State Transfer & Strong Cospectrality in Cayley Graphs

by

Arnbjörg Soffía Árnadóttir

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Combinatorics & Optimization

Waterloo, Ontario, Canada, 2022

Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner: Joy Morris
Professor, Dept. of Mathematics & Computer Science,
University of Lethbridge

Supervisor: Chris Godsil
Professor, Dept. of Combinatorics & Optimization,
University of Waterloo

Internal Member: Penny Haxell
Professor, Dept. of Combinatorics & Optimization,
University of Waterloo

Internal Member: Oliver Pechenik
Professor, Dept. of Combinatorics & Optimization,
University of Waterloo

Internal-External Member: William Slofstra
Professor, Dept. of Pure Mathematics,
University of Waterloo

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis is a study of two graph properties that arise from quantum walks: strong cospectrality of vertices and perfect state transfer. We prove various results about these properties in Cayley graphs.

We consider how big a set of pairwise strongly cospectral vertices can be in a graph. We prove an upper bound on the size of such a set in normal Cayley graphs in terms of the multiplicities of the eigenvalues of the graph. We then use this to prove an explicit bound in cubelike graphs and more generally, Cayley graphs of $\mathbb{Z}_2^{d_1} \times \mathbb{Z}_4^{d_2}$. We further provide an infinite family of examples of cubelike graphs (Cayley graphs of $\mathbb{Z}_2^d$) in which this set has size at least four, covering all possible values of d.

We then look at perfect state transfer in Cayley graphs of abelian groups having a cyclic Sylow-2-subgroup. Given such a group, G, we provide a complete characterization of connection sets $\mathcal{C}$ such that the corresponding Cayley graph for G admits perfect state transfer. This is a generalization of a theorem of Bašić from 2013, where he proved a similar characterization for Cayley graphs of cyclic groups.

Acknowledgements

First and foremost, I am extremely grateful to Chris Godsil, for taking me in, for everything he has taught me, for giving me so much of his time and for all his guidance, patience and support. I could not have imagined a better supervisor.

I thank my committee members: Joy Morris, Penny Haxell, Oliver Pechenik and William Slofstra for reading my thesis and providing discussions and comments that have greatly improved it.

Others among the faculty and staff at the C&O department deserve many thanks. Most notably, I want to thank Karen Yeats and Melissa Cambridge for always handling things that need to be handled.

I would not have made it through this program without the support of my friends. Tina Chen and Sabrina Lato have been solid companions throughout my PhD. Sabrina in particular has not only given me friendship and academic feedback, but also a roof over my head and has always made sure that I'm well fed. I'm grateful to my office mates, in particular my soul mate, Naomi Graham for the many laughs and many tears we shared. I thank Matt Kroeker for his invaluable companionship, kindness, laughter and support, particularly during the last few months of my PhD.

Other friends who have kept me sane for the past five years include Max Levit, Adina Goldberg, Evelyne Smith-Roberge, Sean Kafer, James Davies, Lukas Nabergall, Ben Moore, Tim Miller, Shayla Redlin, Nick Olson-Harris, Fernanda Rivera, Valerie Gilchrist and many others. Special thanks to my friend, Þorsteinn Hjörtur, for all his help and great influence on my life—without him I would never have found myself in Canada in the first place.

I want to thank my brothers Sigurður and Ragnar, and my sisters-in-law, Elín and Karítas for their friendship and my niece Elísa Rán and nephew Árni Kristján who have brought so much joy into my life.

Last but not least, I can barely begin to express my thanks to my parents. They have not only made me the person I am, but even to this day they keep supporting me on all fronts and over long distances. My mother and I have become closer than ever through my PhD experience and my father has provided endless emotional support and good life advice. I owe them everything, and have therefore dedicated this thesis to them.

Dedication

Til mömmu og pabba

To my parents

Table of Contents

List of Figures

Chapter 1

Introduction

Having read the title of this thesis (if you have not, we strongly encourage you to do so), you may already have some questions, including:

1. What is state transfer?

2. What is strong cospectrality?

3. What is a Cayley graph?

4. Why should I care about any of the above?

These are all good questions you have asked and we will attempt to answer some of them in this introduction.

The first section attempts to provide some motivation and background. In the second section we state our main results.

1.1 Background & motivation

Before we can define state transfer, we need to talk about quantum walks. We will do this briefly in our first subsection. The next three subsections explore state transfer, strong cospectrality and Cayley graphs respectively.

1.1.1 Quantum walks

Quantum walks arise in quantum information theory and quantum physics. They have for instance been used in quantum computation to implement algorithms on graphs. These include Grover's search [35], Ambainis' element distinctness [1] and Farhi and Gutmann's algorithm on decision trees [25]. It turns out that these algorithms perform significantly better than their classical analogues.

The first two of these algorithms come from discrete-time quantum walks. Discrete-time quantum walks have received much attention because of their algorithmic applications and are for instance the topic of Zhan's PhD thesis [47]. We are however interested in quantum walks happening in continuous time, and this was first defined by Farhi and Gutmann in the aforementioned paper.

You may already be thinking that the words *quantum* and *algorithm* have appeared rather too often for a thesis in algebraic graph theory, but do not worry. This will be the last use of the word *algorithm*, and one of the goals of this thesis is to convince you that there is no need to fear the word *quantum* as long as you are comfortable with linear algebra.

We will define continuous-time quantum walks properly in the next chapter, but for now, let us think of them as quantum analogues of continuous-time random walks: given a graph, a walker starts at a particular vertex, then moves around the graph in some fashion. At each time t, there is a certain probability that we will find the walker at a given vertex.

In the case of a quantum walk, we think of this system as the Hilbert space, $\mathbb{C}^V$, where V is the vertex set of our graph. A *quantum state* in this system is a 1-dimensional subspace of our Hilbert space, which we typically represent by a unit vector in that subspace. So, for instance, our walker being situated at vertex v of the graph means that the system is in the state represented by the standard basis vector $\mathbf{e}_v$. A quantum walker however, is not necessarily situated at one particular vertex at any given time, but could be in a *superposition* of multiple vertices, since she can live in any 1-dimensional subspace of $\mathbb{C}^V$.

1.1.2 State transfer

We can now return to your first question: what is state transfer? The concept was first introduced by Bose in 2003 [10] and simply refers to a transfer of our system from one state to another. For a graph with adjacency matrix A, we define the matrix $U(t) := e^{itA}$, with $t \in \mathbb{R}$. If u and v are vertices of the graph, the number $|U(t)_{u,v}|^2$ is the probability that the system has been transferred from the state $\mathbf{e}_u$ to $\mathbf{e}_v$ at time t, and we are interested in the case where this probability is equal to one. This is called *perfect state transfer*.

Perfect state transfer is a somewhat strange behaviour of a walk, and it is in fact quite useful in quantum physics. By the *No-Cloning Theorem* [46], a quantum state cannot be copied, and it is therefore significant when it can be efficiently transferred.

Perfect state transfer has been widely studied, the main question being: in which graphs does it occur? It was first proposed by Christandl et al. in 2004 [18], where they proved various results, including that the hypercubes admit perfect state transfer. It turns out, however, that this phenomenon is quite rare, and much of the work that has been done in this area consists of negative results and restrictive necessary conditions.

In 2005 [17], Christandl et al. proved that the path on n vertices, P_n, has perfect state transfer between its end vertices if and only if $n \in \{2, 3\}$, and Godsil later extended this result to show that the paths on at least four vertices admit no perfect state transfer between any pair of vertices [30, Section 14]. Kay wrote several papers on perfect state transfer [37, 38, 39], in one of which he observes that perfect state transfer is a monogamous relationship between two vertices [39, Section III.D]. In 2012, Godsil proved that for any integer k, there are only finitely many connected graphs with maximum degree k that admit perfect state transfer [31]. In another paper from 2012, Godsil also showed that a necessary condition for perfect state transfer to occur between two vertices in a graph is that they are *strongly cospectral* [30].

1.1.3 Strong cospectrality

Vertices u and v in a graph X are *cospectral* if the adjacency matrices of the graphs $X \setminus u$ and $X \setminus v$ have the same spectrum. As you might guess, strong cospectrality is a stronger condition, but for a definition of this, you will have to wait until Chapter 5. The concept was first defined by Fan and Godsil in 2012 [24], and the motivation came from quantum walks.

Strong cospectrality however is not a property of a quantum walk, but rather a spectral property of the graph itself and it has some interesting combinatorial implications. This is explored extensively in a paper by Godsil and Smith [34]. They show for example that if vertices u and v are strongly cospectral, then any graph automorphism that fixes u also fixes v. In the same paper, they ask the question of whether there exist trees that have at least three pairwise strongly cospectral vertices. Coutinho, Juliano and Spier recently gave a negative answer to this question, showing that no such trees exist [22]. More recent work has been done on strongly cospectral vertices by Monterde [41] and by Sin [43].

Being a necessary condition for perfect state transfer to occur, strong cospectrality is an important notion in the study of quantum walks. It is however much weaker than perfect state transfer, which is shown for instance by the fact that perfect state transfer occurs only between a pair of vertices, whereas a set of pairwise strongly cospectral vertices (which we will call a *strongly cospectral set*) can have size larger than two.

The smallest example we have of such a set is in the Cartesian product of P_2 and P_3, where the four vertices of degree two are pairwise strongly cospectral. In fact, by taking Cartesian products of paths, it is possible to construct graphs with arbitrarily large strongly cospectral sets. These graphs however become quite big, and so we can ask the question: in a graph on n vertices, how large can a strongly cospectral set be, as a function of n? Godsil and Smith show in their paper that if the graph is not K_2, then this set cannot contain all the vertices in the graph, but not much more progress has been made on this question.

1.1.4 Cayley graphs

Finally, let us talk about Cayley graphs. A Cayley graph is defined by a group G and a subset $\mathcal{C}$ of this group. It is defined in such a way that many of the graph properties can be studied using group theory and vice versa. These graphs also have a lot of symmetry, and it turns out that they are particularly nice for studying perfect state transfer and strong cospectrality.

In fact, many (if not most) of the known examples of graphs that admit perfect state transfer are Cayley graphs. This includes the hypercubes, as mentioned before, but more generally most cubelike graphs (Cayley graphs for $\mathbb{Z}_2^d$). This is the topic of various papers including [9, 15, 14]. There are also examples of circulants (Cayley graphs of cyclic groups) that admit perfect state transfer as explored by Bašić, Petković and Stevanović in 2009–2010 [8, 6, 7]. In 2013 [5], Bašić gave a complete characterization of such graphs.

Further work has been done on perfect state transfer on Cayley graphs, for example by Tan, Feng and Cao on abelian groups [45], Cao and Feng on dihedral groups [13], Arezoomand, Shafiei and Ghorbani on dicyclic groups [2] and by Sin and Sorci on extraspecial groups [44].

In a Cayley graph, all the vertices are cospectral. They therefore seem like good candidates to look for large strongly cospectral sets. It turns out however that this might not be the case.

1.2 Main results

The thesis is organized as follows. In Chapter 2, we discuss quantum walks in more detail and introduce some of the tools that we need to study them. We further state some preliminary results about perfect state transfer. In Chapter 3 we talk briefly about Cayley graphs and group actions.

Chapter 4 is on association schemes which play an important role in many of our proofs. The results in this chapter are mostly known, but the topic of association schemes is far less popular than it should be and there are some gaps in the literature. In particular, not much has been written on translation schemes and duality which we cover in the last two sections of this chapter. Therefore, some theorems in those sections have not been explicitly stated or proved before.

Chapters 5 and 6 contain our main results and in the last chapter we summarize what we have done and discuss some questions that we have not been able to answer.

1.2.1 Strongly cospectral vertices

In Chapter 5, we look at strongly cospectral vertices in Cayley graphs, trying to answer the question of how big a strongly cospectral set in such a graph can be. We want to thank Ada Chan for bringing this question to our attention.

We show that the vertices that are strongly cospectral to the group identity in the Cayley graph $\mathrm{Cay}(G, \mathcal{C})$ form a subgroup of G and that this subgroup is a largest strongly cospectral set in the graph. Moreover, we show that this subgroup is an elementary abelian 2-group and that if the graph is a *normal* Cayley graph ($\mathcal{C}$ is conjugacy-closed) then the subgroup is contained in the centre of G and is therefore a normal subgroup.

We then give an upper bound on the size of a strongly cospectral set in a normal Cayley graph in terms of the multiplicities of the eigenvalues of the graph with the following theorem.

Theorem (Theorem 5.5.3)**.** *Let $X = \mathrm{Cay}(G, \mathcal{C})$ be a normal Cayley graph, let H be the strongly cospectral subgroup of G with respect to $\mathcal{C}$ and let m be the multiplicity of some eigenvalue of X. Then*

$$|H| \leq \frac{|G|}{m} = \frac{|V(X)|}{m}.$$

We deduce from this that in a normal Cayley graph of a non-abelian group of order n, a strongly cospectral set has size at most $n/4$. We further use the above theorem to show that in a cubelike graph on 2^d vertices, the size of a strongly cospectral set is less than $\sqrt{2^d}$:

Theorem (Theorem 5.6.5)**.** *A strongly cospectral subgroup of $\mathbb{Z}_2^d$, with $d \geq 3$, has order at most $2^{\lceil d/2 \rceil - 1}$.*

We generalize this theorem to Cayley graphs of $\mathbb{Z}_2^{d_1} \times \mathbb{Z}_4^{d_2}$, and use this to show that in a normal Cayley graph, a strongly cospectral set can at most contain a third of the vertices.

Theorem (Theorem 5.7.2)**.** *A strongly cospectral subgroup of $\mathbb{Z}_2^{d_1} \times \mathbb{Z}_4^{d_2}$ with $d := d_1 + 2d_2 \geq 3$ has order at most $2^{\lceil d/2 \rceil - 1}$.*

Theorem (Theorem 5.7.4)**.** *In a normal Cayley graph, $X = \mathrm{Cay}(G, \mathcal{C})$ on at least five vertices, a strongly cospectral set has size at most $|V(X)|/3$.*

Finally, we construct an infinite family of cubelike graphs that contain a strongly cospectral set of size at least four.

1.2.2 Perfect state transfer

Bašić's characterization of perfect state transfer in circulants [5] is an important contribution to the study of perfect state transfer. As mentioned before, finding graphs that admit perfect state transfer has proved hard but Bašić's results make it easy to construct circulants that do.

Moreover, the biggest source of examples we had before were cubelike graphs, but for those, we only have a sufficient condition for perfect state transfer to occur, not a characterization. So the fact that the sufficient condition on the circulants is also necessary is quite significant.

In Chapter 6, we generalize Bašić's characterization to Cayley graphs of abelian groups that have a cyclic Sylow-2-subgroup, thus providing many new examples of graphs admitting perfect state transfer. The main theorem in this chapter is the following.

Theorem (Theorem 6.9.1)**.** *Let G be an abelian group of order $2^d m$ where m is odd and suppose it has a cyclic Sylow-2-subgroup. Let a be the unique element of order two and $b, -b$ the unique pair of elements of order four. For a subset $\mathcal{C}$ of G let $\mathcal{C}_k$ denote the set of elements in $\mathcal{C}$ with order $2^k m'$ where m' is odd. Then the Cayley graph $\mathrm{Cay}(G, \mathcal{C})$ has perfect state transfer if and only if*

(a) $\mathcal{C}$ *is power-closed,*

(b) either a or b is in $\mathcal{C}$ but not both,

(c) $\mathcal{C}_0 = 4(\mathcal{C}_2 \setminus \{-b, b\})$*, and*

(d) $\mathcal{C}_1 \setminus \{a\} = 2(\mathcal{C}_2 \setminus \{-b, b\})$*.*

With these conditions, it is easy to construct a connection set $\mathcal{C}$ such that $\mathrm{Cay}(G, \mathcal{C})$ has perfect state transfer. Further, given a subset, $\mathcal{C}$ of an abelian group with a cyclic Sylow-2-subgroup, it can be checked in polynomial time whether $\mathrm{Cay}(G, \mathcal{C})$ has perfect state transfer.

Whereas Bašić uses number theory to prove his characterization, we take a group theoretic approach using tools from character theory and association schemes. We therefore not only give a generalization of Bašić's result, but also provide new and different proofs of his theorems.

Chapter 2

Quantum Walks

As we have mentioned earlier, quantum walks can be seen as quantum analogues of random walks. However, quantum walks can exhibit behaviours that random walks cannot, and some of these are of particular interest.

In this chapter we give formal definitions of continuous-time quantum walks and of some of these interesting behaviours. We compare quantum walks to random walks with the purpose of showing that they should in fact not be compared. Then, we introduce an important tool called spectral decomposition, and finally we talk about perfect state transfer and state some preliminary results.

2.1 Preliminaries

Let us start with some linear algebra. Denote the conjugate transpose of a complex-valued matrix M by M^*, that is

$$M^* := \overline{M}^T.$$

We call a square matrix, M, *normal* if $MM^* = M^*M$. We say that M is *hermitian* if $M = M^*$, and we call it *unitary* if $MM^* = I$. Note that both hermitian matrices and unitary matrices are normal.

Let A be a hermitian matrix. The *continuous-time quantum walk* on A

at time t is specified by

$$U_A(t) := e^{itA} = \sum_{n \geq 0} \frac{(it)^n}{n!} A^n.$$

Since we will only be concerned with walks happening in continuous time, we will refer to those simply as *quantum walks*. We call $U_A(t)$ the *transition matrix* of the walk at time t. It is a unitary matrix for all $t \in \mathbb{R}$.

We will generally take A to be the adjacency matrix of a graph, X. Note that our graphs will always be finite, so A will always be a finite dimensional matrix. Throughout most of this work, our graphs are undirected and simple, but in the last chapter we will need the notion of a weighted graph. In both cases, the adjacency matrix is real and symmetric and therefore hermitian. If A is the adjacency matrix of a graph, X, we talk about a quantum walk on X and denote the transition matrix by $U_X(t)$ or simply by $U(t)$. We further refer to the eigenvalues and eigenvectors of A as the eigenvalues and eigenvectors of the graph X.

Let X be a graph with adjacency matrix $A(X) = A$ and let u and v be vertices of X. We say that there is *perfect state transfer* from u to v at time τ with phase factor λ if $U(\tau)\mathbf{e}_v = \lambda \mathbf{e}_u$, where $\mathbf{e}_x$ denotes the standard basis vector indexed by x. Since $U(\tau)$ is a unitary matrix, λ will have absolute value one, and so there is perfect state transfer from u to v at time τ if and only if $|U(\tau)_{u,v}| = 1$.

We say that the vertex u is *periodic* at time τ with phase factor λ if $U(\tau)\mathbf{e}_u = \lambda \mathbf{e}_u$, equivalently, $|U(\tau)_{u,u}| = 1$. If every vertex is periodic at the same time τ, we say that the graph X is *periodic* at time τ. The minimum time at which a vertex or a graph is periodic is called the *period* of the vertex or graph, respectively.

Example 2.1.1. Consider the complete graph on two vertices, K_2. It has adjacency matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have $A^{2n} = I$ and $A^{2n+1} = A$ for all n, and so

$$\begin{aligned} U_{K_2}(t) &= \sum_{n\geq 0} \frac{(it)^n}{n!} A^n \\ &= I + itA - \frac{t^2}{2} I - \frac{it^3}{6} A + \frac{t^4}{24} I + \cdots \\ &= \cos(t) I + i \sin(t) A. \end{aligned}$$

In particular, we see that

$$U_{K_2}(\pi/2) = \cos(\pi/2) I + i \sin(\pi/2) A = iA$$

and

$$U_{K_2}(\pi) = \cos(\pi) I + i \sin(\pi) = -I.$$

Thus K_2 has perfect state transfer between its two vertices at time $\pi/2$, with phase factor i and both vertices are periodic (so the graph is periodic) at time π with phase factor -1. We further see that π is the period of K_2 and that perfect state transfer occurs at all odd multiples of $\pi/2$, but no other times.

2.2 Not random walks

We will do a brief comparison with random walks, but the conclusion of this section will be that such a comparison is not very useful. For more on random walks, we refer the reader to [42].

Let us define a random walk. Given a graph X, we call the diagonal matrix with the degrees of the vertices of X on the diagonal the *degree matrix* of X. Let X be a graph with adjacency matrix A and degree matrix D. The *Laplacian* of X is the matrix $L := D - A$. Note that each row sum and column sum of L is zero. Now define the *continuous-time random walk* at time t by $P(t) = e^{tL}$. For each non-negative real number t, it can be shown that $P(t)$ is a doubly stochastic matrix, that is, its entries are non-negative real numbers and the sum of each row and each column is one.

As one might guess, a continuous-time random walk is a random process on a graph that happens in continuous time. We can imagine a walker standing on a vertex with n neighbours. After some time that follows an exponential distribution, the walker jumps to a neighbouring vertex, where the probability of choosing each of the neighbours is $1/n$.

We can then describe what information is stored in the matrix $P(t)$: the uv entry of $P(t)$ is the probability that the walker is standing on vertex v at time t, given that she started at vertex u. It is not too hard to imagine what the evolution of a random walk looks like on a connected graph. As t goes to infinity, the walker will be equally likely to be situated at any vertex, thus each row of $P(t)$ tends to a uniform distribution on the vertices of X.

Now let us compare this to a quantum walk on X. The transition matrix, $U(t)$, is not stochastic but it is unitary and so $U(t)U(t)^* = U(t)U(-t) = I$. Further, since A is the adjacency matrix of a graph, it is symmetric, and so $U(t)$ is also symmetric. Let $\circ$ denote the entry-wise product of matrices and define the *mixing matrix* of the walk by

$$M(t) := U(t) \circ U(-t)$$

We see that the rows and columns of $M(t)$ sum to one, and so $M(t)$ is a doubly stochastic matrix.

Let $\mathbf{e}_u$ denote the standard basis vector indexed by the vertex u. Then $\mathbf{e}_u^T M(t)$ is a probability density on the vertices of X and if we now measure the system using the standard basis, $\mathbf{e}_u^T M(t)\mathbf{e}_v$ is the probability of the walker being at vertex v at time t if the starting state was $\mathbf{e}_u$.

So we have these analogous matrices, $P(t)$ and $M(t)$ for random walks and quantum walks, but what happens to $M(t)$ when t goes to infinity? This is a question we cannot answer in general. In particular, $M(t)$ does not converge to a uniform distribution, as $P(t)$ does if the graph is connected. In fact, $M(t)$ doesn't converge to any kind of steady state. These differences in quantum and random walks have been looked at for example by Childs, Farhi and Gutmann in 2002 [16] and by Gerhardt and Watrous in 2003 [26].

For quantum walks, we are not so much concerned with what happens when t gets large, but whether there is a particular time at which something interesting occurs.

There are several things that are considered "interesting" here, but in this thesis, we are mostly concerned with one of those. In the previous section, we defined perfect state transfer, from vertex u to vertex v, and we now see that an equivalent definition is that

$$\mathbf{e}_u^T M(\tau)\mathbf{e}_v = 1.$$

In other words, if our system starts in state $\mathbf{e}_u$, then upon measurement at time τ, we will find it in state $\mathbf{e}_v$ with probability one. In this definition however, we have lost the phase factor.

We see that even though there is some analogy in the definitions and interpretations of random walks and quantum walks, they behave in very different ways. Thus now we will forget about random walks and never think of them again.

2.3 Spectral decomposition

An important tool we will use in our study of quantum walks is spectral decomposition. Let A be a normal matrix (note that hermitian matrices are normal). Then A is unitarily diagonalizable, that is, there exists a unitary matrix, U and a diagonal matrix, D such that $A = UDU^*$. Furthermore, the diagonal entries of D are the eigenvalues of A and the columns of U are the eigenvectors of A.

Let $\theta_0, \dots \theta_n$ be the distinct eigenvalues of A. Then there are diagonal 01-matrices, $D_0, \dots, D_n$ satisfying

$$\sum_{r=0}^{n} \theta_r D_r = D, \qquad \sum_{r=0}^{n} D_r = I.$$

We see that $D_r D_s = 0$ if $r \neq s$ and $D_r^2 = D_r$. For each $r = 0, \dots, n$, define $E_r = U D_r U^*$. Then

$$A = UDU^* = U\left(\sum_{r=0}^{n} \theta_r D_r\right)U^* = \sum_{r=0}^{n} \theta_r U D_r U^* = \sum_{r=0}^{n} \theta_r E_r.$$

It is easy to see that the matrices E_r are idempotent and pairwise orthogonal and that $AE_r = \theta_r E_r$ for all r. Further, we have $E_r^* = E_r$ and $\sum_r E_r = I$. It follows that the matrix E_r is the orthogonal projection onto the θ_r-eigenspace of A. We call the matrices $E_0, \dots, E_n$ the *spectral idempotents* of A and the identity

$$A = \sum_{r=0}^{n} \theta_r E_r$$

is the *spectral decomposition* of A.

A useful application of the spectral decomposition is the fact that if f is a univariate function defined on the spectrum of A, then

$$f(A) = \sum_{r=0}^{n} f(\theta_r) E_r. \tag{2.1}$$

It follows from this that each E_r is a polynomial in A: define

$$p_r(x) = \prod_{s \neq r} (x - \theta_s)$$

for $r = 0, \dots, n$. Then $p_r(\theta_s) = 0$ if and only if $s \neq r$ and so

$$\frac{1}{p_r(\theta_r)} p_r(A) = \frac{1}{p_r(\theta_r)} \sum_{s=0}^{n} p_r(\theta_s) E_s = E_r.$$

It further follows from Equation 2.1, that if $U(t)$ is the transition matrix of a quantum walk on A, then

$$U(t) = \sum_{r=0}^{n} e^{it\theta_r} E_r.$$

Observe that an immediate consequence of this is that if a graph has integer eigenvalues, then $U(2\pi) = I$, and so every vertex is periodic at time $\tau = 2\pi$, that is, the graph is periodic at time 2π.

2.4 Perfect state transfer

In this section we will state some preliminary results about perfect state transfer and periodicity which will come in handy later on.

Recall that the transition matrix of a walk on a graph is symmetric, since the adjacency matrix is symmetric. Therefore, we have the following lemma.

Lemma 2.4.1 ([30, Lemma 1.1])**.** *Let u and v be vertices of a graph X. If there is perfect state transfer from u to v at time τ, then there is perfect state transfer from v to u at time τ and u and v are both periodic at time 2τ.* □

By Lemma 2.4.1, we can talk about perfect state transfer occurring between two vertices, rather than from one to another. Further, the lemma implies that if there is perfect state transfer in a graph, then there is also periodicity in the graph. The converse is however not true as we will see in our next example, and in fact perfect state transfer is a much stronger property than periodicity.

Example 2.4.2. Consider the complete graph on n vertices, K_n. Its adjacency matrix is $J - I$, where J denotes the all-ones matrix. This matrix has eigenvalues $n-1$ and -1 and the corresponding spectral idempotents are

$$E_0 := \frac{1}{n}J, \quad E_1 := I - \frac{1}{n}J,$$

respectively. Since both eigenvalues are integers, we immediately get that K_n is periodic. We will show that if $n \geq 3$, it cannot have perfect state transfer. Using spectral decomposition, we see that the transition matrix of K_n is

$$U(t) = e^{it(n-1)}E_0 + e^{-it}E_1 = e^{-it}\left(I + \frac{e^{itn}-1}{n}J\right),$$

and so for any vertex u we have

$$U(t)_{u,u} = e^{-it}\left(1 + \frac{e^{itn}-1}{n}\right).$$

It follows that for any time t,

$$|U(t)_{u,u}| \geq 1 - \frac{2}{n},$$

which is strictly positive, given that $n \geq 3$. So if $v \neq u$, then $|U(t)_{u,v}| < 1$ for all t, implying that there is no perfect state transfer on K_n for $n \geq 3$.

Perfect state transfer between vertices in a graph implies a special bond between those vertices, in fact, this is a monogamous relationship. This was first observed by Kay in 2011 [39, Section D]. A proof can also be found in [30].

Lemma 2.4.3 ([30, Corollary 3.2]). *If there is perfect state transfer from u to v in X and also from u to w, then $v = w$.* □

Recall that a graph with integer eigenvalues is periodic. It turns out the the converse is almost true. In 2011, Godsil gave a characterization of periodicity of graphs in terms of their eigenvalues.

Lemma 2.4.4 ([29, Corollary 3.3]). *A graph X is periodic if and only if either*

(a) the eigenvalues of X are integers, or

(b) the eigenvalues of X are rational multiples of $\sqrt{\Delta}$, for some fixed square-free integer Δ. □

Our focus in this thesis is on vertex-transitive graphs, i.e., graphs in which a vertex can be mapped to any other vertex using an automorphism of the graph. The above lemma has strong consequences for vertex-transitive graphs.

Let X be a vertex-transitive graph. If there is perfect state transfer between two vertices in X at time τ, then every vertex of X is involved in perfect state transfer at time τ and the graph is periodic at time 2τ. This is a result of the following theorem of Godsil.

Theorem 2.4.5 ([30, Theorem 6.1]). *Let X be a vertex-transitive graph and let u and v be vertices of X. If there is perfect state transfer from u to v at time τ, then $U(\tau)$ is a scalar multiple of a permutation matrix with order two and no fixed points.* □

The theorem implies that a vertex-transitive graph that admits perfect state transfer has an even number of vertices. In particular, a Cayley graph for a group of odd order has no perfect state transfer. In fact, Theorem 2.4.5 holds for integer-weighted vertex-transitive graphs, and so weighted Cayley graphs for groups of odd order also do not have perfect state transfer.

Theorem 2.4.5 further implies that a vertex-transitive graph with perfect state transfer is periodic, and now the next lemma follows from Lemma 2.4.4 and the fact that vertex-transitive graphs have an integer eigenvalue, namely its degree.

Lemma 2.4.6. *Let X be a vertex-transitive graph. If X admits perfect state transfer, then all its eigenvalues are integers.* □

We call a graph with integer eigenvalues *integral*.

Chapter 3

Group Actions & Cayley Graphs

This is a thesis in combinatorics, but if there is anything that is possibly even better than combinatorics, it is of course group theory.

There are several ways in which we can use group theory to study graphs. Like most mathematical structures, a graph has an automorphism group. The automorphism group acts on the graph and this action can tell us things about the graph itself and vice versa.

Cayley graphs provide another fascinating link between group theory and graph theory. The vertices of a Cayley graph are elements of a group and the edges come from the group operation. Again, there are interesting connections between properties of the group and properties of the graph. In particular, we will see connections to other algebraic properties of the graph, such as its spectrum.

In this chapter we introduce group actions and Cayley graphs and state some preliminary results. We look at concepts from the theory of permutation groups, such as blocks of imprimitivity, and see what implications they have when our group is acting on the vertex set of a graph. As before, all graphs are finite, and so our groups are finite as well.

The results in this chapter are standard. For further reading on group actions, we refer the reader to [23].

3.1 Group actions on graphs

Let G be a group acting on a set, Ω. We denote the image of an element $\alpha \in \Omega$ under $g \in G$ by α^g. We call the action *transitive* if for all elements $\alpha, \beta \in \Omega$, there is an element $g \in G$ such that $\alpha^g = \beta$. We say that the action is *regular* if it is transitive and the stabilizer, $G_\alpha = \{g \in G : \alpha^g = \alpha\}$ is trivial for all $\alpha \in \Omega$.

Suppose the action of G on Ω is transitive. A set $B \subseteq \Omega$ is called a *block (of imprimitivity)* if for each $g \in G$, either $B = B^g$ or $B \cap B^g = \emptyset$. Clearly, the empty set and the singleton sets are blocks and so is the set Ω. These are called *trivial blocks*.

We will be focusing on group actions on the vertex set of a graph. Let X be a graph on n vertices and denote by $\mathrm{Aut}(X)$ the group of automorphisms of X. Each automorphism is a permutation on the vertices of X, so we can think of it as an $n \times n$ permutation matrix. This matrix commutes with the adjacency matrix of X, and in fact the converse is also true.

Lemma 3.1.1. *Let A be the adjacency matrix of a graph X on n vertices and let P be an $n \times n$ permutation matrix. Then P commutes with A if and only if P is an automorphism of X.*

Proof. Let $\varphi : V(X) \to V(X)$ be the map defined by P. Since P is a permutation matrix, the map is bijective with inverse φ^{-1}, represented by the matrix P^T. We want to show that $P^TAP = A$ if and only if φ is an automorphism. For a vertex v of X, we look at the v-th column of P^TAP:

$$(P^TAP)\mathbf{e}_v = P^TA\mathbf{e}_{\varphi(v)} = P^T \sum_{x \in N(\varphi(v))} \mathbf{e}_x.$$

Now, P is an automorphism of X if and only if for all vertices v, we have $N(\varphi(v)) = \varphi(N(v))$. This happens if and only if for all $v \in V(X)$,

$$P^T \sum_{x \in N(\varphi(v))} \mathbf{e}_x = P^T \sum_{x \in \varphi(N(v))} \mathbf{e}_x = \sum_{x \in \varphi(N(v))} P^T\mathbf{e}_x = \sum_{y \in N(v)} \mathbf{e}_y = A\mathbf{e}_v.$$

We have shown that $(P^TAP)\mathbf{e}_v = A\mathbf{e}_v$ for all $v \in V(X)$ if and only if P is an automorphism, and the result follows. □

Let X be a graph and denote by $\mathrm{Aut}(X)$ the group of automorphisms of X. We say that X is *vertex-transitive* if $\mathrm{Aut}(X)$ acts transitively on the vertices of X.

Example 3.1.2. The Petersen graph (shown in Figure 3.1) is vertex-transitive.

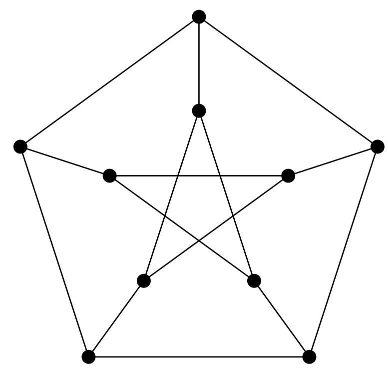


Figure 3.1: Petersen graph

We are particularly interested in a certain class of vertex-transitive graphs, namely Cayley graphs.

3.2 Cayley graphs

Let G be a group and let $\mathcal{C}$ be a subset of G. Define a graph, $X := \mathrm{Cay}(G, \mathcal{C})$ as having vertex set $V(X) = G$ and with vertices g and h adjacent in X if $hg^{-1} \in \mathcal{C}$. We call this the *Cayley graph* for G with respect to $\mathcal{C}$ and refer to $\mathcal{C}$ as the *connection set* of X.

Example 3.2.1.

(a) The cycles are Cayley graphs for the cyclic groups, $\mathbb{Z}_n$ with respect to $\mathcal{C} = \{-1, 1\}$.

(b) The hypercubes are Cayley graphs for $\mathbb{Z}_2^d$ with respect to the standard basis.

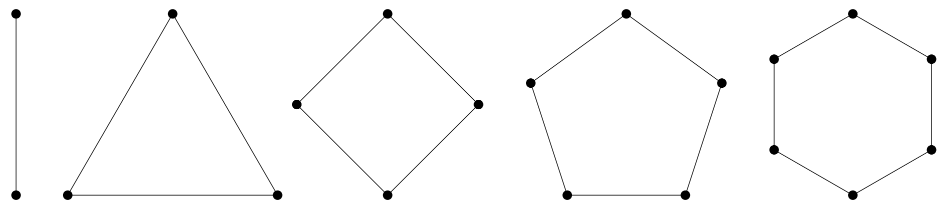


Figure 3.2: Cycles

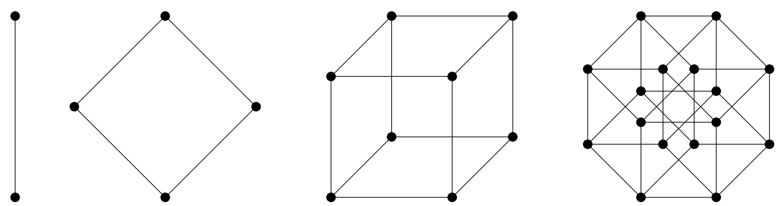


Figure 3.3: Hypercubes

We call a Cayley graph of an abelian group a *translation graph*. In particular, Cayley graphs of cyclic groups are called *circulants* and Cayley graphs of the elementary abelian 2-groups are called *cubelike graphs*. By the example above, we see that the cycles are circulants and the hypercubes are cubelike graphs.

In general, X will be a directed graph, possibly with loops. In fact, it is easy to see that there is a loop on every vertex in X if and only if the group identity, e, is an element of $\mathcal{C}$ (otherwise there are no loops), and that X is undirected if and only if $\mathcal{C}$ is inverse-closed. Further, we see that X is connected if and only if $\mathcal{C}$ generates the group. If this is not the case, then the components of X are isomorphic Cayley graphs of the subgroup generated by $\mathcal{C}$.

Since we are mainly concerned with simple, undirected graphs, we will usually assume that $\mathcal{C}$ is inverse closed and does not contain the identity, but we do not assume in general that $\mathcal{C}$ generates G.

Now let $X = \text{Cay}(G, \mathcal{C})$ be a simple, undirected Cayley graph. The group G acts regularly on itself by right multiplication. We claim that for each $g \in G$, this map is a graph automorphism. Indeed, if $g \in G$ and $x, y \in V(X) = G$, then

$$xg \sim yg \iff (yg)(xg)^{-1} \in \mathcal{C} \iff yx^{-1} \in \mathcal{C} \iff x \sim y.$$

So G acts as a group of automorphisms on X, implying that $G \leq \mathrm{Aut}(X)$, and since this action is regular, in particular transitive, it follows that X is vertex-transitive. In fact, a graph is a Cayley graph if and only if there is a subgroup of the automorphism group that acts regularly on it.

Lemma 3.2.2. *The blocks of the regular action of a group G on a Cayley graph $X = \mathrm{Cay}(G, \mathcal{C})$ are precisely the subgroups of G and their cosets.*

Proof. If H is a subgroup of G, the image of H under $g \in G$ is the right coset Hg, which we know to be either equal to H or disjoint from it. Conversely, if $B' \subseteq G$ is a block, since the action is transitive, some translate, B of B' will contain the group identity, e. Let $x, y \in B$. Then $y = e^y \in B^y$, so $B^y = B$, but this implies that $x^y = xy \in B$. A similar argument shows that $x^{-1} \in B$ and so B is a subgroup of G. □

Corollary 3.2.3. *Let $X = \mathrm{Cay}(G, \mathcal{C})$ be a Cayley graph. Under the action of $\mathrm{Aut}(X)$ on X, every block is a right coset of a subgroup of G. In particular, every block that contains the identity is a subgroup of G.*

Proof. If H_1 and H_2 are groups acting on a set with $H_1 \leq H_2$, then clearly any block under the action of H_2 is also a block under the action of H_1. Therefore, since $G \leq \mathrm{Aut}(X)$, the result follows from Lemma 3.2.2. □

3.3 Normal Cayley graphs

We call a Cayley graph *normal* if the connection set is a union of conjugacy classes. We will see later on that normal Cayley graphs have many convenient properties as they lie in so-called association schemes.

Remark. There are two non-equivalent definitions of a normal Cayley graph in the literature, the other being that the regular subgroup is a normal subgroup of the automorphism group. We will not be concerned with this property here. Our definition is consistent with a paper by Larose et al. from 1998 [40].

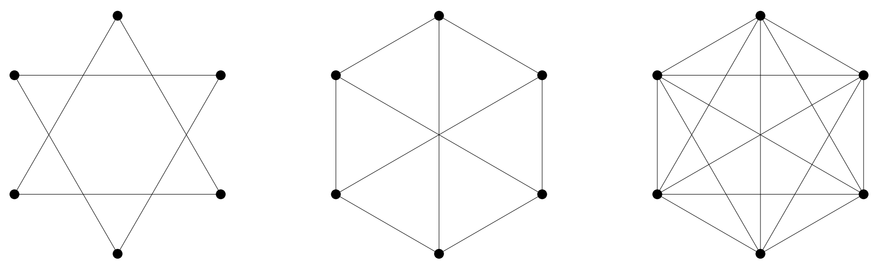
Example 3.3.1.

(a) The graphs in Example 3.2.1 are normal, in fact all Cayley graphs of abelian groups are normal Cayley graphs.

(b) Let $G = S_3$ and define

$$\mathcal{C}_1 = \{(123), (132)\}, \quad \mathcal{C}_2 = \{(12), (13), (23)\}, \quad \mathcal{C}_3 = G \setminus \{e\}.$$

The sets are all inverse-closed, and conjugacy-closed. Thus, the three graphs shown in Figure 3.4 are all normal Cayley graphs of S_3.



(a) $\mathrm{Cay}(S_3, \mathcal{C}_1)$ (b) $\mathrm{Cay}(S_3, \mathcal{C}_2)$ (c) $\mathrm{Cay}(S_3, \mathcal{C}_3)$

Figure 3.4: Cayley graphs of S_3

Note that the three graphs in the figure above are also Cayley graphs for the cyclic group, $\mathbb{Z}_6$. The last one is the complete graph on six vertices. It is not too hard to see that the complete graph on n vertices is a normal Cayley graph for every group of order n.

Let $X = \mathrm{Cay}(G, \mathcal{C})$ be a normal Cayley graph. As before, right multiplication by an element $g \in G$ is an automorphism of X. But now, let's take a look at left multiplication by g. We have for $x, y \in G$

$$\begin{aligned} gx \sim gy &\iff (gy)(gx)^{-1} \in \mathcal{C} \\ &\iff gyx^{-1}g^{-1} \in \mathcal{C} \\ &\iff yx^{-1} \in \mathcal{C} \qquad \text{(since } g^{-1}\mathcal{C}g = \mathcal{C}) \\ &\iff x \sim y. \end{aligned}$$

Then let $g = (g_1, g_2) \in G \times G$ and define an action on G by $x^g = g_1 x g_2^{-1}$. By the above, we see that this is an automorphism of the normal Cayley graph X, and so $\mathrm{Aut}(X)$ contains a subgroup isomorphic to $G \times G$. We now get a lemma analogous to Lemma 3.2.2 for normal Cayley graphs.

Lemma 3.3.2. *The blocks of the action of $G \times G$ described above on a normal Cayley graph $X = \mathrm{Cay}(G, \mathcal{C})$ are precisely the normal subgroups of G and their cosets.*

Proof. Let B be a block under this action and assume $e \in B$. Since $G \times G$ contains G as a subgroup, B is also a block under the regular action so by Lemma 3.2.2, B is a subgroup of G. We claim that it is normal. Indeed, if $g \in G$, then $(g, g) \in G \times G$ and since $e \in B$ and B is a block we have $e^g = e \in B$, so $B^g = B$. But then $gBg^{-1} = B^g = B$, so B is a normal subgroup.

Further we know that any block is of the form $B' = g_1 B g_2^{-1}$ where B is a block containing e. But since B is a normal subgroup we get $B' = g_1(Bg_2^{-1}) = g_1 g_2^{-1} B$, so B' is a coset of a normal subgroup.

Conversely, let xN be a coset of a normal subgroup N of G and let $g = (g_1, g_2) \in G \times G$. Then

$$(xN)^g = g_1 x N g_2^{-1} = g_1 x g_2^{-1} N,$$

so $(xN)^g$ is also a coset of N. Since the cosets form a partition, it is clear that N is a block. $\square$

Chapter 4

Association Schemes

The goal of this chapter is to provide some important tools that we will need later on. This includes theorems about the spectra of normal Cayley graphs and translation graphs.

To do this, we will introduce the topic of association schemes. In particular, we discuss the conjugacy class scheme — the home of normal Cayley graphs, and translation schemes — where translation graphs live. For this discussion, we will need some help from the theory of representations and characters of groups.

Association schemes are a fascinating topic and more on them can be found for example in [12, 27, 28, 32]. The results in this chapter are not new, but Sections 4.6 and 4.7 contain some theorems and proofs that have not been explicitly stated or proved before.

4.1 Preliminaries

Let J denote the all-ones matrix. An *association scheme* (with d classes) is a set of $n \times n$ matrices, $\mathcal{A} = \{A_0, \ldots, A_d\}$ with entries in $\{0, 1\}$ such that

(i) $A_0 = I$,

(ii) $\sum_{r=0}^{d} A_r = J$,

(iii) $A_r^T \in \mathcal{A}$ for all r,

(iv) $A_rA_s = A_sA_r$ for all r, s, and

(v) A_rA_s lies in the span of $\mathcal{A}$ for all r, s.

It is easy to see that the set $\{A_0, \ldots, A_d\}$ is a basis for the vector space it spans. Further, we can see from the conditions that this vector space is a commutative algebra that is closed under the transpose map. This is called the *Bose-Mesner algebra* of the scheme and denoted by $\mathbb{C}[\mathcal{A}]$. The scheme is said to be *symmetric* if each A_r is a symmetric matrix.

For matrices $M, N \in \mathbb{C}[\mathcal{A}]$ define their *Schur product* (sometimes called the *Hadamard product*), $M \circ N$, by

$$(M \circ N)_{r,s} = M_{r,s}N_{r,s}.$$

Since the entries of A_r are 0 and 1, we have $A_r \circ A_r = A_r$, in other words A_r is Schur idempotent for $r = 0, \ldots d$. Moreover, condition (ii) above implies that for all $r \neq s$, we have $A_r \circ A_s = 0$. It follows that $\mathbb{C}[\mathcal{A}]$ is Schur-closed, and it contains the Schur-identity, J. Every Schur idempotent in $\mathbb{C}[\mathcal{A}]$ is a sum of some elements of $\mathcal{A}$, so $A_0, \ldots, A_d$ are the *minimal Schur idempotents* of the scheme. Since the Schur idempotents are 01-matrices, we may view them as adjacency matrices for directed graphs. We refer to an undirected graph whose adjacency matrix lies in $\mathbb{C}[\mathcal{A}]$ as a *graph in the association scheme* $\mathcal{A}$.

A matrix algebra (real or complex) that contains I and J and is closed under Schur product, transpose and complex conjugation is called a *coherent algebra*. Any commutative coherent algebra is the Bose-Mesner algebra of an association scheme. In particular, if $\mathbb{C}[\mathcal{A}]$ is the Bose-Mesner algebra of a scheme $\mathcal{A}$, then any coherent subalgebra of $\mathbb{C}[\mathcal{A}]$ is the Bose-Mesner algebra of some association scheme $\mathcal{B}$. In this case, every minimal Schur idempotent of $\mathcal{B}$ is a Schur idempotent of $\mathcal{A}$ and we say that $\mathcal{B}$ is a *subscheme* of $\mathcal{A}$.

4.2 Matrix idempotents

Recall our definition of a spectral decomposition of a normal matrix in Section 2.3.

Let $\mathcal{A} = \{A_0, \ldots, A_d\}$ be an association scheme. It is clear by definition that the matrices A_r are normal, so they have a spectral decomposition. Let

$\mathcal{E}$ be the set of all possible products of the spectral idempotents of $A_0, \ldots, A_d$. We define a partial ordering on the set $\mathcal{E}$, by letting $E \leq F$ if $EF = E$ for $E, F \in \mathcal{E}$. Since the spectral idempotents of a matrix A are polynomials in A, we see that $\mathcal{E} \subseteq \mathbb{C}[\mathcal{A}]$. In particular, the matrices in $\mathcal{E}$ commute, and equivalently we have $E \leq F$ if $FE = E$.

We call E a *minimal matrix idempotent* of the association scheme if it is a minimal non-zero element of $\mathcal{E}$ with respect to this partial ordering.

Lemma 4.2.1. *The minimal matrix idempotents are a basis for* $\mathbb{C}[\mathcal{A}]$.

Proof. Let $E_0, \ldots, E_m$ be the minimal matrix idempotents of $\mathcal{A}$. First, note that they are pairwise orthogonal since $E_rE_s \leq E_r, E_s$, so E_rE_s must be zero. Then it is easy to see that they are linearly independent: suppose we can write E_r as a linear combination of the rest,

$$E_r = \sum_{s \neq r} \alpha_s E_s.$$

Then, multiplying both sides by E_r gives $E_r = 0$, a contradiction. Therefore, the minimal matrix idempotents are linearly independent.

Since $\mathcal{E}$ contains the spectral idempotents of each $A_r \in \mathcal{A}$ and $\mathcal{A}$ is a basis for $\mathbb{C}[\mathcal{A}]$, we see that $\mathcal{E}$ generates $\mathbb{C}[\mathcal{A}]$. If $\{E_0, \ldots, E_m\}$ is not a basis, we can find $E_{m+1}, \ldots, E_d \in \mathcal{E}$ such that $\{E_0, \ldots, E_d\}$ is an orthogonal basis for $\mathbb{C}[\mathcal{A}]$. But since E_d is not minimal, there is some $r = 0, \ldots, m$ such that $E_rE_d = E_r$, but this is impossible since we chose the basis to be orthogonal.

Therefore, $m = d$ and the minimal matrix idempotents form a basis for $\mathbb{C}[\mathcal{A}]$. □

Now consider these two bases of the Bose-Mesner algebra, $\{A_0, \ldots, A_d\}$ and $\{E_0, \ldots, E_d\}$. They both consist of idempotents, with respect to the Schur product and matrix product, respectively. Moreover, we know that the former has all sorts of properties, by the way it is defined. But it turns out that the latter basis has similar properties. This is summarized in the next lemma. We encourage the reader to compare (i)-(v) in the lemma to (i)-(v) in the definition of an association scheme in the previous section, interchanging Schur product and matrix multiplication.

Lemma 4.2.2. *We have*

(i) $\frac{1}{n}J \in \{E_0, \dots, E_d\}$,

(ii) $\sum_{r=0}^{d} E_r = I$,

(iii) $E_r^T \in \{E_0, \dots, E_d\}$, *for all* r,

(iv) $E_r \circ E_s = E_s \circ E_r$ *for all* r, s, *and*

(v) $E_r \circ E_s$ *lies in the span of* $\{E_0, \dots, E_d\}$.

Proof. Since $J \in \mathbb{C}[\mathcal{A}]$ and $\mathbb{C}[\mathcal{A}]$ is commutative, each matrix in the algebra has constant row and column sum. It follows that for any matrix $M \in \mathbb{C}[\mathcal{A}]$, we have $JM = MJ = kJ$ where k is the row sum of M. In particular, if $E \in \mathcal{E}$ we have that $\frac{1}{n}JE = kJ$ for some k, but this matrix is idempotent, so if E is non-zero we must have $k = \frac{1}{n}$ implying that $\frac{1}{n}J \leq E$, from which part (i) follows.

Since $I \in \mathbb{C}[\mathcal{A}]$, it can be written as a linear combination of the E_r, so

$$I = \sum_{r=0}^{d} \alpha_r E_r.$$

But then the α_r are eigenvalues of I, so they are all one which implies (ii). The last three parts are obvious. □

By convention, we assume that $E_0 = \frac{1}{n}J$.

4.3 Eigenvalues

Let $\mathcal{A} = \{A_0, \dots, A_d\}$ be an association scheme and let $E_0, \dots, E_d$ be the minimal matrix idempotents. Since these are both bases of the Bose-Mesner algebra, we know that each A_r is a linear combination of the minimal matrix idempotents and that each E_r is a linear combination of the minimal Schur idempotents.

Then for $r, s = 0, \dots, d$, there are scalars, $p_r(s)$ and $q_r(s)$ such that

$$A_r = \sum_{s=0}^{d} p_r(s) E_s \quad \text{and} \quad E_r = \frac{1}{n} \sum_{s=0}^{d} q_r(s) A_s.$$

Note that $p_r(s)$ is an eigenvalue of A_r and we call the scalars $p_r(s)$ the *eigenvalues of the scheme*. Define the matrices P and Q by $P_{sr} := p_r(s)$ and $Q_{sr} := q_r(s)$. We call P the *matrix of eigenvalues* of the scheme $\mathcal{A}$ and Q the *matrix of dual eigenvalues* of $\mathcal{A}$. It is not too hard to work out that $PQ = nI$.

We see that $A_r E_s = p_r(s) E_s$ for all r and s. Further, if A is any Schur idempotent of the scheme, it is a sum of some minimal Schur idempotents, say

$$A = \sum_{r \in R} A_r,$$

with $R \subseteq \{0, \ldots, d\}$. Then

$$AE_s = \sum_{r \in R} A_r E_s = \sum_{r \in R} p_r(s) E_s$$

so the eigenvalues of A are $\sum_{r \in R} p_r(s)$ for $s = 0, \ldots, d$. It follows that there is a 01-vector, $\mathbf{x}$ (the characteristic vector of the set R) such that $P\mathbf{x}$ is a vector consisting of the eigenvalues of A.

Recall that since $J \in \mathbb{C}[\mathcal{A}]$, each A_r has constant row sum. Let v_r be the row sum of A_r and define the diagonal matrix Δ_v by $(\Delta_v)_{rr} := v_r$ for $r = 0, \ldots, d$. We call this the *matrix of valencies* of $\mathcal{A}$.

Further, let m_r be the rank (equivalently the trace) of the matrix E_r for all r. Define Δ_m as the diagonal matrix with m_r in the r-th diagonal entry and call this the *matrix of multiplicities*.

It can be shown that for all $r, s = 0, \ldots, d$ we have $p_r(s) m_s = \overline{q_s(r)} v_r$. Equivalently, $\Delta_m P = Q^* \Delta_v$ and using the fact that $PQ = nI$ this implies that $P^* \Delta_m P = n \Delta_v$.

Example 4.3.1. Consider the line graph of the complete graph on n vertices, $L(K_n)$. Let A_1 be its adjacency matrix and define $A_2 := J - I - A_1$ and $A_0 := I$. Then $\mathcal{A} = \{A_0, A_1, A_2\}$ is a symmetric association scheme with three classes. Such association schemes arise from *strongly regular graphs* which we will not define here, but a definition and discussion can be found in [33, Chapter 3].

We can use what is known about strongly regular graphs to calculate the eigenvalues of A_1 and A_2. We find that A_1 has eigenvalues

$$k := 2n - 4, \quad \theta := n - 4 \quad \text{and} \quad \lambda := -2$$

with multiplicities one, $n-1$ and $n(n-1)/2-n$, respectively. Further, A_2 has eigenvalues

$$n(n-1)/2-1-k, \quad -\theta-1 \quad \text{and} \quad -\lambda-1.$$

So the matrix of eigenvalues of $\mathcal{A}$ is

$$P = \begin{pmatrix} 1 & 2n-4 & \frac{n^2-5n+6}{2} \\ 1 & n-4 & 3-n \\ 1 & -2 & 1 \end{pmatrix}.$$

We also find that the matrix of dual eigenvalues is

$$Q = \begin{pmatrix} 1 & n-1 & \frac{(n-3)n}{2} \\ 1 & \frac{(n-4)(n-1)}{2(n-2)} & -\frac{(n-3)n}{2(n-2)} \\ 1 & -\frac{2(n-1)}{n-2} & \frac{n}{n-2} \end{pmatrix},$$

and

$$\Delta_v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2n-4 & 0 \\ 0 & 0 & \frac{n^2-5n+6}{2} \end{pmatrix}, \quad \Delta_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & n-1 & 0 \\ 0 & 0 & \frac{n(n-3)}{2} \end{pmatrix}$$

are the matrix of valencies and multiplicities, respectively.

4.4 Representations & characters

We now briefly discuss representations and characters of groups in the context of association schemes. For definitions and preliminary results we refer the reader to [36].

Let G be a group and let ρ be a representation of G. The *character* of the representation ρ, is the map $\chi : G \to \mathbb{C}$ defined by $\chi(g) = \mathrm{Tr}(\rho(g))$. The *degree* of χ is the dimension of the representation, ρ (equivalently $\chi(e)$). We refer to the characters of the representations of G simply as the characters of G. We say that a character of G is *irreducible* if the corresponding representation is irreducible.

A complex-valued function on G that is constant on the conjugacy classes of G is called a *class function*. The characters of G are class functions, and in fact, the irreducible characters, $\chi_1, \ldots, \chi_n$ of G form a basis for all class functions of G as a vector space over $\mathbb{C}$.

We equip this vector space with an inner product: if $\psi, \varphi : G \to \mathbb{C}$ are class functions, define their inner product by

$$\langle \psi, \varphi \rangle_G := \frac{1}{|G|} \sum_{x \in G} \overline{\psi(x)} \varphi(x).$$

With respect to this inner product, the basis of irreducible characters is orthonormal, that is

$$\langle \chi_r, \chi_s \rangle = \begin{cases} 1 & \text{if } r = s \\ 0 & \text{otherwise.} \end{cases}$$

Further, if ψ is an arbitrary character of G, then there are non-negative integers, $d_1, \ldots, d_n$ such that $\psi = d_1\chi_1 + \cdots + d_n\chi_n$ and $\langle \psi, \chi_r \rangle = d_r$ for all $r = 1, \ldots, n$.

Let $g_1, \ldots, g_n$ be representatives of the conjugacy classes of G. The *character table* of G is the $n \times n$ matrix, M with $M_{rs} = \chi_r(g_s)$. By the orthogonality relations we see that M is invertible.

Let ψ be a character of a group G and φ a character of a group H. Define the *product* of the two characters by

$$\psi\varphi : G \times H \to \mathbb{C}, \quad (g, h) \mapsto \psi(g)\varphi(h).$$

This is a character of the group $G \times H$ and in fact, the irreducible characters of $G \times H$ are precisely the products of the irreducible characters of G and H.

Let G be a group and H a subgroup of G. For a character ψ of G, denote by $\psi{\downarrow}H$ its *restriction* to H, that is $(\psi{\downarrow}H)(h) = \psi(h)$ for all $h \in H$. This is a character of H, but it is not necessarily irreducible even if ψ is an irreducible character of G. For a character φ of H we define the *induced character* $\varphi{\uparrow}G$, of φ to G as follows. Let $\varphi' : G \to \mathbb{C}$ be the map given by

$$\varphi'(g) := \begin{cases} \varphi(g) & \text{if } g \in H \\ 0 & \text{otherwise.} \end{cases}$$

Then we define

$$(\varphi{\uparrow}G)(g) := \frac{1}{|H|} \sum_{x \in G} \varphi'(x^{-1}gx)$$

for all $g \in G$. Again, this is a character of G, although not necessarily irreducible even if φ is irreducible. There is an important relation between the induced and the restricted characters demonstrated by the Frobenius Reciprocity Theorem.

Theorem 4.4.1 ([36, Theorem 21.16], The Frobenius Reciprocity Theorem)**.** *Assume that $H \leq G$, let ψ be a character of G and let φ be a character of H. Then*

$$\langle \psi, \varphi{\uparrow}G \rangle_G = \langle \psi{\downarrow}H, \varphi \rangle_H. \qquad \square$$

A character of a group G is called *linear* if it has degree one. In this case, it is a group homomorphism from G to the multiplicative group of $\mathbb{C}$. If G is an abelian group, all its characters are linear and they form a group under pointwise multiplication. We call this group the *character group* of G and denote it by G^*. We have $G^* \cong G$.

Let ρ be a representation of a group G. The *commutant* of ρ is the commutant of the image of ρ, that is, the set of matrices that commute with $\rho(g)$ for all $g \in G$. We state the following corollary of Schur's Lemma. A proof can be found for example in [32].

Theorem 4.4.2 ([32, Corollary 11.6.2])**.** *The commutant of an irreducible representation consists of scalar multiples of the identity.* □

4.5 Conjugacy class schemes

We are particularly interested in association schemes that have some connections to groups. More details of the following discussion can be found in [32, Chapter 11].

Let G be a group of order n, with conjugacy classes $C_0, \dots, C_d$, where we assume $C_0 = \{e\}$. Define A_r to be the $n \times n$ matrix indexed by the group elements where the (g, h)-entry is one if $hg^{-1} \in C_r$ and zero otherwise. Equivalently, A_r is the adjacency matrix of the (possibly directed) normal Cayley graph, $\mathrm{Cay}(G, C_r)$. The set $\mathcal{A} := \{A_0, \dots, A_d\}$ is called the *conjugacy class scheme* on G and it is an association scheme.

For an irreducible character χ of G, define a matrix E_χ by

$$(E_\chi)_{gh} := \frac{\chi(e)}{n} \chi\left(hg^{-1}\right).$$

If ψ and φ are irreducible characters, it can be shown using the orthogonality relations of characters that

$$\sum_{x\in G} \psi\left(xg^{-1}\right)\varphi\left(hx^{-1}\right) = \begin{cases} n/\psi(e) & \text{if } \varphi = \psi \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the matrices E_χ are idempotent and pairwise orthogonal. We will show that they are matrix idempotents for the conjugacy class scheme on G.

Lemma 4.5.1. *Let $\mathcal{A} = \{A_0, \ldots, A_d\}$ be the conjugacy class scheme on a group G with conjugacy classes $C_0, \ldots, C_d$. Let χ be an irreducible character of G and let $c \in C_r$. Then*

$$A_r E_\chi = \frac{|C_r|\overline{\chi(c)}}{\chi(e)} E_\chi.$$

Proof. Let ρ be the irreducible representation corresponding to χ. Then

$$\begin{aligned}
(A_r E_\chi)_{gh} &= \frac{\chi(e)}{|G|} \sum_{xg^{-1}\in C_r} \chi\left(hx^{-1}\right) \\
&= \frac{\chi(e)}{|G|} \sum_{y\in C_r} \chi\left(hg^{-1}y^{-1}\right) \\
&= \frac{\chi(e)}{|G|} \sum_{y\in C_r} \operatorname{Tr}\left(\rho\left(hg^{-1}y^{-1}\right)\right) \\
&= \frac{\chi(e)}{|G|} \operatorname{Tr}\left(\sum_{y\in C_r} \rho\left(hg^{-1}\right)\rho\left(y^{-1}\right)\right) \\
&= \frac{\chi(e)}{|G|} \operatorname{Tr}\left(\rho\left(hg^{-1}\right)\sum_{y\in C_r} \rho\left(y^{-1}\right)\right).
\end{aligned}$$

Since C_r is closed under conjugation, it is easy to see that the sum of $\rho(y^{-1})$ over C_r lies in the commutant of ρ. Therefore, by Theorem 4.4.2, it is a scalar multiple of the identity matrix, say

$$\sum_{y\in C_r} \rho\left(y^{-1}\right) = \lambda I.$$

Then taking the trace on both sides we see that $|C_r|\chi(c^{-1}) = \lambda\chi(e)$, so the above gives

$$\begin{aligned}(A_r E_\chi)_{gh} &= \frac{\chi(e)}{|G|} \operatorname{Tr}\left(\rho\left(hg^{-1}\right)(\lambda I)\right) \\ &= \frac{|C_r|\chi(c^{-1})}{|G|}\chi(hg^{-1}) \\ &= \frac{|C_r|\overline{\chi(c)}}{\chi(e)}(E_\chi)_{gh},\end{aligned}$$

concluding the proof. □

Corollary 4.5.2. *The eigenvalues of the conjugacy class scheme are*

$$p_r(\chi) = \frac{|C_r|\overline{\chi(c)}}{\chi(e)}$$

where c is some element in C_r. □

We conclude this section with an extremely useful theorem about the spectrum of a normal Cayley graph. Recall that a normal Cayley graph is a Cayley graph whose connection set is conjugacy-closed. Equivalently, it is a graph in the conjugacy class scheme of a group G.

Theorem 4.5.3 ([32, Theorem 11.12.3])**.** *If $X = \mathrm{Cay}(G, \mathcal{C})$ is a normal Cayley graph and χ is an irreducible character of G, then*

$$\theta_\chi = \frac{1}{\chi(e)} \sum_{c \in \mathcal{C}} \overline{\chi}(c)$$

is an eigenvalue of X and every eigenvalue can be obtained in this way for some χ. Moreover, if $\chi_1, \ldots, \chi_k$ are all the irreducible characters such that $\theta_{\chi_r} = \theta$, then θ has multiplicity

$$\sum_{r=1}^{k} \chi_r(e)^2.$$

Proof. Let $A = \sum_{r \in S} A_r$ be the adjacency matrix of X and χ an irreducible character of G. Then by Lemma 4.5.1,

$$\begin{aligned} AE_\chi &= \sum_{r \in S} A_r E_\chi \\ &= \sum_{r \in S} \frac{|C_r|\overline{\chi(c_r)}}{\chi(e)} E_\chi, \quad (c_r \in C_r) \\ &= \frac{1}{\chi(e)} \sum_{c \in \mathcal{C}} \overline{\chi}(c) E_\chi \\ &= \theta_\chi E_\chi \end{aligned}$$

and so θ_χ is an eigenvalue. Now, if $\chi_1, \dots, \chi_k$ are all the characters with eigenvalue θ, then the spectral idempotent for θ is $E := E_{\chi_1} + \cdots + E_{\chi_k}$, so the multiplicity of θ is the rank of E, equivalently the trace of E. Each E_r has constant diagonal $\chi_r(e)^2/n$ and now we see that the multiplicity of θ is

$$\operatorname{rk}(E) = \operatorname{Tr}(E) = \sum_{r=1}^{k} \operatorname{Tr}(E_{\chi_r}) = \sum_{r=1}^{k} \chi_r(e)^2,$$

which concludes the proof. □

Note that since we require graphs to be undirected, $\mathcal{C}$ will be inverse-closed, and so we can drop the complex conjugate of the character in the sum.

4.6 Translation schemes

Let G be an abelian group. Then its conjugacy classes are singletons, and we call the conjugacy class scheme on G the *abelian group scheme*. A subscheme of an abelian group scheme is called a *translation scheme* and a graph in a translation scheme is a *translation graph*. Note that this definition is equivalent to our earlier definition of a translation graph: a Cayley graph for an abelian group.

Recall that the characters of an abelian group are homomorphisms. We can therefore refine Theorem 4.5.3 for translation graphs.

Lemma 4.6.1. *Let $X = \mathrm{Cay}(G, \mathcal{C})$ be a translation graph. Then the irreducible characters of G are eigenvectors of X and the eigenvalue for the character χ is*

$$\chi(\mathcal{C}) := \sum_{c \in \mathcal{C}} \chi(c).$$

Proof. If A is the adjacency matrix of X we have

$$\begin{aligned}(A\chi)_g &= \sum_{xg^{-1} \in \mathcal{C}} \chi(x) \\ &= \sum_{y \in \mathcal{C}} \chi(yg) \\ &= \sum_{y \in \mathcal{C}} \chi(y)\chi(g) \\ &= \chi(g) \sum_{y \in \mathcal{C}} \chi(y).\end{aligned}$$

Therefore $A\chi = \chi(\mathcal{C})\chi$ for all irreducible characters χ of G as required. □

Let $\mathcal{A} = \{A_0, \ldots, A_{n-1}\}$ be the abelian group scheme for G. We will describe a particular subscheme that we are interested in. Consider the subspace of the Bose-Mesner algebra, $\mathbb{C}[\mathcal{A}]$ that is spanned by the matrices with rational entries and rational eigenvalues. This is a commutative algebra and clearly, it contains J. Further, it is closed under transpose, complex conjugation and Schur multiplication so it is a coherent algebra, and therefore the Bose-Mesner algebra of a subscheme, $\mathcal{B} = \{B_0, \ldots, B_d\}$ of $\mathcal{A}$.

It is clear that the integer matrices in $\mathbb{C}[\mathcal{B}]$ have rational eigenvalues, but the eigenvalues of an integer matrix are always algebraic integers. Therefore, the integer matrices in $\mathbb{C}[\mathcal{B}]$ (in particular the Schur idempotents) have integer eigenvalues. We call $\mathcal{B}$ the *integral translation scheme* of G.

Let $X = \mathrm{Cay}(G, \mathcal{C})$ be a translation graph. Define a relation on the group G as follows. We say that g and h are *power-equivalent*, and write $g \approx h$, if they generate the same cyclic subgroup of G. This is an equivalence relation and we call its equivalence classes the *power classes* of G.

Denote the power class of $g \in G$ by $[g]$, and choose representatives, $g_0, \ldots, g_{d'}$ such that $[g_r] \cap [g_s] = \emptyset$ if $r \neq s$ and $G = [g_0] \cup \cdots \cup [g_{d'}]$. It turns out that the integral translation scheme described above is defined by these power classes, that is, we have $d' = d$ and by ordering things correctly, B_r is the adjacency matrix of $X_r := \mathrm{Cay}(G, [g_r])$. This is implied by the following theorem of Bridges and Mena [11].

Theorem 4.6.2 ([11, Theorem 2.4])**.** *Let $\mathcal{A} = \{A_g : g \in G\}$ be the abelian group scheme for the group G and let $\mathcal{B}$ be the integral translation scheme for G. Let $\tau(G)$ denote the number of cyclic subgroups of G. Then $\mathbb{C}[\mathcal{B}]$ has dimension $\tau(G)$ and we have*

$$\sum_{g \in G} a_g A_g \in \mathbb{C}[\mathcal{B}]$$

if and only if $a_g \in \mathbb{Q}$ and $a_g = a_h$ whenever $g \approx h$. □

Let $\mathrm{Cay}(G, \mathcal{C})$ be a translation graph, with adjacency matrix A. Then A is a Schur idempotent in the abelian group scheme, $\{A_g : g \in G\}$, so we can write

$$A = \sum_{g \in G} a_g A_g,$$

with $a_g \in \{0, 1\}$. Therefore, we get the following lemma as a consequence of Theorem 4.6.2.

Lemma 4.6.3 ([11, Corollary 2.5])**.** *The translation graph $X = \mathrm{Cay}(G, \mathcal{C})$ has integer eigenvalues if and only if $\mathcal{C}$ is a union of power classes.* □

It follows immediately that every translation graph of G with integer eigenvalues lies in the integral translation scheme on G.

Recall that G^* denotes the character group of G which is isomorphic to G. We now show that power-equivalent characters have the same eigenvalue.

Lemma 4.6.4. *There is an isomorphism, $G \to G^*$, $g \mapsto \chi_g$ such that $\chi_g(h) = \chi_h(g)$ for all $g, h \in G$ (i.e. the character table is symmetric) and if $g \approx h$, then the characters χ_g and χ_h have the same eigenvalue in $X_r = \mathrm{Cay}(G, [g_r])$ for all r.*

Proof. First, we show that there exists an isomorphism $g \mapsto \chi_g$ such that for all $g, h \in G$ we have $\chi_g(h) = \chi_h(g)$. Since G is abelian, it is a product of cyclic groups, say $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_m}$ with generators $x_1, \ldots, x_m$. Define χ_{x_r} on the generators of G by

$$\chi_{x_r}(x_s) = \begin{cases} e^{2\pi i/n_r}, & \text{if } s = r \\ 1, & \text{otherwise,} \end{cases}$$

and extend it as a homomorphism to G. Then $\chi_{x_1}, \ldots, \chi_{x_m}$ generate G^* and we see that $x_r \mapsto \chi_{x_r}$ extends to an isomorphism from G to G^*. Moreover, $\chi_{x_r}(x_s) = \chi_{x_s}(x_r)$ for all $r, s \in \{1, \ldots, m\}$ and it follows since they are homomorphisms that $\chi_g(h) = \chi_h(g)$ for all $g, h \in G$.

Now let χ be a character of G and fix $k \in \{0, \ldots, d\}$. Then χ has eigenvalue

$$\chi([g_k]) = \sum_{x \in [g_k]} \chi(x)$$

in X_k. Consider the subgroup, $\langle g_k \rangle$. The power class of g_k is the set of generators for this subgroup and it follows that $\langle h_k \rangle := \langle g_k \rangle \setminus [g_k]$ is also a subgroup.

We know that the sum of the values of a character over any subgroup is zero, unless the subgroup is in the kernel of the character. Therefore,

$$\begin{aligned} \chi([g_k]) &= \sum_{x \in [g_k]} \chi(x) \\ &= \sum_{x \in \langle g_k \rangle} \chi(x) - \sum_{x \in \langle h_k \rangle} \chi(x) \\ &= \begin{cases} |\langle g_k \rangle| - |\langle h_k \rangle|, & \text{if } g_k \in \ker(\chi) \\ -|\langle h_k \rangle|, & \text{if } g_k \notin \ker(\chi), h_k \in \ker(\chi) \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We will show that if $g \approx h$, then the kernels of χ_g and χ_h are equal. It then follows from the above that $\chi_g([g_k]) = \chi_h([g_k])$. Suppose $g \approx h$. Then there are integers, s and t such that $g = h^s$ and $h = g^t$. Therefore,

$$\begin{aligned}\chi_g(x) = 1 &\implies \chi_x(g) = 1\\ &\implies \chi_x(g)^t = 1\\ &\implies \chi_x(g^t) = 1\\ &\implies \chi_x(h) = 1\\ &\implies \chi_h(x) = 1\end{aligned}$$

so $x \in \ker(\chi_g)$ implies $x \in \ker(\chi_h)$. Similarly we see that $\chi_h(x) = 1$ implies that $\chi_x(g) = \chi_x(h^s) = 1$ which again implies that $x \in \ker(\chi_g)$.

We conclude that whenever $g \approx h$, we have $\chi_g([g_k]) = \chi_h([g_k])$. □

4.7 Duality

Most of the contents of this section can be found in [28, Chapters 7,8]. The proof of Theorem 4.7.3 is original, although the statement was probably known.

Let $\mathcal{A}$ be an association scheme with matrix of eigenvalues P and matrix of dual eigenvalues Q. We say that $\mathcal{A}$ is *formally self-dual* if $Q = \overline{P}$.

Let $\mathcal{A} = \{A_0, \ldots, A_d\}$ be a formally self-dual association scheme with matrix of eigenvalues $P = (p_r(s))_{sr}$. Define a map $\Theta : \mathbb{C}[\mathcal{A}] \to \mathbb{C}[\mathcal{A}]$ by letting

$$\Theta(A_r) := \sum_s p_r(s) A_s$$

and extending to $\mathbb{C}[\mathcal{A}]$ by linearity. We call Θ the *duality map* of $\mathcal{A}$ and we have the following theorem.

Lemma 4.7.1 ([28, Theorem 7.6.1])**.** *Let $\mathcal{A} = \{A_0, \ldots, A_d\}$ be a formally self-dual association scheme with minimal matrix idempotents $E_0, \ldots, E_d$ and duality map Θ. Then:*

(a) $\Theta(A_r) = n\overline{E}_r$ *for* $r = 0, \ldots, d$.

(b) $\Theta(I) = J$ *and* $\Theta(J) = nI$.

(c) $\Theta(MN) = \Theta(M) \circ \Theta(N)$ *for all* $M, N \in \mathbb{C}[\mathcal{A}]$.

(d) $\Theta(M \circ N) = \frac{1}{n}\Theta(M)\Theta(N)$ *for all* $M, N \in \mathbb{C}[\mathcal{A}]$.

(e) *If* $\mathcal{B}$ *is a subscheme of* $\mathcal{A}$*, then* $\Theta(\mathcal{B})$ *is also a subscheme.* □

We see that the matrix representing this linear map with respect to the basis $\{A_0, \ldots, A_d\}$ is P. This is an invertible matrix, so the map is bijective.

Notice that by part (e), we can restrict Θ to a subscheme $\mathcal{B}$ to get a map $\Theta' : \mathbb{C}[\mathcal{B}] \to \mathbb{C}[\Theta(\mathcal{B})]$. This restriction maps the Schur basis of $\mathcal{B}$ to the idempotent basis of $\Theta(\mathcal{B})$ and so we have a map that has similar properties as Θ, but is defined on an association scheme that is not necessarily formally self-dual. This motivates the notion of duality of schemes.

Let $\mathcal{A} = \{A_0, \ldots, A_d\}$ be an association scheme with matrix of dual eigenvalues $Q_{\mathcal{A}}$. If there exists an association scheme $\mathcal{B} = \{B_0, \ldots, B_d\}$ with matrix of eigenvalues $P_{\mathcal{B}} = \overline{Q}_{\mathcal{A}}$, then we call $\mathcal{B}$ the *dual association scheme* of $\mathcal{A}$. It is trivial to see that the dual of the dual of $\mathcal{A}$ is $\mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are dual association schemes, then we can define the map, $\Theta : \mathbb{C}[\mathcal{A}] \to \mathbb{C}[\mathcal{B}]$ by

$$\Theta(A_r) = \sum_s p_r(s) B_s,$$

and Lemma 4.7.1 still holds, except in part (a), we get $\Theta(A_r) = n\overline{E'}_r$ where E'_r is a minimal matrix idempotent of $\mathcal{B}$.

Not all association schemes have duals; in fact we do not know of any examples of pairs of dual association schemes that are not self-dual and are not translation schemes. Translation schemes, however, always come in dual pairs as we will now see. Sometimes they are self-dual but sometimes the two dual schemes are distinct.

Let $\mathcal{A}$ be the abelian group scheme for a group G of order n and let $\mathcal{B} = \{B_0, \ldots, B_d\}$ be an arbitrary translation scheme of G. The matrix of eigenvalues, $P_{\mathcal{A}}$ for $\mathcal{A}$ is the character table of G. By Lemma 4.6.4, we can order the character table of an abelian group so that it is symmetric, and now it follows from the orthogonality relations of characters that $P_{\mathcal{A}}\overline{P_{\mathcal{A}}} = nI$. Therefore, the abelian group scheme is formally self-dual.

Let $\pi = C_0, \ldots, C_d$ be the partition of G such that B_r is the adjacency matrix of the Cayley graph $\mathrm{Cay}(G, C_r)$. Let S denote the $n \times d$ characteristic matrix of π: the matrix in which the (g, C_r)-entry is one if $g \in C_r$ and zero otherwise. We create a partition, $\pi^* = D_0, \ldots, D_{d'}$ of the dual group, G^* of G, using the matrix $P_{\mathcal{A}}S$. The rows of this matrix are indexed by the characters of G and we let χ_1 and χ_2 be in the same cell of π^* if their corresponding rows of $P_{\mathcal{A}}S$ are the same, and in different cells otherwise.

This defines a translation scheme on the dual group, G^* and since G^* and G are isomorphic it also gives a translation scheme on G. It follows from [27, Theorem 12.7.3] that $d' = d$, so this scheme has the same number of classes as $\mathcal{B}$. In fact it is the dual scheme of $\mathcal{B}$ as we defined it above, and $\mathcal{B}$ is formally self-dual if and only if this scheme is equal to $\mathcal{B}$.

Example 4.7.2. Consider the cyclic group, $G = \langle g : g^6 = 1 \rangle \simeq \mathbb{Z}_6$. Let $\pi = \{C_0, C_1, C_2, C_3\}$ be the following partition of G:

$$C_0 = \{1\}, \quad C_1 = \{g^2\}, \quad C_2 = \{g^4\}, \quad C_3 = \{g, g^3, g^5\},$$

(so $C_1^{-1} = C_2$ and $C_3^{-1} = C_3$) and let $\mathcal{B} = \{B_0, B_1, B_2, B_3\}$ be the corresponding translation scheme. Let $\zeta := e^{2\pi i/6}$ and note that $\zeta^3 = -1$. Denote the character table of G by M and let S be the characteristic matrix for the partition π, thus

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \zeta & \zeta^2 & -1 & -\zeta & -\zeta^2 \\ 1 & \zeta^2 & -\zeta & 1 & \zeta^2 & -\zeta \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\zeta & \zeta^2 & 1 & -\zeta & \zeta^2 \\ 1 & -\zeta^2 & -\zeta & -1 & \zeta^2 & \zeta \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now since $\zeta - \zeta^2 = 1$, we get

$$MS = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & \zeta^2 & -\zeta & 0 \\ 1 & -\zeta & \zeta^2 & 0 \\ 1 & 1 & 1 & -3 \\ 1 & \zeta^2 & -\zeta & 0 \\ 1 & -\zeta & \zeta^2 & 0 \end{pmatrix}$$

The first four rows of MS are all distinct, row five is the same as row two, and row six is the same as row three, thus, the dual partition, π^* of G^* is

$$D_0 = \{1_{G^*}\}, \quad D_1 = \{\chi_1, \chi_4\}, \quad D_2 = \{\chi_2, \chi_5\}, \quad D_3 = \{\chi_3\}.$$

Note that $D_1^{-1} = D_2$ and $D_3^{-1} = D_3$. This partition defines a translation scheme that is not isomorphic to $\mathcal{B}$. Therefore $\mathcal{B}$ is not self-dual.

The following result may have been known, but as far as we know it has not been stated explicitly, and the proof is original.

Theorem 4.7.3. *The integral translation scheme is formally self-dual.*

Proof. Let $\mathcal{B} = \{B_0, \ldots, B_d\}$ be the integral translation scheme for the abelian group G. Since $\mathcal{B}$ is a subscheme of the abelian group scheme of G, which is formally self-dual, Lemma 4.7.1 holds. Therefore, if $\Theta : \mathbb{C}[\mathcal{A}] \to \mathbb{C}[\mathcal{A}]$ is the duality map, then $\Theta(\mathcal{B})$ is an association scheme. We want to show that $\Theta(\mathcal{B})$ is the same scheme as $\mathcal{B}$.

The duality map is bijective and maps the Schur idempotents of $\mathcal{B}$ to the matrix idempotents of $\Theta(\mathcal{B})$. In particular it is injective and so it suffices to show that for each B_r, we have $\Theta(B_r) \in \mathbb{C}[\mathcal{B}]$.

Recall that the integral translation scheme comes from the partition of G into its power classes. Let $g_0, g_1, \ldots, g_d$ be representatives of the power classes of G such that B_r is the adjacency matrix of $\mathrm{Cay}(G, [g_r])$. Then, if $\mathcal{A} = \{A_x : x \in G\}$ is the abelian group scheme of G, we have

$$B_r = \sum_{x \in [g_r]} A_x.$$

According to Lemma 4.6.4, choose an isomorphism $G \to G^*$, $x \mapsto \chi_x$ such that if $x \approx y$, then χ_x and χ_y have the same eigenvalue in all Cayley graphs $\mathrm{Cay}(G, [g_r])$. The eigenvalues of A_x are $\chi_g(x)$ for all $g \in G$. Then

$$\Theta : A_g \mapsto \sum_{x \in G} \chi_x(g) A_x.$$

By linearity, we have

$$\begin{aligned}\Theta(B_r) &= \sum_{g\in[g_r]} \Theta(A_g) \\ &= \sum_{g\in[g_r]} \sum_{x\in G} \chi_x(g) A_x \\ &= \sum_{x\in G} \left(\sum_{g\in[g_r]} \chi_x(g) \right) A_x \\ &= \sum_{x\in G} \chi_x([g_r]) A_x\end{aligned}$$

By Lemma 4.6.4, we can write this as

$$\begin{aligned}\Theta(B_r) &= \sum_{s=0}^{d} \chi_{g_s}([g_r]) \left(\sum_{x\in[g_s]} A_x \right) \\ &= \sum_{s=0}^{d} \chi_{g_s}([g_r]) B_s\end{aligned}$$

which is clearly in $\mathbb{C}[\mathcal{B}]$ for all r, and this concludes the proof. □

Theorem 4.7.3 implies that if P is the matrix of eigenvalues for the integral translation scheme, then $P\overline{P} = nI$. We can now prove a lemma that will become an important tool later on. A proof can also be found in [21].

Lemma 4.7.4 ([21, Lemma 16.6.1])**.** *If G is an abelian group of odd order, n, then any non-empty, integral Cayley graph for G has an odd eigenvalue.*

Proof. Let X be an integral Cayley graph for G. Then X is a graph in the integral translation scheme, $\mathcal{B} = \{B_0, \ldots, B_d\}$, for G. Let P be the matrix of eigenvalues of $\mathcal{B}$. Since $\mathcal{B}$ is self-dual we have $P\overline{P} = nI$. Further, P is real, so its determinant is real, and thus $\det(P) = \det(\overline{P})$. Then

$$\det(P)^2 = \det(P)\det(\overline{P}) = \det(P\overline{P}) = \det(nI) = n^{d+1}.$$

Therefore, $\det(P)$ is odd implying that P is invertible modulo 2.

Since X lies in the scheme $\mathcal{B}$, there is a 01-vector, $\mathbf{x}$ of length $d+1$, such that the entries of $P\mathbf{x}$ are the eigenvalues of X. If the entries of $P\mathbf{x}$ are even, then $P\mathbf{x} \equiv 0 \pmod 2$, but P is invertible modulo 2 and therefore $\mathbf{x}$ is zero modulo 2 implying that X is empty. $\square$

Remark. In Chapter 6 we will introduce integral signed Cayley graphs. We note here that if we allow $\mathbf{x}$ to be a $\{0, \pm 1\}$-vector, the proof of the Lemma still works for such graphs.

Chapter 5

Strongly Cospectral Vertices

Recall that two vertices in a graph are *cospectral* if the graphs obtained by deleting each of them have the same spectrum. This seems like a very natural property to study. Clearly, two vertices that are *similar* (there is an automorphism mapping one to the other) are cospectral, but there are certainly vertices that are cospectral but not similar.

In Chapter 2, we defined perfect state transfer. We saw that this is a very strong property, which is shown by the fact that it is a monogamous relationship between vertices. If perfect state transfer occurs between two vertices in a graph, they must be cospectral but they do not have to be similar. However, something stronger must hold and this is where strong cospectrality comes into play.

Although the motivation for studying strongly cospectral vertices comes from quantum walks, the property itself has nothing to do with quantum walks. It is, much like cospectrality, a spectral property of the graph. Furthermore, it has some interesting combinatorial implications, for example that the vertex stabilizers of two vertices that are strongly cospectral are equal. Consequently, strong cospectrality has in recent years been studied outside the context of quantum walks. In particular, a question of interest is whether there are graphs with large sets of pairwise strongly cospectral vertices.

In this chapter, we prove various results concerning the size of a set of pairwise strongly cospectral vertices in Cayley graphs. There are original proofs in Sections 5.2, 5.3 and 5.4, but our main results can be found in later sections starting from 5.5. The results of this chapter can also be found in our paper [4].

5.1 Preliminaries

Two graphs X and Y are called *cospectral* if they have the same spectrum. Two vertices, u and v in a graph X are said to be *cospectral* if the graphs $X \setminus u$ and $X \setminus v$ are cospectral.

Let X be a graph with adjacency matrix A having spectral decomposition

$$A = \sum_{r=0}^{d} \theta_r E_r.$$

Denote by $\mathbf{e}_u$ the standard basis vector indexed by the vertex u. We say that the vertices u and v are *parallel* if for each $r = 0, 1, \ldots, d$, the projections $E_r\mathbf{e}_u$ and $E_r\mathbf{e}_v$ are parallel. We say that u and v are *strongly cospectral* if they are both cospectral and parallel. Equivalently, u and v are strongly cospectral if $E_r\mathbf{e}_u = \pm E_r\mathbf{e}_v$ for all $r = 0, 1, \ldots, d$.

We call a set of vertices that are all pairwise strongly cospectral a *strongly cospectral set* and we say that it is *non-trivial* if it has size at least two. Clearly, strong cospectrality is an equivalence relation on the vertices of X and its equivalence classes are the maximal strongly cospectral sets of X.

Our main focus in this chapter will be to come up with upper bounds on the size of strongly cospectral sets. One such bound is an immediate consequence of the following lemma of Godsil and Smith [34].

Lemma 5.1.1 ([34, Lemma 10.1])**.** *If all vertices in a graph X are strongly cospectral, then $X = K_2$.* □

So in a graph on $n \geq 3$ vertices, a strongly cospectral set has size at most $n-1$. Although perhaps not a very good bound in general, it is an extremely important one.

5.2 Strongly cospectral subgroups

In Chapter 3 we defined blocks of imprimitivity and discussed them in the context of Cayley graphs. In particular we showed that a block in a Cayley graph of a group G is a coset of a subgroup of G. We will now see that maximal strongly cospectral sets are blocks and therefore also cosets. The following lemma is an observation of Chris Godsil.

Lemma 5.2.1. *Let X be a graph and $G \leq \mathrm{Aut}(X)$ a group acting transitively on its vertices. Then, a maximal strongly cospectral set is a block under this action.*

Proof. Let A be the adjacency matrix of X and let $E_0, \dots, E_d$ denote its spectral idempotents. Let $B \subseteq V(X)$ be a maximal strongly cospectral set and let $g \in G$. Since g is an automorphism of X, we may think of it as a permutation matrix, P_g, mapping the standard basis vector $\mathbf{e}_v$ to $\mathbf{e}_{v^g}$ for all $v \in V(X)$. Recall Lemma 3.1.1, that a permutation matrix commutes with A if and only if it is an automorphism of X. Further, since the spectral idempotents are polynomials in A, this implies that P_g commutes with each of them.

Suppose that there is some vertex u in B such that $u^g \in B$, and let $v \in B$. Then for all $r \in \{0, \dots, d\}$ we have

$$E_r \mathbf{e}_{v^g} = E_r P_g \mathbf{e}_v = P_g E_r \mathbf{e}_v = \pm P_g E_r \mathbf{e}_u = \pm E_r P_g \mathbf{e}_u = \pm E_r \mathbf{e}_{u^g}$$

and so v^g is strongly cospectral to u^g and therefore $v^g \in B$. It follows that B is a block of imprimitivity under the action of G. □

The next corollary follows directly from Lemma 5.2.1 and Corollary 3.2.3.

Corollary 5.2.2. *Every maximal strongly cospectral set in a Cayley graph $\mathrm{Cay}(G, \mathcal{C})$ is a coset of a subgroup of G.* □

One consequence of this is that the size of a maximal strongly cospectral set in a Cayley graph must divide the number of vertices. Since we know by Lemma 5.1.1 that not all the vertices can be pairwise strongly cospectral unless the graph is K_2, we have our first bound.

Corollary 5.2.3. *The size of a maximal strongly cospectral set in a Cayley graph on $n \geq 3$ vertices is at most $n/2$.* □

Since a system of cosets is determined by the subgroup, we will focus on the maximal strongly cospectral set containing the group identity, e. Given a group G and a connection set $\mathcal{C}$ we call the maximal strongly cospectral set of $\mathrm{Cay}(G, \mathcal{C})$ containing e, the *strongly cospectral subgroup* of G with respect to $\mathcal{C}$. Note that this subgroup is unique for a given $\mathcal{C}$. Further, the size of a maximal strongly cospectral set in $\mathrm{Cay}(G, \mathcal{C})$ is equal to the order of the strongly cospectral subgroup of G with respect to $\mathcal{C}$.

Now, instead of asking whether a Cayley graph X has any strongly cospectral vertices, we can restrict ourselves to asking whether there are any non-identity elements that are strongly cospectral to the identity in X.

5.3 Restrictions on the group

Can a Cayley graph of any group have strongly cospectral vertices? The fact that a maximal strongly cospectral set in a Cayley graph is a subgroup gives us a negative answer to this question: take a group of prime order at least three, and recall Lemma 5.1.1, that the vertices cannot all be strongly cospectral. In fact, we will now see that no group of odd order has non-trivial strongly cospectral sets.

Lemma 5.3.1. *Suppose that the vertex $g \in G$ is strongly cospectral to e in a Cayley graph X of G. Then g has order at most two in G.*

Proof. We have seen that for each $h \in G$, the map $x \mapsto xh$ is an automorphism of X. Denote the corresponding permutation matrix by P_h. Then, $P_h \mathbf{e}_x = \mathbf{e}_{xh}$ for all $h, x \in G$.

Let $E_0, \ldots, E_d$ denote the spectral idempotents of the adjacency matrix A of X. As before, P_h commutes with A and consequently with the matrices E_r. Since g is strongly cospectral to e, we have for all r that $E_r \mathbf{e}_g = \varepsilon_r E_r \mathbf{e}_e$, where $\varepsilon_r \in \{\pm 1\}$. This implies

$$\begin{aligned} E_r \mathbf{e}_{g^2} &= P_g E_r \mathbf{e}_g \\ &= P_g(\varepsilon_r E_r \mathbf{e}_e) \\ &= \varepsilon_r E_r \mathbf{e}_g \\ &= \varepsilon_r^2 E_r \mathbf{e}_e \\ &= E_r \mathbf{e}_e \end{aligned}$$

for each r. Then, since the idempotents sum to the identity, we get

$$\mathbf{e}_e = \left(\sum_{r=0}^{d} E_r\right)\mathbf{e}_e = \sum_{r=0}^{d} E_r \mathbf{e}_e = \sum_{r=0}^{d} E_r \mathbf{e}_{g^2} = \mathbf{e}_{g^2},$$

and we have shown that $g^2 = e$. □

Since a group of odd order has no elements of order two, we have shown that its Cayley graphs cannot have any non-trivial strongly cospectral sets. Further, we immediately get the following corollary.

Corollary 5.3.2. *The strongly cospectral subgroup of a group G with respect to any set $\mathcal{C}$ is an elementary abelian 2-group.*

Corollary 5.3.2 gives an upper bound on the size of the strongly cospectral subgroup only in terms of the structure of the group, but there is not much more we can say about Cayley graphs in general. We therefore turn to normal Cayley graphs.

5.4 Normal Cayley graphs

Recall that a normal Cayley graph is a Cayley graph for which the connection set is conjugacy-closed. Let $X = \mathrm{Cay}(G, \mathcal{C})$ be a normal Cayley graph. We have seen that in this case, the group G acts on X by left multiplication. We can use this to get further restrictions on the strongly cospectral subgroup. Lemma 5.4.1 is an observation of Chris Godsil.

Lemma 5.4.1. *Suppose that the vertex $g \in G$ is strongly cospectral to e in the normal Cayley graph $X = \mathrm{Cay}(G, \mathcal{C})$. Then g lies in the centre of G.*

Proof. As in the proof of Lemma 5.3.1, let P_h denote the permutation matrix corresponding to right multiplication by a group element h and now we denote by P'_h the permutation matrix corresponding to left multiplication by h. Then, $P_h \mathbf{e}_x = \mathbf{e}_{xh}$ and $P'_h \mathbf{e}_x = \mathbf{e}_{hx}$ for all $h, x \in G$.

Let $E_0, \dots, E_d$ denote the spectral idempotents of the adjacency matrix A of X. Since g is strongly cospectral to e, we have for all r that $E_r \mathbf{e}_g = \varepsilon_r E_r \mathbf{e}_e$, where $\varepsilon_r \in \{\pm 1\}$. As before, P_h and P'_h commute with the matrices E_r. Now let $h \in G$ be an arbitrary element. Then $P_h E_r \mathbf{e}_g = P_h(\varepsilon_r E_r \mathbf{e}_e)$ implying that

$$\varepsilon_r E_r \mathbf{e}_h = E_r \mathbf{e}_{gh}.$$

But similarly, we have $P'_h E_r \mathbf{e}_g = P'_h(\varepsilon_r E_r \mathbf{e}_e)$ and so

$$\varepsilon_r E_r \mathbf{e}_h = E_r \mathbf{e}_{hg}.$$

Thus $E_r \mathbf{e}_{gh} = E_r \mathbf{e}_{hg}$ for all r. Then, since the idempotents sum to I,

$$\begin{aligned} \mathbf{e}_{gh} &= \left(\sum_{r=0}^{d} E_r\right) \mathbf{e}_{gh} \\ &= \sum_{r=0}^{d} E_r \mathbf{e}_{gh} \\ &= \sum_{r=0}^{d} E_r \mathbf{e}_{hg} \\ &= \mathbf{e}_{hg}. \end{aligned}$$

Therefore, $gh = hg$ for all h and so g lies in the centre of G. □

And now, as we were all hoping, we have the following corollary.

Corollary 5.4.2. *Suppose $X = \mathrm{Cay}(G, \mathcal{C})$ is a normal Cayley graph. Then the strongly cospectral subgroup of X with respect to $\mathcal{C}$ is a normal subgroup of G.*

Proof. By Lemma 5.4.1 it is contained in the centre, thus it is normal. □

Example 5.4.3 (Non-examples)**.**

(a) A normal Cayley graph of the symmetric group on n elements has no non-trivial strongly cospectral sets, since it has trivial centre.

(b) A normal Cayley graph of a simple group has no non-trivial strongly cospectral sets.

(c) If a Cayley graph of a cyclic group $\mathbb{Z}_n$ has non-trivial strongly cospectral sets, then n is even and the sets have size two.

(d) Similarly, if there is a non-trivial strongly cospectral set in a normal Cayley graph of a dihedral group or an extraspecial group, it has size two (an extraspecial group is a p-group, G, with centre Z of order p such that G/Z is a non-trivial elementary abelian p-group).

Example 5.4.4. The Cayley graphs

$$\mathrm{Cay}(\mathbb{Z}_6, \{1,2,4,5\}) \quad \text{and} \quad \mathrm{Cay}(S_3, \{(12),(13),(123),(132)\})$$

are isomorphic and the vertex not in the connection set is strongly cospectral to the identity. Note that the former is a normal Cayley graph, since $\mathbb{Z}_6$ is abelian, but the latter cannot be normal by part (a) of Example 5.4.3. The graph is depicted in Figure 5.1 with the vertices in the strongly cospectral subgroup shown in white.

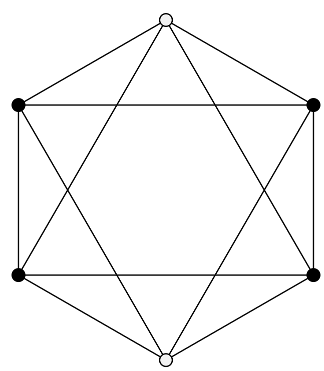


Figure 5.1: A Cayley graph for $\mathbb{Z}_6$ and S_3

5.5 Spectra

The main goal of this whole chapter is to gain some perspective on how big a strongly cospectral subgroup of a group can be. We have already seen some important restrictions. We know that it cannot be the whole group, for instance. We also know that the size of an elementary abelian 2-subgroup is an upper bound and that in the normal case, the size of the centre of the group is an upper bound.

In fact, there is not much more that we can say about this in the general case, and hereafter we will be focusing on normal Cayley graphs. You may recall from earlier chapters that all sorts of magic happens when a Cayley graph is normal. More specifically, a normal Cayley graph lives in an association scheme, and its eigenvalues and their multiplicities can be calculated using the irreducible characters of the group (Theorem 4.5.3).

We want to use this connection to bound the size of a strongly cospectral subgroup in terms of the spectrum of the graph. To do this, we need to characterize strong cospectrality in terms of characters of the group.

Recall from before that given a normal Cayley graph $X = \mathrm{Cay}(G, \mathcal{C})$, we define for each character χ of G,

$$\theta_\chi = \frac{1}{\chi(e)} \sum_{c \in \mathcal{C}} \chi(c),$$

which will be an eigenvalue of X. The following theorem was proved by Sin and Sorci in [44].

Theorem 5.5.1 ([44, Theorem 2.3])**.** *Let $X = \mathrm{Cay}(G, \mathcal{C})$ be a normal Cayley graph. A vertex $g \neq e$ is strongly cospectral to e if and only if g is a central involution in G and for all irreducible characters χ, ψ with $\theta_\chi = \theta_\psi$ we have*

$$\frac{\chi(g)}{\chi(e)} = \frac{\psi(g)}{\psi(e)}. \qquad \square$$

The proof relies on the fact that an irreducible representation of a group maps a central element to a scalar matrix cI. So if g is central and χ is the character corresponding to the irreducible representation ρ, then for any $x \in G$ we have

$$\chi(gx) = \mathrm{Tr}(\rho(gx)) = \mathrm{Tr}(\rho(g)\rho(x)) = \mathrm{Tr}(c\rho(x)) = c\chi(x),$$

in particular, $\chi(g) = c\chi(e)$. If g is also an involution, then cI has order at most two, so $c = \pm 1$ and therefore

$$\frac{\chi(g)}{\chi(e)} = \pm 1.$$

Corollary 5.5.2. *Let $X = \mathrm{Cay}(G, \mathcal{C})$ be a connected, normal Cayley graph and suppose that its complement, $\overline{X} = \mathrm{Cay}(G, \mathcal{D})$ is also connected. Then $g \in G$ is strongly cospectral to e in X if and only if g is strongly cospectral to e in $\overline{X}$.*

Proof. Let g be a vertex strongly cospectral to e in X and let χ and ψ be distinct characters of G with the same eigenvalue λ of $\overline{X}$. Then

$$\lambda = \frac{1}{\chi(e)} \sum_{g \in \mathcal{D}} \chi(g) = \frac{1}{\psi(e)} \sum_{g \in \mathcal{D}} \psi(g).$$

Note that since $\overline{X}$ is connected, the degree is an eigenvalue with multiplicity one, and the corresponding eigenvector is the trivial character. Therefore, χ and ψ are non-trivial characters.

Since $\overline{X}$ is the complement of X, we have $\mathcal{D} = G \setminus (\mathcal{C} \cup \{e\})$. As before, let θ_χ denote the eigenvalue of X for the eigenvector χ. Then

$$\theta_\chi = \frac{1}{\chi(e)} \sum_{g \in \mathcal{C}} \chi(g) = \frac{1}{\chi(e)} \left(\sum_{g \in G} \chi(g) - \sum_{g \in \mathcal{D}} \chi(g) - \chi(e) \right) = -\lambda - 1.$$

Note that the sum over G vanishes since χ is non-trivial. Similarly we get $\theta_\psi = -\lambda - 1$, and now since g is strongly cospectral to e in X, we get by Theorem 5.5.1 that $\chi(g)/\chi(e) = \psi(g)/\psi(e)$, and so again by the theorem, g is strongly cospectral to e in $\overline{X}$. □

We can now give an upper bound on the size of a strongly cospectral subgroup in a normal Cayley graph in terms of the multiplicities of the eigenvalues of the graph.

Theorem 5.5.3. *Let $X = \mathrm{Cay}(G, \mathcal{C})$ be a normal Cayley graph, let H be the strongly cospectral subgroup of G with respect to $\mathcal{C}$ and let m be the multiplicity of some eigenvalue of X. Then*

$$|H| \leq \frac{|G|}{m} = \frac{|V(X)|}{m}.$$

Proof. Let θ be an eigenvalue of X and let $\psi_1, \ldots, \psi_k$ be a complete set of irreducible characters of G satisfying $\theta_{\psi_r} = \theta$. Define d_r to be the degree of ψ_r, i.e. $d_r := \psi_r(e)$. Then by Theorem 4.5.3, the multiplicity of θ is $m_\theta := d_1^2 + \cdots + d_k^2$. Let ℓ denote the index of H in G. We will show that $m_\theta \leq \ell$.

We further let $\psi_{k+1}, \ldots, \psi_n$ be such that $\{\psi_1, \ldots, \psi_n\}$ is a complete set of irreducible characters of G, and define d_r accordingly for $r = k+1, \ldots, n$.

By Theorem 5.5.1, we know that for all $r = 1, \dots, k$, we have

$$c_h := \frac{\psi_r(h)}{d_r} = \frac{\psi_1(h)}{d_1} = \pm 1 \quad \text{for all } h \in H.$$

Let ρ_r denote the irreducible representation corresponding to the character ψ_r, for $r = 1, \dots, k$. As in the discussion following Theorem 5.5.1, we have that $\rho_r(h) = c_h I$ for all $h \in H$. Further, since ρ_r is a homomorphism we see that $c_{h_1} c_{h_2} = c_{h_1 h_2}$ for all $h_1, h_2 \in H$.

Define a function $\chi : H \to \mathbb{C}$ by $h \mapsto c_h$. By the above, this is a homomorphism from H to $\{\pm 1\}$, and since H is an abelian group, it is an irreducible character of H. Further, it is clear by the definition of χ that for each $r = 1, \dots, k$, we have

$$(\psi_r {\downarrow} H) = d_r \chi,$$

where $(\psi {\downarrow} H)$ denotes the restricted character of ψ to H, defined in Section 4.4. This implies that $\langle (\psi_r {\downarrow} H), \chi \rangle_H = d_r$ for $r = 1, \dots, k$. Recall that the values of the induced character, $\tilde{\chi} := \chi {\uparrow} G$, are given by

$$\tilde{\chi}(g) = \frac{1}{|H|} \sum_{x \in G} \chi'(x^{-1} g x), \quad \text{for all } g \in G,$$

where $\chi'(x) = \chi(x)$ if $x \in H$ and $\chi'(x) = 0$ otherwise. In particular, we see that

$$\tilde{\chi}(e) = \frac{|G|}{|H|} = \ell.$$

Since $\tilde{\chi}$ is a character of G, and $\psi_1, \dots \psi_n$ are the irreducible characters of G, we know that $\tilde{\chi}$ can be written uniquely as

$$\tilde{\chi} = d'_1 \psi_1 + \cdots + d'_n \psi_n,$$

where $d'_r := \langle \tilde{\chi}, \psi_r \rangle_G$ are non-negative integers for all $r = 1, \dots, n$. By Theorem 4.4.1 (the Frobenius Reciprocity Theorem), we have for $r = 1, \dots, k$

$$d'_r = \langle \tilde{\chi}, \psi_r \rangle_G = \langle (\psi_r {\downarrow} H), \chi \rangle_H = d_r,$$

and so

$$\tilde{\chi} = d_1 \psi_1 + \cdots + d_k \psi_k + d'_{k+1} \psi_{k+1} + \cdots + d'_n \psi_n.$$

Then, evaluating $\tilde{\chi}$ at e, we get

$$\begin{aligned}\ell &= \tilde{\chi}(e)\\ &= d_1\psi_1(e) + \cdots + d_k\psi_k(e) + d'_{k+1}\psi_{k+1}(e) + \cdots + d'_n\psi_n(e)\\ &= d_1^2 + \cdots + d_k^2 + d'_{k+1}d_{k+1} + \cdots + d'_n d_n\\ &= m_\theta + K\end{aligned}$$

where $K \geq 0$, since $d'_r, d_r \geq 0$ for all r. It follows that $\ell \geq m_\theta$, as required. □

Example 5.5.4. Let $G := \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ and take the connection set $\mathcal{C} := \{(1,0,0), (0,1,0), (0,0,1), (0,0,2)\}$. The Cayley graph $X = \mathrm{Cay}(G, \mathcal{C})$ is normal since G is abelian and it has spectrum

$$\{-3^{(2)}, -1^{(4)}, 0^{(1)}, 1^{(2)}, 2^{(2)}, 4^{(1)}\}$$

(where the superscript denotes the multiplicity). Notice that X has an eigenvalue with multiplicity four, and so by Theorem 5.5.3, the order of the strongly cospectral subgroup is at most $12/4 = 3$. Since it must also be a power of two, it is at most two. Indeed, this graph has a strongly cospectral subgroup of order two, $H = \langle(1,1,0)\rangle$. The graph is shown in Figure 5.2 with the vertices of H in white.

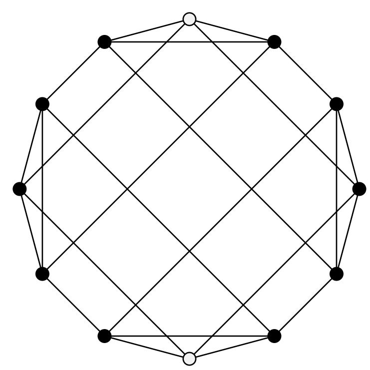


Figure 5.2: Normal Cayley graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$

Now that we have related the size of a strongly cospectral subgroup to the spectrum of the graph, the next question is what information do we have about the spectrum? What can we say about the multiplicities of the eigenvalues of a normal Cayley graph? Or more specifically the largest multiplicity?

These turn out to be tricky questions, so we will restrict ourselves to certain classes of normal Cayley graphs. We will devote the next section to one such class, namely cubelike graphs, but first, we conclude this section with a slightly better bound than before in the non-abelian case.

Corollary 5.5.5. *Let G be a non-abelian group of order n and let $X = \mathrm{Cay}(G, \mathcal{C})$ be a normal Cayley graph. If H is the strongly cospectral subgroup of G with respect to $\mathcal{C}$, then $|H| \leq n/4$.*

Proof. Since G is non-abelian, it has a non-linear character χ of degree $d \geq 2$. Then X has an eigenvalue, θ_χ with multiplicity at least $d^2 \geq 4$, and the result follows from Theorem 5.5.3. □

5.6 Cubelike graphs

In this section, we provide an upper bound on the order of a strongly cospectral subgroup of an elementary abelian 2-group, $\mathbb{Z}_2^d$, only in terms of d.

5.6.1 Why cubelike graphs?

Recall that a Cayley graph of an elementary abelian 2-group is called a *cubelike graph*. Cubelike graphs are translation graphs, so in particular they are normal Cayley graphs. For the elementary abelian 2-groups, we will use additive notation, and thus we call the group identity zero.

We have seen that the strongly cospectral subgroup for a normal Cayley graph must be an elementary abelian 2-subgroup, contained in the centre of the group. The size of such a subgroup in G is therefore an upper bound on the size of our strongly cospectral subgroup. So what can we say in the case where this bound is completely useless?

This is the main motivation for looking at cubelike graphs in particular: the restrictions that we knew previously do not give us any information in the case where the group itself is an elementary abelian 2-group. Another good reason to look at these graphs is that most of them have at least pairs of strongly cospectral vertices. This follows from a result of Bernasconi, Godsil and Severini, [9] and the fact that perfect state transfer between vertices implies that they are strongly cospectral.

Theorem 5.6.1 ([9, Theorem 1]). *A cubelike graph* $\mathrm{Cay}(\mathbb{Z}_2^d, \mathcal{C})$ *satisfying*

$$\sigma := \sum_{c \in \mathcal{C}} c \neq 0$$

has perfect state transfer from 0 *to* σ *at time* $\pi/2$. □

It is therefore tempting to believe that cubelike graphs could have large strongly cospectral subgroups, but we will show that this is not the case, in fact, a strongly cospectral subgroup of $\mathbb{Z}_2^d$ has dimension strictly less than $d/2$ (as a vector space over $\mathbb{Z}_2$) and so it's order will be less than $2^{d/2}$. Thus a maximal strongly cospectral set in a cubelike graph on n vertices has size less than $\sqrt{n}$.

5.6.2 Spectra

Recall that by Lemma 4.6.1, a translation graph $X = \mathrm{Cay}(G, \mathcal{C})$ has the characters of G as its eigenvectors and the eigenvalue for the eigenvector χ is given by

$$\chi(\mathcal{C}) := \sum_{g \in \mathcal{C}} \chi(g).$$

The characters of elementary abelian 2-groups take values in $\{\pm 1\}$ and so it is clear from the above that every eigenvalue of $X = \mathrm{Cay}(\mathbb{Z}_2^d, \mathcal{C})$ with $n := |\mathcal{C}|$ is an integer with the same parity as n and lies on the interval $[-n, n]$. This means that we can write each eigenvalue as $n - 2r$ where $r \in \{0, \ldots, n\}$.

Let $X = \mathrm{Cay}(\mathbb{Z}_2^d, \mathcal{C})$ be a cubelike graph with degree $n = |\mathcal{C}|$. For each $r = 0, \ldots, n$, denote by m_r the multiplicity of the eigenvalue $n - 2r$, with the convention that $m_r = 0$ if $n - 2r$ is not an eigenvalue. Then

$$m_0 + \cdots + m_n = |V(X)| = 2^d.$$

If A is the adjacency matrix of X, consider the matrix A^2. Since X is a regular graph of degree n, it is clear that A^2 has constant diagonal, n. Since the dimension of A^2 is 2^d, we therefore see that the trace of A^2 is $2^d n$. The trace is also the sum of the eigenvalues, and now we have proved the following identity.

Lemma 5.6.2. *Let X be a cubelike graph on 2^d vertices, with degree n and let m_r be the multiplicity of the eigenvalue $n - 2r$ of X. Then*

$$\sum_{r=0}^{n} (n-2r)^2 m_r = 2^d n. \qquad \square$$

It turns out that in a cubelike graph, the eigenvalues that are close to n and $-n$ have small multiplicities, whereas some eigenvalues close to zero will have large multiplicities. We will quantify this in the next section, but first we need one more lemma about the spectrum of graph complements.

Lemma 5.6.3. *Let X be a connected, regular graph and denote by $\overline{X}$ its complement. If θ is an eigenvalue of $\overline{X}$ different from the degree of $\overline{X}$, with multiplicity m, then X has an eigenvalue with multiplicity m.* $\square$

This follows for example from [33, Lemma 8.5.1]. The idea of the proof is that a real, symmetric matrix has an orthogonal basis of eigenvectors and a regular graph and its complement have the same eigenvectors. Then, if $\mathbf{x}$ is an eigenvector of $\overline{X}$ with eigenvalue θ different from the degree of $\overline{X}$, then $\mathbf{x}$ is an eigenvector for X with eigenvalue $-1-\theta$.

5.6.3 A large multiplicity

We will now show that a cubelike graph has an eigenvalue with a large multiplicity. The content of this section is quite technical, and perhaps not very fun to read.

Theorem 5.6.4. *Let $X = \mathrm{Cay}(\mathbb{Z}_2^d, \mathcal{C})$ be a cubelike graph, with $d \geq 3$ and let $q = \frac{d}{2}$. Then X has an eigenvalue with multiplicity larger than 2^q.*

Note that the theorem does not hold for $d = 2$; a 4-cycle does not have an eigenvalue with multiplicity larger than two.

Proof. Suppose first that X is not connected. Then each connected component of X is a cubelike graph on fewer than 2^d vertices and they are all isomorphic. Suppose X has c components, let Y be one of them and let $2^{d'}$ be its number of vertices. Then $2^{d'}c = 2^d$, so $c = 2^{d-d'}$. Further, each eigenvalue of Y with multiplicity m is an eigenvalue of X with multiplicity cm.

If $d' \geq 3$, we may assume inductively, that Y has an eigenvalue with multiplicity $m > 2^{d'/2}$, and so X has an eigenvalue with multiplicity

$$cm > 2^{d-d'} \cdot 2^{d'/2} = 2^{d-d'/2} > 2^{d-d/2} = 2^q.$$

If $d' \in \{0, 1\}$, an eigenvalue of Y with multiplicity one gives an eigenvalue of X with multiplicity $c = 2^{d-d'} > 2^q$. Finally if $d' = 2$, then $Y \in \{C_4, K_4\}$ thus it has an eigenvalue with multiplicity at least two, and so X has an eigenvalue with multiplicity at least

$$2c = 2 \cdot 2^{d-2} = 2^{d-1} > 2^q.$$

Then suppose X is connected. As before, let $n := |\mathcal{C}|$ and let m_r denote the multiplicity of the eigenvalue $n - 2r$. Then

$$\sum_{r=0}^{n} m_r = 2^d$$

and since X is a connected, n-regular graph, n is an eigenvalue with multiplicity one and $-n$ has multiplicity at most one. Therefore, $m_0 = 1$ and $m_n \in \{0, 1\}$. We will assume by way of contradiction that $m_r \leq 2^q$ for all r. Then the above gives

$$2^{2q} = \sum_{r=0}^{n} m_r \leq 2 + 2^q(n-1),$$

which implies $n \geq 2^q - 2/2^q + 1 > 2^q$, (since $q > 1$).

Assume first that both d and n are even, so q is an integer and let $k := \frac{n}{2}$. Recall the identity from Lemma 5.6.2,

$$2^d n = \sum_{r=0}^{n} (n - 2r)^2 m_r.$$

We will see that under the assumption that $m_r \leq 2^q$ for all r, we can use this to derive a quadratic inequality, $n^2 + an + b \leq 0$, with negative discriminant, yielding a contradiction since n is an integer. Notice that

$$\sum_{r=0}^{n}(n-2r)^2 m_r = n^2(1+m_n) + \sum_{r=1}^{k-1}(n-2r)^2(m_r + m_{n-r}).$$

We will split the sum into two parts. Define $t := k - 2^{q-1}$. We see that $0 < t < k-1$, and now Lemma 5.6.2 gives

$$\begin{aligned}
2^d n - n^2 &= n^2 m_n + \sum_{r=1}^{k-1}(n-2r)^2(m_r + m_{n-r}) \\
&= n^2 m_n + \sum_{r=1}^{t}(n-2r)^2(m_r + m_{n-r}) \\
&\quad + \sum_{r=t+1}^{k-1}(n-2r)^2(m_r + m_{n-r}) \\
&\geq (n-2t)^2\left(m_n + \sum_{r=1}^{t}(m_r + m_{n-r})\right) \qquad (5.1)\\
&\quad + \sum_{r=t+1}^{k-1}(n-2r)^2(m_r + m_{n-r}). \qquad (5.2)
\end{aligned}$$

We can rewrite (5.2) as

$$\sum_{r=t+1}^{k-1}(n-2r)^2(m_r + m_{n-r}) = \sum_{r=1}^{2^{q-1}-1}(2r)^2(m_{k-r} + m_{k+r}),$$

and using the fact that $m_1 + \cdots + m_n = 2^d - 1$, we see that (5.1) is equal to

$$\begin{aligned}
&(n-(n-2^q))^2\left(2^d - 1 - m_k - \sum_{r=t+1}^{k-1}(m_r + m_{n-r})\right) \\
&= 2^d(2^d - 1 - m_k) - \sum_{r=1}^{2^{q-1}-1} 2^d(m_{k-r} + m_{k+r}).
\end{aligned}$$

Putting these together, and assuming that $m_r \leq 2^q$ for all r, we get

$$\begin{aligned}
&2^d n - n^2 \\
&\geq 2^d(2^d - 1 - m_k) - \sum_{r=1}^{2^{q-1}-1} \left(2^d - (2r)^2\right)(m_{k-r} + m_{k+r}) \\
&\geq 2^{2d} - 2^d - 2^{d+q} - 2^{q+1} \sum_{r=1}^{2^{q-1}-1} \left(2^{2q} - (2r)^2\right) \\
&= 2^{2d} - 2^d - 2^{d+q} - 2^{q+1} \cdot \frac{2^{q-1}\left(2^{2q+1} - 3 \cdot 2^q - 2\right)}{3} \\
&= 2^{2d} - 2^d + \frac{2^{d+1} - 2^{2d+1}}{3}
\end{aligned}$$

and so

$$n^2 - 2^d n + 2^{2d} - 2^d + \frac{2^{d+1} - 2^{2d+1}}{3} \leq 0. \tag{5.3}$$

This is a quadratic inequality in n with discriminant

$$\begin{aligned}
&2^{2d} - 4\left(2^{2d} - 2^d + \frac{2^{d+1} - 2^{2d+1}}{3}\right) \\
&= -3 \cdot 2^{2d} + 4 \cdot 2^d - \frac{8\left(2^d - 2^{2d}\right)}{3} \\
&= \frac{1}{3}\left(2^{2d}(8-9) + 2^{d+2}(3-2)\right) \\
&= \frac{1}{3}(2^{d+2} - 2^{2d}) \\
&< 0,
\end{aligned}$$

since $d > 2$. Therefore, (5.3) never holds when $d \geq 3$, and we have reached the required contradiction.

Next we consider the case where n is even and d is odd. Here, q is no longer an integer so we need to split our sum differently. We let $q_0 := (d-1)/2$ and define $t = k - 2^{q_0 - 1}$. We use the same technique as before to get

$$n^2 - 2^d n + 2^{2d-1} - 2^{d-1} + \frac{2^{q+q_0+1} - 2^{d+q+q_0}}{3} \leq 0,$$

which has discriminant

$$\begin{aligned}
&2^{2d} - 4\left(2^{2d-1} - 2^{d-1} + \frac{2^{q+q_0+1} - 2^{d+q+q_0}}{3}\right)\\
&= \frac{1}{3}\left(-3\cdot 2^{2d} + 3\cdot 2^{d+1} + 4\cdot 2^{2d-\frac{1}{2}} - 4\cdot 2^{d+\frac{1}{2}}\right)\\
&= \frac{1}{3}\left(2^{2d}(2\sqrt{2}-3) + 2^{d+1}(3-2\sqrt{2})\right)\\
&= \frac{3-2\sqrt{2}}{3}\left(2^{d+1} - 2^{2d}\right)\\
&< 0,
\end{aligned}$$

since $d > 2$, so again we have a contradiction. Note that if $d = 3$ then $t = k - 1$, so the sum in (5.2) is empty, but the same argument still holds.

Now assume that n is odd. Then the complement, $\overline{X}$, of X is a cubelike graph on 2^d vertices with even degree, so by the above it has an eigenvalue with multiplicity $m > 2^q$. If this eigenvalue is different from the degree of $\overline{X}$ then by Lemma 5.6.3, X also has an eigenvalue with multiplicity m.

So suppose that the degree k of $\overline{X}$ is an eigenvalue with multiplicity m. The components of $\overline{X}$ are isomorphic, connected cubelike graphs of degree k. If Y is one such component and θ is an eigenvalue of Y with multiplicity m' then θ is an eigenvalue of $\overline{X}$ with multiplicity cm' where c is the number of components. But k is an eigenvalue of Y with multiplicity one, and so we must have $c = m$. Further, if Y is not the one-vertex graph, then it has another eigenvalue, $\theta \neq k$ and this is an eigenvalue of $\overline{X}$ with multiplicity $m' \geq m$. As before, this gives an eigenvalue of X with multiplicity $m' > 2^q$ by Lemma 5.6.3.

We are left with the case where $\overline{X}$ is edgeless, but then X is complete and has eigenvalue -1 with multiplicity $2^d - 1 > 2^q$ and this concludes the proof. □

5.6.4 The strongly cospectral subgroup

We can now combine Theorems 5.5.3 and 5.6.4 to get an explicit upper bound on the size of a strongly cospectral subgroup of an elementary abelian 2-group in terms of the size of the group.

Theorem 5.6.5. *A strongly cospectral subgroup of $\mathbb{Z}_2^d$, with $d \geq 3$, has order at most $2^{\lceil d/2 \rceil - 1}$.*

Proof. Let $X = \mathrm{Cay}(\mathbb{Z}_2^d, \mathcal{C})$ and let H be the strongly cospectral subgroup of $\mathbb{Z}_2^d$ with respect to $\mathcal{C}$. By Theorem 5.6.4, X has an eigenvalue with multiplicity $m > 2^{d/2}$ and so Theorem 5.5.3 gives

$$|H| \leq \frac{2^d}{m} < 2^{d/2}.$$

But by Corollary 5.3.2, $|H|$ is a power of two, so we get $|H| \leq 2^{\lceil d/2 \rceil - 1}$. □

Corollary 5.6.6. *In a cubelike graph on 2^d vertices, with $d \geq 3$, a strongly cospectral set has size at most $2^{\lceil d/2 \rceil - 1}$.* □

To put this in perspective, let us look at some small values of d. Table 5.1 shows our bound on the strongly cospectral subgroup for cubelike graphs on at most $2^9 = 512$ vertices. The column on the right indicates whether there are examples of the bound being tight.

d	# of vertices	bound	tight?
3	8	2	yes
4	16	2	yes
5	32	4	yes
6	64	4	yes
7	128	8	?
8	256	8	?
9	512	16	?

Table 5.1: Bound on the size of strongly cospectral sets in cubelike graphs

Unfortunately, as shown in the table, we do not know whether our bound is tight in all dimensions. We do, however, have some examples in small dimensions. A few of those can be found below, but we will talk about them in more detail later on.

Example 5.6.7.

(a) For $d \in \{3, 4\}$, we get $2^{\lceil d/2 \rceil - 1} = 2$. In every hypercube, antipodal vertices are strongly cospectral, so the 3-cube and the 4-cube are examples of the bound in Corollary 5.6.6 being tight.

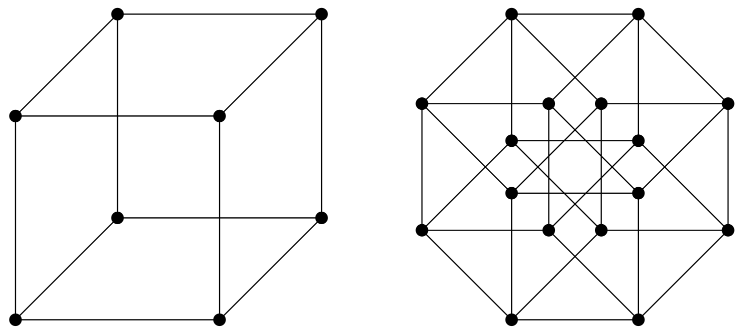


Figure 5.3: Hypercubes in dimensions 3 and 4

(b) Let $d = 5$, then $2^{\lceil d/2 \rceil - 1} = 4$. There are exactly twelve cubelike graphs (in six complementary pairs) on $2^5 = 32$ vertices that have strongly cospectral sets of size four. Two of them are shown in Figure 5.4, with the vertices in the subgroup H shown in white. We will return to these examples in Section 5.8.1.

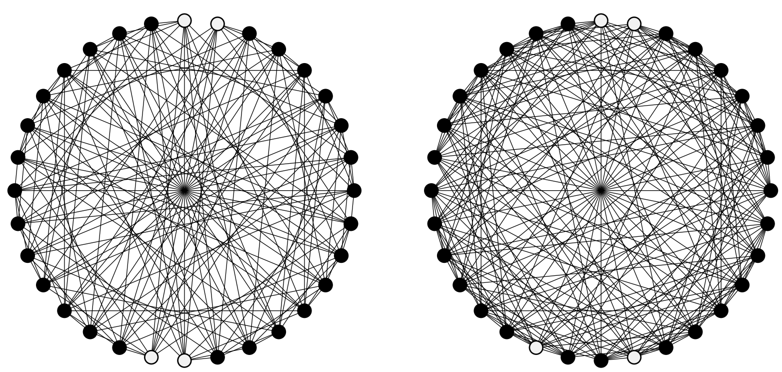


Figure 5.4: Cubelike graphs on 32 vertices with degrees 10 and 14 and a strongly cospectral subgroup of order four.

(c) For $d = 6$, a strongly cospectral set can again have size at most four. There are many such examples, one of which is shown in Figure 5.5

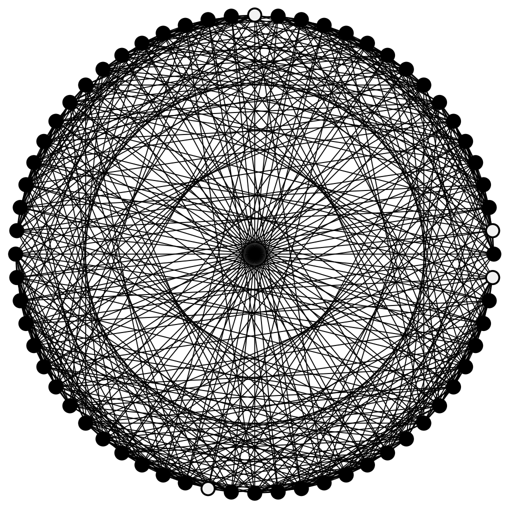


Figure 5.5: Cubelike graph on 64 vertices with degree 16 and a strongly cospectral subgroup of order four.

Note that Figures 5.4 and 5.5, are mainly meant to be aesthetically pleasing rather than informative. The same is true for most of the pictures to follow.

5.7 From $\mathbb{Z}_2^d$ to $\mathbb{Z}_4^d$

We will extend our results on the cubelike graphs to Cayley graphs for groups of the form $\mathbb{Z}_2^{d_1} \times \mathbb{Z}_4^{d_2}$.

Let $G := \mathbb{Z}_4^d$ and $X = \text{Cay}(G, \mathcal{C})$. The elements in G have order at most four, and so its characters take values in $\{\pm 1, \pm i\}$. Let χ be a character of G and consider the corresponding eigenvalue,

$$\chi(\mathcal{C}) = \sum_{g \in \mathcal{C}} \chi(g).$$

If $g \in \mathcal{C}$ is an element of order four, then $g^{-1} \neq g$ and since $\mathcal{C}$ is inverse-closed, we have $g^{-1} \in \mathcal{C}$. Now, if $\chi(g) \in \{\pm i\}$, then $\chi(g^{-1}) = -\chi(g)$, and so in this case, they cancel each other out in the sum $\chi(\mathcal{C})$. Therefore, much as in the case of the cubelike graphs, $\chi(\mathcal{C})$ is a sum of ones and negative ones, and the eigenvalues of X are of the form $|\mathcal{C}| - 2r$ where $r \in \{0, \dots, |\mathcal{C}|\}$.

Clearly, the same holds for groups of the form $\mathbb{Z}_2^{d_1} \times \mathbb{Z}_4^{d_2}$, and now we get analogous bounds to the ones for the cubelike graphs. The proof is identical to the proof of Theorem 5.6.4, so we will omit it.

Theorem 5.7.1. *Let $X := \mathrm{Cay}(\mathbb{Z}_2^{d_1} \times \mathbb{Z}_4^{d_2}, \mathcal{C})$, define $d := d_1 + 2d_2 \geq 3$ and let $q := \frac{d}{2}$. Then X has an eigenvalue with multiplicity larger than 2^q.* □

Theorem 5.7.2. *Let $d_1, d_2 \geq 0$ be integers and suppose $d := d_1 + 2d_2 \geq 3$. Then a strongly cospectral subgroup of $\mathbb{Z}_2^{d_1} \times \mathbb{Z}_4^{d_2}$ has order at most $2^{\lceil d/2 \rceil - 1}$.*

Proof. Let $X := \mathrm{Cay}(\mathbb{Z}_2^{d_1} \times \mathbb{Z}_4^{d_2}, \mathcal{C})$ and let H be the strongly cospectral subgroup with respect to $\mathcal{C}$. By Theorem 5.7.1, X has an eigenvalue with multiplicity $m > 2^{d/2}$ and so by Theorem 5.5.3 we get

$$|H| \leq \frac{2^d}{m} < 2^{d/2}.$$

By Corollary 5.3.2, $|H|$ is a power of two and the result follows. □

Corollary 5.7.3. *A strongly cospectral subgroup of $\mathbb{Z}_4^d$, where $d \geq 2$, has order at most 2^{d-1}.* □

We can now prove a general bound on the size of a strongly cospectral set in normal Cayley graphs, which is slightly better than what we had before.

Theorem 5.7.4. *In a normal Cayley graph, $X = \mathrm{Cay}(G, \mathcal{C})$ on at least five vertices, a strongly cospectral set has size at most $|V(X)|/3$.*

Proof. Let H be the strongly cospectral subgroup of G with respect to $\mathcal{C}$. If $H = G$, all the vertices are pairwise strongly cospectral, but this is impossible by Lemma 5.1.1.

Then, if $|H| > |G|/3$, we are left with the case $|G : H| = 2$. By Corollary 5.5.5, G must be abelian and it contains an elementary abelian 2-subgroup H with index two. Clearly, it follows that $G = \mathbb{Z}_2^{d_1} \times \mathbb{Z}_4^{d_2}$, where $d_2 \in \{0, 1\}$, but by Theorem 5.7.2, such a group does not have a strongly cospectral subgroup of index two. □

5.8 Examples

It is easy to find examples of Cayley graphs with pairs of strongly cospectral vertices. In fact, as we have mentioned before, most cubelike graphs have a strongly cospectral subgroup of order at least two. We did not know however whether there were Cayley graphs with more than two pairwise strongly cospectral vertices.

In this section we will look at examples of cubelike graphs with a strongly cospectral subgroup of order at least four. Some of these examples were discovered through search, and using those examples we were able to construct some infinite families of such graphs. In particular, we will see that for every $d \geq 5$, there exists a cubelike graph of dimension d with a strongly cospectral subgroup of order at least four.

We note here that in a recent paper, Peter Sin has constructed cubelike graphs with arbitrarily large strongly cospectral subgroups [43].

5.8.1 Dimension five

By Theorem 5.6.5, we know that a cubelike graph with a strongly cospectral subgroup of order four has dimension at least five, so this is where we start. Thanks to Gordon Royle, we had access to complete lists of non-isomorphic cubelike graphs on 32 and 64 vertices. There are 1372 non-isomorphic cubelike graphs on 32 vertices and exactly twelve of them, in six complementary pairs, have a strongly cospectral subgroup of order four.

Let $G = \mathbb{Z}_2^5$ and let $\mathbf{e}_1, \ldots, \mathbf{e}_5$ denote the standard basis vectors of G, thinking of G as a vector space over $\mathbb{F}_2$.

Example 5.8.1. Let

$$\begin{aligned}\mathcal{C} := \big\{&\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \mathbf{e}_4,\, \mathbf{e}_5,\\ &\mathbf{e}_1+\mathbf{e}_2,\, \mathbf{e}_1+\mathbf{e}_3,\, \mathbf{e}_1+\mathbf{e}_4,\, \mathbf{e}_1+\mathbf{e}_5,\\ &\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5\big\}.\end{aligned}$$

and let $X = \mathrm{Cay}(G, \mathcal{C})$. Here, $\mathcal{C}$ has ten elements, and so X is a regular graph with degree ten. The strongly cospectral subgroup is

$$H = \langle \mathbf{e}_1,\, \mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5 \rangle.$$

The spectrum of X is

$$\left\{-6^{(1)},\ -4^{(4)},\ -2^{(8)},\ 0^{(8)},\ 2^{(6)},\ 4^{(4)},\ 10^{(1)}\right\}.$$

The graph is depicted in Figure 5.6, with the vertices in the subgroup H shown in white (and you also saw it in Figure 5.4).

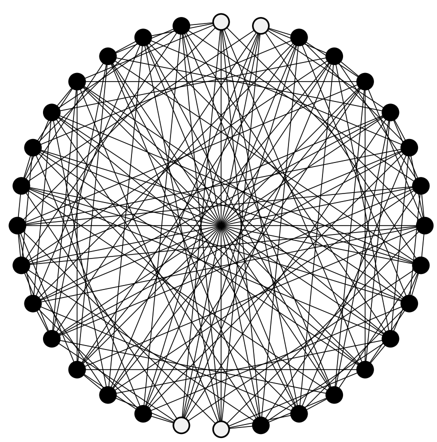
Example 5.8.2. Let

$$\begin{aligned}\mathcal{C} := \{&\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{e}_4,\ \mathbf{e}_5,\\ &\mathbf{e}_1+\mathbf{e}_2,\ \mathbf{e}_1+\mathbf{e}_3,\ \mathbf{e}_2+\mathbf{e}_3,\ \mathbf{e}_4+\mathbf{e}_5,\\ &\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4,\ \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_5\}.\end{aligned}$$

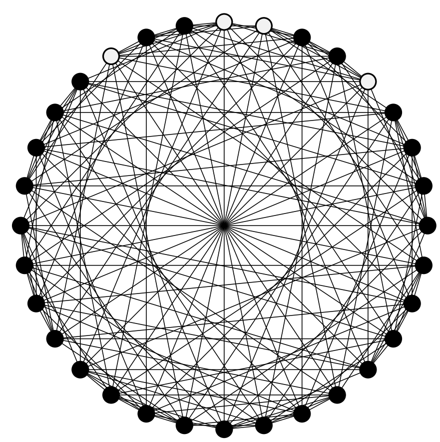
Here, $|\mathcal{C}| = 11$ and so $X = \mathrm{Cay}(G, \mathcal{C})$ has degree 11. The strongly cospectral subgroup is $H = \langle \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\ \mathbf{e}_4+\mathbf{e}_5\rangle$, and the spectrum of X is

$$\left\{-5^{(3)},\ -3^{(6)},\ -1^{(8)},\ 1^{(8)},\ 3^{(4)},\ 5^{(2)},\ 11^{(1)}\right\}.$$

The graph is depicted in Figure 5.6.



(a) Example 5.8.1



(b) Example 5.8.2

Figure 5.6: Cubelike graphs on 32 vertices with degrees 10 and 11 and a strongly cospectral subgroup of order four.

Example 5.8.3. Let

$$\begin{aligned}\mathcal{C} := \big\{&\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \mathbf{e}_4,\, \mathbf{e}_5,\\ &\mathbf{e}_1+\mathbf{e}_2,\, \mathbf{e}_1+\mathbf{e}_3,\, \mathbf{e}_1+\mathbf{e}_4,\, \mathbf{e}_1+\mathbf{e}_5,\, \mathbf{e}_2+\mathbf{e}_3,\\ &\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5,\\ &\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5\big\}.\end{aligned}$$

We have $|\mathcal{C}| = 12$ and the strongly cospectral subgroup is $H = \langle \mathbf{e}_1,\, \mathbf{e}_2+\mathbf{e}_3\rangle$. The spectrum of $X = \mathrm{Cay}(G, \mathcal{C})$ is

$$\left\{-6^{(2)},\, -4^{(3)},\, -2^{(8)},\, 0^{(8)},\, 2^{(6)},\, 4^{(4)},\, 12^{(1)}\right\}.$$

The graph is show in Figure 5.7.

Example 5.8.4. Let

$$\begin{aligned}\mathcal{C} := \big\{&\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \mathbf{e}_4,\, \mathbf{e}_5,\\ &\mathbf{e}_1+\mathbf{e}_2,\, \mathbf{e}_1+\mathbf{e}_3,\, \mathbf{e}_1+\mathbf{e}_4,\, \mathbf{e}_2+\mathbf{e}_3,\\ &\mathbf{e}_1+\mathbf{e}_4+\mathbf{e}_5,\, \mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4,\\ &\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_5,\, \mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5\big\}.\end{aligned}$$

Here, $|\mathcal{C}| = 13$ and the strongly cospectral subgroup is $H = \langle \mathbf{e}_4,\, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3\rangle$. The spectrum of $X = \mathrm{Cay}(G, \mathcal{C})$ is

$$\left\{-5^{(4)},\, -3^{(5)},\, -1^{(8)},\, 1^{(8)},\, 3^{(4)},\, 5^{(2)},\, 13^{(1)}\right\}.$$

The graph is shown in Figure 5.7.

Example 5.8.5. Let

$$\begin{aligned}\mathcal{C} := \big\{&\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \mathbf{e}_4,\, \mathbf{e}_5,\\ &\mathbf{e}_1+\mathbf{e}_2,\, \mathbf{e}_1+\mathbf{e}_3,\, \mathbf{e}_1+\mathbf{e}_4,\, \mathbf{e}_2+\mathbf{e}_5,\, \mathbf{e}_3+\mathbf{e}_5,\, \mathbf{e}_4+\mathbf{e}_5\\ &\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_5,\, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4+\mathbf{e}_5,\, \mathbf{e}_1+\mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5\big\}.\end{aligned}$$

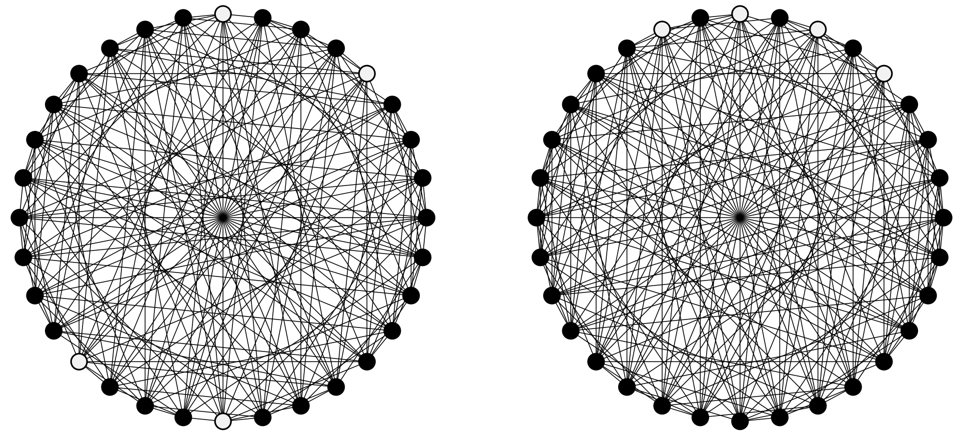
We have $|\mathcal{C}| = 14$ and the strongly cospectral subgroup is

$$H = \langle \mathbf{e}_1+\mathbf{e}_5,\, \mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4\rangle.$$

The spectrum of $X = \mathrm{Cay}(G, \mathcal{C})$ is

$$\left\{-6^{(2)},\, -4^{(4)},\, -2^{(7)},\, 0^{(8)},\, 2^{(6)},\, 4^{(4)},\, 14^{(1)}\right\}.$$

The graph is shown in Figure 5.8 (and you also saw it in Figure 5.4).



(a) Example 5.8.3 (b) Example 5.8.4

Figure 5.7: Cubelike graphs on 32 vertices with degrees 12 and 13 and a strongly cospectral subgroup of order four.

Example 5.8.6. Let

$$\begin{aligned}\mathcal{C} := \big\{&\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \mathbf{e}_4,\, \mathbf{e}_5,\\ &\mathbf{e}_1+\mathbf{e}_2,\, \mathbf{e}_1+\mathbf{e}_3,\, \mathbf{e}_1+\mathbf{e}_4,\, \mathbf{e}_1+\mathbf{e}_5,\\ &\mathbf{e}_2+\mathbf{e}_3,\, \mathbf{e}_2+\mathbf{e}_4,\, \mathbf{e}_2+\mathbf{e}_5,\, \mathbf{e}_3+\mathbf{e}_4,\, \mathbf{e}_3+\mathbf{e}_5,\\ &\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5\big\}.\end{aligned}$$

Here, $|\mathcal{C}| = 15$ and the strongly cospectral subgroup is

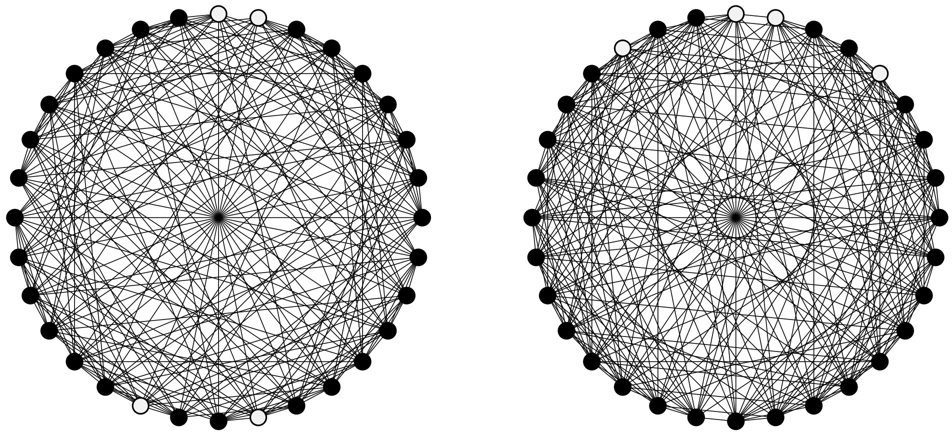
$$H = \langle \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\, \mathbf{e}_4+\mathbf{e}_5\rangle.$$

The spectrum of $X = \mathrm{Cay}(G, \mathcal{C})$ is

$$\left\{-5^{(4)},\, -3^{(6)},\, -1^{(7)},\, 1^{(8)},\, 3^{(4)},\, 5^{(2)},\, 15^{(1)}\right\}.$$

The graph can be found in Figure 5.8.

These six graphs and their complements are the only cubelike graphs on 32 vertices with a strongly cospectral subgroup of order four.



(a) Example 5.8.5 (b) Example 5.8.6

Figure 5.8: Cubelike graphs on 32 vertices with degrees 14 and 15 and a strongly cospectral subgroup of order four.

5.8.2 Dimension six

There are many cubelike graphs on 64 vertices with a strongly cospectral subgroup of order four. We will take a look at a few. Let $G = \mathbb{Z}_2^6$ and again we let $\mathbf{e}_1, \dots, \mathbf{e}_6$ denote the standard basis vectors.

Example 5.8.7. Define

$$\begin{aligned}\mathcal{C} := \big\{&\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \mathbf{e}_4,\, \mathbf{e}_5,\, \mathbf{e}_6,\\ &\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3,\, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_4,\, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_5,\, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_6,\\ &\mathbf{e}_1+\mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5+\mathbf{e}_6\big\}.\end{aligned}$$

The degree of $X = \mathrm{Cay}(G, \mathcal{C})$ is 11, and the strongly cospectral subgroup is $H = \langle \mathbf{e}_1 + \mathbf{e}_2,\, \mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6 \rangle$. The spectrum of X is

$$\left\{-11^{(1)},\, -5^{(5)},\, -3^{(10)},\, -1^{(16)},\, 1^{(16)},\, 3^{(10)},\, 5^{(5)},\, 11^{(1)}\right\}.$$

Observe that the spectrum is symmetric about the origin implying that the graph is bipartite. The graph is shown in Figure 5.9.

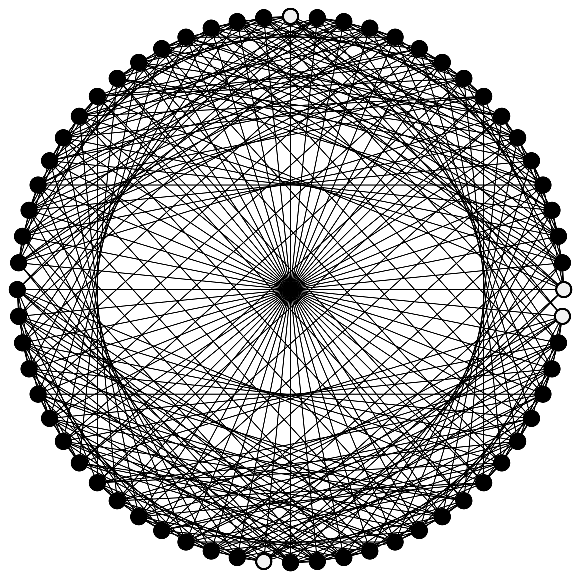


Figure 5.9: Cubelike graph on 64 vertices with degree 11 and a strongly cospectral subgroup of order four.

Example 5.8.8. Define $X = \mathrm{Cay}(G, \mathcal{C})$ with

$$\begin{aligned}\mathcal{C} := \{&\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \mathbf{e}_4,\, \mathbf{e}_5,\, \mathbf{e}_6,\\ &\mathbf{e}_1+\mathbf{e}_2,\, \mathbf{e}_1+\mathbf{e}_3,\, \mathbf{e}_1+\mathbf{e}_4,\, \mathbf{e}_1+\mathbf{e}_5\\ &\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5,\\ &\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4+\mathbf{e}_5\}.\end{aligned}$$

Here, X has degree 12 and the strongly cospectral subgroup is $H = \langle \mathbf{e}_1, \mathbf{e}_6 \rangle$. The spectrum of X is

$$\left\{-6^{(5)},\, -4^{(5)},\, -2^{(16)},\, 0^{(16)},\, 2^{(10)}, 4^{(10)},\, 10^{(1)},\, 12^{(1)}\right\}.$$

This graph is shown in Figure 5.10.

5.8.3 All dimensions

We will now show through construction that for all $d \geq 5$, there exists a cubelike graph on 2^d vertices with a strongly cospectral subgroup of order at least four. The constructions differ depending on the parity of d.

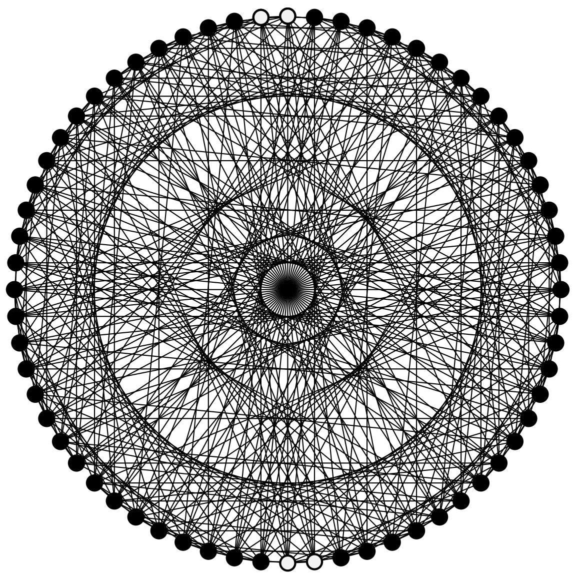


Figure 5.10: Cubelike graph on 64 vertices with degree 12 and a strongly cospectral subgroup of order four.

We start by restating Theorem 5.5.1 for abelian groups.

Theorem 5.8.9 ([21, Lemma 16.2.1])**.** *Let G be an abelian group. The vertices 0 and c are strongly cospectral in $\mathrm{Cay}(G, \mathcal{C})$ if and only if $2c = 0$ and for any two characters φ and ψ, if $\varphi(\mathcal{C}) = \psi(\mathcal{C})$ then $\varphi(c) = \psi(c)$.* □

Construction 5.8.10. Let d be an odd integer with $d \geq 5$. Denote by $\mathbf{e}_1, \dots, \mathbf{e}_d$ the standard basis vectors of $G := \mathbb{Z}_2^d$. Define the sets

$$\begin{aligned} C_1 &:= \{\mathbf{e}_1,\, \mathbf{e}_2, \dots,\, \mathbf{e}_d\}, \\ C_2 &:= \{\mathbf{e}_1 + \mathbf{e}_2,\, \mathbf{e}_1 + \mathbf{e}_3,\, \mathbf{e}_2 + \mathbf{e}_3\}, \\ C_3 &:= \{\mathbf{e}_i + \mathbf{e}_j : 4 \leq i < j \leq d\}, \\ C_4 &:= \{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_i : 4 \leq i \leq d\}, \end{aligned}$$

let $\mathcal{C} := C_1 \cup C_2 \cup C_3 \cup C_4$ and define $X = \mathrm{Cay}(G, \mathcal{C})$. We have

$$|\mathcal{C}| = d + 3 + \binom{d-3}{2} + d - 3 = 2d + \binom{d-3}{2}.$$

Note that when $d = 5$, this is the graph from Example 5.8.2.

Theorem 5.8.11. *Let X be defined as in Construction 5.8.10 and let H be the strongly cospectral subgroup of G with respect to $\mathcal{C}$. Then $|H| \geq 4$.*

Proof. Since d is odd, we see that the sum of $\mathcal{C}$ is $\sigma := \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_d$. By Theorem 5.6.1, there is perfect state transfer between 0 and σ, thus they are strongly cospectral. We will use Theorem 5.8.9 to show that the vertex $g := \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ is strongly cospectral to 0. Note that since $d \geq 5$, we have $g \neq \sigma$ and so this implies that $\langle g, \sigma \rangle$ has order four.

Since every element of G has order two, it suffices to show that if ψ, φ are characters of G with

$$\psi(\mathcal{C}) = \sum_{c \in \mathcal{C}} \psi(c) = \sum_{c \in \mathcal{C}} \varphi(c) = \varphi(\mathcal{C}),$$

then $\psi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \varphi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$. Consider an arbitrary character χ of G. We have

$$\begin{aligned}
\chi(\mathcal{C}) &= \chi(C_1) + \chi(C_2) + \chi(C_3) + \chi(C_4) \qquad (5.4)\\
&= \sum_{i=1}^{d} \chi(\mathbf{e}_i) + \chi(\mathbf{e}_1 + \mathbf{e}_2) + \chi(\mathbf{e}_1 + \mathbf{e}_3) + \chi(\mathbf{e}_2 + \mathbf{e}_3)\\
&\quad + \sum_{4 \leq i < j \leq d} \chi(\mathbf{e}_i + \mathbf{e}_j) + \sum_{i=4}^{d} \chi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_i)\\
&= \sum_{i=1}^{d} \chi(\mathbf{e}_i) + \chi(\mathbf{e}_1)\chi(\mathbf{e}_2) + \chi(\mathbf{e}_1)\chi(\mathbf{e}_3) + \chi(\mathbf{e}_2)\chi(\mathbf{e}_3)\\
&\quad + \sum_{4 \leq i < j \leq d} \chi(\mathbf{e}_i)\chi(\mathbf{e}_j) + \chi(\mathbf{e}_1)\chi(\mathbf{e}_2)\chi(\mathbf{e}_3) \sum_{i=4}^{d} \chi(\mathbf{e}_i).
\end{aligned}$$

Since every element of G has order two, we know that $\chi(g) = \pm 1$ for all $g \in G$. For convenience, let $d' := d - 3$ and consider the set

$$C_1' := \{\mathbf{e}_4, \ldots, \mathbf{e}_d\}.$$

Let p_χ denote the number of elements $c \in C_1'$ such that $\chi(c) = 1$ and n_χ the number of elements $c \in C_1'$ such that $\chi(c) = -1$, i.e.

$$p_\chi = |\chi^{-1}(1) \cap C_1'| \quad \text{and} \quad n_\chi = |\chi^{-1}(-1) \cap C_1'|.$$

Then $p_\chi + n_\chi = d'$ and we have

$$\sum_{i=4}^{d} \chi(\mathbf{e}_i) = p_\chi - n_\chi = 2p_\chi - d'.$$

Furthermore,

$$\begin{aligned}\sum_{4\le i<j\le d} \chi(\mathbf{e}_i)\chi(\mathbf{e}_j) &= \binom{p_\chi}{2} + \binom{n_\chi}{2} - p_\chi n_\chi \\ &= \frac{1}{2}\big(p_\chi(p_\chi - 1) + (d' - p_\chi)(d' - 1 - p_\chi) - 2p_\chi(d' - p_\chi)\big) \\ &= \frac{1}{2}\big(4p_\chi^2 - 4d'p_\chi + d'(d' - 1)\big) \\ &= 2p_\chi^2 - 2d'p_\chi + \frac{d'(d'-1)}{2}.\end{aligned}$$

Putting this together with Equation 5.4, we obtain

$$\begin{aligned}\chi(\mathcal{C}) = \chi(\mathbf{e}_1) &+ \chi(\mathbf{e}_2) + \chi(\mathbf{e}_3) + 2p_\chi - d' \\ &+ \chi(\mathbf{e}_1)\chi(\mathbf{e}_2) + \chi(\mathbf{e}_1)\chi(\mathbf{e}_3) + \chi(\mathbf{e}_2)\chi(\mathbf{e}_3) \\ &+ 2p_\chi^2 - 2d'p_\chi + \frac{d'(d'-1)}{2} \pm (2p_\chi - d')\end{aligned} \tag{5.5}$$

where the $\pm$ depends on the value of $\chi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$. Now let ψ, φ be characters of G such that $\psi(\mathcal{C}) = \varphi(\mathcal{C})$ and suppose by way of contradiction that $\psi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \neq \varphi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$. We may assume without loss of generality that $\psi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = 1$ and $\varphi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = -1$. Define p_ψ and p_φ as before. Since

$$\varphi(\mathbf{e}_1)\varphi(\mathbf{e}_2)\varphi(\mathbf{e}_3) = \varphi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = -1,$$

we have two possibilities: either $\varphi(\mathbf{e}_1) = \varphi(\mathbf{e}_2) = \varphi(\mathbf{e}_3) = -1$ or exactly one out of the two is -1 and the other two are 1. We can see that in both cases, we have

$$\varphi(\mathbf{e}_1) + \varphi(\mathbf{e}_2) + \varphi(\mathbf{e}_3) + \varphi(\mathbf{e}_1)\varphi(\mathbf{e}_2) + \varphi(\mathbf{e}_1)\varphi(\mathbf{e}_3) + \varphi(\mathbf{e}_2)\varphi(\mathbf{e}_3) = 0.$$

Similarly, we have two cases for $\psi(\mathbf{e}_1), \psi(\mathbf{e}_2), \psi(\mathbf{e}_3)$: either they are all one, or exactly two of them are -1. It follows that

$$\psi(\mathbf{e}_1) + \psi(\mathbf{e}_2) + \psi(\mathbf{e}_3) + \psi(\mathbf{e}_1)\psi(\mathbf{e}_2) + \psi(\mathbf{e}_1)\psi(\mathbf{e}_3) + \psi(\mathbf{e}_2)\psi(\mathbf{e}_3) \in \{-2, 6\}.$$

Combining this with Equation 5.5 we get

$$\varphi(\mathcal{C}) = 2p_\varphi^2 - 2d'p_\varphi + \frac{d'(d'-1)}{2}$$

and

$$\psi(\mathcal{C}) = \begin{cases} 2p_\psi^2 - 2d'p_\psi + \frac{d'(d'-1)}{2} + 2(2p_\psi - d') - 2, & \text{or} \\ 2p_\psi^2 - 2d'p_\psi + \frac{d'(d'-1)}{2} + 2(2p_\psi - d') + 6. \end{cases}$$

Now $\psi(\mathcal{C}) = \varphi(\mathcal{C})$ implies

$$2p_\varphi^2 - 2d'p_\varphi - (2p_\psi^2 - 2d'p_\psi + 2(2p_\psi - d')) \in \{-2, 6\}$$

and so

$$p_\varphi^2 - d'p_\varphi - 2p_\psi + d' - p_\psi^2 + d'p_\psi \in \{-1, 3\}.$$

Recall that d is odd, so $d' = d - 3$ is even; let $d' = 2z$ with $z \in \mathbb{Z}$. Further, since σ is strongly cospectral with 0, we know that $\varphi(\sigma) = \psi(\sigma)$ and it follows that p_ψ and p_φ have different parity. Suppose first $p_\psi = 2x$ and $p_\varphi = 2y + 1$, with $x, y \in \mathbb{Z}$. Then

$$\begin{aligned} &(2y+1)^2 - 2z(2y+1) - 4x + 2z - 4x^2 + 4zx \\ &= 4y^2 + 4y + 1 - 4zy - 4x + 4x^2 + 4zx \in \{-1, 3\} \end{aligned}$$

but this implies

$$4(y^2 + y - zy - x + x^2 + zx) \in \{-2, 2\}$$

which is impossible. Then suppose $p_\psi = 2x + 1$ and $p_\varphi = 2y$. In this case,

$$\begin{aligned} &4y^2 - 4zy - 2(2x+1) + 2z - (2x+1)^2 + 2z(2x+1) \\ &= 4y^2 - 4zy - 4x - 2 + 2z - 4x^2 - 4x - 1 + 4zx + 2z \\ &= 4y^2 - 4zy - 8x + 4z - 3 \in \{-1, 3\} \end{aligned}$$

which implies

$$4(y^2 - zy - 2x + z) \in \{2, 6\},$$

again impossible.

We conclude that whenever d is odd and φ and ψ are characters of G such that $\varphi(\mathcal{C}) = \psi(\mathcal{C})$, then $\varphi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \psi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ and so $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ is strongly cospectral to zero in X. Therefore, $\langle g, \sigma \rangle \leq H$ and so $|H| \geq 4$. □

Next we will look at the even dimensions.

Construction 5.8.12. We still let $d \geq 5$ be odd, and we will consider a Cayley graph for $G := \mathbb{Z}_2^{d+1}$. Let $\mathcal{C}$ be defined as in Construction 5.8.10 and let $X = \mathrm{Cay}(G, \mathcal{C}')$ where

$$\mathcal{C}' := \mathcal{C} \cup \{\mathbf{e}_{d+1},\ \mathbf{e}_1 + \cdots + \mathbf{e}_d\}.$$

Observe that $\mathbf{e}_1 + \cdots + \mathbf{e}_d \notin \mathcal{C}$ because $d \geq 5$. Figure 5.11 shows this graph in dimension $d + 1 = 6$.

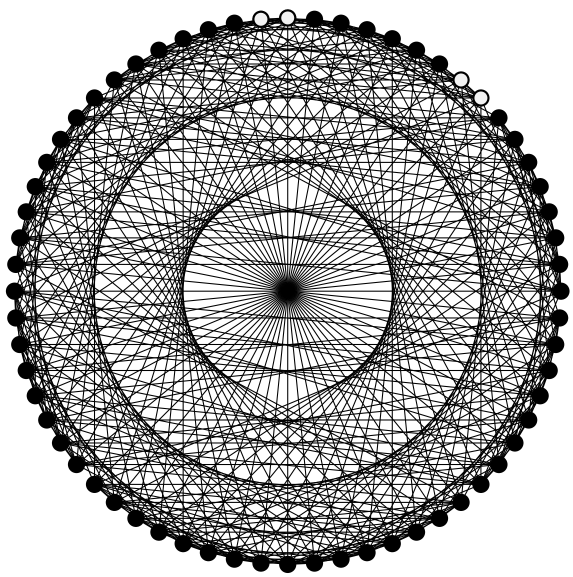


Figure 5.11: Cubelike graph on 64 vertices with degree 13 and a strongly cospectral subgroup of order four.

Recall that the sum of the elements of $\mathcal{C}$ is $\mathbf{e}_1 + \cdots + \mathbf{e}_d$ and so the sum of the elements of $\mathcal{C}'$ is $\mathbf{e}_{d+1}$. Therefore, there is perfect state transfer from 0 to $\mathbf{e}_{d+1}$ in X. For an arbitrary character, χ of G, define as before

$$p_\chi := |\chi^{-1}(1) \cap C_1'| \quad \text{and} \quad n_\chi = |\chi^{-1}(-1) \cap C_1'|$$

where $C_1' := \{\mathbf{e}_4, \ldots, \mathbf{e}_d\}$. We also let $d' := d - 3$ and notice that $d' \equiv d + 1$

(mod 4). Then we can modify equation 5.5 to get

$$\begin{aligned}\chi(\mathcal{C}') = {} & \chi(\mathbf{e}_1) + \chi(\mathbf{e}_2) + \chi(\mathbf{e}_3) + 2p_\chi - d' \\ & + \chi(\mathbf{e}_1)\chi(\mathbf{e}_2) + \chi(\mathbf{e}_1)\chi(\mathbf{e}_3) + \chi(\mathbf{e}_2)\chi(\mathbf{e}_3) \\ & + 2p_\chi^2 - 2d'p_\chi + \frac{d'(d'-1)}{2} \pm (2p_\chi - d') \\ & + \chi(\mathbf{e}_{d+1}) + \prod_{i=1}^{d} \chi(\mathbf{e}_i).\end{aligned} \tag{5.6}$$

We need to consider two cases separately, based on the parity of $(d+1)/2$.

Lemma 5.8.13. *Let X be as in Construction 5.8.12. If $d+1 \equiv 2 \pmod 4$, then $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ is strongly cospectral to zero in X.*

Proof. Let ψ and φ be distinct characters of G such that $\psi(\mathcal{C}') = \varphi(\mathcal{C}')$ and suppose that $\psi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = 1$ and $\varphi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = -1$. Then

$$\varphi(\mathcal{C}') = 2p_\varphi^2 - 2d'p_\varphi + \frac{d'(d'-1)}{2} + \varphi(\mathbf{e}_{d+1}) + \prod_{i=1}^{d} \varphi(\mathbf{e}_i)$$

and

$$\psi(\mathcal{C}') = 2p_\psi^2 - 2d'p_\psi + \frac{d'(d'-1)}{2} + 2(2p_\psi - d') + \psi(\mathbf{e}_{d+1}) + \prod_{i=1}^{d} \psi(\mathbf{e}_i) + \alpha$$

where $\alpha \in \{-2, 6\}$. Since $\mathbf{e}_{d+1}$ is strongly cospectral to 0, we know that $\psi(\mathbf{e}_{d+1}) = \varphi(\mathbf{e}_{d+1})$ and so we get

$$\begin{aligned} & 2p_\varphi^2 - 2d'p_\varphi + \prod_{i=1}^{d} \varphi(\mathbf{e}_i) \\ & \qquad - \left(2p_\psi^2 - 2d'p_\psi + 2(2p_\psi - d') + \prod_{i=1}^{d} \psi(\mathbf{e}_i) \right) \in \{-2, 6\}.\end{aligned}$$

Therefore

$$\begin{aligned} & p_\varphi^2 - d'p_\varphi - p_\psi^2 + d'p_\psi - (2p_\psi - d') \\ & \qquad + \frac{1}{2}\left(\prod_{i=1}^{d} \varphi(\mathbf{e}_i) - \prod_{i=1}^{d} \psi(\mathbf{e}_i) \right) \in \{-1, 3\}.\end{aligned} \tag{5.7}$$

Recall that d' is even and note that since

$$\prod_{i=1}^{d} \varphi(\mathbf{e}_i),\ \prod_{i=1}^{d} \psi(\mathbf{e}_i) \in \{\pm 1\}$$

we have

$$\prod_{i=1}^{d} \varphi(\mathbf{e}_i) - \prod_{i=1}^{d} \psi(\mathbf{e}_i) \in \{-2, 0, 2\}.$$

We consider three cases.

Case 1: p_ψ and p_φ have different parities.

Then, since $\psi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \neq \varphi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$, this implies

$$\prod_{i=1}^{d} \varphi(\mathbf{e}_i) = \prod_{i=1}^{d} \psi(\mathbf{e}_i).$$

Suppose first that p_ψ is even and write $p_\psi = 2x$ and $p_\varphi = 2y+1$ with $x, y \in \mathbb{Z}$. Then Equation 5.7 gives

$$\begin{aligned}
&(2y+1)^2 - d'(2y+1) - (2x)^2 + d'(2x) - (2(2x) - d') \\
&= 4y^2 + 4y + 1 - 2d'y - d' - 4x^2 + 2d'x - 4x + d' \\
&= 4(y^2 + y - x^2 - x) + 2d'(x - y) + 1 \\
&\in \{-1, 3\}
\end{aligned}$$

and so

$$2(y^2 + y - x^2 - x) + d'(x - y) \in \{\pm 1\}$$

which is impossible since d' is even. Then suppose p_ψ is odd and write $p_\psi = 2x + 1$ and $p_\varphi = 2y$. Then Equation 5.7 yields

$$\begin{aligned}
&(2y)^2 - d'(2y) - (2x+1)^2 + d'(2x+1) - (2(2x+1) - d') \\
&= 4y^2 - 2d'y - 4x^2 - 4x - 1 + 2d'x + d' - 4x - 2 + d' \\
&= 4(y^2 - x^2 - 2x) + 2d'(x - y) + 2d' - 3 \\
&\in \{-1, 3\}
\end{aligned}$$

which again implies

$$2(y^2 - x^2 - 2x) + d'(x - y) + d' \in \{1, 3\}.$$

This is impossible since d' is even.

Case 2: p_ψ and p_φ are both even.

Write $p_\psi = 2x$ and $p_\varphi = 2y$. Note that in this case,

$$\prod_{i=1}^{d} \varphi(\mathbf{e}_i) = \varphi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \prod_{i=4}^{d} \varphi(\mathbf{e}_i) = -1(-1)^{d'-p_\varphi} = -1$$

and

$$\prod_{i=1}^{d} \psi(\mathbf{e}_i) = \psi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \prod_{i=4}^{d} \psi(\mathbf{e}_i) = (-1)^{d'-p_\psi} = 1$$

and so

$$\frac{1}{2}\left(\prod_{i=1}^{d} \varphi(\mathbf{e}_i) - \prod_{i=1}^{d} \psi(\mathbf{e}_i)\right) = -1.$$

Therefore, Equation 5.7 gives

$$\begin{aligned}
&(2y)^2 - 2d'y - (2x)^2 + 2d'x - 4x + d' \\
&= 4(y^2 - x^2 - x) + 2d'(x - y) + d' \\
&\in \{0, 4\}.
\end{aligned}$$

It follows that $d' \equiv 0 \pmod 4$.

Case 3: p_ψ and p_φ are both odd.

Let $p_\psi = 2x + 1$ and $p_\varphi = 2y + 1$. Here we have

$$\frac{1}{2}\left(\prod_{i=1}^{d} \varphi(\mathbf{e}_i) - \prod_{i=1}^{d} \psi(\mathbf{e}_i)\right) = 1.$$

So Equation 5.7 gives

$$\begin{aligned}
&(2y+1)^2 - d'(2y+1) - (2x+1)^2 + d'(2x+1) - (2(2x+1) - d') \\
&= 4y^2 + 4y + 1 - 2d'y - d' - 4x^2 - 4x - 1 + 2d'x + d' - 4x - 2 + d' \\
&= 4(y^2 + y - 2x - x^2) + 2d'(x - y) + d' - 2 \\
&\in \{-2, 2\}
\end{aligned}$$

and again we see that $d' \equiv 0 \pmod 4$.

We have shown that if $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ is not strongly cospectral to zero, then $d + 1 \equiv d' \equiv 0 \pmod 4$, thus proving the lemma. □

Lemma 5.8.14. *Let X be as in Construction 5.8.12. If $d+1 \equiv 0 \pmod 4$, then $\mathbf{e}_4 + \cdots + \mathbf{e}_d$ is strongly cospectral to zero in X.*

Proof. Let ψ and φ be distinct characters such that $\psi(\mathcal{C}') = \varphi(\mathcal{C}')$ and suppose that $\psi(\mathbf{e}_4 + \cdots + \mathbf{e}_d) = 1$ and $\varphi(\mathbf{e}_4 + \cdots + \mathbf{e}_d) = -1$. It follows that p_ψ is even and p_φ is odd; write $p_\psi = 2x$ and $p_\varphi = 2y+1$ with $x, y \in \mathbb{Z}$. As before, $\psi(\mathbf{e}_{d+1}) = \varphi(\mathbf{e}_{d+1})$. For $\chi \in \{\psi, \varphi\}$ we define

$$\varepsilon_\chi := \chi(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3).$$

Again we have three cases.

Case 1: $\varepsilon_\psi \neq \varepsilon_\varphi$

First assume $\varepsilon_\psi = 1$ and $\varepsilon_\varphi = -1$. Then, as before we have

$$\psi(\mathbf{e}_1) + \psi(\mathbf{e}_2) + \psi(\mathbf{e}_3) + \psi(\mathbf{e}_1)\psi(\mathbf{e}_2) + \psi(\mathbf{e}_1)\psi(\mathbf{e}_3) + \psi(\mathbf{e}_2)\psi(\mathbf{e}_3) \in \{-2, 6\}$$

and

$$\varphi(\mathbf{e}_1) + \varphi(\mathbf{e}_2) + \varphi(\mathbf{e}_3) + \varphi(\mathbf{e}_1)\varphi(\mathbf{e}_2) + \varphi(\mathbf{e}_1)\varphi(\mathbf{e}_3) + \varphi(\mathbf{e}_2)\varphi(\mathbf{e}_3) = 0.$$

Further,

$$\prod_{i=0}^{d} \psi(\mathbf{e}_i) = \varepsilon_\psi \cdot \psi(\mathbf{e}_4 + \cdots + \mathbf{e}_d) = 1,$$
$$\prod_{i=0}^{d} \varphi(\mathbf{e}_i) = \varepsilon_\varphi \cdot \varphi(\mathbf{e}_4 + \cdots + \mathbf{e}_d) = 1.$$

It follows that

$$\varphi(\mathcal{C}') = 2p_\varphi^2 - 2d'p_\varphi + \frac{d'(d'-1)}{2} + \varphi(\mathbf{e}_{d+1}) + 1, \quad \text{and}$$
$$\psi(\mathcal{C}') = 2p_\psi^2 - 2d'p_\psi + \frac{d'(d'-1)}{2} + 2(2p_\psi - d') + \psi(\mathbf{e}_{d+1}) + 1 + \alpha$$

with $\alpha \in \{-2, 6\}$, and so

$$2p_\varphi^2 - 2d'p_\varphi - 2p_\psi^2 + 2d'p_\psi - 4p_\psi + 2d' \in \{-2, 6\}.$$

This implies

$$
\begin{aligned}
&(2y+1)^2 - d'(2y+1) - (2x)^2 + d'(2x) - 2(2x) + d' \\
&= 4y^2 + 4y + 1 - 2d'y - d' - 4x^2 + 2d'x - 4x + d' \\
&= 4(y^2 + y - x^2 - x) + 2d'(x-y) + 1 \\
&\in \{-1, 3\}.
\end{aligned}
$$

But then

$$2(y^2 + y - x^2 - x) + d'(x-y) \in \{\pm 1\},$$

which is impossible since d' is even. The case where $\varepsilon_\psi = -1$ and $\varepsilon_\varphi = 1$ leads to a similar contradiction.

Case 2: $\varepsilon_\psi = \varepsilon_\varphi = -1$.
In this case, we have

$$\chi(\mathbf{e}_1) + \chi(\mathbf{e}_2) + \chi(\mathbf{e}_3) + \chi(\mathbf{e}_1)\chi(\mathbf{e}_2) + \chi(\mathbf{e}_1)\chi(\mathbf{e}_3) + \chi(\mathbf{e}_2)\chi(\mathbf{e}_3) = 0$$

for $\chi \in \{\psi, \varphi\}$, and so

$$
\begin{aligned}
\psi(\mathcal{C}') &= 2p_\psi - d' + 2p_\psi^2 - 2d'p_\psi + \frac{d'(d'-1)}{2} \\
&\quad - (2p_\psi - d') + \psi(\mathbf{e}_{d+1}) + \prod_{i=0}^{d} \psi(\mathbf{e}_i), \\
&= 2p_\psi^2 - 2d'p_\psi + \frac{d'(d'-1)}{2} + \psi(\mathbf{e}_{d+1}) - 1 \\
&= 2(2x)^2 - 2d'(2x) + \frac{d'(d'-1)}{2} + \psi(\mathbf{e}_{d+1}) - 1 \\
&= 8x^2 - 4d'x + \frac{d'(d'-1)}{2} + \psi(\mathbf{e}_{d+1}) - 1,
\end{aligned}
$$

and

$$\begin{aligned}\varphi(\mathcal{C}') &= 2p_\varphi - d' + 2p_\varphi^2 - 2d'p_\varphi + \frac{d'(d'-1)}{2} \\ &\quad - (2p_\varphi - d') + \varphi(\mathbf{e}_{d+1}) + \prod_{i=0}^{d} \varphi(\mathbf{e}_i) \\ &= 2p_\varphi^2 - 2d'p_\varphi + \frac{d'(d'-1)}{2} + \varphi(\mathbf{e}_{d+1}) + 1 \\ &= 2(2y+1)^2 - 2d'(2y+1) + \frac{d'(d'-1)}{2} + \varphi(\mathbf{e}_{d+1}) + 1 \\ &= 8y^2 + 8y + 2 - 4d'y - 2d' + \frac{d'(d'-1)}{2} + \varphi(\mathbf{e}_{d+1}) + 1.\end{aligned}$$

It follows that

$$\begin{aligned}0 &= \varphi(\mathcal{C}') - \psi(\mathcal{C}') \\ &= 8y^2 + 8y + 2 - 4d'y + 2d' - 8x^2 + 4d'x + 2 \\ &= 4(2y^2 + 2y - d'y - 2x^2 + d'x + 1) + 2d'.\end{aligned} \tag{5.8}$$

Since d' is even we can write $d' = 2z$. Then Equation 5.8 yields

$$0 = 2y^2 + 2y - 2zy - 2x^2 + 2zx + 1 + z$$

which implies

$$z = 2(x^2 - y + zy - y^2 - zx) - 1.$$

Therefore, z is odd, and equivalently, $d + 1 \equiv d' \equiv 2 \pmod 4$.

Case 3: $\varepsilon_\psi = \varepsilon_\varphi = 1$

Then for $\chi \in \{\psi, \varphi\}$, we get

$$\chi(\mathcal{C}') = 2p_\chi^2 - 2d'p_\chi + \frac{d'(d'-1)}{2} + 2(2p_\chi - d') + \chi(\mathbf{e}_{d+1}) + \prod_{i=1}^{d} \chi(\mathbf{e}_i) + \alpha$$

with $\alpha \in \{-2, 6\}$, and so

$$\begin{aligned}&2p_\varphi^2 - 2d'p_\varphi + 4p_\varphi - 2d' - 1 - 2p_\psi^2 + 2d'p_\psi - 4p_\psi + 2d' - 1 \\ &= 2(p_\varphi^2 - d'p_\varphi + 2p_\varphi - p_\psi^2 + d'p_\psi - 2p_\psi - 1) \\ &\in \{-8, 0, 8\}.\end{aligned}$$

We again write $d' = 2z$ with $z \in \mathbb{Z}$. Then, this implies

$$\begin{aligned}&(2y+1)^2 - (2z)(2y+1) + 2(2y+1) - (2x)^2 + (2z)(2x) - 2(2x) - 1\\&= 4y^2 + 4y + 1 - 4zy - 2z + 4y + 2 - 4x^2 + 4zx - 4x - 1\\&= 4(y^2 + 2y - zy - x^2 + zx - x) - 2z + 2\\&\in \{0, \pm 4\}.\end{aligned}$$

Therefore

$$2(y^2 + 2y - zy - x^2 + zx - x) - z \in \{-3, -1, 1\},$$

implying that z is odd and so again, $d + 1 \equiv d' \equiv 2 \pmod 4$.

We have now shown that if $d+1 \equiv 0 \pmod 4$, the vertex $\mathbf{e}_4 + \cdots + \mathbf{e}_d$ is strongly cospectral to zero in X. □

Theorem 5.8.15. *If X is the cubelike graph described in Construction 5.8.12, and H is the strongly cospectral subgroup, then $|H| \geq 4$.*

Proof. Let $g := \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ and $h := \mathbf{e}_4 + \cdots + \mathbf{e}_d$. By our two lemmas, we have either $\langle \mathbf{e}_{d+1},\, g\rangle \leq H$ or $\langle \mathbf{e}_{d+1},\, h\rangle \leq H$. In both cases, $|H| \geq 4$. □

The following theorem now follows immediately from Theorems 5.8.11 and 5.8.15.

Theorem 5.8.16. *If $d \geq 5$, there exists a cubelike graph on 2^d vertices with a strongly cospectral subgroup of order four.* □

5.8.4 Other Cayley graphs

We can use our cubelike examples to construct other Cayley graphs with a strongly cospectral subgroup of order four.

The *Cartesian product* of graphs X and Y with vertex sets $V(X)$ and $V(Y)$, respectively is the graph with vertex set $V(X) \times V(Y)$ in which (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2$ and $y_1 \sim y_2$ or $x_1 \sim x_2$ and $y_1 = y_2$. We denote this graph by $X \square Y$.

Lemma 5.8.17. *Let $X := \mathrm{Cay}(\mathbb{Z}_2^d, \mathcal{C})$ be a cubelike graph, let m be an odd number and denote by C_m the cycle graph with vertices $\{0, \dots, m-1\}$. If the vertices 0 and g are strongly cospectral in X then $(0,0)$ and $(g,0)$ are strongly cospectral in the Cartesian product $Y := X \square C_m$, and this is a Cayley graph for the group $\mathbb{Z}_2^d \times \mathbb{Z}_m$.*

Proof. Denote the vertices of Y by (g,h) where $g \in V(X) = \mathbb{Z}_2^d$ and $h \in V(C_m) = \mathbb{Z}_m$. Note that $V(Y) = \mathbb{Z}_2^d \times \mathbb{Z}_m =: G$. Define $\mathcal{C}_0 := \{(c,0) : c \in \mathcal{C}\}$ and let $\mathcal{C}' := \mathcal{C}_0 \cup \{(0,1),(0,-1)\}$. It is easy to verify that $Y = \mathrm{Cay}(G, \mathcal{C}')$.

Recall that if χ_1 and χ_2 are characters of $\mathbb{Z}_2^d$ and $\mathbb{Z}_m$, respectively, then their product is a character of G, and because 2^d and m are coprime, distinct pairs (χ_1, χ_2) give distinct characters χ of G, and so every character of G can be decomposed uniquely in this way.

Suppose $g \in \mathbb{Z}_2^d$ is strongly cospectral to 0 in X. Let ψ, φ be characters of G such that $\psi(\mathcal{C}') = \varphi(\mathcal{C}')$. By the above, we can write $\varphi = \varphi_1\varphi_2$ and $\psi = \psi_1\psi_2$ where φ_1, ψ_1 are characters of $\mathbb{Z}_2^d$ and φ_2, ψ_2 are characters of $\mathbb{Z}_m$. Then we have

$$\psi(\mathcal{C}') = \sum_{c' \in \mathcal{C}'} \psi(c') = \sum_{c \in \mathcal{C}} \psi_1(c) + \psi_2(1) + \psi_2(-1) = \psi_1(\mathcal{C}) + \zeta_m^k + \zeta_m^{-k},$$

for some $k < m$, where ζ_m is a primitive m-th root of unity. Similarly, there is some $\ell < m$ such that $\varphi(\mathcal{C}') = \varphi_1(\mathcal{C}) + \zeta_m^\ell + \zeta_m^{-\ell}$. Since $\psi(\mathcal{C}') = \varphi(\mathcal{C}')$, this implies that

$$(\zeta_m^k + \zeta_m^{-k}) - (\zeta_m^\ell + \zeta_m^{-\ell}) = \psi_1(\mathcal{C}) - \varphi_1(\mathcal{C}) \in \mathbb{Z},$$

but since m is odd, the only possibility is zero. Therefore, $\psi_1(\mathcal{C}) = \varphi_1(\mathcal{C})$ and so by Theorem 5.8.9, $\psi_1(g) = \varphi_1(g)$. This implies that $\psi(g,0) = \varphi(g,0)$ which again implies that $(g,0)$ is strongly cospectral to $(0,0)$ in Y. □

Theorem 5.8.18. *If n is a positive integer divisible by 32 then there exists a Cayley graph on n vertices with strongly cospectral sets of size at least four.*

Proof. We can write $n = 2^d m$ where $d \geq 5$ and m is odd. By Theorem 5.8.16, there is a cubelike graph on 2^d vertices with strongly cospectral sets of size four. Then, by Lemma 5.8.17, we can build a Cayley graph of $\mathbb{Z}_2^d \times \mathbb{Z}_m$ preserving the strongly cospectral sets of size four. □

Chapter 6

Perfect State Transfer

In the years 2009–2013, Bašić, Petković and Stevanović wrote four papers proving various results on perfect state transfer in circulants [8, 6, 7, 5]. In the last one of these, Bašić completely characterized the connection sets of circulants having perfect state transfer [5].

In this chapter, we will show that Bašić's characterization generalizes to Cayley graphs for abelian groups with a cyclic Sylow-2-subgroup. Such graphs will be called *2-circulants*. Rather than using Bašić's methods, which are largely based in number theory, we take a group theoretic approach, making use of the theory we have developed in previous chapters, including characters and association schemes.

The results of this chapter can also be found in our paper, [3].

6.1 Preliminaries

In this chapter we will need the concepts of weighted and signed Cayley graphs.

A *weighted Cayley graph* is a Cayley graph $\mathrm{Cay}(G, \mathcal{C})$ together with a function, $\omega : \mathcal{C} \to \mathbb{Z}$, with the property that the fibre of each element in $\mathbb{Z}$ is inverse-closed. The weighted adjacency matrix of this graph is given by

$$\sum_{n \in \omega(\mathcal{C})} nA(\mathrm{Cay}(G, \omega^{-1}(n))).$$

Note that we require our graphs to have integer weights. This is because many results that have been proved for unweighted Cayley graphs still hold if the graph has integer (or rational) weights, but break if a graph has real or complex weights.

A *signed Cayley graph* is a weighted Cayley graph for which the image of ω is $\{\pm 1\}$. We will denote by $\mathcal{C}^+$ the subset of elements of $\mathcal{C}$ with positive sign and $\mathcal{C}^-$ the subset with negative sign. Then we see that the signed adjacency matrix of $\mathrm{Cay}(G,\mathcal{C})$ is $A(\mathrm{Cay}(G,\mathcal{C}^+)) - A(\mathrm{Cay}(G,\mathcal{C}^-))$.

We call $\mathrm{Cay}(G,\mathcal{C})$ an *integral signed Cayley graph* if both $\mathrm{Cay}(G,\mathcal{C}^+)$ and $\mathrm{Cay}(G,\mathcal{C}^-)$ are integral. In this case, both the signed and unsigned adjacency matrices have integer eigenvalues.

Recall that the *Cartesian product*, $X \square Y$, of the graphs X and Y is the graph with vertex set $V(X)\times V(Y)$ in which (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1 = x_2$ and $y_1 \sim y_2$ or $x_1 \sim x_2$ and $y_1 = y_2$.

Equivalently, if X and Y have adjacency matrices $A(X)$ and $A(Y)$, respectively, then $X \square Y$ is the graph with adjacency matrix $A(X)\otimes I + I\otimes A(Y)$, where $\otimes$ denotes the Kronecker product of matrices. The matrices $A(X) \otimes I$ and $I \otimes A(Y)$ commute, and so it is not too hard to see that

$$U_{X\square Y}(t) = U_X(t) \otimes U_Y(t).$$

We define the *direct product* of graphs X and Y, denoted by $X \times Y$, as the graph with adjacency matrix $A(X) \otimes A(Y)$. This is the graph with vertex set $V(X) \times V(Y)$ and $(x_1, y_1) \sim (x_2, y_2)$ if and only if both $x_1 \sim x_2$ and $y_1 \sim y_2$.

Throughout this chapter, our groups will be abelian, and we therefore use additive notation.

6.2 Integral translation graphs

In this section, we will give some general results on perfect state transfer in integral translation graphs. In particular, we give a characterization of perfect state transfer in translation graphs in terms of the characters of the underlying group.

Recall that if a vertex-transitive graph admits perfect state transfer, it must have integer eigenvalues (Lemma 2.4.6). The next lemma follows directly from a theorem of Coutinho [19].

Lemma 6.2.1 ([19, Theorem 2.4.4])**.** *Let X be an integral translation graph with distinct eigenvalues $\theta_0 > \cdots > \theta_d$. Define*

$$\delta := \gcd\{\theta_0 - \theta_r : r = 1, \ldots, d\}.$$

Then, if X admits perfect state transfer, it occurs at time π/δ. □

In fact, Coutinho's theorem holds for integer-weighted graphs. In particular our lemma holds for integral signed Cayley graphs.

Let G be an abelian group. Recall from Section 4.6 that the characters of G are eigenvectors for any Cayley graph of G and that the eigenvalue of $\mathrm{Cay}(G, \mathcal{C})$ for the character χ is given by

$$\chi(\mathcal{C}) = \sum_{c \in \mathcal{C}} \chi(c).$$

We can now prove the following.

Lemma 6.2.2 ([21, Lemma 16.3.2])**.** *Let $X = \mathrm{Cay}(G, \mathcal{C})$ be an integral translation graph and let δ be the greatest common divisor of its eigenvalue differences. Then, there is perfect state transfer from 0 to a in X at time π/δ if and only if for each character, χ of G we have*

$$\chi(a) = (-1)^{(|\mathcal{C}| - \chi(\mathcal{C}))/\delta}.$$

Proof. For each character, χ of G, define the $|G| \times |G|$ matrix E_χ by

$$(E_\chi)_{gh} := \frac{1}{|G|}\chi(h - g) = \frac{1}{|G|}\chi(h)\overline{\chi(g)},$$

for all $g, h \in G$ (these are the matrix idempotents defined in Section 4.5). The columns of E_χ are eigenvectors of $A := A(X)$ with eigenvalue $\chi(\mathcal{C})$ and we have

$$A = \sum_{\chi \in G^*} \chi(\mathcal{C}) E_\chi.$$

Let $U(t)$ be the transition matrix at time t. Then

$$U(t) = \sum_{\chi \in G^*} e^{it\chi(\mathcal{C})} E_\chi.$$

In particular,

$$\begin{aligned} U(t)_{0,a} &= \sum_{\chi \in G^*} e^{it\chi(\mathcal{C})} (E_\chi)_{0,a} \\ &= \frac{1}{|G|} \sum_{\chi \in G^*} e^{it\chi(\mathcal{C})} \chi(a) \overline{\chi(0)} \\ &= \frac{1}{|G|} \sum_{\chi \in G^*} e^{it\chi(\mathcal{C})} \chi(a). \end{aligned}$$

Since each term in the sum has absolute value one, we see that $|U(t)_{0,a}| = 1$ if and only if the terms are all equal. This is equivalent to

$$e^{it|\mathcal{C}|} = e^{it\chi(\mathcal{C})} \chi(a)$$

for all characters χ of G. By Lemma 6.2.1, perfect state transfer must occur at time π/δ, and now the result follows. □

It follows from this lemma that perfect state transfer must occur between zero and an element of order two, but we also knew this from Theorem 5.8.9. Since a group with a cyclic Sylow-2-subgroup has a unique element of order two, we see that in a 2-circulant, there is only one possible vertex that could have perfect state transfer with the identity.

6.3 Groups with a cyclic Sylow-2-subgroup

Let G be an abelian group of order $2^d m$ where m is odd, and assume G has a cyclic Sylow-2-subgroup. Then $G \cong \mathbb{Z}_{2^d} \times H$, where $H \leq G$ is an abelian group of order m. This is a simple but extremely important fact. We will use this to decompose a Cayley graph of such a group into smaller graphs.

Let $X = \text{Cay}(G, \mathcal{C})$ be an integral 2-circulant. We partition $\mathcal{C}$ in the following way. Let $\mathcal{C}_r$ denote the subset of $\mathcal{C}$ consisting of elements that have order $2^r m'$ for some odd number m'. It is clear that $\mathcal{C}_0, \ldots, \mathcal{C}_d$ is a partition of $\mathcal{C}$ and so X is an edge-disjoint union of the Cayley graphs $\mathrm{Cay}(G, \mathcal{C}_0), \ldots, \mathrm{Cay}(G, \mathcal{C}_d)$.

Furthermore, since X is integral, $\mathcal{C}$ is power-closed by Lemma 4.6.3, and this clearly implies that $\mathcal{C}_r$ is power-closed (in particular inverse-closed) for all r. We will show that the Cayley graph $X_r := \mathrm{Cay}(G, \mathcal{C}_r)$ is a direct product of two Cayley graphs, one of which has complete bipartite graphs as its components. First, we need two lemmas. Recall that we denote by $[g]$ the power class of g.

Lemma 6.3.1 ([21, Lemma 16.5.2])**.** *The Cayley graph* $\mathrm{Cay}(\mathbb{Z}_{2^d}, [1])$ *is isomorphic to the complete bipartite graph* $K_{2^{d-1},2^{d-1}}$*, for all* $d \geq 1$.

Proof. We see that $[1]$ consists of all the odd numbers, so the even and odd numbers form a bipartition of $\mathrm{Cay}(\mathbb{Z}_{2^d}, [1])$ and the rest is clear. □

Note that in a cyclic group, $[1]$ is the set of elements that generate the whole group.

Lemma 6.3.2. *Let* $X = \mathrm{Cay}(G, \mathcal{C})$ *be a Cayley graph. If* $G \cong H_1 \times H_2$ *and* $\mathcal{C} = \mathcal{D}_1 \times \mathcal{D}_2$*, with* $\mathcal{D}_r \subseteq H_r$*, then* X *is a direct product of Cayley graphs* $Y_1 = \mathrm{Cay}(H_1, \mathcal{D}_1)$ *and* $Y_2 = \mathrm{Cay}(H_2, \mathcal{D}_2)$.

Proof. Since $G \cong H_1 \times H_2$, the vertices of X can be written $g = (g_1, g_2)$ with $g_r \in H_r$ for $r = 1, 2$. Further, vertices $g = (g_1, g_2)$ and $h = (h_1, h_2) \in G$ are adjacent in X if and only if $(h_1 g_1^{-1}, h_2 g_2^{-1}) \in \mathcal{C}$, if and only if $h_1 g_1^{-1} \in \mathcal{D}_1$ and $h_2 g_2^{-1} \in \mathcal{D}_2$. This is equivalent to g and h being adjacent in the direct product $Y_1 \times Y_2$ from which the result follows. □

We can now prove the following theorem.

Theorem 6.3.3. *Let* $X = \mathrm{Cay}(G, \mathcal{C})$ *be an integral signed* 2*-circulant. Suppose the order of* G *is* $2^d m$ *where* m *is odd and* $d \geq 1$*, and let* H *be the unique subgroup of* G *of order* m*. For* $r \geq 0$*, let* $\mathcal{C}_r \subseteq \mathcal{C}$ *be the set of elements in* $\mathcal{C}$ *of order* $2^r m'$ *for some odd number* m' *and let* $X_r := \mathrm{Cay}(G, \mathcal{C}_r)$*. Further, let* $K(d, r)$ *denote the graph on* 2^d *vertices whose components are isomorphic to the complete bipartite graph* $K_{2^{r-1},2^{r-1}}$*, with the convention that* $K(d, 0)$ *is the graph with adjacency matrix* I*. Then*

$$X_r \cong K(d, r) \times Y_r$$

where Y_r *is an integral signed Cayley graph of* H*.*

Proof. First, since $G \cong \mathbb{Z}_{2^d} \times H$, we can write every element of G as (x, h) with $x \in \mathbb{Z}_{2^d}$ and $h \in H$. Fix $r \in \{0, \dots, d\}$ and let $\mathcal{C}_r^+ := \mathcal{C}_r \cap \mathcal{C}^+$ and $\mathcal{C}_r^- := \mathcal{C}_r \cap \mathcal{C}^-$. Since X is an integral signed Cayley graph, $\mathcal{C}^+$ and $\mathcal{C}^-$ are power-closed, thus $\mathcal{C}_r^+$ and $\mathcal{C}_r^-$ are power-closed. We want to show that there are subsets $\mathcal{D}_1 \subseteq \mathbb{Z}_{2^d}$ and $\mathcal{D}_2^+, \mathcal{D}_2^- \subseteq H$ such that $\mathcal{C}_r^+ = \mathcal{D}_1 \times \mathcal{D}_2^+$ and $\mathcal{C}_r^- = \mathcal{D}_1 \times \mathcal{D}_2^-$. We will show that if $(x_1, h_1), (x_2, h_2) \in \mathcal{C}_r$ (respectively $\mathcal{C}_r^+, \mathcal{C}_r^-$) then $(x_1, h_2), (x_2, h_1) \in \mathcal{C}_r$ (respectively $\mathcal{C}_r^+, \mathcal{C}_r^-$).

Note that if $\langle (x_1, h_1) \rangle = \langle (x_2, h_2) \rangle \le G$, then $\langle x_1 \rangle = \langle x_2 \rangle \le \mathbb{Z}_{2^d}$ and $\langle h_1 \rangle = \langle h_2 \rangle \le H$. Let $(x_1, h_1), (x_2, h_2) \in \mathcal{C}_r$. Then x_1 and x_2 have the same order, 2^r, and since they are contained in the cyclic group $\mathbb{Z}_{2^d}$, we have $\langle x_1 \rangle = \langle x_2 \rangle$. But then we have $(x_1, h_2) \in [(x_2, h_2)] \subseteq \mathcal{C}_r$ and $(x_2, h_1) \in [(x_1, h_1)] \subseteq \mathcal{C}_r$, implying that $\mathcal{C}_r = \mathcal{D}_1 \times \mathcal{D}_2$ for some $\mathcal{D}_1, \mathcal{D}_2$.

Similarly we get $\mathcal{C}_r^+ = \mathcal{D}_1' \times \mathcal{D}_2'$ and $\mathcal{C}_r^- = \mathcal{D}_1'' \times \mathcal{D}_2''$. Since $\mathcal{C}_r$ is a disjoint union of $\mathcal{C}_r^+$ and $\mathcal{C}_r^-$, and the elements of $\mathcal{D}_1$ all generate the same subgroup, we must have $\mathcal{D}_1' = \mathcal{D}_1'' = \mathcal{D}_1$ and $\mathcal{D}_2' \cup \mathcal{D}_2'' = \mathcal{D}_2$ with $\mathcal{D}_2'$ and $\mathcal{D}_2''$ disjoint. Thus we have found sets $\mathcal{D}_1 \subseteq \mathbb{Z}_{2^d}$ and $\mathcal{D}_2^+, \mathcal{D}_2^- \subseteq H$ such that $\mathcal{C}_r^+ = \mathcal{D}_1 \times \mathcal{D}_2^+$ and $\mathcal{C}_r^- = \mathcal{D}_1 \times \mathcal{D}_2^-$.

Then by Lemma 6.3.2, we have

$$\begin{aligned}
\mathrm{Cay}(G, \mathcal{C}_r) &\cong \mathrm{Cay}(\mathbb{Z}_{2^d}, \mathcal{D}_1) \times \mathrm{Cay}(H, \mathcal{D}_2),\\
\mathrm{Cay}(G, \mathcal{C}_r^+) &\cong \mathrm{Cay}(\mathbb{Z}_{2^d}, \mathcal{D}_1) \times \mathrm{Cay}(H, \mathcal{D}_2^+),\\
\mathrm{Cay}(G, \mathcal{C}_r^-) &\cong \mathrm{Cay}(\mathbb{Z}_{2^d}, \mathcal{D}_1) \times \mathrm{Cay}(H, \mathcal{D}_2^-).
\end{aligned}$$

Clearly, $\mathcal{D}_2^+$ and $\mathcal{D}_2^-$ are power closed, so $Y_r := \mathrm{Cay}(H, \mathcal{D}_2)$ is an integral signed Cayley graph of H. The signed adjacency matrix of $\mathrm{Cay}(G, \mathcal{C}_r)$ is

$$A(\mathrm{Cay}(\mathbb{Z}_{2^d}, \mathcal{D}_1)) \otimes (A(\mathrm{Cay}(H, \mathcal{D}_2^+)) - A(\mathrm{Cay}(H, \mathcal{D}_2^-))).$$

Further, we see that $\mathcal{D}_1$ consists of all the elements in $\mathbb{Z}_{2^d}$ of order 2^r. This implies that the components of $\mathrm{Cay}(\mathbb{Z}_{2^d}, \mathcal{D}_1)$ are isomorphic to $\mathrm{Cay}(\mathbb{Z}_{2^r}, [1])$, unless $r = 0$ in which case it is the graph with a loop on each vertex and no other edges. Now the result follows from Lemma 6.3.1. □

Note that we have defined a Cayley graph in terms of a connection set that does not contain the identity, since if this were the case, the graph would have a loop on every vertex. However, for the graphs Y_r, we extend the definition to include graphs with loops, since this can indeed happen here, if $r \geq 1$.

Corollary 6.3.4. *For $r \geq 3$, the integral signed graphs $X_r = \text{Cay}(G, \mathcal{C}_r)$ are periodic at time $\pi/2$.*

Proof. The eigenvalues of X_r are $\theta\lambda$ where θ and λ are eigenvalues of $K(d, r)$ and the signed adjacency matrix of Y_r, respectively. The eigenvalues of $K(d, r)$ are $\pm 2^{k-1}$ and 0 and so it follows that all the eigenvalues of X_r are divisible by 2^{r-1}. In particular, if $r \geq 3$ they are divisible by four. Then, using the spectral decomposition of the adjacency matrix of X_r, it is clear that $U(\pi/2) = I$, so X_r is periodic at time $\pi/2$ for $r \geq 3$. □

6.4 Semidirect products

Let $X = \text{Cay}(G, \mathcal{C})$ be an integral signed 2-circulant where G has order $2^d m$ with m odd, and define Y_r as in Theorem 6.3.3 for $r = 0, \ldots, d$. Let A_r denote the signed adjacency matrix of Y_r. We want to construct the adjacency matrix of X using the matrices $A_0, \ldots, A_d$.

Define the matrix $\mathcal{M}(A_0, \ldots, A_d)$ recursively by

$$\mathcal{M}(A_0, A_1) := \begin{pmatrix} A_0 & A_1 \\ A_1 & A_0 \end{pmatrix},$$

and for $r \geq 2$,

$$\begin{aligned} \mathcal{M}(A_0, \ldots, A_r) &:= I_2 \otimes \mathcal{M}(A_0, \ldots, A_{r-1}) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes (J_{2^{r-1}} \otimes A_r) \\ &= \mathcal{M}(\mathcal{M}(A_0, \ldots, A_{r-1}), J_{2^{r-1}} \otimes A_r). \end{aligned}$$

Lemma 6.4.1. *The matrix $\mathcal{M}(A_0, \ldots, A_d)$ is the signed adjacency matrix of X.*

Proof. We proceed by induction. Recall that the adjacency matrix of the direct product of two graphs is the Kronecker product of their adjacency matrices. If $d = 1$, then X is the disjoint union of $\text{Cay}(G, \mathcal{C}_0) = K(d, 0) \times Y_0$ and $\text{Cay}(G, \mathcal{C}_1) = K(d, 1) \times Y_1$, so its adjacency matrix is given by

$$A(X) = I_2 \otimes A_0 + A(K_2) \otimes A_1 = \mathcal{M}(A_0, A_1).$$

(Here, $A(X)$ is the signed adjacency matrix of X.)

Then let $d \geq 2$ and suppose the claim holds for all integral signed 2-circulants on at most $2^{d-1}m$ vertices. Consider the unique subgroup G' of G of index two (which exists since G has a cyclic Sylow-2-subgroup). It is clear that $\mathcal{C}' := \mathcal{C}_0 \cup \cdots \cup \mathcal{C}_{d-1} \subseteq G'$ and that $\langle G' \cup \mathcal{C}_d \rangle = G$.

Then X has a subgraph isomorphic to $X' := \mathrm{Cay}(G', \mathcal{C}')$ which by our inductive hypothesis has adjacency matrix $A(X') := \mathcal{M}(A_0, \ldots, A_{d-1})$.

Further, we know that X is an edge disjoint union of $\mathrm{Cay}(G, \mathcal{C}')$ and $X_d = \mathrm{Cay}(G, \mathcal{C}_d)$, and we see that the former consists of two disjoint copies of X'. Therefore we get

$$A(X) = I \otimes A(X') + A(X_d).$$

Finally, since $X_d = K(d,d) \times Y_d = K_{2^{d-1},2^{d-1}} \times Y_d$ by Theorem 6.3.3, we have

$$A(X_d) = \begin{pmatrix} 0 & J_{2^{d-1}} \\ J_{2^{d-1}} & 0 \end{pmatrix} \otimes A_d$$

and putting all this together we get

$$A(X) = I \otimes \mathcal{M}(A_0, \ldots, A_{d-1}) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes (J_{2^{d-1}} \otimes A_d) = \mathcal{M}(A_0, \ldots, A_d)$$

as required. □

If X and Y are graphs on the same vertex set with adjacency matrices A and B, respectively, we define their *semidirect product*, $X \ltimes Y$, as the graph with adjacency matrix $\mathcal{M}(A, B)$. This graph product is explored in a paper by Coutinho and Godsil [20], where they prove the following theorem.

Theorem 6.4.2 ([20, Theorem 5.2])**.** *Given graphs X and Y on the same vertex set V, with $A = A(X)$ and $B = A(Y)$, the graph $X \ltimes Y$ on vertex set $\{0,1\} \times V$ admits perfect state transfer if and only if one of the following holds.*

(i) For some $\tau \in \mathbb{R}^+, \lambda \in \mathcal{C}$ and $u \in V$, the matrices $A + B$ and $A - B$ are periodic at u at time τ with respective phase factors λ and $-\lambda$. In this case, perfect state transfer is between $(0, u)$ and $(1, u)$.

(ii) *For some $\tau \in \mathbb{R}^+, \lambda \in \mathcal{C}$ and $u \in V$, the matrices $A+B$ and $A-B$ admit uv-perfect state transfer at time τ with the same phase factor λ. In this case, perfect state transfer is between $(0,u)$ and $(0,v)$, and between $(1,u)$ and $(1,v)$.*

(iii) *For some $\tau \in \mathbb{R}^+, \lambda \in \mathcal{C}$ and $u \in V$, the matrices $A+B$ and $A-B$ admit uv-perfect state transfer at time τ with respective phase factors λ and $-\lambda$. In this case, perfect state transfer is between $(0,u)$ and $(1,v)$, and between $(1,u)$ and $(0,v)$.* □

Theorem 6.4.2 is stated only for simple graphs in the original paper, but the proof also works if X and Y are signed graphs.

6.5 Time of perfect state transfer

In this section we will show that if perfect state transfer occurs in a 2-circulant, it must have minimal time $\pi/2$. We will use the decomposition into a semidirect product of Cayley graphs from the previous section. First we need to consider the case where $d = 1$.

Theorem 6.5.1 ([21, Theorem 16.6.2]). *Let $X = \mathrm{Cay}(G, \mathcal{C})$ be a signed Cayley graph for an abelian group G of order $2m$, where m is odd. If X admits perfect state transfer, then $X \cong mK_2$ and the minimal time at which perfect state transfer occurs is $\pi/2$.*

Proof. Suppose that the signed adjacency matrix of X admits perfect state transfer at time τ. Let H be the unique subgroup of G of order m, define $\mathcal{C}_0, \mathcal{C}_1, Y_0$ and Y_1 as in Theorem 6.3.3, and let A_0, A_1 be the signed adjacency matrices for Y_0, Y_1, respectively. Then $X \cong Y_0 \ltimes Y_1$ and so we can apply Theorem 6.4.2.

Recall that Y_0 and Y_1 are Cayley graphs for H, so $A_0 + A_1$ and $A_0 - A_1$ are weighted adjacency matrices of Cayley graphs for a group of odd order. Then, by Theorem 2.4.5, they do not admit perfect state transfer and so, (ii) and (iii) in Theorem 6.4.2 cannot hold. Therefore, $A_0 + A_1$ and $A_0 - A_1$ are periodic at time τ, with phase factors, say, λ and $-\lambda$, respectively. Then $U_{A_0+A_1}(\tau) = \lambda I$ and $U_{A_0-A_1}(\tau) = -\lambda I$, thus we get

$$U_{2A_1}(\tau) = U_{A_0+A_1}(\tau) U_{A_1-A_0}(\tau) = U_{A_0+A_1}(\tau) \overline{U_{A_0-A_1}(\tau)} = -I.$$

This implies that if $\theta_0, \dots, \theta_n$ are the eigenvalues of A_1, then $e^{2i\tau\theta_r} = -1$ for all $r = 0, \dots, n$. So, there are odd integers m_r such that $2\tau\theta_r = m_r\pi$ and it follows that for all $r, s = 0, \dots n$ we have $\theta_r/\theta_s = m_r/m_s$ with m_r and m_s both odd.

Recall that a non-empty integral signed translation graph for a group of odd order has an odd eigenvalue, by Lemma 4.7.4 and the remark following it. Furthermore, a regular graph on an odd number of vertices must have an even degree, which is then an even eigenvalue of the graph and it can be easily verified that this also holds in the signed case.

Now, our graph Y_1 is either a (possibly signed) integral translation graph without loops, or its adjacency matrix may be written as $A_1 = A_1' \pm I$ where A_1' is the (possibly signed) adjacency matrix of an integral translation graph without loops, and so in both cases, unless it is empty, Y_1 contains eigenvalues, θ, θ' of different parities. But then, either the denominator or the enumerator of the reduced fraction θ/θ' has to be even, so we conclude that Y_1 must be empty with loops, i.e., $A_1 = \pm I$. The possible signing will not affect the rest of the proof, so we will drop it.

We now have two possibilities: either Y_0 is empty in which case $X \cong mK_2$ as required, or X is a Cartesian product, $K_2 \square Y_0$ and $A(X) = A(K_2) \otimes I + I \otimes A_0$. Assume for contradiction the latter. Then

$$U_{K_2 \square Y_0}(\tau) = U_{K_2}(\tau) \otimes U_{Y_0}(\tau) = \lambda P,$$

where P is a permutation matrix with zero diagonal and λ is some scalar. Then, both $U_{K_2}(\tau)$ and $U_{Y_0}(\tau)$ must be scalar multiples of permutation matrices, but since Y_0 is a Cayley graph for a group of odd order it does not have perfect state transfer and so we must have $U_{Y_0}(\tau) = \lambda' I$ for some λ'.

We have seen that K_2 has perfect state transfer with minimal time $\pi/2$ and so $\tau = (2k+1)\pi/2$ for some integer k. Let $\theta_0', \dots, \theta_n'$ be the eigenvalues of Y_0. Then the eigenvalues of $U_{Y_0}(\tau)$ are $e^{i\pi(2k+1)\theta_r'/2}$. Since Y_0 has both even and odd eigenvalues, $U_{Y_0}(\tau)$ will have eigenvalues in both $\{\pm 1\}$ and $\{\pm i\}$ and can therefore not be a scalar multiple of I.

We have reached a contradiction and conclude that $X \cong mK_2$, having perfect state transfer with minimal time $\pi/2$. □

We can now prove that the time of perfect state transfer in the case where $d \geq 2$ has to be $\pi/2$.

Theorem 6.5.2 ([21, Theorem 16.7.1]). *Let $X = \mathrm{Cay}(G, \mathcal{C})$ be a 2-circulant. If perfect state transfer occurs on X, it occurs at time $\pi/2$.*

Proof. Let G have order $2^d m$ where m is odd and suppose perfect state transfer occurs on X at time τ. We have seen what happens for $d = 0, 1$, so we assume that $d \geq 2$.

Let $\mathcal{C}_0, \dots, \mathcal{C}_d$ be as before, define the graphs Y_r as in Theorem 6.3.3 and let A_r be their adjacency matrices. Recall that

$$A(X) = \mathcal{M}(A_0, \dots, A_d) = \mathcal{M}(\mathcal{M}(A_0, \dots, A_{d-1}), J_{2^{d-1}} \otimes A_d).$$

We will use Theorem 6.4.2 on the matrices $\mathcal{M}(A_0, \dots, A_{d-1})$ and $J_{2^{d-1}} \otimes A_d$. We know that perfect state transfer must occur between 0 and the unique element of order two in G. This element is contained in the subgroup of G generated by $\mathcal{C}_0 \cup \mathcal{C}_1$. Therefore, part (ii) of Theorem 6.4.2 must apply and we have perfect state transfer at time τ on the matrix

$$\mathcal{M}(A_0, \dots, A_{d-1}) - J_{2^{d-1}} \otimes A_d = \mathcal{M}(A_0 - A_d, \dots, A_{d-1} - A_d).$$

Now, applying the theorem repeatedly, we get that perfect state transfer occurs on the matrix $\mathcal{M}(A_0 - A_2, A_1 - A_2)$ at time τ. This is a signed adjacency matrix of a Cayley graph for a group of order $2m$ and so by Theorem 6.5.1, we have $\tau = \pi/2$ as required. □

6.6 Reducing to a simpler case

In this section we will see how we can reduce the question of perfect state transfer on a graph $X = \mathrm{Cay}(G, \mathcal{C})$ where $G \cong \mathbb{Z}_{2^d} \times H$ to the case where $d = 2$. We start with a lemma.

Lemma 6.6.1. *Let $\mathcal{C}$ be an inverse-closed subset of the abelian group G with a partition into inverse-closed subsets $\mathcal{D}_1$ and $\mathcal{D}_2$. Suppose $\mathrm{Cay}(G, \mathcal{D}_2)$ is periodic with period τ. Then $X := \mathrm{Cay}(G, \mathcal{C})$ admits perfect state transfer at time τ if and only if $\mathrm{Cay}(G, \mathcal{D}_1)$ admits perfect state transfer at time τ.*

Proof. Let A, A_1 and A_2 be the adjacency matrices of $X, \mathrm{Cay}(G, \mathcal{D}_1)$ and $\mathrm{Cay}(G, \mathcal{D}_2)$, respectively. Then $A = A_1 + A_2$ and since G is abelian, A_1 and

A_2 commute. Therefore,

$$\begin{aligned} U_A(t) &= \exp(itA) \\ &= \exp(it(A_1 + A_2)) \\ &= \exp(itA_1)\exp(itA_2) \\ &= U_{A_1}(t)U_{A_2}(t), \end{aligned}$$

for $t \in \mathbb{R}$. Since $\mathrm{Cay}(G, \mathcal{D}_2)$ is periodic at time τ, this implies $U_{A_2}(\tau) = \lambda I$, for some λ and so $U_A(\tau) = \lambda U_{A_1}(\tau)$. The result follows. □

Theorem 6.6.2. *Suppose G is abelian of order $2^d m$ where m is odd and $d \geq 2$, and assume that the Sylow-2-subgroup of G is cyclic. Let G' denote the unique subgroup of G with order $4m$. Then the integral Cayley graph $\mathrm{Cay}(G, \mathcal{C})$ admits perfect state transfer if and only if $\mathrm{Cay}(G', G' \cap \mathcal{C})$ admits perfect state transfer.*

Proof. Note first that $\mathrm{Cay}(G', G' \cap \mathcal{C})$ admits perfect state transfer if and only if $\mathrm{Cay}(G, G' \cap \mathcal{C})$ admits perfect state transfer, since the components of $\mathrm{Cay}(G, G' \cap \mathcal{C})$ are isomorphic to $\mathrm{Cay}(G', G' \cap \mathcal{C})$. Further, by Theorem 6.5.2, if perfect state transfer occurs on any Cayley graph of G, it occurs at time $\pi/2$.

Now define $\mathcal{C}_r$ for $r = 0, \dots, d$ as before. We see that $G' \cap \mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ and that the $\mathcal{C}_r$ are power-closed since X is integral. Recall that by Corollary 6.3.4, the graphs $X_r = \mathrm{Cay}(G, \mathcal{C}_r)$ are periodic at time $\pi/2$, for $r \geq 3$. Since X is an edge-disjoint union of $\mathrm{Cay}(G, G' \cap \mathcal{C})$ and X_r for $r \geq 3$, the result follows from Lemma 6.6.1. □

Theorem 6.6.2 shows that whether or not a 2-circulant has perfect state transfer is determined by the sets $\mathcal{C}_0, \mathcal{C}_1$ and $\mathcal{C}_2$ (as long as the whole connection set is power-closed). We conclude this section with a lemma about the parities of these sets. Note that a group with a cyclic Sylow-2 subgroup has a unique element of order two, and a unique pair of inverse elements of order four.

Lemma 6.6.3. *Let G be an abelian group with a Sylow-2-subgroup that is cyclic and has order at least four. Let $\mathcal{C}$ be a power-closed subset such that $0 \notin \mathcal{C}$ and define $\mathcal{C}_r$ as before, for $r = 0, 1, 2$. Let a be the unique element of order two and $b, -b$ the unique pair of elements of order four. Then* *(a)* $|\mathcal{C}_0|$ *and* $|\mathcal{C}_2|$ *are even,*

(b) $|\mathcal{C}_1|$ *is odd if and only if* $a \in \mathcal{C}$*, and*

(c) $|\mathcal{C}_2|$ *is divisible by four if and only if* $b \notin \mathcal{C}$*.*

Proof. Since $\mathcal{C}_0, \mathcal{C}_1$ and $\mathcal{C}_2$ are power-closed, they are also inverse-closed. The only element in G that is its own inverse is a, and if $a \in \mathcal{C}$, it is in $\mathcal{C}_1$. Therefore, the elements of $\mathcal{C}_0, \mathcal{C}_1$ and $\mathcal{C}_2$ come in pairs, with the possible exception of a, from which (a) and (b) follow.

For (c), write $G = \mathbb{Z}_{2^d} \times H$ where H has odd order and observe that every element in $\mathcal{C}_2$ can be written of the form $c + h$ where $c \in \{b, -b\}$ and $h \in H$. Further, the elements $c + h, -c + h, c - h$ and $-c - h$ all generate the same subgroup, and provided that $h \neq 0$, they are all distinct. Therefore, with the exception of $b, -b$, we can partition $\mathcal{C}_2$ into subsets of size four in this way, and now (c) follows. □

6.7 The necessary conditions

We have shown that perfect state transfer on 2-circulants can be reduced to the case where the underlying group is isomorphic to $\mathbb{Z}_4 \times H$ where H is abelian of odd order. We therefore investigate this case now. In this section we will give some necessary conditions for a Cayley graph of such a group to have perfect state transfer, and in the next section, we will show that these conditions are also sufficient.

Let $G \cong \mathbb{Z}_4 \times H$ where H has odd order m. If $\mathcal{C}$ is a power-closed subset of G, we have $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_r$ is defined as before. Let a be the unique element of order two and $b, -b$ the unique pair of elements of order four.

Lemma 6.7.1. *Let* $G = \mathbb{Z}_4 \times H$ *where* H *has odd order* m *and let* $\mathcal{C}$ *be an inverse-closed subset such that the Cayley graph* $X = \mathrm{Cay}(G, \mathcal{C})$ *admits perfect state transfer. Then either* a *or* b *is in* $\mathcal{C}$*, but not both.*

Proof. Since X has perfect state transfer, it is integral and so $\mathcal{C}$ is power-closed. Further, since a is the unique element of order two in G, perfect state transfer must occur between 0 and a and by Theorem 6.5.2 it occurs at time $\pi/2$. Then, by Lemma 6.2.2 we have for each character χ of G,

$$\chi(a) = (-1)^{(|\mathcal{C}|-\chi(\mathcal{C}))/2}. \tag{6.1}$$

Let φ be the character of G with $\varphi(b) = i$ and $\varphi(h) = 1$ for all $h \in H$. Then $\varphi(a) = -1$, so by Equation (6.1) we know that $(|\mathcal{C}| - \varphi(\mathcal{C}))/2$ is odd. Further, we have $\varphi(-b) = -i$, and now we see that

$$\varphi(\mathcal{C}_0) = |\mathcal{C}_0|, \quad \varphi(\mathcal{C}_1) = -|\mathcal{C}_1| \quad \text{and} \quad \varphi(\mathcal{C}_2) = 0,$$

and therefore

$$|\mathcal{C}| - \varphi(\mathcal{C}) = |\mathcal{C}| - (|\mathcal{C}_0| - |\mathcal{C}_1|) = 2|\mathcal{C}_1| + |\mathcal{C}_2|.$$

Now, combining this with Lemma 6.6.3, we see that if a and b are both in $\mathcal{C}$, or if neither of them is in $\mathcal{C}$, then $2|\mathcal{C}_1| + |\mathcal{C}_2| = |\mathcal{C}| - \varphi(\mathcal{C})$ is divisible by four, contradicting that $(|\mathcal{C}| - \varphi(\mathcal{C}))/2$ is odd. Therefore, we must have that $a \in \mathcal{C}$ or $b \in \mathcal{C}$, but not both. □

Now, think of the elements of G as pairs (c, h) with $c \in \mathbb{Z}_4$ and $h \in H$. The set $\mathcal{C}_0$ consists only of elements $(0, h)$, and so we can view it as a subset of H. The set $\mathcal{C}_1$ has elements of the form (a, h) for $h \in H$, and so there is a power-closed subset, $\mathcal{C}_1^*$ of H such that $\mathcal{C}_1 = \{a\} \times \mathcal{C}_1^*$, and moreover $0 \in \mathcal{C}_1^*$ if and only if $a \in \mathcal{C}$.

Finally, the elements of $\mathcal{C}_2$ have the form $(\pm b, h)$ with $h \in H$ and since $\mathcal{C}_2$ is inverse-closed, $(b, h) \in \mathcal{C}_2$ if and only if $(-b, h) \in \mathcal{C}_2$. Therefore there is a power-closed subset $\mathcal{C}_2^*$ of H such that $\mathcal{C}_2 = \{-b, b\} \times \mathcal{C}_2^*$ and $0 \in \mathcal{C}_2^*$ if and only if $b \in \mathcal{C}$.

We can now prove another necessary condition for perfect state transfer to occur. We will later see that these two conditions along with integrality are also sufficient.

Lemma 6.7.2. *Let $G = \mathbb{Z}_4 \times H$ where H is an abelian group of odd order m and let $\mathcal{C}$ be an inverse-closed subset such that the Cayley graph $X = \operatorname{Cay}(G, \mathcal{C})$ admits perfect state transfer. Then $\mathcal{C}_0 = \mathcal{C}_1^* \setminus \{0\} = \mathcal{C}_2^* \setminus \{0\}$.*

Proof. If X admits perfect state transfer, it must be integral, so $\mathcal{C}$ is power-closed. Further, by Lemma 6.7.1, either a is in $\mathcal{C}$ or b is in $\mathcal{C}$ but not both.

Let $\varphi_0, \varphi_1, \varphi_2$ be characters of $\mathbb{Z}_4$ with $\varphi_0(b) = 1$, $\varphi_1(b) = -1$ and $\varphi_2(b) = i$. Let ψ be an arbitrary character of H and let $\chi_j = \varphi_j\psi : G \to \mathbb{C}^*$ be the character product (as defined in Section 4.4) for $j = 0, 1, 2$. Then

$$\begin{aligned}\chi_j(\mathcal{C}) &= \chi_j(\mathcal{C}_0) + \chi_j(\mathcal{C}_1) + \chi_j(\mathcal{C}_2)\\ &= \psi(\mathcal{C}_0) + \varphi_j(a)\psi(\mathcal{C}_1^*) + (\varphi_j(b) + \varphi_j(-b))\psi(\mathcal{C}_2^*).\end{aligned}$$

Therefore

$$\begin{aligned}\varphi_0\psi(\mathcal{C}) &= \psi(\mathcal{C}_0) + \psi(\mathcal{C}_1^*) + 2\psi(\mathcal{C}_2^*),\\ \varphi_1\psi(\mathcal{C}) &= \psi(\mathcal{C}_0) + \psi(\mathcal{C}_1^*) - 2\psi(\mathcal{C}_2^*),\\ \varphi_2\psi(\mathcal{C}) &= \psi(\mathcal{C}_0) - \psi(\mathcal{C}_1^*).\end{aligned}$$

Since X has perfect state transfer, we have

$$\chi(a) = (-1)^{(|\mathcal{C}|-\chi(\mathcal{C}))/2}$$

for every character $\chi = \varphi\psi$ of G. This means that whenever $\varphi(b) = \pm i$, we have that $(|\mathcal{C}| - \chi(\mathcal{C}))/2$ is odd and otherwise, it is even. In other words, the following holds for all characters, ψ of H:

$$\frac{|\mathcal{C}| - (\psi(\mathcal{C}_0) + \psi(\mathcal{C}_1^*) + 2\psi(\mathcal{C}_2^*))}{2} \quad \text{is even,} \tag{6.2}$$

$$\frac{|\mathcal{C}| - (\psi(\mathcal{C}_0) + \psi(\mathcal{C}_1^*) - 2\psi(\mathcal{C}_2^*))}{2} \quad \text{is even,} \tag{6.3}$$

$$\frac{|\mathcal{C}| - (\psi(\mathcal{C}_0) - \psi(\mathcal{C}_1^*))}{2} \quad \text{is odd.} \tag{6.4}$$

By adding (6.2) and (6.3) together, we find that $|\mathcal{C}| - \psi(\mathcal{C}_0) - \psi(\mathcal{C}_1^*)$ is even and by adding (6.3) and (6.4) we see that $|\mathcal{C}| - \psi(\mathcal{C}_0) + \psi(\mathcal{C}_2^*)$ is odd.

If $a \notin \mathcal{C}$, then $|\mathcal{C}|$ is even, $0 \notin \mathcal{C}_1^*$ and $0 \in \mathcal{C}_2^*$, so this implies that $\psi(\mathcal{C}_0) + \psi(\mathcal{C}_1^*)$ and $\psi(\mathcal{C}_0) + \psi(\mathcal{C}_2^* \setminus \{0\})$ are even for all characters, ψ of H. If $a \in \mathcal{C}$, then we similarly get that $\psi(\mathcal{C}_0) + \psi(\mathcal{C}_1^* \setminus \{0\})$ and $\psi(\mathcal{C}_0) + \psi(\mathcal{C}_2^*)$ are even for all ψ.

Taking $\mathcal{D} := \mathcal{C}_1^* \setminus \{0\}$, we will consider the set $\mathcal{S} = (\mathcal{C}_0 \setminus \mathcal{D}) \cup (\mathcal{D} \setminus \mathcal{C}_0)$. Our aim is to show that this set is empty, implying that $\mathcal{C}_0 = \mathcal{C}_1^* \setminus \{0\}$.

Note that since $\mathcal{C}_0$ and $\mathcal{D}$ are both power-closed, their intersection is as well and then also their difference. Therefore $\mathcal{S}$ is a disjoint union of two power closed sets and so it is itself power closed. Further, we have

$$\psi(\mathcal{S}) = \psi(\mathcal{C}_0 \setminus \mathcal{D}) + \psi(\mathcal{D} \setminus \mathcal{C}_0) = \psi(\mathcal{C}_0) + \psi(\mathcal{D}) - 2\psi(\mathcal{C}_0 \cap \mathcal{D})$$

which by the above is even for all characters, ψ of H.

It follows that the Cayley graph $Y := \mathrm{Cay}(H, \mathcal{S})$ has only even eigenvalues, but Y is an integral Cayley graph for a group of odd order, so by Lemma 4.7.4, we conclude that Y must be empty, implying that $\mathcal{C}_0 = \mathcal{C}_1^* \setminus \{0\}$.

By taking $\mathcal{D}$ to be $\mathcal{C}_2^* \setminus \{0\}$, we similarly get $\mathcal{C}_0 = \mathcal{C}_2^* \setminus \{0\}$, as desired. □

These necessary conditions imply some significant restrictions on the spectrum of a 2-circulant with perfect state transfer. In particular, it will have eigenvalues with multiplicity half the size of the group, as shown in the next corollary.

Corollary 6.7.3. *Let $G \cong \mathbb{Z}_4 \times H$ where H is abelian with odd order and suppose that the Cayley graph $X = \mathrm{Cay}(G, \mathcal{C})$ admits perfect state transfer. If $a \in \mathcal{C}$, then the eigenvalues of X are all odd, and -1 is an eigenvalue with multiplicity $|G|/2$. If $a \notin \mathcal{C}$, all eigenvalues of X are even and 0 is an eigenvalue with multiplicity $|G|/2$.*

Proof. Since X admits perfect state transfer, we have $\mathcal{C}_0 = \mathcal{C}_1^* \setminus \{0\} = \mathcal{C}_2^* \setminus \{0\}$. Let $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ be the four characters of $\mathbb{Z}_4$ defined uniquely by

$$\varphi_0(b) = 1, \quad \varphi_1(b) = -1, \quad \varphi_2(b) = i, \quad \varphi_3(b) = -i.$$

From the proof of Lemma 6.7.2, the eigenvalues of X are given by

$$\varphi_0\psi(\mathcal{C}) = \psi(\mathcal{C}_0) + \psi(\mathcal{C}_1^*) + 2\psi(\mathcal{C}_2^*) = \begin{cases} 4\psi(\mathcal{C}_0) + 1 & \text{if } a \in \mathcal{C} \\ 4\psi(\mathcal{C}_0) + 2 & \text{otherwise} \end{cases}$$

$$\varphi_1\psi(\mathcal{C}) = \psi(\mathcal{C}_0) + \psi(\mathcal{C}_1^*) - 2\psi(\mathcal{C}_2^*) = \begin{cases} 1 & \text{if } a \in \mathcal{C} \\ -2 & \text{otherwise} \end{cases}$$

$$\varphi_2\psi(\mathcal{C}) = \varphi_3\psi(\mathcal{C}) = \psi(\mathcal{C}_0) - \psi(\mathcal{C}_1^*) = \begin{cases} -1 & \text{if } a \in \mathcal{C} \\ 0 & \text{otherwise} \end{cases}$$

for characters ψ of H. We see that all eigenvalues are odd if $a \in \mathcal{C}$ and even otherwise. Further, since there are $|H| = |G|/4$ characters of H, and each of them gives two characters of G with eigenvalue 0 or -1, the result follows. □

6.8 Are also sufficient

We now show that given the necessary conditions from last section, along with integrality, our graphs admit perfect state transfer.

Theorem 6.8.1. *Let $G \cong \mathbb{Z}_4 \times H$ where H is abelian of odd order. Then the Cayley graph $\mathrm{Cay}(G, \mathcal{C})$ admits perfect state transfer if and only if the following conditions hold:*

(a) $\mathcal{C}$ is power-closed,

(b) exactly one of a and b is in $\mathcal{C}$, and

(c) $\mathcal{C}_0 = \mathcal{C}_1^* \setminus \{0\} = \mathcal{C}_2^* \setminus \{0\}$.

Proof. We have seen that a graph must be integral to have perfect state transfer, and that a translation graph is integral if and only if $\mathcal{C}$ is power-closed, thus perfect state transfer implies (a) and by Lemmas 6.7.1 and 6.7.2, it also implies (b) and (c).

Suppose that conditions (a)-(c) of the theorem hold. By (a), X is integral so by Lemma 6.2.2, it suffices show that for every character χ of G we have

$$\chi(a) = (-1)^{(|\mathcal{C}| - \chi(\mathcal{C}))/2}. \tag{6.5}$$

Recall that every character χ of G can be written uniquely as a product of two characters, $\chi = \varphi\psi$ with φ a character of $\mathbb{Z}_4$ and ψ a character of H, where $\chi(c, h) = \varphi(c)\psi(h)$. Moreover, $\chi(a) = -1$ if and only if $\varphi(a) = -1$, which holds if and only if $\varphi(b) = \pm i$.

Let $\chi = \varphi\psi$ and assume first that $a \in \mathcal{C}$. Then $0 \in \mathcal{C}_1^*$ and $0 \notin \mathcal{C}_2^*$, so by condition (c) we have $\mathcal{C}_0 = \mathcal{C}_1^* \setminus \{0\} = \mathcal{C}_2^*$. Therefore,

$$\begin{aligned}
|\mathcal{C}| &= |\mathcal{C}_0| + |\mathcal{C}_1| + |\mathcal{C}_2| \\
&= |\mathcal{C}_0| + |\mathcal{C}_1^* \setminus \{0\}| + 1 + 2|\mathcal{C}_2^*| \\
&= 4|\mathcal{C}_0| + 1.
\end{aligned}$$

Further, we have

$$\begin{aligned}\chi(\mathcal{C}) &= \chi(\mathcal{C}_0) + \chi(\mathcal{C}_1) + \chi(\mathcal{C}_2)\\ &= \psi(\mathcal{C}_0) + \varphi(a)\psi\left(\mathcal{C}_1^*\backslash\{0\}\right) + \varphi(a) + (\varphi(b) + \varphi(-b))\psi(\mathcal{C}_2^*)\\ &= \psi(\mathcal{C}_0) + \varphi(a)\psi\left(\mathcal{C}_0\right) + \varphi(a) + (\varphi(b) + \varphi(-b))\psi(\mathcal{C}_0)\\ &= \psi(\mathcal{C}_0)\big(1 + \varphi(a) + \varphi(b) + \varphi(-b)\big) + \varphi(a). \qquad (*)\end{aligned}$$

Now if $\varphi(b) = \pm i$, then $\varphi(-b) = -\varphi(b)$ and $\varphi(a) = -1$. We see that in this case, $(*)$ becomes $\chi(\mathcal{C}) = \varphi(a) = -1$ and so

$$\begin{aligned}\frac{|\mathcal{C}| - \chi(\mathcal{C})}{2} &= \frac{4|\mathcal{C}_0| + 1 - (-1)}{2}\\ &= 2|\mathcal{C}_0| + 1\end{aligned}$$

which is odd and therefore, Equation (6.5) holds. If $\varphi(b) = -1$ then $\varphi(-b) = -1$ and $\varphi(a) = 1$ and so $(*)$ becomes $\chi(\mathcal{C}) = \varphi(a) = 1$ and

$$\frac{|\mathcal{C}| - \chi(\mathcal{C})}{2} = \frac{4|\mathcal{C}_0|}{2} = 2|\mathcal{C}_0|,$$

which is even, so again Equation (6.5) holds in this case. Finally if $\varphi(b) = 1$, then $(*) = 4\psi(\mathcal{C}_0) + \varphi(a)$, so

$$\frac{|\mathcal{C}| - \chi(\mathcal{C})}{2} = \frac{4|\mathcal{C}_0| - 4\psi(\mathcal{C}_0)}{2} = 2\big(|\mathcal{C}_0| - \psi(\mathcal{C}_0)\big),$$

which is even, as required.

Then suppose $b \in \mathcal{C}$, and therefore $a \notin \mathcal{C}$. Then condition (c) gives $\mathcal{C}_0 = \mathcal{C}_1^* = \mathcal{C}_2^*\backslash\{0\}$, and so

$$\begin{aligned}|\mathcal{C}| &= |\mathcal{C}_0| + |\mathcal{C}_1| + |\mathcal{C}_2|\\ &= |\mathcal{C}_0| + |\mathcal{C}_1^*| + 2|\mathcal{C}_2^*\backslash\{0\}| + 2\\ &= 4|\mathcal{C}_0| + 2.\end{aligned}$$

Moreover,

$$\begin{aligned}\chi(\mathcal{C}) &= \chi(\mathcal{C}_0) + \chi(\mathcal{C}_1) + \chi(\mathcal{C}_2)\\ &= \psi(\mathcal{C}_0) + \varphi(a)\psi\left(\mathcal{C}_1^*\right) + (\varphi(b) + \varphi(-b))\psi(\mathcal{C}_2^*\backslash\{0\}) + \varphi(b) + \varphi(-b)\\ &= \psi(\mathcal{C}_0) + \varphi(a)\psi\left(\mathcal{C}_0\right) + (\varphi(b) + \varphi(-b))\psi(\mathcal{C}_0) + \varphi(b) + \varphi(-b)\\ &= \psi(\mathcal{C}_0)\big(1 + \varphi(a) + \varphi(b) + \varphi(-b)\big) + \varphi(b) + \varphi(-b). \qquad (**)\end{aligned}$$

If $\varphi(b) = \pm i$, then $(**) = 0$, so

$$\frac{|\mathcal{C}| - \psi(\mathcal{C})}{2} = \frac{4|\mathcal{C}_0| + 2}{2} = 2|\mathcal{C}_0| + 1,$$

which is odd. If $\varphi(b) = -1$ then $(**) = -2$ and so

$$\frac{|\mathcal{C}| - \psi(\mathcal{C})}{2} = \frac{4|\mathcal{C}_0| + 2 - 2}{2} = 2|\mathcal{C}_0|,$$

which is even. Finally if $\varphi(b) = 1$ then $(**) = 4\psi(\mathcal{C}_0) + 2$ so

$$\begin{aligned}\frac{|\mathcal{C}| - \psi(\mathcal{C})}{2} &= \frac{4|\mathcal{C}_0| + 2 - (4\psi(\mathcal{C}_0) + 2)}{2}\\ &= \frac{4(|\mathcal{C}_0| - \psi(\mathcal{C}_0))}{2}\\ &= 2(|\mathcal{C}_0| - \psi(\mathcal{C}_0)),\end{aligned}$$

again even. In each case, Equation (6.5) holds, and we have shown that X admits perfect state transfer. □

6.9 The characterization

It now remains to combine Theorems 6.6.2 and 6.8.1 to prove our characterization of connection sets that yield perfect state transfer. We will modify our conditions slightly here. Let $k \in \mathbb{N}$ and let $\mathcal{S}$ be a subset of an abelian group G written additively. Then, we define the set $k\mathcal{S} := \{k \cdot g : g \in \mathcal{S}\}$.

Theorem 6.9.1. *Let G be an abelian group of order $2^d m$ where m is odd and suppose it has a a cyclic Sylow-2-subgroup. Let a be the unique element of order two and $b, -b$ the unique pair of elements of order four. For a subset $\mathcal{C}$ of G let $\mathcal{C}_k$ denote the set of elements in $\mathcal{C}$ with order $2^k m'$ where m' is odd. Then the Cayley graph $\mathrm{Cay}(G, \mathcal{C})$ has perfect state transfer if and only if*

(a) *$\mathcal{C}$ is power-closed,*

(b) *either a or b is in $\mathcal{C}$ but not both,* *(c)* $\mathcal{C}_0 = 4(\mathcal{C}_2 \setminus \{-b, b\})$, *and*

(d) $\mathcal{C}_1 \setminus \{a\} = 2(\mathcal{C}_2 \setminus \{-b, b\})$.

Proof. Let G' be the unique subgroup of order $4m$. Then $G' \cong \mathbb{Z}_4 \times H$ where H has odd order and by Theorem 6.6.2, X admits perfect state transfer if and only if $Y := \mathrm{Cay}(G', G' \cap \mathcal{C})$ does. Note that $G' \cap \mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$. Further, $a, b \in G'$ and a is the unique element of G' of order two and $b, -b$ the unique pair of elements of G' of order four.

Define $\mathcal{C}_1^*, \mathcal{C}_2^*$ as before, so $\mathcal{C}_1 = \{a\} \times \mathcal{C}_1^*$ and $\mathcal{C}_2 = \{-b, b\} \times \mathcal{C}_2^*$. By Theorem 6.8.1, Y has perfect state transfer if and only if conditions (a) and (b) hold and $\mathcal{C}_0 = \mathcal{C}_1^* \setminus \{0\} = \mathcal{C}_2^* \setminus \{0\}$.

Recall that H has odd order. Then, for any $h \in H$, the elements h, $2h$ and $4h$ all generate the same cyclic subgroup because 2 and 4 are coprime to the order of $\langle h \rangle$. Thus, if $h \in \mathcal{C}_0$, we also have $2h, 4h \in \mathcal{C}_0$ and same holds for $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$. Therefore,

$$\begin{aligned} 4(\mathcal{C}_2 \setminus \{-b, b\}) &= \{-4b, 4b\} \times \{4h : h \in \mathcal{C}_2^* \setminus \{0\}\} \\ &= \{0\} \times (\mathcal{C}_2^* \setminus \{0\}), \end{aligned}$$

and

$$\begin{aligned} 2(\mathcal{C}_2 \setminus \{-b, b\}) &= \{-2b, 2b\} \times \{2h : h \in \mathcal{C}_2^* \setminus \{0\}\} \\ &= \{a\} \times (\mathcal{C}_2^* \setminus \{0\}). \end{aligned}$$

We have shown that condition (c) is equivalent to $\mathcal{C}_0 = \mathcal{C}_2^* \setminus \{0\}$ and that condition (d) is equivalent to $\mathcal{C}_1^* \setminus \{0\} = \mathcal{C}_2^* \setminus \{0\}$ and the result follows. □

Now we can state Bašić's characterization as a corollary.

Corollary 6.9.2 ([5, Theorem 22])**.** *An integral circulant,* $\mathrm{Cay}(\mathbb{Z}_n, \mathcal{C})$ *has perfect state transfer if and only if* $n \in 4\mathbb{N}$,

$$\begin{aligned} \mathcal{C}_1 \setminus \{n/2\} &= 2(\mathcal{C}_2 \setminus \{\pm(n/4)\}), \\ \mathcal{C}_0 &= 4(\mathcal{C}_2 \setminus \{\pm(n/4)\}), \end{aligned}$$

and either $n/4 \in \mathcal{C}$ *or* $n/2 \in \mathcal{C}$. □

Example 6.9.3. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ with generators b, g_1 and g_2 and $a = 2b$. Let

$$\begin{aligned}\mathcal{C}_0 &= \{g_1,\, 2g_1,\, g_2,\, 2g_2\}\\ \mathcal{C}_1 &= \{a+g_1,\, a+2g_1,\, a+g_2,\, a+2g_2\}\\ \mathcal{C}_2 &= \{g_1 \pm b,\, 2g_1 \pm b,\, g_2 \pm b,\, 2g_2 \pm b\}\end{aligned}$$

and define $\mathcal{C} := \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{a\}$ and $\mathcal{C}' := \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{-b, b\}$. We see that $4\mathcal{C}_2 = 2\mathcal{C}_1 = \mathcal{C}_0$ and so the graphs $X = \mathrm{Cay}(G, \mathcal{C})$ and $X' = \mathrm{Cay}(G, \mathcal{C}')$ have perfect state transfer from 0 to a. Further, it can be shown by looking at the automorphism groups of these graphs that they are not circulants, and so they do not arise from Basić's theorem.

The graph X is shown in Figure 6.1 with the vertices 0 and a in white.

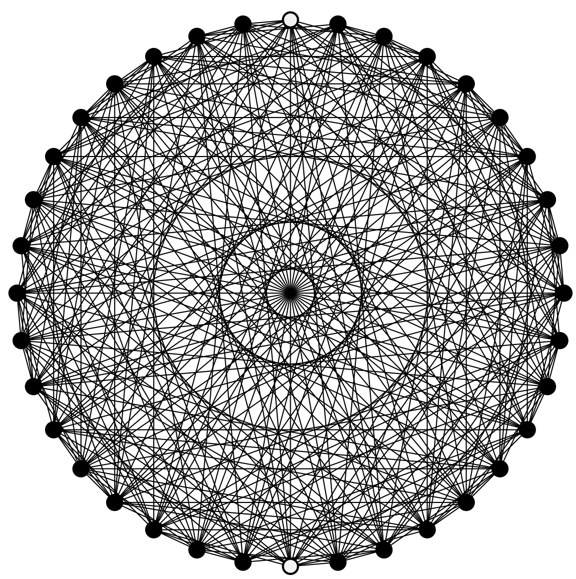


Figure 6.1: Cayley graph of $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ with perfect state transfer

Chapter 7

Summary & Open Problems

In this chapter we give a brief summary of the main results of this thesis and discuss some open questions and directions we could take for further work.

7.1 Strong cospectrality

In Chapter 5, we considered how big a strongly cospectral set in a Cayley graph can be. We gave an upper bound on the size of such a set for normal Cayley graphs in terms of the spectrum of the graph and used this to give an upper bound in Cayley graphs of $\mathbb{Z}_2^{d_1} \times \mathbb{Z}_4^{d_2}$ only in terms of the number of vertices. We further provided some examples of cubelike graphs in which this set has size at least four.

Our results bring up two main questions:

1. Can we get an explicit upper bound in other normal Cayley graphs on the order of the strongly cospectral subgroup in terms of the order of the group?

2. Is our bound for the cubelike graphs tight in all dimensions?

Let us look at the first question. Theorem 5.5.3, gives an upper bound on the size of a strongly cospectral set in terms of the eigenvalue multiplicities for all normal Cayley graphs. So far, we have only been able to apply this to Cayley graphs of groups of the form $\mathbb{Z}_2^{d_1} \times \mathbb{Z}_4^{d_2}$, to get a bound in terms of the size of the graph.

This is because the eigenvalues of these graphs have very simple form, $n-2r$, where n is the degree and $r \in \{0, \dots, n\}$, which doesn't hold for other normal Cayley graphs. However, as we have seen, the spectrum of a normal Cayley graph, $\mathrm{Cay}(G, \mathcal{C})$ has this nice description, where each eigenvalue is given by

$$\frac{1}{\chi(e)} \sum_{c \in \mathcal{C}} \chi(c),$$

for a character χ. It seems like we could use this to derive some lower bound on the maximum multiplicity of an eigenvalue.

In practice, it seems as though Cayley graphs tend to have some eigenvalues with high multiplicities. In particular, since they are vertex transitive, they cannot have all simple eigenvalues. This is encouraging, and we are hopeful that better bounds can be found.

The second question is slightly more frustrating. We have spent quite a bit of time trying to find examples of cubelike graphs on 2^7 and 2^8 vertices with a strongly cospectral subgroup of order eight, both through search and through construction. This has not proved fruitful, but the search has shown that we cannot improve our lower bound on the largest multiplicity (Theorem 5.6.4) in a way that will be useful for this. That is, there are cubelike graphs on 2^7 vertices with largest multiplicity at most 16, so the best bound we can get from Theorem 5.5.3 is $2^7/2^4 = 8$. Therefore, if our bound is not tight, this would have to be shown using different methods.

7.2 State transfer

In Chapter 6, we characterized the connection sets of 2-circulants having perfect state transfer. This is a generalization of Bašić's characterization of circulants and provides many examples of graphs admitting perfect state transfer that were previously unknown. The obvious question here is: can we generalize this further, and how far can we go?

One of the key ingredients in our proofs is that if a 2-circulant admits perfect state transfer, then the minimum time at which it occurs is $\pi/2$.

This is consequence of the Sylow-2-subgroup being cyclic, and does not hold otherwise. In fact, Chan has proved that there are cubelike graphs with arbitrarily fast perfect state transfer [14].

It is therefore unlikely that we can use similar techniques to give a characterization for Cayley graphs with a Sylow-2-subgroup that is not cyclic. However, if we take a group $G \cong \mathbb{Z}_{2^d} \times H$, with $|H|$ odd, do we really need H to be abelian?

We conjecture that if perfect state transfer occurs in a normal Cayley graph of a group with a cyclic Sylow-2-subgroup, then the minimum time is $\pi/2$. Moreover, we conjecture that a version of Theorem 6.9.1 still holds for such graphs.

References

[1] Andris Ambainis. Quantum walk algorithm for element distinctness. *SIAM J. Comput.*, 37(1):210–239, 2007.

[2] Majid Arezoomand, Farideh Shafiei, and Modjtaba Ghorbani. Perfect state transfer on cayley graphs over the dicyclic group. *Linear Algebra and its Applications*, 639:116–134, 2022.

[3] Arnbjörg Soffía Árnadóttir and Chris Godsil. On state transfer in Cayley graphs for abelian groups. https://arxiv.org/abs/2204.09802, 2022.

[4] Arnbjörg Soffía Árnadóttir and Chris Godsil. Strongly cospectral vertices in normal Cayley graphs. https://arxiv.org/abs/2109.07568, 2021.

[5] Milan Bašić. Characterization of quantum circulant networks having perfect state transfer. *Quantum Inf. Process.*, 12(1):345–364, 2013.

[6] Milan Bašić and Marko D. Petković. Some classes of integral circulant graphs either allowing or not allowing perfect state transfer. *Appl. Math. Lett.*, 22(10):1609–1615, 2009.

[7] Milan Bašić and Marko D. Petković. Perfect state transfer in integral circulant graphs of non-square-free order. *Linear Algebra and its Applications*, 433:149–163, 07 2010.

[8] Milan Bašić, Marko D. Petković, and Dragan Stevanović. Perfect state transfer in integral circulant graphs. *Applied Mathematics Letters*, 22(7):1117 – 1121, 2009.

[9] Anna Bernasconi, Chris Godsil, and Simone Severini. Quantum networks on cubelike graphs. *Phys. Rev. A (3)*, 78(5):052320, 5, 2008.

[10] Sougato Bose. Quantum communication through an unmodulated spin chain. *Physical Review Letters*, 91(20), Nov 2003.

[11] William G. Bridges and Roberto A. Mena. Rational G-matrices with rational eigenvalues. *J. Combin. Theory Ser. A*, 32(2):264–280, 1982.

[12] Andries E. Brouwer, Arjeh M. Cohen, and Arnold Neumaier. *Distance-Regular Graphs*. Springer-Verlag, Berlin, 1989.

[13] Xiwang Cao and Keqin Feng. Perfect state transfer on Cayley graphs over dihedral groups. *Linear Multilinear Algebra*, 69(2):343–360, 2021.

[14] Ada Chan. Complex Hadamard matrices, instantaneous uniform mixing and cubes. *Algebr. Comb.*, 3(3):757–774, 2020.

[15] Wang-Chi Cheung and Chris Godsil. Perfect state transfer in cubelike graphs. *ArXiv e-prints*, October 2010.

[16] Andrew M. Childs, Edward Farhi, and Sam Gutmann. An example of the difference between quantum and classical random walks. *Quantum Inf. Process.*, 1(1-2):35–43, 2002.

[17] Matthias Christandl, Nilanjana Datta, Tony C. Dorlas, Artur Ekert, Alastair Kay, and Andrew J. Landahl. Perfect transfer of arbitrary states in quantum spin networks. *Physical Review A*, 71(3), Mar 2005.

[18] Matthias Christandl, Nilanjana Datta, Artur Ekert, and Andrew J. Landahl. Perfect state transfer in quantum spin networks. *Physical Review Letters*, 92(18), May 2004.

[19] Gabriel Coutinho. *Quantum State Transfer in Graphs*. PhD thesis, University of Waterloo, 2014.

[20] Gabriel Coutinho and Chris Godsil. Perfect state transfer in products and covers of graphs. *Linear Multilinear Algebra*, 64(2):235–246, 2016.

[21] Gabriel Coutinho and Chris Godsil. Graph spectra and continuous quantum walks. https://www.math.uwaterloo.ca/~cgodsil/quagmire/pdfs/GrfSpc3.pdf, 2021. Work in progress.

[22] Gabriel Coutinho, Emanuel Juliano, and Thomás Jung Spier. Strong cospectrality in trees. https://arxiv.org/abs/2206.02995, 2022.

[23] John D. Dixon and Brian Mortimer. *Permutation Groups*. Springer-Verlag, New York, 1996.

[24] Xiaoxia Fan and Chris Godsil. Pretty good state transfer on double stars. *Linear Algebra Appl.*, 438(5):2346–2358, 2013.

[25] Edward Farhi and Sam Gutmann. Quantum computation and decision trees. *Phys. Rev. A (3)*, 58(2):915–928, 1998.

[26] Heath Gerhardt and John Watrous. Continuous-time quantum walks on the symmetric group. In *Approximation, randomization, and combinatorial optimization*, volume 2764 of *Lecture Notes in Comput. Sci.*, pages 290–301. Springer, Berlin, 2003.

[27] C. D. Godsil. *Algebraic Combinatorics*. Chapman and Hall Mathematics Series. Chapman & Hall, New York, 1993.

[28] Chris Godsil. Association schemes. https://www.math.uwaterloo.ca/~cgodsil/pdfs/assoc2.pdf, 2010. Lecture notes.

[29] Chris Godsil. Periodic graphs. *Electron. J. Combin.*, 18(1):Paper 23, 15, 2011.

[30] Chris Godsil. State transfer on graphs. *Discrete Math.*, 312(1):129–147, 2012.

[31] Chris Godsil. When can perfect state transfer occur? *Electron. J. Linear Algebra*, 23:877–890, 2012.

[32] Chris Godsil and Karen Meagher. *Erdős-Ko-Rado theorems: algebraic approaches*. Cambridge University Press, Cambridge, 2016.

[33] Chris Godsil and Gordon Royle. *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[34] Chris Godsil and Jamie Smith. Strongly cospectral vertices. https://arxiv.org/pdf/1709.07975, 2017.

[35] Lov K. Grover. A fast quantum mechanical algorithm for database search. In *Proceedings of the Twenty-eighth Annual ACM Symposium on the Theory of Computing (Philadelphia, PA, 1996)*, pages 212–219. ACM, New York, 1996.

[36] Gordon James and Martin Liebeck. *Representations and Characters of Groups*. Cambridge University Press, New York, second edition, 2001.

[37] Alastair Kay. Perfect state transfer: beyond nearest-neighbor couplings. *Phys. Rev. A (3)*, 73(3, part A):032306, 7, 2006.

[38] Alastair Kay. Perfect, efficient, state transfer and its application as a constructive tool. *International Journal of Quantum Information*, 08(04):641–676, Jun 2010.

[39] Alastair Kay. Basics of perfect communication through quantum networks. *Physical Review A*, 84(2), Aug 2011.

[40] Benoit Larose, François Laviolette, and Claude Tardif. On normal Cayley graphs and hom-idempotent graphs. *European J. Combin.*, 19(7):867–881, 1998.

[41] Hermie Monterde. Strong cospectrality and twin vertices in weighted graphs. https://arxiv.org/abs/2111.01265, 2021.

[42] James R. Norris. *Markov Chains*. Cambridge University Press, Cambridge, 1998.

[43] Peter Sin. Large sets of strongly cospectral vertices in cayley graphs. https://arxiv.org/abs/2207.05211, 2022.

[44] Peter Sin and Julien Sorci. Continuous-time quantum walks on Cayley graphs of extraspecial groups. https://arxiv.org/abs/2011.07566, 2020.

[45] Ying-Ying Tan, Keqin Feng, and Xiwang Cao. Perfect state transfer on abelian Cayley graphs. *Linear Algebra Appl.*, 563:331–352, 2019.

[46] W. K. Wootters and W. H. Zurek. A single quantum cannot be cloned. *Nature*, 299(5886):802–803, October 1982.

[47] Zhan, Hanmeng. *Discrete Quantum Walks on Graphs and Digraphs*. PhD thesis, University of Waterloo, 2018.

Index