Integrality theorems for symmetric instantons

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Anti-self-dual (ASD) instantons on $\mathbb{R}^4$ are connections A on $\mathrm{SU}(N)$-vector bundles with curvature $F_A = \mathrm{d}A + A \wedge A$ satisfying the ASD equation $\star F_A = -F_A$. Solutions to this non-linear partial differential equation correspond to certain algebraic data via the celebrated *ADHM correspondence*. While much is known about the space of instantons, it is still difficult to give explicit examples of them, aside from classes of solutions provided by certain ansatze. The perspective in this thesis is that of *symmetry*: by introducing a suitable notion of a nice group action on an instanton, one expects that the condition of 'equivariance with respect to the symmetry group' to reduce the number of parameters present in the ADHM equations, thus allowing for the creation of solutions not visible to existing ansatze.

Through this method of symmetry, the present thesis develops a theory of symmetric instantons and applies it in particular to the case of finite-energy ASD solutions on $\mathbb{R}^4$ with symmetry a compact subgroup of $\mathrm{Spin}(4)$. This theory acts as a framework in which previous work on symmetric instantons (see, for example, [42, 44, 45, 1, 6]) may be realized, and in particular allows for a number of '(algebraic) integrality' results for solutions to the symmetric instanton equations.

Using the equivariant index theorem the '$\mathrm{SU}(2)$ restriction' ansatz used in [42, 44, 45] is proved to give the only non-trivial class of solutions to the symmetric instanton equations for certain symmetry subgroups of $\mathrm{SU}(2)$. Additionally, this thesis resolves in the negative a question of [1] on the existence of a non-trivial instanton with symmetries of the 600-cell occurring at a charge lower than that of the JNR bound of 119.

Finally, this thesis contains ADHM data for two new instantons symmetric under the binary icosahedral group occurring at charges 13 and 23, as well as the software package [48] used to generate them.

Acknowledgements

Acknowledgement must be given primarily to my supervisor, Benoit Charbonneau, whose insight in both mathematics and mathematical communication was immensely valuable in the completion of the present document. At a more fundamental level, it is thanks to Benoit that I find myself in graduate school in the first place, as opposed to a software position at some quantitative trading firm—something for which I cannot express enough gratitude.

In my time at Waterloo I have had the privilege of interacting with a number of mathematicians who have helped shaped my approach to mathematics and have therefore contributed—whether directly or indirectly—to the contents of this thesis.

Spiro Karigiannis taught me most of the linear algebra and differential geometry I learned in my undergraduate degree, and was a constant source of constructive criticism and help during talks in the working seminars held by the geometry and topology group.

Matt Satriano taught me a number of courses in algebra and algebraic geometry, and supervised my second USRA research term. The work I did with Matt and his colleague Dan Edidin during this term helped me develop an interest of representation theory in the abstract—a topic whose influence is manifest in this thesis.

The conversations I had with Dennis The have inspired a number of lines of questioning into the theory of symmetric polytopes that I hope still to resolve over the coming years. Some questions of Dennis lead to interesting applications of the theory developed in this thesis, such as that of Theorem 5.5.5 and its corollary.

To Department of Pure Mathematics at Waterloo I owe a debt of gratitude for their willingness to accomodate for me in the unusual circumstances with which I started this master's thesis. Each interaction I had with the department administration was truly a pleasure.

My parents, while not involved mathematically in the pages to follow, were crucial in the development of this thesis not just as a source of love and encouragement and such, but also really quite concretely as logistical help with housing, visas, transportation and all during the times that I was too busy with work to do so myself.

Finally, I thank my friends Alex, Farbod, Gary, Greg, Matt, and Phlox. It is thanks to their support that this thesis was able to be submitted as late as it was.

Dedication

This thesis is dedicated to my parents.

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$1, 2, 3, \ldots$	irreducible representations of SU(2)
∇	connection on a vector bundle
χ_V	character of the representation V
$\mathrm{conv}(P)$	convex hull of the set $P \subset \mathbb{R}^n$
$(D^{\pm}_{\nabla}), D$	(chiral) dirac operator of a connection ∇
$\mathbb{F}$	a field; either $\mathbb{R}$ or $\mathbb{C}$
$\mathrm{Hom}_G(U, V)$	G-equivariant maps between G-representations U, V
$\mathcal{G}$	group of gauge transformations of a bundle
G^+	orientation-preserving subgroup of a group of orthogonal transformations
G_Δ	the diagonal subgroup of $G \times G$
$\widetilde{G}$	double cover in $\mathrm{Spin}(N)$ of a subgroup of $\mathrm{SO}(N)$
L, R	respectively $2 \boxtimes 1$, $1 \boxtimes 2$; representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{C}^2$ by left and right multiplication
$N_g(\theta)$	subbundle of the normal bundle to X^g where g acts as rotation by θ
$\mathrm{O}(n)$	orthogonal group; the group of isometries of $\mathbb{R}^n$
$P_{\gamma,t}$	parallel transport for time t along γ
$\mathrm{res}^G_H V$	restriction of the G-representation V to a subgroup $H \subset G$
σ_i	the ith elementary symmetric polynomial
$(S^{\pm}), S$	(chiral) spinor bundle
$\mathrm{SO}(n)$	special orthogonal group; the group of orientation-preserving isometries of $\mathbb{R}^n$
$\mathrm{Spin}(n)$	the simply connected universal covering group of $\mathrm{SO}(n)$
$U^\dagger$	conjugate transpose of a quaternionic or complex matrix U
$V \otimes W$	internal tensor product of representations
$V \boxtimes W$	external tensor product of representations
vol	volume form of a Riemannian manifold or Clifford algebra
X^g	set of points in X fixed by g

'It seems very pretty,' she said when she had finished it, 'but it's *rather* hard to understand!'

(You see she didn't like to confess, even to herself, that she couldn't make it out at all.)

'Somehow it seems to fill my head with ideas –only I don't exactly know what they are!'

From *Through the Looking-Glass, and What Alice Found There*
by Lewis Carroll

Chapter 1

Introduction

A *kaleidoscope* is an arrangement of n mirrors in $\mathbb{R}^n$ in which any point has only finitely many distinct images. Reflection through the mirrors in a kaleidoscope generates a group, and the groups generated in this way are called *reflection groups*.

Reflection groups enjoy nice algebraic properties while also generating a significant number of the finite subgroups of the orthogonal groups. For example, in three dimensions, every finite subgroup of the special orthogonal group may be expressed as the orientation-preserving part of a reflection group. For these reasons, reflection groups are a natural playground for finite symmetry.

In this thesis, the objects upon which reflection groups act are *instantons*: connections on $\mathrm{SU}(N)$-bundles over $\mathbb{R}^4$ whose curvature is anti-self-dual (ASD) and hav finite L^2 norm. This L^2 norm, up to a factor of $8\pi^2$, gives a number called the *charge* of an instanton. The ADHM correspondence (originally [3], but see [15] for a survey) describes how instantons correspond to algebraic data and how the ASD equation becomes a system of quadratic polynomials—the *ADHM equations*.

The idea of using finite symmetries to help solve the ADHM equations has been applied for octahedral symmetry in [27], for general platonic symmetry in [29, 42, 44, 45], and finally for symmetry of regular 4-polytopes in [1]. In these references, to study symmetric instantons one needs to choose representations of dimension equal to the charge.

Via clever choices of representations one can solve the symmetric ADHM equations in particular cases. If the symmetry group is a subgroup of $\mathrm{SU}(2)$, one could pick as a

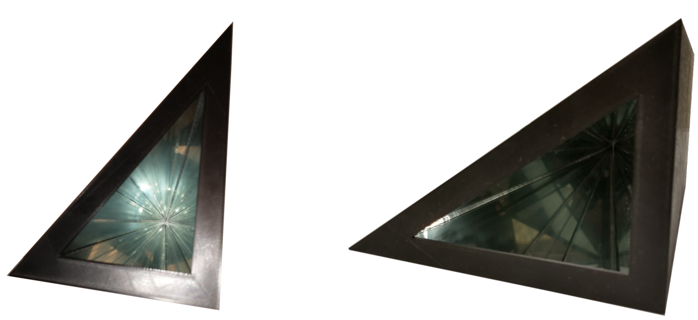


Figure 1.1: A kaleidoscope of icosahedral symmetry formed by three mirrors.

representation the restriction of the unique irreducible SU(2) representation of dimension equal to the charge. However, for other groups there may not necessarily be a canonical choice of representation at a particular dimension. As well, if one's representation does *not* work it is difficult to tell whether there are no symmetric instantons at that charge, or whether another representation may yield solutions.

In this thesis, the requirement to pick a representation to look for symmetric instantons is removed entirely in some cases, and in others the number of representations to be considered is vastly reduced. This outcome is achieved by a careful application of the index theorem. For example, in Theorem 5.5.2 it is shown that the 'SU(2) restriction' ansatz described above is in many cases the *only* choice of representation that has a hope of working at a particular charge.

As well, this method offers definite computational improvements over the procedure of [1] to answer questions on the existence of symmetric instantons at particular charges. In particular, it is shown that no non-trivial instanton of 600-cell symmetry occurs at charge less than the JNR bound of 119. Rather than checking individually all representations of at most this dimension (of which there are 14,886,653,745,942,886,369), instead one can develop a system of inequalities that the multiplicities of irreducible representations must satisfy, and by a dynamic programming approach show that the first dimension larger than 1 at which a solution occurs is 119.

While most results are stated in the case specialized to structure group SU(2) as in the literature, the framework described in this thesis is equipped to handle questions of symmetric instantons for higher rank structure group as well.

Finally, this thesis introduces new solutions to the symmetric instanton problem for the binary icosahedral group in Appendix C as well as a software package [48] that can be used to generate the symmetric ADHM equations at other charges.

The methods used in this thesis are similar to those of [6, Ch. 4], where the case of instantons symmetric under a circle action are studied. While the ideas involved are similar, the theorems and proofs are quite different in flavour. The circle is abelian, while most reflection groups are not. Consequently, irreducible representations of the circle are all one-dimensional, while irreducible representations of reflection groups are generally not.

On the title of this thesis

The title of this thesis, 'Integrality theorems for symmetric instantons', refers to a common theme present in the proofs of results like Proposition 3.4.3, Lemma 5.5.1, and Theorem 5.6.1, where the concept of *algebraic integrality* arises. Recall that an *algebraic integer* is a complex number that occurs as the root of a monic polynomial with integer coefficients—so $2, i$ and $\sqrt{5}$ are algebraic integers, but $\frac{1}{2}$ is not.

In the standard setting of the index theorem, integrality results are manifest. A typical example is Rochlin's theorem, which states that the signature of a compact spin 4-fold is divisible by 16. Even more basic is the underlying fact that the index of an elliptic operator is an integer—or more invariantly, an element of the K-theory of the base manifold.

A general principle is that the introduction of a linear group action weakens 'integrality results' to 'algebraic integrality results'. The equivariant index of a G-operator on a G-bundle is not an integer, but it is an algebraic integer. Invariantly, it is not an element of the

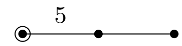
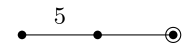
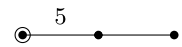
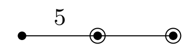
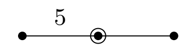
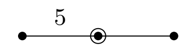
Charge	Polytope	Wythoff	Reference
7	Dodecahedron		[42]
11	Icosahedron		[21]
13	Dodecahedron		C.2.2
17	Truncated icosahedron		[44], [45]
23	Icosidodecahedron		C.2.4
29	Icosidodecahedron		[21]

Table 1.1: A table of icosahedrally symmetric instantons. A detailed version of the table with instantons of other symmetry types is available at Table C.1.

K-theory of the base; but it *is* an element of the equivariant K-theory of the base. These results are consequences of the fact that the character of a linear representation evaluated at a group element is an algebraic integer. A classical example of such an 'algebraic integrality' result at the level of the topology of the base manifold is that of [2], where it is shown, for example, that any smooth non-trivial action by a compact connected Lie group necessitates the vanishing of the $\widehat{A}$-genus. The results of this thesis focus on actions by discrete groups.

A road map

Chapter 2 is a collection of basic results in the theory of reflection groups, polyhedra, orthogonal groups, and spin groups. Chapter 3 is a brief review of the representation theory of finite groups, followed by a description of the algebraic McKay correspondence in Section 3.4, and some novel results in Section 3.4.3 and Section 3.4.4 used later. Chapter 4 describes the equivariant formulation of the Atiyah–Singer index theorem, its specialization to the equivariant spin theorem (also known as the G-spin theorem), and its further specialization to actions of certain subgroups of Spin(4) on $\mathbb{R}^4$. Chapter 5 contains the primary results of the thesis on the theory of symmetric instantons and their structure. Chapter 6 is a collection of remarks and conjectures about the 'polytopal' nature of the level sets of the norm of the curvature of a symmetric instanton. Appendix A describes the finite subgroups of SU(2) and gives character tables and generators for the polyhedral groups. As well, descriptions of particular choices of matrices for representations and invariant maps between them are described for the binary icosahedral group. Appendix B describes the harmonic function ansatz for SU(2) instantons and a generalization to SU($2m$). Appendix C describes the known solutions to the symmetric instanton problem for finite groups of symmetries, including ADHM data for icosahedrally symmetric instantons. New solutions at charge 13 and 23 are presented. Appendix D contains a proof of the Atiyah–Segal–Singer index theorem for the Dirac operator with coefficients in a bundle, following [25].

Chapter 2

Reflections and polyhedra

2.1 Reflection groups

A *reflection* is a non-trivial isometry of $\mathbb{R}^n$ that fixes an $(n-1)$-plane. A *reflection group* is a *finite* group of isometries of $\mathbb{R}^n$ generated by reflections. It is a theorem of Coxeter that every reflection group that does not fix an axis in $\mathbb{R}^n$ has a presentation of the form

$$G = \langle r_1, \dots, r_n | \forall i, j, (r_i r_j)^{m_{ij}} = 1 \rangle,$$

for some symmetric positive integer matrix m_{ij} having all 1's on the diagonal. Equivalently, there exists a generating set comprised of reflections through n hyperplanes in $\mathbb{R}^n$, where the ith and jth mirrors meet at dihedral angle π/m_{ij}.

There is a natural way to associate a weighted graph to a finite n-dimensional reflection group using a presentation of the above form: the vertices of the graph are the generating reflections, and two vertices are connected by an edge of weight m_{ij} as long as $m_{ij} > 2$. The resulting graph is called the *Coxeter diagram* of the group, and by a result of Coxeter ([12]), a different choice of presentation of this form yields an isomorphic graph. By convention, a value of $m_{ij} = 3$ is drawn as an unadorned edge in the Coxeter diagram, and weights are otherwise drawn beside the corresponding edges.

The classification of reflection groups is carried out by a nice argument associating a positive definite form to a Coxeter diagram (see [19, §2.7]), and the result is that each finite reflection group must have a Coxeter diagram in which each connected component

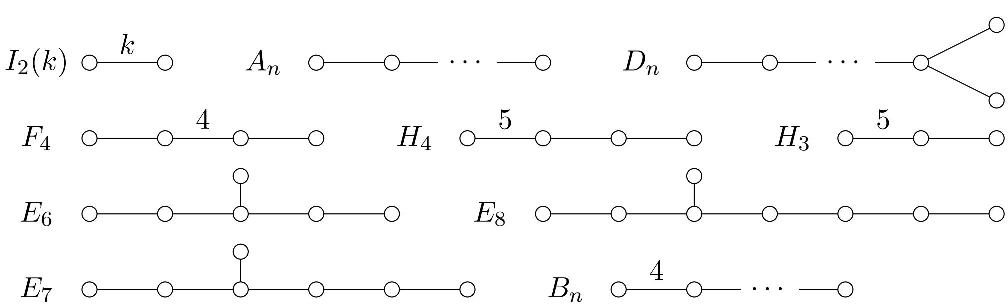


Figure 2.1: A list of the irreducible Coxeter diagrams.

is isomorphic to one of the diagrams of Fig. 2.1. The groups and the associated Coxeter diagrams are referred to by the same names, indicated in the same figure.

Connected Coxeter diagrams are almost identical to the Dynkin diagrams appearing in the analysis of semisimple complex Lie algebras, with the following two differences.

1. A Coxeter diagram 'doesn't know' about roots, and so does not distinguish any vertices as being longer than others.

2. Coxeter diagrams are not subject to the crystallographic restriction of root systems, leaving open the possibility of dihedral angles between mirrors other than $\pi/2, \pi/3, \pi/4, \pi/6$.

In practice, the first point means that the root systems and Dynkin diagrams of type B_n and C_n are identified in the language of Coxeter diagrams, while second point means that Coxeter diagrams allow for some additional geometry. These are reflected in the appearance of Coxeter diagrams of type $I_2(k), H_3$, and H_4; respectively the symmetries of a regular k-gon, an icosahedron, and a 600-cell (hypericosahedron).

2.2 Symmetric spherical polytopes

A *(convex) polytope* is a bounded subset of $\mathbb{R}^n$ defined by the intersection of finitely many half spaces $H_{a,c} = \{x \in \mathbb{R}^n : \langle a, x\rangle \geq c\}$. Equivalently, a convex polytope is the convex hull of a finite set of points in $\mathbb{R}^n$.

Define $H^0_{a,c} = \{x \in \mathbb{R}^n : \langle a, x\rangle = c\}$. Let P be a polytope. If $H_{a,c} \supseteq P$, then $H_{a,c}$ is said to be *valid* for P. If $H_{a,c}$ is valid for P, then $H^0_{a,c} \cap P$ is said to be a *face* of P; the *dimension* of a face is the affine dimension of its affine hull. Equivalently, a face of P is a convex subset F of P such that $P \setminus F$ is convex. The *vertices* of P are the non-empty dimension 0 faces of P, and are denoted by the set $V(P)$; the *edges* of P are the dimension 1 faces of P. Dimension $n-1$ faces of P are called *facets*.

The *centre* (or *centroid*) of P is the average of the vertices of P. If P is contained in a hyperplane, it is said to be *degenerate*. If the centre of P is the origin and the vertices of P are all equidistant from the origin, then P is said to be *spherical*, in the sense that a sphere may be circumscribed at the origin containing all vertices of P.

A basic question one asks about spherical polytopes is under which transformations of $\mathbb{R}^n$ they are preserved. It is immediate that a symmetry of an origin-centered polytope (that is, a bijective isometry $\mathbb{R}^n \to \mathbb{R}^n$ sending P to itself) must be linear, and since it is an isometry it must be an element of the orthogonal group $\mathrm{O}(n)$. Moreover, since any symmetry permutes the vertices, the subgroup of all symmetries of P is a finite group in particular, it is a subgroup of the symmetric group on the vertices of P.

Construction 2.2.1 (Orbit construction for spherical polytopes)**.** Let G be a finite subgroup of the orthogonal group $\mathrm{O}(n)$ and let $x \in \mathbb{R}^n$. Then the orbit Gx is finite, and $\mathrm{conv}(Gx)$ is a spherical polytope whose vertices are in bijection with the coset space G/G_x. Every element of G is a symmetry of $\mathrm{conv}(Gx)$, and the action of G on the vertices of $\mathrm{conv}(Gx)$ is transitive.

Conversely, if P is a spherical polytope whose symmetry group G acts transitively on the vertices of P, then for any vertex x of P it follows that $P = \mathrm{conv}(Gx)$.

Definition 2.2.2. Let P be a spherical polytope. If the symmetries of P act transitively on the vertices of P, then P is said to be *vertex-transitive*.

A *uniform polytope* is defined inductively as a vertex-transitive polytope whose facets are uniform; a uniform polygon is defined to be a regular polygon.

Thus the orbit construction associates to a finite subgroup of $\mathrm{O}(n)$ and a point in $\mathbb{R}^n$ a vertex-transitive polytope. The reverse direction outlined in Construction 2.2.1 is not an inverse to the forward one: certainly for a finite subgroup G of $\mathrm{O}(n)$ and point $x \in \mathbb{R}^n$ the symmetries of $\mathrm{conv}(Gx)$ contain G as a subgroup, but in general $\mathrm{conv}(Gx)$ may exhibit more symmetry than expected. Consider $G = \mathbb{Z}/3\mathbb{Z}$ embedded in $\mathrm{O}(2)$ as a rotation by $2\pi/3$. Then with any non-zero point $x \in \mathbb{R}^2$ chosen $\mathrm{conv}(Gx)$ is an equilateral triangle. But the symmetries of this triangle in $\mathrm{O}(2)$ are isomorphic to a copy of the symmetric group S_3 acting on the vertices.

Since every vertex-transitive spherical polytope arises from the orbit construction of Construction 2.2.1, the study of such objects may be carried out by the study of the structure of finite subgroups of $\mathrm{O}(n)$.

Of particular interest are the reflection groups discussed in Section 2.1. Given a reflection group generated by reflections through hyperplanes $H_i = \{x \in \mathbb{R}^n \colon \langle a_i, x\rangle = 0\}$ for unit vectors a_i with dihedral angles $\arccos\langle a_i, a_j\rangle$ as described in Section 2.1, the *fundamental region* is the cone $\{x \in \mathbb{R}^n \colon \forall i, \langle a_i, x\rangle \geq 0\}$.

To such a group one associates a Coxeter diagram, and to all $2^n - 1$ subsets of the vertices of this diagram but the empty one, Wythoff's construction generates a uniform polytope symmetric under the reflection group.

Theorem 2.2.3 (Wythoff's construction)**.** *Let G be a reflection group of rank n, given as a subgroup of the orthogonal group $\mathrm{O}(n)$. Let X be the set of non-zero points in the fundamental region of G such that if $x \in X$, then $\mathrm{conv}(Gx)$ is a uniform polytope. Then $X/\mathbb{R}^{>0}$ is naturally in bijection with the set of non-empty subsets of the vertices of the Coxeter diagram of G. The map sends a subset S to the ray fixed by each mirror not in S and of equal non-zero distance to each mirror in S.*

Proof. This result is due to Coxeter; see [12]. □

One frequently writes a Wythoff construction as a *decoration* of a Coxeter diagram by drawing a circle around each vertex in the set S of the statement of the theorem. The polytopes arising from Wythoff's construction are called *Wythoffian*.

Example 2.2.4. The reflection groups of rank 2 are all of the form $I_2(k)$ for some k; the three non-empty subsets of the Coxeter diagram correspond to a regular k-gon for a subset of size 1, or a regular $2k$-gon for a subset of size 2. Figure 2.2.4 gives an example of the Wythoff construction for the group $I_2(3)$. The reflecting hyperplanes H_1, H_2 that generate the group bound the fundamental region (shaded in light grey). The orbit construction applied to the points A, B gives a regular triangle. The orbit construction applied to the point C gives a regular hexagon. The orbit construction applied to the point D gives a vertex-transitive but non-uniform (that is, non-regular) hexagon; it has non-zero distance from H_1 and H_2, and these distances are not equal.

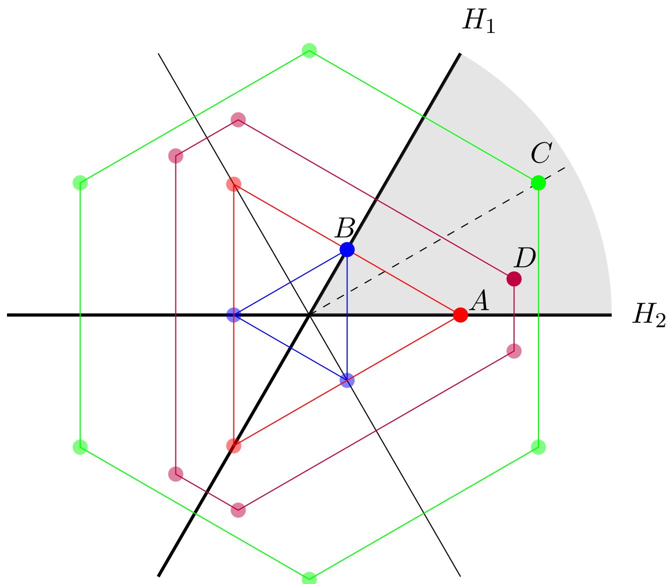


Figure 2.2: An example of the Wythoff construction for the group $I_2(3)$.

With the vertices of the Coxeter diagram being $\{H_1, H_2\}$, the subsets S corresponding to the points A, B, C are respectively $\{H_1\}, \{H_2\}$, and $\{H_1, H_2\}$; that is, S contains precisely the mirrors with non-trivial action on the point.

One reason to look at reflection groups is that Wythoff's construction is already sufficient to generate all the regular polytopes; see [47] for a preprint of the author carrying out this work, and for more background on this material. Furthermore, a combinatorial description of polytopes generated by the Wythoff construction (such as the one offered by Theorem 2.2.5) then enables one to classify fully the regular polytopes. A similar approach to this classification is taken by Coxeter in [13].

While Wythoff's construction provides a method of constructing uniform polytopes, in all dimensions higher than two there exist uniform polytopes not arising in this way. In three dimensions Wythoff's construction misses two uniform polyhedra: the snub cube, and the snub dodecahedron. These polyhedra arise from the orbit construction by using symmetry groups B_3^+ and H_3^+; that is, the orientation-preserving subgroups of B_3 and H_3. In four dimensions all but one uniform polytope arise from the orbit construction applied with either a reflection group of rank four (that is, from Wythoff's construction) or the rotary subgroup of a reflection group of rank four (the 'snub construction'). The remaining four-dimensional uniform polytope is the grand antiprism, which arises from the orbit construction with the group taken to be a particular index 36 maximal subgroup of H_4. This result is due to Conway and Guy [9]; a proof is available in [34].

Wythoffian polytopes are very easy to describe combinatorially. A complete description and proof is given in [47], but the following result gives a basic description.

Theorem 2.2.5 (Champagne–Kjiri–Patera–Sharp)**.** *Let G be a reflection group and S a non-empty subset of the Coxeter diagram, yielding a polytope P by Wythoff's construction.*

The k-faces of P are in correspondence with the Wythoff polytopes generated by $(G_A, S \cap A)$ where A is a size k subset of $1, \ldots, n$, $G_A = \langle r_i | i \in A\rangle$, and each connected component of the Coxeter diagram of G_A contains at least one element of $S \cap A$.

Example 2.2.6. The group H_3 may be given by three generators r_1, r_2, r_3, where r_1, r_3 commute, $(r_1 r_2)^3 = 1$, and $(r_2 r_3)^5 = 1$. Let $S = \{3\}$. To meet the hypotheses of the previous theorem, one must take one of $A = \{1, 2, 3\}$, $A = \{2, 3\}$, or $A = \{3\}$. These correspond to the polytope associated to S, a face, and an edge. For $A = \{2, 3\}$, we have $G_A = I_2(5)$, so that every face of this polyhedron is a pentagon; P is a dodecahedron.

2.3 Orthogonal groups

Reflection groups are distinguished finite subgroups of the larger *orthogonal groups*. The orthogonal group $\mathrm{O}(n)$ consists of all linear transformations of $\mathbb{R}^n$ leaving fixed the norm $|x|^2 = x_1^2 + \cdots + x_n^2$. All orthogonal geometry is captured by understanding reflections, as the following theorem shows.

Theorem 2.3.1 (Cartan–Dieudonné)**.** *Every element of* $\mathrm{O}(n)$ *may be written as a product of at most n reflections.*

Proof. A proof for all n is available in [22, §6.6]. For a more accessible proof in the case $n \leq 4$ (which is the only one required for this thesis), another reference is [10]. □

The orthogonal group is not connected. It consists of two components $\mathrm{O}_+(n) := \{P \in \mathrm{O}(n) \colon \det P = 1\}$ and $\mathrm{O}_-(n) := \{P \in \mathrm{O}(n) \colon \det P = -1\}$. The former is a group, written $\mathrm{SO}(n)$ and called the *special orthogonal group*.

Since the determinant of a reflection is -1, Theorem 2.3.1 implies that each element of $\mathrm{SO}(n)$ is the product of $2m \leq n$ reflections for some m. The product of reflections through hyperplanes with unit normals v, w is a rotation of the 2-plane spanned by v, w by twice the angle between v and w.

Corollary 2.3.2. *Every element of* $\mathrm{SO}(n)$ *may be written as a product of at most* $\lfloor n/2 \rfloor$ *rotations of a 2-plane.*

The low-dimensional cases are particularly salient: every non-identity element of $\mathrm{SO}(3)$ is a rotation of a 2-plane, and so is determined by a line in $\mathbb{R}^3$ and an angle of rotation up to 2π. Likewise, in four dimensions every non-identity rotation is either the rotation of a 2-plane (a *simple rotation*), or a rotation of two *orthogonal* 2-planes (a *double rotation*). That the planes of rotation can be chosen to be orthogonal is a result of the canonical form for an orthogonal transformation: each orthogonal transformation is orthogonally equivalent to a block-diagonal matrix where each block is one of $\begin{bmatrix}1\end{bmatrix}$, $\begin{bmatrix}-1\end{bmatrix}$, or a 2×2 rotation matrix.

2.3.1 The double cover maps

While $\mathrm{SO}(n)$ is connected for all n, it is not simply connected for $n \geq 3$. The fundamental group is $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$, and so by covering space theory it follows that there exists a simply connected Lie group $\widetilde{\mathrm{SO}(n)}$ and a two-to-one surjective Lie group homomorphism $\widetilde{\mathrm{SO}(n)} \to \mathrm{SO}(n)$ making the following sequence short exact:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathrm{SO}(n)} \to \mathrm{SO}(n) \to 1.$$

The group $\widetilde{\mathrm{SO}(n)}$ is called $\mathrm{Spin}(n)$.

An explicit construction of $\mathrm{Spin}(n)$ is useful for computational purposes.

Definition 2.3.3. Let Q be a non-degenerate quadratic form on $\mathbb{R}^n$. The *Clifford algebra* $\mathrm{Cl}(n, Q)$ is defined to be the most general unital associative $\mathbb{R}$-algebra containing $\mathbb{R}^n$ and with the property $x^2 = -Q(x)1$ for $x \in \mathbb{R}^n$.

If $i \colon \mathbb{R}^n \to \mathrm{Cl}(n, Q)$ is the inclusion, this means that for any other unital associative $\mathbb{R}$-algebra A with a map $j \colon \mathbb{R}^n \to A$ satisfying $j(x)j(x) = -Q(x)1$, there must be a unique algebra morphism $\overline{j} \colon \mathrm{Cl}(n, Q) \to A$ so that $j = \overline{j} \circ i$.

By Sylvester's law, a non-degenerate quadratic form on $\mathbb{R}^n$ is determined by its signature (p, q), which are respectively the number of entries that are positive or negative when one diagonalizes the form. Thus $\mathrm{Cl}_{p,q}$ means $\mathrm{Cl}(n, Q)$ when Q is the form of signature (p, q) and $n = p + q$.

As a vector space, the Clifford algebra is isomorphic to the exterior algebra, by the map replacing Clifford multiplication with the wedge product. Let $\{v_1, \ldots, v_n\}$ be a basis of $\mathbb{R}^n$ and $I = (i_1, \ldots, i_k)$ a strictly increasing multi-index $1 \leq i_1 < \cdots < i_k \leq n$. Let $v_I := v_{i_1} \cdots v_{i_k}$; then a basis for $\mathrm{Cl}(n, Q)$ is given by the set of all v_I.

The Clifford algebra comes equipped with a canonical antiautomorphism $v \mapsto v^t$, defined on v_I by

$$v_I^t := (-1)^k v_{i_k} \cdots v_{i_1},$$

and extended to all of $\mathrm{Cl}(n, Q)$ by linearity. Likewise, there is a map $\alpha \colon \mathrm{Cl}(n, Q) \to \mathrm{Cl}(n, Q)$ defined on $\mathbb{R}^n$ by $\alpha(v) = -v$, and extended to the whole algebra by declaring that it is an algebra morphism. Then for $a \in \mathrm{Cl}(n, Q)$, define $a^* = \alpha(a)^t$.

One can verify that if $v \in \mathbb{R}^n$, then $ava^* \in \mathbb{R}^n$ for all $a \in \mathrm{Cl}(n, Q)$, so that $a \mapsto (v \mapsto ava^*)$ is a map $\mathrm{Cl}(n, Q) \to M_{n\times n}(\mathbb{R})$, the $n \times n$ real matrices (endomorphisms of $\mathbb{R}^n$). In the case where a is a unit vector in $\mathbb{R}^n$, then $v \mapsto ava^*$ is reflection through the plane normal to a. Similarly,

Lemma 2.3.4. *Let $v_1, v_2 \in \mathbb{R}^n$ be orthogonal unit vectors. The image in $M_{n\times n}(\mathbb{R})$ of the element $\cos(\theta) + \sin(\theta)v_1v_2$ under the map $a \mapsto (v \mapsto ava^*)$ is a rotation by angle θ in the plane spanned by v_1, v_2.*

Theorem 2.3.5. *In the above notation,* $\mathrm{Spin}(n)$ *is the group generated by all products*

$$x_1 \cdots x_{2k} \in \mathrm{Cl}_{n,0},$$

where each $x_i \in \mathbb{R}^n$ is a unit vector, and the map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ *is $a \mapsto (v \mapsto ava^t)$.*

Proof. See [39, Proposition 4.6]. □

Chapter 3

Rotations and representations

3.1 Basic facts about representations

Let G be a group. A *(linear) representation of G on a (complex) vector space V* is a homomorphism $\rho\colon G \to \mathrm{GL}(V)$. By a common abuse of notation, one says that V *is a representation*, suppressing the map ρ, and for $v \in V$ and $g \in G$ one writes gv instead of $\rho(g)(v)$. The *dimension* of a representation V is $\dim V$, and V is said to be *irreducible* if there is no proper non-trivial subspace W of V such that $gw \in W$ for all $w \in W$ and all $g \in G$. The *character* of V is the function $\chi_V(g) := \mathrm{tr}(\rho(g))$, which computes the trace of g as it acts on V. These characters are *class functions* on G, which means that χ_V is constant on conjugacy classes of G.

If G is finite, then there are only finitely many finite-dimensional irreducible representations of G, and every finite-dimensional representation of G admits a decomposition into a sum of the irreducible representations, unique up to ordering. Moreover, if V is irreducible and W is any representation, then the multiplicity of V in the decomposition of W into irreducible representations is given by

$$\langle \chi_V, \chi_W \rangle := \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_W(g)}.$$

Thus the character of a representation contains sufficient data to reconstruct the representation itself.

The *representation ring* $R(G)$ of G is defined to be the ring generated by the characters of the irreducible representations. It is a space of class functions called *virtual characters*. Virtual characters that may be expressed as a non-negative integer linear combination of the characters of the irreducible representations are called *actual characters*, and correspond to linear representations of the group G.

The number of conjugacy classes of G is equal to the number of irreducible representations of G.

The product structure on the representation ring is given by pointwise multiplication of functions, and on the actual characters corresponds at the vector space level to a tensor product: if $(\rho_1, V_1), (\rho_2, V_2)$ are representations with characters $\chi_1, \chi_2 \in R(G)$, then $\chi_1 \cdot \chi_2$ is an actual character with corresponding representation $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$.

There is a second 'external' tensor product construction on representations, which is used to define a representation on a direct product of groups. If (ρ_1, V_1) is a representation of a group G_1, and (ρ_2, V_2) is a representation of a group G_2, then one may define a representation of the direct product $G_1 \times G_2$ on the tensor product $V_1 \otimes V_2$ via the action

$$(g_1, g_2) \cdot (v_1 \otimes v_2) := \rho_1(g_1)v_1 \otimes \rho_2(g_2)v_2.$$

To emphasize that this construction is distinct from the previous 'internal' tensor product construction, we write $V_1 \boxtimes V_2$ to mean a representation of $G_1 \times G_2$. Thus $\boxtimes$ is a functor taking in a G_1 representation and a G_2 representation and returning a $G_1 \times G_2$ representation, while $\otimes$ is a functor taking in two G representations and returning a third G representation. The character of $V_1 \boxtimes V_2$ is

$$\chi_{V_1 \boxtimes V_2}(g_1, g_2) = \chi_{V_1}(g_1)\chi_{V_2}(g_2).$$

Lemma 3.1.1. *Let G_1, G_2 be finite groups. The irreducible representations of $G_1 \times G_2$ are of the form $V_1 \boxtimes V_2$ for irreducible representations V_1 and V_2 of G_1 and G_2 respectively.*

Proof. An easy computation tells us that the characters of such $V_1 \boxtimes V_2$ are still irreducible, while the number of conjugacy classes of $G_1 \times G_2$ is the product of the number of conjugacy classes of G_1 with that of G_2. □

A map $f\colon V \to W$ between representations of G is called *(G-)equivariant* if $f(gv) = gf(v)$ for all $v \in V$. These maps are the arrows in the category of G-representations, and an isomorphism in this category is also called an *equivalence* of representations.

A basic question in representation theory is to determine the space of equivariant maps between two representations: this set is written $\mathrm{Hom}_G(V, W)$, and it is a complex vector space.

Lemma 3.1.2 (Schur's lemma)**.** *Let V, W be irreducible representations of G. Then V and W are equivalent as representations if and only if $\mathrm{Hom}_G(V, W)$ is non-trivial. In this case, $\mathrm{Hom}_G(V, W)$ is 1-dimensional, and in particular $\mathrm{Hom}_G(V, V) = \{\lambda I : \lambda \in \mathbb{C}\}$.*

Proof. If $f\colon V \to W$ is equivariant and non-zero, then $\ker f$ is a subrepresentation of V which is proper, and so by irreducibility f is injective. Likewise, the image of f is a subrepresentation of W which is non-trivial, so f is surjective. Thus f is an equivalence of representations; then composition with f is a linear isomorphism of $\mathrm{Hom}_G(V, W)$ with $\mathrm{Hom}_G(V, V)$, so to show that $\mathrm{Hom}_G(V, W)$ has dimension one, it suffices to prove that $\mathrm{Hom}_G(V, V)$ consists of the scalar maps.

Suppose that $f\colon V \to V$ is equivariant. The map f has some eigenvalue $\lambda \in \mathbb{C}$, and $f - \lambda I$ is not injective, because it maps the λ-eigenspace to zero. The kernel of $f - \lambda I$ is a subrepresentation of V which is non-trivial, and so it must be all of V; in other words, $f = \lambda I$. □

3.2 The representation theory of $\mathrm{SO}(N)$ and $\mathrm{Spin}(N)$

In the remainder of this chapter, concrete problems of representation theory for the groups $\mathrm{SO}(3), \mathrm{SO}(4), \mathrm{Spin}(3), \mathrm{Spin}(4)$ are treated. Before specializing to $N = 3, 4$, the arbitrary dimension case is treated first in this section.

Let N be fixed, and let $\rho\colon \mathrm{Spin}(N) \to \mathrm{SO}(N)$ denote the double covering homomorphism. If G is a finite subgroup of $\mathrm{SO}(N)$, one may investigate the relation between the representation theory of G and the representation theory of $\widetilde{G} := \rho^{-1}(G)$. The following results are presumably well known, but the author is not aware of a reference containing them.

A first question is to determine the conjugacy classes of $\widetilde{G}$: the motivation for doing so is that irreducible representations of $\widetilde{G}$ are in bijective correspondence with these conjugacy classes. For a finite subset $K \subseteq \mathrm{Spin}(N)$, write $-K := \{-a\colon a \in A\}$.

Lemma 3.2.1. *If K is a conjugacy class of the finite group G, then either $\widetilde{K}$ is a conjugacy class of $\widetilde{G}$, or else it is the union of two conjugacy classes $A, -A$.*

Proof. Take $g \in K$, and let $\widetilde{g}$ denote an element of $\widetilde{G}$ such that $\rho(\widetilde{g}) = g$. If a is any element of $\widetilde{G}$, then $\rho(a\widetilde{g}a^{-1}) = \rho(a)g\rho(a)^{-1} \in K$, and so the conjugacy class A of $\widetilde{g}$ in $\widetilde{G}$ projects to K.

If A contains both lifts $\widetilde{g}, -\widetilde{g}$ of some $g \in G$, then there exists $a \in \widetilde{G}$ so that $a\widetilde{g}a^{-1} = -\widetilde{g}$. Then supposing that $h \in K$ is any other element, pick $b \in G$ so that $bgb^{-1} = h$. Let $\widetilde{b}$ be a lift of b, and one may then verify that $\widetilde{b}\widetilde{g}\widetilde{b}^{-1}, (\widetilde{b}a)\widetilde{g}(\widetilde{b}a)^{-1}$ are the two lifts of h. Since h was arbitrary, $A = \widetilde{K}$.

On the other hand, if A contains only one lift of g for each $g \in K$, then $|A| = |K|$, the set $-A$ is a conjugacy class disjoint from A, $\rho(-A) = K$, and $|-A| = |A| = |K|$. Thus $A \cup -A = \widetilde{K}$. □

The proof also gives a convenient criterion to check whether a conjugacy class splits into two classes of the same order in the double cover or whether it becomes one class of twice the order: the former happens if and only if the lifts of some (hence any) element of the class are not conjugate to each other in $\widetilde{G}$; said differently, if one fixes an element $g \in G$ and a lift $\widetilde{g} \in \widetilde{G}$, the conjugacy class of g splits if and only if there is no $a \in \widetilde{G}$ so that $a\widetilde{g} = -\widetilde{g}a$. In particular, if the centralizer of g is the group generated by g, the conjugacy class of g splits in the double cover; indeed, any $a \in \widetilde{G}$ anticommuting with a lift $\widetilde{g}$ of g yields $\rho(a)g = g\rho(a)$, but then $\rho(a) = g^k$ for some integer k whence $a = \pm\widetilde{g}^k$, so a and $\widetilde{g}$ commute.

The common order of the elements in the lift of a conjugacy class can either stay the same or double: if $r^k = 1$, then $\rho(\widetilde{r})^k = 1$, and so $\widetilde{r}^k = \pm 1$. So if $\widetilde{r}^k = 1$ and k is odd, $-\widetilde{r}$ has order $2k$ and $\widetilde{r}$ has order k. If $\widetilde{r}^k = 1$ and k is even, then $-\widetilde{r}$ has order k. Something similar may be done when instead $\widetilde{r}^k = -1$. What is important is that if a conjugacy class splits in two under the lift, the criterion for the classes to have the same order is that k be even.

Example 3.2.2. An example of interest in this thesis is the binary icosahedral group $\widetilde{H_3^+}$. Since H_3^+ is abstractly the alternating group A_5, we know the conjugacy class structure: there are classes labelled $1, 3, 4, 5A, 5B$ of sizes $1, 20, 15, 12, 12$ respectively; the labelling is explained in Remark A.3.1. The class of label 4 is the only one that does not split. Since the remaining classes have odd common order of elements and split, it follows that the conjugacy classes of $\widetilde{H_3^+}$ and their sizes (in parentheses) are

$$1\ (1),\ 2\ (1),\ 3\ (20),\ 4\ (30),\ 5A\ (12),\ 5B\ (12),\ 6\ (20),\ 10A\ (12),\ 10B\ (12).$$

3.3 The representation theory of $\mathrm{Spin}(4) \to \mathrm{SO}(4)$

Let $\rho\colon \mathrm{Spin}(4) \to \mathrm{SO}(4)$ be the projection map. By Corollary 2.3.2, every element of SO(4) is either the identity, a simple rotation, or a double rotation.

The exceptional isomorphisms described in Section A.1 give $\mathrm{SU}(2) = \mathrm{Spin}(3) = \mathrm{Sp}(1)$ and $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$. Given an element $(p, q) \in \mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$, one can ask what kind of element the projected $\rho(p, q) \in \mathrm{SO}(4)$ is.

Lemma 3.3.1. *Let $(p, q) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$. If $(p, q) \in \{\pm(1, 1)\}$, then $\rho(p, q)$ is the identity. If not, and p, q are conjugate in $\mathrm{SU}(2)$, then $\rho(p, q)$ is a simple rotation. If neither of the two conditions apply, then $\rho(p, q)$ is a double rotation.*

Proof. If p, q are conjugate, say $q = apa^{-1}$, then $(1, a^{-1})(p, q)(1, a) = (p, p)$, which descends to a transformation of $\mathbb{R}^4 = \mathbb{H}$ fixing the real axis. Since this transformation is also an element of SO(4), it must be a simple rotation.

Conversely, if (p, q) is a simple rotation it fixes an axis containing some non-zero a. Then $(p, a^{-1}qa)$ fixes the real axis of $\mathbb{R}^4 = \mathbb{H}$. The transformations fixing the real axis correspond to the diagonal embedding $\mathrm{SU}(2) \to \mathrm{SU}(2) \times \mathrm{SU}(2)$, and so $p = a^{-1}qa$, showing that p, q are conjugate. □

Remark 3.3.2. If $p = aqa^{-1}$, then $(1, a)(p, q)(1, a)^{-1} = (p, p)$. Conversely, if (p, q) is conjugate to an element of the form (r, r), then p and q are conjugate in SU(2). Thus it is equivalent to require that (p, q) be conjugate to such an element, and in this case (p, q) is said to be *diagonal*.

As a result of Lemma 3.1.1, the representation theory of a direct product of groups reduces to that of the representation theory of its factors. In particular, the irreducible representations of $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ are all (external) tensor products of irreducible representations of SU(2), which are classified by the next theorem.

Theorem 3.3.3. *For each integer $n \geq 1$ there is a unique irreducible representation of* SU(2) *of dimension n.*

Proof. See, for example, [40, Theorem 3.32]. □

One realization of the unique n-dimensional irreducible representation is by the complex homogeneous polynomials of degree $n-1$ in two variables. Let W_n denote the set of complex homogeneous polynomials in variables x, y of degree $n - 1$. Then as a complex vector space, W_n has dimension n. It is easy to see that W_n is a representation of SU(2), where one identifies x and y with coordinates of $\mathbb{C}^2$ and has SU(2) act by matrix multiplication. Less obvious is that this representation is actually irreducible; but one may check that it is, and so by Theorem 3.3.3, W_n is the unique n-dimensional irreducible representation of SU(2).

Remark 3.3.4. In light of Theorem 3.3.3, it is common to write just the dimension n to refer to the representation W_n. In this notation, 1 is the trivial representation of SU(2) and 2 is the fundamental representation, and for example one might compute that $3 \otimes 2 = 2 \oplus 4$. This notation is only used for a representation of a particular (not variable) dimension. For example, we write

$$W_n \otimes W_2 = W_{n-1} \oplus W_{n+1}, \quad \text{or} \quad W_n \otimes 2 = W_{n-1} \oplus W_{n+1},$$

but not $n \otimes 2 = (n-1) \oplus (n+1)$; the latter notation has the potential to be misleading as it puts the usual $-$ and $+$ together in an expression with $\oplus$, but a symbolic manipulation like $(n-1) \oplus (n+1) = (n+n) \oplus (1-1) = 2n (= W_{2n})$ is *not* true.

Likewise, when working in $\mathrm{SU}(2) \times \mathrm{SU}(2)$, we interpret $2 \boxtimes 1$ to be the external tensor product of the fundamental and trivial representations of $\mathrm{SU}(2)$.

These representations have a particularly nice formula for their characters. Write $\chi_a(g)$ for the trace of $g \in \mathrm{SU}(2)$ acting on the a-dimensional irreducible representation W_a.

Lemma 3.3.5. *Let g be an element of $\mathrm{SU}(2)$. Then g is diagonalizable by a matrix in $\mathrm{SU}(2)$ to an element of the form $\mathrm{diag}(e^{i\theta}, e^{-i\theta})$, and*

$$\chi_a(g) = \sum_{\ell=0}^{a-1} e^{i(a-1-2\ell)\theta}.$$

Moreover, when θ is not a multiple of π,

$$\chi_a(g) = \sin(a\theta)/\sin(\theta).$$

If θ is a multiple of π, the above statement still holds if one takes a limit; $\chi_a(g) = \lim_{t\to\theta} \sin(at)/\sin(t)$. More generally, if ℓ is an integer and $\ell\theta$ is not a multiple of π, then

$$\chi_a(g^\ell) = \sin(a\ell\theta)/\sin(\ell\theta).$$

Proof. The formulae here follow easily from the explicit description of W_n as complex homogeneous polynomials of degree $n-1$ in two variables (see also [40, 3.32, Exercise 3.22]). The second displayed statement follows easily from the first by summing a geometric series, so long as the denominator $\sin(\theta)$ does not vanish (that is, as long as θ is not a multiple of π). The third displayed statement follows from diagonalizing g using some special unitary A, and noting that $(AgA^{-1})^\ell = Ag^\ell A^{-1}$. □

For any group G, write G_Δ for the diagonal embedding of G in $G \times G$.

Lemma 3.3.6. *Let $G \subseteq \mathrm{SU}(2)$, and $G_\Delta \subseteq \mathrm{SU}(2) \times \mathrm{SU}(2)$ be its diagonal embedding. Let V, W be representations of $\mathrm{SU}(2)$. Under the isomorphism $G \simeq G_\Delta$,*

$$\mathrm{res}_G(V \otimes W) \simeq \mathrm{res}_{G_\Delta}^{G\times G}(\mathrm{res}_G V \boxtimes \mathrm{res}_G W) = \mathrm{res}_{G_\Delta}(V \boxtimes W).$$

Proof. Immediate from the above character computations. □

Example 3.3.7. Let $L = 2 \boxtimes 1$ and $R = 1 \boxtimes 2$ denote the representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{C}^2$, corresponding to multiplication respectively by the left and right factors. For $G \subseteq \mathrm{Spin}(3)$, Lemma 3.3.6 gives

$$\mathrm{res}_{G_\Delta} L = \mathrm{res}_G(2 \otimes 1) = \mathrm{res}_G(1 \otimes 2) = \mathrm{res}_{G_\Delta} R.$$

This shows the obvious relation that L and R are the same representation when restricted to a subgroup on which the left and right elements are always equal.

3.4 Discrete subgroups of SU(2)

3.4.1 The algebraic McKay correspondence

Suppose that G is a finite subgroup of SU(2), and let 2 denote the two-dimensional irreducible representation of SU(2); that is, the standard action of matrix multiplication on $\mathbb{C}^2$. By an abuse of notation, we also denote by 2 the restriction of this representation to G. Let $V_1, \ldots, V_m$ be the irreducible representations of G, and define a quiver $Q(G)$ in the following fashion: create for each V_i a vertex, and let there be a directed edge between V_i and V_j for each time V_j occurs in the decomposition of $V_i \otimes 2$ into irreducible components.

Note that $\chi_2(g) = \sin(2\theta)/\sin(\theta)$ is real, so that

$$\langle \chi_{V_i}, \chi_2 \chi_{V_j} \rangle = \langle \chi_{V_j} \chi_2, \chi_{V_i} \rangle,$$

hence the edges are undirected. In particular, V_i and V_j are joined by an edge if $\langle \chi_{V_i} \chi_2, \chi_{V_j} \rangle \geq 1$. While it is not obvious, one can show that for a finite subgroup of SU(2), the quiver $Q(G)$ contains no multi-edges or loops, and is therefore a graph.

The following result is part of the remarkable *McKay correspondence* of [31]; a proof is available in [43].

Theorem 3.4.1. *The map $G \mapsto Q(G)$ is a bijective correspondence between finite subgroups of* SU(2) *up to conjugation and the quivers of extended affine ADE type.*

The quivers $Q(G)$ and their corresponding groups are given in Table 3.1. The groups are described in terms of three-dimensional reflection groups, where G^+ denotes the subgroup of orientation preserving elements of G, and $\widetilde{G}$ denotes the double cover under the exceptional isomorphism $\mathrm{SU}(2) = \mathrm{Spin}(3)$.

The representations in the quivers of exceptional type are labelled by their dimensions, so for example 6 in the quiver for the binary icosahedral group $\widetilde{H_3^+}$ means the restriction of the six-dimensional irreducible representation of SU(2) to this group. A prime mark denotes a representation not isomorphic to the restriction of the irreducible SU(2) representation of the same dimension. It may be easiest to think of these objects in terms of the McKay correspondence: for example, if $G = \widetilde{H_3^+}$, then $3'$ is the three-dimensional irreducible part of $6 \otimes 2$.

The McKay correspodence is useful for tensor product computations, when combined with the fact that representations decompose uniquely into irreducible parts. Clearly one can use this correspondence to compute products $V \otimes 2$ for various V (the representation is the sum over the neighbours of each irreducible factor of V), but as well it can be used to compute tensor products for different representations. For example, with the binary icosahedral group $\widetilde{H_3^+}$, one has $2 \otimes 2 = 3 \oplus 1$, and so using a purely symbolic 'subtraction', $V \otimes 3 = V \otimes 2 \otimes 2 - V$. For example,

$$4 \otimes 2 \otimes 2 = (3 \oplus 5) \otimes 2 = 2 \oplus 4 \oplus 4 \oplus 6,$$

and then

$$4 \otimes 3 = 4 \otimes 2 \otimes 2 - 4 = 2 \oplus 4 \oplus 4 \oplus 6 - 4 = 2 \oplus 4 \oplus 6.$$

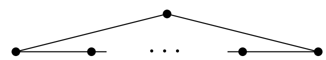
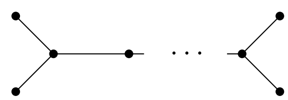
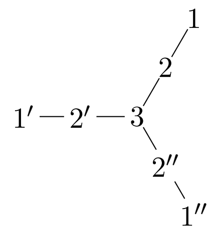
Name	$G \subseteq \mathrm{SU}(2)$	Q(G)
$\widetilde{A_n}$	$\mathbb{Z}/n\mathbb{Z}$	
$\widetilde{D_n}$	D_{2n}	
$\widetilde{E_6}$	$\widetilde{A_3^+}$	
$\widetilde{E_7}$	$\widetilde{B_3^+}$	$2''$ | $1 — 2 — 3 — 4 — 3' — 2' — 1'$
$\widetilde{E_8}$	$\widetilde{H_3^+}$	$3'$ | $1 — 2 — 3 — 4 — 5 — 6 — 4' — 2'$

Table 3.1: Representations of the affine ADE type quivers appearing in the McKay correspondence.

3.4.2 Discrete subgroups of $\mathrm{SU}(2) \times \mathrm{SU}(2)$

For $\mathrm{SU}(2) \times \mathrm{SU}(2)$, a similar result does not hold; while classifications are available (see, for example, [20, 10, 32]), neither an ADE classification nor an "ADE × ADE" classification hold, as many finite subgroups correspond to non-trivial semidirect products of the SU(2) subgroups.

Still, it cannot be said that the geometry of the direct product subgroups is uninteresting: for it is within these direct products that one finds the double covers of the rotational symmetry groups of a number of the four-dimensional regular polytopes: in particular, $\widetilde{H_3^+} \times \widetilde{H_3^+} = \widetilde{H_4^+}$ is the double cover of the rotational symmetries of the 600-cell. In the interest of simplifying the representation theory only direct product discrete subgroups of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and the discrete subgroups arising from the diagonal embedding $\mathrm{SU}(2) \to \mathrm{SU}(2) \times \mathrm{SU}(2)$ are considered in this thesis. However, interesting instantons may lie within the non-trivial semidirect products. For example, in [1] the construction of a symmetric instanton with hypertetrahedral symmetry is performed; the binary symmetry group of the hypertetrahedron is not a direct product of two subgroups of SU(2), but rather it is a twisted embedding of $\widetilde{H_3^+}$ in $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

3.4.3 Stitching for SU(2)-representations

Contained in this section are two mysterious-looking results about a problem that might reasonably be called 'representation stitching': given a description of how a representation decomposes into irreducibles when restricted to cyclic subgroups of a group, what can one say about how the representation decomposes into irreducible representations of the whole group? Of course, in principle the answer is everything, because the character contains all the information about the group and in this setting one can compute the character. But if the restriction to different cyclic subgroups is 'poorly behaved', it may not be easy to generally describe the representation on the whole group. The results presented focus on specific cases where stitching can be performed when the group in question is a compact subgroup of SU(2). These results are useful in the theory of symmetric instantons, where they arise naturally as integrality restrictions on instantons symmetric under these compact subgroups of SU(2).

Proposition 3.4.2. *Suppose κ is a prime number and G is a non-abelian finite subgroup of* $\mathrm{SU}(2)$*. Suppose that V is a κ-dimensional representation of G and that for each $g \in G$ there exist integers $a(g), n(g)$ so that*

$$\mathrm{res}^{G}_{\langle g\rangle} V = \left(\mathrm{res}^{\mathrm{SU}(2)}_{\langle g\rangle} W_{a(g)}\right)^{\oplus n(g)}.$$

Then either $V = \mathrm{triv}^{\oplus\kappa}$ or $V = \mathrm{res}^{\mathrm{SU}(2)}_{G} W_{\kappa}$.

In other words, if V has the property that, restricted to cyclic subgroups, it looks like the restriction of some number of copies of an irreducible representation of SU(2), then at prime dimension this local property 'globalizes'.

Proof. For dimension reasons, for each $g \in G$ one has $\kappa = a(g)n(g)$, thus $a(g) = 1$ or $a(g) = \kappa$. Thus either $\chi_V(g) = \kappa$ or $\chi_V(g) = \chi_\kappa(g)$.[1] The theorem is equivalent to the statement that if one were to create a class function by picking for each conjugacy class representative r either the value κ or $\chi_\kappa(r)$, the only choices that would result in the character of a representation would be to choose all κ or all $\chi_\kappa(r)$—one cannot 'mix' characters in this case.

Recall that in any representation the character takes values of norm bounded above by the dimension of the representation. Moreover, if W is a non-trivial irreducible representation of G and r is an element of order at least three, then $\chi_W(r) \neq \dim W$; this fact follows from a classification of the finite subgroups of SU(2) (see Appendix A, although the character tables for cyclic and dihedral groups are not presented. This statement is not trivial for the latter). Thus if $\chi_V(r) = \kappa$ for any r of order at least 3 (such r exist, as G is non-abelian), the triangle inequality together with the above two points give immediately that $V = \mathrm{triv}^{\oplus\kappa}$, and otherwise the only possible choice is $V = \mathrm{res}^{\mathrm{SU}(2)}_{G} W_\kappa$. □

Proposition 3.4.2 is quite sharp for finite subgroups of SU(2), in the sense that there generally arise representations not corresponding to a divisor in the case that κ is not prime. The abundance of rotations in the full SU(2) allow for a stronger result.

Proposition 3.4.3. *Suppose that V is a representation of* $\mathrm{SU}(2)$ *and that for each $g \in \mathrm{SU}(2)$ there exist integers $a(g), n(g)$ so that*

$$\mathrm{res}^{\mathrm{SU}(2)}_{\langle g\rangle} V = \left(\mathrm{res}^{\mathrm{SU}(2)}_{\langle g\rangle} W_{a(g)}\right)^{\oplus n(g)}.$$

[1] Recall that χ_κ is the character of the κ-dimensional irreducible representation of SU(2).

Then $a(g), n(g)$ do not depend on g, and if $a = a(g), n = n(g)$ denote their constant values, then $V = W_a^{\oplus n}$.

The proof of this proposition relies on the following result from transcendental number theory.

Lemma 3.4.4. *Suppose that $a_1, \ldots, a_k$ are positive, distinct, and algebraic, and θ is algebraic and non-zero. Then $\{\cos(a_i\theta)\}_{i=1}^k, \{\sin(a_i\theta)\}_{i=1}^k$ are both algebraically independent in the sense that, for example, there are no algebraic b_i so that*

$$\sum_{i=1}^k b_i \sin(a_i\theta) = 0,$$

except for the trivial solution $b_1 = \cdots = b_k = 0$.

Proof. This result is an immediate consequence of the Lindemann–Weierstrass theorem (see [4, Theorem 1.4] for the relevant formulation) applied to $\pm\sqrt{-1}a_i\theta$ as $i = 1, \ldots, k$, which asserts that the exponentials of these algebraic numbers are algebraically independent from each other. Apply this algebraic independence to the expressions obtained by writing, for example, that $\sin(a_i\theta) = \frac{1}{2i}(\exp(\sqrt{-1}a_i\theta) - \exp(-\sqrt{1}a_i\theta))$. □

Proof of Proposition 3.4.3. Again, if κ is the dimension of V then $\kappa = a(g)n(g)$ for each g. The first step is to prove that for sufficiently many g, the integer $a(g)$ (and hence $n(g)$) depends only on the angle of rotation of g, and not the axis of rotation.

Since all rotations by the same angle are conjugate in SU(2), they have equal characters $\chi_V(g)$. Then if g, g' are rotations by the same angle θ not a multiple of π, Lemma 3.3.5 gives

$$\begin{aligned} \chi_V(g) &= n(g)\chi_{a(g)}(g) \\ &= \frac{\kappa \sin(a(g)\theta)}{a(g)\sin(\theta)}, \end{aligned}$$

so equating $\chi_V(g) = \chi_V(g')$ and rearranging gives

$$a(g)\sin(a(g')\theta) = a(g')\sin(a(g)\theta).$$

By Lemma 3.4.4, if θ is an algebraic number, this equation for $a(g), a(g')$ has no solutions in positive integers other than the trivial one $a(g) = a(g')$.[2] Thus in the case where θ is algebraic, we write $n(\theta), a(\theta)$ for $n(g), a(g)$.

If ℓ is any integer and g is a rotation by θ, then g^ℓ is a rotation by $\ell\theta$. Then $\langle g^\ell \rangle \subseteq \langle g \rangle$, and so

$$\mathrm{res}^G_{\langle g^\ell\rangle} V = \mathrm{res}^{\langle g\rangle}_{\langle g^\ell\rangle} \mathrm{res}^G_{\langle g\rangle} V = \left(\mathrm{res}^{\mathrm{SU}(2)}_{\langle g^\ell\rangle} W_{a(g)}\right)^{\oplus n(g)}.$$

Thus so long as $g^\ell \neq \pm I$ (that is, so long as $\ell\theta \not\equiv 0 \pmod{\pi}$),

$$n(\ell\theta)\frac{\sin(a(\ell\theta)\ell\theta)}{\sin(\ell\theta)} = \chi_V(g^\ell) = n(\theta)\frac{\sin(a(\theta)\ell\theta)}{\sin(\ell\theta)}.$$

[2] I suspect this statement holds more generally when θ is not a rational multiple of π, but I am not aware of a proof.

Thus

$$n(\ell\theta)\sin(a(\ell\theta)\ell\theta) = n(\theta)\sin(a(\theta)\ell\theta).$$

Since $\ell\theta \neq 0$ and $n(\ell\theta), n(\theta) \neq 0$, then Lemma 3.4.4 gives $a(\ell\theta) = a(\theta)$ and thus $n(\ell\theta) = n(\theta)$.

If m/n is any non-zero rational number, it therefore follows by the fact that π is irrational (so that $m/n \not\equiv 0 \pmod{\pi}$) and that m/n is algebraic that $a(1/n) = a((1/n)\cdot n) = a(1)$, and as well $a(1/n) = a(1/n \cdot m) = a(m/n)$. Thus $a(\theta)$, and hence $n(\theta)$ are constant on $\mathbb{Q}$. Let $a := a(1)$ and $n := n(1)$ denote the constant values of these functions on $\mathbb{Q}$.

When g is a non-identity rational rotation, its character in V agrees with that of $n\chi_a$; but characters are smooth functions of G as a Lie group, and the rational rotations are dense in SU(2). So $\chi_V = n\chi_a$. □

3.4.4 Left-right equivariant maps of SU(2) × SU(2) subgroups

Define L, R to respectively be the representations $2 \boxtimes 1, 1 \boxtimes 2$ of $\mathrm{SU}(2)\times\mathrm{SU}(2)$, or by an abuse of notation their restrictions to some group $G \subseteq \mathrm{SU}(2)\times\mathrm{SU}(2)$. A *left-right equivariant map* for representations V, W of G is a G-equivariant map $L \otimes V \to R \otimes W$. Intuitively, a left-right equivariant map permits the interchanging of the V-action with the W-action, as long as one is willing to switch a multiplication by the left element of $(p, q) \in \mathrm{SU}(2)\times\mathrm{SU}(2)$ with the right element.

Remark 3.4.5. The structure of left-right equivariant maps for G depends heavily on whether or not $G \subseteq \mathrm{SU}(2)_\Delta$. In the former case, L, R are equivalent representations, and every external tensor product may be computed as an internal tensor product (see Lemma 3.3.6). Thus a 'left-right equivariant map' $V \to W$ is just a map $2 \otimes V \to 2 \otimes W$. The existence problem for such maps is solved by the McKay correspondence described in Section 3.4.1. The next lemma treats the case where L, R are not equivalent.

Lemma 3.4.6. *Let V and W be irreducible representations of a finite direct product subgroup of* SU(4)*, corresponding to vertices (v_1, v_2) and (w_1, w_2) of the corresponding Dynkin diagrams. The space of left-right equivariant maps $L \otimes V \to R \otimes W$ is non-zero if and only if v_1 is adjacent to w_1 and v_2 is adjacent to w_2, and in this case it is one-dimensional.*

Proof. Write $V = V_1 \boxtimes V_2$, so that $L \otimes V = [2 \otimes V_1] \boxtimes V_2$. Thus $L \otimes V = \bigoplus_Y Y \boxtimes V_2$, the sum running over all vertices adjacent to V_1 in the Dynkin diagram. Likewise, if $W = W_1 \boxtimes W_2$, then $R \otimes W = \bigoplus_Y W_1 \boxtimes Y$, the sum running over all vertices adjacent to V_2 in the Dynkin diagram.

By Schur's Lemma (Lemma 3.1.2) and Lemma 3.1.1, the space of left-right equivariant maps for V, W is non-zero if and only if there exist Y, Z so that $W_1 \otimes Y$ is equivalent to $Z \otimes V_2$. The only possibility is $Y = V_2, Z = W_1$, and for these terms to appear in the sum requires that W_1 be adjacent to V_1, and W_2 be adjacent to V_2.

Finally, remark by Theorem 3.4.1 that the multiplicity of any irreducible representation in the tensor product $2 \otimes V_1$ or $2 \otimes W_2$ is always 0 or 1, so it cannot be the case that there are multiple copies of $W_1 \boxtimes V_2$ contained in $L \otimes V$ and $R \otimes V$. □

The essential hypothesis in Lemma 3.4.6 is that the group considered is a direct product subgroup of SU(2) × SU(2). In particular, non-trivial direct product subgroups are not contained in the diagonal, $\mathrm{SU}(2)_\Delta$, and so L and R are inequivalent representations.

The situation is thus nice for direct product subgroups of Spin(4): if k is the number of irreducible factors of V, then a left-right equivariant map $L \otimes V \to R \otimes V$ is described by a $k \times k$ matrix, and the entries of this matrix corresponding to pairs having no non-zero left-right equivariant maps between them are set to zero.

Chapter 4

Index theory

One of the crowning achievements of 20th century mathematics is the Atiyah–Singer index theorem: a sweeping generalization of the Riemann–Roch theorem on algebraic curves. This thesis does not endeavour to describe fully its significance, but rather specialize it to some very particular cases which are of use in the symmetric instanton problem. Standard references for the equivariant index theorem in particular are [25, §III.14], [17, §4.5], and [41, §19]; the presentation here closely follows the first.

4.1 Vector bundles and characteristic classes

When not otherwise specified, all work is done in the smooth category, and manifolds are assumed to be paracompact. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$. An $\mathbb{F}$*-vector bundle of rank* n is a smooth submersion $\pi\colon E \to X$ of manifolds that 'locally looks like $X \times \mathbb{F}^n$': that is, the fibres of π have the structure of an n-dimensional $\mathbb{F}$-vector space, there exists an open cover $\{U_\alpha\}_{\alpha \in I}$ such that $\pi^{-1}(U_\alpha)$ is diffeomorphic to $U_\alpha \times \mathbb{R}^n$ via a map f_α, and moreover that these trivializations f_α are compatible in the sense that on the intersection $U_{\alpha\beta} := U_\alpha \cap U_\beta$, the corresponding map $f_\alpha \circ f_\beta^{-1}\colon U_{\alpha\beta} \times \mathbb{F}^n \to U_{\alpha\beta} \times \mathbb{F}^n$ preserves the term in $U_{\alpha\beta}$ and acts as a linear isomorphism on the $\mathbb{F}^n$ factor.

Over a fixed base space X, a map of bundles $(\pi_1\colon E \to X) \to (\pi_2\colon F \to X)$ is a smooth map $\phi\colon E \to F$ such that $\pi_1 = \pi_2 \circ \phi$ and such that for each $x \in X$, the restriction of ϕ to the fibre E_x is a linear map $E_x \to F_x$. With these maps as morphisms, $\mathbb{F}$-vector bundles over X form a category. If E is isomorphic to the bundle $X \times \mathbb{F}^n$ with its obvious linear structure, we say that E is *trivial.*

Given two vector bundles $E_1 \to X, E_2 \to X$, one may form direct sums and tensor products fibrewise to obtain vector bundles $E_1 \oplus E_2 \to X$ and $E_1 \otimes E_2 \to X$. The K*-group* of a manifold, $K(X)$, is an object capturing the kinds of complex vector bundles that can occur over X. One would like to define the K-group as something like the set of all complex vector bundles over X up to isomorphism with $\oplus$ as a group operaton. Since $\operatorname{rank}(E \oplus F) = \operatorname{rank}(E) + \operatorname{rank}(F)$, the identity must be the unique trivial bundle of rank 0, but then there are no additive inverses for non-identity bundles.

The solution is passing to the *Grothendieck group* in which additive inverses are adjoined formally. If E is a vector bundle and $[E]$ denotes the isomorphism class of E, we define a new symbol $-[E]$, which satisfies the following algebraic relation: if E, F are bundles, then

$$[E]-[F]=[E']-[F'] \iff [E]+[F']=[E']+[F] \iff [E\oplus F']=[E'\oplus F].$$

Thus one has that $[E]-[E]=[0]$, the trivial rank zero bundle; it is clear that this operation is associative and commutative, and so the set of all isomorphism classes of complex vector bundles $E \to X$ together with their formal inverses gives an abelian group, denoted $K(X)$.

To a manifold X we associate the *Chern class*: a morphism c from complex vector bundles on X to $H^{2\bullet}(X;\mathbb{Z})$, the integral even cohomology of X.[1] For E a complex vector bundle on X, let $c_i(E)$ denote the part of $c(E)$ in $H^{2i}(X;\mathbb{Z})$. The Chern class exists (see, for example, [33, §14]), and is uniquely defined by the following properties for all complex vector bundles E, F.

1. (Dimension) $c_0(E)=1$, and $c_\ell(E)=0$ for $\ell > \operatorname{rank}(E)$.
2. (Naturality) If $f\colon X\to Y$ is continuous, then $f^*c(E)=c(f^*E)$.
3. (Whitney sum formula) Let $\cup$ denote the cup product in cohomology. Then $c(E\oplus F)=c(E)\cup c(F)$.
4. (Regularization) If γ_n denotes the tautological line bundle on $\mathbb{CP}^n$, then $c_1(\gamma_n)$ is a generator for $H^2(\mathbb{CP}^n;\mathbb{Z})=\mathbb{Z}$.

The Chern class is a topological invariant by the naturality axiom, and it measures the non-triviality of a bundle: a trivial bundle is homotopic to the bundle of rank zero by the canonical 'scrunch' map, and so has Chern class 1; conversely, if the Chern class of a bundle is not 1, it cannot be trivial. The Chern class of a bundle being trivial does not, however, imply triviality of the bundle. It is a *stable* class in the sense that $c(E\oplus F)=c(E)$ whenever F is a trivial bundle, and so if E is a bundle that may be made trivial by the addition of a trivial summand, $c(E)=1$, but E itself may not be trivial (such bundles are called *stably trivial*).

The guiding principle involved in computations with Chern classes is the *splitting principle*, which is the philosophy that "all bundles are a sum of line bundles". While obviously not literally true, if $E\to X$ is any bundle, there exists a space Y and a map $f\colon Y\to X$ such that the pullback f^*E *is* a sum of line bundles over Y, and moreover that the induced map on cohomology is injective (see [25, Proposition III.11.1]). Thus for most computational purposes one may assume that $E=E_1\oplus\cdots\oplus E_r$ decomposes as a sum of line bundles.

Let $x_i=c_1(E_i)$; the x_i are called the *Chern roots* of E, and depend on the choice of decomposition. By the Whitney sum formula, one obtains

$$c(E)=c(E_1)\cdots c(E_r)=(1+x_1)\cdots(1+x_r), \tag{4.1}$$

[1] Chern classes can also be developed purely geometrically through the Chern–Weil theory. The book [39] uses the Chern–Weil description of characteristic classes to state the index theorem. The algebraic perspective makes it clear that Chern classes fall in *integral* cohomology; it also allows for easy handling of the equivariant case, since K-theory can readily be switched with equivariant K-theory. For these reasons, the algebraic approach to Chern classes is followed.

Expanding (4.1), it follows that $c_i(E)$ is equal to the ith elementary symmetric polynomial, σ_i, evaluated at the Chern roots:

$$c_i(E) = \sigma_i(x_1, \ldots, x_r). \tag{4.2}$$

A recipe for new characteristic classes is obtained by picking a smooth function f of r complex variables, totally symmetric under its arguments. Its power series may then be written down up to the terms of degree $\dim X/2$, for the higher terms vanish in cohomology. This truncated power series is now a symmetric polynomial in r variables, and so it may be expressed as a combination of the elementary symmetric polynomials by the fundamental theorem of symmetric functions. In particular, while the choice of Chern roots is not canonical, Equation (4.2) shows that there is a canonical 'evaluation of f at a rank r vector bundle' in the even cohomology by evaluating the power series of f at the r Chern roots, and replacing the resulting elementary symmetric polynomials with Chern classes.

A typical example is the Chern character ch, which is defined as $\mathrm{ch}(E) = \exp(x_1) + \cdots + \exp(x_r)$. The power series expansion is thus

$$\mathrm{ch}(E) = r + \sum_{m=1}^{\infty} \frac{1}{m!}(x_1^m + \cdots + x_r^m),$$

which may be rewritten as a sum of elementary symmetric polynomials by Newton's identities. The first few terms are

$$\mathrm{ch}(E) = \mathrm{rank} E + c_1(E) + \frac{1}{2}(c_1(E)^2 - c_2(E)) + \cdots .$$

Using the Chern roots, we define the following two characteristic classes: first, for some fixed $\theta \in \mathbb{R}$, the $\hat{\mathbf{A}}_\theta$-class is the multiplicative class associated to the power series of $\frac{1}{2\sinh\frac{1}{2}(x+i\theta)}$, which means that one takes

$$f(x_1, \ldots, x_r) = \prod_{j=1}^{r} \frac{1}{2\sinh\frac{1}{2}(x_j + i\theta)} = \prod_{j=1}^{r} \left(-\frac{1}{2} i\csc(\theta) + \frac{1}{4} x_j \cot(\theta)\csc(\theta) + O(x_j^2)\right).$$

The second class required is the $\hat{\mathbf{A}}$-class, associated to the function

$$f(x_1, \ldots, x_r) = \prod_{j=1}^{r} \frac{x_j/2}{\sinh(x_j/2)} = \prod_{j=1}^{r} \left(1 - \frac{x_j^2}{24} + \frac{7x_j^4}{5760} + O(x_j^6)\right).$$

Note that $\hat{\mathbf{A}}(E)$ is not necessarily equal to $\lim_{\theta\to 0} \hat{\mathbf{A}}_\theta(E)$.

4.2 The equivariant Chern character

The next step in the theory is to 'upgrade' the Chern character $\mathrm{ch}(E)$ to keep track of the action by a compact Lie group G. Recall that a *G-bundle* is a vector bundle $E \to X$ with an action of G on E that is assumed to be smooth and sends fibres to fibres linearly and isomorphically. One obtains an action of G on the base X by diffeomorphisms, via the identification of X as the image of the zero section in E. In particular, a G-bundle yields a representation of G on the fibre of any fixed point of the action on the base.

The goal is to develop the *equivariant Chern character* $\mathrm{ch}_G E$, which should, roughly speaking, allow 'coefficients in the representation ring $R(G)$'. The standard Chern character is then the equivariant character over the trivial group: as $R(\{1\}) = \mathbb{Z}$, then $\mathrm{ch}_{\{1\}}(E) = \mathrm{ch}(E)$.

The Grothendieck completion of the set of complex G-bundles over X is denoted $K_G(X)$, and called the *(G-)equivariant K-theory of X*. This K-theory is generally very difficult to compute. In the case where the action on the base manifold is the identity, the situation is much simpler: since the G-action comprises a linear map of each fibre to itself, a G-bundle is a smooth choice of representation over each point of the base manifold. The index theorem requires K-theory data only over submanifolds fixed pointwise by G, so the case of trivial action on X is the only one required.

More formally, let $V_1, \dots$ be a list of the finite-dimensional complex irreducible representations of G. If V is any finite-dimensional representation of G, an easy consequence of Schur's lemma for compact Lie groups is that

$$V \simeq \bigoplus_{i=1}^{\infty} \mathrm{Hom}_G(V_i, V) \otimes V_i.$$

This decomposition is the one of the canonical decomposition theorem for finite dimensional representations of compact groups; see, for example, [40, Theorem 3.19].

Suppose that the entire base manifold is fixed by the G-action; that is, suppose that if $x \in X, g \in G$, then $gx = x$. In this case, *every* fibre of E is a G-representation. Write $\underline{V_i} \to X$ for the trivial G-bundle $X \times V_i$ with the obvious action. Then

$$E \simeq \bigoplus_{i=1}^{k} \mathrm{Hom}_G(\underline{V_i}, E) \otimes V_i. \tag{4.3}$$

This isomorphism of G-bundles extends to an isomorphism $K_G(X) \simeq K(X) \otimes R(G)$. The equivariant Chern character is defined to be

$$\mathrm{ch}_G := \mathrm{ch} \otimes \mathrm{id},$$

and it sends a bundle into $H^{2\bullet}(X; \mathbb{C}) \otimes R(G)$ as desired; that is, one writes down a decomposition of E as above, and applies the usual Chern character to the bundles $\mathrm{Hom}_G(\underline{V_i}, E)$, which are G-bundles with trivial action.

For a particular element $g \in G$, one also defines $\mathrm{ch}_g = \mathrm{ch} \otimes \chi_\cdot(g)$; this expression is obtained from the equivariant Chern character $\mathrm{ch}_G(E)$ by contraction along the $R(G)$ factor. For example, if the canonical decomposition of E is $E \simeq E_1 \otimes V_1 \oplus E_2 \otimes V_2$, then

$$\mathrm{ch}_G(E) = \mathrm{ch}(E_1) \otimes V_1 + \mathrm{ch}(E_2) \otimes V_2,$$

and

$$\mathrm{ch}_g(E) = \chi_{V_1}(g)\mathrm{ch}(E_1) + \chi_{V_2}(g)\mathrm{ch}(E_2).$$

4.3 The topology of rotations fixing the poles of the four-sphere

The integral cohomology of S^4 is zero in all dimensions but 0 and 4, where it is equal to the integers. Bundles over the 4-sphere therefore support a single non-trivial Chern class: c_2.

Of particular interest are the submanifolds of S^4 that arise as fixed-point manifolds when a group acts on it; the actions considered in this thesis are those coming from rotations of $\mathbb{R}^4$, and so one considers $S^4 = \mathbb{R}^4 \cup \{\infty\}$, where ∞ and the origin 0 are fixed points of all elements. Corresponding to the three different kinds of elements in SO(4)—the identity, a simple rotation, and a double rotation—one recovers a fixed manifold of respectively S^4, S^2, and $S^0 = \{0, \infty\}$.

4.3.1 Bundles over S^2

The identification of S^2 with the complex projective line $\mathbb{CP}^1$ enables an easy classification of bundles on S^2. The (complex) line bundles on $\mathbb{CP}^1$ are in correspondence with their first (and only non-trivial) Chern class, and one writes $\mathcal{O}(n)$ to denote the unique line bundle with $c_1(\mathcal{O}(n)) = n \in \mathbb{Z} = H^2(S^2;\mathbb{Z})$. This bundle is the one obtained by covering $\mathbb{CP}^1$ with its standard open cover $U_0 = \{[z:w] \in \mathbb{CP}^1 | z \neq 0\}$, $U_\infty = \{[z:w] \in \mathbb{CP}^1 | w \neq 0\}$ and taking the transition function to be $z \mapsto z^n$. The Birkhoff–Grothendieck theorem (see, for example, [18, Theorem 2.1]) states that bundles on $\mathbb{CP}^1$ may be decomposed as a sum of line bundles.

Given a G-bundle $E \to S^2$, we first decompose it into a sum $\bigoplus_{i=1}^k E_k \otimes V_k$ for bundles E_i with trivial G-action and finite-dimensional irreducible representations V_i of G (see Equation (4.3)). Then by the Birkhoff–Grothendieck theorem, there exists for each $1 \leq i \leq k$ some $a_i \in \mathbb{Z}$ such that for all $1 \leq j \leq a_i$, there is $n_{ij} \in \mathbb{Z}$ so

$$E_i = \bigoplus_{j=1}^{a_i} \mathcal{O}(n_{ij}).$$

We may compute the equivariant Chern character of E via this decomposition. Fix a generator x for $H^2(S^2;\mathbb{Z})$, so that $\mathrm{ch}\mathcal{O}(n) = 1 + nx$. Using the fact that the Chern character distributes over sums, as well as the fact that $x^2 = 0$, one obtains

$$\begin{aligned} \mathrm{ch}_g E &= \sum_{i=1}^{k} \chi_{V_i}(g) \sum_{j=1}^{a_i} (1 + n_{ij} x) \\ &= \sum_{i=1}^{k} \chi_{V_i}(g) \left(a_i + \left(\sum_{j=1}^{a_i} n_{ij} \right) x \right). \end{aligned} \tag{4.4}$$

In the case that E is a restricted bundle from S^4, then one furthermore has $c_1(E) = 0$ by the naturality of Chern classes. In this setting, the vanishing of $c_1(E)$ means that $\sum_{i=1}^k \sum_{j=1}^{a_i} n_{ij} = 0$. A particular case of interest is when E is a rank two bundle. Then there are only two possible decompositions: either E is a trivial bundle with a two-dimensional irreducible representation as its generic fibre, or $E = \mathcal{O}(n) \otimes V_1 \oplus \mathcal{O}(-n) \otimes V_2$, for two one-dimensional (irreducible) representations V_1, V_2. For the former case, $\mathrm{ch}_g(E) = \chi_{E_0}(g)$

is the character of g in the representation given on any fibre of a fixed point, while in the latter case $\mathrm{ch}_g(E) = \chi_{E_0}(g) + n(\mathrm{ch}_{V_1}(g) - \mathrm{ch}_{V_2}(g))x$.

These results are used only in the case where $G = \mathbb{Z}/2m\mathbb{Z}$ is a cyclic group of even order—these are exactly the subgroups of Spin(4) one obtains by lifting the group generated by a simple rotation.

4.3.2 Bundles over S^0

The final case is the bundles occurring over S^0; but as a set of two discrete points, a bundle over it is simply a pair of vector spaces E_0, E_∞. The equivariant Chern character on either connected component is therefore just the representation character.

4.4 The equivariant index theorem

Suppose that G is a compact Lie group, $E, F \to X$ are G-bundles over a smooth manifold, and suppose moreover that $P\colon \Gamma(E) \to \Gamma(F)$ is an elliptic operator commuting with the action of G. In particular, the kernel and cokernel are therefore preserved by the G-action, and thus one may consider the index $[\ker P] - [\mathrm{coker} P]$ as a virtual representation of P. Denote by $\mathrm{ind}_g(P)$ the character of this representation at g—in other words, $\mathrm{ind}_g(P)$ is the trace of g as it acts on $\ker P$, less the trace of g as it acts on $\mathrm{coker} P$.

For $g \in G$, let $X^g = \{x \in X : \forall g \in G, gx = x\}$ be the set of fixed points of g. Let N_g be the normal bundle to X^g inside X. Let $i\colon X^g \to X$ be the inclusion map, σ the principal symbol of P, and $\lambda_{-1}(N_g \otimes \mathbb{C}) = [\Lambda^{\mathrm{even}}(N_g \otimes \mathbb{C})] - [\Lambda^{\mathrm{odd}}(N_g \otimes \mathbb{C})]$ the Thom class of the complexified normal bundle. The equivariant index theorem[2] is stated as follows (see [25, Theorem 14.3]).

Theorem 4.4.1. *In this setting,*

$$\mathrm{ind}_g(P) = (-1)^{\dim Y} \sum_Y \left(\frac{\mathrm{ch}_g(i^*\sigma)}{\mathrm{ch}_g(\lambda_{-1}(N_g \otimes \mathbb{C}))} \cdot \widehat{\mathbf{A}}(TY)^2 \right) [TY].$$

The sum runs over connected components Y of X^g.

4.4.1 The equivariant spin theorem

Suppose that $E \to X$ is a hermitian bundle over a spin manifold, and that $S = S^+ \otimes S^-$ is the spinor bundle of X. The *Dirac operator* $D\colon \Gamma(E \otimes S) \to \Gamma(E \otimes S)$ is an elliptic operator on the sections of the twisted spinor bundle $E \otimes S$. The operator D is self adjoint, and restricts to two operators $D^+_\nabla\colon \Gamma(E \otimes S^+) \to \Gamma(E \otimes S^-)$, and $D^-_\nabla = (D^+_\nabla)^\dagger$.

When $E \otimes S$ is a G-bundle and the Dirac operator is a G-operator, Theorem 4.4.1 applied with $P = D$ is typically called the *G-spin theorem*[3]; in this thesis, it is referred to as the *equivariant spin theorem.*

[2] Also known as the Atiyah–Segal–Singer fixed point formula

[3] A somewhat unwieldy name, if only for the fact that it ostensibly depends upon the group for which the theorem is being applied (although sometimes one ignores it by an abuse of notation). Restricting to cyclic subgroups is an important technique in applications of the theorem, and keeping track of this data in the name quickly becomes unappealing—thus a different name has been chosen for consistency with the equivariant index theorem and notational ease.

Standard texts like [25], [41] cover primarily the case of spinor bundles without coefficients, although the extension of the equivariant spin theorem to coefficients is well-known. The formula suggested for this case in the discussion following [25, Remark III.14.12] contains a typographical error: it would contain a Chern character where there should be an equivariant Chern character.

Let $g \in G$. If $x \in X^g$, then $gx = x$ and so $\mathrm{d}g \colon T_xX \to T_xX$ is a linear isomorphism of vector spaces. It restricts to a map $T_xX^g \to T_xX^g$, and thus $\mathrm{d}g$ induces a map of normal spaces $N_xX^g \to N_xX^g$. By a straightforward computation, the normal bundle is always trivial when $X = S^4$ and $g \in \mathrm{SO}(4)$; as a *real* representation, each fibre is a sum of representations in which $\mathrm{d}g$ acts by rotations through angles $\theta_1, \ldots, \theta_m$ with $0 < \theta_i \leq \pi$. *A priori*, these angles are between 0 and 2π, but as real representations, rotations by θ and $-\theta$ are equivalent. As well, one proves that the angle cannot be 0, or else the exponential map along a normal direction would be a curve fixed by the action of g; see [25, p. 263]. Write $N_g := NX^g = N_g(\theta_1) \oplus \cdots \oplus N_g(\theta_m)$ to denote this decomposition as a real representation.

So long as none of the θ_i in the decomposition of N_g are equal to π, it is possible to use the $\hat{\mathbf{A}}_\theta$-classes to state the equivariant spin theorem.[4]

Theorem 4.4.2 (The equivariant spin theorem)**.** *Let G be a group acting on a bundle E and commuting with the Dirac operator $D^+_\nabla \colon \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$. Then for all $g \in G$, there exists a locally constant integer-valued function d on the connected components of X^g such that*

$$\mathrm{ind}_g(D^+_\nabla) = \sum_Y (-1)^{d(Y)} \left\{ \mathrm{ch}_g(E) \prod_{0<\theta<\pi} \hat{\mathbf{A}}_\theta(N_g(\theta)) \hat{\mathbf{A}}(TY) \right\} [Y]; \tag{4.5}$$

the sum ranges over all connected components Y of the fixed-point manifold X^g.

Proof. The proof of this statement requires a number of standard but technical lemmas about the structure of the spinor bundles not relevant to the present work, and so has been relegated to Appendix D. □

4.4.2 Specializations of the equivariant spin theorem

Let r be a simple rotation in $\mathrm{SO}(4)$ by an angle θ, and lift it to an element $g \in \mathrm{Spin}(4)$ generating a cyclic subgroup $\langle g \rangle$ acting on $X = S^4$. Then $X^g = S^2$, and the normal bundle is rank two, with g acting on it as rotation by θ; the decomposition is $N_g = N_g(\theta)$.

Furthermore, N_g is a trivial S^2-bundle. To see this fact, consider the standard embedding $S^4 \subseteq \mathbb{R}^5$, and by a suitable choice of coordinates identify the copy of S^2 with the points $(p_1, p_2, p_3, 0, 0) \in S^4$. For any point $p \in S^2$, the vectors $(0, 0, 0, 1, 0), (0, 0, 0, 0, 1)$ at p span a two-dimensional subset of the tangent space to S^4 and are clearly orthogonal to the tangent space of S^2, and thus span $(N_g)_p$. Taking this frame over every point $p \in S^2$ gives $N_g = S^2 \times \mathbb{R}^2$ as vector bundles.

[4] Such cases are said to be *good*; one can also define a class $\hat{\mathbf{A}}_\pi$ to handle the non-good case, but determining the sign becomes more complicated. In any case, this case is only relevant if one wishes to compute the index of $-1 \in \mathrm{Spin}(4)$, but in the setting of this thesis, this computation can be done by the usual index theorem, since in this setting $\mathrm{Spin}(4)$ acts by projection onto $\mathrm{SO}(4)$ and thus -1 acts as the identity.

Thus $c_1(N_g) = 0$, and so to compute $\hat{\mathbf{A}}_\theta(N_g)$ we expand a power series for $[2\sinh(x + i\theta)/2]^{-1}$ and evaluate at $x = 0$:

$$\hat{\mathbf{A}}_\theta(N_g) = -\frac{\sqrt{-1}}{2}\csc(\theta/2).$$

Suppose that the connection ∇ has ASD curvature. Then the Lichnerowicz formula for the Dirac operator is given in terms of the rough Laplacian $\nabla^*\nabla$ and the scalar curvature κ as

$$D^-_\nabla D^+_\nabla = \nabla^*\nabla + \frac{\kappa}{4}.$$

Now by Bochner's theorem (see, for example, [39, Theorem 3.10]), the positive scalar curvature implies that $\ker D^-_\nabla D^+_\nabla = 0$, and it follows (since $D^-_\nabla = (D^+_\nabla)^*$ and X is compact) that $\ker D^+_\nabla = 0$. Thus $V := -\mathrm{ind} D^+_\nabla = \mathrm{coker} D^-_\nabla$ is a vector space and representation of G, and $\chi_V(g) = -\mathrm{ind}_g D^+_\nabla$.

Let $\zeta = \exp(\sqrt{-1}\theta/2)$, and for an integer m let V_m be the representation $\langle g\rangle \to \mathbb{C}^*$ sending $g \mapsto \zeta^m$. For $i = 0, \ldots, |g|$, let V_i be the one-dimensional representation of $\langle g\rangle$ given by the map $G \to \mathrm{GL}(\mathbb{C}, 1) = \mathbb{C}^*$ sending $g \mapsto \zeta^i$. By Theorem 4.4.2, with $d = d(S^2)$ then there exist integers $m_1, \ldots, m_k$ so that

$$\begin{aligned}
\chi_V(g) &= -\left((-1)^d \mathrm{ch}_g(E)\hat{\mathbf{A}}_\theta(N_g(\theta))\right)[S^2] \\
&= \frac{(-1)^{d+1}\sqrt{-1}}{2}\csc(\theta/2)\mathrm{ch}_g(E)[S^2] \\
&= \frac{(-1)^{d+1}\sqrt{-1}}{2}\csc(\theta/2)\left(\chi_{E_0}(g) + \sum_{i=1}^{k}\chi_{V_{m_i}}(g)\left(\sum_{j=1}^{n_k} n_{ij}\right) x\right)[S^2] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Equation (4.4))} \\
&= \frac{(-1)^{d+1}\sqrt{-1}}{2}\csc(\theta/2)\sum_{i=1}^{k}\chi_{V_{m_i}}(g)\sum_{j=i}^{a_i} n_{ij}, \\
&\qquad\qquad \text{(since } x[S^2] = 1 \text{ and } c[S^2] = 0 \text{ for every } c \in H^0(S^2;\mathbb{C}))
\end{aligned}$$

Suppose that the rank of E is two.

So restricted to S^2, there exist m_1, m_2 so that $E = \mathcal{O}(n) \otimes V_{m_1} \oplus \mathcal{O}(-n) \otimes V_{m_2}$. Then

$$\chi_V(g) = \frac{(-1)^{d+1}n\sqrt{-1}}{2}\csc(\theta/2)(\chi_{V_{m_1}}(g) - \chi_{V_{m_2}}(g)) = \frac{(-1)^{d+1}n\sqrt{-1}}{2}\csc(\theta/2)(\zeta^{m_1} - \zeta^{m_2}). \tag{4.6}$$

Suppose that the characters of G are all real (this condition holds when G is one of $\widetilde{H_3^+}_\Delta$, $\widetilde{B_3^+}_\Delta$, $\widetilde{H_4^+} = \widetilde{H_3^+} \times \widetilde{H_3^+}$, $\widetilde{B_4^+} = \widetilde{B_3^+} \times \widetilde{B_3^+}$, for example). Then it must be the case that $\zeta^{m_1} - \zeta^{m_2}$ is pure imaginary, so $m_2 = -m_1$. Then

$$\zeta^{m_1} - \zeta^{m_2} = 2\sqrt{-1}\Im(\zeta^{m_1}) = 2\sqrt{-1}\sin(m_1\theta/2). \tag{4.7}$$

The computation of d can be carried out by the method described in [2, p. 20]; much of the work simplifies because the normal bundle has complex rank 1, and the element

$g \in \mathrm{Spin}(4)$ acts on the spinors by Clifford multiplication; writing $g = \cos(\theta/2) + \sin(\theta/2)e_1e_2$ in an appropriate oriented orthonormal basis, Equation (9) on [2, p. 20] becomes

$$(-1)^d 4\cos(\theta/2) = \mathrm{tr}\,(g|_S)\,,$$

and by computing directly the right-hand side one obtains that $(-1)^d = 1$.

Finally, suppose that the representation W is complex, so -1 acts as $-I$. Then since -1 acts as multiplication by -1 on the spinors, it is immediate that $\chi_V(-1) = \chi_V(1)$. So substituting Equation (4.7) into Equation (4.6) yields a formula for the character of V in terms of the character χ_{m_1} of the m_1-dimensional irreducible representation W_{m_1} of SU(2).

$$\chi_V(g) = n\frac{\sin(m_1\theta/2)}{\sin(\theta/2)} = n\chi_{m_1}(\theta/2). \tag{4.8}$$

Remark 4.4.3. For all groups considered (for example, $\mathrm{Spin}(3), \mathrm{Spin}(4), \widetilde{H_3^+}, \widetilde{H_4^+}$) in this thesis, all two-dimensional representations that are not two copies of the trivial representation are complex. This hypothesis may not be satisfied automatically for other groups, and so (4.8) may not be applicable.

When $N = 3$, the irreducible representations are real instead. As a consequence, one can derive that for one of the aforementioned groups, a G-symmetric instanton (in the sense of Chapter 5) for $N = 3$ must have even charge.

Another specialization is in the case where g is a double rotation with angles θ, ϕ and E is of any rank. An analogous computation shows that (see also [17, §4.5.2] for a derivation of the same formula)

$$\chi_V(g) = \pm\frac{1}{4}\csc(\theta/2)\csc(\phi/2)(\chi_0(g) \pm \chi_\infty(g)), \tag{4.9}$$

where $\chi_0(g)$ is the trace of g as it acts on the fibre E_0 of the origin, and $\chi_\infty(g)$ is the trace of g as it acts on the fibre over the point at infinity. The precise choices of sign to be taken in this expression can in principle be computed in the same way as the simple rotation case. These computations are not done in this thesis; Equation (4.9) with its four possible choices of signs is sharp enough for the results presented.

Chapter 5

Symmetric instantons

5.1 Instantons

Let M be a four-dimensional oriented Riemannian manifold, and let $E \to M$ be a complex bundle of rank N with a complex orientation and Hermitian fibre metric, so its structure group is $\mathrm{SU}(N)$. Then $c_1(E) = 0$, and so E is classified topologically by its rank N and its second Chern class, $c_2(E) \in H^4(M;\mathbb{Z})$. A connection A on E is called an *instanton* if it is a minimizer for the Yang–Mills energy functional

$$\mathrm{YM}(A) = \int_M \|F_A\|^2 \mathrm{vol}_M.$$

Given a choice of frame A is locally a $\mathfrak{su}(N)$-valued one-form, and F_A takes the form $F_A = \mathrm{d}A + A \wedge A$.

Let d_A^* be the formal adjoint to the exterior covariant derivative d_A. By a variational argument, the Euler–Lagrange equation for this functional is $\mathrm{d}_A^* F_A = 0$. This non-linear PDE in A is called the Yang–Mills equation.

On forms of pure degree, $\mathrm{d}_A^* = \pm \star \mathrm{d}_A \star$ for some choice of sign depending on the degree and dimension of the base manifold. Thus if $\star F_A = \pm F_A$, it follows from the Bianchi identity $\mathrm{d}_A F_A = 0$ that A is a solution to the Yang–Mills equation. Such solutions are called the self-dual (SD) and anti-self-dual (ASD) connections, respectively as $\star F_A = F_A$ or $\star F_A = -F_A$.

If the base manifold M is equal to $\mathbb{R}^4$, then the minimizers for the Yang–Mills functional are precisely the SD and ASD connections. An *(ASD) instanton* is defined to be an ASD connection A on E with $\mathrm{YM}(A) < \infty$. By a theorem of Uhlenbeck [46], these data are conformally equivalent to solutions to the Yang–Mills equation on S^4.

The purpose of this chapter is to develop a theory of symmetric instantons—first, a more general theory of symmetric connections must be developed.

5.2 Symmetric connections

As in Section 4.2, a vector bundle with a smooth action of a compact Lie group G acting as a linear isomorphism on fibres is called a *G-bundle*.[1] There is a natural action of G on $\Gamma(E)$, given by $(g\sigma)(x) = g \cdot \sigma(g^{-1}x)$ whenever $\sigma \in \Gamma(E)$ and $x \in M$. There is also an action of G on the tangent bundle given by the pushforward.

Definition 5.2.1. A connection ∇ on E is said to be (G-)*symmetric* if $g(\nabla\sigma) = \nabla(g\sigma)$ for all $\sigma \in \Gamma(E)$.

Remark 5.2.2. If ∇ is a G-symmetric connection on E, $X \in \Gamma(TM)$ is a vector field on M, and $\sigma \in \Gamma(E)$ is a section of E, then at each $p \in M$,

$$g(\nabla_{X_p}\sigma) = \nabla_{(\mathrm{d}g)_p X_p}(g\sigma).$$

Proposition 5.2.3. *Let $(E, \nabla) \to M$ be a G-bundle with connection. The following are equivalent.*

1. *The connection ∇ is G-symmetric.*
2. *For all piecewise smooth paths $\gamma\colon x \to y$ in M and all $v \in E_x$, if Π_γ denotes the parallel transport along γ determined by ∇, then*
$$g\Pi_\gamma(v) = \Pi_{g\gamma}(gv).$$
3. *The horizontal distribution $H \subset TE$ determined by ∇ is G-invariant in the sense that $\mathrm{d}g\colon TE \to TE$ sends H to H.*

Proof. The equivalence of (2) and (3) follows from the chain rule: if $\gamma\colon x \to y$ is a piecewise smooth path in M with $\gamma(0) = x, \gamma(1) = y$ and $v \in E_x$, let $P_{\gamma,t}\colon E_x \to E_{\gamma(t)}$ denote the parallel transport along γ for time t, and let $\widehat{\gamma}$ denote the horizontal lift $\widehat{\gamma}(t) = P_{\gamma,t}(v)$. Then $\gamma'(0) \in H_v$ is a horizontal vector.

The chain rule gives $(g\gamma)'(0) = (\mathrm{d}g)(\gamma'(0))$. Assuming (2), then

$$\widehat{g\gamma}(t) = P_{g\gamma,t}(gv) = gP_{\gamma,t}(v),$$

whence $\widehat{g\gamma}'(0) = (\mathrm{d}g)\widehat{\gamma}'(0)$, so g maps $H \to H$. That (3) implies (2) follows from the uniqueness of the horizontal lift; $g\Pi_\gamma(v)$ is a horizontal lift for $g\gamma$ starting at gv, and so $g\Pi_\gamma(v) = \Pi_{g\gamma}(gv)$.

To see that (2) implies (1), use

$$\nabla_{\gamma'(0)}\sigma = \lim_{t\to 0} \frac{P_{\gamma,t}^{-1}(\sigma_{\gamma(t)}) - P_{\gamma,0}^{-1}(\sigma_{\gamma(0)})}{t}$$

[1] This notation is standard but unfortunate, because in the instanton setting for example we require that E be a 'SU(N)-bundle' in a completely different sense of the word [that is, that the associated principal bundle is equipped with a structure group reduction to SU(N)]. A particularly pernicious case is that of a rank two Hermitian bundle with complex orientation and an action of SU(2)—an SU(2)-SU(2)-bundle! I remedy this doublespeak by promising to henceforth mention explicitly if 'G-bundle' means a structure group reduction to G; otherwise, it should be assumed to be the structure of a G-action with the previously mentioned conditions.

$$
\begin{aligned}
&= \lim_{t\to 0} \frac{gP^{-1}_{g^{-1}\gamma,t}(g^{-1}\sigma_{\gamma(t)}) - gP^{-1}_{g^{-1}\gamma,0}(g^{-1}\sigma_{\gamma(0)})}{t} \\
&= g\left(\lim_{t\to 0} \frac{P^{-1}_{g^{-1}\gamma,t}(g^{-1}\sigma_{\gamma(t)}) - P^{-1}_{g^{-1}\gamma,0}(g^{-1}\sigma_{\gamma(0)})}{t}\right) \\
&= g\left(\nabla_{(g^{-1}\gamma)'(0)}g^{-1}\sigma\right),
\end{aligned}
$$

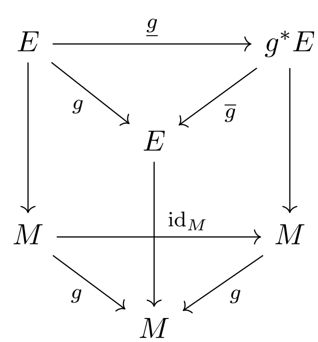
which is true for all γ and so $\nabla = g\nabla g^{-1}$. Likewise, one uses this formula to show that (1) implies (2). □

Proposition 5.2.4. *If (E,∇) and (E',∇') are two G-symmetric connections, then so are $(E\oplus E',\nabla\oplus\nabla')$ and $(E\otimes E',\nabla\otimes\nabla')$ with the induced G-bundle structure.*

Proof. Trivial. □

Remark 5.2.5. By using a gauge transformation that does not commute with the G-action on E, one can transform a connection that is G-symmetric to a connection that is not.

Categorically, the definitions of symmetric connection given thus far are disappointing: to define a 'symmetric connection', one must leave the category of vector bundles over a fixed base (with maps preserving the base). The issue is that the action by $g\in G$ covers the map $g\colon M\to M$ instead of the identity. The solution is to forget the action of G on E, and think instead of G only as a group of diffeomorphisms of M; then one may take the pullback (an isomorphism, since G acts by diffeomorphisms) and recover a map over the identity. One has for each $g\in G$ a commutative diagram



defining $\underline{g}$ and $\overline{g}$.

There are two natural connections one can put on g^*E. The first is the pullback connection $g^*\nabla$; the second is the connection $\underline{g}\nabla\underline{g}^{-1}$ induced by the isomorphism.

Proposition 5.2.6. *The connection ∇ on a G-bundle E is symmetric if and only if $g^*\nabla$ and $\underline{g}\nabla\underline{g}^{-1}$ agree for all $g\in G$.*

Proof. Let $g\in G$ be arbitrary. Recall that $g^*\nabla$ is the unique connection on g^*E with the property that $(g^*\nabla)(g^*\sigma) = g^*(\nabla\sigma)$ for all $\sigma\in\Gamma(E)$. A direct computation (which is also visible from the diagram by taking sections and reversing arrows in the top triangle) shows that $\underline{g}^{-1}g^*\sigma = g^{-1}\sigma$. So if ∇ is G-symmetric, then

$$\underline{g}\nabla\underline{g}^{-1}g^*\sigma = \underline{g}\nabla g^{-1}\sigma$$

$$\begin{aligned} &= \underline{g}g^{-1}\nabla\sigma \\ &= g^*\nabla\sigma, \end{aligned}$$

where the last step is again either a computation or visible from the diagram. Since this holds for all σ, it follows that $g^*\nabla = \underline{g}\nabla\underline{g}^{-1}$.

Conversely, if $g^*\nabla = \underline{g}\nabla\underline{g}^{-1}$, then

$$(g^*\nabla)(g^*\sigma) = \underline{g}\nabla\underline{g}^{-1}g^*\sigma = \underline{g}\nabla g^{-1}\sigma,$$

but also

$$(g^*\nabla)(g^*\sigma) = g^*\nabla\sigma = \underline{g}g^{-1}\nabla\sigma.$$

Since $\underline{g}$ is an isomorphism of vector bundles and this equation holds for all $\sigma \in \Gamma(E)$, it follows that $g^{-1}\nabla = \nabla g^{-1}$ for all $g \in G$, and thus ∇ is G-symmetric. □

Thus there is a categorical definition of a symmetric connection: let $\mathcal{C}(G, M)$ be the category whose objects are (complex) smooth G-bundles over M equipped with a connection ∇, and for which a morphism $f\colon (E, \nabla) \to (E', \nabla')$ is a map $E \to E'$ of vector bundles (that is, linear on fibres and commuting with the projection) such that $f\nabla = \nabla' f$.

Corollary 5.2.7. *An element $(E, \nabla) \in \mathcal{C}(G, M)$ is a symmetric connection if and only if the maps $\underline{g} : (E, \nabla) \to (g^*E, g^*\nabla)$ are morphisms in $\mathcal{C}(G, M)$ for all $g \in G$.*

5.3 Spinors and Dirac operators

For the symmetric instanton problem on $\mathbb{R}^4$, the goal is to ultimately analyze the action on the index of the Dirac operator. The index of the Dirac operator is not generally a virtual representation of G, so for the purposes of this thesis hypotheses are introduced to ensure the Dirac operator is a G-operator; this is natural for instantons on $\mathbb{R}^4$ from the perspective of the ADHM data, but is not necessarily needed for a general definition of 'symmetric instanton'.

When G acts on a spin manifold M by orientation-preserving isometries, say the action is *spin* if it lifts to an action on the spinors $S^{\pm}$. Moreover, say the action is *spin-symmetric* if both (1) the connections on $S^{\pm}$ induced from the Levi-Civita connection on TM are G-symmetric **and** (2) Clifford multiplication is G-equivariant. The condition (2) means that for all tangent vectors v and spinors s at the same point, $((\mathrm{d}g)v) \cdot (gs) = g(v \cdot s)$ (see [6, §4.2.3]).

Definition 5.3.1. A *G-symmetric instanton* is an instanton $(E, \nabla) \to M$ with curvature of finite L^2 norm for which E has the structure of a G-bundle, the action of G on M is spin and spin-symmetric, and ∇ is a G-symmetric connection.

Proposition 5.3.2. *Suppose that the action of G on M is spin and spin-symmetric. Then the Dirac operator D on $S \otimes E$ is G-equivariant.*

Proof. The Dirac operator is the contraction of the tensor connection on $S \otimes E$ with the metric and Clifford multiplication. The tensor connection is symmetric, so the result follows from the facts that G acts by isometries and is compatible with Clifford multiplication. □

In general, it may not be easy to prove that an action is spin-symmetric, or even spin. When M has a unique spin structure and G is finite, we have the following.

Proposition 5.3.3. *Suppose that G is finite and acts by orientation-preserving isometries on a manifold M with unique spin structure. Then the action of G is spin.*

Proof. A spin structure is described by a class in cohomology of the oriented orthonormal frame bundle, and a finite group preserves it if and only if this class is invariant under pullbacks; see [41, pp. 162–163]. But the spin structure is unique. □

Proposition 5.3.4. *Suppose that $M = \mathbb{R}^4$ with its usual metric and Levi-Civita connection, as well as a choice of orientation. If G has a spin action on M and acts as orientation-preserving isometries of the vector space M (that is, the action is given by an embedding $G \to \mathrm{SO}(4)$), then the action of G is spin-symmetric.*

Proof. Since each element of G acts as multiplication by some matrix in $\mathrm{SO}(4)$, it is immediate by (real) linearity of ∇ that $\nabla g = g\nabla$ for all $g \in G$.

The next claim is that the G-action is compatible with Clifford multiplication. Note that there is a natural G-action on the Clifford bundle. This comes from extending the action on the tangent bundle to the tensor algebra of the tangent bundle, and then descending to the Clifford algebra. This descent is possible by the fact that G preserves the metric and hence the ideal defining the Clifford algebra. Since the action of G on $TM \subset \mathrm{Cl}(M)$ by pushforward is the action induced from the Clifford bundle, we rewrite the equivariance condition for $v \in T_xM, s \in S_x$ as

$$g(v \cdot s) = (gv) \cdot (gs).$$

This action descends from the action on the tensor algebra, and so $g(v \otimes s) = (gv) \otimes (gs)$ descends to the displayed statement under the map $\bigoplus_{k=0}^{\infty}(TM)^{\otimes k} \to \mathrm{Cl}(M)$.

Also note that since G is orientation preserving, the volume form, and hence the split into positive and negative spinors, is preserved. □

Corollary 5.3.5. *Let $(E, \nabla) \to \mathbb{R}^4$ be an oriented real G-bundle with symmetric connection for a finite group G that acts on $\mathbb{R}^4$ by orientation-preserving isometries. Then the action is spin and spin-symmetric, and in particular the Dirac operator is G-equivariant.*

Proof. Immediate from Proposition 5.3.3 and Proposition 5.3.4. □

Remark 5.3.6. If $M = \mathbb{R}^4$ and G is a subgroup of $\mathrm{SO}(4)$, then the double cover $\widetilde{G} \subseteq \mathrm{Spin}(4)$ already acts naturally on spinors $S^{\pm}$ by Clifford multiplication, and this action of $\widetilde{G}$ is spin. Moreover, this action is a lift of the natural action of $\mathrm{SO}(4)$ on TM.

Instantons on $\mathbb{R}^4$ with curvature of finite L^2 norm are conformally equivalent to instantons on S^4 by stereographic projection. The conformal factor for stereographic projection is $\mathrm{SO}(4)$-invariant, whence the following result.

Proposition 5.3.7. *If G acts on $\mathbb{R}^4$ by isometries, the S^4-instanton corresponding to a G-symmetric $\mathbb{R}^4$-instanton is again G-symmetric when the action of G is continued to the fibre over infinity.*

The action on the fibre over infinity is defined by the identification of E_∞ with the space of bounded harmonic sections of E (see [14, Lemma (3.3.22)]).

5.4 Descent to ADHM data

Let $(E, \nabla) \to \mathbb{R}^4$ be a rank N complex vector bundle over E with structure group $\mathrm{SU}(N)$, and with ∇ a G-symmetric instanton connection for G a finite subgroup of $\mathrm{Spin}(4)$ lifted from a subgroup of $\mathrm{SO}(4)$.

Lemma 5.4.1. *Suppose that ∇ is ASD with finite L^2 energy. The vector space $V := \ker D^-_\nabla = -\mathrm{ind}(D_\nabla)$ is a finite-dimensional representation of G; its character is computed by the equivariant spin theorem.*

Proof. By Uhlenbeck's theorem [46], ∇ is conformally equivalent to an instanton connection on S^4. By the discussion in Section 4.4.2, the ASD condition and the Lichnerowicz formula imply that the Dirac operator $D^+_{S^4,\nabla}$ has trivial kernel.

The conformal factor for stereographic projection is, up to a constant, $(|x|^2+1)^{-2}$; see, for example, [26, Equation (3.10)]. Thus it has the form $\exp(2f)$ for $f = -\log(|x|^2+1)$. By the formula for change of Dirac operators under conformal change of metric (see [8, Appendix D] for the formula and some history), then

$$D^+_{S^4,\nabla} = \exp\left(-\frac{5}{2}f\right) D^+_\nabla \exp\left(\frac{3}{2}f\right) = (|x|^2+1)^{5/2} D^+_\nabla (|x|^2+1)^{-3/2}.$$

The map sending the kernel of $D_{S^4,\nabla}$ to sections over $\mathbb{R}^4$ by multiplication by $(|x|^2+1)^{-3/2}$ lands in the kernel of D_∇. Moreover, the image of this map is L^2 because sections over S^4 (a compact manifold) are bounded, and $(|x|^2+1)^{-3/2}$ is L^2 on $\mathbb{R}^4$; it is dominated by $|x|^{-3}$.

The inverse map from $\ker D_\nabla \cap L^2$ to sections of S^4 is multiplication by $(|x|^2+1)^{3/2}$. This map is well-defined because elements σ of $\ker D_\nabla$ decay at least as fast as $|x|^{-3}$ and so $(|x|^2+1)^{3/2}\sigma$ is bounded at infinity. By the conformal change formula, the image of this map is in $\ker D_{S^4,\nabla}$. It follows that $\ker D^+_\nabla \cap L^2 \cong \ker D^+_{S^4,\nabla} = 0$.

Since the index of D is finite, $\ker D^-_\nabla$ is finite dimensional, and the same map of multiplication by $(|x|^2+1)^{-3/2}$ is an isomorphism $\ker D^-_\nabla \cap L^2 \cong \ker D^-_{S^4,\nabla}$ of vector spaces. Moreover, since D^-_∇ is equivariant with respect to G, $\ker D^-_\nabla$ is a representation. Since the conformal factor is preserved by G, as it depends only on $|x|^2$, this isomorphism is actually an equivalence of representations. Thus the representation $\ker D^-_\nabla$ is the one computed by the equivariant spin theorem. □

We now have sufficient resources to discuss the ADHM construction for a symmetric connection. This construction takes many different forms: we work through them all.

5.4.1 Donaldson–Kronheimer ADHM data

We very briefly recall the theory required to state the ADHM correspondence: see [14, §3] for a proper coverage of this topic.

Let Λ^+ be the three-dimensional space of self-dual two-forms on $\mathbb{R}^4$, and let $S^\pm$ be the chiral spinor representations (which are isomorphic to $\mathbb{H}$). Let $(e_1, \ldots, e_4)$ be the standard basis of $\mathbb{R}^4$, and let $c(e_i)\colon S^+ \to S^-$ denote Clifford multiplication by e_i. Let $c(e_i)^*$ be the adjoint of Clifford multiplication (which of course is also Clifford multiplication).

The Donaldson–Kronheimer description of ADHM data (see [14, §3.3.3]) consists of four pieces of data:

1. A k-dimensional Hermitian vector space V.
2. An N-dimensional Hermitian vector space W.
3. Self-adjoint maps $T_1, \ldots, T_4 \colon V \to V$.
4. A linear map $P \colon W \to V \otimes S^+$.

These data define for each $x \in \mathbb{R}^4$ a map

$$R_x \colon V \otimes S^- \oplus W \to V \otimes S^+$$

by

$$R_x = \sum (T_i - x_i \mathrm{Id}) \otimes c(e_i) \oplus P.$$

Note that $PP^* \in \mathrm{End}(V \otimes S^+) = \mathrm{End}(V) \otimes \mathrm{End}(S^+)$. Then $\mathrm{End}(S^+)$ contains a summand of $\mathfrak{su}(S^+)$, which is identified with Λ^+. Write $(PP^*)_{\Lambda^+}$ for this part of PP^*. The (closed) ADHM equation is that

$$[T, T]^+ = (PP^*)_{\Lambda^+}.$$

Additionally, one imposes an *open* condition, which is that R_x is surjective for all x. When both the open and closed condition are met, we say that (T, P) are *ADHM data*.

Definition 5.4.2. Suppose there is a spin action of G on $\mathbb{R}^4$. Say that ADHM data (T, P) are *G-symmetric* if there exists a representation $U \colon G \to \mathrm{U}(V)$ and a representation $\rho \colon G \to \mathrm{SU}(W)$ such that

$$R_{gx} \circ (U(g) \otimes g \oplus \rho(g)) = (U(g) \otimes g) \circ R_x \tag{5.1}$$

for all $g \in G$.

Remark 5.4.3. The 'g' appearing in the previous definition as in $U(g) \otimes g$ means the action of Clifford multiplication on spinors. One may also write $c(g)$ to emphasize this Clifford multiplication, so that (5.1) instead reads

$$R_{gx} \circ (U(g) \otimes c(g) \oplus \rho(g)) = (U(g) \otimes c(g)) \circ R_x.$$

In the sequel, both notations are used as is convenient.

Theorem 5.4.4. *There is a one-to-one correspondence between G-symmetric instantons on $\mathbb{R}^4$ modulo gauge transformations commuting with the G-action and G-symmetric ADHM data up to the following equivalence relation: (T_i, P) is considered to be equivalent to (T_i', P') when there exist $A \in \mathrm{U}(V), B \in \mathrm{SU}(W)$ commuting with the G-action such that*

$$T_i' = AT_iA^{-1}, P' = APB^{-1}.$$

More precisely, the last statement means that we consider the set of rotations (A, B) of the ADHM data where $U(g)AU(g)^{-1} = A$ and $\rho(g)B\rho(g)^{-1} = B$ for all $g \in G$. Clearly this set is a group; denote it by $(\mathrm{U}(V) \times \mathrm{SU}(W))^G$, leaving implicit the representations U, ρ, and let $(A, B) \cdot (T, P)$ denote the action thus described.

Most of the work in this theorem is in establishing the bijectiveness of the map from ADHM data modulo $U(V) \times \mathrm{SU}(W)$ to instantons modulo gauge transformations when the G-symmetry conditions are ignored. This part is a hard result called the ADHM correspondence relying on lots of analysis; the interested reader may check the survey [15] or a proof in [14, Theorem 3.3.8] or [35, 11]. The ADHM correspondence was first proved in [3] and generalized by Nahm; see surveys in [8, Ch. 2] and [23].

With this established, we need only show the following statements:

1. A G-symmetric instanton is sent to G-symmetric ADHM data.

2. G-symmetric ADHM data is sent to a G-symmetric instanton.

These statements are proved in Proposition 5.4.7 and Proposition 5.4.8.

Lemma 5.4.5. *Let $g \in \mathrm{Spin}(4)$, and let $\overline{g} \in \mathrm{SO}(4)$ be its image under the double covering map. Then for any $1 \leq i \leq 4$,*

$$c(e_i)^* c(g) = \sum_k c(g)\overline{g}_{ki} c(e_k)^*.$$

Proof. By Remark 5.3.6, the action of $\langle g \rangle$ on spinors by Clifford multiplication is spin-symmetric, and so

$$c(g)c(e_i) = c(\overline{g}e_i)c(g) = \sum_j \overline{g}_{ij} c(e_j)c(g). \tag{5.2}$$

Since Clifford multiplication by unit vectors is orthogonal and $g \in \mathrm{Spin}(4)$, then $c(g)^* = c(g^t) = c(g^{-1})$. For any $w, w' \in S^+$,

$$\begin{aligned}
\langle c(e_i)^* c(g) w, w' \rangle &= \langle c(g)w, c(e_i)w' \rangle \\
&= \langle w, c(g)^* c(e_i) w' \rangle \\
&= \langle w, c(g^{-1}) c(e_i) w' \rangle \\
&= \langle w, c(\overline{g^{-1}} e_i) c(g^{-1}) w' \rangle && \text{(by (5.2))} \\
&= \sum_j (\overline{g}^{-1})_{ij} \langle w, c(e_j) c(g^{-1}) w' \rangle \\
&= \sum_j \overline{g}_{ji} \langle w, c(e_j) c(g^{-1}) w' \rangle && \text{(since } \overline{g} \in \mathrm{SO}(4)) \\
&= \sum_j \overline{g}_{ji} \langle c(g^{-1})^* c(e_j)^* w, w' \rangle \\
&= \sum_j \langle \overline{g}_{ji} c(g) c(e_j)^* w, w' \rangle,
\end{aligned}$$

whence the result. ∎

Let m_i be the endomorphism of $\Gamma(S^- \otimes E)$ defined by

$$m_i(\sigma)(x) = x_i \sigma(x).$$

Lemma 5.4.6. *Let $g \in \mathrm{Spin}(4)$, and let $\overline{g} \in \mathrm{SO}(4)$ be its image under the double covering map. Then $m_i(g\sigma) = \sum_j \overline{g}_{ij} g m_j(\sigma)$.*

Proof. Recall that $(g\sigma)(\overline{g}x) = g(\sigma(\overline{g}^{-1}\overline{g}x) = g(\sigma(x))$. At $x \in \mathbb{R}^4$, one has

$$\begin{aligned} m_i(g\sigma)(\overline{g}x) &= (\overline{g}x)_i(g\sigma)(\overline{g}x) \\ &= \sum_j \overline{g}_{ij} x_j g\sigma(x) \\ &= \sum_j \overline{g}_{ij} g x_j \sigma(x) \\ &= \sum_j \overline{g}_{ij} g\left((m_j(\sigma))(x)\right) \\ &= \sum_j \overline{g}_{ij} (g m_j(\sigma))(\overline{g}x). \end{aligned}$$

The proof is complete. □

Proposition 5.4.7 (Instanton → ADHM)**.** *Suppose $G \subseteq \mathrm{Spin}(4)$ acts on $\mathbb{R}^4$ via the double cover map $\mathrm{Spin}(4) \to \mathrm{SO}(4)$. Then the ADHM construction sends G-symmetric instantons to G-symmetric ADHM data.*

Proof. Let (T, P) be the ADHM data associated to a G-symmetric instanton in S^4. Nahm's construction of the ADHM data gives us precise data, revealed through this proof. There is a natural representation on the kernel V of the chiral Dirac operator D_{∇}^{-} as in Section 4.4.2 given by the action $(g\sigma)(x) = g(\sigma(g^{-1}x))$ on sections of E; this representation is U. Likewise, there is a natural representation on the space W of bounded harmonic sections. This representation is ρ. The action of G on S^4 is spin by hypothesis, so it it remains to check whether the maps R_x satisfy the required 'G-equivariance' property.

Let $(e_1, \ldots, e_4)$ be the standard basis of $\mathbb{R}^4$. Let P_V be the L^2-projection onto V, and let $T_i = P_V \circ m_i$. Fix $g \in G$, and let $\overline{g} \in \mathrm{SO}(4)$ be its image under the double cover map $\mathrm{Spin}(4) \to \mathrm{SO}(4)$. Since V is a unitary G-representation, P_V commutes with the G-action, so by Lemma 5.4.6,

$$T_i(U(g)\sigma) = \sum_j \overline{g}_{ij} U(g) T_j(\sigma). \tag{5.3}$$

Likewise, $P \colon W \to V \otimes S^+$ is G-equivariant as a map of representations. First note one can write any element $\phi \in V$ as

$$\phi(x) = |x|^{-4} c(x) \widehat{\phi} + O(|x|^{-4})$$

for some $\widehat{\phi} \in W \otimes S^+$; see [8, §1.2]. Then $F \colon \phi \to \widehat{\phi}/2$ is a map $V \to W \otimes S^+$. Use a complex skew-form ω to identify S^+ with $(S^+)^*$; then define P to be the adjoint to $(\omega \otimes 1) \circ (1 \otimes F) \colon S^+ \otimes V \to W$ (see [8, p. 18]).

By choosing ω to be invariant with respect to G (for example, by using compactness to integrate over the action) it suffices to show that F is G-equivariant. For this, note that G acts by isometries, so

$$(g\phi)(x) = (g\phi)(\overline{g}^{-1}x)$$

$$
\begin{aligned}
&= g(|\overline{g}^{-1}x|^{-4}c(\overline{g}^{-1}x)\widehat{\phi}(\overline{g}^{-1}x) + O(|\overline{g}^{-1}x|^{-4})) \\
&= |x|^{-4}g(c(\overline{g}^{-1}x)\widehat{\phi}(\overline{g}^{-1}x)) + O(|x|^{-4})) && \text{(since } g \text{ is an isometry)} \\
&= |x|^{-4}c(x)g(\widehat{\phi}(\overline{g}^{-1}x)) + O(|x|^{-4}) && \text{(by spin-symmetry)} \\
&= |x|^{-4}c(x)(g\widehat{\phi})(x) + O(|x|^{-4}).
\end{aligned}
$$

Thus $\widehat{g\phi} = g\widehat{\phi} = (\rho(g) \otimes g)\widehat{\phi}$. In other words, the map $F : V \to W \otimes S^+$ is G-equivariant, and by G-invariance of ω, then P is also.

To check that R satisfies the claimed G-equivariance property, we only need to focus on the $S^- \otimes V \to W$ portion of it since P is G-equivariant. One computes for $x \in \mathbb{R}^4, v \in V, s \in S^-$, and $g \in G$ that

$$
\begin{aligned}
\sum_i (T_i(U(g)v) - (gx)_iU(g)v) \otimes c(e_i)^*c(g)s &= \sum_{i,j,k} \left(\overline{g}_{ij}U(g)T_jv - \overline{g}_{ji}x_jU(g)v\right) \otimes c(g)\overline{g}_{ki}c(e_k)^*s \\
& \qquad \text{(by Lemma 5.4.5, (5.3))} \\
&= (U(g) \otimes g) \sum_{i,j,k} \overline{g}_{ij}\overline{g}_{ki}(T_jv - x_jv) \otimes c(e_k)^*s \\
&= (U(g) \otimes g) \sum_{j,k} \delta_{jk}(T_jv - x_jv) \otimes c(e_k)^*s \\
& \qquad \text{(by orthogonality of } \overline{g}\text{)} \\
&= (U(g) \otimes g) \sum_j (T_jv - x_jv) \otimes c(e_j)^*s.
\end{aligned}
$$

This equation, combined with the G-equivariance of P, says exactly that for all $(v \otimes s, w) \in V \otimes S^- \oplus W$,

$$(U(g) \otimes g)R_x(v \otimes s, w) = R_{gx}((U(g) \otimes g)(v \otimes s), \rho(g)w),$$

which was to be shown. □

Proposition 5.4.8 (ADHM → Instanton)**.** *If (T, P) are G-symmetric ADHM data, the instanton $A(T, P)$ it generates are G-symmetric. Two pairs (T, P) and (T', P') of ADHM data generate the same G-symmetric instanton if and only if $(T', P') = (A, B) \cdot (T, P)$ for some $(A, B) \in (\mathrm{U}(V) \times \mathrm{SU}(W))^G$.*

Proof. The bundle and connection in this case are quite simple to describe: the open condition guarantees that the dimension of the kernel of R_x does not depend on x, and so $\ker R_x$ is therefore a sub-bundle of the trivial bundle over $\mathbb{R}^4$ with generic fibre $V \otimes S^+ \oplus W$. There is a natural G-action on this bundle, since there is an action given on $V \otimes S^+ \oplus W$, and if $R_xv = 0$ then $0 = gR_xv = R_{gx}(gv)$ so gv is in the fibre over gx.

To define a connection on this bundle, one uses the trivial connection on the trivial bundle and then projects it to the kernel of R using the metric. Since the action of g on $V \otimes S^+ \oplus W$ preserves the metric and sends the fibre over x to the fibre over gx isomorphically, it follows that it commutes with the projection and hence the connection. So this bundle is G-symmetric.

On the one hand, the usual ADHM construction gives that the gauge-equivalent instantons to $A(T,P)$ are those obtained by a rotation (A,B) with $A \in \mathrm{U}(V), B \in \mathrm{SU}(W)$. But it is evident that these data are only G-symmetric if $(A,B) \in (\mathrm{U}(V) \times \mathrm{SU}(W))^G$. □

To summarize what has been shown, the ADHM construction carries G-symmetric instantons to G-symmetric ADHM data, and G-symmetric ADHM data to G-symmetric instantons. Moreover, a pair of G-symmetric ADHM data corresponds uniquely up to an equivalence class of G-symmetric instantons modulo gauge equivalence commuting with the group action up to a rotation of the data commuting with the representations. Theorem 5.4.4 is now proved. □

5.4.2 Quaternionic form

A particularly special case of the ADHM data is when $N = 2$. In this case, one may make the identification of many spaces with the quaternions: for $W = \mathbb{C}^2 = \mathbb{H}$, and as well $\mathbb{R}^4 = \mathbb{H}$. The ADHM data then take a very compact form, being described by one symmetric $k \times k$ quaternion matrix and one k-dimensional column vector of quaternions, called respectively L and M. Let $\widehat{M} = \begin{bmatrix} L & M \end{bmatrix}$. The ADHM equations are the conditions that $\widehat{M}^\dagger \widehat{M}$ is real and invertible.

This identification engenders a complex structure on $\mathbb{R}^4$, and so reduces the ADHM equations from a complex and real equation (respectively, (3.3.11) and (3.3.12) of [14]) to just a complex one. Accordingly, gauge-equivalence classes of instantons are in correspondence with solutions to this complex equation modulo the action of $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})$ rather than $U(k) \times \mathrm{SU}(2)$. More details on these maps are available in [14, §3.3.2].

Rather than reproving the symmetric ADHM correspondence, we describe how to translate from the Donaldson–Kronheimer ADHM data to this quaternionic form, and the correspondence follows from Theorem 5.4.4.

Clearly the goal is to identify L with the map P, and M with the map $\sum T_i \otimes c(e_i)^*$. To do this, recall that $S^+ = \mathbb{H}$, and so pick a basis (e_1, e_2, e_3, e_4) so that $c(e_1)^* = 1, c(e_2)^* = i, c(e_3)^* = j, c(e_4)^* = k$. As well, pick a bases on V and W allowing one to rewrite R_x as a map $\mathbb{C}^k \otimes \mathbb{C}^2 \oplus \mathbb{C}^2 \to \mathbb{C}^k \otimes \mathbb{H} = \mathbb{H}^k$. Identifying both copies $\mathbb{C}^2$ with $\mathbb{H}$ and then $\mathbb{C}^k \otimes \mathbb{H}$ with $\mathbb{H}^k$ means that $R_x \colon \mathbb{H}^k \oplus \mathbb{H} \to \mathbb{H}^k$, and under this identification one has $P = L$ and $\sum T_i \otimes c(e_i)^* = M$. The map R_x is then $(M - x \otimes I) \oplus L$. Now use the fact that $R_{gx}(gv) = gR_x(v)$ to see that

$$M(U(g) \otimes g) = (U(g) \otimes g)M, \quad L\rho(g) = (U(g) \otimes g)L.$$

One case in which these equations take a nicer form is when G is a subgroup of $\mathrm{Sp}(1) = \mathrm{SU}(2) = \mathrm{Spin}(3)$ and then acts on $\mathbb{H}$ by quaternion multiplication. In this case, each g can be thought of as a unit quaternion, and

$$MU(g)g = U(g)gM, \quad L\rho(g) = U(g)gL$$

make sense as equations of quaternion matrices.

More generally, we must be more careful; G is a subgroup of $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2) = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$, and so an element $g \in G$ is described by a pair (ℓ, r) of unit quaternions.

The action on a quaternion x is then $gx = \ell x r^{-1}$. So the symmetry condition says that

$$MU(g)r = U(g)\ell M, \quad U(g)\ell L r^{-1} = L\rho(g);$$

these equations are the ones used in [1].

Example 5.4.9. Finally we can construct our first examples of symmetric instantons: given k distinct quaternions $x_1, \ldots, x_k$, let $L = (1, \ldots, 1)$, and M be the diagonal matrix with entries $x_1, \ldots, x_k$. Suppose moreover that the action of $G \subset \mathrm{SU}(2)$ on quaternions by conjugation sends the set $\{x_1, \ldots, x_k\}$ to itself. Let $U(g)$ be the inverse of the corresponding permutation matrix, so $gMg^{-1} = U(g)^{-1}MU(g)$; the map U is clearly a representation of G. Multiplying on the left by $U(g)$ and on the right by g and using that $U(g), g$ commute (recall that $U(g)g$ is really $U(g) \otimes g = (U(g) \otimes 1)(1 \otimes g) = (1 \otimes g)(U(g) \otimes 1)$), it follows that $U(g)gM = MU(g)g$, so the first symmetric equation holds.

Take $\rho(g) = g$ to be the action of multiplication by g. Since L is real, $gL = Lg$ for each $g \in G$, so

$$g^{-1}L\rho(g) = g^{-1}Lg = g^{-1}gL = L.$$

As well, since each $U(g)$ is a permutation matrix and L has all entries equal to 1, $U(g)L = L$. Thus $U(g)L = g^{-1}L\rho(g)$, whence $L\rho(g) = gU(g)L = U(g)gL$, with $U(g)$ and g commuting as before.

Thus associated to any set $\{x_1, \ldots, x_k\}$ of distinct quaternions mapped onto itself by the action of G, there is a charge k instanton, which we call the *'t Hooft symmetric instanton* associated to these points. This example is a manifestation of a more general phenomenon explored in Section 5.4.4.

Data of this form can be used to give a number of other examples of G-symmetric ADHM data. Appendix C contains a number of SU(2)-instantons symmetric under the binary icosahedral group $\widetilde{H_3^+}$, given in terms of quaternionic ADHM data.

5.4.3 General matrix form

When $\mathrm{rank}(E) > 2$, one obviously cannot make the identification $W = \mathbb{H}$, but a choice of bases still gives the identification

$$R_x : \mathbb{C}^k \otimes \mathbb{C}^2 \oplus \mathbb{C}^N \to \mathbb{C}^k \otimes \mathbb{C}^2.$$

Here the action of $G \subseteq \mathrm{SU}(2) \times \mathrm{SU}(2)$ on each of these factors is as $U(g)$ on each $\mathbb{C}^k$, as $\rho(g)$ on $\mathbb{C}^N$, and as $(\ell, r) \cdot x = \ell x r^{-1}$ on each $x \in \mathbb{C}^2$, with $\mathbb{C}^2$ identified with $\mathbb{H}$ and SU(2) identified with Sp(1).

Said differently, the ADHM data are given by maps $B_1, B_2 : \mathbb{C}^k \to \mathbb{C}^k$ and $C_1, C_2 : \mathbb{C}^N \to \mathbb{C}^k$, which stitch together to give, for $x = (z_1, z_2) \in \mathbb{R}^4 = \mathbb{C}^2$,

$$R_x = \begin{bmatrix} C_1 & B_1 - z_1 & B_2 - z_2 \\ C_2 & -B_2^\dagger + \overline{z_2} & B_1^\dagger - \overline{z_1} \end{bmatrix}.$$

More commonly, one works with $C_2^\dagger : \mathbb{C}^k \to \mathbb{C}^N$ instead of C_2, and relabels $(I, J) := (C_1, C_2^\dagger)$. It is also common to write $\Delta(x)$ instead of R_x, so

$$\Delta(x) = \begin{bmatrix} I & B_1 - z_1 & B_2 - z_2 \\ J^\dagger & -B_2^\dagger + \overline{z_2} & B_1^\dagger - \overline{z_1} \end{bmatrix}.$$

The ADHM equations are then that the off-diagonal blocks of $\Delta(x)\Delta(x)^\dagger$ vanish, and what remains is invertible, Hermitian, and block-diagonal with two equal blocks. In other words, that there exists a function $M = M(x)$ valued in $k \times k$ invertible Hermitian matrices so that

$$\Delta(x)\Delta(x)^\dagger = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}. \tag{5.4}$$

As in the quaternion case of the previous section, equivalence classes of ADHM data are in correspondence with equivalence classes of solutions to these equations under the action of $\mathrm{GL}(k, \mathbb{C}) \times \mathrm{GL}(N, \mathbb{C})$.

One could also work directly with the complex and real equations [14, (3.3.11), (3.3.12)]

$$[B_1, B_2] + IJ = 0 \tag{5.5}$$

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0 \tag{5.6}$$

modulo the action of $U(k) \times \mathrm{SU}(N)$ described in the statement of Theorem 5.4.4.

Analogous to the quaternion case, writing C and B for the two parts of Δ (so that $\Delta(x) = \begin{bmatrix} C & B - x \otimes I_k \end{bmatrix}$) one obtains the equations

$$(U(g) \otimes \ell)B = B(U(g) \otimes r), \tag{5.7}$$

$$(U(g) \otimes \ell)C = C(\rho(g) \otimes r), \tag{5.8}$$

or equivalently

$$\Delta(gx) = (U(g) \otimes \ell)\Delta(x) \left(\begin{bmatrix} \rho(g) & 0 \\ 0 & U(g) \end{bmatrix} \otimes r \right)^{-1}. \tag{5.9}$$

In these equations, $\otimes$ denotes the Kronecker product of matrices.

5.4.4 The symmetric harmonic function ansatz

Appendix B details the harmonic function ansatz; this method can be used to generate a large subspace of the moduli space of SU(2) instantons of a particular charge. What is interesting about the harmonic function ansatz is that it linearizes the ADHM equations: for if ρ_1, ρ_2 are two positive harmonic functions, then so is $\rho_1 + \rho_2$, and one therefore obtains a new SU(2)-instanton as the sum of these two. Then, given a harmonic function ρ, there is a canonical way to symmetrize it under a compact group $G \subset \mathrm{Spin}(4)$: one takes

$$\widetilde{\rho}(x) := \frac{1}{\mathrm{Vol}(G)} \int_G \rho(gx) \mathrm{d}g.$$

Since G acts on $\mathbb{R}^4$ as orientation-preserving isometries, $\widetilde{\rho}$ is positive. If it is still harmonic (for example, if G is finite, or if ρ was already constant on orbits of G), it may be used to generate a new instanton via the harmonic function ansatz; call it $(\widetilde{E}, \widetilde{\nabla})$.

Proposition 5.4.10. *The instanton $(\widetilde{E}, \widetilde{\nabla})$ is G-symmetric.*

Proof. The proof is a direct computation from Theorem B.1.1, using the fact that G preserves $\widetilde{\rho}$. □

One has therefore a *symmetric harmonic function ansatz* (SHFA)[2], which is the map $\rho \mapsto (\widetilde{E}, \widetilde{\nabla})$ from positive harmonic functions on $\mathbb{R}^4$ to G-symmetric ASD instantons with structure group SU(2) on $\mathbb{R}^4$.

This ansatz can be used to construct a large family of symmetric instantons. For example, if $G \subset \mathrm{Spin}(4)$ is finite and $x \in \mathbb{R}^4$, then there exists a G-symmetric instanton corresponding to the 't Hooft and JNR ansatze (Constructions B.1.2 and B.1.3) with singularities placed at points on the orbit of Gx.

The analytic behaviour of the action density of these instantons is a good introduction to the more general theory tackled later: for in this case, the action density may be expressed simply as a negative constant—which is ignored for simplicity—multiplied by $\Delta\Delta \log \rho$. Consider the JNR case (the 't Hooft case is similar): then for $y \in \mathbb{R}^4$ define $\rho(x) := |x-y|^{-2}$. So

$$\widetilde{\rho}(x) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{|gx-y|^2} = \frac{1}{|G|}\sum_{g\in G}\frac{1}{|x-gy|^2}$$

since G acts by isometries. Then by the orbit-stabilizer theorem, up to multiplication by the constant $\frac{1}{|Gy|}$ (which is also suppressed),

$$\widetilde{\rho}(x) = \sum_{z\in Gy}\frac{1}{|z-x|^2}.$$

Let $z \in Gy$ be fixed, and consider

$$c := \sum_{z'\in Gy\setminus z}\frac{1}{|z'-z|^2}.$$

For x close to z, then $\widetilde{\rho}(x)$ is approximately $\frac{1}{|z-x|^2}+c$. Up to a scaling by c, this function is exactly the one giving the basic instanton of charge $k=1$ and centre z via the harmonic function ansatz. Thus close to singularities, the action density looks like that of the basic ASD instanton. In particular, the level sets of the action density near a singularity are approximately spheres.

Figure 5.1 shows level sets of the action density for a charge 11 icosahedral instanton constructed through the SHFA, by letting $\widetilde{G}$ be the binary icosahedral group acting on $\mathbb{R}^4$ fixing an axis, and letting the initial point chosen have icosahedral orbit. It is displayed in three dimensions by showing only a slice perpendicular to a fixed axis. In an imprecise sense, this instanton is 'icosahedral'. More formally, the maxima of the action density $|F|^2$ are precisely the vertices of a regular icosahedron.

5.4.5 A classification of symmetric 't Hooft instantons

The 't Hooft ansatz is a special case of the harmonic function ansatz, with superpotential

$$\rho(x) = 1 + \sum_{i=1}^{\kappa}\frac{\lambda_i}{|x-x_i|^2}$$

[2]One might consider appending the names of Coxeter and Wythoff to the list of those responsible for the harmonic function ansatz, yielding the somewhat more obtuse 'CCFtHJNRWW ansatz', which the author has elected to not use.

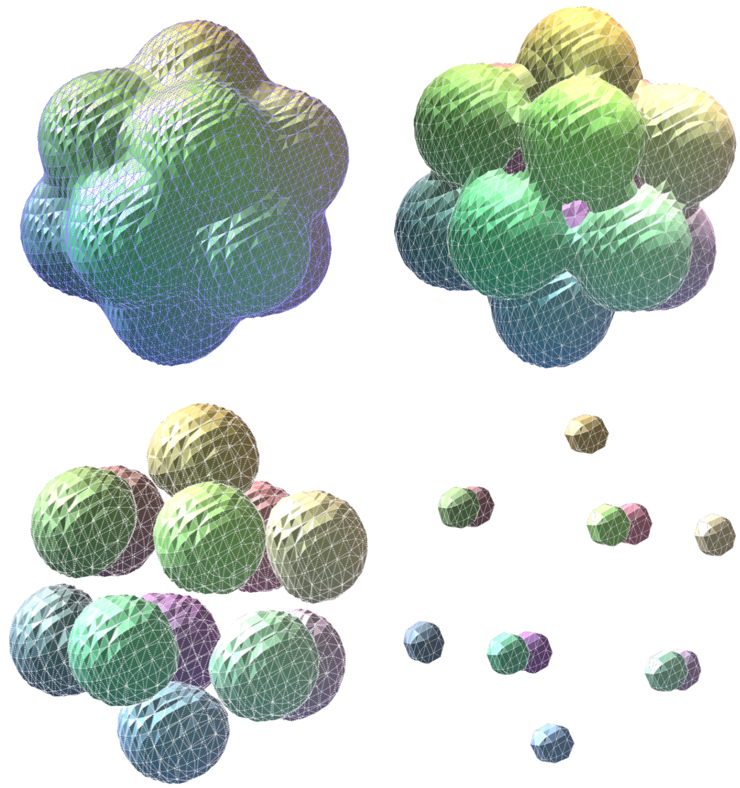


Figure 5.1: Progressively higher energy level sets of the energy density of an 11-instanton. These surfaces should be thought of as smooth—the polyhedral appearance is an artefact of the method used to generate the plots, described in Appendix C.

for points $x_1, \ldots, x_\kappa \in \mathbb{R}^4$ and scalars $\lambda_1, \ldots, \lambda_\kappa \in \mathbb{R}$. As long as the x_i/λ_i are distinct, the ADHM data for the corresponding instanton of structure group SU(2) and charge κ may be given in quaternionic form by the map

$$\Delta(x) = \begin{bmatrix} \lambda_1 & x_1 - x & 0 & \cdots & 0 \\ \lambda_2 & 0 & x_2 - x & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_k & 0 & 0 & \cdots & x_k - x \end{bmatrix}.$$

In the previous section it was shown that the corresponding instanton was G-symmetric if and only if $(x_1, \lambda_1), \ldots, (x_\kappa, \lambda_\kappa)$ are permuted by the action of G on $\mathbb{R}^4$. This suggests an immediate decomposition of a symmetric 't Hooft instanton into G-orbits: let $y_1, \ldots, y_r$ be a set of unique orbit representatives of the x_i, and then write $\{x_1, \ldots, x_\kappa\} = Gy_1 \cup \cdots \cup Gy_\ell$. The ADHM data for this instanton is gauge-equivalent to the 'sum' of that for each of the Gy_i, under the operation

$$\begin{bmatrix} L_1 & M_1 \end{bmatrix} \boxplus \begin{bmatrix} L_2 & M_2 \end{bmatrix} = \begin{bmatrix} L_1 & M_1 & 0 \\ L_2 & 0 & M_2 \end{bmatrix}.$$

From the above discussion one has

Theorem 5.4.11. *Each G-symmetric 't Hooft instanton is gauge-equivalent to a sum of 't Hooft instantons whose singularities have a transitive and free G-action.*

5.5 Integrality for diagonal groups

The purpose of this section is to work out certain integrality theorems for subgroups of $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ that are obtained via the diagonal embedding $\mathrm{SU}(2) \to \mathrm{SU}(2) \times \mathrm{SU}(2)$ of a subgroup of $\mathrm{SU}(2)$. In their natural action on $\mathbb{R}^4$, these are (up to conjugacy) the groups fixing an axis. By Example 3.3.7, in this setting the 'left' and 'right' representations L and R are equivalent representations, and so ADHM data comprise of a map $W \oplus 2 \otimes V \to 2 \otimes V$.

Let G be a compact diagonal subgroup of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ with a real character table. Since diagonal elements act as simple rotations (Lemma 3.3.1), it follows by Lemma 3.3.5 and the work of Section 4.4.2 when $N = 2$ that for any $g \in G$ of order $2k > 2$, there exist integers a, n so that

$$\chi_V(g^r) = rn \frac{\sin(ar\pi/k)}{\sin(r\pi/k)}.$$

On the other hand, the right-hand side is exactly the character of n copies of the irreducible representation W_a of $\mathrm{SU}(2)$ of dimension a, restricted to the cyclic subgroup generated by g. To emphasize this result, it is stated below as Lemma 5.5.1.

Lemma 5.5.1. *Let G be a compact diagonal subgroup of* $\mathrm{SU}(2) \times \mathrm{SU}(2)$*, and let (E, ∇) be a G-symmetric instanton with structure group* $\mathrm{SU}(2)$*. Suppose that $\chi_V(g)$ is real for every $g \in G$ and every finite-dimensional representation V of G, and suppose that the representation W is complex.*

Let $g \in G$ be arbitrary. If $\langle g \rangle$ denotes the cyclic subgroup generated by g—which is identified with a subgroup of $\mathrm{SU}(2) \simeq \mathrm{SU}(2)_\Delta$*—then there exist integers $n(g), a(g)$ so that*

$$\mathrm{res}^G_{\langle g \rangle} V = (\mathrm{res}^{\mathrm{SU}(2)}_{\langle g \rangle} W_{a(g)})^{\oplus n(g)}. \tag{5.10}$$

Notice in particular that one has $\chi_V(1) = an$ for any such decomposition on a cyclic subgroup. Since a, n are both integers, this says that the decomposition of the restriction of V to a cyclic subgroup depends only on a choice of divisor of the charge κ of the instanton. This result does not easily globalize, in the sense that the factorization $\kappa = an$ does, *a priori*, vary with the choice of cyclic subgroup H. If $G \subseteq \mathrm{SU}(2)$, the entire character may be computed via this lemma; clearly letting $V = (\mathrm{res}^{\mathrm{SU}(2)}_G W_a)^{\oplus n}$ yields a representation satisfying (5.10) for a choice of factorization $\kappa = an$, but these are not generally all the possible solutions for V. By checking with a computer, it may be shown that the number of solutions for V when G is the binary icosahedral group exceeds the number of divisors of κ when $\kappa = 25$, for example. This check is done by first computing the characters (restricted to $\widetilde{H_3^+}$) of $\mathrm{triv}^{\oplus 25}, W_5^{\oplus 5}, W_{25}$, generating a list of all 3^9 possible 'stitched' characters by picking one of the three representations for each conjugacy class of $\widetilde{H_3^+}$, and then checking which elements of this list decompose into a sum of irreducible representations with integer multiplicities by taking inner products. There are four solutions: the characters of the three representations given, and one obtained by a non-trivial stitching.

Theorem 5.5.2. *Let $G = \widetilde{H_3^+}$ or $\widetilde{B_3^+}$. Suppose that (E, ∇) is a G-symmetric instanton with structure group* SU(2)*. If the charge κ of E is a prime number, then the representation V of G is either κ copies of the trivial representation, or else it is the restriction to G of the κ-dimensional irreducible representation of* SU(2)*.*

Proof. Immediate from Lemma 5.5.1 and Proposition 3.4.2. □

This proof relies on the fact that the character tables of $\widetilde{H_3^+}$ and $\widetilde{B_3^+}$ are real, and would not be sufficient for $\widetilde{A_3^+}$, for example.

Theorem 5.5.2 demonstrates the power of the index theorem for solving symmetric instanton existence problems. In [42, 44] the index theorem is not used, but instead the choice $V = \mathrm{res}_G^{\mathrm{SU}(2)} W_\kappa$ (for the case $G = \widetilde{H_3^+}$) is made for $\kappa = 7, 17$; Theorem 5.5.2 shows that since 7, 17 are prime, this choice of representation is not just a clever one but in fact the *only* interesting one.

Remark 5.5.3. It is a curious fact that all the odd numbers one encounters when studying symmetric instantons of icosahedral symmetry are prime:

$$7, 11, 13, 17, 19, 23, 37, 59, 67, 97.$$

As a result, Theorem 5.5.2 seems almost unreasonably effective at constructing interesting symmetric instantons when the symmetry group is $\widetilde{H_3^+}$. The one exception to this pattern is $|\widetilde{H_3^+}| - 1 = 119$. Note, however, that there is no 'irreducible' instanton at charge 119 in the 't Hooft sense of Theorem 5.4.11, because the action of $\widetilde{H_3^+}$ on $\mathbb{R}^4$ is not faithful; -1 and 1 have the same image, and the largest orbit has size 60.

Analogously, Proposition 3.4.3 gives an integrality result when the symmetry group is a diagonally embedded SU(2).

Theorem 5.5.4. *Let $G =$ SU(2), embedded diagonally in SU(2) × SU(2). If (E, ∇) is a G-symmetric instanton of charge κ with structure group* SU(2)*, then there exist integers a, n with $an = \kappa$ such that V is n copies of the unique a-dimensional irreducible representation of $G =$* SU(2)*.*

Proof. The character formula of Lemma 3.3.5 implies that the character table of SU(2) is real, so this result follows immediately from Lemma 5.5.1 and Proposition 3.4.3. □

Due to this restriction on V, instantons symmetric under SU(2) are exceptionally rare, at least in low structure group rank.

Theorem 5.5.5. *The* SU(2)*-symmetric instantons with structure group* SU(2) *(that is, for which $N = 2$) are exactly the 't Hooft instantons in which weighted singularities fall at distinct points on the real axis of $\mathbb{H}$.*

Proof. The 't Hooft instantons with weighted singularities at distinct points on the real axis of $\mathbb{H}$ are clearly symmetric, because the action of the diagonal SU(2) is to rotate the imaginary part of $\mathbb{H}$ and leave invariant the real part.

In the $N = 2$ setting the ADHM data involves an invariant map $W \to V \otimes 2$. But $V = nW_a$ for some n, a and thus $V \otimes 2 = nW_{a-1} \oplus nW_{a+1}$. On the other hand, $W = W_2$ since the G-action is not trivial and SU(2) has no other non-trivial 2-dimensional representation, and by Schur's lemma we must have $a - 1 = 2$ or $a + 1 = 2$; that is, either $a = 1$ or $a = 3$. The solutions for $a = 1$ are precisely 't Hooft instantons where the set of points selected consists entirely of real numbers; thus we assume that $a = 3$.

The quaternionic form of these data are described by a vector ℓ of n real numbers for L and a skew-symmetric[3] $n \times n$ matrix m for M; then

$$L = \ell \otimes \begin{bmatrix} i & j & k \end{bmatrix}, M = m \otimes \begin{bmatrix} 0 & k & -j \\ -k & 0 & i \\ j & -i & 0 \end{bmatrix},$$

where $\otimes$ here denotes the Kronecker product of matrices. The matrices in this expression were obtained by imposing symmetry under the quaternion subgroup of SU(2) generated by i, j, k; if any symmetric data is to exist, it must take this form.

The ADHM equations require that $L^\dagger L + M^\dagger M$ be real and invertible. Then by the mixed product property of the Kronecker product and because ℓ is real,

$$L^\dagger L = \ell^T \ell \otimes \begin{bmatrix} 1 & -k & j \\ k & 1 & -i \\ -j & i & 1 \end{bmatrix},$$

the right factor being the outer product $\begin{bmatrix} i & j & k \end{bmatrix}^\dagger \begin{bmatrix} i & j & k \end{bmatrix}$. Likewise,

$$\begin{aligned} M^\dagger M &= m^T m \otimes \begin{bmatrix} 2 & -k & j \\ k & 2 & -i \\ -j & i & 2 \end{bmatrix} \\ &= m^T m \otimes \left(I_3 + \begin{bmatrix} 1 & -k & j \\ k & 1 & -i \\ -j & i & 1 \end{bmatrix} \right) \\ &= m^T m \otimes I_3 + m^T m \otimes \begin{bmatrix} 1 & -k & j \\ k & 1 & -i \\ -j & i & 1 \end{bmatrix} \end{aligned}$$

Thus

$$L^\dagger L + M^\dagger M = m^T m \otimes I_3 + (\ell^T \ell + m^T m) \otimes \begin{bmatrix} 1 & -k & j \\ k & 1 & -i \\ -j & i & 1 \end{bmatrix}.$$

Since m, ℓ are real, for this to be real it must be the case that $\ell^T \ell + m^T m = 0$. But m is skew-symmetric, and so $(m^T m)_{ii} \geq 0$; it is the norm of the ith row. Likewise, $(\ell^T \ell)_{ii} = \ell_i^2$, and so to have $\ell^T \ell + m^T m = 0$ implies in particular that $\ell = 0$. Thus $m^T m = 0$, and $L^\dagger L + M^\dagger M = 0$ is not invertible. □

[3]Recall that M must be symmetric; this condition is equivalent to the skew-symmetry of m.

While the purpose of this section is to treat only the diagonal instantons, a classification for a four-dimensional group is so close that it would be a shame to not mention it here.

Corollary 5.5.6. *The only* Spin(4)*-symmetric instanton with structure group* SU(2) *is the basic instanton of charge 1 centered at the origin.*

Proof. The result follows from Theorem 5.4.11 and Theorem 5.5.5, when one considers that a Spin(4)-symmetric instanton must certainly also be symmetric under $\mathrm{SU}(2) \subset \mathrm{Spin}(4)$. □

Remark 5.5.7. This result is very well known. For a recent example it appears as [28, Proposition 2.4]. The proof in this thesis is somewhat more abstract, with uniqueness arising from the representation theory of SU(2) as opposed to an explicit analysis to the solutions of an ODE.

5.6 Integrality theorems for four-dimensional instantons

The discrete subgroups of Spin(4) can be rather wild if no further conditions are imposed. Rather than a systematic study of all these subgroups, we look instead at just the direct product subgroups of $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$: that is, those subgroups of the form $G \times H$ obtained from subgroups G, H of SU(2). There are a number of reasons to do so. For one, the representation theory of these groups is particularly simple by Lemma 3.1.1 and Theorem 3.4.1. Another is that almost all symmetries of the 4-dimensional regular polytopes arise in this way: the only exception are the symmetries of the 5-cell, which are isomorphic to a twisted embedding of the binary icosahedral group in $\mathrm{SU}(2) \times \mathrm{SU}(2)$. This case is discussed in [1], where the JNR instanton with singularities at the vertices of a 5-cell is constructed through the representation theory of the symmetry group of the 5-cell. Finally, it is easy to write down a large number of double rotations in direct product groups. Since by Lemma 3.3.1 an element of the form $(g, 1)$ or $(1, g)$ is a double rotation when $g \neq 1$, the subgroups $G \times \{1\}$ and $\{1\} \times H$ of $G \times H$ consist entirely of double rotations apart from the identity; one applies the index theorem to these subgroups to obtain the claimed integrality theorems for the four-dimensional symmetric instantons.

Let N be fixed, let $G = G_1 \times G_2$ be a direct product subgroup of $\mathrm{SU}(2) \times \mathrm{SU}(2)$, and suppose that $E \to \mathbb{R}^4$ is a G-symmetric instanton of charge k with structure group $\mathrm{SU}(N)$, and let (V, W) be the representations obtained by the ADHM correspondence. Fix $g \in G_1 \setminus \{1\}$, and suppose it acts on the imaginary quaternions as a simple rotation with angle θ. Let $V_1, \ldots, V_m$ be the N-dimensional representations of G, and let $\alpha_{ij}^{\pm} = \chi_{V_i}(g) \pm \chi_{V_j}(g)$.

Theorem 5.6.1 (Integrality for direct-product subgroups)**.** *In the above setting,* $\chi_V(g, 1) = \pm\frac{1}{4}\csc(\theta/2)^2 \alpha_{ij}^{\pm}$ *for some* i, j *and independent choices of sign, and* $\chi_V(g)$ *is also an algebraic integer.*

Proof. That characters are algebraic integers is a basic fact of representation theory; the claimed formula for $\chi_V(g, 1)$ is a consequence of Equation (4.9) since $(g, 1)$ is a double rotation with angles θ, θ. □

This theorem clearly holds for elements of the form $(1, g)$, with the necessary notational changes made.

The utility of this theorem is most apparent when N is small, so the number of N-dimensional representations of G is small also. As a result, the number of possible values the character of V can take on for a particular $(g, 1)$ are severely limited.

Another way of thinking about this situation is that the character of V at $(g, 1)$ is the same as the character of $\operatorname{res}^G_{G_1} V$ at g. The problem has now been converted to a simpler one: what are the representations of G_1 with characters at non-identity g taking the form in Theorem 5.6.1? Suppose that the solutions for some dimension k are $X_1, \ldots, X_\ell$. Do the same for G_2, getting k-dimensional representations $Y_1, \ldots, Y_{\ell'}$. Then what k-dimensional representations V of G have the property that $\operatorname{res}^G_{G_1} V = X_i, \operatorname{res}^G_{G_2} V = Y_{i'}$ for some i, i'?

Combining this integrality restriction with the need for a non-zero equivariant map $V_L \to V_R$, a huge number of representations can be thrown out immediately. Recall that restrictions on such equivariant maps were, at least for the case of irreducible representations, developed in Section 3.4.4, under the name of 'left-right equivariant maps'.

To demonstrate the power of this method, consider the conjecture of Allen and Sutcliffe (see [1]) that the lowest charge SU(2)-symmetric instanton for a polytope is the one from the JNR ansatz if and only if the polytope in question has only triangles as 2-faces. Using these methods, one of the hardest cases of the reverse direction may be obtained.

Theorem 5.6.2. *There is no $\widetilde{H_4^+}$-symmetric instanton with charge in* $(1, 119)$ *and structure group* SU(2). *In particular, the smallest instanton with the symmetries of the 600-cell instanton aside from the basic one centered at the origin is the charge* 119 *instanton of the symmetric harmonic function ansatz.*

Proof. This proof is done mostly by computer. In particular, we must enumerate the representations $X_1, \ldots, X_\ell$ mentioned above. Because both factors of $\widetilde{H_4^+} = \widetilde{H_3^+} \times \widetilde{H_3^+}$ are identical, the representations $Y_1, \ldots, Y_{\ell'}$ are identical. Let $\tau = \frac{1+\sqrt{5}}{2}$ be the golden ratio. To find the X_i, note that for $N = 2$ and $G = \widetilde{H_4^+}$, the values for θ are always among $0, \pi/2, 2\pi/3, 2\pi/5$, and by computing all $\alpha^\pm_{ij}$ for a particular $g \in G$, one obtains that the only way $\pm\frac{1}{4}\csc^2(\theta/2)\alpha^\pm_{ij}$ is an algebraic integer is when it is of the form $a + b\tau$ for some $-1 \le a, b \le 1$. So by Theorem 5.6.1, for each $g \neq 1 \in G$ there exist $-1 \le a, b \le 1$ such that $\chi_V(g, 1) = a + b\tau$.

Now suppose that X is a representation of $G_1 = \widetilde{H_3^+}$ with the property that for all g, there exist $-1 \le a, b, \le 1$ so that $\chi_V(g, 1) = a + b\tau$. Writing the restriction of V to G_1 as $m_1V_1 \oplus m_2V_2 \oplus m_{2'}V_{2'} \oplus \cdots \oplus m_6V_6$, one can read restrictions on the multiplicities m_i from the character table for $\widetilde{H_3^+}$ (see Section A.3.3). For example, the restriction from the conjugacy class **3** is $|m_1 + m_4 + m_{4'} - m_5 - m_2 - m_{2'}| \le 1$. Similarly, the classes **5A**, **5B**, **10A**, **10B** all give two restrictions: one for the integer part and one for the τ part. Now a brute-force search with pruning on these conditions is sufficient to enumerate all the X_i on a computer.

A dynamic programming approach is used to generate the list of all representations of $\widetilde{H_4^+}$ that restrict in left and right factors to one of the X_i, by writing a $\widetilde{H_4^+}$ representation as a 9×9 integer matrix, by picking particular X_i, X_j, thinking of them as vectors of their multiplicities, and then looking for non-negative integer matrices whose column sums are X_i and whose row sums are X_j. This step does not depend on the group being $\widetilde{H_4^+}$, and could be used for other product groups given the list of representations X_i, Y_j of the factors. This computation results in the claimed theorem.

The precise implementation used is available at [48]. □

The program described in the previous proof outputs all possible representations V for a given dimension, but the restricted number of choices is never quite as dramatic as in Theorem 5.5.2. Even the JNR 600-cell instanton is $\widetilde{H_4^+}$ symmetric at charge 119, which is not a prime number. Still, one can make the obvious restriction (as in the proof of Corollary 5.5.6) of $\widetilde{H_4^+}$ to the diagonal subgroup isomorphic to $\widetilde{H_3^+}$ acting by simple rotations.

Corollary 5.6.3. *If κ is prime, $(\widetilde{H_3^+})_\Delta$ is embedded diagonally in $\widetilde{H_4^+} = \widetilde{H_3^+} \times \widetilde{H_3^+}$, and (E, ∇) is a $\widetilde{H_4^+}$-symmetric instanton of charge κ with structure group* SU(2)*, then the representation V of $\widetilde{H_4^+}$ satisfies either*

$$\operatorname{res}^{\widetilde{H_4^+}}_{(\widetilde{H_3^+})_\Delta} V = \operatorname{res}^{\mathrm{SU}(2)}_{\widetilde{H_3^+}} W_\kappa \quad \text{or} \quad \operatorname{res}^{\widetilde{H_4^+}}_{(\widetilde{H_3^+})_\Delta} V = \mathrm{triv}^{\oplus\kappa}.$$

An analogous result holds with $\widetilde{H_3^+}, \widetilde{H_4^+}$ replaced by $\widetilde{B_3^+}, \widetilde{B_4^+}$.

5.7 The moduli space of symmetric instantons

Fix G to be a compact subgroup of Spin(4), let W and V be representations of G of dimension N, κ respectively, and consider the set of all pairs of G-invariant maps $C\colon W \to V_L, B\colon V_R \to V_L$; by imposing upon these maps the open and closed ADHM equations, and reducing modulo gauge transformations, one obtains a moduli space of G-symmetric instantons, called henceforth $\mathcal{M}(G, V, W)$, or, by an abuse of notation, just $\mathcal{M}(V)$ if W, G are understood. The goal is to understand this moduli space, and hence understand all the symmetric instantons associated to some initial data (G, V, W).

Example 5.7.1 (Dodecahedral 7-instanton)**.** In Section C.2.1 a family of 7-instantons with dodecahedral symmetry ($G = \widetilde{H_3^+}$) is described. For this family, $V = 3' \oplus 4'$ and $W = 2'$. There is a two-dimensional family of solutions with a non-zero parameter corresponding to the freedom to rescale all the ADHM data and a translation parameter corresponding to the freedom to translate the data along the axis fixed by G as it acts on $\mathbb{R}^4$. Thus $\mathcal{M}\left(\widetilde{H_3^+}, 3' \oplus 4', 2'\right) = \mathbb{R}^* \times \mathbb{R}$.

5.7.1 Quasi-irreducibility

As Theorem 5.4.11 revealed, there is some notion by which symmetric ’t Hooft type-instantons can be “irreducible”, or unable to be broken into smaller parts. The goal in analyzing the moduli space of symmetric instantons is to recover something akin to this theorem, although to do so is much more difficult due to the non-linearity of the ADHM equations. Since the data parametrizing a family of symmetric instantons is a representation (when $W, \widetilde{G}$ are fixed), the word ‘irreducible’ should not be overloaded—but we introduce the following notion of *quasi-irreducibility* to capture the desired behaviour.

Definition 5.7.2. Let (G, V, W) be data giving a moduli space of G-symmetric instantons where G acts by orientation-preserving isometries on $\mathbb{R}^4$, and suppose that ℓ is the number of axes in $\mathbb{R}^4$ fixed by G.[4] If $\mathcal{M}(G, V, W) \simeq \mathbb{R}^* \times \mathbb{R}^\ell$, say that (G, V, W) are (or by an abuse of notation, that V is) *quasi-irreducible*.

Remark 5.7.3. What V being quasi-irreducible means is that:

1. there are symmetric instantons corresponding to this representation; and,
2. the only instantons corresponding to this representation are formed from a generic member of the moduli space by the operations of scaling by a non-zero constant, or by translating in a direction fixed by the group.

Example 5.7.4. There is not generally a relationship between irreducibility and quasi-irreducibility; one is not stronger than the other. For example let $G = \widetilde{H_3^+}$ be the binary icosahedral group, and let $W = 2'$. Then *no* irreducible representation is quasi-irreducible. Conversely, Example 5.7.1 demonstrated that $\mathrm{res}_G W_7$ is quasi-irreducible, and indeed for this G, W, one has that $\mathrm{res}_G W_\kappa$ is quasi-irreducible at least when $\kappa \in \{7, 11, 13, 17, 19, 23\}$.

Of course, choice of G, W changes the notion of quasi-irreducibility: in the above example, if instead $W = 2$, then the basic instanton is quasi-irreducible, corresponding to the trivial representation, and $\mathrm{res}_G W_7 = 3' \oplus 4'$ is not quasi-irreducible, because $\mathcal{M}(\widetilde{H_3^+}, 3' \oplus 4', 2)$ is empty for a lack of $\widetilde{H_3^+}$-invariant maps $2 \to (3' \oplus 4') \otimes 2$; any solution would necessarily be singular.

Quasi-irreducibility is an attempt to generalize the condition of 'having a single G-orbit' from the 't Hooft case (Section 5.4.5), which does not otherwise make sense in the general one—a 'decomposition into orbits' does not exist *a priori*. If (B, C) are the ADHM data (in the sense of Section 5.4.3) of an element of $\mathcal{M}(V, W, G)$, let $\Delta = C \oplus B$ be the map $W \oplus V_R \to V_L$ defined by $\Delta(u, v) = Cu + Bv$. Then $\Delta\Delta^\dagger$ is a map $V_L \to V_L$. Denote by Δ_* its real part: that is, the real map $V \to V$ resulting from the identification $L = \mathbb{C}^2 = \mathbb{H}$. A first consequence of quasi-irreducibility is the following:

Proposition 5.7.5. *If (G, V, W) is quasi-irreducible, then Δ_* is a scalar multiple of the identity for all (B, C) ADHM data representing an element in $\mathcal{M}(G, V, W)$.*

Proof. A computation shows that multiplication by Δ_* preserves the ADHM equations and the origin; so if (G, V, W) is quasi-irreducible, then it must be the case that it is scalar multiplication. □

Remark 5.7.6. As well as a concrete description of how instantons in the moduli space $\mathcal{M}(V)$ are related to each other, quasi-irreducibility is a natural condition from the representation-theoretic point of view also: it is, by Proposition 5.7.5, exactly the condition one needs for a Schur's lemma-style result to hold

[4] Any rotation fixing three axes is already the identity by Corollary 2.3.2, so the possibilities are $\ell = 0, 1, 2, 4$; 4 occurs for the trivial action on $\mathbb{R}^4$ and 2 for a rotation fixing a plane. Of interest are the cases $\ell = 1$, corresponding to a diagonally embedded polyhedral subgroup of SU(2), and the generic case $\ell = 0$, like the symmetries of a four-dimensional regular polytope.

Question 5.7.7. Is the implication of Proposition 5.7.5 an equivalence? In other words, if Δ_* is a scalar multiple of the identity for each (B, C) ADHM data representing an element in $\mathcal{M}(G, V, W)$, then are (G, V, W) quasi-irreducible?

Chapter 6

Action and instantopes

Level sets of the action density (that is, the pointwise norm of the curvature) of symmetric instantons 'look like' polytopes symmetric under the group of symmetries. For example, the images displayed in the top-left corner of the even arabic-numbered pages of this thesis look roughly like dodecahedra. In this chapter the polytope associated to an instanton is defined, which provides a formal way to speak of an instanton being 'dodecahedral'.

This definition, which looks at the critical points of the action density, is motivated by the images displayed on the top left- and right-hand corners of the pages; for by flipping through these images quickly, one may convince themselves that the global maxima of the action density ought to be vertices of the 'obvious' polytope. Figure 5.1 provides another example; here due to the simple nature of the action density for JNR instantons (see Theorem B.1.1) it is not so difficult to compute that the icosahedral 11-instanton indeed has the global maxima of its action density at the vertices of the icosahedron used to generate the instanton.

To compute more generally the action density is exceptionally difficult (generally requiring mixed fourth logarithmic derivatives of the determinant of a matrix whose rows and columns number equal to the charge), and so in many cases the relation between polytope and instanton remains conjectural—although numerically it can be verified to very high accuracy. In this chapter we investigate some basic properties of the action density, and present a conjecture about the structure of the critical points of the action density.

6.1 Critical points of the action density

Let (I, J, B_1, B_2) be ADHM data in the general matrix form of Section 5.4.3. Given $x \in \mathbb{R}^4$, write it as an element $(z_1, z_2) \in \mathbb{C}^2$ and set

$$R(x) := \Re(II^\dagger + (B_1 - z_1 I)(B_1 - z_1 I)^\dagger + (B_2 - z_2 I)(B_2 - z_2 I)^\dagger).$$

This matrix is the real part of the top-left block of the matrix $\Delta(x)\Delta(x)^\dagger$. Let $\nabla^2 = \sum_{i=1}^4 \partial_i^2$ denote the usual Laplacian on $\mathbb{R}^4$. The norm of the curvature of the instanton A described by the ADHM data (I, J, B_1, B_2) is given by

$$|F_A|^2(x) = -\frac{1}{16\pi^2}\nabla^2\nabla^2 \log\det R(x). \tag{6.1}$$

For an instanton A, let $C(A)$ be the set of critical points of the action density.

Lemma 6.1.1. *If (I, J, B_1, B_2) are ADHM data for an instanton A, then for all $a \neq 0 \in \mathbb{R}$, and A' is the instanton corresponding to ADHM data (aI, aJ, aB_1, aB_2), then*

$$C(A') = \{ax \colon x \in C(A)\}.$$

Likewise, if $b \in \mathbb{R}$ and A'' is the instanton corresponding to ADHM data $(I, J, B_1 + bI, B_2)$, then

$$C(A'') = \{x + (b, 0, 0, 0) \colon x \in C(A)\}.$$

Proof. The two statements are similar, so only the first is proved. Let $\Delta(x)$ denote the ADHM matrix for A and $\Delta'(x)$ that for A'; then $\frac{1}{a}\Delta'(ax) = \Delta(x)$, so $\Delta'(x) = a\Delta(x/a)$. Then by Equation (6.1),

$$\begin{aligned} |F_{A'}|^2(x) &= -\frac{\pi^2}{16}\nabla^2\nabla^2 \log\det(a^2 R(x/a)) \\ &= -\frac{\pi^2}{16}\nabla^2\nabla^2(\log\det R(x/a) + \log a^{2k}) \\ &= \frac{1}{a^4}|F_A|^2(x/a) \end{aligned}$$

by four applications of the chain rule, whence the result. □

6.2 Instantopes

An instanton A has an associated set $C(A)$ of critical points of the action density, and thus an associated convex hull $P(A) = \operatorname{conv}(C(A))$. Numerical evidence suggests that this set $P(A)$ is actually a polytope; that is, that the set $C(A)$ is finite.

In particular, if (G, V, W) are quasi-irreducible and $P(A)$ is a polytope for some $A \in \mathcal{M}(G, V, W)$, then $P(A)$ is a polytope for all $A \in \mathcal{M}(G, V, W)$, and $P(A)$ is well-defined up to scaling and translation by Lemma 6.1.1.

Definition 6.2.1. A *G-instantope* is a polytope P such that there is a G-symmetric instanton A of charge κ such that $P = P(A)$ and $|V(P)| < \kappa - 1$.

Clearly if A is G-symmetric, then so is $\|F_A\|^2$ and hence so is its set of critical points. Thus if $P(A)$ is a polytope, it is vertex-transitive with respect to the G-action. Conversely, any polytope P arising from the orbit construction (Construction 2.2.1) invoked with G occurs as $P(A)$ for an instanton A of charge $|V(P)| - 1$; this instanton is constructed by the JNR ansatz (Construction B.1.3) with a singularity at each vertex of P. The condition that a polytope P be an instantope is more restrictive; it stipulates that there exists an instanton A of charge *lower* than the trivial JNR bound $|V(P)| - 1$ such that $P(A) = P$.

Say that a polytope is *gyral* if it is uniform and furthermore the action by rotational symmetries is transitive on vertices. Say that a polytope is a *deltope*[1] if all its two-faces are triangles. With this language, we state a somewhat more refined version of the conjecture of [1], and prove half of it.

[1] So the notion of a deltope is the n-dimensional generalization of the notion of a *deltahedron*.

Conjecture 6.2.2. *Let d be equal to three or four, and let H be a finite reflection group in d dimensions. A uniform polytope P of dimension d is a $\widetilde{H^+}$-instantope if and only if it is gyral but not a deltope.*

Theorem 6.2.3. *The forwards direction of Conjecture 6.2.2 holds. That is, in the setting of the conjecture, if P is an instantope then it is gyral but not a deltope.*

Proof. It follows from the symmetry conditions that $\widetilde{H^+}$ (and so in particular H^+) act transitively on the vertices of any instantope, so all instantopes must be gyral.

The only polytopes of dimension three or four which are gyral and deltopes for $G = \widetilde{H^+}$ for some Coxeter group H of rank three or four may be computed by the algorithm of [7], and are as follows:

- In dimension three, the tetrahedron, octahedron, and icosahedron
- In dimension four, the 5-cell, the 8-cell, the 24-cell, and the 600-cell.

It was shown in [1] that the 5-cell, 8-cell, and 24-cell are not instantopes. The results of Chapter 5 can be used to show that none of the tetrahedron, octahedron, icosahedron, and 600-cell (see Theorem 5.6.2) are instantopes; this agrees with the work also in [1] that excludes these possibilities (except for the 600-cell) within their framework for symmetric ADHM data. □

Remark 6.2.4. With a particular reflection group in mind, the converse to Theorem 6.2.3 could theoretically be checked by a computation. For example, when $H = H_3$, the polytopes that are gyral and not deltopes are a dodecahedron, a truncated icosahedron, an icosidodecahedron, and a truncated dodecahedron. It is reasonable to expect, given the action density plots of Appendix C, that the dodecahedron, truncated icosahedron, and icosidodecahedron are all $\widetilde{H_3^+}$-instantopes, with instantons witnessing as much occurring at (for example) charges 7, 17, and 23 respectively.

However, the action density plots do not consist of a proof that the critical points of the action density *actually* fall at the vertices for such polytopes.

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Appendices

Appendix A

SU(2) and its discrete subgroups

A.1 Low-dimensional isomorphisms

The group SU(2) consists of the 2×2 unitary matrices with unit determinant; that is, those 2×2 matrices U satisfying $U^\dagger = U^{-1}$ and $\det U = 1$. There is an exceptional isomorphism of real compact Lie groups, $\mathrm{SU}(2) \times \mathrm{SU}(2) = \mathrm{Spin}(4)$.

There are also exceptional isomorphisms $\mathrm{SU}(2) = \mathrm{Spin}(3) = \mathrm{Sp}(1)$, the final group being the unit quaternions under multiplication. Let i, j, k be the standard basis for the imaginary part of the quaternions, satisfying the relations $i^2 = j^2 = k^2 = ijk = -1$. One makes the identification $\mathbb{H} = \mathbb{C}^2$ via $a + bi + cj + dk \mapsto (a + bi, c + di)$, and then from $\mathbb{C}^2$ into 2×2 complex matrices by $F\colon (z_1, z_2) \mapsto \begin{bmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{bmatrix}$.

Lemma A.1.1. *If $\alpha + \beta j$ is a unit quaternion, then $F(\alpha, \beta)$ is special unitary, and restricted to the group* Sp(1) *of unit quaternions $F\colon \mathrm{Sp}(1) \to \mathrm{SU}(2)$ is an isomorphism.*

Proof. Easy computation. □

With Sp(1) denoting the group of unit quaternions it therefore follows that $\mathrm{Sp}(1) = \mathrm{SU}(2)$. The map $\rho\colon \mathrm{Spin}(4) \to \mathrm{SO}(4)$ under the identification $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2) = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ is particularly simple: if one considers $\mathbb{R}^4 = \mathbb{H} = \mathbb{C}^2$ by the maps above, then the double covering becomes $(p, q) \mapsto (x \mapsto pxq^{-1})$. Likewise, the double cover map $\mathrm{Sp}(1) = \mathrm{Spin}(3) \to \mathrm{SO}(3)$ is given under the identification of $\mathbb{R}^3$ with the imaginary quaternions by the map $q \mapsto (x \mapsto qxq^{-1})$.

A.2 Classification of discrete subgroups

A good reference for this section is [10, §3]. The notation used by Conway and Smith is a bit idiosyncratic; there is no agreement on standard notation to denote these groups, so we recall here their arguments and at the same time introduce the notation to be used throughout the present document.

Under the double cover map $\rho\colon \mathrm{Spin}(3) \to \mathrm{SO}(3)$ of the previous section a finite subgroup of $\widetilde{G}$ of Spin(3) projects to a finite subgroup of G of SO(3). In fact, it is shown in [10, §3] that the projected group has the form $G' \cap \mathrm{SO}(3)$, where G' is a reflection subgroup of O(3).

Such groups were classified in [12], and there are five kinds: the first three are the polyhedral groups: the group of reflective symmetries of an icosahedron is H_3, of a cube is B_3, and of a tetrahedron is A_3.[1] To denote their subgroups consisting of rotations, a superscript + is used (as this subgroup is the one where the determinant is positive): H_3^+, B_3^+, A_3^+. The remaining two groups fix an axis; their rotational parts are the dihedral group D_n (the symmetries of an origin-centered regular n-gon drawn in a plane in 3-space) and the cyclic group $\mathbb{Z}/n\mathbb{Z}$ (the subgroup of D_n fixing an axis pointwise).

One then argues that if a discrete subgroup G of SO(3) contains an order two rotation, a lift of it squares to -1 in SU(2), and so the discrete subgroup of SU(2) that projects to G is a double cover, $\widetilde{G}$. There is one additional case: if G contains no order two rotation, then by the above classification it must be $\mathbb{Z}/n\mathbb{Z}$ for odd n. This group lifts to SU(2) without doubling. Thus in this notation, the discrete subgroups of SU(2) are $\widetilde{H_3^+}, \widetilde{B_3^+}, \widetilde{A_3^+}$ (called respectively the binary icosahedral group, the binary octahedral group, and the binary tetrahedral group) along with the cyclic groups $\mathbb{Z}/n\mathbb{Z}$ for all n and the binary dihedral groups $\widetilde{D_n}$. Each of these groups except the odd-order cyclic groups arise as the lift of a discrete subgroup of SU(2).

The story for $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is much more complicated, although similar work can be done (see [20], [10, §4] or [32] for example).

A.3 The finite subgroups of SU(2) and their representations

The following sections contain a brief description of the exceptional finite subgroups of SU(2) and their character tables.

Remark A.3.1 (Conventions for character tables)**.** If G is a group, $k_1, k_2, g \in G$ and $k_1 = gk_2g^{-1}$, then for all positive integers n one has $k_1^n = gk_2^ng^{-1}$, so that conjugate elements of G have the same order: $|k_1| = |k_2|$. We therefore adopt the convention that a conjugacy class is labelled by the common order of the elements it contains.

Frequently, but not always, this common order uniquely identifies the conjugacy class in G. In the event that there exist multiple classes of elements of the same order, a letter suffix is appended to the conjugacy class to distinguish it from others of the same order. For example, a class labeled **4** would contain all the elements of order 4, while classes labeled **5A**, **5B** would be non-conjugate and contain elements of order 5 only.

Labels of conjugacy classes are recorded on the top row of the character table and printed in bold. Every finite subgroup of SU(2) that lifts from a subgroup of SO(3) has two distinguished conjugacy classes: **1** and **2**, the classes of 1 and -1 respectively. Since the image of 1 under a representation must be the identity matrix, the value of the characters of this column are the dimensions of the irreducible representations of the group.

The character evaluated at -1 is equal to the dimension up to a sign. If the sign is positive (so the character evaluated -1 is equal to the dimension), the representation is real, and factors through the double cover map $\mathrm{SU}(2) \to \mathrm{SO}(3)$. If the sign is negative (so the

[1]The letter 'A' appearing in this context is an unfortunate bit of notation; it refers to the Coxeter–Dynkin diagram of type A, and not the alternating group on 3 symbols. We keep the notation A_n to refer to both the Coxeter group of type A_n and the alternating group on n symbols, but with the convention that it always refers to the former unless stated otherwise in apposition.

character evaluated -1 is negative the dimension), the representation is complex, and does not factor through the double cover map $\mathrm{SU}(2) \to \mathrm{SO}(3)$. We adopt the convention that *real representations are shown first, followed by complex representations.* Thus, for example, for the binary icosahedral group $\widetilde{H_3^+}$, the rows are given in the order $1, 3, 3', 4', 5, 2, 2', 4, 6$; the representations $1, 3, 3', 4', 5$ are real (coming from the representation theory of the alternating group $A_5 = H_3^+$), while the representations $2, 2', 4, 6$ are complex. The character of a representation V is written as χ_V, while the representation itself is written ρ_V if not left implicit.

The first column of the character table contains names for the representations, corresponding to the McKay quivers of Table 3.1.

All character tables and generators in this appendix were obtained from the nLab ([38, 37, 36]), and were verified with GAP ([16]).

A.3.1 The binary tetrahedral group $\widetilde{A_3^+}$

1 Description The full symmetry group of the regular tetrahedron is the symmetric group S_4, so its rotational symmetries are the alternating group A_4, and thus $\widetilde{A_3^+}$ is an order 24 subgroup of Sp(1).

2 Generators A choice of generators is

$$g_1 = i, \quad g_2 = \frac{1}{2}(1 + i + j + k), \quad g_3 = \frac{1}{2}(1 + i + j - k).$$

These generators satisfy the relations

$$(g_1 g_2)^3 = (g_1 g_3)^3 = (g_2 g_3)^4 = 1.$$

3 Character table Let $\zeta = \exp(2\pi i/3)$ be a primitive third root of unity.

	1	**2**	**4**	**3A**	**3B**	**6A**	**6B**
χ_1	1	1	1	1	1	1	1
$\chi_{1'}$	1	1	1	ζ	ζ^2	ζ^2	ζ
$\chi_{1''}$	1	1	1	ζ^2	ζ	ζ	ζ^2
χ_3	3	3	-1	0	0	0	0
$\chi_{2''}$	2	-2	0	ζ	ζ^2	$-\zeta^2$	$-\zeta$
$\chi_{2'}$	2	-2	0	ζ^2	ζ	$-\zeta$	$-\zeta^2$
χ_2	2	-2	0	1	1	-1	-1

The labelling of representations $\rho_{1'}, \rho_{1''}$ is arbitrary, depending on the choice of primitive third root of unity.

A.3.2 The binary octahedral group $\widetilde{B_3^+}$

1 Description The full symmetry group of the cube has order 48, and the rotational symmetries have order 24. The rotational symmetry group is isomorphic to the symmetric group S_4, acting on the diagonals through the centre of the cube by permutations. The binary octahedral group $\widetilde{B_3^+}$ is therefore an order 48 subgroup of Sp(1).

2 Generators A choice of generators is

$$g_1 = \frac{1}{\sqrt{2}}(i+j), \quad g_2 = \frac{1}{2}(1+i+j+k), \quad g_3 = \frac{1}{\sqrt{2}}(1+i).$$

These generators satisfy the relations

$$(g_1 g_2)^8 = (g_1 g_3)^3 = (g_2 g_3)^4 = 1.$$

3 Character table

	1	**2**	**4A**	**6**	**3**	**4B**	**4C**	**8**
χ_1	1	1	1	1	1	1	1	1
$\chi_{1'}$	1	1	1	1	1	-1	-1	-1
$\chi_{2''}$	2	2	2	-1	-1	0	0	0
χ_3	3	3	-1	0	0	1	1	-1
$\chi_{3'}$	3	3	-1	0	0	-1	-1	1
χ_2	2	-2	0	1	-1	$\sqrt{2}$	$-\sqrt{2}$	0
$\chi_{2'}$	2	-2	0	1	-1	$-\sqrt{2}$	$\sqrt{2}$	0
χ_4	4	-4	0	-1	1	0	0	0

A.3.3 The binary icosahedral group $\widetilde{H_3^+}$

1 Generators The full symmetry group of the icosahedron has order 120, and the rotational symmetry group is isomorphic to A_5. The double cover is thus an order 120 subgroup of Sp(1), which can be described using the golden ratio $\tau = (1+\sqrt{5})/2$ and the following three generators.

$$g_1 = i, \quad g_2 = j, \quad g_3 = -\frac{1}{2}(i + \tau j - \tau^{-1} k).$$

These generators satisfy the relations

$$(g_1 g_2)^4 = (g_1 g_3)^6 = (g_2 g_3)^{10} = 1.$$

2 Character table

	1	**2**	**4**	**3**	**6**	**5A**	**5B**	**10A**	**10B**
χ_1	1	1	1	1	1	1	1	1	1
χ_3	3	3	-1	0	0	$1-\tau$	τ	τ	$1-\tau$
$\chi_{3'}$	3	3	-1	0	0	τ	$1-\tau$	$1-\tau$	τ
$\chi_{4'}$	4	4	0	1	1	-1	-1	-1	-1
χ_5	5	5	1	-1	-1	0	0	0	0
χ_2	2	-2	0	-1	1	$\tau-1$	$-\tau$	τ	$1-\tau$
$\chi_{2'}$	2	-2	0	-1	1	$-\tau$	$\tau-1$	$1-\tau$	τ
χ_4	4	-4	0	1	-1	-1	-1	1	1
χ_6	6	-6	0	0	0	1	1	-1	-1

3 Explicit representations We present explicit choices of matrices for the real representations of these groups; these matrices are identical to the ones appearing in [44].

$$\rho_3(g_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \rho_3(g_2) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \rho_3(g_3) = -\frac{1}{2} \begin{bmatrix} 1 & -\tau & \tau^{-1} \\ -\tau & -\tau^{-1} & 1 \\ \tau^{-1} & 1 & \tau \end{bmatrix}.$$

The matrices for $\rho_{3'}$ are identical, with the substitution $\tau \mapsto -\tau^{-1}$ made in the matrix associated to g_3.

$$\rho_{4'}(g_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \rho_{4'}(g_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\rho_{4'}(g_3) = \frac{1}{4} \begin{bmatrix} -1 & \sqrt{5} & -\sqrt{5} & -\sqrt{5} \\ \sqrt{5} & 3 & 1 & 1 \\ -\sqrt{5} & 1 & -1 & 3 \\ -\sqrt{5} & 1 & 3 & -1 \end{bmatrix}, \quad \rho_5(g_1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\rho_5(g_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \rho_5(g_3) = \frac{1}{4} \begin{bmatrix} -1 & \sqrt{2} & -\sqrt{3} & \sqrt{2} & -\sqrt{8} \\ \sqrt{2} & 0 & -\sqrt{6} & 2 & 2 \\ -\sqrt{3} & -\sqrt{6} & 1 & \sqrt{6} & 0 \\ \sqrt{2} & 2 & \sqrt{6} & 2 & 0 \\ -\sqrt{8} & 2 & 0 & 0 & 2 \end{bmatrix}.$$

4 Invariant quaternionic maps The two-dimensional complex representations $2, 2'$ of $\widetilde{H_3^+}$ can be considered as one-dimensional quaternionic representations, where g_1, g_2, g_3 act as quaternion multiplication (on 2) and as quaternion multiplication after the substitution $\tau \mapsto -\tau^{-1}$) (on $2'$). If V is a real representation, then $V \otimes 2$ can be considered as a quaternionic representations with the real representation on V and described representation on the quaternionic factor.

Under this identification, these are the most general invariant quaternion maps $V \otimes 2 \to W \otimes 2$ as V, W range over irreducible representations of $\widetilde{H_3^+}$, and named $B_{W,V}$. By Schur's lemma, all the invariant maps are scalar multiples of these when $V \neq W$. When $V = W$ are real representations, maps $B_{V,V}$ are multiples of the identity, plus one additional dimension of maps if $V = 3, 5, 4'$; we use B_V to denote this additional map, so that an invariant map takes the form $aI + bB_V$ for some real parameters a, b. Note also that if $V \neq W$, then one may choose $B_{V,W} = B_{W,V}^T$. The maps B_V are skew-symmetric.

$$B_{5,3'} = \begin{bmatrix} i & j & -2k \\ 0 & -\sqrt{2}\tau^{-1}k & \sqrt{2}\tau j \\ -\sqrt{3}i & \sqrt{3}j & 0 \\ \sqrt{2}\tau^{-1}j & -\sqrt{2}\tau i & 0 \\ -\sqrt{2}\tau k & 0 & \sqrt{2}\tau^{-1}i \end{bmatrix},$$

$$B_{5,3} = \begin{bmatrix} i & -(\tau+1)j & \tau k \\ 0 & \sqrt{2}\tau k & \sqrt{2}\tau j \\ \left(\frac{2\sqrt{3}}{3} + \sqrt{\frac{5}{3}}\right) i & \left(\frac{\sqrt{3}}{6} - \frac{1}{2}\sqrt{\frac{5}{3}}\right) j & -\left(\frac{5\sqrt{3}}{6} + \frac{1}{2}\sqrt{\frac{5}{3}}\right) k \\ -\left(\sqrt{\frac{5}{2}} + \frac{\sqrt{2}}{2}\right) j & -\left(\sqrt{\frac{5}{2}} + \frac{\sqrt{2}}{2}\right) i & 0 \\ -\left(\sqrt{\frac{5}{2}} + \frac{\sqrt{2}}{2}\right) k & 0 & -\left(\sqrt{\frac{5}{2}} + \frac{\sqrt{2}}{2}\right) i \end{bmatrix},$$

$$B_{5,4'} = \begin{bmatrix} 0 & (1-3\sqrt{5})i & (1+3\sqrt{5})j & -2k \\ -2\sqrt{10}i & 0 & \sqrt{2}(3+\sqrt{5})k & \sqrt{2}(3-\sqrt{5})j \\ 0 & -\sqrt{3}(1+\sqrt{5})i & \sqrt{3}(1-\sqrt{5})j & 2\sqrt{15}k \\ 2\sqrt{10}k & -\sqrt{2}(3+\sqrt{5})j & -\sqrt{2}(3-\sqrt{5})i & 0 \\ 2\sqrt{10}j & -\sqrt{2}(3-\sqrt{5})k & 0 & -\sqrt{2}(3+\sqrt{5})i \end{bmatrix},$$

$$B_{4',3'} = \begin{bmatrix} i & j & k \\ 0 & \tau k & \tau^{-1} j \\ \tau^{-1} k & 0 & \tau i \\ \tau j & \tau^{-1} i & 0 \end{bmatrix}.$$

Note that $B_{5,4'}$ appeared in [44] with a typographical error in the $(4,3)$ entry—the corrected version is given above. The error was detected using the Sage library of the author [48], which automatically verifies the relations of [44, Equation (4.22)] when run.

$$B_3 = \begin{bmatrix} 0 & k & -j \\ -k & 0 & i \\ j & -i & 0 \end{bmatrix}, \quad B_{4'} = \begin{bmatrix} 0 & i & j & k \\ -i & 0 & -k & j \\ -j & k & 0 & -i \\ -k & -j & i & 0 \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 0 & 2\sqrt{2}(2+\sqrt{5})\tau^{-1} i & 0 & \frac{\sqrt{2}(5+\sqrt{5})}{2}\tau^{-1} k & -\frac{\sqrt{10}-\sqrt{2}}{2}\tau^{-1} j \\ -2\sqrt{2}(2+\sqrt{5})\tau^{-1} i & 0 & \sqrt{6}\tau^{-1} i & 2j & -2k \\ 0 & -\sqrt{6}\tau^{-1} i & 0 & \sqrt{6}k & -\sqrt{6}\tau j \\ -\frac{\sqrt{2}(5+\sqrt{5})}{2}\tau^{-1} k & -2j & -\sqrt{6}k & 0 & -2i \\ -\frac{\sqrt{10}-\sqrt{2}}{2}\tau^{-1} j & 2k & \sqrt{6}\tau j & 2i & 0 \end{bmatrix}.$$

The last sort of map between real representations is $B_{1,3} = \begin{bmatrix} i & j & k \end{bmatrix}$. Additionally, we give maps $C_{W,V} \colon V \otimes 2 \to W$; these are the other kinds of maps that appear when constructing quaternionic ADHM data.

$$C_{2',4'} = \begin{bmatrix} 1 & i & j & k \end{bmatrix}, C_{2,3} = B_{1,3}.$$

Appendix B

Ansatze for the ADHM equations

By an *ansatz* (plural: *ansatze*) for the ASD Yang–Mills equations, we mean a particular sort of guess made to make the problem tractable. Such ansatze have long been known for SU(2) instantons, and have been described in papers such as [21]. The purpose of this section is to briefly describe the most general of these ansatze, called either the Jackiw–Nohl–Rebbi (JNR) ansatz, or the Corrigan–Fairlie–'t Hooft–Wilczek (CFtHW) ansatz.

The basic idea in this ansatz is that the connection (and hence the curvature) should be derived from a 'scalar superpotential' $\rho\colon \mathbb{R}^4 \to \mathbb{R}$, and from it one writes down the ASD Yang–Mills equations as a PDE whose solutions may now be observed easily.

B.1 The 't Hooft and JNR ansatze

An excellent presentation of this material is available in [24], which we follow. The key ingredient is the exceptional isomorphism between the Lie algebra $\mathfrak{su}(2)$ and the Lie algebra of imaginary quaternions with quaternion multiplication as a bracket. One identifies $\mathbb{R}^4 = \mathbb{H}$ by sending $(x_0, x_1, x_2, x_3) \to x_0 + ix_1 + jx_2 + kx_3$. Then, define the differential operator $\frac{\partial}{\partial x}$ on functions $\mathbb{H} \to \mathbb{H}$ by $2\frac{\partial}{\partial x} = \partial_0 + i\partial_1 + j\partial_2 + k\partial_3$. Likewise, define

$$2\frac{\partial}{\partial \overline{x}} = \partial_0 - i\partial_1 - j\partial_2 - k\partial_3.$$

Finally, define corresponding differential forms $\mathrm{d}x = \mathrm{d}x^0 + i\mathrm{d}x^1 + j\mathrm{d}x^2 + k\mathrm{d}x^3$ and $\mathrm{d}\overline{x} = \mathrm{d}x^0 - i\mathrm{d}x^1 - j\mathrm{d}x^2 - k\mathrm{d}x^3$. The harmonic function ansatz then takes the following form

Theorem B.1.1 (Harmonic function ansatz). *Let $\rho\colon \mathbb{R}^4 \to \mathbb{R}$ be a positive smooth superpotential function. The connection defined by*

$$A = -\Im\left(\frac{\partial}{\partial \overline{x}}(\log \rho)\,\mathrm{d}\overline{x}\right)$$

is ASD if and only if ρ is harmonic. The action density $|F_A|^2$ of this connection is proportional to $\Delta\Delta \log \rho$.

Proof. See [24, Theorem 1] for one proof. □

Construction B.1.2 ('t Hooft ansatz)**.** For distinct points $x_1, \ldots, x_k \in \mathbb{R}^4$ and non-zero real numbers $\lambda_1, \ldots, \lambda_k$, the function

$$\rho(x) := 1 + \sum_{i=1}^{k} \frac{\lambda_i}{|x - x_i|^2}$$

is harmonic. The instanton corresponding to the superpotential ρ by Theorem B.1.1 is called the *'t Hooft instanton* associated to the data λ_i, x_i.

The ADHM data for a 't Hooft instanton is given in quaternion form (see Section 5.4.2) by letting $L = \begin{bmatrix}\lambda_1 & \cdots & \lambda_k\end{bmatrix}$ and $M = \operatorname{diag}(x_1, \ldots, x_k)$ under the identification $\mathbb{R}^4 = \mathbb{H}$. Thus the 't Hooft instanton associated to the data λ_i, x_i has charge k.

Construction B.1.3 (Corrigan–Fairlie–'t Hooft–Wilczek / Jackiw–Nohl–Rebbi ansatz)**.** For distinct points $x_1, \ldots, x_k \in \mathbb{R}^4$ and non-zero real numbers $\lambda_1, \ldots, \lambda_k$, the function

$$\rho(x) := \sum_{i=1}^{k} \frac{\lambda_i}{|x - x_i|^2}$$

is harmonic. The instanton corresponding to the superpotential ρ by Theorem B.1.1 is called the *JNR instanton* associated to the data λ_i, x_i, and was described in [21]. This ansatz was also described in independent unpublished reports by 't Hooft, Corrigan–Fairlie, and Wilczek; see the bibliography in [21].

The ADHM data for a JNR instanton is more complicated than the 't Hooft case, but still admits a description in terms of the quaternionic form. See [1, §8] for a description and example computations with ADHM data of JNR instantons.

B.2 The harmonic function ansatz for structure group $\mathrm{SU}(2m)$

The harmonic function ansatz offers a natural generalization to higher rank structure groups—to which the author has no reference, although it is presumably well-known. Before presenting it, it is easiest to motivate by looking at the ADHM data: the harmonic function ansatz is roughly equivalent to the linearization of the ADHM equations.

Consider the presentation in Section 5.4.3. Then ADHM data is given by four matrices (I, J, B_1, B_2), and expanding the left-hand side of Equation (5.4) gives an alternate form for the ADHM equations:

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0 \quad \text{and} \quad [B_1, B_2] + IJ = 0.$$

To find special solutions to these equations, one then supposes that the commutators $[B_1, B_1^\dagger], [B_2, B_2^\dagger]$, and $[B_1, B_2]$ all vanish. For example, in the 't Hooft ansatz for structure group $\mathrm{SU}(2)$, B_1 is diagonal and B_2 is the zero matrix, so the brackets vanish.

With this ansatz made, the ADHM equations reduce to the two equations $IJ = 0, II^\dagger = J^\dagger J$. Additionally, the non-singularity condition implies that the matrix $C = \begin{bmatrix} I \\ J^\dagger \end{bmatrix}$ has rank N. Let $v_1, \ldots, v_k$ be the rows of I, and $w_1, \ldots, w_k$ the columns of J. Then with $\cdot$ the standard hermitian inner product on $\mathbb{C}^N$, the equation $IJ = 0$ says that $v_i \cdot w_j = 0$ for all i, j;

in other words, the span of the v_i is orthogonal as a subspace of $\mathbb{C}^N$ to the span of the w_i. Since C has rank N, it follows that I, J gives an orthogonal splitting $\mathbb{C}^N = \mathbb{C}^r \oplus \mathbb{C}^{N-r}$ of $\mathbb{C}^N$ into the subspaces spanned by the v_i and w_i respectively. Thus up to a unitary rotation of the ADHM data, one may assume that the v_i are zero in the last $N-r$ components, and the w_i are zero in the first r components.

The second equation, $II^\dagger = J^\dagger J$ states that $v_i \cdot v_j = w_i \cdot w_j$ for all i, j. Choose a subset of the rows of I that form a basis for the subspace it spans; the matrix of the inner products of all pairs of these rows are then invertible. The corresponding subset of the columns of J has the same (invertible) matrix of inner products, and is thus linearly independent. The dimension of the span of the columns of J is therefore at least as large as the dimension of the span of the rows of V. In other words, $N - r \geq r$; but reversing this argument, one has also that $N - r \leq r$ and so N is even and $r = \frac{N}{2}$. Moreover, to preserve the inner products it then follows that there is some $r \times r$ unitary matrix U and $k \times r$ matrix A so that

$$I = \begin{bmatrix} A & 0 \end{bmatrix}, J = \begin{bmatrix} 0 \\ UA^\dagger \end{bmatrix}.$$

This situation is analogous to the ADHM data for the 't Hooft ansatz in SU(2): in this case A is a $k \times 1$ matrix; that is, the list of scale parameters $\lambda_1, \ldots, \lambda_k$. For the SU($2m$) case, instead one has 'multi-weights' for each $i = 1, \ldots, k$, consisting of m different complex parameters. The right generalization should therefore replace the single harmonic function ρ of the harmonic function ansatz with m harmonic functions $\rho_1, \ldots, \rho_m$, and use these to build the connection one-form. The key to this process is the inclusion $\mathrm{SU}(2m) \supset \mathrm{Sp}(m)$, which at the Lie algebra level implies that one may choose m different orthogonal mutually commuting copies of the Lie algebra $\mathfrak{su}(2)$ within $\mathfrak{su}(2m)$. Call these subalgebras $\mathfrak{g}^1, \ldots, \mathfrak{g}^m$, and pick for each $\mathfrak{g}^\mu$ elements $\mathfrak{g}^\mu_i, \mathfrak{g}^\mu_j, \mathfrak{g}^\mu_k$ satisfying

$$[\mathfrak{g}^\mu_i, \mathfrak{g}^\mu_j] = \mathfrak{g}^\mu_k, \quad [\mathfrak{g}^\mu_j, \mathfrak{g}^\mu_k] = \mathfrak{g}^\mu_i, \quad [\mathfrak{g}^\mu_i, \mathfrak{g}^\mu_k] = -\mathfrak{g}^\mu_j.$$

One may do this because each $\mathfrak{g}^\mu$ is isomorphic to the imaginary quaternions, with the bracket being quaternion multiplication and projection to the imaginary part.

Define as before for each μ a symbol $\mathfrak{g}^\mu_1$, and extend the bracket so that $[\mathfrak{g}^\mu_1, \cdot] = 0$. Then define the differential operator $\frac{\partial_\mu}{\partial_\mu \overline{x}}$, identifying the real span of $\mathfrak{g}^\mu_1, \mathfrak{g}^\mu_i, \mathfrak{g}^\mu_j, \mathfrak{g}^\mu_k$ with $\mathbb{H}$. Likewise, define as before the $\mathfrak{g}^\mu$-valued one-form $\mathrm{d}_\mu \overline{x}$.

Theorem B.2.1 (Generalized harmonic function ansatz)**.** *Let $\rho_1, \ldots, \rho_m : \mathbb{R}^4 \to \mathbb{R}$ be positive smooth functions. The $\mathfrak{su}(2m)$-valued connection defined by*

$$A = -\sum_{\mu=1}^{m} \Im \left(\frac{\partial_\mu}{\partial_\mu \overline{x}} (\log \rho_\mu) \, \mathrm{d}_\mu \overline{x} \right)$$

is ASD if and only if each ρ_μ is harmonic. The action density is proportional to $\sum_\mu \Delta\Delta \log \rho_\mu$.

Proof. The proof is immediate from Theorem B.1.1 and the fact that the $\mathfrak{g}^\mu$ mutually commute and are direct summands; for it is these conditions that linearize the ASD equation. To be precise, if one writes A^μ for $-\Im\left(\frac{\partial_\mu}{\partial_\mu \overline{x}} (\log \rho_\mu) \, \mathrm{d}_\mu \overline{x}\right)$, then one uses the fact that $[g^\mu, g^\nu] = 0$ for $\mu \neq \nu$ to get

$$F_A = \mathrm{d}A + \frac{1}{2}[A, A]$$

$$
\begin{aligned}
&= \sum_\mu \mathrm{d}A^\mu + \frac{1}{2}\left[\sum_\mu A^\mu, \sum_\mu A^\mu\right] \\
&= \sum_\mu \left(\mathrm{d}A^\mu + \frac{1}{2}\left[A^\mu, A^\mu\right]\right) \\
&= \sum_\mu F_{A^\mu}.
\end{aligned}
$$

Clearly if each F_{A^μ} is ASD, then F_A is also. Conversely, if F_A is ASD, then the g^μ part of $\star F_A$ is given by $\star F_{A^\mu}$, because F_{A^μ} is $\mathfrak{g}^\mu$-valued, and the sum $\sum_\mu \mathfrak{g}^\mu$ is direct by hypothesis. Thus $\star F_{A^\mu} = -F_{A^\mu}$ for each μ.

By Theorem B.1.1, each F_{A^μ} being ASD holds if and only if all the ρ_μ are harmonic. The statement about the action density (that is, $|F_A|^2$) likewise follows from Theorem B.1.1 together with the orthogonality of the $\mathfrak{g}^\mu$. □

The generalization of Construction B.1.3 is formed by picking a set of distinct points $x_\mu \in \mathbb{R}^4$, and to each point x_μ a set of m real weights $\lambda_1^\mu, \ldots, \lambda_m^\mu$ not all equal to zero, and using the harmonic functions

$$\rho_\mu(x) = \sum_{i=1}^{m} \frac{(\lambda_i^\mu)^2}{|x_i - x|^2}.$$

Example B.2.2. Let $\mathfrak{g}_i = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \mathfrak{g}_j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \mathfrak{g}_k = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. These matrices form a basis for $\mathfrak{su}(2)$ which satisfies the relations $\mathfrak{g}_i^2 = \mathfrak{g}_j^2 = \mathfrak{g}_k^2 = \mathfrak{g}_i\mathfrak{g}_j\mathfrak{g}_k = -1$. With the addition of the matrix $\mathfrak{g}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the real span of $\{\mathfrak{g}_1, \mathfrak{g}_i, \mathfrak{g}_j, \mathfrak{g}_k\}$ is a subalgebra of $\mathfrak{gl}(\mathbb{C})$ isomorphic to the quaternions with the multiplication $p \cdot q := \Im(pq)$.

For $\mathfrak{su}(2m)$, one may take $\mathfrak{g}_\nu^\mu$ to be the $2m \times 2m$ block diagonal matrix with $\mathfrak{g}_\nu$ in the μ'th 2×2 block on the diagonal, and then $\mathfrak{g}^\mu$ to be the subalgebra spanned by $\mathfrak{g}_i^\mu, \mathfrak{g}_j^\mu, \mathfrak{g}_k^\mu$. Defined in this way, the algebras $\mathfrak{g}^\mu$ for $\mu = 1, \ldots, m$ are direct summands that mutually commute, and so may be used to generate SU($2m$)-instantons via Theorem B.2.1.

Appendix C

Solutions to the symmetric ADHM equations

All action density plots in this thesis were made using the implicit plot functionality of MAPLE [30] and the native C libraries of the author available at [48].

C.1 Instantons symmetric under finite groups

Table C.1 contains a table of all instantons symmetric under finite groups of Spin(4) that appear in the literature and do not result from the JNR ansatz (Construction B.1.3). In addition to these non-JNR instantons are displayed a selection of symmetric JNR instantons formed by placing singularities at the vertices of a polytope of the corresponding symmetry. If a particular JNR instanton has been explicitly discussed in the literature, a reference to this work is provided; else, the reference row contains the original JNR paper [21]. References of the form **X.Y.Z** are internal to this thesis.

The 'notes' column in Table C.1 uses the following codes:

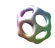
PC 'polytope conjectural'; the entry in the 'polytope' column is conjectured to be the convex hull of the instantope based on numerical approximations to the action density (in the sense of Section 6.2), but this relation has not been formally shown

LC 'lowest charge'; this instanton is the lowest charge instanton at which the given polytope appears as the (conjecturally) associated instanton

JNR 'Jackiw–Nohl–Rebbi'; this instanton results from the JNR ansatz of [21] (Construction B.1.3).

C.2 Explicit solutions to icosahedral ADHM equations

In this appendix, special solutions (that is, those not arising from the harmonic function ansatz) to the symmetric ADHM equations are presented. For all presented instantons the group of symmetries is $G = \widetilde{H_3^+}$, and the notation of Appendix A.3.3 is used.



Group	Charge	Polytope	Wythoff	Reference	Notes
$\widetilde{A_3^+}$	3	Tetrahedron	⊙—•—•	[29]	JNR, LC
$\widetilde{B_3^+}$	4	Cube	⊙—4—•—•	[27]	LC, PC
$\widetilde{B_3^+}$	5	Octahedron	•—4—•—⊙	[21]	JNR, LC
$\widetilde{H_3^+}$	7	Dodecahedron	⊙—5—•—•	[42]	LC, PC
$\widetilde{H_3^+}$	11	Icosahedron	•—5—•—⊙	[21]	JNR, LC
$\widetilde{H_3^+}$	13	Dodecahedron	⊙—5—•—•	C.2.2	PC
$\widetilde{H_3^+}$	17	Truncated icosahedron	•—5—⊙—⊙	[44], [45]	LC, PC
$\widetilde{H_3^+}$	23	Icosidodecahedron	•—5—⊙—•	C.2.4	LC, PC
$\widetilde{H_3^+}$	29	Icosidodecahedron	•—5—⊙—•	[21]	JNR
$\widetilde{A_4^+}$	4	5-cell	•—•—•—⊙	[1]	JNR, LC
$\widetilde{B_4^+}$	7	16-cell	•—4—•—•—⊙	[1]	JNR, LC
$\widetilde{F_4^+}$	23	24-cell	•—•—4—•—⊙	[1]	JNR, LC
$\widetilde{H_4^+}$	119	600-cell	•—5—•—•—⊙	5.6.2	JNR, LC

Table C.1: A table of all known non-JNR instantons and a selection of JNR instantons symmetric under finite subgroups of Spin(4). The column 'Wythoff' contains the Wythoff symbol as described in the discussion following Theorem 2.2.3; the circled vertices indicate the set S to be taken in the theorem.

C.2.1 The dodecahedral 7-instanton

This solution first appeared in [42]. At prime charge 7, one uses the representation $V = 3' \oplus 4'$, and $W = 2'$. The resulting family of solutions is two-dimensional, with a non-singular instanton for each $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$ given by ADHM data

$$L = a\begin{bmatrix}0 & C_{2',4'}\end{bmatrix}, \quad M = a\begin{bmatrix} bI_4 & B^T_{4',3'} \\ B_{4',3'} & bI_3 \end{bmatrix}.$$

A plot of a level set of the action density is shown in Figure C.1. Additionally, plots of various level sets of the action density of this instanton are displayed in the top-right corner of even arabic-numbered pages.

C.2.2 The dodecahedral 13-instanton

This solution is new, to the extent of the author's knowledge. At prime charge 13, one uses the representation $V = \mathrm{res}_G(W_{13}) = 1 \oplus 3 \oplus 5 \oplus 4'$, and $W = 2'$. Let $\alpha := \sqrt{\frac{1}{5}(3-\sqrt{5})} \approx$

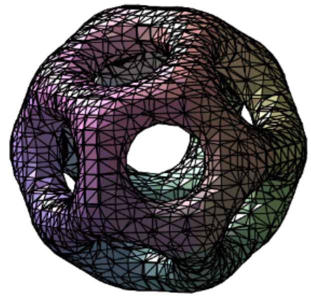


Figure C.1: A level set of the action density for the dodecahedral 7-instanton

0.3908 .... The resulting family of solutions is two-dimensional, with a non-singular instanton for each $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$ given by ADHM data

$$L = a \begin{bmatrix} 0 & 0 & 0 & C_{2',4'} \end{bmatrix},$$

$$M = a \begin{bmatrix} bI_1 & \sqrt{4/3}B_{1,3} & 0 & 0 \\ \sqrt{4/3}B_{3,1} & bI_3 & \alpha B_{3,5} & 0 \\ 0 & \alpha B_{5,3} & bI_5 & \frac{1}{\sqrt{40}}B_{5,4'} \\ 0 & 0 & \frac{1}{\sqrt{40}}B_{4',5} & bI_4 \end{bmatrix}.$$

C.2.3 The truncated dodechaderal 17-instanton

This solution first appeared in [44, p. 9], and the presentation given here replicates Sutcliffe's solution nearly identically (in the notation of this thesis). At prime charge 17, one uses the representation $V = \text{res}_G(W_{17}) = 3' \oplus 4' \oplus 5 \oplus 5$, and $W = 2'$. The resulting family of solutions is three-dimensional, depending on parameters $a \in \mathbb{R}^*, b \in \mathbb{R}$, and $\theta \in S^1$. Two instantons in this family (a, b, θ) and (a', b', θ') are gauge-equivalent if and only if $a' = a$ and $b' = b$, so modulo gauge equivalence the family of solutions is two-dimensional, making this representation quasi-irreducible. Let

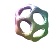
$$\alpha_1 = \frac{2}{3}, \quad \alpha_2 = \frac{\sqrt{2}}{3}\sin\theta, \quad \alpha_3 = \frac{\sqrt{2}}{3}\cos\theta$$

$$\alpha_4 = -\frac{1}{6\sqrt{2}}\sin\theta, \quad \alpha_5 = \frac{1}{3}, \quad \alpha_6 = \frac{1}{6\sqrt{2}}\cos\theta.$$

The general form of the ADHM data is

$$L = a \begin{bmatrix} 0 & C_{2',4'} & 0 & 0 \end{bmatrix},$$

$$M = a \begin{bmatrix} bI_{3'} & \alpha_1 B_{3',4'} & \alpha_2 B_{3',5} & \alpha_3 B_{3',5} \\ \alpha_1 B_{4',3'} & bI_{4'} & \alpha_6 B_{4',5} & \alpha_4 B_{4',5} \\ \alpha_2 B_{5,3'} & \alpha_6 B_{5,4'} & bI_5 & \alpha_5 B_5 \\ \alpha_3 B_{5,3'} & \alpha_4 B_{5'4'} & \alpha_5 B_5 & bI_5 \end{bmatrix}.$$



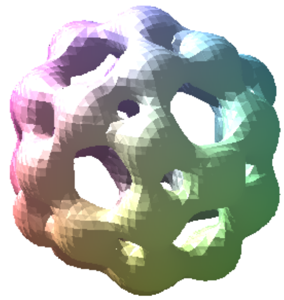


Figure C.2: A level set of the action density for the icosidodecahedral 23-instanton

Since the gauge equivalence class of the instanton obtained does not depend on θ, one can make a choice of, for example, $\theta = 0$ so that $\alpha_2 = \alpha_4 = 0$ and $\alpha_3 = \frac{\sqrt{2}}{3}, \alpha_6 = \frac{1}{6\sqrt{2}}$. As well, taking $a = 1$ and $b = 0$ to get a particular member of this family, then the ADHM data are

$$L = \begin{bmatrix} 0 & C_{2',4'} & 0 & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} 0 & \frac{2}{3}B_{3',4'} & 0 & \frac{\sqrt{2}}{3}B_{3',5} \\ \frac{2}{3}B_{4',3'} & 0 & \frac{1}{6\sqrt{2}}B_{4',5} & 0 \\ 0 & \frac{1}{6\sqrt{2}}B_{5,4'} & 0 & \frac{1}{3}B_5 \\ \frac{\sqrt{2}}{3}B_{5,3'} & 0 & \frac{1}{3}B_5 & 0 \end{bmatrix}.$$

C.2.4 The icosidodecahedral 23-instanton

This solution is new, to the extent of the author's knowledge. At prime charge 17, one uses the representation $V = \mathrm{res}_G(W_{23}) = 3 \oplus 3 \oplus 5 \oplus 5 \oplus 3' \oplus 4'$, and $W = 2'$. As in the case of the truncated icosahedral instanton of [44], there is some freedom in the family of solutions, although all solutions are gauge equivalent up to an overall scale and translation parameter, making this representation quasi-irreducible. Let $\alpha_1 = -\sqrt{\frac{3}{5(3+\sqrt{5})}}$ and $\alpha_2 = \frac{1}{3\sqrt{5}}$. The ADHM data for a particular member of the family (the one used to generate the Figure C.2) are

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & B_{2',4'} \end{bmatrix},$$

$$M = \begin{bmatrix} 0 & B_3 & 0 & \alpha_1 B_{3,5} & 0 & 0 \\ -B_3 & 0 & \alpha_1 B_{3,5} & 0 & 0 & 0 \\ 0 & \alpha_1 B_{5,3} & 0 & \frac{1}{6}B_5 & 4\alpha_2 B_{5,3'} & 0 \\ \alpha_1 B_{5,3} & 0 & -\frac{1}{6}B_5 & 0 & 0 & \alpha_2 B_{5,4'} \\ 0 & 0 & 4\alpha_2 B_{3',5} & 0 & 0 & -\frac{1}{3}B_{3',4'} \\ 0 & 0 & 0 & \alpha_2 B_{4',5} & -\frac{1}{3}B_{4',3'} & 0 \end{bmatrix}.$$

A plot of a level set of the action density is shown in Figure C.2. Additionally, plots of various level sets of the action density of this instanton are displayed in the top-left corner of odd arabic-numbered pages.

Appendix D

A proof of the equivariant spin theorem

The purpose of this appendix is to prove Theorem 4.4.2. Throughout, X^{2n} denotes a compact oriented spin manifold. The material of this appendix is contained entirely within [25], although not in the order presented. To elucidate the proof of the equivariant spin theorem, all the required results are collected below with references to where they may be found in the aforementioned text.

D.1 Proof of the equivariant spin theorem

Theorem D.1.1. *Let G be a group of spin-structure preserving isometries of X. Then for all $g \in G$, there exists a locally constant integer-valued function d on the connected components of X^g such that*

$$\operatorname{ind}_g(D^+_\nabla) = \sum_Y (-1)^{d(Y)} \left(\prod_{0<\theta<\pi} \hat{\mathbf{A}}_\theta(N_g(\theta)) \hat{\mathbf{A}}(TY) \right) [Y]; \tag{D.1}$$

Proof. This result is [25, III, Theorem 14.11]; the equivariant spin theorem without bundle coefficients. □

Theorem D.1.2. *Let M be compact, and $\pi\colon E \to M$ an oriented real vector bundle with a spin structure. Let $S^+(E)$ and $S^-(E)$ denote the complex spinors of E, and let μ denote Clifford multiplication. Then*

$$\mathbf{s}(E) := [\pi^* S^+(E), \pi^* S^-(E); \mu] \in K_{\mathrm{cpt}}(E)$$

is a K-theory orientation for E.

Proof. This result is [25, Theorem C.12]. □

Lemma D.1.3. *The principal symbol of the (positive) Dirac operator on a spin manifold is a generator of the Thom isomorphism. That is,*

$$\sigma = \mathbf{s}(TX).$$

Proof. The first statement is [25, §III.1.8]. The second is a consequence of Theorem D.1.2. □

Lemma D.1.4. *The principal symbol of the (positive) Dirac operator on* $\pi\colon TX \otimes E \to X$ *for* E *a hermitian bundle is*

$$\sigma_E = \sigma \cdot \pi^* E = \mathbf{s}(E) \cdot \pi^* E \in K_{\mathrm{cpt}}(TX \otimes E).$$

Here σ *is the principal symbol of the positive Dirac operator on* TX*.*

Proof. As in [25, III, Equation (12.16)], one has for S^+, S^- the spinor bundles of X and μ Clifford multiplication that

$$\sigma \cdot \pi^* E = \mathbf{s}(TX) \cdot \pi^* E = [\pi^* S^+ \otimes \pi^* E, \pi^* S^- \otimes \pi^* E; \mu \otimes \mathrm{id}].$$

The first equality is by Lemma D.1.3, while the second follows from multiplicativity in the Atiyah–Bott–Shapiro construction, which is obvious from the definition (see, for example, [25, §I.9]).

Let S_E^+ and S_E^- be the spinor bundles with E coefficients, and μ_E Clifford multiplication in this case. Then $S_E^+ = S^+ \otimes E$ and $S_E^- = S^- \otimes E$ (see [25, III, Construction 11.23], together with the fact that μ_E does not act on the E term of a tensor [25, p. 121]). Then

$$[\pi^* S_E^+, \pi^* S_E^-; \mu_E] = [\pi^* S^+ \otimes \pi^* E, \pi^* S^- \otimes \pi^* E; \mu \otimes \mathrm{id}].$$

On the other hand, by [25, III, Equation (1.7)],

$$\sigma_E = [\pi^* S^+ \otimes \pi^* E, \pi^* S^- \otimes \pi^* E; \mu \otimes \mathrm{id}] = \sigma \cdot \pi^* E$$

as desired. □

From the above, the proof of the equivariant spin theorem with coefficients follows easily.

Proof of Theorem 4.4.2. Let $\pi\colon TX \otimes E \to X$ be projection, and let σ_E, σ denote respectively the principal symbols of the Dirac operator with coefficients in E and the Dirac operator with no bundle coefficients, as elements of $K_{\mathrm{cpt}}(TX \otimes E)$. By Lemma D.1.4, $\sigma_E = \sigma \cdot \pi^* E$. With $i\colon X \to TX \otimes E$ given by the identification of X with the zero section, then $i^* \pi^* = \mathrm{id}$ yields

$$\mathrm{ch}_g(i^* \sigma_E) = \mathrm{ch}_g(i^*(\sigma \cdot \pi^* E)) = \mathrm{ch}_g(i^* \sigma \cdot E) = \mathrm{ch}_g(E)\mathrm{ch}_g(i^* \sigma).$$

Now apply Theorem 4.4.1:

$$\begin{aligned} \mathrm{ind}_g(D_E) &= \sum_Y (-1)^{\dim Y} \left(\frac{\mathrm{ch}_g(i^* \sigma_E)}{\mathrm{ch}_g(\lambda_{-1}(N_g \otimes \mathbb{C}))} \hat{\mathbf{A}}(TY)^2 \right) [TY] \\ &= \sum_Y (-1)^{\dim Y} \left(\mathrm{ch}_g(E) \frac{\mathrm{ch}_g(i^* \sigma)}{\mathrm{ch}_g(\lambda_{-1}(N_g \otimes \mathbb{C}))} \hat{\mathbf{A}}(TY)^2 \right) [TY] \end{aligned}$$

On the other hand,

$$\mathrm{ind}_g(D) = \sum_Y (-1)^{\dim Y} \left(\frac{\mathrm{ch}_g(i^* \sigma)}{\mathrm{ch}_g(\lambda_{-1}(N_g \otimes \mathbb{C}))} \hat{\mathbf{A}}(TY)^2 \right) [TY].$$

So the index of D_E is obtained by multiplying in cohomology by $\mathrm{ch}_g(E)$ before evaluating at a fundamental class. Thus Theorem 4.4.2 follows from Theorem D.1.1. More formally, one may follow the proof at [25, p. 267] to rewrite

$$\frac{\mathrm{ch}_g(i^*\sigma)}{\mathrm{ch}_g(\lambda_{-1}(N_g \otimes \mathbb{C}))}\hat{\mathbf{A}}(TY)^2 = \pm \prod_{0<\theta\leq\pi} \hat{\mathbf{A}}_\theta(N_g(\theta))\hat{\mathbf{A}}(TY),$$

whence the result. □