Brands of cumulants in non-commutative probability, and relations between them

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapter 3 contains results from my single-authored paper [Per21]. Chapter 4 and part of Chapter 6 contain results from [CEFN+21], a paper that I co-authored with Adrian Celestino, Kurusch Ebrahimi-Fard, Alexandru Nica, and Leon Witzman. The other chapters are the sole work of Daniel Perales Anaya and constitute a general presentation of results in this area, as well as some extensions of Chapter 4.

Abstract

The study of non-commutative probability revolves around the different notions of independeces, such as free, Boolean and monotone. To each type of independence one can associate a notion of cumulants that linearize the addition of independent random variables. These notions of cumulants are a clear analogue of classic cumulants that linearize the addition of independent random variables. The family of set partitions $\mathcal{P}$ plays a key role in the combinatorial study of probability because several formulas relating moments to a brand of cumulants can be expressed as a sum indexed by set partitions. An intriguing fact observed in the recent research literature was that non-commutative cumulants sometimes have applications to other areas of non-commutative probability than the one they were designed for. Thus one wonders if there are nice combinatorial formulas to directly transition from one brand of cumulants to another. This thesis is concerned with the study of interrelations between different brands of cumulants associated to classic, Boolean, free and monotone independences. My development is naturally divided in four topics.

For the first topic I focus on free cumulants, and I use them in the study of the distribution of the anti-commutator $ab + ba$ of two free random variables a and b. This follows up on some questions raised a while ago by Nica and Speicher in the 1990's.

I next consider the notion of convolution in the framework of a family of lattices, which goes back to work of Rota and collaborators in the 1970's. I focus on the lattices of non-crossing partitions $\mathcal{NC}$ and put into evidence a certain group of semi-multiplicative functions, which encapsulate the moment-cumulant formulas and the inter-cumulant transition formulas for several known brands of cumulants in non-commutative probability.

I next extend my considerations to the framework of $\mathcal{P}$, which allows us to include the classical cumulants in the picture. Moreover, I do this in a more structured fashion by introducing a notion of iterative family of partitions $\mathcal{S} \subseteq \mathcal{P}$. This unifies the considerations related to $\mathcal{NC}$ and $\mathcal{P}$, and also provides a whole gallery of new examples.

Finally, it is important to make the observation that the group of semi-multiplicative functions which arise in connection to an iterative family is in fact a dual structure. Namely, it appears as the group of characters $\mathcal{S}^\circ$ for a certain Hopf algebra $\widehat{\mathcal{S}}$ over the partitions in $\mathcal{S}$. In particular, the antipode of $\widehat{\mathcal{S}}$ promises to serve as a universal inversion tool for moment-cumulant formulas and for transition formulas between different brands of cumulants.

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Dedication

Para mi amada
Karina De León,

y mis queridos padres,
Guillermo Perales,
Ma. Isabel Anaya.

Table of Contents

List of Figures

Chapter 1

Introduction

The family $\mathcal{P}(n)$ of set partitions of $[n] := \{1, \dots, n\}$ plays a key role in the combinatorial study of probability because it indexes the moment-cumulant formula. Specifically, if we have the sequence of moments $(\varphi_n)_{n\in\mathbb{N}}$ and the sequence of cumulants $(c_n)_{n\in\mathbb{N}}$ of some random variable, then these two sequences are always related via the formula

$$\varphi_n = \sum_{\pi\in\mathcal{P}(n)} \prod_{i=1}^{k} c_{|V_i|},$$

where the product runs over the blocks of the partition $\pi = \{V_1, V_2, \dots, V_k\} \in \mathcal{P}(n)$. These *classical cumulants* have a long history going back to 1889 when Thiele studied them under the name of semi-invariants. For more details on classical cumulants we refer to the book of Shiryaev [Shi16, Chapter II, Section 12]. For us, the relevance of cumulants stems from the fact that they linearize the addition of independent random variables. Namely, if $(c_n(a))_{n\in\mathbb{N}}$ and $(c_n(b))_{n\in\mathbb{N}}$ are the cumulants of two independent random variables a and b, then the cumulants of their sum $a + b$ are given by

$$c_n(a + b) = c_n(a) + c_n(b), \qquad \forall n \in \mathbb{N}.$$

In the 1990's, Speicher [Spe94] introduced the notion of *free cumulants* in connection to Voiculescu's developement of free probability [VDN92]. The analogy to classical cumulants becomes transparent when we look at the free moment-cumulant formula. Denoting by $(r_n)_{n\in\mathbb{N}}$ the sequence of free cumulants, one has the universal formula

$$\varphi_n = \sum_{\pi\in\mathcal{NC}(n)} \prod_{i=1}^{k} r_{|V_i|},$$

where $\mathcal{NC}(n)$ is a subset of $\mathcal{P}(n)$ consisting of the so-called *non-crossing* partitions. The relevant property of free cumulants is that they linearize the addition of freely independent random variables. Namely, if $(r_n(a))_{n\in\mathbb{N}}$ and $(r_n(b))_{n\in\mathbb{N}}$ are the free cumulants of two random variables a and b, that are freely independent, then the free cumulants of their sum are given by

$$r_n(a+b) = r_n(a) + r_n(b), \qquad \forall n \in \mathbb{N}.$$

Since the 1980s, when Voiculescu introduced free probability, some other notions of non-commutative independence have emerged, such as Boolean independence [SW97] and monotone independence [Mur00], together with their corresponding notion of Boolean cumulants [SW97] and monotone cumulants [HS11]. Moreover, one can define a continuous interpolation between Boolean and free cumulants, which we refer to as t-Boolean cumulants, and arise from the work of Bożejko and Wysoczanski [BW01] – the case $t = 0$ gives Boolean cumulants and the case $t = 1$ gives free cumulants. Cumulants of all sorts are an indispensable ingredient for investigations in non-commutative probability, because their effectiveness is comparable to the one of the available analytical tools. This is unlike the situation from classical probability where one can use very sharp analytic tools coming from Fourier analysis.

An intriguing fact observed in the recent research literature was that non-commutative cumulants sometimes have applications to other areas of non-commutative probability than the one they were designed for. For instance, Boolean cumulants were successfully used to study the free analogue of infinitely divisible distributions, see e.g. [BN08]. In connection to this, it is natural to ask if there are nice combinatorial formulas to directly transition from one brand of cumulants to another. This goes back to Lehner [Leh02], and was thoroughly pursued in [AHLV15], where explicit relations among classic, Boolean, free and monotone cumulants were studied in detail. The study of interrelations between different brands of cumulants also is a highlight of the present thesis. More precisely, the thesis is concerned with four main topics involving cumulants, as described below.

• For the first topic I focus on free cumulants, and I use them in the study of the distribution of the anti-commutator $ab + ba$ of two free random variables a and b. This follows up on some questions raised a while ago in [NS98]. A more precise description of this topic appears in Section 1.1 below and then in Chapter 3.

• In order to actually go to interrelations between different brands of cumulants, I next consider the notion of convolution in the framework of a family of lattices, which goes back to work of Rota and collaborators [DRS72, JR79] in the 1970's. I focus on the lattices of non-crossing partitions $\mathcal{NC}(n)$, and put into evidence a certain group $SM^{\mathcal{NC}}$ of *semi-multiplicative functions* on $\mathcal{NC} = \cup_{n=1}^{\infty}\mathcal{NC}(n)$, which encapsulate the moment-cumulant formulas and the inter-cumulant transition formulas for several known brands of cumulants in non-commutative probability. A more precise description of this topic appears in Section 1.2 below and then in Chapter 4.

• I next extend my considerations to the framework of the set $\mathcal{P} = \cup_{n=1}^{\infty} \mathcal{P}(n)$ of all partitions, which allows us to include the classical cumulants in the picture. Moreover, I do this in a more structured fashion by introducing a notion of *iterative family of partitions* $\mathcal{S} = \cup_{n=1}^{\infty} \mathcal{S}(n)$. This unifies the considerations related to $\mathcal{NC}$ and to $\mathcal{P}$, and also provides a whole gallery of new examples. A more precise description of this topic appears in Section 1.3 below and then in Chapter 5.

• Finally, an important observation which one should make at this point is that the group of semi-multiplicative functions which arise in connection to an iterative families $\mathcal{S}$ is in fact a dual structure. Namely, it appears as the group of characters for a certain Hopf algebra $\widehat{\mathcal{S}}$, which is constructed in the spirit of the so-called incidence Hopf algebras of Schmitt [Sch94]. In particular, the antipode of $\widehat{\mathcal{S}}$ promises to serve as a universal inversion tool for moment-cumulant formulas and for transition formulas between different brands of cumulants. A more precise description of this topic appears in Section 1.4 below and then in Chapter 6.

1.1 The problem of the anti-commutator

Speicher's free cumulants $(r_n)_{n\geq 1}$ are not only useful to study the addition $a+b$ of two free random variables a and b, they can also be used to study the product ab of free random variables [NS96]. It is then natural that one studies some other polynomials in two free random variables a and b. Due to the non-commutative nature of a and b, it makes sense that the next in line should be the commutator $i(ab-ba)$ and the anti-commutator $ab+ba$. It turns out that the free cumulants are also useful to study these cases. The combinatorial study of $i(ab-ba)$ was done by Nica and Speicher [NS98]. The main observation is that the minus sign leads to a massive cancellation of terms in the expansions of the free cumulants of $i(ab-ba)$, leading to a tractable formula. On the other hand for $ab+ba$ there are no cancellations to be followed, and the combinatorial approach becomes much more involved.

In Chapter 3, following the presentation in [Per21], we provide a formula to compute the free cumulants of the anti-commutator $ab+ba$ as a sum indexed by some non-crossing partitions, and where each term consist of a product of free cumulants of a and free cumulants of b. The key idea is to carefully examine a certain graph $\Gamma(\pi)$ that can be associated to each non-crossing partition π. These graphs had already appeared in [MS12], see also Section 4.4 of [MS17]. Our main contribution, Theorem 3.1.4, is a formula that expresses the cumulants of the anti-commutator as a sum indexed by

$$\mathcal{X}(2n) := \{\pi \in \mathcal{NC}(2n) : \Gamma(\pi) \text{ is connected and bipartite}\}.$$

As a direct application, we can use our formula to study the distribution, ν of the anti-commutator $ab + ba$, where a, b are free and both have a Marchenko-Pastur distribution of parameter 1, which is the free analogue of a Poisson distribution. In this case, the cumulants of the anti-commutator $ab + ba$ are just given by twice the size of $\mathcal{X}(2n)$. In Section 3.2, we will use an important transformation on non-crossing partitions called the Kreweras complementation map to transform $\mathcal{X}(2n)$ into a more appealing set $\mathcal{Y}(2n)$ that consists of partitions that separate odd elements and where blocks with only even elements have even size. The nice structure of the sets $\mathcal{Y}(2n)$ allows us to give recursive formulas (on n) for their cardinality. In Section 3.3 we find a recursion for the values $|\mathcal{Y}(2n)|$ and use it to find the compositional inverse of the moment series of ν:

$$M_\nu^{\langle -1\rangle}(z) = \frac{-7z - 6 + 3\sqrt{(z+2)(9z+2)}}{4(z+2)^2(z+1)}. \tag{1.1}$$

In Section 3.4 we analyze what kind of connected and bipartite graphs $\Gamma(\pi)$ can be obtained from a partition π. We observe that the graphs have a very specific form that can be identified as a bipartite cactus graph. If we keep track of the orientation of the graphs, we can also count how many partitions give the same graph. This allows us to give a formula for the anti-commutator where the sum is indexed solely by graphs, this is the content of Theorem 3.4.6. Moreover, the graph theoretic approach suggests a very natural generalization which is useful to provide a formula for the cumulants of every quadratic form:

$$a := \sum_{1\leq i,j\leq k} w_{i,j} a_i a_j,$$

where $w_{i,j} \in \mathbb{R}$ and $w_{j,i} = w_{i,j}$ for $1 \leq i \leq j \leq k$. In this generality we just make use of oriented cacti graphs, the only difference is that instead of bipartite graphs we need to consider k-colored graphs, whose edges have weights given by the values $(w_{i,j})_{1\leq i,j\leq k}$, see Theorem 3.5.2. These results can be seen as a generalization of other results that have been recently discovered in relation with sums of commutators and anti-commutators. For instance, we can retrieve Proposition 4.5 of [EL17] (see also [EL21]), the details are given in Corollary 3.5.3.

1.2 Transitions between different brands of cumulants as an action of semi-multiplicative functions on $\mathcal{NC}$

As mentioned earlier, there are several brands of cumulants used in non-commutative probability, including free, Boolean, monotone cumulants, and t-Boolean cumulants. A common thing to all these notions of cumulants is that the moment-cumulant formulas can be expressed as a sum over non-crossing partitions, if we allow the use of certain complex coefficients. If we denote by $\{\kappa_n\}_{n\in\mathbb{N}}$ the sequence of cumulants for some type of independence, then the moment-cumulant formula is

$$\varphi_n = \sum_{\pi\in\mathcal{NC}(n)} \alpha_\pi \prod_{i=1}^{k} \kappa_{|V_i|}, \tag{1.2}$$

where the α_π are complex coefficients indexed by partitions that depend on the given notion of independence. For instance, if $\alpha_\pi = 1$ for every non-crossing partitions, we get Speicher's free moment-cumulant formula. If $\alpha_\pi = 1$ for interval partitions, and 0 otherwise, then we are looking at Boolean moment-cumulant formula.

A natural question, first asked by [Leh02], is if there are nice combinatorial formulas to directly transition from one sequence of cumulants to another. This question was studied in detail in [AHLV15], where each transition from one cumulant to another was approached with a different method. This prompts the question if there is a common framework where we can study all the transitions at the same time. The objective of Chapter 4 is to provide this framework by using *semi-multiplicative functions* over the incidence algebra of $\mathcal{NC}$, this chapter follows the presentation of the paper [CEFN+21].

As hinted by the name, semi-multiplicative functions are a weaker version of the multiplicative functions studied in [Spe94], where the free cumulants were introduced. The idea of studying the convolution of multiplicative functions defined on the set of all intervals of a "coherent" collection of lattices $(\mathcal{L}_n)_{n=1}^{\infty}$ goes back to the 1970's work of Rota and collaborators, e.g. in [DRS72]. The phenomenon which prompts this study is that, in a number of important examples: for every $\pi \leq \sigma$ in an $\mathcal{L}_n$, the sublattice $[\pi, \sigma] := \{\rho \in \mathcal{L}_n \mid \pi \leq \rho \leq \sigma\}$ of $\mathcal{L}_n$ is canonically isomorphic to a direct product,

$$[\pi, \sigma] \approx \mathcal{L}_1^{p_1} \times \cdots \times \mathcal{L}_n^{p_n}, \quad \text{with } p_1, \ldots, p_n \geq 0. \tag{1.3}$$

A function $f : \sqcup_{n=1}^{\infty}\{(\pi, \sigma) \mid \pi, \sigma \in \mathcal{L}_n,\, \pi \leq \sigma\} \to \mathbb{C}$ is declared to be multiplicative when there exists a sequence of complex numbers $(\alpha_n)_{n=1}^{\infty}$ such that, for π, σ and non-negative integers $p_1, \ldots, p_n$ as in (1.3), one has $f(\pi, \sigma) := \alpha_1^{p_1} \cdots \alpha_n^{p_n}$.

In Chapter 4 we focus on the case where $\mathcal{L}_n$ is the lattice $\mathcal{NC}(n)$ of non-crossing partitions of $[n]$, endowed with the *partial order by reverse refinement* $\leq$. It was found by Speicher [Spe94] that, when considered in connection to the $\mathcal{NC}(n)$'s, the convolution of multiplicative functions plays an essential role in the combinatorial development of free probability. For the purposes of the present paper it is convenient to focus on the set $MF^{\mathcal{NC}}$ consisting of multiplicative functions on the $\mathcal{NC}(n)$'s where the sequence $(\alpha_n)_{n=1}^{\infty}$ defining the function has $\alpha_1 = 1$. Then $MF^{\mathcal{NC}}$ is a group under a certain convolution operation $*$.

In the case of the lattices $\mathcal{NC}(n)$, the canonical isomorphism indicated in (1.3) is obtained by combining two kinds of lattice isomorphisms, the first one writes every interval $[\pi, \sigma]$ as a product of intervals of the form $[\tau, 1_n]$, were 1_n is the maximal partition, and the second one reduces intervals of the form $[\tau, 1_n]$ to products of smaller lattices $\mathcal{NC}(k)$. These isomorphisms will be reviewed precisely as soon as the notation is set for them, cf. Subsection 2.2.2 below. The important point is that it is worth studying the convolution for the set $SM^{\mathcal{NC}}$ of functions that are only required to be multiplicative with respect to the first kind of isomorphism mentioned above. It turns out that $SM^{\mathcal{NC}}$ is a group under convolution. This group and some of its subgroups (in particular the group $MF^{\mathcal{NC}} \subset SM^{\mathcal{NC}}$ of multiplicative functions) are the main players in the considerations of Chapter 4.

Consider now the framework of a non-commutative probability space $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$ and $\varphi\colon \mathcal{A} \to \mathbb{C}$ is a linear functional such that the algebra unit is mapped to one ($\varphi(1_{\mathcal{A}}) = 1$), and look at

$$\mathfrak{M}_{\mathcal{A}} := \{\underline{\psi} \mid \underline{\psi} = (\psi_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}, \text{ where } \psi_n \text{ is an } n\text{-linear functional}\}.$$

The group $SM^{\mathcal{NC}}$ has a natural action on $\mathfrak{M}_{\mathcal{A}}$, which is discussed in detail in Section 4.5. This action captures the transitions between moment functionals and the different brands of cumulants, such as free, Boolean, monotone and t-Boolean. As a consequence of the action of $SM^{\mathcal{NC}}$ on $\mathfrak{M}_{\mathcal{A}}$, our group $SM^{\mathcal{NC}}$ captures the formulas for direct transitions between two such brands of cumulants, providing an efficient framework for streamlining calculations related to various moment-cumulant and inter-cumulant formulas. It is in fact possible to identify precisely some notions of what it means for a function $h \in SM^{\mathcal{NC}}$ to be of *cumulant-to-moment type*, and what it means for a $g \in SM^{\mathcal{NC}}$ to be of *cumulant-to-cumulant* type, this is done in Section 4.6.

In Section 4.8 we will identify a 1-parameter subgroup $(u_q)_{q\in\mathbb{R}}$ of $SM^{\mathcal{NC}}$ generated by the function which encodes transition between free cumulants and Boolean cumulants. This subgroup is an important component of the transition formula between t-Boolean cumulants and moments, and in particular gives an easy way (cf. Corollary 4.8.5 below) to write the transition formula between s-Boolean cumulants and t-Boolean cumulants for distinct values $s, t \in \mathbb{R}$.

In Section 4.9 we prove that every u_q belongs to the normalizer of $MF^{\mathcal{NC}}$:

$$\left(q \in \mathbb{R},\ f \in MF^{\mathcal{NC}}\right) \ \Rightarrow\ u_q^{-1} * f * u_q \in MF^{\mathcal{NC}}. \tag{1.4}$$

This is a non-trivial fact, as there is no obvious direct connection between $MF^{\mathcal{NC}}$ and the u_q's. This result obtained in (1.4) can be used in order to give a neat explanation of the intriguing fact that the multiplication of freely independent random variables is nicely described in terms of Boolean cumulants (who aren't a priori meant to be related to free probability). Furthermore, we show that the multiplication is nicely described in terms of the whole family of t-Boolean cumulants, that interpolate between free and Boolean cumulants.

1.3 Iterative families of partitions

The group of semi-multiplicative functions over $\mathcal{NC}$ has a very natural analogue, where instead of considering the poset $\mathcal{NC}$ we consider the poset $\mathcal{P}$. From the fact that $\mathcal{NC} \subset \mathcal{P}$, there is a natural way to include the group $SM^{\mathcal{NC}}$ of semi-multiplicative functions over $\mathcal{NC}$ as a subgroup of $SM^{\mathcal{P}}$, the *group of semi-multiplicative functions over* $\mathcal{P}$. Moreover, this bigger structure allows us to include the classical moment-cumulant formula in the picture. Having more formulas under the same umbrella enables us to systematically study their interactions.

Furthermore, we can identify several other *iterative families* $\mathcal{S} \subset \mathcal{P}$ that, same as $\mathcal{NC}$, have sufficient structure to admit a group $SM^{\mathcal{S}}$ of *semi-multiplicative functions over* $\mathcal{S}$, the iterative families are introduced in Section 5.1, and the semi-multiplicative functions in Section 5.3. Roughly speaking, a family $\mathcal{S} \subset \mathcal{P}$ is iterative if it is closed under block substitutions, that is, whenever we take a partition $\pi \in \mathcal{S}$ and we substitute each block $V \in \pi$ with a partition in $\mathcal{S}$, then we obtain another partition in $\mathcal{S}$. In Section 5.2 we will show that several well-known families of partitions are iterative, including the set of non-crossing, interval, connected, irreducible, and non-crossing irreducible partitions; the definitions of these families can be found in Section 2.2. The group of semi-multiplicative functions over $\mathcal{S}$ can be naturally included as a subgroup of the semi-multiplicative functions over $\mathcal{P}$.

Another remarkable fact is that for each iterative family of partitions $\mathcal{S}$ we construct a partial order $\leq_{\mathcal{S}}$ in $\mathcal{P}$. The order is such that $\pi \leq_{\mathcal{S}} \sigma$ whenever $\pi \leq \sigma$ in the reverse refinement order, together with the extra condition that the restrictions of π to the blocks $W \in \sigma$ are required to be in $\mathcal{S}$, see Notation 2.2.8. With this recipe we can create several partial orders in $\mathcal{P}$. As particular cases we obtain two important orders that have appeared in connection to non-commutative probability, $\ll$ and $\sqsubseteq$. The order $\ll$ was introduced in [BN08], in connection to the study of the so-called Boolean Bercovici-Pata bijection, while $\sqsubseteq$ was introduced and studied in [JV15], see Subsection 2.2.5 for the definition of these orders.

1.4 Hopf algebraic structures on partitions

The group $SM^{\mathcal{P}}$ of semi-multiplicative functions in $\mathcal{P}$ is in fact a dual structure, namely it can be identified as the group of characters $\mathcal{P}^{\circ}$ of a Hopf algebra $\widehat{\mathcal{P}}$. It is known since the 1980's that Hopf algebras can be used to streamline the study of combinatorial objects. Therefore, it is better to focus on the study of Hopf algebras first, and then the convolution of semi-multiplicative functions appears as the convolution of characters of $\widehat{\mathcal{P}}$. Then, the properties of the group of functions are naturally obtained from dualizing properties of the Hopf algebraic structures. For instance, the group of multiplicative functions on incidence algebras has been studied since the 1970's [DRS72], but there was not until the 1990's that it was identified as the group of characters of an incidence Hopf algebra [Sch94].

The use of Hopf algebras in the study of free probability was first done in [MN10], where the group of multiplicative functions $MF^{\mathcal{NC}}$ was naturally identified as the group of characters of the Hopf algebra Sym of symmetric functions. When combined with the log map for characters of Sym, this identification retrieves the celebrated S-transform of Voiculescu [Voi87], which is the most efficient tool for computing distributions of products of free random variables.

In analogy to that, for each iterative family $\mathcal{S}$ we will construct a Hopf algebra $\widehat{\mathcal{S}}$, done in such a way that the character group $\mathcal{S}^{\circ}$ is naturally isomorphic to $SM^{\mathcal{S}}$, the group of semi-multiplicative functions over $\mathcal{S}$. More specifically, we can use the order $\leq_{\mathcal{S}}$ to create a Hopf algebraic structure of commutative polynomials with indeterminates $\widehat{\pi}$ indexed by partitions $\pi \in \mathcal{S}$, and where the main source of combinatorial structure comes from defining the coproduct $\Delta : \widehat{\mathcal{S}} \to \widehat{\mathcal{S}} \otimes \widehat{\mathcal{S}}$ as

$$\Delta(\widehat{\pi}) = \sum_{\substack{\sigma \in \mathcal{S}, \\ \sigma \geq_{\mathcal{S}} \pi}} \left(\prod_{W \in \sigma} \widehat{\pi_W} \right) \otimes \widehat{\sigma}.$$

The Hopf algebras $\widehat{\mathcal{S}}$ are the main players of Chapter 6, where we will discuss several properties and find connections between them. $\widehat{\mathcal{S}}$ can be identified as an incidence Hopf algebra, cf. [Sch94, Ein10]. In the case of non-crossing partitions, the Hopf algebra $\widehat{\mathcal{NC}}$ is also closely related to one of the Hopf algebras studied in the recent paper [EFFKP20]. Moreover, the inclusion of groups $MF^{\mathcal{NC}} \subseteq SM^{\mathcal{NC}}$ is precisely (in view of the canonical isomorphisms $MF^{\mathcal{NC}} \approx \mathrm{Sym}^{\circ}$ and $SM^{\mathcal{NC}} \approx \mathcal{NC}^{\circ}$) the dual $\Psi^{\circ} : \mathrm{Sym}^{\circ} \to \mathcal{NC}^{\circ}$ of a natural bialgebra homomorphism $\Psi : \widehat{\mathcal{NC}} \to \mathrm{Sym}$ provided by the Kreweras complementation map.

An advantage of working with Hopf algebras is that their antipode map can, in principle, serve as a universal tool for inversion in formulas that relate moments to cumulants, or relate different brands of cumulants living in the $\mathcal{P}$ framework. The issue of performing such inversions is constantly present in the literature on cumulants. Indeed, it is typical that cumulants (of one brand or another) are introduced via some simple formulas which are deemed to express moments in terms of the desired cumulants, like in Equation (1.2); these simple formulas then need to be inverted, if one wants to see explicit formulas describing cumulants in terms of moments. In such a situation, the tool that is typically used for inversion is the Möbius function of some underlying poset which luckily turns out to be related to the cumulants in question.

The considerations on the Hopf algebric approach suggest an alternate method which can provide a unified way of treating the inversions of various cumulant-to-moment formulas, and also for doing inversions in cumulant-to-cumulant formulas. Since moment-cumulant formulas are encoded by characters $g \in \mathcal{P}^{\circ}$, the inverse of these formulas can be obtained from the knowledge of the antipode S by simply computing $g^{-1} = g \circ S$. In Section 6.5 we examine the antipode of $\mathcal{S}$ and in particular we identify (Theorem 6.5.10) a cancellation-free formula for how the antipode works, described in terms of a suitable notion of "efficient chains" in the poset $\mathcal{S}$.

1.5 Organization of the thesis

Besides this introductory chapter, the thesis has six other chapters. In Chapter 2 we survey all the preliminaries required for the rest of the thesis. The problem of the free anti-commutator is treated in Chapter 3. The semi-multiplicative functions on $\mathcal{NC}$ are the content of Chapter 4. The more general approach of iterative families is presented in Chapter 5. The Hopf algebras over partitions $\widehat{\mathcal{S}}$ are constructed and studied in Chapter 6. Finally, Chapter 7 draws some conclusions and proposes some future work.

$$
\begin{tikzcd}
 & \text{Introduction} \arrow[d] & \\
 & \text{Chapter 2} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{Chapter 3} & \text{Chapter 4} \arrow[r, dashed] & \text{Chapter 5} \arrow[d] \\
 & & \text{Chapter 6} \arrow[d] \\
 & & \text{Chapter 7}
\end{tikzcd}
$$

The thesis can be roughly divided into three parts that are independent of each other and can be read in any order. The dashed arrow from Chapter 4 to Chapter 5 is to indicate that is recommended to read the former before proceeding to the latter. The reason is that the development for iterative families $\mathcal{S}$ follows the same line of study than the one done in Chapter 4 for the particular case of $\mathcal{NC}$, thus this chapter contains interesting examples that are useful to keep in mind while reading Chapter 5, these examples are also useful in Chapter 6 when constructing Hopf algebras. Chapters 5, 6, and 7 must be read in order as they build up on ideas given in previous chapters.

Chapter 2

Preliminaries

In this chapter we will introduce some basic notation and definitions needed for the rest of the manuscript. We first explain some basic concepts of graphs. Then we define set partitions, we present some of their main properties and introduce several special families of set partitions. Then we give a brief introduction to moments and cumulants. The last section is devoted to some basic notions on Hopf algebras.

2.1 Graphs

A *simple graph* $G = (\mathrm{Vrt}_G, \mathrm{Edg}_G)$ consists of a finite set of vertices Vrt_G, that we will usually take to be $[k] := \{1, 2, \ldots, k\}$ for $k = |\mathrm{Vrt}_G|$, together with a set of edges Edg_G, that connect two distinct vertices. The edges $e \in \mathrm{Edg}_G$ are represented by undirected pairs $(i, j) = (j, i)$ with $1 \leq i < j \leq k$. We denote by $|G|$ the number of vertices of G (in this case k). The unique graph with 1 vertex will be denoted by $\bullet$.

In Chapter 3, we will use other types of graphs, that allow *loops*, namely edges with both ends in the same vertex (i, i). We will also allow *multiple edges* between the same pair of vertices, thus we will label the edges $e_1, e_2, \ldots$ to distinguish between two edges e_a, e_b that connect the same pair of vertices (i, j). At the end of Chapter 3 we will also use directed graphs, where $(i, j) \neq (j, i)$, to emphasize the direction we will use the notation $\overleftarrow{(i, j)} = \overrightarrow{(j, i)} \neq \overleftarrow{(j, i)} = \overrightarrow{(i, j)}$. All these special concepts will be clarified before we use them. From now on, unless stated otherwise, we will work with simple graphs and refer to them only as graphs.

Two graphs G and H are *isomorphic*, if there is a bijection $\sigma : \mathrm{Vrt}_G \to \mathrm{Vrt}_H$ mapping the vertices of G to the vertices of H, such that $(i, j) \in \mathrm{Edg}_G$ if and only if $(\sigma(i), \sigma(j)) \in \mathrm{Edg}_H$. We say that G and H are *rooted-isomorphic* if we have identified $\mathrm{Vrt}_G, \mathrm{Vrt}_H$ as $[k]$ and further require that $\sigma(1) = 1$.

Notation 2.1.1. We want to distinguish between 3 main different classes of graphs.

1. For every $k \in \mathbb{N}$ we denote by $\mathcal{G}(k)$ the set of *numbered or labeled graphs* with k vertices, where we assume that the set of vertices is $[k] := \{1, 2, \ldots, k\}$. We write $\mathcal{G} := \bigcup_{k=1}^{\infty} \mathcal{G}(k)$. Notice that every graph with a total order on their vertices can be naturally made into a numbered graph.

2. We denote by $\mathcal{G}_r(k)$ the set of classes of rooted-homomorphic graphs with k vertices, and we write $\mathcal{G}_r := \bigcup_{k=1}^{\infty} \mathcal{G}_r(k)$. The distinguished vertex 1 is referred as the *root*.

3. We denote by $\mathcal{G}_\sim(k)$ the set of homomorphism classes of graphs with k vertices, and we write $\mathcal{G}_\sim := \bigcup_{k=1}^{\infty} \mathcal{G}_\sim(k)$.

To simplify the explanations, when we consider an element G of $\mathcal{G}_r(k)$ or $\mathcal{G}_\sim(k)$, we will think of G as a representative of the class, thus will label their vertices with $[k]$.

Definition 2.1.2 (Restriction of graphs)**.** Let $G \in \mathcal{G}(k)$ be a graph with $k \in \mathbb{N}$ vertices. Given a subset of m vertices $W \subset [k]$, we denote by $G_W \in \mathcal{G}(m)$ the *restriction graph* of G to W obtained by keeping the m vertices in W and all the edges between them, and then relabelling the vertices (following the total order in W) to make it a graph labelled by $[m]$. More specifically, if $W = \{w_1, w_2, \ldots, w_m\}$ with $w_1 < w_2 < \cdots < w_m$, then $(i, j) \in E_{G_W}$ if and only if $(w_i, w_j) \in E_G$.

2.1.1 Families of graphs

For $G \in \mathcal{G}(k)$, a *path* of length r between two vertices $i, j \in [k]$ is a sequence $i = i_0, i_2, \ldots, i_r = j$ in $[k]$ such that $(i_{s-1}, i_s) \in E$ for all $s = 1, \ldots, r$. If such a path exists, we say that i, j are *connected*. The *distance* between two connected vertices u, v is the length of the smallest path connecting them, we denote it by "$d(u, v)$". A *simple cycle* is a path $i_1, i_2, \ldots, i_r$ with 3 or more vertices, all of them distinct, such that $(i_1, i_r) \in E$.

Definition 2.1.3. We say that $G \in \mathcal{G}(k)$ is a *connected graph* G if there exists a path between every two vertices $i, j \in [k]$. We denote by $\mathcal{CG} \subset \mathcal{G}$ the family of connected graphs. Since connectedness is preserved by isomorphisms, it makes sense to define $\mathcal{CG}_\sim$.

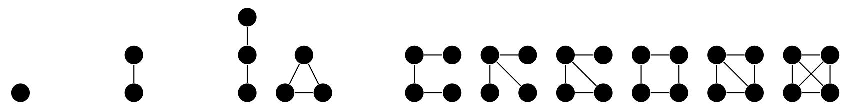


Figure 2.1: Classes of isomorphisms of connected graphs with up to 4 vertices

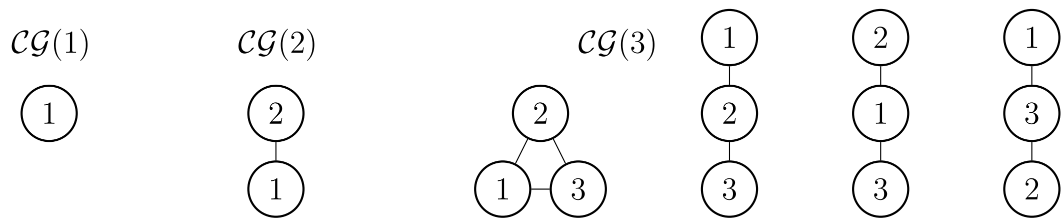


Figure 2.2: Connected graphs with up to 3 vertices

It is a standard result in graph theory that if a graph G is not connected, then its vertex set $[m]$ can be partitioned into sets $U_1, \dots, U_k$ such that the restriction G_{U_i} is connected for every $1 \le i \le k$. If such sets are maximal (under containment) they are called the *connected components* of G.

Definition 2.1.4. A graph $G \in \mathcal{G}$ is called a *tree* if it is connected and do not contains any simple cycle. We denote by $\mathcal{T} \subset \mathcal{G}$ the family of trees. Since trees are preserved by isomorphisms, it makes sense to define $\mathcal{T}_\sim$ and $\mathcal{T}_r$.

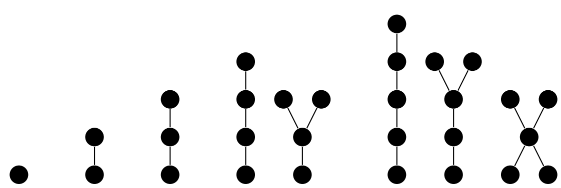


Figure 2.3: Classes of isomorphisms of trees up to 5 vertices

Remark 2.1.5. It is standard result that the trees $T \in \mathcal{T}(m)$ with m vertices, have precisely $m-1$ edges. Notice that all graphs from Figure 2.2 belong to $\mathcal{T}$, except for the cycle of length 3.

Notice that in a rooted tree $G \in \mathcal{T}_r$, every vertex v distinct from the root has a unique *parent* u such that $(u, v) \in E$ and $d(v_r, u) = d(v_r, v) - 1$. v is said to be a *child* of u.

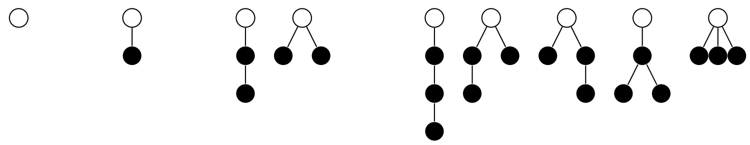
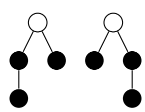


Figure 2.4: Planar rooted trees with up to 4 vertices

A *planar rooted tree* has the added condition that for every vertex v, its set of children has a total order. We denote this set by $\mathcal{PT}$. Notice that given a tree $G \in \mathcal{T}(k)$ we can naturally think of it as a planar tree by taking 1 to be the root, and for every parent v we make the order in its children coincide with the total order of the vertices $[k]$. When drawing trees we will put the root 1 at the top and then "branch" or "hang" each other vertex to its parent.

Notice that $\mathcal{T}_r(4)$ has one tree less than $\mathcal{PT}(4)$ because we identify two trees



Definition 2.1.6. The *tree factorial* $T!$ of a finite rooted tree T is recursively defined as follows. Let T be a rooted tree with k vertices. If T consists of a single vertex, set $T! = 1$. Otherwise T can be decomposed into its root vertex and branches $T_1, T_2, \ldots, T_r$ and we define recursively the number

$$T! = k \cdot T_1!T_2! \cdots T_r!.$$

2.2 Partitions

Our main objects of study are the set partitions, that we will simply call partitions. We begin with a brief introduction to the basics of partitions. For detailed discussion of all of these concepts we refer the reader to [Sta12].

Definition 2.2.1. A *partition* π *of a finite set* S is a set of the form $\pi = \{V_1, \ldots, V_k\}$ where $V_1, \ldots, V_k \subset S$ are pairwise disjoint non-empty subsets of S such that $V_1 \cup \cdots \cup V_k = S$. The subsets $V_1, \ldots, V_k$ are called *blocks* of π, and we write $|\pi|$ for the number of blocks of π, in this case k. Unless otherwise stated, we will assume that the blocks of π are ordered by their minimum element, namely that $\min(V_1) < \min(V_2) < \cdots < \min(V_k)$.

We denote by $\mathcal{P}(n)$ the family of all partitions of $[n] := \{1, \ldots, n\}$. And we denote by $\mathcal{P} = \cup_{n\in\mathbb{N}}\mathcal{P}(n)$ the set of all possible partitions. We can also organize the partitions by their number of blocks, we will denote by $\mathcal{P}_k := \{\pi \in \mathcal{P} : |\pi| = k\}$ the set of partitions with k blocks.

Every partition is naturally associated to an equivalence relation $\sim_\pi$ on $[n]$. Given $\pi \in \mathcal{P}(n)$, we say that $i \sim_\pi j$ holds if and only if there is a block $V \in \pi$ such that $i, j \in V$.

We will represent the partitions using diagrams in the plane where the elements $1, 2, \ldots,$ n are written on a horizontal line and we connect elements in the same block with lines in the upper half plane. The partitions $\pi, \sigma, \tau \in \mathcal{P}(9)$ are depicted in Figure 2.5.

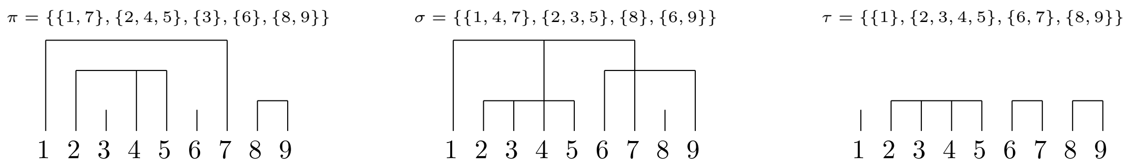
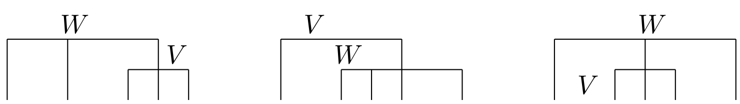


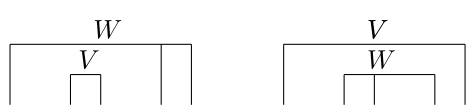
Figure 2.5: Set partitions

Notation and Remark 2.2.2 (Relative position of blocks)**.** Given two distinct blocks V, W of a partition, they can be in one of the following three relative positions:

1. V and W *cross (or have a crossing)*, namely there exist elements $1 \le i_1 < j_1 < i_2 < j_2 \le n$ such that $i_1, i_2 \in V$ and $j_1, j_2 \in W$.



2. V and W are *nested*, which can happen in two possible ways. One option is that V *nests inside* W, denoted as "$V \prec W$", meaning that there are $i, j \in W$ consecutive (within the block) such that $i < v < j$ for every $v \in V$. The other option is the opposite, that V *is an outer block of* W, denoted as "$V \succ W$", meaning that W is nested inside V.



3. V and W are in *interval position*, meaning that either V is to the left of W, namely $\max(V) \leq \min(W)$, or that V is to the right of W, namely $\max(W) \leq \min(V)$.

Remark 2.2.3. Observe that no matter how you draw the diagram of a partition (in the upper half plane), two blocks that cross will always have an intersection in the diagram. Now, depending on how one draws the diagram, there could be one or more intersections. There are several ways to define the number of crossings between a pair of blocks. In this thesis we do not care how many times they cross, we just distinguish if they cross or do not cross. In the latter case, the blocks must be nested or in interval position.

When drawing the diagram of a partition we can potentially create unnecessary crossings between two nested partitions, say, if for $V \prec W$ we draw V taller than W. We will not create unnecessary crossings to avoid miss-leading diagrams, this can always be done by drawing W taller than V whenever $W \succ V$.

Definition 2.2.4 (Non-crossing partitions $\mathcal{NC}$)**.** We say that $\pi \in \mathcal{P}$ is a *non-crossing partition* if there do not exist two distinct blocks of π that cross. In other words, the diagram of π can be drawn without crossings. We denote this family by $\mathcal{NC}$.

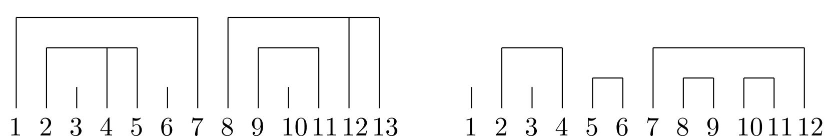


Figure 2.6: Non-crossing partitions

Concerning the sets of non-crossing partitions $\mathcal{NC}(n)$, we will follow standard notation commonly used in the literature, as presented for instance in Lectures 9 and 10 of [NS06]. Here is a quick review of some notational highlights.

Remark and Notation 2.2.5. Let us fix a partition $\pi \in \mathcal{NC}$. Observe that the relation $\prec$ defines a partial order on the blocks of π. For $V, W \in \pi$ we say that W is the *parent* of V if $W \succ V$ and there is no other block $U \in \pi$ such that $W \succ U \succ V$.

A maximal block $V \in \pi$ in this order (which is not nested into anything else) is called *outer block*, all the other blocks are called *inner blocks*. We will use the notation $\mathrm{inner}(\pi)$ and $\mathrm{outer}(\pi)$ for the numbers of inner respectively outer blocks of π. We thus have $|\pi| = \mathrm{inner}(\pi) + \mathrm{outer}(\pi)$, with $\mathrm{inner}(\pi) \geq 0$ and $\mathrm{outer}(\pi) \geq 1$.

Notice that $V \in \pi$ is minimal with respect to $\prec$ (no other $W \in \pi$ is nested inside V) if and only if V is of the form $V = \{i, i+1, \ldots, i+j\}$ for some positive integers $i \leq i+j$, this type of blocks are called *interval blocks*.

Definition 2.2.6. A partition $\pi \in \mathcal{P}$ is an *interval partition* if all the blocks $V \in \pi$ are intervals, and we denote the set of interval partitions by $\mathcal{I}$. Observe that

$$\mathcal{I} := \big\{\pi \in \mathcal{NC} \mid \mathrm{inner}(\pi) = 0\big\} \tag{2.1}$$

Figure 2.7: Interval partitions

Remark 2.2.7. Notice that if $\pi \in \mathcal{I}$, then every pair of blocks $V, W \in \pi$ is in interval position. Actually this is an equivalent definition of an interval partition.

2.2.1 Restriction of partitions

Since throughout this manuscript we will work extensively with restrictions of non-crossing partitions, we take a moment to state clearly what is our notation for how this works.

Notation 2.2.8 (Restriction of partitions)**.** Given a partition $\pi \in \mathcal{P}(n)$ and a set $W = \{p_1, \ldots, p_m\} \subseteq [n]$ where $1 \leq m \leq n$ and $p_1 < \cdots < p_m$ then "π_W" is the partition of $[m] := \{1, \ldots, m\}$ described as follows: for $i, j \in [m]$ we have

$$\begin{pmatrix} i \text{ and } j \text{ belong to} \\ \text{the same block of } \pi_W \end{pmatrix} \Leftrightarrow \begin{pmatrix} p_i \text{ and } p_j \text{ belong to} \\ \text{the same block of } \pi \end{pmatrix}.$$

It is immediate that the hypothesis of π being non-crossing (respectively interval) implies that $\pi_W \in \mathcal{NC}(m)$ (respectively $\pi_W \in \mathcal{I}(m)$).[1]

[1]It would be possible to introduce and use here the notion of "non-crossing or interval partition of W". After weighing the pros and cons of doing so, I decided to rather let π_W be a partition in $\mathcal{NC}(m)$ or $\mathcal{I}(m)$, for $m = |W|$.

For instance, if we restrict the partition $\pi = \{\{1,7\},\{2,4,5\},\{3\},\{6\},\{8,9\}\}$, to the set $W = \{2,3,4,5,7,8,9\}$ we obtain the partition $\pi = \{\{1,3,4\},\{2\},\{5\},\{6,7\}\}$ as depicted in Figure 2.8.

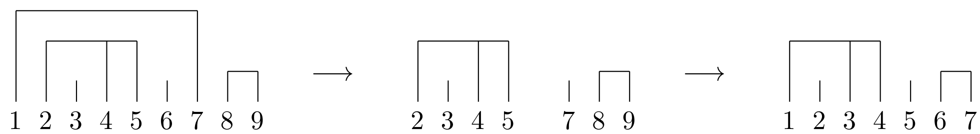


Figure 2.8: Restriction of partitions

2.2.2 Reverse refinement partial order in $\mathcal{P}(n)$

Notation 2.2.9 (Partial order in $\mathcal{P}(n)$)**.** The main partial order we consider on $\mathcal{P}(n)$ is the one given by reverse refinement: for $\pi, \sigma \in \mathcal{P}(n)$, we write "$\pi \leq \sigma$" to mean that every block of σ is a union of blocks of π. The maximal element of partially ordered set (poset) $(\mathcal{P}(n), \leq)$ is $1_n := \{\{1,\dots,n\}\}$, the partition of $[n]$ with only one block, and the minimal element is $0_n := \{\{1\},\{2\},\dots,\{n\}\}$, the partition of $[n]$ with n blocks.

Throughout this thesis it is convenient to combine all the posets $(\mathcal{P}(n), \leq)$ into one single poset $(\mathcal{P}, \leq)$, where $\mathcal{P} = \cup_{n\in\mathbb{N}} \mathcal{P}(n)$ and two partitions $\pi, \sigma \in \mathcal{P}$ are incomparable whenever they do not partition the same number of elements.

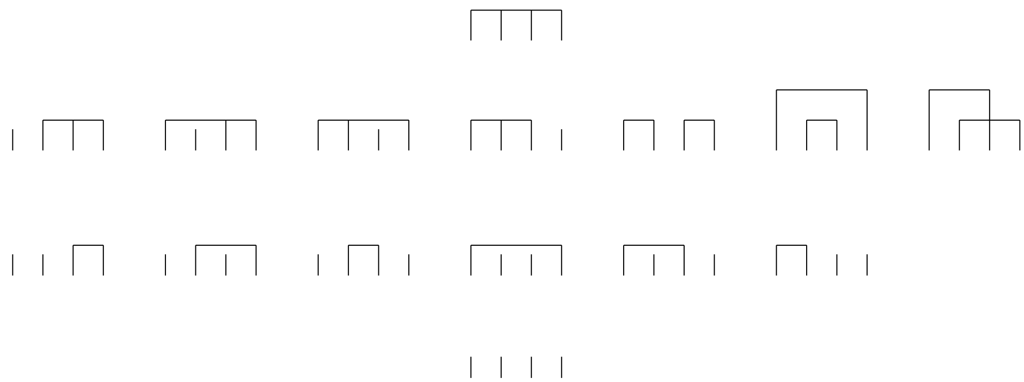
The families $\mathcal{NC}$ and $\mathcal{I}$ naturally inherit the poset structure from $(\mathcal{P}, \leq)$. We will also make occasional use of two other partial orders on $\mathcal{NC}$, denoted $\ll$ and $\sqsubseteq$, which are reviewed in Subsection 2.2.5.

Remark and Notation 2.2.10. Given a poset $(\mathcal{L}, \leq)$, we say it is a *lattice* if every two elements $\pi, \sigma \in \mathcal{L}$ have:

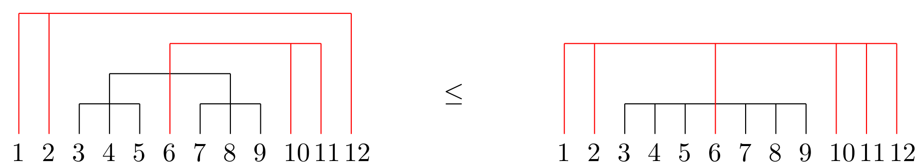
- a *join* "$\pi \vee_{\mathcal{L}} \sigma$", defined as the smallest element that is bigger than both π and σ;
- and a *meet* "$\pi \wedge_{\mathcal{L}} \sigma$", defined as the largest element that is smaller than π and σ.

It is not hard to show that for every $n \in \mathbb{N}$ we get that $(\mathcal{P}(n), \leq)$, $(\mathcal{NC}(n), \leq)$ and $(\mathcal{I}(n), \leq)$ are lattices, see for instance [NS06, Proposition 9.17]. Moreover, we can rank the partition depending on the number of blocks.

The set $\mathcal{P}(4)$ contains 15 partitions ranked as follows:



Below we show an example of two comparable partitions in $(\mathcal{P}(12), \leq)$



Remark 2.2.11 (Common meet and properties of the joins)**.** It can be seen that the meet operation is the same for our three main lattices, and it can be given explicitly. That is, for $\pi, \sigma \in \mathcal{NC}(n) \subset \mathcal{P}(n)$ with $\pi = \{V_1, \ldots, V_j\}$ and $\sigma = \{W_1, \ldots, W_k\}$ one has that

$$\pi \wedge_{\mathcal{P}} \sigma = \pi \wedge_{\mathcal{NC}} \sigma = \{V_r \cap W_s : 1 \leq r \leq j, 1 \leq s \leq k, \text{ and } V_r \cap W_s \neq \emptyset\}.$$

Also, if $\pi, \sigma \in \mathcal{I}(n) \subset \mathcal{NC}(n)$, then we one obtains $\pi \wedge_{\mathcal{I}} \sigma = \pi \wedge_{\mathcal{NC}} \sigma$, so we can denote the meet (with respect to any the three lattices) simply by $\wedge$.

On the other hand, the join really depends on which lattice we are working with. However, in this thesis we are only concerned with a very special case, where $\pi \in \mathcal{NC}(2n)$ and $\sigma = I_{2n} := \{\{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}\} \in \mathcal{I}(2n)$ is an interval partition. In this case, the joins in $\mathcal{NC}$ and $\mathcal{P}$ coincide: $\pi \vee_{\mathcal{P}} \sigma = \pi \vee_{\mathcal{NC}} \sigma$. Furthermore, for every two partitions $\pi, \sigma \in \mathcal{P}(n)$, it is known that $\pi \vee_{\mathcal{P}} \sigma = 1_n$ if and only if for every two elements $i, j \in [n]$ we can find elements $i = i_0, i_1, i_2, \ldots, i_{2k+1} = j$ such that for $l = 0, \ldots, k$, we have $i_{2l} \sim_\sigma i_{2l+1}$ and $i_{2l+1} \sim_\pi i_{2l+2}$. A detailed discussion on these and other properties of the join can be found in Lecture 9 of [NS06].

For $\pi \leq \sigma$ in $\mathcal{P}(n)$, we use the standard notation $[\pi, \sigma] := \{\tau \in \mathcal{P}(n) : \pi \leq \tau \leq \sigma\}$ for the intervals in a poset. A crucial property of the family of partitions $\mathcal{P}$ is that any interval is canonically isomorphic, as a poset, to a direct product of intervals of the form $[\pi, 1_n]$. Due to the importance of this result, we will record it as a Lemma.

Lemma 2.2.12 (Useful lattice factorization). *For every $\pi \leq \sigma$ in $\mathcal{P}$, it holds that*

$$[\pi, \sigma] \cong \prod_{W \in \sigma} [\pi_W, 1_{|W|}]. \tag{2.2}$$

If in this factorization we fix $\sigma \in \mathcal{P}(n)$ and take $\pi = 0_n$, then for any elements X_1, X_2, $\dots$, X_n in some ring R one has the following identity

$$\prod_{W \in \sigma} \left(\sum_{\tau \in \mathcal{P}(|W|)} \prod_{V \in \tau} X_{|V|} \right) = \sum_{\substack{\tau \in \mathcal{P}, \\ \tau \leq \sigma}} \prod_{V \in \tau} X_{|V|}, \tag{2.3}$$

Remark 2.2.13. Observe that if we instead consider $[\pi, \sigma]_{\mathcal{NC}} := \{\tau \in \mathcal{NC}(n) : \pi \leq \tau \leq \sigma\}$ to be the interval of non-crossing partitions then it is straightforward to adapt the previous lemma to the family $\mathcal{NC}$, as this property is just inherited from poset structure of $\mathcal{P}$.

An interesting thing to observe, is what happens with a second isomorphism that is usually taken to follow the one from (2.2). In general this isomorphism breaks intervals of the form $[\pi, 1_n]$ into intervals of the form $[0_n, 1_n] = \mathcal{P}(n)$. Turns out that, in the case of $\mathcal{P}$ the isomorphism is very simple, as $[\pi, 1_n] \cong [0_{|\pi|}, 1_{|\pi|}] = \mathcal{P}(|\pi|)$. The same happens for interval partitions, as $[\pi, 1_n]_{\mathcal{I}} \cong [0_{|\pi|}, 1_{|\pi|}]_{\mathcal{I}} = \mathcal{I}(|\pi|)$. On the other hand, for $\mathcal{NC}$ one needs to do a more involved procedure: for every $k \geq 1$ and $\theta \in \mathcal{NC}(k)$ one has

$$[\theta, 1_k] \cong [0_k, \mathrm{Kr}(\theta)] \cong \prod_{U \in \mathrm{Kr}(\theta)} [0_{|U|}, 1_{|U|}] = \prod_{U \in \mathrm{Kr}(\theta)} \mathcal{NC}(|U|), \tag{2.4}$$

where Kr is the Kreweras complementation map that will be defined in detail in Subsection 2.2.4.

It is immediate how the two kinds of isomorphisms discussed above work together to yield the fact that for any $n \geq 1$ and $\pi \leq \sigma$ in $\mathcal{NC}(n)$ one has a canonical isomorphism

$$[\pi, \sigma] \cong \mathcal{NC}(1)^{p_1} \times \mathcal{NC}(2)^{p_2} \times \cdots \times \mathcal{NC}(n)^{p_n} \text{ for some } p_1, \dots, p_n \geq 0. \tag{2.5}$$

For a detailed discussion on these isomorphisms we refer to [NS06, pages 148-153 in Lecture 9]. It may be re-assuring to know that, more than being canonical, the exponents $p_2, \dots, p_n$ in (2.5) are in fact uniquely determined – cf. [NS06, Proposition 9.38]. (The exponent p_1 in (2.5) is not uniquely determined, since $|\mathcal{NC}(1)| = 1$.)

2.2.3 Closure operators

Given a poset P, a map $\bar{\cdot} : P \to P$ is a *closure operator* if it is:

1. **Increasing**, $x \leq \overline{x}$ for every $x \in P$.
2. **Order preserving**, if $x \leq y$ then $\overline{x} \leq \overline{y}$;
3. **Idempotent**, $\overline{\overline{x}} = \overline{x}$.

There is a natural way to define closure operations on $\mathcal{P}(n)$ using the sublattices $\mathcal{I}(n)$ and $\mathcal{NC}(n)$.

Definition 2.2.14. Let $n \in \mathbb{N}$ and consider a subset $\mathcal{L}(n) \subset \mathcal{P}(n)$ which is stable under the meet operation $\wedge$. We define the $\mathcal{L}$*-closure of* $\pi \in \mathcal{P}(n)$, as the smallest partition $\overline{\pi}^{\mathcal{L}} \in \mathcal{L}$ that is bigger than π.

It is not hard to convince ourselves that for $\mathcal{L} = \mathcal{NC}$ or $\mathcal{L} = \mathcal{I}$ we indeed get a closure operator. We can now use these two closure operators to define two new families of partitions in $\mathcal{P}$.

Definition 2.2.15 (Connected partitions)**.** A partition π is *connected*[2] if its non-crossing closure is equal to the maximal partition 1_n for some $n \in \mathbb{N}$. The set of connected partitions is denoted by $\mathcal{CON}$.

Definition 2.2.16 (Irreducible partitions)**.** A partition $\pi \in \mathcal{P}(n)$ is *irreducible* if its interval closure is equal to the maximal partition 1_n. The family of irreducible partitions is denoted by $\mathcal{IRR}$.

Another very important family of partitions that we will use is $\mathcal{NCIRR} := \mathcal{NC} \cap \mathcal{IRR}$ and consists of all the non-crossing irreducible partitions.

Notation and Remark 2.2.17. *(Concatenation and irreducibility).*

1^o Given $n_1, n_2 \geq 1$ and $\pi_1 \in \mathcal{NC}(n_1)$, $\pi_2 \in \mathcal{NC}(n_2)$, we denote by $\pi_1 \diamond \pi_2$ the non-crossing partition in $\mathcal{NC}(n_1 + n_2)$ which is obtained by placing π_1 on the points $1, \ldots, n_1$ and π_2 on the points $n_1 + 1, \ldots, n_1 + n_2$.

[2] We already used the term connected to refer to a property of graphs. The reason for the name here is that a partition π is connected if the diagram of π is connected, this equivalence will be made precise in Subsection 5.2.3, where we show that the crossing graph of a connected partition is a connected graph

2^o Notice that a non-crossing partition $\pi \in \mathcal{NC}(n)$ is *irreducible* precisely when it cannot be written in the form $\pi = \pi_1 \diamond \pi_2$ with $\pi_1 \in \mathcal{NC}(n_1)$ and $\pi_2 \in \mathcal{NC}(n_2)$ for some $n_1, n_2 \geq 1$ with $n_1 + n_2 = n$. This condition is easily seen to be equivalent to the fact that the numbers 1 and n belong to the same block of π, i.e., outer$(\pi) = 1$.

3^o Every $\pi \in \mathcal{NC}(n)$ can be written as a concatenation of irreducible partitions. This is best understood by referring to the outer blocks of π. Indeed, it is straightforward to check that these outer blocks can be listed as $W_1, \ldots, W_k$, with

$$\min(W_1) = 1, \min(W_2) = 1 + \max(W_1), \ldots, \min(W_k) = 1 + \max(W_{k-1}), \max(W_k) = n.$$

For every $1 \leq i \leq k$, consider the interval $J_i := \{m \in \mathbb{N} \mid \min(W_i) \leq m \leq \max(W_i)\}$, which is a union of blocks of π, and consider the restricted partition $\pi_{_{J_i}} \in \mathcal{NC}(|J_i|)$. The concatenation $\pi_{_{J_1}} \diamond \cdots \diamond \pi_{_{J_k}}$ will then give back the π we started with, and every $\pi_{_{J_i}}$ is an irreducible partition in $\mathcal{NC}(|J_i|)$. Notice that the interval partition $\{J_1, \ldots, J_k\}$ is precisely $\overline{\pi}^{\mathcal{I}}$, the interval closure of π.

Throughout this thesis we will always use the convention that if $\mathcal{S} \subset \mathcal{P}$ is a family of partitions then $\mathcal{S}_k \subset \mathcal{P}_k$ are the partition of the family with k blocks, while $\mathcal{S}(n) \subset \mathcal{P}(n)$ are the partitions of the set $[n]$.

2.2.4 The Kreweras complement in $\mathcal{NC}$

Throughout this thesis we are going to encounter some well studied subsets of $\mathcal{NC}(n)$. We will exploit the fact that some of these subsets are well behaved when applying the Kreweras complementation map. This subsection is devoted to fix the notation for all these subsets and discuss the Kreweras complementation map.

Definition 2.2.18. Given a partition $\pi \in \mathcal{P}(2n)$ we say that

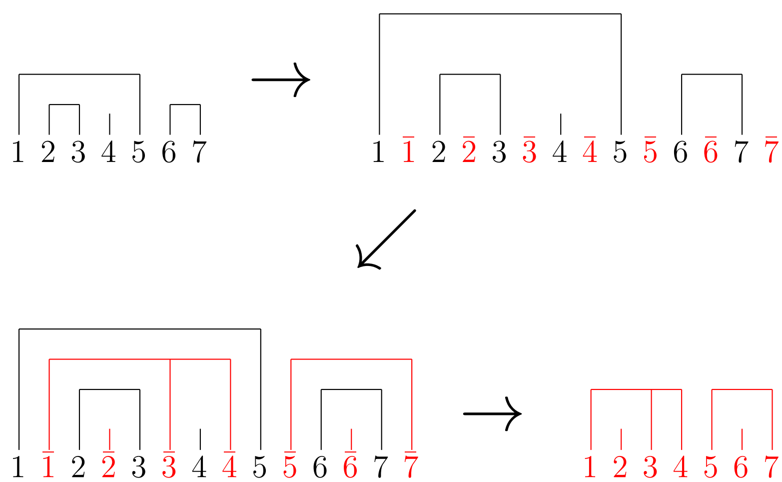
- π is *parity preserving* if every block $V \in \pi$ is contained either in $\{1, 3, \ldots, 2n-1\}$ or in $\{2, 4, \ldots, 2n\}$. We denote the set of parity preserving non-crossing partitions of $[2n]$ by $\mathcal{NC}^{\text{pa.pr}}(2n)$.
- π is *even* if every block $V \in \pi$ has even size. We denote the set of non-crossing even partitions of $[2n]$ by $\mathcal{NC}^{\text{even}}(2n)$.
- π is *a pairing (or a pair partition)* if every block $V \in \pi$ has size $|V| = 2$. We denote the set of non-crossing pair partitions of $[2n]$ by $\mathcal{NC}^{\text{pair}}(2n)$.

Notice that the above definitions work in general for every partition, not necessarily non-crossing, and in general we can adopt notation such as $\mathcal{P}^{even}$. Since we will not use these families here, we just fix the notation for $\mathcal{NC}$.

Remark and Notation 2.2.19. Observe that given two partitions $\sigma_1, \sigma_2 \in \mathcal{NC}(n)$, there is a unique partition $\sigma \in \mathcal{P}(2n)$ such that σ is parity preserving and the restrictions to odd and even elements satisfy $\sigma_{\{1,3,\ldots,2n-1\}} = \sigma_1$ and $\sigma_{\{2,4,\ldots,2n\}} = \sigma_2$, we denote this partition σ by $\langle \sigma_1, \sigma_2 \rangle$.

Notice that σ_1, σ_2 being non-crossing is a necessary but not sufficient condition for $\langle \sigma_1, \sigma_2 \rangle$ to be non-crossing. The Kreweras complement of a fixed partition σ_1 is actually the biggest partition $\sigma_2 \in \mathcal{NC}(n)$ such that $\langle \sigma_1, \sigma_2 \rangle$ is non-crossing.

Definition 2.2.20 (Kreweras complement)**.** Given $\pi \in \mathcal{NC}(n)$, its *Kreweras complement*, $\mathrm{Kr}_n(\pi) \in \mathcal{NC}(n)$, is defined as the biggest partition of $\mathcal{NC}(n)$ such that $\langle \pi, Kr_n(\pi) \rangle \in \mathcal{NC}(2n)$. We will omit the sub-index n in Kr_n whenever it is clear from the context. Below we depict an example on how to get the Kreweras complement of a partition.



Occasionally it is useful to consider the more general notion of *relative Kreweras complement of π in σ*, defined for any $\pi \leq \sigma$ in $\mathcal{NC}(n)$, and where the "usual" Kreweras complement corresponds to the special case $\sigma = 1_n$; see the discussion on pages 288-291 in Lecture 18 of [NS06].

Remark 2.2.21. Since $\langle \pi, 0_n \rangle$ is always non-crossing, the set of non-crossing partitions such that $\langle \pi, \mathrm{Kr}_n(\pi) \rangle \in \mathcal{NC}(2n)$ is not empty. Since $\mathcal{NC}(n)$ is a lattice, the latter set has a well-defined join, which can be shown to still belong to the set. This is a justification for the fact that the Kreweras complement is well-defined. Moreover, the map $\mathrm{Kr}_n : \mathcal{NC}(n) \to \mathcal{NC}(n)$ is actually a lattice anti-isomorphism of $\mathcal{NC}(n)$. In particular, for every $\pi, \sigma \in \mathcal{NC}(n)$ we have that $\mathrm{Kr}_n(\pi \vee_{\mathcal{NC}} \sigma) = Kr_n(\pi) \wedge \mathrm{Kr}_n(\sigma)$ and $\mathrm{Kr}_n(\pi \wedge \sigma) = Kr_n(\pi) \vee_{\mathcal{NC}} \mathrm{Kr}_n(\sigma)$. Another important property of this map is that $|Kr_n(\pi)| = n + 1 - |\pi|$ for all $\pi \in \mathcal{NC}(n)$, which generalizes the immediate fact that $\mathrm{Kr}_n(1_n) = 0_n$ and $\mathrm{Kr}_n(0_n) = 1_n$.

We are also going to use the inverse of the Kreweras complement Kr_n^{-1}, which for $\sigma \in \mathcal{NC}(n)$ can be explicitly defined as the biggest partition $\mathrm{Kr}_n^{-1}(\sigma) \in \mathcal{NC}(n)$ such that $\left\langle \mathrm{Kr}_n^{-1}(\sigma), \sigma \right\rangle \in \mathcal{NC}(2n)$.

Remark 2.2.22. An interesting fact is that the Kreweras complement of an even partition is a parity preserving partition, and vice-versa. Namely we have that $\mathrm{Kr}_{2n}(\mathcal{NC}^{\mathrm{even}}(2n)) = \mathcal{NC}^{\mathrm{pa.pr}}(2n)$ and $\mathrm{Kr}_{2n}(\mathcal{NC}^{\mathrm{pa.pr}}(2n)) = \mathcal{NC}^{\mathrm{even}}(2n)$, see Exercise 9.42 of [NS06].

Notice that $\mathcal{NC}^{\mathrm{pair}}(2n) \subset \mathcal{NC}^{\mathrm{even}}(2n)$, and this implies that

$$\mathrm{Kr}_{2n}(\mathcal{NC}^{\mathrm{pair}}(2n)) \subset \mathrm{Kr}_{2n}(\mathcal{NC}^{\mathrm{even}}(2n)) = \mathcal{NC}^{\mathrm{pa.pr}}(2n).$$

Moreover, the pair partitions are in some sense the minimal even partitions, and thus when taking the Kreweras complement we get the maximal parity preserving partitions. Specifically, the image of the set of pair partitions under the Kreweras complement map can be described as

$$\mathrm{Kr}_{2n}(\mathcal{NC}^{\mathrm{pair}}(2n)) = \{\langle \pi, \mathrm{Kr}_n(\pi) \rangle \in \mathcal{NC}(2n) : \pi \in \mathcal{NC}(n)\}.$$

Vice-versa, the Kreweras complement of partitions of this type gives again a pair partition.

2.2.5 The partial orders $\ll$ and $\sqsubseteq$ on $\mathcal{NC}(n)$

We will make use of two partial order relations on $\mathcal{NC}(n)$ which are finer than reverse refinement, and are defined as follows.

Notation 2.2.23. Let $n \in \mathbb{N}$ and $\pi, \sigma \in \mathcal{NC}(n)$.

1^o We will write "$\pi \ll \sigma$" to mean that $\pi \leq \sigma$ in the reverse refinement order and that, in addition, the following happens:

$$\left\{ \begin{array}{l} \text{For every block } W \in \sigma \text{ there exists a block} \\ V \in \pi \text{ such that } \min(W), \max(W) \in V. \end{array} \right. \tag{2.6}$$

2^o We will write "$\pi \sqsubseteq \sigma$" to mean that $\pi \leq \sigma$ in the reverse refinement order and that, in addition, the following happens:

$$\left\{ \begin{array}{l} \text{Suppose } W \in \sigma \text{ and } i_1 < i_2 < i_3 \text{ are elements of } W. \\ \text{Suppose moreover that } i_1 \text{ and } i_3 \text{ belong to the same block } V \in \pi. \\ \text{Then it follows that } i_2 \in V \text{ as well.} \end{array} \right. \tag{2.7}$$

Remark 2.2.24. Both partial orders $\ll$ and $\sqsubseteq$ have been considered before: $\ll$ was introduced in [BN08], in connection to the study of the so-called Boolean Bercovici-Pata bijection, while $\sqsubseteq$ was introduced and studied in [JV15].

We mention that the recent paper [BJV19] generalizes $\ll$ and $\sqsubseteq$ to the setting of Coxeter groups and puts into evidence the fact that these two partial orders are, in a certain sense, dual to each other. A special case of this duality, which we use in Section 4.8, is reviewed in Remark 2.2.26 below.

Remark 2.2.25. 1^o Let $n \in \mathbb{N}$ and $\pi \in \mathcal{NC}(n)$, and let us record what happens when in Notation 2.2.23 we put $\sigma = 1_n$. We note that:

– "$\pi \ll 1_n$" means that 1 and n are in the same block of π, i.e. that π is irreducible.

– "$\pi \sqsubseteq 1_n$" means precisely that π is an interval partition.

2^o More generally, let $n \in \mathbb{N}$ and let $\pi, \sigma \in \mathcal{NC}(n)$ be such that $\pi \leq \sigma$. One can construe the latter inequality as saying that π is obtained out of σ by taking, one by one, the blocks of σ, and replacing them by a non-crossing partition of the same size. From this perspective: the relation $\pi \ll \sigma$ amounts to the fact that π is obtained by replacing every block of σ by an *irreducible* partition, while the relation $\pi \sqsubseteq \sigma$ amounts to the fact that π is obtained by replacing every block of σ by an *interval* partition.

Remark 2.2.26. We record here two facts about the order relations $\ll$ and $\sqsubseteq$ that will be used later on in the paper.

1^o The Kreweras complementation map Kr on $\mathcal{NC}(n)$ provides us with a bijection

$$\big\{\pi \in \mathcal{NC}(n) \mid \pi \text{ is irreducible}\big\} \ni \tau \mapsto \mathrm{Kr}(\tau) \in \big\{\sigma \in \mathcal{NC}(n) \mid \{n\} \text{ is a block of } \sigma\big\}, \tag{2.8}$$

which is a *poset anti-isomorphism* when the set on the left-hand side of (2.8) is endowed with the partial order $\ll$, while the set on the right-hand side is endowed with $\sqsubseteq$. For reference, see e.g. [FMNS19, Lemma 2.10].

2^o For an irreducible partition $\pi \in \mathcal{NC}(n)$, the upper ideal $\{\sigma \in \mathcal{NC}(n) \mid \sigma \gg \pi\}$ has cardinality $2^{|\pi|-1}$. And more precisely: for $\pi \in \mathcal{NC}(n)$ and every $1 \leq k \leq |\pi|$, one has that

$$\Big| \{\sigma \in \mathcal{NC}(n) \mid \sigma \gg \pi, \, |\sigma| = k\} \Big| = \binom{|\pi| - 1}{k - 1}. \tag{2.9}$$

For reference, see e.g. [BN08, Proposition 2.13].

2.2.6 Terminology for monotone partitions

To finish this section we introduce the families of monotone partitions.

Definition 2.2.27. An *ordered partition* is a pair (π, λ) of a set partition π and a bijection $\lambda : \pi \to |\pi|$. Notice that λ generates a total order $<_\lambda$ of the blocks of π.

A *monotone partition* is an ordered partition (π, λ) with $\pi \in \mathcal{NC}$ such that, for $V, W \in \pi$, $\lambda(V) > \lambda(W)$ whenever $V \prec W$.

Ordered partitions are not properly a subset of $\mathcal{P}$, but we do not really need to deal with this type of partitions, we just need to know how many monotone labelings there are for a non-crossing partition π. For this we will use [AHLV15, Proposition 3.3], which counts monotone labelings in terms of the nesting forest associated to the partition π.

Definition 2.2.28. Let $\pi \in \mathcal{NCIRR}$, its *nesting tree* $\tau(\pi) \in \mathcal{PT}$ is the planar rooted tree, whose vertices are the blocks of π, the root being the unique outer block, and we draw an edge between V and W if V covers W or W covers V in the $\prec$ order. The order of the children can be inherited from the total order of blocks given by the minimum element. If $\pi \in \mathcal{NC}$, its nesting forest $\tau(\pi)$ consists of the nesting trees of the irreducible components of π.

Proposition 2.2.29 (Proposition 3.3 of [AHLV15])**.** *The number $m(\pi)$ of monotone labellings of a non-crossing irreducible partition π depends only on its nesting tree $\tau(\pi)$ and is given by*

$$m(\pi) = \frac{|\pi|!}{\tau(\pi)!},$$

where the tree factorial $\tau(\pi)!$ is as in Definition 2.1.6. Moreover, for a general partition $\pi \in \mathcal{NC}$, not necessarily irreducible, we have

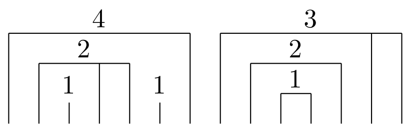
$$\frac{m(\pi)}{|\pi|!} = \prod_{i=1}^{k} \frac{1}{\tau(\pi_i)!}, \tag{2.10}$$

where $\pi_1, \pi_2, \dots, \pi_k$ are the irreducible components of π.

Remark 2.2.30. The reciprocal of the quantity that appeared in (2.10) can be defined directly in terms of π as follows:

$$\tau(\pi)! := \prod_{V\in\pi} \left| \left\{ W \in \pi : W \preceq V \right\} \right|, \tag{2.11}$$

That is, for every block of $V \in \pi$ we count all the blocks nested inside V (including it) and we multiply all these numbers. We exemplify this procedure with a diagram:

$\pi =$ 

$$\tau(\pi)! = 4 \cdot 2 \cdot 1 \cdot 1 \cdot 3 \cdot 2 \cdot 1 = 48$$

2.3 Moment-cumulant formulas

Relations between cumulants in non-commutative probability are the motivation for this work, and the source of all our applications. Thus, we will give a brief introduction to the combinatorial aspects of non-commutative probability.

A *non-commutative probability space* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional, such that $\varphi(1_{\mathcal{A}}) = 1$. The *n-th multivariate moment* is the multilinear functional $\varphi_n : \mathcal{A}^n \to \mathbb{C}$, such that $\varphi_n(a_1, \dots, a_n) := \varphi(a_1 \cdots a_n) \in \mathbb{C}$, for elements $a_1, \dots, a_n \in \mathcal{A}$. In this framework, given subalgebras $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ with certain conditions, one can define different notions of non-commutative independence. Usually if one knows all the mixed moments with variables in $\mathcal{A}_1$ and all the mixed moments with variables in $\mathcal{A}_2$, then $\mathcal{A}_1$ and $\mathcal{A}_2$ being independent provides a unique universal rule to compute the mixed moments of products of elements in $\mathcal{A}_1$ and $\mathcal{A}_2$. The first such rule, known as freeness was introduced by Voiculescu [VDN92]. Other two important rules include Boolean and monotone independence. In this work we just care about the combinatorial approach, that can be expressed in terms of cumulants. In order to present it, we first introduce a preliminary notation.

Notation 2.3.1. Given a family of multilinear functionals $\{f_m : \mathcal{A}^m \to \mathbb{C}\}_{m\geq 1}$ and a partition $\pi \in \mathcal{P}(n)$, we define $f_\pi : \mathcal{A}^n \to \mathbb{C}$ to be the map

$$f_\pi(a_1, \dots, a_n) := \prod_{V\in\pi} f_{|V|}(a_V), \qquad \forall a_1, \dots, a_n \in \mathcal{A},$$

where for every block $V := \{v_1, \cdots, v_k\}$ of π (such that $v_1 < \cdots < v_k$ are in natural order) we use the notation $f_{|V|}(a_V) := f_k(a_{v_1}, \dots, a_{v_k})$.

With this notation in hand we can define several brands of cumulants.

Definition 2.3.2 (Cumulants)**.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. The *classical cumulants* $\{c_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$, *free cumulants* $\{r_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$, *Boolean cumulants* $\{b_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$, and *monotone cumulants* $\{h_n : \mathcal{A}^n \to \mathbb{C}\}_{n\geq 1}$ are families of multilinear functionals recursively defined by requiring that for all $n \in \mathbb{N}$ and $a_1, \dots .a_n \in \mathcal{A}$ the following formulas hold:

$$\varphi_n(a_1, \dots, a_n) = \sum_{\pi \in \mathcal{P}(n)} c_\pi(a_1, \dots, a_n), \tag{2.12}$$

$$\varphi_n(a_1, \dots, a_n) = \sum_{\pi \in \mathcal{NC}(n)} r_\pi(a_1, \dots, a_n), \tag{2.13}$$

$$\varphi_n(a_1, \dots, a_n) = \sum_{\pi \in \mathcal{I}(n)} b_\pi(a_1, \dots, a_n), \tag{2.14}$$

$$\varphi_n(a_1, \dots, a_n) = \sum_{\pi \in \mathcal{NC}(n)} \frac{m(\pi)}{|\pi|!} h_\pi(a_1, \dots, a_n), \tag{2.15}$$

where $m(\pi)$ stands for the number of monotone labellings of the non-crossing partition π, as defined in Subsection 2.2.6.

Remark 2.3.3. These defining formulas are known as *moment-cumulant formulas*. It will be convenient to think of all our formulas as having sums over all partitions $\mathcal{P}(n)$ for some $n \in \mathbb{N}$. In order to do this, we introduce some coefficients that are allowed to be 0. Denoting by $\{\kappa_n\}_{n\in\mathbb{N}}$, the sequence of cumulants for some arbitrary type of non-commutative independence, then the moment-cumulant formula is of the form

$$\varphi_n = \sum_{\pi \in \mathcal{P}(n)} \alpha(\pi) \prod_{i=1}^{k} \kappa_{|V_i|}, \tag{2.16}$$

where the $\alpha(\pi)$ are complex coefficients indexed by partitions that depend on the notion of independence. Observe that the transition formula that writes the sequence of moments $\{\varphi_n\}_{n\in\mathbb{N}}$ in terms of the sequence of some cumulants $\{\kappa_n\}_{n\in\mathbb{N}}$ is determined by the family of values $\{\alpha(\pi) \in \mathbb{C} : \pi \in \cup_{n\in\mathbb{N}}\mathcal{P}(n)\}$. Thus, to every notion of independence we can associate a family of values $\{\alpha(\pi)\}_{\pi\in\mathcal{P}} \subset \mathbb{C}$.

As mentioned in the Introduction, the relevance of cumulants is that they linearize the addition of two independent random variables. In the case of classic, free and Boolean probabilities this follows from the stronger fact that independence is equivalent to the vanishing of mixed cumulants. Since we will only use this fact in the framework of free probability, we provide the precise statement in this case.

Theorem 2.3.4 (Vanishing of mixed cumulants, see e.g. [NS06], Lecture 11)**.** *Given a non-commutative probability space $(\mathcal{A}, \varphi)$ and $a, b \in \mathcal{A}$ two random variables. Then the following two statements are equivalent:*

1. *a and b are free.*
2. *Every mixed free cumulant vanishes. Namely, for every $n \geq 2$ and $a_1, \ldots, a_n \in \{a, b\}$ which are not all equal, we have that $r_n(a_1, \ldots, a_n) = 0$.*

In [NS96] it was observed that free cumulants can also be used to study the product of free random variables, via the following formula

$$r_n(ab) = \sum_{\tau \in \mathcal{NC}(n)} \left(\prod_{V \in \tau} r_{|V|}(a) \prod_{W \in \mathrm{Kr}(\tau)} r_{|W|}(b) \right). \tag{2.17}$$

This equation is a special case of an efficient formula for a cumulant whose entries are products of the underlying algebra $\mathcal{A}$. The efficient formula allows us to write this cumulant as a sum over cumulants with more entries, where the products are now separated into different entries. The general formula was found in [KS00] and is known as the products as arguments formula. Here we will just use a particular case.

Theorem 2.3.5 (Products as arguments formula)**.** *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and fix $n \in \mathbb{N}$. Let $a_1, a_2, \ldots, a_{2n} \in \mathcal{A}$ be random variables, and consider $I_{2n} := \{\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}\}$ the unique interval pair partition. Then we have that*

$$r_n(a_1 a_2, a_3 a_4, \ldots, a_{2n-1} a_{2n}) = \sum_{\substack{\pi \in \mathcal{NC}(2n) \\ \pi \vee_{\mathcal{NC}} I_{2n} = 1_{2n}}} r_\pi(a_1, a_2, a_3, a_4, \ldots, a_{2n-1}, a_{2n}). \tag{2.18}$$

2.3.1 Operator-valued case

In this subsection we will introduce the topic of operator-valued non-commutative probability. For a more detailed introduction to the general theory of operator-valued free probability we refer the reader to the Section 9.2 of [MS17].

The triple $(\mathcal{A}, \mathcal{B}, E)$ is called an *operator-valued probability space* if $\mathcal{A}$ is a unital algebra, $\mathcal{B}$ is a unital subalgebra of $\mathcal{A}$ and $E : \mathcal{A} \to \mathcal{B}$ is a *conditional expectation*, namely, a linear map satisfying

$$E(b) = b \quad \text{and} \quad E(bab') = bE(a)b' \qquad \forall b, b' \in \mathcal{B},\ a \in \mathcal{A}.$$

For a given random variable $x \in \mathcal{A}$, the *operator-valued distribution* of x is the set of all operator-valued moments

$$E(xb_1xb_2 \cdots b_{n-1}xb_nx) \in \mathcal{B}$$

where $n \geq 0$ and $b_1, \cdots, b_n \in \mathcal{B}$, and we denote it by μ_x (see [Voi95]). In this more general context, we can also define the notions of free, Boolean and monotone independence and we can define operator-valued cumulants. Now, to keep track of the information provided by the moments, or cumulants, we need to study multiplicative families of multilinear maps from $\mathcal{A}^n$ to $\mathcal{B}$.

Definition 2.3.6. Let $\mathcal{B} \subset \mathcal{A}$ be unital subalgebras and fix an $n \in \mathbb{N}$. We say that a $\mathbb{C}$-multilinear map $\psi_n : \mathcal{A}^n \to \mathcal{B}$ is *balanced* if for all $a_1, a_2, \ldots, a_n \in \mathcal{A}$ and $b \in \mathcal{B}$, it satisfies that following properties

$$\psi_n(ba_1, a_2, \ldots, a_{n-1}, a_n) = b\psi_n(a_1, a_2, \ldots, a_{n-1}, a_n),$$

$$\psi_n(a_1, a_2, \ldots, a_{n-1}, a_nb) = \psi_n(a_1, a_2, \ldots, a_{n-1}, a_n)b,$$

and for $k = 1, \ldots, n-1$ one has

$$\psi_n(a_1, \ldots, a_kb, a_{k+1}, \ldots, a_{n-1}, a_n) = \psi_n(a_1, \ldots, a_k, ba_{k+1}, \ldots, a_{n-1}, a_n).$$

Definition 2.3.7. Let $\mathcal{B} \subset \mathcal{A}$ be unital subalgebras and let $\{\psi_n\}_{n\geq 1}$ be a sequence of balanced $\mathbb{C}$-multilinear maps: $\psi_n : \mathcal{A}^n \to \mathcal{B}$. Then for every $\pi \in \mathcal{NC}(n)$ we define the maps $\psi_\pi : \mathcal{A}^n \to \mathcal{B}$ recursively as follows:

1. For $\pi = 1_n$, we set $\psi_\pi = \psi_n$.
2. For $\pi \in \mathcal{NC}(n)$ we pick $V = \{l+1, \ldots l+k\} \in \pi$ a interval block of π (there always exists one) and define

   $$\psi_\pi(x_1, \ldots, x_n) = \psi_{\pi'}(x_1, \ldots, x_l\psi_k(x_{l+1}, \ldots, x_{l+k}), x_{l+k+1}, \ldots, x_n),$$

   where $\pi' = \pi \backslash V \in \mathcal{NC}(n-k)$ is the partition obtained by deleting the block V from the partition π.

For every π it can be seen that ψ_π is balanced and it does not depend on how we pick the interval block V. A family $\underline{\psi} = \{\psi_\pi\}_{\pi\in\mathcal{NC}}$ created in this way is called *multiplicative*, and we denote by $\mathfrak{M}^{ov}_{\mathcal{A},\mathcal{B}}$ the set of multiplicative families.

Remark 2.3.8. In Definition 2.3.7 we did not need to assume the existence of a conditional expectation $E : \mathcal{A} \to \mathcal{B}$. If that would be the case, and we would thus be dealing with an operator-valued non-commutative probability space $(\mathcal{A}, \mathcal{B}, E)$, then the set $\mathfrak{M}^{ov}_{\mathcal{A},\mathcal{B}}$ would get to have a special element $\underline{E} = (E_\pi : \mathcal{A}^n \to \mathcal{B})_{\pi\in\mathcal{NC}}$ obtained by enlarging the sequence

$$E_n(x_1, \dots, x_n) := E(x_1 \cdots x_n), \quad \forall\, x_1, \dots, x_n \in \mathcal{A}. \tag{2.19}$$

Example 2.3.9. For $\pi = \{\{1,5,6\},\{2,4\},\{3\},\{7\}\} \in \mathcal{NC}(7)$ we have that $E_\pi(a_1, \dots, a_7)$ is given by

$$E_3(a_1E_2(a_2E_1(a_3), a_4), a_5, a_6)E_1(a_7) = E(a_1E(a_2E(a_3)a_4)a_5a_6)E(a_7).$$

Notice that E_π is indeed balanced as we have that it is also equal to

$$E_3(a_1, E_2(a_2E_1(a_3), a_4)a_5, a_6E_1(a_7)) = E_3(a_1, E_2(a_2, E_1(a_3)a_4)a_5, a_6E_1(a_7)).$$

To finish this section, we define the operator-valued cumulants. We want to highlight the fact that the combinatorial structure of the moment-cumulant formulas remains the same, no matter if we use the simple case of univariate cumulants, or more general frameworks such as infinitesimal cumulants (see Section 4.6.4 below) or operator-valued cumulants.

Definition 2.3.10. The *operator-valued free cumulants* $\{r_n^{\mathcal{B}} : \mathcal{A}^n \to \mathcal{B}\}_{n\geq 1}$, the *operator-valued Boolean cumulants* $\{\beta_n^{\mathcal{B}} : \mathcal{A}^n \to \mathcal{B}\}_{n\geq 1}$ and the *operator-valued monotone cumulants* $\{h_n^{\mathcal{B}} : \mathcal{A}^n \to \mathcal{B}\}_{n\geq 1}$ are recursively defined via the following moment-cumulant formulas:

$$E_n = \sum_{\pi\in\mathcal{NC}(n)} r_\pi^{\mathcal{B}}, \tag{2.20}$$

$$E_n = \sum_{\beta\in I(n)} \beta_\pi^{\mathcal{B}}, \tag{2.21}$$

$$E_n = \sum_{\beta\in\mathcal{NC}(n)} \frac{m(\pi)}{|\pi|!} h_\pi^{\mathcal{B}}. \tag{2.22}$$

2.4 Hopf algebras

In this section we follow [Swe69] to introduce the basic theory of Hopf algebras that will be used in Chapters 6 and 7.

2.4.1 Basic definitions

We will begin by presenting a "diagramatic" way to describe the notion of a unital algebra over $\mathbb{C}$. We say that $\mathcal{B}$ is an algebra if it is a vector space over $\mathbb{C}$, that has a

- **Multiplication:** bilinear map $m : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$
- **Unit map:** a linear map $u : \mathbb{C} \to \mathcal{B}$.

And these maps satisfy the following properties:

- **Associativity of** m, that is $m \circ (m \otimes id) = m \circ (id \otimes m)$.
- **Unit property**, $m \circ (u \otimes id) = id = m \circ (id \otimes u)$.

Equivalently we can rephrase these two properties in terms of commutative diagrams:

$$
\begin{tikzcd}
& \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \arrow[dl, "m\otimes id"'] \arrow[dr, "id\otimes m"] & \\
\mathcal{B} \otimes \mathcal{B} \arrow[dr, "m"'] & & \mathcal{B} \otimes \mathcal{B} \arrow[dl, "m"] \\
& \mathcal{B} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{C} \otimes \mathcal{B} \cong \mathcal{B} \cong \mathcal{B} \otimes \mathbb{C} \arrow[dl, "u\otimes id"'] \arrow[d, "id"] \arrow[dr, "id\otimes u"] & \\
\mathcal{B} \otimes \mathcal{B} \arrow[r, "m"'] & \mathcal{B} & \mathcal{B} \otimes \mathcal{B} \arrow[l, "m"]
\end{tikzcd}
$$

Figure 2.9: Associativity and Unitary property

Notice that the previous way of looking at an algebra is just a recast of the common definition of unital associative algebra. For instance, $m(a \otimes b)$ is usually denoted simply as ab, so the associativity is expressed as $(ab)c = a(bc)\ \forall a, b, c \in \mathcal{B}$. On the other hand, the unit $1_{\mathcal{B}} \in \mathcal{B}$, satisfies that $1_{\mathcal{B}} b = b = b 1_{\mathcal{B}}\ \forall b \in \mathcal{B}$. In the previous definition, u is a map, rather than an element of $\mathcal{B}$, but notice that if we denote $1_{\mathcal{B}} := u(1)$ then u is completely determined (because it is linear), so we must have $u(\lambda) = \lambda 1_{\mathcal{B}}$. Moreover, the unitary property is precisely saying that $u(1)b = b = bu(1)\ \forall b \in \mathcal{B}$. The reason for giving the previous definition is that the coalgebra structure can be easily defined by taking the dual of the previous definition, or equivalently, turning around all the arrows.

Definition 2.4.1. Let $\mathcal{B}$ be a vector space over $\mathbb{C}$. We say that $\mathcal{B}$ is a **coalgebra** if it has:

- **Comultiplication (or coproduct)**, linear map $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$.
- **Counit**, linear map $\varepsilon : \mathcal{B} \to \mathbb{C}$.

And these maps satisfy the following properties:

- **Coassociativity**, $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$.
- **Counit property**, $(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta$.

Equivalently we can rephrase these two properties in terms of commutative diagrams:

$$
\begin{tikzcd}
& \mathcal{B}\otimes\mathcal{B}\otimes\mathcal{B} & \\
\mathcal{B}\otimes\mathcal{B} \arrow[ur, "\Delta\otimes id"] & & \mathcal{B}\otimes\mathcal{B} \arrow[ul, "id\otimes\Delta"'] \\
& \mathcal{B} \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{C}\otimes\mathcal{B}\cong\mathcal{B}\cong\mathcal{B}\otimes\mathbb{C} & \\
\mathcal{B}\otimes\mathcal{B} \arrow[ur, "\varepsilon\otimes id"] & & \mathcal{B}\otimes\mathcal{B} \arrow[ul, "id\otimes\varepsilon"'] \\
& \mathcal{B} \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] \arrow[uu, "id"] &
\end{tikzcd}
$$

Figure 2.10: Coassociativity and counit property

Notice that we just turned around all the arrows from Figure 2.9.

Example 2.4.2 (Unraveling coassociativity)**.** Given $b \in \mathcal{B}$, let b_1', b_1'', ..., b_n', b_n'' be some elements in $\mathcal{B}$ such that

$$\Delta(b) = \sum_{i=1}^{n} b_i' \otimes b_i''. \tag{2.23}$$

In this notation the coassociativity of Δ means that

$$\sum_{i=1}^{n} \Delta(b_i') \otimes b_i'' = \sum_{i=1}^{n} b_i' \otimes \Delta(b_i''),$$

and the counit property means that

$$\sum_{i=1}^{n} \varepsilon(b_i')b_i'' = b = \sum_{i=1}^{n} b_i'\varepsilon(b_i'').$$

Definition 2.4.3. $\mathcal{B}$ is a *bialgebra* if it is an algebra and a coalgebra such that Δ and ε are algebra morphisms (or equivalently m and u are coalgebra morphisms)

We further say the $\mathcal{B}$ is a *Hopf algebra (HA)* if it is a bialgebra with an *antipode*, which is a linear map $S : \mathcal{B} \to \mathcal{B}$ satisfying

$$m \circ (S \otimes id) \circ \Delta = u \circ \varepsilon = m \circ (id \otimes S) \circ \Delta.$$

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dl, "\Delta"'] \arrow[dd, "\varepsilon"] \arrow[dr, "\Delta"] & \\
\mathcal{B} \otimes \mathcal{B} \arrow[dd, "S\otimes id"'] & & \mathcal{B} \otimes \mathcal{B} \arrow[dd, "id\otimes S"] \\
 & \mathbb{C} \arrow[dd, "u"] & \\
\mathcal{B} \otimes \mathcal{B} \arrow[dr, "m"'] & & \mathcal{B} \otimes \mathcal{B} \arrow[dl, "m"] \\
 & \mathcal{B} &
\end{tikzcd}$$

Figure 2.11: Antipode property

It can be proved that if a bialgebra has an antipode, then it must be unique. Most of the times, finding a concrete formula to define the antipode is not easy. Fortunately there is an important situation when we can check that a bialgebra has an antipode without the need to compute it precisely.

Definition 2.4.4. We say that a bialgebra $\mathcal{B}$ is *graded* when it has a direct sum decompoasition (as a vector space) $\mathcal{B} = \oplus_{n=0}^{\infty} \mathcal{B}_n$, where the *grading* is compatible with the

- Product: $\mathcal{B}_m \mathcal{B}_n \subset \mathcal{B}_{m+n}$,
- Coproduct: $\Delta(\mathcal{B}_n) \subset \bigoplus_{m=0}^{n} \mathcal{B}_m \otimes \mathcal{B}_{n-m}$.
- Counit: $\varepsilon(1_{\mathcal{B}}) = 1$ and $\varepsilon(b) = 0$ for all $b \in \mathcal{B}_n$ with $n \geq 1$.

We say that $\mathcal{B}$ is *connected*[3] if the zeroth level of the grading consists of a copy of our base field $\mathcal{B}_0 = \mathbb{C}$.

[3]This is the third appearance of the term *connected*, we also use the same word to refer to a family of graphs and to a family of partitions. Since all three are well-established terms within their area, I decide to keep the same name. This should not pose any problem as it will be clear from the context, if we are referring to a graph, partition or Hopf algebra

Theorem 2.4.5. *If $\mathcal{B}$ is a graded, connected, bialgebra, then it is a Hopf algebra.*

The proof of this result uses the grading of $\mathcal{B}$ to recursively define S and can be found in any book of basic theory on Hopf algebras, see for instance [Man04, Section II.3].

2.4.2 Other structures

Definition 2.4.6 (sub-Hopf-Algebra)**.** Given a subset $\mathcal{C}$ of $\mathcal{B}$, we say that $\mathcal{C}$ is a *sub-Hopf-Algebra* of $\mathcal{B}$ if

- $\mathcal{C}$ is a subalgebra of $\mathcal{B}$, namely it is closed under multiplication $m(\mathcal{C} \otimes \mathcal{C}) \subset \mathcal{C}$ and contains the unit $u(\mathbb{C}) \subset \mathcal{C}$.
- $\mathcal{C}$ is a subcoalgebra of $\mathcal{A}$, namely it is closed under comultiplication $\Delta(\mathcal{C}) \subset (\mathcal{C} \otimes \mathcal{C})$.
- $S(\mathcal{C}) \subset \mathcal{C}$.

Definition 2.4.7. Given two Hopf algebras $\mathcal{B}, \mathcal{C}$, and a linear map $\Psi : \mathcal{B} \to \mathcal{C}$ we say that Ψ is an *(algebra) homomorphism* if it

- Preserves multiplication: $m_{\mathcal{C}} \circ (\Psi \otimes \Psi) = \Psi \circ m_{\mathcal{B}}$.
- And preserves the unit: $u_{\mathcal{C}} = \Psi \circ u_{\mathcal{B}}$.

We say that $\Psi : \mathcal{B} \to \mathcal{C}$ is a *coalgebra homomorphism* if it

- Preserves comultiplication: $\Delta_{\mathcal{C}} \circ \Psi = (\Psi \otimes \Psi) \circ \Delta_{\mathcal{B}}$.
- And preserves the counit: $\varepsilon_{\mathcal{C}} \circ \Psi = \varepsilon_{\mathcal{B}}$.

We say that $\Psi : \mathcal{C} \to \mathcal{B}$ is a *Hopf algebra homomorphism* if it is an algebra and coalgebra homomorphism that also preserves the antipode: $S_{\mathcal{C}} \circ \Psi = S_{\mathcal{B}}$.

Remark 2.4.8. Notice that if $\mathcal{B}, \mathcal{C}$ are graded connected Hopf algebra, and $\Psi : \mathcal{B} \to \mathcal{C}$ is an algebra and coalgebra homomorphism, then instead of showing that Ψ preserves the antipode, it is enough to show that Ψ preserves the grading ($\deg_{\mathcal{C}} \circ \Psi = \deg_{\mathcal{B}}$) in order to prove that Ψ is Hopf algebra homomorphism.

Definition 2.4.9 (Right comodule and coaction)**.** If $\mathcal{B}$ is a coalgebra, we define a *right* $\mathcal{B}$*-comodule* to be a vector space V together with a *coaction* $\Gamma : V \to V \otimes \mathcal{B}$ which satisfies the following two properties:

- **Coassociative**, $(\Gamma \otimes id) \circ \Gamma = (id \otimes \Delta) \circ \Gamma$.
- **Counit**, $(id \otimes \varepsilon) \circ \Gamma = id$.

Equivalently, we can rephrase these two properties in terms of commutative diagrams:

$$\begin{tikzcd}
 & M \otimes \mathcal{B} \otimes \mathcal{B} & \\
M \otimes \mathcal{B} \arrow[ur, "\Gamma\otimes id"] & & M \otimes \mathcal{B} \arrow[ul, "id\otimes\Delta"'] \\
 & M \arrow[ul, "\Gamma"] \arrow[ur, "\Gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \cong M \otimes \mathbb{C} & \\
 & M \otimes \mathcal{B} \arrow[ul, "id\otimes\epsilon"'] \\
M \arrow[uu, "id"] \arrow[ur, "\Gamma"'] &
\end{tikzcd}$$

Figure 2.12: Coassociative and counit of a comodule.

As the reader may have guessed, if we turn around all the arrows we get the usual definitions of module and action.

2.4.3 Group of characters under convolution

The main advantage of working with a coalgebra structure is that it nicely encodes a convolution operation on the set of functionals over $\mathcal{B}$. More generally, given a coalgebra $\mathcal{B}$ and a unital algebra $\mathcal{A}$ over $\mathbb{C}$, we can define a convolution on the set of linear functions from $\mathcal{B}$ to $\mathcal{A}$.

Notation 2.4.10. For $\mathcal{A}$ and $\mathcal{B}$ vector spaces over $\mathbb{C}$, we will denote

$$L(\mathcal{B}, \mathcal{A}) = \{f : \mathcal{B} \to \mathcal{A} | f \text{ is linear}\},$$

and write $L(\mathcal{B}) := L(\mathcal{B}, \mathcal{B})$.

Definition 2.4.11. Suppose that $\mathcal{A}$ is an algebra and $\mathcal{B}$ is a coalgebra. For $f, g \in L(\mathcal{B}, \mathcal{A})$ we define the *convolution* $f * g \in L(\mathcal{B}, \mathcal{A})$ as follows. If $b \in \mathcal{B}$ and write $\Delta(b)$ as in (2.23), then we put

$$f * g(b) = \sum_{i=1}^{n} f(b_i')g(b_i'').$$

In short we have $f * g = m \circ (f \otimes g) \circ \Delta$. Or in a diagram:

$$\begin{tikzcd}
& \mathcal{B}\otimes\mathcal{B} \arrow[r, "f\otimes g"] & \mathcal{A}\otimes\mathcal{A} \arrow[dr, "m"] & \\
\mathcal{B} \arrow[ur, "\Delta"] \arrow[rrr, "f*g"] & & & \mathcal{A}
\end{tikzcd}$$

Figure 2.13: Convolution

Remark 2.4.12. It is easy to see that the convolution operation "$*$" on $L(\mathcal{B},\mathcal{A})$ is well-defined. From the coassociativity of $\mathcal{B}$ it follows that the convolution $*$ is associative. Moreover, if we define $\varepsilon_{\mathcal{A}} \in L(\mathcal{B},\mathcal{A})$ by $\varepsilon_{\mathcal{A}}(b) = \varepsilon(b)1_{\mathcal{A}}$ then we have that it is the (necessarily unique) unit for the semigroup $(L(\mathcal{B},\mathcal{A}),*)$, namely

$$\varepsilon_{\mathcal{A}} * f = f = f * \varepsilon_{\mathcal{A}} \qquad \forall f \in L(\mathcal{B},\mathcal{A}).$$

We are mainly interested in working with linear maps going to two special algebras, $\mathcal{A} = \mathcal{B}$ and $\mathcal{A} = \mathbb{C}$.

When we consider $\mathcal{A} = \mathcal{B}$, the identity map $\mathrm{Id} \in L(\mathcal{B})$ is sure to be an invertible element of $(L(\mathcal{B}),*)$. This follows from general considerations on graded connected bialgebras – see for instance [Man04, Corollary II.3.2]. Then, the inverse of Id in $(L(\mathcal{B}),*)$ is precisely the antipode S of $\mathcal{B}$. Furthermore, if $\mathcal{B}$ is commutative, general bialgebra considerations yield the fact that $S : \mathcal{B} \to \mathcal{B}$ is a unital algebra homomorphism (cf. [Man04, Proposition I.7.1]). This implies in particular that S is completely determined by how it acts on a set $H \subset \mathcal{B}$ which generates $\mathcal{B}$ as an algebra.

When we consider $\mathcal{A} = \mathbb{C}$ and define a convolution as before, we get a special group under convolution, known as the group of characters.

Definition 2.4.13. A *character* is a unital homomorphism from $\mathcal{B}$ to $\mathbb{C}$. The *group of characters of* $\mathcal{B}$ will be denoted as

$$\mathcal{B}^\circ := \{f \in L(\mathcal{B},\mathbb{C}) \mid f(1_{\mathcal{B}}) = 1, f \text{ is multiplicative}\}.$$

Most of our applications are done at the level of duals. The general idea is that whenever we have a result at the level of Hopf algebras, that can be converted right away to a dual result at the level of group of characters.

Remark 2.4.14. The fact that $\mathcal{B}^\circ$ is a group follows from the coalgebra structure. The product is defined in terms of the coproduct and the associativity follows from the coassociativity if Δ. The unit, is simply the counit ε, and the inverse of $f \in \mathcal{B}^\circ$ can be computed used the antipode, $f^{-1} = f \circ S$.

Now, if $\mathbb{C}$ is a sub-Hopf-Algebra of $\mathcal{B}$, then we can identify $\mathcal{C}^\circ$ as a subgroup of $\mathcal{B}^\circ$ by taking the functions that are supported on $\mathcal{C}$. More specifically, to every $g \in \mathcal{C}^\circ$ we associate the function $f \in \mathcal{B}^\circ$ determined by taking $f(b) = g(b)$ if $b \in \mathcal{C}$ and $f(b) = 0$ otherwise.

If $\Psi : \mathcal{B} \to \mathcal{C}$ Hopf algebra homomorphism, this naturally yields group homomorphism $\Psi^\circ : \mathcal{C}^\circ \to \mathcal{B}^\circ$ determined by requiring $g \mapsto f := g \circ \Psi$. The fact that it preserves multiplication and unit follows from the fact that Ψ preserves the comultiplication and the counit. Moreover, if $\Psi : \mathcal{B} \to \mathcal{C}$ is surjective, then Ψ° is injective.

Finally, if M is a right-$\mathcal{B}$-comodule with coaction $\Gamma : M \to M \otimes \mathcal{B}$, then this defines a right action $\Gamma^\circ : M^\circ \otimes \mathcal{B}^\circ \to M^\circ$ of the group $\mathcal{B}^\circ$ on the space of linear maps $M^\circ := L(M, \mathbb{C})$ by taking $(\psi, f) \mapsto (\psi \otimes f) \circ \Gamma$.

2.4.4 Our prototypical way of constructing a Hopf algebra

Throughout this document we will use a specific way of constructing Hopf algebras. For this type of Hopf algebras, several constructions and results that will be used follow very similar procedures. Therefore, the purpose of this section is to serve as a general guideline on the notation and the details that we will use in subsequent sections.

Example 2.4.15. Our starting point will a set $\mathcal{H}$ which is going to be either a family of partitions or a family of graphs. Thus $\mathcal{H}$ is going to be an infinite set expressed as a countable disjoint union of finite sets.

Algebraic structure. Our main object will be $\widehat{\mathcal{H}} = \mathbb{C}[\mathcal{H}]$, the commutative algebra of polynomials over $\mathbb{C}$ which uses a countable collection of indeterminates indexed by the elements of $\mathcal{H}$. The indeterminate associated to $p \in \mathcal{H}$ will be denoted as $\widehat{p}$ or simply as p when it is clear from the context. The unit of $\widehat{\mathcal{H}}$ will be denoted as $1_{\widehat{\mathcal{H}}}$ and will be identified with $\widehat{p}$ for some special elements $p \in \mathcal{H}$, either the one vertex graph, or the one block partitions. Equivalently, $u : \mathbb{C} \to \widehat{\mathcal{H}}$ is given by $u(\lambda) = \lambda 1_{\widehat{\mathcal{H}}}$.

Therefore, a linear basis of $\widehat{\mathcal{H}}$ consists of the set of monomials:

$$M := \{1_{\widehat{\mathcal{H}}}\} \cup \{\widehat{p_1}\widehat{p_2} \cdots \widehat{p_k} \mid k \in \mathbb{N},\ p_1, \ldots, p_k \in \mathcal{H}\}.$$

Thus, all the elements of $\widehat{\mathcal{H}}$ have the form

$$\sum_{m \in S} \alpha_m m,$$

where S is a finite subset of M and $\alpha_m \in \mathbb{C}$.

An immediate consequence of our previous definition is that $\widehat{\mathcal{H}}$ has a universality property.

Proposition 2.4.16 (Universality property of $\widehat{\mathcal{H}}$)**.** *If $\mathcal{A}$ is a unital commutative algebra over $\mathbb{C}$ and we are given elements $\{a_p\}_{p\in\mathcal{H}}$ in $\mathcal{A}$, (with $a_p = 1_{\mathcal{A}}$ for the special elements p in $\mathcal{H}$ such that $\widehat{p} = 1_{\widehat{\mathcal{H}}}$), then there exists a unital algebra homomorphism $\Phi : \widehat{\mathcal{H}} \to \mathcal{A}$, uniquely determined, such that $\Phi(\widehat{p}) = a_p$ for all $p \in \mathcal{H}$.*

We will exploit this property to define the comultiplication and counit of $\widehat{\mathcal{H}}$, (recall that in order to have a bialgebra these functions must be algebra homomorphisms.)

Coalgebra stucture. The counit of $\widehat{\mathcal{H}}$ will be the unital algebra homomorphism $\varepsilon : \widehat{\mathcal{H}} \to \mathbb{C}$, uniquely determined (due to the universality property of $\widehat{\mathcal{H}}$), such that

$$\varepsilon(\widehat{p}) = 0, \quad \forall\, p \in \mathcal{H} \text{ such that } \widehat{p} \neq 1_{\widehat{\mathcal{H}}}. \tag{2.24}$$

Notice that since ε is a unital algebra homomorphism, then $\varepsilon(1_{\widehat{\mathcal{H}}}) = 1$ and $\varepsilon(m) = 0$ for every other monomial $m \in M \backslash \{1_{\widehat{\mathcal{H}}}\}$.

Finally, the comultiplication is going to be the only structure that really depends on what Hopf algebra we want to consider. Thus, it is going to be the main distinction between the different Hopf algebras that we are going to work with. However, there are certain properties that all comultiplications will share. We always want the comultiplication of $\widehat{\mathcal{H}}$ to be a unital algebra homomorphism $\Delta : \widehat{\mathcal{H}} \to \widehat{\mathcal{H}} \otimes \widehat{\mathcal{H}}$, thus by the universality property of $\widehat{\mathcal{H}}$, it is going to be uniquely determined by its values on $\widehat{p}$ for $p \in \mathcal{H}$. Moreover, we want Δ to be linear on the right, i.e., of the form

$$\Delta(\widehat{p}) = \sum_{q\in S_p} m_q \otimes \widehat{q}, \tag{2.25}$$

where S_p is some subset of $\mathcal{H}$ depending on p and $m_q \in M$ is a monomial depending on q.

Grading. For the grading of $\widehat{\mathcal{H}} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$, we will take it to be $\deg(\widehat{p}) = |p| - 1$, that is, the number of blocks minus 1 in the case of partitions, and the number of vertices minus 1, in the case of graphs. Since we need it to be compatible with the product, this means there is only one way to extend it to a monomial $m = \widehat{p_1} \cdots \widehat{p_k}$ by taking $\deg(m) = \deg(\widehat{p_1} \cdots \widehat{p_k}) = \deg(\widehat{p_1}) + \cdots + \deg(\widehat{p_k})$.

Chapter 3

The free anti-commutator

As mentioned in the introduction, Speicher's free cumulants are a combinatorial tool to study the addition $a + b$ of two free random variables a and b, [Spe94]. Free cumulants can also be used to study the product ab [NS96], the commutator $i(ab - ba)$ [NS98], and the anti-commutator [NS98]. However, for the anti-commutator the first results obtained are only valid for the case where a and b have symmetric distributions, that is, all their odd moments (and therefore all their odd cumulants) are zero. For the general case, the combinatorial approach becomes much more involved.

More recently, a formula to compute the Boolean cumulants of the free anti-commutator $ab+ba$ in terms of the individual Boolean cumulants of a and b was provided in [FMNS20]. On the other hand, Ejsmont and Lehner studied quadratic forms of even variables [EL21] (see also [EL17]) and the limiting distribution that arises from sums of commutators and anti-commutators [EL20]. There also exist plainly analytic approaches to compute the anti-commutator. In [Vas03], various systems of equations are given, which can in principle be used to calculate the Cauchy transform of $ab + ba$, while [BMS17] uses the linearization trick to provide an algorithm which can be used to obtain the distribution of a free anti-commutator, see also [HMS18].

In this chapter, we provide a general formula to compute the free cumulants of the anti-commutator $ab+ba$ in terms of the free cumulants of a and b. The chapter is a compilation of the results from [Per21], and it is divided into five sections. In Section 3.1 we prove our main result (Theorem 3.1.4) that provides a formula for the cumulants of the anti-commutator expressed as a sum indexed by certain partitions. In Section 3.2, we study in detail the set $\mathcal{Y}(2n)$ that is the dual of $\mathcal{X}(2n)$ under the Kreweras complementation map. The results of this section are used for the applications in Section 3.3. In Section 3.4, we explore a graph theoretic approach to give a formula for the anti-commutator just in terms of cacti graphs. This graph theoretic approach is naturally generalized to study quadratic forms in Section 3.5.

3.1 A formula for the anti-commutator

The key idea in our approach to the anti-commutator problem is to carefully examine certain graphs associated to non-crossing partitions. These graphs have already appeared in [MS12], see also Section 4.4 of [MS17]. In the present manuscript, the relevant feature we seek in these graphs is whether they are connected and bipartite.

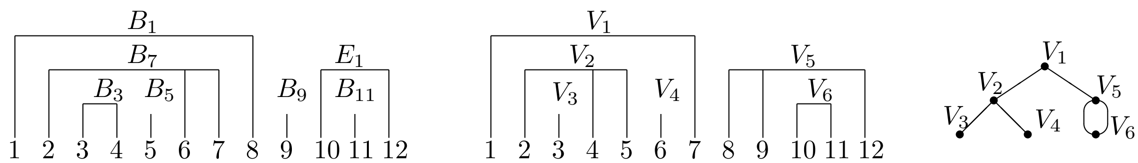
Definition 3.1.1. Let $\mathcal{NC}(2n)$ be the set of non-crossing partitions of $[2n]$.

1. Given a partition $\pi \in \mathcal{NC}(2n)$, we construct the graph $\Gamma(\pi)$ as follows:
   - The vertices of $\Gamma(\pi)$ are going to be the blocks of π, thus $\mathrm{Vrt}_{\Gamma(\pi)} = \pi$.
   - For $k = 1, 2, \ldots, n$ we draw an undirected edge between the block containing element $2k-1$ and the block containing element $2k$. We allow for loops and multiple edges, thus $\Gamma(\pi)$ has exactly n edges.
2. We denote
$$\mathcal{X}(2n) := \{\pi \in \mathcal{NC}(2n) : \Gamma(\pi) \text{ is connected and bipartite}\}.$$
   A partition $\pi \in \mathcal{X}(2n)$ has a natural *bipartite decomposition* $\pi = \pi' \sqcup \pi''$. Denoting by V_1 the block of π which contains the number 1, we have that π' consists of the blocks of π which are at even distance from V_1 in the graph $\Gamma(\pi)$, while π'' consists of the blocks of π which are at odd distance from V_1 in that graph.
3. For every $n \in \mathbb{N}$, we denote by $\mathcal{Y}(2n)$ the set of non-crossing partitions of the form
$$\sigma = \{B_1, B_3, \ldots, B_{2n-1}, E_1, \ldots, E_r\}, \qquad \text{with } r \geq 0,$$
   such that $i \in B_i$ for $i = 1, 3, \ldots, 2n-1$ and $|E_j|$ even for $j \leq r$.

   In plain words, $\sigma \in \mathcal{Y}(2n)$ if it separates odd elements and the blocks of σ containing just even elements have even size. The notation is set to remind us that B_i is the *block containing* i, while E_j is of *even size* and also happens to contain only even elements, as $\{1, 3, \ldots, 2n-1\} \subset B_1 \cup B_3 \cup \cdots \cup B_{2n-1}$.

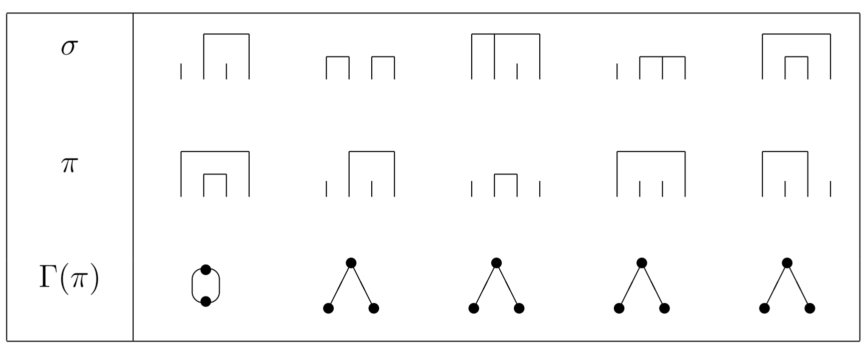
In Proposition 3.2.1 from next section we will show that the two sets that we just defined satisfy the relation $\mathcal{X}(2n) = \mathrm{Kr}(\mathcal{Y}(2n))$, where Kr is the Kreweras complementation map from Definition 2.2.20.

Example 3.1.2. Here is an example of a partition $\sigma \in \mathcal{Y}(12)$, on the left. In the middle we have $\pi = \mathrm{Kr}(\sigma) \in \mathcal{X}(12)$, and on the right we have the graph $\Gamma(\pi)$.



The bipartite decomposition of $\Gamma(\pi)$ has $\pi' = \{V_1, V_3, V_4, V_6\}$ and $\pi'' = \{V_2, V_5\}$.

Example 3.1.3. In the table below we list the 5 partitions σ of $\mathcal{Y}(4)$ with their corresponding $\pi := \mathrm{Kr}(\sigma) \in \mathcal{X}(4)$ and the graph $\Gamma(\pi)$:



Our main contribution is a formula that expresses the cumulants of the anti-commutator as a sum indexed by $\mathcal{X}(2n)$, or equivalently $\mathcal{Y}(2n)$ (as the Kreweras map is a bijection).

Theorem 3.1.4. *Consider two free random variables a and b, and let $(r_n(a))_{n\geq 1}$, $(r_n(b))_{n\geq 1}$ and $(r_n(ab+ba))_{n\geq 1}$ be the sequence of free cumulants of a, b and $ab+ba$, respectively. Then, for every $n \geq 1$ one has*

$$r_n(ab+ba) = \sum_{\substack{\pi \in \mathcal{X}(2n) \\ \pi = \pi' \sqcup \pi''}} \left(\prod_{V\in\pi'} r_{|V|}(a) \prod_{W\in\pi''} r_{|W|}(b) + \prod_{V\in\pi'} r_{|V|}(b) \prod_{W\in\pi''} r_{|W|}(a) \right). \tag{3.1}$$

This theorem will be proved at the end of this section. Equation (3.1) takes a simpler form when a and b have the same distribution, which means that $\varphi(a^n) = \varphi(b^n)$ for all $n \in \mathbb{N}$, or equivalently that $r_n(a) = r_n(b)$ for all $n \in \mathbb{N}$. This phenomenon was also observed in [FMNS20], for the formula in terms of Boolean cumulants.

Corollary 3.1.5. *Consider two free random variables, a and b, with the same distribution, and let us denote $\lambda_n := r_n(a) = r_n(b)$. Then, one has*

$$r_n(ab + ba) = 2 \cdot \sum_{\pi \in \mathcal{X}(2n)} \prod_{V \in \pi} \lambda_{|V|} \qquad \forall n \in \mathbb{N}. \tag{3.2}$$

Corollary 3.1.6. *If a and b are free random variables, both with free Poisson distribution of parameter λ, namely $r_n(a) = r_n(b) = \lambda$ for all $n \in \mathbb{N}$, formula (3.2) specializes to*

$$r_n(ab + ba) = 2 \cdot \sum_{\pi \in \mathcal{X}(2n)} \lambda^{|\pi|} = 2 \cdot \sum_{\pi \in \mathcal{Y}(2n)} \lambda^{2n+1-|\pi|} \qquad \forall n \in \mathbb{N}.$$

In the special case where a, b are both free Poisson of parameter 1, the cumulants are just given by twice the size of $\mathcal{Y}(2n)$. In Theorem 3.2.2 below we will obtain a recursive formula for the sizes $|\mathcal{Y}(2)|, |\mathcal{Y}(4)|, |\mathcal{Y}(6)|, \ldots$ and this allows us to get the compositional inverse of the moment series of the anti-commutator.

Theorem 3.1.7. *Let ν be the distribution of the free anti-commutator of two free Poisson of parameter 1. The compositional inverse of the moment series of ν is given by*

$$M_\nu^{\langle -1 \rangle}(z) = \frac{-7z - 6 + 3\sqrt{(z+2)(9z+2)}}{4(z+2)^2(z+1)}. \tag{3.3}$$

Remark 3.1.8. To get a better grasp of Theorem 3.1.4, we now identify in Equation (3.1) all those terms that also appear in the expansion of $r_n(ab)$ from formula (2.17). It is immediate that all non-crossing pair-partitions $\mathcal{NC}^{\text{pair}}(2n)$ belong to $\mathcal{Y}(2n)$; indeed, the non-crossing condition forces every pair of a $\sigma \in \mathcal{NC}^{\text{pair}}(2n)$ to contain one odd number and one even number, hence the pairs $V_1, \ldots, V_n$ of σ can be renamed as $B_1, B_3, \ldots, B_{2n-1}$ with $i \in B_i$ for $i = 1, 3, \ldots, 2n-1$.

As a consequence, it follows that

$$\mathcal{X}(2n) \supseteq \mathrm{Kr}(\mathcal{NC}^{\text{pair}}(2n)) = \{\langle \tau, \mathrm{Kr}(\tau) \rangle \mid \tau \in \mathcal{NC}(n)\},$$

where for every $\tau \in \mathcal{NC}(n)$ we denote by $\langle \tau, \mathrm{Kr}(\tau) \rangle$ the partition in $\mathcal{NC}(2n)$ obtained by placing τ on positions $\{1, 3, \ldots, 2n-1\}$ and its Kreweras complement on positions $\{2, 4, \ldots, 2n\}$, as detailed in Section 2.2.

When we look at the summation over $\mathcal{X}(2n)$ indicated on the right-hand side of Equation (3.1), and check the terms of that summation which are indexed by partitions $\langle \tau, \mathrm{Kr}(\tau) \rangle$ with $\tau \in \mathcal{NC}(n)$, we run into a double copy of the summation shown in (2.17), giving the free cumulants $r_n(ab)$ and $r_n(ba)$.

The rest of this section is devoted to prove our main result, Theorem 3.1.4. The approach relies on first performing standard computations using linearity of the cumulants and products as arguments formula, to get a sum indexed by partitions π, this is the content of Proposition 3.1.10. Then, the key idea is that the graph $\Gamma(\pi)$ puts into evidence many partitions π that do not really contribute to the previous sum, this is done in Proposition 3.1.13. This ultimately yields that the sum is indexed just by $\mathcal{X}(2n)$.

We are going to fix two free random variables a, b, and a natural number $n \in \mathbb{N}$. Our ultimate goal is to describe the n-th free cumulant of the anti-commutator $r_n(ab + ba)$ in terms of the cumulants of a and b. By multilinearity of the cumulants this amounts to study n-th cumulants with entries given by ab or ba. To keep track of this kind of expressions we fix the following notation.

Notation 3.1.9. Given non-commutative variables a, b, we use the notation $(ab)^1 := ab$ and $(ab)^* := ba$. Given a n-tuple $\varepsilon \in \{1, *\}^n$, we denote by

$$(ab)^\varepsilon := ((ab)^{\varepsilon(1)}, (ab)^{\varepsilon(2)}, \dots, (ab)^{\varepsilon(n)})$$

the n-tuple with entries ab or ba dictated by the entries of ε. Furthermore, we will denote by $(a, b)^\varepsilon$ the $2n$-tuple obtained from separating the a's from the b's in the n-tuple $(ab)^\varepsilon$. To keep track of the entries in $(a, b)^\varepsilon$ that contain an a we use the notation

$$A(\varepsilon) := \{2i - 1 | 1 \le i \le n, \varepsilon(i) = 1\} \cup \{2i | 1 \le i \le n, \varepsilon(i) = *\}.$$

Then the entries in $(a, b)^\varepsilon$ that contain a b are given by $B(\varepsilon) := [2n] \backslash A(\varepsilon)$.

As an example let us consider $\varepsilon = (1, *, *, 1, *, 1) \in \{1, *\}^6$, this means that we get

$$(ab)^\varepsilon = (ab, ba, ba, ab, ba, ab)$$

and if we split each entry we get

$$(a, b)^\varepsilon = (a, b, b, a, b, a, a, b, b, a, a, b).$$

This means that

$$A(\varepsilon) = \{1, 4, 6, 7, 10, 11\}, \qquad \text{and} \qquad B(\varepsilon) = \{2, 3, 5, 8, 9, 12\}.$$

Using the products as entries formula together with the vanishing of mixed cumulants, we can easily rewrite the n-th cumulant of the anti-commutator. This is a combination of ideas that is commonly found in arguments involving the combinatorics of free probability.

Proposition 3.1.10. *The free cumulants of the anti-commutator $ab+ba$ of two free random variables a, b satisfy the following formula for all $n \in \mathbb{N}$:*

$$r_n(ab+ba) = \sum_{\substack{\pi \in \mathcal{NC}(2n) \\ \pi \vee_{\mathcal{NC}} I_{2n} = 1_{2n}}} \sum_{\substack{\varepsilon \in \{1,*\}^n \\ \{A(\varepsilon), B(\varepsilon)\} \geq \pi}} \left(\prod_{\substack{V \in \pi, \\ V \subset A(\varepsilon)}} r_{|V|}(a) \right) \left(\prod_{\substack{W \in \pi, \\ W \subset B(\varepsilon)}} r_{|W|}(b) \right), \tag{3.4}$$

where $I_{2n} := \{\{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}\} \in \mathcal{I}(2n)$ is the unique interval pair partition of $[2n]$, and 1_{2n} is the maximum partition.

Proof. For a fixed $\varepsilon \in \{1,*\}^n$, the products as arguments formula (2.18) asserts that

$$r_n((ab)^\varepsilon) = \sum_{\substack{\pi \in \mathcal{NC}(2n) \\ \pi \vee_{\mathcal{NC}} I_{2n} = 1_{2n}}} r_\pi((a,b)^\varepsilon).$$

Therefore, if we sum over all possible $\varepsilon \in \{1,*\}^n$ we get that

$$r_n(ab+ba, \ldots, ab+ba) = \sum_{\varepsilon \in \{1,*\}^n} \sum_{\substack{\pi \in \mathcal{NC}(2n) \\ \pi \vee_{\mathcal{NC}} I_{2n} = 1_{2n}}} r_\pi((a,b)^\varepsilon) = \sum_{\substack{\pi \in \mathcal{NC}(2n) \\ \pi \vee_{\mathcal{NC}} I_{2n} = 1_{2n}}} \sum_{\varepsilon \in \{1,*\}^n} r_\pi((a,b)^\varepsilon), \tag{3.5}$$

where in the second equality we just changed the order of the sums. Finally, since a and b are free, every mixed cumulant will vanish and thus we require that $\{A(\varepsilon), B(\varepsilon)\} \geq \pi$ in order to get $r_\pi((a,b)^\varepsilon) \neq 0$. In this case we actually get

$$r_\pi((a,b)^\varepsilon) = \left(\prod_{\substack{V \in \pi, \\ V \subset A(\varepsilon)}} r_{|V|}(a) \right) \left(\prod_{\substack{W \in \pi, \\ W \subset B(\varepsilon)}} r_{|W|}(b) \right).$$

After substituting this into (3.5) we get the desired result. □

A key idea is that the right-hand side of (3.4) can be greatly simplified by observing that there are very few ε satisfying $\{A(\varepsilon), B(\varepsilon)\} \geq \pi$. Here is where the graph $\Gamma(\pi)$ from Definition 3.1.1 becomes very useful. Actually, the reason to construct the edges of $\Gamma(\pi)$ by joining the blocks containing $2i-1$ and $2i$ is specifically to store information regarding $\pi \vee_{\mathcal{NC}} I_{2n}$. This is made precise in the next lemma.

Lemma 3.1.11. *Fix a partition $\pi \in \mathcal{NC}(2n)$. Then $\pi \vee_{\mathcal{NC}} I_{2n} = 1_{2n}$ if and only if $\Gamma(\pi)$ is connected.*

Proof. Recall from Remark 2.2.11 that $\pi \vee_{\mathcal{NC}} I_{2n} = 1_{2n}$ if and only if $\pi \vee_{\mathcal{P}} I_{2n} = 1_{2n}$, and this is equivalent to the fact that for every two elements $i, j \in [2n]$ we can find elements $i = i_0, i_1, i_2, \ldots, i_{2k+1} = j$ such that for $l = 0, \ldots, k$, the pair i_{2l}, i_{2l+1} is in the same block of I_{2n} and the pair i_{2l+1}, i_{2l+2} is in the same block V_l of π. Observe that i_{2l}, i_{2l+1} being in the same block of I_{2n}, means that (i_{2l}, i_{2l+1}) is an edge in $\Gamma(\pi)$ and since $i_{2l+1}, i_{2l+2} \in V_l \in \pi$, then edges (i_{2l}, i_{2l+1}) and (i_{2l+2}, i_{2l+3}) have the vertex V_l in common. Thus, to every sequence we can associate a path in $\Gamma(\pi)$ between the block containing i and the block containing j.

Therefore if we assume that $\pi \vee_{\mathcal{P}} I_{2n} = 1_{2n}$ and want to prove that $\Gamma(\pi)$ is connected, we take two arbitrary blocks $V, W \in \pi$ and some representatives $i \in V$ and $j \in W$. Since $\pi \vee_{\mathcal{P}} I_{2n} = 1_{2n}$ we have a sequence $i = i_0, i_1, i_2, \ldots, i_{2k+1} = j$ and thus a path in $\Gamma(\pi)$ between the V and W, so $\Gamma(\pi)$ is connected.

Conversely, assume that $\Gamma(\pi)$ is connected, and fix $i, j \in [2n]$. Then we consider the blocks $V, W \in \pi$ such that $i \in V$ and $j \in W$. Since $\Gamma(\pi)$ is connected there exists a path connecting V and W in $\Gamma(\pi)$, which in turn gives us a sequence connecting i and j, and thus $\pi \vee_{\mathcal{P}} I_{2n} = 1_{2n}$. $\square$

Now we can use $\Gamma(\pi)$ to test when $\{A(\varepsilon), B(\varepsilon)\} \geq \pi$ for a fixed $\pi \in \mathcal{NC}(n)$ and a fixed $\varepsilon \in \{1, *\}^n$. Turns out that $\Gamma(\pi)$ must be bipartite.

Remark 3.1.12 (Bipartite graphs)**.** Let $G = (V_G, E_G)$ be an undirected graph (where loops and multiedges are allowed), we say that G is *bipartite* if there exist a partition $\{V_G', V_G''\}$ of the set of vertices V_G such that there are no edges E_G connecting two vertices in V_G' or connecting two vertices in V_G''. In other words, all edges connect a vertex in V_G' with a vertex in V_G''.

Moreover, if G is connected and bipartite, then the bipartition $\{V_G', V_G''\}$ is unique (up to the permutation $\{V_G'', V_G'\}$). This is because once we identify a vertex $v \in V_G'$, then the set of any other vertex $u \in V_G$ is determined by its distance to v. If the distance is even, then $u \in V_G'$, and if the distance is odd, then $u \in V_G''$.

Proposition 3.1.13. *Let $\pi \in \mathcal{NC}(2n)$ such that $\pi \vee_{\mathcal{NC}} I_{2n} = 1_{2n}$ and consider $\Gamma(\pi)$.*

- *If there exists an $\varepsilon \in \{1, *\}^n$ such that $\{A(\varepsilon), B(\varepsilon)\} \geq \pi$, then $\Gamma(\pi)$ is bipartite.*

- *Moreover, if $\Gamma(\pi)$ is bipartite, then there are exactly two tuples $\varepsilon \in \{1, *\}^n$ satisfying $\{A(\varepsilon), B(\varepsilon)\} \geq \pi$. Furthermore, these tuples are completely opposite, that is, they do not coincide in any entry.*

Proof. For the first part, the existence of an $\varepsilon \in \{1, *\}^n$ such that $\{A(\varepsilon), B(\varepsilon)\} \geq \pi$ implies that we can write $\pi = \{A_1, \dots, A_r, B_1, \dots, B_s\}$ such that $A_i \subset A(\varepsilon)$ for $i = 1, \dots, r$ and $B_j \subset B(\varepsilon)$ for $j = 1, \dots, s$. If we consider $V' = \{A_1, \dots, A_r\}$ and $V'' = \{B_1, \dots, B_s\}$, then $\{V', V''\}$ is a bipartition of $\Gamma(\pi)$. Indeed, by construction of $\Gamma(\pi)$, if $e \in E_{\Gamma(\pi)}$, then e connects the blocks containing elements $2k-1$ and $2k$ for some $k = 1, \dots, n$. However, by definition of $A(\varepsilon)$ and $B(\varepsilon)$, one must contain $2k-1$ while the other contains $2k$. Thus e connects vertices from different sets and $\{V', V''\}$ is actually a bipartition of $\Gamma(\pi)$.

For the second part, since $\pi \vee_{\mathcal{NC}} I_{2n} = 1_{2n}$ we know from Lemma 3.1.11 that $\Gamma(\pi)$ is connected. Then, if $\Gamma(\pi)$ is also bipartite, by Remark 3.1.12 there exist a unique bipartition $\{\pi', \pi''\}$ of the vertices of $\Gamma(\pi)$ (blocks of π), say $\pi' = \{V_1, \dots, V_r\}$ and $\pi'' = \{W_1, \dots, W_s\}$, where V_1 contains the element 1. Then we must have $A(\varepsilon) = V_1 \cup \dots \cup V_r$ or $A(\varepsilon) = W_1 \cup \dots \cup W_s$. This clearly determines ε, since $\varepsilon(i) = 1$ if and only if $2i - 1 \in A(\varepsilon)$. Furthermore, since $(V_1 \cup \dots \cup V_r) \cap (W_1 \cup \dots \cup W_s) = \emptyset$, then the two possible ε do not coincide in any entry. $\square$

Notation 3.1.14. In light of the previous result, given a $\pi \in \mathcal{NC}(2n)$ such that $\Gamma(\pi)$ is connected and bipartite, we will denote by ε_π the (unique) tuple such that $\{A(\varepsilon_\pi), B(\varepsilon_\pi)\} \geq \pi$ and $1 \in A(\varepsilon)$. And we denote by ε'_π the other possible tuple, which actually satisfies that $A(\varepsilon'_\pi) = B(\varepsilon_\pi)$ and $B(\varepsilon'_\pi) = A(\varepsilon_\pi)$.

Remark 3.1.15. Recall from Definition 3.1.1 that every $\pi \in \mathcal{X}(2n)$ is naturally decomposed as $\pi := \pi' \sqcup \pi''$ where (π', π'') is the bipartition of $\Gamma(\pi)$. From the previous proof we can observe that we have the equalities $\pi' = \pi_{{}_{A(\varepsilon_\pi)}}$ and $\pi'' = \pi_{{}_{B(\varepsilon_\pi)}}$.

We are now ready to prove our main result.

Proof of Theorem 3.1.4. Our starting point is Equation (3.4) from Proposition 3.1.10. Using Lemma 3.1.11, it can be rephrased as

$$r_n(ab + ba) = \sum_{\substack{\pi \in \mathcal{NC}(2n) \\ \Gamma(\pi) \text{ is connected}}} \sum_{\substack{\varepsilon \in \{1,*\}^n \\ \{A(\varepsilon), B(\varepsilon)\} \geq \pi}} \left(\prod_{\substack{V \in \pi, \\ V \subset A(\varepsilon)}} r_{|V|}(a) \right) \left(\prod_{\substack{W \in \pi, \\ W \subset B(\varepsilon)}} r_{|W|}(b) \right). \tag{3.6}$$

By Proposition 3.1.13 the condition $\{A(\varepsilon), B(\varepsilon)\} \geq \pi$ is only true if $\Gamma(\pi)$ is bipartite. Furthermore, in this case ε can only be one of ε_π or ε'_π. Therefore, the right-hand side of (3.6) can be simplified to

$$\sum_{\substack{\pi \in \mathcal{NC}(2n), \\ \Gamma(\pi) \text{ is connected}, \\ \Gamma(\pi) \text{ is bipartite}}} \left(\prod_{\substack{V \in \pi, \\ V \subset A(\varepsilon_\pi)}} r_{|V|}(a) \prod_{\substack{W \in \pi, \\ W \subset B(\varepsilon_\pi)}} r_{|W|}(b) + \prod_{\substack{V \in \pi, \\ V \subset A(\varepsilon'_\pi)}} r_{|V|}(a) \prod_{\substack{W \in \pi, \\ W \subset B(\varepsilon'_\pi)}} r_{|W|}(b) \right).$$

Finally, from Remark 3.1.15 we have that $\pi' = \{V \in \pi : V \subset A(\varepsilon_\pi)\} = \{W \in \pi : W \subset B(\varepsilon'_\pi)\}$ and similarly $\pi'' = \{W \in \pi : W \subset B(\varepsilon_\pi)\} = \{V \in \pi : V \subset A(\varepsilon'_\pi)\}$, so we obtain the desired formula (3.1). $\square$

3.2 Detailed study of the set $\mathcal{Y}_{2n}$

In order to apply Theorem 3.1.4 it is useful to have an easy way to determine for which π the graph $\Gamma(\pi)$ is connected and bipartite. So far, the only way to check this is to actually construct the graph and analyze it. Here is where the fact that $\mathrm{Kr}(\mathcal{Y}(2n)) = \mathcal{X}(2n)$ anticipated in the previous section, proves to be very useful, as the set $\mathcal{Y}(2n)$ has a very simple description. We begin this section by proving this fundamental result, then we give some basic remarks and collect various facts concerning the set $\mathcal{Y}(2n)$. The goal is to have a better understanding of this set, which leads to a better understanding of $\mathcal{X}(2n)$ and simplifies the computations when applying Theorem 3.1.4. The main result in this section is Theorem 3.2.2, that provides a recursive formula to compute the values $|\mathcal{Y}(2n)|$.

Proposition 3.2.1. *For every $n \in \mathbb{N}$, we have that $\mathcal{X}(2n) = \mathrm{Kr}(\mathcal{Y}(2n))$.*

Proof. Let us fix $\sigma \in \mathcal{NC}(2n)$ and $\pi := \mathrm{Kr}(\sigma)$. the proof can be separated in two independent statements:

(A) $\Gamma(\pi)$ is connected if and only if σ separates odd elements (there is no block $W \in \sigma$ containing two odd elements).

Indeed, from Lemma 3.1.11, $\Gamma(\pi)$ is connected if and only $\pi \vee_{\mathcal{NC}} I_{2n} = 1_{2n}$. By properties of the Kreweras complement $\pi \vee_{\mathcal{NC}} I_{2n} = 1_{2n}$ if and only if $\mathrm{Kr}^{-1}(\pi) \wedge \mathrm{Kr}^{-1}(I_{2n}) = \mathrm{Kr}^{-1}(1_{2n}) = 0_{2n}$. Since $\mathrm{Kr}^{-1}(I_{2n}) = \{\{1, 3, \ldots, 2n-1\}, \{2\}, \{4\}, \ldots, \{2n\}\}$, for the latter to hold it is necessary and sufficient that all the odd elements of $\sigma = \mathrm{Kr}^{-1}(\pi)$ are in different blocks.

(B) $\Gamma(\pi)$ is bipartite if and only if $|W|$ is even for every block $W \in \sigma$ such that $W \subset \{2, 4, \ldots, 2n\}$.

In this case it is simpler to show that the negations of previous statements are equivalent. By standard arguments in graph theory (see for instance [Bol13]) we know that $\Gamma(\pi)$ is non-bipartite if and only if $\Gamma(\pi)$ contains an odd cycle which in turn is equivalent to $\Gamma(\pi)$ containing a simple odd cycle. Thus, to complete the proof we need to show that $\Gamma(\pi)$ contains a simple odd cycle if and only if σ contains a block W with $|W|$ odd and $W \subset \{2, 4, \ldots, 2n\}$.

For the if statement take a simple cycle of $\Gamma(\pi)$ with odd length j. Assume that the cycle has (in that order) the vertices $V_1, \ldots, V_j \in \pi$ and edges $e_1, e_2, \ldots, e_j$ corresponding to $(2i_1 - 1, 2i_1), (2i_2 - 1, 2i_2), \ldots, (2i_j - 1, 2i_j)$. This means that for $k = 1, \ldots j$, we have that $2i_{k-1}, 2i_k - 1 \in V_k$ (with $i_0 := i_j$). Then, the previous restrictions are equivalent to the fact that $W := \{2i_1, 2i_2, \ldots, 2i_j\}$ is a block of σ. Indeed, those elements are in the same block of σ since $V_1, \ldots, V_j$ are different blocks of π (because the cycle is simple). Also W do not contains any other element since this would generate a crossing with one of the arches $(2i_j, 2i_1 - 1), (2i_1, 2i_2 - 1), \ldots, (2i_{j-1}, 2i_j - 1)$ of π.

Conversely, if σ contains the block $W = \{2i_1, 2i_2, \ldots, 2i_j\}$ of odd size and completely contained in $\{2, 4, \ldots, 2n\}$. Then each of the arches $(2i_j, 2i_1 - 1)$, $(2i_1, 2i_2 - 1)$, $\ldots$, $(2i_{j-1}, 2i_j - 1)$ belongs to a different block of π and this implies that $\Gamma(\pi)$ contains an odd cycle with edges $(2i_1 - 1, 2i_1), (2i_2 - 1, 2i_2), \ldots, (2i_j - 1, 2i_j)$.

Once we have **(A)** and **(B)**, we know that $\pi \in \mathcal{X}(2n)$ if and only if σ separates odd elements and $|W|$ is even for every block $W \in \sigma$ such that $W \subset \{2, 4, \ldots, 2n\}$. Then σ has blocks $B_1, B_3, \ldots, B_{2n-1}$ with $i \in B_i$ for $i = 1, 3, \ldots, 2n-1$, and since the rest of the blocks $E_1, \ldots, E_r$, are contained in $[2n] \backslash (B_1 \cup B_3 \cup \cdots \cup B_{2n-1}) \subset \{2, 4, \ldots, 2n\}$ they must have even size. Therefore we conclude that $\mathrm{Kr}(\sigma) = \pi \in \mathcal{X}(2n)$ if and only if $\sigma \in \mathcal{Y}(2n)$. $\square$

Now we turn to study the size of the set $\mathcal{Y}(2n)$. A direct enumeration of this set looks rather complicated, but we can give a recurrence relation. For this we need to introduce the sets $\mathcal{Y}_{2n-1}$ for $n \in \mathbb{N}$, that (analogously to the even case) consist of all non-crossing partitions of $[2n-1]$ that can be written as

$$\sigma = \{B_1, B_3, \ldots, B_{2n-1}, E_1, \ldots, E_r\}, \quad \text{with } r \geq 0,$$

where $i \in B_i$ for $i = 1, 3, \ldots, 2n-1$ and $|E_j|$ even for $j \leq r$.

Theorem 3.2.2. *For every $n \in \mathbb{N}$, denote $\alpha_n := |\mathcal{Y}(2n)|$ and $\beta_n := |\mathcal{Y}(2n-1)|$. Then, we have that $\alpha_1 = \beta_1 = 1$ and the following relations hold:*

$$\alpha_n = \sum_{\substack{1\le s\le n \\ s \text{ is even}}} \sum_{\substack{r_1+\cdots+r_s=n \\ r_1,\dots,r_s\ge 1}} \beta_{r_1}\beta_{r_2}\cdots\beta_{r_s} + \sum_{j=1}^{n} \beta_j\beta_{n+1-j} \qquad \forall n \ge 2, \tag{3.7}$$

$$\beta_n = \sum_{s=1}^{n} \sum_{\substack{r_1+\cdots+r_s=n, \\ r_1,\dots,r_s\ge 1}} \beta_{r_1}\beta_{r_2}\cdots\beta_{r_{s-1}}\alpha_{r_s}, \qquad \forall n \ge 2. \tag{3.8}$$

Remark 3.2.3. Notice that if we denote $\gamma_k := |\mathcal{Y}(k)|$ for $k \in \mathbb{N}$, then the previous result gives a recursive way to compute $(\gamma_k)_{k\ge 1}$. Since the formulas are different for even and odd values, the notation with α_n and β_n turns out to be simpler, and that is the reason we opt for it.

Proof. The fact that $\alpha_1 = \beta_1 = 1$ is straightforward since sets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ just contain one partition. In order to obtain the recursive formulas, the general idea is to represent $\mathcal{Y}(m)$ as a disjoint union of products of sets $\mathcal{Y}(k)$ with $k < m$. We will base this study in the following standard bijection between non-crossing partitions:

Given $\sigma \in \mathcal{NC}(m)$, assume that $W = \{w_1, w_2, \dots, w_s\} \in \sigma$ is the block containing the last element, namely $w_s = m$. Consider the intervals $J_1 = \{1, 2 \dots, w_1 - 1\}, J_2 = \{w_1+1, w_1+2, \dots, w_2-1\}, \dots, J_s = \{w_{s-1}+1, \dots, m-1\}$ created after removing W from $[m]$, and let $k_i := |J_i|$ for $i = 1, \dots, s$, note that if $w_i + 1 = w_{i+1}$, then $J_i = \emptyset$ and $k_j = 0$. Then, we have the following bijection

$$\Psi : \mathcal{NC}(m) \to \bigcup_{s=1}^{m} \bigcup_{\substack{k_1+\cdots+k_s=m-s \\ k_1,\dots,k_s\ge 0}} \mathcal{NC}(k_1) \times \cdots \times \mathcal{NC}(k_s),$$
$$\sigma \mapsto \Psi(\sigma) := (\sigma|J_1, \sigma|J_2, \dots, \sigma|J_s),$$

where if $k_j = 0$, we assume that $\mathcal{NC}(0) = \{\emptyset\}$ contains an abstract *empty partition* and thus $|\mathcal{NC}(0)| = 1$. To reconstruct σ from an element $(\sigma_1, \sigma_2, \dots, \sigma_s) \in \mathcal{NC}(k_1) \times \cdots \times \mathcal{NC}(k_s)$ with $k_1 + \cdots + k_s = m - s$, we consider $w_i = k_1 + \cdots + k_i + i$ for $i = 1, \dots, s$ and define σ as the (unique) partition in $\mathcal{NC}(m)$ such that $\{w_1, w_2, \dots, w_s\} \in \sigma$ and $\sigma|J_i = \sigma_i$ for $i = 1, \dots, s$.

Going back to our proof, here we are only concerned with the restriction of Ψ to the set $\mathcal{Y}(m)$. Let us separate in the cases $m = 2n - 1$ and $m = 2n$.

Case $m = 2n - 1$. Let $\sigma = \{B_1, B_3, \dots, B_m, E_1, \dots, E_r\} \in \mathcal{Y}(m)$ and take the block $B_m = \{w_1, w_2, \dots, w_{s-1}, m\}$ containing the last element. Since σ separates odd elements, then $w_1, w_2, \dots, w_{s-1}$ must be even. If we consider the intervals $J_1, \dots, J_s$ left by W with sizes $k_1, \dots, k_s$ respectively, we have that $k_1, k_2, k_3, \dots, k_{s-1}$ are odd, while k_s is even. Moreover, since w_i is even for $i = 1, \dots, s-1$, the restrictions $\sigma|J_i$ separate odd elements and the blocks with only even elements are of even size. Therefore, we have that $\sigma|J_i \in \mathcal{Y}(k_i)$ for $i = 1, \dots, s$ and we obtain that

$$\Psi(\mathcal{Y}(m)) = \bigcup_{s=1}^{m} \bigcup_{\substack{k_1+\cdots+k_s=m-s \\ k_1,k_2,\dots,k_{s-1}\geq 1 \text{ odd} \\ k_s \geq 0 \text{ even}}} \mathcal{Y}(k_1) \times \cdots \times \mathcal{Y}(k_s). \tag{3.9}$$

Now, let us write $k_s = 2r_s - 2$, such that $r_s \geq 1$ and $|\mathcal{Y}(k_s)| = \alpha_{r_s - 1}$, we also write $k_j = 2r_j - 1$ such that $|\mathcal{Y}(k_j)| = \beta_{r_j}$ for $j = 1, 2, \dots, s-1$. This means that the condition in the indices is $2(r_1 + \cdots + r_s) - s - 1 = k_1 + \cdots + k_s = m - s = 2n - 1 - s$, or equivalently $r_1 + \cdots + r_s = n$. Finally, since Ψ is bijection and on the right-hand side of (3.9) the union is disjoint, we obtain (3.8).

Case $m = 2n$. Let $\sigma = \{B_1, B_3, \dots, B_{m-1}, E_1, \dots, E_r\} \in \mathcal{Y}(m)$, then we have two options for the last block $W := \{w_1, w_2, \dots, w_{s-1}, m\}$. Either $W = E_j$ for some $j = 1, \dots, r$ (without loss of generality we take $j = 1$), or $W = B_i$ for some $i = 1, 3, \dots, m-1$.

First subcase. Assume that $W = E_1$ and let $\mathcal{Y}^E(m) \subset \mathcal{Y}(m)$ be the set of partitions $\sigma \in \mathcal{Y}(m)$ such that their last block, W, has only even elements. In this case, $w_1, \dots, w_{s-1}, s$ are all even, and then k_i is odd for all $i = 1, \dots, s$. Proceeding as in the previous case we get that

$$\Psi(\mathcal{Y}^E(m)) = \bigcup_{\substack{1\leq s\leq m \\ s \text{ is even}}} \bigcup_{\substack{k_1+\cdots+k_s=m-s \\ k_1,k_2,\dots,k_s\geq 1 \text{ odd}}} \mathcal{Y}(k_1) \times \cdots \times \mathcal{Y}(k_s). \tag{3.10}$$

Let us now write $k_i = 2r_i - 1$ such that $|\mathcal{Y}(k_i)| = \beta_{r_i}$ for $i = 1, \dots, s$. The condition on the indices is $2(r_1 + \cdots + r_s) - s = k_1 + \cdots + k_s = m - s = 2n - s$, which can be restated as $r_1 + \cdots + r_s = n$, and actually this implies that $s \leq n$. Therefore, we obtain

$$|\mathcal{Y}^E(m)| = |\Psi(\mathcal{Y}^E(m))| = \sum_{\substack{1\leq s\leq n \\ s \text{ is even}}} \sum_{\substack{r_1+\cdots+r_s=n \\ r_1,\dots,r_s\geq 1}} \beta_{r_1}\beta_{r_2}\cdots\beta_{r_s}. \tag{3.11}$$

Second subcase. Assume that $W = B_i$ for an odd element i, and let $\mathcal{Y}^{B_i}(m) \subset \mathcal{Y}(m)$ be the set of partitions where W is of the form B_i. We could proceed as in the previous cases, but now there is a simpler approach. We notice that the restrictions $\sigma_1 := \sigma_{\{1,2,\dots,i\}}$

and $\sigma_2 := \sigma_{\{i,i+1,\dots m-1\}}$ satisfy that $\sigma_1 \in \mathcal{Y}(i)$ and $\sigma_2 \in \mathcal{Y}(m-i)$. Moreover we have a bijection

$$\Phi : \mathcal{Y}^{B_i}(m) \to \mathcal{Y}(i) \times \mathcal{Y}(m-i),$$
$$\sigma \mapsto (\sigma_1, \sigma_2).$$

Notice that given $(\sigma_1, \sigma_2) \in \mathcal{Y}(i) \times \mathcal{Y}(m-i)$ such that $\sigma_1 = \{V_1, \dots, V_r\}$ with $i \in V_r$, and $\sigma_2 = \{W_1, \dots, W_s\}$ with $1 \in W_1$. Then the inverse function is given by

$$\Phi^{-1}(\sigma_1, \sigma_2) = \{V_r \cup W_1' \cup \{m\}, V_1, V_2, \dots, V_{r-1}, W_2', W_3', \dots, W_s'\},$$

where for $k = 1, \dots, s$ the sets $W_k' := W_k + i - 1$ are the translations by $i-1$ of the blocks of σ_2. Finally if we sum over all possible odd numbers $i = 2j-1$ between 1 and $m = 2n-1$ we obtain that

$$\sum_{\substack{1 \le i \le m \\ i \text{ odd}}} |\mathcal{Y}^{B_i}(m)| = \sum_{j=1}^{n} \beta_j \beta_{n+1-j}, \tag{3.12}$$

If we add together (3.11) and (3.12) we obtain the desired formula (3.7). □

Now we want to give a more careful look to the set $\mathcal{Y}(2n)$ and study the cardinality of the subsets where partitions $\sigma \in \mathcal{Y}(2n)$ have a fixed size $|\sigma| = k$. We begin by checking which are these possible sizes k.

Remark 3.2.4. Recall that every $\sigma \in \mathcal{Y}(2n)$ can be written as

$$\sigma = \{B_1, \dots, B_{2n-1}, E_1, \dots, E_r\}.$$

Then the size of the partition is $|\sigma| = n + r$ with $r \ge 0$. Moreover, since $|E_j|$ is even we have $|E_j| \ge 2$ for $j = 1, \dots, r$, and since $E_1 \cup \dots \cup E_r \subset \{2, 4, \dots, 2n\}$, we have that $|E_1| + \dots + |E_r| \le n$. Thus, we conclude that $r \le \frac{n}{2}$ and we get a bound for the size of partitions in $\mathcal{Y}(2n)$:

$$n \le |\sigma| \le \frac{3n}{2}.$$

Notation 3.2.5. For $\sigma \in \mathcal{Y}(2n)$, we will say that $r := |\sigma| - n$ is the *level* of σ and use the notation $\mathcal{Y}^{(r)}(2n) := \{\sigma \in \mathcal{Y}(2n) : |\sigma| = n + r\}$, for $r = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$.

It turns out that we can give closed formulas for the sizes $|\mathcal{Y}^{(r)}(2n)|$ in the extreme cases when the level, r, is equal to 0 or $\lfloor \frac{n}{2} \rfloor$. Interestingly enough, both formulas are related in a different way to the sequence of Catalan numbers $(\mathrm{Cat}_n)_{n \ge 0} = 1, 1, 2, 5, 14, \dots$.

Proposition 3.2.6. *With the previous notation,* $|\mathcal{Y}^{(0)}(2n)| = 2^{n-1}\mathrm{Cat}_n$.

Proof. Notice that $\mathcal{Y}^{(0)}(2n)$ is the set of non-crossing partitions where every block contains exactly one odd element. We similarly define $\mathcal{Y}^{(0)}(2n-1)$ as the set of non-crossing partitions of $[2n-1]$ where every block contains exactly one odd element. And we use the notation $\theta_m := |\mathcal{Y}^{(0)}(m)|$. We follow the recurrence relation ideas from the proof of Theorem 3.2.2 to inductively show that $\theta_{2n} = 2^{n-1}\mathrm{Cat}_n$, and $\theta_{2n+1} = 2^n\mathrm{Cat}_n$. The induction hypothesis is trivially true, since $|\mathcal{Y}^{(0)}(1)| = |\mathcal{Y}^{(0)}(2)| = 1$. For the inductive step we separate in two cases.

First consider the case $m = 2n$. Let $\sigma = \mathcal{Y}^{(0)}(2n+1)$ and take the last block $W \in \sigma$ (namely $m \in W$). Opposed to the proof of Theorem 3.2.2, now we cannot have the first subcase $W = E_1$, and only need to check the second subcase $W = B_i$. Moreover, when doing the restrictions we still have exactly one odd element in each block. So we obtain only partitions with level 0. Thus, in this case the bijection Φ gives

$$\Phi(\mathcal{Y}^{(0)}(2n)) = \bigcup_{\substack{1\leq k\leq 2n \\ k \text{ odd}}} \mathcal{Y}^{(0)}(k) \times \mathcal{Y}^{(0)}(2n-k).$$

Thus if we take $k = 2j+1$, then the induction hypothesis an the recursive relation satisfied by the Catalan numbers yield that

$$\theta_{2n} = \sum_{j=0}^{n-1} \theta_{2j+1}\theta_{2n-2j-1} = \sum_{j=0}^{n-1} \ 2^j \ \mathrm{Cat}_j \ \cdot \ 2^{n-j-1}\mathrm{Cat}_{n-j-1}$$

$$= 2^{n-1}\sum_{j=0}^{n-1} \mathrm{Cat}_{j-1}\mathrm{Cat}_{n-j} = 2^{n-1}\mathrm{Cat}_n.$$

For the case $m = 2n+1$. Let $\sigma = \mathcal{Y}^{(0)}(2n+1)$ and consider the block $W \in \sigma$ with $m \in W$. We separate in 2 cases. If $W = \{m\}$, then we have that $\sigma_{\{1,\dots,m-1\}} \in \mathcal{Y}^{(0)}(m-1)$ and we obtain $\theta_{2n} = 2^{n-1}\mathrm{Cat}_n$ partitions in this subcase. Otherwise, we consider $s = min(W)$, which has to be even (as $s \neq m$ and σ separates odd elements). Then we separate σ into the partitions $\sigma_1 = \sigma_{\{1,\dots,s-1\}}$ and $\sigma_2 = \sigma_{\{s+1,\dots,m\}}$. Then, the map Φ' sending σ to

$$\Phi'(\sigma) = (\sigma_1, \sigma_2) \in \bigcup_{\substack{1\leq s\leq m \\ s \text{ even}}} \mathcal{Y}^{(0)}(s-1) \times \mathcal{Y}^{(0)}(m-s),$$

is a bijection. Thus, if we take $s = 2j+2$, we count

$$\sum_{j=0}^{n-1} \theta_{2j+1}\theta_{2n-2j-1} = 2^{n-1}\mathrm{Cat}_n$$

partitions in this subcase. Adding both subcases, we conclude that $\theta_{2n+1} = 2^n \mathrm{Cat}_n$. $\square$

Proposition 3.2.7. *With the previous notation, for $k \in \mathbb{N}$ one has*

$$|\mathcal{Y}^{(k)}(4k)| = \mathrm{Cat}_k \qquad \text{and} \qquad |\mathcal{Y}^{(k)}(4k+2)| = (2k+1)\mathrm{Cat}_{k+1}.$$

Proof. For the case $2n = 4k$, we want to count partitions that can be written as $\sigma = \{B_1, \dots, B_{4k-1}, E_1, \dots, E_k\}$, but this implies that $|E_1| = |E_2| = \cdots = |E_k| = 2$ and $|B_1| = \cdots = |B_{4k-1}| = 1$. Then, the whole partition is determined by the partition $\{E_1, \dots, E_k\} \in \mathcal{NC}(\{2, 4, \dots, 4k\})$ and its only condition is to be a pair partition. Thus, partitions of the form $\sigma = \{B_1, \dots, B_{4k-1}, E_1, \dots, E_k\}$ are clearly in bijection with the set $\mathcal{NC}^{\mathrm{pair}}(2k)$, and we conclude that $|\mathcal{Y}^{(k)}(4k)| = |\mathcal{NC}^{\mathrm{pair}}(2k)| = \mathrm{Cat}_k$.

For the case $2n = 4k+2$ we use the same kind of bijections we have been using throughout all this section. We want to count partitions $\sigma = \{B_1, \dots, B_{4k+1}, E_1, \dots, E_k\}$, but this implies that $|E_1| = |E_2| = \cdots = |E_k| = 2$ and $|B_1| = \cdots = |B_{4k-1}| = 1$, except for one block $B_i = \{e, i\}$ that consist of its corresponding odd element i and one even element e. Then, the block B_i separates the partition σ into two independent partition σ_1 and σ_2 contained in the pockets $J_1 = \{e+1, \dots, i-1\}$ and $J_2 = \{i+1, \dots, e-1\}$ (with elements taken mod $4k$). Moreover, the even elements of σ_1 form a pair partition, so $|\sigma_1| := 4j$ should be multiple of 4, and we actually have that $\sigma_1 \in \mathcal{Y}^{(j)}(4j)$ and $\sigma_2 \in \mathcal{Y}^{(k-j)}(4k-4j)$. Thus, there are $2k+1$ ways to choose the odd element i, and for each of them we have

$$\sum_{j=0}^{k} |\mathcal{Y}^{(j)}(4j)| \cdot |\mathcal{Y}^{(k-j)}(4k-4j)| = \sum_{j=0}^{k} \mathrm{Cat}_j \mathrm{Cat}_{k-j} = \mathrm{Cat}_{k+1}$$

ways to construct the partitions σ_1 and σ_2. Thus, we conclude the desired equality $|\mathcal{Y}^{(k)}(4k+2)| = (2k+1)\mathrm{Cat}_{k+1}$. $\square$

Remark 3.2.8. At this point, one may wonder if the sizes of the sets $|\mathcal{Y}^{(r)}(2n)|$ for $r = 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor - 1$, have some other interesting formulas. We can surely generalize the ideas used in Theorem 3.2.2, Proposition 3.2.6 and Proposition 3.2.7 to get recursive formulas. However, the sets $\mathcal{Y}^{(r)}(2n)$ (and thus the recursions needed) become much more involved and it is not clear if there is a nice closed formula. We now point out an algorithm that can be used to recursively compute each value. Observe that $|\mathcal{Y}^{(1)}(2n)|$ can be written in terms of $|\mathcal{Y}^{(1)}(m)|$ and $|\mathcal{Y}^{(0)}(m)|$ for $m < 2n$. Since we know the values for the level $r = 0$ we can recursively compute all the values for the level $r = 1$. In general, $|\mathcal{Y}^{(r)}(2n)|$ can be written in terms of $|\mathcal{Y}^{(j)}(m)|$ for $m < 2n$ and $j \leq r$. Thus we can do a recursion on r to compute all the values of $|\mathcal{Y}^{(r)}(m)|$ (where for each r we first do the recursion on m).

The sizes of the sets $|\mathcal{Y}(2n)|$ and $|\mathcal{Y}^{(r)}(2n)|$ for small values of n and r are given in the following table:

$2n$	2	4	6	8	10
$\|\mathcal{Y}(2n)\|$	1	5	26	155	987
$\|\mathcal{Y}^{(0)}(2n)\|$	1	4	20	112	672
$\|\mathcal{Y}^{(1)}(2n)\|$		1	6	41	290
$\|\mathcal{Y}^{(2)}(2n)\|$				2	25

As the reader may have noticed, it is useful to consider the sizes $|\mathcal{Y}(m)|$ for all integers m, rather than just the even values. Thus, we also provide a table including these values (the level in the odd case is defined in the same way as in the even case).

m	1	2	3	4	5	6	7	8	9	10	11
$\|\mathcal{Y}(m)\|$	1	1	2	5	9	26	48	155	287	987	1834
$\|\mathcal{Y}^{(0)}(m)\|$	1	1	2	4	8	20	40	112	224	672	1344
$\|\mathcal{Y}^{(1)}(m)\|$				1	1	6	8	41	61	290	460
$\|\mathcal{Y}^{(2)}(m)\|$								2	2	25	30

Remark 3.2.9. Since $|\sigma| + |\mathrm{Kr}(\sigma)| = 2n + 1$, all the previous discussion directly gives information on the size of the levels of $\mathcal{X}(2n)$. In particular, for every $\pi \in \mathcal{X}(2n)$ we have that

$$n + 1 \geq |\pi| \geq \left\lceil \tfrac{n}{2} \right\rceil + 1.$$

And for a fixed $k \in \mathbb{N}$ we have that $\mathrm{Kr}(\mathcal{Y}^{(n+1-k)}(2n)) = \{\sigma \in \mathcal{X}(2n) : |\sigma| = k\}$, and thus their cardinalites concide.

To finish this section we give a possible line of study that will help to further understand the set $\mathcal{Y}(2n)$.

Remark 3.2.10. Fix a $\sigma \in \mathcal{Y}(2n)$ and consider its restriction $\sigma_{even} := \sigma_{\{2,4,\ldots,2n\}}$ to the even elements. This gives us a non-crossing partition of $[n]$. Moreover, it is easy to observe that for every $\pi \in \mathcal{NC}(n)$ we can find a $\sigma \in \mathcal{Y}(2n)$ such that $\sigma_{even} = \pi$. Then, for a fixed $\pi \in \mathcal{NC}(n)$, an interesting question is which are the values $q_\pi := \#\{\sigma \in \mathcal{Y}(2n) : \sigma_{even} = \pi\}$. This information will give a much better understanding of $\mathcal{Y}(2n)$ and $\mathcal{X}(2n)$. For instance if $\pi = 0_n$, the partitions of $\sigma \in \mathcal{Y}(n)$ such that $\sigma_{even} = 0_n$ are exactly the pair partitions. Thus, in this case we get $q_{0_n} = |\mathcal{NC}(n)| = \mathrm{Cat}_n$ partitions. We now give some basic ideas towards finding the values q_π.

In general, given $\pi \in \mathcal{NC}(n)$, a possible algorithm to construct a $\sigma \in \mathcal{Y}(2n)$ such that $\sigma_1 = \pi$ is the following. We take $[2n]$ and draw π in the even elements $\{2, 4, \dots, 2n\}$. Then we choose a subset $S = \{V_1, \dots, V_r\}$ of the blocks of π, and attach to each block a distinct odd element. For the resulting partition to be in $\mathcal{Y}(2n)$ we need to make sure of two things. First, that all the blocks of odd size in π are in S. The second and more tricky part, is that we do not generate any crossing when doing the attachment of the odd elements.

An attachment that works for every partition is to put the element $min(V) - 1$ in each $V \in S$. Alternatively, we can attach $min(V) + 1$ to each $V \in S$. Thus there are at least two options for each subset S. Since there are $2^{\#\{V \in \pi \,:\, |V| \text{ is even}\}}$ possible ways to pick the subset S we get the following lower bound:

$$q_\pi \geq 2^{\#\{V \in \pi \,:\, |V| \text{ is even}\}+1}.$$

This bound is tighter when π has various even blocks. However, if π has only blocks of odd size, such as when $\pi = 0_n$, the right-hand side becomes 2 and the bound does not really helps. Knowing the values q_π, or at least improving this simple bound is a useful step towards a better understanding of $\mathcal{Y}(2n)$.

3.3 Applications

This section is divided in two parts, each makes use of Theorem 3.1.4 and some results from Section 3.2 to deal with different applications. In Section 3.3.1 we study the case of two free Poisson of parameter 1 and prove Theorem 3.1.7. Then, in Section 3.3.2 we use our method to retrieve the formula from [NS98], where distributions are assumed to be symmetric.

3.3.1 Anti-commutator of two free Poisson with parameter 1

The objective of this section is to study the distribution ν of the anti-commutator $ab + ba$ in the particular case where a, b are distributed as a free Poisson distribution (also known as Marchenko-Pastur distribution) of parameter 1, and in particular prove Theorem 3.1.7. The reason why the free Poisson is very special, is because all cumulants are equal to 1, namely $r_n(a) = r_n(b) = 1$ for all $n \in \mathbb{N}$.

Remark 3.3.1. Notice that Corollary 3.1.6 directly yields that

$$r_n(ab + ba) = 2|\mathcal{X}(2n)| = 2|\mathcal{Y}(2n)|, \qquad \forall n \in \mathbb{N}.$$

Therefore, we can use formulas (3.7) and (3.8) from Theorem 3.2.2 to recursively compute the cumulants of the anti-commutator $ab + ba$. The first few cumulants are

$$2,\ 10,\ 52,\ 310,\ 1974,\ 13176,\ 90948,\ 643918,\ 4650382, \dots$$

Therefore, the moment-cumulant formula yields that the first few moments are

$$2,\ 14,\ 120,\ 1182,\ 12586,\ 141160,\ 1642584,\ 19646558,\ 240050838, \dots$$

We now want to compute the moment series of ν. In order to do this we first rewrite Theorem 3.2.2 as a relation between the formal power series associated to $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$, the sequences of values of $\mathcal{Y}(m)$ for even and odd m respectively.

Notation 3.3.2. Let us denote by A and B the formal power series on x associated to the sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$, respectively. That is

$$A(x) := \sum_{n=1}^{\infty} \alpha_n x^n = \sum_{n=1}^{\infty} |\mathcal{Y}(2n)| x^n, \qquad B(x) := \sum_{n=1}^{\infty} \beta_n x^n = \sum_{n=1}^{\infty} |\mathcal{Y}(2n-1)| x^n.$$

Proposition 3.3.3. *The formal power series $A(x)$ and $B(x)$ satisfy the following relations*

$$A(x) = \frac{B^2(x)}{1 - B^2(x)} + \frac{B^2(x)}{x}, \tag{3.13}$$

$$B(x) = \frac{(A(x) + 1)x}{1 - B(x)}, \tag{3.14}$$

Proof. First we observe that basic operations of power series tell us that

$$\frac{1}{1 - B(x)} = 1 + B(x) + B^2(x) + \cdots = 1 + \sum_{n=1}^{\infty} x^n \sum_{s=1}^{n} \sum_{\substack{r_1 + \cdots + r_s = n \\ r_1, r_2, \dots, r_s \geq 1}} \beta_{r_1} \beta_{r_2} \dots \beta_{r_s}.$$

To handle the sum appearing in (3.11) (where tuples must be of even size) we use the following formal power series:

$$E(x) := \sum_{n=1}^{\infty} x^n \sum_{\substack{1\le s\le n \\ s \text{ is even}}} \sum_{\substack{r_1+\cdots+r_s=n \\ r_1,r_2,\ldots,r_s\ge 1}} \beta_{r_1}\beta_{r_2}\ldots\beta_{r_s}.$$

A straightforward comparison of coefficients yields $E(x) = B^2(x)(E(x)+1)$. Solving for E we get

$$E(x) = \frac{B^2(x)}{1-B^2(x)}.$$

Then, the n-th coefficient of $A(x)$ is the left hand side of (3.7), while the right-hand side is exactly the sum of n-th coefficient of $E(x)$ plus the n-th coefficient of $\frac{B^2(x)}{x}$, thus we get formula (3.13):

$$A(x) = E(x) + \frac{B^2(x)}{x} = \frac{B^2(x)}{1-B^2(x)} + \frac{B^2(x)}{x}.$$

On the other hand if we multiply $(\frac{1}{1-B(x)})(A(x)+1)(x)$ (where x merely just adjust the coefficients), then the n-th coefficient of this product is given by the right-hand side of (3.8), since the left hand side is n-th coefficient of $B(x)$ we obtain (3.14). $\square$

Since we have two power series, satisfying two equations, (3.13) and (3.14), we can manipulate them to obtain a functional equation containing only $A(x)$ or only $B(x)$. After these manipulations we obtain the following.

Proposition 3.3.4. *The formal power series $A(x)$ and $B(x)$ satisfy the following equations:*

$$4(A(x)+1)^4x^2+7(A(x)+1)^3x-4(A(x)+1)^2x-2(A(x)+1)^2+(A(x)+1)+1=0, \quad (3.15)$$

$$B(x)(1-2B(x))(1-B(x))(1+B(x)) = x. \quad (3.16)$$

Proof. To simplify notation we omit the dependence on x whenever we are just evaluating on x, that is, we write A, B instead of $A(x), B(x)$. We also use the notation $A_1 := A(x)+1$. By using (3.14) and then (3.13) we get that

$$B - B^2 = A_1 x = \left(\frac{B^2}{1-B^2} + \frac{B^2}{x} + 1\right)x = \left(\frac{1}{1-B^2} + \frac{B^2}{x}\right)x = \frac{x}{1-B^2} + B^2.$$

Multiplying by $1-B^2$ and solving for x yields (3.16). On the other hand, from (3.14) and (3.16) we obtain

$$\frac{1}{A_1} = \frac{x}{B(1-B)} = (1-2B)(1+B) = 1 - B - 2B^2.$$

Also, directly from (3.14) we know that $A_1 x = B - B^2$, a linear combination of both equations yields

$$\frac{1}{A_1} - 2A_1 x = 1 - B - 2B^2 - 2B + 2B^2 = 1 - 3B.$$

Solving for B we obtain

$$B = \frac{1}{3}\left(1 - \frac{1}{A_1} + 2A_1 x\right) = \frac{2A_1^2 x + A_1 - 1}{3A_1}.$$

Replacing this on (3.14) we obtain an expression entirely on A_1:

$$\begin{aligned} A_1 x &= \left(\frac{-2A_1^2 x + 2A_1 + 1}{3A_1}\right)\left(\frac{2A_1^2 x + A_1 - 1}{3A_1}\right) \\ &= \frac{-4A_1^4 x^2 - 2A_1^3 x + 2A_1^2 x + 4A_1^3 x + 2A_1^2 - 2A_1 + 2A_1^2 x + A_1 - 1}{9A_1^2}. \end{aligned}$$

Multiplying by $9A_1^2$ and simplifying we obtain (3.15). $\square$

We are now ready to prove Theorem 3.1.7, the main result of this section. We use the standard notation

$$R_\nu(z) := \sum_{n=1}^{\infty} r_n(ab+ba)z^n \qquad \text{and} \qquad M_\nu(z) := \sum_{n=1}^{\infty} \varphi((ab+ba)^n)z^n,$$

for the R-transform and moment series of ν, respectively, where ν is the distribution of the free anti-commutator of two free Poisson distributions.

Proof of Theorem 3.1.7. We want to show that the compositional inverse of the moment series of ν is given by

$$M_\nu^{\langle -1\rangle}(z) = \frac{-7z - 6 + 3\sqrt{(z+2)(9z+2)}}{4(z+2)^2(z+1)}. \tag{3.17}$$

For this, we first observe that the R_ν is simply $2 \cdot A$. This implies that $2(A(x)+1) = R_\nu(x)+2$. Therefore, if we multiply (3.15) by 8 (to avoid fractions on the coefficients) and write it in terms of the R-transform we get

$$\begin{aligned}
0 &= 2(R_\nu(x)+2)^4x^2 + 7(R_\nu(x)+2)^3x - 4(R_\nu(x)+2)^2x - 8(R_\nu(x)+2)^2 \\
&\quad + 4(R_\nu(x)+2) + 8 \\
&= \Big(2(R_\nu(x)+2)^4\Big)x^2 + \Big((R_\nu(x)+2)^2(7R_\nu(x)+6)\Big)x - \Big(4R_\nu^2(x) + 12R_\nu(x)\Big).
\end{aligned}$$

Let us simplify this expression by taking $y := (R_\nu(x)+2)^2x$, this yields

$$0 = 2y^2 + (7R_\nu(x)+6)y - 4(R_\nu^2(x) + 3R_\nu(x)).$$

This is a polynomial in y of degree 2 so by the quadratic formula we know that

$$y = \frac{-(7R_\nu(x)+6) \pm \sqrt{(7R_\nu(x)+6)^2 + 4\cdot 2\cdot 4(R_\nu^2(x)+3R_\nu(x))}}{2\cdot 2}.$$

The discriminant simplifies to

$$49R_\nu^2(x) + 84R_\nu(x) + 36 + 32R_\nu^2(x) + 96R_\nu(x) = 9(R_\nu(x)+2)(9R_\nu(x)+2),$$

and we obtain that

$$x = \frac{y}{(R_\nu(x)+2)^2} = \frac{-(7R_\nu(x)+6) \pm 3\sqrt{(R_\nu(x)+2)(9R_\nu(x)+2)}}{4(R_\nu(x)+2)^2}.$$

Since we want $R_\nu(0) = 0$ to hold, we pick the positive sign in the previous equation. Thus the compositional inverse $R^{\langle -1\rangle}(z)$ of the R-transform is

$$R^{\langle -1\rangle}(z) = \frac{-7z - 6 + 3\sqrt{(z+2)(9z+2)}}{4(z+2)^2}.$$

And by a basic relation between $R^{\langle -1\rangle}(z)$ and $M^{\langle -1\rangle}(z)$, (see formula (16.31) of [NS06]), we conclude that

$$M^{\langle -1\rangle}(z) = \frac{R^{\langle -1\rangle}(z)}{(1+z)} = \frac{-7z - 6 + 3\sqrt{(z+2)(9x+2)}}{4(z+2)^2(z+1)},$$

as desired. □

Remark 3.3.5. If we consider the Cauchy transform $G := G_\nu(z)$ and use its relation to the R-transform or the Moment series of μ, we can obtain that G is the solution to a polynomial equation of degree six. For instance, from (3.15) and the relation $zG(z)+1 = 2A(G(z))+2$ we can directly obtain

$$2z^4G^6+8z^3G^5+12z^2G^4+8zG^3+2G^2+7z^3G^4+13z^2G^3+5zG^2-G-4z^2G^2-4zG+8=0.$$

Remark 3.3.6 (Anti-commutator of two free Poisson with parameter λ)**.** Let a, b be free Poisson of parameter $\lambda > 0$. Notice that our discussion from Section 3.2 together with Corollary 3.1.6 yield that $r_n(ab+ba) = P_n(\lambda)$ is a polynomial on λ of degree $n+1$, with principal coefficient $2^n\mathrm{Cat}_n$, and where 0 is a root of multiplicity $\left\lceil \frac{n}{2} \right\rceil + 1$. More specifically, if we let $h := \left\lfloor \frac{n}{2} \right\rfloor$ we have that

$$P_n(\lambda) = \lambda^{n+1-h}\Big(d_0\lambda^h + d_1\lambda^{h-1} + \cdots + d_h\Big),$$

where $d_r := 2|\mathcal{Y}(2n)^{(r)}|$ for $r = 0, 1, \ldots, h$. Since we already computed this numbers for small values of n (see Remark 3.2.8), we get that the first few cumulants are given by

$$\begin{aligned}
r_1(ab+ba) &= 2\lambda^2, \\
r_2(ab+ba) &= 8\lambda^3 + 2\lambda^2, \\
r_3(ab+ba) &= 40\lambda^4 + 12\lambda^3, \\
r_4(ab+ba) &= 224\lambda^5 + 82\lambda^4 + 4\lambda^3, \\
r_5(ab+ba) &= 1344\lambda^6 + 580\lambda^5 + 50\lambda^4.
\end{aligned}$$

3.3.2 Anti-commutator of two even elements

In this section we apply Theorem 3.1.4 to the special case where both a and b are even elements. Recall that a is an *even element* if all its odd moments are 0, namely $\varphi(a^n) = 0$ for every odd n. This case was already studied by Nica and Speicher in connection with the free commutator (see Theorem 15.20 of [NS06]). Here we recover this formula using our main result and some nice combinatorial observations.

Theorem 3.3.7 (Theorem 15.20 of [NS06])**.** *Let a and b two free even random variables. Then the odd free cumulants of the anti-commutator are 0 while the even free cumulants are given by the following formula*

$$r_{2n}(ab+ba) = 2\sum_{\pi_1\in\mathcal{NC}(n)}\left(\prod_{V\in\pi_1} r_{2|V|}(a)\right)\sum_{\substack{\pi_2\in\mathcal{NC}(n)\\ \pi_2\le \mathrm{Kr}(\pi_1)}}\left(\prod_{W\in\pi_2} r_{2|W|}(b)\right), \qquad \forall n\in\mathbb{N}. \quad (3.18)$$

Proof. The approach is to use our main theorem, to express the cumulant of the anti-commutator as sum indexed by $\mathcal{X}(2n)$, that can be further restricted to even partitions. If we apply Kr^{-1} to this index set, we obtain partitions in $\mathcal{Y}(2n)$ that are parity preserving. These partitions have a simple description, so when we apply the Kreweras complement we retrieve the result. Let us begin by applying Theorem 3.1.4 to rewrite $r_n(ab+ba)$ as

$$\sum_{\substack{\pi\in\mathcal{X}(2n)\\ \pi=\pi'\sqcup\pi''}} \left(\prod_{V\in\pi'} r_{|V|}(a) \prod_{W\in\pi''} r_{|W|}(b) + \prod_{V\in\pi'} r_{|V|}(b) \prod_{W\in\pi''} r_{|W|}(a) \right). \tag{3.19}$$

Recall that by the moment-cumulant formula, since a and b are even, this readily implies that all its odd cumulants vanish. Thus, whenever we have some block $V\in\pi$ such that $|V|$ is odd, the whole product will vanish. As a result, if n is odd, all the products will have such a block and we get that $r_n(ab+ba)=0$. Moreover when n is even, the partition π must be even for the product not to vanish. Thus the index of the sums in (3.19) can be restricted to $\mathcal{X}(2n)\cap\mathcal{NC}^{\mathrm{even}}(2n)$. Since we understand better $\mathcal{Y}(2n)$, let us take inverse of the Kreweras complement map. Using Remark 2.2.22 we get that

$$\mathrm{Kr}^{-1}(\mathcal{X}(2n)\cap\mathcal{NC}^{\mathrm{even}}(2n)) = \mathrm{Kr}^{-1}(\mathcal{X}(2n))\cap\mathrm{Kr}^{-1}(\mathcal{NC}^{\mathrm{even}}(2n)) = \mathcal{Y}(2n)\cap\mathcal{NC}^{\mathrm{pa.pr}}(2n).$$

Notice that every partition $\sigma\in\mathcal{Y}(2n)\cap\mathcal{NC}^{\mathrm{pa.pr}}$ can be written as $\sigma=\{B_1,\dots,B_{2n-1}, E_1,\dots,E_r\}$ with $i\in B_i$ and $|E_j|$ even. Moreover, the blocks of σ have the same parity. Therefore, $B_i\subset\{1,3,\dots,2n-1\}$ and this implies that $B_i=\{i\}$ for $i=1,3,\dots,2n-1$. On the other hand, we have that $E_1\cup\cdots\cup E_r=\{2,4,\dots,2n\}$ and since all have even size, we get that $\sigma_{even}:=\sigma_{\{2,4,\dots,2n\}}$ is an even partition. Therefore, $\sigma=\langle 0_n,\sigma_{even}\rangle$ for an even partition σ_{even} (see Notation 2.2.19), and we obtain the simpler description

$$\mathcal{Y}(2n)\cap\mathcal{NC}^{\mathrm{pa.pr}}(2n) = \{\langle 0_n,\sigma'\rangle\in\mathcal{NC}(2n) : \sigma'\in\mathcal{NC}^{\mathrm{even}}(n)\}.$$

Now we apply the Kreweras complement to this set in order to retrieve $\mathcal{X}(2n)\cap\mathcal{NC}^{\mathrm{even}}(2n)$. Let us denote by $I'_{2n}:=\{\{2,3\},\{4,5\},\dots,\{2n-2,2n-1\},\{1,2n\}\}$ the partition which is the Kreweras complement of $\langle 0_n,1_n\rangle=\{\{1\},\{3\},\dots,\{2n-1\},\{2,4,\dots,2n\}\}$. For $\sigma=\langle 0_n,\sigma_{even}\rangle$ we observe that if $\sigma\le\langle 0_n,1_n\rangle$ then $\mathrm{Kr}(\sigma)\ge I'_{2n}$. Moreover the fact that $\sigma_{even}\in\mathcal{NC}^{\mathrm{even}}(n)$ implies that $(\mathrm{Kr}(\sigma))_{\{2,4,\dots,2n\}}\in\mathcal{NC}^{\mathrm{pa.pr}}(n)$.

Let us now consider the 'fattening' map $\mathrm{Fat}:[n]\to I'_{2n}$ that sends every $j\in[n]$ to the pair $\mathrm{Fat}(j)=\{2j,2j+1\}\in I'_{2n}$ (where we assume $2n+1=1$). We can naturally extend this map to blocks $V=\{j_1,\dots,j_k\}\subset[n]$ by taking $\mathrm{Fat}(V)=\{2j_1,2j_1+1,\dots,2j_k,2j_k+1\}\subset$

$[2n]$, and we can extend it further to partitions $\tau \in \mathcal{NC}(n)$ by taking $\mathrm{Fat}(\tau) = \{\mathrm{Fat}(V) : V \in \tau\} \in \mathcal{NC}(2n)$. Then we have a bijection

$$\mathrm{Fat} : \mathcal{NC}(n) \to \{\pi \in \mathcal{NC}(2n) : \pi \geq I'_{2n}\},$$

and we obtain that partitions $\pi \in \mathcal{X}(2n) \cap \mathcal{NC}^{\mathrm{even}}(2n) = \mathrm{Kr}(\mathcal{Y}(2n) \cap \mathcal{NC}^{\mathrm{pa.pr}}(2n))$ can be described as $\pi = \mathrm{Fat}(\tau)$ for some $\tau \in \mathcal{NC}^{\mathrm{pa.pr}}(n)$. Therefore in (3.19) we can alternatively sum over $\mathrm{Fat}(\mathcal{NC}^{\mathrm{pa.pr}}(n))$. Moreover, the tuple ε_π associated to a partition $\pi \in \mathrm{Fat}(\mathcal{NC}^{\mathrm{pa.pr}}(n))$ satisfies that $A(\varepsilon_\pi) = \{\mathrm{Fat}(V) \subset [2n] : V \subset \{1, 3, \ldots, n-1\}\}$ and $B(\varepsilon_\pi) = \{\mathrm{Fat}(V) \subset [2n] : V \subset \{2, 4, \ldots, n\}\}$. Thus, for $\pi = \mathrm{Fat}(\tau)$ we have

$$\prod_{\substack{V\in\pi,\\ V\subset A(\varepsilon_\pi)}} r_{|V|}(a) \prod_{\substack{W\in\pi,\\ W\subset B(\varepsilon_\pi)}} r_{|V|}(b) = \prod_{\substack{V'\in\tau,\\ V'\subset\{1,3,\ldots,n-1\}}} r_{2|V'|}(a) \prod_{\substack{W'\in\tau,\\ W'\subset\{2,4,\ldots,n\}}} r_{2|W'|}(b).$$

Finally, if we write $n = 2m$ we have the description $\mathcal{NC}^{\mathrm{pa.pr}}(n) : \{\langle \pi_1, \pi_2 \rangle \in \mathcal{NC}(n) : \pi_1, \pi_2 \in \mathcal{NC}(m),\ \pi_2 \leq \mathrm{Kr}(\pi_1)\}$, and we obtain the desired result. □

3.4 Cacti graphs

This section is divided in two parts. The first part is devoted to prove the main result of this section, Theorem 3.4.6, which is a formula for the anti-commutator where the summation is now indexed by cacti graphs. In the second part we discuss rigid cacti and their connection to the case of even variables.

3.4.1 Rephrasing in terms of cacti graphs

The purpose of this section is to explain how our main formula (3.1) from section 3.1 can be also phrased solely in terms of graphs.

Definition 3.4.1. Let G be a graph, where we allow loops and multiple edges.

1. A *cycle* is a finite sequence of vertices and edges $(v_1, e_1, v_2, e_2, \ldots, v_j, e_j)$ with vertices $v_1, \ldots, v_j \in V_G$ and where $e_i \in E_G$ connects v_i with v_{i+1} for $i = 1, \ldots, j$ (assuming $j+1 = 1$). We say that the cycle is *simple* if there is no proper subset $\{v_{i_1}, \ldots, v_{i_k}\} \subset \{v_1, \ldots, v_j\}$ such that $v_{i_1}, \ldots, v_{i_k}$ are the vertices of another cycle.[1]

[1]Since here we allow multiple edges, an edge is no longer determined by its two ending vertices, so that is why we also include the edges in the cycle.

2. A *cactus* is a connected graph in which every edge $e \in E_G$ belongs to at most one simple cycle. We say that e is *rigid* if it belongs to a simple cycle, and we say that e is *flexible* if it does not belong to a simple cycle. The term cactus has already appeared in Traffic Freeness in connection with Free Probability, but our notion is slightly different, see Remark 3.4.16 for details.

3. An *outercycle (or orientation)* of a cactus graph G is a cycle C that passes exactly once through every rigid edge and twice through every flexible edge. Every cactus graph has an outercycle, see Remark 3.4.11 below. An *oriented cactus graph* is a pair (G, C) where G is a cactus graph and C is an outercycle. We denote by $\mathcal{OCG}(n)$ the set of oriented cacti graphs with exactly n edges.

Note: two orientations $C = (v_1, e_1, v_2, e_2, \dots, v_j, e_j)$ and $C' = (v'_1, e'_1, v'_2, e'_2, \dots, v'_j, e'_j)$ of a cactus graph G are considered to be the same if there exists an automorphism g of the graph G such that $g(v_i) = v'_i$ and $g(e_i) = e'_i$ for $i = 1, \dots, j$.

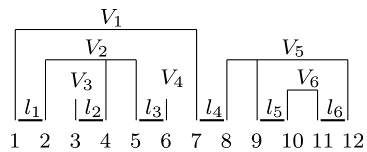
Now let us see what kind of graph $\Gamma(\pi)$ can be.

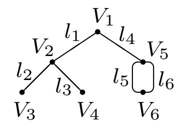
Proposition 3.4.2. *Let $\pi \in \mathcal{NC}(2n)$ be such that $\Gamma(\pi)$ is connected. Then, $\Gamma(\pi)$ is a cactus graph.*

In particular, the set $\{\Gamma(\pi) | \pi \in \mathcal{X}(2n)\}$ consists of bipartite cacti graphs. We next want to study the size of $\{\pi \in \mathcal{X}(2n) : \Gamma(\pi) = G\}$ for a given cactus graph G. This can be done effectively, once we have an outercycle.

Notation and Remark 3.4.3. Consider a partition $\pi \in \mathcal{NC}(2n)$ such that $\Gamma(\pi)$ is connected. It is possible to construct a canonical outercycle C_π of the cactus $\Gamma(\pi)$. This outercycle starts in the block V_1 of π that contains the number 1 and 'goes around' $\Gamma(\pi)$ in counterclockwise direction, it is concisely defined as follows. Let S_π be the permutation of the symmetric group in $2n$ elements whose orbits are the blocks of π, where the elements of each block are run in increasing order. Namely, if $V = \{y_1 < y_2 < \dots < y_r\}$ is a block of π then $S_\pi(y_i) = y_{i+1}$ for $i = 1, \dots, r$ (considering $r + 1 = 1$). Also, let $\tau := (1, 2)(3, 4) \dots (2n-1, 2n)$ be the permutation that transposes pairs of consecutive elements. Then consider the orbit of 1 in $S_\pi \circ \tau$, and denote it as $x_1, x_2, \dots, x_j$ with $x_1 = 1$. If we let v_i be the block containing the element x_i, and e_i be the edge joining the blocks containing elements x_i and $\tau(x_i)$, then the outercycle is given by $C_\pi = (v_1, e_1, v_2, e_2, \dots, v_j, e_j)$.

Example 3.4.4. We now explain how to obtain the canonical outercycle C_π of the graph $\Gamma(\pi)$ associated to the partition π from the first part of Example 3.1.2:





To the left we have the partition π with 6 extra segments l_i that remind us of the pairs of consecutive elements $2i-1$ and $2i$ that will form the edges of the graph. Notice that $\Gamma(\pi)$, which is depicted to the right, is indeed a cactus graph. It has four flexible edges l_1, l_2, l_3, l_4, and two rigid edges l_5, l_6 coming from a multiple edge that forms a simple cycle of size 2. For this partition π, we have that $S_\pi = (1,7)(2,4,5)(3)(6)(8,9,12)(10,11)$ and if we alternately apply τ and S_π to 1 we get

$$1,2,\ 4,3,\ 3,4,\ 5,6,\ 6,5,\ 2,1,\ 7,8,\ 9,10,\ 11,12,\ 8,7,\ 1,2,\dots \tag{3.20}$$

Therefore, the orbit of 1 in $S_\pi \circ \tau$ is $1,4,3,5,6,2,7,9,11,8$, and the canonical outercycle is

$$C_\pi = (V_1, l_1, V_2, l_2, V_3, l_2, V_2, l_3, V_4, l_3, V_2, l_1, V_4, l_4, V_5, l_5, V_6, l_6, V_5, l_4).$$

There is a simple way to obtain C_π from the drawing of π together with the six segments. Imagine that the lines of the diagram represent the walls of a house seen from above, and that we are standing between 1 and 2, touching the wall l_1 from the south (from below) with the left hand. Then we start walking around the house always touching the walls with the left hand. If we record in order the blocks V_i and lines l_j that we pass when going around, we end up with C_π. Moreover, if we record the numbers we encounter while going around we get the sequence (3.20). The first 1 and 2 (or l_1) we encounter is our starting point (to the south of the wall). The second time we reach l_1, we are on the other side of the wall (to the north) of our starting point, the last time we reach 1 and 2 we are back in our starting point. The reason why we did not get 10 and 12 in the orbit, is because there is a 'secret' chamber enclosed by V_5, V_6, l_5 and l_6. This chamber corresponds to the unique simple cycle of $\Gamma(\pi)$. Notice that in the planar drawing of $\Gamma(\pi)$ this amounts for going around the outer (unbounded) face in counterclockwise direction. So C_π is indeed a cycle that surrounds $\Gamma(\pi)$. Furthermore, notice C_π passes once through every rigid edge and twice through every flexible edge. We can get other outercycles of $\Gamma(\pi)$ by cyclically permuting the entries of C_π, but these are not considered the canonical cycle corresponding to π.

We now present a concise formula for the size of the preimage of an oriented cactus graph.

Proposition 3.4.5. *Let $(G, C) \in \mathcal{OCG}(n)$ with $C = (v_1, e_1, v_2, \ldots, v_j, e_j)$ and denote by f_C the number of flexible edges in (G, C) distinct from e_1. Namely, if e_1 is rigid, then f_C is simply the number of flexible edges in G and if e_1 is flexible, f_C is the number of flexible edges of G minus 1. Then,*

$$|\{\pi \in \mathcal{NC}(2n) : (\Gamma(\pi), C_\pi) = (G, C)\}| = 2^{f_C}.$$

In other words, there are exactly 2^{f_C} partitions $\pi \in \mathcal{NC}(2n)$ such that $\Gamma(\pi) = G$ and whose canonical cycle is C.

The previous results allow us to rewrite Theorem 3.1.4 just in terms of graphs.

Theorem 3.4.6. *Consider two free random variables a and b, and let $(r_n(a))_{n\geq 1}$, $(r_n(b))_{n\geq 1}$ and $(r_n(ab + ba))_{n\geq 1}$ be the sequence of free cumulants of a, b and $ab + ba$, respectively. Then, for every $n \geq 1$ one has:*

$$r_n(ab + ba) = \sum_{\substack{(G,C)\in\mathcal{OCG}(n) \\ G \text{ is bipartite}}} 2^{f_C} \left(\prod_{v\in V'_G} r_{d(v)}(a) \prod_{w\in V''_G} r_{d(w)}(b) + \prod_{v\in V'_G} r_{d(v)}(b) \prod_{w\in V''_G} r_{d(w)}(a) \right), \tag{3.21}$$

where f_C is the number of flexible edges in (G, C) without counting the first edge of C; (V'_G, V''_G) is the unique bipartition of the vertices V_G such that V'_G contains the first vertex of C; and $d(v)$ is the degree of the vertex v in G.

Again, if a and b have the same distribution the formula simplifies.

Corollary 3.4.7. *Consider two free random variables, a and b, with the same distribution, and let us denote $\lambda_n := r_n(a) = r_n(b)$. Then, one has*

$$r_n(ab + ba) = \sum_{\substack{(G,C)\in\mathcal{OCG}(n) \\ G \text{ is bipartite}}} 2^{f_C+1} \prod_{v\in V_G} \lambda_{d(v)}, \qquad \forall n \in \mathbb{N}. \tag{3.22}$$

Another application of Theorem 3.4.6 concerns the free anti-commutator with a semicircular element.

Proposition 3.4.8. *Consider two free random variables a and s, where s is semicircular. This means that $r_2(s) = 1$ and $r_n(s) = 0$ for every $n \neq 2$, and let us denote $\lambda_n := r_n(a)$. Then, for odd $m \in \mathbb{N}$ one has $r_m(as + sa) = 0$, while for even $m := 2n$, one has*

$$r_{2n}(as + sa) = \sum_{(G,C)\in\mathcal{OCG}(n)} 2^{g_C+1} \prod_{v\in V_G} \lambda_{d(v)}, \tag{3.23}$$

where if the first edge, e_1, of C is flexible then $g_C := 2f_C + 1$, while if the first edge of C is rigid then $g_C := 2f_C$.

Notice that Equation (3.22) is similar to Equation (3.23), except that the former is restricted to bipartite graphs and the exponents of 2 differ. Compare this to the analogue equations where we pick cumulants of products instead of anti-commutators. Namely, if a, b, s are free random variables, with a, b having the same distribution and s being semicircular, then it follows from Equation (2.17) that $r_m(as) = 0$ for m odd and $r_{2n}(as) = r_n(ab)$ for every $n \in \mathbb{N}$. Thus, for the cumulants of products these two formulas coincide.

3.4.2 Proof of the formula in terms of cacti graphs

Through this section it is useful to consider a directed version of the graph $\Gamma(\pi)$ from Definition 3.1.1.

Definition 3.4.9. Given a $\pi \in \mathcal{NC}(2n)$, we denote by $\overrightarrow{\Gamma(\pi)}$ the directed graph with vertices given by the blocks of π and n edges, where for the direction of the k-th edge, we simply select the block containing element $2k-1$ as the outgoing vertex and the block containing element $2k$ as the ingoing vertex.

For a directed graph $\overrightarrow{G}$, we use the notation $\overrightarrow{(v, w)}$ to mean that v is the outgoing vertex and w the ingoing vertex. We say that a cycle $v_1, \dots, v_j$ is *oriented* if all its edges are in the same direction, namely $\overrightarrow{(v_i, v_{i+1})}$ for $i = 1, \dots, j$.

We begin by proving Proposition 3.4.2, which asserts that graphs $\Gamma(\pi)$ coming from a non-crossing partition π must be cacti. Moreover, we notice that the cycles of $\overrightarrow{\Gamma(\pi)}$ are oriented.

Proof of Proposition 3.4.2. Let π be a non-crossing partition and take a block $V \in \pi$. The other blocks $W \in \pi$ can be naturally separated in two groups. Either $W \prec V$, namely W is nested inside of V (see Notation 2.2.2), otherwise we say W is outside V. Let us denote

$$\mathcal{N}_V := \{W \in \pi : W \prec V\} \qquad \text{and} \qquad \mathcal{O}_V := \pi \backslash (\mathcal{N}_V \cup \{V\}).$$

Notice that there is no edge in $\Gamma(\pi)$ that connects a vertex from $\mathcal{N}_V$ with a vertex from $\mathcal{O}_V$. Indeed, for the sake of contradiction assume that $Y \in \mathcal{N}_V$, $W \in \mathcal{O}_V$ and the edge $e_i = (2i-1, 2i)$ connects Y and W, this means that Y and W have elements that differ by 1, but this is impossible as $Y \subset \{\min(V)+1,\dots,\max(V)-1\}$ while $W \subset \{1,\dots \min(v)-1\} \cup \{\max(V)+1,\dots,2n\}$, a contradiction.

If we consider a simple cycle $V_1,\dots,V_j$ of $\Gamma(\pi)$, the previous observation yields that for $i=1,\dots,j$ if we take the vertex V_i, then the other vertices $V_1,\dots,V_{i-1},V_{i+1},\dots,V_j$ all belong either to $\mathcal{N}_{V_i}$ in which case we say V_i is principal, or else they all belong to $\mathcal{O}_{V_i}$. It is easy to check that there must be at least one principal block, otherwise we can order the blocks $V_1 < V_2 < \cdots < V_j$ in such a way that $\max(V_i)+1 = \min(V_{i+1})$ for $i = 1,\dots j-1$. But we also need that $\max(V_j)+1 = \min(V_1)$ which is impossible. Moreover, if a vertex V is principal, then the other blocks are nested inside it and no other vertex can be principal, so in every cycle there is exactly one principal vertex. Assume without loss of generality that V_1 is the principal vertex, this forces that $\max(V_i)+1 = \min(V_{i+1})$ for $i = 2,\dots j-1$, and that $\min(V_2)-1$ and $\max(V_j)+1$ are two consecutive elements of V_1. Notice that if we consider the directed version $\overrightarrow{\Gamma(\pi)}$, this implies that the cycle is oriented, thus every simple cycle of $\overrightarrow{\Gamma(\pi)}$ must be oriented.

Observe that a principal vertex V is a block of the partition π, thus its elements $V = \{x_1,\dots,x_r\}$ naturally leave $r-1$ pockets $P_s := \{x_s+1,\dots x_{s+1}-1\}$ for $s=1,\dots,r-1$. Then, if an edge (W,Y) is on a simple cycle with principal vertex V, this means that both blocks are contained on the same pocket, say P_i. Moreover, the only possible cycle within P_i that has V_1 as principal vertex must consist of those blocks covered by V in P_i (namely, the outer blocks of π_{P_i}). This implies that given a vertex V and an edge (W,Y) of a graph $\Gamma(\pi)$, there is at most one simple cycle that has V as principal vertex and (W,Y) as an edge.

To conclude that every $\Gamma(\pi)$ is a cactus we proceed by contradiction. Assume that the edge (Y,W) belongs to two different simple cycles s_1 and s_2, and consider the principal vertex V_1 and V_2 of each cycle. Then V_1 and V_2 both cover Y (and W) so $V_1 = V_2$, but this further implies that $s_1 = s_2$. □

Remark 3.4.10. Recall that for a planar graph G, Euler's formula states that $v-e+f=2$, where $v := |V_G|$, $e := |E_G|$ and f are the number of vertices, edges and faces, respectively, of the graph G. This formula has a very nice interpretation in terms of partitions. Consider a partition $\sigma \in \mathcal{Y}(2n)$, its Kreweras complement $\pi := \mathrm{Kr}(\sigma) \in \mathcal{X}(2n)$ and the graph $\Gamma(\pi)$. First, the vertices of $\Gamma(\pi)$ are the blocks of π, so $v = |\pi|$. Secondly, we have a fixed number of edges n, so $e = n$. Finally, we can write σ as $\{B_1, B_3,\dots,B_{2n-1},E_1,\dots,E_r\}$, and we noticed in Section 3.1 that each of the even blocks E_j corresponds to a simple cycle of $\Gamma(\pi)$, or equivalently, to an inner face of its planar representation. If we add the outer face we get that $f = r+1 = |\sigma| - n + 1$. Thus, Euler's formula asserts that $|\pi| - n + |\sigma| - n + 1 = 2$ or equivalenty that $|\pi| + |\sigma| = 2n+1$. But this is a well known fact of the Kreweras complement. Thus, in this case the formula satisfied by the Kreweras complement, $|\sigma| + |Kr(\sigma)| = 2n + 1$ is just a recast of Euler's formula.

Next we study the size of $\{\pi \in \mathcal{NC}(2n) : \Gamma(\pi) = G\}$ for a given cactus graph G. We begin by explaining why every cactus graph G has at least one outercycle C (possibly several).

Remark 3.4.11. Cacti graphs are outerplanar, that is, they admit a planar representation where all vertices belong to the outer (unbounded) face. In such a planar representation, each simple cycle corresponds to an inner face, and for every edge $e \in E_G$ there are exactly two faces that have e as an edge. If e is rigid, one of the faces is the simple cycle it belongs to, while the other must be the outer face. On the other hand, if e is flexible, the 'two' faces turn out to be the same face, which has to be the outer face. Thus, if we fix an edge and start moving around the contour of the outer face in counter-clockwise direction, we will obtain a cycle $C = (v_1, e_1, v_2, e_2, \dots, v_j, e_j)$, that passes once through every rigid edge and twice through every flexible edge, thus C is an outercycle. Notice that the outercycle completely determines the planar representation of G, thus it determines G, as we can retrieve the graph by drawing the edges in order in counter-clockwise direction, being careful that we may need to return to every vertex several times. The number of outercycles of each cactus G depends on the number of planar representations. Furthermore, for each planar representation we may have various outercycles, depending on which edge we start, and some of these outercycles may turn out to be the same, if there is an automorphism of G sending one to another.

After this discussion, we are ready to prove Proposition 3.4.5, which asserts $\#\{\pi \in \mathcal{NC}(2n) : (\Gamma(\pi), C_\pi) = (G, C)\} = 2^{f_C}$, where f_C is the number flexible edges without counting the first edge of C.

Proof of Proposition 3.4.5. The main idea is that if we consider a $\pi \in \mathcal{NC}(2n)$, then we already observed that $\overrightarrow{\Gamma(\pi)}$ is a cactus whose all simple cycles are oriented. Moreover, C_π was constructed in a way that when a rigid edge $e_i = (v_1, v_{i+1})$ appears in C_π it is pointing to the right, to remember this, we use the notation $\overrightarrow{e_i}$. Thus in order to go from the pair $(\overrightarrow{\Gamma(\pi)}, C_\pi)$ to $(\Gamma(\pi), C_\pi)$ we just need to forget the direction of all the edges, except $\overrightarrow{e_1}$ which is oriented to the right by construction. Then, to get the preimage we have a pair (G, C) and want to reconstruct $(\vec{G}_\pi, C_\pi)$, but this amounts to remembering (choosing) the direction of the flexible edges different from e_1, since the directions of the rigid edges and of e_1 are determined by C. And this can be done in 2^{f_C} different ways. The detailed proof consists of two main steps:

First, we transform the undirected graph G into a directed graph $\overrightarrow{G}$. For this, we pick a direction for the edges E_G. We will do this by assigning a direction to the edges $e_1, \dots, e_j$ of C. For $e_r = (v_r, v_{r+1})$ we use the notation $\overrightarrow{e_i} = \overrightarrow{(v_r, v_{r+1})}$ or $\overleftarrow{e_i} = \overleftarrow{(v_r, v_{r+1})}$. We begin by assigning the simplest direction, that is $\overrightarrow{e_1}, \dots, \overrightarrow{e_j}$. Since every rigid vertex appears once in C, this direction is well defined for these edges. Moreover, since we draw C in counter-clockwise direction, this implies that every simple cycle will be also oriented in counter-clockwise direction. On the other hand, flexible edges appear twice in C, and actually the direction we assigned is always inconsistent. For instance if the flexible edge e appears in C as $e_r = (v_r, v_{r+1})$ and as $e_s = (v_s, v_{s+1})$ for $1 \leq r < s \leq j$, then we must have that $v_{r+1} = v_s$ and $v_r = v_{s+1}$. This is because e is not in a simple cycle, and thus if we remove e from G we disconnect the graph, (separating v_r from v_{r+1}) and since $v_{r+1}, v_{r+2}, \dots, v_s$ is a path in the disconnected graph, this implies that $v_{r+1} = v_s$ and thus $v_r = v_{s+1}$. Thus we either need to assign $\overrightarrow{e_r}$ and $\overleftarrow{e_s}$ or take $\overleftarrow{e_r}$ and $\overrightarrow{e_s}$. If e_1 is a flexible edge, we always pick $\overrightarrow{e_1}$, since C_π starts with the block containing element 1, and goes to the block containing element 2. For any other flexible edge, we can go for any of the two choices. After this procedure, in our sequence $\overrightarrow{e_1}, \overrightarrow{e_2}, \overleftarrow{e_3}, \dots, \overrightarrow{e_j}$ of edges of C we end up with n of them pointing to the right, one for each rigid edge and one for each flexible edge. Meanwhile, the remaining $j - n$ edges of C, one for each flexible edge, are pointing to the left. Choosing a direction for each flexible edge different from e_1 can be done in two ways, so in total we have 2^{f_C} possible directed graphs $\overrightarrow{G}$ obtained from G.

The second part of the proof consists on observing that each directed oriented cactus $\overrightarrow{G}$ determines a unique partition $\pi \in \mathcal{NC}(n)$ such that $(\Gamma(\pi), C_\pi) = (G, C)$. To retrieve the partition we consider in order the n edges of C that are pointing to the right: $\overrightarrow{e_{i_1}}, \overrightarrow{e_{i_2}}, \dots, \overrightarrow{e_{i_n}}$ (with $1 = i_1 < i_2 < \cdots < i_n \leq j$). Then, we set $\overrightarrow{l_r} := \overrightarrow{e_{i_r}}$ to be the edge going from the block containing $2r-1$ to the block containing $2r$. Namely, if $\overrightarrow{l_r} = \overrightarrow{(u, v)}$ this means that $u \ni 2r - 1$ and $v \ni 2r$. Notice that this uniquely determines a non-crossing partition. This is because if we have a vertex V with outcoming edges $l_{k_1}, \dots, l_{k_r}$ and incoming edges $l_{m_1}, \dots, l_{m_s}$, this implies that the block V has elements $2k_1 - 1, \dots, 2k_r - 1$ and $2m_1, \dots, 2m_s$. $\square$

Remark 3.4.12. Notice that as a corollary of the result we just proved, we can give a new proof of Proposition 3.2.6, which states that $|\mathcal{Y}^{(0)}(2n)| = 2^{n-1}\mathrm{Cat}_n$. Recall that $\mathcal{Y}^{(0)}(2n)$ consist of those partitions of the form $\sigma = \{B_1, B_3, \dots, B_{2n-1}\}$ that separate odd elements and do not have 'even' blocks. But in terms of the graph of its Kreweras complement $\pi := \mathrm{Kr}(\sigma)$, this means that $\Gamma(\pi)$ is a cactus graph with no cycles. This is better known as a tree. Let OTG_n denote the set of oriented tree graphs, which are simply those oriented cacti graphs $(G, C) \in \mathcal{OCG}(n)$ such that G is a tree. Notice that OTG_n is in bijection with planar rooted trees with n edges, which are counted by the Catalan number Cat_n. Indeed, given a planar rooted tree, we take G to be the tree itself, and for the orientation C we start in the root and go around in counter-clockwise direction. Moreover, in a tree all the edges are flexible, so $f_C = n-1$ (as the first edge of the cycle do not counts). Thus, we can conclude that $|\mathcal{Y}^{(0)}(2n)| = |\{\pi \in \mathcal{NC}(2n) : (\Gamma(\pi), C_\pi) \in OTG_n\}| = 2^{n-1}\mathrm{Cat}_n$.

We can also give a new proof of Proposition 3.2.7, which asserts that

$$|\mathcal{Y}^{(k)}(4k)| = \mathrm{Cat}_k \qquad \text{and} \qquad |\mathcal{Y}^{(k)}(4k+2)| = (2k+1)\mathrm{Cat}_{k+1}, \qquad \forall k \in \mathbb{N}.$$

In this case we look for cacti graphs G with the largest amount of cycles, in the even case when the cactus has $2k$ edges, the largest amount of cycles can be k, all of which should have size 2, and all edges must be rigid. But cycles of size two are just two edges joining the same two vertices. And if we 'thin' the cactus by identifying the edges in each pair, we end up with a tree on k edges. Each orientation of the resulting tree gives one possible orientation of the associated cactus and we directly conclude that $|\mathcal{Y}^{(k)}(4k)| = \mathrm{Cat}_k$. In the odd case, when the cactus has $2k+1$ edges, the largest amount of cycles can be k, all of which should have size 2. In this case, exactly one of the edges is flexible, we call it f. We can flatten the cactus again, keeping track of the flexible edge, and we obtain a tree on $k+1$ edges with an special edge f. Each orientation of the tree gives an orientation of the cactus, and in one of the orientations f is the first edge and thus $f_C = 1-1 = 0$. However, in the other k orientations, f is not the first edge and $f_C = 1$. Thus, each oriented tree on $k+1$ edges, was obtained from $2k+1$ cacti and we conclude $|\mathcal{Y}^{(k)}(4k+2)| = (2k+1)\mathrm{Cat}_{k+1}$.

We now proceed to prove Theorem 3.4.6, which is a purely graph theoretic formula for the anti-commutator, where the sum runs over bipartite cacti graphs.

Proof of Theorem 3.4.6. Recall that in Theorem 3.1.4 we found the following formula for the anti-commutator:

$$r_n(ab+ba) = \sum_{\substack{\pi \in \mathcal{X}(2n) \\ \pi = \pi' \sqcup \pi''}} \left(\prod_{V \in \pi'} r_{|V|}(a) \prod_{W \in \pi''} r_{|W|}(b) + \prod_{V \in \pi'} r_{|V|}(b) \prod_{W \in \pi''} r_{|W|}(a) \right). \tag{3.24}$$

From Proposition 3.4.2 we know that $\Gamma(\pi)$ is always a cactus, and we already saw in Section 3.1 that $\Gamma(\pi)$ must be bipartite. In other words, all its simple cycles are of even size. So we can break the sum depending on which bipartite cactus graph $\Gamma(\pi)$ is formed by π, moreover we consider the canonical orientation C_π. Then, $r_n(ab+ba)$ is expressed as

$$\sum_{\substack{(G,C) \in \mathcal{OCG}(n) \\ G \text{ is bipartite}}} \sum_{\substack{\pi \in \mathcal{X}(2n) \\ (\Gamma(\pi), C_\pi) = (G,C) \\ \pi = \pi' \sqcup \pi''}} \left(\prod_{V \in \pi'} r_{|V|}(a) \prod_{W \in \pi''} r_{|W|}(b) + \prod_{V \in \pi'} r_{|V|}(b) \prod_{W \in \pi''} r_{|W|}(a) \right). \tag{3.25}$$

But we know that $(\pi', \pi'') = (V'_G, V''_G)$ is the unique bipartition of G. And we can observe that for $V \in \pi$ the size of the block $|V|$ corresponds to the degree of V as a vertex of G. Thus, once we fix a graph G, no matter which partition it came from, the term of the sum is given by:

$$\prod_{v \in V'_G} r_{|v|}(a) \prod_{w \in V''_G} r_{|w|}(b) + \prod_{v \in V'_G} r_{|v|}(b) \prod_{w \in V''_G} r_{|w|}(a).$$

Moreover, from Proposition 3.4.5 we know that $\#\{\pi \in \mathcal{NC}(2n) : (\Gamma(\pi), C_\pi) = (G, C)\} = 2^{f_C}$, and this allows us to conclude that

$$r_n(ab + ba) = \sum_{\substack{(G,C) \in \mathcal{OCG}(n) \\ G \text{ is bipartite}}} 2^{f_C} \left(\prod_{v \in V'_G} r_{|v|}(a) \prod_{w \in V''_G} r_{|w|}(b) + \prod_{v \in V'_G} r_{|v|}(b) \prod_{w \in V''_G} r_{|w|}(a) \right).$$

□

A direct application of Theorem 3.4.6 is Corollary 3.4.7 concerning the case where the variables have the same distribution. Another interesting application is Proposition 3.4.8 which studies the case when one variable is semicircular. We present the proof of the latter as it is not straightforward as the former.

Proof of Proposition 3.4.8. Recall that Equation (3.21) asserts that for every $m \in \mathbb{N}$

$$r_m(as + sa) = \sum_{\substack{(G,C) \in \mathcal{OCG}(m) \\ G \text{ is bipartite}}} 2^{f_C} \left(\prod_{v \in V'_G} r_{d(v)}(a) \prod_{w \in V''_G} r_{d(w)}(s) + \prod_{v \in V'_G} r_{d(v)}(s) \prod_{w \in V''_G} r_{d(w)}(a) \right).$$

Recall that $r_{d(v)}(s) = 0$ unless $d(v) = 2$, so the whole product will vanish unless $d(v) = 2$ for all V'_G. Let us say that V'_G and V''_G are 2-regular if all its vertices have degree 2 in G. This means that our formula simplifies to

$$r_m(as + sa) = \sum_{\substack{(G,C) \in \mathcal{OCG}(m) \\ G \text{ is bipartite} \\ V'_G \text{ is 2-regular}}} 2^{f_C} \prod_{w \in V''_G} r_{d(w)}(a) + \sum_{\substack{(G,C) \in \mathcal{OCG}(2n) \\ G \text{ is bipartite} \\ V''_G \text{ is 2-regular}}} 2^{f_C} \prod_{v \in V'_G} r_{d(v)}(a).$$

Notice that since (V'_G, V''_G) is a bipartition of the graph G which has m edges, this implies that $\sum_{v \in V'_G} d(v) = m = \sum_{v \in V''_G} d(v)$. But, if V'_G or V''_G are 2-regular this forces m to be even. Thus, the product vanishes whenever m is odd and we get $r_m(as + sa) = 0$. For the even case $m = 2n$, let us focus on the second summation. Notice that given a bipartite graph $(G,C) \in \mathcal{OCG}(2n)$ such that V''_G is 2-regular we can construct a graph $(G',C') \in \mathcal{OCG}(n)$, by 'erasing' all the vertices $w \in V''_G$ but keeping track of the 2 edges ending on w, say $e = (u,w) \in E_G$ and $f = (w,v) \in E_G$, by merging them into the edge $g = (u,v) \in E_{G'}$. Formally, we let $V_{G'} = V'_G$, and draw and edge between $u, v \in V'_G$ if there exist a vertex $w \in V''_G$ such that $(u,w),(w,v) \in E_G$ and such that $(u,w) \neq (v,w)$.

Given an outercycle $C = (v_1, e_1, w_1, f_1, v_2, e_2, w_2, f_2, \ldots, w_j, f_j)$ of G, where $v_i \in V'_G$ and $w_i \in V''_G$ for $i = 1, \ldots, j$, there is a natural way to construct an outercycle $C' = (v_1, g_1, v_2, g_2, \ldots, v_j, g_j)$ of G', by letting $g_i \in E_{G'}$ be the edge obtained by merging edges e_i and f_i of G. Notice that for each $w \in V''_G$, either both (u,w) and (w,v) are flexible edges of G and thus (u,v) is flexible edge of G' or both are rigid edges of G and (u,v) is a rigid edge of G'. This implies that that the flexible edges of G' are half the flexible edges of G, thus we have that if e_1 is flexible, then $f_C = 2f_{C'} + 1 = g_{C'}$, and if e_1 is rigid, then $f_C = 2f_{C'} = g_{C'}$.

The previous procedure is actually a bijection between $\mathcal{OCG}(n)$ and

$$\{(G,C) \in \mathcal{OCG}(2n) : G \text{ is bipartite}, V''_G \text{ is 2-regular}\}.$$

This means that we have

$$\sum_{\substack{(G,C)\in\mathcal{OCG}(2n)\\ G \text{ is bipartite}\\ V''_G \text{ is 2-regular}}} 2^{f_C} \prod_{v\in V_{G'}} r_{d(v)}(a) = \sum_{(G',C')\in\mathcal{OCG}(n)} 2^{g_{C'}} \prod_{v\in V_{G'}} r_{d(v)}(a) \tag{3.26}$$

With a similar, but slightly more involved procedure, we can biject $\mathcal{OCG}(n)$ with

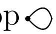
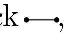
$$\{(G,C) \in \mathcal{OCG}(2n) : G \text{ is bipartite}, V'_G \text{ is 2-regular}\},$$

and we again obtain the right-hand side of (3.26) when on the left-hand side we take V'_G to be 2-regular instead of V''_G. Adding both cases we conclude Equation (3.23). $\square$

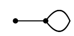
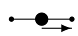
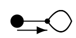
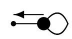
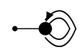
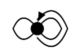
Example 3.4.13. We will compute $r_m(as + sa)$ for small values of m, in the case where a, s are free, s has semicircular distribution and a has Marchenko-Pastur distribution. This case is discussed from an analytical point of view in Example 5.1 of [BMS17] and in Example 6.17 of [HMS18]. Recall that $r_j(a) = 1$ for all $j \in \mathbb{N}$, then Proposition 3.4.8 asserts that the odd cumulants are 0 and that

$$r_{2n}(as + sa) = \sum_{(G,C)\in\mathcal{OCG}(n)} 2^{g_C+1}.$$

Thus, we will analyze the small oriented cacti graphs, and the value of g_C for each orientation.

- For $n = 1$, there are two cacti graphs with one edge, the loop , and the stick , each has a unique possible orientation C. For the loop we compute $f_C = 0$ and $g_C = 0$. And for the stick we get $f_C = 0$ and $g_C = 1$. Therefore $r_2(as + sa) = 2^1 + 2^2 = 6$.

- For $n = 2$, there are four cacti graphs and seven oriented cacti graphs. Their values f_C and g_C are summarized in the table below. In the second row, the first vertex of the orientation C is enlarged, and the first edge is indicated with an arrow, the rest of C is obtained by going around G in counterclockwise direction.

G							
(G, C)							
f_C	0	1	1	0	0	1	0
g_C	0	3	3	1	1	2	0

Then, if we add 2^{g_C+1} for each graph we conclude that

$$r_4(as + sa) = 2 + 16 + 16 + 4 + 4 + 8 + 2 = 52.$$

- For the case $n = 3$, there are 10 cacti graphs with 3 edges. And if we also consider the orientations, it can be seen that $|\mathcal{OCG}(3)| = 30$. After computing the value g_C for each orientation, one gets $r_6(as + sa) = 582$.

In the table below we compare the values $r_{2m}(as + sa)$ that we just obtain, with the values of $r_m(ab + ba)$ from Remark 3.3.1, where we assume that b is free from a and both have Marchenko-Pastur distribution.

m	1	2	3
$r_{2m}(as + sa)$	6	52	582
$r_m(ab + ba)$	2	10	52

The rationale behind this comparison is that for the cumulants of the product we have $r_{2m}(as) = r_m(ab)$ for every $m \in \mathbb{N}$. Thus, the cumulants coincide, whereas for the anti-commutator, $r_{2m}(as + sa)$ rapidly starts to grow much faster than $r_m(ab + ba)$.[2] This

[2]The fact that $r_4(as + sa) = 52 = r_3(ab + ba)$ seems to be just a coincidence.

somehow measures the asymmetry that it is naturally introduced when considering the anti-commutator of two variables, instead of just the product.

3.4.3 Rigid cacti and even variables

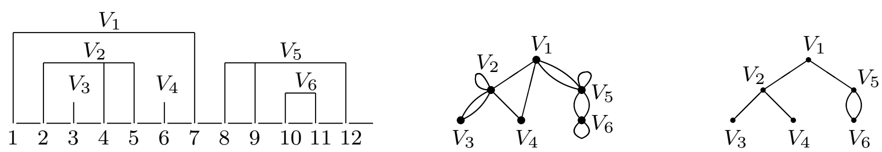
Here we discuss how the even case looks in the cacti graphs approach. This case is particularly interesting as the graphs appearing here are the same that appear in the study of Traffic Freeness.

Definition 3.4.14. We say that a cactus graph G is *rigid*, if all its edges are rigid. Namely, all its edges belong to exactly one simple cycle.

Remark 3.4.15. Observe that a cactus graph is rigid if and only if all its vertices have even degree. Indeed, if it is rigid, then for each vertex v we can count to how many simple cycles it belongs, say c and then its degree has to be $d(v) = 2c$. For the other direction, we assume, for the sake of contradiction, that G is a cactus graph where all its vertices have even degree and it has a flexible edge e. If we consider the graph G' obtained by deleting e from G, then G' has two connected components, let A be one of these components and let v be the unique vertex of A that was an endpoint of e. Then the degree of v in G' is odd (as $d(v)$ in G is even and we removed e), since the sum of the degrees of the vertices in A is even, there must be another vertex u that has odd degree, but then $d(u)$ in G is odd, a contradiction.

Remark 3.4.16. The term 'cactus' has already appeared in the study of Traffic Freeness in connection to Free Probability, see for instance [Mal20], [AM20] and the references therein. In [AM20], a cactus graph has all its edges on a simple cycle, thus it is what we call rigid cactus. What we call cactus, in [AM20] appears under the name of quasi-cactus. Although Au and Male consider graphs with extra structure necessary for handling traffics, some of the underlying combinatorial techniques look similar. The similarities and differences between these two alike objects of study can be explained if we look at how the graphs are constructed from a non-crossing partition. In [AM20], this is done via a quotient graph called C^π, which in our approach can be described as the graph $\Gamma(\pi)$ where n extra edges are drawn, each joining the blocks containing elements $2i$ and $2i+1$ for $i = 1, \ldots, n$. Adding these extra edges forces the cactus graph to be rigid.

In the picture below, to the left we have non-crossing partition, in the middle is the graph C^π as it is constructed in [AM20], notice that it is rigid. For comparison, to the right is our $\Gamma(\pi)$.



The study of the anti-commutator via graphs, in the special case where the variables are even is governed by rigid cacti, as pointed out by the following result.

Corollary 3.4.17. *Consider two free even random variables a and b. Then, for every $n \geq 1$*

$$r_n(ab+ba) = \sum_{\substack{(G,C)\in\mathcal{OCG}(n)\\ G\text{ is bipartite}\\ G\text{ is rigid}}} \left(\prod_{v\in V'_G} r_{d(v)}(a) \prod_{w\in V''_G} r_{d(w)}(b) + \prod_{v\in V'_G} r_{d(v)}(b) \prod_{w\in V''_G} r_{d(w)}(a) \right). \quad (3.27)$$

Proof. We can apply our formula (3.21) for the anti-commutator in terms of graphs. But, since we deal with even variables, we can restrict the sum to graphs with only even degrees, the conclusion follows from Remark 3.4.15. □

Remark 3.4.18. Recall that in Theorem 3.3.7 we already had a formula for the anti-commutator of even variables. The reason why the right-hand side of formula (3.18) is the same as the right-hand side of (3.27), follows from the fact that oriented rigid cacti graphs with $2n$ edges are in bijection with $\mathcal{NC}(n)$. This bijection should be clear from the development in [AM20] and previous work on Traffic Freeness related to rigid cacti graphs. Below we just give an idea on a precise bijection that uses oriented tree graphs.

Given an oriented rigid cactus graph G consider the tree T_G that has vertices given by the vertices and faces (simple cycles) of G, we draw an edge between a vertex v and a face f of G if v is contained in f. It is ease to see that T_G is a tree with n edges, and where the degree of v in T_G is now half the degree of v in G. The outercycle $C = (v_1, e_1, v_2, \dots)$ of G naturally gives an outercycle C' of T_G that starts in v_1 and whenever we had the string (v_i, e_i, v_{i+1}) in C, we replace it by the string $(v_i, e'_i, f_i, e'_{i+1}, v_{i+1})$ of C' that instead of going directly from v_i to v_{i+1} first goes to the face f_i that contains e_i. On the other hand, non-crossing partitions are in bijection with oriented tree graphs as follows. Let $\pi \in \mathcal{NC}(n)$ and consider the partition $\hat{\pi} = \langle \pi, Kr(\pi)\rangle \in \mathcal{NC}(2n)$, then $(G_{\hat{\pi}}, C_{\hat{\pi}})$ is an oriented tree with n edges, (it must be a tree as it has $n+1$ vertices). Moreover, in the bipartite decomposition $\hat{\pi} = \hat{\pi}' \sqcup \hat{\pi}''$ the degrees of the vertices of $\hat{\pi}'$ are exactly the sizes of the blocks of π. Putting together these bijections, gives a combinatorial explanation on why the right-hand sides of (3.18) and (3.27) are the same.

3.5 Generalization to quadratic forms

The graph approach from last section suggests a very natural generalization which is useful to study quadratic forms:

$$a := \sum_{1\le i,j\le k} w_{i,j} a_i a_j,$$

where $a_1, \ldots, a_k$ are free random variables, $w_{ij} \in \mathbb{R}$ and we require $w_{j,i} = w_{i,j}$ for $1 \le i \le j \le k$, so that a is self-adjoint. Even in this generality we just make use of cacti graphs, the difference is that instead of bipartite graphs we need to consider k-colored graphs.

Notation 3.5.1. We will denote by $\mathcal{OCG}^{(k)}(n)$, the set of k-colored oriented cacti graphs with n edges. Namely, we take an oriented cactus $(G, C) \in \mathcal{OCG}(n)$ together with a coloring $\lambda : V_G \to [k]$ of the vertices of G. For $i = 1, \ldots, k$, we denote by $Q_i =: \lambda^{-1}(i) \subset V_G$ the subset of vertices that has color i. Given a sequence of weights $(w_{i,j})_{1\le i,j\le k}$, and a graph $G \in \mathcal{OCG}^{(k)}(n)$ we denote by

$$w_G := \prod_{(v,u)\in E_G} w_{\lambda(v),\lambda(u)}$$

the product of the weights of each edge of the graph G.

Theorem 3.5.2. *Consider free random variables $a_1, \ldots, a_k$ and let*

$$a := \sum_{1\le i,j\le k} w_{i,j} a_i a_j,$$

where $w_{i,j} \in \mathbb{R}$ and $w_{j,i} = w_{i,j}$ for $1 \le i \le j \le k$. Then, the cumulants of a are given by

$$r_n(a) = \sum_{(G,C)\in\mathcal{OCG}^{(k)}(n)} 2^{f_C} w_G \prod_{i=1}^{k} \Big(\prod_{v_i\in Q_i} r_{|v_i|}(a_i) \Big). \tag{3.28}$$

Proof. The approach is to adapt the ideas from Section 3.1, to this more general setting. In most of the cases the generalization is straightforward, so we can directly provide a proof of this result. We will restrict our attention to introducing the new necessary notation and pointing out how we can modify our previous results to fit the new setting.

Consider a $2n$-tuple $\varepsilon = (\varepsilon(1), \ldots, \varepsilon(2n)) \in [k]^{2n}$ with entries $\varepsilon(i)$ in $[k] := \{1, \ldots, k\}$. And for $i \in [k]$, let $A_i(\varepsilon) := \{t \in [2n] : \varepsilon(t) = i\}$ denote the entries of ε that are equal to

i. Then, these sets together, $A(\varepsilon) = \{A_1(\varepsilon), \dots, A_k(\varepsilon)\}$, form an ordered partition of $[2n]$. For every tuple $\varepsilon \in [k]^{2n}$, we denote its weight as

$$w(\varepsilon) := \prod_{r=1}^{n} w_{\varepsilon(2r-1),\varepsilon(2r)}.$$

With this notation, formula (3.4) from Proposition 3.1.10 is easily generalized to

$$r_n(a) = \sum_{\substack{\pi \in \mathcal{NC}(2n) \\ \pi \vee_{\mathcal{NC}} I_{2n} = 1_{2n}}} \sum_{\substack{\varepsilon \in [k]^{2n} \\ A(\varepsilon) \geq \pi}} w(\varepsilon) \prod_{i=1}^{k} \left(\prod_{\substack{V \in \pi, \\ V \subset A_i(\varepsilon)}} r_{|V|}(a) \right). \tag{3.29}$$

Then, if we fix a $\pi \in \mathcal{NC}(2n)$ such that $\pi \vee_{\mathcal{NC}} I_{2n} = 1_{2n}$. From Lemma 3.1.11 we know that $\Gamma(\pi)$ is connected, and proceeding as in Proposition 3.1.13 we further obtain that for every colored graph $\Gamma(\pi) \in \mathcal{OCG}^{(k)}(n)$, we can construct a unique tuple $\varepsilon \in [k]^{2n}$ such that $A(\varepsilon) = (Q_1, \dots, Q_k)$ (as ordered partitions). If replace this in (3.29) we obtain a general version of our main formula

$$r_n(a) = \sum_{\substack{\pi \in \mathcal{NC}(2n), \\ \Gamma(\pi) \text{ is connected}}} w(\varepsilon) \prod_{i=1}^{k} \left(\prod_{\substack{V \in \pi, \\ V \subset A_i(\varepsilon)}} r_{|V|}(a) \right).$$

Notice that we no longer require the graph to be bipartite, as this was a restriction coming from the fact that the anti-commutator is the case $k = 2$ (bicolored) with the extra requirement that $w_{1,1} = w_{2,2} = 0$, which forces the graph to be bipartite. Finally, we can break the sum depending on the oriented cactus graph formed by each π. Proceeding in the same way we did for the proof of Theorem 3.4.6, we conclude the desired formula (3.28). $\square$

As a direct corollary of Theorem 3.5.2 we can restrict to the case where the variables $a_1, \dots, a_k$ are all even. A formula for the cumulants of a quadratic form in this case was already provided by Ejsmont and Lehner in Proposition 4.5 of [EL17] (see also [EL21]). The combinatorial link between their formula and the one presented below is again explained by the bijection between rigid cacti and non-crossing partitions mentioned in Remark 3.4.18.

Corollary 3.5.3. *Consider k free even random variables $a_1, \dots, a_k$ and let a be quadratic form on these variables*

$$a := \sum_{1 \leq i,j \leq k} w_{ij} a_i a_j,$$

where $w_{ij} \in \mathbb{R}$ and $w_{ji} = w_{ij}$ for $1 \le i \le j \le k$. Then, the cumulants of a are given by

$$r_n(a) = \sum_{\substack{(G,C) \in \mathcal{OCG}(n)^{(k)} \\ G \text{ is rigid}}} w_G \prod_{i=1}^{k} \Big(\prod_{v_i \in Q_i} r_{|v_i|}(a_i) \Big). \tag{3.30}$$

To finish this section, we study the quadratic form given by a sum of anti-commutators. As some weights are equal to 0, this implies that the terms for some graphs vanish and the sum to the right-hand side of (3.28) simplifies, and has a nice interpretation as k-partite graphs.

Example 3.5.4. To study the sum of the anti-commutators of k free random variables $a_1, \dots, a_k$, namely

$$a := \sum_{1 \le i < j \le k} a_i a_j + a_j a_i,$$

this amounts to considering the weights $w_{ii} = 0$ for $1 \le i \le k$ and $w_{ij} = 1$ for $1 \le i, j \le k$ with $i \ne j$. Therefore, we can restrict our attention to the subset $\mathcal{ACSG}^{(k)}(n) \subset \mathcal{OCG}^{(k)}(n)$ (standing for *anti-commutator sums graphs*) where the coloring $(Q_1, \dots, Q_k)$ is k-partition of G. Then, the cumulants of a are given by

$$r_n(a) = \sum_{(G,C) \in \mathcal{ACSG}^{(k)}(n)} 2^{f_C} \prod_{i=1}^{k} \Big(\prod_{v_i \in Q_i} r_{|v_i|}(a_i) \Big). \tag{3.31}$$

Notice that if just consider semicircular variables in this formula, then the terms in the sum will vanish for all graphs except when G is the cycle graph of size n. This graph has only one possible orientation C, this cycle is the graph G itself. Moreover G has only rigid edges, so $f_C = 0$. Thus $r_n(a)$ is just the number of k-partitions of the cycle of size n. This retrieves the formulas presented in Section 6.1 of [EL20] (see also Remark 2.6 of [EL21]).

Chapter 4

Multiplicative and semi-multiplicative functions on $\mathcal{NC}$

As mentioned earlier, there are several brands of cumulants in non-commutative probability, including free, Boolean and monotone cumulants, and t-Boolean cumulants that are an interpolation of the first two. A common thing to all these notions of cumulants is that the moment-cumulant formulas can be expressed as a sum over non-crossing partitions, if we allow the use of certain complex coefficients. Denoting by $\{\kappa_n\}_{n\in\mathbb{N}}$, the sequence of cumulants for some arbitrary type of non-commutative independence, then the moment-cumulant formula is of the form

$$\varphi_n = \sum_{\pi\in\mathcal{NC}(n)} \alpha_\pi \prod_{i=1}^{k} \kappa_{|V_i|}, \tag{4.1}$$

where the α_π are complex coefficients indexed by partitions that depend on the notion of independence. For instance, if $\alpha_\pi = 1$ for every non-crossing partitions, we get Speicher's free moment-cumulant formula. If $\alpha_\pi = 1$ for interval partitions, and 0 otherwise, then we are looking at Boolean moment-cumulant formula. To obtain the monotone moment-cumulant formula we take $\alpha_\pi = \frac{1}{\tau(\pi)!}$ for every $\pi \in \mathcal{NC}$, where $\tau(\pi)!$ is the tree factorial of the nesting forest of π, as reviewed in Subsection 2.2.6.

An intriguing fact observed in the recent research literature was that non-commutative cumulants sometimes have applications to other areas of non-commutative probability than the one they were designed for. For instance, Boolean cumulants were successfully used to study the free analogue of infinitely divisible distributions, see e.g. [BN08]. In connection to this, it is natural to ask if there are nice combinatorial formulas to directly transition from one brand of cumulants to another. This goes back to Lehner [Leh02], and was thoroughly pursued in [AHLV15], where explicit relations among classic, Boolean, free and monotone cumulants were studied in detail. In that paper, the authors use different methods to study the different cumulant to cumulant formulas. The objective of this chapter is to study all the interrelations between different brands of cumulants using the same method, in the framework of semi-multiplicative functions over the incidence algebra of $\mathcal{NC}$. The contents of this chapter follow the presentation of the paper [CEFN$^+$21] that I coauthored with Celestino, Ebrahimi-Fard, Nica and Witzman.

Following to a short Section 4.1 which outlines the results, the chapter is roughly divided in three parts:

In the first part, we establish some basic relevant facts concerning the group $(SM^{\mathcal{NC}}, *)$. More precisely: after giving a brief presentation of the results in Section 4.1, we introduce $SM^{\mathcal{NC}}$ in Section 4.2. Then in Section 4.3 (Theorem 4.3.3) we prove that $SM^{\mathcal{NC}}$ is indeed a group under convolution. The review of the smaller group $MF^{\mathcal{NC}}$ and some discussion around the inclusion $MF^{\mathcal{NC}} \subset SM^{\mathcal{NC}}$ appears in Section 4.4.

In the second part, we demonstrate the relevance of $SM^{\mathcal{NC}}$ to the study of non-commutative cumulants. This comes into the picture via a natural action which a function $g \in SM^{\mathcal{NC}}$ has on sequences of multilinear functionals on a non-commutative probability space. This action is presented in Section 4.5. Then in Section 4.6 we look at specific examples of cumulants and, based on them, we identify what it means for $g \in SM^{\mathcal{NC}}$ to encode transitions of "moment-to-cumulant" type or of "cumulant-to-cumulant" type. In Section 4.7, we identify that the functions of "cumulant-to-cumulant" type form a subgroup of $SM^{\mathcal{NC}}$ that has the set of "moment-to-cumulant" functions as a right coset.

In the third part, we study the interpolation between free and Boolean cumulants. Section 4.8 discusses the 1-parameter subgroup $\{u_q \mid q \in \mathbb{R}\} \subseteq SM^{\mathcal{NC}}$ and its applications to t-Boolean cumulants. In Section 4.9 we prove that the u_q's normalize $\mathcal{G}$, and in Section 4.10 we explain how to derive the description of the multiplication of free random variables in terms of t-Boolean cumulants.

4.1 Presentation of the results in this chapter

We begin with an overview of the results that will serve as an introduction to the topic as well as general guideline of the organization of this chapter.

4.1.1 The group $MF^{\mathcal{NC}}$ of unitized multiplicative functions on $\mathcal{NC}$.

The idea of studying the convolution of multiplicative functions defined on the set of all intervals of a "coherent" collection of lattices $(\mathcal{L}_n)_{n=1}^{\infty}$ goes back to the 1960's work of Rota and collaborators, e.g. in [DRS72]. The phenomenon which prompts this study is that, in a number of important examples: for every $\pi \leq \sigma$ in an $\mathcal{L}_n$, the sublattice $[\pi, \sigma] := \{\rho \in \mathcal{L}_n \mid \pi \leq \rho \leq \sigma\}$ of $\mathcal{L}_n$ is canonically isomorphic to a direct product,

$$[\pi, \sigma] \approx \mathcal{L}_1^{p_1} \times \cdots \times \mathcal{L}_n^{p_n}, \quad \text{with } p_1, \ldots, p_n \geq 0. \tag{4.2}$$

A function $f : \sqcup_{n=1}^{\infty}\{(\pi, \sigma) \mid \pi, \sigma \in \mathcal{L}_n,\ \pi \leq \sigma\} \to \mathbb{C}$ is declared to be multiplicative when there exists a sequence of complex numbers $(\alpha_n)_{n=1}^{\infty}$ such that, for π, σ and non-negative integers $p_1, \ldots, p_n$ as in (4.2), one has $f(\pi, \sigma) := \alpha_1^{p_1} \cdots \alpha_n^{p_n}$.

In the present paper we are interested in the case when $\mathcal{L}_n$ is the lattice $\mathcal{NC}(n)$ of non-crossing partitions of $\{1, \ldots, n\}$, endowed with the partial order by reverse refinement. In the 1990's it was found by Speicher [Spe94] that, when considered in connection to the $\mathcal{NC}(n)$'s, the convolution of multiplicative functions plays an essential role in the combinatorial development of free probability. For the purposes of the present paper it is convenient to focus on the set $MF^{\mathcal{NC}}$ consisting of multiplicative functions on the $\mathcal{NC}(n)$'s where the sequence $(\alpha_n)_{n=1}^{\infty}$ defining the function has $\alpha_1 = 1$. Then $MF^{\mathcal{NC}}$ is a group under convolution. While this is a self-standing structure, which can be considered without any knowledge of what is a non-commutative probability space, it nevertheless turns out that the group operation of $MF^{\mathcal{NC}}$ encapsulates the combinatorics of the multiplication of free random variables; for a detailed presentation of how this goes, we refer to Lectures 14 and 18 of the monograph [NS06].

4.1.2 The group $SM^{\mathcal{NC}}$ of unitized semi-multiplicative functions on $\mathcal{NC}$.

In the case of the lattices $\mathcal{NC}(n)$, the canonical isomorphism indicated in (4.2) is obtained by combining two kinds of lattice isomorphisms, as follows.

First kind of isomorphism: one observes that for every $\pi \leq \sigma$ in some $\mathcal{NC}(n)$, the interval $[\pi, \sigma] \subseteq \mathcal{NC}(n)$ is canonically isomorphic to a direct product of intervals of the form $[\theta, 1_k]$, with $\theta \in \mathcal{NC}(k)$ for some $1 \leq k \leq n$, and where 1_k is the maximal element of $\mathcal{NC}(k)$, i.e. it is the partition of $\{1, \ldots, k\}$ into a single block.

Second kind of isomorphism: for every $k \geq 1$ and $\theta \in \mathcal{NC}(k)$ one finds $[\theta, 1_k]$ to be canonically isomorphic to a direct product $\mathcal{NC}(1)^{q_1} \times \cdots \times \mathcal{NC}(k)^{q_k}$, with $q_1, \ldots, q_k \geq 0$.

A precise review of these two kinds of isomorphisms can be found in Remark 2.2.13. Our main point is this:

> It is worth studying convolution for functions $g : \{(\pi, \sigma) \mid \pi, \sigma \in \mathcal{NC}, \pi \leq \sigma\} \to \mathbb{C}$ which are only required to be multiplicative with respect to the *first kind* of isomorphism mentioned above. (4.3)

We will use the term *semi-multiplicative* for a function g as in (4.3), and we will denote

$$SM^{\mathcal{NC}} := \left\{ g : \{(\pi, \sigma) \mid \pi \leq \sigma \text{ in } \mathcal{NC}\} \to \mathbb{C} \;\middle|\; \begin{array}{l} g \text{ is semi-multiplicative and} \\ g(\pi, \pi) = 1, \ \forall\, \pi \in \mathcal{NC} \end{array} \right\}. \tag{4.4}$$

It turns out that $SM^{\mathcal{NC}}$ is a group under convolution. This group and some of its subgroups (in particular the group $MF^{\mathcal{NC}} \subset SM^{\mathcal{NC}}$ from Section 1.1) are the main players in the considerations of the present chapter. The benefits that come from studying $SM^{\mathcal{NC}}$ are presented in the next subsections.

4.1.3 Relations of $SM^{\mathcal{NC}}$ with moments and some brands of cumulants.

Consider now the framework of a non-commutative probability space $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$ and $\varphi \colon \mathcal{A} \to \mathbb{C}$ is a linear functional such that the algebra unit is mapped to one ($\varphi(1_{\mathcal{A}}) = 1$), and look at

$$\mathfrak{M}_{\mathcal{A}} := \{\underline{\psi} \mid \underline{\psi} = (\psi_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}, \text{ where } \psi_n \text{ is an } n\text{-linear functional}\}.$$

In $\mathfrak{M}_{\mathcal{A}}$ we have a special element $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty}$ called *family of moment functionals* of $(\mathcal{A}, \varphi)$, where $\varphi_n : \mathcal{A}^n \to \mathbb{C}$ is defined by putting $\varphi_n(a_1, a_2, \ldots, a_n) := \varphi(a_1 a_2 \cdots a_n)$ for all $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$. Then in $\mathfrak{M}_{\mathcal{A}}$ there also are several families of *cumulant functionals* which relate to $\underline{\varphi}$ via summation formulas over non-crossing partitions, and receive constant attention in the research literature on non-commutative probability: free cumulants, Boolean cumulants, monotone cumulants (see e.g. [AHLV15]). In this chapter we also devote some attention to a continuous interpolation between Boolean and free cumulants, which we refer to as t-Boolean cumulants, and are arising from the work of

Bożejko and Wysoczanski [BW01] – the case $t = 0$ gives Boolean cumulants and the case $t = 1$ gives free cumulants.

The group $SM^{\mathcal{NC}}$ has a natural action on $\mathfrak{M}_{\mathcal{A}}$, which is discussed in detail in Section 6 of the paper. This action captures the transitions between moment functionals and the brands of cumulants mentioned above, and as a consequence it also captures the formulas for direct transitions between two such brands of cumulants. We mention that the study of direct transitions between different brands of cumulants goes back to the work of Lehner [Leh02], and was thoroughly pursued in [AHLV15]. The benefit of using the group $SM^{\mathcal{NC}}$ is that it offers an efficient framework for streamlining calculations related to various moment-cumulant and inter-cumulant formulas.

It is in fact possible to identify precisely some notions of what it means for a function $h \in SM^{\mathcal{NC}}$ to be of *cumulant-to-moment type*, and what it means for a $g \in SM^{\mathcal{NC}}$ to be of *cumulant-to-cumulant* type, this is done in Section 4.6. Denoting

$$SM^{\mathcal{NC}}_{\mathrm{c-m}} = \{h \in SM^{\mathcal{NC}} \mid h \text{ is of cumulant-to-moment type}\} \text{ and}$$

$$SM^{\mathcal{NC}}_{\mathrm{c-c}} = \{g \in SM^{\mathcal{NC}} \mid g \text{ is of cumulant-to-cumulant type}\},$$

we prove in Section 4.7 that $SM^{\mathcal{NC}}_{\mathrm{c-c}}$ is a subgroup of $(SM^{\mathcal{NC}}, *)$, while $SM^{\mathcal{NC}}_{\mathrm{c-m}}$ is a right coset of $SM^{\mathcal{NC}}_{\mathrm{c-c}}$. The latter statement means that we have

$$SM^{\mathcal{NC}}_{\mathrm{c-m}} = SM^{\mathcal{NC}}_{\mathrm{c-c}} * h := \{g * h \mid g \in SM^{\mathcal{NC}}_{\mathrm{c-c}}\}, \tag{4.5}$$

for no matter what $h \in SM^{\mathcal{NC}}_{\mathrm{c-m}}$ we choose to fix. An easy choice is to fix the h which is identically equal to 1; this is indeed a function in $SM^{\mathcal{NC}}_{\mathrm{c-m}}$, and encodes the transition from free cumulants to moment functionals. However, as pointed out in Subsection 4.7.2 of the paper, it seems to be more advantageous (both for writing proofs and for finding applications) if in (4.5) we use a different choice for h, and pick the function which encodes the transition from Boolean cumulants to moments.

4.1.4 The 1-parameter subgroup $\{u_q \mid q \in \mathbb{R}\}$ of $SM^{\mathcal{NC}}_{\mathrm{c-c}}$

The method we use for proving (4.5) draws attention to the subgroup of $SM^{\mathcal{NC}}$ generated by the function which encodes transition between free cumulants and Boolean cumulants. In the notation system used throughout the paper, the latter function is denoted as $g_{\mathrm{fc-bc}}$. The subgroup $\{g^p_{\mathrm{fc-bc}} \mid p \in \mathbb{Z}\} \subseteq SM^{\mathcal{NC}}_{\mathrm{c-c}}$ can be naturally incorporated into a continuous 1-parameter subgroup of $SM^{\mathcal{NC}}_{\mathrm{c-c}}$, which we denote as $\{u_q \mid q \in \mathbb{R}\}$ (thus $u_q = g^q_{\mathrm{fc-bc}}$

for $q \in \mathbb{Z}$). Working with the u_q's nicely streamlines the various formulas involving t-Boolean cumulants, and in particular gives an easy way (cf. Corollary 4.8.5 below) to write the transition formula between s-Boolean cumulants and t-Boolean cumulants for distinct values $s, t \in \mathbb{R}$.

In Section 10 we prove that every u_q belongs to the normalizer of $MF^{\mathcal{NC}}$:

$$\Big(q \in \mathbb{R},\ f \in MF^{\mathcal{NC}} \Big) \ \Rightarrow \ u_q^{-1} * f * u_q \in MF^{\mathcal{NC}}. \tag{4.6}$$

This is a non-trivial fact, as the u_q's are coming from $SM^{\mathcal{NC}}_{\text{c−c}}$, and there is no obvious direct connection between $SM^{\mathcal{NC}}_{\text{c−c}}$ and $MF^{\mathcal{NC}}$ – it is, in any case, easy to check that the intersection $MF^{\mathcal{NC}} \cap SM^{\mathcal{NC}}_{\text{c−c}}$ only contains the unit e of $SM^{\mathcal{NC}}$, while the intersection of $MF^{\mathcal{NC}}$ with the coset $SM^{\mathcal{NC}}_{\text{c−m}}$ only contains the function h which is constantly equal to 1.

4.1.5 Multiplication of free random variables, in terms of t-Boolean cumulants.

The result obtained in (4.6) can be used in order to give a neat explanation of the intriguing fact that the multiplication of freely independent random variables is nicely described in terms of Boolean cumulants (who aren't a priori meant to be related to free probability).

We find it convenient to place the discussion in the more general framework of t-Boolean cumulants. So let us consider a non-commutative probability space $(\mathcal{A}, \varphi)$, let x, y be two freely independent elements of $\mathcal{A}$, and let t be a parameter with values in $\mathbb{R}$. What happens is that the formula describing the t-Boolean cumulants of the product xy in terms of the separate t-Boolean cumulants of x and of y is *one and the same*, no matter what value of t we are using. More precisely: denoting the family of t-Boolean cumulants as $\underline{b}^{(t)} = (b_n^{(t)} : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$, the formula for t-Boolean cumulants of xy says that:

$$b_n^{(t)}(xy, \ldots, xy) = \sum_{\pi \in \mathcal{NC}(n)} b_\pi^{(t)}(x, \ldots, x) \cdot b_{\mathrm{Kr}(\pi)}^{(t)}(y, \ldots, y), \quad \forall\, n \geq 1. \tag{4.7}$$

Equation (4.7) contains some notation that has to be clarified (such as what is the multilinear functional $b_\pi^{(t)} : \mathcal{A}^n \to \mathbb{C}$ associated to a partition $\pi \in \mathcal{NC}(n)$, and the fact that every $\pi \in \mathcal{NC}(n)$ has a complement $\mathrm{Kr}(\pi) \in \mathcal{NC}(n)$). All the necessary notation will be reviewed in the body of the paper; the reason for giving the formula (4.7) at this point is so that we can explain our way of proving it.

Our approach can be summarized as follows. For every $t \in \mathbb{R}$, consider the statement:

$$\text{(Statement } t\text{)} \qquad \left\{ \begin{array}{c} \text{The formula (4.7) holds true for this } t \\ \text{and for any freely independent elements } x, y \text{ in} \\ \text{some non-commutative probability space } (\mathcal{A}, \varphi) \end{array} \right\}.$$

The action by conjugation of the u_q's on multiplicative functions allows us to prove the following fact:

Fact. If there exists a $t_o \in \mathbb{R}$ for which (Statement t_o) is true, then it follows that (Statement t) is true for all $t \in \mathbb{R}$.

But it has been known since the 1990's that (Statement t_o) is true for $t_o = 1$ – this is the very basic description of multiplication of free random variables in terms of free cumulants, cf. [NS06, Theorem 14.4]. The above "Fact" then assures us that (Statement t) is indeed true for all t; in particular, at $t = 0$ we retrieve the result (first found in [BN08] via a direct combinatorial analysis) about how multiplication of free random variables is described in terms of Boolean cumulants.

4.2 Definition of $SM^{\mathcal{NC}}$

4.2.1 Framework of the incidence algebra on $\mathcal{NC}$

Definition 4.2.1. We denote

$$\mathcal{NC}^{(2)} := \sqcup_{n=1}^{\infty} \{(\pi, \sigma) \mid \pi, \sigma \in \mathcal{NC}(n),\ \pi \leq \sigma\}. \tag{4.8}$$

The set of functions from $\mathcal{NC}^{(2)}$ to $\mathbb{C}$ goes under the name of *incidence algebra of non-crossing partitions*. This set of functions carries a natural associative operation of *convolution*, denoted as "$*$", where for any $f, g : \mathcal{NC}^{(2)} \to \mathbb{C}$ and any $\pi \leq \sigma$ in an $\mathcal{NC}(n)$ one puts

$$f * g\,(\pi, \sigma) = \sum_{\substack{\rho \in \mathcal{NC}(n), \\ \pi \leq \rho \leq \sigma}} f(\pi, \rho) \cdot g(\rho, \sigma). \tag{4.9}$$

In the next remark we collect a few relevant facts concerning the above mentioned convolution operation. The reader is referred to [NS06, Lecture 10] (cf. pages 155-158 there) for a more detailed presentation. The general framework of incidence algebras comes from work of Rota and collaborators, e.g. in [DRS72]; a detailed presentation of this appears in Chapter 3 of [Sta12].

Remark 4.2.2. It is easy to verify that the convolution operation "$*$" defined by (4.9) is associative and unital, where the unit is the function $e : \mathcal{NC}^{(2)} \to \mathbb{C}$ given by

$$e(\pi, \sigma) = \begin{cases} 1, & \text{if } \pi = \sigma, \\ 0, & \text{otherwise.} \end{cases} \tag{4.10}$$

For a function $f : \mathcal{NC}^{(2)} \to \mathbb{C}$ one has (see e.g. [NS06, Proposition 10.4]) that

$$\left(\begin{array}{c} f \text{ is invertible} \\ \text{with respect to "}*\text{"} \end{array} \right) \Leftrightarrow \Big(f(\pi, \pi) \neq 0, \quad \forall\, \pi \in \mathcal{NC} \Big). \tag{4.11}$$

Moreover, if f is invertible with respect to "$*$", then upon writing explicitly what it means to have $f * f^{-1}(\pi, \pi) = e(\pi, \pi) = 1$, one immediately sees that the inverse f^{-1} satisfies

$$f^{-1}(\pi, \pi) = \frac{1}{f(\pi, \pi)}, \qquad \forall\, \pi \in \mathcal{NC}. \tag{4.12}$$

A reader who is matrix-inclined may choose to take the point of view that a function $f : \mathcal{NC}^{(2)} \to \mathbb{C}$ is just an upper triangular matrix with rows and columns indexed by $\mathcal{NC}$, and where the values $f(\pi, \sigma)$ appear as certain entries of the matrix. Then the operation "$*$" amounts to matrix multiplication, and the formulas (4.10), (4.11), (4.12) have obvious meanings in that language as well.

Notation and Remark 4.2.3. *(Unitized functions on $\mathcal{NC}^{(2)}$.)* We denote

$$\mathcal{F} := \{ f : \mathcal{NC}^{(2)} \to \mathbb{C} \mid f(\pi, \pi) = 1, \quad \forall\, \pi \in \mathcal{NC} \}. \tag{4.13}$$

The observations made in (4.11), (4.12) show that every $f \in \mathcal{F}$ is invertible under convolution, where the inverse f^{-1} still belongs to $\mathcal{F}$. It is also immediate that if $f, g \in \mathcal{F}$ then $f * g \in \mathcal{F}$, since for every $\pi \in \mathcal{NC}$ the formula defining $f * g\, (\pi, \pi)$ boils down to just $f * g\, (\pi, \pi) = f(\pi, \pi) \cdot g(\pi, \pi) = 1$. Thus $(\mathcal{F}, *)$ is a group.

4.2.2 Unitized semi-multiplicative functions on $\mathcal{NC}^{(2)}$.

We now proceed, as promised, to looking at functions on $\mathcal{NC}^{(2)}$ which are (only) required to be multiplicative with respect to the first kind of isomorphism indicated in Section 4.1.

Definition 4.2.4. We will denote by $SM^{\mathcal{NC}}$ the set of functions $g : \mathcal{NC}^{(2)} \to \mathbb{C}$ which have $g(\pi, \pi) = 1$ for all $\pi \in \mathcal{NC}$ and satisfy the following condition:

$$\left\{ \begin{array}{l} \text{For every } \pi \leq \sigma \text{ in } \mathcal{NC} \text{ one has the factorization} \\ \\ \qquad\qquad g(\pi, \sigma) = \prod_{W \in \sigma} g(\pi_W, 1_{|W|}) \end{array} \right. \tag{4.14}$$

where π_W is restriction of π to W as described in Notation 2.2.8. We will refer to the condition (4.14) by calling it *semi-multiplicativity*, in contrast with the stronger *multiplicativity* condition from the work of Speicher [Spe94], which also considers the second kind of isomorphism reviewed in Remark 2.2.13.

From (4.14) it is obvious that a function $g \in SM^{\mathcal{NC}}$ is completely determined when we know the values $g(\pi, 1_n)$ for all $n \geq 1$ and $\pi \in \mathcal{NC}(n)$. It is hence clear that the map indicated in (4.15) below is injective. This map turns out to also be surjective; it thus identifies $SM^{\mathcal{NC}}$, as a set, with the countable direct product of copies of $\mathbb{C}$ denoted as "$\mathcal{Z}$" in the next proposition.

Proposition 4.2.5. *Let us denote* $\mathcal{Z} := \big\{ \underline{z} \mid \underline{z} : \sqcup_{n=1}^{\infty} NC(n) \setminus \{1_n\} \to \mathbb{C} \big\}$.

1^o *One has a bijection* $SM^{\mathcal{NC}} \ni g \mapsto \underline{z} \in \mathcal{Z}$, *with* $\underline{z}$ *obtained out of* g *by putting*

$$\underline{z}(\pi) = g(\pi, 1_n) \text{ for every } n \geq 1 \text{ and } \pi \in \mathcal{NC}(n) \setminus \{1_n\}. \tag{4.15}$$

2^o *The inverse of the bijection from (4.15) is described as follows. Given a* $\underline{z} \in \mathcal{Z}$, *we "fill in" values* $\underline{z}(1_n) = 1$ *for all* $n \geq 1$, *and then define* $g : \mathcal{NC}^{(2)} \to \mathbb{C}$ *by*

$$g(\pi, \sigma) := \prod_{W \in \sigma} \underline{z}(\pi_W), \quad \forall\, (\pi, \sigma) \in \mathcal{NC}^{(2)}. \tag{4.16}$$

Then $g \in SM^{\mathcal{NC}}$, *and is sent by the map from (4.15) onto the* $\underline{z}$ *we started with.*

Proof. Let $\underline{z} \in \mathcal{Z}$ be given and let $g : \mathcal{NC}^{(2)} \to \mathbb{C}$ be defined as in (4.16). Then (4.15) is satisfied, because it is the special case "$\sigma = 1_n$" of (4.16). Upon combining (4.16) and (4.15), we thus see that g satisfies the factorization condition indicated in (4.14). We have moreover that $g(\pi, \pi) = \prod_{W \in \pi} z(1_{|W|}) = 1, \quad \forall\, \pi \in \mathcal{NC}$, so we conclude that $g \in SM^{\mathcal{NC}}$. Clearly, this g is sent by the map (4.15) into the $\underline{z} \in \mathcal{Z}$ that we started with.

The argument in the preceding paragraph covers at the same time the surjectivity which was left to check in part 1^o of the proposition, and the inverse description in part 2^o. $\square$

Remark 4.2.6. Recall that, in parallel with $1_n \in \mathcal{NC}(n)$, one uses the notation "0_n" for the partition in $\mathcal{NC}(n)$ which has n blocks of cardinality 1. We warn the reader that 0_n and 1_n do not play symmetric roles in the study of $SM^{\mathcal{NC}}$. Indeed, it is immediate that if $g \in SM^{\mathcal{NC}}$ corresponds to a $\underline{z} \in \mathcal{Z}$ in the way described in Proposition 4.2.5, then we have

$$g(0_n, \sigma) = \prod_{W \in \sigma} g(0_{|W|}, 1_{|W|}) = \prod_{W \in \sigma} \underline{z}(0_{|W|}), \quad \forall\, n \geq 1 \text{ and } \sigma \in \mathcal{NC}(n), \tag{4.17}$$

quite different from the Equation (4.15) giving the values $g(\pi, 1_n)$.

4.3 $SM^{\mathcal{NC}}$ is a group under convolution

In this section we prove that $SM^{\mathcal{NC}}$ is a subgroup of the convolution group $(\mathcal{F}, *)$ considered in Remark 4.2.3. We start by observing that the semi-multiplicativity condition (4.14) has an automatic upgrade to a "local" version, shown in the next lemma (where the special case $U = \{1, \ldots, n\}$ retrieves the original definition of semi-multiplicativity).

Lemma 4.3.1. (Local semi-multiplicativity.) *Let $n \geq 1$ and $\pi, \sigma \in \mathcal{NC}(n)$ be such that $\pi \leq \sigma$. Let U be a non-empty subset of $\{1, \ldots, n\}$ which is a union of blocks of σ. For every $g \in SM^{\mathcal{NC}}$ one has:*

$$g(\pi_{_U}, \sigma_{_U}) = \prod_{\substack{W \in \sigma, \\ W \subseteq U}} g(\pi_{_W}, 1_{_{|W|}}), \tag{4.18}$$

where all the restrictions ($\pi_{_U}$ and such) are in the sense of Notation 2.2.8.

Proof. We write explicitly $U = W_1 \cup \cdots \cup W_k$ with blocks $W_1, \ldots, W_k \in \sigma$, and for every $1 \leq i \leq k$ we write $W_i = W_{i,1} \cup \cdots \cup W_{i,p_i}$ with blocks $W_{i,1}, \ldots, W_{i,p_i} \in \pi$. It follows that the restriction $\sigma_{_U}, \pi_{_U} \in \mathcal{NC}(|U|)$ are of the form

$$\sigma_{_U} = \{T_1, \ldots, T_k\} \text{ and } \pi_{_U} = \{T_{1,1}, \ldots, T_{1,p_1}, \ldots, T_{k,1}, \ldots, T_{k,p_k}\},$$

where for every i, j, T_i and $T_{i,j}$ are just the re-labelings of W_i and $W_{i,j}$ to make the partitions of U become partitions of $|U|$. Then we have that $T_i = T_{i,1} \cup \cdots \cup T_{i,p_i} \subseteq \{1, \ldots, |U|\}$ for every $1 \leq i \leq k$, and following the restrictions and re-labelings we get that for every $1 \leq i \leq k$ we have

$$(\pi_{_U})_{T_i} = \pi_{_{W_i}} \text{ (equality of partitions in } \mathcal{NC}(n_i)\text{, with } n_i = |W_i| = |T_i|). \tag{4.19}$$

When applied to the partitions $\pi_{_U} \leq \sigma_{_U}$ in $\mathcal{NC}(|U|)$, the original semi-multiplicativity condition from (4.14) says that

$$g(\pi_{_U}, \sigma_{_U}) = \prod_{i=1}^{k} g\big((\pi_{_U})_{T_i}, 1_{_{|T_i|}} \big).$$

On the right-hand side of the latter equation we replace $(\pi_{_U})_{T_i}$ by $\pi_{_{W_i}}$ and $1_{|T_i|}$ by $1_{|W_i|}$, and the required formula (4.18) follows. ☐

As an application of local semi-multiplicativity, we get the following fact.

Lemma 4.3.2. *Let $n \geq 1$, let $\pi, \rho, \sigma \in \mathcal{NC}(n)$ with $\pi \leq \rho \leq \sigma$, and let $g \in SM^{\mathcal{NC}}$. One has*

$$g(\pi, \rho) = \prod_{U \in \sigma} g(\pi_U, \rho_U). \tag{4.20}$$

Proof. Every block $U \in \sigma$ is a union of blocks of ρ, hence Lemma 4.3.1 can be invoked in connection to this U and the partitions $\pi \leq \rho$ to infer that

$$g(\pi_U, \rho_U) = \prod_{\substack{W \in \rho, \\ W \subseteq U}} g(\pi_W, 1_{|W|}). \tag{4.21}$$

We thus find that

$$\begin{aligned}\prod_{U \in \sigma} g(\pi_U, \rho_U) &= \prod_{U \in \sigma} \Big(\prod_{\substack{W \in \rho, \\ W \subseteq U}} g(\pi_W, 1_{|W|}) \Big) \text{ (by (4.21))} \\ &= \prod_{W \in \rho} g(\pi_W, 1_{|W|}) \Big) \\ &= g(\pi, \rho) \quad \text{(by definition of semi-multiplicativity),}\end{aligned}$$

and the required formula (4.20) is obtained. $\square$

Theorem 4.3.3. *$SM^{\mathcal{NC}}$ is a subgroup of $(\mathcal{F}, *)$.*

Proof. 1^o *We pick two functions $g_1, g_2 \in SM^{\mathcal{NC}}$, and we prove that $g_1 * g_2$ is in $SM^{\mathcal{NC}}$ as well.*

The function $g_1 * g_2 : \mathcal{NC}^{(2)} \to \mathbb{C}$ can be in any case considered as an element of the larger group $\mathcal{F}$. Proposition 4.2.5 gives us a function $g \in SM^{\mathcal{NC}}$ such that

$$g(\pi, 1_n) = g_1 * g_2\,(\pi, 1_n) \;\text{ for every } n \geq 1 \text{ and } \pi \in \mathcal{NC}(n). \tag{4.22}$$

We will prove that, for this g, we actually have

$$g(\pi, \sigma) = g_1 * g_2\,(\pi, \sigma) \;\text{ for every } n \geq 1 \text{ and } \pi \leq \sigma \text{ in } \mathcal{NC}(n). \tag{4.23}$$

This will imply in particular that $g_1 * g_2 = g \in SM^{\mathcal{NC}}$, as required.

So let us fix an $n \geq 1$ and some $\pi \leq \sigma$ in $\mathcal{NC}(n)$, for which we will verify that (4.23) holds. We write explicitly $\sigma = \{W_1, \ldots, W_k\}$, and we calculate as follows:

$$\begin{aligned} g(\pi,\sigma) &= \prod_{i=1}^{k} g(\pi_{W_i}, 1_{|W_i|}) \quad \text{(by semi-multiplicativity)} \\ &= \prod_{i=1}^{k} g_1 * g_2\,(\pi_{W_i}, 1_{|W_i|}) \quad \text{(by (4.22))} \end{aligned}$$

$$= \prod_{i=1}^{k} \Big(\sum_{\substack{\rho_i \in \mathcal{NC}(|W_i|), \\ \rho_i \geq \pi_{W_i}}} g_1(\pi_{W_i}, \rho_i) \cdot g_2(\rho_i, 1_{|W_i|}) \Big) \text{ (by the def. of “$*$”)}$$

$$= \sum_{\substack{\rho_1 \geq \pi_{W_1} \in \mathcal{NC}(|W_1|), \ldots \\ \ldots, \rho_k \geq \pi_{W_k} \in \mathcal{NC}(|W_k|)}} \Big(\prod_{i=1}^{k} g_1(\pi_{W_i}, \rho_i) \Big) \cdot \Big(\prod_{i=1}^{k} g_2(\rho_i, 1_{|W_i|}) \Big), \tag{4.24}$$

where the latter equality is obtained by expanding the product from the preceding line.

But now, one has a natural order-preserving bijection

$$\left\{ \begin{array}{ccc} \{\rho \in \mathcal{NC}(n) \mid \rho \leq \sigma\} & \longrightarrow & \mathcal{NC}(|W_1|) \times \cdots \times \mathcal{NC}(|W_k|), \\ \rho & \mapsto & (\rho_{W_1}, \ldots, \rho_{W_k}), \end{array} \right. \tag{4.25}$$

which in particular sends

$$\{\rho \in \mathcal{NC}(n) \mid \pi \leq \rho \leq \sigma\} \longrightarrow \left(\begin{array}{c} (\rho_1, \ldots, \rho_k) \in \mathcal{NC}(|W_1|) \times \cdots \times \mathcal{NC}(|W_k|), \\ \rho_1 \geq \pi_{W_1}, \ldots, \rho_k \geq \pi_{W_k} \end{array} \right).$$

This bijection can be used in order to perform a “change of variable” in the summation from (4.24), in order to turn it into a summation over the set $\{\rho \in \mathcal{NC}(n) \mid \pi \leq \rho \leq \sigma\}$. As a result of this change of variable, we arrive to the formula

$$g(\pi,\sigma) = \sum_{\substack{\rho \in \mathcal{NC}(n), \\ \pi \leq \rho \leq \sigma}} \Big(\prod_{i=1}^{k} g_1(\pi_{W_i}, \rho_{W_i}) \Big) \cdot \Big(\prod_{i=1}^{k} g_2(\rho_{W_i}, 1_{|W_i|}) \Big). \tag{4.26}$$

At this point we recognize the products on the right-hand side of (4.26) as

$$\begin{cases} \prod_{i=1}^{k} g_1(\pi_{W_i}, \rho_{W_i}) = g_1(\pi, \rho) \quad \text{(by Lemma 4.3.2 for } g_1\text{), and} \\ \\ \prod_{i=1}^{k} g_2(\rho_{W_i}, 1_{|W_i|}) = g_2(\rho, \sigma) \quad \text{(by plain semi-multiplicativity for } g_2\text{).} \end{cases} \tag{4.27}$$

Upon substituting (4.27) into (4.26), we arrive to

$$g(\pi, \sigma) = \sum_{\substack{\rho \in \mathcal{NC}(n), \\ \pi \leq \rho \leq \sigma}} g_1(\pi, \rho) \cdot g_2(\rho, \sigma) = g_1 * g_2\,(\pi, \sigma), \text{ as required in (4.23).}$$

2^o *We pick a $g \in SM^{\mathcal{NC}}$ and we prove that g^{-1} (inverse under convolution) is in $SM^{\mathcal{NC}}$ as well.*

The inverse g^{-1} of g can be in any case considered in the larger group $\mathcal{F}$. Our task here is to prove that g^{-1} belongs in fact to $SM^{\mathcal{NC}}$.

For every $n \geq 1$ we define a family of complex numbers $\{\underline{z}(\pi) \mid \pi \in \mathcal{NC}(n)\}$ in the way described as follows. We first put $\underline{z}(1_n) = 1$, then for $\pi \in \mathcal{NC}(n) \setminus \{1_n\}$ we proceed by induction on the number $|\pi|$ of blocks of π and put

$$\underline{z}(\pi) := - \sum_{\substack{\sigma \in \mathcal{NC}(n), \\ \sigma \geq \pi,\ \sigma \neq \pi}} g(\pi, \sigma)\underline{z}(\sigma). \tag{4.28}$$

Note that all the values $\underline{z}(\sigma)$ invoked on the right-hand side of (4.28) can indeed be used in this inductive definition, since the conditions $\sigma \geq \pi,\ \sigma \neq \pi$ imply that $|\sigma| < |\pi|$.

Proposition 4.2.5 gives us a function $h \in SM^{\mathcal{NC}}$ such that $h(\pi, 1_n) = \underline{z}(\pi)$ for every $n \geq 1$ and $\pi \in \mathcal{NC}(n)$. It is immediate that, with h so defined, Equation (4.28) can be read as saying that

$$g * h\,(\pi, 1_n) = 0, \quad \text{for every } n \geq 1 \text{ and } \pi \in \mathcal{NC}(n) \setminus \{1_n\}. \tag{4.29}$$

Now, in view of part 1^o of the present proposition, we have that $g * h \in SM^{\mathcal{NC}}$. Equation (4.29) states that $g * h$ agrees with the unit e of $SM^{\mathcal{NC}}$ on all couples $(\pi, 1_n)$ with $n \geq 1$ and $\pi \in \mathcal{NC}(n) \setminus \{1_n\}$. Since an element of $SM^{\mathcal{NC}}$ is uniquely determined by its values on such couples $(\pi, 1_n)$, we conclude that $g * h = e$.

Upon reading the equality $g * h = e$ in the larger group $\mathcal{F}$, we see that $h = g^{-1}$. Hence $g^{-1} = h \in SM^{\mathcal{NC}}$, as we had to prove. □

4.4 Multiplicative vs semi-multiplicative: the inclusion $MF^{\mathcal{NC}} \subseteq SM^{\mathcal{NC}}$

In this section we briefly review the situation when a function $g \in SM^{\mathcal{NC}}$ is also required to respect the second kind of isomorphism reviewed in Remark 2.2.13, and is thus a *multiplicative function on non-crossing partitions* in the sense considered by Speicher [Spe94]. It is easily seen that in order to upgrade to this situation, it suffices to require g to be well-behaved with respect to the isomorphism $[\theta, 1_k] \approx [0_k, \mathrm{Kr}(\theta)]$ mentioned at the beginning of the line in (2.4). We can therefore go with the following concise definition.

Definition 4.4.1. Consider the group of semi-multiplicative functions $SM^{\mathcal{NC}}$ discussed in Sections 3 and 4. A function $g \in SM^{\mathcal{NC}}$ will be said to be *multiplicative* when it has the property that

$$g(\pi, 1_n) = g(0_n, \mathrm{Kr}(\pi)), \quad \forall\, \pi \in \mathcal{NC}, \tag{4.30}$$

where Kr is the Kreweras complementation map on $\mathcal{NC}(n)$. We will denote

$$MF^{\mathcal{NC}} := \{g \in SM^{\mathcal{NC}} \mid g \text{ satisfies the condition } (4.30)\}. \tag{4.31}$$

Remark 4.4.2. Let g be a function in $MF^{\mathcal{NC}}$ and let us denote

$$g(0_n, 1_n) =: \lambda_n, \quad n \geq 1. \tag{4.32}$$

Upon combining (4.30) with the formula for $g(0_n, \sigma)$ that had been recorded in Remark 4.2.6, we find that for every $n \geq 1$ and $\pi \in \mathcal{NC}(n)$ we have

$$g(\pi, 1_n) = \prod_{W \in \mathrm{Kr}(\pi)} \lambda_{|W|}. \tag{4.33}$$

This generalizes to

$$g(\pi, \sigma) = \prod_{W \in \mathrm{Kr}_\sigma(\pi)} \lambda_{|W|}, \quad \forall\, \pi \leq \sigma \text{ in } \mathcal{NC}, \tag{4.34}$$

where $\mathrm{Kr}_\sigma(\pi)$ is the relative Kreweras complement of π in σ. Indeed, Equation (4.34) follows easily from (4.33) when we invoke the semi-multiplicativity factorization (4.14) and then take into account that $\mathrm{Kr}_\sigma(\pi)$ is obtained by performing in parallel Kreweras complementation on all the restricted partitions π_{W}, with W running among the blocks of σ.

In connection to the above, we have the following statement.

Proposition 4.4.3. *Let $(\lambda_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$, with $\lambda_1 = 1$. There exists a multiplicative function $g \in MF^{\mathcal{NC}}$, uniquely determined, such that $g(0_n, 1_n) = \lambda_n$ for all $n \geq 1$.*

The uniqueness part of Proposition 4.4.3 is clearly implied by the formula (4.34). For the existence part, one *defines* g by using the formula (4.34), and then proves (via a discussion very similar to the one on pages 164-167 of [NS06, Lecture 10]) that $g \in MF^{\mathcal{NC}}$.

Remark 4.4.4. It turns out that $MF^{\mathcal{NC}}$ is in fact a *subgroup* of $SM^{\mathcal{NC}}$. For the proof of this fact we refer to [NS06, Theorem 18.11]. Due to some basic symmetry properties enjoyed by the Kreweras complementation map it turns out, moreover, that $MF^{\mathcal{NC}}$ (unlike $SM^{\mathcal{NC}}$) is a commutative group – see [NS06, Corollary 17.10].

4.5 The action of $SM^{\mathcal{NC}}$ on sequences of multilinear functionals

The relevance of the group $SM^{\mathcal{NC}}$ for non-commutative probability considerations stems from a natural action that this group has on certain sequences of multilinear functionals. In order to describe this action, it is convenient to introduce the following notation.

Notation 4.5.1. Let $\mathcal{A}$ be a vector space over $\mathbb{C}$. We denote

$$\mathfrak{M}_{\mathcal{A}} := \{\underline{\psi} \mid \underline{\psi} = (\psi_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}, \text{ where } \psi_n \text{ is an } n\text{-linear functional}\}. \tag{4.35}$$

Remark and Notation 4.5.2. 1^o In Notation 4.5.1 we did not need to assume that $\mathcal{A}$ is an algebra, or that it comes endowed with an expectation functional $\varphi : \mathcal{A} \to \mathbb{C}$. If that would be the case, and we would thus be dealing with a non-commutative probability space $(\mathcal{A}, \varphi)$, then the set $\mathfrak{M}_{\mathcal{A}}$ would get to have a special element $\underline{\varphi} = (\varphi_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$ where

$$\varphi_n(x_1, \ldots, x_n) := \varphi(x_1 \cdots x_n), \quad \forall\, x_1, \ldots, x_n \in \mathcal{A}. \tag{4.36}$$

Such a $\underline{\varphi}$ is called "family of moment functionals" of $(\mathcal{A}, \varphi)$.

2^o Given a $\underline{\psi} = (\psi_n)_{n=1}^{\infty}$ as in (4.35), there is a standard way of enlarging $\underline{\psi}$ by adding to it some multilinear functionals indexed by non-crossing partitions. More precisely: for any $n \geq 1$ and $\pi \in \mathcal{NC}(n)$, it is customary to denote as $\psi_\pi : \mathcal{A}^n \to \mathbb{C}$ the multilinear functional which acts by

$$\psi_\pi(x_1, \ldots, x_n) = \prod_{V \in \pi} \psi_{|V|}(\,(x_1, \ldots, x_n) \mid V\,), \quad x_1, \ldots, x_n \in \mathcal{A}. \tag{4.37}$$

[A concrete example: if we have, say, $n = 5$ and $\pi = \{\, \{1, 2, 5\},\ \{3, 4\}\, \} \in \mathcal{NC}(5)$, then the formula defining ψ_π becomes $\psi_\pi(x_1, \ldots, x_5) := \psi_3(x_1, x_2, x_5) \cdot \psi_2(x_3, x_4)$.]

The convention for how to enlarge $\underline{\psi}$ is useful when we introduce the following notation.

Notation 4.5.3. Let $\mathcal{A}$ be a vector space over $\mathbb{C}$. For every $\underline{\psi} = (\psi_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ and $g \in SM^{\mathcal{NC}}$, we denote by "$\underline{\psi} \cdot g$" the element $\underline{\theta} = (\theta_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ defined by putting

$$\theta_n = \sum_{\pi \in \mathcal{NC}(n)} g(\pi, 1_n)\psi_\pi, \quad \forall\, n \geq 1. \tag{4.38}$$

The right-hand side of (4.38) has a linear combination done in the vector space of n-linear functionals from $\mathcal{A}^n$ to $\mathbb{C}$, where the ψ_π are as defined in Notation 4.5.2.

We will prove that the map introduced in Notation 4.5.3 is a group action. It is convenient to first record an extension of the formula used to define $\underline{\psi} \cdot g$.

Lemma 4.5.4. *Let $\underline{\psi}, \underline{\theta} \in \mathfrak{M}_{\mathcal{A}}$ and $g \in SM^{\mathcal{NC}}$ be such that $\underline{\theta} = \underline{\psi} \cdot g$. Consider the extended families of multilinear functionals $\{\psi_\pi \mid \pi \in \mathcal{NC}\}$ and $\{\theta_\pi \mid \pi \in \mathcal{NC}\}$ that are obtained out of $\underline{\psi}$ and $\underline{\theta}$, respectively, in the way indicated in Notation 4.5.2. Then for every $n \geq 1$ and $\sigma \in \mathcal{NC}(n)$ one has*

$$\theta_\sigma = \sum_{\substack{\pi \in \mathcal{NC}(n), \\ \pi \leq \sigma}} g(\pi, \sigma)\psi_\pi. \tag{4.39}$$

Proof. Let us write explicitly $\sigma = \{W_1, \ldots, W_k\}$. Then for every $x_1, \ldots, x_n \in \mathcal{A}$ we have

$$\begin{aligned}
\theta_\sigma(x_1, \ldots, x_n) &= \prod_{j=1}^{k} \theta_{|W_j|}\big(\, (x_1, \ldots, x_n) \mid W_j \,\big) \text{ (by the definition of } \theta_\sigma) \\
&= \prod_{j=1}^{k} \Big(\sum_{\pi_j \in \mathcal{NC}(|W_j|)} g(\pi_j, 1_{|W_j|})\psi_{\pi_j}\big(\, (x_1, \ldots, x_n) \mid W_j \,\big) \Big) \text{ (by Eqn. (4.38))}.
\end{aligned}$$

Upon expanding the latter product of k factors, we find $\theta_\sigma(x_1, \ldots, x_n)$ to be equal to

$$\sum_{\substack{\pi_1 \in \mathcal{NC}(|W_1|), \ldots \\ \ldots, \pi_k \in \mathcal{NC}(|W_k|)}} \Big(\prod_{j=1}^{k} g(\pi_j, 1_{|W_j|}) \Big) \cdot \Big(\prod_{j=1}^{k} \psi_{\pi_j}\big(\, (x_1, \ldots, x_n) \mid W_j \,\big) \Big). \tag{4.40}$$

But now, one has a natural bijection

$$\left\{ \begin{array}{rcl} \{\pi \in \mathcal{NC}(n) \mid \pi \leq \sigma\} & \longrightarrow & \mathcal{NC}(|W_1|) \times \cdots \times \mathcal{NC}(|W_k|), \\ \pi & \mapsto & (\pi_{W_1}, \ldots, \pi_{W_k}), \end{array} \right. \tag{4.41}$$

where the partitions $\pi_{W_j} \in \mathcal{NC}(|W_j|)$ are restrictions of π (cf. Notation 2.2.8). When we use this bijection in order to perform a change of variables in the summation from (4.40), the semi-multiplicativity property of g assures us that the product $\prod_{j=1}^{k} g(\pi_j, 1_{|W_j|})$ is converted into just "$g(\pi, \sigma)$". On the other hand, it is easily checked that the said change of variable transforms $\prod_{j=1}^{k} \psi_{\pi_j}\big((x_1, \ldots, x_n) \mid W_j \big)$ into "$\psi_\pi(x_1, \ldots, x_n)$". Hence our computation of what is $\theta_\sigma(x_1, \ldots, x_n)$ has lead to $\sum_{\pi \leq \sigma} g(\pi, \sigma) \cdot \psi_\pi(x_1, \ldots, x_n)$, as required. □

Proposition 4.5.5. *Let $\mathcal{A}$ be a vector space over $\mathbb{C}$. The formula (4.38) from Notation 4.5.3 defines an action of the group $SM^{\mathcal{NC}}$ on the set $\mathfrak{M}_{\mathcal{A}}$. That is, one has*

$$(\underline{\psi} \cdot g_1) \cdot g_2 = \underline{\psi} \cdot (g_1 * g_2), \quad \forall \underline{\psi} \in \mathfrak{M}_{\mathcal{A}} \text{ and } g_1, g_2 \in SM^{\mathcal{NC}}. \tag{4.42}$$

Proof. We denote $\underline{\psi} \cdot g_1 =: \underline{\theta} = (\theta_n)_{n=1}^{\infty}$ and $(\underline{\psi} \cdot g_1) \cdot g_2 =: \underline{\eta} = (\eta_n)_{n=1}^{\infty}$. Our goal for the proof is to verify that $\underline{\eta} = \underline{\psi} \cdot (g_1 * g_2)$, i.e. that we have

$$\eta_n = \sum_{\pi \in \mathcal{NC}(n)} g_1 * g_2\,(\pi, 1_n)\, \psi_\pi, \quad \forall\, n \geq 1, \tag{4.43}$$

where $\{\psi_\pi \mid \pi \in \mathcal{NC}\}$ is the extension of $\underline{\psi}$. We thus fix an $n \geq 1$ for which we will verify that (4.43) holds. We write the formula given for η_n by the relation $\underline{\eta} = \underline{\theta} \cdot g_2$ and then we invoke Lemma 4.5.4 in connection to the relation $\underline{\theta} = \underline{\psi} \cdot g_1$, to find that:

$$\eta_n = \sum_{\sigma \in \mathcal{NC}(n)} g_2(\sigma, 1_n)\theta_\sigma = \sum_{\sigma \in \mathcal{NC}(n)} g_2(\sigma, 1_n) \Big(\sum_{\substack{\pi \in \mathcal{NC}(n), \\ \pi \leq \sigma}} g_1(\pi, \sigma)\, \psi_\pi \Big).$$

Changing the order of summation in the latter double sum then leads to:

$$\eta_n = \sum_{\pi \in \mathcal{NC}(n)} \Big(\sum_{\substack{\sigma \in \mathcal{NC}(n), \\ \pi \leq \sigma}} g_1(\pi, \sigma) g_2(\sigma, 1_n) \Big)\, \psi_\pi. \tag{4.44}$$

The interior sum in (4.44) is equal to $g_1 * g_2\,(\pi, 1_n)$, and we have thus obtained the required Equation (4.43). □

Remark 4.5.6. Throughout this section we have considered, for the sake of simplicity, only multilinear functionals with values in $\mathbb{C}$. We invite the reader to take a moment to observe that the whole discussion could have been pursued, without any change, in the framework where we consider multilinear functionals with values in a unital commutative algebra $\mathcal{C}$ over $\mathbb{C}$. Indeed, suppose we have fixed such a $\mathcal{C}$. Then Notation 4.5.1 is adjusted by putting

$$\mathfrak{M}_{\mathcal{A}}^{(\mathcal{C})} := \left\{ \underline{\psi} \;\middle|\; \begin{array}{l} \underline{\psi} = (\psi_n : \mathcal{A}^n \to \mathcal{C})_{n=1}^{\infty}, \text{ where every } \psi_n \\ \text{is a } \mathbb{C}\text{-multilinear functional} \end{array} \right\}. \tag{4.45}$$

Given $\underline{\psi} \in \mathfrak{M}_{\mathcal{A}}^{(\mathcal{C})}$ and $g \in SM^{\mathcal{NC}}$, we define what is $\underline{\psi} \cdot g \in \mathfrak{M}_{\mathcal{A}}^{(\mathcal{C})}$ by the very same formula as in (4.38) of Notation 4.5.3. The proof of Proposition 4.5.5 goes through without any changes, to show that in this way we obtain a right group action of $SM^{\mathcal{NC}}$ on $\mathfrak{M}_{\mathcal{A}}^{(\mathcal{C})}$.

In the rest of the paper we will stick everywhere to the basic case when $\mathcal{C} = \mathbb{C}$, with only one exception: Section 4.6.4 will have an occurrence of the case where $\mathcal{C}$ is the Grassmann algebra $\mathbb{G} := \{\alpha + \varepsilon\beta \mid \alpha, \beta \in \mathbb{C}\}$, with multiplication defined by

$$(\alpha_1 + \varepsilon\beta_1) \cdot (\alpha_2 + \varepsilon\beta_2) = \alpha_1\alpha_2 + \varepsilon(\alpha_1\beta_2 + \alpha_2\beta_1), \quad \text{for } \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{C}.$$

4.6 Cumulant-to-moment type, and cumulant-to-cumulant type

There are several brands of cumulants which live naturally in the universe of non-crossing partitions, and are commonly used in the non-commutative probability literature. Each such brand of cumulants has its own "moment-cumulant" summation formula, and there also exist useful summation formulas that connect different brands of cumulants. The action of $SM^{\mathcal{NC}}$ on sequences of multilinear functionals that was observed in Section 6 offers an efficient way to do calculations related to these moment-cumulant and inter-cumulant formulas. In connection to that, we next put into evidence: a *factorization property* which seems to always be fulfilled when one considers functions $g \in SM^{\mathcal{NC}}$ involved in moment-cumulant formulas; and a *vanishing property* fulfilled by functions $g \in SM^{\mathcal{NC}}$ which are involved in inter-cumulant formulas. Both these properties are phrased in connection to the operation "$\diamond$" of concatenation of non-crossing partitions, and to the notion of irreducibility with respect to concatenation, as reviewed in Notation 2.2.17.

Definition 4.6.1. 1^o A function $g \in SM^{\mathcal{NC}}$ will be said to be of *cumulant-to-moment* type when it has the property that

$$g(\pi_1 \diamond \pi_2, 1_{n_1+n_2}) = g(\pi_1, 1_{n_1}) \cdot g(\pi_2, 1_{n_2}), \tag{4.46}$$

holding for all $n_1, n_2 \geq 1$ and $\pi_1 \in \mathcal{NC}(n_1)$, $\pi_2 \in \mathcal{NC}(n_2)$. We denote

$$SM^{\mathcal{NC}}_{\mathrm{c-m}} := \{g \in SM^{\mathcal{NC}} \mid g \text{ is of cumulant-to-moment type}\}.$$

2^o A function $g \in SM^{\mathcal{NC}}$ will be said to be of *cumulant-to-cumulant* type when it satisfies

$$g(\pi_1 \diamond \pi_2, 1_{n_1+n_2}) = 0, \quad \forall\, n_1, n_2 \geq 1 \text{ and } \pi_1 \in \mathcal{NC}(n_1),\ \pi_2 \in \mathcal{NC}(n_2). \tag{4.47}$$

We denote

$$SM^{\mathcal{NC}}_{\mathrm{c-c}} := \{g \in SM^{\mathcal{NC}} \mid g \text{ is of cumulant-to-cumulant type}\}.$$

Remark 4.6.2. 1^o A function $g \in SM^{\mathcal{NC}}_{\mathrm{c-c}}$ is completely determined when we know its values $g(\pi, 1_n)$ with $\pi \in \mathcal{NC}(n)$ irreducible. Indeed, the condition on g stated in (4.47) just says that if $\pi \in \mathcal{NC}(n)$ is not irreducible, then $g(\pi, 1_n) = 0$. So we know the values $g(\pi, 1_n)$ for *all* $n \geq 1$ and $\pi \in \mathcal{NC}(n)$, which determines g (cf. Proposition 4.2.5).

2^o Consider now a function $g \in SM^{\mathcal{NC}}_{\mathrm{c-m}}$. An easy induction shows that for every $k \geq 1$, $n_1, \ldots, n_k \geq 1$ and $\pi_1 \in \mathcal{NC}(n_1), \ldots, \pi_k \in \mathcal{NC}(n_k)$, one has:

$$g(\pi_1 \diamond \cdots \diamond \pi_k, 1_{n_1+\cdots+n_k}) = \prod_{j=1}^{k} g(\pi_j, 1_{n_j}). \tag{4.48}$$

Since every non-crossing partition can be written as a concatenation of irreducible ones, we conclude that our $g \in SM^{\mathcal{NC}}_{\mathrm{c-m}}$ can be completely reconstructed if we know its values $g(\pi, 1_n)$ with $\pi \in \mathcal{NC}(n)$ irreducible – indeed, Equation (4.48) then tells us what is $g(\pi, 1_n)$ for *all* $n \geq 1$ and $\pi \in \mathcal{NC}(n)$, and Proposition 4.2.5 can be applied.

In Section 4.7 we will examine $SM^{\mathcal{NC}}_{\mathrm{c-m}}$ and $SM^{\mathcal{NC}}_{\mathrm{c-c}}$ from the group structure point of view, within the group $(SM^{\mathcal{NC}}, *)$. For now we only want to show, by example, what is the rationale for the terms "cumulant-to-moment" and "cumulant-to-cumulant" used in Definition 4.6.1. This is an opportunity to review a few salient examples of cumulants, and to display some of the functions in $SM^{\mathcal{NC}}$ which encode transition formulas from these cumulants to moments, or encode transition formulas between two different brands of cumulants.

Throughout the rest of this section we fix a non-commutative probability space $(\mathcal{A}, \varphi)$, we look at

$$\mathfrak{M}_{\mathcal{A}} := \{\underline{\psi} \mid \underline{\psi} = (\psi_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty},\ \text{where } \psi_n \text{ is an } n\text{-linear functional}\},$$

and we consider the family of moment functionals $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ which was introduced in Notation 4.5.2.

4.6.1 Free and Boolean cumulants.

Definition and Remark 4.6.3. Recall from Definition 2.3.2 that the family of free cumulant functionals of $(\mathcal{A}, \varphi)$ is the family $\underline{r} = (r_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ defined via the requirement that for every $n \geq 1$ and $x_1, \ldots, x_n \in \mathcal{A}$ one has:

$$\varphi(x_1 \cdots x_n) = \sum_{\pi \in \mathcal{NC}(n)} \prod_{V \in \pi} r_{|V|}\big((x_1, \ldots, x_n) \mid V \big). \tag{4.49}$$

This requirement can be re-phrased as follows: let $g_{\mathrm{fc-m}} : \mathcal{NC}^{(2)} \to \mathbb{C}$ be[1] defined by

$$g_{\mathrm{fc-m}}(\pi, \sigma) = 1, \qquad \forall\, \pi \leq \sigma \text{ in } \mathcal{NC}(n). \tag{4.50}$$

It is immediate that $g_{\mathrm{fc-m}} \in SM^{\mathcal{NC}}$ and that it fulfills the factorization condition (4.46), hence $g_{\mathrm{fc-m}} \in SM^{\mathcal{NC}}_{\mathrm{c-m}}$. The "moment-cumulant" formula (4.49) can be read as an instance of the group action from Section 4.5, it just says that

$$\underline{\varphi} = \underline{r} \cdot g_{\mathrm{fc-m}}. \tag{4.51}$$

Indeed, (4.49) asks for the equality of n-linear functionals $\varphi_n = \sum_{\pi \in \mathcal{NC}(n)} r_\pi$, holding for every $n \geq 1$, and with r_π's defined as in Notation 4.5.2; but the latter equality is the same as (4.51).

We next repeat the same moment-cumulant formulation in connection to *Boolean cumulants*, where we now refer to interval partitions.

Definition and Remark 4.6.4. Recall from Definition 2.3.2 that the family of Boolean cumulant functionals of $(\mathcal{A}, \varphi)$ is the family $\underline{b} = (b_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ defined via the requirement that for every $n \geq 1$ and $x_1, \ldots, x_n \in \mathcal{A}$ one has:

$$\varphi(x_1 \cdots x_n) = \sum_{\pi \in \mathcal{I}(n)} \prod_{J \in \pi} b_{|J|}\big((x_1, \ldots, x_n) \mid J \big). \tag{4.52}$$

Now, consider the function $g_{\mathrm{bc-m}} \in SM^{\mathcal{NC}}$ defined via the requirement that for every $n \geq 1$ and $\pi \in \mathcal{NC}(n)$ we have

$$g_{\mathrm{bc-m}}(\pi, 1_n) = \begin{cases} 1, & \text{if } \pi \in \mathcal{I}(n), \\ 0, & \text{otherwise.} \end{cases} \tag{4.53}$$

[1] In $g_{\mathrm{fc-m}}$, the subscript "fc-m" is a reminder that we are doing a transition from free cumulants to moments. Similar conventions will be used for other such special functions, e.g. "$g_{\mathrm{bc-m}}$" for the function in $SM^{\mathcal{NC}}_{\mathrm{c-m}}$ which encodes the transition from Boolean cumulants to moments, or "$g_{\mathrm{fc-bc}}$" for the function in $SM^{\mathcal{NC}}_{\mathrm{c-c}}$ which encodes the transition from free cumulants to Boolean cumulants.

Such a function does indeed exist and is unique, as guaranteed by Proposition 4.2.5. We see moreover that $g_{\mathrm{bc-m}}$ is a function of cumulant-to-moment type: indeed, given any $n_1, n_2 \geq 1$ and $\pi \in \mathcal{NC}(n_1)$, $\pi_2 \in \mathcal{NC}(n_2)$, it is immediate that

$$g_{\mathrm{bc-m}}(\pi_1 \diamond \pi_2, 1_{n_1+n_2}) = g_{\mathrm{bc-m}}(\pi_1, 1_{n_1}) \cdot g_{\mathrm{bc-m}}(\pi_2, 1_{n_2}) = \begin{cases} 1, & \text{if both } \pi_1 \text{ and } \pi_2 \text{ are} \\ & \quad \text{interval partitions,} \\ 0, & \text{otherwise.} \end{cases}$$

Hence $g_{\mathrm{bc-m}} \in SM^{\mathcal{NC}}_{\mathrm{c-m}}$ and (exactly as we did for free cumulants in Remark 4.6.3) we see that the moment-cumulant formula (4.52) amounts to just:

$$\underline{\varphi} = \underline{b} \cdot g_{\mathrm{bc-m}}. \tag{4.54}$$

Remark 4.6.5. It was convenient to introduce the function $g_{\mathrm{bc-m}}$ by just postulating its values $g_{\mathrm{bc-m}}(\pi, 1_n)$, and then by invoking Proposition 4.2.5. It is not hard to actually write down the formula for the values taken by $g_{\mathrm{bc-m}}$ on general couples in $\mathcal{NC}^{(2)}$; this is found by using the semi-multiplicativity property, and comes out (immediate verification) as

$$g_{\mathrm{bc-m}}(\pi, \sigma) = \left\{ \begin{array}{ll} 1, & \text{if } \pi \sqsubseteq \sigma, \\ 0, & \text{otherwise.} \end{array} \right\}, \qquad \forall\, (\pi, \sigma) \in \mathcal{NC}^{(2)}, \tag{4.55}$$

where $\sqsubseteq$ is one of the partial order relations reviewed in Chapter 2

4.6.2 An interpolation between free and Boolean: t-Boolean cumulants.

In this subsection we continue to use the notation from Section 4.6.1, where $\underline{\varphi} \in \mathfrak{M}_{\mathcal{A}}$ is the family of moment functionals of the non-commutative probability space $(\mathcal{A}, \varphi)$, and $\underline{r}, \underline{b} \in \mathfrak{M}_{\mathcal{A}}$ are the families of free and respectively Boolean cumulants of the same space. Our goal for the subsection is to review a 1-parameter interpolation between $\underline{b}$ and $\underline{r}$, arising from the work of Bożejko and Wysoczanski [BW01], and defined in the way described as follows.

Definition 4.6.6. Let $t \in \mathbb{R}$ be a parameter. We will use the name *t-Boolean cumulant functionals* of $(\mathcal{A}, \varphi)$ to refer to the sequence of multilinear functionals $\underline{b}^{(t)} = (b_n^{(t)})_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ defined via the requirement that for every $n \geq 1$ and $x_1, \ldots, x_n \in \mathcal{A}$ one has:

$$\varphi(x_1 \cdots x_n) = \sum_{\pi \in \mathcal{NC}(n)} t^{\mathrm{inner}(\pi)} \prod_{V \in \pi} b^{(t)}_{|V|}\big((x_1, \ldots, x_n) \mid V \big). \tag{4.56}$$

Recall that $\mathrm{inner}(\pi)$ is our notation for the number of inner blocks of $\pi \in \mathcal{NC}(n)$.

Remark 4.6.7. It is clear that for $t = 1$ one gets $\underline{b}^{(1)} = \underline{r}$. On the other hand, for $t = 0$ one gets that $\underline{b}^{(0)} = \underline{b}$, because in this case the right-hand side of Equation (4.56) reduces to a sum over $\mathcal{I}(n)$ (cf. (2.1) in the review of background).

Notation and Remark 4.6.8. For every $t \in \mathbb{R}$, let $g^{(t)}_{\mathrm{bc-m}}$ be the function in $SM^{\mathcal{NC}}$ defined via the requirement that

$$g^{(t)}_{\mathrm{bc-m}}(\pi, 1_n) := t^{\mathrm{inner}(\pi)}, \quad \text{for all } n \geq 1 \text{ and } \pi \in \mathcal{NC}(n). \tag{4.57}$$

As an immediate consequence of the obvious fact that

$$\mathrm{inner}(\pi_1 \diamond \pi_2) = \mathrm{inner}(\pi_1) + \mathrm{inner}(\pi_2), \quad \forall\, \pi_1, \pi_2 \in \mathcal{NC},$$

one has that $g^{(t)}_{\mathrm{bc-m}}$ is a function of cumulant-to-moment type. The formula (4.56) used to define the t-Boolean cumulant functionals can be concisely re-written in the form

$$\underline{\varphi} = \underline{b}^{(t)} \cdot g^{(t)}_{\mathrm{bc-m}}. \tag{4.58}$$

This is a common generalization of the formulas (4.49) and (4.52) observed for free and for Boolean cumulants – the latter formulas are obtained by setting the parameter to $t = 1$ and to $t = 0$, respectively.

Remark 4.6.9. When using the action of the group $SM^{\mathcal{NC}}$, one sees very clearly how to combine moment-cumulant formulas for two different brands of cumulants in order to get a direct connection between the cumulants themselves. We illustrate how this works when we want to go from s-Boolean cumulants to t-Boolean cumulants for two distinct parameters $s, t \in \mathbb{R}$. We have $\underline{b}^{(t)} \cdot g^{(t)}_{\mathrm{bc-m}} = \underline{\varphi} = \underline{b}^{(s)} \cdot g^{(s)}_{\mathrm{bc-m}}$, hence:

$$\underline{b}^{(t)} = \underline{\varphi} \cdot (g^{(t)}_{\mathrm{bc-m}})^{-1} = (\underline{b}^{(s)} \cdot g^{(s)}_{\mathrm{bc-m}}) \cdot (g^{(t)}_{\mathrm{bc-m}})^{-1} = \underline{b}^{(s)} \cdot \big(g^{(s)}_{\mathrm{bc-m}} * (g^{(t)}_{\mathrm{bc-m}})^{-1} \big). \tag{4.59}$$

In short: the transition from s-Boolean cumulants to t-Boolean cumulants is encoded by the function $g^{(s)}_{\mathrm{bc-m}} * (g^{(t)}_{\mathrm{bc-m}})^{-1} \in SM^{\mathcal{NC}}_{\mathrm{c-c}}$. The values of this function turn out to have a nice explicit description (cf. Remark 4.8.2.1 and Corollary 4.8.5 below), where in particular we find that for $n \geq 1$ and $\pi \in \mathcal{NC}(n)$ we have:

$$g^{(s)}_{\mathrm{bc-m}} * (g^{(t)}_{\mathrm{bc-m}})^{-1}\, (\pi, 1_n) = \begin{cases} (s-t)^{|\pi|-1}, & \text{if } \pi \text{ is irreducible,} \\ 0, & \text{otherwise.} \end{cases} \tag{4.60}$$

Hence, when spelled out explicitly, the transition formula (4.59) says this: for every $n \geq 1$ and $x_1, \ldots, x_n \in \mathcal{A}$ we have

$$b^{(t)}_n(x_1, \ldots, x_n) = \sum_{\substack{\pi \in \mathcal{NC}(n), \\ \text{irreducible}}} (s-t)^{\mathrm{inner}(\pi)} \prod_{V \in \pi} b^{(s)}_{|V|}\big((x_1, \ldots, x_n) \mid V \big). \tag{4.61}$$

In the special case when $s = 1$ and $t = 0$, Equation (4.61) becomes the transition formula from free cumulants to Boolean cumulants, which is well-known since the work of Lehner [Leh02]. When swapping the role of the parameters and putting $s = 0$ and $t = 1$, one finds the inverse transition formula which writes free cumulants in terms of Boolean cumulants, and is also well-known (cf. [BN08, Proposition 3.9], [AHLV15, Section 4]).

4.6.3 Monotone cumulants.

Another family of cumulant functionals associated to $(\mathcal{A}, \varphi)$ that gets constant attention in the research literature on non-commutative probability is the family of monotone cumulant functionals which were introduced in [HS11]. Recall from Definition 2.3.2 that the family $\underline{h} = (\rho_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$ of *monotone cumulants* of $(\mathcal{A}, \varphi)$ is defined via the requirement that for every $n \geq 1$ and $x_1, \ldots, x_n \in \mathcal{A}$ one has

$$\varphi(x_1 \cdots x_n) = \sum_{\pi \in \mathcal{NC}(n)} \frac{m(\pi)}{|\pi|!} \cdot \prod_{V \in \pi} \rho_{|V|}\big((x_1, \ldots, x_n) \mid V \big). \tag{4.62}$$

Notation and Remark 4.6.10. In order to re-phrase the preceding definition in terms of the action of $SM^{\mathcal{NC}}$ on $\mathfrak{M}_{\mathcal{A}}$, we let $g_{\mathrm{mc-m}}$ be the function in $SM^{\mathcal{NC}}$ defined via the requirement that

$$g_{\mathrm{mc-m}}(\pi, 1_n) = \frac{m(\pi)}{|\pi|!}, \quad \forall\, n \geq 1 \text{ and } \pi \in \mathcal{NC}(n). \tag{4.63}$$

An elementary counting argument (see Proposition 2.2.29 [AHLV15, Proposition 3.3]) shows that $g_{\mathrm{mc-m}}$ satisfies the factorization condition stated in (4.46), and is therefore of cumulant-to-moment type. The formula (4.62) from the preceding definition gets to be re-phrased as

$$\underline{\varphi} = \underline{h} \cdot g_{\mathrm{mc-m}}, \tag{4.64}$$

in close analogy to how the definitions of $\underline{r}, \underline{b}, \underline{b}^{(t)}$ were re-phrased in the preceding subsections.

4.6.4 Infinitesimal cumulants.

There exists an "infinitesimal" extension of the notion of non-commutative probability space, which has been considered primarily for the purpose of pinning down an infinitesimal version of the notion of free independence for non-commutative random variables (cf. [BPV12], and the follow-up in [Shl18] relating this topic to random matrix theory). An *infinitesimal non-commutative probability space* is a triple $(\mathcal{A}, \varphi, \varphi')$ where $(\mathcal{A}, \varphi)$ is a non-commutative probability space in the usual sense and one also has a second linear functional $\varphi' : \mathcal{A} \to \mathbb{C}$ such that $\varphi'(1_{\mathcal{A}}) = 0$. To such a space one associates:

- a sequence of free infinitesimal cumulant functionals $\underline{r}' = (r'_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$;
- a sequence of Boolean infinitesimal cumulant functionals $\underline{b}' = (b'_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$;
- a sequence of monotone infinitesimal cumulant functionals $\underline{h}' = (h'_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$.

The infinitesimal free cumulants $\underline{r}'$ were introduced in [FN10], while $\underline{b}', \underline{h}'$ were introduced in [Has11] (see also the detailed study of all these notions appearing in the recent paper [CEFP21]).

We note that $\underline{r}', \underline{b}', \underline{h}'$ belong to $\mathfrak{M}_{\mathcal{A}}$, the set bearing the action of $SM^{\mathcal{NC}}$ from Section 4.5. The definitions of these infinitesimal cumulants can be described in terms of a variation of this action of $SM^{\mathcal{NC}}$, going now on $\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}}$. The occurrence of $\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}}$ comes from the fact that the summations over lattices $\mathcal{NC}(n)$ used to describe infinitesimal cumulants have terms which depend on *both* of the linear functionals φ, φ' considered on $\mathcal{A}$. The details are as follows.

Notation 4.6.11. Let $\mathcal{A}$ be a vector space over $\mathbb{C}$ and let $\mathfrak{M}_{\mathcal{A}}$ be the set of sequences of multilinear functionals introduced in Notation 4.5.1. Suppose we are given a couple $(\underline{\psi}^{(1)}, \underline{\psi}^{(2)}) \in \mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}}$, where $\underline{\psi}^{(1)} = (\psi_n^{(1)})_{n=1}^{\infty}$ and $\underline{\psi}^{(2)} = (\psi_n^{(2)})_{n=1}^{\infty}$, and suppose we are also given a function $g \in SM^{\mathcal{NC}}$. We then denote

$$(\underline{\psi}^{(1)}, \underline{\psi}^{(2)}) \overset{\text{inf}}{\odot} g := (\underline{\theta}^{(1)}, \underline{\theta}^{(2)}) \in \mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}}, \tag{4.65}$$

where $\underline{\theta}^{(1)} = \underline{\psi}^{(1)} \cdot g$ (exactly as in Notation 4.5.3) and $\underline{\theta}^{(2)} = (\theta_n^{(2)})_{n=1}^{\infty}$ is defined by the requirement that for every $n \geq 1$ and $x_1, \ldots, x_n \in \mathcal{A}$ we have

$$\theta_n^{(2)}(x_1, \ldots, x_n) = \tag{4.66}$$

$$\sum_{\pi \in \mathcal{NC}(n)} g(\pi, 1_n) \cdot \sum_{W \in \pi} \Big(\psi_{|W|}^{(2)}(\,(x_1, \ldots, x_n) \mid W\,) \cdot \prod_{\substack{V \in \pi, \\ V \neq W}} \psi_{|V|}^{(1)}(\,(x_1, \ldots, x_n) \mid V\,) \Big).$$

Remark 4.6.12. Let $\mathcal{A}$ be as in Notation 4.6.11. What hides behind (4.65) and (4.66) is the fact that we have a canonical identification:

$$\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}} \ni (\underline{\psi}^{(1)}, \underline{\psi}^{(2)}) \mapsto \widetilde{\underline{\psi}} \in \mathfrak{M}_{\mathcal{A}}^{(\mathbb{G})}, \tag{4.67}$$

where $\mathbb{G}$ is the Grassmann algebra and $\mathfrak{M}_{\mathcal{A}}^{(\mathbb{G})}$ is as considered in Remark 4.5.6 at the end of Section 4.5. That is: given $\underline{\psi}^{(1)} = (\psi_n^{(1)})_{n=1}^{\infty}$ and $\underline{\psi}^{(2)} = (\psi_n^{(2)})_{n=1}^{\infty}$ in $\mathfrak{M}_{\mathcal{A}}$, we create a sequence of $\mathbb{C}$-multilinear functionals $\widetilde{\underline{\psi}} = (\widetilde{\psi}_n : \mathcal{A}^n \to \mathbb{G})_{n=1}^{\infty}$ by simply putting

$$\widetilde{\psi}_n(x_1, \ldots, x_n) = \psi_n^{(1)}(x_1, \ldots, x_n) + \varepsilon \psi_n^{(2)}(x_1, \ldots, x_n), \quad \forall\, x_1, \ldots, x_n \in \mathcal{A}.$$

As explained in Remark 4.5.6, the group $SM^{\mathcal{NC}}$ acts on the right on $\mathfrak{M}_{\mathcal{A}}^{(\mathbb{G})}$. The explicit formula for $(\underline{\psi}^{(1)}, \underline{\psi}^{(2)}) \overset{\text{inf}}{\odot} g$ shown in the preceding notation is just the conversion of the formula for $\widetilde{\underline{\psi}} \cdot g \in \mathfrak{M}_{\mathcal{A}}^{(\mathbb{G})}$, via the identification (4.67).

As a byproduct of the connection with the Grassmann algebra, one also gets an immediate proof of the following fact.

Proposition 4.6.13. *Let $\mathcal{A}$ be a vector space over $\mathbb{C}$. Then "$\overset{\text{inf}}{\odot}$" from Notation 4.6.11 is an action on the right of the group $SM^{\mathcal{NC}}$ on $\mathfrak{M}_{\mathcal{A}} \times \mathfrak{M}_{\mathcal{A}}$. That is, one has*

$$\big((\underline{\psi}^{(1)}, \underline{\psi}^{(2)}) \overset{\text{inf}}{\odot} g \big) \overset{\text{inf}}{\odot} h = (\underline{\psi}^{(1)}, \underline{\psi}^{(2)}) \overset{\text{inf}}{\odot} (g * h), \quad \forall\, \underline{\psi}^{(1)}, \underline{\psi}^{(2)} \in \mathfrak{M}_{\mathcal{A}} \text{ and } g, h \in SM^{\mathcal{NC}}, \tag{4.68}$$

where on the right-hand side we use the convolution operation of $SM^{\mathcal{NC}}$. $\square$

Remark 4.6.14. Consider an infinitesimal non-commutative probability space $(\mathcal{A}, \varphi, \varphi')$, and let us spell out how the infinitesimal cumulants $\underline{r}', \underline{b}', \underline{h}' \in \mathfrak{M}_{\mathcal{A}}$ mentioned at the beginning of this subsection are described in terms of the action $\overset{\text{inf}}{\odot}$ of the group $SM^{\mathcal{NC}}$. To that end, let $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty}$ and $\underline{\varphi}' = (\varphi_n')_{n=1}^{\infty}$ be the sequences of moment functionals associated to φ and to φ'; that is, for every $n \geq 1$ the n-linear functionals $\varphi_n, \varphi_n' : \mathcal{A}^n \to \mathbb{C}$ act by

$$\varphi_n(x_1, \ldots, x_n) = \varphi(x_1 \cdots x_n) \text{ and } \varphi_n'(x_1, \ldots, x_n) = \varphi'(x_1 \cdots x_n), \quad \forall\, x_1, \ldots, x_n \in \mathcal{A}.$$

The infinitesimal cumulants we are interested in are determined by the "moment-cumulant" equations

$$(\underline{\varphi}, \underline{\varphi}') = (\underline{r}, \underline{r}') \overset{\text{inf}}{\odot} g_{\text{fc}-\text{m}} = (\underline{b}, \underline{b}') \overset{\text{inf}}{\odot} g_{\text{bc}-\text{m}} = (\underline{h}, \underline{h}') \overset{\text{inf}}{\odot} g_{\text{mc}-\text{m}}, \tag{4.69}$$

where $g_{\text{fc-m}}, g_{\text{bc-m}}, g_{\text{mc-m}} \in SM^{\mathcal{NC}}$ are the functions of cumulant-to-moment type that appeared in the preceding subsections (cf. Equations (4.50), (4.55) and (4.63), respectively). So for instace the sequence of free infinitesimal cumulants $\underline{r}'$ is found by picking the second component in the formula

$$(\underline{r}, \underline{r}') = (\underline{\varphi}, \underline{\varphi}') \overset{\text{inf}}{\odot} g_{\text{fc-m}}^{-1}. \tag{4.70}$$

We note that the $\underline{r}$ appearing in (4.69), (4.70) is precisely the sequence of free cumulant functionals of $(\mathcal{A}, \varphi)$, as one sees by picking the first component in (4.70) and by taking into account that on the first component of $\overset{\text{inf}}{\odot}$ we have the "usual" action of $SM^{\mathcal{NC}}$ on $\mathfrak{M}_{\mathcal{A}}$.

4.7 $SM_{\text{c-c}}^{\mathcal{NC}}$ is a subgroup, and $SM_{\text{c-m}}^{\mathcal{NC}}$ is a right coset

In this section we follow up on the subsets $SM_{\text{c-c}}^{\mathcal{NC}}, SM_{\text{c-m}}^{\mathcal{NC}} \subseteq SM^{\mathcal{NC}}$ introduced in Definition 4.6.1. We will prove that:

(i) $SM_{\text{c-c}}^{\mathcal{NC}}$ is a *subgroup* of $(SM^{\mathcal{NC}}, *)$, and

(ii) $SM_{\text{c-m}}^{\mathcal{NC}}$ is a *right coset* of the subgroup $SM_{\text{c-c}}^{\mathcal{NC}}$.

The statement (ii) means that we can write

$$SM_{\text{c-m}}^{\mathcal{NC}} = SM_{\text{c-c}}^{\mathcal{NC}} * h = \{g * h \mid g \in SM_{\text{c-c}}^{\mathcal{NC}}\}, \tag{4.71}$$

for no matter what $h \in SM_{\text{c-m}}^{\mathcal{NC}}$ we choose to fix. The easiest choice for h is to pick $h(\pi, \sigma) = 1$ for all $(\pi, \sigma) \in \mathcal{NC}^{(2)}$, that is, let h be the special function $g_{\text{fc-m}}$ from Definition 4.6.3. However, as we will see in Section 8.2 below, it may be more advantageous for proofs and applications if we go instead with $h = g_{\text{bc-m}}$, picked from Definition 4.6.4.

4.7.1 Proof that $SM_{\text{c-c}}^{\mathcal{NC}}$ is a subgroup of $(SM^{\mathcal{NC}}, *)$.

We first record a straightforward extension of the vanishing condition postulated in Equation (4.47), in the definition of $SM_{\text{c-c}}^{\mathcal{NC}}$.

Lemma 4.7.1. *Let $n \geq 1$ and let $\pi \leq \sigma$ be two partitions in $\mathcal{NC}(n)$, where:*

$$\begin{cases} \text{there exists a block } W_o \text{ of } \sigma \text{ such that} \\ \min(W_o) \text{ and } \max(W_o) \text{ belong to different blocks of } \pi. \end{cases} \tag{4.72}$$

Then $g(\pi, \sigma) = 0$ for all $g \in SM_{\text{c-c}}^{\mathcal{NC}}$.

Proof. For any $g \in SM_{\text{c-c}}^{\mathcal{NC}}$ we write the factorization $g(\pi, \sigma) = \prod_{W \in \sigma} g(\pi_W, 1_{|W|})$ provided by semi-multiplicativity, and we observe that the factor $g(\pi_{W_o}, 1_{|W_o|})$ of this factorization is sure to be equal to 0, since the partition $\pi_{W_o} \in \mathcal{NC}(|W_o|)$ is not irreducible. $\square$

Proposition 4.7.2. 1^o *Let g_1, g_2 be in $SM_{\text{c-c}}^{\mathcal{NC}}$. Then $g_1 * g_2$ is in $SM_{\text{c-c}}^{\mathcal{NC}}$ as well.*

2^o *Let g be in $SM_{\text{c-c}}^{\mathcal{NC}}$. Then g^{-1} (inverse under convolution) is in $SM_{\text{c-c}}^{\mathcal{NC}}$ as well.*

Proof. 1^o Let $n \geq 1$ and let $\pi \in \mathcal{NC}(n)$ which is not irreducible, that is, 1 and n belong to distinct blocks of π. We want to prove that $g_1 * g_2\,(\pi, 1_n) = 0$. We have

$$g_1 * g_2\,(\pi, 1_n) = \sum_{\sigma \in \mathcal{NC}(n),\, \sigma \geq \pi} g_1(\pi, \sigma) \cdot g_2(\sigma, 1_n),$$

and we will argue that every term of the latter sum is equal to 0. Indeed, for a $\sigma \in \mathcal{NC}(n)$ such that $\sigma \geq \pi$ there are two possible cases.

Case 1: σ is not irreducible. In this case $g_2(\sigma, 1_n) = 0$, and thus $g_1(\pi, \sigma) \cdot g_2(\sigma, 1_n) = 0$.

Case 2: σ is irreducible. In this case the numbers 1 and n belong to the same block of σ, but belong to different blocks of π. Lemma 4.7.1 applies, and tells us that $g_1(\pi, \sigma) = 0$. It thus follows that $g_1(\pi, \sigma) \cdot g_2(\sigma, 1_n) = 0$ in this case as well.

2^o We fix an $n \geq 1$, for which we prove that:

$$g^{-1}(\pi, 1_n) = 0 \text{ for every } \pi \in \mathcal{NC}(n) \text{ which is not irreducible.} \tag{4.73}$$

We will verify (4.73) by induction on $|\pi| \in \{1, \ldots, n\}$. The base-case $|\pi| = 1$ holds vacuously, since the only partition with $|\pi| = 1$ is $\pi = 1_n$, which is irreducible. In the remaining part of the proof we discuss the induction step: we pick an $m \in \{2, \ldots, n\}$, we assume that (4.73) is true whenever $|\pi| < m$, and we verify that (4.73) is also true when $|\pi| = m$.

So consider a partition $\pi \in \mathcal{NC}(n)$ which is not irreducible and has $|\pi| = m$. We have $g * g^{-1}\,(\pi, 1_n) = e(\pi, 1_n) = 0$, and upon writing explicitly what is $g * g^{-1}\,(\pi, 1_n)$ we find (very similar to Equation (4.28) in the proof of Theorem 4.3.3) that

$$g^{-1}(\pi, 1_n) = - \sum_{\substack{\sigma \in \mathcal{NC}(n) \\ \sigma \geq \pi,\, \sigma \neq \pi}} g(\pi, \sigma)\, g^{-1}(\sigma, 1_n). \tag{4.74}$$

In order to arrive to the desired conclusion that $g^{-1}(\pi, 1_n) = 0$, we now verify that every term in the sum on the right-hand side of (4.74) is equal to 0. In reference to the partition σ which indexes the terms of that sum, we distinguish two cases.

Case 1: σ is not irreducible. Since $|\sigma| < |\pi| = m$ (as implied by the conditions $\sigma \geq \pi$, $\sigma \neq \pi$), the induction hypothesis applies to σ, and tells us that $g^{-1}(\sigma, 1_n) = 0$. Hence $g(\pi, \sigma)\, g^{-1}(\sigma, 1_n) = 0$, as we wanted.

Case 2: σ is irreducible. In this case the numbers 1 and n belong to the same block of σ, but belong to different blocks of π. Lemma 4.7.1 tells us that $g(\pi, \sigma) = 0$, and we thus get that $g(\pi, \sigma)\, g^{-1}(\sigma, 1_n) = 0$ in this case as well. □

4.7.2 Proof that $SM^{\mathcal{NC}}_{\text{c-m}}$ is a right coset of $SM^{\mathcal{NC}}_{\text{c-c}}$.

The claim about $SM^{\mathcal{NC}}_{\text{c-m}}$ that we want to prove is as stated in Equation (4.71) at the beginning of the section, where on the right-hand side we have to choose a suitable "representative" h picked from $SM^{\mathcal{NC}}_{\text{c-m}}$. As mentioned immediately following to (4.71), the coset representative we will go with is the function $g_{\text{bc-m}}$ introduced in Equation (4.53) of the preceding section. In connection to it, we first prove a lemma.

Lemma 4.7.3. *Let g be in $SM^{\mathcal{NC}}_{\text{c-c}}$. Then: 1^o $g * g_{\text{bc-m}} \in SM^{\mathcal{NC}}_{\text{c-m}}$.*

2^o One has

$$g * g_{\text{bc-m}}\,(\pi, 1_n) = g(\pi, 1_n) \ \text{ for every } n \geq 1 \text{ and } \pi \in \mathcal{NCIRR}(n). \tag{4.75}$$

Proof. We start by recording the general fact that

$$g * g_{\text{bc-m}}\,(\pi, 1_n) = \sum_{\sigma \in \mathcal{I}(n),\, \sigma \geq \pi} g(\pi, \sigma), \quad \forall\, n \geq 1 \text{ and } \pi \in \mathcal{NC}(n). \tag{4.76}$$

This is obtained directly from the definition of the convolution operation, when we take into account the specifics of what is $g_{\text{bc-m}}$.

Proof of 1^o. We take a partition $\pi = \pi_1 \diamond \pi_2 \in \mathcal{NC}(n)$ where $n = n_1 + n_2$ with $n_1, n_2 \geq 1$ and where $\pi_1 \in \mathcal{NC}(n_1)$, $\pi_2 \in \mathcal{NC}(n_2)$. Our goal here is to verify that

$$g * g_{\text{bc-m}}\,(\pi, 1_n) = \big(g * g_{\text{bc-m}}\,(\pi_1, 1_{n_1})\big) \cdot \big(g * g_{\text{bc-m}}\,(\pi_2, 1_{n_2})\big). \tag{4.77}$$

Observe that we clearly have

$$\{\sigma \in \mathcal{I}(n) \mid \sigma \geq \pi\} \supseteq \left\{ \sigma_1 \diamond \sigma_2 \;\middle|\; \begin{array}{l} \sigma_1 \in \mathcal{I}(n_1),\, \sigma_1 \geq \pi_1, \\ \sigma_2 \in \mathcal{I}(n_2),\, \sigma_2 \geq \pi_2 \end{array} \right\}. \tag{4.78}$$

By starting from (4.76), we can thus write

$$g * g_{\mathrm{bc-m}}(\pi, 1_n) = \Big(\sum_{\substack{\sigma_1 \in \mathcal{I}(n_1),\, \sigma_1 \geq \pi_1 \\ \sigma_2 \in \mathcal{I}(n_2),\, \sigma_2 \geq \pi_2}} g(\pi, \sigma_1 \diamond \sigma_2) \Big) + \sum_{\sigma \in \mathcal{J}} g(\pi, \sigma), \tag{4.79}$$

where $\mathcal{J}$ denotes the difference of the two sets indicated in (4.78). But note that $\mathcal{J}$ can be described as

$$\mathcal{J} = \Big\{ \sigma \in \mathcal{I}(n) \;\Big|\; \begin{array}{l} \sigma \geq \pi \text{ and there exists a block } W_o \text{ of } \sigma \\ \text{such that } \min(W_o) \leq n_1,\ \max(W_o) > n_1 \end{array} \Big\};$$

a direct application of Lemma 4.7.1 then gives that $g(\pi, \sigma) = 0$ for every $\sigma \in \mathcal{J}$. So the second sum on the right-hand side of Equation (4.79) is actually equal to 0. Concerning the first sum appearing there, we observe that its general term can be written as

$$g(\pi, \sigma_1 \diamond \sigma_2) = g(\pi_1 \diamond \pi_2, \sigma_1 \diamond \sigma_2) = g(\pi_1, \sigma_1) \cdot g(\pi_2, \sigma_2),$$

with the factorization at the second equality sign coming from the semi-multiplicativity of g. When we put these observations together, we find that (4.79) leads to the factorization

$$g * g_{\mathrm{bc-m}}(\pi, 1_n) = \Big(\sum_{\sigma_1 \in \mathcal{I}(n_1),\, \sigma_1 \geq \pi_1} g(\pi_1, \sigma_1) \Big) \cdot \Big(\sum_{\sigma_2 \in \mathcal{I}(n_2),\, \sigma_2 \geq \pi_2} g(\pi_2, \sigma_2) \Big). \tag{4.80}$$

Finally, upon specializing the general Equation (4.76) to the case of the partitions π_1 and π_2, we identify the right-hand side of (4.80) as being the product that had been announced on the right-hand side of (4.77). This completes the verification that had to be done in this part of the proof.

Proof of 2^o. Let $n \geq 1$ and let $\pi \in \mathcal{NC}(n)$ be irreducible. It is immediate that, since 1 and n belong to the same block of π, the only $\sigma \in \mathcal{I}(n)$ such that $\sigma \geq \pi$ is $\sigma = 1_n$. Hence, in this special case, the sum on the right-hand side of (4.76) has only 1 term, which is equal to $g(\pi, 1_n)$. It follows that $g * g_{\mathrm{bc-m}}(\pi, 1_n) = g(\pi, 1_n)$, as stated in (4.75). □

Proposition 4.7.4. $SM^{\mathcal{NC}}_{\mathrm{c-m}} = \{ g * g_{\mathrm{bc-m}} \mid g \in SM^{\mathcal{NC}}_{\mathrm{c-c}} \}$.

Proof. "$\supseteq$": This inclusion is provided by Lemma 4.7.3.1.

"$\subseteq$": Let a function $h \in SM^{\mathcal{NC}}_{\mathrm{c-m}}$ be given. We have to prove that h can be written in the form $h = g * g_{\mathrm{bc-m}}$, with $g \in SM^{\mathcal{NC}}_{\mathrm{c-c}}$.

Proposition 4.2.5 assures us that there exists $g \in SM^{\mathcal{NC}}$, uniquely determined, such that for every $n \geq 1$ and $\pi \in \mathcal{NC}(n)$ we have

$$g(\pi, 1_n) = \begin{cases} h(\pi, 1_n), & \text{if } \pi \text{ is irreducible,} \\ 0, & \text{otherwise.} \end{cases} \tag{4.81}$$

Since Equation (4.81) includes the fact that $g(\pi, 1_n) = 0$ whenever π is not irreducible, we know that $g \in SM^{\mathcal{NC}}_{\text{c-c}}$. Let $\widetilde{h} := g * g_{\text{bc-m}}$. Then $\widetilde{h} \in SM^{\mathcal{NC}}_{\text{c-m}}$, by Lemma 4.7.3.1. Moreover, for every $n \geq 1$ and irreducible partition $\pi \in \mathcal{NC}(n)$ we have

$$\begin{aligned} \widetilde{h}(\pi, 1_n) &= g(\pi, 1_n) \quad \text{(by Lemma 4.7.3.2)} \\ &= h(\pi, 1_n) \quad \text{(by (4.81)).} \end{aligned}$$

We thus get to have two functions h and $\widetilde{h}$ in $SM^{\mathcal{NC}}_{\text{c-m}}$, such that $h(\pi, 1_n) = \widetilde{h}(\pi, 1_n)$ whenever $\pi \in \mathcal{NC}(n)$ is irreducible. As observed in Remark 4.6.2.2, this implies $h = \widetilde{h}$. In particular we have obtained $h = g * g_{\text{bc-m}}$ with $g \in SM^{\mathcal{NC}}_{\text{c-c}}$, and this concludes the proof. $\square$

Proposition 4.7.4 has established the required claim that $SM^{\mathcal{NC}}_{\text{c-m}}$ is a right coset of the subgroup $SM^{\mathcal{NC}}_{\text{c-c}} \subseteq SM^{\mathcal{NC}}$. It is useful to also record the following fact, which gives a converse to Lemma 4.7.3.2, and was implicitly included in our method of deriving Proposition 4.7.4.

Corollary 4.7.5. *Suppose that $g \in SM^{\mathcal{NC}}_{\text{c-c}}$, $h \in SM^{\mathcal{NC}}_{\text{c-m}}$ and it holds true that*

$$g(\pi, 1_n) = h(\pi, 1_n), \quad \forall\, n \geq 1 \text{ and irreducible } \pi \in \mathcal{NC}(n). \tag{4.82}$$

Then it follows that $h = g * g_{\text{bc-m}}$.

Proof. We have $g * g_{\text{bc-m}} \in SM^{\mathcal{NC}}_{\text{c-m}}$, by Proposition 4.7.4. We are thus required to prove an equality between two functions in $SM^{\mathcal{NC}}_{\text{c-m}}$. To this end, we know (cf. Remark 4.6.2.2) it is sufficient to verify that the two functions in question, h and $g * g_{\text{bc-m}}$, agree on every couple $(\pi, 1_n)$ where $n \geq 1$ and $\pi \in \mathcal{NC}(n)$ is irreducible. And indeed, for such $(\pi, 1_n)$ we have

$$\begin{aligned} g * g_{\text{bc-m}}\,(\pi, 1_n) &= g(\pi, 1_n) \text{ (by Lemma 4.7.3.2)} \\ &= h(\pi, 1_n) \text{ (by hypothesis).} \end{aligned}$$

$\square$

4.7.3 An application: why Boolean cumulants are the easiest to connect to

In this subsection we show how the method of proof used in Proposition 4.7.4 can be invoked to retrieve a known "rule of thumb", which says that when given a cumulant-to-moment summation formula, it is usually immediate to write down the corresponding cumulant-to-(Boolean cumulant) summations: one uses *the very same coefficients* as in the description of moments, only that the summations are now restricted to non-crossing partitions that are irreducible. The precise statement of this fact goes as follows.

Proposition 4.7.6. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Suppose we are given a sequence of multilinear functionals $\underline{\lambda} = (\lambda_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{A}}$ and a family of complex coefficients $(c(\pi))_{\pi \in \mathcal{NC}}$ such that for every $n \geq 1$ and $x_1, \ldots, x_n \in \mathcal{A}$ we have*

$$\varphi(x_1 \cdots x_n) = \sum_{\pi \in \mathcal{NC}(n)} c(\pi) \prod_{V \in \pi} \lambda_{|V|}\big((x_1, \ldots, x_n) \mid V \big). \tag{4.83}$$

Suppose moreover that, in relation to the operation "$\diamond$" of concatenating non-crossing partitions, the coefficients $c(\pi)$ have the property that

$$c(\pi_1 \diamond \pi_2) = c(\pi_1) \cdot c(\pi_2), \quad \forall\, \pi_1, \pi_2 \in \mathcal{NC}. \tag{4.84}$$

Then: denoting by $\underline{b} = (b_n)_{n=1}^{\infty}$ the Boolean cumulant functionals of $(\mathcal{A}, \varphi)$, we have

$$b_n(x_1, \ldots, x_n) = \sum_{\pi \in \mathcal{NCIRR}(n)} c(\pi) \prod_{V \in \pi} \lambda_{|V|}\big((x_1, \ldots, x_n) \mid V \big), \tag{4.85}$$

holding for every $n \geq 1$ and every $x_1, \ldots, x_n \in \mathcal{A}$.

Proof. Let h be the function in $SM^{\mathcal{NC}}$ which is defined via the requirement that $h(\pi, 1_n) = c(\pi)$, for all $n \geq 1$ and $\pi \in \mathcal{NC}(n)$. The factorization hypothesis (4.84) satisfied by the coefficients $c(\pi)$ tells us that h is a function of cumulant-to-moment type. On the other hand: Equation (4.83) can be re-written concisely in terms of h, in the form of the relation $\underline{\varphi} = \underline{\lambda} \cdot h$, where $\underline{\varphi}$ is the family of moment functionals of the space $(\mathcal{A}, \varphi)$. We can therefore write that

$$\begin{aligned} \underline{b} &= \underline{\varphi} \cdot g_{\text{bc-m}}^{-1} \text{ (Boolean cumulants expressed in terms of moments)} \\ &= (\underline{\lambda} \cdot h) \cdot g_{\text{bc-m}}^{-1} = \underline{\lambda} \cdot (h * g_{\text{bc-m}}^{-1}) = \underline{\lambda} * g, \end{aligned}$$

where we denoted $g := h * g_{\text{bc-m}}^{-1}$.

Since $h \in SM^{\mathcal{NC}}_{\mathrm{c-m}}$, Proposition 4.7.4 implies that $g \in SM^{\mathcal{NC}}_{\mathrm{c-c}}$. Moreover, since g and h are related by the convolution $h = g * g_{\mathrm{bc-m}}$, Lemma 4.7.3.2 tells us that we have

$$g(\pi, 1_n) = h(\pi, 1_n), \quad \forall\, n \geq 1 \text{ and irreducible } \pi \in \mathcal{NC}(n).$$

By taking into account that $h(\pi, 1_n) = c(\pi)$, we thus come to the conclusion that for every $n \geq 1$ and $\pi \in \mathcal{NC}(n)$ we have

$$g(\pi, 1_n) = \begin{cases} c(\pi), & \text{if } \pi \text{ is irreducible,} \\ 0, & \text{otherwise.} \end{cases} \tag{4.86}$$

Finally, for a fixed $n \geq 1$ we obtain that

$$b_n = \sum_{\pi \in \mathcal{NC}(n)} g(\pi, 1_n)\, \lambda_\pi = \sum_{\pi \in \mathcal{NCIRR}(n)} c(\pi)\, \lambda_\pi,$$

where the first equality is just spelling out the meaning of "$\underline{b} = \underline{\lambda} \cdot g$", and the second equality makes use of (4.86). The formula for b_n obtained in this way is precisely the one stated in Equation (4.85). □

Here is how the preceding proposition applies to some examples discussed in Section 4.5. A rather general result of this kind, going in a framework of cumulant constructions related to trees, appears as Lemma 7.6 of [JL19], in Section 7.3 we will briefly explain how these constructions related to trees also fit in our framework.

Example 4.7.7. *(Boolean cumulants in terms of t-Boolean cumulants.)*

Let $t \in \mathbb{R}$ be a parameter, and in Proposition 4.7.6 let us make $\underline{\lambda}$ be the family of t-Boolean cumulant functionals of $(\mathcal{A}, \varphi)$: $\underline{\lambda} = \underline{b}^{(t)} = (b_n^{(t)})_{n=1}^{\infty}$, when the coefficients of interest are $c(\pi) = t^{\mathrm{inner}(\pi)}$ and Equation (4.83) becomes the moment-cumulant formula recorded in Definition 4.6.6. The factorization condition from (4.84) is holding; this corresponds precisely to the fact that the function $g^{(t)}_{\mathrm{bc-m}}$ introduced in Notation 4.6.8 is of cumulant-to-moment type. Thus Proposition 4.7.6 applies, and yields a formula expressing Boolean cumulants in terms of t-Boolean cumulants:

$$\beta_n = \sum_{\pi \in \mathcal{NCIRR}(n)} t^{|\pi|-1} b_\pi^{(t)}, \quad n \geq 1, \tag{4.87}$$

where on the right-hand side we took into account that an irreducible partition π has $\mathrm{inner}(\pi) = |\pi| - 1$, and therefore has $c(\pi) = t^{|\pi|-1}$. A generalization of this formula appears in Corollary 4.8.5 of the next section.

Example 4.7.8. *(Boolean cumulants in terms of monotone cumulants.)*

In Proposition 4.7.6 let us make $\underline{\lambda}$ be the family of monotone cumulant functionals of $(\mathcal{A}, \varphi)$: $\underline{\lambda} = \underline{\gamma} = (\gamma_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$, when the coefficients of interest are

$$c(\pi) = \frac{m(\pi)}{|\pi|!}, \quad \pi \in \mathcal{NC},$$

and Equation (4.83) becomes the moment-cumulant formula recorded in Definition 2.3.2. The factorization condition from (4.84) is holding; this corresponds precisely to the fact that the function $g_{\mathrm{mc-m}}$ introduced in Subsection 2.2.6 is of cumulant-to-moment type. Thus Proposition 4.7.6 applies, and yields a formula expressing Boolean cumulants in terms of monotone cumulants:

$$\beta_n = \sum_{\pi \in \mathcal{NCIRR}(n)} \frac{m(\pi)}{|\pi|!} \cdot \gamma_\pi, \quad n \geq 1. \tag{4.88}$$

This retrieves Equation (1.6) of [AHLV15], which is the beginning of the analysis done in that paper on how to relate monotone cumulants to other brands of cumulants.

Equation (4.88) is equivalent to a formula that gives an explicit description of the function $g_{\mathrm{mc-bc}} := g_{\mathrm{mc-m}} * g_{\mathrm{bc-m}}^{-1} \in SM_{\mathrm{c-c}}^{\mathcal{NC}}$, encoding the transition from monotone cumulants to Boolean cumulants. We mention that it is an interesting and non-trivial issue, addressed in [AHLV15] and in the recent paper [CEFPP21], to describe explicitly the inverse

$$g_{\mathrm{mc-bc}}^{-1} = \left(g_{\mathrm{mc-m}} * g_{\mathrm{bc-m}}^{-1} \right)^{-1} = g_{\mathrm{bc-m}} * g_{\mathrm{mc-m}}^{-1} \in SM_{\mathrm{c-c}}^{\mathcal{NC}},$$

which encodes the reverse transition from Boolean to monotone cumulants. In Subsection 7.1.4 we will explain how we can address this problem using our framework and Hopf algebras of graphs.

4.8 The 1-parameter subgroup $\{u_q \mid q \in \mathbb{R}\}$ of $SM^{\mathcal{NC}}$, and its action on $g_{\mathrm{bc-m}}^{(t)}$'s

The method used for studying the right coset $SM_{\mathrm{c-m}}^{\mathcal{NC}}$ in Section 4.7.2 draws attention to a 1-parameter family of functions in $SM_{\mathrm{c-c}}^{\mathcal{NC}}$, defined as follows.

Notation 4.8.1. For every $q \in \mathbb{R}$, we denote by u_q the function in $SM^{\mathcal{NC}}$ which is determined via the requirement that for all $n \geq 1$ and $\pi \in \mathcal{NC}(n)$ we have

$$u_q(\pi, 1_n) = \begin{cases} q^{|\pi|-1}, & \text{if } \pi \text{ is irreducible} \\ 0, & \text{otherwise.} \end{cases} \tag{4.89}$$

Clearly, this is a function of cumulant-to-cumulant type.

Remark 4.8.2. 1^o In the case $q = 0$, the usual conventions apply to yield that $u_0(1_n, 1_n) = 1$ and $u_0(\pi, 1_n) = 0$ for every $\pi \neq 1_n$ in $\mathcal{NC}(n)$. This implies that $u_0 = e$, the unit of $SM^{\mathcal{NC}}$.

2^o The formula for the values taken by u_q on general couples in $\mathcal{NC}^{(2)}$ is determined from (4.89) by using the semi-multiplicativity property; we leave it as an easy exercise to the reader to check that for every $n \geq 1$ and $\pi \leq \sigma$ in $\mathcal{NC}(n)$ one gets:

$$u_q(\pi, \sigma) = \begin{cases} q^{|\pi|-|\sigma|}, & \text{if } \pi \ll \sigma, \\ 0, & \text{otherwise,} \end{cases} \tag{4.90}$$

where $\ll$ is one of the partial order relations reviewed in Subsection 2.2.5.

Proposition 4.8.3. *The u_q's form a 1-parameter subgroup of $SM^{\mathcal{NC}}$:*

$$u_{q_1} * u_{q_2} = u_{q_1+q_2}, \quad \textit{for all } q_1, q_2 \in \mathbb{R}. \tag{4.91}$$

Proof. We fix $q_1, q_2 \in \mathbb{R}$ for which we will prove that (4.91) holds. The case when $q_1 = 0$ or $q_2 = 0$ is clear, so we will assume that $q_1 \neq 0 \neq q_2$.

Since both $u_{q_1} * u_{q_2}$ and $u_{q_1+q_2}$ are in $SM^{\mathcal{NC}}_{\text{c-c}}$, in order to prove their equality it will suffice (cf. Remark 4.6.2) to check that

$$\begin{cases} u_{q_1} * u_{q_2}\,(\pi, 1_n) = u_{q_1+q_2}(\pi, 1_n) \\ \text{for every } n \geq 1 \text{ and every irreducible } \pi \in \mathcal{NC}(n). \end{cases} \tag{4.92}$$

For the rest of the proof, we fix an $n \geq 1$ and an irreducible $\pi \in \mathcal{NC}(n)$ for which we will verify that Equation (4.92) holds.

The right-hand side of (4.92) is, directly from the definitions, equal to $(q_1 + q_2)^{|\pi|-1}$. So our job is to verify that the left-hand side of (4.92) is equal to that same quantity.

We compute:

$$u_{q_1} * u_{q_2}\,(\pi, 1_n) = \sum_{\substack{\sigma \in \mathcal{NC}(n), \\ \sigma \geq \pi}} u_{q_1}(\pi, \sigma) \cdot u_{q_2}(\sigma, 1_n) = \sum_{\substack{\sigma \in \mathcal{NC}(n), \\ \sigma \gg \pi}} q_1^{|\pi|-|\sigma|} \cdot q_2^{|\sigma|-1},$$

where at the second equality sign we used (4.90) and also the fact that every $\sigma \geq \pi$ in $\mathcal{NC}(n)$ is irreducible, thus has $u_{q_2}(\sigma, 1_n) = q_2^{|\sigma|-1}$. In the latter summation over σ we sort out the terms according to what is $|\sigma|$. As reviewed in Remark 2.2.26, one has

$$\left| \{\sigma \in \mathcal{NC}(n) \mid \sigma \gg \pi,\, |\sigma| = k\} \right| = \left(\begin{array}{c} |\pi| - 1 \\ k-1 \end{array} \right), \quad \forall\, k \in \{1, \ldots, |\pi|\}.$$

Hence our evaluation of the left-hand side of Equation (4.92) continues as follows:

$$u_{q_1} * u_{q_2}(\pi, 1_n) = \sum_{k=1}^{|\pi|} \binom{|\pi|-1}{k-1} q_1^{|\pi|-k} q_2^{k-1} = \sum_{\ell=0}^{|\pi|-1} \binom{|\pi|-1}{\ell} q_1^{(|\pi|-1)-\ell} q_2^{\ell} = (q_1+q_2)^{|\pi|-1},$$

which is precisely the value we wanted to obtain. $\square$

We now consider the functions of cumulant-to-moment type $g^{(t)}_{\mathrm{bc-m}}$ introduced in Section 4.6.2, which encode moment-cumulant formulas for t-Boolean cumulants, and we look at how our 1-parameter subgroup of u_q's acts on $g^{(t)}_{\mathrm{bc-m}}$'s, by left translations.

Proposition 4.8.4. *One has*

$$u_q * g^{(t)}_{\mathrm{bc-m}} = g^{(q+t)}_{\mathrm{bc-m}}, \quad \textit{for all } q, t \in \mathbb{R}. \tag{4.93}$$

Proof. We first verify the special case of (4.93) when $t = 0$. Since $g^{(0)}_{\mathrm{bc-m}}$ is just the function $g_{\mathrm{bc-m}}$ from Definition 4.6.4, this case amounts to checking that for every $q \in \mathbb{R}$ we have

$$u_q * g_{\mathrm{bc-m}} = g^{(q)}_{\mathrm{bc-m}}. \tag{4.94}$$

And indeed, let us notice that: u_q is in $SM^{\mathcal{NC}}_{\mathrm{c-c}}$, $g^{(t)}_{\mathrm{bc-m}}$ is in $SM^{\mathcal{NC}}_{\mathrm{c-m}}$, and they are such that

$$g^{(q)}_{\mathrm{bc-m}}(\pi, 1_n) = q^{\mathrm{inner}(\pi)} = q^{|\pi|-1} = u_q(\pi, 1_n), \quad \forall\, \pi \in \mathcal{NC}(n), \text{ irreducible.}$$

Thus Equation (4.94) does hold, as a special case of Corollary 4.7.5.

Going to general $q, t \in \mathbb{R}$ we can then write:

$$\begin{aligned} u_q * g^{(t)}_{\mathrm{bc-m}} &= u_q * (u_t * g_{\mathrm{bc-m}}) \text{ (by (4.94))} \\ &= (u_q * u_t) * g_{\mathrm{bc-m}} \\ &= u_{q+t} * g_{\mathrm{bc-m}} \text{ (by Proposition 4.8.3)} \\ &= g^{(q+t)}_{\mathrm{bc-m}} \text{ (by (4.94))}, \end{aligned}$$

yielding the required Equation (4.93). $\square$

The preceding proposition yields, in particular, the explicit transition formula from s-Boolean cumulants to t-Boolean cumulants which was anticipated in Remark 4.6.9 of Section 4.6.2. Recall that, in the said Remark 4.6.9, the point that remained to be justified was the validity of Equation (4.60); this is now very easy to fill in.

Corollary 4.8.5. (A repeat of Equation (4.60).)

Let s and t be real parameters, and consider the functions $g^{(s)}_{\mathrm{bc-m}}, g^{(t)}_{\mathrm{bc-m}} \in SM^{\mathcal{NC}}_{\mathrm{c-c}}$. For every $n \geq 1$ and $\pi \in \mathcal{NC}(n)$ one has:

$$g^{(s)}_{\mathrm{bc-m}} * (g^{(t)}_{\mathrm{bc-m}})^{-1} \, (\pi, 1_n) = \begin{cases} (s-t)^{|\pi|-1}, & \text{if } \pi \text{ is irreducible,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Proposition 4.8.4 says that $g^{(s)}_{\mathrm{bc-m}} = u_{s-t} * g^{(t)}_{\mathrm{bc-m}}$, which implies that

$$g^{(s)}_{\mathrm{bc-m}} * (g^{(t)}_{\mathrm{bc-m}})^{-1} = u_{s-t}. \tag{4.95}$$

We evaluate both sides of (4.95) at $(\pi, 1_n)$, then we refer to the formula for $u_{s-t}(\pi, 1_n)$ which comes from Equation (4.89), and the corollary follows. $\square$

4.9 The action of $\{u_q \mid q \in \mathbb{R}\}$, by conjugation, on multiplicative functions

The goal of the present section is to prove the following result.

Theorem 4.9.1. *Let $\{u_q \mid q \in \mathbb{R}\}$ be as in the preceding section and let $MF^{\mathcal{NC}}$ be the subgroup of $SM^{\mathcal{NC}}$ which consists of multiplicative functions, as reviewed in Section 5. One has that:*

$$\Big(q \in \mathbb{R} \text{ and } f \in MF^{\mathcal{NC}} \Big) \ \Rightarrow \ u_q^{-1} * f * u_q \in MF^{\mathcal{NC}}. \tag{4.96}$$

Remark 4.9.2. Recall from Remark 4.4.2 and Proposition 4.4.3 that a function $f \in MF^{\mathcal{NC}}$ is completely determined by the sequence of complex numbers $(\lambda_n)_{n=1}^{\infty}$, where $\lambda_n := f(0_n, 1_n)$, $n \geq 1$. If we accept Theorem 4.9.1, then it follows that $u_q^{-1} * f * u_q$ must be determined in a similar way by the sequence of η_n's where $\eta_n := u_q^{-1} * f * u_q \, (0_n, 1_n)$ for $n \geq 1$. It is easy to write down the explicit formula which gives η_n in terms of $\lambda_1, \ldots, \lambda_n$ and q, this is:

$$\eta_n = \sum_{\mathcal{NCIRR}(n)} q^{|\pi|-1} \prod_{V \in \pi} \lambda_{|V|}. \tag{4.97}$$

This gives for instance:

$$\eta_1 = \lambda_1 = 1,\ \eta_2 = \lambda_2,\ \eta_3 = \lambda_3 + q\lambda_2,\ \eta_4 = \lambda_4 + 2q\lambda_3 + q\lambda_2^2 + q^2\lambda_2.$$

Verification of (4.97): use the definition of the convolution operation "$*$" to find that

$$\eta_n = u_q^{-1} * f * u_q\,(0_n, 1_n) = \sum_{\substack{\sigma,\tau \in \mathcal{NC}(n),\\ \sigma \leq \tau}} u_q^{-1}(0_n, \sigma)\, f(\sigma, \tau)\, u_q(\tau, 1_n), \tag{4.98}$$

then notice that $u_q^{-1}(0_n, \sigma) = 0$ for every $\sigma \neq 0_n$ in $\mathcal{NC}(n)$ (since $u_q^{-1} = u_{-q}$ and we can invoke the formula (4.90)). Thus σ in (4.98) is forced to be 0_n, and we continue with

$$= \sum_{\tau \in \mathcal{NC}(n)} f(0_n, \tau) u_q(\tau, 1_n),$$

which yields (4.97) upon replacing $f(0_n, \tau)$ by $\prod_{V \in \tau} \lambda_{|V|}$ and $u_q(\tau, 1_n)$ from (4.89).

Deriving the formula (4.97) for η_n does not, however, substitute for a proof of Theorem 4.9.1. We still need to evaluate $u_q^{-1} * f * u_q\,(\pi, 1_n)$ for general $\pi \in \mathcal{NC}(n)$, and to suitably express the resulting value as a product of η_m's. In order to achieve this we will prove two factorization formulas, presented in Lemmas 4.9.5 and 4.9.8 below.

For Lemma 4.9.5 we will need the following notation.

Notation and Remark 4.9.3. *(Irreducible cover of a non-crossing partition.)*

Let $n \geq 1$ and let π be in $\mathcal{NC}(n)$.

1^o It is easy to see that there exists a partition $\overline{\pi}^{\mathrm{irr}} \in \mathcal{NC}(n)$, uniquely determined, with the following properties:

$$\begin{cases} \text{(i)} & \overline{\pi}^{\mathrm{irr}} \text{ is irreducible and } \overline{\pi}^{\mathrm{irr}} \geq \pi; \\ \text{(ii)} & \text{If } \sigma \in \mathcal{NC}(n) \text{ is irreducible and } \sigma \geq \pi, \text{ then } \sigma \geq \overline{\pi}^{\mathrm{irr}}. \end{cases} \tag{4.99}$$

We will refer to $\overline{\pi}^{\mathrm{irr}}$ as the *irreducible cover* of π. For its explicit description we distinguish two cases.

Case 1. π is irreducible. Then, clearly, $\overline{\pi}^{\mathrm{irr}} = \pi$.

Case 2. π is not irreducible. Then the blocks $V_{\mathrm{left}}, V_{\mathrm{right}} \in \pi$ which contain the numbers 1 respectively n are such that $V_{\mathrm{left}} \neq V_{\mathrm{right}}$. In this case, $\overline{\pi}^{\mathrm{irr}}$ is obtained out of π by merging together V_{left} and V_{right}. (It is easy to see that the said merger is sure to give a partition which is still non-crossing and has all the properties required in (4.99).)

2^o In what follows, we will need at some point to deal with the relative Kreweras complement of π in $\overline{\pi}^{\mathrm{irr}}$. In the case when $\overline{\pi}^{\mathrm{irr}} = \pi$, the complement $\mathrm{Kr}_{\overline{\pi}^{\mathrm{irr}}}(\pi)$ is just 0_n. In the case when $\overline{\pi}^{\mathrm{irr}} \neq \pi$ (i.e. in the Case 2 indicated above), the complement $\mathrm{Kr}_{\overline{\pi}^{\mathrm{irr}}}(\pi)$ has 1 block with 2 elements and $n-2$ blocks with 1 element. Upon drawing a picture which features the outer blocks of π, the reader should have no difficulty to check that, in Case 2, the unique 2-element block of $\mathrm{Kr}_{\overline{\pi}^{\mathrm{irr}}}(\pi)$ is of the form $\{m, n\}$, with m described as follows:

$$\begin{cases} m = \max(V_{\mathrm{left}}) = \min(W_{\mathrm{right}}), \text{ where} \\ V_{\mathrm{left}} \text{ is the block of } \pi \text{ which contains the number } 1, \text{ and} \\ W_{\mathrm{right}} \text{ is the block of } \mathrm{Kr}(\pi) \text{ which contains the number } n. \end{cases} \tag{4.100}$$

3^o The drawing of the outer blocks of π that was recommended above also reveals that the block $W_{\mathrm{right}} \in \mathrm{Kr}(\pi)$ can be explicitly written in the form

$$W_{\mathrm{right}} = \{\max(U_1), \ldots, \max(U_k)\}, \tag{4.101}$$

where $U_1, \ldots, U_k$ are the outer blocks of π (in particular, $U_1 = V_{\mathrm{left}}$). A consequence of (4.101) which will be needed in the sequel is this: any partition $\sigma \in \mathcal{NC}(n)$ such that $\sigma \gg \pi$ is sure to produce the same "W_{right}" as π. This is because the relation "$\gg$" forces σ to have the same maximal elements of outer blocks as π does.

Notation and Remark 4.9.4. In Lemma 4.9.5 we will use three sequences of numbers $(\alpha_n)_{n=1}^{\infty}$, $(\widehat{\alpha}_n)_{n=1}^{\infty}$ and $(\widetilde{\alpha}_n)_{n=1}^{\infty}$, where the $\widehat{\alpha}_n$'s and $\widetilde{\alpha}_n$'s are obtained out of the α_n's via summation formulas, as follows:

$$\widehat{\alpha}_n = \sum_{\pi \in \mathcal{NC}(n)} \prod_{V \in \pi} \alpha_{|V|} \ \text{ and } \ \widetilde{\alpha}_n = \sum_{\pi \in \mathcal{NCIRR}(n)} \prod_{V \in \pi} \alpha_{|V|}, \quad n \geq 1. \tag{4.102}$$

One can also write summation formulas which give a direct relation between the two derived sequences $(\widehat{\alpha}_n)_{n=1}^{\infty}$ and $(\widetilde{\alpha}_n)_{n=1}^{\infty}$. For future reference, we record here one such formula (which is not hard to verify via direct calculation) saying that

$$\widetilde{\alpha}_n = \sum_{\rho \in \mathcal{I}(n)} (-1)^{|\rho|+1} \prod_{J \in \rho} \widehat{\alpha}_{|J|}, \quad \forall\, n \geq 1. \tag{4.103}$$

Lemma 4.9.5. (A factorization formula.) *Consider sequences of numbers as in Notation 4.9.4, and on the other hand let us pick an $n \geq 1$ and a partition $\pi \in \mathcal{NC}(n)$. We consider the Kreweras complement $\mathrm{Kr}(\pi)$ and, same as in Remark 4.9.3, we denote by W_{right} the block of $\mathrm{Kr}(\pi)$ which contains the number n. Then:*

$$\sum_{\substack{\sigma \in \mathcal{NC}(n), \\ \sigma \geq \overline{\pi}^{\mathrm{irr}}}} \Big(\prod_{U \in \mathrm{Kr}_{\sigma}(\pi)} \alpha_{|U|} \Big) = \widetilde{\alpha}_{|W_{\mathrm{right}}|} \cdot \prod_{\substack{W \in \mathrm{Kr}(\pi), \\ W \not\ni n}} \widehat{\alpha}_{|W|}. \tag{4.104}$$

Proof. On the left-hand side of (4.104) we perform the change of variable "$\tau = \mathrm{Kr}_\sigma(\pi)$". When σ runs in the interval $[\overline{\pi}^{\mathrm{irr}}, 1_n] \subseteq \mathcal{NC}(n)$, the relative Kreweras complement τ runs in the interval $[\mathrm{Kr}_{\overline{\pi}^{\mathrm{irr}}}(\pi), \mathrm{Kr}_{1_n}(\pi)]$, where $\mathrm{Kr}_{1_n}(\pi)$ is just $\mathrm{Kr}(\pi)$. For a discussion of this nice behaviour of the partition $\mathrm{Kr}_\sigma(\pi)$ viewed as a function of σ (and with π fixed) see [NS06, Lemma 18.9].

Let us also recall, from Remark 4.9.3.2, that the inequality $\tau \geq \mathrm{Kr}_{\overline{\pi}^{\mathrm{irr}}}(\pi)$ amounts to requesting that τ connects m with n, where $m = \min(W_{\mathrm{right}})$. Our processing of the left-hand side of Equation (4.104) has thus taken us to:

$$\sum_{\substack{\tau \in \mathcal{NC}(n),\, \tau \leq \mathrm{Kr}(\pi) \\ \text{and } \tau \text{ connects } m \text{ with } n}} \prod_{U \in \tau} \alpha_{|U|}. \tag{4.105}$$

Now let us write explicitly $\mathrm{Kr}(\pi) = \{W_1, \ldots, W_p\}$, with the blocks listed such that $W_p = W_{\mathrm{right}}$. A standard decomposition argument shows that a partition $\tau \in \mathcal{NC}(n)$ such that $\tau \leq \mathrm{Kr}(\pi)$ is bijectively identified to the tuple

$$(\tau_{W_1}, \ldots, \tau_{W_p}) \in \mathcal{NC}(|W_1|) \times \cdots \times \mathcal{NC}(|W_p|)$$

which records the restrictions of τ to the blocks $W_1, \ldots, W_p$. At the level of the tuple $(\tau_{W_1}, \ldots, \tau_{W_p})$, the requirement that "$\tau$ connects m with n" (where m and n are the minimal and maximal elements of the block $W_{\mathrm{right}} = W_p$) is transformed into the requirement that $\tau_{W_p} \in \mathcal{NC}(|W_p|)$ is irreducible. We leave it as a straightforward exercise to the reader to check that, upon performing the change of variable $\tau \leftrightarrow (\tau_{W_1}, \ldots, \tau_{W_p})$ in the summation from (4.105), one gets precisely the product of p separate summations which is indicated on the right-hand side of (4.104). $\square$

Notation and Remark 4.9.6. The second factorization formula that we want to use is presented in Lemma 4.9.8. We find it convenient to first prove this lemma in a special case, stated separately as Lemma 4.9.7. In these lemmas we use two sequences of complex numbers, $(\gamma_n)_{n=1}^\infty$ and $(\widehat{\gamma}_n)_{n=1}^\infty$, with $\gamma_1 = \widehat{\gamma}_1 \neq 0$, and where the $\widehat{\gamma}_n$'s are expressed in terms of γ_n's by summations over interval partitions, as follows:

$$\widehat{\gamma}_n = \sum_{\rho \in \mathcal{I}(n)} \Big(\prod_{J \in \rho} \gamma_{|J|} \Big), \quad \forall\, n \geq 1. \tag{4.106}$$

Lemma 4.9.7. *Consider the framework of Notation 4.9.6, and on the other hand consider an $n \geq 1$ and an irreducible partition $\pi \in \mathcal{NC}(n)$. Then:*

$$\sum_{\substack{\sigma \in \mathcal{NC}(n), \\ \sigma \gg \pi}} \Bigg(\prod_{\substack{U \in \mathrm{Kr}(\sigma), \\ U \not\ni n}} \gamma_{|U|} \Bigg) = \prod_{\substack{W \in \mathrm{Kr}(\pi), \\ W \not\ni n}} \widehat{\gamma}_{|W|}. \tag{4.107}$$

Proof. Due the hypothesis that π is irreducible, the Kreweras complement $\mathrm{Kr}(\pi)$ has a singleton block $\{n\}$. The same is true for any Kreweras complement $\mathrm{Kr}(\sigma)$ with $\sigma \geq \pi$ (since σ will have to be irreducible as well). Hence when we multiply the left-hand side of (4.107) by γ_1 and the right-hand side of (4.107) by $\widehat{\gamma}_1$, where $\gamma_1 = \widehat{\gamma}_1 \neq 0$, we find that (4.107) is equivalent to

$$\sum_{\substack{\sigma \in \mathcal{NC}(n), \\ \sigma \gg \pi}} \Big(\prod_{U \in \mathrm{Kr}(\sigma)} \gamma_{|U|} \Big) = \prod_{W \in \mathrm{Kr}(\pi)} \widehat{\gamma}_{|W|}. \tag{4.108}$$

It is thus all right if we prove (4.108) instead of (4.107).

We now recall the poset anti-isomorphism (2.8) given by Kreweras complementation between $\ll$ and $\sqsubseteq$, where $\ll$ is considered on irreducible partitions in $\mathcal{NC}(n)$ while $\sqsubseteq$ is considered on non-crossing partitions which have $\{n\}$ as a 1-element block. Since the set $\{\sigma \in \mathcal{NC}(n) \mid \sigma \gg \pi\}$ only contains irreducible partitions, we can use (2.8) as a change of variable in the summation on the left-hand side of (4.108), which is thus transformed into

$$\sum_{\substack{\tau \in \mathcal{NC}(n), \\ \tau \sqsubseteq \mathrm{Kr}(\pi)}} \Big(\prod_{U \in \tau} \gamma_{|U|} \Big). \tag{4.109}$$

From here on we proceed with a variation of the argument that finalized the proof of Lemma 4.9.5: we list explicitly the blocks of $\mathrm{Kr}(\pi)$ as $W_1, \ldots, W_p$, and we use the fact that a $\tau \in \mathcal{NC}(n)$ with $\tau \leq \mathrm{Kr}(\pi)$ is bijectively identified to the tuple

$$(\tau_{W_1}, \ldots, \tau_{W_p}) \in \mathcal{NC}(|W_1|) \times \cdots \times \mathcal{NC}(|W_p|).$$

At the level of the latter tuple, the requirement "$\tau \sqsubseteq \mathrm{Kr}(\pi)$" amounts to asking that $\tau_{W_1}, \ldots, \tau_{W_p}$ are interval partitions. Performing the change of variable $\tau \leftrightarrow (\tau_{W_1}, \ldots, \tau_{W_p})$ in the summation from (4.109) thus takes us to a summation over $\mathcal{I}(|W_1|) \times \cdots \times \mathcal{I}(|W_p|)$. From Lemma 2.2.12, the latter summation factors as the product of p separate summations over $\mathcal{I}(|W_1|), \ldots, \mathcal{I}(|W_p|)$, and one obtains in this way the product indicated on the right-hand side of (4.108). $\square$

Lemma 4.9.8. *The factorization formula stated in Equation* (4.107) *of Lemma 4.9.7 holds even if we do not assume the partition π to be irreducible.*

Proof. Consider the canonical decomposition $\pi = \pi_1 \diamond \cdots \diamond \pi_k$ with $\pi_1 \in \mathcal{NC}(n_1), \ldots, \pi_k \in \mathcal{NC}(n_k)$ irreducible, as reviewed in Remark 2.2.17.3. The specifics of the partial order $\ll$

force that we have

$$\{\sigma \in \mathcal{NC}(n) \mid \sigma \gg \pi\} = \left\{ \sigma_1 \diamond \cdots \diamond \sigma_k \;\middle|\; \begin{array}{r} \sigma_1 \gg \pi_1 \text{ in } \mathcal{NC}(n_1), \ldots, \\ \sigma_k \gg \pi_k \text{ in } \mathcal{NC}(n_k) \end{array} \right\}. \tag{4.110}$$

For a partition $\sigma = \sigma_1 \diamond \cdots \diamond \sigma_k$ as in (4.110) we note that $\sigma_1, \ldots, \sigma_k$ are all irreducible, and an examination of the relevant Kreweras complements leads to the formula

$$\prod_{\substack{U \in \mathrm{Kr}(\sigma), \\ U \not\ni n}} \gamma_{|U|} = \prod_{j=1}^{k} \Big(\prod_{\substack{U \in \mathrm{Kr}(\sigma_j), \\ U \not\ni n_j}} \gamma_{|U|} \Big). \tag{4.111}$$

The observations from (4.110), (4.111) and a straightforward conversion of sum into product then imply that we have:

$$\sum_{\substack{\sigma \in \mathcal{NC}(n), \\ \sigma \gg \pi}} \prod_{\substack{U \in \mathrm{Kr}(\sigma), \\ U \not\ni n}} \gamma_{|U|} = \prod_{j=1}^{k} \Big(\sum_{\substack{\sigma_j \in \mathcal{NC}(n_j), \\ \sigma_j \gg \pi_j}} \prod_{\substack{U \in \mathrm{Kr}(\sigma_j), \\ U \not\ni n_j}} \gamma_{|U|} \Big). \tag{4.112}$$

But now, Lemma 4.9.7 can be applied to each of $\pi_1, \ldots, \pi_k$. When we do this, we find that Equation (4.112) can be continued with

$$= \prod_{j=1}^{k} \Big(\prod_{\substack{W \in \mathrm{Kr}(\pi_j), \\ W \not\ni n_j}} \widehat{\gamma}_{|W|} \Big) = \prod_{\substack{W \in \mathrm{Kr}(\pi), \\ W \not\ni n}} \widehat{\gamma}_{|W|},$$

where at the second equality sign we used the counterpart of (4.111) in connection to the numbers $\widehat{\gamma}_i$, and for the decomposition $\pi = \pi_1 \diamond \cdots \diamond \pi_k$. $\square$

4.9.9. Proof of Theorem 4.9.1. We fix a $q \in \mathbb{R}$ and an $f \in MF^{\mathcal{NC}}$ for which we will prove that $u_q^{-1} * f * u_q \in MF^{\mathcal{NC}}$. The case when $q = 0$ is clear, since $u_0^{-1} * f * u_0 = f$, so we assume $q \neq 0$.

Let us denote $\lambda_n := f(0_n, 1_n)$, $n \geq 1$, and let $(\eta_n)_{n=1}^{\infty}$ be the sequence of complex numbers obtained out of the λ_n's by using the formula (4.97) from Remark 4.9.2. As anticipated in that remark, we will obtain the desired conclusion about $u_q^{-1} * f * u_q$ by proving that

$$u_q^{-1} * f * u_q \, (\pi, 1_n) = \prod_{W \in \mathrm{Kr}(\pi)} \eta_{|W|}, \quad \forall \, n \geq 1 \text{ and } \pi \in \mathcal{NC}(n). \tag{4.113}$$

From now on and until the end of the proof we fix an $n \geq 1$ and a $\pi \in \mathcal{NC}(n)$ for which we will verify that (4.113) holds. We divide the argument into several steps.

Step 1. Write explicitly what is $u_q^{-1} * f * u_q\,(\pi, 1_n)$, as a double sum "over σ and τ".

Similarly to the derivation of Equation (4.98), we start from

$$u_q^{-1} * f * u_q\,(\pi, 1_n) = \sum_{\substack{\sigma,\tau \in \mathcal{NC}(n) \\ \text{such that } \pi \leq \sigma \leq \tau}} u_q^{-1}(\pi, \sigma)\, f(\sigma, \tau)\, u_q(\tau, 1_n). \tag{4.114}$$

We have

$$u_q^{-1}(\pi, \sigma) = u_{-q}(\pi, \sigma) = \begin{cases} (-q)^{|\pi|-|\sigma|}, & \text{if } \pi \ll \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

We plug this into the right-hand side of (4.114), where we also replace the values of $f(\sigma, \tau)$ and of $u_q(\tau, 1_n)$ from (4.34) and (4.89), respectively. In this way we arrive to the formula:

$$u_q^{-1} * f * u_q\,(\pi, 1_n) = \sum_{\substack{\sigma \in \mathcal{NC}(n), \\ \sigma \gg \pi}} (-q)^{|\pi|-|\sigma|} \Big(\sum_{\substack{\tau \in \mathcal{NC}(n),\, \tau \geq \sigma \\ \text{and } \tau \text{ irreducible}}} \big(\prod_{W \in \mathrm{Kr}_\tau(\sigma)} \lambda_{|V|} \big) \cdot q^{|\tau|-1} \Big). \tag{4.115}$$

In the summation over τ performed in (4.115), the conditions "$\tau \geq \sigma$" and "τ irreducible" are consolidated in the requirement that $\tau \geq \overline{\sigma}^{\mathrm{irr}}$. Let us also re-arrange the factor $q^{|\tau|-1}$ appearing in that summation: we have (cf. [NS06, Exercise 18.23]) $|\sigma| + |\mathrm{Kr}_\tau(\sigma)| = |\tau| + n$, which implies that $q^{|\tau|-1} = q^{|\mathrm{Kr}_\tau(\sigma)|} \cdot q^{|\sigma|-(n+1)}$. With these changes, we thus arrive to

$$u_q^{-1} * f * u_q\,(\pi, 1_n) = \sum_{\substack{\sigma \in \mathcal{NC}(n), \\ \sigma \gg \pi}} \frac{(-q)^{|\pi|-|\sigma|}}{q^{(n+1)-|\sigma|}} \cdot \Big(\sum_{\substack{\tau \in \mathcal{NC}(n), \\ \tau \geq \overline{\sigma}^{\mathrm{irr}}}} \prod_{W \in \mathrm{Kr}_\tau(\sigma)} (q\lambda_{|V|}) \Big). \tag{4.116}$$

Step 2. Use the factorization formula from Lemma 4.9.5.

Here we must first clarify what are the input sequences "α_k, $\widehat{\alpha}_k$, $\widetilde{\alpha}_k$" that we plan to use in Lemma 4.9.5. We go as follows: start from the sequence $(\lambda_k)_{k=1}^\infty$ which was fixed from the beginning of the proof and put $\alpha_k := q\lambda_k$ for every $k \geq 1$; after that, define sequences $(\widehat{\alpha}_k)_{k=1}^\infty$ and $(\widetilde{\alpha}_k)_{k=1}^\infty$ via the formulas (4.102) given in Notation 4.9.4.

In view of what are our α_k's, we re-write (4.116) in the form

$$u_q^{-1} * f * u_q\,(\pi, 1_n) = \sum_{\substack{\sigma\in\,\mathcal{NC}(n),\\ \sigma\gg\pi}} \frac{(-1)^{|\pi|-|\sigma|}\, q^{|\pi|-|\sigma|}}{q^{(n+1)-|\sigma|}} \cdot \Big(\sum_{\substack{\tau\in\,\mathcal{NC}(n),\\ \tau\geq\overline{\sigma}^{\mathrm{irr}}}} \prod_{W\in \mathrm{Kr}_\tau(\sigma)} \alpha_{|W|} \Big),$$

and we invoke Lemma 4.9.5 in order to continue with

$$= \sum_{\substack{\sigma\in\,\mathcal{NC}(n),\\ \sigma\gg\pi}} (-1)^{|\pi|}\ (-1)^{|\sigma|}\, q^{|\pi|-(n+1)} \cdot \Big(\widetilde{\alpha}_{|W_{\mathrm{right}}|} \cdot \prod_{\substack{W\in\mathrm{Kr}(\sigma),\\ W\not\ni n}} \widehat{\alpha}_{|W|} \Big). \tag{4.117}$$

In the expression we arrived to, note that we can pull to the front of the sum the factors $(-1)^{|\pi|}$, $q^{|\pi|-(n+1)}$ and $\widetilde{\alpha}_{|W_{\mathrm{right}}|}$. The justification for pulling out the latter factor comes from Remark 4.9.3.3 – the block W_{right} is the same for all the partitions σ with $\sigma \gg \pi$. Thus from (4.117) we go on with

$$= (-1)^{|\pi|}\, q^{|\pi|-(n+1)}\, \widetilde{\alpha}_{|W_{\mathrm{right}}|} \cdot \sum_{\substack{\sigma\in\,\mathcal{NC}(n),\\ \sigma\gg\pi}} (-1)^{|\sigma|} \cdot \Big(\prod_{\substack{W\in\mathrm{Kr}(\sigma),\\ W\not\ni n}} \widehat{\alpha}_{|W|} \Big). \tag{4.118}$$

Step 3. Use the factorization formula from Lemma 4.9.8.

Here we must clarify what are the input sequences "γ_k and $\widehat{\gamma}_k$" that we plan to use in Lemma 4.9.8. We go as follows: put $\gamma_k = -\widehat{\alpha}_k$ for all $k \geq 1$; after that, define the sequence $(\widehat{\gamma}_k)_{k=1}^{\infty}$ via the formula (4.106) indicated in Notation 4.9.6. Observe that the common value of γ_1 and $\widehat{\gamma}_1$ is equal to $-q$ (this is found by backtracking in the definitions: $\gamma_1 = -\widehat{\alpha}_1 = -\alpha_1 = -q\lambda_1 = -q$). Since it is assumed that $q \neq 0$, we are thus in a situation where the hypotheses of Lemma 4.9.8 are satisfied.

Next observation: in (4.118), the factor $(-1)^{|\sigma|}$ can be written as

$$(-1)^n \cdot (-1)^{n-|\sigma|} = (-1)^n \cdot (-1)^{|\mathrm{Kr}(\sigma)|-1},$$

where at the second equality we use the fact that one always has $|\sigma| + |\mathrm{Kr}(\sigma)| = n+1$. The $(-1)^{|\mathrm{Kr}(\sigma)|-1}$ can be absorbed into the product of $\widehat{\alpha}_{|W|}$'s (which has $|\mathrm{Kr}(\sigma)| - 1$ factors), and therefore (4.118) continues with

$$= (-1)^{|\pi|}\, q^{|\pi|-(n+1)}\, \widetilde{\alpha}_{|W_{\mathrm{right}}|} \cdot \sum_{\substack{\sigma\in\,\mathcal{NC}(n),\\ \sigma\gg\pi}} (-1)^n \cdot \Big(\prod_{\substack{W\in\mathrm{Kr}(\sigma),\\ W\not\ni n}} (-\widehat{\alpha}_{|W|}) \Big)$$

$$= (-1)^{n-|\pi|}\, q^{|\pi|-(n+1)}\, \widetilde{\alpha}_{|W_{\text{right}}|} \cdot \sum_{\substack{\sigma\in\, \mathcal{NC}(n), \\ \sigma\gg\pi}} \Big(\prod_{\substack{W\in \mathrm{Kr}(\sigma), \\ W\not\ni n}} \gamma_{|W|} \Big). \tag{4.119}$$

The sum over $\sigma \gg \pi$ in (4.119) is precisely the one to which Lemma 4.9.8 applies, and in this way we arrive to the conclusion of Step 3, which is that we have

$$u_q^{-1} * f * u_q\,(\pi, 1_n) = (-1)^{n-|\pi|}\, q^{|\pi|-(n+1)}\, \widetilde{\alpha}_{|W_{\text{right}}|} \cdot \prod_{\substack{U\in \mathrm{Kr}(\pi), \\ U\not\ni n}} \widehat{\gamma}_{|U|}. \tag{4.120}$$

Step 4. Identify the factors in the product found in (4.120).

It is convenient to re-write the right-hand side of (4.120) in the form

$$\big(\frac{1}{q}\widetilde{\alpha}_{|W_{\text{right}}|}\big) \cdot \prod_{\substack{U\in \mathrm{Kr}(\pi), \\ U\not\ni n}} \big(-\frac{1}{q}\widehat{\gamma}_{|U|}\big), \tag{4.121}$$

with the pre-factor $(-1)^{n-|\pi|}\, q^{|\pi|-(n+1)}$ distributed among the $(n+1)-|\pi|$ blocks of $\mathrm{Kr}(\pi)$.

We are then left to chase through the formulas used in Steps 2 and 3, and verify that the product over blocks of $\mathrm{Kr}(\pi)$ that appears in (4.121) is the same as the one on the right-hand side of our target Equation (4.113) indicated at the beginning of the proof. It is visible that everything would be in place if we had that:

$$\widetilde{\alpha}_k = q\eta_k \ \text{ and } \ \widehat{\gamma}_k = -q\eta_k, \ \ \forall\, k \geq 1. \tag{4.122}$$

We will argue that the desirable relations stated in (4.122) are indeed holding.

The first relation (4.122) comes out by direct comparison of the formulas defining $\widetilde{\alpha}_k$ and η_k. Indeed, upon replacing $\alpha_{|V|} = q\lambda_{|V|}$ in the formula (4.102) which defines $\widetilde{\alpha}_k$, we find that

$$\widetilde{\alpha}_k = \sum_{\substack{\rho\in\mathcal{NC}(k), \\ \text{irreducible}}} \prod_{V\in\rho} (q\lambda_{|V|}) = \sum_{\substack{\rho\in\mathcal{NC}(k), \\ \text{irreducible}}} q^{|\rho|} \prod_{V\in\rho} \lambda_{|V|} = q\,\eta_k,$$

where at the third equality sign we refer to the formula (4.97) for η_k.

For the second relation (4.122) it suffices to check that $\widehat{\gamma}_k = -\widetilde{\alpha}_k$. We have

$$\begin{aligned}\widehat{\gamma}_k &= \sum_{\rho\in\mathcal{I}(k)} \prod_{J\in\rho} \gamma_{|J|} \text{ (by the definition of } \widehat{\gamma}_k\text{, in Equation (4.106))}\\ &= \sum_{\rho\in\mathcal{I}(k)} \prod_{J\in\rho} (-\widehat{\alpha}_{|J|}) \text{ (by the definition of } \gamma_{|J|} \text{ in Step 3)}\\ &= \sum_{\rho\in\mathcal{I}(k)} (-1)^{|\rho|} \prod_{J\in\rho} \widehat{\alpha}_{|J|} = -\widetilde{\alpha}_k,\end{aligned}$$

where the latter equality follows from Equation (4.103) of Remark 4.9.4. □

4.10 An application: multiplication of free random variables, in terms of t-Boolean cumulants

As explained in Section 1.5 of the Introduction, the multiplication of free random variables has a nice description in terms of t-Boolean cumulants, by a formula which is actually the same for all values of t. In the present section we show how this fact can be neatly derived by using the 1-parameter subgroup $\{u_q \mid q \in \mathbb{R}\}$ of $SM^{\mathcal{NC}}_{\text{c}-\text{c}}$.

Notation and Remark 4.10.1. *(Framework, and discussion of what we will prove.)*

We fix for the whole section a non-commutative probability space $(\mathcal{A}, \varphi)$ and two unital subalgebras $\mathcal{M}, \mathcal{N} \subseteq \mathcal{A}$ which are freely independent with respect to φ. For every $t \in \mathbb{R}$ we consider the family $\underline{b}^{(t)} = (b_n^{(t)} : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$ of t-Boolean cumulants of $(\mathcal{A}, \varphi)$; we also consider the standard enlargement of $\underline{b}^{(t)}$ to $(b_\pi^{(t)} : \mathcal{A}^n \to \mathbb{C})_{n\geq 1,\, \pi\in\mathcal{NC}(n)}$, as discussed in Notation 4.5.2. It will be convenient to aim for a formula slightly more general than what was announced in Equation (4.7) of the Introduction, and which is stated as follows:

$$\left\{ \begin{array}{c} \text{One has } b_n^{(t)}(x_1 y, \ldots, x_n y) = \sum_{\pi\in\mathcal{NC}(n)} b_\pi^{(t)}(x_1, \ldots, x_n) \cdot b_{\mathrm{Kr}(\pi)}^{(t)}(y, \ldots, y), \\ \text{holding for every } n \in \mathbb{N},\ x_1, \ldots, x_n \in \mathcal{M},\ y \in \mathcal{N} \text{ and } t \in \mathbb{R}. \end{array} \right. \tag{4.123}$$

Our approach to (4.123) is this: we note that for fixed y and t, the family of equalities stated in (4.123) is equivalent to one equation concerning the action of the group $SM^{\mathcal{NC}}$ on the space $\mathfrak{M}_{\mathcal{M}}$ of sequences of multilinear functionals on $\mathcal{M}$. The latter equation can then be treated by using results from Sections 9 and 10, particularly Theorem 4.9.1.

In order for the trick of fixing a y to play smoothly into the setting from Sections 9 and 10, it is good to arrange that $\varphi(y) = 1$. We start by pointing out that, without loss of generality, we can make this assumption.

Lemma 4.10.2. *Assume it is true that (4.123) holds whenever $y \in \mathcal{N}$ has $\varphi(y) = 1$. Then (4.123) is sure to hold with $y \in \mathcal{N}$ arbitrary.*

Proof. We first extend the validity of (4.123) to the case when $\varphi(y) \neq 0$. If $\varphi(y) = \lambda \neq 0$ then for every $t \in \mathbb{R}$, $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{M}$ we have

$$
\begin{aligned}
b_n^{(t)}(x_1 y, \ldots, x_n y) &= b_n^{(t)}\big((\lambda x_1) \cdot (\lambda^{-1} y), \ldots, (\lambda x_n) \cdot (\lambda^{-1} y) \big) \\
&= \sum_{\pi \in \mathcal{NC}(n)} b_\pi^{(t)}(\lambda x_1, \ldots, \lambda x_n) \cdot b_{\mathrm{Kr}(\pi)}^{(t)}(\lambda^{-1} y, \ldots, \lambda^{-1} y) \\
&\quad \text{(by hypothesis, since } \varphi(\lambda^{-1} y) = 1) \\
&= \sum_{\pi \in \mathcal{NC}(n)} \lambda^n b_\pi^{(t)}(x_1, \ldots, x_n) \cdot \lambda^{-n} b_{\mathrm{Kr}(\pi)}^{(t)}(y, \ldots, y) \\
&= \sum_{\pi \in \mathcal{NC}(n)} b_\pi^{(t)}(x_1, \ldots, x_n) \cdot b_{\mathrm{Kr}(\pi)}^{(t)}(y, \ldots, y), \quad \text{as required.}
\end{aligned}
$$

Now consider a $y \in \mathcal{N}$ with $\varphi(y) = 0$. From the fact proved in the preceding paragraph, it follows that: for every $t \in \mathbb{R}$, $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{M}$, one has

$$
b_n^{(t)}\big(x_1(y + \varepsilon 1_{\mathcal{A}}), \ldots, x_n(y + \varepsilon 1_{\mathcal{A}})\big) \tag{4.124}
$$

$$
= \sum_{\pi \in \mathcal{NC}(n)} b_\pi^{(t)}(x_1, \ldots, x_n) \cdot b_{\mathrm{Kr}(\pi)}^{(t)}(y + \varepsilon 1_{\mathcal{A}}, \ldots, y + \varepsilon 1_{\mathcal{A}}), \quad \forall\, \varepsilon \neq 0 \text{ in } \mathbb{C}.
$$

It is easy to check that the two sides of (4.124) depend continuously (in fact polynomially) on ε. We can thus make $\varepsilon \to 0$ in (4.124), to conclude that (4.123) holds for this y as well. $\square$

Notation 4.10.3. 1^o For the remaining part of this section we fix an element $y \in \mathcal{N}$ with $\varphi(y) = 1$, in connection to which we will prove that (4.123) is holding.

2^o It is convenient that, by using the y which was fixed, we introduce some sequences of multilinear functionals on $\mathcal{M}$, as follows: for every $t \in \mathbb{R}$ and $n \in \mathbb{N}$, let $\gamma_n^{(t,\mathcal{M})} : \mathcal{M}^n \to \mathbb{C}$ be defined by

$$
\gamma_n^{(t,\mathcal{M})}(x_1, \ldots, x_n) = b_n^{(t)}(x_1 y, \ldots, x_n y), \quad \forall\, x_1, \ldots, x_n \in \mathcal{M}. \tag{4.125}
$$

Clearly, we have that $\underline{\gamma}^{(t,\mathcal{M})} := (\gamma_n^{(t,\mathcal{M})})_{n=1}^\infty \in \mathfrak{M}_{\mathcal{M}}$, where $\mathfrak{M}_{\mathcal{M}}$ is defined exactly as in Notation 4.5.1, but by using $\mathcal{M}$ instead of $\mathcal{A}$.

In the same vein, it is convenient that for every $t \in \mathbb{R}$ and $n \geq 1$ we use the notation $\beta_n^{(t,\mathcal{M})} : \mathcal{M}^n \to \mathbb{C}$ for the restriction of the multilinear functional $b_n^{(t)} : \mathcal{A}^n \to \mathbb{C}$ to the subspace $\mathcal{M}^n$. Then $\underline{b}^{(t,\mathcal{M})} := (\beta_n^{(t,\mathcal{M})})_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{M}}$, and (as immediately verified) it is the family of t-Boolean cumulants of the non-commutative probability space $(\mathcal{M}, \varphi|\mathcal{M})$.

3^o Recall from Section 5 that every sequence $(\alpha_n)_{n=1}^{\infty}$ of complex numbers, with $\alpha_1 = 1$, defines a multiplicative function $f \in MF^{\mathcal{NC}}$ via the requirement that $f(0_n, 1_n) = \alpha_n$ for all $n \geq 1$. For every $t \in \mathbb{R}$ we can therefore consider a multiplicative function $f_t \in MF^{\mathcal{NC}}$ defined via the requirement that

$$f_t(0_n, 1_n) = b_n^{(t)}(y, \ldots, y), \quad \forall\, n \geq 1, \tag{4.126}$$

where y is the element of $\mathcal{N}$ fixed in part 1^o of this notation. Note that when defining f_t we use the fact that $\varphi(y) = 1$, which ensures that the sequence of numbers proposed on the right-hand side of (4.126) does indeed start with $b_1^{(t)}(y) = \varphi(y) = 1$.

For every $t \in \mathbb{R}$ and for general $\pi \leq \sigma$ in some $\mathcal{NC}(n)$, an explicit formula giving $f_t(\pi, \sigma)$ is then obtained out of (4.126), in the way reviewed in Remark 4.4.2. Recall, in particular, that for every $n \geq 1$ and $\pi \in \mathcal{NC}(n)$ we have

$$f_t(\pi, 1_n) = \prod_{W \in \mathrm{Kr}(\pi)} f_t(0_{|W|}, 1_{|W|}) = \prod_{W \in \mathrm{Kr}(\pi)} b_{|W|}^{(t)}(y, \ldots, y) = b_{\mathrm{Kr}(\pi)}^{(t)}(y, \ldots, y). \tag{4.127}$$

In terms of the notation just introduced, we can give an equivalent form of (4.123), which is stated as follows.

Lemma 4.10.4. *For every $t \in \mathbb{R}$, one has:*

$$\left(\begin{array}{c} \textit{Formula (4.123) holds for our} \\ \textit{fixed } y \textit{ and this particular value of } t \end{array} \right) \Leftrightarrow \left(\begin{array}{c} \underline{\gamma}^{(t,\mathcal{M})} = \underline{b}^{(t,\mathcal{M})} \cdot f_t \\ \textit{(an equality in } \mathfrak{M}_{\mathcal{M}}\textit{)} \end{array} \right). \tag{4.128}$$

Proof. The equality stated on the right-hand side of the equivalence is spelled out as follows:

$$\left\{ \begin{array}{l} \gamma_n^{(t,\mathcal{M})}(x_1, \ldots, x_n) = \sum_{\pi \in \mathcal{NC}(n)} f_t(\pi, 1_n)\, \beta_\pi^{(t,\mathcal{M})}(x_1, \ldots, x_n), \\ \\ \qquad\qquad\qquad \text{holding for every } n \geq 1 \text{ and } x_1, \ldots, x_n \in \mathcal{M}. \end{array} \right. \tag{4.129}$$

We leave it as an immediate exercise to the reader to replace the various quantities mentioned in (4.129) by their definition from Notation 4.10.3, and to verify that what comes out is indeed equivalent to the instance of (4.123) referring to our fixed y and t. $\square$

We next examine how one can connect two instances of the equation appearing on the right-hand side of the equivalence (4.128), considered for two different values $s, t \in \mathbb{R}$. This is done by using the 1-parameter subgroup $\{u_q \mid q \in \mathbb{R}\}$ from the preceding sections, both in reference to $\beta^{(t,\mathcal{M})}, \gamma^{(t,\mathcal{M})}$ (in Lemma 4.10.5) and in reference to f_t (in Lemma 4.10.6).

Lemma 4.10.5. *For every $s, t \in \mathbb{R}$ we have*

$$\underline{b}^{(t,\mathcal{M})} = \underline{b}^{(s,\mathcal{M})} \cdot u_{s-t} \text{ and } \underline{\gamma}^{(t,\mathcal{M})} = \underline{\gamma}^{(s,\mathcal{M})} \cdot u_{s-t}. \tag{4.130}$$

Proof. The first formula (4.130) is a direct consequence of Corollary 4.8.5, written in connection to the non-commutative probability space $(\mathcal{M}, \varphi \mid \mathcal{M})$.

The second formula (4.130) also follows from Corollary 4.8.5. Indeed, for every $n \geq 1$ and $x_1, \ldots, x_n \in \mathcal{M}$ we can write

$$\gamma_n^{(t,\mathcal{M})}(x_1, \ldots, x_n) \;=\; b_n^{(t)}(x_1 y, \ldots, x_n y) \;=\; \sum_{\substack{\pi \in \mathcal{NC}(n), \\ \text{irreducible}}} (s-t)^{|\pi|-1} \cdot \beta_\pi^{(s)}(x_1 y, \ldots, x_n y),$$

where at the second equality sign we use Equation (4.61) from Remark 4.6.9. An inspection of the definition of the functionals $\beta_\pi^{(s)}$ and $\gamma_\pi^{(s)}$ shows that in the latter expression we can replace $\beta_\pi^{(s)}(x_1 y, \ldots, x_n y)$ with $\gamma_\pi^{(s,\mathcal{M})}(x_1, \ldots, x_n)$; hence what we got is

$$\gamma_n^{(t,\mathcal{M})}(x_1, \ldots, x_n) \;=\; \sum_{\substack{\pi \in \mathcal{NC}(n), \\ \text{irreducible}}} (s-t)^{|\pi|-1} \cdot \gamma_\pi^{(s)}(x_1, \ldots, x_n),$$

where the right-hand side is indeed the value at $(x_1, \ldots, x_n)$ of the n-th functional in the family $\underline{\gamma}^{(s,\mathcal{M})} \cdot u_{s-t} \in \mathfrak{M}_{\mathcal{M}}$. ∎

Lemma 4.10.6. *Let t, q be in $\mathbb{R}$. One has that $u_q^{-1} * f_t * u_q = f_{t-q}$.*

Proof. We have that f_{t-q} is multiplicative (by definition, cf. Notation 4.10.3.3) and $u_q^{-1} * f_t * u_q$ is multiplicative as well (due to Theorem 4.9.1); so in order to prove their equality, it suffices to verify that

$$u_q^{-1} * f_t * u_q \,(0_n, 1_n) = f_{t-q}(0_n, 1_n), \quad \forall\, n \geq 1. \tag{4.131}$$

The right-hand side of (4.131) is, by definition, equal to $\beta_n^{(t-q)}(y, \ldots, y)$. For the left-hand side of the same equation we resort to Equation (4.97) of Remark 4.9.2, which says that

$$u_q^{-1} * f_t * u_q\,(0_n, 1_n) = \sum_{\substack{\pi \in \mathcal{NC}(n), \\ \text{irreducible}}} q^{|\pi|-1} \prod_{V \in \pi} f_t(0_{|V|}, 1_{|V|}). \tag{4.132}$$

Upon replacing $f_t(0_{|V|}, 1_{|V|})$ from its definition, the right-hand side of (4.132) becomes

$$\sum_{\substack{\pi \in \mathcal{NC}(n), \\ \text{irreducible}}} q^{|\pi|-1} \prod_{V \in \pi} \beta_{|V|}^{(t)}(y, \ldots, y),$$

and this is indeed equal to $\beta_n^{(t-q)}(y, \ldots, y)$, thanks to Equation (4.61) of Remark 4.6.9. $\square$

Lemma 4.10.7. *Suppose there exists a value $t_o \in \mathbb{R}$ for which it is true that $\underline{\gamma}^{(t_o,\mathcal{M})} = \underline{b}^{(t_o,\mathcal{M})} \cdot f_{t_o}$. Then it follows that $\underline{\gamma}^{(t,\mathcal{M})} = \underline{b}^{(t,\mathcal{M})} \cdot f_t$ for all $t \in \mathbb{R}$.*

Proof. Fix a $t \in \mathbb{R}$. We use Lemmas 4.10.5 and 4.10.6, with $q := t_o - t$, to replace $\underline{b}^{(t,\mathcal{M})} = \underline{b}^{(t_o,\mathcal{M})} \cdot u_{t_o-t}$ and $f_t = u_{t_o-t}^{-1} * f_{t_o} * u_{t_o-t}$, and thus get:

$$\underline{b}^{(t,\mathcal{M})} \cdot f_t = (\underline{b}^{(t_o,\mathcal{M})} \cdot u_{t_o-t}) \cdot (u_{t_o-t}^{-1} * f_{t_o} * u_{t_o-t}) = (\underline{b}^{(t_o,\mathcal{M})} \cdot f_{t_o}) \cdot u_{t_o-t}.$$

In the latter expression we can replace $\underline{b}^{(t_o,\mathcal{M})} \cdot f_{t_o}$ with $\underline{\gamma}^{(t_o,\mathcal{M})}$ (by hypothesis), then we can invoke Lemma 4.10.5 to conclude that $\underline{\gamma}^{(t_o,\mathcal{M})} \cdot u_{t_o-t} = \underline{\gamma}^{(t,\mathcal{M})}$. In this way we obtain that $\underline{b}^{(t,\mathcal{M})} \cdot f_t = \underline{\gamma}^{(t,\mathcal{M})}$, as required. $\square$

4.10.8. Proof of the statement (4.123). In view of Lemma 4.10.2, it suffices to prove (4.123) in connection to the element $y \in \mathcal{N}$ with $\varphi(y) = 1$ which was fixed since Notation 4.10.3.

The special case $t_o = 1$ of (4.123) concerns the description of multiplication of free elements in terms of free cumulants. This is a basic result in the combinatorics of free probability, which is not hard to obtain via a suitable grouping of terms in the moment-cumulant formula for free cumulants, followed by an application of Möbius inversion. For the details of how this goes, see for instance [NS06, Theorem 14.4].

We therefore accept the case $t_o = 1$ in (4.123). The equivalence noticed in Lemma 4.10.4 then tells us that that the equality $\underline{b}^{(t_o,\mathcal{M})} \cdot f_{t_o} = \underline{\gamma}^{(t_o,\mathcal{M})}$ holds for $t_o = 1$. This puts us in the position to invoke Lemma 4.10.7, in order to conclude that the equality $\underline{b}^{(t,\mathcal{M})} \cdot f_t = \underline{\gamma}^{(t,\mathcal{M})}$ holds for every $t \in \mathbb{R}$. Finally, the equivalence noticed in Lemma 4.10.4 is used again (this time in the direction from right to left) to conclude that (4.123) holds for all values $t \in \mathbb{R}$, as required. $\square$

Chapter 5

Iterative families of set partitions

The group of semi-multiplicative functions over $\mathcal{NC}$ has a very natural analogue, where instead of considering functions over the incidence algebra of the poset $\mathcal{NC}$ we consider functions over the incidence algebra of the poset $\mathcal{P}$. From the fact that $\mathcal{NC} \subset \mathcal{P}$, there is a natural way to include the group $SM^{\mathcal{NC}}$ of semi-multiplicative functions over $\mathcal{NC}$ as a subgroup of the group $SM^{\mathcal{P}}$ of semi-multiplicative functions over $\mathcal{P}$. Moreover, this bigger structure allows us to include the classical moment-cumulant formula in the picture. Furthermore, we will identify several other *iterative families* $\mathcal{S} \subset \mathcal{P}$, that same as $\mathcal{NC}$, have sufficient structure to admit a notion of semi-multiplicative functions over $\mathcal{S}$.

5.1 Basic definitions

Recall that the family of set partitions can be separated by number of blocks $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n$ where $\mathcal{P}_n := \{\pi \in \mathcal{P} : |\pi| = n\}$ is the set of partitions with n blocks. To ease notation, we will denote by $\mathcal{P}_{\geq 2} := \mathcal{P} \setminus \mathcal{P}_1 = \bigcup_{n=2}^{\infty} \mathcal{P}_n$ the family of set partitions with 2 or more blocks.

Definition 5.1.1. Consider a subset $\mathcal{S} \subset \mathcal{P}$ of partitions.

1. We define the $\mathcal{S}$*-block-substitution* relation in $\mathcal{P}$ by saying that "$\pi \leq_{\mathcal{S}} \sigma$" if $\pi \leq \sigma$ in the reversed refinement order and for every block $V \in \sigma$, the restriction of π to V is in $\mathcal{S}$, namely $\pi_V \in \mathcal{S}$.

2. We say that a subset $\mathcal{S} \subseteq \mathcal{P}$ is *iterative* if it contains all the partitions with one block ($\mathcal{P}_1 \subset \mathcal{S}$), and satisfies the following *iterative property*:

$$\left\{ \text{Whenever } \sigma \in \mathcal{S} \text{ and } \pi \leq_{\mathcal{S}} \sigma, \text{ this implies that } \pi \in \mathcal{S}. \right. \tag{5.1}$$

Notation 5.1.2. Similar to the set of all partitions $\mathcal{P}$, we can sort the partitions of $\mathcal{S}$ by the size of the set they are partitioning $\mathcal{S}(n) := \mathcal{S} \cap \mathcal{P}(n)$, or by the number of blocks in the partition, $\mathcal{S}_k := \mathcal{S} \cap \mathcal{P}_k$. We also use the notation $\mathcal{S}_{\geq 2} := \mathcal{S} \backslash \mathcal{P}_1 = \mathcal{S} \cap \mathcal{P}_{\geq 2}$ to refer to the set of partitions in $\mathcal{S}$ with two or more blocks.

Remark 5.1.3. The iterative property (5.1) says that in order for $\mathcal{S}$ to be iterative, we must have that $\mathcal{S}$ is closed under block substitution, namely, whenever we have a partition in $\mathcal{S}$ and every block is replaced by an element of $\mathcal{S}$ of the same size, we end up with another element of $\mathcal{S}$. The reason to call the families *iterative* is from the intuitive fact that whenever we have an arbitrary subset $R \subset \mathcal{P}$ and want to find an iterative subset containing R, then in order to get a set $\mathcal{S}$ that contains R and is closed under block-substitution one should iteratively apply block substitutions to the elements of R to get new partitions that enlarge the set. If at some point this process stabilizes, this means we have reached an iterative family of partitions. This intuitive fact will be clarified in Proposition 5.1.9.

Another easy fact that is important to keep in mind is that the partitions π that belong to an iterative subset $\mathcal{S}$ are precisely the partitions satisfying $\pi \leq_{\mathcal{S}} 1_n$ for some $n \in \mathbb{N}$.

Proposition 5.1.4. *If $\mathcal{S} \subset \mathcal{P}$ is iterative, then $\leq_{\mathcal{S}}$ is a partial order of $\mathcal{P}$, finer than reversed refinement order $\leq$. We call this the $\mathcal{S}$-*block-substitution order*.*

Proof. For $\leq_{\mathcal{S}}$ to be a partial order we need to check that it is reflexive, antisymmetric and transitive. Reflexivity follows directly from the fact that $1_n \in \mathcal{S}$, $\forall n \in \mathbb{N}$, while the antisymmetry follows from the antisymmetry of $\leq$. Thus, we just need to focus on the transitivity.

First we do the simpler case where the larger partition only has one block, that is, we show that if $\pi \leq_{\mathcal{S}} \sigma \leq_{\mathcal{S}} 1_n$ then $\pi \leq_{\mathcal{S}} 1_n$. Recall that $\sigma \leq_{\mathcal{S}} 1_n$ is equivalent to $\sigma \in \mathcal{S}$. Since $\pi \leq_{\mathcal{S}} \sigma$ then from the iterative property of $\mathcal{S}$ it follows that $\pi \in \mathcal{S}$ or equivalently $\pi \leq_{\mathcal{S}} 1_n$.

For the general case when $\pi \leq_{\mathcal{S}} \sigma \leq_{\mathcal{S}} \tau$ we use the fact that the interval $[\pi, \tau]$ is isomorphic to the Cartesian product $\prod_{W \in \tau} [\pi_W, 1_{|W|}]$. This bijection implies that the condition $\pi \leq_{\mathcal{S}} \sigma \leq_{\mathcal{S}} \tau$ is equivalent to $\pi_W \leq_{\mathcal{S}} \sigma_W \leq_{\mathcal{S}} 1_{|W|}$ for every $W \in \tau$, and from the previous case this implies $\pi_W \leq_{\mathcal{S}} 1_{|W|}$ for every $W \in \tau$, which is equivalent to $\pi \leq_{\mathcal{S}} \tau$.

Observe that $\pi, \sigma \in \mathcal{S}$ are related in $\leq_{\mathcal{S}}$ if and only if they are comparable in the order $\leq$. This readily implies that, $\leq_{\mathcal{S}}$ is finer than $\leq$. $\square$

Remark 5.1.5. It is not hard to notice that the converse of Proposition 5.1.4 holds:

If $\leq_{\mathcal{S}}$ is a partial order of $\mathcal{P}$, then $\mathcal{S}$ is iterative.

This means that we have the alternative definition that $\mathcal{S}$ is an iterative set if and only if $\leq_{\mathcal{S}}$ is a partial order in $\mathcal{P}$.

Remark 5.1.6. Notice that a direct implication of $\leq_{\mathcal{S}}$ being finer than $\leq$, is that for $\pi, \sigma \in \mathcal{S}$ to be comparable, they must partition the same number of elements, that is, $\sigma, \pi \in \mathcal{S}(n)$ for some $n \in \mathbb{N}$. This will be used implicitly for the rest of the manuscript. Moreover this also implies that the Lemma 2.2.12 also applies if we replace the order $\leq$ by the order $\leq_{\mathcal{S}}$.

Two straightforward examples of iterative sets are $\mathcal{P}_1$ and $\mathcal{P}$.

Example 5.1.7 (Partitions with one block $\mathcal{P}_1$)**.** Notice that $\mathcal{P}_1$ is trivially an iterative subset, in fact it is the smallest possible one, as by definition every iterative set must contain $\mathcal{P}_1$. The partial order $\leq_{\mathcal{P}_1}$ is the simplest possible, where $\pi \leq \sigma$ if and only $\pi = \sigma$, this means that every two distinct partitions are incomparable.

Example 5.1.8 (All partitions $\mathcal{P}$)**.** It is easy to check that $\mathcal{P}$ is an iterative subset, it clearly contains $\mathcal{P}_1$ and for every $\pi \leq_{\mathcal{P}} \sigma$ we know that $\pi \in \mathcal{P}$. Moreover, $\pi \leq_{\mathcal{P}} \sigma$ means that $\pi \leq \sigma$ and for very $V \in \sigma$ we have that $\pi_V \in \mathcal{P}$. But the latter condition is always true. So, the partial order $\leq_{\mathcal{P}}$ coincides with the order $\leq$.

Interestingly enough, we can always create new examples by noticing that for every subset $R \subset \mathcal{P}$ we can construct a minimal iterative subset containing R as stated below.

Proposition 5.1.9. *If $(\mathcal{S}^{(i)})_{i\in I}$ is a family of iterative subsets of $\mathcal{P}$, then their intersection $\mathcal{S} := \bigcap_{i\in I} \mathcal{S}^{(i)}$ is also iterative.*

This implies that for every subset $R \subset \mathcal{P}$, there exists a minimal (under inclusion) iterative subset $\mathcal{S}$ that contains R.

Proof. The first claim is straightforward from the definition of an iterative set. For the second claim, we first recall that $\mathcal{P} \supset R$ is iterative, and then use the standard trick of taking $\mathcal{S}$ to be the (non-empty) intersection $\cap\{\mathcal{T} \supset R \mid \mathcal{T} \text{ is iterative}\}$, which is iterative by the previous part. $\square$

Definition 5.1.10. In light of previous proposition, for every subset $R \subset \mathcal{P}$, we will denote as $\overline{R}^{\text{iter}} := \cap\{\mathcal{T} \supset R \mid \mathcal{T} \text{ is iterative}\}$ the iterative family generated by R.

5.2 A gallery of examples

In this section we provide several important examples of iterative families of partitions and study the partial order that they generate. These examples include the non-crossing partitions $\mathcal{NC}$, the interval partitions $\mathcal{I}$, the connected partitions $\mathcal{CON}$, the irreducible partitions $\mathcal{IRR}$, and the non-crossing irreducible partitions $\mathcal{NCIRR}$.

5.2.1 Non-crossing partitions $\mathcal{NC}$

Proposition 5.2.1. *The set of non-crossing partitions $\mathcal{NC}$ is iterative.*

Proof. Clearly $\mathcal{P}_1 \subset \mathcal{NC}$ as every 1_n is non-crossing. To prove the iterative property we consider $\pi \leq \sigma$ with $\sigma \in \mathcal{NC}$ and $\pi_{_W} \in \mathcal{NC}$ for every $W \in \sigma$. For the sake of contradiction we assume that $\pi \notin \mathcal{NC}$, and that two crossing blocks are $U, V \in \pi$, this means that there exist $u_1, u_2 \in U$ and $v_1, v_2 \in V$ with $u_1 < v_1 < u_2 < v_2$. Then we consider two cases:

1) U, V are contained in the same block $W \in \sigma$. In this case the image of the blocks U, V under the restriction $\pi_{_W}$ will create a crossing, implying $\pi_{_W} \notin \mathcal{NC}$, which is a contradiction.

2) U, V are not contained in the same block of σ. In this case $U \subset U' \in \sigma$ and $V \subset V' \in \sigma$ but then $u_1, u_2 \in U'$ and $v_1, v_2 \in V'$ with $u_1 < v_1 < u_2 < v_2$, implying that $U', V' \in \sigma$ have a crossing and $\sigma \notin \mathcal{NC}$ which is a contradiction.

Therefore, we conclude that $\mathcal{NC}$ is an iterative set. $\square$

Notice that $\mathcal{NC}$ has the special feature that the iterative property is an if and only if, namely we also have that:

$$\textit{If } \pi \leq \sigma \textit{ are both in } \mathcal{NC}, \textit{ then } \pi_{_W} \in \mathcal{NC} \textit{ for all } W \in \sigma.$$

This claim can be corroborated by contradiction, and follows directly from the fact that a crossing in a restriction $\pi_{_W}$ to some subset W implies there is a crossing in π. This special feature implies that the poset $(\mathcal{NC}, \leq_{\mathcal{NC}})$ can be obtained by simply restricting the poset $(\mathcal{P}, \leq)$ to the subset $\mathcal{NC}$, and somehow explains why one just uses the order $\leq$ in $\mathcal{NC}$. This order is commonly used in the combinatorics of free probability and was extensively discussed in the monograph [NS06].

5.2.2 Interval partitions $\mathcal{I}$

Recall that the family of interval partitions, $\mathcal{I}$, consists of partitions that only have interval blocks, that is, all the blocks must consist of consecutive elements. The family of interval partitions $\mathcal{I}$ has been well studied in connection to Boolean cumulants.

Proposition 5.2.2. *The set of interval partitions, $\mathcal{I}$, is iterative.*

Proof. Since $[n]$ is an interval block, then $1_n \in \mathcal{I}$ for all $n \in \mathbb{N}$. Suppose next that $\pi \leq \sigma$ with $\sigma \in \mathcal{I}$ and $\pi_W \in \mathcal{I}$ for every $W \in \mathcal{I}$. Then, the blocks of π are all intervals, because $\pi_W \in \mathcal{I}$ and the restricting map does not separates any element (because W is an interval block). So we get that $\pi \in \mathcal{I}$ and conclude that $\mathcal{I}$ is iterative. $\square$

The partial order $\leq_{\mathcal{I}}$ is commonly denoted as $\sqsubseteq$ (see Subsection 2.2.5) and was introduced in [JV15], a generalization to Coxeter groups can be found in [BJV19].

Notice that same as $\mathcal{NC}$, $\mathcal{I}$ has the special feature that the iterative property is an if and only if, namely we also have that if $\pi \leq \sigma$ are both in $\mathcal{I}$ then $\pi_V \in \mathcal{I}$ for all $V \in \sigma$. This means that the poset $(\mathcal{I}, \leq_{\mathcal{I}})$ is simply the poset $(\mathcal{P}, \leq)$ restricted to $\mathcal{I}$. Again, this explains why the poset $(\mathcal{I}, \leq)$ does not requires a new order.

5.2.3 Connected partitions $\mathcal{CON}$

Recall from Definition 2.2.15 that the set $\mathcal{CON}$ of connected partitions consists of those partitions $\pi \in \mathcal{P}(n)$ such that $\overline{\pi}^{\mathcal{NC}} = 1_n$, for some $n \in \mathbb{N}$.

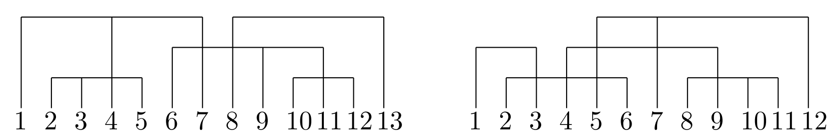


Figure 5.1: Connected partitions

We also mentioned, without proof, that the set of connected partitions consists of those partitions whose diagram is connected. We will begin this section by providing a concise statement and proof of this fact. This is best done in terms of the crossing graph.

Definition 5.2.3 (Definition 6.1 of [AHLV15])**.** The *crossing graph* of $\pi \in \mathcal{P}$ is the graph $\mathrm{CrG}(\pi)$ whose vertices are the blocks of π and we draw an edge between the vertices (blocks) V, W if and only if they cross.

Observe that $\pi \in \mathcal{NC}$ if and only if its crossing graph $\mathrm{CrG}(\pi)$ does not contain any edge. With this definition in hand we can give a precise statement.

Proposition 5.2.4. *The set $\mathcal{CON}$ consists precisely of the partitions π such that their crossing graph $CrG(\pi)$ is connected. That is, for $\pi \in \mathcal{P}$ we have that $\pi \in \mathcal{CON}$ if and only if for every two blocks $U, V \in \pi$ there exists a sequence of blocks $U = U_0, U_1, \ldots, U_k = V$ such that U_{i-1} and U_i have a crossing for all $i = 1, \ldots, k$. We will refer to this sequence as a connecting path.*

Before proving this result we will first show the following useful lemma.

Lemma 5.2.5. *Let $\pi \le \sigma$ be two partitions, and let $V, W \in \sigma$ be two crossing blocks. If the restrictions of π to these blocks are connected $\pi_V, \pi_W \in \mathcal{CON}$, then there exist two crossing blocks $V_{left}, W_{right} \in \pi$ with $V_{left} \subset V$ and $W_{right} \subset W$.*

Proof. Since V, W cross, there are sure to exist consecutive elements v_l, v_r in V and consecutive elements w_l, w_r in W such that $v_l < w_l < v_r < w_r$. Then, we claim there is a block in π_V whose convex hull contains the images of v_l and v_r under restriction. Since, v_l, v_r are consecutive in V, then their image under restriction to V are consecutive integers, say $k, k+1$, in $[|V|]$. Now, since $\pi_V \in \mathcal{CON}$, there is a path connecting $V' \ni 1$ and $V'' \ni |V|$ (the blocks containing 1 and $|V|$, respectively), which means that there is a sequence $V' = X_0, X_1, \ldots, X_j = V''$ such that $\max(X_i) \le \min(X_{i-1})$ for $i = 1, \ldots, k$. Since we also have $\min(X_0) = \min(V') = 1$ and $\max(X_j) = \max(V'') = |V|$, this means that the union of the convex hull of the blocks in the path is the whole set $[|V|]$, implying that the convex hull of one of the blocks must contain $k, k+1$. To be precise, if you consider the smallest i such that $k+1 \le \max(X_i)$, then you have $\min(X_i) \le \max(X_{i-1}) \le k < k+1 \le \max(X_i)$. Then, if we let $V_{left} \in \pi$ (contained in V) be the block whose restriction to V gives V_i, we get that $\min(V_{left}) \le v_l < w_l < v_r \le \max(V_{left})$. With the exact same procedure, we can find a block $W_{right} \in \pi$ contained in W such that $\min(W_{right}) \le w_l < v_r < w_r \le \max(W_{right})$. And we conclude that V_{left} and W_{right} are the two crossing blocks in π that we were looking for. $\square$

Proof of Proposition 5.2.4. First we prove that $\mathrm{CrG}(\pi)$ being connected implies that $\pi \in \mathcal{CON}$. Thus, we assume that $\mathrm{CrG}(\pi)$ is connected and for the sake of contradiction we also assume that $\overline{\pi}^{\mathcal{NC}} = \sigma \neq 1_n$, then $\sigma := \{W_1, \dots W_k\}$ with $k \geq 2$. Let us consider two blocks $V_1, V_2 \in \pi$ such that $V_1 \subset W_1$ and $V_2 \subset W_2$. Observe that if V_1 and V_2 cross, this readily implies that W_1 and W_2 cross, but this cannot happen since σ is non-crossing. Thus, there is no edge in $\mathrm{CrG}(\pi)$ connecting a block $V \in \pi$ with $V \subset W_1$ with a block of π contained in W_2. Similarly, there are no edges connecting blocks of π that are contained in different blocks of σ. Therefore, we conclude that there is no connecting path between $V_1 \subset W_1$ and $V_2 \subset W_2$, contradicting the assumption that $\mathrm{CrG}(\pi)$ is connected. We conclude that $\overline{\pi}^{\mathcal{NC}} = 1_n$, so $\pi \in \mathcal{CON}$.

Now we show that $\pi = \{U_1, \dots, U_m\} \in \mathcal{CON}$ implies that $\mathrm{CrG}(\pi)$ is connected. For the sake of contradiction let us assume that $\mathrm{CrG}(\pi)$ is not connected, and thus it has at least two connected components. Then we can partition the vertices of $\mathrm{CrG}(\pi)$ into the sets $V_1, \dots, V_k$ corresponding to its connected components. Let us now consider the partition $\sigma := \{W_1, \dots W_k\}$ such that $W_i = \cup_{U \in V_i} U$, we want to prove that $\sigma = \overline{\pi}^{\mathcal{NC}}$. Since σ groups together blocks $V \in \pi$ we directly get that $\sigma \geq \pi$. Notice that for every $i = 1, \dots, k$ the crossing graph of the restricted partition π_{W_i} is precisely the restricted graph $\mathrm{CrG}(\pi)_{V_i}$. Thus, since $\mathrm{CrG}(\pi)_{V_i}$ is connected, from the previous part we get that $\pi_{W_i} \in \mathcal{CON}$. After this observation we can invoke the contra-positive of 5.2.5 to show that $\sigma \in \mathcal{NC}$. Indeed, notice that if two blocks $W_i, W_j \in \sigma$ have a crossing, then since $\pi_{W_i}, \pi_{W_j} \in \mathcal{CON}$ we can use Lemma 5.2.5 to obtain two crossing blocks $U, U' \in \pi$ with $U \subset W_i$ and $U' \subset W_j$, but this means that the vertices $U \in V_i$ and $U' \in V_j$ are connected in $\mathrm{CrG}(\pi)$, contradicting the fact that V_i and V_j are connected components of $\mathrm{CrG}(\pi)$. $\square$

Returning to the main topic of the section let us now show the following.

Theorem 5.2.6. *The family of connected partitions $\mathcal{CON}$ is iterative.*

Proof. Clearly $\mathcal{P}_1 \subset \mathcal{CON}$ because partition 1_n has only one block. To prove the iterative property we consider $\pi \leq \sigma$ with $\sigma \in \mathcal{CON}$ and $\pi_W \in \mathcal{CON}$ for every $W \in \sigma$, and we want to show that $\pi \in \mathcal{CON}$. In order to do this, we fix two distinct blocks $U, V \in \pi$ and find a connecting path between them. There are two cases:

1) U, V are contained in the same block $W \in \sigma$. In this case, we consider the blocks $U', V' \in \pi_W$ obtained after restricting U, V to W, since $\pi_W \in \mathcal{CON}$, there exists a connecting path $U' = U'_0, U'_1, \dots, U'_k = V'$ in π_W. If we consider the blocks $U_i \in \pi$ whose restriction to W gives $U'_i \in \pi_W$ we obtain the connecting path $U = U_0, U_1, \dots, U_k = V$.

2) U, V are not contained in the same block of σ. Consider the two blocks of σ that contain U and V, namely $U \subset W^{(0)} \in \sigma$ and $V \subset W^{(j)} \in \sigma$. Since $\sigma \in \mathcal{CON}$ then there exists a connecting path $W^{(0)}, W^{(1)}, \ldots, W^{(j)}$ in σ. Now, since $W^{(i-1)}, W^{(i)}$ cross, by Lemma 5.2.5 there exist blocks $W^{(i-1)}_{left}, W^{(i)}_{right} \in \pi$ with $W^{(i-1)}_{left} \subset W^{(i-1)}$ and $W^{(i)}_{right} \subset W^{(i)}$. Moreover, since $\pi_{W^{(i)}} \in \mathcal{CON}$, by the previous case there is a path $W^{(i)}_{right} = W^{(i)}_0, W^{(i)}_1, \ldots, W^{(i)}_{left}$ connecting any two blocks of π contained in $W^{(i)}$. Thus, we can concatenate all the paths to get a path connecting U and V as follows:

$$U, U_1, \ldots, W^{(0)}_{left}, W^{(1)}_{right}, W^{(1)}_1, W^{(1)}_2, \ldots, W^{(1)}_{left}, W^{(2)}_{right}, \ldots\ldots, W^{(j)}_{right}, \ldots, V.$$

Since in all cases we get a connecting path, we obtain that $\pi \in \mathcal{CON}$, and conclude that $\mathcal{CON}$ is an iterative set. $\square$

To the best of our knowledge, the partial order $\leq_{\mathcal{CON}}$ has not been studied. Recall that by definition, $\pi \leq_{\mathcal{CON}} \sigma$ if we can sort the blocks of π in "connected components" and we glue each connected component to form a block of σ. If we want to identify two comparable partitions in this order, this can be done easily after drawing the planar diagrams associated to the the two partitions, or drawing the corresponding crossing graphs.

Notice that, opposed to the previous examples, $\mathcal{CON}$ does not have the special feature that if $\pi \leq \sigma$ are both in $\mathcal{CON}$ then $\pi_V \in \mathcal{CON}$ for all $V \in \sigma$. This means that in this case we get a poset $(\mathcal{CON}, \leq_{\mathcal{CON}})$ that is different from the poset $(\mathcal{CON}, \leq)$ obtained by simply restricting the order. An easy example of this difference is if we take $\pi = \{\{1,3\},\{2,5\},\{4,6\}\}$, $\sigma = \{\{1,3,4,6\},\{2,5\}\}$ in $\mathcal{CON}(6)$ and $V = \{1,3,4,6\} \in \sigma$. In this case $\pi_V = \{\{1,2\},\{3,4\}\}, \notin \mathcal{CON}$. Thus we get that $\pi \leq \sigma$ but π is not comparable to σ in the order $\leq_{\mathcal{CON}}$.

5.2.4 Irreducible partitions $\mathcal{IRR}$

Recall from Definition 2.2.16 that the family $\mathcal{IRR}$ of irreducible partitions, consists of those $\pi \in \mathcal{P}$ whose interval closure belongs to $\mathcal{P}_1$. Alternatively, it is not hard to observe that $\pi \in \mathcal{P}(n)$ is irreducible if and only if there exists a path connecting the blocks $V_0 \ni 1$ and $V_k \ni n$ (the blocks containing the smallest and largest element). This means that there exist $V_0, V_1, \ldots, V_k \in \pi$ such that $\min(V_1) = 1$, $\min(V_i) < \max(V_{i-1})$ for $i = 1, \cdots, k$, and $\max(V_k) = n$.

Notice that $\mathcal{CON} \subset \mathcal{IRR}$, because in a connected partition, there are connecting paths between every two blocks, in particular, between the blocks containing the first and last element. When drawing a partition π on the plane, we know that $\pi \in \mathcal{IRR}$ if and only if all vertical lines intersect at least one block.

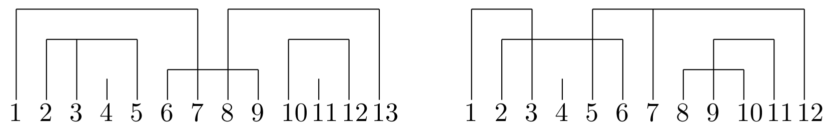


Figure 5.2: Irreducible partitions

Theorem 5.2.7. *The family of irreducible partitions $\mathcal{IRR}$ is iterative.*

Proof. Clearly $\mathcal{P}_1 \subset \mathcal{IRR}$ because the interval closure of 1_n is 1_n for all $n \in \mathbb{N}$. For the iterative property, consider $\pi \leq \sigma$ with $\sigma \in \mathcal{IRR}(n)$ and $\pi_W \in \mathcal{IRR}$ for every $W \in \sigma$. Then, there exist a connecting path $W_0, W_1, \ldots, W_k \in \sigma$ where $W_0 \ni 1$ and $W_k \ni n$. Using a similar procedure to the proof that connected partitions are iterative, we can check that since W_{i-1}, W_i cross, and $\pi_{W_{i-1}}, \pi_{W_i} \in \mathcal{IRR}$, then there exist crossing blocks $W_{left}, W_{right} \in \pi$ with $W_{left} \subset W_{i-1}$ and $W_{right} \subset W_i$, moreover we can assume that the image of W_{left} under restriction is in the connecting sequence of $\pi_{W_{i-1}}$ that goes from 1 to $|W_{i-1}|$. Thus we can concatenate all the connecting paths to get a path in π connecting $V_0 \ni 1$ to $V_k \ni n$. $\square$

To the best of our knowledge, the partial order $\leq_{\mathcal{IRR}}$ has not been studied, except in the case where we just consider non-crossing partitions, which is done in Section 5.2.5 below. Notice that, same as with $\mathcal{CON}$, the poset $(\mathcal{IRR}, \leq_{\mathcal{IRR}})$ is not the same as the restricted poset $(\mathcal{IRR}, \leq)$. As an example of this difference we can consider the same partitions we used for $\mathcal{CON}$, $\pi = \{\{1,3\},\{2,5\},\{4,6\}\}$, $\sigma = \{\{1,3,4,6\},\{2,5\}\}$ in $\mathcal{IRR}(6)$ and $V = \{1,3,4,6\} \in \sigma$. In this case $\pi_V = \{\{1,2\},\{3,4\}\} \notin \mathcal{IRR}$.

In the previous subsection we observed that the property of being a connected partition was naturally captured by checking if its crossing graph was connected. Let us take a moment to define another type of graph in terms of a partition that plays an analogue role for the irreducible partitions.

Definition 5.2.8 (Definition 6.1 of [AHLV15])**.** Let $\pi \in \mathcal{P}$, the *anti-interval graph* of π has vertices given by the blocks of π and an edge joins the vertices V, W if the blocks cross or are nested (if they are not in interval position).

Remark 5.2.9. Observe that in terms of this graph, $\pi \in \mathcal{I}$ if and only if its anti-interval graph has no edges. On the other hand, one can corroborate that $\pi \in \mathcal{IRR}$ if and only if its anti-interval graph is connected. We omit this proof, but the interested reader should not have problem to obtain this proof by adapting the proof of Proposition 5.2.4.

5.2.5 Non-crossing irreducible partitions $\mathcal{NCIRR}$

As suggested by its name, the set of *non-crossing irreducible partitions* consists of partitions π that are both, irreducible and non-crossing, the set is denoted as $\mathcal{NCIRR} := \mathcal{IRR} \cap \mathcal{NC}$.

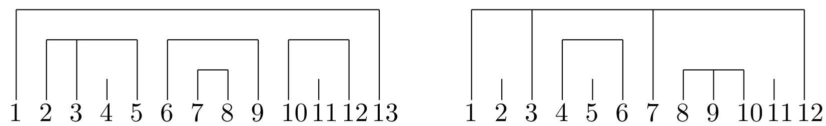


Figure 5.3: Non-crossing irreducible partitions

The family of non-crossing irreducible partitions $\mathcal{NCIRR}$ has been well studied in connection to combinatorics of free probability [Leh02, AHLV15].

Remark 5.2.10. Observe that $\mathcal{NCIRR}(n)$ consist of all non-crossing partitions with exactly one outer block V that contains 1 and n. Indeed, If $\pi \in \mathcal{NCIRR}(n) = \mathcal{IRR}(n) \cap \mathcal{NC}(n)$ and $V_1 \ni 1$, $V_k \ni n$ are the blocks containing the first and last elements, then there is a connecting path $V_1, V_2, \dots, V_k$. But $\pi \in \mathcal{NC}$, so there cannot be crossing blocks, which means that the only possible way to have a connecting path is if $V_1 = V_k$, so $1, n$ are in the same block.

Proposition 5.2.11. *The set of non-crossing irreducible partitions, $\mathcal{NCIRR}$, is iterative.*

Proof. We already know that $\mathcal{IRR}$ and $\mathcal{NC}$ are iterative families, and we know that the intersection of iterative families is iterative. Thus we readily get that $\mathcal{NCIRR} = \mathcal{IRR} \cap \mathcal{NC}$ is iterative. □

The partial order $\leq_{\mathcal{NCIRR}}$ is better known as $\ll$ and was introduced in [BN08] in connection to the study of the so-called Boolean Bercovici-Pata bijection.

Remark 5.2.12. We also have the interesting fact that $\overline{\mathcal{NCIRR}_2}^{\text{iter}} = \mathcal{NCIRR}$, meaning that the iterative family $\mathcal{NCIRR}$ can be generated from the set $\mathcal{NCIRR}_2$ of partitions with two blocks V, W that are nested.

5.2.6 Why these are the natural examples

Before presenting other iterating families, let us take a moment to get a better grasp of iterative families and convince ourselves that the previous examples of iterative families are very natural from a purely combinatorial point of view.

The first thing to observe is that if we apply a block substitution on a partition π we obtain a new partition σ with more blocks than π. Intuitively this means that if all the partitions in R have a lot of blocks, then the partitions π in $\mathcal{S} := \overline{R}^{\text{iter}}$ will also have a lot of blocks, We can pin down this idea very concretely by observing that:

If $\mathcal{S}$ is iterative, then for every n, the set $\mathcal{P}_1 \cup \mathcal{S}_{\geq n} := \mathcal{P}_1 \cup (\cup_{k\geq n}\mathcal{S}_k)$ is iterative.

Another intuitive observation is that if R contains partitions with two blocks, then when doing block-substitutions we will have to include several partition in $\mathcal{S}$. This can be expressed concretely as:

If $\pi = \{V, W\} \in \mathcal{P}_2$ is a partition with two blocks and $\pi \notin R$ then $\pi \notin \mathcal{S}$.

The proof follows easily from observing that in order to create a partition $\pi \in \mathcal{P}_2(n)$ with only two blocks using block-substitution we must use the partition π. Either by substituting π into 1_n or by substituting $1_{|V|}$ and $1_{|W|}$ the partition $\pi = \{V, W\}$. Therefore we can think of the two-block partitions $\mathcal{P}_2$ as the "basic building blocks" of the iterative sets.

With this idea in mind, it makes sense to study which iterative families are generated if we take certain interesting subsets $R \subset \mathcal{P}_2$. From Notation and Remark 2.2.2 we can identify three natural subsets of two-block partitions, the subset $C \subset \mathcal{P}_2$ of partitions where the two blocks are crossing, the subset $N \subset \mathcal{P}_2$ of partitions where the two blocks are nested and the subset $I \subset \mathcal{P}_2$ of partitions where the two blocks are in interval position. It is not hard to see that the following holds:

- $\overline{C}^{\text{iter}} = \mathcal{CON}$. Notice that in particular this means that $C = \mathcal{CON}_2$.
- $\overline{N}^{\text{iter}} = \mathcal{NCIRR}$. In particular we can observe that $N = \mathcal{NCIRR}_2$.
- $\overline{I}^{\text{iter}} = \mathcal{I}$. In particular one has that $I = \mathcal{I}_2$.

Now, if we check what happens when we take two type of relative position, we observe that

- $\overline{C \cup N}^{\text{iter}} = \mathcal{IRR}$. Notice that in particular this means that $C \cup N = \mathcal{IRR}_2$.
- $\overline{N \cup I}^{\text{iter}} = \mathcal{NC}$. In particular we can observe that $N \cup I = \mathcal{NC}^{\text{pair}}$.

- We will not study the set $\overline{I \cup C}^{\text{iter}}$, but the interested reader can corroborate that it consists of those partitions $\pi \in \mathcal{P}$ such that $\overline{\pi}^{\mathcal{NC}} \in \mathcal{I}$.

Finally, it is easy to check that if we take all of them together $C \cup N \cup I = \mathcal{P}_2$ then $\overline{\mathcal{P}_2}^{\text{iter}} = \mathcal{P}$. In a certain sense, that will be made very explicit in Section 6.6, we just "factorized" $\mathcal{P}$ into the families $\mathcal{CON}$, $\mathcal{NCIRR}$ and $\mathcal{I}$.

5.2.7 Further examples

As mentioned before, for every subset $S \subset \mathcal{P}$ we can define an iterative set generated by S, this opens the door to construct several iterative subsets. Of course, the skeptical reader may wonder if there are only a small number of iterative families and that despite the infinite number of subsets $R \subset \mathcal{P}$ all of them will generate just one of these few examples. This is not the case, and the objective of this section is to provide enough examples to convince the said skeptical reader. These examples will not be used later in the thesis, but it is an interesting problem for future work to find a meaningful classification of all the iterative families.

Example 5.2.13 (Min-max partitions MinMax)**.** The subset MinMax $\subset \mathcal{P}$ consists of partitions π where the smallest and largest element are contained in the same block. Namely, $\pi \in \mathcal{P}(n)$ is a min-max partition if there exists $V \in \pi$ with $1, n \in V$.

Notice that the definition readily implies that MinMax $\subset \mathcal{IRR}$.

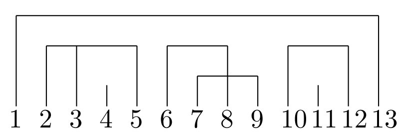


Figure 5.4: Min-max partitions

It is easy to check that the set MinMax is iterative.

To the best of our knowledge, the partial order $\leq_{\text{MinMax}}$ has not been studied, except in the case where we just consider non-crossing partitions, see Section 5.2.5 above.

Example 5.2.14 (Similar to the min-max)**.** There are other interesting iterative families that we can obtain with a slightly modification to the definition of the min-max partitions. For instance we can consider the $1 \sim 2$ *partitions* π, characterized by having a block V

that contains the elements 1 and 2. Or the $(\max -1) \sim \max$ partitions that have a block containing the last two elements. The proof that these families are iterative follows the same lines that for MinMax. In general, if we consider $\mathcal{S}$ to be the set of partitions where the first k elements must be in the same block, we get an iterative set. Similarly if we ask for the last k elements to be in the same block.

Strangely enough, it can be checked that the family of "$2 \sim 3$ partitions", characterized by having a block V containing elements 2 and 3, is not an iterative family.

Example 5.2.15 (Number of blocks congruent to $1 \mod k$)**.** For a fixed $k \in \mathbb{N}$, we can consider $\mathcal{S}$ to be the set of partitions that satisfy $|\pi| \equiv 1 \mod k$. It is not hard to see that when doing an $\mathcal{S}$-block substitution, the number of blocks increases my a multiple of k and thus we get a partition in $\mathcal{S}$. In particular, if $k = 2$, then set of partitions with an odd number of blocks is iterative. This is not the case if we consider partitions with an even number of blocks, even if we artificially add the partitions with one block.

Example 5.2.16 (Size of blocks multiple of k)**.** For a fixed $k \in \mathbb{N}$, the set $\mathcal{S} = \mathcal{P}^{(k)} \cup \mathcal{P}_1$ is iterative, where $\mathcal{P}^{(k)}$ consist of partitions π that satisfy that $|V|$ divides k for all the blocks $V \in \pi$. Notice that here we need to artificially add the set $\mathcal{P}_1$ as some 1_n partitions are not included automatically.

Example 5.2.17 (Elements of a block have same congruence mod k)**.** For a fixed $k \in \mathbb{N}$, the set $\mathcal{P}^{(\text{mod } k)} \cup \mathcal{P}_1$ is iterative, where $\mathcal{P}^{(\text{mod } k)}$ consist of partitions π such that for every $V \in \pi$ the elements of V have all the same congruence mod k. The key observation here is that block-substitution does not change the mod of the elements. As a particular interesting case we can consider $k = 2$, meaning that $\pi \in \mathcal{P}^{(\text{mod } 2)}$ when for every $V \in \pi$ the elements of V must have the same parity.

Example 5.2.18 (Size of block divides n)**.** If we consider $\mathcal{S}$ where $\mathcal{S}(n)$ is set of partitions $\pi \in \mathcal{P}(n)$ such that $|V|$ divides n for all $V \in \pi$, then we obtain a very curious iterative set.

Example 5.2.19 (Subset of some $\mathcal{P}(n)$)**.** For a fixed $n \in \mathbb{N}$, and whichever subset $R \subset \mathcal{P}(n)$, then $\mathcal{S} = \mathcal{P}_1 \cup R$ is iterative. This is somehow a trivial example, because it exploits the fact that $\pi \leq_{\mathcal{S}} \sigma$ can only happen when $\pi = \sigma$ or $\sigma = 1_n$, generating the simple order where the elements are only comparable with 1_n.

5.3 Semi-multiplicative functions on $\mathcal{S}$

Recall that in Section 4.2 we did a thorough study of semi-multiplicative functions on $\mathcal{NC}$. The purpose of this section is to adapt the results from that section to make them work for an arbitrary iterative family $\mathcal{S}$. Since most of the justifications done in Section 4.2 can be easily adapted to an arbitrary iterative family, we will omit some proofs.

The first important clarification regarding the incidence algebra framework is that we want to work with intervals of the poset $(\mathcal{S}, \leq_{\mathcal{S}})$, where we use the finer order $\leq_{\mathcal{S}}$ instead of $\leq$.

Definition 5.3.1. We denote

$$\mathcal{S}^{(2)} := \sqcup_{n=1}^{\infty}\{(\pi,\sigma) \mid \pi,\sigma \in \mathcal{S}(n),\ \pi \leq_{\mathcal{S}} \sigma\}. \tag{5.2}$$

The set of functions from $\mathcal{S}^{(2)}$ to $\mathbb{C}$ will be called *incidence algebra of* $\mathcal{S}$. This set of functions carries a natural associative operation of *convolution*, denoted as "$*$", where for any $f, g : \mathcal{S}^{(2)} \to \mathbb{C}$ and any $\pi \leq_{\mathcal{S}} \sigma$ in an $\mathcal{S}(n)$ one puts

$$f * g\,(\pi,\sigma) = \sum_{\substack{\rho\in\mathcal{S}(n),\\ \pi\leq_{\mathcal{S}}\rho\leq_{\mathcal{S}}\sigma}} f(\pi,\rho)\cdot g(\rho,\sigma). \tag{5.3}$$

In the next remark we collect a few relevant facts concerning the above mentioned convolution operation. We now proceed to repeat, pretty much word-by-word, the developement that was shown in Section 4.2.

Remark 5.3.2. It is easy to verify that the convolution operation "$*$" defined by (5.3) is associative and unital, where the unit is the function $e : \mathcal{S}^{(2)} \to \mathbb{C}$ given by

$$e(\pi,\sigma) = \begin{cases} 1, & \text{if } \pi = \sigma, \\ 0, & \text{otherwise.} \end{cases} \tag{5.4}$$

For a function $f : \mathcal{S}^{(2)} \to \mathbb{C}$ one has that

$$\left(\begin{array}{c} f \text{ is invertible} \\ \text{with respect to "}*\text{"} \end{array} \right) \Leftrightarrow \Big(f(\pi,\pi) \neq 0, \quad \forall\, \pi \in \mathcal{S} \Big). \tag{5.5}$$

Moreover, if f is invertible with respect to "$*$", then upon writing explicitly what it means to have $f * f^{-1}(\pi,\pi) = e(\pi,\pi) = 1$, one immediately sees that the inverse f^{-1} satisfies

$$f^{-1}(\pi,\pi) = \frac{1}{f(\pi,\pi)}, \qquad \forall\, \pi \in \mathcal{S}. \tag{5.6}$$

We now proceed to define (unitized) semi-multiplicative functions. This is done by looking at functions on $\mathcal{S}^{(2)}$, which are required to be multiplicative with respect to the isomorphism indicated in the version of Lemma 2.2.12 that is adapted to $(\mathcal{S}, \leq_{\mathcal{S}})$, as pointed out in Remark 5.1.6.

Definition 5.3.3. We will denote by $SM^{\mathcal{S}}$ the set of functions $g : \mathcal{S}^{(2)} \to \mathbb{C}$ which have $g(\pi, \pi) = 1$ for all $\pi \in \mathcal{S}$ and satisfy the following condition:

$$\begin{cases} \text{For every } \pi \leq_{\mathcal{S}} \sigma \text{ in } \mathcal{S} \text{ one has the factorization} \\ \qquad\qquad g(\pi, \sigma) = \prod_{W \in \sigma} g(\pi_W, 1_{|W|}) \end{cases} \tag{5.7}$$

We will refer to the condition (5.7) by calling it *semi-multiplicativity.*

From (5.7) it is obvious that a function $g \in SM^{\mathcal{S}}$ is completely determined when we know the values $g(\pi, 1_n)$ for all $n \geq 1$ and $\pi \in \mathcal{S}(n)$. It is hence clear that the map indicated in (5.8) below is injective. This map turns out to also be surjective; it thus identifies $SM^{\mathcal{S}}$, as a set, with the countable direct product of copies of $\mathbb{C}$. This result is the analogue to Proposition 4.2.5.

Proposition 5.3.4. *Let us denote* $\mathcal{Z} := \left\{ \underline{z} \mid \underline{z} : \sqcup_{n=1}^{\infty} \mathcal{S}(n) \setminus \{1_n\} \to \mathbb{C} \right\}$.

1^o *One has a bijection* $SM^{\mathcal{S}} \ni g \mapsto \underline{z} \in \mathcal{Z}$, *with* $\underline{z}$ *obtained out of* g *by putting*

$$\underline{z}(\pi) = g(\pi, 1_n) \text{ for every } n \geq 1 \text{ and } \pi \in \mathcal{S}(n) \setminus \{1_n\}. \tag{5.8}$$

2^o *The inverse of the bijection from* (5.8) *is described as follows. Given a* $\underline{z} \in \mathcal{Z}$, *we "fill in" values* $\underline{z}(1_n) = 1$ *for all* $n \geq 1$, *and then define* $g : \mathcal{S}^{(2)} \to \mathbb{C}$ *by*

$$g(\pi, \sigma) := \prod_{W \in \sigma} \underline{z}(\pi_W), \quad \forall\, (\pi, \sigma) \in \mathcal{S}^{(2)}. \tag{5.9}$$

Then $g \in SM^{\mathcal{S}}$, *and is sent by the map from* (5.8) *onto the* $\underline{z}$ *we started with.*

Same as with $SM^{\mathcal{NC}}$, it can be shown that $SM^{\mathcal{S}}$ is a group under convolution.

Theorem 5.3.5. $(SM^{\mathcal{S}}, *)$ *is a group.*

We will not prove this result as it follows the same reasoning used to prove Theorem 4.3.3. We just want to point out that the proof makes use of two lemmas that are useful to have in mind. First, we can observe that the semi-multiplicativity condition (5.7) has an automatic upgrade to a "local" version, shown in the next lemma (where the special case $U = \{1, \ldots, n\}$ retrieves the original definition of semi-multiplicativity).

Lemma 5.3.6. (Local semi-multiplicativity.) *Let $n \geq 1$ and $\pi, \sigma \in \mathcal{S}(n)$ be such that $\pi \leq_{\mathcal{S}} \sigma$. Let U be a non-empty subset of $\{1, \ldots, n\}$ which is a union of blocks of σ. For every $g \in SM^{\mathcal{S}}$ one has:*

$$g(\pi_{U}, \sigma_{U}) = \prod_{\substack{W \in \sigma, \\ W \subseteq U}} g(\pi_{W}, 1_{|W|}), \tag{5.10}$$

where all the restrictions (π_{U} and such) are in the sense of Notation 2.2.8.

As an application of local semi-multiplicativity, we get the following fact.

Lemma 5.3.7. *Let $n \geq 1$, let $\pi, \rho, \sigma \in \mathcal{S}(n)$ with $\pi \leq_{\mathcal{S}} \rho \leq_{\mathcal{S}} \sigma$, and let $g \in SM^{\mathcal{S}}$. One has*

$$g(\pi, \rho) = \prod_{U \in \sigma} g(\pi_{U}, \rho_{U}). \tag{5.11}$$

We will now put on hold the study of semi-multiplicative functions. In the next Chapter we will construct a Hopf algebra $\widehat{\mathcal{S}}$ for every iterative set. In Section 6.2 we will show that the group $SM^{\mathcal{S}}$ is isomorphic to the character group of $\widehat{\mathcal{S}}$. After that, we will continue our study of semi-multiplicative functions, but now as characters in the framework of Hopf algebras.

Chapter 6

Hopf algebras of set partitions

The use of Hopf algebras in the study of free probability was first done in [MN10], where the group of multiplicative functions $MF^{\mathcal{NC}}$ was naturally identified as the group of characters of the Hopf algebra Sym of symmetric functions. When combined with the log map for characters of Sym, this identification retrieves the celebrated S-transform of Voiculescu [Voi87], which is the most efficient tool for computing distributions of products of free random variables.

In analogy to that, in this chapter we present a unified method to construct a Hopf algebra $\widehat{\mathcal{S}}$ associated to an iterative family of set partitions $\mathcal{S}$. The construction is done in such a way that the character group $\mathcal{S}^\circ$ is naturally isomorphic to the group of semi-multiplicative functions $SM^{\mathcal{S}}$. The structure $\widehat{\mathcal{S}}$ can be identified as an incidence Hopf algebra, cf. [Sch94, Ein10].

In the special case when $\mathcal{S}$ is the family $\mathcal{NC}$ of non-crossing partitions, the Hopf algebra $\widehat{\mathcal{NC}}$ was studied in [CEFN+21] and is also closely related to one of the Hopf algebras studied in the recent paper [EFFKP20]. Part of this chapter follows the presentation in [CEFN+21] to show that the inclusion of groups $MF^{\mathcal{NC}} \subseteq SM^{\mathcal{NC}}$ proved in Section 4.4 is precisely (in view of the canonical isomorphisms $MF^{\mathcal{NC}} \approx \mathrm{Sym}^\circ$ and $SM^{\mathcal{NC}} \approx \mathcal{NC}^\circ$) the dual $\Psi^\circ : \mathrm{Sym}^\circ \to \mathcal{NC}^\circ$ of a natural bialgebra homomorphism $\Psi : \widehat{\mathcal{NC}} \to \mathrm{Sym}$ provided by the Kreweras complementation map.

A promising feature of the Hopf algebraic approach is that the antipode map of $\mathcal{S}$ can, in principle, serve as a universal tool for inversion in formulas that relate moments to cumulants, or relate different brands of cumulants whose transition formulas are captured by semi-multiplicative functions on $\mathcal{S}$. In Section 6.5 we examine the antipode of $\widehat{\mathcal{S}}$ and in particular we identify (Theorem 6.5.10) a cancellation-free formula for how the antipode works, described in terms of a suitable notion of "efficient chains" in the poset $\mathcal{S}$.

6.1 The Hopf algebra $\widehat{\mathcal{S}}$

The goal of this section is to show how, given an iterative family $\mathcal{S}$ we make use of its associated order $\leq_{\mathcal{S}}$ to construct a Hopf algebra $\widehat{\mathcal{S}}$. The resulting Hopf algebraic structure can be identified as an incidence Hopf algebra in the sense of Schmitt [Sch94], and this follows from the fact that the intervals $[\pi, \sigma]$ of the poset $(\mathcal{S}, \leq_{\mathcal{S}})$ form an "hereditary family of posets" in the sense of that paper. At the end of this section, in Remark 6.1.7, we provide a brief outline of this connection.

Definition 6.1.1 (Algebraic structure)**.** We let $\widehat{\mathcal{S}} = \mathbb{C}[\mathcal{S}_{\geq 2}]$ be the commutative algebra of polynomials over $\mathbb{C}$ which uses a countable collection of indeterminates indexed by the set partitions $\mathcal{S}_{\geq 2} := \mathcal{S} \backslash \mathcal{P}_1$. The indeterminate associated to $\pi \in \mathcal{S}$ will be denoted as $\widehat{\pi}$, and the unit of $\widehat{\mathcal{S}}$ is denoted as $1_{\mathcal{S}}$.

Notice that we can alternatively define $\widehat{\mathcal{S}} = \mathbb{C}[\mathcal{S}]/\sim$, where $\sim$ is the relation obtained by identifying $\widehat{1_n} = 1_{\mathcal{S}}$ for every $n \in \mathbb{N}$.

Recall that as an immediate consequence of our previous definition $\widehat{\mathcal{S}}$ has a universality property, see Proposition 2.4.16. Thus, if $\mathcal{A}$ is a unital commutative algebra over $\mathbb{C}$ and we are given elements $\{a_\pi\}_{\pi \in \mathcal{S}_{\geq 2}}$ in $\mathcal{A}$, then there exists a unital algebra homomorphism $\Phi : \widehat{\mathcal{S}} \to \mathcal{A}$, uniquely determined, such that $\Phi(\widehat{\pi}) = a_\pi$ for all $\pi \in \mathcal{S}_{\geq 2}$.

The most important combinatorial object is the coalgebra structure that we want to impose on $\widehat{\mathcal{S}}$.

Definition 6.1.2 (Coalgebra structure)**.** We endow $\widehat{\mathcal{S}}$ with a coalgebra structure as follows:

- The comultiplication of $\widehat{\mathcal{S}}$ is the unital algebra homomorphism $\Delta : \widehat{\mathcal{S}} \to \widehat{\mathcal{S}} \otimes \widehat{\mathcal{S}}$, uniquely determined such that for every $\pi \in \mathcal{S}_{\geq 2}$ we have

$$\Delta(\widehat{\pi}) = \sum_{\substack{\sigma \in \mathcal{S}, \\ \sigma \geq_{\mathcal{S}} \pi}} \left(\prod_{W \in \sigma} \widehat{\pi_W} \right) \otimes \widehat{\sigma}, \tag{6.1}$$

with restrictions $\pi_W \in \mathcal{S}(|W|)$ considered in the sense of Notation 2.2.8.

- The counit of $\widehat{\mathcal{S}}$ is the unital algebra homomorphism $\varepsilon : \widehat{\mathcal{S}} \to \mathbb{C}$, uniquely determined, such that
$$\varepsilon(\widehat{\pi}) = 0, \qquad \forall \pi \in \mathcal{S}_{\geq 2}.$$

Definition 6.1.3 (Grading)**.** $\widehat{\mathcal{S}}$ has a natural grading, where $\deg(\widehat{\pi}) = |\pi| - 1$ and we extend this to monomials by setting $\deg(\widehat{\pi}_1 \widehat{\pi}_2 \cdots \widehat{\pi}_k) = |\pi_1| + |\pi_2| + \cdots + |\pi_k| - k$. Therefore, if we denote by $\widehat{\mathcal{S}}_n$ the linear span of all monomials "of degree n", this gives a direct sum decomposition $\widehat{\mathcal{S}} = \bigoplus_{n=0}^{\infty} \widehat{\mathcal{S}}_n$, which we will refer to as "the grading of $\widehat{\mathcal{S}}$".

Notation 6.1.4. For every $\pi \leq_{\mathcal{S}} \sigma$ in $\mathcal{S}$ let us denote
$$\widehat{\pi}|_{\widehat{\sigma}} := \prod_{W \in \sigma} \widehat{\pi_W} \in \widehat{\mathcal{S}}. \tag{6.2}$$

Note that if $\sigma = 1_n$ for some integer $n \geq 1$, then the monomial $\widehat{\pi}|_{\widehat{\sigma}}$ consists of only one factor, so we get
$$\widehat{\pi}|_{\widehat{1_n}} = \widehat{\pi}, \qquad \forall\, n \geq 1 \text{ and } \pi \in \mathcal{S}(n).$$
At the other extreme, setting $\sigma = \pi$ makes $\widehat{\pi}|_{\widehat{\sigma}}$ consist of factors $\widehat{1_{|V|}}$ with V running among the blocks of π, and we thus get
$$\widehat{\pi}|_{\widehat{\pi}} = 1_{\mathcal{S}}, \qquad \forall \pi \in \mathcal{S}.$$

In terms of the monomials $\widehat{\pi}|_{\widehat{\sigma}}$, the formula (6.1) defining the comultiplication of $\widehat{\mathcal{S}}$ takes the more appealing form
$$\Delta(\widehat{\pi}) = \sum_{\substack{\sigma \in \mathcal{S} \\ \sigma \geq_{\mathcal{S}} \pi}} \widehat{\pi}|_{\widehat{\sigma}} \otimes \widehat{\sigma}, \qquad \forall\, \widehat{\pi} \in \widehat{\mathcal{S}}. \tag{6.3}$$

It is easy to further extend this, in the way indicated in the next lemma.

Lemma 6.1.5. *Let $\pi, \tau \in \mathcal{S}$ be such that $\pi \leq_{\mathcal{S}} \tau$. Then*
$$\Delta(\widehat{\pi}|_{\widehat{\tau}}) \;=\; \sum_{\substack{\sigma \in \mathcal{S} \\ \pi \leq_{\mathcal{S}} \sigma \leq_{\mathcal{S}} \tau}} \widehat{\pi}|_{\widehat{\sigma}} \otimes \widehat{\sigma}|_{\widehat{\tau}}. \tag{6.4}$$

Proof. Let us write explicitly $\tau = \{U_1, \ldots, U_k\}$. The left-hand side of Equation (6.4) then becomes
$$\Delta\Big(\prod_{j=1}^{k} \widehat{\pi_{U_j}}\Big) = \prod_{j=1}^{k} \Delta(\widehat{\pi_{U_j}}) = \prod_{j=1}^{k} \Big(\sum_{\substack{\sigma_j \in \mathcal{S}, \\ \sigma_j \geq_{\mathcal{S}} \pi_{U_j}}} \widehat{\pi_{U_j}}|_{\widehat{\sigma_{U_j}}} \otimes \widehat{\sigma_{U_j}} \Big).$$

Expanding the product over j in the latter expression takes us to

$$\sum_{\substack{\sigma_1,\sigma_2,\ldots,\sigma_k\in\mathcal{S},\\ \sigma_1\geq_{\mathcal{S}}\pi_1,\ldots,\sigma_k\geq_{\mathcal{S}}\pi_k}} \Big(\prod_{j=1}^{k} \widehat{\pi_{U_j}}|_{\widehat{\sigma_{U_j}}}\Big) \otimes \Big(\prod_{j=1}^{k} \widehat{\sigma_{U_j}}\Big). \tag{6.5}$$

Notice that the right-hand side of the tensor is

$$\prod_{j=1}^{k} \widehat{\sigma_{U_j}} = \widehat{\sigma}|_{\widehat{\tau}},$$

while in the left-hand side we just need to notice that the index set for the sum in (6.5) can be identified as the bijective image of the set $\{\sigma \in \mathcal{S}(n) \mid \pi \leq_{\mathcal{S}} \sigma \leq_{\mathcal{S}} \tau\}$, via the map

$$\sigma \mapsto (\sigma_{U_1}, \ldots, \sigma_{U_k}). \tag{6.6}$$

Therefore, using the bijection (6.6) to make a change of variable in the summation from (6.5), we obtain that

$$\prod_{j=1}^{k} \widehat{\pi_{U_j}}|_{\widehat{\sigma_{U_j}}} = \prod_{j=1}^{k} \prod_{W\in\sigma_k} \widehat{\pi_W} = \prod_{W\in\sigma} \widehat{\pi_W} = \widehat{\pi}|_{\widehat{\sigma}}.$$

Putting all pieces together we conclude the formula (6.4) claimed by the lemma. $\square$

Theorem 6.1.6. *When endowed with the algebra, coalgebra and grading structures introduced in Definitions 6.1.1, 6.1.2, and 6.1.3, $\widehat{\mathcal{S}}$ becomes a graded connected Hopf algebra.*

Proof. The proof consists of three verifications, pertaining to comultiplication, counit and grading, respectively.

(i) Verification that Δ is coassociative, i.e. that $(\mathrm{Id}\otimes\Delta)\circ\Delta = (\Delta\otimes\mathrm{Id})\circ\Delta$. Since both sides of this equality are unital algebra homomorphisms from $\widehat{\mathcal{S}}$ to $\widehat{\mathcal{S}}\otimes\widehat{\mathcal{S}}\otimes\widehat{\mathcal{S}}$, it suffices to check that they agree on every generator $\widehat{\pi}$ of $\widehat{\mathcal{S}}$. We thus pick a $\pi \in \mathcal{S}$, and we will verify that both $\mathrm{Id}\otimes\Delta\big(\Delta(\widehat{\pi})\big)$ and $\Delta\otimes\mathrm{Id}\big(\Delta(\widehat{\pi})\big)$ are equal to

$$\sum_{\substack{\sigma,\tau\in\mathcal{S}\\ \tau\geq_{\mathcal{S}}\sigma\geq_{\mathcal{S}}\pi}} \widehat{\pi}|_{\widehat{\sigma}} \otimes \widehat{\sigma}|_{\widehat{\tau}} \otimes \widehat{\tau}. \tag{6.7}$$

Indeed, if in the double sum of (6.7) we first sum over τ, then we get

$$\sum_{\substack{\sigma\in\mathcal{S}\\ \sigma\geq_{\mathcal{S}}\pi}} \widehat{\pi}|_{\widehat{\sigma}} \otimes \Big(\sum_{\substack{\tau\in\mathcal{S}\\ \tau\geq_{\mathcal{S}}\sigma}} \widehat{\sigma}|_{\widehat{\tau}} \otimes \widehat{\tau} \Big) = \sum_{\substack{\sigma\in\mathcal{S}\\ \sigma\geq_{\mathcal{S}}\pi}} \widehat{\pi}|_{\widehat{\sigma}} \otimes \Delta(\widehat{\sigma}) = \mathrm{Id}\otimes\Delta\left(\Delta(\widehat{\pi})\right).$$

While if in (6.7) we first sum over σ, then we get

$$\sum_{\substack{\tau\in\mathcal{S},\\ \tau\geq\pi}} \Big(\sum_{\substack{\sigma\in\mathcal{S},\\ \pi\leq_{\mathcal{S}}\sigma\leq_{\mathcal{S}}\tau}} \widehat{\pi}|_{\widehat{\sigma}} \otimes \widehat{\sigma}|_{\widehat{\tau}} \Big) \otimes \widehat{\tau} = \sum_{\substack{\tau\in\mathcal{S},\\ \tau\geq_{\mathcal{S}}\pi}} \Delta(\widehat{\pi}|_{\widehat{\tau}}) \otimes \widehat{\tau} \quad \text{(by Lemma 6.1.5)},$$

and the latter quantity is precisely equal to $\Delta\otimes\mathrm{Id}\left(\Delta(\widehat{\pi})\right)$.

(ii) Verification of the counit property, i.e. that $(\mathrm{Id}\otimes\epsilon)\circ\Delta = \mathrm{Id} = (\epsilon\otimes\mathrm{Id})\circ\Delta$.

Here again it suffices to focus on a generator $\widehat{\pi}$. Upon chasing through the definitions, we see that what needs to be verified is this: given $n\geq 2$ and $\pi\in\mathcal{S}(n)$, check that

$$\sum_{\sigma\geq\pi} \epsilon(\widehat{\sigma}) \cdot \prod_{W\in\sigma} \widehat{\pi_W} = \widehat{\pi} = \sum_{\sigma\geq\pi} \prod_{W\in\sigma} \epsilon\big(\widehat{\pi_W}\big) \cdot \widehat{\sigma}. \tag{6.8}$$

And indeed: the first of the two equalities (6.8) holds because the only non-zero contribution to the sum occurs for $\sigma = 1_n$, when $\prod_{W\in 1_n} \widehat{\pi_W} = \widehat{\pi}$. The second equality (6.8) also holds, with the only non-zero contribution now coming from the term indexed by π:

$$\Big(0 \neq \prod_{W\in\sigma} \epsilon(\widehat{\pi_W})\Big) \;\Leftrightarrow\; \Big(\pi_W = 1_{|W|},\ \forall\, W\in\sigma\Big) \;\Leftrightarrow\; (\sigma=\pi).$$

(iii) Verifications related to the grading and conectedness. The fact that $\widehat{\mathcal{S}}$ is connected follows from noticing that $\deg\widehat{\pi} = |\pi|-1>0$ for every $\pi\in\mathcal{S}_{\geq 2}$. The fact that the grading respects products follows from the definition of the grading. Thus the only non-obvious item to verify is that the coproduct respects the grading. Namely, that for every $m\geq 1$ and $\widehat{\pi}\in\widehat{\mathcal{S}}_m$ (equivalently $|\pi| = m+1$) one has that $\Delta(\widehat{\pi}) \in \sum_{i=0}^{m} \widehat{\mathcal{S}}_i \otimes \widehat{\mathcal{S}}_{m-i}$. In order to verify this fact, recall that the coproduct is given by sums of the form $\widehat{\pi}|_{\widehat{\sigma}}\otimes\widehat{\sigma}$, so we need to check that $\deg(\widehat{\pi}|_{\widehat{\sigma}}) + \deg(\widehat{\sigma}) = m-1$ Indeed, recall that $\deg(\widehat{\sigma}) = |\sigma|-1$ and $\widehat{\pi}|_{\widehat{\sigma}}$ is a monomial, so its degree is given by

$$\deg(\widehat{\pi}|_{\widehat{\sigma}}) = \sum_{W\in\sigma} \deg(\pi_W) = \sum_{W\in\sigma} |\pi_W| - \sum_{W\in\sigma} 1 = |\pi| - |\sigma| = m+1-|\sigma|.$$

So both degrees add to m, as required. $\square$

Remark 6.1.7. As mentioned at the beginning of the section, the Hopf algebra $\widehat{\mathcal{S}}$ can be treated as an *incidence Hopf algebra* in the sense of Schmitt [Sch94]. The present remark gives a brief outline of how this happens.

For every $n \geq 1$ and $\pi \leq \sigma$ in $\mathcal{S}(n)$ let us denote

$$[\pi, \sigma] = \{\rho \in \mathcal{S}(n) \mid \pi \leq_{\mathcal{S}} \rho \leq_{\mathcal{S}} \sigma\} \ \ \text{(a sub-poset of } (\mathcal{S}(n), \leq_{\mathcal{S}})) \tag{6.9}$$

and let $P_{\mathcal{S}}$ denote the collection of all the posets $[\pi, \sigma]$ considered in (6.9). Notice that, at the level of posets, the partial order $\leq_{\mathcal{S}}$ satisfies that

$$[\pi, \sigma] \cong \prod_{W \in \sigma} [\pi_W, 1_{|W|}], \ \ \text{for every } [\pi, \sigma] \in P_{\mathcal{S}}. \tag{6.10}$$

Namely, the interval $[\pi, \sigma]$ is isomorphic to the Cartesian product of the intervals $[\pi_W, 1_{|W|}]$ for every block $W \in \sigma$. This property is inherited from the fact that $\leq_{\mathcal{S}}$ is finer than $\leq$.

Then, on $P_{\mathcal{S}}$ we can introduce the equivalence relation "$\sim$" such that

$$[\pi, \sigma] \sim \prod_{W \in \sigma} [\pi_W, 1_{|W|}], \ \ \text{for every } [\pi, \sigma] \in P_{\mathcal{S}}.$$

This equivalence relation produces a commutative quotient monoid $P_{\mathcal{S}}/\sim$ generated by

$$\Big\{ \widetilde{[\pi, 1_n]} \mid n \geq 1 \text{ and } \pi \in \mathcal{S}(n) \setminus \{1_n\} \Big\},$$

where we use the notation "$\widetilde{[\pi, \sigma]}$" for the image of $[\pi, \sigma]$ under the quotient map $P_{\mathcal{S}} \to P_{\mathcal{S}}/\sim$. When plugged into the general machinery described in [Sch94, Sections 2-4], the monoid algebra $\mathbb{C}[P_{\mathcal{S}}/\sim]$ becomes a Hopf algebra, which turns out to be naturally isomorphic, as Hopf algebras, to $\widehat{\mathcal{S}}$ from Theorem 6.1.6, via the unital algebra homomorphism defined by requiring that

$$\widehat{\mathcal{S}} \ni \widehat{\pi} \mapsto \widetilde{[\pi, 1_n]} \in \mathbb{C}[P_{\mathcal{S}}/\sim], \ \ \forall\, n \geq 1 \text{ and } \pi \in \mathcal{S}(n) \setminus \{1_n\}.$$

6.2 The isomorphism $\mathcal{S}^\circ \cong SM^{\mathcal{S}}$

The objective of this section is to observe that the group of characters $\mathcal{S}^\circ$ is isomorphic to the group of semi-multiplicative functions $SM^{\mathcal{S}}$. As we already saw in Section 4.5, the relevance of this group for non-commutative probability considerations stems from the fact that some transition formulas can be encoded using these characters.

Recall from Definition 2.4.13 that the set of characters of $\widehat{\mathcal{S}}$ is defined as

$$\mathcal{S}^\circ = \{\chi \in L(\widehat{\mathcal{S}}, \mathbb{C}) \mid \chi(1_{\mathcal{S}}) = 1, \chi \text{ is multiplicative}\}.$$

Observe that if $\chi \in \mathcal{S}^\circ$ then f is completely determined by its values $\chi(\widehat{\pi})$ for every $\pi \in \mathcal{S}_{\geq 2}$. Indeed, we know that $\chi(1_{\mathcal{S}}) = 1$. Then, from the multiplicativity of f, the values on the monomials are just the product of the values for each partition: $\chi(\widehat{\pi}_1\widehat{\pi}_2 \cdots \widehat{\pi}_k) = \chi(\widehat{\pi}_1)\chi(\widehat{\pi}_2) \cdots \chi(\widehat{\pi}_k)$. Finally, the values of χ on a polynomial $p \in \widehat{\mathcal{S}}$ are just the linear extension of the values for the monomials.

The characters $\mathcal{S}^\circ$ form a group under the convolution $*$, which is defined as the dual operation to the coproduct of $\widehat{\mathcal{S}}$. Specifically, we have that if $\chi_1, \chi_2 \in \mathcal{S}^\circ$ then $\chi_1 * \chi_2 : \widehat{\mathcal{S}} \to \mathbb{C}$ is the map uniquely determined by

$$\chi_1 * \chi_2(\widehat{\pi}) = \sum_{\substack{\sigma \in \mathcal{S} \\ \sigma \geq_{\mathcal{S}} \pi}} \left(\prod_{W \in \sigma} \chi_1(\widehat{\pi_W}) \right) \chi_2(\widehat{\sigma}). \tag{6.11}$$

The main result of this section is the fact that the semi-multiplicative functions $SM^{\mathcal{S}}$ are in bijection with the characters $\mathcal{S}^\circ$, which means that the group $SM^{\mathcal{S}}$ is the dual structure of the Hopf algebra $\widehat{\mathcal{S}}$.

Theorem 6.2.1. 1^o *For every $g \in SM^{\mathcal{S}}$ there exists a character $\chi_g \in \mathcal{S}^\circ$, uniquely determined, such that*

$$\chi_g(\widehat{\pi}) = g(\pi, 1_n), \qquad \forall\, \pi \in \mathcal{S}. \tag{6.12}$$

2^o *The map $SM^{\mathcal{S}} \ni g \mapsto \chi_g \in \mathcal{S}^\circ$ is a group isomorphism, i.e. it is bijective and has*

$$\chi_{g_1 * g_2} = \chi_{g_1} * \chi_{g_2}, \qquad \forall\, g_1, g_2 \in SM^{\mathcal{S}}. \tag{6.13}$$

Proof. The universality property noted in Proposition 2.4.16 implies that the characters of $\mathcal{S}$ are in bijective correspondence with families of complex numbers of the form $\{z(\pi) \mid \pi \in \mathcal{S}_{\geq 2}\}$, where the family of numbers corresponding to $\chi \in \mathcal{S}^\circ$ is simply obtained by putting $z(\pi) = \chi(\widehat{\pi})$ for all $\pi \in \mathcal{S}_{\geq 2}$. When considered in conjunction with the Proposition 5.3.4 about functions in $SM^{\mathcal{S}}$, this immediately implies the statement 1^o of the theorem, and also the fact that the map $SM^{\mathcal{S}} \ni g \mapsto \chi_g \in \mathcal{S}^\circ$ is a bijection.

We are left to check that (6.13) holds. In order to establish the equality of the characters $\chi_{g_1} * \chi_{g_2}$ and $\chi_{g_1 * g_2}$ it suffices to verify that they agree on every generator $\widehat{\pi}$ of $\mathcal{S}$. We thus fix a $\pi \in \mathcal{S}_{\geq 2}$, and we compute:

$$
\begin{aligned}
\chi_{g_1} * \chi_{g_2}(\widehat{\pi}) &= \sum_{\substack{\sigma \in \mathcal{S} \\ \sigma \geq \pi}} \left(\prod_{W \in \sigma} \chi_{g_1}(\widehat{\pi_W}) \right) \cdot \chi_{g_2}(\widehat{\sigma}) && \text{(by (6.11))} \\
&= \sum_{\substack{\sigma \in \mathcal{S} \\ \sigma \geq \pi}} \left(\prod_{W \in \sigma} g_1(\pi_W, 1_{|W|}) \right) \cdot g_2(\sigma, 1_n) && \text{(by formulas defining } \chi_{g_1} \text{ and } \chi_{g_2}) \\
&= \sum_{\substack{\sigma \in \mathcal{S} \\ \sigma \geq \pi}} g_1(\pi, \sigma) \cdot g_2(\sigma, 1_n) && \text{(by Eqn.(5.7) in Definition 5.3.3)} \\
&= g_1 * g_2(\pi, 1_n) && \text{(by the definition of } * \text{ in } SM^{\mathcal{S}}) \\
&= \chi_{g_1 * g_2}(\widehat{\pi}) && \text{(by the formula defining } \chi_{g_1 * g_2}).
\end{aligned}
$$

Therefore we conclude the desired equality in (6.13). □

Notation 6.2.2. In view of the isomorphism $\mathcal{S}^\circ \cong SM^{\mathcal{S}}$, from now on if we have some semi-multiplicative function $g \in SM^{\mathcal{S}}$, we will do some abuse of notation and let g also be considered as the element of $\mathcal{S}^\circ$ that satisfies $g(\widehat{\pi}) = g(\pi, 1)$ and $g(\widehat{\pi}|_{\widehat{\sigma}}) = g(\pi, \sigma)$ for every $n \geq 1$ and $\pi \leq_{\mathcal{S}} \sigma$ in $\mathcal{S}(n)$.

6.3 A surjective Hopf algebra homomorphism from $\widehat{\mathcal{P}}$ to $\widehat{\mathcal{S}}$

Recall that iterative sets $\mathcal{S}$ are, by definition, contained in the larger iterative set $\mathcal{P}$. This makes it tempting to consider the natural embedding of $\widehat{\mathcal{S}}$ into $\widehat{\mathcal{P}}$ by simply associating $\widehat{\pi}^{\mathcal{S}}$ with $\widehat{\pi}^{\mathcal{P}}$ for every $\pi \in \mathcal{S}$. Here the superscripts are just to emphasize that $\widehat{\pi}^{\mathcal{S}}$ is viewed as an element on $\widehat{\mathcal{S}}$, while $\widehat{\pi}^{\mathcal{P}}$ is viewed as an element of $\widehat{\mathcal{P}}$. The problem with such an embedding is that it does not works well with the coalgebra structure. Indeed, if $\pi \in \mathcal{S}$, then every $\sigma \in \mathcal{P}$ with $\sigma \geq \pi$ will appear in the coproduct $\Delta(\widehat{\pi})$, but in most of the cases σ will not belong to $\mathcal{S}$. This means that there is no natural way to think of $\widehat{\mathcal{S}}$ as a sub-Hopf-algebra of $\widehat{\mathcal{P}}$. Still there is another way to relate $\widehat{\mathcal{S}}$ to $\widehat{\mathcal{P}}$ via a surjective Hopf algebra homomorphsim from $\widehat{\mathcal{P}}$ into $\widehat{\mathcal{S}}$.

Theorem 6.3.1. *Let $\mathcal{S}$ be an iterative subset of $\mathcal{P}$. We define the unital algebra homomorphism $\iota : \widehat{\mathcal{P}} \to \widehat{\mathcal{S}}$ to be the map uniquely dermined by the fact that $\iota(\widehat{\pi}^{\mathcal{P}}) = \widehat{\pi}^{\mathcal{S}}$ if $\pi \in \mathcal{S}$ and $\iota(\widehat{\pi}^{\mathcal{P}}) = 0$ if $\pi \in \mathcal{P} \setminus \mathcal{S}$. Then ι is a surjective Hopf algebra homomorphsim.*

Proof. It is straightforward that the map is surjective, and it is easy to check that ι respects the algebra structure, as well as the counit property. Thus the main property that we want to corroborate is that ι respects the coproduct, $(\iota \otimes \iota) \circ \Delta_{\mathcal{P}} = \Delta_{\mathcal{S}} \circ \iota$. Let $\pi \in \mathcal{P}$ and consider two cases:

- If $\pi \notin \mathcal{S}$ we have that $\Delta_{\mathcal{S}}(\iota(\widehat{\pi})) = \Delta_{\mathcal{S}}(0) = 0$. On the other hand we can see that for every $\sigma \geq \pi$, either $\sigma \notin \mathcal{S}$ or there exist a $W \in \sigma$ such that $\pi_W \notin \mathcal{S}$. Indeed, this is just the contrapositive of the iterative property of $\mathcal{S}$: if $\sigma \in \mathcal{S}$ and for every $W \in \sigma$ we have that $\pi_W \in \mathcal{S}$, then π must be in $\mathcal{S}$, which is not the case. From the fact that $\iota(\widehat{\pi}^{\mathcal{P}}) = 0$ if $\pi \notin \mathcal{S}$ we conclude that

$$(\iota \otimes \iota) \circ \Delta_{\mathcal{P}}(\widehat{\pi}) = \sum_{\substack{\sigma \in \mathcal{P}, \\ \sigma \geq \pi}} \iota(\widehat{\pi}|_{\widehat{\sigma}}) \otimes \iota(\widehat{\sigma}) = 0 = \Delta_{\mathcal{S}}(\iota(\widehat{\pi})).$$

- If $\pi \in \mathcal{S}$, then applying $\iota \otimes \iota$ to the coproduct $\Delta_{\mathcal{P}}(\widehat{\pi})$ will make several terms vanish in the sum over $\sigma \geq \pi$ that is involved. For a term to survive we need that $\sigma \in \mathcal{S}$ and that for every $W \in \sigma$ we have that $\pi_W \in \mathcal{S}$, which is equivalent to $\sigma \geq_{\mathcal{S}} \pi$. Thus, we conclude that

$$(\iota \otimes \iota) \circ \Delta_{\mathcal{P}}(\widehat{\pi}) = \sum_{\substack{\sigma \in \mathcal{P}, \\ \sigma \geq \pi}} \iota(\widehat{\pi}|_{\widehat{\sigma}}) \otimes \iota(\widehat{\sigma}) = \sum_{\substack{\sigma \in \mathcal{S}, \\ \sigma \geq_{\mathcal{S}} \pi}} \widehat{\pi}|_{\widehat{\sigma}} \otimes \widehat{\sigma} = \Delta_{\mathcal{S}}(\iota(\widehat{\pi})).$$

Therefore we conclude that ι is a surjective Hopf algebra homomorphsim from $\widehat{\mathcal{P}}$ onto $\widehat{\mathcal{S}}$. □

As byproduct of this result we obtain the following:

Corollary 6.3.2. *In the group $\mathcal{P}^{\circ}$ we consider the subset of functions $Supp_{\mathcal{S}} := \{f \in \mathcal{P}^{\circ} | f(\widehat{\pi}) = 0$ whenever $\pi \notin \mathcal{S}\}$ that are "supported" on the set $\mathcal{S}$. Then $Supp_{\mathcal{S}}$ is a subgroup that is isomorphic to $\mathcal{S}^{\circ}$, with the natural identification. That is, it is given by the map $Supp_{\mathcal{S}} \ni f \mapsto g \in \mathcal{S}^{\circ}$ where we take $g(\widehat{\pi}^{\mathcal{S}}) = f(\widehat{\pi}^{\mathcal{P}})$ for every $\mathcal{S}$.*

6.4 Right-comodule structures in $\widehat{\mathcal{S}}$

As we already saw in Section 4.5, the best way to think of the transition formulas between moments and different brands of cumulants, is as an action of semi-multiplicative functions on multilinear functionals. These actions arise naturally from a right-comodule structure in $\widehat{\mathcal{S}}$, in the sense of Definition 2.4.9. In this section we will introduce two such structures on $\widehat{\mathcal{S}}$.

6.4.1 The comodule $\mathcal{X}$ and the action of $\mathcal{S}^{\circ}$ on sequences

Definition 6.4.1. Let us denote by $\mathcal{X} := \mathbb{C}[X_1, X_2, X_3, \dots]$ the commutative algebra of polynomials over the countable set of indeterminates $X_1, X_2, X_3, \dots$.

We define the map $\gamma : \mathcal{X} \to \mathcal{X} \otimes \widehat{\mathcal{S}}$ to be the unital algebra homomorphism, uniquely determined such that for every $n \in \mathbb{N}$ we have

$$\gamma(X_n) = \sum_{\tau \in \mathcal{S}(n)} \left(\prod_{V \in \tau} X_{|V|} \right) \otimes \widehat{\tau}. \tag{6.14}$$

Theorem 6.4.2. *$\mathcal{X}$ with the map γ is a right comodule of $\widehat{\mathcal{S}}$.*

Proof. We just need to fix an $n \in \mathbb{N}$ and check that the counit and coassociative properties hold.

Counit, $(id \otimes \varepsilon_{\mathcal{S}}) \circ \gamma = id$. We start computing

$$(id \otimes \varepsilon_{\mathcal{S}}) \circ \gamma(X_n) = \sum_{\substack{\tau \in \mathcal{S}, \\ \tau \leq_{\mathcal{S}} \pi}} id \left(\prod_{V \in \tau} X_{|V|} \right) \otimes \varepsilon_{\mathcal{S}}(\widehat{\tau}),$$

then we observe that $\varepsilon_{\mathcal{S}}(\widehat{\tau}) = 0$ whenever $\widehat{\tau} \neq 1_{\mathcal{S}}$. Therefore all the terms in the sum vanish except for the corresponding to $\tau = 1_n$, so we get

$$(id \otimes \varepsilon_{\mathcal{S}}) \circ \gamma(X_n) = X_n \otimes 1_{\mathcal{S}},$$

and conclude that $(id \otimes \varepsilon_{\mathcal{S}}) \circ \gamma = id$.

Coassociative, $(\gamma \otimes id) \circ \gamma = (id \otimes \Delta) \circ \gamma$. Here we fix an X_n and show that both $(\gamma \otimes id) \circ \gamma(X_n)$ and $(id \otimes \Delta) \circ \gamma(X_n)$ are equal to

$$\sum_{\substack{\tau,\sigma \in \mathcal{S}(n),\\ \tau \leq_{\mathcal{S}} \sigma}} \left(\prod_{V \in \tau} X_{|V|} \right) \otimes \widehat{\tau}|_{\widehat{\sigma}} \otimes \widehat{\sigma}. \tag{6.15}$$

Indeed, on one hand we compute

$$\begin{aligned}
(\gamma \otimes id) \circ \gamma(X_n) &= \sum_{\sigma \in \mathcal{S}(n)} \gamma\left(\prod_{W \in \sigma} X_{|W|} \right) \otimes id(\widehat{\sigma}) \\
&= \sum_{\sigma \in \mathcal{S}(n)} \prod_{W \in \sigma} \left(\sum_{\tau \in \mathcal{S}(|W|)} \left(\prod_{V \in \tau} X_{|V|} \right) \otimes \widehat{\tau} \right) \otimes \widehat{\sigma} \\
&= \sum_{\sigma \in \mathcal{S}(n)} \sum_{\substack{\tau \in \mathcal{S}(n),\\ \tau \leq_{\mathcal{S}} \sigma}} \left(\prod_{V \in \tau} X_{|V|} \right) \otimes \widehat{\tau}|_{\widehat{\sigma}} \otimes \widehat{\sigma},
\end{aligned}$$

which coincides with (6.15). On the other hand, we obtain

$$\begin{aligned}
(id \otimes \Delta) \circ \gamma(X_n) &= \sum_{\tau \in \mathcal{S}(n)} id\left(\prod_{V \in \tau} X_{|V|} \right) \otimes \Delta(\widehat{\tau}) \\
&= \sum_{\tau \in \mathcal{S}(n)} \left(\prod_{V \in \tau} X_{|V|} \right) \otimes \left(\sum_{\substack{\sigma \in \mathcal{S},\\ \sigma \geq_{\mathcal{S}} \tau}} \widehat{\tau}|_{\widehat{\sigma}} \otimes \widehat{\sigma} \right),
\end{aligned}$$

and if we merge the two sums we also get (6.15), which concludes the proof. □

We now turn to the study of the dual of γ. Consider $\mathcal{X}^\circ$ the character group of $\mathcal{X}$. Recall that an element $\psi \in \mathcal{X}^\circ$ is completely determined by its values at the generators $X_1, X_2, \ldots$. So $\psi \in \mathcal{X}^\circ$ can be viewed as a function $\psi : \mathbb{N} \to \mathbb{C}$ or as sequence of complex numbers $(\psi_n)_{n \in \mathbb{N}} \subset \mathbb{C}$, where $\psi_n := \psi(n)$ for $n \in \mathbb{N}$. We make use of the coaction γ to create a left action of $\mathcal{S}^\circ$ on set $\mathcal{X}^\circ$.

Notation 6.4.3. For every $f \in \mathcal{S}^\circ$ and every $\psi = (\psi_n)_{n \in \mathbb{N}} \in \mathcal{X}^\circ$. We denote by "$\psi \cdot f$" the sequence $\theta = \{\theta_n\}_{n \in \mathbb{N}} \in \mathcal{X}^\circ$ defined as $\theta = (\psi \otimes f) \circ \gamma$, where we think of θ and ψ as functions and we use the natural identification $\mathcal{X} \otimes \mathbb{C} = \mathcal{X}$. In other words,

$$\theta_n = \sum_{\pi \in \mathcal{S}(n)} f(\widehat{\pi}) \prod_{V \in \pi} \psi_{|V|} \qquad \forall n \in \mathbb{N}.$$

Corollary 6.4.4. *The operation $\cdot$ from the previous notation, defines a left action of the group $\mathcal{S}^\circ$ on the set $\mathcal{X}^\circ$. That is, one has*

$$(\psi \cdot f) \cdot g = \psi \cdot (f * g), \qquad \forall \psi \in \mathcal{X}^\circ \text{ and } f, g \in \mathcal{S}^\circ. \tag{6.16}$$

Proof. This follows directly from Theorem 6.4.2. Indeed,

$$(\psi \cdot f) \cdot g = (\psi \otimes f \otimes g) \circ ((\gamma \circ id) \circ \gamma) = (\psi \otimes f \otimes g) \circ ((id \circ \Delta) \circ \gamma) = \psi \cdot (f * g). \quad \square$$

Let us now give a quick example on how this action look in the context of a non-commutative probability. In the previous section we observed that semi-multiplicative functions of an arbitrary iterative set $\mathcal{S}$ can all be seen as part of $\mathcal{P}^\circ$, so we can just focus on the group $\mathcal{P}^\circ$. Recall that in Section 4.6 we already studied some elements of $\mathcal{P}^\circ$ in the context of semi-multiplicative functions on $\mathcal{NC}$. In that section we studied the transition formulas $g_{\mathrm{fc-m}}, g_{\mathrm{bc-m}}, g_{\mathrm{mc-m}} \in SM^{\mathcal{NC}}$ that go from different brands of cumulants to moments. Viewing these functions as characters in $\mathcal{P}^\circ$, as described in Notation 6.2.2, we have that the characters $g_{\mathrm{fc-m}}, g_{\mathrm{bc-m}}, g_{\mathrm{mc-m}} \in \mathcal{P}^\circ$ are determined by

$$g_{\mathrm{fc-m}}(\widehat{\pi}) = 1, \text{ if } \pi \in \mathcal{NC}, \text{ and } 0 \text{ otherwise}, \tag{6.17}$$

$$g_{\mathrm{bc-m}}(\widehat{\pi}) = 1, \text{ if } \pi \in \mathcal{I}, \text{ and } 0 \text{ otherwise}, \tag{6.18}$$

$$g_{\mathrm{mc-m}}(\widehat{\pi}) = \frac{m(\pi)}{|\pi|!}, \text{ if } \pi \in \mathcal{I}, \text{ and } 0 \text{ otherwise}. \tag{6.19}$$

Since now we can study semi-multiplicative functions on all partitions, we can include the classical moment-cumulant formula in the picture.

Definition 6.4.5. The character $g_{\mathrm{cc-m}} \in \mathcal{P}^\circ$ is uniquely determined by requiring

$$g_{\mathrm{cc-m}}(\widehat{\pi}) = 1, \qquad \forall \pi \in \mathcal{P}. \tag{6.20}$$

Then, the action $\cdot$ of these elements of $\mathcal{P}^\circ$ represents the transition from a sequence of some brand of cumulants to the sequence of moments. Given a non-commutative probability space $(\mathcal{A}, \varphi)$, and a variable $a \in \mathcal{A}$, let us consider the sequences of moments $\underline{\varphi}(a) = (\varphi_n(a))_{n=1}^\infty \in \mathcal{X}^\circ$, classical cumulants $\underline{c}(a) = (c_n(a))_{n=1}^\infty \in \mathcal{X}^\circ$, free cumulants $\underline{r}(a) = (r_n(a))_{n=1}^\infty \in \mathcal{X}^\circ$, Boolean cumulants $\underline{b}(a) = (b_n(a))_{n=1}^\infty \in \mathcal{X}^\circ$, and monotone cumulants $\underline{h}(a) = (h_n(a))_{n=1}^{\infty} \in \mathcal{X}^{\circ}$. Then we have that

$$\begin{aligned}
\underline{\varphi}(a) &= \underline{c}(a) \cdot g_{\mathrm{cc-m}}, \\
\underline{\varphi}(a) &= \underline{b}(a) \cdot g_{\mathrm{bc-m}}, \\
\underline{\varphi}(a) &= \underline{r}(a) \cdot g_{\mathrm{fc-m}}, \\
\underline{\varphi}(a) &= \underline{h}(a) \cdot g_{\mathrm{mc-m}}.
\end{aligned}$$

6.4.2 The comodule $V_{\mathcal{S}}$ and the action of $\mathcal{S}^{\circ}$ on families of multilinear functionals

In this subsection we point out that the right-comodule structure $(\mathcal{X}, \gamma)$ on $\widehat{\mathcal{S}}$, is intimately related with another right-comodule on $\widehat{\mathcal{S}}$ which is somehow more robust. In this case, we consider $V_{\mathcal{S}}$ to be the vector space with a linear basis given by $\{v_\pi : \pi \in \mathcal{S}\}$. As a consequence, we will thus find that the action of $\mathcal{S}^{\circ}$ not only works for sequences of complex numbers, but it can also be defined on sequences of multilinear functionals. Thus, our machinery allows us to deal with families of cumulant functionals that are used for multivariate cumulants, and families of functions over algebras, that are used in the operator-valued case.

Definition 6.4.6. Let $\Gamma : V_{\mathcal{S}} \to V_{\mathcal{S}} \otimes \widehat{\mathcal{S}}$ be the linear map uniquely determined such that for every $\pi \in \mathcal{S}$ we have

$$\Gamma(v_\pi) = \sum_{\substack{\tau \in \mathcal{S}, \\ \tau \leq_{\mathcal{S}} \pi}} v_\tau \otimes \widehat{\tau}|_{\widehat{\pi}}, \tag{6.21}$$

Notice that unlike the coproduct Δ, where we sum over partitions $\sigma \geq_{\mathcal{S}} \pi$, in the map Γ we sum over partitions $\tau \leq_{\mathcal{S}} \pi$. With this definition we have the following result.

Proposition 6.4.7. *$V_{\mathcal{S}}$ with the map Γ is a right comodule of $\widehat{\mathcal{S}}$.*

Proof. The proof of this result is similar to the proof of Theorem 6.4.2.

Since Γ is a linear map we just need to fix a partition $\pi \in \mathcal{S}$ and check that the counit and coassociative properties hold when applied to the partition π.

Counit, $(id \otimes \varepsilon_{\mathcal{S}}) \circ \Gamma = id$. We start computing

$$(id \otimes \varepsilon_{\mathcal{S}}) \circ \Gamma(v_\pi) = \sum_{\substack{\tau \in \mathcal{S}, \\ \tau \leq_{\mathcal{S}} \pi}} id(v_\tau) \otimes \varepsilon_{\mathcal{S}}(\widehat{\tau}|_{\widehat{\pi}}),$$

then we observe that $\varepsilon_{\mathcal{S}}(\widehat{\tau}|_{\widehat{\pi}})=0$ if and only if $\widehat{\tau}|_{\widehat{\pi}}\neq 1_{\mathcal{S}}$ which happen if and only if $\tau\neq\pi$. Therefore all the terms in the sum vanish except for the corresponding to $\tau=\pi$, so we get

$$\sum_{\substack{\tau\in\mathcal{S},\\ \tau\leq_{\mathcal{S}}\pi}} id(v_\tau)\otimes\varepsilon_{\mathcal{S}}(\widehat{\tau}|_{\widehat{\pi}})=v_\pi\otimes 1,$$

and conclude that $(id\otimes\varepsilon_{\mathcal{S}})\circ\Gamma=id$.

Coassociative, $(\Gamma\otimes id)\circ\Gamma=(id\otimes\Delta)\circ\Gamma$. Here we fix a $\pi\in\mathcal{S}$ and want to check that both, $(\Gamma\otimes id)\circ\Gamma(v_\pi)$ and $(id\otimes\Delta)\circ\Gamma(v_\pi)$, are equal to

$$\sum_{\substack{\tau,\sigma\in\mathcal{S},\\ \tau\leq_{\mathcal{S}}\sigma\leq_{\mathcal{S}}\pi}} v_\tau\otimes\widehat{\tau}|_{\widehat{\sigma}}\otimes\widehat{\sigma}|_{\widehat{\pi}}. \tag{6.22}$$

Indeed, for the left hand side we compute

$$\begin{aligned}(\Gamma\otimes id)\circ\Gamma(v_\pi)&=\sum_{\substack{\sigma\in\mathcal{S},\\ \sigma\leq_{\mathcal{S}}\pi}}\Gamma(v_\sigma)\otimes id(\widehat{\sigma}|_{\widehat{\pi}})\\ &=\sum_{\substack{\sigma\in\mathcal{S},\\ \sigma\leq_{\mathcal{S}}\pi}}\left(\sum_{\substack{\tau\in\mathcal{S},\\ \tau\leq_{\mathcal{S}}\sigma}} v_\tau\otimes\widehat{\tau}|_{\widehat{\sigma}}\right)\otimes\widehat{\sigma}|_{\widehat{\pi}},\end{aligned}$$

which coincides with (6.22). On the other hand, for the right hand side we compute

$$\begin{aligned}(id\otimes\Delta)\circ\Gamma(\widehat{\pi})&=\sum_{\substack{\tau\in\mathcal{S},\\ \tau\leq_{\mathcal{S}}\pi}} id(v_\tau)\otimes\Delta(\widehat{\tau}|_{\widehat{\pi}})\\ &=\sum_{\substack{\tau\in\mathcal{S},\\ \tau\leq_{\mathcal{S}}\pi}} v_\tau\otimes\left(\sum_{\substack{\sigma\in\mathcal{S},\\ \tau\leq_{\mathcal{S}}\sigma\leq_{\mathcal{S}}\pi}}\widehat{\tau}|_{\widehat{\sigma}}\otimes\widehat{\sigma}|_{\widehat{\pi}}\right), \qquad \text{(using Lemma 6.1.5)}\end{aligned}$$

If we merge the two sums we also get (6.22), which concludes the proof. □

We now turn to the study of the dual picture. Let us consider M to be some vector space over $\mathbb{C}$. Recall from Notation 2.4.10 that we denote by $L(V_{\mathcal{S}},M)$ the set of linear functions from $V_{\mathcal{S}}$ to M. Notice that an element $\underline{\psi}\in L(V_{\mathcal{S}},M)$ is completely determined by its values at $(v_\pi)_{\pi\in\mathcal{S}}$. Thus denoting $\psi_\pi:=\underline{\psi}(v_\pi)$, we can think of $\underline{\psi}=(\psi_\pi)_{\pi\in\mathcal{S}}\subset M$ as a family of elements of M indexed by partitions in $\mathcal{S}$.

We now dualize the coaction Γ to obtain an action of $\mathcal{S}^\circ$ on set $L(V_{\mathcal{S}},M)$.

Notation 6.4.8. Fix M is some vector space over $\mathbb{C}$. Let $f \in \mathcal{S}^\circ$ and let $\underline{\psi} = \{\psi_\pi\}_{\pi \in \mathcal{S}} \in L(V_\mathcal{S}, M)$. We denote by "$\underline{\psi} \cdot f$" the element family $\underline{\theta} = (\theta_\pi)_{\pi \in \mathcal{S}} \in L(V_\mathcal{S}, M)$ defined as $\underline{\theta} = (\underline{\psi} \otimes f) \circ \Gamma$, (with the natural identification $M \otimes \mathbb{C} = M$). In other words,

$$\theta_\pi = \sum_{\substack{\tau \in \mathcal{S}, \\ \tau \leq_\mathcal{S} \pi}} f(\widehat{\tau}|_{\widehat{\pi}}) \psi_\tau \qquad \forall\, \pi \in \mathcal{S}.$$

Observe that $f(\widehat{\tau}|_{\widehat{\pi}})$ is just a complex coefficient.

Proposition 6.4.9. *The formula $\cdot$ from the previous notation defines a left action of the group $\mathcal{S}^\circ$ on the set $L(V_\mathcal{S}, M)$. That is, one has*

$$(\underline{\psi} \cdot f) \cdot g = \underline{\psi} \cdot (f * g), \qquad \forall \underline{\psi} \in L(V_\mathcal{S}, M) \text{ and } f, g \in \mathcal{S}^\circ. \tag{6.23}$$

Proof. This follows directly from the fact that Γ is a coaction. Indeed,

$$(\underline{\psi} \cdot f) \cdot g = (\underline{\psi} \otimes f \otimes g) \circ ((\Gamma \circ id) \circ \Gamma) = (\underline{\psi} \otimes f \otimes g) \circ ((id \circ \Delta) \circ \Gamma) = \underline{\psi} \cdot (f * g). \qquad \square$$

Observe that we have a lot of flexibility to choose the vector space M, so we can define actions on very general structures, allowing us to deal with several examples. For instance, we can take M to be sequences of complex numbers, or sequences of multilinear functionals, or sequences of multilinear maps.

The study of the action of $\widehat{\mathcal{P}}$ on sequences of complex numbers, was already done in the previous section.

If we take M to be a sequence of multilinear functionals, after some considerations we end up with the same action that was studied in Section 4.5. The difference is that we can now extend it to an action of $\mathcal{P}$ instead of just $\mathcal{NC}$. Since the development is very similar we will omit this example.

For the rest of the section we will focus on the more general case of an action on sequences of multilinear maps. This allows us to work with transition formulas at the level of multivariable cumulants, arising in the framework of operator-valued non-commutative probability spaces. In order to do this we consider M to be the vector space of multilinear maps from $\mathcal{A}^n$ to $\mathcal{B}$ for some $n \in \mathbb{N}$. We also take $\mathcal{S}$ to be the largest possible iterative family $\mathcal{P}$, as we know this contains all the other families.

Definition 6.4.10 (Action of $\mathcal{P}^\circ$ on M)**.** For every $f \in \mathcal{P}^\circ$ and $\underline{\psi} \in L(\widetilde{\mathcal{P}}, M)$ we follow Notation 6.4.8 to define $\underline{\psi} \cdot f$ as the family of multilinear functionals $\underline{\theta} \in L(\widetilde{\mathcal{P}}, M)$ satisfying $\underline{\theta} := (\underline{\psi} \otimes f) \circ \Gamma$.

The relevance of this action is that it is closed on multiplicative families when we restrict our attention to $\mathcal{NC}^\circ$. Recall from Subsection 2.3.1 that if $\mathcal{A}$ is a unital algebra and $\mathcal{B}$ is a unital subalgebra of $\mathcal{A}$ we denote by $\mathfrak{M}^{ov}_{\mathcal{A},\mathcal{B}}$ the set of multiplicative families of multilinear maps from $\mathcal{A}^n$ to $\mathcal{B}$ for some $n \in \mathbb{N}$. Notice that the definition of multiplicative families is only done for partitions in $\mathcal{NC}$, because it relies on the fact that the blocks of a partition do not cross. In fact, in Definition 2.3.7, the construction of the maps ψ_π heavily relies on the nesting structure of the blocks of π. For this reason we will restrict our attention to the group $\mathcal{NC}^\circ$.

We will show that $\cdot$ is a left action of the group $\mathcal{NC}^\circ$ on the set $\mathfrak{M}^{ov}_{\mathcal{A},\mathcal{B}}$. This result was proved in [PT21], and follows from generalizing the proof of Proposition 2.1.7 from [Spe98].

Proposition 6.4.11 (Lemma 6.1, [PT21])**.** *If $f \in \mathcal{NC}^\circ$ and $\underline{\psi} \in \mathfrak{M}^{ov}_{\mathcal{A},\mathcal{B}}$, then $\underline{\psi} \cdot f$ from Definition 6.4.10 is multiplicative. In other words, $\cdot$ defines a right action of the group $\mathcal{NC}^\circ$ on the set $\mathfrak{M}^{ov}_{\mathcal{A},\mathcal{B}}$.*

Proof. Let $\underline{\theta} := \underline{\psi} \cdot f$. To corroborate that $\underline{\theta}$ is multiplicative, we only need to check that it satisfies the recurrence from Definition 2.3.7. So take $\pi \in \mathcal{NC}(n)$, pick $V = \{l+1, \dots l+k\} \in \pi$ a interval block of π, and denote $\pi_0 := \pi \backslash V \in \mathcal{NC}(n-k)$, $a = (a_1, \dots, a_n)$ and $a_V = (a_{l+1}, \dots, a_{l+k})$. By substituting, $\theta_{\pi'}, \theta_k$ and using linearity we obtain:

$$\begin{aligned}
&\theta_{\pi'}(a_1, \dots, a_l \theta_k(a_V), a_{l+k+1}, \dots, a_n) \\
&= \sum_{\sigma_2 \in \mathcal{NC}(n-k)} f(\widehat{\sigma_2}) \psi_{\sigma_2}\left(a_1, \dots, a_l \sum_{\sigma_1 \in \mathcal{NC}(k)} f(\widehat{\sigma_1}) \psi_{\sigma_1}(a_V), a_{l+k+1}, \dots, a_n \right) \\
&= \sum_{\substack{\sigma_1 \in \mathcal{NC}(k) \\ \sigma_2 \in \mathcal{NC}(n-k)}} f(\widehat{\sigma_2}) f(\widehat{\sigma_1}) \psi_{\sigma_2}(a_1, \dots, a_l \psi_{\sigma_1}(a_{l+1}, \dots, a_{l+k}), a_{l+k+1}, \dots, a_n).
\end{aligned}$$

Then we construct $\sigma \in \mathcal{NC}(n)$ as the unique partition such that $\sigma \leq \{V, [n] \backslash V\}$ and with restrictions $\sigma_{{}_V} = \sigma_1$ and $\sigma_{{}_{[n]\backslash V}} = \sigma_2$. Thus we get

$$\sum_{\substack{\sigma \in \mathcal{NC}(n) \\ \sigma \leq \pi}} \prod_{V \in \pi} f(\widehat{\sigma_{{}_V}}) \psi_\sigma(a) = \sum_{\substack{\sigma \in \mathcal{NC}(n) \\ \sigma \leq \pi}} f(\widehat{\sigma}|_{\widehat{\pi}}) \psi_\sigma(a) = \theta_\pi(a),$$

as desired. $\square$

Remark 6.4.12. The fact that $\cdot$ defines a left action of $\mathcal{NC}^\circ$ on $\mathfrak{M}^{ov}_{\mathcal{A},\mathcal{B}}$ allows us to conclude that when studying the transition formulas between moments and different brands of cumulants we just need to understand the group $\mathcal{NC}^\circ$. In other words, we just care about the combinatorial structure provided by the Hopf algebra $\widehat{\mathcal{NC}}$, and it does not really matter if we are working in the simple framework of moment-cumulant formulas for one random variable a, or with the far more general framework of operator valued non-commutative probability.

6.5 The antipode of $\widehat{\mathcal{S}}$

The antipode of the Hopf algebra $\widehat{\mathcal{S}}$ deserves special attention due to its potential use as a tool for inversion in formulas that relate moments to cumulants using a sum over partitions in $\mathcal{S}$. The issue of performing such inversions is constantly present in the literature on cumulants. Indeed, it is typical that cumulants (of one brand or another) are introduced via some simple formulas which are deemed to express moments in terms of the desired cumulants; these simple formulas then need to be inverted, if one wants to see explicit formulas describing cumulants in terms of moments. In such a situation, the tool that is typically used for inversion is the Möbius function of some underlying poset which luckily turns out to be related to the cumulants in question.

The considerations on the Hopf algebra $\widehat{\mathcal{S}}$ suggest an alternate method which can provide a unified way of treating the inversions of various cumulant-to-moment formulas, and also for doing inversions in cumulant-to-cumulant formulas. Since moment-cumulant formulas are encoded by characters $g \in \mathcal{S}^{\circ}$, the inverse of these formulas can be obtained from the knowledge of the antipode S by simply computing $g^{-1} = g \circ S$

Since $\widehat{\mathcal{S}}$ is commutative, general bialgebra considerations yield the fact that $S : \widehat{\mathcal{S}} \to \widehat{\mathcal{S}}$ is a unital algebra homomorphism (cf. [Man04, Proposition I.7.1]). This implies in particular that S is completely determined by how it acts on the generators $\widehat{\pi}$ of $\widehat{\mathcal{S}}$. In Proposition 6.5.2 below we state some formulas which allow recursive calculations of values $S(\widehat{\pi})$, and go under the name of *Bogoliubov formulas.*

Notation 6.5.1. For $\pi, \sigma \in \mathcal{S}$, we will write "$\pi <_{\mathcal{S}} \sigma$" to mean that $\pi \leq_{\mathcal{S}} \sigma$ and that $\pi \neq \sigma$.

Proposition 6.5.2. (Bogoliubov formulas.) *Let $\mathcal{S}$ be an iterative subset of $\mathcal{P}$. For $n \geq 1$ and $\pi \in \mathcal{S}(n) \setminus \{1_n\}$ one has:*

$$S(\widehat{\pi}) = -\widehat{\pi} - \sum_{\substack{\sigma \in \mathcal{S}, \\ \pi <_{\mathcal{S}} \sigma <_{\mathcal{S}} 1_n}} \widehat{\pi}|_{\widehat{\sigma}}\, S(\widehat{\sigma}), \tag{6.24}$$

and also that

$$S(\widehat{\pi}) = -\widehat{\pi} - \sum_{\substack{\sigma\in\mathcal{S},\\ \pi<_{\mathcal{S}}\sigma<_{\mathcal{S}}1_n}} S(\widehat{\pi}|_{\widehat{\sigma}})\,\widehat{\sigma}, \tag{6.25}$$

where the monomials $\widehat{\pi}|_{\widehat{\sigma}}$ are as introduced in Notation 6.1.4.

Proof. The relation $\mathrm{Id} * S = \varepsilon_{\mathcal{S}}$ implies in particular that $\mathrm{Id} * S\,(\widehat{\pi}) = \varepsilon_{\mathcal{S}}(\widehat{\pi})1_{\mathcal{S}} = 0$. But on the other hand, the explicit description (6.1) used for $\mathrm{Id} * S$ says that:

$$\mathrm{Id} * S\,(\widehat{\pi}) = \sum_{\sigma\geq\pi} \widehat{\pi}|_{\widehat{\sigma}}\,S(\widehat{\sigma}) = \widehat{\pi}|_{\widehat{\pi}}\,S(\widehat{\pi}) + \widehat{\pi}|_{\widehat{1_n}}\,S(\widehat{1_n}) + \sum_{\pi<\sigma<1_n} \widehat{\pi}|_{\widehat{\sigma}}\,S(\widehat{\sigma}).$$

Upon recalling (cf. Remark 6.1.4) that $\widehat{\pi}|_{\widehat{\pi}} = 1_{\mathcal{S}}$ and $\widehat{\pi}|_{\widehat{1_n}} = \widehat{\pi}$, we thus find that

$$0 = S(\widehat{\pi}) + \widehat{\pi} + \sum_{\pi<\sigma<1_n} \widehat{\pi}|_{\widehat{\sigma}}\,S(\widehat{\sigma}), \tag{6.26}$$

where separating the term $S(\widehat{\pi})$ on the right-hand side leads to the formula (6.24). The derivation of (6.25) is analogous, where we now start from the fact that $S * \mathrm{Id} = \varepsilon_{\mathcal{S}}$. □

Remark 6.5.3. In the statement of Proposition 6.5.2 we excluded the case when $\pi = 1_n$. In that case we have $\widehat{\pi} = 1_{\mathcal{S}}$ and taking the antipode just gives $S(\widehat{1_n}) = S(1_{\mathcal{S}}) = 1_{\mathcal{S}}$. Note also that, in the case when $\pi \in \mathcal{S}_2$, namely $|\pi| = 2$, the sum over $\{\sigma \in \mathcal{S} \mid \pi <_{\mathcal{S}} \sigma <_{\mathcal{S}} 1_n\}$ is an empty sum. In that case, either (6.24) or (6.25) gives that $S(\widehat{\pi}) = -\widehat{\pi}$.

Both (6.24) and (6.25) can be used for a recursive computation of values $S(\widehat{\pi})$, but the setting of the recursion is different in the two situations. Formula (6.24) works when we fix an $n \in \mathbb{N}$, taken in isolation, and compute $S(\widehat{\pi})$ for $\pi \in \mathcal{S}(n)$, by induction on $|\pi|$. Formula (6.25) works when we already know how S works on some partitions from $\mathcal{S}(m)$ with $m < n$ – for instance, this works neatly when we fix an $\ell \geq 1$ and we are interested in $S(\widehat{\pi})$ for all $\pi \in \mathcal{S}$ such that every block V of π has $|V| \leq \ell$.

At the end of this section we will present a detailed example illustrating, in the case $\mathcal{S} = \mathcal{NC}$, how these two recursive methods work towards computing $S(\widehat{0_n})$ for some small values of n.

In preparation for our main result, we will present Proposition 6.5.5, which is a special case of a result of Schmitt [Sch87] holding in the general framework of an incidence Hopf algebra. For the proof of Proposition 6.5.5 (which is, essentially, an induction on $|\pi|$ based on the recursion formula (6.24)) we refer to [Sch87, Theorem 6.1] or [Sch94, Theorem 4.1].

Definition 6.5.4. Let n be a positive integer and let $\pi, \sigma \in \mathcal{S}$ be such that $\pi <_{\mathcal{S}} \sigma$. A $\mathcal{S}$-*chain* from π to σ is a tuple

$$c = (\pi_0, \pi_1, \ldots, \pi_k), \text{ where } \pi = \pi_0 <_{\mathcal{S}} \pi_1 <_{\mathcal{S}} \cdots <_{\mathcal{S}} \pi_k = \sigma. \tag{6.27}$$

The number k appearing in (6.27) is called the *length* of c.

For a chain c as in (6.27) it will be convenient to use the shorthand notation

$$\widehat{c} := \widehat{\pi}_0|_{\widehat{\pi_1}} \cdot \widehat{\pi}_1|_{\widehat{\pi_2}} \cdots \widehat{\pi_{k-1}}|_{\widehat{\pi_k}} \in \widehat{\mathcal{S}}. \tag{6.28}$$

Proposition 6.5.5. *For $n \geq 1$ and $\pi \in \mathcal{S}(n) \setminus \{1_n\}$ one has:*

$$S(\widehat{\pi}) = \sum_{\substack{c=(\pi_0,\pi_1,\ldots,\pi_k),\\ \mathcal{S}-\text{chain from } \pi \text{ to } 1_n}} (-1)^k \widehat{c}. \tag{6.29}$$

Remark 6.5.6. We will next show how one can re-structure the summation over chains from (6.29) in order to obtain a *cancellation-free* summation formula. This will be done by pruning the index set used in (6.29) to a smaller collection of chains in $\mathcal{S}(n)$, which we call "efficient chains" – cf. Definition 6.5.7, Theorem 6.5.10.

We mention that our identifying of the notion of efficient chain retrieves a special case of a notion identified in the thesis [Ein10], in the general framework of incidence Hopf algebras, where the terms of the cancellation-free summations arrive to be described by objects called "forests of lattices" (cf. [Ein10, Chapter 5]). While it would be possible to review the fairly substantial background and terminology developed in [Ein10] and then invoke the result from there, we find it easier to write down a direct inductive argument which covers the special case needed in Theorem 6.5.10.

We would also like to signal that another path towards obtaining a cancellation-free summation formula for the antipode of $\widehat{\mathcal{S}}$ is offered by the work in [MP18]. This would exploit the fact that $\widehat{\mathcal{S}}$ is an example of so-called *left-handed polynomial Hopf algebra*, a term which refers to the fact that the formula (6.1) defining comultiplication merely has an "$\widehat{\sigma}$" (rather than a product of $\widehat{\sigma}$'s) on the right side of the tensor product.

Definition 6.5.7. Let n be a positive integer and let $\pi, \sigma \in \mathcal{S}$ be such that $\pi <_{\mathcal{S}} \sigma$.

1. To every $\mathcal{S}$-chain $c = (\pi_0, \pi_1, \ldots, \pi_k)$ from π to σ we associate two collections of subsets of $\{1, \ldots, n\}$, as follows:

$$\begin{aligned} \text{Blocks}(c) &:= \{V \subseteq \{1, \ldots, n\} \mid \exists\, 0 \leq j \leq k \text{ such that } V \in \pi_j\}, \text{ and} \\ \text{Blocks}^+(c) &:= \{V \in \text{Blocks}(c) \mid V \notin \pi_0\}. \end{aligned}$$

2. A chain $c = (\pi_0, \pi_1, \ldots, \pi_k)$ from π to σ will be said to be *efficient* when it satisfies:

$$\begin{cases} \text{For every set } V \in \text{Blocks}^+(c) \text{ there exists} \\ \text{a } \textit{unique } j \in \{1, \ldots, k\} \text{ such that } V \text{ is a block of } \pi_j. \end{cases}$$

3. We denote by $\text{EC}_{\mathcal{S}}(\pi, \sigma)$ the set of all efficient $\mathcal{S}$-chains from π to σ.

Remark 6.5.8. 1. In order to explain the term "efficient" used in the preceding definition, let $\pi <_{\mathcal{S}} \sigma$ be as above and let $c = (\pi_0, \pi_1, \ldots, \pi_k)$ be a chain from π to σ. Pick an $m \in \{1, \ldots, n\}$ and for every $0 \leq j \leq k$ let us denote by $V^{(j)}$ the block of π_j which contains the number m. Then we have

$$V^{(0)} \subseteq V^{(1)} \subseteq \cdots \subseteq V^{(k)} \quad (\text{subsets of } \{1, \ldots, n\}), \tag{6.30}$$

where some of the inclusions in (6.30) may actually be equalities. The property of c described in Definition 6.5.7.2 amounts to the fact that once we run into an inclusion $V^{(i-1)} \subseteq V^{(i)}$ which is strict, all the subsequent inclusions $V^{(j-1)} \subseteq V^{(j)}$ with $j \geq i$ have to be strict as well – in a certain sense, one "moves efficiently" towards the last set $V^{(k)}$ indicated in that list.

2. Given $\pi <_{\mathcal{S}} \sigma$ in $\mathcal{S}$ and a chain $c = (\pi_0, \pi_1, \ldots, \pi_k) \in \text{EC}_{\mathcal{S}}(\pi, \sigma)$ we will be interested in the quantity

$$(-1)^{|\text{Blocks}^+(c)|}\widehat{c} = (-1)^{|\text{Blocks}^+(c)|}\widehat{\pi}_0|_{\widehat{\pi_1}}\widehat{\pi}_1|_{\widehat{\pi_2}} \cdots \widehat{\pi_{k-1}}|_{\widehat{\pi_k}}, \tag{6.31}$$

which will be featured in Theorem 6.5.10 below. To understand the intuition behind this term we refer the reader to example at the end of this section.

3. Let $\pi <_{\mathcal{S}} \sigma$ be in $\mathcal{S}$ and consider a chain $c = (\pi_0, \pi_1, \ldots, \pi_k) \in \text{EC}_{\mathcal{S}}(\pi, \sigma)$. Upon tallying what indeterminates "$\widehat{\rho}$" are taken into the monomials $\widehat{\pi}_0|_{\widehat{\pi_1}}, \widehat{\pi}_1|_{\widehat{\pi_2}}, \ldots, \widehat{\pi_{k-1}}|_{\widehat{\pi_k}}$ multiplied in (6.31), one finds the following interpretation for the cardinality of the set $\text{Blocks}^+(c)$: it counts the total number of factors $\widehat{\rho}$ when the product $\widehat{\pi}_0|_{\widehat{\pi_1}}\widehat{\pi}_1|_{\widehat{\pi_2}} \cdots \widehat{\pi_{k-1}}|_{\widehat{\pi_k}}$ is simply treated as a product of $\widehat{\rho}$'s, and we eliminate the units "$\widehat{1_m}$" which may have appeared in the description of the monomials $\widehat{\pi_{j-1}}|_{\widehat{\pi_j}}$.

The observation made in the preceding paragraph ensures that the summation formula stated in Theorem 6.5.10 is cancellation-free! Indeed, if two chains appearing on the right-hand side of that summation formula turn out to have the same "$\widehat{c}$" contribution, then they will also have the same sign in the "$(-1)^{|\text{Blocks}^+(c)|}$" part of the formula; hence the terms indexed by the two chains in question will not cancel, but will rather add up.

For a concrete example, suppose we make $n = 7$ and we consider the chains

$$c' := \Big(0_7,\ \{\{1,2\},\ \{3\},\ \{4,5,6\},\ \{7\}\},\ \{\{1,2,3\},\{4,5,6,7\}\},\ 1_7\Big) \text{ and}$$

$$\begin{aligned} c'' \ := \ \Big(&0_7, \{\{1\},\{2\},\{3\},\{4,5\},\{6\},\{7\}\}, \{\{1\},\{2\},\{3\},\{4,5,6\},\{7\}\}, \\ &\qquad\qquad\qquad\qquad \{\{1,2,3\},\{4,5,6,7\}\},\ 1_7\Big), \end{aligned}$$

which are efficient chains going from 0_7 to 1_7. In the summation formula (6.35) of Theorem 6.5.10, c' and c'' have identical contributions, of $(-1)^5 \widehat{\rho_1} \cdots \widehat{\rho_5}$ where

$$\rho_1 = \big\{\{1,2,3\},\{4,5,6,7\}\big\},\ \rho_2 = \big\{\{1,2,3\},\{4\}\big\},\ \rho_3 = \big\{\{1,2\},\{3\}\big\},\ \rho_4 = 0_3,\ \rho_5 = 0_2.$$

The point to note is that the contributions of c' and c'' to the right-hand side of (6.35) do not cancel each other, but rather get to be added together, as mentioned above.

The proof of Theorem 6.5.10 is based on the following lemma.

Lemma 6.5.9. *Let π, σ be in $\mathcal{S}(n)$ for some $n \geq 1$, such that $\pi <_{\mathcal{S}} \sigma <_{\mathcal{S}} 1_n$, and where we write explicitly $\sigma = \{V_1, \ldots, V_r\}$. Consider the set of* [1] *efficient chains*

$$\widetilde{EC_{\mathcal{S}}} := \{c \in EC_{\mathcal{S}}(\pi, 1_n) \mid c = (\pi^{(k)}, \pi^{(k-1)}, \ldots, \pi^{(1)}, 1_n) \text{ with } k \geq 2 \text{ and } \pi^{(1)} = \sigma\}. \tag{6.32}$$

One has a bijection

$$\widetilde{EC_{\mathcal{S}}} \ni c \mapsto (c_1, \ldots, c_r) \in \prod_{s=1}^{r} EC_{\mathcal{S}}(\pi_{_{V_s}}, 1_{|V_s|}) \tag{6.33}$$

where, for $c = (\pi^{(k)}, \pi^{(k-1)}, \ldots, \pi^{(1)}, 1_n) \in \widetilde{EC_{\mathcal{S}}}$ and $1 \leq r \leq s$, we put $c_s := (\pi^{(k)}_{_{V_s}}, \ldots, \pi^{(1)}_{_{V_s}})$. (If it happens that we have $\pi_{_{V_2}} = \pi^{(k)}_{_{V_s}} = \pi^{(k-1)}_{_{V_s}} = \cdots = \pi^{(j)}_{_{V_s}}$ for some $1 \leq j \leq k-1$, then $(\pi^{(k)}_{_V}, \ldots, \pi^{(2)}_{_V}, \pi^{(1)}_{_V})$ is not properly a chain, so we rather take $c_s = (\pi_{_V}, \pi^{(j-1)}_{_V}, \ldots, \pi^{(1)}_{_V})$.)
Furthermore, for $c \mapsto (c_1, \ldots, c_r)$ as in (6.33), one has

$$(-1)^{|\mathrm{Blocks}^{+}(c)|}\widehat{c} = -\widehat{\pi^{(1)}} \prod_{s=1}^{r} \Big((-1)^{|\mathrm{Blocks}^{+}(c_s)|}\widehat{c_s} \Big). \tag{6.34}$$

[1] Note that in the chain c indicated in (6.32) the partition π appears as $\pi^{(k)}$. We chose this way of denoting c because it simplifies the write-up of the proof of the lemma.

Proof. We first prove that $c_1, \ldots, c_r$ from (6.33) are efficient chains. Pick an $s \in \{1, \ldots, r\}$ and, for the sake of contradiction, assume that c_s is not efficient. This implies that there exist a block $W \in \text{Blocks}^+(c_s)$ and indices $1 \leq i < j \leq k$ such that $W \in \pi_V^{(j)}$ and $W \in \pi_V^{(j)}$. Since $\pi^{(j)} < \pi^{(i)} \leq \pi^{(1)}$, this implies that $W \in \text{Blocks}^+(c)$ with $W \in \pi^{(i)}$ and $W \in \pi^{(j)}$, contradicting the fact that c is efficient. Therefore, $c' \in \text{EC}_{\mathcal{S}}(\pi_V, 1_{|V|})$ for all $c \in \text{EC}_{\mathcal{S}}(\pi, 1_n)$ and $V \in \pi^{(1)}$.

In order to prove that the map indicated in (6.33) is bijective, we will describe how its inverse works. For this, suppose we have an r-tuple of chains, $c_s = (\pi_s^{(j_s)}, \ldots, \pi_s^{(1)}) \in \text{EC}_{\mathcal{S}}(\pi_{V_s}, 1_{|V_s|})$. To reconstruct the chain $c \in \widetilde{\text{EC}_{\mathcal{S}}}$ which corresponds to $(c_1, \ldots, c_r)$, we first consider the size of the largest chain $j := \max_{1 \leq s \leq r} j_s$. Then, we enlarge the other chains so that all have the largest size, by denoting $\pi_s^{(i)} := \pi_{V_s}$ for every $s = 1, \ldots, r$ and $j_s < i \leq j$. Then, for every $i = 1, \ldots, j$, we construct the partition $\pi^{(i)} \in \mathcal{S}(n)$ uniquely determined by the fact that $\pi^{(i)} \leq \sigma$ and $\pi_{V_s}^{(i)} = \pi_s^{(i)}$ for $s = 1, \ldots, r$. Finally, we define $c := (\pi^{(k)}, \pi^{(k-1)}, \ldots, \pi^{(1)}, 1_n)$. It is not hard to show that this c is in $\widetilde{\text{EC}_{\mathcal{S}}}$, is mapped by (6.33) into the $(c_1, \ldots, c_r)$ we started from, and is uniquely determined by this property.

Finally, Equation (6.34) follows easily from the bijection (6.33). Indeed, for the equality of signs we break c by taking apart $(\pi^{(1)}, 1_n)$, and then regroup the remaining chain in terms of the blocks of $\pi^{(1)}$. Since $\text{Blocks}^+(\pi^{(1)}, 1_n) = 1$ we get that

$$\text{Blocks}^+(c) = \text{Blocks}^+(\pi^{(k)}, \pi^{(k-1)}, \ldots, \pi^{(1)}) + 1 = 1 + \sum_{s=1}^{r} \text{Blocks}^+(c_r).$$

For the term $\widehat{c}$, the idea is the same, although the computation is a bit more involved:

$$\begin{aligned}\widehat{c} &= \widehat{\pi^{(1)}}|_{\widehat{1_n}} \prod_{i=1}^{k-1} \widehat{\pi^{(i+1)}}|_{\widehat{\pi^{(i)}}} = \widehat{\pi^{(1)}} \prod_{i=1}^{k-1} \prod_{s=1}^{r} \widehat{\pi_{V_s}^{(i+1)}}|_{\widehat{\pi_{V_s}^{(i)}}} = \widehat{\pi^{(1)}} \prod_{s=1}^{r} \prod_{i=1}^{k-1} \widehat{\pi_{V_s}^{(i+1)}}|_{\widehat{\pi_{V_s}^{(i)}}} \\ &= \widehat{\pi^{(1)}} \prod_{s=1}^{r} \prod_{i=1}^{j_s} \widehat{\pi_s^{(i+1)}}|_{\widehat{\pi_s^{(i)}}} = \widehat{\pi^{(1)}} \prod_{s=1}^{r} \widehat{c_s}.\end{aligned}$$

Putting together the sign and the computation for $\widehat{c}$ yields Equation (6.34). $\square$

Theorem 6.5.10. *For every $\pi \in \mathcal{S}_{\geq 2}$ one has:*

$$S(\widehat{\pi}) = \sum_{c \in EC_{\mathcal{S}}(\pi, 1_n)} (-1)^{|\text{Blocks}^+(c)|} \widehat{c}. \tag{6.35}$$

Proof. The proof is by induction on $|\pi|$. For the base case: consider a $\pi \in \mathcal{S}_2$. In this case we know that $S(\widehat{\pi}) = -\widehat{\pi}$. On the other hand, the set $\mathrm{EC}_{\mathcal{S}}(\pi, 1)$ consists of only one chain, namely $c = (\pi, 1)$, which has $|\mathrm{Blocks}^+(c)| = 1$ and $\widehat{c} = \widehat{\pi}|_{\widehat{1}} = \widehat{\pi}$; hence the right-hand side of Equation (6.35) also comes out as $-\widehat{\pi}$, as required.

For the inductive step we fix a $j \geq 3$, we assume that the formula (6.35) holds for every $\sigma \in \cup_{m=1}^{j-1} \mathcal{S}_m$, and we prove that the same formula also holds for a $\pi \in \mathcal{S}_j$.

By the Bogoliubov recursion indicated in Equation (6.25), we have

$$
\begin{aligned}
S(\widehat{\pi}) &= -\widehat{\pi} - \sum_{\substack{\sigma \geq \pi \\ \pi \neq \sigma \neq 1_n}} \left(\prod_{V \in \sigma} S(\widehat{\pi|_V}) \right) X_\sigma &\quad (6.36)\\
&= -\widehat{\pi} - \sum_{\substack{\sigma = \{V_1, \ldots, V_r\} \\ 1_n > \sigma > \pi}} X_\sigma \prod_{s=1}^{r} \left(\sum_{c_s \in \mathrm{EC}_{\mathcal{S}}(\pi_{V_s}, 1)} (-1)^{|\mathrm{Blocks}^+(c_s)|} \widehat{c_s} \right), &\quad (6.37)
\end{aligned}
$$

where for the latter equality we used the induction hypothesis on $S(\widehat{\pi_{V_s}})$, for each $V_s \in \sigma$. Finally, from the bijection in Lemma 6.5.9, equation (6.37) can be concisely written as

$$
-\widehat{\pi} + \sum_{\substack{\sigma = \{V_1, \ldots, V_r\} \\ 1_n > \sigma > \pi}} \sum_{c \in \widetilde{\mathrm{EC}_{\mathcal{S}}}^{\sigma}} \widehat{\pi^{(1)}} (-1)^{|\mathrm{Blocks}^+(c)|} \widehat{c}, \tag{6.38}
$$

where the notation $\widetilde{\mathrm{EC}_{\mathcal{S}}}^{\sigma}$ is just to acknowledge that the set $\widetilde{\mathrm{EC}_{\mathcal{S}}}$ from Lemma 6.5.9 depends on the partition σ.

The conclusion follows from observing that the sum in (6.38) is a sum over all chains in $\mathrm{EC}_{\mathcal{S}}(\pi, 1_n)$ and thus coincides with the right hand side of (6.35). Indeed, given a chain $c \in \mathrm{EC}_{\mathcal{S}}(\pi, 1_n)$, we either have $c = (\pi, 1_n)$, in which case we get the term $-\widehat{\pi}$, or else we have $c = (\pi^{(k)}, \pi^{(k-1)}, \ldots, \pi^{(1)}, 1_n)$ with $k \geq 2$ and $\pi^{(1)} = \sigma$ for some $1_n > \sigma > \pi$, implying that $c \in \widetilde{\mathrm{EC}_{\mathcal{S}}}^{\sigma}$. $\square$

6.5.1 A detailed example in the case $\mathcal{S} = \mathcal{NC}$

Let us consider the iterative subset $\mathcal{NC}$. Recall that $0_n \in \mathcal{NC}(n)$ is the partition with n singleton blocks. From Remark 6.5.3 we infer that $S(\widehat{0_1}) = 1_{\mathcal{S}}$ (because $\widehat{0_1} = 1_{\mathcal{S}}$) and that $S(\widehat{0_2}) = -\widehat{0_2}$ because $\widehat{0_2} \in \mathcal{NC}_2$.

For the computation of $S(\widehat{0_3})$, let us record that the set of intermediate partitions $\{\sigma \in \mathcal{NC}(3) \mid 0_3 <_{\mathcal{NC}} \sigma <_{\mathcal{NC}} 1_3\}$ consists of $\sigma_1, \sigma_2, \sigma_3$, where:

$$\sigma_1 = \big\{ \{1\}, \{2,3\} \big\}, \ \sigma_2 = \big\{ \{1,3\}, \{2\} \big\}, \ \sigma_3 = \big\{ \{1,2\}, \{3\} \big\}.$$

For every $1 \leq i \leq 3$ we have that $S(\widehat{\sigma_i}) = -\widehat{\sigma_i}$, because $|\sigma_i| = 2$, and (directly from the definition of the monomials $\widehat{\pi}|_{\widehat{\sigma}}$) we see that $\widehat{0_3}|_{\widehat{\sigma_i}} = \widehat{0_2}$. Based on this information, either of the two Bogoliubov formulas shown in Proposition 6.5.2 leads to:

$$S(\widehat{0_3}) = -\widehat{0_3} + \widehat{0_2}\big(\widehat{\sigma_1} + \widehat{\sigma_2} + \widehat{\sigma_3}\big). \tag{6.39}$$

Notice that the sum of 4 terms that appeared on the right-hand side of Equation (6.39) can be viewed as a sum indexed by all possible chains that go from 0_3 to 1_3 in the poset $\mathcal{NC}(3)$, as stated in Proposition 6.5.5.

Let us now compute what is $S(\widehat{0_4})$. Proposition 6.5.5 gives an explicit formula for this antipode, as a sum indexed by chains in $\mathcal{NC}(4)$ which go from 0_4 to 1_4. There are 29 such chains:

- 1 chain of length 1, the chain $c = (0_4, 1_4)$;
- 12 chains of length 2, of the form $c = (0_4, \sigma, 1_4)$ with $\sigma \in \mathcal{NC}(4) \setminus \{0_4, 1_4\}$;
- 16 chains of length 3, of the form $c = (0_4, \sigma, \sigma', 1_4)$ where $\sigma \in \mathcal{NC}_3(4)$ and $\sigma' \in \mathcal{NC}_2(4)$.

Hence Proposition 6.5.5 gives $S(\widehat{0_4})$ written as a sum of 29 terms.

Now, recall that Proposition 6.5.5 is based on the Bogoliubov formula (6.24), which "has S-factors on the right". The computation of $S(\widehat{0_4})$ can also be done by using the formula (6.25), which has S-factors on the left:

$$S(\widehat{0_4}) = -\widehat{0_4} - \sum_{\substack{\sigma \in \mathcal{NC}(4), \\ 0_4 <_{\mathcal{NC}} \sigma <_{\mathcal{NC}} 1_4}} S(\widehat{0_4}|_{\widehat{\sigma}})\, \widehat{\sigma}. \tag{6.40}$$

It is immediate that, for every $\sigma \in \mathcal{NC}(4)$ with $0_4 <_{\mathcal{NC}} \sigma <_{\mathcal{NC}} 1_4$, the monomial $\widehat{0_4}|_{\widehat{\sigma}}$ is a product of factors $\widehat{0_2}$ and $\widehat{0_3}$; so, consequently, $S(\widehat{0_4}|_{\widehat{\sigma}})$ can be computed explicitly by using the formulas for $S(\widehat{0_2}), S(\widehat{0_3})$ that we already computed. In this way, the right-hand side of (6.40) is turned into an explicit formula for $S(\widehat{0_4})$. After writing this formula down, we discover the interesting detail that it only has 25 terms (instead of 29, as we got from applying Proposition 6.5.5).

- 1 term associated to the chain of length 1, $c = (0_4, 1_4)$;
- 12 term associated to chains of length 2, of the form $c = (0_4, \sigma, 1_4)$ with $\sigma \in \mathcal{NC}(4) \setminus \{0_4, 1_4\}$;
- 12 term associated to chains of length 3 (instead of 16 we had earlier)

This provides an example that the formula (6.29) isn't generally *cancellation-free*. In the case at hand, of $\pi = 0_4$, we can pin down precisely where it is that the cancellations in (6.29) take place. There are two terms disappearing because the chains of length 3

$$\begin{cases} c' = \big(0_4,\ \{\{1,2\},\{3\},\{4\}\},\ \{\{1,2\},\{3,4\}\},\ 1_4\big) \text{ and} \\ c'' = \big(0_4,\ \{\{1\},\{2\},\{3,4\}\},\ \{\{1,2\},\{3,4\}\},\ 1_4\big) \end{cases} \tag{6.41}$$

have the same contribution (but with opposite sign) as the shorter chain

$$(0_4, \{\{1,2\},\{3,4\}\}, 1_4).$$

Then there are two other terms that disappear, in a similar way, in connection to the chain $(0_4, \{\{1,4\},\{2,3\}\}, 1_4)$.

The chains c', c'' of length 3 shown in (6.41) are not efficient: for instance for the first of them we find that the set $V = \{1,2\} \in \mathrm{Blocks}^+(c')$ belongs to both partitions π_1 and π_2 of c', where $\pi_1 = \{\{1,2\},\{3\},\{4\}\}$ and $\pi_2 = \{\{1,2\},\{3,4\}\}$. On the other hand:

$$c := (0_4,\ \{\{1,2\},\{3,4\}\},\ 1_4) \text{ is efficient, with } \mathrm{Blocks}^+(c) = \big\{\ \{1,2\},\ \{3,4\},\ \{1,2,3,4\}\ \big\}.$$

When plugged into the summation on the right-hand side of (6.29), both c' and c'' have contributions of $-\widehat{0_2}\widehat{\sigma}$, for $\sigma = \{\{1,2\},\ \{3,4\}\}$, while c has a contribution of $+\widehat{0_2}\widehat{\sigma}$. (Cancellation!) In the formula featured in Theorem 6.5.10, the chains c' and c'' no longer appear, while c appears with a contribution of $-\widehat{0_2}\widehat{\sigma}$; we note the different sign in the contribution of c (coming from the fact that $|\mathrm{Blocks}^+(c)|$ is of different parity than the length of c), and accounting for the cancellations "$(-1) + (-1) + 1 = -1$" that we had encountered before.

Let us now turn to the study of the number of terms in the cancellation-free summation giving $S(\widehat{0_n})$, $n \geq 1$, we denote this number by t_n. In view of Theorem 6.5.10, t_n can also be viewed as $|\mathrm{EC}_{\mathcal{NC}}(0_n, 1_n)|$, the number of efficient chains from 0_n to 1_n in $\mathcal{NC}(n)$.

We will derive a recursion satisfied by the numbers t_n. It is possible (left as exercise to the reader) to do so by examining the method of proof used for Theorem 6.5.10, and by extracting out of it a recursion among the cardinalities of the sets $\mathrm{EC}_{\mathcal{NC}}(0_n, 1_n)$. Here we will take the alternative path of getting the desired recurrence for the t_n's via a direct analysis of the Bogoliubov formula (6.25), which says that

$$S(\widehat{0_n}) = -\widehat{0_n} - \sum_{\substack{\sigma \in \mathcal{NC}(n), \\ 0_n < \sigma < 1_n}} S(\widehat{0_n}|_{\widehat{\sigma}})\, \widehat{\sigma}.$$

Every monomial $\widehat{0_n}|_{\widehat{\sigma}}$ is equal, by definition, to $\prod_{W \in \sigma} \widehat{0_{|W|}}$. Since S is multiplicative, we thus find that

$$S(\widehat{0_n}) = -\widehat{0_n} - \sum_{\substack{\sigma \in \mathcal{NC}(n), \\ 0_n < \sigma < 1_n}} \Big(\prod_{W \in \sigma} S(\widehat{0_{|W|}}) \Big)\, \widehat{\sigma}. \tag{6.42}$$

Suppose that on the right-hand side of (6.42) we write every $S(\widehat{0_{|W|}})$ as a cancellation-free sum of $t_{|W|}$ terms, then cross-multiply these sums. For every $\sigma \in \mathcal{NC}(n) \setminus \{0_n, 1_n\}$ we thus get a sum of $\prod_{W \in \sigma} t_{|W|}$ terms, which (very importantly) get to be also multiplied by an additional factor of $\widehat{\sigma}$. Now, the latter factor of $\widehat{\sigma}$ appears only once in the whole expression on the right-hand side of (6.42). Multiplying with it will therefore prevent any cancellations with terms that appear from the analogous discussion related to some other $\sigma' \in \mathcal{NC}(n) \setminus \{0_n, 1_n\}$.

Altogether, the discussion in the preceding paragraph shows how on the right-hand side of (6.42) we arrive to a cancellation-free summation, where we can count the terms, in order to arrive to the conclusion that

$$t_n = 1 + \sum_{\substack{\sigma \in \mathcal{NC}(n), \\ 0_n < \pi < 1_n}} \prod_{W \in \sigma} t_{|W|}, \quad n \geq 1 \tag{6.43}$$

(the empty sums appearing for $n = 1$ and $n = 2$ correspond to the fact that $t_1 = t_2 = 1$). Equation (6.43) is the recursion we wanted for the numbers t_n. If we read the separate term of 1 on the right-hand side as t_1^n, and we add on both side a term of t_n, we arrive to the nicer form

$$2t_n = \sum_{\sigma \in \mathcal{NC}(n)} \prod_{W \in \sigma} t_{|W|}, \quad n \geq 2. \tag{6.44}$$

Finally, Equation (6.44) strongly suggests using the functional equation of the R-transform from free probability (very closely related to free cumulants – cf. [NS06, Lecture 16]), in order to find an equation satisfied by the generating function

$$T(z) := \sum_{n=1}^{\infty} t_n z^n = z + z^2 + 4z^3 + 25z^4 + \cdots \tag{6.45}$$

More precisely: let $\mu : \mathbb{C}[X] \to \mathbb{C}$ be the linear functional which has $\mu(1) = 1$ and has its sequence of free cumulants equal to $(t_n)_{n=1}^{\infty}$, hence has R-transform $R_\mu(z)$ equal to the above $T(z)$. From (6.44) it follows that the moment series $M_\mu(z) = \sum_{n=1}^{\infty} \mu(X^n) z^n$ is then equal to $2T(z) - z$. The functional equation of the R-transform says that

$$R_\mu\big(z(1 + M_\mu(z)\big) = M_\mu(z) \quad \text{(cf. [NS06, Remark 16.18])},$$

which becomes

$$T\big(z(2T(z) - z + 1)\big) = 2T(z) - z. \tag{6.46}$$

It is nicer to record this equation in terms of the series

$$U(z) := 2T(z) - z + 1 = 1 + z + 2z^2 + 8z^3 + 50z^4 + \cdots, \tag{6.47}$$

which satisfies:

$$U\big(zU(z)\big) = (2 - z)U(z) - 1. \tag{6.48}$$

The first few t_n's come out as $1, 1, 4, 25, 206, 2060, 23920, 314065, 4582300, \ldots$ This, unfortunately, doesn't seem to match the beginning of some sequence previously recorded in the research literature.

6.6 Two interesting factorizations

In Section 4.6 we identified that the cumulant to cumulant functions in non-commutative probability all belong to the subgroup $SM^{\mathcal{NC}}_{\mathrm{c-c}}$ of semi-multiplicative functions, while the cumulant to moment functions belong to a coset $SM^{\mathcal{NC}}_{\mathrm{c-m}}$ of that subgroup. Since $SM^{\mathcal{NC}} \cong \mathcal{NC}^\circ \subset \mathcal{P}^\circ$, this interaction between the subgroup and the coset still holds in our larger group $\mathcal{P}^\circ$. However, if we want to include the classical moment-cumulant function in the picture, turns out that the associated character $g_{\mathrm{cc-m}}$ does not belong to $SM^{\mathcal{NC}}_{\mathrm{c-c}}$ and thus it makes sense to enlarge our notion of cumulant to cumulant subgroup. The purpose of this section is to identify two other interesting subgroups and cosets that interact in a similar fashion to $SM^{\mathcal{NC}}_{\mathrm{c-c}}$ and $SM^{\mathcal{NC}}_{\mathrm{c-m}}$. Since these relations in $\mathcal{P}^\circ$ are of a combinatorial nature, we will discuss them from a purely combinatorial point of view, and then point out why one such structure is a natural candidate to receive the name of cumulant-to-cumulant subgroup and moment-to-cumulant coset.

We will begin by analyzing some interesting factorizations of indicator functions in $\mathcal{P}^\circ$ that only assign values 1 or 0 to the partitions.

Notation 6.6.1. Given an iterative subset $\mathcal{S} \subset \mathcal{P}$, let us denote by $I_{\mathcal{S}} \in \mathcal{P}^{\circ}$ the indicator function of the set $\mathcal{S}$, that is, $I_{\mathcal{S}}$ is determined by the fact that $I_{\mathcal{S}}(\widehat{\pi}) = 1$ if $\pi \in \mathcal{S}$ and 0 otherwise.

Recall that $I_{\mathcal{S}}$ can be also seen as an element in $\mathcal{S}^{\circ}$. Although these functions are rather simple, they are very important as they frequently appear in the transition formulas between moments and cumulants. For instance $I_{\mathcal{P}} = g_{\mathrm{cc-m}}$, $I_{\mathcal{NC}} = g_{\mathrm{fc-m}}$, and $I_{\mathcal{I}} = g_{\mathrm{bc-m}}$. Interestingly enough, $I_{\mathcal{P}}$ can be factored very nicely as follows.

Proposition 6.6.2. *Using Notation 6.6.1 we have that $I_{\mathcal{P}} = I_{\mathcal{CON}} * I_{\mathcal{NC}}$ and $I_{\mathcal{NC}} = I_{\mathcal{NCIRR}} * I_{\mathcal{I}}$. If we bring them together we obtain*

$$I_{\mathcal{P}} = I_{\mathcal{CON}} * I_{\mathcal{NCIRR}} * I_{\mathcal{I}}.$$

Proposition 6.6.2 is just a recast of some results of Lehner [Leh02], the proof below follows the original proofs and can be seen more as a translation to our framework of the original approach. The main advantage of using our framework is that we can naturally extend this result to obtain some interesting applications.

Proof. To check that $I_{\mathcal{P}} = I_{\mathcal{CON}} * I_{\mathcal{NC}}$ we first observe that for every $\pi \in \mathcal{P}$ there exists a unique $\sigma \in \mathcal{NC}$ such that $\pi \leq_{\mathcal{CON}} \sigma$. Indeed, the partition we are looking for is $\sigma = \overline{\pi}^{\mathcal{NC}}$, the non-crossing closure of π. A nice way to observe this is by looking at the crossing graph of π. If $\sigma \geq \pi$ this means that σ partitions the blocks of π, moreover, since $\sigma \geq_{\mathcal{CON}} \pi$, every block of σ should be contained in a connected component of the graph. Finally, since $\sigma \in \mathcal{NC}$ and bigger than π by definition it has to be bigger than $\overline{\pi}^{\mathcal{NC}}$ but every partition strictly bigger than $\overline{\pi}^{\mathcal{NC}}$ will not have all their restrictions connected.

Then we get that for every $\pi \in \mathcal{P}$ we have

$$I_{\mathcal{CON}} * I_{\mathcal{NC}}(\widehat{\pi}) = \sum_{\substack{\sigma \in \mathcal{P} \\ \sigma \geq \pi}} I_{\mathcal{CON}}(\widehat{\pi}|_{\widehat{\sigma}}) * I_{\mathcal{NC}}(\widehat{\sigma}) = \sum_{\substack{\sigma \in \mathcal{NC} \\ \sigma \geq_{\mathcal{CON}} \pi}} 1 = 1 = I_{\mathcal{P}}(\widehat{\pi}).$$

The second product follows from an analogous procedure, but now using the interval closure of $\pi \in \mathcal{NC}$ to obtain that there is a unique $\sigma \in \mathcal{I}$ such that $\pi \leq_{\mathcal{NCIRR}} \sigma$. $\square$

Remark 6.6.3. Observe that in the previous factorization into three functions also gives us another factorization of $I_{\mathcal{P}}$ as $I_{\mathcal{P}} = (I_{\mathcal{CON}} * I_{\mathcal{NCIRR}}) * I_{\mathcal{I}}$, with some computations we can identify that $I_{\mathcal{CON}} * I_{\mathcal{NCIRR}} = I_{\mathcal{IRR}}$ is precisely the indicator function on irreducible partitions.

Remark 6.6.4. Notice that if we think of the indicator functions as semi-multiplicatice functions in $SM^{\mathcal{P}}$ then we directly get that for every pair of partitions $\pi \leq \sigma$ there exists a unique τ such that $\pi \leq_{\mathcal{CON}} \tau \leq_{NC} \sigma$. Similarly, for every pair of non-crossing partitions $\tau \leq \sigma$ there exists a unique γ such that $\tau \leq_{\mathcal{NCIRR}} \gamma \leq_{\mathcal{I}} \sigma$. If we put both of them together, we can see that for every pair of partitions $\pi \leq \sigma$ there exist unique τ and γ such that

$$\pi \leq_{\mathcal{CON}} \tau \leq_{\mathcal{NCIRR}} \gamma \leq_{\mathcal{I}} \sigma.$$

Now we extend our previous result to a more appealing and useful result. The main idea is that instead of just looking at the product $I_{\mathcal{CON}} * I_{\mathcal{NC}}$, we can take an arbitrary element in $\mathcal{CON}^\circ$ and multiply it by $I_{\mathcal{NC}}$ from the left, and this operation preserves the structure of the connected components.

Definition 6.6.5. 1. Let $g \in \mathcal{P}^\circ$, we say that g is *multiplicative on connected components* if for every $\pi \in \mathcal{P}$, with connected components $\pi_1, \dots, \pi_j$, it holds that $g(\widehat{\pi}) = g(\widehat{\pi}_1)g(\widehat{\pi}_2)\cdots g(\widehat{\pi}_j)$. We denote this set of functions as $M_{\mathcal{CON}}$.

2. Let $g \in \mathcal{P}^\circ$, we say that g is *multiplicative on irreducible components* if for every $\pi \in \mathcal{P}$, with irreducible components $\pi_1, \dots, \pi_j$, it holds that $g(\widehat{\pi}) = g(\widehat{\pi}_1)g(\widehat{\pi}_2)\cdots g(\widehat{\pi}_j)$. We denote this set of functions as $M_{\mathcal{IRR}}$.

Remark 6.6.6. Recall that the connected components of π are precisely the restrictions of π to the blocks of $\overline{\pi}^{\mathcal{NC}}$, thus the condition given in the previous definition can be concisely rephrased as $g(\widehat{\pi}) = g(\widehat{\pi}|_{\widehat{\overline{\pi}^{\mathcal{NC}}}})$.

Observe that the previous definition readily implies that if g is multiplicative on connected components, then its values are completely determined by the values of $g(\widehat{\pi})$ for $\pi \in \mathcal{CON}$. Moreover, if $\pi \in \mathcal{NC}$ then $g(\widehat{\pi}) = g(\widehat{\pi}|_{\widehat{\pi}}) = 1$, so g is equal to 1 on non-crossing partitions.

Similarly, if g is multiplicative on irreducible components, the condition can be condensed as $g(\widehat{\pi}) = g(\widehat{\pi}|_{\widehat{\overline{\pi}^{\mathcal{I}}}})$, and the values of g are completely determined by the values $g(\widehat{\pi})$ for $\pi \in \mathcal{IRR}$. Also, g must be equal to 1 on interval partitions.

Theorem 6.6.7. *1. $M_{\mathcal{CON}}$ is a right coset of $\mathcal{CON}^\circ$ in $\mathcal{P}^\circ$. Moreover, if $g \in \mathcal{CON}^\circ$, then $g * I_{\mathcal{NC}}$ is the unique function in $M_{\mathcal{CON}}$ determined by the fact that $g * I_{\mathcal{NC}}(\pi) = g(\pi)$ for all $\pi \in \mathcal{CON}$.*

2. $M_{\mathcal{IRR}}$ is a right coset of $\mathcal{IRR}^\circ$. Moreover, if $g \in \mathcal{IRR}^\circ$, then $g * I_{\mathcal{I}}$ is the unique function in $M_{\mathcal{IRR}}$ determined by the fact that $g * I_{\mathcal{I}}(\pi) = g(\pi)$ for all $\pi \in \mathcal{IRR}$.

Proof. For the proof of 1 we use the fact showed in the proof of Proposition 6.6.2 that for every $\pi \in \mathcal{P}$ the partition $\sigma = \overline{\pi}^{\mathcal{NC}}$ is the unique partition satisfying $\sigma \in \mathcal{NC}$ and $\pi \leq_{\mathcal{CON}} \sigma$. If we fix $\pi \in \mathcal{P}$ and $g \in \mathcal{CON}^{\circ}$ we obtain that

$$g * I_{\mathcal{NC}}(\widehat{\pi}) = \sum_{\substack{\sigma \in \mathcal{P} \\ \sigma \geq \pi}} g(\widehat{\pi}|_{\widehat{\sigma}}) * I_{\mathcal{NC}}(\widehat{\sigma}) = g(\widehat{\pi}|_{\widehat{\overline{\pi}^{\mathcal{NC}}}}) * I_{\mathcal{NC}}(\widehat{\overline{\pi}^{\mathcal{NC}}}) = g(\widehat{\pi}|_{\widehat{\overline{\pi}^{\mathcal{NC}}}}) = g(\widehat{\pi}_1)g(\widehat{\pi}_2)\cdots g(\widehat{\pi}_j),$$

where $\pi_1, \dots, \pi_j$ are the connected components of π. Therefore, if $\pi \in \mathcal{CON}$ we get that $g * I_{\mathcal{NC}}(\widehat{\pi}) = g(\widehat{\pi})$, while for a general π we get that

$$g * I_{\mathcal{NC}}(\widehat{\pi}) = g(\widehat{\pi}_1)g(\widehat{\pi}_2)\cdots g(\widehat{\pi}_j) = g * I_{\mathcal{NC}}(\widehat{\pi}_1) g * I_{\mathcal{NC}}(\widehat{\pi}_2) \cdots g * I_{\mathcal{NC}}(\widehat{\pi}_j).$$

So we conclude that $g * I_{\mathcal{NC}} \in M_{\mathcal{CON}}$ and $g * I_{\mathcal{NC}}(\pi) = g(\pi)$ for all $\pi \in \mathcal{CON}$. The proof of 2 is analogue to the proof of 1. $\square$

Corollary 6.6.8. *If $f, g \in M_{\mathcal{IRR}}$ then $f * g^{-1} \in \mathcal{IRR}^{\circ}$. Similarly, if $f, g \in M_{\mathcal{CON}}$ then $f * g^{-1} \in \mathcal{CON}^{\circ}$.*

Proof. Since $f, g \in M_{\mathcal{IRR}}$ and $M_{\mathcal{IRR}}$ is a right coset of $\mathcal{IRR}^{\circ}$, then there exist $f_1, g_1 \in \mathcal{IRR}^{\circ}$ such that $f = f_1 * I_{\mathcal{I}}$ and $g = g_1 * I_{\mathcal{I}}$. Thus we get that

$$f * g^{-1} = f_1 * I_{\mathcal{I}} * (g_1 * I_{\mathcal{I}})^{-1} = f_1 * I_{\mathcal{I}} * (I_{\mathcal{I}}^{-1} * g_1^{-1}) = f_1 * g_1^{-1}.$$

Since $\mathcal{IRR}^{\circ}$ is a subgroup and $f_1, g_1 \in \mathcal{IRR}^{\circ}$ we conclude that $f * g^{-1} \in \mathcal{IRR}^{\circ}$. The proof for $M_{\mathcal{CON}}$ follows the same lines. $\square$

To finish this section we will observe that the moment-cumulant formulas that usually appear in classical probability and non-commutative probability are multiplicative on connected components or multiplicative in irreducible components.

First notice that the function giving the transition from classical cumulants to moments is $g_{\mathrm{cc-m}} = I_{\mathcal{P}}$. This trivially implies that $g_{\mathrm{cc-m}} \in M_{\mathcal{CON}}$ and $g_{\mathrm{cc-m}} \in M_{\mathcal{IRR}}$. In this case the factorizations provided by Theorem 6.6.7 yield the two factorizations that we already knew, $I_{\mathcal{CON}} * I_{\mathcal{NC}} = I_{\mathcal{P}}$ and $I_{\mathcal{IRR}} * I_{\mathcal{I}} = I_{\mathcal{P}}$.

The function giving the transition from free cumulants to moments is $g_{\mathrm{fc-m}} = I_{\mathcal{NC}}$. Let us check that $I_{\mathcal{NC}} \in M_{\mathcal{CON}}$ by considering two cases. First, if $\pi \in \mathcal{NC}$ then the smallest non-crossing partition that is bigger than π is π itself, thus we trivially get $I_{\mathcal{NC}}(\widehat{\pi}|_{\widehat{\overline{\pi}^{\mathcal{NC}}}}) = I_{\mathcal{NC}}(\widehat{\pi}|_{\widehat{\pi}}) = 1 = I_{\mathcal{NC}}(\widehat{\pi})$. Secondly, if $\pi \notin \mathcal{NC}$ then $\overline{\pi}^{\mathcal{NC}}$ is strictly bigger than π and there exists a block $W \in \overline{\pi}^{\mathcal{NC}}$ such that π_W has a crossing (otherwise π will not have any crossings). Therefore we get that $I_{\mathcal{NC}}(\pi_W) = 0$ and $I_{\mathcal{NC}}(\pi) = 0 = I_{\mathcal{NC}}(\widehat{\pi}|_{\overline{\overline{\pi}}\widehat{\mathcal{NC}}})$. Observe that Theorem 6.6.7 assures us that there is a unique function $f \in \mathcal{CON}^\circ$ that coincides with $I_{\mathcal{NC}}$ for every connected partition. Since for every n, the only connected partition that is non-crossing is 1_n we get that $f = \varepsilon$ is the identity function in $\mathcal{P}^\circ$. In this case Theorem 6.6.7 trivially tells us that $\varepsilon * I_{\mathcal{NC}} = I_{\mathcal{NC}}$.

The function giving the transition from Boolean cumulants to moments formula is $g_{\mathrm{bc-m}} = I_{\mathcal{I}}$. Since $I_{\mathcal{I}}(\pi)$ is not equal to 1 for all $\pi \in \mathcal{NC}$, by Remark 6.6.6 it cannot be multiplicative on connected components. However, it is easy to see that $I_{\mathcal{I}} \in M_{\mathcal{IRR}}$. This follows from the fact that $\varepsilon * I_{\mathcal{I}} = I_{\mathcal{I}}$ and $\varepsilon \in \mathcal{IRR}^\circ$.

The function giving the transition from monotone cumulants to moments is $g_{\mathrm{mc-m}}$, and satisfies that $g_{\mathrm{mc-m}}(\pi) = \frac{1}{tree(\pi)!}$ if $\pi \in \mathcal{NC}(n)$ and 0 otherwise. Since this function is not equal to 1 for all $\pi \in \mathcal{NC}$, by Remark 6.6.6 it cannot be multiplicative on connected components. However, it can be seen that $g_{\mathrm{mc-m}} \in M_{\mathcal{NCIRR}} \subset M_{\mathcal{IRR}}$. This follows from the fact that tree factorial can be factored in irreducible components, see Proposition 2.2.29.

Remark 6.6.9. Since all cumulant-moment functions belong to the set $M_{\mathcal{IRR}}$, including the transition from classical cumulants to moments, then it makes sense to call it the "set of cumulant-moment functions", and rename $SM^{\mathcal{NC}}_{\mathrm{c-m}}$ as the "set of non-commutative cumulant-moment functions".

Also, from Corollary 6.6.8 and the fact that all the cumulant to moment functions belong to the coset $M_{\mathcal{IRR}}$ we readily get the interesting fact that the transition formulas between cumulants all belong to $\mathcal{IRR}^\circ$. Thus it makes sense to call $\mathcal{IRR}^\circ$ the "set of cumulant-cumulant functions", and rename $SM^{\mathcal{NC}}_{\mathrm{c-c}}$ as the "set of non-commutative cumulant-cumulant functions".

6.7 The Hopf algebra Sym and the subgroup of multiplicative functions

In [MN10] it was shown that the group of multiplicative functions on the lattice $\mathcal{NC}$ can be recognized as the group of characters Sym° of the Hopf algebra Sym of symmetric functions. The group of multiplicative functions is naturally a subgroup of $\mathcal{NC}^\circ$, the group of semi-multiplicative functions on $\mathcal{NC}$. In the present section we put into evidence a natural Hopf algebra homomorphism $\Psi : \widehat{\mathcal{NC}} \to \mathrm{Sym}$, such that the dual group homomorphism $\Psi^\circ : \mathrm{Sym}^\circ \to \mathcal{NC}^\circ$ corresponds in a canonical way to this inclusion of subgroups.

One of the main ingredients that make $\mathcal{NC}$ special in this regard is that we have a Kreweras complement map $\mathrm{Kr} : \mathcal{NC} \to \mathcal{NC}$ that provides an anti-isomorphism of the lattice $(\mathcal{NC}, \leq_{\mathcal{NC}})$ into itself, that is, $\mathrm{Kr}(\pi) \leq_{\mathcal{NC}} \mathrm{Kr}(\sigma)$ whenever $\pi \geq_{\mathcal{NC}} \sigma$. Notice that $\mathcal{P}$ does not posses such kind of symmetry.

6.7.1 Review of the Hopf algebra Sym.

We use the incarnation of the Hopf algebra Sym as a commutative algebra of polynomials

$$\mathrm{Sym} = \mathbb{C}\big[Y_2, Y_3, \ldots, Y_n, \ldots\big] \tag{6.49}$$

where $Y_2, Y_3, \ldots, Y_n, \ldots$ are the so-called *parking-function symmetric functions*. As usual, we will also denote

$$Y_1 := 1_{\mathrm{Sym}} \quad \text{(the unit of Sym)}. \tag{6.50}$$

A description of how the Y_n's relate to other (more commonly used) families of generators of Sym can e.g. be found in [Sta97, Proposition 2.2]. But here the only thing we need to know about the Y_n's is how they relate to the coalgebra structure:

- The coproduct $\Delta : \mathrm{Sym} \to \mathrm{Sym} \otimes \mathrm{Sym}$ is given by

$$\Delta(Y_n) = \sum_{\pi \in \mathcal{NC}(n)} \Big(\prod_{V \in \pi} Y_{|V|} \Big) \otimes \Big(\prod_{W \in \mathrm{Kr}(\pi)} Y_{|W|} \Big), \quad \forall\, n \geq 1. \tag{6.51}$$

- The counit of Sym, as usual is the character $\varepsilon : \mathrm{Sym} \to \mathbb{C}$ uniquely determined by the requirement that $\varepsilon(Y_n) = 0$ for all $n \geq 2$.
- The grading of Sym is determined by the fact that Y_n has degree $n-1$, for every $n \geq 1$, with the usual follow-up defining the degree of a monomial $Y_{n_1} \cdots Y_{n_k}$ to be $n_1 + \cdots + n_k - k$.

The original motivation for featuring the Y_n's in [MN10] was that the formula giving $\Delta(Y_n)$ follows the same pattern as (2.17) in connection to the multiplication of free elements.

Example 6.7.1. For instance, $\Delta(Y_3) = Y_3 \otimes Y_1^3 + 3Y_1Y_2 \otimes Y_1Y_2 + Y_1^3 \otimes Y_3 = Y_3 \otimes 1_{\mathrm{Sym}} + 3Y_2 \otimes Y_2 + 1_{\mathrm{Sym}} \otimes Y_3$, a sum of 5 terms, corresponding to the 5 partitions in $\mathcal{NC}(3)$.

Notation and Remark 6.7.2. It is convenient that for every $\pi \in \mathcal{NC}$ we denote

$$Y_\pi := \prod_{V \in \pi} Y_{|V|} \qquad \text{(a monomial in the algebra Sym).} \tag{6.52}$$

[For example, $\pi = \big\{\{1,2,6\},\{3,4\},\{5\},\{7,8\}\big\} \in \mathcal{NC}(8)$ has $Y_\pi = Y_3 Y_2 Y_1 Y_2 = Y_3 Y_2^2$.]

The formula (6.51) describing comultiplication can then be written concisely as

$$\Delta(Y_n) = \sum_{\pi \in \mathcal{NC}(n)} Y_\pi \otimes Y_{\mathrm{Kr}(\pi)}, \quad n \geq 1. \tag{6.53}$$

By using the fact that Δ is an algebra homomorphism, it is easy (see [MN10, Lemma 3.3]) to extend (6.53) to

$$\Delta(Y_\sigma) = \sum_{\substack{\pi \in \mathcal{NC}(n) \\ \pi \leq \sigma}} Y_\pi \otimes Y_{\mathrm{Kr}_\sigma(\pi)}, \quad \forall\, n \geq 1 \text{ and } \sigma \in \mathcal{NC}(n), \tag{6.54}$$

where $\mathrm{Kr}_\sigma(\pi)$ stands, for the relative Kreweras complement of π in σ.

Remark 6.7.3. Now let us look at the group $MF^{\mathcal{NC}}$ of multiplicative functions and at the group Sym° of characters of Sym. For every $f \in MF^{\mathcal{NC}}$ one can consider a character $\widetilde{\chi}_f \in \mathrm{Sym}^\circ$, defined by requiring that

$$\widetilde{\chi}_f(Y_n) = f(0_n, 1_n), \quad \forall\, n \geq 1. \tag{6.55}$$

It is clear that the map $MF^{\mathcal{NC}} \ni f \mapsto \widetilde{\chi}_f \in \mathrm{Sym}^\circ$ is bijective, and it is easy to check that

$$\widetilde{\chi}_{f_1 * f_2} = \widetilde{\chi}_{f_1} * \widetilde{\chi}_{f_2}, \quad \forall\, f_1, f_2 \in MF^{\mathcal{NC}},$$

where on the left-hand side we invoke the convolution operation on $MF^{\mathcal{NC}}$, while on the right-hand side we use the convolution of characters of Sym. Thus $f \mapsto \widetilde{\chi}_f$ gives a group isomorphism $MF^{\mathcal{NC}} \approx \mathrm{Sym}^\circ$, analogous to the isomorphism $SM^{\mathcal{NC}} \approx \mathcal{NC}^\circ$ from Theorem 6.2.1.

6.7.2 The surjective homomorphism $\Psi : \widehat{\mathcal{NC}} \to \mathrm{Sym}$

Consider now the Hopf algebra $\widehat{\mathcal{NC}}$ and recall that it enjoys a universality property, stated in Proposition 2.4.16, which makes it very easy to define unital algebra homomorphisms having $\widehat{\mathcal{NC}}$ as domain. We use that to make the following definition.

Definition 6.7.4. We let $\Psi : \widehat{\mathcal{NC}} \to \mathrm{Sym}$ be the unital algebra homomorphism defined by using the universality property and the requirement that

$$\Psi(\widehat{\pi}) = Y_{\mathrm{Kr}(\pi)} = \prod_{W \in \mathrm{Kr}(\pi)} Y_{|W|}, \quad \forall\, \pi \in \mathcal{NC}. \tag{6.56}$$

Note: in order for the universality property of $\widehat{\mathcal{NC}}$ to apply, the right-hand side of (6.56) must be equal to 1_{Sym} whenever $\pi = 1_n$ for some $n \geq 1$. This is indeed the case, since $\mathrm{Kr}(1_n) = 0_n$ and $Y_{0_n} = Y_1^n = 1_{\mathrm{Sym}}$.

Remark 6.7.5. For every $n \geq 1$, the definition of Ψ gives $\Psi(\widehat{0_n}) = Y_{1_n} = Y_n$. This immediately implies that the homomorphism Ψ is surjective.

The point about Ψ is that it also respects the coalgebra structure, as we show next.

Theorem 6.7.6. *The map Ψ introduced in Definition 6.7.4 is a homomorphism of graded connected bialgebras.*

Proof. We have to check that Ψ respects: (i) comultiplications; (ii) counits; (iii) gradings on the Hopf algebras $\widehat{\mathcal{NC}}$ and Sym.

For (i): we have to verify the equality

$$\Delta_{\mathrm{Sym}} \circ \Psi = (\Psi \otimes \Psi) \circ \Delta_{\widehat{\mathcal{NC}}}, \tag{6.57}$$

where Δ_{Sym} and $\Delta_{\widehat{\mathcal{NC}}}$ are the comultiplications of Sym and of $\widehat{\mathcal{NC}}$, respectively. Since both sides of (6.57) are unital algebra homomorphisms from $\widehat{\mathcal{NC}}$ to $\mathrm{Sym} \otimes \mathrm{Sym}$, it suffices to check their agreement on a generator $\widehat{\pi}$ of $\widehat{\mathcal{NC}}$, with $\pi \in \mathcal{NC}_{\geq 2}$.

Let us then pick a $\widehat{\pi}$ as mentioned above, plug it into the left-hand side of (6.57), and compute:

$$\begin{aligned}
\big(\Delta_{\mathrm{Sym}} \circ \Psi\big)(\widehat{\pi}) &= \Delta_{\mathrm{Sym}}(Y_{\mathrm{Kr}(\pi)}) \\
&= \sum_{\rho \leq \mathrm{Kr}(\pi)} Y_\rho \otimes Y_{\mathrm{Kr}_{\mathrm{Kr}(\pi)}(\rho)} \text{ (by Equation (6.54))}.
\end{aligned}$$

We next do the same on the right-hand side of (6.57):

$$\begin{aligned}
\big((\Psi \otimes \Psi) \circ \Delta_{\widehat{\mathcal{NC}}}\big)(\widehat{\pi}) &= (\Psi \otimes \Psi)\big(\Delta_{\widehat{\mathcal{NC}}}(\widehat{\pi})\big) \\
&= (\Psi \otimes \Psi)\Big(\sum_{\sigma \geq \pi} \big(\prod_{W \in \sigma} \widehat{\pi_W}\big) \otimes \widehat{\sigma}\Big) \\
&= \sum_{\sigma \geq \pi} \big(\prod_{W \in \sigma} \Psi(\widehat{\pi_W})\big) \otimes \Psi(\widehat{\sigma}) \\
&= \sum_{\sigma \geq \pi} \big(\prod_{W \in \sigma} Y_{\mathrm{Kr}(\pi_W)}\big) \otimes Y_{\mathrm{Kr}(\sigma)},
\end{aligned}$$

with the restrictions $\pi_{_W}$ as in Notation 2.2.8. In the latter summation over σ: when we put together the Kreweras complements of all the partitions $\pi_{_W}$ with W running in σ, what comes out is the relative Kreweras complement of π in σ. Thus the conclusion for the right-hand side of (6.57) reads:

$$\big((\Psi \otimes \Psi) \circ \Delta_{\widehat{\mathcal{NC}}}\big)(\widehat{\pi}) = \sum_{\sigma \geq \pi} Y_{\mathrm{Kr}_\sigma(\pi)} \otimes Y_{\mathrm{Kr}(\sigma)}. \tag{6.58}$$

In order to reconcile the results of our calculations on the two sides of (6.57), we perform the change of variable $\rho := \mathrm{Kr}_\sigma(\pi)$ on the right-hand side of (6.58). It fits perfectly to invoke here the considerations on relative Kreweras complements from [NS06, Lecture 18], and specifically Lemma 18.9 from that lecture, which tells us that:

$$\left\{ \begin{array}{l} \text{if } \sigma \text{ runs in the interval } [\pi, 1_n] \text{ of } \mathcal{NC}(n), \\ \text{then } \rho = \mathrm{Kr}_\sigma(\pi) \text{ runs (bijectively) in the interval } [0_n, \mathrm{Kr}(\pi)] \text{ of } \mathcal{NC}(n), \\ \text{ and one has the relation } \mathrm{Kr}(\sigma) = \mathrm{Kr}_{\mathrm{Kr}(\pi)}(\rho). \end{array} \right.$$

The change of variable from σ to ρ thus transforms (6.58) into

$$\big((\Psi \otimes \Psi) \circ \Delta_{\widehat{\mathcal{NC}}}\big)(\widehat{\pi}) = \sum_{\rho \leq \mathrm{Kr}(\pi)} Y_\rho \otimes Y_{\mathrm{Kr}_{\mathrm{Kr}(\pi)}(\rho)};$$

this brings us to precisely the same expression as we found when we processed the left-hand side of (6.57).

For (ii): we must check that $\varepsilon_{\mathrm{Sym}} \circ \Psi = \varepsilon_{\widehat{\mathcal{NC}}}$, where $\varepsilon_{\mathrm{Sym}}$ and $\varepsilon_{\widehat{\mathcal{NC}}}$ are the counits of Sym and of $\widehat{\mathcal{NC}}$, respectively. Both $\varepsilon_{\mathrm{Sym}} \circ \Psi$ and $\varepsilon_{\widehat{\mathcal{NC}}}$ are unital algebra homomorphisms from $\widehat{\mathcal{NC}}$ to $\mathbb{C}$, hence it suffices to check that they agree on every $\widehat{\pi}$ with $\pi \in \mathcal{NC}_{\geq 2}$. But for any such π we have that

$$(\varepsilon_{\mathrm{Sym}} \circ \Psi)(\widehat{\pi}) = 0 = \varepsilon_{\widehat{\mathcal{NC}}}(\widehat{\pi}). \tag{6.59}$$

Indeed, the second equality (6.59) holds by the definition of $\varepsilon_{\widehat{\mathcal{NC}}}$; while for the first equality (6.59) we write, for $\pi \in \mathcal{NC}(n) \setminus \{1_n\}$:

$$\pi \neq 1_n \Rightarrow \ \mathrm{Kr}(\pi) \neq 0_n \ \Rightarrow \ \exists\, W_o \in \mathrm{Kr}(\pi) \text{ with } |W_o| \geq 2 \text{ and hence with } \varepsilon_{\mathrm{Sym}}(Y_{|W_o|}) = 0$$
$$\Rightarrow \ (\varepsilon_{\mathrm{Sym}} \circ \Psi)(\widehat{\pi}) = \varepsilon_{\mathrm{Sym}}(Y_{Kr(\pi)}) = \prod_{W \in \mathrm{Kr}(\pi)} \varepsilon_{\mathrm{Sym}}(Y_{|W|}) = 0.$$

For (iii): since Ψ is a unital algebra homomorphism, it will suffice to check that

$$\deg_{\mathrm{Sym}}(\Psi(\widehat{\pi})) = \deg_{\widehat{\mathcal{NC}}}(\widehat{\pi}), \ \ \forall\, n \geq 1 \text{ and } \pi \neq 1_n \text{ in } \mathcal{NC}(n), \tag{6.60}$$

where $\deg_{\mathrm{Sym}}$ and $\deg_{\widehat{\mathcal{NC}}}$ denote the degree functions for Sym and $\widehat{\mathcal{NC}}$, respectively. And indeed, direct computation yields that both sides of (6.60) are equal to $|\pi| - 1$, where on the left-hand side we first write that $\deg_{\mathrm{Sym}}(Y_{\mathrm{Kr}(\pi)})) = n - |Kr(\pi)|$, and then we invoke the known fact that $|Kr(\pi)| = n + 1 - |\pi|$. $\square$

Corollary 6.7.7. *Let* $\Psi : \widehat{\mathcal{NC}} \to \mathrm{Sym}$ *be as above, and consider the groups of characters* Sym° *and* $\mathcal{NC}^\circ$ *of the Hopf algebras* Sym *and* $\widehat{\mathcal{NC}}$*.*

1^o *One has an injective group homomorphism* $\Psi^\circ : \mathrm{Sym}^\circ \to \mathcal{NC}^\circ$ *defined by*

$$\Psi^\circ(\chi) := \chi \circ \Psi, \ \ \chi \in \mathrm{Sym}^\circ\,. \tag{6.61}$$

2^o *Consider the identifications* $\mathrm{Sym}^\circ = \{\widetilde{\chi}_f \mid f \in MF^{\mathcal{NC}}\}$ *from Remark 6.7.3 and* $\mathcal{NC}^\circ = \{\chi_g \mid g \in SM^{\mathcal{NC}}\}$ *from Theorem 6.2.1. In terms of these identifications, the group homomorphism* Ψ° *is just the inclusion of* $MF^{\mathcal{NC}}$ *into* $SM^{\mathcal{NC}}$*; that is, one has*

$$\Psi^\circ(\,\widetilde{\chi}_f\,) = \chi_f, \ \ \forall\, f \in MF^{\mathcal{NC}}. \tag{6.62}$$

Proof. The property of Ψ° of being a group homomorphism is a general Hopf algebra fact and the injectivity of Ψ° is implied, in particular, by (6.62). We are thus left to fix an $f \in MF^{\mathcal{NC}}$ and to verify that the two characters $\chi_f, \widetilde{\chi}_f \circ \Psi \in \mathbb{X}(\widehat{\mathcal{NC}})$ are equal to each other. To that end, it suffices to also fix an $n \geq 1$ and a $\pi \in \mathcal{NC}(n) \setminus \{1_n\}$, and to check that the two characters in question agree on the generator $\widehat{\pi}$ of $\widehat{\mathcal{NC}}$. We know that

$$\chi_f(\widehat{\pi}) = f(\pi, 1_n) = \prod_{W \in \mathrm{Kr}(\pi)} f(0_{|W|}, 1_{|W|}),$$

where the second equality sign uses the fact that f is multiplicative. On the other hand, we have

$$\big(\widetilde{\chi}_f \circ \Psi\big)(\widehat{\pi}) = \widetilde{\chi}_f(Y_{\mathrm{Kr}(\pi)}) = \prod_{W \in \mathrm{Kr}(\pi)} \widetilde{\chi}_f(Y_{|W|}) = \prod_{W \in \mathrm{Kr}(\pi)} f(0_{|W|}, 1_{|W|}),$$

and this completes the required verification. $\square$

Chapter 7

Outlook and future work

In the previous two chapters most of the effort was devoted to defining the notion of iterative family and constructing the Hopf algebraic notions associated to it, paving the way to a unified and systematic way of studying transition formulas in non-commutative probability. An ongoing project is to put all this machinery to work towards two main objectives: retrieving previous work in non-commutative probability by adapting it to this new framework, and use the intuition provided by this framework to obtain new results in the realm of non-commutative probability.

The purpose of this last chapter is to provide a survey of the ongoing and future projects that I have in connection to the topics discussed in this thesis. These projects can be separated in three topics. The first one is the study of Hopf algebras over graphs and of a *graph-simplification map* that allows us to reduce the study of Hopf algebras of partitions to the study of Hopf algebras of graphs. The second project aims to study 1-parameter continuous subgroups of $\mathcal{P}^\circ$ and apply them in the study of interpolation between different notions of cumulants, the approach is to parallel the work from Section 4.8 where we studied an interpolation between free and Boolean cumulants. Finally, the third project concerns the so-called T-free cumulants of Jekel and Liu [JL19], their transitions from T-free cumulants to moments can be included into our framework, as elements in $\mathcal{P}^\circ$ that arise from the character group of a Hopf algebra of trees $\mathcal{T}^\circ$.

7.1 Hopf algebras over graphs

The importance of Hopf algebras on graphs has risen due to their applications. Some remarkable works include the Hopf algebra on trees of Grossman and Larson [GL89] and its applications to the field of numerical analysis, as well as the seminal findings of Connes and Kreimer in the context of the process of renormalization in perturbative quantum field theory [CK99]. In these two works, it is the notion of Hopf algebra defined on rooted trees that characterizes genuine combinatorial aspects of the underlying problems.

We now explain how to construct a Hopf algebra structure on connected graphs $\widehat{\mathcal{CG}}$ that is compatible with our Hopf algebras $\widehat{\mathcal{S}}$ on iterative families of partitions. The compatibility is in the sense that there are natural Hopf algebra homomorphisms from $\widehat{\mathcal{S}}$ to $\widehat{\mathcal{CG}}$. Although this Hopf algebra structure on graphs $\widehat{\mathcal{CG}}$, was introduced long ago in [Sch94], we will rediscover this structure from a non-commutative probability perspective. For our development, we draw inspiration on some maps from partitions to graphs that have been used before to study relations between cumulants. Specifically, in [AHLV15] there are three different types of graphs that are associated to partitions: the crossing graph, the anti-interval graph, and the nesting forest (check Definitions 2.2.28, 5.2.3, and 5.2.8 respectively). Although these graphs are constructed in different ways, the three types of graphs follow the same principle:

- The vertices of the graph are labeled by the blocks of the partition.
- We draw an edge between two distinct blocks when the partition obtained by merging them together has some specified properties.

We will describe a method to construct a graph $\Gamma_{\mathcal{S}}(\pi)$ using a given order $\leq_{\mathcal{S}}$ in $\mathcal{P}$, and we will see how the three graphs mentioned above arise naturally from the partial orders $\leq_{\mathcal{S}}$ associated to some iterative sets. In some cases, the map from partitions $\mathcal{S}$ yields only connected graphs $\mathcal{CG}$ and it can be naturally extended to a Hopf algebra homomorphism from $\widehat{\mathcal{S}}$ to $\widehat{\mathcal{CG}}$. In terms of their groups of characters, we get a inclusion of $\mathcal{CG}^\circ$ in $\mathcal{S}^\circ$, which in turn is included in $\mathcal{P}^\circ$. This approach looks promising as it means that we can mimic the semi-multiplicative functions over partitions with semi-multiplicative functions over graphs, allowing us to perform computations at the level of graphs, rather than with partitions. The main advantage is that typically several (usually infinitely many) partitions are represented by the same graph, greatly reducing the computations.

7.1.1 The Hopf algebraic structure

We will construct a Hopf algebra $\widehat{\mathcal{CG}}$ by starting from the commutative algebra of polynomials over indeterminates indexed by connected graphs. And then we will look at several

Hopf algebra homomorphisms that relate Hopf algebras on partitions with Hopf algebras on graphs.

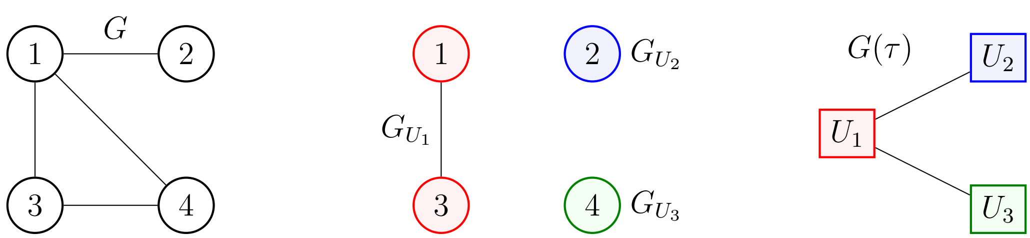
The idea is to define a coalgebra structure on $\mathcal{CG}$ in a way that resembles the coalgebra structure on partitions. In order to do this, given a graph G we need to define a notion of *bigger* graphs H, that are obtained by gluing together vertices of G. We will do this by taking a partition on the vertices and then contracting the graph.

Definition 7.1.1. Let G be a graph in $\mathcal{G}(m)$, that is, the set of vertices of G is $\mathrm{Vrt}_G = [m]$ for some $m \in \mathbb{N}$. Given a partition $\tau = \{U_1, \ldots, U_k\} \in \mathcal{P}(m)$ of the vertices of G, with the blocks ordered by their minimum element. Then the *contraction graph* $G(\tau) \in \mathcal{G}(k)$, is the graph with vertices $[k]$, and for $1 \leq i < j \leq k$ we draw an edge between two vertices i and j, if there exist elements $a \in U_i$ and $b \in U_j$ such that $(a, b) \in E_G$.

Notice that for a connected graph $G \in \mathcal{CG}(m)$ and a partition $\tau = \{U_1, \ldots, U_k\} \in \mathcal{P}(m)$, the contraction graph $G(\tau) \in \mathcal{G}(k)$ is also connected. However, the k restriction graphs $G_{U_1}, \ldots, G_{U_k}$ (from Definition 2.1.2) that are associated to the blocks of τ may not be connected. For the coalgebra structure, we want to focus on those partitions such that the restriction graphs are also connected, so we fix the following notation.

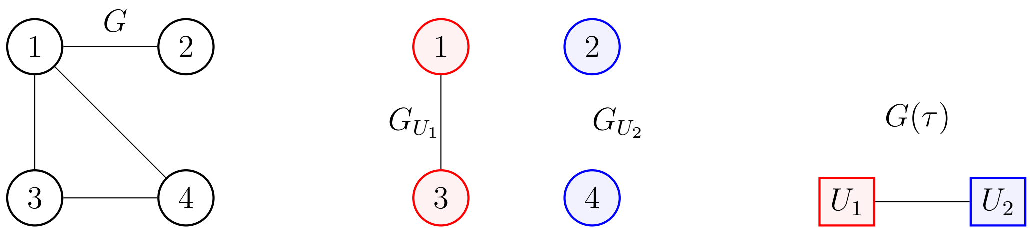
Notation 7.1.2. Given a connected graph $G \in \mathcal{CG}(m)$ We denote by $\mathcal{P}_{\mathrm{Conn}}(G) \subset \mathcal{P}(m)$ the family of partitions τ of the vertices of G such that the restriction graph to each block is connected, $G_U \in \mathcal{CG}$ for all $U \in \tau$.

Example 7.1.3. As a simple example, let us consider the graph $G \in \mathcal{CG}(4)$ with edges $E = \{(1, 2), (1, 3), (1, 4), (3, 4)\}$. If we take $\tau = \{U_1, U_2, U_3\} \in \mathcal{P}(4)$ to be the partition with blocks $U_1 = \{1, 3\}$, $U_2 = \{2\}$, $U_3 = \{4\}$ then the contraction graph $G(\tau)$ and restrictions G_{U_1}, G_{U_2} and G_{U_3} are the following:



Observe that all the restriction graphs are connected, so we get that $\tau \in \mathcal{P}_{\mathrm{Conn}}(k)$.

On the other hand, if we consider $G \in \mathcal{CG}(4)$ the same graph as before, and now we take $\tau = \{U_1, U_2\} \in \mathcal{P}(4)$ to be the partition with blocks $U_1 = \{1, 3\}$, $U_2 = \{2, 4\}$, then the contraction graph $G(\tau)$ and restrictions G_{U_1}, and G_{U_2} are:



Observe that in this case the restriction graph G_{U_2} has two vertices that are not connected, thus it is a disconnected graph and $\tau \notin \mathcal{P}_{\text{Conn}}(k)$.

Now that we have all this notation in hand, we are ready to define the coalgebra structure on graphs.

Definition 7.1.4. Let $\widehat{\mathcal{CG}}$ be the commutative algebra of polynomials over $\mathbb{C}$ which uses a countable collection of indeterminates indexed by the connected graphs $\mathcal{CG}$, where $\bullet$, the unique graph with one vertex, is identified with the unit 1 of $\widehat{\mathcal{CG}}$. The indeterminate associated to $G \in \mathcal{CG}$ will be denoted by $\widehat{G}$. We endow $\widehat{\mathcal{CG}}$ with the following coalgebra structure:

Comultiplication. Is the unital algebra homomorphism $\Delta : \widehat{\mathcal{CG}} \to \widehat{\mathcal{CG}} \otimes \widehat{\mathcal{CG}}$ uniquely determined by

$$\Delta(\widehat{G}) = \sum_{\pi \in \mathcal{P}_{\text{Conn}}(G)} \left(\prod_{V \in \tau} \widehat{G_V} \right) \otimes \widehat{G(\tau)}. \tag{7.1}$$

Counit. As usual, it is the unital algebra homomorphism $\varepsilon : \widehat{\mathcal{S}} \to \mathbb{C}$ determined by

$$\varepsilon(\widehat{G}) = 0, \qquad \forall G \in \mathcal{CG} \setminus \{\bullet\}.$$

Degree. We define the degree of $G \in \mathcal{CG}$ to be its number of vertices minus 1, $\deg(G) = |G| - 1$. As usual, we extend this grading multiplicatively to all monomials.

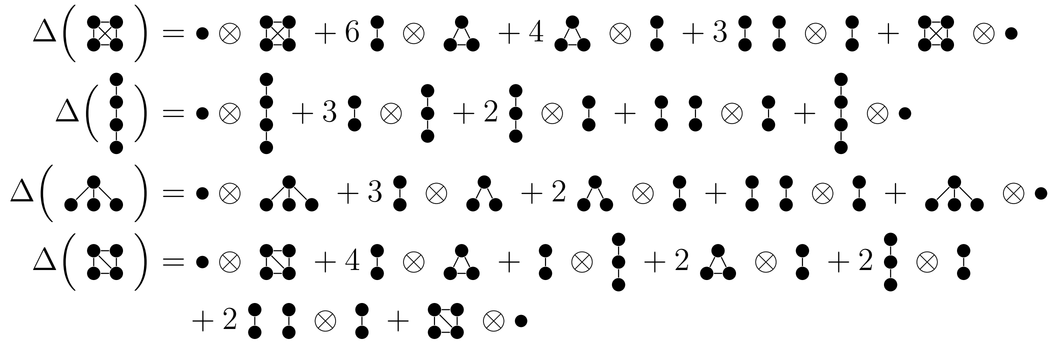
Remark 7.1.5. To emphasize the analogy to the case of partitions, we can define a partial order $\leq_{\mathcal{CG}}$ on the set of connected graphs $\mathcal{CG}$, by taking $G \leq_{\mathcal{CG}} H$ whenever there exist a $\tau \in \mathcal{P}_{\text{Conn}}(G)$ such that $G(\tau) = H$. With this partial order, $\widehat{\mathcal{CG}}$ can be seen as an incidence Hopf algebra.

Remark 7.1.6. Although this construction is for connected graphs with an ordered vertex set $[k]$ for some $k \in \mathbb{N}$, if we forget the order of the vertices, the construction can be carried out without any problem to define a Hopf algebra $\widehat{\mathcal{CG}_\sim}$ with indeterminates indexed by classes of isomorphisms of connected graphs. Moreover, we can naturally define a surjective Hopf algebra homomorphism $\Psi_\sim : \widehat{\mathcal{CG}} \to \widehat{\mathcal{CG}_\sim}$ where G is mapped to its class of isomorphism. To the best of our knowledge, this Hopf algebra on classes of isomorphisms in graphs was introduced in [Sch94, Section 14], although its description is different from the one presented here.

Similarly we can define a Hopf algebra on rooted graphs, $\widehat{\mathcal{CG}_r}$, and a surjective Hopf algebra homomorphism $\Psi_r : \widehat{\mathcal{CG}} \to \widehat{\mathcal{CG}_r}$ where G is mapped to its class of rooted-isomorphism.

Similar to the Hopf algebras of partitions, we can study functions on graphs using a convolution inherited from the coproduct. Specifically, $\mathcal{CG}^\circ$, the group of characters of $\widehat{\mathcal{CG}}$, can be identified with the group of unitary functions on graphs, where the unitary means that we have the extra condition $f(\bullet) = 1$.

Example 7.1.7. Below we compute the coproduct for some small graphs in $\widehat{\mathcal{CG}_\sim}$



Example 7.1.8. Instead of looking at all connected graphs, we can restrict out attention to certain subsets of graphs, and we obtain interesting sub-Hopf-Algebras. Examples of such subsets are:

- **Complete graphs $\widehat{\mathcal{K}}$.** Let us restrict the Hopf algebra $\widehat{\mathcal{CG}}$ to the subset $\mathcal{K} = \{K_n | n \in \mathbb{N}\}$ where K_n is the complete graph on n vertices, that is, all the $\binom{n}{2}$ pairs of distinct vertices in K_n have a edge between them. Notice that there is only one element in the class of isomorphisms of the complete graph, so $\mathcal{K}_\sim \cong \mathcal{K}$. It is not hard to observe that the formula for the coproduct from (7.1) simplifies to

$$\Delta(\widehat{K_m}) = \sum_{\tau \in \mathcal{P}(m)} \left(\prod_{V \in \tau} \widehat{K_{|V|}} \right) \otimes \widehat{K_{|\tau|}}. \tag{7.2}$$

This Hopf algebra was first studied by Doubilet [Dou74] and it is known as the Faà di Bruno Hopf algebra, a name coined in [JR79]. For more details on this Hopf algebra we refer the reader to [Sch94, Example 14.1].

- **Ladder/path graphs $\widehat{\mathcal{L}_\sim}$.** The (class of isomorphisms of a) *ladder (or path) graph* on m vertices, denoted $L_m \in \mathcal{G}_\sim$, has as a representative the graph with vertices $1, 2, \ldots, m$ and edges $(i, i+1)$ for $i = 1, \ldots, m-1$. That is, L_m is simply a path from vertex 1 to vertex m going through all the other vertices in order. We denote by $\mathcal{L}_\sim = \{L_m | m \in \mathbb{N}\}$ the set of all (classes of isomorphisms of) ladder graphs.

  If we restrict $\widehat{\mathcal{CG}_\sim}$ to ladder graphs we obtain the sub-Hopf-Algebra $\widehat{\mathcal{L}_\sim}$. The algebra structure consists of commutative polynomials over $\mathcal{L}$ where $L_1 = \bullet$ is identified with the unit. The formula for the coproduct (7.1) simplifies to

$$\Delta(\widehat{L_m}) = \sum_{\tau \in \mathcal{I}(m)} \left(\prod_{V \in \tau} \widehat{L_{|V|}} \right) \otimes \widehat{L_{|\tau|}}. \tag{7.3}$$

  This is an incidence Hopf algebra of a hereditary family of graphs in the sense of [Sch94]. For more details on this Hopf algebra we refer the reader to [Sch94, Example 14.2].

- **Star graphs $\{S_n\}_{n\in\mathbb{N}}$.** It is not hard to verify that one also get a sub-Hopf-algebra of $\widehat{\mathcal{CG}_\sim}$ if we restrict our generators to the set of star graphs $\{S_n\}_{n\in\mathbb{N}}$, where the representative S_n has vertex set $[n]$ and edges $\mathrm{Edg}_{S_n} = \{(1, i) \mid i = 2, \ldots, n\}$. Moreover, it can be shown that this Hopf algebra is isomorphic to $\widehat{\mathcal{L}_\sim}$, simply by mapping $\widehat{S_n} \mapsto \widehat{L_n}$.

- **The Hopf algebra of trees $\widehat{\mathcal{T}}$.** With some effort it can be seen that $\widehat{\mathcal{T}}$ is sub-Hopf-algebra of $\widehat{\mathcal{CG}}$ with the formula for the coproduct is given by

$$\Delta(\widehat{T}) = \sum_{\tau \in \mathcal{P}_{\mathrm{Conn}}(T)} \left(\prod_{V \in \tau} \widehat{T_{|V|}} \right) \otimes \widehat{T(\tau)}. \tag{7.4}$$

  Moreover, $(\mathcal{P}_{\mathrm{Conn}}(T), \leq)$ is anti-isomorphic to the lattice of subsets of Edg_T, meaning that it has a Boolean structure. More especifically, given $T \in \mathcal{T}(m)$ for some $m \in \mathbb{N}$, then there is a one-to-one correspondance between $\tau \in \mathcal{P}_{\mathrm{Conn}}(T)$ and subsets $W \subset \mathrm{Edg}_T$, such that $|\tau| = |W| + 1$. In particular, one has $|\{\tau \in \mathcal{P}_{\mathrm{Conn}}(T) : |\tau| = k\}| = \binom{m-1}{k-1}$ and the $|\mathcal{P}_{\mathrm{Conn}}(T)| = 2^{m-1}$.

Recall that in Section 2.1 we presented three variations of families of trees, the rooted trees $\mathcal{T}_r$, the planar rooted trees $\mathcal{PT}$ and the classes of isomorphisms of trees $\mathcal{T}_\sim$. We can create a Hopf algebra with the same structure for each of the other types of trees, and the difference is just whether we want to distinguish a root, order the children of each vertex, or order all the vertices. The Hopf-Algebra $\widehat{\mathcal{T}_r}$ of (non-planar) rooted tress was studied in [CEFM11] under the name of contraction-deletion Hopf algebra. In that paper it was proved that there is a $\widehat{\mathcal{T}_r}$-bicomodule structure on the famous Connes-Kreimer Hopf algebra [CK99].

7.1.2 The link between partitions and graphs

Let $\mathcal{S}$ be an iterative subset of $\mathcal{P}$ and consider its associated partial order $\leq_{\mathcal{S}}$. We want to point out that there is a natural way to associate a graph to each element of $\mathcal{S}$ in such a way that we keep track of which blocks can be joined together to get a new partition in the order $\leq_{\mathcal{S}}$.

Definition 7.1.9. Let $\mathcal{S}$ be an iterative subset of $\mathcal{P}$.

1. Given a partition $\pi \in \mathcal{S}$ and two blocks $U, V \in \pi$, we denote by $\pi(U \cup V)$ the partition $(\pi \setminus \{U, V\}) \cup \{U \cup V\}$ obtained by gluing the blocks U and V together. Notice that $\pi(U \cup V) \geq \pi$.

2. We define the $\mathcal{S}$-*graphing map* as the function $\Gamma_{\mathcal{S}} : \mathcal{S} \to \mathcal{G}$ that assigns to every $\pi \in \mathcal{S}$ a graph $\Gamma_{\mathcal{S}}(\pi)$ with vertices given by the blocks of π, ordered by their minimum element. And we draw an edge between two distinct blocks U and V if it holds that $\pi(U \cup V) \in \mathcal{S}$ and $\pi(U \cup V) \geq_{\mathcal{S}} \pi$. Notice that the second condition is equivalent to $\{U, V\}|_{U \cup V} \in \mathcal{S}$.

3. The graphing map $\Gamma_{\mathcal{S}}$ can be extended to a unital algebra homomorphism $\Gamma_{\mathcal{S}} : \widehat{\mathcal{S}} \to \widehat{\mathcal{G}}$, uniquely determined by requiring $\Gamma_{\mathcal{S}}(\widehat{\pi}) = \widehat{\Gamma_{\mathcal{S}}(\pi)}$.

4. We say that $\widehat{\mathcal{S}}$ has a *graph simplification* if $\Gamma_{\mathcal{S}}(\mathcal{S}) \subset \mathcal{CG}$ and $\Gamma_{\mathcal{S}} : \widehat{\mathcal{S}} \to \widehat{\mathcal{CG}}$ is a homomorphism of Hopf algebras.

Example 7.1.10. Observe that this unified way of associating graphs to partitions has some known graphs as particular cases. The map $\Gamma_{\mathcal{NCIRR}} : \mathcal{NCIRR} \to \mathcal{G}$ is precisely the nesting tree of a non-crossing partition, the map $\Gamma_{\mathcal{CON}} : \mathcal{CON} \to \mathcal{G}$ is the crossing graph of a connected partition, and the map $\Gamma_{\mathcal{IRR}} : \mathcal{IRR} \to \mathcal{G}$ is the anti-interval graph of an irreducible partition.

There are several important examples of iterative sets $\mathcal{S}$ that have a graph simplifications. For instance, for the subsets $\mathcal{P}$, $\mathcal{CON}$, $\mathcal{I}$, $\mathcal{IRR}$ and $\mathcal{NCIRR}$. We will now explore with more detail these examples.

Example 7.1.11 (All partitions $\mathcal{P}$ and complete graphs)**.** It is not hard to check that $\Gamma_{\mathcal{P}}(\pi) = K_m$ is the complete graph on $m = |\pi|$ vertices. Moreover, with some effort it can be checked that $\Gamma_{\mathcal{P}} : \widehat{\mathcal{P}} \to \widehat{\mathcal{K}}$ is a surjective homomorphism of Hopf algebras.

An interesting thing to notice here is that the group $\mathcal{K}^\circ$ of characters of $\widehat{\mathcal{K}}$ is isomorphic to the group $MF^{\mathcal{P}}$ of *multiplicative functions on the lattice* $\mathcal{P}$. This group can be defined in a similar fashion to how $MF^{\mathcal{NC}}$ was defined in Section 4.4, by recalling from Lemma 2.2.12 and Remark 2.2.13 the fact that in the lattice $\mathcal{P}$ we have two isomorphism of intervals given by $[\pi, \sigma] \cong \prod_{W \in \sigma} [\pi_W, 1_{|W|}]$ and $[\pi, 1_n] \cong [0_{|\pi|}, 1_{|\pi|}] = \mathcal{P}(|\pi|)$, for every $\pi, \sigma \in \mathcal{P}(n)$. Then the functions in $MF^{\mathcal{P}}$ factorize respecting the two isomorphisms, and are a subgroup of $SM^{\mathcal{P}}$ which are the functions that only respect the first isomorphism.

As a conclusion we obtain that the map $\Gamma_{\mathcal{P}} : \widehat{\mathcal{P}} \to \widehat{\mathcal{K}}$, provides us with a natural way to retrieve the inclusion $MF^{\mathcal{P}} \subset SM^{\mathcal{P}}$ by taking the dual $\Gamma^\circ_{\mathcal{P}} : \mathcal{K}^\circ \to \mathcal{P}^\circ$. Notice that this is analogue to the development for $\mathcal{NC}$ done in Section 6.7.

Example 7.1.12 (Interval partitions $\mathcal{I}$ and path graphs)**.** It can be seen that for $\pi \in \mathcal{I}$ one gets that $\Gamma_{\mathcal{I}}(\pi) = L_m$ is the ladder graph with $m = |\pi|$ vertices. Moreover, it can be seen that the algebra and coalgebra structures are compatible and thus we get that $\Gamma_{\mathcal{I}} : \widehat{\mathcal{I}} \to \widehat{\mathcal{L}}$ is a surjective homomorphism of Hopf algebras. Same as with connected graphs, an interesting thing to notice here is that the group $\mathcal{L}^\circ$ of characters of $\widehat{\mathcal{L}}$ is isomorphic to the group $MF^{\mathcal{I}}$ of *multiplicative functions on the lattice* $\mathcal{I}$. This group can be defined in a similar fashion to $MF^{\mathcal{NC}}$ and $MF^{\mathcal{P}}$. Thus, we obtain that if we dualize the map $\Gamma_{\mathcal{I}} : \widehat{\mathcal{I}} \to \widehat{\mathcal{L}}$, we obtain $\Gamma^\circ_{\mathcal{I}} : \mathcal{L}^\circ \to \mathcal{I}^\circ$, which is a recast of the inclusion $MF^{\mathcal{I}} \subset SM^{\mathcal{I}}$.

Example 7.1.13 (Connected partitions $\mathcal{CON}$ and the crossing graph)**.** As mentioned before, the map $\Gamma_{\mathcal{CON}}$ is precisely the crossing graph from Definition 5.2.3. A remarkable fact is that $\Gamma_{\mathcal{CON}} : \widehat{\mathcal{CON}} \to \widehat{\mathcal{CG}}$ is a homomorphism of Hopf algebras, and thus $\widehat{\mathcal{CON}}$ has a graph simplification.

Example 7.1.14 (Irreducible partitions $\mathcal{IRR}$ and the anti-interval graph)**.** As mentioned before, the map $\Gamma_{\mathcal{IRR}}$ is precisely the anti-interval graph from Definition 5.2.8. Turns out that $\mathcal{IRR}$ also has a graph simplification provided by the fact that $\Gamma_{\mathcal{IRR}} : \widehat{\mathcal{IRR}} \to \widehat{\mathcal{CG}}$ is a homomorphism of Hopf algebras.

Example 7.1.15 (Non-crossing irreducible partitions $\mathcal{NCIRR}$ and the nesting tree)**.** Consider now the set of non-crossing irreducible partitions $\mathcal{NCIRR}$. It is not hard to check that the map $\Gamma_{\mathcal{NCIRR}}$ gives us the nesting tree from Definition 2.2.28. Moreover, the map $\Gamma_{\mathcal{NCIRR}} : \widehat{\mathcal{NCIRR}} \to \widehat{\mathcal{T}}$ is a surjective homomorphism of Hopf algebras.

Remark 7.1.16. It can be seen that $\Gamma_{\mathrm{MinMax}} : \widehat{\mathrm{MinMax}} \to \widehat{\mathcal{G}}$ is not a Hopf algebra homomorphism, so MinMax does not have a graph simplification. Similarly, $\mathcal{NC}$ does not have a graph simplification.

7.1.3 The dual $\mathcal{CG}^\circ$ as a subgroup of $\mathcal{P}^\circ$

As we have seen before, the applications we are more interested involve the groups of characters. So we can dualize the examples from the previous section to obtain several interesting subgroups of $\mathcal{P}^\circ$. This can be sumarized in the following result.

Theorem 7.1.17. *If $\mathcal{S}$ is an iterative family of partitions and $\mathcal{H}$ is a family of graphs such that $\Gamma_{\mathcal{S}} : \widehat{\mathcal{S}} \to \widehat{\mathcal{H}}$ is a surjective homomorphism of Hopf algebras, then there is an injecitve group isomorphism $\Psi^\circ_{\mathcal{S}} : \mathcal{H}^\circ \to \mathcal{S}^\circ$ that maps $f \to f \circ \Gamma_{\mathcal{S}}$. Moroever, since there is an inclusion $\mathcal{S}^\circ \to \mathcal{P}^\circ$, then there is a copy of $\mathcal{H}^\circ$ in $\mathcal{P}^\circ$ that consists of functionals $f : \mathcal{P} \to \mathbb{C}$ that satisfy:*

1. *$f(\pi) = f(\sigma)$ whenever $\pi, \sigma \in \mathcal{S}$ have the same associated graph $\Gamma_{\mathcal{S}}(\pi) = \Gamma_{\mathcal{S}}(\sigma)$.*
2. *$f(\pi) = 0$ for $\pi \notin \mathcal{S}$.*

Example 7.1.18 (Subgroups of $\mathcal{P}^\circ$ governed by graphs)**.** In light of the previous Theorem, the previous section provides us with several new subgroups of $\mathcal{P}^\circ$:

- The dual of the map $\Gamma_{\mathcal{S}}$ yields a subgroup of *size-says-all* functions in $\mathcal{P}^\circ$ that consists of functionals $f \in \mathcal{P}^\circ$ that satisfy that $f(\pi) = f(\sigma)$ whenever $|\pi| = |\sigma|$.
- The dual of the map $\Gamma_{\mathcal{CON}}$ yields a subgroup of *crossings-say-all* functions in $\mathcal{P}^\circ$ that consists of functionals $f \in \mathcal{P}^\circ$ that satisfy that $f(\pi) = f(\sigma)$ whenever $\pi, \sigma \in \mathcal{CON}$ have the same crossing graph and $f(\pi) = 0$ for $\pi \notin \mathcal{CON}$.
- The dual of the map $\Gamma_{\mathcal{I}}$ yields a group of *interval-size-says-all* functions that consists of functionals $f \in \mathcal{P}^\circ$ that satisfy that $f(\pi) = f(\sigma)$ whenever $\pi \in \sigma \in \mathcal{I}$ with $|\pi| = |\sigma|$, and $f(\pi) = 0$ for $\pi \notin \mathcal{I}$. Notice that this is similar to the group of size-says-all functions except that $f \in \mathcal{P}^\circ$ only takes values on interval partitions.

- The dual of the map $\Gamma_{\mathcal{IRR}}$ yields a group of *anti-interval-says-all* functions in $\mathcal{P}^\circ$ that consists of functionals f that satisfy that $f(\pi) = f(\sigma)$ whenever $\pi, \sigma \in \mathcal{IRR}$ have the same anti-interval graph, and $f(\pi) = 0$ for $\pi \notin \mathcal{IRR}$.

- Finally, the dual of the map $\Gamma_{\mathcal{NCIRR}}$ yields a group of *nesting-says-all* functions in $\mathcal{P}^\circ$ that consists of functionals f that satisfy that $f(\pi) = f(\sigma)$ whenever $\pi, \sigma \in \mathcal{NCIRR}$ have the same nesting tree, and $f(\pi) = 0$ for $\pi \notin \mathcal{NCIRR}$.

7.1.4 An application: inverting the monotone-to-Boolean inter-cumulant formula

Let us take a moment to explain a direct application of our development with graphs. Recall from Example 4.7.8 that the transition formula expressing Boolean cumulants in terms of monotone cumulants formula is captured by the function $g_{\mathrm{mc-bc}} \in \mathcal{P}^\circ$, that is defined as:

$$g_{\mathrm{mc-bc}}(\widehat{\pi}) = \frac{m(\pi)}{|\pi|!}, \quad \forall \pi \in \mathcal{NCIRR} \text{ and } 0 \text{ otherwise.} \tag{7.5}$$

The question we want to address is how to invert $g_{\mathrm{mc-bc}}$ to obtain the function $g_{\mathrm{bc-mc}} := g_{\mathrm{mc-bc}}^{-1}$ that transitions from Boolean cumulants to monotone cumulants.

This question was partially solved in [AHLV15] for the case of univariate cumulants, and the multivariate case remained open for some years until it was fully solved in [CEFPP21] using the pre-Lie Magnus expansion coming from the shuffle algebras development of [EFP15]. In order to present the result we need the following.

Definition 7.1.19. Let $\pi \in \mathcal{NCIRR}$. We will write $w_k(\pi)$ for the number of increasing k-colorings of π. That is, the number of functions $\lambda : \pi \to [k]$ such that if $V \prec W$ then $\lambda(V) < \lambda(W)$. Then we define

$$w(\pi) := \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} w_k(\pi). \tag{7.6}$$

Theorem 7.1.20. *The function $g_{\mathrm{bc-mc}} \in \mathcal{P}^\circ$ that transitions from Boolean cumulants to monotone cumulants satisfies that*

$$g_{\mathrm{bc-mc}}(\widehat{\pi}) = w(\pi) \quad \forall \pi \in \mathcal{NCIRR} \text{ and 0 otherwise.} \tag{7.7}$$

We now propose a concise method to invert $g_{\mathrm{mc-bc}}$ and retrieve this theorem using our framework.

Idea of the proof. First notice that we can think of $g_{\text{mc}-\text{bc}}$ as a member of $\mathcal{NCIRR}^{\circ}$, implying that we can just perform the inversion in this subgroup using the antipode of $\widehat{\mathcal{NCIRR}}$. However, this can be further simplified by noticing that Proposition 2.2.29 allows us to rephrase the values of $g_{\text{bc}-\text{mc}} \in \mathcal{NCIRR}^{\circ}$ in terms of the tree factorial of the nesting tree τ (that is now renamed as $\Gamma_{\mathcal{NCIRR}}$) as follows

$$g_{\text{mc}-\text{bc}}(\widehat{\pi}) = \frac{1}{\Gamma_{\mathcal{NCIRR}}(\widehat{\pi})!}, \qquad \forall \pi \in \mathcal{NCIRR}. \tag{7.8}$$

Since the values just depend on the nesting tree, we readily get that it is nesting-says-all function. In particular, if we consider the Hopf algebra of rooted trees $\widehat{\mathcal{T}_r}$ there exists a character $f_{\text{mc}-\text{bc}} \in \mathcal{T}_r^{\circ}$ that satisfies $g_{\text{mc}-\text{bc}} = f_{\text{mc}-\text{bc}} \circ \Gamma_{\mathcal{NCIRR}}$.

Thus, we just need to invert $f_{\text{mc}-\text{bc}}$ in the group $\mathcal{T}_r^{\circ}$. Turns out that this was already done in [CEFM11, Theorem 17]. There it was shown that the inverse of $f_{\text{mc}-\text{bc}}$ is given by *Murua's ω-map* that is defined as follows. If $T \in \mathcal{T}_r$, we write $\omega_k(T)$ for the number of increasing k-colorings of the vertices of T. That is, the number of functions $\lambda : \mathrm{Vrt}_T \to [k]$ such that if v is a descendant of w then $\lambda(v) < \lambda(w)$. Then we define $\omega \in \mathcal{T}_r^{\circ}$ as the map satisfying

$$\omega(T) := \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} w_k(T) \qquad \forall T \in \mathcal{T}_r. \tag{7.9}$$

As a result we obtain that $f_{\text{mc}-\text{bc}}^{-1} = \omega$. Then, going back to partitions, we get that

$$g_{\text{bc}-\text{mc}} = g_{\text{mc}-\text{bc}}^{-1} = (f_{\text{mc}-\text{bc}} \circ \Gamma_{\mathcal{NCIRR}})^{-1} = f_{\text{mc}-\text{bc}}^{-1} \circ \Gamma_{\mathcal{NCIRR}} = \omega \circ \Gamma_{\mathcal{NCIRR}}.$$

Finally, we just need to notice that $w = \omega \circ \Gamma_{\mathcal{NCIRR}}$ because $\Gamma_{\mathcal{NCIRR}}$ maps the values in Equation 7.6 to the values in Equation 7.9, and we conclude that

$$g_{\text{bc}-\text{mc}}(\widehat{\pi}) = w(\pi) \qquad \forall \pi \in \mathcal{NCIRR}.$$

□

Remark 7.1.21. Same as we did in this particular example. Several transition formulas between different brands of cumulants can be simplified to characters in Hopf algebras on graphs. An interesting line of research is to study how the antipode for a Hopf algebra on graphs can streamline inter-cumulant calculations.

7.2 1-parameter continuous subgroups and interpolations

In Section 4.8, the 1-parameter subgroup $\{u_q\}_{q\in\mathbb{R}}$ played a fundamental role for understanding the transition formula from t-Boolean cumulants to $(q+t)$-Boolean cumulants. The reason to study this subgroup came from the already developed theory on t-Boolean cumulants that linearize a t-Boolean convolution. In this section, we will give a purely combinatorial justification on how this 1-parameter subgroup naturally arises from the function $I_{\mathcal{NCIRR}}$ and then we will replicate this procedure with $I_{\mathcal{CON}}$ to suggest a 1-parameter subgroup interpolating between free and classical probability.

7.2.1 The 1-parameter continuous subgroup arising from $I_{\mathcal{NCIRR}}$

Recall that $I_{\mathcal{NCIRR}}$ is the transition formula from free cumulants to Boolean cumulants. And we can think of ε as the transition formula from Boolean cumulants to themselves. If we want to make sense of a transition formula from t-Boolean cumulant to Boolean cumulant, it stands to reason to look for a definition that works well with our convolution $*$. The first step is to study $\{I^{*m}_{\mathcal{NCIRR}} | m \in \mathbb{Z}\}$, which is the subgroup of $\mathcal{P}^\circ$ generated by $I_{\mathcal{NCIRR}}$. To be clear, by $I^{*m}_{\mathcal{NCIRR}}$ we mean $I^{*0}_{\mathcal{NCIRR}} := \varepsilon$, $I^{*(-1)}_{\mathcal{NCIRR}} := I^{-1}_{\mathcal{NCIRR}}$ and for $m \in \mathbb{N}$ we have

$$I^{*m}_{\mathcal{NCIRR}} := I_{\mathcal{NCIRR}} * I_{\mathcal{NCIRR}} * \cdots * I_{\mathcal{NCIRR}}, \quad \text{and} \quad I^{*-m}_{\mathcal{NCIRR}} := I^{-1}_{\mathcal{NCIRR}} * \cdots * I^{-1}_{\mathcal{NCIRR}},$$

where we take the convolution of $I_{\mathcal{NCIRR}}$ with itself m times.

Lemma 7.2.1. *The function $I^{*m}_{\mathcal{NCIRR}}$ is determined by its values $I^{*m}_{\mathcal{NCIRR}}(\widehat{\pi}) = m^{|\pi|-1}$ if $\pi \in \mathcal{NCIRR}$ and 0 otherwise.*

Notice that the lemma gives us a concrete picture of what are the powers under convolution of $I_{\mathcal{NCIRR}}$. Although it does not tell us anything about the powers between 0 and 1, it provides the key idea to construct a continuous family interpolating between 0 and 1.

Definition 7.2.2. For every $q \in \mathbb{R}$, we denote by $I^{*q}_{\mathcal{NCIRR}}$ the unique function in $\mathcal{P}^\circ$ that satisfies $I^{*q}_{\mathcal{NCIRR}}(\widehat{\pi}) = q^{|\pi|-1}$ if $\pi \in \mathcal{NCIRR}$ and 0 otherwise.

Remark 7.2.3. Notice that we can alternatively define $I^{*q}_{\mathcal{NCIRR}}(\widehat{\pi}) := q^{\deg(\widehat{\pi})}$, and this formula nicely extends to monomials. For instance, we have

$$I^{*q}_{\mathcal{NCIRR}}(\widehat{\pi}|_{\widehat{\sigma}}) = \prod_{W\in\sigma} q^{\deg(\widehat{\pi_W})} = q^{\deg(\widehat{\pi}|_{\widehat{\sigma}})} = q^{|\pi|-|\sigma|}.$$

Proposition 7.2.4. *The group $\{I^{*q}_{\mathcal{NCIRR}}\}_{q\in\mathbb{R}}$ is a 1-parameter continuous subgroup of $\mathcal{P}^\circ$:*

$$I^{*q_1}_{\mathcal{NCIRR}} * I^{*q_2}_{\mathcal{NCIRR}} = I^{*(q_1+q_2)}_{\mathcal{NCIRR}}, \qquad \forall q_1, q_2 \in \mathbb{R}.$$

Notice that this result is just a recast of Proposition 4.8.3, and the proof relies on the fact that the $(\mathcal{NCIRR}(n), \leq_{\mathcal{NCIRR}})$ is a Boolean lattice. Therefore, this statement can be easily generalized to the following:

Theorem 7.2.5. *Let $\mathcal{S} \subset \mathcal{P}$ be an iterative subset satisfying that $(\mathcal{S}(n), \leq_{\mathcal{S}})$ is a Boolean lattice for every $n \in \mathbb{Z}$. For every $q \in \mathbb{R}$ we define $I^{*q}_{\mathcal{S}}$ to be the unique family satisfying $I^{*q}_{\mathcal{S}}(\pi) = q^{|\pi|-1}$ if $\pi \in \mathcal{S}$ and 0 otherwise. Then, the family $\{I^{*q}_{\mathcal{S}} | q \in \mathbb{R}\}$ is a 1-parameter continuous subgroup of $\mathcal{P}^\circ$:*

$$I^{*q_1}_{\mathcal{S}} * I^{*q_2}_{\mathcal{S}} = I^{*(q_1+q_2)}_{\mathcal{S}}, \qquad \forall q_1, q_2 \in \mathbb{R}.$$

Remark 7.2.6. Since $(\mathcal{I}(n), \leq_{\mathcal{I}})$ is a boolean lattice for every $n \in \mathbb{N}$, as a corollary we get that $\{I^{*q}_{\mathcal{I}} | q \in \mathbb{R}\}$ is 1-parameter continuous subgroup of $\mathcal{P}^\circ$. Notice that $I^{*0}_{\mathcal{I}} = \varepsilon$ can be interpreted as the moment to moment transition formula and $I^{*1}_{\mathcal{I}} = I_{\mathcal{I}}$ can be interpreted as the Boolean cumulant to moment transition formula. Thus $I^{*q}_{\mathcal{I}}$ is transition formula that goes from an interpolation between moments and Boolean cumulants to moments.

7.2.2 The 1-parameter continuous subgroup arising from $I_{\mathcal{CON}}$

In the previous subsection, we gave a purely combinatorial approach to define a continuous interpolation between Boolean additive convolution and free additive convolution, and the only piece of information that we required was the free to Boolean cumulant formula. The same algorithm can be used to produce interpolations between additive convolutions of any two types of independence from the sole knowledge of the transition function between their corresponding cumulants.

We want to focus on finding an interpolation between free additive convolution and classical additive convolution. There are several approaches that have been proposed [Nic95, Bia97, Ans01, MP22, BGCG21]. Following these approaches one can define, at an algebraic level, a q-convolution of measures $\boxplus_q$ for every $q \in [0,1]$. However, in none of them it is clear if the proposed q-convolution is positive for all $q \in [0,1]$, i.e. that for every two measures μ, ν one has that $\mu \boxplus_q \nu$ is also a measure.

A future project is to use the algorithm of the previous subsection to propose a new interpolation between free and classical cumulants. In this case our starting point is $I_{\mathcal{CON}}$, the classical cumulant to free cumulant transition formula.

Unlike the last section where the values of $I^{*q}_{\mathcal{NCIRR}}(\widehat{\pi})$ are just powers of q thanks to the fact that $(\mathcal{NCIRR}(n), \leq_{\mathcal{NCIRR}})$ is a Boolean lattice, the lattice $(\mathcal{CON}(n), \leq_{\mathcal{CON}})$ of connected partitions is not Boolean, and this results in much more involved formulas for $I^{*q}_{\mathcal{CON}}(\widehat{\pi})$.

Let us begin by analyzing $I^{*m}_{\mathcal{CON}} := I_{\mathcal{CON}} * \cdots * I_{\mathcal{CON}}$ where we take the convolution of $I_{\mathcal{CON}}$ with itself $m \in \mathbb{N}$ times. Since $\mathcal{CON}$ is an iterative subset of $\mathcal{P}$, we know that $I^{*m}_{\mathcal{CON}} \in \mathcal{CON}^{\circ}$ so we just need to find the values on the connected partitions, the rest being 0.

Since $I^{*(m+1)}_{\mathcal{CON}} := I_{\mathcal{CON}} * I^{*m}_{\mathcal{CON}}$, using the formula for the convolution, we know that for $\pi \in \mathcal{CON}$

$$\begin{aligned} I^{*(m+1)}_{\mathcal{CON}}(\widehat{\pi}) &= \sum_{\substack{\sigma \in \mathcal{CON} \\ \sigma \geq_{\mathcal{CON}} \pi}} I_{\mathcal{CON}}(\widehat{\pi}|_{\widehat{\sigma}}) * I^{*m}_{\mathcal{CON}}(\widehat{\sigma}) \\ &= \sum_{\substack{\sigma \in \mathcal{CON} \\ \sigma \geq_{\mathcal{CON}} \pi}} I^{*m}_{\mathcal{CON}}(\widehat{\sigma}), \end{aligned}$$

where in the second equality we used the fact that $I_{\mathcal{CON}}(\widehat{\pi}|_{\widehat{\sigma}}) = 1$ for every $\sigma \in \mathcal{CON}$ and $\sigma \geq_{\mathcal{CON}} \pi$.

If for each $\pi \in \mathcal{CON}$ we think of the values $I^{*(m)}_{\mathcal{CON}}(\widehat{\pi})$ as a function of m, then we can use the previous formula recursively to find all the values. More precisely, let us denote by $f_1(m) := I^{*(m)}_{\mathcal{CON}}(1) = 1$ for all $m \in \mathbb{N}$, and $f_\pi(m) := I^{*(m)}_{\mathcal{CON}}(\widehat{\pi})$ for $|\pi| \geq 2$. Then, we have that

$$f_\pi(m+1) = f_\pi(m) + \sum_{\substack{\sigma \in \mathcal{CON}(n) \\ \sigma >_{\mathcal{CON}} \pi}} f_\sigma(m),$$

where all the σ appearing in the sum are strictly bigger than π. Since $|\sigma| < |\pi|$ for all σ in the sum, we can recursively compute f_π from the knowledge of f_σ with $|\sigma| < |\pi|$.

Let us compute f_π for π with small number of blocks.

- If $|\pi| = 1$, then $\pi = 1_n = 1$ and f_1 is the constant function 1.
- If $|\pi| = 2$, then the smallest connected partition with two blocks is given by $\pi_0 := \{\{1,3\},\{2,4\}\}$. Even though π can have several choices the function will be the same for every partition with 2 blocks. Indeed,

$$f_\pi(m+1) = f_\pi(m) + f_{1_n}(m) = f_\pi(m) + 1, \qquad \forall m \in \mathbb{N}.$$

By induction on m it readily follows that $f_\pi(m) = m$ for all $m \in \mathbb{N}$ and π a connected partition with two blocks.

- If $|\pi| = 3$, then the value of f_π is no longer the same for all connected partitions with 3 blocks, but it depends on $\{\sigma \in \mathcal{CON}(n) | \sigma \geq_{\mathcal{CON}} \pi\}$. It turns out that there are two possible outcomes and both can be observed when partitioning $n = 6$ elements, these two outcomes correspond to the two connected graphs with 3 vertices. One case is with the partition $\pi_1 = \{\{1,4\},\{2,5\},\{3,6\}\}$ and the other one with the partition $\pi_2 = \{\{1,3\},\{2,5\},\{4,6\}\}$. Notice that the connected partitions that are larger than π_1 in the $\geq_{\mathcal{CON}}$ order are

$$\{\{1,2,4,5\},\{3,6\}\}, \quad \{\{1,3,4,6\},\{2,5\}\}, \quad \{\{1,4\},\{2,3,5,6\}\}, \quad 1_6.$$

Therefore we have that

$$f_{\pi_1}(m+1) - f_{\pi_1}(m) = 3f_{\pi_0}(m) + f_{1_n}(m) = 3m + 1.$$

By induction on m this means that

$$f_{\pi_1}(m) = \sum_{i=0}^{m-1} (3m+1) = 3\frac{(m-1)m}{2} + m = \frac{(3m-1)m}{2}.$$

On the other hand, the connected partitions that are larger than π_2 in the $\geq_{\mathcal{CON}}$ order are only

$$\{\{1,2,3,5\},\{4,6\}\}, \quad \{\{1,3\},\{2,4,5,6\}\}, \quad 1_6.$$

Notice that, despite $\pi_2 \leq \{\{2,5\},\{1,3,4,6\}$, we have that $\pi_2|_{\{1,3,4,6\}} = \{1,3\},\{4,6\}$ is not connected, so they are not comparable in the $\geq_{\mathcal{CON}}$ order. Then we get that

$$f_{\pi_2}(m+1) - f_{\pi_2}(m) = 2f_{\pi_0}(m) + f_{1_n}(m) = 2m + 1.$$

And by induction on m this means that

$$f_{\pi_2}(m) = \sum_{i=0}^{m-1} (2m+1) = 2\frac{(m-1)m}{2} + m = m^2.$$

- For $|\pi| = 4$ the cases become much more complicated, and their study remains for future work. An observation that will hopefully simplify this study is that $I_{\mathcal{CON}}$ is a crossings-say-all functions, in the sense of Example 7.1.18, this means that instead of looking at the powers of the function $I_{\mathcal{CON}}$ in the character group $\mathcal{CON}^\circ$ we can instead study the powers of the function $I_{\mathcal{CG}}$ in the character group $\mathcal{CG}^\circ$. As a byproduct we obtain that the function f_π only depends on the crossing graph of π.

7.3 Subgroup of nesting-say-all functions and Jekel and Liu's T-free cumulants

Recall from Example 7.1.18 that there is a group of *nesting-says-all* functions in $\mathcal{P}^\circ$ that consists of functionals f that satisfy that $f(\pi) = f(\sigma)$ whenever $\pi, \sigma \in \mathcal{NCIRR}$ have the same nesting tree, and $f(\pi) = 0$ for $\pi \notin \mathcal{NCIRR}$. I mention here that a vast family of nesting-say-all functions is provided by the transition formulas that write moments in terms of the T-free cumulants of Jekel and Liu [JL19].

In [JL19], for a fixed $N \geq 2$ the authors consider subtrees T of the infinite N-regular tree $T_{N,free}$. For each subtree they define a notion of T-free independence and their corresponding notion of T-free cumulants [JL19, Definition 7.5]. More specifically, given an operator-valued probability space $(\mathcal{A}, \mathcal{B}, E)$ we can define the family of T-free cumulants $\{r_{T,n} : \mathcal{A}^n \to \mathcal{B}\} n \in \mathbb{N}$ as the unique maps satisfying the moment-cumulant formula

$$E[a_1 \dots a_n] = \sum_{\pi} \in \mathcal{NC}(n) \alpha_T(\widehat{\pi}) r_{T,n}(a_1, \dots, a_n),$$

where α_T is an element of $\mathcal{NC}^\circ$ that depends on the tree T.

With some effort it can be shown that α_T is a nesting-says-all function. Moreover if we consider the associated element $\beta_T \in \mathcal{T}^\circ$, that satisfies $\Gamma^\circ_{\mathcal{NCIRR}} \circ \beta_T = \alpha_T$, then this element has an interesting description in terms of the number "compatible colorings $C_T(U)$" of an arbitrary tree U and the subtree T associated to the independence.

The advantage of studying the α_T or β_T in our framework is that we can apply all our standard machinery to study them. For instance, the transition formula that writes T-free cumulants in terms of moments is simply the inverse of α_T which can be expressed as $\alpha_T^{-1} := I_{\mathcal{I}}^{-1} * (\Gamma^\circ_{\mathcal{NCIRR}} \circ \beta_T^{-1})$. Moreover, we can compute $\beta_T^{-1} = \beta_T \circ S_{\mathcal{PT}}$ using the antipode on $\mathcal{PT}$.

As a byproduct, we get that given two subtrees T, S, the transition formula that writes T-cumulants in terms of S-cumulants is the nestings-say-all function associated to $f_S * f_T^{-1} \in \mathcal{PT}^\circ$.

One important feature of [JL19] is that it generalizes the notion of free, Boolean and monotone cumulants. So the last corollary can be seen as an extension of the fact that $g_{\mathrm{cc-bc}}$, $g_{\mathrm{fc-bc}}$ and $g_{\mathrm{mc-bc}}$ are all nestings-say-all functions.

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