which is given by the coefficient of $q^2$ in (6.1.20). Indeed, expanding this polynomial one finds that the index vanishes.

The domain wall with $n = 1$ (and $n = 3$, which is the anti-wall of $n = 1$) is addressed below.

### 6.4.5 Minimal Wall for Arbitrary Gauge Group

The $n = 1$ domain wall theory for $4d$ $\mathcal{N} = 1$ SYM with arbitrary $G$ is proposed to be $G_{-1}$ Chern-Simons theory.

As $G$ is simply-connected, the theory is naturally bosonic, and the trace $\operatorname{tr}_{G_{-1}}(-1)^F$ computes the dimension of the Hilbert space, that is, the number of integrable representations at level 1. In other words, the trace is the number of solutions to (6.4.4) with $k = 1$, namely

$$\sum_{i=0}^{r} \lambda_i a_i^\vee = 1\,, \tag{6.4.71}$$

which, as in (6.2.59), requires $\lambda_i = 1$ for some $i$ with $a_i^\vee = 1$, and $\lambda_j = 0$ for all $j \neq i$. Therefore, the trace is

$$\operatorname{tr}_{G_{-1}}(-1)^F = m_1 \tag{6.4.72}$$

where $m_1$ denotes the number of nodes in the Dynkin diagram of $G$ that have comark equal to 1. This clearly reproduces the ultraviolet index (6.2.60), as required.

For simply-laced $G$, $G_{-1}$ Chern-Simons theory is in fact an abelian TQFT, and all the lines generate one-form symmetries. The number of lines is the number of one-form symmetries, that is, the order of $\Gamma$, which indeed agrees with $m_1$. Equivalently, it is known that simply-laced theories at level 1 admit a $K$-matrix representation, where one can take $K$ as the Cartan matrix of $\mathfrak{g}$. The number of states is indeed $\det(K) \equiv |\Gamma|$.[117]

One can define a zero-form twisted index for $(E_6)_{-1}$. The only node with comark 1 preserved by the charge conjugation symmetry of this theory is the extended node, and thus

$$\mathrm{tr}_{(E_6)_{-1}}(-1)^F = 1\,. \tag{6.4.73}$$

Since $(E_6)_{-1}$ and $(E_7)_{-1}$ are abelian, twisting by a one-form symmetry has no fixed points and

$$\begin{aligned} &\mathrm{tr}_{(E_6)_{-1}}((-1)^F \mathsf{g}) = 0 \qquad \mathsf{g} \in \Gamma = \mathbb{Z}_3 \\ &\mathrm{tr}_{(E_7)_{-1}}((-1)^F \mathsf{g}) = 0 \qquad \mathsf{g} \in \Gamma = \mathbb{Z}_2\,. \end{aligned} \tag{6.4.74}$$

These reproduce the twisted Witten indices on the walls (6.2.63).

## 6.5 Concluding Remarks and Open Questions

In this paper we have proposed explicit $3d$ topological field theories on the domain walls of $4d$ $\mathcal{N}=1$ SYM with gauge group $G$. We have found precise agreement between computations carried out in terms of the ultraviolet $4d$ degrees of freedom (gluons and gluinos) and the conjectured infrared topological $3d$ degrees of freedom. We have highlighted the importance in identifying the infrared of the domain wall theories of studying the Hilbert space of spin TQFTs, in particular the partition function in the R-R sector and identifying the fermionic states in the Hilbert space, as opposed to merely counting states. The nontrivial matching of the twisted Witten indices provides strong support for our proposal.

A heuristic argument can be made in favor of our proposal that the $n$-domain wall in $4d$ $\mathcal{N}=1$ SYM with gauge group $G$ is the infrared of $3d$ $\mathcal{N}=1$ $G_{h/2-n}$ SYM (see equation (6.1.3)).[118] Consider $4d$ SYM on $\mathbb{R}^3\times S^1$ with the YM $\theta$-angle linear in the $S^1$ coordinate and winding number $n$ around the circle. This theory can be defined while preserving half of the supersymmetry.[119] When the radius of the circle is large one can expect the theory to be gapped everywhere except at the location of the wall $\mathrm{W}_n$. For small radius, the theory reduces to $3d$ $\mathcal{N}=1$ $G_{-n}$ SYM with an adjoint real multiplet (the scalar is compact, as it arises from reducing the gauge field along a circle). It was argued in [127] that with a suitable superpotential for the real multiplet, the multiplet gaps out and flows to

[117] Note that abelian systems typically have a very large number of zero-form symmetries [4], most of which are emergent in our picture, inasmuch as the ultraviolet theory only has $\mathsf{C}$ as its zero-form symmetry group.

[118] We would like to thank D. Gaiotto for an interesting discussion regarding this point.

[119] The Lagrangian of this theory can be written as $\mathcal{L}=\int d^2\vartheta\ XW_\alpha W^\alpha$, where $X$ is a background chiral multiplet and $W_\alpha$ the chiral gauge field strength. $\mathrm{Re}(X)$ determines the gauge coupling and $\mathrm{Im}(X)$ the $\theta$-angle. The background $\mathrm{Im}(X)\propto nx_3$ with $F_X\propto in$ preserves half of the supersymmetries. The background for $F_X$ induces a mass term for the gaugino $\propto in\bar\lambda\gamma_{(5)}\gamma^3\lambda$, where $\lambda$ is Majorana.

$3d$ $\mathcal{N}=1$ $G_{h/2-n}$ SYM, where the shift is induced by integrating out the massive fermion in the real multiplet. Assuming that there is no phase transition as the size of the circle is reduced leads to the proposal (6.1.3). However, the lack of control over the superpotential upon reduction makes the argument suggestive but heuristic.

The $n>1$ domain wall theories for the groups $G=F_4, E_6, E_7$ and $E_8$ remain to be discovered. Equivalently, the phase diagram of the corresponding $3d$ $\mathcal{N}=1$ $G_k$ SYM with $k<h/2-1$ remains elusive. We collect in section 6.7 the twisted partition functions computed in the ultraviolet for future reference. One strategy towards the identification of the infrared domain wall theory is to search for novel level-rank dualities in $G_k$ Chern-Simons theories that go beyond the ones that follow from conformal embeddings. In general, level-rank dualities follow from embeddings into holomorphic theories (theories with only one state), and this approach could lead to suitable level-rank dualities and in turn to explicit proposals for the remaining $3d$ $\mathcal{N}=1$ $G_k$ SYM phase diagrams (and associated $4d$ domain walls).

In this paper we have made an intriguing connection between the Hilbert space of Chern-Simons theories on the torus and the Hilbert space of fermions in $0+1$ dimensions labeled by the extended Dynkin diagram $\mathfrak{g}^{(1)}$ corresponding to a Lie group $G$. That is, the fermionic Hilbert space $\mathcal{H}_{\mathrm{F}}^{n}$ with energy $n$ is isomorphic as super-vector spaces to the R-R Hilbert space of a suitable spin TQFT, which we denote by $\mathrm{TQFT}_n$

$$\mathcal{H}_{\mathrm{F}}^{n} \simeq \mathcal{H}_{\text{R-R}}^{\mathrm{TQFT}_n} \,. \tag{6.5.1}$$

Consequently, the partition functions with periodic and antiperiodic boundary conditions on the time circle also match. Specifically we have established the correspondence (the $A_{N-1}^{(1)}$ case was studied by Douglas in [322])[120]

$$\begin{aligned}
A_{N-1}^{(1)} &\longleftrightarrow \mathrm{U}(n)_{N-n,N}\\
B_{N}^{(1)} &\longleftrightarrow \mathrm{O}(n)^1_{2N-1-n,2N-n+2}\\
C_{N}^{(1)} &\longleftrightarrow \mathrm{Sp}(n)_{N+1-n}\\
D_{N}^{(1)} &\longleftrightarrow \mathrm{O}(n)^1_{2N-2-n,2N-n+1}\\
G_{2}^{(1)} &\longleftrightarrow \mathrm{U}(2)_{3n,2-n}\,.
\end{aligned} \tag{6.5.2}$$

Another route to constructing the domain walls for $G=F_4, E_6, E_7$ and $E_8$ is to identify the TQFT whose R-R Hilbert space on the torus is that of the collection of free fermions based on the corresponding affine Dynkin diagram $\mathfrak{g}^{(1)}$.

## 6.6 Chern-Simons with Unitary and Orthogonal groups

In this section we compute several traces on the torus Hilbert space of Chern-Simons theories over simply-connected Lie groups. These traces are useful when studying more complicated

[120] In writing this we use the duality $(G_2)_1 \leftrightarrow \mathrm{U}(2)_{3,1}$ and the notation $\mathrm{U}(2)_{6,0} \equiv \mathrm{SO}(3)_3 \times S^1$.

theories over non-simply-connected groups.

### 6.6.1 $G = \mathrm{SU}(n)$

Consider the algebra $A_{n-1} = \mathfrak{su}_n$. The comarks are all $a_i^\vee = 1$. Plugging this into (6.4.11) we get the generating function as

$$Z(\mathrm{SU}(n), q) = (1-q)^{-n}\,, \tag{6.6.1}$$

and, by expanding, the untwisted trace

$$\mathrm{tr}_{\mathrm{SU}(n)_k}(\mathbf{1}) = \binom{n+k-1}{k}\,. \tag{6.6.2}$$

This is the number of integrable representations of $\mathrm{SU}(n)_k$, that is, the dimension of the torus Hilbert space of this Chern-Simons theory. This result will be useful when we discuss the Chern-Simons theory over the unitary group $\mathrm{U}(n)$, see section 6.4.2.

### 6.6.2 $G = \mathrm{Spin}(2n+1)$

Consider the algebra $B_n = \mathfrak{so}_{2n+1}$. The comarks are $a_i^\vee = 1$ for $i = 0, 1, n$, and $a_i^\vee = 2$ for $i = 2, \ldots, n-1$. Plugging this into (6.4.11) we get the generating function as

$$Z(\mathrm{Spin}(2n+1), q) = (1-q)^{-3}(1-q^2)^{-(n-2)}\,, \tag{6.6.3}$$

and, by expanding, the untwisted trace

$$\begin{aligned}\mathrm{tr}_{\mathrm{Spin}(2n+1)_{2k}}(\mathbf{1}) &= \binom{n+k}{k} + 3\binom{n+k-1}{k-1}\,,\\ \mathrm{tr}_{\mathrm{Spin}(2n+1)_{2k+1}}(\mathbf{1}) &= \binom{n+k-1}{k-1} + 3\binom{n+k}{k}\,.\end{aligned} \tag{6.6.4}$$

This is the number of integrable representations of $\mathrm{Spin}(2n+1)_k$, that is, the dimension of the torus Hilbert space of this Chern-Simons theory. For future reference, it is also useful to break up the states into the tensors and spinors. In other words, we shall be interested in knowing how many of the states of $\mathrm{Spin}(2n+1)$ are tensorial representations, and how many are spinorial representations. These are defined by $\lambda_n =$ even and $\lambda_n =$ odd, respectively, which yields the following:

$$\begin{aligned}N_{\text{tensor}}^{\mathrm{Spin}(2n+1)_{2k}} &= \binom{n+k}{k} + \binom{n+k-1}{k-1}\,,\\ N_{\text{spinor}}^{\mathrm{Spin}(2n+1)_{2k}} &= 2\binom{n+k-1}{k-1}\,,\\ N_{\text{tensor}}^{\mathrm{Spin}(2n+1)_{2k+1}} &= 2\binom{n+k}{k}\,,\\ N_{\text{spinor}}^{\mathrm{Spin}(2n+1)_{2k+1}} &= \binom{n+k}{k} + \binom{n+k-1}{k-1}\,,\end{aligned} \tag{6.6.5}$$

so that

$$\mathrm{tr}_{\mathrm{Spin}(2n+1)_k}(\mathbf{1}) \equiv N_{\text{tensor}}^{\mathrm{Spin}(2n+1)_k} + N_{\text{spinor}}^{\mathrm{Spin}(2n+1)_k}\,. \tag{6.6.6}$$

For a more interesting example, let us now compute the partition function of $\mathrm{SO}(2n+1)_k = \mathrm{Spin}(2n+1)_k/\mathbb{Z}_2$, which corresponds to the algebra $\mathfrak{so}_{2n+1}$ extended by the simple current $\chi = [0,k,0,\dots,0]$. This current has spin $h_\chi = k/2$, and so the extension is fermionic for odd $k$. The current acts on a given representation $[\lambda_0,\lambda_1,\dots,\lambda_n]$ as $\lambda_0 \leftrightarrow \lambda_1$.

Consider first the case of even $k$, so that $\mathrm{SO}(2n+1)_k$ makes sense as a bosonic theory. The extension has two effects: first, it projects out all the spinors, and second, it organizes the tensors into $\mathbb{Z}_2$-orbits. Such an orbit may have length two or one; the latter corresponds to a fixed-point under spectral flow, i.e., to a tensor with $\lambda_0 = \lambda_1$, which splits into two primaries in the quotient. The number of fixed-points corresponds to the number of solutions to $\lambda_0+\lambda_1+2(\lambda_2+\cdots+\lambda_{n-1})+\lambda_n = k$ with $\lambda_0 = \lambda_1$ and $\lambda_n$ even, i.e., $\binom{n+k/2-1}{k/2}$. Therefore, the number of conformal blocks is

$$\mathrm{tr}_{\mathrm{SO}(2n+1)_{2k}}(\mathbf{1}) = \frac{1}{2}\left(N_{\text{tensor}}^{\mathrm{Spin}(2n+1)_{2k}} - \binom{n+k-1}{k}\right) + 2\binom{n+k-1}{k}\,. \tag{6.6.7}$$

Let now $k$ be odd, which makes $\mathrm{SO}(2n+1)_k$ a spin theory. The total number of states is the same on every spin structure, so we shall count the bosons and fermions in the Ramond sector (which is the richest case, as only this sector may contain fermions). The total number of states is the sum, while the Witten index is the difference. In the Ramond sector, the quotient projects out the tensors, and it organizes the spinors into $\mathbb{Z}_2$-orbits. The bosons are the length-two orbits, and the fermions are the fixed-points. The latter are the representations with $\lambda_0+\lambda_1+2(\lambda_2+\cdots+\lambda_{n-1})+\lambda_n = k$ with $\lambda_0 = \lambda_1$ and $\lambda_n$ odd, which has $\binom{n+(k-1)/2-1}{(k-1)/2}$ solutions. Thus, the number of bosons and fermions is

$$\begin{aligned} N_{\text{boson}}^{\mathrm{SO}(2n+1)_{2k+1}} &= \frac{1}{2}\left(N_{\text{spinor}}^{\mathrm{Spin}(2n+1)_{2k+1}} - \binom{n+k-1}{k}\right), \\ N_{\text{fermion}}^{\mathrm{SO}(2n+1)_{2k+1}} &= \binom{n+k-1}{k}, \end{aligned} \tag{6.6.8}$$

from where it follows that

$$\begin{aligned} \mathrm{tr}_{\mathrm{SO}(2n+1)_{2k+1}}(\mathbf{1}) = \mathrm{tr}_{\mathrm{SO}(2n+1)_{2k+1}}(-1)^F &= N_{\text{boson}}^{\mathrm{SO}(2n+1)_{2k+1}} + N_{\text{fermion}}^{\mathrm{SO}(2n+1)_{2k+1}} \\ &= \binom{n+k}{k}, \end{aligned} \tag{6.6.9}$$

for all spatial spin structures, except for the odd structure for which

$$\begin{aligned} \mathrm{tr}_{\mathrm{SO}(2n+1)_{2k+1}}(-1)^F &= N_{\text{boson}}^{\mathrm{SO}(2n+1)_{2k+1}} - N_{\text{fermion}}^{\mathrm{SO}(2n+1)_{2k+1}} \\ &= \binom{n+k}{k} - 2\binom{n+k-1}{k}. \end{aligned} \tag{6.6.10}$$

We see that $\mathrm{tr}(\mathbf{1})$ is invariant under $n \leftrightarrow k$, as required by level-rank duality. Similarly, $\mathrm{tr}(-1)^F$ is invariant up to a sign, which is due to the difference in the framing anomalies (i.e., the precise level-rank duality [112] is $\mathrm{SO}(2n{+}1)_{2k+1} \leftrightarrow \mathrm{SO}(2k{+}1)_{-2n-1} \times \mathrm{SO}((2n{+}1)(2k{+}1))_1$, with the invertible factor contributing with a global factor of $(-1)^{(2n+1)(2k+1)} \equiv -1$ to the trace, cf. (6.3.25)).

### 6.6.3 $G = \mathrm{Spin}(2n)$

Consider the algebra $D_n = \mathfrak{so}_{2n}$. The comarks are $a_i^\vee = 1$ for $i = 0, 1, n-1, n$, and $a_i^\vee = 2$ for $i = 2, \dots, n-2$. Plugging this into (6.4.11) we get the generating function as

$$Z(\mathrm{Spin}(2n), q) = (1-q)^{-4}(1-q^2)^{-(n-3)}\,, \tag{6.6.11}$$

and, by expanding, the untwisted trace

$$\begin{aligned} \mathrm{tr}_{\mathrm{Spin}(2n)_{2k}}(\mathbf{1}) &= \binom{n+k}{k} + 6\binom{n+k-1}{k-1} + \binom{n+k-2}{k-2}\,, \\ \mathrm{tr}_{\mathrm{Spin}(2n)_{2k+1}}(\mathbf{1}) &= 4\binom{n+k}{k} + 4\binom{n+k-1}{k-1}\,. \end{aligned} \tag{6.6.12}$$

This is the number of integrable representations of $\mathrm{Spin}(2n)_k$, that is, the dimension of the torus Hilbert space of this Chern-Simons theory. For future reference, it is also useful to break up the states into the tensors and spinors. In other words, we shall be interested in knowing how many of the states of $\mathrm{Spin}(2n)$ are tensorial representations, and how many are spinorial representations. These are defined by $\lambda_{n-1} + \lambda_n = \text{even}$ and $\lambda_{n-1} + \lambda_n = \text{odd}$, respectively, which yields the following:

$$\begin{aligned} N_{\text{tensor}}^{\mathrm{Spin}(2n)_{2k}} &= \binom{n+k}{k} + 2\binom{n+k-1}{k-1} + \binom{n+k-2}{k-2}\,, \\ N_{\text{spinor}}^{\mathrm{Spin}(2n)_{2k}} &= 4\binom{n+k-1}{k-1}\,, \\ N_{\text{tensor}}^{\mathrm{Spin}(2n)_{2k+1}} = N_{\text{spinor}}^{\mathrm{Spin}(2n)_{2k+1}} &= 2\binom{n+k}{k} + 2\binom{n+k-1}{k-1}\,, \end{aligned} \tag{6.6.13}$$

so that

$$\mathrm{tr}_{\mathrm{Spin}(2n)_k}(\mathbf{1}) \equiv N_{\text{tensor}}^{\mathrm{Spin}(2n)_k} + N_{\text{spinor}}^{\mathrm{Spin}(2n)_k}\,. \tag{6.6.14}$$

For a more interesting example, let us now compute the partition function $\mathrm{SO}(2n)_k = \mathrm{Spin}(2n)_k/\mathbb{Z}_2$, which corresponds to the algebra $\mathfrak{so}_{2n}$ extended by the simple current $\chi = [0, k, 0, \dots, 0]$. This current has spin $h_\chi = k/2$, and so the extension is fermionic for odd $k$. The current acts on a given representation $[\lambda_0, \lambda_1, \dots, \lambda_n]$ as $\lambda_0 \leftrightarrow \lambda_1$ and $\lambda_{n-1} \leftrightarrow \lambda_n$.

Consider first the case of even $k$, so that $\mathrm{SO}(2n)_k$ makes sense as a bosonic theory. The extension has two effects: first, it projects out all the spinors, and second, it organizes the tensors into $\mathbb{Z}_2$-orbits. Such an orbit may have length two or one; the latter corresponds to a fixed-point under spectral flow, i.e., to a tensor with $\lambda_0 = \lambda_1$ and $\lambda_{n-1} = \lambda_n$, which splits into two primaries in the quotient. The number of fixed-points corresponds to the number of solutions to $\lambda_0 + \lambda_1 + 2(\lambda_2 + \cdots + \lambda_{n-2}) + \lambda_{n-1} + \lambda_n = k$ with $\lambda_0 = \lambda_1$ and $\lambda_{n-1} = \lambda_n$, i.e., $\binom{n+k/2-2}{k/2}$. Therefore, the number of conformal blocks is

$$\mathrm{tr}_{\mathrm{SO}(2n)_{2k}}(\mathbf{1}) = \frac{1}{2}\left(N_{\text{tensor}}^{\mathrm{Spin}(2n)_{2k}} - \binom{n+k-2}{k}\right) + 2\binom{n+k-2}{k}. \tag{6.6.15}$$

Let now $k$ be odd, which makes $\mathrm{SO}(2n)_k$ a spin theory. The number of states is the same on every spin structure, so we shall count the bosons and fermions in the Ramond sector (which is the richest case, as only this sector may contain fermions). The total number of states is the sum, while the Witten index is the difference. In the Ramond sector, the quotient projects out the tensors, and it organizes the spinors into $\mathbb{Z}_2$-orbits. The bosons are the length-two orbits, and the fermions are the fixed-points. Note that the spinors have $\lambda_{n-1} + \lambda_n =$ odd, which is incompatible with the fixed-point condition $\lambda_{n-1} = \lambda_n$, and so there are no fixed-points. Thus, the number of bosons and fermions is

$$\begin{aligned} N_{\text{boson}}^{\mathrm{SO}(2n)_{2k+1}} &= \frac{1}{2} N_{\text{spinor}}^{\mathrm{Spin}(2n)_{2k+1}}, \\ N_{\text{fermion}}^{\mathrm{SO}(2n)_{2k+1}} &= 0, \end{aligned} \tag{6.6.16}$$

from where it follows that

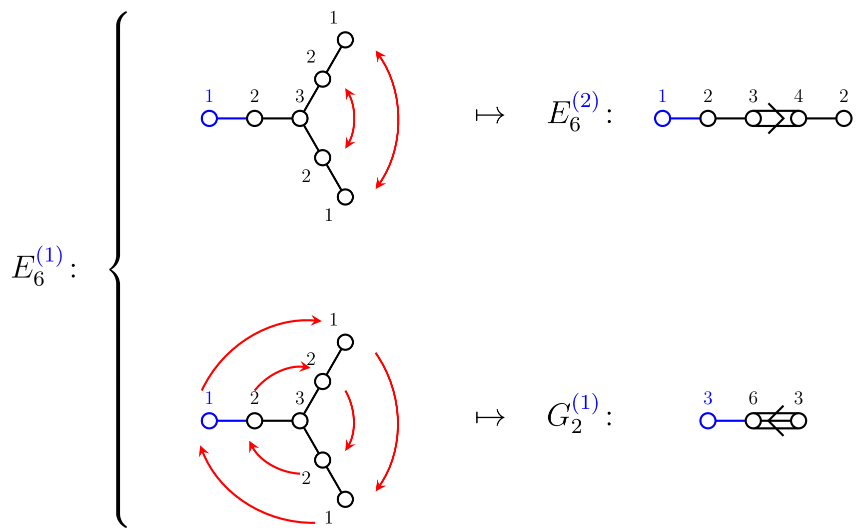
$$\mathrm{tr}_{\mathrm{SO}(2n)_{2k+1}}(\mathbf{1}) = \mathrm{tr}_{\mathrm{SO}(2n)_{2k+1}}(-1)^F = N_{\text{boson}}^{\mathrm{SO}(2n)_{2k+1}}, \tag{6.6.17}$$

for all spatial spin structures. Note that the equality of $\mathrm{tr}(\mathbf{1}), \mathrm{tr}(-1)^F$ on all spin structures was in fact expected from the level-rank duality $\mathrm{SO}(2n)_{2k+1} \leftrightarrow \mathrm{SO}(2k+1)_{-2n}$, the *r.h.s.* being fermionic only due to a trivial $\mathrm{SO}(2n(2k+1))_1 = \{\mathbf{1}, \psi\}$ factor (which contains an even number of fermions, so not even the sign of $\mathrm{tr}(-1)^F$ may depend on the spin structure).

## 6.7 The Exceptional Groups

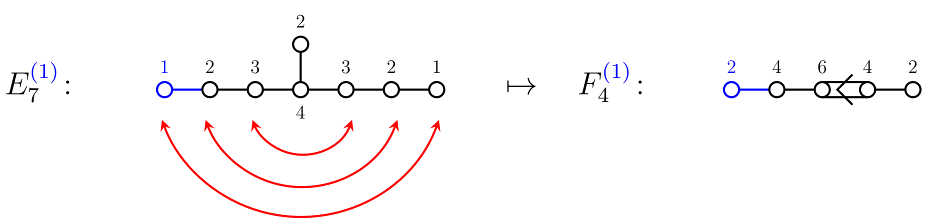
In this section we gather the different indices for the exceptional groups, whose domain wall theory is yet to be identified. Any given proposal for the dynamics of such walls ought to be consistent with the indices below. By particle-hole symmetry, the indices satisfy $I_n^{\mathsf{s}} = \pm I_{h-n}^{\mathsf{s}}$, and therefore we only show the first $\lceil h/2 \rceil$ indices, so as to avoid repetition.

We compute the untwisted indices, and the indices twisted by the zero-form and one-form symmetries (see table 6.1). The symmetries $\mathsf{c} \in \mathsf{C} = \mathbb{Z}_2$ and $\mathsf{g} \in \Gamma = \mathbb{Z}_3$ act on the Dynkin diagram of $E_6$ as follows:




(6.7.1)

The symmetry $\mathtt{g} \in \Gamma$ acts on $E_7$ as follows:




(6.7.2)

Using these diagrams we find:

- $E_6$:

$$\begin{aligned} Z(q) &= 1 - 3q + 7q^3 - 3q^4 - 6q^5 + \cdots \\ Z^{\mathtt{c}}(q) &= 1 - q - 2q^2 + q^3 + q^4 + 2q^5 + \cdots \\ Z^{\mathtt{g}}(q) &= 1 - 2q^3 + \cdots \end{aligned} \tag{6.7.3}$$

- $E_7$:

$$\begin{aligned} Z(q) &= 1 - 2q - 2q^2 + 4q^3 + 3q^4 - 7q^6 - 4q^7 + 5q^8 + 4q^9 + \cdots \\ Z^{\mathtt{g}}(q) &= 1 - 2q^2 - q^4 + 3q^6 + q^8 + \cdots \end{aligned} \tag{6.7.4}$$

- $E_8$:

$$\begin{aligned} Z(q) = 1 &- q - 2q^2 + q^4 + 4q^5 + q^6 - 3q^8 \\ &- 6q^9 - q^{10} + 4q^{12} + 5q^{13} + 5q^{14} + \cdots \end{aligned} \tag{6.7.5}$$

- $F_4$:

$$Z(q) = 1 - 2q - q^2 + 3q^3 + q^4 + \cdots \tag{6.7.6}$$

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# Appendices.

# Appendix A

# Review and background.

In this appendix we review some basic facts about QFT that will be useful for the rest of this thesis.

## A.1 Fermions in various dimensions.

We begin by collecting the properties of fermions in various dimensions. A useful reference is [323] (see also [324]).

In general dimension $d = s + t$, where $s$ is the number of space dimensions and $t$ is the number of time dimensions, the different fields are classified by the finite-dimensional irreducible representations of the Lorentz group $\mathrm{Spin}(s,t)$. Fermions are always assumed to transform according to the smallest spinorial representation, unless specified otherwise. Here, *spinorial* means that the representation is odd with respect to the central element $(-1)^F \in \mathrm{Spin}(s,t)$, i.e., that it is charged under the $\mathbb{Z}_2$ subgroup that defines the extension $\mathrm{SO}(s,t) \equiv \mathrm{Spin}(s,t)/\mathbb{Z}_2$.

We take the mostly plus metric, with the first $t$ components carrying $-1$ sign and the other $s$ components a $+1$ sign. Classical fermions are valued in the algebra of Grassmann-odd numbers; consequently, quantum-mechanical fermions are operator fields that satisfy anti-commutation relations.

**One dimension.** In $d = 0 + 1$ (i.e., $t = 1, s = 0$) the Lorentz group is $\mathrm{Spin}(1) = \mathbb{Z}_2$. There are two irreducible representations, both one-dimensional. The trivial representation classifies bosonic fields, and the sign representation fermionic fields. Therefore, fermions in $d = 1$ are described by one-dimensional spinors, $\psi(t)$. Lorentz transformations are just $(-1)^F\colon \psi(t) \mapsto -\psi(t)$, a map that commutes with the condition $\psi(t) = \psi(t)^*$. Therefore, the reality (Majorana) condition is compatible with Lorentz transformations, which means that spinors can be taken to be real.

In conclusion,

The minimal spinor in $d = 1 + 0$ is a real-valued one-dimensional field $\psi(t)$.

As $\mathrm{Spin}(1, 0)$ is identical to $\mathrm{Spin}(0, 1)$, a euclidean fermion $\psi(x)$ is formally identical to a Lorentzian one $\psi(t)$.

**Two dimensions.** In $d = 1 + 1$ (i.e., $t = s = 1$), the Lorentz group is $\mathrm{Spin}(1, 1) \cong \mathbb{R}$, hence irreducible representations are one-dimensional. These representations are labelled by a *weight* $w \in \mathbb{R}$ such that, under a group element $\eta \in \mathbb{R}$, a field transforms as $\mathcal{O} \mapsto e^{w\eta}\mathcal{O}$. Here $\eta$ is called the *rapidity* of the Lorentz transformation, and is related to the more common notation as $\cosh\eta \equiv 1/\sqrt{1-\beta^2}$, with $\beta$ the speed.

Bosons are, by definition, fields with integral weight; and fermions are fields with half-integral weight. The minimal half integers are $\pm 1/2$, hence we take fermions to transform as

$$\psi \mapsto e^{\pm\eta/2}\psi \tag{A.1.1}$$

If a spinor transforms with minus sign, it is said to be a *left-mover*, and if with plus sign, a *right-mover*.

Note that (A.1.1) commutes with complex conjugation, which means that fermions can be taken to be real.

In conclusion,

There are two types of minimal spinors in $d = 1+1$: left-moving fermions and right-moving fermions. They are both real-valued.

One can also derive these results via the standard gamma matrix approach (Clifford algebra). A basis of gamma matrices in $d = 1 + 1$ can be taken to be

$$\gamma^0 = i\sigma_y, \qquad \gamma^1 = \sigma_x \tag{A.1.2}$$

which satisfy $(\gamma^0)^2 = -1$, $(\gamma^1)^2 = +1$ and $\gamma^0\gamma^1 + \gamma^1\gamma^0 = 0$. These matrices act on two-dimensional fermions as

$$\begin{aligned}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} &\mapsto e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\\ &= \begin{pmatrix}e^{-\omega/2} & 0\\ 0 & e^{+\omega/2}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\end{aligned} \tag{A.1.3}$$

Note that this transformation commutes with the "third" gamma matrix $\gamma_\star = \gamma_0\gamma_1 \equiv \sigma_z$. Therefore, one may project to eigenspaces of $\gamma_\star$ in a way that is consistent with Lorentz transformations. As $\gamma_\star^2 = +1$, the eigenvalues are $\pm 1$, and the reducibility condition reads $\gamma_\star\psi = \pm\psi$. A spinor with sign $+1$ is said to be a left-mover, and one with sign $-1$ a right-mover. We thus recover the previous classification under the renaming $\psi_1 \to \psi_L$, $\psi_2 \to \psi_R$, with $\omega \equiv \eta$.

The euclidean Lorentz group $d = 2 + 0$ (i.e., $t = 0, s = 2$) is instead $\mathrm{Spin}(2) \cong \mathrm{U}(1)$. Representations are still labelled by a weight $w \in \mathbb{R}$, with transformation law $\mathcal{O} \mapsto e^{i\omega\eta}\mathcal{O}$. Left-movers are fields with $w = -1/2$, and right-movers fields with $w = +1/2$. Note that Lorentz transformations no longer commute with complex conjugation, and therefore these two types of fermions are now complex. Naturally, one can break up these fermions into their real and imaginary parts, and then the minimal fermions are real and two-dimensional.

The euclidean gamma matrices can be taken as

$$\gamma^0 = \sigma_y, \qquad \gamma^1 = \sigma_x \tag{A.1.4}$$

where now both matrices square to $+1$. A Lorentz transformation now becomes

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \begin{pmatrix} e^{i\omega/2} & 0 \\ 0 & e^{-i\omega/2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{A.1.5}$$

and the chirality matrix is $\gamma_\star = i\gamma^0\gamma^1 = \sigma_z$. Hence, the irreducible spinors satisfy $\gamma_\star\psi = \pm\psi$, and we recover the left- and right-movers above. Instead, one can work in a basis where the gamma matrices are real, for example

$$\gamma^0 = \sigma_x, \qquad \gamma^1 = \sigma_z \tag{A.1.6}$$

in which case Lorentz transformations read

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\omega/2 & \sin\omega/2 \\ -\sin\omega/2 & \cos\omega/2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{A.1.7}$$

This now commutes with complex conjugation, and therefore the spinor can be taken to be real. But in this basis the chirality matrix is $\gamma_\star = \sigma_y$, whose eigenvectors are complex, and therefore one cannot impose the chirality condition $\gamma_\star\psi = \pm\psi$ at the same time as the reality condition $\psi = \psi^*$. Euclidean spinors in $d = 2$ can be taken to be either Weyl or Majorana, but not both.

**Three dimensions.** In $d = 2 + 1$ (i.e., $t = 1$, $s = 2$), the Lorentz group is $\mathrm{Spin}(2, 1) \cong \mathrm{SL}(2, \mathbb{R})$, hence the irreducible representations are labelled by a half-integer $j \in \mathbb{Z}_{\geq 0}$, such that the dimension is $2j + 1$. The minimal representation has $j = 1/2$, i.e., it is two-dimensional. Under an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ (with $ad - bc = 1$), a spinor transforms as

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{A.1.8}$$

Note that this is real, hence one can impose the reality condition $\psi = \psi^*$ consistently with Lorentz transformations.

In conclusion,

The minimal spinor in $d = 2+1$ has two components, and it is real-valued.

One can also establish these conclusions by looking at the gamma matrices. As $d$ is odd, one can obtain a basis of gamma matrices by adjoining the chirality matrix $\gamma_\star$ to the $d = 1+1$ matrices, i.e., one can take

$$\gamma^0 = i\sigma_y, \qquad \gamma^1 = \sigma_x, \qquad \gamma^2 = \sigma_z \tag{A.1.9}$$

(These matrices generate the algebra $\mathfrak{su}(1,1)$; this is nothing but the statement that $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)$.) A Lorentz transformation now reads

$$\begin{aligned} \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} &\mapsto e^{-\frac{1}{4}\omega_{\mu\nu}\gamma^\mu\gamma^\nu} \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \\ &= \left[\cosh\Omega\,\mathbf{1}_2 + \frac{\sinh\Omega}{\Omega}(\omega_{2,0}\sigma_x + i\omega_{2,1}\sigma_y - \omega_{1,0}\sigma_z)\right]\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \end{aligned} \tag{A.1.10}$$

where $\Omega := \sqrt{\omega_{1,0}^2 + \omega_{2,0}^2 - \omega_{2,1}^2}$. Note that this matrix has unit determinant and three free parameters $\omega_{1,0}, \omega_{2,0}, \omega_{2,1}$, hence it can be written as $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, as above.

If we look instead at the euclidean Lorentz group $\mathrm{Spin}(3) = \mathrm{SU}(2)$, the minimal spinor is the two-dimensional representation of $\mathrm{SU}(2)$, which is pseudo-real. One cannot impose a reality condition on the fermion (such a condition can be imposed on the direct sum of two fermions, hence the field effectively has four real degrees of freedom).

**Four dimensions.** In $d = 3+1$ (i.e., $t = 1$, $d = 3$), the Lorentz group is $\mathrm{Spin}(3,1) \cong \mathrm{SL}(2,\mathbb{C})$, hence the irreducible representations are labelled by two half-integers $(j_1, j_2) \in \mathbb{Z}^2_{\geq 0}$, whose dimension is $(2j_1+1)(2j_2+1)$. The minimal representations are $(j_1, j_2) = (1/2, 0)$ and $(j_1, j_2) = (0, 1/2)$, both two-dimensional. Under an element $X = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$ (with $ad - bc = 1$), the two irreducible spinors transform as

$$\psi \mapsto X\psi, \qquad \text{or} \qquad \psi \mapsto X^*\psi \tag{A.1.11}$$

(The map $X \mapsto X^*$ is the unique outer automorphism of $\mathrm{SL}(2,\mathbb{C})$; it interchanges $j_1 \leftrightarrow j_2$. It is more common to let the second action be $\psi \mapsto (X^\dagger)^{-1}\psi$, which is an equivalent automorphism obtained by conjugating by the epsilon tensor, i.e., $\sigma_y X^* \sigma_y \equiv (X^\dagger)^{-1}$.)

The matrix $X$ is generically complex, and therefore Lorentz transformations do not commute with a reality condition $\psi^* = \psi$; the fermions are necessarily complex. That being said, note that if $\psi$ transforms according to $\psi \mapsto X\psi$, then $\psi^*$ transforms as $\psi^* \mapsto X^*\psi$. In this sense, we do not need to keep track of the two possibilities above; either can be obtained from the other by complex-conjugating the fermion. In particle-physics terminology, one can write down $3+1$ dimensional theories using left-handed spinors alone.

In conclusion,

The minimal spinor in $d = 3+1$ has two components, and it is complex-valued.

Naturally, one can break such spinor into its real and imaginary parts, so the minimal spinor can also be taken to have four real-valued components.

In terms of the gamma matrix approach, this is reproduced as follows. The gamma matrices can be taken as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \tag{A.1.12}$$

where $\sigma^\mu = (\mathbf{1}_2, \vec\sigma)$ and $\bar\sigma^\mu = (\mathbf{1}_2, -\vec\sigma)$. The "fifth" gamma matrix is $\gamma_\star = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \mathbf{1}_2 \oplus (-\mathbf{1}_2)$.

As $\gamma_\star^2 = +1$, we can consider the eigenspaces $\gamma_\star = \pm 1$, which yield two-component complex fermions. Instead, if we change basis via the unitary matrix $\frac{1}{2}(\sigma_z \otimes \sigma_z - i\sigma_z \otimes \sigma_x - i\sigma_x \otimes \sigma_x - \sigma_x \otimes \sigma_z)$, the gamma matrices become purely imaginary, whence Lorentz transformations are real, and the Majorana condition $\psi^* = \psi$ can be imposed.

If we look instead at euclidean spinors (i.e., $t = 0, s = 4$), the Lorentz group becomes $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. This group is just two copies of the Lorentz group in $d = 3+0$ dimensions, so the properties of spinors are immediate.

## A.2 Fermion kinetic term.

Here we discuss a few important properties of the fermion kinetic term

$$\mathcal{L} = i\bar\psi\gamma^\mu\partial_\mu\psi \tag{A.2.1}$$

where $\bar\psi = \psi^\dagger\gamma^0$. We only consider Lorentzian fermions here.

The gamma matrices satisfy $\bar\gamma^\mu = \gamma^\mu$ (i.e., the time-like component is anti-hermitian and the space-like components are hermitian); therefore, $\bar\psi\gamma^\mu\partial_\mu\psi$ is anti-hermitian and the inclusion of $i$ renders the Lagrangian real-valued, as it should be.

Note that in $d = 0+1$ and $d = 3+1$ we could write the gamma matrices as purely imaginary, while in $d = 1+1$ and $d = 2+1$ we could find matrices that were purely real. In general we write $\gamma^{\mu *} = \eta\gamma^\mu$, where $\eta = \pm 1$ depends on the number of spacetime dimensions. (This identity is valid in a suitable basis where $\gamma^\mu$ is either real or imaginary; for general bases the condition is replaced by $\gamma^{\mu *} = \eta B\gamma^\mu B^{-1}$ for a certain matrix $B$ [323].)

**Time-reversal.** A symmetry of (A.2.1) that shall play a special role in this work is time-reversal. Such transformation takes $t \to -t$, hence it is anti-hermitian (as $iHt$ must be invariant). The most general anti-linear transformation is

$$\mathsf{T}\colon \psi(t) \mapsto M\psi(-t) \tag{A.2.2}$$

for some unitary matrix $M$. Performing this transformation in the fermion Lagrangian yields the condition $M^\dagger\gamma^{0*}\gamma^{\mu*}M = \gamma^\mu\gamma^0$. In the Majorana basis where $\gamma^{\mu*} = \eta\gamma^\mu$ this becomes $M^\dagger\gamma^0\gamma^\mu M = \gamma^\mu\gamma^0$, from where it follows that $\gamma^0 M$ commutes with $\gamma^0\gamma^\mu$. In odd dimensions this requires $\gamma^0 M$ to be proportional to the unit matrix, while in even dimensions it must be a linear combination of the identity matrix and the chirality matrix $\gamma_\star$.

Multiplication by the chirality matrix constitutes an independent, unitary symmetry by itself, so if we count unitary symmetries separately, we do not need to count both the unit matrix and the chirality matrix in $\mathsf{T}$; either of them can be obtained from the other by composing with the unitary symmetry generated by the chirality matrix. Therefore, for even and odd $d$, the basic time-reversal transformation is $M \propto \gamma^0$. Finally, in order to preserve the reality properties of $\psi$ one must have $M^* = M$, i.e., $M = \pm\eta^{1/2}\gamma^0$. Again, multiplication by $-1$ is nothing but the unitary symmetry $(-1)^F\colon \psi \mapsto -\psi$, so if we count this separately we can take without loss of generality, say, the plus sign. In conclusion,

Time-reversal symmetry acts on fermions as

$$\mathsf{T}\colon \psi(t) \mapsto \eta^{1/2}\gamma^0\psi(-t) \tag{A.2.3}$$

Note that $\mathsf{T}^2 = \gamma^{0*}\gamma^0 \equiv -\eta$. Therefore, if $\eta = 1$ this becomes $\mathsf{T}^2 = (-1)^F$ while if $\eta = -1$ it reads $\mathsf{T}^2 = 1$.

Of course, if the spinors carry internal indices (either flavor or gauge), then the most general transformation involves an orthogonal transformation on those indices.

**Commutation relations.** Another piece of important information contained in the fermion kinetic term is the canonical anti-commutation relations. In fact, these relations depend only on the time-derivatives, so it is enough to look at the first term only:

$$\mathcal{L} = i\psi^\dagger\partial_t\psi + \cdots \tag{A.2.4}$$

As far as the canonical formalism is concerned, we can focus without loss of generalization on the one-dimensional case $d = 1$; the generalization to higher $d$ just requires adding the spatial dependence $\vec{x}$ throughout, and suitable spinorial indices. Finally, by breaking complex fermions into their real and imaginary parts, we can take all fermions to be real, $\psi^* = \psi$. Real fields are often rescaled as $\psi \to \psi/\sqrt{2}$ so that the kinetic term carries the standard factor of $1/2$ in front. With this in mind, the Lagrangian reads

$$\mathcal{L} = \frac{i}{2}\psi\partial_t\psi \tag{A.2.5}$$

The conjugate momentum is

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \frac{i}{2}\psi \tag{A.2.6}$$

(Here we use a right-derivative.) As we cannot use this relation to solve for $\dot{\psi}$, it is to be regarded as a constraint, [325]

$$\chi = \pi - \frac{i}{2}\psi \tag{A.2.7}$$

which must vanish on-shell, $\chi \approx 0$.

Note that the Poisson bracket of the constraints is $\{\chi, \chi\} = -i$, and hence the Dirac bracket [326] is

$$\{\psi, \psi\}_D \equiv \{\psi, \psi\} - \{\psi, \chi\}\frac{1}{-i}\{\chi, \psi\} \equiv -i \tag{A.2.8}$$

The quantum theory is obtained by promoting the Dirac bracket into an anti-commutator,

$$\{\psi, \psi\} \to i\{\psi, \psi\}_D \equiv 1 \tag{A.2.9}$$

The generalization to higher dimensions is straightforward:

The canonical anti-commutation relations for free fermions is

$$\{\psi_{\alpha,i}(t, \vec{x}), \psi_{\beta,j}(t, \vec{y})\} = \delta_{\alpha,\beta}\delta_{ij}\delta(\vec{x} - \vec{y}) \tag{A.2.10}$$

where $\alpha, \beta$ are spinor indices and $i, j$ are flavor indices. Complex fields can be written as the sum of two real fields, $\Psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$, hence their canonical anti-commutators read

$$\{\Psi_{\alpha,i}(t, \vec{x}), \Psi_{\beta,j}(t, \vec{y})\} = 0, \qquad \{\Psi_{\alpha,i}(t, \vec{x}), \Psi^*_{\beta,j}(t, \vec{y})\} = \delta_{\alpha,\beta}\delta_{ij}\delta(\vec{x} - \vec{y}) \tag{A.2.11}$$

**Canonical quantization.** Here we describe the quantization of a system described by real fermions $\psi_i$ subject to the canonical anti-commutation relations

$$\{\psi_i, \psi_j\} = \delta_{ij} \tag{A.2.12}$$

where $i, j = 1, 2, \ldots, n$. We can think of the fields $\psi_i$ as describing actual $0 + 1$ dimensional fermions, or the space of zero-modes for higher-dimensional fermions on a compact spatial manifold. This is also the rank-$n$ real Clifford algebra so the fermions can also be regarded as the gamma matrices in $n$ euclidean dimensions.

We begin by considering the case $n = 1$. Here the algebra is $\psi^2 = 1/2$, which has two one-dimensional representations, namely

$$\psi = \pm 1/\sqrt{2} \tag{A.2.13}$$

They are clearly inequivalent, and any other representation is a direct sum of these. Note also that neither of these two representations admits an action of $(-1)^F\colon \psi \mapsto -\psi$. The operator $(-1)^F$ should be represented as a matrix that anti-commutes with $\psi$, which is impossible for $1 \times 1$ representations. This means that the vector space that realizes either of the $1 \times 1$ dimensional representations of the canonical algebra is not a $\mathbb{Z}_2$-graded vector space; the system does not admit a good notion of fermion parity. One could consider, instead, the direct sum of these two representations, which now does admit an action of $(-1)^F$, but the corresponding vector space is not an irreducible module for the canonical algebra.

Consider now the $n = 2$ case, with algebra $\psi_1^2 = \psi_2^2 = 1/2$ and $\psi_1\psi_2 + \psi_2\psi_1 = 0$. It is clear that there are no $1 \times 1$ representations, but there is an obvious candidate for a $2 \times 2$ representation, namely

$$\psi_1 = \frac{1}{\sqrt{2}}\sigma_x, \qquad \psi_2 = \frac{1}{\sqrt{2}}\sigma_y \tag{A.2.14}$$

These are hermitian matrices that square to $1/2$ and anti-commute with each other. There are more options, such as using $\sigma_z$ instead of $\sigma_x$, but it is not hard to convince oneself that any other representation is equivalent to the one above. Furthermore, this representation admits an action of $(-1)^F$, namely $(-1)^F = \sigma_z$.

We can now tackle the general case. As the fermions are all independent, the Hilbert space of $n + m$ fermions is the tensor product of the Hilbert space of $n$ fermions and the Hilbert space of $m$ fermions. Hence, given a representation of $n$ fermions, we can obtain one for $n + 2$ fermions as follows. The first $n$ matrices can be taken as

$$\psi_i^{(n+2)} = \psi_i^{(n)} \otimes A, \qquad i = 1, 2, \ldots, n \tag{A.2.15}$$

while the last two as

$$\psi_{n+1}^{(n+2)} = \mathbf{1} \otimes B, \qquad \psi_{n+2}^{(n+2)} = \mathbf{1} \otimes C \tag{A.2.16}$$

for some matrices $A, B, C$ that realize a representation of the $n = 2$ case, e.g., $A = \sigma_z$, $B = \frac{1}{\sqrt{2}}\sigma_x$, $C = \frac{1}{\sqrt{2}}\sigma_y$. It is easily checked that the matrices $\psi_i^{(n+2)}$ satisfy the canonical anti-commutation relations if $\psi_i^{(n)}$ do. Finally, given a representation for $(-1)^F$ on the $n$ fermions, one can obtain one for the $n + 2$ fermions via

$$(-1)^F\big|^{(n+2)} = (-1)^F\big|^{(n)} \otimes D \tag{A.2.17}$$

for some matrix $D$. Requiring that $(-1)^F$ anti-commutes with all $\psi^{(n+2)}$ and squares to $+1$, we get the conditions

$$D^2 = 1, \qquad [A, D] = 0, \qquad \{D, B\} = \{D, C\} = 0 \tag{A.2.18}$$

with solution $D = \sigma_z$. Note that $A, B, C, D$ are $2 \times 2$ matrices, which means that going from $n$ to $n + 2$ doubles the number of states; and, as $D = \sigma_z$ has one $+1$ eigenvalue and one $-1$ eigenvalue, $n \to n + 2$ adds as many bosons as it adds fermions.

The conclusion is that, if a suitable representation of $n$ fermions exists, then there is also one for $n + 2$ fermions, whose dimension is twice as large. The converse is obviously also true, just by looking at a subset of the operators. By counting degrees of freedom it is not hard to convince oneself that these are irreducible representations, and with a little bit more work one can show that the construction exhausts all the representations, up to unitary equivalence. So to summarize,

- The real Clifford algebra of odd rank has two inequivalent irreducible representations, of dimension $2^{(n-1)/2}$. Neither of these representations admits an action of $(-1)^F$ on it, so it is not a $\mathbb{Z}_2$-graded vector space. The direct sum of these two representations does admit an action of $(-1)^F$, but the module does not furnish an irreducible representation of the algebra.

- The real Clifford algebra of even rank has a unique irreducible representation, of dimension $2^{n/2}$. This representation admits an action of $(-1)^F$ on it, so the module is a $\mathbb{Z}_2$-graded vector space. The module contains $2^{n/2-1}$ even states (bosons) and $2^{n/2-1}$ odd states (fermions).

In either case, a possible choice for the representation matrices is

$$\begin{aligned} \psi^{(1)} &= \pm\frac{1}{\sqrt{2}} & \psi_1^{(2)} &= \frac{1}{\sqrt{2}}\sigma_x & \psi_i^{(n+2)} &= \psi_i^{(n)} \otimes \sigma_z, \qquad i \le n \\ & & \psi_2^{(2)} &= \frac{1}{\sqrt{2}}\sigma_y & \psi_{n+1}^{(n+2)} &= \frac{1}{\sqrt{2}}\mathbf{1} \otimes \sigma_x \\ & & & & \psi_{n+2}^{(n+2)} &= \frac{1}{\sqrt{2}}\mathbf{1} \otimes \sigma_y \end{aligned} \tag{A.2.19}$$

with $(-1)^F\big|^{(n+2)} = (-1)^F\big|^{(n)} \otimes \sigma_z$, if $n$ is even.

It is useful to rederive these conditions from a different point of view. When the number of fermions is even, one can break them up into two groups and use them to define complex fermions, say

$$\Psi_i := \frac{1}{\sqrt{2}}(\psi_{2i-1} + i\psi_{2i}), \qquad i = 1, 2, \ldots, n/2 \tag{A.2.20}$$

whose commutators are

$$\{\Psi_i, \Psi_j\} = 0, \qquad \{\Psi_i, \Psi_j^*\} = \delta_{ij} \tag{A.2.21}$$

Note that now $\Psi_i^2 = 0$, which means that it is consistent to define the vacuum $|0\rangle$ as the state annihilated by all the fermions:

$$\Psi_i|0\rangle = 0 \tag{A.2.22}$$

The rest of states are now defined by acting with $\Psi_i^*$ on $|0\rangle$. In particular, given an arbitrary state $|v\rangle$ created by these $n$ fermions, the space of states for $n+2$ fermions is either of the form $|v\rangle$ or $\Psi_{n/2+1}^*|v\rangle$, which explains why the dimension is doubled when going from $n$ to $n+2$, and why half of the new states are bosons and the other half are fermions. Furthermore, these two sets of states are identical as far as the first $n$ fermions is concerned, except that $\Psi_{n/2+1}^*$

anti-commutes with such fermions, which explains the general structure $\psi_i^{(n+2)} = \psi_i^{(n)} \otimes \sigma_z$; similarly, given that the last two fermions can be written as $\psi_{n+1} = \frac{1}{\sqrt{2}}(\Psi_{n/2+1} + \Psi^*_{n/2+1})$ and $\psi_{n+2} = \frac{1}{i\sqrt{2}}(\Psi_{n/2+1} - \Psi^*_{n/2+1})$, this also explains the appearance of the Pauli matrices $\sigma_x, \sigma_y$ in $\psi_{n+1}^{(n+2)} = \frac{1}{\sqrt{2}}\mathbf{1} \otimes \sigma_x$ and $\psi_{n+2}^{(n+2)} = \frac{1}{\sqrt{2}}\mathbf{1} \otimes \sigma_y$. Finally, these considerations also explain why it is impossible to define a suitable Hilbert space for odd $n$: the complex fields $\Psi$ are only defined when $n$ is even.

One final perspective that is also quite illuminating is to try and solve the system via the path integral. Generically, a path integral on a time-circle with radius $\beta$ and anti-periodic boundary conditions around it computes the trace $\mathrm{tr}(e^{-\beta H})$, while the same path integral with periodic boundary conditions computes $\mathrm{tr}((-1)^F e^{-\beta H})$. In our system the Hamiltonian vanishes and therefore these path integrals compute the total number of states in the Hilbert space, and the number of bosons minus the number of fermions, respectively.

The path integral is easily evaluated:

$$\mathrm{Pf}(i\partial_t)^n = \prod_\lambda \lambda^n \tag{A.2.23}$$

where $\lambda$ are the eigenvalues of $i\partial_t$ with the appropriate boundary conditions. In particular, $\lambda = k + 1/2$ or $\lambda = k$, with $k \in \mathbb{Z}_{\geq 0}$, if the boundary conditions are anti-periodic or periodic, respectively. For periodic boundary conditions the product vanishes due to the zero mode $k = 0$. For anti-periodic boundary conditions the product can be computed using zeta-regularization,

$$\prod_\lambda \lambda^n := e^{-n\zeta'_{i\partial_t}(0)}, \qquad \zeta_{i\partial_t}(s) := \sum_\lambda \lambda^{-s} \tag{A.2.24}$$

where in our case $\zeta_{i\partial_t}(s) = \sum_{k\geq 0}(k+1/2)^{-s} \equiv (2^s - 1)\zeta(s)$, whence $\zeta'_{i\partial_t}(0) = -\frac{1}{2}\log 2$. Collecting results, we conclude that

$$\mathrm{tr}(1) = 2^{n/2}, \qquad \mathrm{tr}(-1)^F = 0 \tag{A.2.25}$$

This is consistent with our previous observations. The Hilbert space is only well-defined when $n$ is even, in which case the total number of states is $2^{n/2}$, and half of these are bosons and the other half fermions. For odd $n$ the would-be dimension $2^{n/2}$ is fractional.

In order to get a non-zero result for the periodic case we must insert fermions to compensate for the zero-modes. In particular, the (unnormalized) correlation function

$$\langle \psi_1 \psi_2 \cdots \psi_n \rangle \tag{A.2.26}$$

is non-vanishing. For $n$ even this is a bosonic operator, so a non-zero vacuum expectation value is perfectly fine; but for $n$ odd this is a fermion, so its vacuum expectation value is inconsistent with the symmetry $(-1)^F$, under which it is charged, which means that this symmetry is not realized on the Hilbert space.

**Fermion Hilbert space.** We just reviewed the construction of the Hilbert space for a system of $n$ free $d=1$ Majorana fermions. One thing we noticed was that, when $n$ is odd, this Hilbert space is somewhat ill-defined. Here we would like to be more explicit about what exactly goes wrong. Note that this is a question of practical interest, since given any system in $d$ dimensions that has non-trivial fermionic zero-modes – and these are surprisingly common – the full Hilbert space will contain a subsector isomorphic $d=1$ free fermions, and hence the lack of Hilbert space is inherited to the full theory.

The general conclusion from the previous discussion was that, when $n$ is odd, there are two inequivalent irreducible representations of the canonical algebra, and neither admits an action of $(-1)^F$, i.e., an operator that anti-commutes with all the fermions. The direct sum of these two representations does admit an action, but this representation is no longer irreducible. This is not a good thing, for several reasons. The one that is most pertinent given the theme of this thesis is the fact that an action on the canonical operators does not induce a unique action on the Hilbert space. In other words, if we define some operation via its action on the fields, $\mathcal{O}\psi\mathcal{O}^{-1} := \psi'$, and the Hilbert space realizes an irreducible representation of the canonical algebra, the action of $\mathcal{O}$ on the Hilbert space is uniquely specified by this information, while if the representation is reducible, the action of $\mathcal{O}$ on it is not unique.

This is easily illustrated by considering the action of time-reversal, which is a recurring symmetry in this thesis, so it is useful to use it again here. Time-reversal is defined by its action on the fermions as $\mathsf{T}\psi(t)\mathsf{T}^{-1} := \psi(-t)$ (cf. (A.2.3)). When $n$ is even, this fixes an action of $\mathsf{T}$ on the Hilbert space uniquely, up to an irrelevant global phase. For example, for $n=2$ we showed that the irreducible representation could be taken as $\psi_1 = \sigma_x/\sqrt{2}$ and $\psi_2 = \sigma_y/\sqrt{2}$. Time-reversal should act as a certain $2\times 2$ matrix $\tau$ which, by definition, satisfies $\tau\psi_i = \psi_i^*\tau$, which has solution $\tau \propto \sigma_x$, which is unique (up to an overall phase). On the other hand, when $n$ is odd, this definition of time-reversal does not specify an operator on the Hilbert space. For example, when $n=1$, the irreducible representations of the Hilbert space were $\psi = \pm 1/\sqrt{2}$, and the reducible representation that admits an action of $(-1)^F$ was the direct sum of these, namely $\psi = \sigma_z/\sqrt{2}$. Time-reversal should satisfy $\tau\psi = \psi^*\tau$, which now has general solution $\tau = \mathrm{diag}(z_1, z_2)$, where $z_i$ are two arbitrary phases. Here we see that the operation is not uniquely defined (and the undetermined parameters are not just a global phase).

Extrapolating to the general case, an action on the algebra of operators induces a unique action on the Hilbert space if and only if such space realizes an irreducible representation of the canonical commutation relations. In the case of $d=1$ fermions (or zero-modes for general $d$), the symmetries have a unique action on the Hilbert space if and only if the number of fermions is even. When this number is odd, the action depends on several arbitrary choices.

That being said, there are some aspects of the action of $\mathsf{T}$ (or other operations in the general case) that are in fact specified, even if the representation is reducible. For example, in the $n=1$ case above, we saw that $\tau = \mathrm{diag}(z_1, z_2)$, where $z_i$ are two arbitrary phases.

Note that $\mathsf{T}^2 = \tau\tau^* \equiv 1$, and therefore time-reversal squares to $+1$ for any choice of $z_i$. So even if the action of $\mathsf{T}$ is not uniquely defined, its square is. On the other hand, this system has $(-1)^F = \sigma_x$, and one can check that, if we choose $z_1 = \pm z_2$, then $(-1)^F\mathsf{T} = \pm\mathsf{T}(-1)^F$. Therefore, whether time-reversal commutes or anti-commutes with fermion parity can be fixed by an arbitrary choice. For arbitrary odd $n$, the pattern is

$$\begin{array}{c|c|c} & [\mathsf{T},(-1)^F]_\pm & \mathsf{T}^2 \\ \hline n=1 \mod 8 & \pm1 & +1 \\ n=3 \mod 8 & \pm1 & \pm1 \\ n=5 \mod 8 & \pm1 & -1 \\ n=7 \mod 8 & \pm1 & \mp1 \end{array} \tag{A.2.27}$$

where $[A,B]_\pm = AB \pm BA$. For a given $n$, the sign on the second column can be chosen arbitrarily, but this choice then fixes the sign in the third column (i.e., the two signs in a given row are not independent). A common choice is to let $\mathsf{T}$ and $(-1)^F$ always commute, in which case one gets $\mathsf{T}^2 = +1$ for $n = 1,7 \mod 8$, and $\mathsf{T}^2 = -1$ for $n = 3,5 \mod 8$ (see e.g. [242]). Another useful choice (which we implicitly make in chapter 1) is $\mathsf{T}(-1)^F = (-1)^{(n-1)/2}(-1)^F\mathsf{T}$, in which case $\mathsf{T}^2 = (-1)^{(n-1)(n-3)/8}$; this is useful because, in this case, the second column measures the second binary digit of $n$ and the third column its first binary digit. In other words, given $n = n_0 + 2n_1 + 4n_2 \mod 8$, the digit $n_0$ tells us whether $(-1)^F$ exists; $n_1$ tells us whether time-reversal commutes or anti-commutes with $(-1)^F$; and $n_2$ whether time-reversal squares to $+1$ or $-1$.

# Appendix B

# Useful data for Lie groups and representations.

In this appendix we collect some useful results concerning simple Lie groups and their representations. These groups arise as either the symmetry group or gauge group of a certain theory, and the representation specifies how it acts on the fields. Specifically, given a group $G$ and a representation $R$, the group element $g \in G$ acts on a field $\psi$ as $\psi \mapsto R(g)\psi$, where we think of $\psi$ as a column vector and $R(g)$ as a matrix.

A very useful physics-oriented reference is [150]. A more technical but still readable source is [327]. For quick computations one can use e.g. the Mathematica package LieART [196] or simply look at the extensive tables in [197].

It is a key result of Lie-representation theory that an arbitrary irreducible finite-dimensional representation can be labelled via $\operatorname{rank}(G)$ non-negative integers, known as the *Dynkin labels* of the representation. These integers $(\lambda_1, \lambda_2, \ldots, \lambda_{\operatorname{rank}(G)})$ are the coordinates of the highest weight of $R$ with respect to the basis of fundamental weights. One can also label the representation via a Young diagram $Y = Y(\lambda)$, which is a diagram that has $\lambda_i$ columns of $i$ boxes, aligned on top and ordered by column height from left (largest column) to right (smallest column). For example the labels $\lambda = (2, 0, 1)$ yield a diagram with two columns with a single box, and one column with three boxes, i.e., $Y = \ydiagram{3,1,1}$.

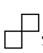
Given a representation $R$, two useful group-theoretic objects are its dimension $\dim(R)$ and its Dynkin index $T(R)$. The dimension is just the size of the matrix $R(g)$, i.e., the number of components of $\psi$. The Dynkin index is defined as $\delta^2 \operatorname{tr} R(g) = T(R)\delta^2 \operatorname{tr} g$, where $\delta$ denotes the differential at the origin $g = 1$, and tr the matrix trace. In general it will be useful to count real fields, so $\dim(R)$ and $T(R)$ are rescaled by a factor of 2 if the field $\psi$ is complex.

To be concrete we shall mostly concentrate on $G = \mathrm{SU}(N), \mathrm{SO}(N), \mathrm{Sp}(N)$, although we will make some remarks about other groups below. As these groups admit a natural definition in terms of matrices, they have an obvious representation, viz. $R(g) = g$, the *fundamental* representation. There is also the *anti-fundamental* representation $R(g) = g^*$, although it is often useful to think of "anti" fields as being row vectors instead of column, i.e., fundamental indices are upper indices and anti-fundamental indices are lower indices; in this convention, the anti-fundamental representation acts from the right and it reads $R(g) = g^\dagger$.

The representation $R(g) = g$ is clearly faithful, hence any representation of $G = \mathrm{SU}(N), \mathrm{SO}(N), \mathrm{Sp}(N)$ is contained in the tensor product of some number $r_1$ of fundamentals, and $r_2$ anti-fundamentals. The number $r = r_1 + r_2$ is the *rank* of the representation. In other words, an arbitrary representation is of the form

$$\begin{aligned} R(g)\colon \psi^{i_1 i_2 \ldots i_{r_1}}{}_{i'_1 i'_2 \ldots i'_{r_2}} \mapsto \quad & g^{i_1}{}_{j_1}\, g^{i_2}{}_{j_2} \cdots g^{i_{r_1}}{}_{j_{r_1}} \\ & \times (g^\dagger)^{j'_1}{}_{i'_1}\, (g^\dagger)^{j'_2}{}_{i'_2} \cdots (g^\dagger)^{j'_{r_2}}{}_{i'_{r_2}} \\ & \times \psi^{j_1 j_2 \ldots j_{r_1}}{}_{j'_1 j'_2 \ldots j'_{r_2}} \end{aligned} \tag{B.0.1}$$

where $g \in G$. (By contrast, the fundamental representation of $\mathrm{Spin}(N)$ is not faithful and thus it does not generate the set of spinorial representations; in general the lack of faithfulness of a given representation is conveniently captured by its charge under the center of the group.)

In the case of $\mathrm{SO}(N)$ and $\mathrm{Sp}(N)$ we have the invariant symbols $\delta^{ij}$ and $\Omega^{ij}$ which allow us to raise and lower indices, which means that we can take without loss of generality $r_2 = 0$, so that all indices are upper indices. The group $\mathrm{SU}(N)$ has invariant symbol $\delta^i_j$, which does not raise or lower indices, so fundamental and anti-fundamental indices are truly distinct. That being said, this group also has invariant symbol $\epsilon^{i_1 i_2 \cdots i_N}$, whence we can exchange $r$ anti-symmetrized fundamental indices for $N - r$ anti-symmetrized anti-fundamental ones. In other words, we can replace in its Young diagram any column with $N + \ell$ boxes for one with $\ell$ boxes. It is notationally convenient to extend this to negative $\ell$ so, for example, a column of $N - 1$ boxes is equivalent to one with "$-1$ boxes", which are drawn below the Young diagram. Specifically, we draw the usual Young diagram for the upper indices, and an inverted diagram for the lower indices, such that for example the representation $\psi^i{}_{i'}$ is drawn as , with the "anti-box" on the left standing for a column with $N - 1$ regular boxes.

A given representation can be either complex or self-conjugate. The latter can be further subdivided into real and pseudo-real representations. We denote the field over which the representation is defined as $\mathbb{K}$. Complex, real, and pseudo-real representations have $\mathbb{K} = \mathbb{C}, \mathbb{R}, \mathbb{H}$, respectively. Note that complex representations occur only when the Dynkin diagram of $G$ has a $\mathbb{Z}_2$ reflection symmetry, i.e., $A_n$ and $D_n$ (and also $E_6$). This outer automorphism acts on the Dynkin labels of the representation as it acts on the nodes of the diagram.

In order for $R$ to be irreducible we must impose suitable (anti)symmetrization on all the indices. Finally, we must subtract traces using invariant symbols, if possible. We next summarize the basic properties of all the representations of $\mathrm{SU}(N), \mathrm{SO}(N), \mathrm{Sp}(N)$ up to rank-4.

## B.1 Summary of representations.

**SU($N$):** For a given representation with Dynkin labels $(\lambda_1, \lambda_2, \dots, \lambda_{N-1})$, the complex conjugate representation is the representation with reversed Dynkin labels: $(\lambda_{N-1}, \lambda_{N-2}, \dots, \lambda_1)$. The representation is self-conjugate if and only if it is equal to its conjugate representation (i.e., if the Dynkin labels form a palindrome). A representation has the same dimension and index as its conjugate. A self-conjugate representation is pseudo-real if $N = 2 \mod 4$ and $\lambda_{N/2}$ is odd; otherwise it is real. (In what follows $\mathbb{K}$ denotes the "generic case", i.e., valid for most $N$; for some specific $N$ a representation labelled as $\mathbb{C}$ may in fact be $\mathbb{R}$ or $\mathbb{H}$, such as e.g. the rank-$r$ anti-symmetric for $N = 2r$).

The center of the group is $Z(SU(n)) = \mathbb{Z}_N$, and it acts on an arbitrary representation as $\psi \mapsto \mu^{r_1 - r_2}\psi$, which can also be written in terms of Dynkin labels as $r_1 - r_2 = \lambda_1 + 2\lambda_2 + 3\lambda_3 + \cdots + (N-1)\lambda_{N-1} \mod N$; this number also agrees with the number of boxes minus the number of anti-boxes.

**Rank 1:** The representation is the fundamental representation:

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
| $\square$ | $(1, 0, \dots, 0)$ | $2N$ | $1$ | $\mathbb{C}$ |

Under $\mathfrak{su}_{A+B} \to \mathfrak{su}_A \oplus \mathfrak{su}_B$, this representation decomposes as

$$\square \mapsto \square \otimes \bullet + \bullet \otimes \square \tag{B.1.1}$$

**Rank 2:** We have the symmetric and anti-symmetric, which satisfy $\psi^{ij} = +\psi^{ji}$ and $\psi^{ij} = -\psi^{ji}$:

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
| $\square\square$ | $(2, 0, 0, \dots, 0)$ | $N(N+1)$ | $N+2$ | $\mathbb{C}$ |
| $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ | $(0, 1, 0, \dots, 0)$ | $N(N-1)$ | $N-2$ | $\mathbb{C}$ |

We also have a traceless representation, namely the adjoint, which satisfies $(\psi^i{}_j)^\dagger = \psi^j{}_i$ and $\psi^i{}_i = 0$:

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
| $\begin{smallmatrix} & \square\\\square & \end{smallmatrix}$ | $(1, 0, \dots, 0, 1)$ | $N^2 - 1$ | $N$ | $\mathbb{R}$ |

Under $\mathfrak{su}_{A+B} \to \mathfrak{su}_A \oplus \mathfrak{su}_B$, these representations decompose as

$$\begin{aligned} \ydiagram{2} &\mapsto \ydiagram{2} \otimes \bullet + \square \otimes \square + \bullet \otimes \ydiagram{2} \\ \ydiagram{1,1} &\mapsto \ydiagram{1,1} \otimes \bullet + \square \otimes \square + \bullet \otimes \ydiagram{1,1} \\ \ydiagram{1+1,1} &\mapsto \ydiagram{1+1,1} \otimes \bullet + \bar{\square} \otimes \square + \bullet \otimes \bullet + \square \otimes \bar{\square} + \bullet \otimes \ydiagram{1+1,1} \end{aligned} \tag{B.1.2}$$

**Rank 3:** We have the symmetric, anti-symmetric and mixed. The first one is completely symmetric with respect to its three indices, the second one completely anti-symmetric, and the third one satisfies $\psi^{ijk} = \psi^{jik}$ and $\psi^{ijk} + \psi^{kji} + \psi^{ikj} = 0$:

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
| $\ydiagram{3}$ | $(3,0,0,0,\dots,0)$ | $\frac{1}{3}N(N+1)(N+2)$ | $\frac{1}{2}(N+2)(N+3)$ | $\mathbb{C}$ |
| $\ydiagram{1,1,1}$ | $(0,0,1,0,\dots,0)$ | $\frac{1}{3}N(N-1)(N-2)$ | $\frac{1}{2}(N-2)(N-3)$ | $\mathbb{C}$ |
| $\ydiagram{2,1}$ | $(1,1,0,0,\dots,0)$ | $\frac{2}{3}N(N+1)(N-1)$ | $N^2-3$ | $\mathbb{C}$ |

We also have two traceless representations:

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
| $\ydiagram{1+2,1}$ | $(2,0,0,\dots,0,1)$ | $N(N-1)(N+2)$ | $\frac{1}{2}(N+2)(3N-1)$ | $\mathbb{C}$ |
| $\ydiagram{1+1,1+1,1}$ | $(0,1,0,\dots,0,1)$ | $N(N-2)(N+1)$ | $\frac{1}{2}(N-2)(3N+1)$ | $\mathbb{C}$ |

These satisfy ${\psi^{ij}}_k = +{\psi^{ji}}_k$ and ${\psi^{ij}}_k = -{\psi^{ji}}_k$, respectively, together with ${\psi^{ij}}_j = 0$.

Under $\mathfrak{su}_{A+B} \to \mathfrak{su}_A \oplus \mathfrak{su}_B$, these representations decompose as

$$
\begin{aligned}
\ydiagram{3} &\mapsto \ydiagram{3} \otimes \bullet + \ydiagram{2} \otimes \ydiagram{1} + \ydiagram{1} \otimes \ydiagram{2} + \bullet \otimes \ydiagram{3} \\
\ydiagram{1,1,1} &\mapsto \text{previous line, transposed} \\
\ydiagram{2,1} &\mapsto \ydiagram{2,1} \otimes \bullet \\
&\quad + \ydiagram{1,1} \otimes \ydiagram{1} + \ydiagram{1} \otimes \ydiagram{1,1} \\
&\quad + \ydiagram{2} \otimes \ydiagram{1} + \ydiagram{1} \otimes \ydiagram{2} \\
&\quad + \bullet \otimes \ydiagram{2,1} \\
\ydiagram{1+1,1+1,1} &\mapsto \ydiagram{1+1,1+1,1} \otimes \bullet \\
&\quad + \ydiagram{1+1,1} \otimes \ydiagram{1} + \ydiagram{1} \otimes \bullet + \bullet \otimes \ydiagram{1} + \ydiagram{1} \otimes \ydiagram{1+1,1} \\
&\quad + \ydiagram{1,1} \otimes \bar{\ydiagram{1}} + \bar{\ydiagram{1}} \otimes \ydiagram{1,1} \\
&\quad + \bullet \otimes \ydiagram{1+1,1+1,1} \\
\ydiagram{1+2,1} &\mapsto \text{previous line, transposed}
\end{aligned}
\tag{B.1.3}
$$

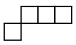
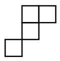
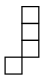
**Rank 4:** The representations are as follows:

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
| $\ydiagram{4}$ | $(4,0,0,0,0,\ldots,0)$ | $\frac{1}{12}N(N+1)(N+2)(N+3)$ | $\frac{1}{6}(N+2)(N+3)(N+4)$ | $\mathbb{C}$ |
| $\ydiagram{3,1}$ | $(2,1,0,0,0,\ldots,0)$ | $\frac{1}{4}N(N-1)(N+1)(N+2)$ | $\frac{1}{2}(N+2)(N^2+N-4)$ | $\mathbb{C}$ |
| $\ydiagram{2,1,1}$ | $(1,0,1,0,0,\ldots,0)$ | $\frac{1}{4}N(N-2)(N-1)(N+1)$ | $\frac{1}{2}(N-2)(N^2-N-4)$ | $\mathbb{C}$ |
| $\ydiagram{2,2}$ | $(0,2,0,0,0,\ldots,0)$ | $\frac{1}{6}N^2(N-1)(N+1)$ | $\frac{1}{3}N(N-2)(N+2)$ | $\mathbb{C}$ |
| $\ydiagram{1,1,1,1}$ | $(0,0,0,1,0,\ldots,0)$ | $\frac{1}{12}N(N-1)(N-2)(N-3)$ | $\frac{1}{6}(N-4)(N-3)(N-2)$ | $\mathbb{C}$ |

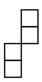
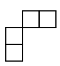
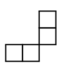
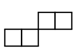
which satisfy

$$
\begin{aligned}
&\psi^{ijk\ell} = \psi^{(ijk\ell)} \\
&\psi^{ijk\ell} = \psi^{(ijk)\ell}, \quad \psi^{ijk\ell} + \psi^{\ell jki} + \psi^{i\ell kj} + \psi^{ij\ell k} = 0 \\
&\psi^{ijk\ell} = \psi^{[ijk]\ell}, \quad \psi^{ijk\ell} - \psi^{\ell jki} - \psi^{i\ell kj} - \psi^{ij\ell k} = 0 \\
&\psi^{ijk\ell} = \psi^{[ij]k\ell} = \psi^{ij[k\ell]} = \psi^{k\ell ij}, \quad \psi^{ijk\ell} + \psi^{ik\ell j} + \psi^{i\ell jk} = 0 \\
&\psi^{ijk\ell} = \psi^{[ijk\ell]}
\end{aligned}
\tag{B.1.4}
$$

respectively. We also have representations with one trace removed:

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
|  | $(3,0,0,0,\ldots,0,1)$ | $\frac{1}{3}N(N-1)(N+1)(N+3)$ | $\frac{2}{3}(N+2)(N+3)(2N-1)$ | $\mathbb{C}$ |
|  | $(1,1,0,0,\ldots,0,1)$ | $\frac{2}{3}N^2(N-2)(N+2)$ | $\frac{8}{3}N(N-2)(N+2)$ | $\mathbb{C}$ |
|  | $(0,0,1,0,\ldots,0,1)$ | $\frac{1}{3}N(N-3)(N-1)(N+1)$ | $\frac{2}{3}(N-3)(N-2)(2N+1)$ | $\mathbb{C}$ |

which satisfy the symmetry properties of the corresponding rank-3 representation (such as e.g. $\psi^{ijk}{}_{\ell} = \psi^{(ijk)}{}_{\ell}$ and $\psi^{ijk}{}_{i} = 0$ for the first representations), and representations with two traces removed:

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
|  | $(0,1,0,\ldots,0,1,0)$ | $\frac{1}{4}N^2(N-3)(N+1)$ | $N^2(N-3)$ | $\mathbb{R}$ |
|  | $(2,0,0,\ldots,0,1,0)$ | $\frac{1}{2}(N-2)(N-1)(N+1)(N+2)$ | $2N(N-2)(N+2)$ | $\mathbb{C}$ |
|  | $(0,1,0,\ldots,0,0,2)$ | $\frac{1}{2}(N-2)(N-1)(N+1)(N+2)$ | $2N(N-2)(N+2)$ | $\mathbb{C}$ |
|  | $(2,0,0,\ldots,0,0,2)$ | $\frac{1}{4}N^2(N-1)(N+3)$ | $N^2(N+3)$ | $\mathbb{R}$ |

which satisfy the symmetry properties of the corresponding pair of rank-2 representations, together with the appropriate hermiticity condition, if real. (For example, the first representation satisfies $\psi^{ij}{}_{k\ell} = \psi^{[ij]}{}_{k\ell} = \psi^{ij}{}_{[k\ell]}$ and $\psi^{ij}{}_{i\ell} = 0$, together with $(\psi^{ij}{}_{k\ell})^{\dagger} = \psi^{k\ell}{}_{ij}$).

Under $\mathfrak{su}_{A+B} \to \mathfrak{su}_A \oplus \mathfrak{su}_B$, these representations decompose as

$$
\begin{aligned}
\ydiagram{4} &\mapsto \ydiagram{4} \otimes \bullet + \ydiagram{3} \otimes \ydiagram{1} + \ydiagram{2} \otimes \ydiagram{2} + \ydiagram{1} \otimes \ydiagram{3} + \bullet \otimes \ydiagram{4} \\
\ydiagram{1,1,1,1} &\mapsto \text{previous line, transposed} \\
\ydiagram{3,1} &\mapsto \ydiagram{3,1} \otimes \bullet \\
&\quad + \ydiagram{3} \otimes \ydiagram{1} + \ydiagram{2} \otimes \ydiagram{2} + \ydiagram{1} \otimes \ydiagram{3} \\
&\quad + \ydiagram{2,1} \otimes \ydiagram{1} + \ydiagram{1,1} \otimes \ydiagram{2} + \ydiagram{2} \otimes \ydiagram{1,1} + \ydiagram{1} \otimes \ydiagram{2,1} \\
&\quad + \bullet \otimes \ydiagram{3,1} \\
\ydiagram{2,1,1} &\mapsto \text{previous line, transposed} \\
\ydiagram{2,2} &\mapsto \ydiagram{2,2} \otimes \bullet + \ydiagram{2,1} \otimes \ydiagram{1} + \ydiagram{2} \otimes \ydiagram{2} + \ydiagram{1,1} \otimes \ydiagram{1,1} + \ydiagram{1} \otimes \ydiagram{2,1} + \bullet \otimes \ydiagram{2,2} \\
\ydiagram{1+3,1} &\mapsto \ydiagram{1+3,1} \otimes \bullet \\
&\quad + \ydiagram{3} \otimes \bar{\ydiagram{1}} + \ydiagram{2} \otimes \bullet + \ydiagram{1} \otimes \ydiagram{1} + \bullet \otimes \ydiagram{2} + \bar{\ydiagram{1}} \otimes \ydiagram{3} \\
&\quad + \ydiagram{1+2,1} \otimes \ydiagram{1} + \ydiagram{1+1,1} \otimes \ydiagram{2} + \ydiagram{2} \otimes \ydiagram{1+1,1} + \ydiagram{1} \otimes \ydiagram{1+2,1} \\
&\quad + \bullet \otimes \ydiagram{1+3,1} \\
\ydiagram{1+1,1+1,1+1,1} &\mapsto \text{previous line, transposed} \\
\ydiagram{1+2,1+1,1} &\mapsto \ydiagram{1+2,1+1,1} \otimes \bullet \\
&\quad + \ydiagram{2,1} \otimes \bar{\ydiagram{1}} + \ydiagram{2} \otimes \bullet + \ydiagram{1} \otimes \ydiagram{1} + \bullet \otimes \ydiagram{2} + \bar{\ydiagram{1}} \otimes \ydiagram{2,1} \\
&\quad + \ydiagram{1+1,1+1,1} \otimes \ydiagram{1} + \ydiagram{1+1,1} \otimes \ydiagram{2} + \ydiagram{1} \otimes \ydiagram{1} + \ydiagram{2} \otimes \ydiagram{1+1,1} + \ydiagram{1} \otimes \ydiagram{1+1,1+1,1} \\
&\quad + \ydiagram{1,1} \otimes \ydiagram{1+1,1} + \ydiagram{1} \otimes \ydiagram{1+2,1} + \ydiagram{1+2,1} \otimes \ydiagram{1} + \ydiagram{1+1,1} \otimes \ydiagram{1,1} \\
&\quad + \ydiagram{1,1} \otimes \bullet + \bullet \otimes \ydiagram{1,1} \\
&\quad + \bullet \otimes \ydiagram{1+2,1+1,1} \qquad\qquad\qquad\qquad \text{(B.1.5)} \\
\ydiagram{1+1,1+1,1,1} &\mapsto \ydiagram{1+1,1+1,1,1} \otimes \bullet \\
&\quad + \ydiagram{1+1,1,1} \otimes \ydiagram{1} + \overline{\ydiagram{1,1}} \otimes \ydiagram{1,1} + \bar{\ydiagram{1}} \otimes \ydiagram{1} + \bullet \otimes \bullet + \ydiagram{1} \otimes \bar{\ydiagram{1}} + \ydiagram{1,1} \otimes \overline{\ydiagram{1,1}} + \ydiagram{1} \otimes \ydiagram{1+1,1,1} \\
&\quad + \ydiagram{1+1,1+1,1} \otimes \bar{\ydiagram{1}} + \ydiagram{1+1,1} \otimes \bullet + \ydiagram{1+1,1} \otimes \ydiagram{1+1,1} + \bullet \otimes \ydiagram{1+1,1} + \bar{\ydiagram{1}} \otimes \ydiagram{1+1,1+1,1} \\
&\quad + \bullet \otimes \ydiagram{1+1,1+1,1,1} \\
\ydiagram{2+2,2} &\mapsto \text{previous line, transposed} \\
\ydiagram{1+2,1,1} &\mapsto \ydiagram{1+2,1,1} \otimes \bullet \\
&\quad + \ydiagram{1+1,1,1} \otimes \ydiagram{1} + \overline{\ydiagram{1,1}} \otimes \ydiagram{2} + \ydiagram{2} \otimes \overline{\ydiagram{1,1}} + \ydiagram{1} \otimes \ydiagram{1+1,1,1} \\
&\quad + \ydiagram{1+2,1} \otimes \bar{\ydiagram{1}} + \ydiagram{1+1,1} \otimes \ydiagram{1+1,1} + \bar{\ydiagram{1}} \otimes \ydiagram{1+2,1} \\
&\quad + \ydiagram{1+1,1} \otimes \bullet + \bar{\ydiagram{1}} \otimes \ydiagram{1} + \ydiagram{1} \otimes \bar{\ydiagram{1}} + \bullet \otimes \ydiagram{1+1,1} \\
&\quad + \bullet \otimes \ydiagram{1+2,1,1} \\
\ydiagram{2+1,2+1,2} &\mapsto \text{previous line, transposed}
\end{aligned}
$$

**SO($N$):** Consider a representation $(\lambda_1, \lambda_2, \ldots, \lambda_{\lfloor N/2 \rfloor})$ of Spin($N$). It is self-conjugate except if $N = 4n+2$, in which case its conjugate is obtained by permuting the last two labels: $(\lambda_1, \lambda_2, \ldots, \lambda_{2n-1}, \lambda_{2n+1}, \lambda_{2n})$. A self-conjugate representation is pseudo-real if $N = 3, 5 \mod 8$ and $\lambda_{(N-1)/2}$ is odd, or if $N = 2 \mod 8$ and $\lambda_{(N/2)-1} + \lambda_{N/2}$ is odd; otherwise it is real. A representation of Spin($N$) descends to a representation of SO($N$) if and only if $N$ is odd and $\lambda_{(N-1)/2}$ is even, or if $N$ is even and $\lambda_{(N/2)-1} + \lambda_{N/2}$ is even.

The symmetry properties of the tensors is the same as in the SU($N$) case, except that we subtract a trace for every pair of symmetrized indices. All the representations are real. Also, SO($N$) has no center if $N$ is odd, and a $\mathbb{Z}_2$ center if $N$ is even. In the latter case it acts as $\psi \mapsto (-1)^r \psi$; this number also agrees with the number of boxes.

**Rank 1:** The representation is the fundamental representation:

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
| $\square$ | $(1, 0, \ldots, 0)$ | $N$ | $1$ | $\mathbb{R}$ |

**Rank 2:** We have the symmetric and anti-symmetric, which satisfy $\psi^{ij} = +\psi^{ji}$ and $\psi^{ij} = -\psi^{ji}$, respectively. Also, the symmetric is traceless: $\psi^{ij}\delta_{ij} = 0$. With this, the representations are as follows:

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
| $\square\square$ | $(2, 0, 0, \ldots, 0)$ | $\frac{1}{2}(N-1)(N+2)$ | $N+2$ | $\mathbb{R}$ |
| $\begin{array}{c}\square\\\square\end{array}$ | $(0, 1, 0, \ldots, 0)$ | $\frac{1}{2}N(N-1)$ | $N-2$ | $\mathbb{R}$ |

**Rank 3:** We have the symmetric and anti-symmetric, and mixed:

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
| $\square\square\square$ | $(3, 0, 0, 0, \ldots, 0)$ | $\frac{1}{6}N(N-1)(N+4)$ | $\frac{1}{2}(N+1)(N+4)$ | $\mathbb{R}$ |
| $\begin{array}{ll}\square&\square\\\square&\end{array}$ | $(1, 1, 0, 0, \ldots, 0)$ | $\frac{1}{3}N(N-2)(N+2)$ | $(N-2)(N+2)$ | $\mathbb{R}$ |
| $\begin{array}{c}\square\\\square\\\square\end{array}$ | $(0, 0, 1, 0, \ldots, 0)$ | $\frac{1}{6}N(N-1)(N-2)$ | $\frac{1}{2}(N-2)(N-3)$ | $\mathbb{R}$ |

which satisfy the same symmetry properties as the SU($N$) representations, but we also impose tracelessness for every pair of symmetrized indices (e.g., the first representation satisfies $\psi^{ijk} = \psi^{(ijk)}$ and $\psi^{ijk}\delta_{ij} = 0$).

**Rank 4:** We have

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
| $\ydiagram{4}$ | $(4,0,0,0,0,\ldots,0)$ | $\frac{1}{24}N(N-1)(N+1)(N+6)$ | $\frac{1}{6}(N+1)(N+2)(N+6)$ | $\mathbb{R}$ |
| $\ydiagram{3,1}$ | $(2,1,0,0,0,\ldots,0)$ | $\frac{1}{8}(N-1)(N-2)(N+1)(N+4)$ | $\frac{1}{2}(N-2)(N+1)(N+4)$ | $\mathbb{R}$ |
| $\ydiagram{2,1,1}$ | $(1,0,1,0,0,\ldots,0)$ | $\frac{1}{8}N(N-1)(N-3)(N+2)$ | $\frac{1}{2}(N-2)(N-3)(N+2)$ | $\mathbb{R}$ |
| $\ydiagram{2,2}$ | $(0,2,0,0,0,\ldots,0)$ | $\frac{1}{12}N(N-3)(N+1)(N+2)$ | $\frac{1}{3}(N-3)(N+1)(N+2)$ | $\mathbb{R}$ |
| $\ydiagram{1,1,1,1}$ | $(0,0,0,1,0,\ldots,0)$ | $\frac{1}{24}N(N-1)(N-2)(N-3)$ | $\frac{1}{6}(N-2)(N-3)(N-4)$ | $\mathbb{R}$ |

which again satisfy the same symmetry properties of the corresponding SU($N$) representation, with traces removed.

The representations decompose under $\mathfrak{so}_{A+B} \to \mathfrak{so}_A \oplus \mathfrak{so}_B$ the same as in $\mathfrak{su}_N$, with the following extra representations:

$$\begin{aligned}
\ydiagram{2} &\mapsto \cdots + \bullet \otimes \bullet \\
\ydiagram{3} &\mapsto \cdots + \square \otimes \bullet + \bullet \otimes \square \\
\ydiagram{2,1} &\mapsto \cdots + \square \otimes \bullet + \bullet \otimes \square \\
\ydiagram{4} &\mapsto \cdots + \ydiagram{2} \otimes \bullet + \square \otimes \square + \bullet \otimes \bullet + \bullet \otimes \ydiagram{2} \\
\ydiagram{3,1} &\mapsto \cdots + \ydiagram{2} \otimes \bullet + \square \otimes \square + \bullet \otimes \ydiagram{2} \\
&\quad + \ydiagram{1,1} \otimes \bullet + \square \otimes \square + \bullet \otimes \ydiagram{1,1} \\
\ydiagram{2,1,1} &\mapsto \cdots + \ydiagram{1,1} \otimes \bullet + \square \otimes \square + \bullet \otimes \ydiagram{1,1} \\
\ydiagram{2,2} &\mapsto \cdots + \ydiagram{2} \otimes \bullet + \square \otimes \square + \bullet \otimes \bullet + \bullet \otimes \ydiagram{2}
\end{aligned} \tag{B.1.6}$$

**Sp($N$):** All the representations $(\lambda_1, \lambda_2, \ldots, \lambda_N)$ are self-conjugate. A given representation is pseudo-real if $\lambda_1 + \lambda_3 + \lambda_5 + \ldots$ is odd, and real otherwise; equivalently, it is pseudo-real if its rank is odd, and real otherwise. The representations are as follows:

**Rank 1:** The representation is the fundamental representation:

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
| $\square$ | $(1,0,\ldots,0)$ | $2N$ | $1$ | $\mathbb{H}$ |

**Rank 2:** We have the symmetric and anti-symmetric. The latter is $\Omega$-traceless: $\psi^{ij}\Omega_{ij} = 0$. Thus,

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
| $\ydiagram{2}$ | $(2,0,0,\ldots,0)$ | $N(2N+1)$ | $N+1$ | $\mathbb{R}$ |
| $\ydiagram{1,1}$ | $(0,1,0,\ldots,0)$ | $(N-1)(2N+1)$ | $N-1$ | $\mathbb{R}$ |

**Rank 3:** We have the symmetric, anti-symmetric, and mixed:

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
| $\ydiagram{3}$ | $(3,0,0,0,\ldots,0)$ | $\frac{2}{3}N(N+1)(2N+1)$ | $(N+1)(2N+3)$ | $\mathbb{H}$ |
| $\ydiagram{2,1}$ | $(1,1,0,0,\ldots,0)$ | $\frac{8}{3}N(N-1)(N+1)$ | $4(N-1)(N+1)$ | $\mathbb{H}$ |
| $\ydiagram{1,1,1}$ | $(0,0,1,0,\ldots,0)$ | $\frac{2}{3}N(N-2)(2N+1)$ | $(N-2)(2N-1)$ | $\mathbb{H}$ |

which have the same symmetry properties as the $\mathrm{SU}(N)$ representations, but with a tracelesssness condition on every pair of anti-symmetrized indices (e.g., the last representation satisfies $\psi^{ijk} = \psi^{[ijk]}$ and $\psi^{ijk}\Omega_{ij} = 0$).

**Rank 4:**

| $Y$ | $\lambda$ | $\dim_{\mathbb{R}}(\lambda)$ | $T_{\mathbb{R}}(\lambda)$ | $\mathbb{K}$ |
|---|---|---|---|---|
| $\ydiagram{4}$ | $(4,0,0,0,0,\ldots,0)$ | $\frac{1}{6}N(N+1)(2N+1)(2N+3)$ | $\frac{1}{3}(N+1)(N+2)(2N+3)$ | $\mathbb{R}$ |
| $\ydiagram{3,1}$ | $(2,1,0,0,0,\ldots,0)$ | $\frac{1}{2}N(N-1)(2N+1)(2N+3)$ | $(N-1)(N+1)(2N+3)$ | $\mathbb{R}$ |
| $\ydiagram{2,1,1}$ | $(1,0,1,0,0,\ldots,0)$ | $\frac{1}{2}(N-2)(N+1)(2N-1)(2N+1)$ | $(N-2)(N+1)(2N-1)$ | $\mathbb{R}$ |
| $\ydiagram{2,2}$ | $(0,2,0,0,0,\ldots,0)$ | $\frac{1}{3}N(N-1)(2N-1)(2N+3)$ | $\frac{1}{3}(N-1)(2N-1)(2N+3)$ | $\mathbb{R}$ |
| $\ydiagram{1,1,1,1}$ | $(0,0,0,1,0,\ldots,0)$ | $\frac{1}{6}N(N-3)(2N-1)(2N+1)$ | $\frac{1}{3}(N-3)(N-1)(2N-1)$ | $\mathbb{R}$ |

where again the symmetry properties are identical to those of the $\mathrm{SU}(N)$ representations, with traced removed as appropriate.

The representations decompose under $\mathfrak{sp}_{A+B} \to \mathfrak{sp}_A \oplus \mathfrak{sp}_B$ the same as in $\mathfrak{su}_N$, with the following extra representations:

$$
\begin{aligned}
\ydiagram{1,1} &\mapsto \cdots + \bullet \otimes \bullet \\
\ydiagram{1,1,1} &\mapsto \cdots + \square \otimes \bullet + \bullet \otimes \square \\
\ydiagram{2,1} &\mapsto \cdots + \square \otimes \bullet + \bullet \otimes \square \\
\ydiagram{1,1,1,1} &\mapsto \cdots + \ydiagram{1,1} \otimes \bullet + \square \otimes \square + \bullet \otimes \bullet + \bullet \otimes \ydiagram{1,1} \\
\ydiagram{3,1} &\mapsto \cdots + \ydiagram{2} \otimes \bullet + \square \otimes \square + \bullet \otimes \ydiagram{2} \\
\ydiagram{2,1,1} &\mapsto \cdots + \ydiagram{2} \otimes \bullet + \square \otimes \square + \bullet \otimes \ydiagram{2} \\
&\qquad + \ydiagram{1,1} \otimes \bullet + \square \otimes \square + \bullet \otimes \ydiagram{1,1} \\
\ydiagram{2,2} &\mapsto \cdots + \ydiagram{1,1} \otimes \bullet + \square \otimes \square + \bullet \otimes \bullet + \bullet \otimes \ydiagram{1,1}
\end{aligned}
\tag{B.1.7}
$$

## B.2 Group characters.

In this section we write the characters of the basic representations of $\mathrm{SU}(N), \mathrm{SO}(N), \mathrm{Sp}(N)$. The group characters are defined as

$$
\chi_R(g) \equiv \mathrm{tr}_R(g) := \mathrm{tr}\, R(g) \tag{B.2.1}
$$

where in the last expression tr denotes the regular matrix trace. Note that, by definition, characters are class functions, i.e., they only depend on the conjugacy class of $g$. In other words, we can assume without loss of generality that $g$ is inside a maximal torus of $G$, which in practice essentially means that it is diagonal. Hence, $\chi_R(g)$ can be expressed as a certain polynomial (of degree $r$) in the eigenvalues of $g$.

The most fundamental character is the character of the fundamental representation, namely $\chi_\square(g) \equiv \mathrm{tr}\, g$. This character can be expressed as a sum of eigenvalues of $g$. Any other character can be expressed as a certain linear combination of monomials of the form $\chi_\square(g^m)^n$ for integers $m, n$. In practice, one can use e.g. the Weyl character formula [327], which gives an explicit expression of $\chi_R$ in terms of the eigenvalues of $g$, which can then be reorganized as a linear combination of the monomials $\chi_\square(g^m)^n$. In what follows we present the end result of this little exercise for the first few representations of $G$.

• **SU($N$)**

$$
\begin{aligned}
\mathrm{tr}_{\ydiagram{1}}(g) &= \mathrm{tr}(g) \\
\mathrm{tr}_{\ydiagram{2}}(g) &= \frac{1}{2}(+\,\mathrm{tr}(g^2) + \mathrm{tr}(g)^2) \\
\mathrm{tr}_{\ydiagram{1,1}}(g) &= \frac{1}{2}(-\,\mathrm{tr}(g^2) + \mathrm{tr}(g)^2) \\
\mathrm{tr}_{\ydiagram{3}}(g) &= \frac{1}{3!}(+2\,\mathrm{tr}(g^3) + 3\,\mathrm{tr}(g^2)\,\mathrm{tr}(g) + \mathrm{tr}(g)^3) \\
\mathrm{tr}_{\ydiagram{2,1}}(g) &= \frac{1}{3}(-\,\mathrm{tr}(g^3) + \mathrm{tr}(g)^3) \\
\mathrm{tr}_{\ydiagram{1,1,1}}(g) &= \frac{1}{3!}(+2\,\mathrm{tr}(g^3) - 3\,\mathrm{tr}(g^2)\,\mathrm{tr}(g) + \mathrm{tr}(g)^3) \\
\mathrm{tr}_{\ydiagram{4}}(g) &= \frac{1}{4!}(+6\,\mathrm{tr}(g^4) + 8\,\mathrm{tr}(g^3)\,\mathrm{tr}(g) + 3\,\mathrm{tr}(g^2)^2 \\
&\quad + 6\,\mathrm{tr}(g^2)\,\mathrm{tr}(g)^2 + \mathrm{tr}(g)^4) \\
\mathrm{tr}_{\ydiagram{3,1}}(g) &= \frac{1}{8}(-2\,\mathrm{tr}(g^4) + 2\,\mathrm{tr}(g^2)\,\mathrm{tr}(g)^2 - \mathrm{tr}(g^2)^2 + \mathrm{tr}(g)^4) \\
\mathrm{tr}_{\ydiagram{2,2}}(g) &= \frac{1}{12}(-4\,\mathrm{tr}(g^3)\,\mathrm{tr}(g) + 3\,\mathrm{tr}(g^2)^2 + \mathrm{tr}(g)^4) \\
\mathrm{tr}_{\ydiagram{2,1,1}}(g) &= \frac{1}{8}(+2\,\mathrm{tr}(g^4) - 2\,\mathrm{tr}(g^2)\,\mathrm{tr}(g)^2 - \mathrm{tr}(g^2)^2 + \mathrm{tr}(g)^4) \\
\mathrm{tr}_{\ydiagram{1,1,1,1}}(g) &= \frac{1}{4!}(-6\,\mathrm{tr}(g^4) + 8\,\mathrm{tr}(g^3)\,\mathrm{tr}(g) + 3\,\mathrm{tr}(g^2)^2 \\
&\quad - 6\,\mathrm{tr}(g^2)\,\mathrm{tr}(g)^2 + \mathrm{tr}(g)^4)
\end{aligned}
\tag{B.2.2}
$$

We also have the representations with traces removed:

$$
\begin{aligned}
\operatorname{tr}_{\ydiagram{1+1,1}}(g) &= \operatorname{tr}(g)\operatorname{tr}(g^{-1}) - 1 \\
\operatorname{tr}_{\ydiagram{1+2,1}}(g) &= \frac{1}{2}(+\operatorname{tr}(g)^2\operatorname{tr}(g^{-1}) + \operatorname{tr}(g^2)\operatorname{tr}(g^{-1}) - 2\operatorname{tr}(g)) \\
\operatorname{tr}_{\ydiagram{1+1,1+1,1}}(g) &= \frac{1}{2}(+\operatorname{tr}(g)^2\operatorname{tr}(g^{-1}) - \operatorname{tr}(g^2)\operatorname{tr}(g^{-1}) - 2\operatorname{tr}(g)) \\
\operatorname{tr}_{\ydiagram{1+3,1}}(g) &= \frac{1}{6}(+\operatorname{tr}(g)^3\operatorname{tr}(g^{-1}) + 3\operatorname{tr}(g^2)\operatorname{tr}(g)\operatorname{tr}(g^{-1}) \\
&\quad + 2\operatorname{tr}(g^3)\operatorname{tr}(g^{-1}) - 3\operatorname{tr}(g)^2 - 3\operatorname{tr}(g^2)) \\
\operatorname{tr}_{\ydiagram{1+2,1+1,1}}(g) &= \frac{1}{3}(+\operatorname{tr}(g)^3\operatorname{tr}(g^{-1}) - \operatorname{tr}(g^3)\operatorname{tr}(g^{-1}) - 3\operatorname{tr}(g)^2) \\
\operatorname{tr}_{\ydiagram{1+1,1+1,1+1,1}}(g) &= \frac{1}{6}(+\operatorname{tr}(g)^3\operatorname{tr}(g^{-1}) - 3\operatorname{tr}(g^2)\operatorname{tr}(g)\operatorname{tr}(g^{-1}) \\
&\quad + 2\operatorname{tr}(g^3)\operatorname{tr}(g^{-1}) + 3\operatorname{tr}(g^2) - 3\operatorname{tr}(g)^2) \\
\operatorname{tr}_{\ydiagram{1+1,1+1,1,1}}(g) &= \frac{1}{4}(+\operatorname{tr}(g)^2\operatorname{tr}(g^{-1})^2 - \operatorname{tr}(g^2)\operatorname{tr}(g^{-1})^2 - \operatorname{tr}(g)^2\operatorname{tr}(g^{-2}) \\
&\quad - 4\operatorname{tr}(g)\operatorname{tr}(g^{-1}) + \operatorname{tr}(g^2)\operatorname{tr}(g^{-2})) \\
\operatorname{tr}_{\ydiagram{1+2,1,1}}(g) &= \frac{1}{4}(-\operatorname{tr}(g)^2\operatorname{tr}(g^{-1})^2 - \operatorname{tr}(g^2)\operatorname{tr}(g^{-1})^2 + \operatorname{tr}(g)^2\operatorname{tr}(g^{-2}) \\
&\quad + \operatorname{tr}(g^2)\operatorname{tr}(g^{-2}) + 4\operatorname{tr}(g)\operatorname{tr}(g^{-1})) - 1 \\
\operatorname{tr}_{\ydiagram{2+2,2}}(g) &= \frac{1}{4}(+\operatorname{tr}(g)^2\operatorname{tr}(g^{-1})^2 + \operatorname{tr}(g^2)\operatorname{tr}(g^{-1})^2 + \operatorname{tr}(g)^2\operatorname{tr}(g^{-2}) \\
&\quad + \operatorname{tr}(g^2)\operatorname{tr}(g^{-2}) - 4\operatorname{tr}(g)\operatorname{tr}(g^{-1}))
\end{aligned}
\tag{B.2.3}
$$

As a simple check, it is easy to see that these characters are consistent with the decompositions under $\mathfrak{su}_{A+B} \to \mathfrak{su}_A \oplus \mathfrak{su}_B$ in (B.1.3), e.g.,

$$
\operatorname{tr}_{\ydiagram{2}}(g_1 \oplus g_2) = \operatorname{tr}_{\ydiagram{2}}(g_1) + \operatorname{tr}_{\ydiagram{1}}(g_1)\operatorname{tr}_{\ydiagram{1}}(g_2) + \operatorname{tr}_{\ydiagram{2}}(g_2) \tag{B.2.4}
$$

as expected from $\ydiagram{2} \mapsto \ydiagram{2} \otimes \bullet + \ydiagram{1} \otimes \ydiagram{1} + \bullet \otimes \ydiagram{2}$. Similarly, the characters are also consistent with the dimensions of the representations, e.g.,

$$
\operatorname{tr}_{\ydiagram{2,1}}(\mathbf{1}) = \frac{1}{3}(-N + N^3) \tag{B.2.5}
$$

as expected from $\dim_{\mathbb{R}}(\ydiagram{2,1}) = \frac{2}{3}N(N+1)(N-1)$.

We can also use the character to compute the index. For example, if we denote by $\delta$ the differential at $g = 1$, then

$$
\delta^2 \operatorname{tr}_{\ydiagram{2,1}}(1) = \frac{1}{3}(-9 + 3\operatorname{tr}(1)^2)\delta^2 \operatorname{tr}(1) \tag{B.2.6}
$$

whence $T_{\mathbb{R}}(\ydiagram{2,1}) = -3 + N^2$, as expected.

- $\boldsymbol{\mathrm{SO}(N)}$

The result is identical to the $\mathrm{SU}(N)$ case, with the following extra terms:

$$\begin{aligned}
\mathrm{tr}_{\ydiagram{2}}(g) &= \cdots - 1 \\
\mathrm{tr}_{\ydiagram{3}}(g) &= \cdots - \mathrm{tr}(g) \\
\mathrm{tr}_{\ydiagram{2,1}}(g) &= \cdots - \mathrm{tr}(g) \\
\mathrm{tr}_{\ydiagram{4}}(g) &= \cdots - \frac{1}{2}(\mathrm{tr}(g)^2 + \mathrm{tr}(g^2)) \\
\mathrm{tr}_{\ydiagram{3,1}}(g) &= \cdots - (\mathrm{tr}(g)^2 - 1) \\
\mathrm{tr}_{\ydiagram{2,2}}(g) &= \cdots - \frac{1}{2}(\mathrm{tr}(g)^2 + \mathrm{tr}(g^2)) \\
\mathrm{tr}_{\ydiagram{2,1,1}}(g) &= \cdots - \frac{1}{2}(\mathrm{tr}(g)^2 - \mathrm{tr}(g^2))
\end{aligned} \tag{B.2.7}$$

- $\boldsymbol{\mathrm{Sp}(N)}$

The result is identical to the $\mathrm{SU}(N)$ case, with the following extra terms:

$$\begin{aligned}
\mathrm{tr}_{\ydiagram{1,1}}(g) &= \cdots - 1 \\
\mathrm{tr}_{\ydiagram{2,1}}(g) &= \cdots - \mathrm{tr}(g) \\
\mathrm{tr}_{\ydiagram{1,1,1}}(g) &= \cdots - \mathrm{tr}(g) \\
\mathrm{tr}_{\ydiagram{3,1}}(g) &= \cdots - \frac{1}{2}(\mathrm{tr}(g)^2 + \mathrm{tr}(g^2)) \\
\mathrm{tr}_{\ydiagram{2,2}}(g) &= \cdots - \frac{1}{2}(\mathrm{tr}(g)^2 - \mathrm{tr}(g^2)) \\
\mathrm{tr}_{\ydiagram{2,1,1}}(g) &= \cdots - (\mathrm{tr}(g)^2 - 1) \\
\mathrm{tr}_{\ydiagram{1,1,1,1}}(g) &= \cdots - \frac{1}{2}(\mathrm{tr}(g)^2 - \mathrm{tr}(g^2))
\end{aligned} \tag{B.2.8}$$

- $\boldsymbol{G_2}$

The maximal torus sits inside $\mathrm{SO}(7)$, say with

$$g = \mathcal{R}(\theta_1) \oplus \mathcal{R}(\theta_2) \oplus \mathcal{R}(\theta_3) \oplus 1, \qquad \theta_1 + \theta_2 + \theta_3 = 0 \tag{B.2.9}$$

where $\mathcal{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. As $G_2$ has rank 2, the characters are polynomials in two variables. These are non-unique, as we can always add a polynomial that vanishes at $g$ above, e.g., $-\frac{1}{6}\operatorname{tr}(g)^3+\frac{1}{2}\operatorname{tr}(g)^2+\frac{1}{2}\operatorname{tr}(g^2)\operatorname{tr}(g)+\operatorname{tr}(g)+\frac{1}{2}\operatorname{tr}(g^2)-\frac{1}{3}\operatorname{tr}(g^3)$. All the expressions below are just one of the infinitely many possibilities.

The fundamental representation $(1,0)$ has real dimension 7, and character

$$\chi_{(1,0)}=\operatorname{tr}(g)=2(\cos\theta_1+\cos\theta_2+\cos\theta_3)+1 \tag{B.2.10}$$

The next few representations have

$$\begin{aligned}
\chi_{(0,1)}&=\frac{1}{2}\operatorname{tr}(g)^2-\operatorname{tr}(g)-\frac{1}{2}\operatorname{tr}(g^2)\\
\chi_{(2,0)}&=\frac{1}{2}\operatorname{tr}(g)^2+\frac{1}{2}\operatorname{tr}(g^2)-1\\
\chi_{(1,1)}&=\frac{1}{3}\operatorname{tr}(g)^3-\operatorname{tr}(g)^2-\frac{1}{3}\operatorname{tr}(g^3)+1\\
\chi_{(3,0)}&=\frac{1}{2}\operatorname{tr}(g)^2+\operatorname{tr}(g^2)\operatorname{tr}(g)+\frac{1}{2}\operatorname{tr}(g^2)\\
\chi_{(0,2)}&=\frac{1}{4}\operatorname{tr}(g)^4-\operatorname{tr}(g)^3-\frac{1}{2}\operatorname{tr}(g^2)\operatorname{tr}(g)^2\\
&\quad-\frac{1}{2}\operatorname{tr}(g)^2+\operatorname{tr}(g)+\frac{1}{4}\operatorname{tr}(g^2)^2-\frac{1}{2}\operatorname{tr}(g^2)\\
\chi_{(2,1)}&=\frac{1}{4}\operatorname{tr}(g)^4-\operatorname{tr}(g)^3-\frac{1}{2}\operatorname{tr}(g)^2-\operatorname{tr}(g^2)\operatorname{tr}(g)\\
&\quad+2\operatorname{tr}(g)-\frac{1}{4}\operatorname{tr}(g^2)^2+\frac{1}{2}\operatorname{tr}(g^2)\\
\chi_{(4,0)}&=-\frac{1}{4}\operatorname{tr}(g)^4+\frac{4}{3}\operatorname{tr}(g)^3+\operatorname{tr}(g^2)\operatorname{tr}(g)^2-3\operatorname{tr}(g)\\
&\quad+\frac{1}{4}\operatorname{tr}(g^2)^2-2\operatorname{tr}(g^2)+\frac{2}{3}\operatorname{tr}(g^3)
\end{aligned} \tag{B.2.11}$$

all of which are real, and have real dimensions $14,27,64,77,77,189,182$, respectively. For completeness, their indices are $4,9,32,44,55,144,156$, respectively.

- $\boldsymbol{F_4}$

The maximal torus is free of rank 4, and it sits inside SO(26), say with

$$g_4=\mathcal{R}(0)\oplus\left(\bigoplus_{i=1}^{4}\mathcal{R}(\theta_i)\right)\oplus\left(\bigoplus_{\pm}\mathcal{R}(\tfrac{1}{2}(\theta_1\pm\theta_2\pm\theta_3\pm\theta_4))\right) \tag{B.2.12}$$

where the second sum runs over the $2^3=8$ choices of sign, for a total of $2\times(8+4+1)=26$ matrix elements. One can also realise the torus inside e.g. SO(52), with

$$g_1=g_4\oplus\mathcal{R}(0)\oplus\left(\bigoplus_{i<j,\pm}\mathcal{R}(\theta_i\pm\theta_j)\right) \tag{B.2.13}$$

where the sum has $2\times 6=12$ terms, for a total of $26+2\times(12+1)=52$.

These two realizations correspond to two of the four fundamental weights $(0,0,0,1)$, $(1,0,0,0)$, respectively, whose associated characters are

$$\begin{aligned}\chi_{(0,0,0,1)} &= \operatorname{tr}(g_4) = 2\left(1+\sum_{i=1}^{4}\cos\theta_i + \sum_{\pm}\cos\tfrac{1}{2}(\theta_1\pm\theta_2\pm\theta_2\pm\theta_4)\right)\\ \chi_{(1,0,0,0)} &= \operatorname{tr}(g_1) = \operatorname{tr}(g_4) + 2\left(1+\sum_{i<j,\pm}\cos(\theta_i\pm\theta_j)\right)\end{aligned} \tag{B.2.14}$$

The next few representations are

$$\begin{aligned}
\operatorname{tr}_{(1,0,0,0)} &= \operatorname{tr}(g_1)\\
\operatorname{tr}_{(0,0,0,1)} &= \operatorname{tr}(g_4)\\
\operatorname{tr}_{(0,1,0,0)} &= -\operatorname{tr}(g_1) + \frac{1}{2}\operatorname{tr}(g_1)^2 - \frac{1}{2}\operatorname{tr}(g_1^2)\\
\operatorname{tr}_{(0,0,1,0)} &= \frac{1}{2}(\operatorname{tr}(g_4)^2 - 2\operatorname{tr}(g_1) - \operatorname{tr}(g_4^2))\\
\operatorname{tr}_{(0,0,0,2)} &= \frac{1}{2}(-2 - 2\operatorname{tr}(g_4) + \operatorname{tr}(g_4)^2 + \operatorname{tr}(g_4^2))\\
\operatorname{tr}_{(1,0,0,1)} &= \frac{1}{6}(-3\operatorname{tr}(g_4)^2 + \operatorname{tr}(g_4)^3 + 12\operatorname{tr}(g_1) - 3\operatorname{tr}(g_1)^2\\
&\qquad + 3\operatorname{tr}(g_4^2) - 3\operatorname{tr}(g_4)\operatorname{tr}(g_4^2) + 3\operatorname{tr}(g_1^2) + 2\operatorname{tr}(g_4^3))\\
\operatorname{tr}_{(2,0,0,0)} &= \frac{1}{2}(2\operatorname{tr}(g_4) - \operatorname{tr}(g_4)^2 + \operatorname{tr}(g_1)^2 - \operatorname{tr}(g_4^2) + \operatorname{tr}(g_1^2))\\
\operatorname{tr}_{(0,0,0,3)} &= \frac{1}{6}(-6\operatorname{tr}(g_4)^2 + \operatorname{tr}(g_4)^3 + 6\operatorname{tr}(g_1)\\
&\qquad + 3\operatorname{tr}(g_4)\operatorname{tr}(g_4^2) + 2\operatorname{tr}(g_4^3))\\
\operatorname{tr}_{(0,0,1,1)} &= 1 - \operatorname{tr}(g_4) - \frac{1}{2}\operatorname{tr}(g_4)^2 + \frac{1}{6}\operatorname{tr}(g_4)^3 - 2\operatorname{tr}(g_1)\\
&\qquad + \frac{1}{2}\operatorname{tr}(g_1)^2 - \frac{1}{2}\operatorname{tr}(g_4^2) + \frac{1}{2}\operatorname{tr}(g_4)\operatorname{tr}(g_4^2)\\
&\qquad - \frac{1}{2}\operatorname{tr}(g_1^2) - \frac{2}{3}\operatorname{tr}(g_4^3)\\
\operatorname{tr}_{(1,0,1,0)} &= \operatorname{tr}(g_4) + \operatorname{tr}(g_4)^2 - \frac{1}{3}\operatorname{tr}(g_4)^3 + \frac{1}{24}\operatorname{tr}(g_4)^4\\
&\qquad - \frac{1}{2}\operatorname{tr}(g_1)^2 - \frac{1}{4}\operatorname{tr}(g_4)^2\operatorname{tr}(g_4^2) + \frac{1}{8}\operatorname{tr}(g_4^2)^2\\
&\qquad - \frac{1}{2}\operatorname{tr}(g_1^2) + \frac{1}{3}\operatorname{tr}(g_4^3) + \frac{1}{3}\operatorname{tr}(g_4)\operatorname{tr}(g_4^3) - \frac{1}{4}\operatorname{tr}(g_4^4)
\end{aligned} \tag{B.2.15}$$

• ***$E_6$***

The maximal torus is free of rank 6, and can be realised inside SU(27), with

$$g_1 = \operatorname{diag}(e^{-\frac{2}{3}\theta_6}, e^{\frac{1}{3}\theta_6\pm\theta_i}, e^{\frac{1}{2}(\pm\theta_1\pm\theta_2\pm\theta_3\pm\theta_4\pm\theta_5)-\frac{1}{6}\theta_6}) \tag{B.2.16}$$

where in the last term we take all combinations of signs such that there is an odd number of negative signs. In total, there are $1+2\times 5+16=27$ terms.

Another realisation of the maximal torus is inside SU(78), with

$$g_6 = \mathbf{1}_6 + \mathrm{diag}(e^{\pm\theta_i\pm\theta_j}, e^{\frac{1}{2}(\pm\theta_1\pm\theta_2\pm\theta_3\pm\theta_4\pm\theta_5\pm\theta_6)}) \tag{B.2.17}$$

where $i<j$ in the second term, and we take all signs combinations in the third term, such that the number of negative signs is odd. In total, there are $6+10\times 4+32=78$ terms.

The first few characters are as follows:

$$\begin{aligned}
\chi_{(1,0,0,0,0,0)} &= \mathrm{tr}(g_1)\\
\chi_{(0,1,0,0,0,0)} &= \frac{1}{2}\,\mathrm{tr}(g_1)^2 - \frac{1}{2}\,\mathrm{tr}(g_1^2)\\
\chi_{(0,0,1,0,0,0)} &= \frac{1}{6}(\mathrm{tr}(g_1)^3 - 3\,\mathrm{tr}(g_1^2)\,\mathrm{tr}(g_1) + 2\,\mathrm{tr}(g_1^3))\\
\chi_{(0,0,0,0,0,1)} &= \mathrm{tr}(g_6)\\
\chi_{(0,0,0,0,2,0)} &= \frac{1}{2}(\mathrm{tr}(g_1^{-1})^2 + \mathrm{tr}(g_1^{-2}) - 2\,\mathrm{tr}(g_1))\\
\chi_{(1,0,0,0,1,0)} &= \mathrm{tr}(g_1)\,\mathrm{tr}(g_1^{-1}) - \mathrm{tr}(g_6) - 1\\
\chi_{(1,0,0,0,0,1)} &= \frac{1}{2}\,\mathrm{tr}(g_1)\,\mathrm{tr}(g_3)^2 - \frac{1}{2}\,\mathrm{tr}(g_1)\,\mathrm{tr}(g_3^2)\\
&\quad - \frac{1}{6}\,\mathrm{tr}(g_1)^4 + \frac{1}{2}\,\mathrm{tr}(g_1^2)\,\mathrm{tr}(g_1)^2 - \frac{1}{3}\,\mathrm{tr}(g_1^3)\,\mathrm{tr}(g_1)\\
&\quad - \mathrm{tr}(g_1) + \frac{1}{2}\,\mathrm{tr}(g_1^{-2}) - \frac{1}{2}\,\mathrm{tr}(g_1^{-1})^2\\
\chi_{(0,0,0,0,0,2)} &= -\frac{1}{2}\,\mathrm{tr}(g_1)\,\mathrm{tr}(g_3)^2 + \mathrm{tr}(g_1)\,\mathrm{tr}(g_3) + \frac{1}{2}\,\mathrm{tr}(g_1)\,\mathrm{tr}(g_3^2) + \mathrm{tr}(g_3^2)\\
&\quad - \mathrm{tr}(g_1)\,\mathrm{tr}(g_1^{-1}) + \frac{1}{6}\,\mathrm{tr}(g_1)^4 + \frac{1}{6}\,\mathrm{tr}(g_1)^3 - \frac{1}{2}\,\mathrm{tr}(g_1^2)\,\mathrm{tr}(g_1)\\
&\quad - \frac{1}{2}\,\mathrm{tr}(g_1^2)\,\mathrm{tr}(g_1)^2 + \frac{1}{3}\,\mathrm{tr}(g_1^3)\,\mathrm{tr}(g_1) + \frac{1}{3}\,\mathrm{tr}(g_1^3) + 2\,\mathrm{tr}(g_3)\\
\chi_{(3,0,0,0,0,0)} &= \frac{1}{6}(-6\,\mathrm{tr}(g_1)\,\mathrm{tr}(g_1^{-1}) + \mathrm{tr}(g_1)^3 + 3\,\mathrm{tr}(g_1^2)\,\mathrm{tr}(g_1)\\
&\quad + 2\,\mathrm{tr}(g_1^3) + 6\,\mathrm{tr}(g_3))
\end{aligned} \tag{B.2.18}$$

- $\boldsymbol{E_7}$

The maximal torus is free of rank 7, and can be realised inside Sp(28), with

$$g_6 = \mathrm{diag}(e^{\theta_i}, e^{\theta_i-\theta_7}, e^{\frac{1}{2}(\theta_1+\theta_2+\theta_3+\theta_4+\theta_5+\theta_6-3\theta_7)}, e^{\frac{1}{2}(\pm\theta_1\pm\theta_2\pm\theta_3\pm\theta_4\pm\theta_5\pm\theta_6+\theta_7)}) \oplus \text{c.c.} \tag{B.2.19}$$

where $i=1,\ldots,6$ and, in the last term, we take all combinations of signs such that there are exactly two "+" signs. All in all, $6+6+1+15=28$ terms.

One can also realise the maximal torus inside SO(133), with

$$
\begin{aligned}
g_1 = \mathbf{1}_7 &\oplus \mathcal{R}(\theta_7) \oplus \mathcal{R}(\theta_i - \theta_j) \oplus \mathcal{R}(\theta_i + \theta_j - \theta_7) \\
&\oplus \mathcal{R}(\tfrac{1}{2}(\pm\theta_1 \pm \theta_2 \pm \theta_3 \pm \theta_4 \pm \theta_5 \pm \theta_6 + \theta_7)) \\
&\oplus \mathcal{R}(\tfrac{1}{2}(\pm\theta_1 \pm \theta_2 \pm \theta_3 \pm \theta_4 \pm \theta_5 \pm \theta_6 + 3\theta_7))
\end{aligned}
\tag{B.2.20}
$$

where in the first line $1 \leq i < j \leq 6$, and in the second line we take all combinations of signs such that there are either one or three "+" signs, and in the third line exactly one "+" sign. All in all, $7 + 2 \times (1 + 2 \times 15 + 26 + 6) = 133$ elements.

Finally, it is also convenient to introduce a second pseudo-real realisation of the torus, sitting inside Sp(456), this time using the seventh fundamental representation:

$$
\begin{aligned}
g_7 = g_6^{\oplus 3} &\oplus \\
&\mathrm{diag}(e^{\theta_i \pm \theta_j \pm \theta_k}, e^{\theta_i + \theta_j + \theta_k - 2\theta_7}, e^{\pm\theta_i \pm \theta_j \pm \theta_k + \theta_7}, e^{\frac{1}{2}(\theta_1 + \cdots + \theta_6 - 5\theta_7)}, \\
&\quad e^{\frac{1}{2}(\pm\theta_1 \pm \cdots \pm \theta_6 + 3\theta_7)}, e^{\frac{1}{2}(\pm\theta_1 \pm \cdots \widehat{\pm\theta_i} \pm \cdots \pm \theta_6 + 3\theta_7 - 3\theta_j)}, e^{\frac{1}{2}(\pm\theta_1 \pm \cdots \widehat{\pm\theta_i} \pm \cdots \pm \theta_6 - \theta_7 + 3\theta_j)}, \\
&\quad e^{\frac{1}{2}(\theta_1 + \cdots + \theta_6 - \theta_7) - 2\theta_j}, e^{\frac{1}{2}(\theta_1 + \cdots + \theta_6 - \theta_7)}, e^{\frac{1}{2}(\pm\theta_1 \pm \cdots \pm \theta_6 + \theta_7)}) \oplus \text{c.c.}
\end{aligned}
\tag{B.2.21}
$$

where $1 \leq i < j < k \leq 6$, and we take all the sign combinations for which the number of "+"s is as follows: in the first term, either zero or one; in the third term, either zero or one; in the fifth term, two; in the sixth term, one; in the seventh term, two; and finally in the tenth term, four. All in all, the number of terms is $3 \times 56 + 3 \times 20 + 1 \times 20 + 4 \times 20 + 1 + 15 + 6 \times 5 + 6 \times 10 + 6 + 1 + 15 = 456$.

The rest of characters can easily be expressed in terms of these. For example,

$$
\begin{aligned}
\chi_{(0,0,0,0,0,1,0)} &= \operatorname{tr}(g_6) \\
\chi_{(1,0,0,0,0,0,0)} &= \operatorname{tr}(g_1) \\
\chi_{(0,0,0,0,0,0,1)} &= \operatorname{tr}(g_7) \\
\chi_{(0,0,0,0,0,2,0)} &= +\frac{1}{2}\operatorname{tr}(g_6^2) + \frac{1}{2}\operatorname{tr}(g_6)^2 - \operatorname{tr}(g_1) \\
\chi_{(0,0,0,0,1,0,0)} &= -\frac{1}{2}\operatorname{tr}(g_6^2) + \frac{1}{2}\operatorname{tr}(g_6)^2 - 1 \\
\chi_{(1,0,0,0,0,1,0)} &= +\operatorname{tr}(g_6)\operatorname{tr}(g_1) - \operatorname{tr}(g_6) - \operatorname{tr}(g_7) \\
\chi_{(2,0,0,0,0,0,0)} &= +\frac{1}{2}\operatorname{tr}(g_6^2) - \frac{1}{2}\operatorname{tr}(g_6)^2 + \frac{1}{2}\operatorname{tr}(g_1^2) + \frac{1}{2}\operatorname{tr}(g_1)^2 \\
\chi_{(0,1,0,0,0,0,0)} &= -\frac{1}{2}\operatorname{tr}(g_1^2) + \frac{1}{2}\operatorname{tr}(g_1)^2 - \operatorname{tr}(g_1) \\
\chi_{(0,0,0,0,0,3,0)} &= +\frac{1}{3}\operatorname{tr}(g_6^3) + \frac{1}{2}\operatorname{tr}(g_6^2)\operatorname{tr}(g_6) - \operatorname{tr}(g_6)\operatorname{tr}(g_1) + \frac{1}{6}\operatorname{tr}(g_6)^3 + \operatorname{tr}(g_7) \\
\chi_{(0,0,0,1,0,0,0)} &= +\frac{1}{3}\operatorname{tr}(g_6^3) - \frac{1}{2}\operatorname{tr}(g_6^2)\operatorname{tr}(g_6) + \frac{1}{6}\operatorname{tr}(g_6)^3 - \operatorname{tr}(g_6) \\
\chi_{(0,0,0,0,0,1,1)} &= +\frac{1}{2}\operatorname{tr}(g_6^2) + \operatorname{tr}(g_6)\operatorname{tr}(g_7) - \frac{1}{2}\operatorname{tr}(g_6)^2 + \frac{1}{2}\operatorname{tr}(g_1^2) - \frac{1}{2}\operatorname{tr}(g_1)^2 + 1 \\
\chi_{(0,0,1,0,0,0,0)} &= +\frac{1}{3}\operatorname{tr}(g_1^3) - \frac{1}{2}\operatorname{tr}(g_1^2)\operatorname{tr}(g_1) + \frac{1}{6}\operatorname{tr}(g_1)^3 - \operatorname{tr}(g_1)^2 + \operatorname{tr}(g_1)
\end{aligned}
\tag{B.2.22}
$$

of dimension $56, 133, 912, 1463, 1539, 6480, 7371, 8645, 24320, 27664, 40755, 365750$, respectively.

- $\boldsymbol{E_8}$ The maximal torus is free of rank 8. We will not discuss this group here.