Local Perspectives on Planar Colouring

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In 1994, Thomassen famously proved that every planar graph is 5-choosable, resolving a conjecture initially posed by Vizing and, independently, Erdős, Rubin, and Taylor in the 1970s. Later, Thomassen proved that every planar graph of girth at least five is 3-choosable. In this thesis, we introduce the concept of a *local girth list assignment*: a list assignment wherein the list size of a vertex depends not on the girth of the graph, but rather on the length of the shortest cycle in which the vertex is contained. We state and prove a local list colouring theorem unifying the two theorems of Thomassen mentioned above. In particular, we show that if G is a planar graph and L is a list assignment for G such that $|L(v)| \geq 3$ for all $v \in V(G)$; $|L(v)| \geq 4$ for every vertex v contained in a 4-cycle; and $|L(v)| \geq 5$ for every vertex v contained in a triangle, then G admits an L-colouring.

Next, we generalize a framework of list colouring results to *correspondence colouring*. Correspondence colouring is a generalization of list colouring wherein we localize the meaning of the colours available to each vertex. As pointed out by Dvořák and Postle, both of Thomassen's theorems on the 5-choosability of planar graphs and 3-choosability of planar graphs of girth at least five carry over to the correspondence colouring setting. In this thesis, we show that the family of graphs that are critical for 5-correspondence colouring as well as the family of graphs of girth at least five that are critical for 3-correspondence colouring form *hyperbolic families*. Analogous results for list colouring were shown by Postle and Thomas. Using results on hyperbolic families proved by Postle and Thomas, we show further that this implies that locally planar graphs are 5-correspondence colourable; and, using results of Dvořák and Kawarabayashi, that there exist linear-time algorithms for the decidability of 5-correspondence colouring for embedded graphs. We show analogous results for 3-correspondence colouring graphs of girth at least five.

Finally we show that, in general, slightly stronger hyperbolicity theorems imply that the associated family of planar graphs have exponentially many colourings. The existence of exponentially many colourings has been studied before for list-colouring: for instance, Thomassen showed (without using hyperbolicity) that planar graphs have exponentially many 5-list colourings, and that planar graphs of girth at least five have exponentially many 3-list colourings. Using our stronger hyperbolicity theorems, we prove that planar graphs of girth at least five have exponentially many 3-correspondence colourings, and that planar graphs have exponentially many 5-correspondence colourings. This latter result proves a conjecture of Langhede and Thomassen. As correspondence colouring generalizes list colouring, our theorems also provide new, independent proofs that there are exponentially many 5-list colourings of planar graphs, and 3-list colourings of planar graphs of girth at least five.

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Chapter 1

Introduction

Section 1.1 introduces the basic terminology and notation that will be used throughout the thesis. Readers familiar with graph theory basics may wish to skip Section 1.1, and refer to it only when necessary. Section 1.2 introduces the main topics and results in the thesis, and provides an overview of the history of the subjects and related results. Sections 1.3, 1.4, and 1.5 explore the ideas and challenges surrounding the proofs of the most important results in Chapters 2, 3, and 4, respectively. Section 1.6 provides an outline of the rest of the thesis.

1.1 Notation and Nomenclature

We begin by introducing basic terminology and notation used throughout the thesis. For further information, the reader may wish to consult the standard textbook of Diestel [9]; the PhD thesis of Postle [32] for additional history and background on embedded graphs not provided in this thesis; and the textbook of Thomassen and Mohar [30] for more information on embedded graphs and surfaces.

A *graph* G is a set $V(G)$ of objects called *vertices* together with a set $E(G)$ of objects called *edges*. An edge $e \in E(G)$ is a subset of $V(G)$ of size exactly two. We typically represent vertices as points, and each edge as a line joining the two points it contains. For simplicity, we use the notation uv (equivalently, vu) to denote the edge $\{u, v\}$. If there exists an edge uv in a graph G, we say the vertices u and v are *adjacent*, and that they are each *incident* to uv (and similarly that uv is incident to each of u and v). If two vertices are adjacent, we say also that they are *neighbours*. The *neighbourhood* of a vertex is the set of vertices adjacent to that vertex. The *degree* of a vertex v is the number of edges incident to v, and is denoted $\deg(v)$. We say two graphs G and H are *isomorphic* if there exists a bijection $f : V(G) \to V(H)$ such that for every pair of vertices $u, v \in V(G)$, we have that $f(u)f(v) \in E(H)$ if and only if $uv \in E(G)$.

Given a non-negative integer k, a *path of length* k is a graph isomorphic to the graph with vertex-set $\{v_0, v_1, \cdots, v_k\}$ and edge-set $\{v_i v_{i+1} : i \in \{0, \cdots, k-1\}\}$. We denote the path with vertex-set $\{v_0, v_1, \cdots, v_k\}$ and edge-set $\{v_i v_{i+1} : i \in \{0, \cdots, k-1\}\}$ by $v_0 v_1 \cdots v_k$, and generalize this notation to other paths in the natural way: for instance, a path with vertex-set $\{v, u, w\}$ and edge-set $\{vu, vw\}$ is denoted vuw. Given an integer $k \geq 3$, a *cycle of length* k is a graph isomorphic to the graph with vertex set $\{v_0, v_1, \cdots, v_{k-1}\}$ and edge-set $\{v_i v_{i+1} : i \in \{0, \cdots k-1\}\}$ where the indices are taken modulo k. We denote this cycle as $v_0 v_1 \cdots v_k v_0$, and generalize this notation to other cycles in the natural way: for instance, the cycle with vertex-set $\{u, v, w\}$ and edge-set $\{uv, vw, wu\}$ is denoted $uvwu$. A *walk* in a graph G is a sequence of alternating edges and vertices $v_0 e_1 v_1 e_2 v_3 \cdots v_k$ where $\{v_0, \cdots, v_k\} \subseteq V(G)$; where $\{e_1, \ldots, e_k\} \subseteq E(G)$; and where for each $i \in \{1, 2, \ldots, k\}$, we have that $e_i = v_{i-1} v_i$. A walk is *closed* if $v_0 = v_k$. The *length* of the walk is the number of edges (not necessarily distinct) that appear in the walk.

We say a graph is *connected* if there exists a path between every pair of vertices. Otherwise, it is *disconnected*. Given a graph G, a *subgraph* of G is a graph H with $V(H) \subseteq V(G)$, and with $E(H) \subseteq E(G)$. A maximal connected subgraph is a *component*. Given a graph G, a *cutvertex* of G is a vertex v in G whose removal increases the number of components in the graph. We denote graph obtained by the removal of v as $G - v$ or $G \setminus \{v\}$. A graph is *2-connected* if it is connected, contains at least three vertices, and contains no cutvertices.

A *surface* is a connected, compact, 2-dimensional manifold. By the Classification Theorem for surfaces, every surface can be obtained from the sphere by adding a handles and b crosscaps for non-negative integers a and b. The precise definition of handle and crosscap will not be required for this thesis; again, we encourage the reader to consult [32] and [30] for further information. If a surface Σ is obtained from the sphere by adding a handles and b crosscaps, the *(Euler) genus* of Σ is defined as $2a + b$. Given a surface Σ, an *arc in* Σ is the image of a continuous injective function $f : [0, 1] \to \Sigma$. A graph G is *embedded in* Σ if the vertices in $V(G)$ are distinct elements in Σ, the edges in $E(G)$ are arcs in Σ, and for each arc in $E(G)$, its interior is disjoint from the other arcs in $E(G)$ as well as the vertices in $V(G)$. An *embedded graph* is a pair (G, Σ), where Σ is a surface and G is a graph embedded in Σ. A topological space X is *arcwise connected* if there exists an arc between every two elements in S. An *arcwise connected component of* X is a maximal arcwise connected subspace of X. Given an embedded graph (G, Σ), a *face* of G is an arcwise connected component of $\Sigma - G$. The *boundary walk* of a face is the shortest closed walk (or, if the boundary of the face contains more than one component, the smallest *set* of shortest closed walks) containing every vertex and edge in the boundary of the face.

1.2 Context and Main Results

All graphs considered in this thesis are simple and finite. A *colouring* of a graph G is a function $\phi : V(G) \to C$ that assigns to each vertex v of G a *colour* $\phi(v) \in C$ such that for every edge $uv \in E(G)$, $\phi(u) \neq \phi(v)$. If $|C| \leq k$, ϕ is called a *k-colouring*; and if G has a k-colouring, we say it is *k-colourable.* The *chromatic number* of G is the minimum number k such that G is k-colourable. We say a graph is *planar* if it can be embedded in the plane such that no vertices overlap, and edges meet only at their endpoints. A *plane graph* is a planar graph together with a fixed embedding in the plane. We refer the reader to [30] and [32] for definitions not covered here.

In 1976, Appel and Haken [2, 3] proved the following result, settling a conjecture over a century old. It is arguably the most famous theorem in the field of graph colouring.

Theorem 1.2.1 (Four Colour Theorem [2, 3])**.** *Every planar graph is 4-colourable.*

The Four Colour Theorem is of course tight: there exist planar graphs that are not 3-colourable (for instance K_4, the complete graph on four vertices). However, in 1959 Grötzsch [21] showed that a stronger bound on the chromatic number of planar graphs is obtained by forbidding the triangle as a subgraph.

Theorem 1.2.2 (Grötzsch's Theorem [21])**.** *Every triangle-free planar graph is 3-colourable.*

This thesis further investigates the problem of colouring planar graphs: in particular, of *list colouring* and *correspondence colouring* planar graphs.

List colouring is a generalization of colouring introduced in the late 1970s by Vizing [46] and, independently, by Erdős, Rubin, and Taylor [17].

Definition 1.2.3. Given a graph G, a *list assignment* L for G is a function that assigns to each $v \in V(G)$ a list $L(v)$ of colours. L is a *k-list assignment* if $|L(v)| \geq k$ for every $v \in V(G)$. An *L-colouring* of G is a colouring ϕ such that $\phi(v) \in L(v)$ for each vertex $v \in V(G)$. We say G is *L-colourable* if there exists an L-colouring of G, and that G is *k-choosable* (or *k-list-colourable*) if G is L-colourable for every k-list assignment L for G.

As compared to ordinary colouring, we think of a list assignment as *localizing* the possible images of the colouring function to each vertex: i.e., of localizing the available colours for the vertices in the graph. This notion arises quite naturally in many graph colouring proofs wherein we colour part of the graph, delete it, and later colour the remainder of the graph without creating colour conflicts between the remaining and deleted vertices.

Correspondence colouring is a natural generalization of list colouring introduced by Dvořák and Postle in 2018 [15]. It is defined as follows.

Definition 1.2.4. Let G be a graph. A *k-correspondence assignment for G* is a k-list assignment L together with a function M that assigns to every edge $e = uv \in E(G)$ a partial matching M_e between $\{u\} \times L(u)$ and $\{v\} \times L(v)$. An (L, M)-colouring of G is a function φ that assigns to each vertex $v \in V(G)$ a colour $\varphi(v) \in L(v)$ such that for every $e = uv \in E(G)$, the vertices $(u, \varphi(u))$ and $(v, \varphi(v))$ are non-adjacent in M_e. We say that G is (L, M)-colourable if such a colouring exists, and that G is *k-correspondence-colourable* if G is (L, M)-colourable for every k-correspondence assignment (L, M) for G.

A correspondence assignment can be thought of as a further localization of colouring: just as list colouring localizes the notion of what colours are available at a vertex, a correspondence assignment localizes the *meaning* of these colours.

The following notation will be useful in proving results about correspondence assignments.

Definition 1.2.5. Given a graph G with correspondence assignment (L, M), if $uv \in E(G)$ and $(u, d)(v, c) \in M_{uv}$ we write $d = u[v, c]$ and $c = v[u, d]$. We say $d \in L(u)$ *corresponds to* $c \in L(v)$ and symmetrically $c \in L(v)$ *corresponds to* $d \in L(u)$. Given $c \in L(v)$, if there does not exist a colour $d \in L(u)$ with $(u, d)(v, c) \in M_{uv}$, we write $u[v, c] = \emptyset$.

As correspondence colouring generalizes list colouring generalizes ordinary vertex colouring, it is perhaps unsurprising that many theorems that hold for ordinary colouring do not hold for list colouring; and similarly that many list colouring results do not hold for correspondence colouring. We note a few important instances in which they differ: though even cycles have list chromatic (and ordinary chromatic) number two, their correspondence chromatic number is three. (In Figure 1.1, we demonstrate a 2-correspondence assignment (L, M) for a 4-cycle G such that G does not admit an (L, M)-colouring.) In [4], Bernshteyn and Kostochka give an example of a planar bipartite graph with correspondence chromatic number four; and by a result of Alon and Tarsi [1], every planar bipartite graph has list chromatic number at most three. Perhaps most relevant to this thesis: in 1993, Voigt [47] showed that the Four Colour Theorem does not carry over directly to list colouring by constructing a planar graph that is not 4-choosable. However, Thomassen [40] showed in 1994 that lists of size five suffice, thus answering a conjecture posed by Vizing [46] and Erdős, Rubin, and Taylor [17] in the 1970s.

Theorem 1.2.6 (Thomassen, [40]). *Every planar graph is 5-choosable.*

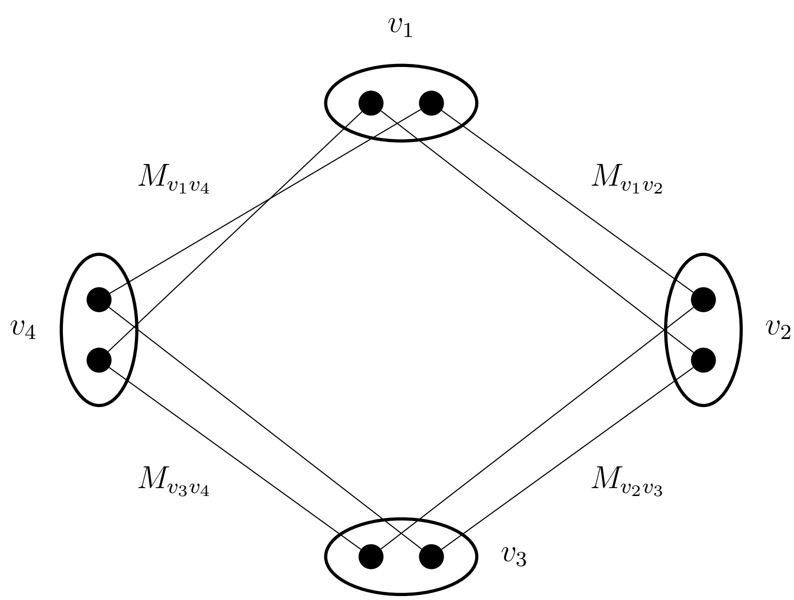

Figure 1.1: An illustration of a 4-cycle $v_1v_2v_3v_4v_1$ together with a 2-correspondence (L, M) for which the 4-cycle does not admit an (L, M)-colouring. Each oval represents a vertex in the 4-cycle. The dots within these vertices represent the colours in vertices' lists. The edges between vertices correspond to the matchings in M.

In the paper in which they introduce correspondence colouring, Dvořák and Postle [15] point out that Thomassen's proof of the choosability result above also holds for correspondence colouring.

Theorem 1.2.7 (Dvořák and Postle, [15]). *Every planar graph is 5-correspondence-colourable.*

A natural question to ask is the following: by ruling out certain substructures in planar graphs, can we further restrict vertices' list sizes and still obtain list or correspondence colourings of the resulting class of graphs? For instance: what if we rule out certain short cycles? As noted prior, this paradigm works for ordinary colouring: triangle-free planar graphs can be 3-coloured (Theorem 1.2.2). As we will explain below, something similar holds for list and correspondence colouring. Recall that the *girth* of a graph G is the minimum number k such that G contains a cycle of length k. Since planar graphs of girth at least four (i.e. triangle-free planar graphs) are 3-degenerate[1], a simple greedy argument shows that they are 4-choosable (and indeed 4-correspondence-colourable). Voigt [48] showed in 1995 that there are triangle-free planar graphs that are not 3-choosable: thus Grötzsch's Theorem does not carry over directly to list colouring. However, Thomassen [41] showed in 1995 that if we also exclude 4-cycles, lists of size three suffice as follows.

Theorem 1.2.8 (Thomassen, [41])**.** *Every planar graph of girth at least five is 3-choosable.*

Theorem 1.2.8 carries over to correspondence colouring, as pointed out by Dvořák and Postle in [15]. Thomassen later gave a shorter proof of Theorem 1.2.8 in 2003 [43].

Though these bounds on the choosability number are tight, there is hope of strengthening Thomassen's theorems by *allowing larger list sizes only where they are required in the graph*. In Chapter 2 we explore this idea. We introduce the concept of *local girth choosability*, wherein the list size of a vertex depends not on the girth of the graph, but rather on the length of the shortest cycle in which that vertex is contained. We provide a formal definition below, following a few other necessary definitions.

Definition 1.2.9. Let G be a graph. The *girth of a vertex* $v \in V(G)$ is denoted $g_G(v)$ and is defined as the minimum number k such that v is contained in a k-cycle. If the graph G is clear from context, we will often omit the subscript and write $g(v)$ instead of $g_G(v)$. If v is not contained in a cycle in G, we set $g_G(v) = \infty$.

Note that if $G' \subseteq G$ and $v \in V(G')$, then $g_{G'}(v) \geq g_G(v)$. We define a *local girth list assignment* as follows.

Definition 1.2.10. Let G be a planar graph. A *local girth list assignment* for G is a list assignment L such that:

- $|L(v)| \geq 3$ for all $v \in V(G)$ with $g(v) \geq 5$,
- $|L(v)| \geq 4$ for all $v \in V(G)$ with $g(v) = 4$, and
- $|L(v)| \geq 5$ for all $v \in V(G)$ with $g(v) = 3$.

[1] A graph is d-degenerate if every subgraph contains a vertex of degree at most d.

We say a planar graph G is *local girth choosable* if G admits an L-colouring for every local girth list assignment L.

Our first main result (and the topic of Chapter 2) is the following theorem.

Theorem 1.2.11. *Every planar graph is local girth choosable.*

We note that Theorem 1.2.11 is a joint strengthening of Theorems 1.2.6 and 1.2.8.

The idea of restricting the list size of vertices based on the structure that surrounds them is not a new one. Indeed, many list-colouring theorems admit a *local* version: that is, a version where list sizes depend on the local structure rather than a global property of the graph. For instance, Borodin, Kostochka and Woodall [6] proved a local version of Galvin's Theorem [20] that the List Colouring Conjecture holds for bipartite graphs, where the size of an edge's list depends on the maximum degree of its endpoints. In a similar vein, Bonamy, Delcourt, Lang, and Postle [5] proved a local asymptotic version of Kahn's Theorem [24] on list-edge colouring. In [26], Kelly and Postle proved a local epsilon version of Reed's Conjecture [38], where list sizes are lower-bounded by a linear combination of vertices' degrees and the size of the largest clique in which they are contained. In [7], Davies, de Joannis de Verclos, Kang, and Pirot gave an asymptotic theorem for list-colouring triangle-free graphs, where again vertices' list sizes are bounded by a function of their degree.

In Section 1.3, we will discuss the main ideas and innovations behind the proof of Theorem 1.2.11.

Our next main result concerns correspondence colouring embedded graphs, which is the main topic of Chapter 3. The precise avenue of research is inspired by the theory of hyperbolic families developed by Postle and Thomas [37].

As a graph is planar if and only if it embeds in the sphere, Theorem 1.2.7 can be rephrased as follows.

Theorem 1.2.12 (Theorem 1.2.7, rephrased)**.** *There does not exist a graph that embeds in the sphere and is not 5-correspondence-colourable.*

Phrased this way, Theorem 1.2.7 suggests a fresh avenue of study. Namely: given a surface Σ other than the sphere, what are the colouring properties of the set of graphs that embed in Σ and are not 5-correspondence-colourable? To partially answer this question, we required the following definitions.

Definition 1.2.13. A *non-contractible cycle* in a surface is a cycle that cannot be continuously deformed to a single point. An embedded graph is *ρ-locally planar* if every cycle (in the graph) that is non-contractible (in the surface) has length at least ρ.

This is closely related to the concept of *edge-width*. Recall that the edge-width of an embedded graph is the length of the shortest non-contractible cycle; thus if a graph is ρ-locally planar, it has edge-width at least ρ.

Since the study of locally planar graphs is a straightforward generalization of the study of planar graphs to graphs that embed in other surfaces, it is therefore natural to wonder whether results that hold for planar graphs similarly hold for locally planar graphs. In 2006, DeVos, Kawarabayashi, and Mohar [8] showed that for every surface Σ, there exists a constant $\rho = 2^{O(g)}$, where g is the Euler genus of Σ, such that every ρ-locally planar graph that embeds in Σ is 5-list-colourable. A similar result for 5-colourability (rather than choosability) was proved by Thomassen in 1993 [39]. Per the work of Postle and Thomas [37], analogous results for 5-choosability, 4-choosability of graphs of girth at least four, and 3-choosability of graphs of girth at least five with $\rho = \Omega(\log(g))$ for are implied by the *hyperbolicity* of certain associated families of graphs. *Hyperbolicity* is defined below.

Definition 1.2.14. Let $\mathcal{F}$ be a family of embedded graphs. We say that F is *hyperbolic* if there exists a constant $c > 0$ such that if $(G, \Sigma) \in \mathcal{F}$ is an embedded graph, then for every closed curve $\eta : S^1 \to \Sigma$ that bounds an open disk Δ and intersects G only in vertices, if Δ includes a vertex of G, then the number of vertices of G in Δ is at most $c(|\{x \in S^1 : \eta(x) \in V(G)\}| - 1)$. We say that c is a *Cheeger constant* for $\mathcal{F}$.

In [37], Postle and Thomas give a theorem known as the *hyperbolic structure theorem*, which characterises the structure of graphs in hyperbolic families. We state the theorem below informally to help give the reader intuition regarding hyperbolicity, and to better explain the implications of hyperbolicity for locally planar graphs. For more information (and a more formal description of what is meant below), we encourage the reader to consult [37].

Theorem 1.2.15 (Theorem 6.29, [37] (informally stated))**.** *Let $\mathcal{F}$ be a hyperbolic family of embedded graphs, and let $(G, \Sigma) \in \mathcal{F}$. Let g be the Euler genus of Σ. The graph G decomposes into a graph with $O(g)$ vertices together with a set of $O(g)$ cylinders of edge-width $O(1)$.*

This theorem together with the hyperbolicity of certain families of graphs is enough to prove that, given a surface Σ with genus g, there exists an integer ρ with $\rho = O(g)$ such that ρ-locally planar graphs embeddable in Σ are list-colourable. To explain this further, we again require a few definitions.

Definition 1.2.16. Let G be a graph, and k a positive integer. We say G is *critical for k-colouring* if every proper subgraph of G is k-colourable, but G itself is not.

The study of critical graphs was instigated by Dirac in 1951 [10], and since then, critical graphs have attracted much attention [18, 19, 23, 28, 27, 31]. As every graph that is not k-colourable contains a subgraph that is critical for k-colouring, the study of critical graphs is a very natural way to approach the study of non-k-colourable graphs. Indeed, to decide whether or not a graph G is k-colourable, it thus suffices to show that G contains no k-critical subgraph.

For list and correspondence colouring, one could also define critical graphs as being minimal graphs for which there exists a list (or correspondence) assignment such that the graphs are not colourable. However, since when proving list colouring results we typically work with fixed list assignments, it is more natural and useful to define critical graphs as follows.

Definition 1.2.17. Let G be a graph, k a positive integer, and L a k-list assignment for G. We say G is *L-critical* if every proper subgraph of G admits an L-colouring, but G itself does not. If there exists a k-list assignment L' such that G is L'-critical, we say G is *critical for k-list colouring*.

Definition 1.2.18. Let G be a graph, k a positive integer, and (L, M) a k-correspondence assignment for G. We say G is *(L, M)-critical* if every proper subgraph of G admits an (L, M)-colouring, but G itself does not. If there exists a k-correspondence assignment (L', M') such that G is (L', M')-critical, we say G is *critical for k-correspondence colouring*.[2]

In 2013, Dvořák and Kawarabayashi [12] showed the family of embedded graphs of girth at least five that are critical for 3-choosability is hyperbolic. Postle and Thomas showed the same for the family of embedded graphs of girth at least four that are critical for 4-choosability [37]; and in 2016 [35], for the family of embedded graphs that are critical for 5-choosability.

We now demonstrate how Theorem 1.2.15 can be used to show that, given a surface Σ with Euler genus g, there exists an integer ρ with $\rho = O(g)$ such that ρ-locally planar

[2]Note that a graph that is critical for k-list colouring may contain a proper subgraph that is also critical for k-list colouring. Similarly, a graph that is critical for k-correspondence colouring may contain a proper subgraph that is also critical for k-correspondence colouring. This is not the case for graphs that are critical for k-colouring. To decide whether G is k-list-colourable (or k-correspondence-colourable), it is enough to show G contains no *minimal* subgraph that is critical for k-list colouring (or k-correspondence colouring). To decide if G is colourable for a fixed list or correspondence assignment, it suffices to show G contains no subgraph that is critical with respect to this specific assignment.

graphs embeddable in Σ are 5-choosable. To see this, let (G, Σ) be an embedded graph that is not 5-choosable. Then G contains a subgraph H that is critical for 5-choosability. By the work of Postle and Thomas [35], the family $\mathcal{F}$ of embedded graphs that are critical for 5-choosability is hyperbolic, and so by the Hyperbolic Structure Theorem (Theorem 1.2.15), H can be decomposed into a graph with $O(g)$ vertices together with a set of $O(g)$ cylinders of edge-width $O(1)$. If this set is nonempty, then H has edge-width $O(1)$ and hence so does G; otherwise, H has at most $O(g)$ vertices and since $\mathcal{F}$ contains no plane graphs by the definition of hyperbolic, it follows that H has edge-width $O(g)$ and hence so does G. In either case, G has edge-width $O(g)$.

Thus locally planar embedded graphs are 5-choosable. Since the associated families of critical graphs are hyperbolic, it follows similarly that locally planar embedded graphs of girth at least four are 4-choosable, and locally planar embedded graphs of girth at least five are 3-choosable [37]. In [37], Postle and Thomas showed that with more work, hyperbolicity in fact implies analogous results with $\rho = \Omega(\log(g))$ instead of $O(g)$. As discussed in [37], this bound is best possible.

We are interested in generalizing these results to the framework of correspondence colouring. The main result of Chapter 3 is a technical theorem (Theorem 3.4.7) that implies the following.

Theorem 1.2.19. *The family of embedded graphs that are critical for 5-correspondence colouring is hyperbolic.*

Theorem 3.4.7—which implies Theorem 1.2.19—uses similar ideas to that of the analogous theorem for list colouring of Postle and Thomas (Theorem 4.6, [35]); however, a number of new ideas and reductions are needed in order to make the proof go through in the correspondence colouring framework.

Per the work of Postle and Thomas [37], Theorem 1.2.19 implies the following.

Theorem 1.2.20. *For every surface Σ, there exists a constant $\rho > 0$ such that every ρ-locally planar graph that embeds in Σ is 5-correspondence-colourable.*

We note that this result is new, and without hyperbolicity, it is unclear how one would prove it. For the theorem above, $\rho = \Omega(\log(g))$ where g is the Euler genus of Σ. See Chapter 3 for further details. We note this bound for ρ is best possible: since (as noted above) this bound is best possible for list colouring, it immediately follows that it is best possible for correspondence colouring.

We draw a connection between this and our previously discussed work: recall that our local girth choosability theorem (Theorem 1.2.11) implies that in the case of planar graphs, the structures that influence how many colour choices are needed to ensure colourability are *local* structures. In the same vein, Theorem 1.2.20 further suggests that for graphs that appear at least locally to be planar, it is again only the local structures that dictate the colouring properties of the graphs.

Before discussing other implications of hyperbolicity, we pause briefly to discuss k-list colouring (and k-correspondence colouring) embedded graphs when $k \geq 6$. Recall that Postle and Thomas showed that the family of embedded graphs that are critical for 5-choosability is hyperbolic [35]. It follows from the definition of critical for k-choosability and k-list assignment that every graph that is critical for k-choosability is also critical for ℓ-choosability for each positive integer $\ell < k$, since a k-list assignment is also an ℓ-list assignment: thus it follows from the work of Postle and Thomas that, for each $k \geq 6$, the family of embedded graphs that are critical for k-choosability is also a hyperbolic family. In the same vein, Theorem 1.2.19 also implies that for each $k \geq 6$ the family of embedded graphs that are critical for k-correspondence colouring is a hyperbolic family.

The theory of hyperbolic families developed by Postle and Thomas [37] has many interesting implications aside from those for locally planar graphs. We highlight two more of these below.

In [35], Postle and Thomas show that the list-colouring analogue to Theorem 3.4.7 has implications for the *precolouring extension problem* for planar graphs. The problem can be stated as follows: given a planar graph G with a list assignment L, a subgraph $C \subseteq G$ and an L-colouring ϕ of C, when does ϕ extend to G? (Similarly: given a planar graph G with a correspondence assignment (L, M), a subgraph $C \subseteq G$ and an (L, M)-colouring ϕ of C, when does ϕ extend to G?)

Alternatively, we might phrase this without ϕ as part of the input: given a planar graph G with list (or correspondence) assignment L (or (L, M)) and subgraph C of G, when does an arbitrary L (or (L, M)) colouring of C extend to G?

One way to approach this problem is to describe the structure of C that ensures an arbitrary colouring of C will to extend to a colouring of the whole graph. For instance, Thomassen showed (see Theorem 2.1.9) that if G is a plane graph of girth at least five with 3-list assignment L and C is a path of length at most three in the outer cycle of G, then every L-colouring of C extends to an L-colouring of G.

A different, natural way to approach the problem is to try to quantify the amount of computation required to determine whether or not the colouring will extend. In particular: can we bound the size of a subgraph H with $H \subseteq G$ such that every colouring of C that extends to H also extends to G? Note that a subgraph H with this property always exists: G is such a subgraph. To limit the computation required to answer the decidability question presented above, it is useful to study the minimal subgraphs H with this property. These subgraphs serve as small certificates for the decidability problem.

Postle and Thomas show the following, settling a conjecture of Dvořák et al. [14].

Theorem 1.2.21 (Postle and Thomas, [35])**.** *Let G be a plane graph with outer cycle C, let L be a 5-list assignment for G, and let H be a minimal subgraph of G such that every L-colouring of C that extends to an L-colouring of H also extends to an L-colouring of G. Then H has at most $19|V(C)|$ vertices.*

In 1997, Thomassen [42] proved a similar theorem for ordinary colouring, showing $|V(H)| \leq 5^{|V(C)|^3}$. In 2010, Yerger [49] improved Thomassen's bound to $O(|V(C)|^3)$. We note that a linear bound in terms of the number of vertices in the precoloured subgraph is asymptotically best possible.

In 2011, Dvořák and Kawarabayashi gave an analogous theorem to Theorem 1.2.21 for 3-choosability below.

Theorem 1.2.22 (Dvořák and Kawarabayashi, [11])**.** *Let G be a plane graph of girth at least five and with outer cycle C, let L be a 3-list assignment for G, and let H be a minimal subgraph of G such that every L-colouring of C that extends to an L-colouring of H also extends to an L-colouring of G. Then H has at most $\frac{37}{3}|V(C)|$ vertices.*

These theorems suggest that, given these graphs and list assignments, there is a small subgraph H that encodes the answer to the precolouring extension problem for cycles: that is, if a cycle C in a plane graph G is precoloured and we wish to determine whether this colouring extends to G, there exists a small subgraph H such that it suffices to check whether the colouring extends to H. Like Theorem 1.2.11, this too suggests that in the case of planar graphs, local structure alone is enough to glean valuable colouring information.

We show in Chapter 3 that Theorem 3.4.7 implies the following result.

Theorem 1.2.23. *Let G be a plane graph with outer cycle C, let (L, M) be a 5-correspondence assignment for G, and let H be a minimal subgraph of G such that every (L, M)-colouring of C that extends to an (L, M)-colouring of H also extends to an (L, M)-colouring of G. Then H has at most $51|V(C)|$ vertices.*

The final implications of hyperbolicity that will be discussed in Chatper 3 involve algorithms for the decidability of the colouring problem for embedded graphs: Dvořák and Kawarabayashi [12] also gave linear-time[3] algorithms for the decidability of 3-choosability

[3]The algorithms' running times are linear with respect to the number of vertices in the graph.

of embedded graphs of girth at least five. Their algorithms can be modified to allow the precolouring of a subgraph H, at the cost of increasing the time complexity of the algorithm to $O(|V(G)|^{k(g+s)+1})$ where k is some absolute constant, g is the genus of the surface in which the graph is embedded, and s is the number of components in H. This modification ensures the algorithms find a colouring, should it exist. Theorem 1.2.15 helps guide the structure of the algorithms: the algorithms roughly attempt to decompose embedded graphs into subgraphs as described in Theorem 1.2.15, and find colourings that extend to these subgraphs via dynamic programming. For details, see [12]. The algorithms rely on Theorem 1.2.22; and per [12], these algorithms can be adapted to other settings where a linear bound analogous to that in Theorem 1.2.22 holds.[4]

In particular, Theorem 1.2.21 thus implies the existence of linear algorithms for deciding the 5-choosability of embedded graphs, and Theorem 1.2.23 implies the following.

Theorem 1.2.24. *Let Σ be a fixed surface. There exists a linear-time algorithm that takes as input an embedded graph (G, Σ) and 5-correspondence assignment (L, M) for G with lists of bounded size and determines whether or not G is (L, M)-colourable.*

Theorem 1.2.25. *Let Σ be a fixed surface. There exists a linear-time algorithm that takes as input an embedded graph (G, Σ) and determines whether or not G is 5-correspondence-colourable.*

Note that in Theorem 1.2.24 the correspondence assignment (L, M) is fixed, whereas in Theorem 1.2.25 it is not. We note that these algorithmic results are new; and in fact, prior to this thesis, it was not known whether there existed poly-time algorithms (let alone linear algorithms) for the decidability of 5-correspondence colouring embedded graphs.

As mentioned prior, we obtain Theorem 1.2.23 as a consequence of a more technical theorem (Theorem 3.4.7), the proof of which constitutes the bulk of Chapter 3. We delay the statement of Theorem 3.4.7 until Section 3.4, when we will have built up the necessary background and terminology.

We further observe in Chapter 3 that the embedded graphs G of girth at least five that are critical for 3-correspondence colouring form a hyperbolic family. This follows from observing that the proof for list colouring in [34] also holds for correspondence colouring with only minor modifications. This is discussed further in Chapter 3.

As discussed above, the hyperbolicity of such a family of graphs (as well as related theorems) has many interesting implications. As in the case for 5-correspondence colouring, we highlight the following three.

[4] This is explained further in Chapter 3 (in particular, in Section 3.6), which contains a brief description of the algorithms.

Theorem 1.2.26. *Let Σ be a fixed surface. There exists a linear-time algorithm that takes as input an embedded graph of girth at least five (G, Σ) and a 3-correspondence assignment (L, M) for G with lists of bounded size and determines whether or not G is (L, M)-colourable.*

Theorem 1.2.27. *Let Σ be a fixed surface. There exists a linear-time algorithm that takes as input an embedded graph of girth at least five (G, Σ) and determines whether or not G is 3-correspondence-colourable.*

Theorem 1.2.28. *For every surface Σ, there exists a constant $\rho > 0$ such that every ρ-locally planar graph of girth at least five that embeds in Σ is 3-correspondence-colourable.*

In Section 1.4, we discuss the main ideas behind the proof of Theorem 1.2.23.

As demonstrated above, many questions in the field of graph colouring involve determining whether or not a graph with specific structure is colourable (for some notion of colouring). Changing track, we might ask the following: given that a graph *is* colourable, how easy is it to find a colouring?

One way to answer this question is to investigate how many distinct colourings of the graph there are, which brings us to the topic of Chapter 4: counting correspondence colourings of planar graphs. This sort of question has already been studied extensively for list and ordinary colourings. For instance, Thomassen [44] proved in 2007 the theorem below. The proof is approximately ten pages, and required new ideas and insights to those used for the proof of Theorem 1.2.6 (which says that planar graphs are 5-choosable).

Theorem 1.2.29 (Thomassen, [44])**.** *If G is a planar graph with 5-list assignment L, then G has at least $2^{\frac{|V(G)|}{9}}$ distinct L-colourings.*

In a later paper, Thomassen proved the following. The proof is roughly eight pages, and required substantially new ideas to those present in [44] and those used for the proof of Theorem 1.2.8 (which says that planar graphs of girth at least five are 3-choosable).

Theorem 1.2.30 (Theorem 4.3, [45])**.** *If G is a planar graph of girth at least five and L is a 3-list assignment for G, then G has at least $2^{\frac{|V(G)|}{10000}}$ distinct L-colourings.*

In [34] and [37], Postle and Thomas generalized these results to other surfaces as follows.

Theorem 1.2.31 ([34], [37]). *There exist constants $\varepsilon, \alpha > 0$ such that the following holds. Let G be a graph embedded in a surface Σ of genus g, and let H be a proper subgraph of G. If either*

- *L is a 5-list-assignment for G, or*
- *G has girth at least 5 and L is a 3-list assignment for G,*

then if ϕ is an L-colouring of H that extends to an L-colouring of G, then ϕ extends to at least $2^{\varepsilon(|V(G)|-\alpha(g+|V(H)|))}$ distinct L-colourings of G.

In 2018, Kelly and Postle later proved an analogous theorem for embedded triangle-free graphs with 4-list assignments.

Theorem 1.2.32 (Theorem 7, [25]). *There exist constants $\varepsilon, \alpha > 0$ such that the following holds. Let G be a triangle-free graph embedded in a surface Σ of genus g, and let L be a 4-list-assignment for G. If H is a proper subgraph of G and ϕ is an L-colouring of H that extends to an L-colouring of G, then ϕ extends to $2^{\varepsilon(|V(G)|-\alpha(g+|V(H)|))}$ distinct L-colourings of G.*

Finally, in 2021, Langhede and Thomassen [29] prove a result analogous to Theorem 1.2.29 for $\mathbb{Z}_5$-colouring, defined below. Recall that an orientation of an edge uv is a mapping f of uv to an element of $\{(u,v),(v,u)\}$. If $f(uv) = (u,v)$, we say u is *directed towards* v.

Definition 1.2.33. Let G be a graph together with an orientation of its edges. Let $\phi : E(G) \to \{0,1,2,3,4\}$ be a labelling of the edges of G. Let (L, M_ϕ) be a 5-correspondence assignment for G, where $L(v) = \{0,1,2,3,4\}$ for each $v \in V(G)$ and where M_ϕ is defined as follows: for each edge uv where u is directed towards v, we let $M_{uv} = \{(u,a)(v,b) : a - b \equiv \phi(uv)\}$, where the equivalence is taken modulo 5. We call (L, M_ϕ) a $\mathbb{Z}_5$-assignment. We say G is $\mathbb{Z}_5$-colourable if G admits an (L, M_ϕ)-colouring for every labelling ϕ.

Theorem 1.2.34 (Langhede and Thomassen, [29]). *If G is a planar graph with $\mathbb{Z}_5$-assignment (L, M_ϕ), then G has at least $2^{\frac{|V(G)|}{9}}$ distinct (L, M_ϕ)-colourings.*

It follows from their definitions that 5-correspondence colouring generalizes $\mathbb{Z}_5$-colouring; thus if one could show that planar graphs have exponentially many[5] 5-correspondence colourings, this would imply the same for $\mathbb{Z}_5$-colourings. Indeed, Langhede and Thomassen conjectured the following.

[5]Unless otherwise specified, we mean by this that the number of colourings is exponential in the number of vertices in the graph.

Conjecture 1.2.35 (Langhede and Thomassen, [29])**.** *Planar graphs have exponentially many 5-correspondence colourings.*

In Chapter 4, we prove Conjecture 1.2.35. In fact, another contribution of this thesis is to show that Theorems 3.4.7 and 3.7.2—our technical theorems that imply our results on hyperbolicity—are themselves enough to prove that there are exponentially many correspondence colourings of planar graphs. That these results follow from Theorems 3.4.7 and 3.7.2 suggests an overall *technique* for tackling these sorts of colouring-counting problems, rather than the more ad hoc approaches used by Thomassen in [44] and [45].

In particular, we prove in Chapter 4 that Theorem 3.4.7 implies the theorem below.

Theorem 1.2.36. *If G is a planar graph with at least three vertices and (L, M) is a 5-correspondence assignment for G, then G has at least $2^{\frac{|V(G)|+306}{67}}$ distinct (L, M)-colourings.*

For planar graphs of girth at least five, we show the following as a consequence of Theorem 3.7.2.

Theorem 1.2.37. *If G is a planar graph with at least two vertices and girth at least five and (L, M) is a 3-correspondence assignment for G, then G has at least $2^{\frac{|V(G)|+890}{292}}$ distinct (L, M)-colourings.*

As correspondence colouring generalizes list colouring, this improves upon the bound given for list colouring by Thomassen in 2007 (Theorem 1.2.30). In Section 1.5, we discuss the main ideas behind the proofs in Chapter 4.

Postle and Thomas show in [37] that, given a graph G with outer cycle C and 5-list assignment L, if ϕ is an L-colouring of C that extends to an L-colouring of G, then there are in fact exponentially many extensions of ϕ to G. Their proof relies as a base case on a technical theorem of Thomassen (Theorem 4, [44]) which implies Theorem 1.2.29. One of our main theorems of Chapter 4 —Theorem 4.2.6, which implies Theorem 1.2.36 —also implies that there are exponentially many extensions of ϕ to C, but completely sidesteps the use of this theorem of Thomassen. In fact, we show that it is enough that Theorem 3.4.7 (which implies that the family of graphs that are critical for 5-correspondence colouring is hyperbolic) holds. Indeed, our theorem gives *an independent proof of Thomassen's result that there are exponentially many 5-list colourings of planar graphs.*

Analogously for 3-list colouring, Postle shows in [34] that given a graph G of girth at least five with outer cycle C and 3-list assignment L, if ϕ is an L-colouring of C that extends to an L-colouring of G, then there are in fact exponentially many extensions of ϕ

to G. Again, this proof relies on a technical theorem of Thomassen which implies Theorem 1.2.30. Our other main theorem of Chapter 4 —Theorem 4.3.5, which implies Theorem 1.2.37 —also implies that there are exponentially many extensions of ϕ to C, but sidesteps the use of this theorem of Thomassen. We show that it is enough that Theorem 3.7.2 (which implies that the family of graphs of girth at least five that are critical for 3-correspondence colouring is hyperbolic) holds.

As mentioned above, Theorems 1.2.36 and 1.2.37—along with being the first results showing there are exponentially many correspondence colourings of planar graphs —also provide independent proofs that there are exponentially many 5-list colourings of planar graphs, and 3-list colourings of planar graphs of girth at least five, respectively. Moreover, that our results follow from Theorems 3.4.7 and 3.7.2 further motivates the study of hyperbolic families.

Finally, we note that an easy consequence of the definition of hyperbolicity is that hyperbolic families do not contain planar graphs: for instance, the family of graphs that are critical for 5-correspondence choosability does not contain a planar graph. As every graph that is not 5-correspondence-colourable contains a subgraph that is critical for 5-correspondence colouring, it follows that every planar graph is 5-correspondence-colourable. This motivates the following question.

Question 1.2.38. *Is the family of embedded graphs that are critical for local girth choosability (or local girth correspondence colouring) hyperbolic?*

In particular, we ask the following.

Question 1.2.39. *Does there exist an analogous theorem to Theorem 3.4.7 for local girth correspondence colouring?*

A positive answer to Question 1.2.39 would simultaneously imply all the results in this thesis (though perhaps with worse constants in the case of the theorems on hyperbolicity and counting colourings in Chapters 3 and 4), as well as all implications of the hyperbolicity of the family of 5-correspondence critical graphs; of 4-correspondence critical graphs of girth at least four; and of 3-correspondence critical graphs of girth at least 5. Moreover, it would of course imply results analogous to these for local girth correspondence colouring, and local girth choosability. This problem, however, seems difficult: as evidence, we highlight the intricacies involved in the proof of Theorem 3.4.7 for 5-correspondence colouring as well as the fact that even the proof of Theorem 1.2.11 for local girth choosability does not seem to admit any obvious modification to allow an analogous theorem for correspondence colouring. However —though no doubt difficult —given the results proved in this thesis, proving a theorem like that described in Question 1.2.39 now seems possible. For more open questions inspired by the work in this thesis, we encourage the reader to consult Chapter 5.

Sections 1.3, 1.4, and 1.5 elaborate on some of the challenges faced in the hardest proofs of Chapters 2, 3, and 4, respectively. Section 1.6 gives an outline of the rest of the thesis.

1.3 Local Girth Colouring

The proof of Theorem 1.2.11 is inspired by those of Theorems 1.2.6 (that planar graphs are 5-choosable) and 1.2.8 (that planar graphs of girth at least five are 3-choosable). To prove those theorems, Thomassen instead proved more technical theorems (Theorems 2.1.7 and 2.1.9, given in Chapter 2) involving precolouring paths on the outer face boundary of plane graphs, and extending these to colourings to the whole graph. In Theorems 2.1.7 and 2.1.9, list sizes are restricted still further than in Theorems 1.2.6 and 1.2.8: in particular, vertices on the outer face boundary of the graphs have smaller lists than the others. It is precisely these stronger restrictions that allow the theorems to be proven inductively: a subset of the vertices of a vertex-minimum counterexample to each theorem are coloured, their colours removed from the lists of neighbouring vertices, and the coloured vertices deleted. If this is done carefully, the resulting graph will still satisfy the premises of the technical theorem, allowing the proof to be completed by induction. Like Thomassen, we also prove our main colouring result (Theorem 1.2.11) via a more technical theorem. Our main technical theorem is Theorem 2.1.6, which implies both of Thomassen's technical theorems, Theorems 2.1.7 and 2.1.9, and hence implies both of Theorems 1.2.6 and 1.2.8.

Unfortunately, the proofs of Theorems 1.2.6 and 1.2.8 cannot be readily combined to prove Theorem 1.2.11. The proof of Theorem 2.1.7 (which implies Theorem 1.2.6) relies on the fact that we may delete up to two colours in the lists of vertices not on the outer face boundary of the graph without making their lists too small. The same is not true for a graph with a local girth list assignment. The proof of Theorem 2.1.9 (which implies Theorem 1.2.8) of course relies on the fact that every vertex in the graph has girth at least five: this ensures that in colouring and deleting a path of length at most one on the outer face boundary of the graph, the vertices that were adjacent to these deleted vertices form an independent set (a condition required for the technical inductive statement to go through).

Though many of our lemmas are similar in spirit to those used by Thomassen, the main colouring argument is quite different: in our proof, we colour and delete an *arbitrarily long* path on the outer face boundary of the graph. The path in question is determined by the lists of the vertices on the outer face boundary. Since the structure of the graph under study is more complex than that of a simple near-triangulation or a graph of girth at least five, a stronger inductive statement and more structural lemmas are needed than for either of the proofs of Theorems 1.2.6 or 1.2.8. Indeed for our proof, we even have to generalize Thomassen's characterization of when a precoloured path of length two does not extend to a colouring (Theorem 2.1.8) to the local setting. Unfortunately then, the stronger inductive statement given in Theorem 2.1.6 gives rise to a number of exceptional graphs which render the analysis more difficult: when working through inductive arguments, we will have to argue why these exceptional cases do not arise.

It is perhaps surprising that there is a single theorem that unifies both Theorems 1.2.6 and 1.2.8. After all, the proofs of the Four Colour Theorem and Grötzsch's Theorem (Theorem 1.2.2)—which are in a sense the analogous theorems for ordinary vertex colouring—are very different. That Theorem 1.2.11 (a local, unified version for list colouring) holds hints perhaps at something fundamental about planar graphs and the relation between colouring and short cycles.

1.4 Hyperbolicity

One of the main results of Chapter 3 is Theorem 1.2.23, which implies that the family of graphs that are critical for 5-correspondence colouring is hyperbolic. Our proof of Theorem 1.2.23 follows the basic framework laid out by Postle and Thomas in [35] to prove the analogous theorem for list colouring: we focus on a stronger, more technical theorem —Theorem 3.4.7—which bounds the number of vertices in terms of the sum of the sizes of large faces (a notion Postle and Thomas call "deficiency"). The proof of Theorem 3.4.7 also involves counting the number of vertices that share an edge or a face with vertices in the outer cycle of the graph. In keeping track of these quantities, we are able to perform various reductions, showing a minimum counterexample to Theorem 3.4.7 must have a very specific structure and ultimately that a minimum counterexample cannot exist.

The bulk of the arguments present in the proof of Postle and Thomas' list colouring version of Theorem 3.4.7 carry over to correspondence colouring with only minor modifications. This is largely due to the fact that many of the arguments are structural, and do not rely on the specific list assignment. However, there are a few key points at which the arguments fail for correspondence colouring. In particular, Claims 5.23 and 5.24 in [35] argue that the lists of specific vertices in a minimum counterexample are subsets of one another. For a triangle ux_2z_2u in a minimum counterexample with list assignment

S, Claim 5.23 shows that $S(u) \subseteq S(x_2)$. Claims 5.27 and 5.28 then use the fact that $S(z_2) \setminus (S(x_2) \cup S(u)) = S(z_2) \setminus S(x_2)$. This (along with an argument showing $S(z_2) \setminus S(x_2)$ is non-empty) implies that it is possible to colour z_2 from $S(z_2)$ while avoiding the lists of both x_2 and u. This argument crucially does not hold for correspondence colouring: an analogous argument to that in Claim 5.23 shows merely that for a correspondence assignment (S, M), we have $|M_{x_2u}| = |S(u)|$, which of course implies nothing about M_{z_2u}. As a consequence of this, we are unable to use the reductions found in [35], and must instead develop an entirely new set of reductions that can be performed in the correspondence colouring framework. This adds considerable length and intricacy to the proof.

A more detailed overview of the proof will be given in Chapter 3.

1.5 Counting Colourings

The main results in Chapter 4 are Theorems 4.2.6 and 4.3.5, both of which count the number of extensions of a precoloured subgraph S of a planar graph G to a colouring of G itself. Theorem 4.2.6 concerns extending a 5-correspondence colouring of S to G; Theorem 4.3.5 concerns extending a 3-correspondence colouring of S to a graph G of girth at least five. The proofs of both theorems use several key results from a recent paper of Postle [33], in which the theory of hyperbolicity is extended by the introduction of the notion of *deletable subgraphs*. This concept will be defined in Chapter 4. We note the results from [33] used were originally written for list colouring and adapted in this thesis to the correspondence colouring framework.

Both proofs proceed by induction on $|V(G)| - |V(S)|$, and use the key observation (Claims 39 and 40, respectively) that there does not exist a subgraph H with $S \subsetneq H \subsetneq G$ and $|V(S)| < |V(H)| < |V(G)|$ such that every colouring of H extends to a colouring of G. Otherwise —the proof goes —by our choice of G, since $|V(S)| < |V(H)| < |V(G)|$ and $S \subsetneq H \subsetneq G$, we have that both that $|V(H)| - |V(S)| < |V(G)| - |V(S)|$ and that $|V(G)| - |V(H)| < |V(G)| - |V(S)|$. We then prove the main counting theorem by first counting the extensions of the colouring of S to H, and then of each colouring of H to G.

Both proofs require separate, careful analysis of the cases where $|V(G)| - |V(S)|$ is at most two. As will become apparent to the reader, for both theorems the case where $|V(G)| - |V(S)| = 1$ is what determines the denominator in the exponent of the theorem statements.

1.6 Thesis Outline

In each chapter of this thesis, we will study a particular type of object called a *canvas*. A canvas is a tuple consisting of a graph G, a list (or correspondence) assignment for G, and a selection subgraphs of G that we wish to keep track for various reasons (e.g. a subgraph we think of as being *precoloured*, a subgraph whose vertices have otherwise restricted lists, etc.). The precise definition of canvas varies depending on the chapter. The working definition will be given explicitly in the chapter introduction.

In Chapter 2, we introduce the notion of local girth colouring and give a proof of Theorem 1.2.11. We prove Theorem 1.2.11 via a more technical theorem —Theorem 2.1.6 —which involves extending a precolouring of a short path or cycle in the outer face boundary walk of a planar graph. The majority of the chapter is dedicated to the proof of Theorem 2.1.6.

In Chapter 3, we pivot our attention to correspondence colouring. The main result is Theorem 3.4.7, a technical theorem which in turn implies Theorem 1.2.23, Theorem 3.6.4, and —perhaps most importantly —Theorem 3.6.5, that the family of graphs that are critical for 5-correspondence colouring is hyperbolic. Using the theory of hyperbolic families developed by Postle and Thomas [37], we then show that for every surface Σ, there exists a constant $\rho > 0$ such that locally planar graphs are 5-correspondence-colourable (Theorem 1.2.20). Using the ideas of Dvořák and Kawarabayashi in [12], we further observe that there exist linear-time algorithms for the decidability of the 5-correspondence colouring for embedded graphs. In addition, in Chapter 3 we state Observation 3.7.3, which implies that the family of graphs of girth at least five that are critical for 3-correspondence colouring is hyperbolic (Corollary 3.7.4). This implies that locally planar graphs of girth at least five are 3-correspondence-colourable (Corollary 1.2.28). Observation 3.7.3 also implies the existence of linear-time algorithms for the decidability of the 3-correspondence colouring problem for embedded graphs of girth at least five (again using the ideas in [12]). Observation 3.7.3 follows from Observation 3.7.2, the proof of which is nearly identical to that of an analogous theorem of Postle (Theorem 3.9, [34]) for list colouring. We include a short discussion of why precisely this theorem of Postle goes through for correspondence colouring, when the analogous theorem of Postle and Thomas for 5-choosability (Theorem 4.6, [35]), which uses very similar tools and ideas, does not.

In Chapter 4, we show that Theorem 3.4.7 has another interesting implication beyond those discussed in Chapter 3: namely, that if G is a planar graph with 5-correspondence assignment (L, M), then G has at least $2^{\frac{|V(G)|+306}{67}}$ distinct (L, M)-colourings (Theorem 1.2.36). Similarly, we show that Observation 3.7.2 can be used to show that if G is a planar graph of girth at least five and (L, M) is a 3-correspondence assignment for G, then G has at least $2^{\frac{|V(G)|+890}{292}}$ distinct (L, M)-colourings (Theorem 1.2.37).

The introduction of Chapter 2, Chapter 3, and Chapter 4 each contain a more detailed overview of the contents of the chapter, including overviews of each chapter section.

In Chapter 5, we give a brief summary of our main results and leave the reader with some open problems suggested by the work in this thesis.

Chapter 2

Local Girth Colouring Planar Graphs

2.1 Introduction

Subsection 2.1.1 contains an overview of the main results of this chapter, along with necessary definitions. Subsection 2.1.2 provides an outline of the remainder of the chapter.

2.1.1 Results

The main result of this chapter is Theorem 1.2.11, restated below for convenience.

Theorem 1.2.11. *Every planar graph is local girth choosable.*

As described in Sectioin 1.3, we prove Theorem 1.2.11 via a more technical theorem —Theorem 2.1.6 —involving precolouring paths on the outer face boundary of plane graphs, and extending these to colourings to the whole graph. In Theorem 2.1.6, vertices' lists are restricted further still than a local girth list assignment. In particular, vertices on the outer face boundary of the graphs have smaller lists than the others. Before we present Theorem 2.1.6, we give a few required definitions.

The *outer* face of a plane graph is its infinite face. Let G be a plane graph, and let E and V be the set of edges and vertices, respectively, in the boundary walk of the outer face of G. Let H be the subgraph of G with $V(H) = V$ and $E(H) = E$. We say a path (or cycle) S is *on the outer face boundary of* G if $S \subseteq H$.

Definition 2.1.1. Let G be a plane graph, and let $S = v_1v_2 \dots v_k$ be a path in G. We say S is an *acceptable path* if one of the following holds.

- $k \leq 3$, or
- $k = 4$, $g(v_2) \geq 4$, and $g(v_3) \geq 4$, or
- $k = 4$, and either $g(v_2) \geq 5$ or $g(v_3) \geq 5$.

An *acceptable cycle* is a cycle S in G where S contains an edge e such that $S - e$ is an acceptable path in G.

Note that in the above definition, we allow S to be the empty path. Moreover, note that if S is an acceptable path in a graph G, S_1 is a subpath of S, and G_1 is a subgraph of G that contains S_1, then S_1 is an acceptable path of G_1.

Theorem 2.1.6 below —the proof of which constitutes the bulk of this chapter —characterizes precisely when a colouring of an acceptable path or cycle extends to a list colouring of the whole graph. The list assignment described in the theorem is a restriction of a local girth list assignment. Under this list assignment, the list sizes depend in part on whether or not the vertices are on the outer face boundary of the graph. For this reason (among others), the following definition is useful.

Definition 2.1.2. Let G be a plane graph, and let $C \subseteq G$ be a cycle. We define $\text{Int}(C)$ as the subgraph of G induced by the vertices of G in the interior of C. Similarly, we define $\text{Int}[C]$ to be the subgraph of G containing precisely the vertices and edges inside and on C.

If the graph under study contains one of a specific set of subgraphs, a colouring of an acceptable path is only guaranteed to extend to a colouring of the whole graph if the acceptable path is short. Among these problematic subgraphs are the *broken* and *generalized wheels*, defined below. We take the definitions and associated terminology from Thomassen in [44].

Definition 2.1.3. A *broken wheel* is either a cycle $v_1v_2v_3v_1$ or the graph formed by a single cycle $v_1v_2 \ldots v_qv_1$ together with edges $v_2v_4, v_2v_5, \ldots, v_2v_q$. The path $v_1v_2v_3$ is called the *principal path* of the broken wheel. A *wheel* is a cycle $v_1v_2 \ldots v_qv_1$ together with a single vertex v and edges $vv_1, vv_2, \ldots, vv_q$. Again, we say $v_1v_2v_3$ is the *principal path* of the wheel. If $v_1v_2v_3$ is the principal path of a wheel or broken wheel, we call v_1v_2 and v_2v_3 the principal edges.

Definition 2.1.4. A *generalized wheel* is defined recursively as follows. Every wheel and broken wheel is a generalized wheel. If W is a generalized wheel with principal path $v_1v_2v_3$

and W' is a generalized wheel with principal path $u_1u_2u_3$, then the graph obtained from W and W' by identifying a principal edge of W with a principal edge in W' in such a way that v_2 and u_2 are identified is also a generalized wheel. Its principal path is the path formed by the principal edges in W and W' that were not identified.

Note that if W is a generalized wheel and C is the outer cycle of W, then either C has a chord, W is a wheel, or W is a 3-cycle. Moreover, note that as there may be multiple ways to construct a generalized wheel W, there maybe several possible principal paths for W. Unless W is a triangle or a wheel, these paths are edge-disjoint. Finally, note that every vertex in a generalized wheel has girth 3.

As mentioned earlier, Theorem 2.1.6 describes when a list colouring of an acceptable path S in a graph G extends to a list colouring of the entire graph. Below, we define a *canvas*, which will allow us to concisely keep track of the graph, acceptable path, and list assignment.

Definition 2.1.5. Let G be a plane graph, and let C be the subgraph whose vertex- and edge-set are precisely those of the outer face boundary of G. We say (G, L, S, A) is a *canvas* if S is a subgraph of C, $A \setminus V(S)$ is an independent set of vertices with $A \subseteq V(C) \setminus V(S)$ such that $g(v) \geq 5$ for each $v \in A$, and L is a list assignment whose domain contains $V(G)$ such that:

- $|L(v)| \geq 1$ for all $v \in V(S)$,
- $|L(v)| = 2$ for all $v \in A \setminus V(S)$,
- $|L(v)| \geq 3$ for all $v \in V(G) \setminus (A \cup V(S))$,
- $|L(v)| \geq 4$ for all $v \in V(G) \setminus V(C)$ such that $g(v) = 4$, and
- $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$ such that $g(v) = 3$.

We say $(H, L, S \cap H, A \cap V(H))$ is a *subcanvas* of a canvas $K = (G, L, S, A)$ if $H \subseteq G$ and $S \cap H$ is connected. In this case, we denote $(H, L, S \cap H, A \cap V(H))$ by $K[H]$. Note that $K[H]$ is a canvas.

We think of the vertices of S as being *precoloured*, and we say a canvas $K = (G, L, S, A)$ admits an L-colouring if G admits an L-colouring. It might seem more natural to require that A (and not $A \setminus V(S)$) is an independent set of vertices with lists of size two; this definition is more convenient, since when working through inductive arguments the precoloured vertices are sometimes elements of A. Having established the required definitions, we give our technical theorem below.

Theorem 2.1.6. *Let G be a plane graph, and let S be either an acceptable path $v_1v_2\cdots v_k$ or acceptable cycle $v_1v_2\cdots v_kv_1$. If (G,L,S,A) is a canvas, then every L-colouring ϕ of $G[V(S)]$ extends to an L-colouring of G unless one of the following occurs:*

(i) *$k=4$; there exists a vertex $u \in A \setminus V(S)$ that is adjacent to both v_1 and v_4; and $L(u) = L(v_1) \cup L(v_4)$, or*

(ii) *$k=4$; there exists a vertex $u \in A \setminus V(S)$ such that, up to reversing the names of the vertices of S, u is adjacent to v_4; there exists a vertex $w \notin V(S)$ on the outer face boundary of G such that $uw \in E(G)$; v_1v_2w is the principal path of a generalized wheel W where the vertices on the outer cycle of W are on the outer face boundary of G; every vertex v in the outer cycle of W except v_1 and v_2 has $|L(v)|=3$, or*

(iii) *$k=3$; S is the principal path of a generalized wheel W where the vertices on the outer cycle of W are on the outer face boundary of G; every vertex v in the outer cycle of W except v_1, v_2, and v_3 has $|L(v)|=3$.*

We say (G,L,S,A) is an *exceptional canvas of type (i), type (ii),* or *type (iii)*, if it is described by (i), (ii), or (iii), respectively, above. If a canvas is not an exceptional canvas of any type, we say it is *unexceptional*. Note that an exceptional canvas of type (ii) or (iii) might still admit a colouring, depending on L. Examples of exceptional canvases that do not admit an L-colouring are given in Figure 2.1, below. We note that there is precisely one case where (G,L,S,A) is unexceptional and there exists an L-colouring of S but not necessarily of $G[V(S)]$: this is the case where S is a path of length three and $V(S)$ induces a 4-cycle.

While Theorem 2.1.6 is not an if and only if statement, it is rather straightforward to characterize the list assignments of exceptional canvases that do not admit a colouring (see [44] and [36] for such details).

As a corollary to Theorem 2.1.6, we immediately obtain Theorem 1.2.11: to see this, note that if (G,L,S,A) is a canvas, then by definition L is a restriction of a local girth list assignment. Moreover, if $|V(S)| \leq 2$ (that is, if at most two vertices on the outer face boundary of G are precoloured), then (G,L,S,A) is unexceptional and so G admits an L-colouring.

We highlight the similarity between our technical theorem and the technical theorems used by Thomassen in proving Theorems 1.2.6 and 1.2.8: these technical theorems are rephrased below using our terminology.

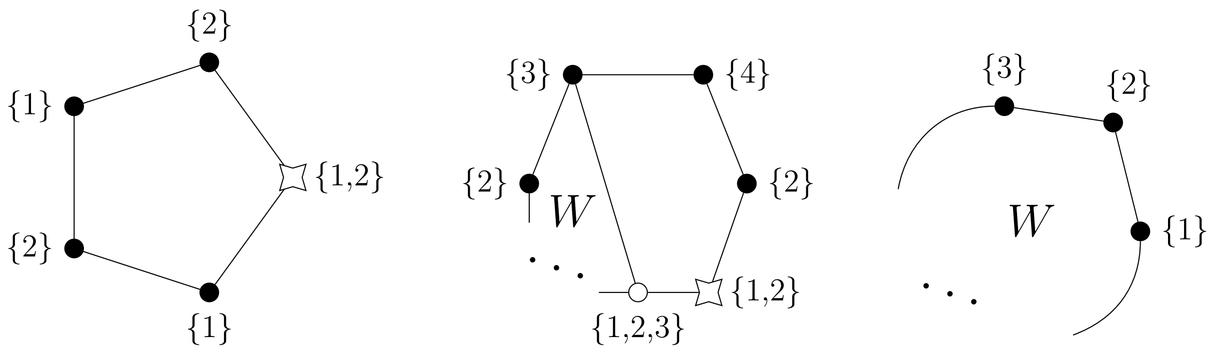

Figure 2.1: Exceptional canvases of type (i), (ii), and (iii), together with potential partial list assignments. Vertices in S are black; vertices in A are drawn as four-pointed stars. W indicates the presence of a generalized wheel subgraph.

Theorem 2.1.7 (Thomassen [40])**.** *If (G, L, S, A) is a canvas where G is a near-triangulation*[1] *and S is a path of length at most one in the outer face boundary of G, then G admits an L-colouring.*

This theorem implies Theorem 1.2.6, that every planar graph is 5-choosable, and is implied by Theorem 2.1.6. We note that in a near-triangulation, every vertex has girth three. Consequently, in the above theorem $A = \emptyset$ by definition of A.

Thomassen also characterized the set of canvases (G, L, S, A) where G is a near-triangulation and S a path of length two that do *not* necessarily admit an L-colouring. Thomassen's original theorem is Theorem 3 in [44]. We state it below in the language of canvases.

Theorem 2.1.8 (Thomassen [44])**.** *If $K = (G, L, S, A)$ is a canvas where G is a near-triangulation and S is a path of length two in the outer face boundary of G, then G admits an L-colouring unless K is an exceptional canvas of type (iii).*

We note that Theorem 2.1.8 is also a special case of Theorem 2.1.6 where G is a near-triangulation (and so $A = \emptyset$).

One technical theorem that may be used to establish Theorem 1.2.8 is as follows.

[1]That is, a plane graph where each face with the possible exception of the outer face has a boundary walk of length three.

Theorem 2.1.9. *If $K = (G, L, S, A)$ is a canvas where S is an acceptable path and every vertex in G has girth at least 5, then every L-colouring of S extends to an L-colouring of G unless K is an exceptional canvas of type (i).*

Thomassen's true technical theorem forbids edges between S and A, but allows S to have up to six vertices. However, as mentioned in the conclusion of [43], this is essentially equivalent to Theorem 2.1.9. Note that Theorem 2.1.9 is a special case of Theorem 2.1.6, and it implies Theorem 1.2.8 (and moreover Grötzsch's Theorem, as is shown in [43]).

A specific type of exceptional canvases of type (iii) appears frequently in the induction arguments in this chapter. The following lemma will allow us to deal with them painlessly. It is Lemma 1 by Thomassen in [44].

Lemma 2.1.10 (Thomassen [44])**.** *Let W be a generalized wheel but not a broken wheel with outer cycle $C = v_1v_2v_2 \dots v_q$. If L is a list assignment for W such that every vertex $v \in \mathrm{Int}(C)$ has $|L(v)| \geq 5$ and every very vertex $v \in V(C) \setminus \{v_1, v_2, v_3\}$ has $|L(v)| \geq 3$, then there is at most one colouring of v_1, v_2, v_3 that does not extend to an L-colouring of G.*

Note that Lemma 2.1.10 and Theorem 2.1.6 together give us the following lemma. It is an analogous lemma to Lemma 2.1.10 for local girth list assignments.

Lemma 2.1.11. *Let G be a plane graph, and let $S = v_1v_2v_3$ be a path on the outer face boundary of G. Suppose G contains a subgraph W that is a generalized wheel whose principal path is $v_1v_2v_3$ such that the vertices on the outer cycle of W are on the outer face boundary of G. Let (G, L, S, A) be a canvas. If W is not a broken wheel, then there is at most one L-colouring of $G[V(S)]$ that does not extend to an L-colouring of G.*

We will not use Theorem 2.1.11 in this thesis: our inductive arguments will allow us to use Theorem 2.1.10 directly.

We require one more definition.

Definition 2.1.12. Let G be a connected plane graph with outer face boundary walk C. Let P be a path with endpoints u, v, where $\{u, v\} \subseteq V(C)$. We say P *separates G into two plane graphs G_1 and G_2* if the following hold.

- G_1 and G_2 inherit the embedding of G,
- $G_1 \cap G_2 = P$,

- $G_1 \cup G_2 = G$,
- for each $i \in \{1, 2\}$, we have that $V(G) \setminus V(G_i) \neq \emptyset$,
- both G_1 and G_2 are connected, and
- P is in the outer face boundary walk of both G_1 and G_2.

2.1.2 Outline of Chapter

Theorem 2.1.6 is proved in Section 2.3, by supposing the existence of a minimum counterexample. To that end, for the remainder of the chapter we let $K = (G, L, S, A)$ be a counterexample to Theorem 2.1.6, chosen such that $|V(G)|$ is minimized over all counterexamples of Theorem 2.1.6; and subject to that, such that $\sum_{v \in V(G)} |L(v)|$ is minimized. In Section 2, we establish necessary structural properties of K. Note that by our choice of K, it follows that S contains at least one vertex. We may assume that there exists at least one colouring of $G[V(S)]$ that does not extend to a colouring of G, as otherwise there is nothing to prove. Since K is chosen to minimize $\sum_{v \in V(G)} |L(v)|$, it follows that $|L(v)| = 1$ for each vertex $v \in V(S)$. To derive a contradiction (and hence prove Theorem 2.1.6), it thus suffices to show that G admits an L-colouring.

2.2 Describing Our Minimum Counterexample

Recall that $K = (G, L, S, A)$ is a counterexample to Theorem 2.1.6. We may assume without loss of generality that $V(S) \cap A = \emptyset$. Let C be the subgraph of G whose vertex set and edge set is precisely that of the outer face boundary walk of G. It is straightforward to verify the theorem holds if $|V(G)| = |V(S)|$, so we assume $|V(G)| > |V(S)|$. We note the following.

Observation 2.2.1. *If K is an unexceptional canvas and K' is a subcanvas of K, then K' is also unexceptional.*

This follows from the fact that whether or not a canvas is exceptional depends in part on L, and that every vertex in a generalized wheel has girth three: thus if the outer face boundary of K' contains a vertex v not in the outer face boundary of K with $g(v) = 3$, then $|L(v)| \geq 5$. We will use this observation repeatedly throughout this chapter.

2.2.1 Section Overview

We now give an overview of the results of this section. As this subsection is purely a summary of this section's results compiled for the reader's convenience, the reader should feel free to skip ahead to Subsection 2.2.2 if desired.

In Subsection 2.2.2, we give a few lemmas establishing basic properties of K: namely, Lemma 2.2.2 shows that G is 2-connected (and thus, that C is a cycle); and Lemma 2.2.3, that its outer cycle C is chordless. These first two lemmas allow us to make useful assumptions about the structure of G later on.

The majority of Subsection 2.2.3 describes the interiors of certain cycles in G of length up to six. In particular, Lemma 2.2.4 shows that G does not contain cycles of length three with vertices in their interior. Observation 2.2.5 argues that we may assume no vertex in C has more than three colours in its list. This allows us to argue explicitly about the lists themselves in Subsection 2.2.6 and in our final colouring argument in Section 2.3. Lemma 2.2.6 shows that G does not contain cycles of length four with vertices in their interior. Observation 2.2.7 shows that $|V(S)| \geq 3$. Lemma 2.2.8 shows that if $H \subseteq G$ is a 5-cycle with at least one vertex in its interior, then all vertices in $V(H)$ have girth three; and finally, Lemma 2.2.9 shows that if $H \subseteq G$ is a 6-cycle with at least one vertex in its interior, then no vertex in $V(H)$ has girth at least five. These lemmas are used to establish more complicated structural properties of G (in particular, to rule out the existence of certain generalized wheel subgraphs in Subsection 2.2.5).

In Subsection 2.2.4, we characterize the set of short paths in G whose endpoints lie in C and whose interior vertices do not. We think of these paths as *separating* paths: such a path neatly divides G into two subgraphs whose intersection is the path (see Definition 2.1.12). In particular, we show in Lemma 2.2.10 that G does not contain a separating path of length two that has at least one vertex of girth at least four, and in Lemma 2.2.11 we describe exactly the structure surrounding separating paths $u_1u_2u_3u_4$ where both u_2 and u_3 have girth at least five.

Subsection 2.2.5 uses the structure established in previous subsections to rule out the existence of certain generalized wheel subgraphs in G, and to bound the size of others. We note that whether or not a generalized wheel subgraph can be ruled out depends not only on its structure, but also its list assignment. Lemma 2.2.12 shows that if Q is a separating path of length two whose vertices all have girth three, then Q is the principal path of a broken wheel in G. Lemma 2.2.13 shows that upon identifying certain vertices in a generalized wheel, the girth of the remaining vertices in the graph does not change too much: every vertex with girth at least five in G still has girth at least five after the identification, and every vertex with girth four in G still has girth at least four after the identification. This allows us in Lemma 2.2.14 to rule out some large broken wheels in G. Lemma 2.2.15 rules out yet another type of broken wheel in G, and will be used in Section 3 to show that, after performing our main reductions, what remains is not an exceptional canvas of type (iii).

Finally, Subsection 2.2.6 establishes the remainder of the structure required for the proof of Theorem 2.1.6 in the next section. In particular, Lemma 2.2.16 shows that C contains at least three vertices other than those in S, and Lemma 2.2.17 shows that if there is a vertex in A adjacent to a vertex in S, then this lower bound can be raised to four. Lemma 2.2.20 describes the lists of these non S-vertices. Lastly, we give the definition of a particular type of path contained in C called a *deletable path*: this path comprises the main reducible configuration for the proof of Theorem 2.1.6 in Section 2.1.6. We end the section by showing that K contains a deletable path.

2.2.2 Groundwork Lemmas

We begin by showing that G is 2-connected. Note that G is connected: otherwise, since K is a minimum counterexample to Theorem 2.1.6 we obtain an L-colouring of G by applying Theorem 2.1.6 to each component of G. Moreover, since K was chosen to minimize $\sum_{v\in V(G)}|L(v)|$, we may assume without loss of generality that $|V(S)| \geq 2$. Since $G[V(S)]$ has an L-colouring by assumption, we may further assume that $|V(G)| \geq 3$.

Lemma 2.2.2. *G is 2-connected.*

Proof. Suppose not. Since $|V(G)| \geq 3$, there exists a cut vertex $u \in V(G)$ such that the path u separates G into two graphs G_1 and G_2 (as in Definition 2.1.12). For $i \in \{1, 2\}$, let S_i be the subgraph of S contained in G_i. Suppose without loss of generality that $|V(S_1)| \geq |V(S_2)|$. Since S is a path or cycle in the outer face boundary of G, it follows that either $S_1 \cap S_2 = \emptyset$, or $S_1 \cap S_2 = u$. By Observation 2.2.1, $K[G_1]$ is an unexceptional canvas. By the minimality of K, G_1 thus admits an L-colouring ϕ_1. Let L' be a list assignment for G_2 obtained by setting $L'(v) = L(v)$ for all $v \in V(G_2) \setminus \{u\}$ and setting $L'(u) = \{\phi(u)\}$. Note that since $S_2 \subset S$ and S is an acceptable path or cycle, $S_2 \cup u$ is an acceptable path. Since $|V(S_1)| \geq |V(S_2)|$, it follows that $|V(S_2)| \leq 2$ and so that $(G_2, L', S_2 \cup u, A \cap V(G_2))$ is an unexceptional canvas. Thus G_2 admits an L'-colouring ϕ_2. By construction, ϕ_1 and ϕ_2 agree on u, and so $\phi_1 \cup \phi_2$ is an L-colouring of G, a contradiction. □

Since G is 2-connected by Lemma 2.2.2, it follows that C is a cycle. We now show this cycle is chordless.

Lemma 2.2.3. *C is chordless.*

Proof. Suppose not, and let uw be a chord of C. Thus the path uw separates G into two graphs G_1 and G_2. Suppose S is a cycle. Since C has a chord, $|V(C)| \geq 4$. Thus S is a 4-cycle. But then at least one vertex in S has girth at least four by the definition of acceptable cycle. Thus C is chordless—a contradiction. It follows that S is a path.

Suppose now that $S \subseteq G_1$. Note that since K is unexceptional, it follows from Observation 2.2.1 that $K[G_1]$ is also unexceptional. Since K is a minimum counterexample to Theorem 2.1.6 and $|V(G_1)| < |V(G)|$, it follows further that $K[G_1]$ admits an L-colouring ϕ. Let L' be a list assignment for G_2 obtained from L by setting $L'(v) = L(v)$ for all $v \in V(G_2) \setminus \{u, w\}$, and $L'(v) = \{\phi(v)\}$ for $v \in \{u, w\}$. Note that $S \cap V(G_2) \subseteq \{u, w\}$, and so that $K_2 = (G_2, L', uw, A \cap V(G_2))$ is a canvas. Moreover, uw is an acceptable path for G_2 since it contains only two vertices. This also implies that K_2 is unexceptional. Since $|V(G_2)| < |V(G)|$, we have by the minimality of K that G_2 admits an L'-colouring ϕ'. This is a contradiction since $\phi \cup \phi'$ is an L-colouring of G.

Thus $S \not\subseteq G_1$, and symmetrically $S \not\subseteq G_2$. Suppose there is no vertex of S contained in $V(G_2) \setminus V(G_1)$. Then since $S \not\subseteq G_1$, it follows that there is an edge e of S in $E(G_2) \setminus E(G_1)$ but that both endpoints of S are in G_1. Since $G_1 \cap G_2 = uv$, it follows that $e = uv$, a contradiction since $uv \in E(G_1)$. Thus there exists a vertex of S contained in $V(G_1) \setminus V(G_2)$, and symmetrically, a vertex of S contained in $V(G_2) \setminus V(G_1)$. It follows that $|V(S)| \geq 3$, and one of u and w is an internal vertex of the path S. We may assume without loss of generality that w is an internal vertex of S. Note that u is not contained in $V(S)$ since otherwise $|V(S)| = 4$ and S contains a subpath of length 2 containing only vertices of girth 3, contradicting the definition of acceptable path. Let S_1 and S_2 be subpaths of S such that $|V(S_1)| \leq |V(S_2)|$; $S_1 \cup S_2 = S$; and $S_1 \cap S_2 = w$. Note that both S_1 and S_2 are acceptable paths in G, since a subpath of an acceptable path is itself acceptable. We may assume without loss of generality that $S_1 \subseteq G_1$ and $S_2 \subseteq G_2$.

First suppose that $|L(u)| \geq 4$. Since $K[G_1]$ is a subcanvas of K and K is unexceptional, it follows from Observation 2.2.1 that $K[G_1]$ is also unexceptional. By the minimality of K, $K[G_1]$ admits an L-colouring ϕ_1. Let L' be a list assignment for G_1 obtained by setting $L'(v) = L(v)$ for all $v \in V(G_1) \setminus \{u\}$, and setting $L'(u) = L(u) \setminus \{\phi_1(u)\}$. Since $|L(u)| \geq 4$, it follows that $|L'(u)| \geq 3$ and so that $(G_1, L', S_1, A \cap V(G_1))$ is a canvas. Note that since $|V(S_1)| \leq |V(S_2)|$ and $|V(S)| \leq 4$, we have that $|V(S_1)| \leq 2$. Thus $(G_1, L', S_1, A \cap V(G_1))$ is unexceptional. By the minimality of K, there exists an L'-colouring ϕ_2 of $(G_1, L', S_1, A \cap$

$V(G_1)$). Let L'' be a list assignment for G_2 obtained by setting $L''(v) = L(v)$ for all $v \in V(G_2) \setminus \{u\}$, and $L''(u) = \{\phi_1(u), \phi_2(u), \phi_1(w)\}$. Note that since S is an acceptable path and w is an interior vertex of S, either $|V(S_2)| = 2$, or $|V(S_2)| = 3$ and S_2 contains a vertex of girth at least four. In either case, $(G_2, L'', S_2, A \cap V(G_2))$ is an unexceptional canvas and, by the minimality of K, admits an L''-colouring ϕ. Note that $\phi(u) \neq \phi_1(w)$, since ϕ is an L''-colouring. If $\phi(u) = \phi_1(u)$, then $\phi \cup \phi_1$ forms an L-colouring of G. Otherwise, $\phi(u) = \phi_2(u)$ and so $\phi \cup \phi_2$ forms an L-colouring of G. In either case, this is a contradiction.

We may therefore assume that $|L(u)| \leq 3$. For each $i \in \{1, 2\}$, since $K[G_i]$ is a subcanvas of an unexceptional canvas, by Observation 2.2.1 $K[G_i]$ admits an L-colouring φ_i. Let L_i be the list assignment for G_i obtained by setting $L_i(v) = L(v)$ for all $v \in V(G_i) \setminus \{u\}$, and $L_i(u) = \{\varphi_{3-i}(u)\}$. For each $i \in \{1, 2\}$, let S_i' be the path obtained from S_i by adding the edge uw, and let $K_i = (G_i, L_i, S_i', A \cap V(G_i))$. Note that both K_1 and K_2 are canvases. Suppose there exists some $i \in \{1, 2\}$ such that K_i is unexceptional. Then G_i admits an L_i-colouring ψ_i, and $\psi_i \cup \varphi_{3-i}$ together form an L-colouring of G —a contradiction. Note that since $|V(S_1)| \leq |V(S_2)|$, it follows that $|V(S_1')| \leq 3$ and so that K_1 is not an exceptional canvas of type (i) or (ii).

We may therefore assume that K_1 is an exceptional canvas of type (iii). If K_2 is also an exceptional canvas of type (iii), then since $|L(u)| \leq 3$, we have that K is an exceptional canvas of type (iii) —a contradiction. If K_2 is an exceptional canvas of type (i), then since K_1 is an exceptional canvas of type (iii) we have that K is an exceptional canvas of type (ii) —a contradiction. Thus we may assume that K_2 is an exceptional canvas of type (ii) and thus contains a generalized wheel subgraph W with principal path $u_1u_2u_3$, where u_2 and one vertex in $\{u_1, u_3\}$ are contained in S_2', and the remaining vertex in $\{u_1, u_3\}$ forms a chord with u_2 in the outer cycle of G_2. Note that since K_2 is an exceptional canvas of type (ii), we have that $|V(S_2)| = 3$. Recall that every vertex in a generalized wheel has girth three. Suppose uw is not a subpath of $u_1u_2u_3$. Then every vertex in $V(S_2) \setminus \{w\}$ has girth three. Since K_1 is an exceptional canvas of type (iii), every vertex in $V(S_1)$ has girth three. But then S is not an acceptable path, as it contains four vertices of girth three. Thus we may assume that uw is a subpath of $u_1u_2u_3$. But then K is an exceptional canvas of type (ii), a contradiction. □

2.2.3 Separating Cycle Lemmas

In Lemmas 2.2.4-2.2.9, we describe a set of reducible configurations for K: together, Lemmas 2.2.4 and 2.2.6 show that G does not contain a cycle of length at most four with vertices in its interior. Lemmas 2.2.8 and 2.2.9 show that every cycle of length at most six with vertices in its interior is composed of vertices of relatively low girth.

Lemma 2.2.4. *If T is a triangle in G, then* $\mathrm{Int}(T) = \emptyset$.

Proof. Suppose not, and let $T = u_1u_2u_3u_1$ be a counterexample. By the minimality of K, we have that $G - \mathrm{Int}(T)$ admits an L-colouring ϕ. Let G' be the graph obtained from $\mathrm{Int}[T]$ by deleting u_3. Let C' be the outer cycle of G'. Let L' be the list assignment obtained from L by removing the colour $\phi(u_3)$ from the lists of all neighbours of u_3 in $\mathrm{Int}(C)$. Finally, let $L'(u_i) = \{\phi(u_i)\}$ for $i \in \{1, 2\}$. Note that every vertex $v \in V(C') \setminus \{u_1, u_2\}$ with $g(v) \in \{3, 4\}$ has $|L'(v)| \geq 3$, since $|L(v)| \geq 4$. Let A' be the set of vertices $v \in V(C')$ with $|L'(v)| = 2$. By the above, every vertex in A' has girth at least five. Moreover, A' is an independent set, since every vertex in A' is adjacent to u_3 in G. Thus since u_1u_2 is an acceptable path for G' and contains only two vertices, it follows that (G', L', u_1u_2, A') is an unexceptional canvas. Since $|V(G')| < |V(G)|$, by the minimality of K we have that G' has an L'-colouring ϕ'. But $\phi \cup \phi'$ is an L-colouring of G, a contradiction. □

Our colouring arguments will be more straightforward if we assume that for every vertex $v \in V(C) \setminus (A \cup V(S))$, we have $|L(v)| = 3$. Unfortunately, deleting extra colours from lists in $V(C) \setminus (A \cup V(S))$ might result in the creation of an exceptional canvas of type (ii) or (iii). We show below that this does not occur.

Observation 2.2.5. *Every vertex in S has a list of size one. Every vertex in A has a list of size two. Every vertex in $V(C) \setminus (V(S) \cup A)$ has a list of size three.*

Proof. Suppose not, and let $v \in V(G)$ be a counterexample. Since K was chosen to minimize $\sum_{u \in V(G)} |L(u)|$, it follows that $|L(u)| = 1$ for every $u \in V(S)$. Since $A \cap V(S) = \emptyset$, by the definition of canvas every vertex in A has a list of size two. Thus $v \in V(G) \setminus (V(S) \cup A)$, and so $|L(v)| \geq 4$. Since K was chosen to minimize $\sum_{v \in V(G)} |L(v)|$, it follows that the canvas K' obtained from K by deleting a colour from the list of v is an exceptional canvas. Since C is chordless, K' is not an exceptional canvas of type (ii). Since $|L(v)| \geq 4$, we have further that K' is not an exceptional canvas of type (i). Thus K' is an exceptional canvas of type (iii), and so contains a subgraph W that is a generalized wheel with principal path S such that the vertices in the outer cycle of W are in the outer face boundary of G and have lists of size at most 3. Since C is chordless by Lemma 2.2.3, W is a wheel and moreover the outer cycle of W is the outer cycle of G. Since every triangle in G has no vertices in its interior by Lemma 2.2.4, it follows that $W = G$. Since $v \in V(C)$ and W is a wheel, we have further that $\deg(v) = 3$. By the minimality of K, $G - v$ admits an L-colouring ϕ.

But then since $|L(v)| \geq 4$ and $\deg(v) = 3$, it follows that ϕ extends to an L-colouring of G by choosing $\phi(v) \in L(v) \setminus \{\phi(u) : u \in N(v)\}$, a contradiction. □

This observation allows us more easily to argue explicitly about the list assignment L in Subsection 2.2.6 and Section 3.

In a similar vein as Lemma 2.2.4, we have the following.

Lemma 2.2.6. *If T is a 4-cycle in G, then* $\text{Int}(T) = \emptyset$.

Proof. Suppose not, and let $T = u_1u_2u_3u_4u_1$ be a counterexample. By the minimality of K, we have that $G - \text{Int}(T)$ admits an L-colouring ϕ. Let G' be the graph obtained from $\text{Int}[T]$ by deleting u_3 and u_4, and let C' be the boundary walk of the outer face of G'. Let L' be the list assignment obtained from L by removing the colour $\phi(u_i)$ from the lists of all neighbours of u_i in $\text{Int}(T)$, for $i \in \{3, 4\}$. Finally, let $L'(u_j) = \{\phi(u_j)\}$ for $j \in \{1, 2\}$. Note that every vertex $v \in V(C') \setminus \{u_1, u_2\}$ with $g(v) = 3$ has $|L'(v)| \geq 3$, since $|L(v)| \geq 5$. Moreover, every vertex v in $V(C') \setminus \{u_1, u_2\}$ with $g(v) = 4$ has $|L'(v)| \geq 3$, since $|L(v)| \geq 4$ and every such vertex v is adjacent to at most one of u_3 and u_4 in G. Finally, let A' be the set of vertices $v \in V(C') \setminus \{u_1, u_2\}$ with lists of size at most two. By the above, $g(v) \geq 5$ for every $v \in A'$. It follows that very vertex in A' is adjacent to exactly one of u_3 and u_4 in G, and moreover that A' forms an independent set. Thus $|L'(v)| = 2$ for all $v \in A'$, since $|L(v)| \geq 3$ for all $v \in A'$. It follows that $K' = (G', L', u_1u_2, A')$ is a canvas. Since u_1u_2 contains only two vertices, u_1u_2 is an acceptable path and moreover K' is unexceptional. By the minimality of K we have that G' has an L'-colouring ϕ'. By construction, $\phi \cup \phi'$ is an L-colouring of G, a contradiction. □

Note that by Theorems 2.2.4 and 2.2.6, we have immediately that S is not a cycle, and that $V(C) \setminus V(S) \neq \emptyset$. Thus S is a path; for the remainder of the chapter, let $S = v_1v_2 \ldots v_k$, and let $C = v_1 \ldots v_kv_{k+1} \ldots v_qv_1$. As noted in Subsection 2.2, we may assume without loss of generality that $k \geq 2$. Below, we show that in fact $k \geq 3$.

Observation 2.2.7. *We may assume that S contains at least three vertices.*

Proof. Suppose not. Thus $|V(S)| = 2$. First suppose $|V(C)| \leq 4$. Then every vertex in $V(C)$ has girth at most four, and so $A = \emptyset$. By definition of canvas, we have that $|L(v)| \geq 3$ for all $v \in V(C) \setminus V(S)$. Since C is chordless by Lemma 2.2.3, it then follows that $G[V(C)]$ has an L-colouring φ. Since $V(\text{Int}(C)) = \emptyset$ by Lemmas 2.2.4 and 2.2.6 with $T = C$, it follows further that $G = C$ and so that φ is an L-colouring of G, a contradiction.

Thus we may assume that $|V(C)| \geq 5$. Let c be a colour in $L(v_3) \setminus L(v_2)$. Let L' be a list assignment for G defined by $L(v_3) = \{c\}$ and $L'(v) = L(v)$ for all $v \in V(G) \setminus \{v_3\}$. Since S has only two vertices by assumption, $S + v_2v_3$ is an acceptable path. Moreover, $K' = (G, L', S + v_2v_3, A)$ is a canvas. Since K was chosen to minimize $\sum_{v \in V(G)} |L(v)|$, it follows that K' is an exceptional canvas; and since $|V(S + v_2v_3)| = 3$, K' is an exceptional canvas of type (iii), and thus contains a subgraph W that is a generalized wheel with principal path $S + v_2v_3$ such that the vertices on the outer cycle of W are on the outer cycle of G and have lists of size at most three under L'. Since C is chordless by Lemma 2.2.3 and $|V(C)| \geq 5$, it follows that W is a wheel and moreover that the outer cycle of W is the outer cycle of G. Since every triangle in G has no vertices in its interior by Lemma 2.2.4, we have that $W = G$. Let w be the unique vertex not on the outer face boundary of G. Note that $N(v_3) = \{v_2, v_4, w\}$.

Since v_3 is in a wheel, $g(v_3) = 3$ and so $|L(v_3)| = 3$. Let $X \subseteq L(v_3) \setminus L(v_2)$ be a set of size two, and let L' be a list assignment for $G - v_3$ obtained by setting $L'(v) = L(v)$ for all $v \in V(G - v_3) \setminus \{w\}$ and $L'(w) = L(w) \setminus X$. Note that since $w \in V(\text{Int}(C))$ and $g(w) = 3$, it follows that $|L(w)| \geq 5$ and so that $|L'(w)| \geq 3$. Thus $K'' = (G - v_3, L', S, A)$ is a canvas. Since $|V(S)| = 2$, it is moreover unexceptional. By the minimality of K, we have that K'' admits an L'-colouring ϕ. But ϕ extends to an L-colouring of G by setting $\phi(v_3) \in X \setminus \{\phi(v_4)\}$, a contradiction. □

The following two lemmas show that only certain types of 5- and 6-cycles with vertices in their interior exist in G.

Lemma 2.2.8. *If P is a 5-cycle in G and $\text{Int}(P)$ contains a vertex, then every vertex in $V(P)$ has girth three.*

Proof. Suppose not, and let $P \subseteq G$ be a counterexample with $P = u_1u_2u_3u_4u_5u_1$. Suppose without loss of generality that $g(u_3) \geq 4$. By the minimality of K, we have that $G - \text{Int}(P)$ admits an L-colouring ϕ. Let G' be the graph obtained from $\text{Int}[P]$ by deleting the vertices u_4 and u_5, and let C' be the outer face boundary of G'. Let L' be the list assignment obtained from L by removing the colour $\phi(u_i)$ from the lists of all neighbours of u_i in $\text{Int}(P)$ for $i \in \{4, 5\}$, and setting $L'(u_j) = \{\phi(u_j)\}$ for $j \in \{1, 2, 3\}$. Note that every vertex $v \in V(C') \setminus \{u_1, u_2, u_3\}$ with $g(v) \leq 4$ has $|L'(v)| \geq 3$, since vertices in $\text{Int}(C)$ of girth three have $|L(v)| \geq 5$, and vertices in $\text{Int}(C)$ of girth four have $|L(v)| \geq 4$ and are adjacent to at most one of u_4 and u_5. Finally, let A' be the set of vertices $v \in V(C') \setminus \{u_1, u_2, u_3\}$ with lists of size at most two under L'. By the above, $g(v) \geq 5$ for every $v \in A'$. It follows that that A' forms an independent set, and moreover that each vertex in A' is adjacent to at most one of u_4 and u_5. Thus $|L'(v)| = 2$ for each $v \in A'$. Note that $u_1u_2u_3$ is an acceptable path for G', as it has only three vertices. By the minimality of K, we have that $(G', L', u_1u_2u_3, A')$ is a canvas. Since $g(u_3) \geq 4$, it is unexceptional and thus G' has an L'-colouring ϕ'. By construction, $\phi \cup \phi'$ is an L-colouring of G, a contradiction. □

We end this subsection with the following lemma, characterizing the separating cycles of length six.

Lemma 2.2.9. *If H is a 6-cycle in G and $\text{Int}(H)$ contains a vertex, then $V(H)$ does not contain a vertex of girth at least five.*

Proof. Suppose not, and let $H \subseteq G$ be a counterexample with $H = u_1u_2 \cdots u_6u_1$. Suppose without loss of generality that $g(u_3) \geq 5$. By the minimality of K, we have that $G - \text{Int}(H)$ admits an L-colouring ϕ. The argument proceeds in a way similar to that of Lemmas 2.2.4-2.2.8: we aim to delete a path of vertices in H and argue about the resulting graph. Which vertices we delete from H will depend on the structure of $\text{Int}(H)$.

First, suppose that there is no vertex in $V(\text{Int}(H))$ adjacent to u_4, u_5, and u_6; that there is no vertex w in $V(\text{Int}(H))$ with $g(w) = 4$ such that w is adjacent to both u_4 and u_6; and that there do not exist vertices w_1, w_2 in $V(\text{Int}(H))$ such that $g(w_1) \geq 5$ and $g(w_2) \geq 5$ and $\{w_1u_4, w_2u_6, w_1w_2\} \subseteq E(G)$. In this case, let G' be the graph obtained from $\text{Int}[H]$ by deleting vertices u_4, u_5 and u_6, and let C' be the outer cycle of G'. Let L' be the list assignment obtained from L by removing the colour $\phi(u_i)$ from the lists of all neighbours of u_i in $\text{Int}(H)$ for $i \in \{4, 5, 6\}$. Let $L'(u_i) = \{\phi(u_i)\}$ for $i \in \{1, 2, 3\}$. Note that every vertex $v \in V(C') \setminus \{u_1, u_2, u_3\}$ with $g(v) = 3$ has $|L'(v)| \geq 3$, since vertices in $\text{Int}(C)$ of girth three have $|L(v)| \geq 5$ and are adjacent to at most two of u_4, u_5, and u_6 by assumption. Moreover, every vertex $v \in V(C') \setminus \{u_1, u_2, u_3\}$ with $g(v) = 4$ has $|L'(v)| \geq 3$ since vertices in $\text{Int}(C)$ of girth four have $|L(v)| \geq 4$ and are adjacent to at most one of u_4, u_5, and u_6 by assumption. Finally, let A' be the set of vertices $v \in V(C') \setminus \{u_1, u_2, u_3\}$ with lists of size at most two under L'. By the above, $g(v) \geq 5$ for every $v \in A'$. It follows that every vertex in A' is adjacent to exactly one of u_4, u_5, and u_6, and so that $|L'(v)| = 2$ for each $v \in A'$ (since $|L(v)| \geq 3$). Finally, we note that A' forms an independent set by assumption. Thus $(G', L', u_1u_2u_3, A')$ is a canvas. Since $u_1u_2u_3$ contains only three vertices, it is an acceptable path; and since $g(u_3) \geq 5$, we have that $(G', L', u_1u_2u_3, A')$ is unexceptional. By the minimality of K, we have further that G'admits an L'-colouring ϕ'. By construction, $\phi \cup \phi'$ is an L-colouring of G, a contradiction.

Thus we may assume that there is a vertex in $V(\text{Int}(H))$ adjacent to u_4, u_5, and u_6; or that there is a vertex w in $V(\text{Int}(H))$ with $g(w) = 4$ such that w is adjacent to both u_4 and u_6; or finally that there exist vertices w_1, w_2 in $V(\text{Int}(H))$ such that $g(w_1) \geq 5$ and

$g(w_2) \geq 5$ and $\{w_1u_4, w_2u_6, w_1w_2\} \subseteq E(G)$. We break into cases depending on which of these occur.

Case 1: Either there is a vertex w in $V(\text{Int}(H))$ with $g(v) = 4$ such that w is adjacent to both u_4 and u_6, or there exist vertices w_1, w_2 in $V(\text{Int}(H))$ such that $g(w_1) \geq 5$ and $g(w_2) \geq 5$ and $\{w_1u_4, w_2u_6, w_1w_2\} \subseteq E(G)$. Note that by the planarity of G and by Lemmas 2.2.6-2.2.8, it follows that there does not exist a vertex in $V(\text{Int}(H))$ adjacent to u_5. In this case, let G' be the graph obtained from $\text{Int}[H]$ by deleting u_1, u_5, and u_6, and let C' be the outer cycle of G'. Let L' be the list assignment obtained from L by removing the colour $\phi(u_i)$ from the lists of all neighbours of u_i in $\text{Int}(H)$ for $i \in \{1, 5, 6\}$. Let $L'(u_i) = \{\phi(u_i)\}$ for $i \in \{2, 3, 4\}$. Note that since no vertex in $\text{Int}(H)$ is adjacent to u_5, there does not exist a vertex w in $\text{Int}(H)$ adjacent to u_1, u_5, and u_6. Similarly, there does not exist a vertex w in $\text{Int}(H)$ with $g(w) = 4$ such that w is adjacent to u_1 and u_5. Finally, there do not exist vertices w_1, w_2 in $\text{Int}(H)$ such that $g(w_1) \geq 5$ and $g(w_2) \geq 5$ and $\{w_1u_1, w_2u_5, w_1w_2\} \subseteq E(G)$. Let A' be the set of vertices $v \in V(C')$ with $g(v) \geq 5$ and $|L'(v)| = 2$. It follows as in the previous cases that A' is an independent set, and moreover that $(G', L', u_2u_3u_4, A')$ is a canvas. Since $g(u_3) \geq 5$, it is unexceptional. By the minimality of K, we have that G' admits an L'-colouring ϕ. By construction, $\phi \cup \phi'$ is an L-colouring of G, a contradiction.

Case 2: there is a vertex w in $\text{Int}(H)$ adjacent to u_4, u_5, and u_6. First suppose w is not adjacent to u_1. In this case, as in Case 1 we delete u_1, u_5, and u_6. The argument is nearly identical as that for Case 1, with one caveat: w is adjacent to u_5. However, by Lemma 2.2.4, it is the only vertex in $\text{Int}(H)$ adjacent to u_5, and since $g(w) = 3$, we have that $|L(w)| \geq 5$. This ensures $|L'(w)| \geq 3$ in the argument for Case 1. We may thus assume w is adjacent to u_1. In this case, by Lemmas 2.2.4 and 2.2.8 (noting $g(u_3) \geq 5$), it follows that w is the only vertex in the interior of H. By the minimality of K, we have that $G - w$ admits an L-colouring ϕ. Since $g(u_3) \geq 5$, we have that w is adjacent to only u_1, u_6, u_5, and u_4 on H. As $|L(w)| \geq 5$, it follows that ϕ extends to an L-colouring of G, a contradiction. □

2.2.4 Separating Path Lemmas

The proofs of many of our lemmas take the following basic shape: we colour and delete vertices in $V(C) \setminus V(S)$, modifying lists where appropriate, and argue about the structure of the resulting canvas (G', L', S', A'). The following lemma shows that in doing this, as long as the set of precoloured vertices does not change (i.e. as long as $S' = S$), we do not create an exceptional canvas of type (i). Moreover, it shows that there are no edges between vertices in A' and vertices in A. We will use this lemma extensively throughout the remainder of the chapter.

Lemma 2.2.10. *If v is a vertex in $V(\text{Int}(C))$ that is adjacent to two vertices u, w in $(V(C) \setminus V(S)) \cup \{v_1, v_k\}$, then $g(x) = 3$ for every $x \in \{u, v, w\}$.*

Proof. Suppose not. The path uvw separates G into two graphs G_1 and G_2 (note that since $g(x) \geq 4$ for at least one $x \in \{u, v, w\}$, it follows that for each $i \in \{1, 2\}$, we have that $V(G_i) \setminus V(G_{3-i}) \neq \emptyset$). Since neither u nor w is an internal vertex of the path S, we may assume without loss of generality that $S \subseteq G_1$.

Note that since K is unexceptional, it follows from Observation 2.2.1 that $K[G_1]$ is an unexceptional canvas. By the minimality of K, we have that G_1 admits an L-colouring ϕ. Let L' be a list assignment for G_2 defined by $L'(x) = \phi(x)$ for $x \in \{u, v, w\}$, and $L'(v) = L(v)$ for $x \in V(G_2) \setminus \{u, v, w\}$. Since uvw has only three vertices, it is an acceptable path for G_2. Note that $(G_2, L', uvw, A \cap V(G_2))$ is a canvas, and since $g(x) \geq 4$ for at least one vertex $x \in \{u, v, w\}$, it is moreover unexceptional. Since $|V(G_2)| < |V(G)|$ and K is a minimum counterexample to Theorem 2.1.6, G_2 admits an L'-colouring ϕ'. But then $\phi \cup \phi'$ is an L-colouring of G, a contradiction. □

In a similar spirit, we have the following lemma which partially describes the structure of G surrounding separating paths of length three whose inner vertices have girth at least five.

Lemma 2.2.11. *If v and w are vertices in $V(\text{Int}(C))$ such that:*

- *$g(v) \geq 5$ and $g(w) \geq 5$,*
- *$vw \in E(G)$,*
- *v is adjacent to a vertex $v' \in (V(C) \setminus V(S)) \cup \{v_1, v_k\}$, and*
- *w is adjacent to a vertex $w' \in (V(C) \setminus V(S)) \cup \{v_1, v_k\}$,*

then there exists a vertex $u \in A$ that is adjacent to both v' and w'.

Proof. The proof is nearly identical to that of Lemma 2.2.10: suppose not. Note that $w' \neq v'$ and $w'v' \notin E(G)$ since $g(v) \geq 5$. Thus the path $w'wvv'$ separates G into two graphs G_1 and G_2. Since neither w' nor v' is an internal vertex of the path S, we may assume without loss of generality that $S \subseteq G_1$.

Since K is unexceptional, by Observation 2.2.1 so too is $K[G_1]$. By the minimality of K, we have that G_1 admits an L-colouring ϕ. Let L' be a list assignment for G_2 defined by $L'(x) = \phi(x)$ for $x \in \{w', w, v, v'\}$, and $L'(x) = L(x)$ for $x \in V(G_2) \setminus \{w', w, v, v'\}$. Note that since $g(x) \geq 5$ for both $x \in \{v, w\}$, it follows that $w'wvv'$ is an acceptable path for G_2. Note that $K_2 = (G_2, L', w'wvv', A \cap V(G_2))$ is a canvas. If K_2 is an exceptional canvas of type (i), then there is a vertex $u \in A$ adjacent to both v' and w', a contradiction. Moreover, K_2 is not an exceptional canvas of type (ii) since both w and v have girth at least five and every vertex in a generalized wheel has girth three. Finally, K_2 is not an exceptional canvas of type (iii) since its precoloured path has four vertices. Since $|V(G_2)| < |V(G)|$ and K is a minimum counterexample to Theorem 2.1.6, G_2 admits an L'-colouring ϕ'. But then $\phi \cup \phi'$ is an L-colouring of G, a contradiction. □

2.2.5 Broken Wheel Lemmas

The next lemma restricts the set of possible generalized wheels contained in G.

Lemma 2.2.12. *Suppose u is a vertex in $\text{Int}(C)$ that is adjacent to two distinct vertices v_i and v_j on C, where $\{i, j\} \cap \{2, \ldots, k-1\} = \emptyset$ (so that neither v_i nor v_j is an internal vertex of the path S). Let Q be the path in C with endpoints v_i and v_j containing no edges of S. Then G contains a subgraph W which is a broken wheel with principal path v_iuv_j such that Q is in the outer face boundary of W.*

Proof. Suppose not. Note that by Lemma 2.2.10, v_i, u, and v_j all have girth three. The path v_iuv_j separates G into two subgraphs G_1 and G_2. (By Observation 2.2.7, S contains at least three vertices and so it follows that $V(G_i) \setminus V(G_{3-i}) \neq \emptyset$ for each $i \in \{1, 2\}$.) Since neither v_i nor v_j is an internal vertex of the path S, we may assume without loss of generality that $S \subseteq G_1$. By Observation 2.2.1, $K[G_1]$ is an unexceptional canvas. By the minimality of K, it follows that G_1 admits an L-colouring ϕ. By assumption, G_2 does not contain a subgraph W that is a broken wheel with principal path v_iuv_j such that the vertices on the outer cycle of W are on the outer cycle of G_2. Suppose G_2 does not contain a generalized wheel W with principal path v_iuv_j such that the vertices on the outer cycle of W are in the outer cycle of G_2. Let L' be a list assignment for G_2 obtained from L by setting $L'(v) = \{\phi(v)\}$ for $v \in \{v_i, u, v_j\}$, and $L'(v) = L(v)$ otherwise. Then $K_2 = (G_2, L', v_iuv_j, A \cap V(G_2))$ is a canvas. Note that since v_iuv_j has only three vertices, it is an acceptable path. If K_2 is unexceptional, then G_2 admits an L'-colouring ϕ'; but then $\phi \cup \phi'$ forms an L-colouring of G, a contradiction. Thus K_2 is exceptional, and since $v_i u v_j$ has only three vertices, K_2 is an exceptional canvas of type (iii).

It follows that G_2 contains a subgraph W that is a generalized wheel but not a broken wheel such that the vertices on the outer cycle of W are on the outer cycle of G_2. By Lemma 2.2.3, every chord of the outer cycle of G_2 has u as an endpoint. Thus all edges on the outer cycle of W are on the outer cycle of G_2. By Lemma 2.2.4, every triangle in G_2 has no vertices in its interior. It follows that $W = G_2$, and so that G_2 is a near-triangulation. By Theorem 2.1.10, there is at most one colouring of $v_i u v_j$, say φ, that does not extend to a colouring of G_2. Let $L^\star(u) = L(u) \setminus \varphi(u)$, and $L^\star(v) = L(v)$ for all $v \in V(G) \setminus \{u\}$. Note that since $g(u) = 3$ and $u \in V(\text{Int}(C))$, we have that $|L(u)| \geq 5$. It follows that $|L^\star(u)| \geq 4$.

Since $K[G_1]$ is unexceptional and $|L^\star(u)| \geq 4$, it follows that $(G_1, L^\star, S, A \cap V(G_1))$ is an unexceptional canvas. By the minimality of K, we have that G_1 admits an $L^\star$-colouring $\varphi^\star$. Let $L^{\star\star}$ be the list assignment for G_2 obtained by setting $L^{\star\star}(v) = \{\varphi^\star(v)\}$ for all $v \in \{v_i, u, v_j\}$ and $L^{\star\star}(v) = L(v)$ otherwise. By Theorem 2.1.10 and the fact that $\varphi^\star(u) \neq \varphi(u)$, we have that $(G_2, L^{\star\star}, v_i u v_j, A \cap V(G_2))$ admits an $L^{\star\star}$-colouring $\varphi^{\star\star}$. As $\varphi^\star \cup \varphi^{\star\star}$ is an L-colouring of G, this is a contradiction.

□

The following lemma provides some insight into the structure surrounding the broken wheels described by Lemma 2.2.12. It will be useful in bounding the size of certain broken wheels in G.

Lemma 2.2.13. *Suppose $W \subseteq G$ is a broken wheel with outer cycle $w w_1 w_2 \dots w_t w$ and principal path $w_1 w w_t$ such that $w \in V(\text{Int}(C))$; $V(W) \setminus \{w\} \subseteq (V(C) \setminus V(S)) \cup \{v_1, v_k\}$; and $t \geq 3$. Let $1 \leq j \leq t-2$, and let G' be the graph obtained from G by identifying w_j and w_{j+2} to a new vertex z and deleting w_{j+1}. Let $x \in V(G') \setminus \{z\}$. The following both hold.*

1. *If $g_G(x) \geq 5$, then $g_{G'}(x) \geq 5$.*
2. *If $g_G(x) = 4$, then $g_{G'}(x) \geq 4$.*

Proof. Suppose not. Then $g_G(x) \geq 4$, and $g_{G'}(x) < g_G(x)$. Moreover, $x \notin V(W)$, since every vertex in a broken wheel has girth three. By Lemma 2.2.3, every chord of the outer cycle of W has w as an endpoint. Thus all edges on the outer cycle of W other than ww_1 and ww_t are in $E(C)$. Moreover, every triangle in G —and so in particular, every triangle in W —has no vertex in its interior by Lemma 2.2.4. Let H' be a smallest cycle in G' containing x. Note that H' is induced. Since $g_{G'}(x) < g_G(x)$, it follows that $z \in V(H')$. First suppose that $w \in V(H')$. Then since $z \in V(H')$ and H' is an induced cycle it follows that $zw \in E(H')$. Let $w' \neq w$ be a neighbour of z in H'. Since z is the identification of w_j and w_{j+2}, it follows that G contains a cycle P obtained from H' by replacing the path wzw' with one of ww_jw' and $ww_{j+2}w'$. This is a contradiction, since then $|V(P)| = |V(H')|$, but by assumption $|V(H')| < g_G(x)$.

Thus we may assume that $w \notin V(H')$. It follows that $\{w_1, \ldots, w_t\} \setminus \{w_j, w_{j+1}w_{j+2}\} \subseteq V(H')$. Let H be the cycle in G obtained from H' by replacing z by the path $w_jw_{j+1}w_{j+2}$. Since every triangle in W has no vertices in its interior and $w \notin V(H)$, it follows that $w \in V(\mathrm{Int}(H))$. Note that $|V(H')| \geq 3$, and so $|V(H)| \geq 5$. If $|V(H)| = 5$, this contradicts Lemma 2.2.8 since $g_G(x) \geq 4$ by assumption and $x \in V(H)$. Thus $|V(H)| \geq 6$, and so that that $|V(H')| \geq 4$. If $|V(H)| \geq 7$, then $|V(H')| \geq 5$ and so $g_{G'}(x) \geq 5$, a contradiction. Thus we may assume that $|V(H)| = 6$, and so that $|V(H')| = 4$. It follows that $g_G(x) \geq 5$. But then this contradicts Lemma 2.2.9, since $x \in V(H)$.

□

We are now equipped to bound the size of certain broken wheels in G.

Lemma 2.2.14. *Suppose that $W \subseteq G$ is a broken wheel with outer cycle $ww_1w_2 \cdots w_tw$ and principal path w_1ww_t such that $w \in V(\mathrm{Int}(C))$; $V(W) \setminus \{w\} \subseteq V(C) \setminus V(S)$; and $t \geq 3$. There does not exist an index $1 \leq j \leq t-2$ such that $L(w_j) = L(w_{j+2})$.*

Proof. Suppose not. Let j be an index with $1 \leq j \leq t-2$ and $L(w_j) = L(w_{j+2})$. By Lemma 2.2.3, every chord of the outer cycle of W has w as an endpoint. Thus every edge on the outer cyce of W other than ww_1 and ww_t is in $E(C)$. Moreover, note that every triangle in W has no vertices in its interior by Lemma 2.2.4. Let G' be the graph obtained from G by identifying w_j and w_{j+2} to a new vertex z and deleting w_{j+1}. Note that G' is planar, and inherits the embedding of G. Set $L(z) = L(w_j)$. Since w_j is in a broken wheel in G, $g_G(w_j) = 3$. Since $w_j \notin V(S)$, it follows from Observation 2.2.5 that $|L(w_j)| = |L(z)| = 3$. (Similarly, $|L(w_{j+1})| = |L(w_{j+2})| = 3$.) By Lemma 2.2.13, we have that for all $x \in V(G') \setminus \{z\}$, if $g_G(x) \geq 5$, then $g_{G'}(x) \geq 5$; and that if $g_G(x) = 4$, then $g_{G'}(x) = 4$.

It follows from this that $K' = (G', L, S, A)$ is a canvas. Moreover, it follows from Lemma 2.2.13 that S is an acceptable path in G'. If K' is unexceptional, then since K is a minimum counterexample it follows that G' admits an L-colouring ϕ. This is a contradiction, since ϕ extends to an L-colouring of G by setting $\phi(w_j) = \phi(w_{j+2}) = \phi(z)$, and choosing $\phi(w_{j+1}) \in L(w_{j+1}) \setminus \{\phi(w_j), \phi(w)\}$. (Since $g(w_{j+1}) = 3$, it follows that $|L(w_{j+1})| = 3$ and so that w_{j+1} receives a colour.)

Thus K' is an exceptional canvas. Since C is chordless by Lemma 2.2.3, it follows that K' is not an exceptional canvas of type (i) or (ii). Thus we may assume that K' is an exceptional canvas of type (iii), and so that G' contains a generalized wheel W' with principal path S such that the vertices on the outer cycle of W' are on the outer cycle of G'. Again because C is chordless it follows that the outer cycle of W' is the outer cycle of G', and that the outer cycle of G' is also chordless. Since the outer cycle of every generalized wheel that is not a wheel or a triangle has a chord, we have that W' is a wheel. It follows that every triangle in G' corresponds to a triangle in G (replacing z by w_j or w_{j+2} where appropriate). By Lemma 2.2.4, we have that $W' = G'$; and since $w \in V(\text{Int}(C))$, it follows further that w is the only vertex in W' not in the outer cycle of G'. But then G too is a wheel with principal path S, and since $|L(w_{j+1})| = 3$ by Observation 2.2.5, it follows that K is an exceptional canvas of type (iii) —a contradiction. □

To close this subsection, we give one last lemma: it restricts still further the set of broken wheel subgraphs in G. It is used in Section 3 to argue concisely that upon performing our main reductions (colouring and deleting a subset of the vertices of G, and adjusting lists where appropriate), what remains is not an exceptional canvas of type (iii).

Lemma 2.2.15. *If S has length two and contains only vertices of girth three and $|V(C)| \geq 5$, then $V(\text{Int}(C))$ does not contain a vertex w adjacent to v_3, v_4, and v_5.*

Proof. Suppose not. By Lemma 2.2.4, triangles in G have no vertices in their interior, and so $\text{Int}(wv_3v_4w) = \emptyset$ and $\text{Int}(wv_4v_5w) = \emptyset$. Thus $N(v_4) = \{v_3, v_5, w\}$. Since v_4 has girth three, it is not contained in A. Thus by Observation 2.2.5, we have that $|L(v_3)| = 1$ and $|L(v_4)| = 3$. We claim $L(v_3) \subset L(v_4)$. To see this, suppose not. Let G^* be the graph obtained from G by deleting v_3, and let L^* be a list assignment for G^* obtained by setting $L^*(v) = L(v)$ for all $v \in V(G) \setminus N_G(v_3)$, and $L^*(v) = L(v) \setminus L(v_3)$ for all $v \in N_G(v_3)$. Let C^* be the graph whose vertex- and edge-set are precisely those of the outer face boundary of G^*. Let A^* be the set of vertices with lists of size two under L^*. We now show that $K^* = (G^*, L^*, S - v_3, A^*)$ is a canvas. To see this, note that $|L^*(v_4)| = |L(v_4)|$ by assumption, and that every vertex $v \in V(C^*) \setminus V(C)$ satisfies $|L^*(v)| \geq |L(v)| - 1$. Thus every vertex v in $A^* \setminus A$ has $|L(v)| = 3$ and hence has girth at least 5 in G. Since every vertex in $A^* \setminus A$ is adjacent to v_3, follows from Lemma 2.2.10 that A^* is an independent set. Finally, $S - v_3$ has only two vertices; hence $S - v_3$ is an acceptable path in G^* and K^* is unexceptional. By the minimality of K there is an L-colouring φ of K^* which extends to an L-colouring of K by setting $\varphi(v_3) \in L(v_3)$, a contradiction. This proves the claim.

Thus $L(v_3) \subset L(v_4)$. By assumption, $g(v_4) = 3$ and so $|L(v_4)| = 3$. Without loss of generality we may assume that $L(v_3) = \{1\}$ and that $L(v_4) = \{1, 2, 3\}$. Let L' be the list assignment obtained from L by setting $L'(w) = L(w) \setminus \{2, 3\}$, and $L'(v) = L(v)$ for all $v \in V(G) \setminus \{w\}$. Let C' be the graph with vertex- and edge-set precisely those of the outer face boundary walk of $G - v_4$. (Thus C' is the cycle obtained from C by replacing the path $v_3v_4v_5$ by the path v_3wv_5.) Note that since $w \in V(\text{Int}(C))$ and $g(w) = 3$, it follows that $|L(w)| \geq 5$ and so that $|L'(w)| \geq 3$. Thus $K' = (G - v_4, L', S, A)$ is a canvas.

If K' is unexceptional, then by the minimality of K we have that $G - v_4$ admits an L'-colouring ϕ which extends to an L-colouring of G by choosing $\phi(v_4) \in L(v_4) \setminus \{\phi(v_3), \phi(v_5)\}$, a contradiction.

Thus we may assume that K' is an exceptional canvas. Since S is a path of length two, we have further that K' is an exceptional canvas of type (iii). Thus $G - v_4$ contains a subgraph W that is a generalized wheel with principal path S such that the vertices on the outer cycle of W are on C'. Since $V(C') \setminus \{w\} \subset V(C)$ and K is unexceptional, we have that $w \in V(W)$ and moreover that w is in the outer cycle of W. Let u be the neighbour of w in the outer cycle W with $u \neq v_3$. Note that $u \in V(C)$. Thus the path v_3wu separates G into two graphs G_1 and G_2 where without loss of generality $S \subseteq G_1$, and since C is chordless by Lemma 2.2.3, the outer cycle of G_1 is the outer cycle of W. By Lemma 2.2.4, every triangle in W (and thus in G_1) has no vertex in its interior and so G_1 is a near-triangulation. By Lemma 2.2.12, w is adjacent to every vertex in the subpath of C with endpoints v_3 and u containing no edges of S and hence by Lemma 2.2.4, G_2 is a near-triangulation. Thus G is a near-triangulation. Since K is not exceptional, we have by Theorem 2.1.8 that G admits an L-colouring, a contradiction.

□

2.2.6 Towards a Deletable Path

In this subsection, we give several lemmas necessary for establishing the existence of our main reducible configuration, called a *deletable path*. We end this subsection with the precise definition of the path. Recall that $S = v_1v_2 \dots v_k$, and that $C = v_1v_2 \cdots v_k \cdots v_qv_1$. Moreover, we have assumed that $G[V(S)]$ has an L-colouring. By Lemmas 2.2.4 and 2.2.6 and the fact that $k \leq 4$ by definition, it follows that S is not a cycle: that is, $q \neq k$. Lemma 2.2.16 shows that in fact C contains several non-S vertices: in particular, that $q \geq k + 3$. In Lemma 2.2.17, we show that if $v_{k+1} \in A$, then in fact $q \geq k + 4$. Lemma 2.2.20 partially describes the list assignment of v_{k+1}, v_{k+2}, and v_{k+3}. Corollary 2.2.21

restricts $A \cap \{v_{k+1}, v_{k+2}, v_{k+3}, v_{k+4}\}$ and is used in showing that G contains a deletable path, formally defined in Definition 2.2.22.

Lemma 2.2.16. *$q \geq k+3$.*

Proof. Suppose not. As noted above, $k \neq q$. We claim that $G[V(C)]$ admits an L-colouring. To see this, suppose that $V(C) \setminus V(S)$ contains only vertices with lists of size two. Then by definition $V(C) \setminus V(S) \subseteq A$ —and since A is an independent set in G, it follows that $|V(S)| = 4$, and A contains only a single vertex that is adjacent to both v_1 and v_4. This is a contradiction, since K is not an exceptional canvas of type (i). It follows that $V(C) \setminus V(S)$ contains a vertex with a list of size three. Since C is chordless by Lemma 2.2.3, we have then that $G[V(C)]$ is L-colourable. Therefore since G does not admit an L-colouring, $V(\text{Int}(C)) \neq \emptyset$.

First assume $q = k+1$. Note that $k = 4$ since otherwise one of Lemmas 2.2.4 and 2.2.6 gives a contradiction. By Lemma 2.2.8, every vertex in C (and so in particular, every vertex in $V(S)$) has girth three. But then S is not an acceptable path, a contradiction.

Next, assume $q = k+2$. Similar to the previous case, if $k = 1$ then C contradicts Lemma 2.2.4. If $k = 2$, then C is a 4-cycle with a vertex in its interior, contradicting Lemma 2.2.6. If $k = 3$, then since C is a 5-cycle with a vertex in its interior, by Lemma 2.2.8 every vertex in S has girth three. Note that since C is chordless by Lemma 2.2.3, C admits an L-colouring. Let ϕ be an L-colouring of C, and let G' be the graph obtained from G by deleting v_4 and v_5. Let C' be the graph whose vertex- and edge-set are precisely those of the outer face boundary of G'. Let L' be the list assignment for G' obtained from L by setting $L'(v) = \{\phi(v)\}$ for every $v \in V(S)$ and $L'(v) = L(v) \setminus \{\phi(x) : x \in N_G(v) \cap \{v_4, v_5\}\}$ for all $v \in V(G') \setminus V(S)$.

Let A' be the set of vertices in $V(C') \setminus V(S)$ with lists of size at most two under L'. Note that every vertex $v \in V(C') \setminus V(S)$ of girth at most four has $|L'(v)| \geq 3$, since vertices in $\text{Int}(C)$ of girth three had lists of size at least five and lost at most two colours, and vertices in $\text{Int}(C)$ of girth four had lists of size at least four and lost at most one colour. Thus the vertices in A' all have girth at least five, and so A' is an independent set. Note no vertex in A' is adjacent to both v_4 and v_5 in G for this reason: thus $|L'(v)| = 2$ for all $v \in A'$. Furthermore, (G', L', S, A') is a canvas and does not admit an L'-colouring ϕ', as otherwise $\phi \cup \phi'$ forms an L-colouring of G, a contradiction. By the minimality of K, it follows that (G', L', S, A') is an exceptional canvas of type (iii): in other words, G' contains a generalized wheel W with principal path S such that the vertices on the outer cycle of W are on the outer face boundary of G', and no vertex in the outer cycle of W has a list of size more than three. By the definition of generalized wheel, all vertices in the outer face boundary of W have girth three. It follows that all vertices in the outer cycle of W that are not in $V(C)$ are in $\text{Int}(C)$, have girth three, and (as they have lists of size at most three under L') are adjacent to both v_4 and v_5. Since triangles in G have no vertices in their interiors by Lemma 2.2.4, it follows that there exists exactly one vertex $u \in V(\text{Int}(C))$ that is adjacent to both u_4 and u_5, and therefore only u is in the outer cycle of W and not in $V(C)$. Thus u is adjacent v_1, v_3, v_4 and v_5 in G. Since 4-cycles in G have no vertices in their interiors by Lemma 2.2.6 and $g(v_2) = 3$ since v_2 is in a generalized wheel W, it follows that u is also adjacent to v_2. But then G is a wheel, and K is an exceptional canvas of type (iii) —a contradiction.

Thus we may assume $k = 4$, and so that $q = 6$. In this case, let ϕ be an L-colouring of C. Let G' be the graph obtained from G by deleting v_6 and v_5, and let C' be the outer cycle of G'. Let L' be the list assignment for G' obtained from L by setting $L'(v) = \{\phi(v)\}$ for every $v \in V(S)$ and $L'(v) = L(v) \setminus \{\phi(x) : x \in N_G(v) \cap \{v_5, v_6\}\}$ for all $v \in V(G') \setminus V(S)$.

Note that every vertex $v \in V(C') \setminus V(S)$ of girth at most four has $|L'(v)| \geq 3$: to see this, note that if v has girth three in G, then $|L(v)| \geq 5$; similarly, if v has girth four in G, then $|L(v)| \geq 4$ and v is adjacent to at most one of v_5 and v_6 in G.

Let $A' \subseteq V(C') \setminus V(S)$ be the set of vertices v with $|L'(v)| \leq 2$. Note that A' contains only vertices of girth at least five by the foregoing paragraph. It follows that A' is an independent set, since every vertex in A' is adjacent to one of v_5 and v_6 in G. Furthermore, since every vertex in A' has girth at least five, $|L'(v)| = 2$ for all $v \in A'$. Thus (G', L', S, A') is a canvas. Finally, no vertex in A' is adjacent to either of v_1 or v_4, again because every vertex in A' has girth at least five and is adjacent to one of v_5 and v_6. Thus (G', L', S, A') is not exceptional of type (i) or (ii), and since $|V(S)| = 4$, (G', L', S, A') is not exceptional of type (iii). By the minimality of K, it follows that G' admits an L'-colouring ϕ'. By construction, $\phi' \cup \phi$ is an L-colouring of G, a contradiction. □

If $v_{k+1} \in A$, then Lemma 2.2.16 can be strengthened. This is shown below.

Lemma 2.2.17. *If $v_{k+1} \in A$, then $q \neq k + 3$.*

Proof. Suppose not. Note that we may assume $|V(S)| = 4$, as otherwise since $g(v_{k+1}) \geq 5$, we obtain a contradiction to one of Lemmas 2.2.6-2.2.9. We break into cases, depending on whether or not v_{k+3} is in A.

Case 1: v_{k+3} is in A. In this case, let ϕ be a colouring of C. Note that ϕ exists, since C is chordless by Lemma 2.2.3 and $v_{k+2} \notin A$. Thus $V(\text{Int}(C)) \neq \emptyset$. Let G' be the graph obtained from G by deleting v_{k+1}, v_{k+2}, and v_{k+3}. Let L' be the list assignment for G'

obtained from L by setting $L'(v) = \{\phi(v)\}$ for every $v \in V(S)$ and $L'(v) = L(v) \setminus \{\phi(x) : x \in N_G(v) \cap \{v_{k+1}, v_{k+2}, v_{k+3}\}\}$ for every $v \in V(G') \setminus V(S)$. Let C' be the graph whose vertex- and edge-set are precisely those of the outer face boundary of G', and let A' be the set of vertices in $V(C') \setminus V(S)$ with lists of size at most two under L'. We now show that $K' = (G', L', S, A')$ is a canvas. Note that every vertex $x \in V(\text{Int}(C))$ is adjacent to at most one of v_{k+1}, v_{k+2}, and v_{k+3}: this follows immediately from the fact that both v_{k+1} and v_{k+3} are in A, and so by definition have girth at least five. Thus $|L'(x)| \geq 4$ for every vertex x of girth three in $V(C') \setminus V(S)$. Similarly, $|L'(x)| \geq 3$ for every vertex x of girth four in $V(C') \setminus V(S)$. Thus A' contains only vertices of girth at least five, and moreover every vertex v in A' satisfies $|L'(v)| = 2$. It remains only to show that A' is an independent set. Suppose not: then there exist vertices a_1, a_2 in A' such that $\{a_1a_2, a_1v_{k+3}, a_2v_{k+1}\} \subset E(G)$. But by Lemma 2.2.8, $V(\text{Int}(a_1a_2v_{k+1}v_{k+2}v_{k+3}a_1)) = \emptyset$ since $g(a_1) \geq 5$. This implies that $\deg(v_{k+2}) = 2$, which is of course a contradiction since $|L(v_{k+2})| = 3$ and K is a vertex-minimum counterexample. Thus K' is a canvas. We now show it is unexceptional. It is not exceptional of type (iii) since $|V(S)| = 4$. Moreover, it is not exceptional of type (i) since every vertex in A' has girth at least five and is adjacent to one of v_{k+1}, v_{k+2}, and v_{k+3} in G. Finally, as noted above each vertex $x \in V(C') \setminus V(S)$ with $g(x) = 3$ has $|L'(x)| \geq 4$. Since K is unexceptional, it thus follows that K' is not an exceptional canvas of type (ii).

Case 2: v_{k+3} is not in A. Since A is an independent set and $v_{k+1} \in A$, it follows that $v_{k+2} \notin A$ and so that $|L(v_{k+2})| = 3$. Thus v_{k+2} contains a colour c not in the list of v_{k+1}. Let ϕ be a colouring of v_{k+2} and v_{k+3}, where $\phi(v_{k+2}) = c$ and $\phi(v_{k+3}) \in L(v_{k+3}) \setminus (\{c\} \cup L(v_1))$. Let G' be obtained from G by deleting v_{k+2} and v_{k+3}. Let L' be the list assignment for G' obtained from L by setting $L'(v) = \{\phi(v)\}$ for every $v \in V(S)$ and $L'(v) = L(v) \setminus \{\phi(x) : x \in N_G(v) \cap \{v_{k+2}, v_{k+3}\}\}$ for every $v \in V(G') \setminus V(S)$. Note that by our choice of ϕ, we have that $L(v_{k+1}) = L'(v_{k+1})$. Let A' be the set of vertices $v \in V(G') \setminus V(S)$ with $|L'(v)| \leq 2$, and let C' be the graph whose vertex- and edge-set are precisely those of the outer face boundary of G'. As in the previous case, we will argue that (G', L', S, A') is an unexceptional canvas. First, note that since every vertex $v \in V(C') \setminus V(S)$ of girth three has $|L(v)| \geq 5$, it follows that for each such vertex $|L'(v)| \geq 3$. Similarly, since every vertex $v \in V(C') \setminus V(S)$ with $g(v) = 4$ is adjacent to at most one of v_{k+2} and v_{k+3} in G, it follows that each such vertex has $|L'(v)| \geq 4$. Thus every vertex in A' has girth at least five. Note furthermore that every vertex in A' has at least two colours in its list under L', since $c \notin L(v_{k+1})$ and every vertex in A' is adjacent to at most one of v_{k+2} and v_{k+3}. We claim moreover that A' is an independent set: again, this follows easily from the fact that every vertex in A' has girth five and is adjacent to one of v_{k+2} and v_{k+3}.

Thus (G', L', S, A') is a canvas; it remains only to show it is unexceptional. That it is not exceptional of type (i) follow from the facts that $v_{k+1} \in A'$ and C is chordless (thus $v_1 v_{k+1} \notin E(G)$), and that no vertex in $A' \setminus A$ is adjacent to both v_1 and v_4 since every vertex in $A' \setminus A$ is adjacent to one of v_{k+2} and v_{k+3}. Suppose now that (G', L', S, A') is exceptional of type (ii). In this case, there exists a vertex $v \in V(\text{Int}(C))$ of girth three that is adjacent to a vertex u in A' and $|L'(v)| = 3$. Since $|L(v)| \geq 5$, it follows that v is adjacent to both v_{k+3} and v_{k+2}. Since every vertex in A' is adjacent to one of v_{k+3} and v_{k+2} in G and has girth at least five, this is a contradiction. Finally, (G', L', S, A') is not an exceptional canvas of type (iii) since S contains four vertices. By the minimality of K and the fact that $|V(G')| < |V(G)|$, we have that G' admits an L'-colouring ϕ'. This again is a contradiction, since $\phi \cup \phi'$ is an L-colouring of G. □

We make the following definitions for convenience.

Definition 2.2.18. Let k' be an index defined as follows: if $v_{k+1} \in A$, then $k' = k + 1$. Otherwise, $k' = k$.

Definition 2.2.19. The *available colour* c at $v_{k'}$ is defined as follows: if $k = k'$, then $c \in L(v_{k'})$, and if $k' = k + 1$, then $c \in L(v_{k'}) \setminus L(v_k)$.

By Observation 2.2.5, $|L(v_k)| = 1$ and if $v_{k+1} \in A$, then $|L(v_{k'})| = 2$. We will show below in Lemma 2.2.20 (1) that the available colour at $v_{k'}$ is uniquely determined.

By Lemmas 2.2.16 and 2.2.17, there exist vertices $v_{k'+1}, v_{k'+2}, v_{k'+3} \in V(C) \setminus V(S)$. The following lemma partially describes their list assignments.

Lemma 2.2.20. *The following hold.*

(1) $L(v_k) \subset L(v_{k+1})$,

(2) $L(v_{k'+1}) \setminus \{c\} \subseteq L(v_{k'+2})$, *where* c *is the available colour at* $v_{k'}$, *and*

(3) $L(v_{k'+2}) \subseteq L(v_{k'+3})$.

Proof of (1). Suppose not. That is, suppose $L(v_k) \not\subset L(v_{k+1})$. Recall that by Observation 2.2.5, $|L(v_k)| = 1$. Let $G' = G - v_k$, and let L' be a list assignment for G' defined by $L'(v) = L(v) \setminus L(v_k)$ for all $v \in N_G(v_k)$, and $L'(v) = L(v)$ for all $v \in V(G') \setminus N_G(v_k)$. Note that $L(v_{k+1}) = L'(v_{k+1})$ since $L(v_k) \not\subset L(v_{k+1})$. Let A' be the set of vertices in $V(G') \setminus V(S)$ that have lists of size at most two under L'. Since $L(v_{k+1}) = L'(v_{k+1})$, it follows that $v_{k+1} \notin A' \setminus A$. Since C is chordless by Lemma 2.2.3, it follows that $A' \setminus A \subseteq V(\text{Int}(C))$. Thus every vertex $v \in A' \setminus A$ is a neighbour of v_k in G and has $|L(v)| = 3$, and so every vertex $v \in A' \setminus A$ satisfies both $|L'(v)| = 2$ and $g(v) \geq 5$. Since every vertex in A has girth at least five by definition, it follows that every vertex in A' has girth at least five. Since every vertex in $A' \setminus A$ is adjacent to v_k in G and is in $V(\text{Int}(C))$, it follows from Lemma 2.2.10 that A' is an independent set. Thus $(G', L', S - v_k, A')$ is a canvas. Note that $S - v_k$ is an acceptable path in G', since a subpath of an acceptable path is itself acceptable. Furthermore, note that $|V(S - v_k)| \leq 3$; and if $|V(S - v_k)| = 3$, then by the definition of acceptable path $S - v_k$ contains a vertex of girth at least four. It follows that $(G', L', S - v_k, A')$ is unexceptional. By the minimality of K, we have that G' admits an L'-colouring ϕ. But ϕ extends to an L-colouring of G by setting $\phi(v_k) \in L(v_k)$, a contradiction. □

Proof of (2). Suppose not. That is, suppose there exists a colour $c' \in L(v_{k'+1}) \setminus (\{c\} \cup L(v_{k'+2}))$. Let $G' = G - v_{k'+1}$, and let L' be a list assignment for G' defined by $L'(v) = L(v) \setminus \{c'\}$ for all $v \in N_G(v_{k'+1}) \setminus \{v_{k'}\}$, and $L'(v) = L(v)$ for all other $v \in V(G')$. Let A' be the set of vertices in $V(G') \setminus V(S)$ that have lists of size at most two under L'. Since $c' \notin L(v_{k'+2})$, it follows that $L'(v_{k'+2}) = L(v_{k'+2})$. Since C is chordless by Lemma 2.2.3, we have further that $A' \setminus A \subseteq V(\text{Int}(C))$. We claim $K' = (G', L', S, A')$ is a canvas; the argument is identical to that in the proof of (1).

First suppose K' is unexceptional. Then by the minimality of G we have that G' admits an L'-colouring ϕ. By the definition of available colour at $v_{k'}$, it follows that $\phi(v_{k'}) = c$. Since $c \neq c'$, we have that ϕ extends to an L-colouring of G by setting $\phi(v_{k'+1}) = c'$, a contradiction. Thus we may assume that K' is exceptional.

Suppose next that K' is an exceptional canvas of type (i). Then there exists a vertex $u \in A'$ adjacent to v_1 and v_4. Since K is unexceptional, we have that $u \notin A$. Thus $u \in A' \setminus A$; but $v_1 u v_4$ contradicts Lemma 2.2.10, since $u \in V(\text{Int}(C))$ and $g(u) \geq 5$.

Next, suppose (G', L', S, A') is an exceptional of type (ii). Then there exists a vertex $v \in V(G') \setminus V(S)$ with $g(v) = 3$ and $|L'(v)| = 3$ such that v is adjacent to one of v_2 and v_3. Since C is chordless, $v \in V(\text{Int}(C))$, and so $|L(v)| \geq 5$. But then $|L'(v)| \geq 4$, a contradiction.

Thus we may assume that K' is an exceptional canvas of type (iii), and therefore that G' contains a generalized wheel W such that the vertices on the outer cycle of W are on the outer cycle of G' and have lists of size at most three under L'. Since K is unexceptional, it follows that there exists a vertex w on the outer cycle of W that is not on the outer cycle of G. Since $w \in V(W)$, we have that $g(w) = 3$ and so $|L(w)| \geq 5$. But then $|L'(w)| \geq 4$, a contradiction. □

Proof of (3). Suppose not. That is, suppose there exists a colour $c' \in L(v_{k'+2}) \setminus L(v_{k'+3})$. Let ϕ be an L-colouring of $v_{k'+2}$ and $v_{k'+1}$ where $\phi(v_{k'+2}) = c'$ and where $\phi(v_{k'+1}) \in L(v_{k'+1}) \setminus \{c, c'\}$, where c is the available colour at $v_{k'}$. Note that since A is an independent set, $|L(v_{k'+1})| = 3$ and so ϕ exists as described.

Let G' be the graph obtained from G by deleting $v_{k'+1}$ and $v_{k'+2}$, and let C' be the subgraph of G' with vertex-set and edge-set equal to that of the outer face boundary walk of G'. Let L' be a list assignment obtained from L by setting $L'(v_{k'}) = L(v_{k'})$, and $L'(v) = L(v) \setminus \{\phi(v_i) : i \in \{k'+1, k'+2\}$ and $v_i \in N(v)\}$ for all $v \in V(G') \setminus \{v_{k'}\}$. Let A' be the set of vertices in $V(G') \setminus V(S)$ with lists of size at most two under L'. Note that by assumption, $L'(v_{k'+3}) = L(v_{k'+3})$ and so since C is chordless by Lemma 2.2.3 it follows that every vertex $v \in V(C) \setminus \{v_{k'+1}, v_{k'+2}\}$ satisfies $L(v) = L'(v)$. We claim that $K' = (G', L', S, A')$ is a canvas. To see this, note that every vertex $v \in V(\text{Int}(C))$ with $g(v) = 3$ satisfies $|L(v)| \geq 5$ and thus $|L'(v)| \geq 3$. Similarly, every vertex $v \in V(\text{Int}(C))$ of girth four is adjacent to at most one of $v_{k'+1}$ and $v_{k'+2}$ in G and so satisfies $|L'(v)| \geq 3$. It follows that every vertex $v \in A'$ has girth at least five and so has exactly two colours in its list under L'. Finally, we note that A' is an independent set. To see this, observe that every vertex in $A' \setminus A$ has girth at least five and is adjacent to one of $v_{k'+1}$ and $v_{k'+2}$ in G. Thus $A' \setminus A$ is an independent set, and it follows from Lemma 2.2.10 that there are no edges between vertices in $A' \setminus A$ and vertices in A. This proves the claim that K' is a canvas.

First suppose that K' is unexceptional. By the minimality of K, we have that G' admits an L'-colouring ϕ'. But then $\phi \cup \phi'$ is an L-colouring of G, a contradiction. Thus we may assume K' is exceptional.

Suppose next that K' is an exceptional canvas of type (i). Then there exists a vertex $u \in A'$ adjacent to v_1 and v_4. Since K is unexceptional, we have that $u \notin A$. Thus $u \in A' \setminus A$; but v_1uv_4 contradicts Lemma 2.2.10 since $u \in V(\text{Int}(C))$ and $g(u) \geq 5$.

Next, suppose that K' is an exceptional canvas of type (ii). Then there exists a vertex $w \in V(G') \setminus V(S)$ with $g(w) = 3$ and $|L'(w)| = 3$ such that w is adjacent to one of v_2 and v_3 and to a vertex $u \in A'$. Since C is chordless, $w \in V(\text{Int}(C))$. Since $|L'(w)| = 3$ and $w \in V(\text{Int}(C))$, it follows that w is a neighbour of both $v_{k'+1}$ and $v_{k'+2}$ in G. If $u \in A$, then $uwv_{k'+1}$ contradicts Lemma 2.2.10 since $g(u) \geq 5$. Thus $u \in A' \setminus A$, and so it follows that u is adjacent to a vertex v in $\{v_{k'+1}, v_{k'+2}\}$. This is a contradiction, since $g(u) \geq 5$ and $uvwu$ is a cycle of length 3.

Thus we may assume K' is an exceptional canvas of type (iii). Then $k = 3$, and S contains only vertices of girth three. Moreover, G' contains a generalized wheel W such that the vertices on the outer cycle of W are on the outer cycle of G' and have lists of size at most three under L'. Let $v_1v_2v_3w_1\dots w_tv_1$ be the outer face boundary walk of W. Since $|L'(w_1)| = 3$, it follows that $|L(w_1)| \geq 3$. Thus $w_1 \notin A$. Moreover, $w_1 \neq v_{k'+1}$ since $v_{k'+1} \notin V(G')$. It follows that $w_1 \neq v_4$, and so that $w_1 \in V(\text{Int}(C))$. Thus $|L(w_1)| \geq 5$; and since $|L'(w_1)| = 3$, we have that w_1 is adjacent to both $v_{k'+1}$ and $v_{k'+2}$ in G. Since $5 \in \{k'+1, k'+2\}$, we have that $w_1v_5 \in E(G)$. Since w_1 is adjacent to v_3, it follows from Lemma 2.2.12 that w_1 is also adjacent to v_4 in G. This contradicts Lemma 2.2.15. □

We require one final corollary, which will be used to show that G contains our main reducible configuration. The corollary shows that $A \cap \{v_{k+1}, v_{k+2}, v_{k+3}, v_{k+4}\} \notin \{\{v_{k+1}, v_{k+4}\}, \{v_{k+3}\}\}$.

Corollary 2.2.21. $v_{k'+3} \notin A$.

Proof. Suppose not. Since $v_{k'+3} \in A$, it follows that $|L(v_{k'+3})| = 2$. Since A is an independent set, $v_{k'+2} \notin A$ and so by Observation 2.2.5 we have that $|L(v_{k'+2})| = 3$. But this contradicts Lemma 2.2.20 (3).

□

Since A is an independent set, we thus have that if $v_{k+1} \in A$, then $A \cap \{v_{k+2}, v_{k+3}, v_{k+4}\} \in \{\emptyset, \{v_{k+3}\}\}$. Similarly, if $v_{k+1} \notin A$, then $A \cap \{v_{k+1}, v_{k+2}, v_{k+3}\} \in \{\emptyset, \{v_{k+2}\}\}$. To summarize: $A \cap \{v_{k'+1}, v_{k'+2}, v_{k'+3}\} \in \{\emptyset, \{v_{k'+2}\}\}$.

We now define our main reducible configuration. It follows from Lemmas 2.2.16 and 2.2.17 that $q \geq k'+3$. Recall that by Observation 2.2.5, every vertex in $V(C) \setminus (V(S) \cup A)$ has a list of size three.

Definition 2.2.22. Let $P = v_{k'+1}v_{k'+2}\dots v_j$ be a subpath of C satisfying the following set of conditions:

1. $j \in \{k'+3, \dots, q\}$,
2. either $V(P) \cap A = \{v_{k'+2}\}$ or $V(P) \cap A = \emptyset$,
3. $L(v_{i-1}) \subseteq L(v_i)$ for all $k'+3 \leq i \leq j$, and
4. $L(v_j) \not\subseteq L(v_{(j \mod q)+1})$.

We call P a deletable path.

By Corollary 2.2.21, we have that $A \cap \{v_{k'+1}, v_{k'+2}, v_{k'+3}\} \in \{\emptyset, \{v_{k'+2}\}\}$. Thus we immediately have the following.

Corollary 2.2.23. *G contains a deletable path.*

2.3 Proving Theorem 2.1.6

Before proceeding, we give a brief overview of this section. This section contains a proof of our main technical theorem, Theorem 2.1.6. Recall that $K = (G, L, S, A)$ is a counterexample to Theorem 2.1.6, where $|V(G)|$ is minimized over all counterexamples to the theorem, and subject to that, where $\sum_{v \in V(G)} |L(v)|$ is minimized. By Corollary 2.2.23, G contains a deletable path $P = v_{k'+1}v_{k'+2} \dots v_j$.

Most of the lemmas in this section take the following basic shape: we colour and delete P, and argue about the structure of what remains. In Lemma 2.3.1, we show that $V(P) \cap A = \{v_{k'+2}\}$ and argue that there exists a separating path as described in Lemma 2.2.11, where the endpoints of the path are $v_{k'+1}$ and $v_{k'+3}$.

From there, Lemma 2.3.2 establishes that no vertex in $\text{Int}(C)$ is adjacent to two vertices on P at distance at least two in P. This will be useful in arguing that upon deleting P and modifying lists where appropriate, what remains is a canvas: that is, no list loses too many colours. Finally, in Lemma 2.3.3, we colour and delete P as well as the separating path described in Lemma 2.3.1 and argue via a series of claims that what remains is an unexceptional canvas. By induction, this canvas admits an L-colouring —and so G admits an L-colouring. This shows that K is not a counterexample to Theorem 2.1.6, completing the proof.

We begin with the following lemma.

Lemma 2.3.1. *$V(P) \cap A = \{v_{k'+2}\}$, and there exist distinct vertices $u_1, u_2 \in V(\text{Int}(C))$ with $g(u_1) \geq 5$ and $g(u_2) \geq 5$ such that u_1 is adjacent to $v_{k'+1}$, u_2 is adjacent to $v_{k'+3}$, and u_1 is adjacent to u_2.*

Proof. Suppose not. Let ϕ be an L-colouring of P such that: $\phi(v_j) \notin L(v_{(j \mod q)+1})$; $\phi(v_{k'+1}) \in L(v_{k'+1}) \setminus \{c\}$, where c is the available colour at $v_{k'}$; if $v_{k'+2} \notin A$, then ϕ is a 2-colouring of $P - v_{k'+1}$; and if $v_{k'+2} \in A$, then ϕ is a 2-colouring of $P - v_{k'+1} - v_{k'+2}$. Note that ϕ exists: by Observation 2.2.5 and the definition of deletable path we have that $L(v_{k'+3}) = L(v_i)$ for all $i \in \{k'+4, \dots, j\}$, and if $v_{k'+2} \notin A$, then $L(v_{k'+2}) = L(v_{k'+3})$ as well. By Observation 2.2.5 and Lemma 2.2.20 (1), the available colour at $v_{k'}$ is unique.

Let G' be the graph obtained from G by deleting $V(P)$, and let L' be the list assignment for G' obtained from L by setting $L'(v_{k'}) = L(v_{k'})$ and $L'(v) = L(v) \setminus \{\phi(x) : x \in V(P) \cap N_G(v)\}$ for all $v \in V(G') \setminus \{v_{k'}\}$. Let C' be the subgraph of G' whose vertex- and edge-set are precisely those of the outer face boundary walk of G', and let A' be the set of vertices in $V(G') \setminus V(S)$ with lists of size at most two under L'.

We now show the following.

Claim 1. (G', L', S, A') *is a canvas.*

Proof. Suppose not. Note that by our choice of L', every vertex $v \in V(S)$ satisfies $|L'(v)| = 1$. First suppose there exists a vertex $v \in V(C') \setminus V(S)$ with $g(v) = 3$ and $|L'(v)| < 3$. Note that if $v \in V(C) \setminus V(S)$, then since C is chordless by Lemma 2.2.3 and $\phi(v_j) \notin L(v_{(j \mod q)+1})$ by our choice of ϕ, it follows that $|L'(v)| \geq 3$, a contradiction. Thus we may assume $v \in V(C') \setminus V(C)$, and so that $v \in V(\text{Int}(C))$. Since $g(v) = 3$, we have by the definition of canvas that $|L(v)| \geq 5$. Since $|L'(v)| < 3$, it follows that v is adjacent to at least three vertices in $V(P)$; and in particular, since ϕ is a 2-colouring of $P - v_{k'+2} - v_{k'+1}$, that either $vv_{k'+1} \in E(G)$ or $vv_{k'+2} \in E(G)$. Let ℓ be the smallest index such that $vv_\ell \in E(G)$ and $\ell \in \{k'+1, k'+2\}$. Let m be the largest index such that $v_m \in N_G(v) \cap V(P)$. Since v neighbours at least three vertices in P, it follows that $m \geq \ell + 2$. By Lemma 2.2.12, since v is adjacent to v_ℓ and v_m, we have that v is also adjacent to $v_{\ell+1}, v_{\ell+2}, \ldots, v_{m-1}$. Thus for each $i \in \{\ell, \ldots, m\}$, we have that $g(v_i) = 3$. Since $i \leq k' + 2 < m$, it follows that $g(v_{k'+2}) = 3$ and so that $v_{k'+2} \notin A$. Thus by the definition of deletable path, $L(v_{k'+2}) = L(v_{k'+3})$. It then follows from Lemma 2.2.14 that $\ell = k' + 1$ and $m = k' + 3$.

Hence v is adjacent to $v_{k'+1}$, $v_{k'+2}$, and $v_{k'+3}$. Note that by Lemma 2.2.4, $V(\text{Int}(vv_{k'+1}v_{k'+2}v)) = V(\text{Int}(vv_{k'+2}v_{k'+3}v)) = \emptyset$. Let G'' be the graph obtained from G by identifying $v_{k'+1}$ and $v_{k'+3}$ to a new vertex z and deleting $v_{k'+2}$. Let $L(z) = L(v_{k'+1})$. Note that by Lemma 2.2.20 (2) and (3), we have that $L(v_{k'+1}) \setminus \{c\} \subseteq L(v_{k'+3})$, where c is the available colour at $v_{k'}$. By Lemma 2.2.13, for every vertex $x \in V(G'')$, if $g_G(x) \geq 5$, then $g_{G''}(x) \geq 5$, and similarly if $g_G(x) = 4$, then $g_{G''}(x) \geq 4$. It follows that S is an acceptable path in G''. Note that $K'' = (G'', L, S, A)$ is a canvas; in particular, $|L(z)| = 3$.

First suppose K'' is unexceptional. By the minimality of K, we have that K'' admits an L-colouring φ. Note that by definition of available colour, $\varphi(z) \neq c$, where c is the available colour at $v_{k'}$. But then $\varphi(z) \in L(v_{k'+3})$, and so φ extends to an L-colouring of G by setting $\varphi(v_{k'+1}) = \varphi(v_{k'+3}) = \varphi(z)$ and $\varphi(v_{k'+2}) \in L(v_{k'+2}) \setminus \{\varphi(z), \varphi(v)\}$, a contradiction.

Thus we may assume that K'' is an exceptional canvas. Since C is chordless by Lemma 2.2.3, $z \notin A$, and K is unexceptional, we have that K'' is not an exceptional canvas of type (i) or (ii). We may therefore assume that K'' is an exceptional canvas of type (iii), and thus that G'' contains a subgraph W that is a generalized wheel with principal path S such that the vertices on the outer cycle of W are on the outer face boundary of G'' and all have lists of size at most three under L. Again because C is chordless it follows that the outer cycle of W is the outer cycle of G'', and that the outer cycle of G'' is also chordless. Since every generalized wheel that is neither a triangle nor a wheel has a chord in its outer cycle, it follows that W is either a triangle or a wheel. Note that since K is unexceptional, we have that $z \in V(W)$. Since $|V(W)| \geq |V(S)| + |\{z\}| = 4$, we have that W is a wheel. It follows that every triangle in G'' corresponds to a triangle in G (replacing z by $v_{k'+1}$ or $v_{k'+3}$ where appropriate). By Lemma 2.2.4, we have that $W = G''$; and since $v \in V(\text{Int}(C))$, it follows further that v is the only vertex in W not in the outer cycle of G''. But then G too is a wheel with principal path S, and since $|L(v_{k'+2})| = 3$ by Observation 2.2.5, it follows that K is an exceptional canvas of type (iii), a contradiction. Thus we may assume that every vertex in $V(C') \setminus V(S)$ of girth three has a list of size at least three under L'.

Next suppose that there exists a vertex $v \in V(C') \setminus V(S)$ with $|L'(v)| \leq 2$ and $g(v) = 4$. Note that by our choice of ϕ and the fact that C is chordless by Lemma 2.2.3, we have that $L'(v) = L(v)$ for all $v \in V(C)$. Thus we may assume that $v \in V(C') \setminus V(C)$, and so that $v \in V(\text{Int}(C))$. Thus $|L(v)| \geq 4$. By Lemma 2.2.10, since $g(v) = 4$ we have that v is adjacent to at most one vertex in $V(P)$ and thus since $|L(v)| \geq 4$, it follows that $|L'(v)| \geq 3$, a contradiction. Thus we may assume that every vertex in $V(C') \setminus V(S)$ of girth four has a list of size at least three under L'.

It follows from the above that every vertex in A' has girth at least five. Suppose now that there exits a vertex $v \in A'$ with $|L'(v)| \leq 1$. Note that if $v \in V(C)$, then by our choice of ϕ and the fact that C is chordless by Lemma 2.2.3 it follows that $L(v) = L'(v)$. Thus we may assume that $v \in V(C') \setminus V(C)$, and so that $v \in V(\text{Int}(C))$. By Lemma 2.2.10, since $g(v) \geq 5$ we have that v is adjacent to only one vertex in $V(P)$ and so that $|L'(v)| \geq 2$, a contradiction. Thus we may assume that every vertex in $V(C') \setminus V(S)$ of girth at least five has a list of size at least two under L'.

Since we assumed K' is not a canvas, it follows that A' is not an independent set. Thus there exist distinct vertices $u_1, u_2 \in A'$ such that $u_1u_2 \in E(G)$. (Note that $A' \cap V(P) = \emptyset$, and so that in particular $v_{k'+2} \notin \{u_1, u_2\}$.) Since C is chordless, $\{u_1, u_2\} \not\subseteq A$. Suppose that exactly one of u_1 and u_2 is in A. This contradicts Lemma 2.2.10, since $g(u_i) \geq 5$ for each $i \in \{1, 2\}$ and every vertex in $A' \setminus A$ is adjacent to a vertex in $V(P)$. Thus we may assume that $\{u_1, u_2\} \subseteq A' \setminus A$, and so that $\{u_1, u_2\} \subseteq V(\text{Int}(C))$. Recall that by the definition of deletable path, either $V(P) \cap A = \emptyset$ or $V(P) \cap A = \{v_{k'+2}\}$. Since $g(u_1) \geq 5$ and $g(u_2) \geq 5$ and each of u_1 and u_2 is adjacent to a vertex in $V(P)$, it follows from Lemma 2.2.11 that $V(P) \cap A = \{v_{k'+2}\}$, and that, up to relabelling u_1 and u_2, both $u_1v_{k'+1} \in E(G)$ and $u_2v_{k'+3} \in E(G)$. Thus the hypotheses of Lemma 2.3.1 hold, a contradiction. □

Thus $K' = (G', L', S, A')$ is a canvas by Claim 1. First suppose K' is unexceptional. Then by the minimality of K we have that K' admits an L'-colouring ϕ'. But then $\phi \cup \phi'$ is an L-colouring of G, a contradiction. We may therefore assume that K' is an exceptional canvas.

Suppose next that K' is an exceptional canvas of type (i). Then there exists a vertex

$u \in A'$ adjacent to v_1 and v_4. Since K is unexceptional, $u \notin A$. Thus $u \in A' \setminus A$, and so $u \in V(\text{Int}(C))$. But then v_1uv_4 contradicts Lemma 2.2.10, since $g(u) \geq 5$.

Next suppose that K' is exceptional of type (ii). Then $|V(S)| = 4$, and moreover G' contains a vertex $w \in V(G') \setminus V(S)$ of girth three with $|L'(w)| = 3$ adjacent to one of v_2 and v_3, and a vertex $u \in A'$ adjacent to w and one of v_4 and v_1. Since $|L'(w)| = 3$ and C is chordless by Lemma 2.2.3, it follows that $w \in V(C') \setminus V(C)$, and so that w is adjacent in G to at least two vertices on P. If $u \in A$, then w contradicts Lemma 2.2.10 since w is adjacent to u and a vertex on P and $g(u) \geq 5$. Thus we may assume $u \in A' \setminus A$. But then since u is adjacent to one of v_1 and v_4, again this contradicts Lemma 2.2.10, since every vertex in $A' \setminus A$ is adjacent to a vertex on P and $g(u) \geq 5$.

Thus we may assume K' is an exceptional canvas of type (iii). Then $k = 3$, and S contains only vertices of girth three. Moreover, G' contains a generalized wheel W such that the vertices on the outer cycle of W are on the outer cycle of G' and have lists of size at most three under L'. Let $v_1v_2v_3w_1 \dots w_tv_1$ be the outer cycle of W. Since $|L'(w_1)| = 3$, it follows that $|L(w_1)| \geq 3$. Thus $w_1 \notin A$. Moreover, $w_1 \neq v_{k'+1}$ since $v_{k'+1} \notin V(G')$. It follows that $w_1 \neq v_4$, and so since C is chordless by Lemma 2.2.3 we have that $w_1 \in V(\text{Int}(C))$. Since $g(w_1) = 3$, we have further that $|L(w_1)| \geq 5$; and since $|L'(w_1)| = 3$, we have that w_1 is adjacent to at least two vertices in $V(P)$. Since w_1 is also adjacent to v_3, it follows from Lemma 2.2.12 that w is adjacent to v_4 and v_5 in G. This contradicts Lemma 2.2.15. □

Following Lemma 2.3.2, our final lemma—Lemma 2.3.3—will also involve colouring and deleting P, restricting lists where appropriate, and arguing that what remains is an unexceptional canvas. Lemma 2.3.2 shows that in doing so, the vertices in $\text{Int}(C)$ lose at most one colour from their list.

Lemma 2.3.2. *There does not exist a vertex $v \in V(\text{Int}(C))$ such that v is adjacent to two vertices in $V(P)$ at distance at least two in P.*

Proof. Suppose not, and let v be a vertex adjacent to two vertices of P at distance at least two. Let i be the smallest index with $k'+1 \leq i \leq j-2$ such that $v_i \in N_G(v) \cap V(P)$. Let m be the largest index with $k'+3 \leq m \leq j$ such that $v_m \in N_G(v) \cap V(P)$. By Lemma 2.2.12, v is adjacent to every vertex in $\{v_i, v_{i+1}, \dots, v_m\}$: that is, $vv_iv_{i+1} \dots v_mv$ is the outer cycle of a broken wheel with principal path v_mvv_i. By Lemma 2.3.1, $V(P) \cap A = \{v_{k'+2}\}$. Since $g(v_{k'+2}) \geq 5$, it follows that $i \geq k'+3$. By the definition of P, we have that $L(v_m) = L(v_i)$. Since $m \geq i+2$, this contradicts Lemma 2.2.14. □

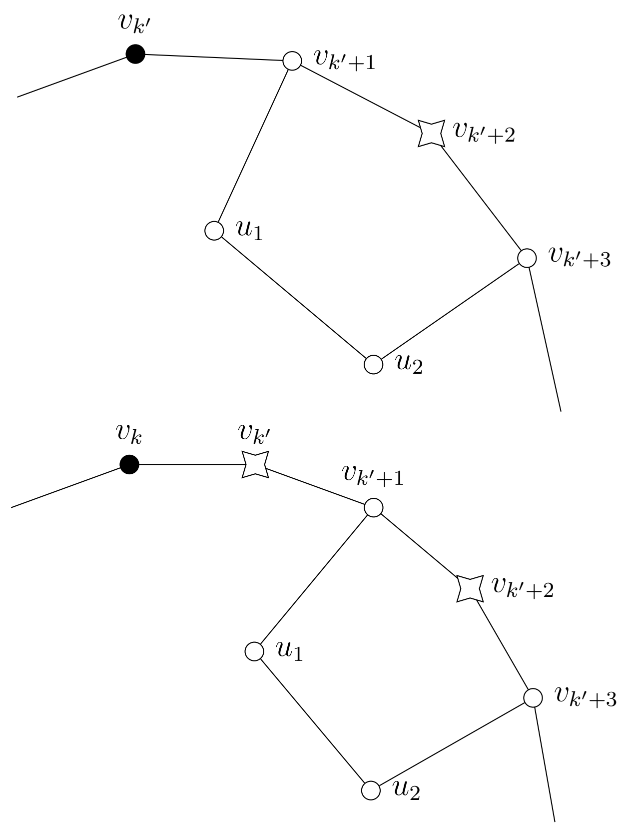

Figure 2.2: The final cases considered in Lemma 2.3.3. Vertices in S are black; vertices in A are drawn as four-pointed stars. On the left, $v_{k+1} \notin A$ and so $k = k'$. On the right, since $v_{k+1} \in A$, it follows that $k' = k + 1$. Recall that by definition, the deletable path P begins at $v_{k'+1}$; recall moreover that by Lemma 2.2.8, since $g(u_1) = 5$ it follows that the 5-cycles in the figure have no vertices in their interior.

The following lemma concludes the proof of Theorem 2.1.6 (and moreover this chapter of the thesis). See Figure 2.2 (below) for an illustration of the cases considered in Lemma 2.3.3.

Lemma 2.3.3. *K is not a counterexample to Theorem 2.1.6.*

Proof. Suppose not. By Corollary 2.2.21, G contains a deletable path P, and by Lemma 2.3.1, $V(P) \cap A = \{v_{k'+2}\}$ and there exists an edge u_1u_2 in $E(\text{Int}(C))$ such that both u_1 and u_2 have girth at least five, and $\{u_1v_{k'+1}, u_2v_{k'+3}\} \subset E(G)$. By Lemma 2.2.8, we have that $\mathrm{Int}(u_1u_2v_{k'+3}v_{k'+2}v_{k'+1}u_1) = \emptyset$, and so that $N_G(v_{k'+2}) = \{v_{k'+1}, v_{k'+3}\}$. We first show the following.

Claim 2. *There exists an L-colouring ϕ of $G[\{u_1, u_2\} \cup V(P) \cup V(S)]$ such that:*

- *$\phi(v_j) \notin L(v_{(j \mod q)+1})$, and*
- *$\phi(v_{k'+1})$ is not the available colour at $v_{k'}$.*

Proof. Note that by Lemma 2.2.10, since $g(u_1) \geq 5$ it follows that $v_{k'+1}$ is the unique neighbour of u_1 in $V(P) \cup \{v_1, v_k\}$. Similarly, since $g(u_2) \geq 5$, we have that $v_{k'+3}$ is the unique neighbour of u_2 in $V(P) \cup \{v_1, v_k\}$. Since each of u_1 and u_2 have girth at least five, it follows that at most one of u_1 and u_2 has a neighbour in $V(S)$.

Since $|L(v_{k'+1})| = 3$ and C is chordless by Lemma 2.2.3, we have that $G[V(P) \cup V(S)]$ admits an L-colouring ϕ as described in the statement of the claim (by first colouring S and then the vertices of P in decreasing order of index). Moreover, as argued above there exists $i \in \{1, 2\}$ such that u_i has degree at most two in $H = G[\{u_1, u_2\} \cup V(P) \cup V(S)]$. After colouring $G[V(P) \cup V(S)]$, we then colour u_{3-i} with a colour $\phi(u_{3-i}) \in L(u_{3-i}) \setminus \{\phi(x) : x \in N_H(u_{3-i})\}$. Note that u_{3-i} has at most two neighbours in $H - u_i$, and so since $|L(u_{3-i})| \geq 3$, we have that $\phi(u_{3-i})$ exists. Finally, we colour u_i with a colour in $L(u_i) \setminus \{\phi(x) : x \in N_H(u_i)\}$. Since u_i has at most two neighbours in H and $|L(u_i)| \geq 3$, this is possible. □

Let ϕ be as in the statement of Claim 2, and let G' be the graph obtained from G by deleting $V(P) \cup \{u_1, u_2\}$. Note that by Lemma 2.2.8 and the fact that $g(u_1) \geq 5$, we have that $\mathrm{Int}(v_{k'+1}v_{k'+2}v_{k'+3}u_2u_1v_{k'+1}) = \emptyset$. Let C' be the graph whose vertex- and edge-set are precisely those of the outer face boundary of G'. Let L' be the list assignment obtained from L by setting $L'(v_{k'}) = L(v_{k'})$ and $L'(v) = L(v) \setminus \{\phi(x) : x \in (V(P) \cup \{u_1, u_2\}) \cap N(v)\}$ for all $v \in V(G') \setminus \{v_{k'}\}$. Let A' be the set of vertices in $V(G') \setminus V(S)$ with lists of size at most two under L'.

Claims 3 and 4 will be used repeatedly to argue that (G', L', S, A') is a canvas.

Claim 3. *Every vertex $v \in V(C')$ satisfies $L(v) = L'(v)$.*

Proof. Let $v \in V(C')$. Note that that since $u_1v_{k'+1} \in E(G)$ and $g(u_1) \geq 5$, it follows from Lemma 2.2.10 that u_1 is not adjacent to v. Similarly, u_2 is not adjacent to v. By Lemma 2.2.3, C is chordless and so no internal vertex of P is adjacent to v. Moreover, note that $L'(v_{k'}) = L(v_{k'})$ by definition of L'. Thus we may assume that $v = v_{(j \mod q)+1}$, as otherwise

$L(v) = L'(v)$. By Claim 2, $\phi(v_j) \notin \phi(v_{(j \mod q)+1})$, and so $L(v_{(j \mod q)+1}) = L'(v_{(j \mod q)+1})$, as desired. $\square$

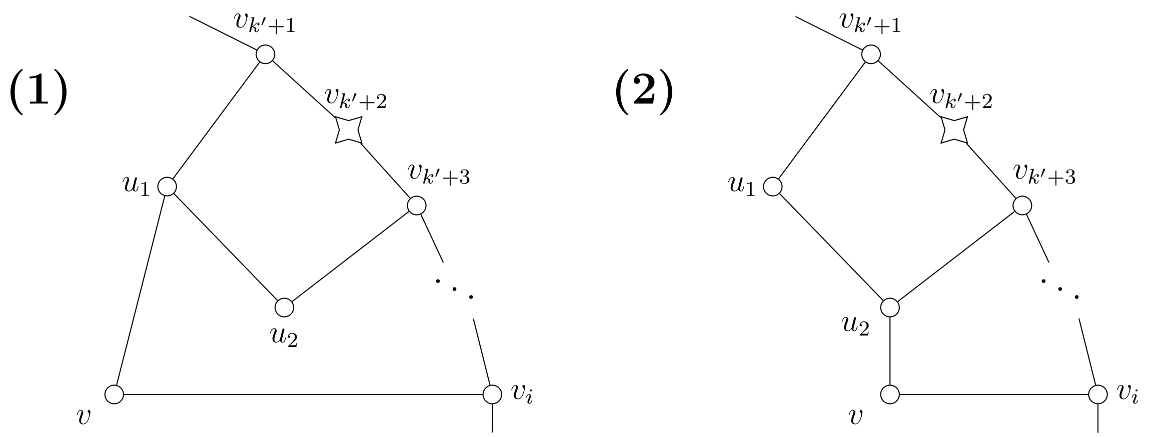

Figure 2.3: Cases considered in Claim 4. Vertices in A are drawn as four-pointed stars. Both u_1 and u_2 have girth five. Recall that by Lemma 2.2.8, since $g(u_1) = 5$ it follows that the 5-cycles in all figures have no vertices in their interior.

Claim 4. *There does not exist a vertex in* $V(\text{Int}(C)) \setminus \{u_1, u_2\}$ *adjacent to a vertex in* $V(P)$ *and a vertex in* $\{u_1, u_2\}$.

Proof. Suppose not, and let $v \in V(\text{Int}(C)) \setminus \{u_1, u_2\}$ be a counterexample. Let i be an index such that $k' + 1 \leq i \leq j$ and v is adjacent to v_i. Since $g(u_2) \geq 5$ and v is adjacent to one of u_1 and u_2, it follows that $i \geq k' + 4$. See Figure 2.3 (below) for an illustration of the cases considered in this claim.

First suppose v is adjacent to u_1. In this case, the path $v_{k'+1}u_1vv_i$ separates G into two graphs G_1 and G_2 where without loss of generality $S \subseteq G_1$. By Observation 2.2.1, $K[G_1]$ is an unexceptional canvas. By the minimality of K, it follows that G_1 admits an L-colouring ψ. Let L'' be a list assignment for G_2 obtained from L by setting $L''(x) = \{\psi(x)\}$ for $x \in \{v_{k'+1}, u_1, v, v_i\}$, and $L''(x) = L(x)$ for $x \in V(G_2) \setminus \{v_{k'+1}, u_1, v, v_i\}$. Since $g(u_1) \geq 5$, it follows that $v_{k'+1}u_1vv_i$ is an acceptable path for G_2. Note that $K_2 = (G_2, L'', v_{k'+1}u_1vv_i, A \cap V(G_2))$ is a canvas. If K_2 is unexceptional, it follows from the minimality of K that G_2 admits an L''-colouring ψ'. But then $\psi \cup \psi'$ forms an L-colouring of G, a contradiction. Thus we may assume K_2 is an exceptional canvas.

First suppose K_2 is an exceptional canvas of type (i). Since $A \cap V(G_2) = \{v_{k'+2}\}$, it follows that $v_{k'+2}$ is adjacent to v_i. This is a contradiction, since $i \geq k' + 4$ and C is chordless by Lemma 2.2.3.

Next, suppose K_2 is an exceptional canvas of type (ii). Note that $g(u_1) \geq 5$, and every vertex in a generalized wheel has girth three. Thus there exists a vertex $w \notin \{u_1, v_{k'+1}\}$ such that G_2 contains a subgraph W that is a generalized wheel with principal path wvv_i such that the vertices on the outer cycle of W are on the outer face boundary of G_2 and have lists of size at most three under L''. Moreover, there exists a vertex in $A \cap V(G_2)$ adjacent to w and $v_{k'+1}$. Recall that $A \cap V(G_2) = \{v_{k'+2}\}$, and $N_G(v_{k'+2}) = \{v_{k'+1}, v_{k'+3}\}$. It follows that $w = v_{k'+3}$. But $vv_{k'+3}u_2u_1v$ is a cycle of length four and $g(u_2) = 5$, a contradiction.

Thus we may assume that K_2 is an exceptional canvas of type (iii). But this too is a contradiction, since $v_{k'+1}u_1vv_i$ is a path of length three.

Thus we may assume instead that v is adjacent to u_2. In this case, the path $v_{k'+3}u_2vv_i$ separates G into two graphs G_1 and G_2 as above. Note here that $i \geq k'+5$ since $g(u_2) \geq 5$. The argument is the same as in the previous case, except that here $V(G_2) \cap A = \emptyset$, and so we have immediately that the canvas $(G_2, L'', v_{k'+3}u_2vv_i, A \cap V(G_2))$ is unexceptional and thus that G_2 admits an L''-colouring, a contradiction. □

We now prove (G', L', S, A') is an unexceptional canvas via the following claims.

Claim 5. *Every vertex $v \in V(C') \setminus V(S)$ with $g(v) = 3$ satisfies $|L'(v)| \geq 3$.*

Proof. Suppose not, and let $v \in V(C') \setminus V(S)$ be a counterexample. By Claim 3, we have that $v \in V(\text{Int}(C))$ and so that $|L(v)| \geq 5$. Since $|L'(v)| \leq 2$, it follows that v is adjacent to at least three vertices in $V(P) \cup \{u_1, u_2\}$. It follows from Lemma 2.3.2 that v is adjacent to at least one of u_1 and u_2. But this contradicts Claim 4. □

Claim 6. *Every vertex $v \in V(C') \setminus V(S)$ with $g(v) = 4$ satisfies $|L'(v)| \geq 3$.*

Proof. Suppose not, and let $v \in V(C') \setminus V(S)$ be a counterexample. It follows from Claim 3 that $v \in V(\text{Int}(C))$ and so that $|L(v)| \geq 4$. Since $|L'(v)| \leq 2$, it follows further that v is adjacent to at least two vertices in $V(P) \cup \{u_1, u_2\}$. By Lemma 2.3.2, since $g(v) > 3$ we have that v is not adjacent to two vertices in $V(P)$. Since $g(v) = 4$, it follows that v is adjacent to exactly one of u_1 and u_2 and a vertex in $V(P)$. This contradicts Claim 4. □

From Claims 5 and 6, it follows that every vertex in A' has girth at least five. The following lemma completes the proof that (G', L', S, A') is a canvas. After this, it will remain only to show that it is unexceptional.

Claim 7. *A' is an independent set, and if $v \in A'$, then $|L'(v)| = 2$.*

Proof. First, we show that every vertex in A' has two colours in its list under L'. Suppose not; let $v \in A'$ have a list of size at most one. It follows from Claim 3 that $v \in A' \setminus A$, and so that $v \in V(\text{Int}(C))$ and $|L(v)| \geq 3$. Since $|L'(v)| \leq 1$, we have that v is adjacent in G to at least two vertices in $V(P) \cup \{u_1, u_2\}$. Since $g(v) \geq 5$, we have that v is adjacent to at most one of u_1 and u_2. By Claim 2.2.10, we have furthermore that v is adjacent to at most one vertex in $V(P)$. Thus v is adjacent to exactly one of u_1 and u_2, and exactly one vertex in $V(P)$. This contradicts Claim 4.

It remains to show that A' is independent. Suppose not: let v, w be two vertices in A' with $vw \in E(G)$. Since A is an independent set, it follows that at least one of v and w is in $A' \setminus A$. Without loss of generality, let $v \in A' \setminus A$.

First suppose $w \in A$. Note that v is not adjacent to a vertex in $V(P)$, as otherwise v contradicts Lemma 2.2.10. Since every vertex in $A' \setminus A$ is adjacent to a vertex in $V(P) \cup \{u_1, u_2\}$, it follows that v is adjacent to a vertex $u \in \{u_1, u_2\}$. If $u = u_1$, define $Q = wvu_1v_{k'+1}$. If $u = u_2$, define $Q = wvu_2v_{k'+3}$. In either case, since $g(u_1) \geq 5$ and $g(u_2) \geq 5$, by Lemma 2.2.11 there exists a vertex $x \in A$ adjacent to both endpoints of Q. This is a contradiction, since A is an independent set and $w \in A$ is an endpoint of Q.

Thus we may assume that both v and w are in the set $A' \setminus A$. Note that if each of v and w is adjacent to a vertex in $V(P)$, then by Lemma 2.2.11 we have that without loss of generality v is adjacent to $v_{k'+1}$ and w is adjacent to $v_{k'+3}$. Since $g(v_{k'+2}) \geq 5$, we have by Lemma 2.2.8 that $\text{Int}(u_1v_{k'+1}v_{k'+2}v_{k'+3}u_2u_1) = \emptyset$. Thus $\{u_1, u_2\} \subseteq V(\text{Int}(uv_{k'+1}v_{k'+2}v_{k'+3}w))$. This is a contradiction, since $g(v_{k'+2}) \geq 5$ and so by Lemma 2.2.8 we have that $\text{Int}(wuv_{k'+1}v_{k'+2}v_{k'+3}w) = \emptyset$.

Thus at least one of v and w is adjacent to a vertex in $\{u_1, u_2\}$; moreover, since $g(v) \geq 5$ and $g(w) \geq 5$, exactly one of v, w is adjacent to a vertex in $\{u_1, u_2\}$. Without loss of generality, we may assume v is adjacent to one of u_1 and u_2, and so that there exists an index i such that w is adjacent to a vertex $v_i \in V(P)$.

Case 1. $i \leq k' + 4$**.** Note that in this case, we may assume $i = k' + 4$. To see this, suppose not. First assume v is adjacent to u_2. If $i = k' + 1$, then since $g(u_1) \geq 5$ we have by Lemma 2.2.8 that $\text{Int}(vwv_{k'+1}u_1u_2v) = \emptyset$. Since $\text{Int}(u_2u_1v_{k'+1}v_{k'+2}v_{k'+3}u_2) = \emptyset$ by the same lemma, it follows that $\deg(u_1) = 2$. This is a contradiction, since $|L(u_1)| \geq$

3 and K is a vertex-minimum counterexample. It further follows from the fact that $\mathrm{Int}(u_2u_1v_{k'+1}v_{k'+2}v_{k'+3}u_2) = \emptyset$ that $i \neq k'+2$. Thus we may assume $i = k'+3$. But this is a contradiction, since $vwv_{k'+3}u_2v$ is a 4-cycle and $g(v) \geq 5$. Symmetrical arguments show that if v is adjacent to u_1, then $i \notin \{k'+3, k'+2, k'+1\}$.

We may therefore assume that $i = k'+4$. If v is adjacent to u_1, then by Lemma 2.2.9 we have that $\mathrm{Int}(u_1vwv_{k'+4}v_{k'+3}u_2u_1) = \emptyset$ since $g(u_1) \geq 5$. By Lemma 2.2.8, $\mathrm{Int}(u_2u_1v_{k'+1}v_{k'+2}v_{k'+3}u_2) = \emptyset$, and so u_2 has degree two. Since $|L(u_2)| \geq 3$ and K is a vertex-minimal counterexample, this is a contradiction since every L-colouring of $G - u_2$ extends to an L-colouring of G. Thus we may assume that v is adjacent to u_2. See Figure 2.4 (1), below, for an illustration of this case.

Let φ be an L-colouring of $v_{k'+1}$ and $v_{k'+2}$, where $\varphi(v_{k'+1})$ is not the available colour at $v_{k'}$. Let $G'' = G - v_{k'+1} - v_{k'+2}$, and let L'' be a list assignment for G'' obtained from L by setting $L''(v_{k'}) = L(v_{k'})$, and $L''(v) = L(v) \setminus \{\varphi(v_i) : i \in \{k'+1, k'+2\} \text{ and } v_i \in N_G(v)\}$ for all $v \in V(G'') \setminus \{v_{k'}\}$. Let A'' be the set of vertices in $V(G'') \setminus V(S)$ with lists of size at most two under L''. Note that $A'' \setminus A \subseteq N_G(v_{k'+1}) \cup \{v_{k'+3}\}$ and by definition of P, we have that $v_{k'+3} \in A''$.

By definition of P, since $i = k'+4$ we have that $v_{k'+4} \notin A$. Moreover, since C is chordless by Lemma 2.2.3 there are no edges between $v_{k'+3}$ and vertices in $A \cup \{v_1, v_4\}$. Since $g(v_{k'+3}) \geq 5$ and $v_{k'+4} \notin A$, it follows from Lemma 2.2.10 that A'' is an independent set, and so that $K'' = (G'', L'', S, A'')$ is a canvas. It further follows from Lemma 2.2.10 that there are no edges between $A'' \setminus A$ and $\{v_1, v_4\}$.

First suppose that K'' is unexceptional. Since $|V(G'')| < |V(G)|$, we have by the minimality of K that G'' admits an L''-colouring φ''. But then $\varphi \cup \varphi''$ is an L-colouring of G, a contradiction. Thus we may assume K'' is exceptional.

Suppose that K'' is an exceptional canvas of type (i), and so that there exists a vertex $u \in A''$ adjacent to both v_1 and v_4. Since K is unexceptional it follows that $u \in A'' \setminus A$. This is a contradiction, since as noted above there are no edges between $A'' \setminus A$ and $\{v_1, v_4\}$.

Suppose now that K'' is an exceptional canvas of type (ii). Then there exists a vertex $u \in A''$ adjacent to one of v_1 and v_4 and to a vertex y such that either v_4v_3y or v_1v_2y is the principal path of a generalized wheel W_1 where the vertices on the outer cycle of W_1 are on the outer face boundary of G'' and have lists of size at most three under L''. Note that every vertex in a generalized wheel has girth three, and for every vertex $x \in V(G'')$ with $g_G(x) = 3$, if $|L(x)| > 3$ then $|L(x)| \geq 5$ and so $|L''(x)| \geq 4$. It follows that every vertex in the outer cycle of W_1 is in the outer cycle of G. Since K is unexceptional, we thus have that $u \in A'' \setminus A$. This is a contradiction, since there are no edges between $A'' \setminus A$ and $\{v_1, v_4\}$.

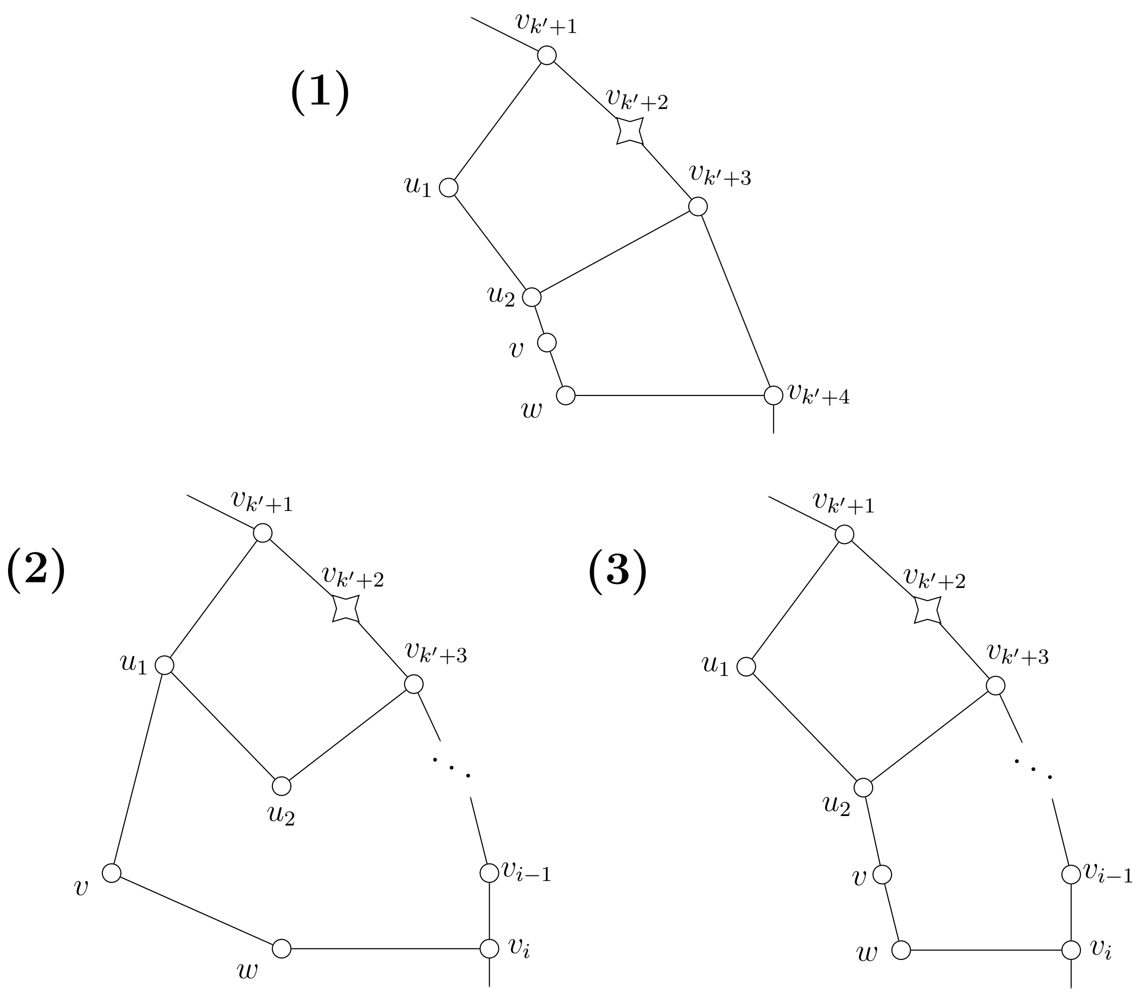

Figure 2.4: Cases considered in Claim 7 of Lemma 2.3.3. Vertices in A are drawn as four-pointed stars. The vertices v, w, u_1, and u_2 all have girth at least five. Recall that by Lemma 2.2.8, since $g(u_1) = 5$ it follows that the 5-cycles in all figures have no vertices in their interior.

Thus we may assume that K'' is an exceptional canvas of type (iii) and thus that there exists a subgraph W_2 that is a generalized wheel with principal path S such that the vertices on the outer cycle of W_2 are on the outer cycle of G'' and have lists of size at most three under L''. As noted above, for every vertex $x \in V(G'')$ with $g_G(x) = 3$, if $|L(x)| > 3$ then $|L(x)| \geq 5$ and so $|L''(x)| \geq 4$. It follows that every vertex x in the outer cycle of

W_2 is in the outer cycle of G and has $|L(x)| = |L''(x)|$. Since K is unexceptional, this is a contradiction.

Case 2. $i \geq k' + 5$**.** See Figure 2.4 (2) and (3), above, for an illustration of these cases. If v is adjacent to u_1, define $Q = v_{k'+1}u_1vwv_i$. If v is adjacent to u_2, define $Q = v_{k'+3}u_2vwv_i$. In either case, the path Q separates G into two graphs H_1 and H_2 where without loss of generality $S \subseteq H_1$. By Observation 2.2.1, $K[H_1]$ is an unexceptional canvas. By the minimality of K, we therefore have that H_1 admits an L-colouring φ.

Let L'' be a list assignment defined by $L''(x) = L(x)$ for all $x \in V(H_2) \setminus V(Q)$, and $L''(x) = \{\varphi(x)\}$ for all $x \in V(Q)$. Note that H_2 is 2-connected. Let C_2 be the outer cycle of H_2. By Claim 2.3.2 and the fact that $g(w) \geq 5$, we have that $N_G(w) \cap V(P) = \{v_i\}$. Similarly, $N_G(u_1) \cap V(P) = \{v_{k'+1}\}$ and $N_G(u_2) \cap V(P) = \{v_{k'+3}\}$. Since v is adjacent to one of u_1 and u_2, it follows from Lemma 4 that that $N_G(v) \cap V(P) = \emptyset$. Finally, since $g(x) \geq 5$ for all $x \in \{u_1, u_2, v, w\}$, it follows that C_2 is chordless. Since $i \geq k' + 5$ and $V(P) \cap A = \{k' + 2\}$, it follows from Observation 2.2.5 that $|L(v_{i-1})| = 3$.

Thus $H_2[V(C_2)]$ admits an L''-colouring φ' obtained by extending φ to C_2 by colouring the vertices of $V(C) \cap V(C_2)$ in increasing order of index. Let H_2' be obtained from H_2 by deleting the vertices in $V(C_2) \cap V(C)$. Let C_2' be the graph whose vertex- and edge-set are precisely those of the outer face boundary of H_2', and let $S' = Q - C$. Let L''' be the list assignment obtained from L by setting $L'''(v) = L(v) \setminus \{\varphi(x) : x \in N_{H_2}(v)\}$ for all $v \in V(H_2') \setminus V(S')$, and $L'''(u) = \{\varphi'(u)\}$ for all $u \in V(S')$. Let $A_2 \subseteq V(C_2') \setminus V(S')$ be the set of vertices with lists of size at most two under L'''. Note that S' is an acceptable path, since it has exactly three vertices. We claim (H_2', L''', S', A_2) is a canvas. To see this, note that since every vertex $x \in V(C_2') \setminus V(S')$ of girth three has $|L(x)| \geq 5$ and every vertex $y \in V(C_2') \setminus V(S')$ has $|L(y)| \geq 4$, it follows from Lemma 2.3.2 that every vertex $u \in V(C_2') \setminus V(S')$ with $g_G(u) \in \{3, 4\}$ has $|L'''(u)| \geq 3$. Thus A_2 contains only vertices of girth at least five in G. Note that $A_2 \cap A = \emptyset$, and hence $|L(u)| \geq 3$ for every $u \in A_2$. It therefore follows from Lemma 2.3.2 that every vertex in A_2 has a list of size exactly two under L'''. It remains to show that A_2 is an independent set. To see this, suppose not. Then there exists an edge $y_1y_2 \in E(G_2)$ with $\{y_1, y_2\} \subseteq A_2$. Note that every vertex in A_2 is adjacent in G to a vertex in $V(P)$. Since $g_G(y_1) \geq 5$ and $g_G(y_2) \geq 5$, it follows from Lemma 2.2.11 and the fact that $V(P) \cap A = \{v_{k'+2}\}$ that one of y_1 and y_2 is adjacent to $v_{k'+1}$. But since $g_G(u_1) \geq 5$, by Lemma 2.2.8 we have that $V(\mathrm{Int}(u_1u_2v_{k'+3}v_{k'+2}v_{k'+1})) = \emptyset$. Thus no vertex in A_2 is adjacent to $v_{k'+1}$ in G, a contradiction.

Thus K_2 is a canvas. Recall that $w \in V(S')$ has girth at least five. Since $V(S') = 3$, it follows that K_2 is unexceptional. By the minimality of K, we have that H_2' admits an L'''-colouring φ''. But then $\varphi'' \cup \varphi' \cup \varphi$ is an L-colouring of G, a contradiction. □

By the previous claims, we have that $K' = (G', L', S, A')$ is a canvas. We now show it is unexceptional.

Claim 8. *K' is unexceptional.*

Proof. Suppose not. First suppose K' is an exceptional canvas of type (iii). Then G' contains a generalized wheel W such that the vertices on the outer cycle of W are on the outer cycle of G' and have lists of size at most three under L'. Let $v_1v_2v_3w_1 \dots w_tv_1$ be the outer cycle of W. Since $|L'(w_1)| = 3$, it follows that $|L(w_1)| \geq 3$. Thus $w_1 \notin A$. Moreover, $w_1 \neq v_{k'+1}$ since $v_{k'+1} \notin V(G')$. It follows that $w_1 \neq v_4$, and so that $w_1 \in V(\text{Int}(C))$. Thus $|L(w_1)| \geq 5$, and so we have that w_1 is adjacent to at least two vertices in $V(P) \cup \{u_1, u_2\}$. Since both u_1 and u_2 have girth at least five, it follows that w_1 is adjacent to a vertex v_ℓ in $V(P)$, with $\ell \geq k' + 3$. Since w_1 is adjacent to v_3, by Lemma 2.2.12 we have that w_1 is also adjacent to $v_{k'+1}, v_{k'+2}, \dots, v_{\ell-1}$. This contradicts the fact that $v_{k'+2} \in A$ by Lemma 2.3.1, and thus $g(v_{k'+2}) \geq 5$.

Next, suppose that K' is an exceptional canvas of type (i). Then there exists a vertex $u \in A'$ adjacent to v_1 and v_4. Since K is unexceptional, $u \notin A$. Thus $u \in A' \setminus A$. By Claim 3, every vertex $x \in V(C) \setminus V(S)$ has $L(x) = L'(x)$, and hence $u \notin V(C)$. It follows that $u \in V(\text{Int}(C))$. But then v_1uv_4 contradicts Lemma 2.2.10.

We may thus assume that K' is exceptional of type (ii): and in particular, that there exists a vertex $u \in A'$ such that u is adjacent to either v_1 or v_4; and such that u is adjacent to a vertex $w \in V(C') \setminus V(S)$ where v_4v_3w or v_1v_2w is the principal path of a generalized wheel W where the vertices on the outer cycle of W are on the outer cycle of G' and all have lists of size at most three under L'. Note that since C is chordless by Lemma 2.2.3, it follows that $w \notin V(C)$ and so that $w \in V(\text{Int}(C))$.

First suppose that $u \in A$. Since $g(w) = 3$ and $w \in V(\text{Int}(C))$, it follows that $|L(w)| \geq 5$: thus in G, w is adjacent to two vertices in $V(P) \cup \{u_1, u_2\}$. Note that since each of u_1 and u_2 have girth at least five, it follows that w is adjacent to at most one of u_1 and u_2, and thus that w is adjacent to a vertex v_i in P. But since $u \in A$ and w is adjacent to both v_i and u, this contradicts Lemma 2.2.10.

Thus we may assume that $u \in A' \setminus A$, and so that u is adjacent to a vertex $x \in V(P) \cup \{u_1, u_2\}$. If $x \in V(P)$, this contradicts Lemma 2.2.10 since u is also adjacent to one of v_1 and v_4. Thus u is adjacent to one of u_1 and u_2.

First suppose u is adjacent to u_1. If u is adjacent to v_1, then by Lemma 2.2.11 applied to u and u_1 we have that v_1 is adjacent to $v_{k'+2}$. Thus $q = k' + 2$, a contradiction to either Lemma 2.2.16 or Lemma 2.2.17. Thus we may assume that u is adjacent to v_4, and by

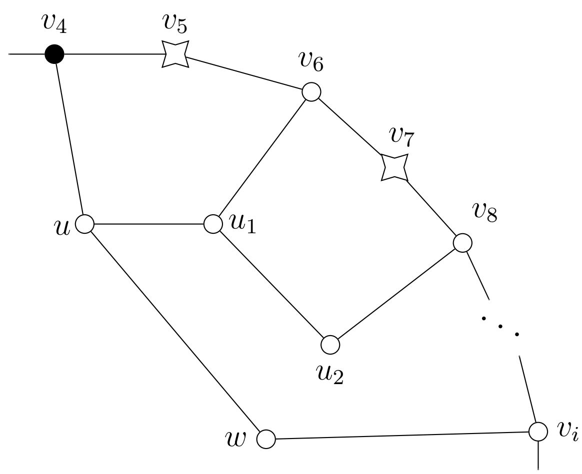

Figure 2.5: A case considered in Claim 8. Vertices in A are drawn as four-pointed stars. Both u_1 and u_2 have girth five. Recall that by Lemma 2.2.8, since $g(u_1) = 5$ it follows that the 5-cycles in all figures have no vertices in their interior.

Lemma 2.2.11 applied to u and u_1, we have that $v_5 \in A$. Thus in this case $k' = k + 1 = 5$. See Figure 2.5, below. Note that since $|L'(w)| = 3$, we have that in G, w is adjacent to two vertices in $V(P) \cup \{u_1, u_2\}$. Since u_1 has girth five and u is adjacent to u_1, it follows that w is adjacent to a vertex $v_i \in V(P)$. We claim $i \geq 8$: this follows from the fact that u_2 has girth five. The path v_4uwv_i separates G into two graphs G_1 and G_2, where without loss of generality $S \subset G_1$. By Observation 2.2.1, $K[G_1]$ is an unexceptional canvas. By the minimality of K, it follows that G_1 admits an L-colouring φ. Let L'' be the list assignment for G_2 obtained from L by setting $L''(v) = L(v)$ for all $v \in V(G_2) \setminus \{v_4, u, w, v_i\}$ and setting $L''(v) = \{\varphi(v)\}$ for all $v \in \{v_4, u, w, v_i\}$. Note that since $g(u) \geq 5$, we have that v_4uwv_i is an acceptable path for G_2. Moreover, $K_2 = (G_2, L'', v_4uwv_i, A \cap V(G_2))$ is a canvas. Since $A \cap V(G_2) = \{v_5, v_7\}$ and $i \geq 8$, it follows from the fact that C is chordless (Lemma 2.2.3) that K_2 is not an exceptional canvas of type (i). Since G is planar, we have that w is not adjacent to v_6. Since $g(u_1) \geq 5$, we have furthermore that u is not adjacent to v_6. It follows that K_2 is not an exceptional canvas of type (ii). Finally, K_2 is trivially not an exceptional canvas of type (iii) since v_4uwv_i has four vertices. Since $|V(G_2)| < |V(G)|$, it follows from the minimality of K that K_2 admits an L''-colouring φ''. As $\varphi'' \cup \varphi$ is an L-colouring of G, this is a contradiction.

We may thus assume u is adjacent to u_2. Recall that w is adjacent to one of v_2 and v_3, and so since G is planar u is not adjacent to v_1. Thus u is adjacent to v_4. By Lemma

2.2.11 applied to u_2 and u, there exists a vertex in A adjacent to v_4 and $v_{k'+3}$. Since C is chordless by Lemma 2.2.3, this is a contradiction. □

Since K is a minimum counterexample to Theorem 2.1.6 and $|V(G')| < |V(G)|$, it follows that G' admits an L'-colouring ϕ'. Recall that by the properties of ϕ described in Claim 2, we have that $\phi(v_{k'+1})$ is not the available colour at $v_{k'}$, and so $\phi'(v_{k'}) \neq \phi(v_{k'+1})$. But then $\phi \cup \phi'$ is an L-colouring of G, contradicting that K is a counterexample to Theorem 2.1.6. □

Chapter 3

Hyperbolicity Theorems for Correspondence Colouring

Subsection 3.1.1 lists the main results of this chapter. In Subsection 3.1.2, we outline the content of the chapter. Finally, in Subsection 3.1.3 we give a brief overview of the main proof of Chapter 3: the proof of Theorem 3.4.7.

3.1 Introduction

3.1.1 Results

The main result of this chapter is Theorem 3.4.7. We will delay the statement of Theorem 3.4.7 until later on in the chapter, when we will have defined the necessary terminology. As covered in Chapter 1, Theorem 3.4.7 implies several other results; for instance, the theorem below (which will be proved in Section 3.6).

Theorem 1.2.23. *Let G be a plane graph with outer cycle C, let (L, M) be a 5-correspondence assignment for G, and let H be a minimal subgraph of G such that every (L, M)-colouring of C that extends to an (L, M)-colouring of H also extends to an (L, M)-colouring of G. Then H has at most $51|V(C)|$ vertices.*

Theorem 1.2.23 is the correspondence colouring analogue to the following theorem of Postle and Thomas [35], which settled a conjecture of Dvořák et al. [14]

Theorem 1.2.21 (Postle and Thomas, [35])**.** *Let G be a plane graph with outer cycle C, let L be a 5-list assignment for G, and let H be a minimal subgraph of G such that every L-colouring of C that extends to an L-colouring of H also extends to an L-colouring of G. Then H has at most $19|V(C)|$ vertices.*

Theorem 3.4.7 also implies that the family of embedded graphs G that are 5-correspondence critical is hyperbolic (see Definition 1.2.14). We will delay the proof of this fact until Section 3.6, where we will also discuss the implications of the hyperbolicity of this family of graphs. In particular, we will show that locally planar graphs are 5-correspondence-colourable (Theorem 1.2.20), and observe that there exist linear-time algorithms for the decidability of 5-correspondence colouring of embedded graphs (Theorems 1.2.24 and 1.2.25).

Finally, in Section 3.7 we also observe that the family of embedded graphs G of girth at least five that are critical for 3-correspondence colouring is hyperbolic (Theorem 3.7.4). This follows from observing that the analogous proof for list colouring in [34] also holds for correspondence colouring with very minor modifications. This in turn has several interesting implications: we observe as in the girth three case that locally planar graphs of girth at least five are 3-correspondence-colourable (Theorem 1.2.28), and that there exist linear-time algorithms for the decidability of 3-correspondence colouring of embedded graphs of girth at least five (Theorems 1.2.26 and 1.2.27).

Chapters 3 and 4 make use of the following theorem, due to Thomassen.

Theorem 3.1.1 (Thomassen [40])**.** *Let G be a planar graph with outer face boundary walk C. Let S be a path of length at most one contained in C. Let (L, M) be a correspondence assignment for G where $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$, and where $|L(v)| \geq 3$ for all $v \in V(C) \setminus V(S)$. Every (L, M)-colouring of S extends to an (L, M)-colouring of G.*

Thomassen originally stated this for list colouring (see Theorem 2.1.7). However, as pointed out by Dvořák and Postle in [15], the proof carries over to correspondence colouring.

3.1.2 Chapter Outline

Subsection 3.1.3 gives a brief overview of the proof of the main theorem of this chapter (Theorem 3.4.7). In Section 3.2, we establish a few basic results and definitions used in the proof of Theorem 3.4.7. Section 3.3 introduces the notion of *deficiency*, a critical measurement that will be used throughout the rest of the thesis. Section 3.4 establishes yet more useful definitions and results, and concludes with the statement of Theorem 3.4.7. Many proofs in these sections are taken from [35], where they were originally written for list colouring. We include them for completeness, adapting them to the correspondence colouring framework when required. Section 3.5 contains the proof of Theorem 3.4.7. We note that Section 3.5 itself contains several subsections; Section 3.5 will begin with an overview of the contents of each subsection.

3.1.3 Main Proof Overview

The proof of Theorem 3.4.7 constitutes the bulk of this chapter. The proof 3.4.7 has two main parts: the first involves purely the structure of a minimum counterexample G to Theorem 3.4.7, and the second involves arguing about the specific matchings $\{M_e : e \in E(G)\}$ of the correspondence assignment (L, M) of G. Several of our structural results are taken directly from the analogous theorems for list colouring in [35]. Some of these arguments involve the set of vertices X_1 that have at least three neighbours in the outer cycle C of G, as well as the set of vertices X_2 with at least three neighbours in $V(C) \cup X_1$ and at least one neighbour in X_1. Informally, we think of the sets X_1 and X_2 as "layers" near the outer cycle C. These two layers alone do not provide us with enough freedom to force a contradiction in the second part of the proof, unlike in the proof of the analogous theorem for list colouring given in [35]. Our analysis thus involves moving one layer further into the graph, and considering the structure surrounding vertices in the set X_3 of vertices with at least three neighbours in $V(C) \cup X_1 \cup X_2$ and at least one neighbour in X_2. The proof before this point is very similar to that of Postle and Thomas in [35]. It is from this point on —the introduction of this third "layer", X_3 —that the proof diverges substantially.

In the second part of the proof, we argue about the matchings in the correspondence assignment. In particular, Claim 37 establishes very precisely the matchings between vertices $x_1 \in X_1$, $x_2 \in X_2$, and $x_3 \in X_3$ as well as their other neighbours in the graph. We use this claim to finish the proof, showing that for one edge $e \in E(G)$, we have that M_e is not a matching. This contradicts the definition of correspondence assignment, and thus dispels the existence of a minimum counterexample to Theorem 3.4.7.

3.2 Critical Subgraphs

In this section, we establish a few basic definitions and results that will be used throughout the rest of the chapter. As mentioned above, several results in this section are already proved in [35] for list colouring instead of correspondence colouring. In all cases, the results are easily adapted for correspondence colouring. For cohesion, we have included the proofs here as well, along with a few proofs that were omitted from [35]. Whenever a proof is taken directly from [35], we will say so explicitly.

We will need the following definitions.

Definition 3.2.1. Let G be a graph. For a set $X \subseteq V(G)$, we denote by $N(X)$ the set $\left(\bigcup_{v\in X} N(v)\right) \setminus X$.

Definition 3.2.2 (S-critical)**.** Let G be a graph, $S \subseteq G$ a subgraph of G, and (L, M) a correspondence assignment for G. For an (L, M)-colouring ϕ of S, we say that ϕ *extends to an* (L, M)*-colouring* of G if there exists an (L, M)-colouring ψ of G such that $\phi(v) = \psi(v)$ for all $v \in V(S)$. The graph G is S*-critical with respect to* (L, M) if $G \neq S$ and for every proper subgraph $G' \subset G$ such that $S \subseteq G'$, there exists an (L, M)-colouring of S that extends to an (L, M)-colouring of G', but does not extend to an (L, M)-colouring of G. If the list assignment is clear from the context, we shorten this and say that G is S-critical.

Note that the following definition of *canvas* differs from that used in Chapter 2. We will use the definition given below until Section 3.7.

Definition 3.2.3. We say the triple $(G, C, (L, M))$ is a *canvas* if G is a 2-connected plane graph, C is its outer cycle, and (L, M) is a correspondence assignment for the vertices of G such that $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$ and there exists an (L, M)-colouring of C. We say a canvas $(G, C, (L, M))$ is *critical* if G is C-critical with respect to the correspondence assignment (L, M).

These definitions match those given for list colouring in [35], with the appropriate adjustments for correspondence colouring instead of list colouring.

In addition to being used below to establish helpful corollaries regarding subgraphs of critical canvases, the following lemma will be used in Section 3.6 to show that the family of embedded graphs that are critical for 5-correspondence colouring form a hyperbolic family.

Lemma 3.2.4 (Proof taken from Lemma 2.3, [35])**.** *Let T be a subgraph of a graph G that is T-critical with respect to the correspondence assignment (L, M). Let $G = (A, B)$ be a separation of G such that $T \subseteq A$ and $B \neq \emptyset$. Then $G[V(B)]$ is $A[V(A) \cap V(B)]$-critical.*

Proof. Let $G' = G[V(B)]$ and $S = A[V(A) \cap V(B)]$. Since G is T-critical, every isolated vertex of G belongs to T, and thus every isolated vertex of G' belongs to S. Suppose for a contradiction that G' is not S-critical. Then, there exists an edge $e \in E(G') \setminus E(S)$ such that every (L, M)-colouring of S that extends to $G' \setminus e$ also extends to G'. Note that $e \notin E(T)$. Since G is T-critical, there exists a colouring Φ of T that extends to an (L, M)-colouring ϕ of $G \setminus e$, but does not extend to an (L, M)-colouring of G. However, by the choice of e, the restriction of ϕ to S extends to an (L, M)-colouring ϕ' of G'. Let ϕ'' be the colouring that matches ϕ' on $V(G')$ and ϕ on $V(G) \setminus V(G')$. Observe that ϕ'' is an (L, M)-colouring of G extending Φ, which is a contradiction. □

Our main theorem characterises planar graphs that are outer cycle-critical (and so planar graphs whose outer face boundary walk is bounded by a cycle). Note that though canvases are 2-connected by definition, the same is not true for critical graphs. The observation below motivates restricting our attentions to 2-connected graphs.

Observation 3.2.5 (Proof taken from Lemma 2.5, [35]). *Let G be a plane graph with outer cycle C, and let (L, M) be a correspondence assignment for G such that G is C-critical with respect to (L, M). Then $(G, C, (L, M))$ is a canvas.*

Proof. By the definition of canvas, it suffices to show that G is 2-connected. Suppose not. Then G contains two subgraphs A and B such that $A \cup B = G$, $C \subseteq A$, $|V(A \cap B)| \leq 1$, and $V(B) \setminus V(A) \neq \emptyset$. By Lemma 3.2.4, we have that $G[V(B)]$ is $A[V(A) \cap V(B)]$-critical. This contradicts Theorem 3.1.1. □

Before stating the implications of Lemma 3.2.4, we give the following definition.

Definition 3.2.6. Let $T = (G, C, (L, M))$ be a canvas, and let G' be a plane graph obtained from G by adding a (possibly empty) set of edges. If C' is a cycle in G', we let $G\langle C'\rangle$ denote the subgraph of $G \cup C'$ contained in the closed disk bounded by C'. We let $T\langle C'\rangle$ denote the canvas $(G\langle C'\rangle, C', (L, M))$. Similarly, if G' is a subgraph of G and f is a face of G', we denote by $G\langle f\rangle$ the subgraph of G contained in the closed disk given by the boundary walk of f, and let $T\langle f\rangle = (G\langle f\rangle, C_f, (L, M))$, where C_f is the cycle given by the boundary walk of f.

Note that the boundary walk of f is indeed a cycle since T is a canvas (and is thus 2-connected). The following useful corollary follows from Lemma 3.2.4.

Corollary 3.2.7 (Proof taken from Corollary 2.7, [35]). *Let $T = (G, C, (L, M))$ be a critical canvas. If C' is a cycle in G such that $G\langle C'\rangle \neq C'$, then $T\langle C'\rangle$ is a critical canvas.*

Proof. Let $B = G\langle C'\rangle$ and $A = G \setminus (B \setminus C')$. By applying Lemma 3.2.4, it follows that $G\langle C'\rangle$ is C'-critical. □

We will require the following definition.

Definition 3.2.8. Let $T = (G, C, (L, M))$ be a canvas and $G' \subseteq G$ such that $C \subseteq G'$ and G' is 2-connected. We define the *subcanvas* of T induced by G' to be $(G', C, (L, M))$ and we denote it by $T[G']$.

Note that in the above definition, the outer cycle of G' is the outer cycle of G.

The proof of the following helpful fact is omitted from [35]; we have included it here for completeness.

Proposition 3.2.9 (Proposition 2.9, [35])**.** *Let $T = (G, C, (L, M))$ be a canvas such that there exists a proper (L, M)-colouring of C that does not extend to G. Then T contains a critical subcanvas.*

Proof. Let ϕ be an (L, M)-colouring of C that does not extend to G. Let X be the set of subgraphs H of G containing C such that ϕ does not extend to H. Note that X is non-empty, since $G \in X$. Let $G' \in X$ be chosen to minimize $|V(G')|$ and subject to that, to minimize $|E(G')|$. By our choice of G', we have that ϕ extends to every proper subgraph of G' but not to G' itself, and so that G' is C-critical. Thus $(G', C, (L, M))$ is a critical subcanvas of T. □

Below, we establish some of the structure of critical canvases.

Theorem 3.2.10 (Proof taken from Theorem 2.10, [35])**.** *(Chord or Tripod Theorem) If $T = (G, C, (L, M))$ is a critical canvas, then either*

1. *C has a chord in G, or*
2. *there exists a vertex of G with at least three neighbours on C, and at most one of the internal faces of $G[\{v\} \cup V(C)]$ includes a vertex or edge of G.*

Proof. Suppose C does not have a chord. Let X be the set of vertices with at least three neighbours in $V(C)$. Let G' be the subgraph of G defined by $V(G') := C \cup X$ and $E(G') := E(G[C \cup X]) \setminus E(G[X])$. We claim that if f is a face of G' such that f is incident with at most one vertex of X, then f does not include a vertex or edge of G. To see this, suppose not. Let C' be the boundary of f. Since C has no chords and every edge with one end in X and the other in C is in $E(G')$, it follows that C' has no chords. Since T is critical, there exists an (L, M)-colouring ϕ of $G \setminus (V(G\langle C'\rangle) \setminus V(C'))$ which does not extend to G. Hence the restriction of ϕ to C' does not extend to $V(G\langle C'\rangle)$.

Let $G'' := G\langle C'\rangle \setminus V(C)$, let $S = V(C') \setminus V(C)$, let $L'(v) = \{\phi(v)\}$ for $v \in S$ and $L'(v) = L(v) \setminus \{v[x, \phi(x)] : x \in V(C) \cap N(v)\}$ for $v \in V(G'') \setminus S$. Note that $|L'(v)| \geq 3$ for all $v \notin S$ by definition of X. By Theorem 3.1.1, there exists an (L', M)-colouring ϕ' of G''. But then $\phi' \cup \phi$ is an (L, M)-colouring of G, a contradiction. This proves the claim.

Since T is critical, $G \neq C$. Since C has no chords, it follows from the claim above that $X \neq \emptyset$. Let $\mathcal{F}$ be the set of internal faces of G' incident with at least two elements of X. Consider the graph H whose vertices are $X \cup \mathcal{F}$, where a vertex $x \in X$ is adjacent to $f \in \mathcal{F}$ if x is incident with f. By planarity, H is a tree. Let v be a leaf of H. By the definition of H, we have that $v \in X$. Hence at most one of the faces of $G[\{v\} \cup V(C)]$ is incident with another vertex of X. Yet all other faces of $G[\{v\} \cup V(C)]$ are incident with only one element of X, namely v, and so by the claim above these faces do not include a vertex or edge of C, as desired. □

The following easy facts are very useful and will be used throughout the proof of Theorem 3.4.7. The proofs are omitted from [35], but we have included them here for completeness.

Proposition 3.2.11 (Proposition 2.11, [35]). *If $T = (G, C, (L, M))$ is a critical canvas, then*

1. *for every cycle C' of G of length at most four, $V(G\langle C'\rangle) = V(C')$, and*
2. *every vertex in $V(G) \setminus V(C)$ has degree at least five.*

Proof. We begin with the first statement. Suppose not, and let C' be a cycle in G of length at most four such that $V(G\langle C'\rangle) \neq V(C)$. Let $C' = v_1v_2v_3v_1$ if C' is a triangle, and $C' = v_1v_2v_3v_4v_1$ if C' is a 4-cycle. Since G is C-critical, there exists an (L, M)-colouring ϕ of C that extends to every proper subgraph of G but not to G itself. Then ϕ extends to an (L, M)-colouring ϕ' of $G \setminus V(\text{Int}(C'))$. Let (L', M') be a list assignment for $G' = G[V(\text{Int}[C']) \setminus \{v_3, v_4\}]$ where $L'(v_i) = \phi(v_i)$ for $i \in \{1, 2\}$, where $L'(v) = L(v) \setminus \{v[u, \phi(u)] : u \in \{v_3, v_4\} \cap N(v)\}$ for all $v \in V(G') \setminus V(C')$. Then by Theorem 3.1.1, G' admits an (L', M)-colouring ϕ''. But $\phi'' \cup \phi'$ is an extension of ϕ to G, a contradiction.

We now prove that every vertex in $V(G) \setminus V(C)$ has degree at least 5. To see this, suppose not: let $v \in V(G) \setminus V(C)$ have degree at most 4. Since G is C-critical, there exists an (L, M)-colouring ϕ of C extends to every proper subgraph of G but not to G itself. Thus ϕ extends to an (L, M)-colouring ϕ' of $G - v$. Since $|L(v)| \geq 5$ and $\deg(v) \leq 4$, it follows that ϕ' extends to v, a contradiction.

□

3.3 Deficiency

This section introduces *deficiency*, a measure defined by Postle and Thomas in [35]. Our main theorem —Theorem 3.4.7 —concerns the deficiency of critical canvases.

Definition 3.3.1. Let G be a plane graph, and let C be the subgraph of G whose edge- and vertex-set are precisely those of the outer face boundary walk of G. We call a vertex $v \in V(G)$ *internal* if $v \in V(G) \setminus V(C)$. We denote by $v(G)$ the number of internal vertices of G. If $T = (G, C, (L, M))$ is a canvas, we define $v(T) = v(G)$. We denote by $\mathcal{F}(G)$ the set of finite faces of G; given a face f of G, we denote by $|f|$ the length of the boundary walk of f.

Definition 3.3.2. Let G be a plane graph, and $H \subseteq G$. The *deficiency of G with respect to H* is defined as $\text{def}(G|H) := |E(G) \setminus E(H)| - 3|V(G) \setminus V(H)|$. When H is clear from context, we sometimes omit it and speak only of the *deficiency of G*, denoted $\text{def}(G)$. Given a canvas $T = (G, C, (L, M))$, we define $\text{def}(T) := \text{def}(G|C) = |E(G) \setminus E(C)| - 3v(G)$.

Lemma 3.3.3 (Proof taken from Lemma 3.4, [35])**.** *Let G be a 2-connected plane graph with outer cycle C, and let G' be a 2-connected subgraph of G containing C. Then*

$$\text{def}(G) = \text{def}(G') + \sum_{f \in \mathcal{F}(G')} \text{def}(G\langle f \rangle).$$

Proof. The lemma follows from the fact that every internal vertex of G is an internal vertex of exactly one of G' and $G\langle f \rangle$ for each $f \in \mathcal{F}(G')$, and the same holds for edges not incident with the outer face. □

Theorem 3.3.4 (Proof taken from Theorem 3.5, [35])**.** *If $T = (G, C, (L, M))$ is a critical canvas, then* $\text{def}(T) \geq 1$.

Proof. We proceed by induction on the number of vertices of G. We consider each outcome of Theorem 3.2.10 applied to T: first suppose that C has a chord, e. Let C_1 and C_2 be the cycles of $C + e$ other than C. Hence $|V(C_1)| + |V(C_2)| = |V(C)| + 2$. For $i \in \{1, 2\}$, let $T_i = T\langle C_i \rangle = (G_i, C_i, (L, M))$. By Lemma 3.3.3 applied to $G' = C + e$, we have that $\text{def}(T) = \text{def}(T_1) + \text{def}(T_2) + 1$. By Corollary 3.2.7, for $i \in \{1, 2\}$ either T_i is critical or $G_i = C_i$. If $G_i \neq C_i$, then $\text{def}(T_i) \geq 1$ by induction. If $G_i = C_i$, then $\text{def}(T_i) = 0$ by definition. In either case, $\text{def}(T) \geq 0 + 0 + 1 \geq 1$, as desired.

We may therefore assume that C is chordless, and so by Theorem 3.2.10 there exists an internal vertex of G with at least three neighbours in $V(C)$ such that at most one of the faces of $G[V(C) \cup \{v\}]$ contains an edge or vertex of G. Let $G' = G[V(C) \cup \{v\}]$. First suppose that none of the faces of G' contains an edge or vertex of G, and hence that $V(G) = V(C) \cup \{v\}$. Since G is C-critical, it follows from Proposition 3.2.11 (2) that v has degree at least 5. Thus $\text{def}(T) \geq 5 - 3 \cdot 1 = 2$, as desired. Thus we may assume that exactly one of the faces of G' contains a vertex or edge of G. Let C' be the cycle of G' bounding that face. Note that by Corollary 3.2.7, $T\langle C' \rangle$ is a critical canvas, and so by induction $\text{def}(T\langle C' \rangle) \geq 1$. Moreover, we claim $\text{def}(T) \geq \text{def}(T\langle C' \rangle)$. To see this, note that by definition $\text{def}(T\langle C' \rangle) = |E(G') \setminus E(C')| - 3v(G') \leq |E(G) \setminus E(C)| - 3 - (\deg(v) - 3(v(G)+1)$ since v has at least three neighbours in $V(C)$. Again using the definition of deficiency, we have that $\text{def}(T\langle C' \rangle) \geq \text{def}(T)$, and so combining these results $\text{def}(T) \geq \text{def}(T\langle C' \rangle) \geq 1$, as desired. □

The following inequality will be helpful in dealing with critical canvases with at most seven internal vertices.

Lemma 3.3.5 (Proof taken from Lemma 3.6, [35]). *Let $T = (G, C, (L, M))$, where G is a 2-connected plane graph with outer cycle C and every internal vertex of G has degree at least five. Then*

$$\text{def}(T) \geq 2v(G) - |E(G \setminus V(C))|,$$

with equality if and only if every vertex of G has degree exactly five.

Proof. Note that

$$\begin{aligned}\text{def}(T) &= |E(G) \setminus E(C)| - 3v(G) \\ &\geq 5v(G) - |E(G \setminus V(C))| - 3v(G) \quad \text{as internal vertices have degree} \geq 5 \text{ by assumption} \\ &= 2v(G) - |E(G \setminus V(C))|,\end{aligned}$$

with equality if and only if every vertex of G has degree exactly five. □

3.4 Linear Bound for Cycles

In this section, we build towards stating Theorem 3.4.7, the proof of which constitutes the bulk of this chapter. We begin with a few necessary definitions.

Definition 3.4.1. Let G be a plane graph. We say vertices u and v in $V(G)$ are *cofacial* if there exists a face f of G that is incident to both u and v.

Definition 3.4.2. Let G be a 2-connected plane graph with outer cycle C. We define the *boundary* of G, denoted $B(G)$, as $N(V(C))$. We define the *quasi-boundary* of T, denoted by $Q(T)$, as the set of vertices not in C that are cofacial with at least one vertex of C (and so $B(T) \subseteq Q(T)$). We let $b(t) := |B(T)|$ and $q(T) := |Q(T)|$. If $T = (G, C, (L, M))$ is a canvas, then we extend the above notions to T in the obvious way, defining $B(T) := B(G)$ and $Q(T) := Q(G)$.

For the remainder of the chapter, let ε and α be fixed positive real numbers. Theorem 3.4.7 depends on ε and α and holds as long as these two numbers satisfy three inequalities listed in the theorem statement. In Section 3.6, we will make a specific choice of ε and α in order to optimize the constant in Theorem 1.2.23. Before proceeding, we need one final definition.

Definition 3.4.3. Let G be a 2-connected plane graph with outer cycle C. We define $s(G) := \varepsilon \cdot v(G) + \alpha(b(G) + q(G))$ and $d(G) := \text{def}(G|C) - s(G)$. If $T = (G, C, (L, M))$ is a canvas, we extend these notions to T in the obvious way, defining $s(T) := s(G)$ and $d(T) := d(G)$.

Below, we establish useful properties of the quantities introduced in the above definitions.

Proposition 3.4.4 (Proof taken from Proposition 4.3, [35])**.** *Let G be a 2-connected plane graph with outer cycle C, and let G' be a 2-connected subgraph of G containing C as a subgraph.*

- $v(G) = v(G') + \sum_{f \in F(G')} v(G\langle f \rangle)$,
- $b(G) \leq b(G') + \sum_{f \in F(G')} b(G\langle f \rangle)$,
- $q(G) \leq q(G') + \sum_{f \in F(G')} q(G\langle f \rangle)$,
- $s(G) \leq s(G') + \sum_{f \in F(G')} s(G\langle f \rangle)$,
- $d(G) \geq d(G') + \sum_{f \in F(G')} d(G\langle f \rangle)$.

Proof. For $f \in \mathcal{F}(G')$, let C_f denote the cycle bounding f. The first assertion follows from the fact that every vertex of $V(G) \setminus V(C)$ is in exactly one of $G' \setminus V(C)$ and $G\langle f \rangle \setminus V(C_f)$ for $f \in \mathcal{F}(G')$, and every vertex in one of those sets is in $V(G) \setminus V(C)$. The second assertion follows from the claim that $B(G) \subseteq B(G') \cup \bigcup_{f \in \mathcal{F}(G')} B(G\langle f \rangle)$. To see this claim, suppose that $v \in B(G)$. Now $v \in B(G)$ if and only if v has a neighbour u in $V(C)$. If $v \in V(G')$, then $v \in B(G')$. So we may assume that v is a vertex of $G\langle f\rangle \setminus V(C_f)$ for some $f \in \mathcal{F}(G')$. Then $u \in V(C_f)$ and hence $v \in B(G\langle f\rangle)$. The third assumption follows from the claim that $Q(T) \subseteq Q(G') \cup \bigcup_{f\in\mathcal{F}(G')} Q(G\langle f\rangle)$. That claim follows from the same argument as above, except that $u \in V(C)$ is cofacial with v instead of a neighbour of v. The fourth statement follows from the first three. The fifth statement follows from the fourth and Lemma 3.3.3. □

As a corollary to this, we obtain the following.

Corollary 3.4.5 (Proof taken from Corollary 4.4, [35]). *Let G be a 2-connected plane graph with outer cycle C. If e is a chord of C and C_1, C_2 are the cycles of $C + e$ other than C, then*

$$d(G) \geq d(G\langle C_1\rangle) + d(G\langle C_2\rangle) + 1.$$

If v is a vertex with two neighbours $u_1, u_2 \in V(C)$ and C_1, C_2 are cycles such that $C_1 \cap C_2 = u_1vu_2$ and $C_1 \cup C_2 = C + u_1vu_2$, then

$$d(G) \geq d(G\langle C_1\rangle) + d(G\langle C_2\rangle) - 1 - (2\alpha + \varepsilon).$$

Proof. Both formulas follows from Proposition 3.4.4 applied to $G' := C_1 \cup C_2$. □

It will be convenient to be able to refer to the facts below, which follow directly from the definition of $d(\cdot)$.

Proposition 3.4.6 (Proposition 4.5, [35]). *Let $T = (G, C, (L, M))$ be a canvas.*

(i) If $v(G) = 0$, then $d(T) = |E(G) \setminus E(C)|$.

(ii) If $v(G) = 1$, then $d(T) = |E(G) \setminus E(C)| - 3 - (2\alpha + \varepsilon)$.

Having defined all necessary quantities, we are now equipped to state the main theorem of this chapter. We remind the reader that the definition of *canvas* used in Chapter 3 differs from that in Chapter 2: see Definition 3.2.3.

Theorem 3.4.7. *Let $\varepsilon, \alpha, \gamma > 0$ satisfy the following:*

(I1) $2\varepsilon \leq \alpha$

(I2) $14\alpha + 7\varepsilon \leq \gamma$, and

(I3) $\gamma + 6\alpha + 3\varepsilon \leq 1$

If $T = (G, C, (L, M))$ is a critical canvas and $v(G) \geq 2$, then $d(T) \geq 3 - \gamma$.

3.5 Proof of Theorem 3.4.7

This section contains a proof of Theorem 3.4.7. Throughout this section, let $T = (G, C, (L, M))$ be a counterexample to Theorem 3.4.7 such that $|E(G)|$ is minimum; subject to that, such that $\sum_{v \in V(G)} |L(v)|$ is minimum; and subject to that, such that $\sum_{e \in E(G)} |M_e|$ is maximum. Recall that by Lemma 3.2.11, there is no cycle C' in G of length at most four with $G\langle C' \rangle \neq C'$; and moreover that $\deg(v) \geq 5$ for all internal vertices v of G.

The following claim establishes that G contains at least eight internal vertices. A similar claim (showing $v(G) \geq 5$ instead of $v(G) \geq 8$) can be found in [35] as Claim 5.1.

Claim 9. $v(T) \geq 8$.

Proof. Suppose not. Note that $s(G) \leq v(G)(\varepsilon + 2\alpha)$, and hence $d(G) \geq \text{def}(G) - 7(2\alpha + \varepsilon)$. Since $14\alpha + 7\varepsilon \leq \gamma$ by (I2) and T is a counterexample to Theorem 3.4.7, it follows that $\text{def}(G) < 3$. Since deficiency is integral, $\text{def}(G) \leq 2$.

Let $m = |E(G \setminus V(C))|$. By Lemma 3.3.5, $\text{def}(G) \geq 2v(G) - m$, and if equality holds every vertex of G has degree exactly 5. With this in mind, we consider the inequality $2 \geq \text{def}(G) \geq 2v(G) - m$ for each value of $v(G) \in \{2, 3, \ldots, 7\}$.

Since $2 \geq \text{def}(G) \geq 2v(G) - m$, if $v(G) = 2$ this implies $m \geq 2$. This is a contradiction, since G is a simple graph. Similarly, since $2 \geq \text{def}(G) \geq 2v(G) - m$, if $v(T) = 3$ this implies $m \geq 4$. This too is a contradiction.

Thus we may assume that $v(G) \geq 4$. If $v(G) = 4$, then $2 \geq 2 \cdot 4 - m$ and so $m \geq 6$. This implies $G \setminus V(C)$ is a complete graph; and singe G is planar, the outer face boundary walk of $G \setminus V(C)$ is a triangle. This contradicts Proposition 3.2.11 (1).

If $v(T) = 5$, then $2 \geq 2 \cdot 5 - m$ and so $m \geq 8$. If $m \geq 9$, then $G \setminus V(C)$ is a triangulation, contradicting Proposition 3.2.11 (1). Thus we may assume that $m = 8$. It follows that the outer face boundary walk of $G \setminus V(C)$ is not a 5-cycle, as otherwise $m \leq 7$. If it is a cycle of length at most 4, this contradicts Proposition 3.2.11 (1). Thus the outer face boundary walk of $G \setminus V(C)$ is not a cycle; but then $m \leq 6$, a contradiction.

Thus we may assume that $v(G) \in \{6, 7\}$. If $v(T) = 6$, then $2 \geq 2 \cdot 6 - m$ and so $m \geq 10$. If $m \geq 11$, the outer face boundary walk of $G \setminus V(C)$ is a 4-cycle, contradicting Proposition 3.2.11 (1). Thus we may assume $m = 10$, and moreover that the outer face boundary walk of $G \setminus V(C)$ is not a cycle of length at most 4. If it is a cycle of length 6, then $m \leq 9$, a contradiction. If the outer face boundary walk of $G \setminus V(C)$ is not a cycle, then $m \leq 7$, again a contradiction. Thus we may assume the outer face boundary walk of $G \setminus V(C)$ is a 5-cycle; and by Lemma 3.3.5, every vertex of G has degree exactly 5. Thus $G \setminus V(C)$ is a 5-wheel. We claim that every (L, M)-colouring of C extends to G. To see this, fix an (L, M)-colouring ϕ of C. By adding a (possibly empty) set of edges to matchings in M, we may assume that $|M_{uv}| = \min\{|L(u)|, L(v)|\}$ as this only makes the task of extending a colouring harder. Let the outer cycle of the 5-wheel $G \setminus V(C)$ be $v_1v_2v_3v_4v_5v_1$, and the central vertex v_6. Let $S(v_1) := L(v_1) \setminus \{d : (v_1, d)(u, \phi(u)) \in M_{uv_1}$ and $u \in N(v_1) \cap V(C)\}$. By our choice of counterexample T and since v_1 has degree 5 in G, it follows that $|S(v_1)| = 3$. Thus there exists a choice of colour $c \in L(v_6)$ such that $(c, v_6)(d, v_1) \notin M_{v_1v_6}$ for all $d \in S(v_1)$. But then ϕ extends to G by first colouring v_6 with c, and then colouring v_2, v_3, v_4, v_5, and v_1 in that order. This contradicts the fact that T is critical.

We may therefore assume that $v(T) = 7$. Then $2 \geq 2 \cdot 7 - m$, and so $m \geq 12$. By Proposition 3.2.11 (1), the outer face boundary walk of of $G \setminus V(C)$ is not a cycle of length at most 4. Suppose first that it is a 5-cycle. Then $m \leq 13$. But since $m \geq 12$ and G is planar, at least one internal vertex of $G \setminus V(C)$ does not have degree 5, contradicting Proposition 3.2.11 (2). Next, suppose the outer face boundary walk of $G \setminus V(C)$ is a 6-cycle. Them $m \leq 12$, and so $m = 12$. By Lemma 3.3.5, every vertex in $G \setminus C$ has degree exactly 5. But then $m \leq 11$, a contradiction. Suppose now the outer face boundary walk of $G \setminus V(C)$ is a 7-cycle. Then $m \leq 11$, a contradiction. Finally, suppose the outer face boundary walk of $G \setminus V(C)$ is not a cycle. Then $m \leq 10$, again a contradiction. □

3.5.1 Proper Critical Subgraphs

Many of our proofs will involve passing to a smaller canvas whose underlying graph and outer cycle are strictly contained within G. It is useful for inductive purposes to be able to bound the deficiency and $d(\cdot)$ of one such canvas in terms of another; the following lemma allows us to do this.

Claim 10 (Proof taken from Claim 5.2, [35])**.** *Suppose $T_0 = (G_0, C_0, (L_0, M_0))$ is a critical canvas with $|E(G_0)| \leq |E(G)|$ and $v(G_0) \geq 2$, and let G' be a proper subgraph of G_0 such that for some correspondence assignment (L', M'), the tuple $(G', C_0, (L', M'))$ is a critical canvas. Then*

(1) $d(T_0) \geq 4 - \gamma$, and

(2) $d(T_0) \geq 4 - 2(2\alpha + \varepsilon)$ if $|E(G_0 \setminus E(G')| \geq 2$ and $|E(G') \setminus E(C_0)| \geq 2$, and

(3) $d(T_0) \geq 5 - 2\alpha - \varepsilon - \gamma$ if $|E(G_0) \setminus E(G')| \geq 2$ and $|E(G') \setminus E(C_0)| \geq 2$ and $v(G_0) \geq 3$.

Proof. Note that the inequality in (3) implies the inequality in (2) implies the inequality in (1) by inequalities (I2) and (I3). Given Proposition 3.4.6 and the fact that T is a minimum counterexample, it follows that for each $f \in \mathcal{F}(G')$, we have $d(T_0\langle f\rangle) \geq 0$ and moreover that if f includes a vertex or edge of G_0 then $d(T_0\langle f\rangle) \geq 1$. In addition, since G' is a proper subgraph of G_0 there exists at least one $f \in \mathcal{F}(G')$ such that f includes a vertex or edge of G_0. Furthermore, if $|E(G_0) \setminus E(G')| \geq 2$, then either there exist two such fs or $d(T_0\langle f\rangle) \geq 2 - (2\alpha + \varepsilon)$ for some $f \in \mathcal{F}(G')$ by Proposition 3.4.6.

Now $d(T_0) \geq d(G') + \sum_{f \in \mathcal{F}(G')} d(T_0\langle f\rangle)$ by Proposition 3.4.4. But as noted above, $\sum_{f \in \mathcal{F}(G')} d(T_0\langle f\rangle)$ is at least 1 and is at least $2 - (2\alpha + \varepsilon)$ if $|E(G_0) \setminus E(G')| \geq 2$. Furthermore, if $v(T_0\langle f\rangle) \geq 2$ then $d(T_0\langle f\rangle) \geq 3 - \gamma$ by the minimality of T.

Assume first that $v(G') \geq 2$. Then $d(G') \geq 3 - \gamma$ as T is a minimum counterexample and $|E(G')| < |E(G_0)| \leq |E(G)|$. Hence $d(T_0) \geq 4 - \gamma$ if $|E(G_0) \setminus E(G')| = 1$ and (1) holds as desired. Otherwise $d(T_0) \geq 5 - (2\alpha + \varepsilon) - \gamma$ and (2) and (3) hold as desired.

So we may assume that $v(G') \leq 1$. Suppose $v(G') = 1$. Then $d(G') \geq 2 - (2\alpha + \varepsilon)$ by Proposition 3.4.6 and criticality. Moreover, there exists $f \in \mathcal{F}(G')$ such that $v(T_0\langle f\rangle) \geq 1$. If $v(T_0\langle f\rangle) \geq 2$, then $d(T_0\langle f\rangle) \geq 3 - \gamma$ as T is a minimum counterexample. As above, $d(T_0) \geq d(G') + d(T_0\langle f\rangle) \geq 5 - (2\alpha + \varepsilon)$ and (3) holds, as desired. If $v(T_0\langle f\rangle) \geq 1$, then $d(T_0\langle f\rangle) \geq 2 - (2\alpha + \varepsilon)$ by Proposition 3.4.6. As above, $d(T_0) \geq d(G') + d(T\langle f\rangle \geq 2(2 - (2\alpha - \varepsilon)) = 4 - 2(2\alpha + \varepsilon)$ and (1) and (2) hold, as desired. Yet if $v(T_0) \geq 3$, there must be two such faces if no face has at least two internal vertices. In that case then, $d(T_0) \geq 3(2 - (2\alpha + \varepsilon)) = 6 - (3(2\alpha + \varepsilon)$ which is at least $5 - (2\alpha + \varepsilon) - \gamma$ by (I2) and so (3) holds, as desired.

So suppose $v(G') = 0$. As $G' \neq C_0$, we have that $d(T_0[G']) \geq |E(G') \setminus E(C_0)|$ by Proposition 3.4.6. As $v(T_0) \geq 2$, either there exists $f \in \mathcal{F}(G')$ such that $v(T_0\langle f\rangle) \geq 2$, or there exist faces $f_1, f_2 \in \mathcal{F}(G')$ such that $v(T_0\langle f_i\rangle) \geq 1$ for $i \in \{1, 2\}$. Suppose the first case holds. Then $d(T_0\langle f\rangle) \geq 3 - \gamma$ as T is a minimum counterexample. Hence $d(T_0) \geq |E(G') \setminus E(C_0)| + 3 - \gamma$. Since $|E(G') \setminus E(C_0)| \geq 1$, we have that $d(T_0) \geq 4 - \gamma$ and so (1) holds, as desired. If $|E(G') \setminus E(C_0)| \geq 2$, then $d(T_0) \geq 5 - \gamma$ and (2) and (3) hold, as desired. So suppose the latter case holds. Then $d(T_0\langle f_i\rangle) \geq 2 - (2\alpha + \varepsilon)$ for $i \in \{1, 2\}$, and $d(T_0) \geq 1 + 2(2 - (2\alpha + \varepsilon)) = 5 - 2(2\alpha + \varepsilon)$ and all three statements hold as desired, since $2\alpha + \varepsilon \leq \gamma$ by (I2). □

Claim 11 (Proof taken from Claim 5.3, [35]). *There does not exist a proper C-critical subgraph G' of G.*

Proof. This follows from Claim 10 applied to $T_0 = T$. □

In addition, we have the following claim which will simplify the colouring arguments in Section 3.5.3.

Claim 12. *If $uv \in E(G) \setminus E(C)$, then $|M_{uv}| = \min\{|L(v)|, |L(u)|\}$.*

Proof. Suppose not. Then there exist colours $c_1 \in L(u)$ and $c_2 \in L(v)$ such that both c_1 and c_2 are unmatched in M_{uv}. Let M' be obtained from M by setting $M'_e = M_e$ for all $e \neq uv$, and setting $M'_{uv} = M_{uv} \cup \{(u, c_1)(v, c_2)\}$. Let $T' = (G, C, (L, M'))$. Note that $|V(T)| = |V(T')|$, and that the sum of the list sizes of the vertices in T' is the same as that in T. Since $(G, C, (L, M))$ was chosen to maximize $\sum_{e \in E(G)} |M_e|$, it follows that T' is not a counterexample to Theorem 3.4.7. Otherwise, since $\text{def}(T') = \text{def}(T)$, we have that T' contradicts our choice of T. Thus T' is not a critical canvas. Since T is C-critical, there exists an (L, M')-colouring of C that does not extend to G. By Proposition 3.2.9, T' contains a critical subcanvas $(G', C, (L, M'))$; and since T' is not critical, G' is a proper subgraph of G. But this contradicts Claim 11. □

Claim 13 (Proof taken from Claim 5.4, [35])**.** *There does not exist a chord of C.*

Proof. Suppose there exists a chord e of C. Let $G' = C \cup e$. As $v(T) \neq 0$, it follows that G' is a proper subgraph of G. Yet G' is C-critical, contradicting Claim 11. □

3.5.2 Dividing Vertices

In this subsection, we prove a few claims regarding *dividing vertices*, defined below. Namely, we show that if G is a critical canvas with at most $|E(G)|$ edges, if G contains a true dividing vertex (Claim 14) or a strong dividing vertex (Claim 15) then $d(G)$ is relatively high. These claims will be useful in the following section in showing that certain canvases obtained from T (called *relaxations*) do not contain true or strong dividing vertices. This in turn is useful in arguments showing that when passing to these relaxations, the size of the boundaries and quasiboundaries of the resulting canvases are at least $b(G)$ and $q(G)$.

Definition 3.5.1. Let G_0 be a 2-connected plane graph with outer cycle C_0. Let v be a vertex in $V(G) \setminus V(C)$, and suppose there exist two distinct faces $f_1, f_2 \in F(G_0)$ such that for $i \in \{1, 2\}$ the boundary of f_i includes v and a vertex of C_0, say u_i. Let us assume that $u_1 \neq u_2$, and let G' be the plane graph obtained from G_0 by adding the edges u_1v, u_2v if they are not present in G_0. Consider the cycles C_1, C_2 of G', where $C_1 \cap C_2 = u_1vu_2$ and $C_1 \cup C_2 = C_0 \cup u_1vu_2$. If for both $i \in \{1, 2\}$, $|E(T\langle C_i \rangle) \setminus E(C_i)| \geq 2$, then we say that v is a *dividing* vertex. If for both $i \in \{1, 2\}$ $|V(T\langle C_i \rangle) \setminus V(C_i)| \geq 1$, we say v is a *strong* *dividing* vertex. If v is a dividing vertex and the edges u_1v, u_2v are in G, then we say that v is a *true dividing* vertex. If $T_0 = (G_0, C_0, (L_0, M_0))$ is a canvas, then by a *dividing* vertex of T_0 we mean a dividing vertex of G_0, and similarly for strong and true dividing vertices.

Claim 14 (Proof taken from Claim 5.6, [35]). *Suppose $T_0 = (G_0, C_0, (L_0, M_0))$ is a critical canvas with $e(G_0) \leq e(G)$ and $v(G_0) \geq 2$. If G_0 contains a true dividing vertex, then*

1. *$d(T_0) \geq 3 - 2(2\alpha + \varepsilon)$, and*
2. *$d(T_0) \geq 4 - 2\alpha - \varepsilon - \gamma$ if $v(G_0) \geq 3$.*

Proof. Note the inequality in (2) implies the inequality in (1) by (I3). Let u_1, u_2, C_1, C_2, and v be as in the definition of true dividing vertex. Since v is a true dividing vertex, u_1v and u_2v belong to G_0. Let $G' = C_1 \cup C_2$. Hence G', C_1, and C_2 are subgraphs of G_0. Thus $d(T_0[G']) = -2 - (2\alpha + \varepsilon)$ by Proposition 3.4.6 (ii).

Note that by Corollary 3.2.7, both $T_0\langle C_1\rangle$ and $T_0\langle C_2\rangle$ are critical canvases. If $v(T_0\langle C_1\rangle) = 0$, then $d(T_0\langle C_1\rangle) \geq 2$ by Proposition 3.4.6 (i), because $|E(T_0\langle C_1\rangle) \setminus E(C_1))| \geq 2$ by Proposition 3.4.6 (ii). If $v(T_0\langle C_1\rangle) \geq 2$, then $d(T_0\langle C_1\rangle) \geq 3 - \gamma$ as T is a minimum counterexample. In any case, $d(T_0\langle C_1\rangle) \geq 2 - (2\alpha + \varepsilon)$ as $\gamma \leq 1 + (2\alpha + \varepsilon)$ by (I3). Similarly, $d(T_0\langle C_\rangle) \geq 2 - (2\alpha + \varepsilon)$.

By Proposition 3.4.4, $d(T_0) \geq d(T_0[G']) + d(T_0\langle C_1\rangle) + d(T_0\langle C_2\rangle)$. Now let us choose v such that $a = \min\{v(T\langle C_1\rangle), v(T_0\langle C_2\rangle)\}$ is minimized. Note then that $a \neq 1$, as otherwise there exists another true dividing vertex, contradicting the minimality of a. First suppose that $a \geq 2$ and hence

$$d(T_0) \geq (-1 - (2\alpha + \varepsilon)) + 2(3 - \gamma) = 5 - 2\gamma - (2\alpha + \varepsilon).$$

Then (1) and (2) hold by inequality (I3), as desired. So we may assume that $a = 0$. Hence

$$d(T_0) \geq (-1 - (2\alpha + \varepsilon)) + 2 + 2 - (2\alpha + \varepsilon) = 3 - 2(2\alpha + \varepsilon),$$

and so (1) holds, as desired. Yet if $v(T_0) \geq 3$, then

$$d(T_0) \geq (-1 - (2\alpha + \varepsilon)) + 2 + 3 - \gamma = 4 - \gamma - (2\alpha + \varepsilon),$$

and (2) holds, as desired. □

The following proof is nearly identical to the analogous result in [35], with a few minor changes to the colouring arguments to ensure they hold for correspondence colouring.

Claim 15 (Proof adapted from Claim 5.7, [35])**.** *Suppose $T_0 = (G_0, C_0, (L_0, M_0))$ is a critical canvas with $|E(G_0)| \leq |E(G)|$. If G_0 contains a strong dividing vertex, then $d(T_0) \geq 4 - 2\gamma$.*

Proof. Let u_1, u_2, C_1, C_2, and v be as in the definition of strong dividing vertex. As v is a strong dividing vertex, it follows that $v(T_0) \geq 3$ and $v(T_0\langle C_1\rangle) \geq 1$. If $v(T_0\langle C_1\rangle) = 1$, then the unique vertex in $V(T_0\langle C_1\rangle) \setminus V(C_1)$ is a true dividing vertex and hence

$$d(T_0) \geq 4 - 2\alpha - \varepsilon - \gamma \geq 4 - 2\gamma$$

by Claim 14 and (I2), as desired. So we may assume that $v(T_0\langle C_1\rangle) \geq 2$ and similarly that $v(T_0\langle C_2\rangle) \geq 2$.

Let G_0' be the graph obtained from G_0 by adding vertices z_1, z_2 not in $V(G)$ and edges u_1z_1, z_1v, vz_2, z_2u_2. Similarly let G' be the graph obtained from C_0 by adding vertices $v, z_1,$, and z_2 and edges u_1z_2, z_1v, vz_2, and z_2v_2. Let $L_0'(x) = L_0(x)$ for all $x \in V(G_0)$, and let $L_0'(z_1) = L_0'(z_2) = R$, where R is a set of five new colours. Let M_0' be defined as $(M_0')_{uv} = (M_0)_{xy}$ for all $xy \in E(G_0')$ with $\{z_1, z_2\} \cap \{x, y\} = \emptyset$, and $(M_0')_{xy} = \emptyset$ for all $xy \in E(G_0')$ with $\{z_1, z_2\} \cap \{x, y\} \neq \emptyset$. Let $T_0' = (G_0', C_0, (L_0', M_0'))$. Now

$$\text{def}(G') = |E(G')| - |E(C_0)| - 3v(G') = 4 - 3 \cdot 3 = -5.$$

Since G_0 is C_0-critical, there exists an (L_0, M_0)-colouring ϕ_0 of C_0 that does not extend to G_0. By Claim 10 the graph G_0 does not have a proper C_0 critical subgraph, and hence ϕ_0 extends to every proper subgraph of G_0 by Proposition 3.2.9. For every $c \in L_0(v)$, let $\phi_c(v) = c$, $\phi_c(z_1) = \phi_c(z_2) \in R$, and $\phi_c(x) = \phi_0(x)$ for all $x \in C_0$. Let C_1' and C_2' be the two facial cycles of G' other than C_0. Since ϕ_0 does not extend to an (L_0, M_0)-colouring of G_0, for every $c \in L_0(v)$ the colouring ϕ_c does not extend to an (L_0', M_0')-colouring of either $G_0'\langle C_1'\rangle$ or $G_0'\langle C_2'\rangle$. Since $|L_0(v)| \geq 5$, there exists $i \in \{1, 2\}$ such that there exist at least three colours $c \in L_0(v)$ such that ϕ_c does not extend to an (L_0', M_0')-colouring of G_0'. We may assume without loss of generality that $i = 1$. Let $\mathcal{C}$ be the set of all colours $c \in L(v)$ such that ϕ_c does not extend to an (L_0, M_0)-colouring of $G_0'\langle C_1'\rangle$. Thus $|\mathcal{C}| \geq 3$.

Let G_1' be the graph obtained from $G_0'\langle C_1'\rangle$ by adding the edge z_1z_2 inside the outer face of $G_0'\langle C_1'\rangle$. Let $C_1'' = (C_1' \setminus v) + z_1z_2$. Let $L'(z_1) = \{c_1\}$, $L'(z_2) = \{c_2\}$, and $L'(x) = L_0(x)$ for every $x \in V(G_0) \setminus \{v\}$. Let $L'(v) = \mathcal{C} \cup \{c_1, c_2\}$. Let M' be defined as follows: we set $(M_{xy}' = M_0)$ for all $xy \in E(G_0)$ with $\{x, y\} \cap \{z_1, z_2\} = \emptyset$; we set $M_{xy}' = \emptyset$ for all $xy \in E(G_0)$ with $v \notin \{x, y\}$ and $\{x, y\} \cap \{z_1, z_2\} \neq \emptyset$; and finally we set $M_{vz_i}' = \{(v, c_i)(z_i, c_i)\}$ for $i \in \{1, 2\}$.

We claim that the canvas $T_1 = (G'_1, C''_1, (L', M'))$ is a critical canvas. To see this, let H be a proper subgraph of G'_1 that includes C''_1 as a subgraph. Let us extend ϕ_0 by defining $\phi_0(z_1) := c_1$ and $\phi_0(z_2) := c_2$. We will show that (the restriction to C''_1 of) ϕ_0 extends to H but not to G'_1. If ϕ_0 extended to G'_1, then $\phi_0(v) \notin \mathcal{C}$ by definition of $\mathcal{C}$ and $\phi_0(v) \notin \{c_1, c_2\}$ because v is adjacent to both z_1 and z_2, a contradiction. Thus ϕ_0 does not extend to G'_1. To show that ϕ_0 extends to H assume first that $H \setminus \{z_1, z_2\}$ is a proper subgraph of $G_0\langle C'_1\rangle$. Then $(H \setminus \{z_1, z_2\}) \cup G_0\langle C'_2\rangle$ is a proper subgraph of G_0, and hence ϕ_0 extends to it, as desired. So we may assume that $H \setminus \{z_1, z_2\} = G_0\langle C'_1\rangle$. Since H is a proper subgraph of G'_1 we may assume from the symmetry that $vz_1 \in E(H)$. Now ϕ_0 extends to an (L', M')-colouring of $G_0 \setminus \{v\}$. Letting $\phi_0(v) = c_1$ shows that ϕ_0 extends to H, as desired. This proves the claim that T_1 is critical. As $v(T_1) \geq 2$, we find that $d(T_1) \geq 3 - \gamma$ by the minimality of T. Similarly, since $v(T'_0\langle C'_2\rangle) \geq 2$, we find that $d(T'_0\langle C'_2\rangle) \geq 3 - \gamma$. Let us now count deficiencies. By Lemma 3.3.4,

$$\mathrm{def}(T'_0) = \mathrm{def}(T'_0[G'_0]) + \mathrm{def}(T'_0\langle C'_1\rangle) + \mathrm{def}(T'_0\langle C'_2\rangle) = -5 + \mathrm{def}(T'_0\langle C'_1\rangle) + \mathrm{def}(T'_0\langle C'_2\rangle).$$

Yet $\mathrm{def}(T_0) = \mathrm{def}(T'_0) + 2$. Furthermore, $\mathrm{def}(T'_0\langle C'_1\rangle) = \mathrm{def}(T_1) + 1$. Hence,

$$\mathrm{def}(T_0) = \mathrm{def}(T'_0\langle C'_1\rangle) + \mathrm{def}(T'_0\langle C'_2\rangle) - 3 = \mathrm{def}(T_1) + \mathrm{def}(T'_0\langle C'_2\rangle) - 2.$$

Next we count the function s. We claim that $s(T_0) \leq s(T_1) + s(T'_0\langle C'_2\rangle)$. This follows as every vertex of $V(G_0) \setminus V(C_0)$ is either in $V(G'_1) \setminus V(C''_1)$ or $V(G'_0\langle C'_2\rangle) \setminus V(C'_2)$. Moreover every vertex of $B(T_0)$ is either in $B(T_1)$ or $B(T'_0\langle C'_2\rangle)$ and similarly every vertex of $Q(T_0)$ is either in $Q(T_1)$ or $Q(T'_0\langle C'_2\rangle)$.

Finally putting it all together, we find that

$$d(T_0) \geq d(T_1) + d(T'_0\langle C'_2\rangle) - 2 \geq 2(3 - \gamma) - 2 = 4 - 2\gamma,$$

as desired. □

3.5.3 Tripods

An easy consequence of Theorem 3.1.1 is that if $(G, C, (L, M))$ is a C-critical canvas with $v(G) \geq 2$, then there exists a vertex in $V(G) \setminus V(C)$ with at least three neighbours in $V(C)$. Otherwise, every (L, M)-colouring of C extends to an (L, M)-colouring of G, contradicting the fact that G is C-critical. Much of our analysis will involve performing certain reductions around specific vertices in G that have precisely three neighbours in C, arguing about the deficiency and $d(\cdot)$ of the resulting canvas, and extrapolating from that about $\mathrm{def}(T)$ and $d(T)$. To that end, it is convenient to define the following.

Definition 3.5.2. Let G_0 be a plane graph with outer cycle C_0, and let $v \in V(G_0) \setminus V(C_0)$ have at least three neighbours in C_0. Let $u_1, u_2, \cdots, u_k$ be all the neighbours of v in C_0 listed in a counterclockwise order of appearance on C_0. Assume that at most one face of $G_0[V(C_0) \cup \{v\}]$ includes an edge or vertex of G_0, and that if such a face exists, then it is incident with u_1 and u_k. If $k = 3$, then we say that v is a *tripod* of G_0, and if $k \geq 4$, then we say that v is a *quadpod* of G_0. The tripod or quadpod v is *regular* if there exists a face of $G_0[V(C_0) \cup \{v\}]$ that includes an edge or vertex of G_0. If such a face exists, then we say that $u_1, u_2, \cdots, u_k$ are listed in *standard (counterclockwise) order*. Note that every tripod of degree at least four is regular.

If v is a regular tripod or quadpod, we let $C_0 \oplus v$ denote the boundary of the face of $G_0[V(C_0) \cup \{v\}]$ that includes an edge or vertex of G_0, and we define $G_0 \oplus v := G_0\langle C_0 \oplus v\rangle$. If X is a set of tripods or quadpods of G_0 and there exists a face of $G_0[V(C_0) \cup X]$ that includes an edge or vertex of G_0, then we let $C_0 \oplus X$ denote the boundary of such a face and define $G_0 \oplus X := G_0\langle C_0 \oplus X\rangle$.

If $T_0 = (G_0, C_0, (L_0, M_0))$ is a canvas, then we extend all the above terminology to T_0 in the natural way: thus we can speak of tripods or quadpods of T_0, we define $T_0 \oplus X = T_0[G_0 \oplus X]$, etc.

Claim 16 (Proof taken from Claim 5.9, [35].)**.** *Let $T_0 = (G_0, C_0, (L_0, M_0))$ be a canvas with $v(G_0) \geq 2$, and let $v \in V(G_0) \setminus V(C_0)$ have at least three neighbours in C_0. Then v is either a regular tripod of T_0, or a true dividing vertex of T_0.*

Proof. Let $u_1, u_2, \ldots, u_k$ be all the neighbours of v in C_0 listed in their order of appearance on C_0 and numbered such that the face f of $G_0[V(C_0) \cup \{v\}]$ incident with u_1 and u_k includes a vertex of G_0. If another face of $G_0[V(C_0) \cup \{v\}]$ includes an edge or vertex of G_0 or if $k \geq 4$, then by considering the vertices u_1 and u_k we find that v_0 is a true dividing vertex of T_0. Thus we may assume that $k = 0$ and that f is the only face of $G_0[V(C_0) \cup \{v\}]$ that contains a vertex or edge of G_0. It follows that v is a tripod, as desired. □

Definition 3.5.3. Let $T_0 = (G_0, C_0, (L_0, M_0))$ be a canvas. We say that T_0 is a 0-relaxation of T_0. Let $k > 0$ be an integer, T_0' be a $(k-1)$-relaxation of T_0, and v be a regular tripod for T_0'. Then we say that $T_0' \oplus v$ is a *k-relaxation* of T_0.

By Proposition 3.2.11 (2), every tripod of a critical canvas is regular. As a consequence, if v is a tripod of a critical canvas T_0, then $T_0 \oplus v$ is well-defined. Moreover, $T_0 \oplus v$ is itself a critical canvas by Corollary 3.2.7, and $v(T_0 \oplus v) \geq 2$ by Proposition 3.2.11 (2). Thus for all $k \geq 1$, a k-relaxation T_0' of a critical canvas is well-defined; T_0' is critical; and $v(T_0') \geq 2$. Claims 17 and 18, below, establish useful results about canvas relaxations.

Claim 17 (Proof taken from Claim 5.11, [35]). *If T_0' is a k-relaxation of a canvas T_0, then $d(T_0) \geq d(T_0') - k(2\alpha + \varepsilon)$.*

Proof. We proceed by induction on k. We may assume that $k \geq 1$ (as otherwise the claim trivially holds) and that the claim holds for all integers strictly smaller than k. Let T_{k-1} be a $(k-1)$-relaxation of T_0 and v a regular tripod of T_{k-1} such that T_0' is a 1-relaxation of T_{k-1}. By induction, $d(T_0) \geq d(T_{k-1}) - (k-1)(2\alpha + \varepsilon)$. Yet $\text{def}(T_{k-1}) = \text{def}(T_0')$ while $v(T_{k-1}) = v(T_0') + 1$, $b(T_{k-1}) \leq b(T_0') + 1$, and $q(T_{k-1}) \leq q(T_0') + 1$. Thus $d(T_{k-1}) \geq d(T_0') - (2\alpha + \varepsilon)$ and the claim follows. □

The following claim is analogous to Claim 5.12 in [35]. However, Claim 5.12 in [35] makes no mention of the case when $k \geq 3$, or of $k \geq 2$ for strong dividing vertices, as these cases were not necessary for the resulting analysis for list colouring.

Claim 18. *Let $k \in \{0, 1, 2, 3, 4\}$ and let T' be a k-relaxation of T. Then T' does not have a true dividing vertex, and if $k \leq 3$, it does not have a strong dividing vertex.*

Proof. Suppose not. By Corollary 3.2.7, T' is critical, and $v(T') \geq 3$ by Claim 9. If T' is a true dividing vertex, then by Claim 17 we have that $d(T) \geq d(T') - 4(2\alpha + \varepsilon)$. By Claim 14 (2), we have moreover that $d(T') \geq 4 - 2\alpha - \varepsilon - \gamma$. Thus

$$\begin{aligned} d(T) &\geq d(T) - 4(2\alpha + \varepsilon) \\ &\geq 4 - 2\alpha - \varepsilon - \gamma - 8\alpha - 4\varepsilon \\ &= 3 - \gamma + (1 - 10\alpha - 7\varepsilon) \\ &> 3 - \gamma \end{aligned}$$

where the last line follows because $10\alpha + 7\varepsilon < 1$ by inequalities (I2) and (I3) (which imply together that $20\alpha + 10\epsilon \leq 1$). Thus we may assume that $k \leq 3$ and T' has a strong dividing vertex. By Claim 17 we have that $d(T) \geq d(T') - 3(2\alpha + \varepsilon)$. By Claim 15, we have that $d(T') \geq 4 - 2\gamma$. Thus

$$\begin{aligned} d(T) &\geq d(T') - 3(2\alpha + \varepsilon) \\ &\geq 4 - 2\gamma - 6\alpha - 3\varepsilon \\ &= 3 - \gamma + (1 - \gamma - 6\alpha - 3\varepsilon). \end{aligned}$$

But by (I3), $\gamma + 6\alpha + 3\varepsilon \leq 1$. Thus $d(T) \geq 3 - \gamma$, a contradiction. □

Claim 19 (Proof taken from Claim 5.13, [35]). *If x_1 is a tripod of T, then letting $T' = T \oplus x_1$, either*

1. $\deg(x_1) = 5$, *or,*

2. $\deg(x_1) = 6$, *the neighbours of* x_1 *not in* C *form a path of length two and the ends of that path are in* $B(T)$, $b(T) = b(T')$, $q(T) = q(T')$ *and* $d(T) \geq d(T') - \varepsilon$.

Proof. Note that as $v(T) \geq 8$ by Claim 9, then $v(T') \geq 7$. By Proposition 3.2.11 (2), $\deg(x_1) \geq 5$. If $\deg(x_1) = 5$, then (1) holds as desired, so we may assume that $\deg(x_1) \geq 6$. By Claim 3.2.7, T' is a critical canvas, and so by the minimality of T, we have that $d(T') \geq 3 - \gamma$. Moreover, $v(T) = v(T') + 1$ and since x_1 is a tripod of T, $\text{def}(T') = \text{def}(T)$. Thus $d(T) = d(T') - \varepsilon + \alpha(b(T') - b(T) + q(T') - q(T)$. Let c_1, c_2, and c_3 be the neighbours of x_1 in $V(C)$ listed in standard order, and let $c_1, c_2, c_3, q_1, \ldots, q_2$ be all the neighbours of x_1 listed in their cyclic order around x_1.

Let $R = N(x_1) \setminus \{c_1, c_2, c_3, q_1, q_2\}$. We claim that $R \cap Q(T) = \emptyset$. Suppose not, and let $q \in R \cap Q(T)$. Then q is a dividing vertex of T'. Given the presence of q_1 and q_2, q is a strong dividing vertex of T', contrary to Claim 18. This proves that $R \cap Q(T) = \emptyset$, and implies that $R \cap B(T) = \emptyset$ as well since $B(T) \subseteq Q(T)$.

Note that $R \subseteq B(T') \subseteq Q(T')$. Thus $q(T') \geq q(T) + |R| - 1$, and $b(T') \geq b(T) + |R| - 1$. Hence if $|R| \geq 2$, then $d(T) \geq d(T') - \varepsilon + 2\alpha \geq d(T') \geq 3 - \gamma$ since $2\alpha \geq \varepsilon$ by (I1), a contradiction. So $|R| = 1$ and so $\deg(x_1) = 6$. Thus $q(T') \geq q(T)$ and $b(T') \geq b(T)$. Now it follows that $q(T) = q(T')$ and $b(T) = b(T')$ as otherwise $d(T) \geq d(T') - \varepsilon + \alpha \geq 3 - \gamma$ since $\alpha \geq \varepsilon$ by (I1), a contradiction. Hence $d(T) \geq d(T') - \varepsilon$.

Let $q \in R$. The conclusions above imply that $Q(T') \setminus \{q\} = Q(T) \setminus \{x_1\}$ and $B(T') \setminus \{q\} = B(T) \setminus \{x_1\}$. The latter implies that $\{q_1, q_2\} \subseteq B(T)$. The former implies that $q_1 q q_2$ for a path, for otherwise there would exist a vertex other than q, q_1, and q_2 that is cofacial with x_1 and therefore belongs to $Q(T')$; yet that vertex also then belongs to $Q(T)$ and so is a strong dividing vertex of T', a contradiction as above. Thus (2) holds as desired. □

The following claim is analogous to Claim 5.14 in [35]; however, the result in [35] is weaker in that $k \leq 3$. The proof is nearly identical.

Claim 20. *For* $k \in \{0, 1, 2, 3, 4, 5, 6\}$, *if* T' *is a* k*-relaxation of* T, *then there does not exist a proper critical subcanvas of* T'.

Proof. Suppose not. By Claim 9, $v(T') \geq 1$. Then by Claim 10, we have that $d(T') \geq 4 - \gamma$. By Claim 17, $d(T) \geq 4 - \gamma - 6(2\alpha + \varepsilon)$. By (I2), this is at least $3 - \gamma$, a contradiction. □

For the remainder of this chapter, let X_1 be the set of internal vertices of G with at least three neighbours in C. The following claim combines Claims 5.15, 5.16, and 5.17 in [35], and establishes several some of the structure surrounding vertices in X_1.

Claim 21 (Proofs taken from Claims 5.15-5.17, [35])**.** *The following all hold.*

1. *$X_1 \neq \emptyset$ and every member of X_1 is a tripod of T.*

2. *$T \oplus X_1$ is well-defined and is a critical canvas.*

3. *The graph $G \oplus X_1$ does not have a chord of $C \oplus X_1$.*

Proof. We begin by proving (1). By Claim 13, there does not exist a chord of C, and hence $X_1 \neq \emptyset$ by Theorem 3.2.10. By Claims 16 and 18 every member of X_1 is a tripod of T.

Next, we prove (2). By Proposition 3.2.11 (2) every tripod of G is regular, and hence $T \oplus X_1$ is well-defined. It is critical by Corollary 3.2.7.

Finally, we prove (3). Suppose not. Let v_1v_2 be a chord of $C \oplus X_1$. As C has no chord by Claim 13, we may assume without loss of generality that $v_1 \notin V(C)$. Thus v_1 is a tripod of C. Hence v_2 is also a tripod, as otherwise v_1 is not a tripod. But then v_2 is a true dividing vertex of $T \oplus v_1$ because $v(T) \geq 4$ by Claim 9, contradicting Claim 18. □

For the remainder of this chapter, let X_2 be the set of vertices $v \in V(G) \setminus (V(C) \cup X_1)$ with at least three neighbours in $C \oplus X_1$. The first part of the following claim corresponds to Claim 5.19 in [35]; the following two parts are analogous to parts (2) and (3) of Claim 21, above.

Claim 22. *The following all hold.*

1. *(Claim 5.19, [35].) We have $X_2 \neq \emptyset$. Furthermore, let $x_2 \in X_2$, and let $u_1, u_2, \cdots, u_k$ be the neighbours of x_2 in $C \oplus X_1$ listed in standard order. Then $k = 3$, and $u_2 \in V(C)$. In particular, every member of X_2 is a tripod of $C \oplus X_1$.*

2. *$(T \oplus X_1) \oplus X_2$ is well-defined and is a critical canvas.*

3. *The graph $(G \oplus X_1) \oplus X_2$ does not have a chord of $(C \oplus X_1) \oplus X_2$.*

Proof. We begin by proving (1). By Claim 21 (3), there does not exist a chord of $C \oplus X_1$ and hence from Claim 21 (2) and Theorem 3.2.10 it follows that $X_2 \neq \emptyset$. Let $x_2 \in X_2$, and $u_1, u_2, \ldots, u_k$ be as stated. Let $i \in \{2, 3, \cdots, k-1\}$. If $u_i \in X_1$, then since u_i has no neighbours in X_1 by Claim 21 (3), we have that u_i has three neighbours in C and is adjacent to x_2 but has no other neighbours, contrary to Proposition 3.2.11 (2). Thus $u_i \in V(C)$. We may assume that $k \geq 4$, as otherwise (1) holds. Since $x_2 \notin X_1$, we may assume from the symmetry that $u_1 \in X_1$. By considering the vertices u_1 and u_4 we find that x_2 is a true dividing vertex of either $T \oplus u_1$ (if $u_4 \in V(C)$) or $T \oplus \{u_1, u_4\}$ (if $u_4 \notin V(C)$), either case contradicting Claim 18.

We now prove (2). By Claim 21 (2), we have that $T \oplus X_1$ is well-defined. By Claim 22 (1), every member of X_2 is a tripod of $C \oplus X_1$, and so by Proposition 3.2.11 we have that $(T \oplus X_1) \oplus X_2$ is well-defined. It is a critical canvas by Claim 3.2.7.

Finally, we prove (3). Suppose not. Let v_1v_2 be a chord of $(C \oplus X_1) \oplus X_2$. By Claims 13 and 21 (2), we may assume without loss of generality that $v_1 \in X_2$. Since every member of X_2 is a tripod of $C \oplus X_1$ by (2), it follows that $v_2 \in X_2$. Let u_1, u_2, u_3 be the neighbours of v_1 in $C \oplus X_1$ listed in standard order, and let w_1, w_2, w_3 be the neighbours of v_2 in $C \oplus X_1$ listed in standard order. Since v_1 and v_2 are not in X_1, at least one of u_1 and u_3 (say u_1) is in X_1; and similarly, at least one of w_1 and w_3 is in X_1. Let $Y = \{w_1, w_3\} \cap X_1$. First suppose that $\{w_1, w_3\} \cap \{u_1, u_3\} = \emptyset$. Recall that by (1), u_2 and w_2 are in $V(C)$.Then since $\deg(v_1) \geq 5$ and $\deg(v_2) \geq 5$ by Proposition 3.2.11 (2), it follows that v_1 is a true dividing vertex of $T \oplus u_1 \oplus Y \oplus v_2$, contradicting Claim 18.

Thus we may assume that $\{w_1, w_3\} \cap \{u_1, u_3\} \neq \emptyset$. Without loss of generality, suppose that $u_3 = w_3$. Then again since $\deg(v_1) \geq 5$ and $\deg(v_2) \geq 5$ by Proposition 3.2.11 (2), it follows that v_1 is a true dividing vertex of $T \oplus Y \oplus v_2$ (if $u_1 \notin X_1$) or $T \oplus Y \oplus u_1 \oplus v_2$ (if $u_1 \in X_1$), again contradicting Claim 18. □

It follows from the definition of critical canvas that there exists an (L, M)-colouring of C that does not extend to an (L, M)-colouring of G. For the remainder of the proof, let ϕ be one such fixed (L, M)-colouring of C.

Claim 23 (Claim 5.20, [35]). *The colouring ϕ extends to every proper subgraph of G that contains C as a subgraph.*

Proof. This follows from Proposition 3.2.9 and Claim 11. □

For $v \in V(G) \setminus V(C)$, we let $S(v) := L(v) \setminus \{v[u, \phi(u)] \mid u \in N(v) \cap V(C)\}$. The following claim is analogous to Claim 5.21 in [35], with the appropriate changes for correspondence colouring.

Claim 24 (Claim 5.21, [35]). *For all $v \in V(G) \setminus V(C)$, $|L(v)| = 5$ and $|S(v)| = 5 - |N(v) \cap V(C)|$.*

Proof. Suppose for a contradiction that $|L(v)| \geq 6$ for some $v \in V(G) \setminus V(C)$. Let $c \in L(v)$, and let L' be defined by $L'(v) : L(v) \setminus \{c\}$ and $L'(x) := L(x)$ for all $x \in V(C) \setminus \{v\}$. Then $(G, C, (L', M))$ is a canvas, and ϕ clearly does not extend to an (L', M)-colouring of G. By Proposition 3.2.9 the canvas $(G, C, (L', M)))$ has a critical subcanvas $(G', C, (L', M))$. Since T was chosen to minimize $\sum_{v \in V(G)} |L(v)|$ subject to $|E(G)|$ being as small as possible, it follows that G' is a proper subgraph of G. That contradicts Claim 10 applied to $T_0 = T$ and G'.

To prove the second statement, suppose for a contradiction that $|S(v)| \geq 5 - |N(v) \cap V(C)|$ for some $v \in V(G) \setminus V(C)$. Thus by our choice of minimum counterexample, it follows that either v has a neighbour w_1 such that $v[w_1, \phi(w_1)] = \emptyset$, or that v has two distinct neighbours w_1 and w_2 in $V(C)$ such that $v[w_1, \phi(w_1)] = v[w_2, \phi(w_2)]$. But then ϕ does not extend to $G \setminus vw_1$, contradicting Claim 23. $\square$

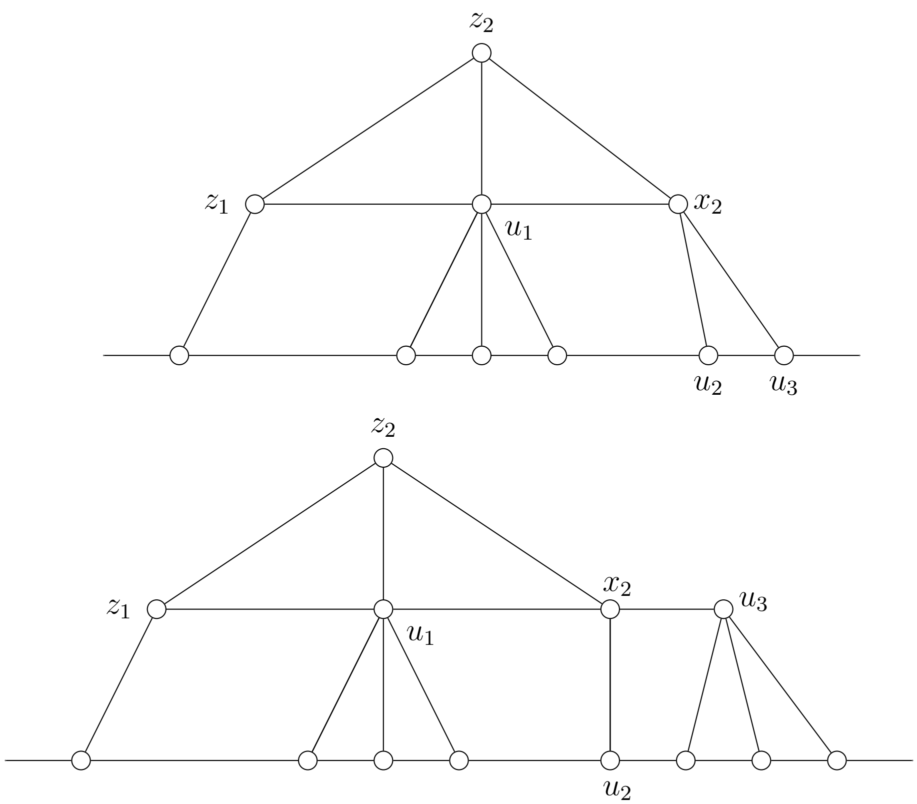

Figure 3.1: Vertices z_1, z_2 as described in Claim 34.

The following claim is nearly identical to Claim 5.22 in [35], and establishes some of the structure surrounding X_1 vertices that neighbour X_2 vertices. See Figure 3.1 for an illustration of the vertices described in the claim.

Claim 25 (Claim 5.22, [35])**.** *Let $x \in X_2$, and let $U = N(x) \cap X_1$. If $u \in U$, then $\deg(u) = 6$ and there exist adjacent vertices $z_1, z_2 \notin V(C)$ such that z_1 is adjacent to u and is in $B(T)$, and z_2 is adjacent to u and x.*

Proof. By Claim 19 applied to T and u, we find that $\deg(u) \leq 6$ and the claim follows unless $\deg(u) = 5$. So suppose for a contradiction that $\deg(u) = 5$.

Let $C' = (C \oplus U) \oplus x$, and $T\langle C'\rangle = (G', C', (L, M))$. Let $z \in V(G') \setminus V(C')$ be a neighbour of u. We claim that $G' \setminus uz$ has a C'-critical subgraph. To see this, we extend ϕ to an (L, M)-colouring ϕ' of $G \setminus uz$ as follows. For $v \in V(C)$, let $\phi'(v) = \phi(v)$. Since $x \notin X_1$, we have that $|S(x)| \geq 3$, and $|S(u)| = 2$ by Claim 24 since $u \in X_1$ and every vertex in X_1 is a tripod of T by Claim 21 (1). We may therefore choose $\phi'(x)$ with $u[x, \phi'(x)] \notin S(u)$. Now if ϕ' extends to an (L, M)-colouring ϕ'' of $G' \setminus uz$, then by re-defining $\phi''(u)$ to be a colour in $S(u) \setminus u[z, \phi''(z)]$, we obtain an extension of ϕ to G, a contradiction. Thus ϕ' does not extend to an (L, M)-colouring of $G \setminus uz$, and so by Proposition 3.2.9 this proves our claim that $G' \setminus uz$ contains a C'-critical subgraph, say G''. But G'' is a proper C'-critical subgraph of G', contradicting Claim 20. □

For the remainder of this chapter, let X_3 be the set of vertices $v \in V(G) \setminus (V(C) \cup X_1 \cup X_2)$ with at least three neighbours in $(C \oplus X_1) \oplus X_2$. Having established that every vertex $u \in N(X_2) \cap X_1$ has degree 6, we are now equipped to prove the following claim. The claim is analogous to Claim 5.19 in [35] (or equivalently Claim 22 (1), above), with X_3 playing the role of X_2. Note that none of the proofs in [35] require the introduction of this third layer of tri- and quadpods X_3.

Claim 26. *We have $X_3 \neq \emptyset$. Furthermore, let $x_3 \in X_3$, and let $u_1, u_2, \ldots, u_k$ be all the neighbours of $x_3 \in (C \oplus X_1) \oplus X_2$ listed in standard order. Then $k = 3$, and $u_2 \notin X_2$.*

Proof. By Claim 22 (3), there does not exist a chord of $(C \oplus X_1) \oplus X_2$, and by Claim 22 (2), $(T \oplus X_1) \oplus X_2$ is a critical canvas. Thus by Theorem 3.2.10 it follows that $X_3 \neq \emptyset$. Let $x_3 \in X_3$ and $u_1, \ldots, u_k$ be as stated. Let $i \in \{2, \ldots, k-1\}$. Suppose that $u_i \in X_2$. If both u_{i-1} and u_{i+1} are in X_2, then since there are no edges in $E(G)$ with both endpoints in X_2 by Claim 22, it follows that $\deg(u_i) \leq 4$ as u_i is adjacent to x_3 and u_i has exactly three neighbours in $C \oplus X_1$ by Claim 22 (1). Thus at least one of u_{i-1} and u_{i+1} is not in X_2. If neither u_{i-1} nor u_{i+1} is in X_2, then given the existence of u_i and the fact that $\deg(x_3) \geq 5$ by

Proposition 3.2.11 (2) it follows that x_3 is a true dividing vertex of $T \oplus (\{u_{i-1}, u_{i+1}\} \cap X_1)$, contradicting Claim 18. Thus exactly one of u_{i-1} and u_{i+1} is in X_2. By symmetry, we may assume $u_{i-1} \in X_2$. Let $W = N(u_{i+1}) \cap X_1$. Note that $|W| \leq 2$ by Claim 22 (1). Then again we find that x_3 is a true dividing vertex: either of $T \oplus W \oplus u_{i-1} \oplus u_{i+1}$ (if $u_{i+1} \in X_1$) or of $T \oplus W \oplus u_{i-1}$ (if $u_{i+1} \in V(C)$), either way contradicting Claim 18.

We may assume that $k \geq 4$, as otherwise the claim holds. Since $x_3 \notin X_2$, we may assume by symmetry that $u_1 \in X_2$. By above, $u_2, \ldots, u_{k-1} \in V(C) \cap X_1$. Recall that by Claim 21 (3) and (1), there are no edges with both endpoints in X_1 and every member of X_1 has exactly three neighbours in $V(C)$. In addition, by Proposition 3.2.11 (2) every vertex in X_1 has degree at least 5. By Claim 25, if a vertex $u \in X_1$ is adjacent to a vertex in X_2, then it follows that $\deg(u) = 6$. Thus $\{u_2, u_3, \ldots, u_{k-1}\} \subseteq V(C)$, and moreover, $|N(u_1) \cap X_1| = 1$. Similarly, if $u_k \in X_2$ then $|N(u_k)| = 1$. Then since $\deg(u_1) \geq 5$ and $\deg(u_k) \geq 5$ by Proposition 3.2.11 (2) and every vertex of X_2 is a tripod of $C \oplus X_1$, it follows that x_3 is a true dividing vertex of either $(T \oplus (N(u_1) \cap X_1)) \oplus u_1$ (if $u_k \in V(C)$); or of $((T \oplus (N(u_1) \cap X_1)) \oplus u_1) \oplus u_k$ (if $u_k \in X_1$); or finally of $(((T \oplus (N(u_1) \cap X_1)) \oplus u_1) \oplus (N(u_k) \cap X_1)) \oplus u_k$ (if $u_k \in X_2$). Since as argued above $|N(u_1) \cap X_1| = 1$ and if $u_k \in X_2$ then $|N(u_k) \cap X_1| = 1$, it follows that x_3 is a true dividing vertex of a k-relaxation with $k \leq 4$, contradicting Claim 18. □

Let $X_0 := V(C)$. It will be convenient to be able to succinctly describe the structure surrounding vertices in X_2 and X_3. To that end, we make the following definition.

Definition 3.5.4. Let $i \in \{2, 3\}$, and let $x_i \in X_i$. Let $u_1, u_2, u_3, \ldots$ be the neighbours of x_i listed in standard counterclockwise order. We say x_i is of type (j, k, ℓ) with $j, k, \ell \in \{0, 1, 2\}$ if $u_1 \in X_j$, $u_2 \in X_k$, and $u_3 \in X_\ell$.

By Claim 26, there exists a vertex $x_3 \in X_3$. For the remainder of the proof, we will fix such an x_3. Since $x_3 \notin X_2$, it follows that x_3 has at least one neighbour in X_2. By Claim 26, if x_3 is of type (j, k, ℓ), then $k \neq 2$. Up to reflection of the graph, we may assume that x_3 is of one of the following types: (2,0,0), (2,0,1), (2,1,0), (2,1,1), (2,0,2), (2,1,2). The following claims allow us to rule out several of these types.

Claim 27. *The vertex x_3 is not of type $(2, 0, 1)$.*

Proof. Suppose not. Let $x_2, x_0, x_1, \ldots$ be the neighbours of x_3 in standard counterclockwise order. Note that by Claim 21 (1), x_1 is a tripod of T, and since $x_2 \notin X_1$, it follows that x_2 has at most two neighbours in $V(C)$. Thus by Claim 24, $|S(x_2)| \geq 3$ and $|S(x_1)| = 2$. Moreover, since x_3 is adjacent to a vertex in $V(C)$, we have that $|S(x_3)| = 4$. Thus there exists a colour $c_1 \in S(x_1)$ and a colour $c_2 \in S(x_2)$ such that $x_3[x_1, c_1] = x_3[x_1, c_2]$. Set $\phi(x_i) = c_i$ for $i \in \{1, 2\}$. Let $U = N(x_2) \cap X_1$, and let $T' = T\langle((C \oplus U) \oplus x_2) \oplus x_1\rangle = (G', C', (L, M))$. We claim that $G' - x_3x_2$ has a C'-critical subgraph. To see this, note that if ϕ extends to an (L, M)-colouring of G, then by our choice of $\phi(x_1)$ and $\phi(x_2)$ this extension is also an (L, M)-colouring of G. Thus ϕ does not extend to an (L, M)-colouring of $G' - x_3x_2$, and so by Proposition 3.2.9 we have that $G' - x_3x_2$ contains a C'-critical subgraph. But this subgraph is a proper subgraph of G'; and by Claim 22 (2), $|U| \leq 2$ and so T' is a k-relaxation of T with $k \leq 4$, contradicting Claim 20. □

Claim 28. *The vertex x_3 is not of type* $(2, 1, 0)$.

Proof. Suppose not. Let $x_2, x_1, x_0, \ldots$ be the neighbours of x_3, listed in standard counterclockwise order. Note that x_1 has exactly three neighbours in $V(C)$ since x_1 is a tripod of T by Claim 21 (1). If x_2 is adjacent to x_1, then since G is planar and x_1 is adjacent to x_3, it follows that $\deg(x_1) = 5$. But this contradicts Claim 25 since $x_1x_2 \in E(G)$. Thus we may assume that x_2 is not adjacent to x_1, and so since x_1 has degree at least five by Proposition 3.2.11 (2), there exists a vertex $x_1' \in X_1$ with $x_1'x_2 \in E(G)$ and $x_1'x_1 \in E(G)$. But this contradicts Claim 21 (3), since G does not contain an edge with both endpoints in X_1. □

Claim 29. *The vertex x_3 is not of type* $(2, 1, 1)$.

Proof. Suppose not. Let $x_2, x_1, x_1', \ldots$ be the neighbours of x_3 listed in standard counterclockwise order. Note that x_1 has exactly three neighbours in $V(C)$ by Claim 22 (1), and that x_1 is not adjacent to x_1' by Claim 21 (3). If x_1 is not adjacent to x_2, then since x_1 is adjacent to x_3 it follows that $\deg(x_1) \leq 4$, contradicting Proposition 3.2.11 (2). Thus we may assume that x_1 is adjacent to x_2. But then $\deg(x_2) \leq 5$, contradicting Claim 25. □

Claim 30. *The vertex x_3 is not of type* $(2, 0, 2)$.

Proof. Suppose not. Let $x_2, x_0, x_2', \ldots$ be the neighbours of x_3 listed in standard counterclockwise order. Note that x_2 is of type (1,0,0) and x_2' is of type (0,0,1), as otherwise a vertex in X_1 adjacent to either x_2 or x_2' has degree at most 4, contradicting Claim 25. Let x_1 and x_1' be the unique neighbours of x_2 and x_2', respectively, in X_1. By Claim 24, $|S(x_2)| = |S(x_2')| \geq 3$. Moreover, since x_3 is adjacent to a vertex in $V(C)$, we have that $|S(x_3)| = 4$. Thus there exists an extension ϕ' of ϕ to $C \cup \{x_2, x_2'\}$ where $x_3[x_2, \phi'(x_2)] = x_3[x_2', \phi'(x_2')]$. Let $T' = (G', C', (L, M)) = T\langle((C \oplus \{x_1, x_1'\}) \oplus \{x_2, x_2'\})\rangle$. We claim $G' - x_3x_2$ has a C'-critical subgraph. To see this, note that if ϕ' extends to an (L, M)-colouring of G, then by our choice of $\phi(x_2)$ and $\phi(x_2')$ this extension is also an

(L, M)-colouring of G that extends ϕ. Thus ϕ' does not extend to an (L, M)-colouring of $G' - x_3x_2$, and so by Proposition 2.9 $G' - x_3x_2$ contains a C'-critical subgraph. But this subgraph is a proper subgraph of G', and G' is a 4-relaxation of G, contradicting Claim 20. □

Claim 31. *The vertex x_3 is not of type $(2, 1, 2)$.*

Proof. Suppose not. Let $x_2, x_1, x_2', \ldots$ be the neighbours of x_3 listed in standard counter-clockwise order. By Claim 21 (1), x_1 has exactly three neighbours in $V(C)$. By Proposition 3.2.11 (2), x_1 is adjacent to at least one of x_2 and x_2'. But then by Claim 25, it follows that x_1 is adjacent to both x_2 and x_2'. By Claim 24, $|S(x_2)| \geq 3$, $|S(x_2')| \geq 3$, and $|S(x_1)| = 2$. Thus there exists an extension ϕ' of ϕ to $C \cup \{x_2, x_2'\}$ where $x_1[x_2, \phi'(x_2)] \notin S(x_1)$ and similarly $x_1[x_2', \phi'(x_2')] \notin S(x_1)$. Let $U = (N(x_2) \cup N(x_2')) \cap X_1$, and let $T' = (G', C', (L, M)) = T\langle C \oplus U \oplus \{x_2, x_2'\}\rangle$. We claim $G' - x_3x_1$ has a C'-critical subgraph. To see this, note that if ϕ' extends to an (L, M)-colouring of G, then redefining $\phi'(x_1) \in S(x_1) \setminus \phi'(x_3)$ we obtain an (L, M)-colouring of G that extends ϕ. By Proposition 3.2.9 $G' - x_3x_1$ contains a C'-critical subgraph. But this subgraph is a proper subgraph of G', and G' is a k-relaxation of G with $k \leq 5$ (since $x_1 \in N(x_2) \cap N(x_2')$), contradicting Claim 20.

It follows from Claims 27-31 that x_3 is a vertex of type $(2, 0, 0)$. For the remainder of the proof, let x_2 be the neighbour of x_3 in X_2.

Claim 32. *The vertex x_2 is of type $(1, 0, 0)$.*

Proof. Suppose not. Note that since $x_2 \notin X_1$, we have that x_2 is adjacent to at least one vertex in X_1. By Claim 22 (2), since x_2 is not of type $(1, 0, 0)$, it follows that x_2 is of type $(1, 0, 1)$. Let x_1 and x_1' be the neighbours of x_2 in X_1. Since G is planar, one of x_1 and x_1' has degree at most 5, since it is adjacent to three vertices in $V(C)$ by Claim 21 (1); to x_2; and possibly to x_3. This contradicts Claim 25. □

□

For the remainder of the proof, let x_1 be the neighbour of x_2 in X_1, let $T_1 = T \oplus x_1$, and let $T_2 = T_1 \oplus x_2$. The following claim is similar to Claim 19, and establishes some of the structure of T_2.

Claim 33. *Either:*

- $\deg(x_2) = 5$, *or*

- $\deg(x_2) = 6$, *the neighbours of* x_2 *not in* X_1 *or* C *form a path of length two with the ends in* $B(T \oplus x_1)$, $b(T_2) = b(T)$, $q(T_2) = q(T)$, *and* $d(T) \geq d(T_2) - 2\varepsilon$.

Proof. Suppose not. Note that we may assume $\deg(x_2) \geq 6$, since every vertex in $V(G) \setminus V(C)$ has degree at least five by Proposition 3.2.11 (2) and $\deg(x_2) \neq 5$ by assumption. Moreover, by Claim 9 we have that $v(T) \geq 8$ and so $v(T_2) \geq 6$. By Claim 3.2.7, T_2 is a critical canvas, and so by the minimality of T it follows that $d(T_2) \geq 3 - \gamma$. In addition, since x_1 is a tripod of T and x_2 is a tripod of $T \oplus x_1$, we have that $\text{def}(T) = \text{def}(T_2)$, and $v(T) = v(T_2) + 2$. It follows that

$$d(T) = d(T_2) - 2\varepsilon + \alpha(b(T_2) - b(T) + q(T_2) - q(T)).$$

By Claim 25, $\deg(x_1) = 6$ and there exist adjacent vertices $z_1, z_2 \notin V(C)$ such that z_1 is adjacent to x_1 and is in $B(T)$, and z_2 is adjacent to x_1 and x_2. Let $x_1, u_2, u_3, q_1, \cdots, z_2$ be the neighbours of x_2 listed in their cyclic order around x_2. Let $R = N(x_2) \setminus \{x_1, u_2, u_3, q_1, z_2\}$. We claim that $R \cap Q(T_1) = \emptyset$. To see this, suppose not: let q be a vertex in $R \cap Q(T_1)$. Then given the presence of the path $z_1z_2x_2$ and the fact that $z_1 \in B(T)$, it follows that q is cofacial with a vertex in $V(C)$. Given the existence of q_1, this implies that q is a strong dividing vertex of T_2. Since T_2 is a 2-relaxation of T, this contradicts Claim 18. Thus $R \cap Q(T_1) = \emptyset$; and since $B(T_2) \subseteq Q(T_2)$, it follows that $R \cap B(T_1) = \emptyset$. Since $\deg(x_1) = 6$ by Claim 25, Claim 19 implies that $b(T) = b(T_1)$, $q(T) = q(T_1)$, and $d(T) \geq d(T_1) - \varepsilon$. Since $R \cap Q(T_2) = \emptyset$, it follows that $q(T_2) \geq q(T_1) + |R| - 1$, and $b(T_2) \geq b(T_1) + |R| - 1$. Moreover, $\text{def}(T_2) = \text{def}(T_1)$, and $v(T_1) = v(T_2) + 1$. Thus

$$d(T_1) = d(T_2) - \varepsilon + \alpha(b(T_2) - b(T_1) + q(T_2) - q(T_1)), \tag{3.5.1}$$

and so

$$\begin{aligned} d(T_1) &\geq d(T_2) - \varepsilon + \alpha(b(T_1) + |R| - 1 - b(T_1) + q(T_1) + |R| - 1 - q(T_1)) \\ &\geq d(T_2) - \varepsilon + \alpha(2|R| - 2). \end{aligned}$$

Since $d(T) \geq d(T_1) - \varepsilon$ by Claim 19, it follows that

$$d(T) \geq d(T_2) - 2\varepsilon + \alpha(2|R| - 2)$$

If $|R| \geq 2$, then $d(T) \geq d(T_2) - 2\varepsilon + 2\alpha$. This is a contradiction, since by the minimality of T we have that $d(T_2) \geq 3 - \gamma$, and $\varepsilon < \alpha$ by (I1), implying that $d(T) \geq 3 - \gamma$. Thus $|R| = 1$, and so $\deg(x_2) = 6$, $q(T_2) \geq q(T_1)$, and $b(T_2) \geq b(T_1)$. If equality does not hold in both of these expressions, then in Equation (3.5.1) we have

$$\begin{aligned} d(T_1) &> d(T_2) - \varepsilon + \alpha \\ d(T) \geq d(T_1) - \varepsilon &> d(T_2) - 2\varepsilon + \alpha \text{ since } d(T) \geq d(T_1) - \varepsilon \text{ by Claim 19} \\ &> 3 - \gamma - 2\varepsilon + \alpha, \end{aligned}$$

where the last line follows from the fact that by the minimality of T, we have $d(T_2) \geq 3-\gamma$. This is a contradiction, since $\alpha \geq 2\varepsilon$ by (I1). Hence $q(T_1) = q(T_2)$, and $b(T_1) = b(T_2)$. Note this implies $b(T) = b(T_2)$ and $q(T) = q(T_2)$ by Claims 25 and 19. Moreover, $d(T_2) \geq d(T_1) - \varepsilon$; and since $d(T) \geq d(T_1) - \varepsilon$ by Claim 19, it follows that $d(T) \geq d(T_2) - 2\varepsilon$.

Now, let $q \in R$. From the above, $Q(T_2) \setminus \{q\} = Q(T_1) \setminus \{x_2\}$, which implies that $q_1 q q_2$ forms a path as otherwise there would exist a vertex $x \notin \{q, q_1, q_2\}$ with $x \in Q(T_2) \cap Q(T_1)$. But then x is a strong dividing vertex of T_1, contradicting Claim 18. Similarly, $B(T_2) \setminus \{q\} = B(T_1) \setminus \{x_2\}$, implying that $\{q_1, q_2\} \subseteq B(T_1)$. □

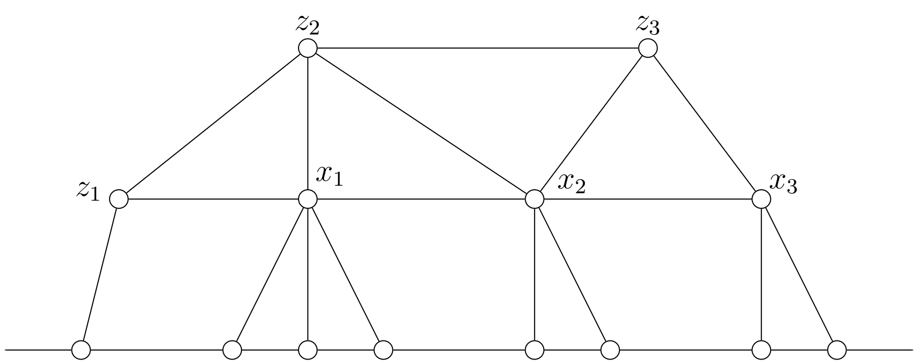

Figure 3.2: Vertices z_1, z_2, and z_3 as described in Claims 34 and 36. For each $i \in \{1, 2, 3\}$, the vertex x_i is in X_i. Moreover, x_2 is of type (1,0,0), and x_3 is of type (2,0,0).

The following claim establishes that $\deg(x_2) = 6$. See Figure 3.2 for an illustration of the vertices described in the claim. For the remainder of the proof, let $T_3 = T_2 \oplus x_3$, with $T_3 = (G_3, C_3, (L, M))$.

Claim 34. *The following both hold.*

- $\deg(x_2) = 6$, *and*

- *there exist adjacent vertices $z_2, z_3 \notin V(C)$ such that z_2 is adjacent to x_2 and x_1, and z_3 is adjacent to x_2 and x_3.*

Proof. By Claim 33, this holds unless $\deg(x_2) = 5$. Let $z \in V(G_3) \setminus V(C_3)$ be a neighbour of x_2. We claim $G \setminus \{x_2 z\}$ has a C_3-critical subgraph. To see this, we start by extending ϕ to a partial (L, M)-colouring of $C \cup C_3$ as follows: first, note that since $x_1 \in X_1$ is a tripod of T by Claim 21 (1), Claim 24 implies that $|S(x_1)| = 2$. Similarly, since x_2 is of type (1,0,0) by Claim 32, we have that Claim 24 implies that $|S(x_2)| = 3$. Since x_3 is of type (2,0,0), again Claim 24 implies that $|S(x_3)| = 3$. By Claim 12, $|M_{x_1 x_2}| = |M_{x_2 x_3}| = 5$. Thus there exists a colour $c_1 \in S(x_1)$ and a colour $c_3 \in S(x_3)$ such that $x_2[x_1, c_1] = x_2[x_3, c_3]$. Set $\phi(x_i) = c_i$ for $i \in \{1, 3\}$. If ϕ extends to an (L, M)-colouring ϕ' of $G_3 \setminus x_2 z$, then ϕ' extends to an (L, M)-colouring of G by redefining $\phi'(x_2)$ to be a colour in $S(x_2) \setminus \{x_2[x_1, c_1], x_2[z, \phi'(z)]\}$. Since $|S(x_2)| = 3$, such a choice exists —a contradiction, since ϕ does not extend to G. Thus ϕ does not extend to an (L, M)-colouring of $G_3 \setminus x_2 z$, and thus by Proposition 3.2.9, we have that $G_3 \setminus x_2 z$ has a C_3-critical subgraph, G_3'. But then G_3' is a proper C_3-critical subgraph of G_3 (since $x_2 z \notin E(G_3')$). Since T_3 is a 3-relaxation of T, this contradicts Claim 20. □

For the remainder of the proof, let z_1, z_2, and z_3 be as in Claims 25 and 34.

Claim 35. *Neither z_2 nor z_3 have a neighbour in $V(C)$.*

Proof. Note that $N(z_2) \cap V(C) = \emptyset$, as otherwise given the existence of z_1 and z_3, we have that z_2 is a strong dividing vertex of T_1, contradicting Claim 18. Moreover, note that $\deg(x_3) \geq 5$ by Proposition 3.2.11 (2); and so, given that x_3 is a tripod of T_2 by Claim 26, there exists a vertex $q_1 \in N(x_3)$ such that q_1, z_3, x_2 is part of the cyclic ordering of the neighbours of x_3 and $q_1 \notin V(C)$. Thus $N(z_3) \cap V(C) = \emptyset$ by Claim 35, as otherwise given the existence of z_2 and q_1, we have that z_3 is a strong dividing vertex of T_3, contradicting Claim 18. □

The following claim bounds $d(T)$ in terms of $d(T_3)$ and establishes that $\deg(x_3) = 6$.

Claim 36. $\deg(x_3) = 6$ *and* $d(T) \geq d(T_3) - 3\varepsilon$.

Proof. Suppose not. First suppose that $\deg(x_3) \geq 6$. Note that since x_1 is a tripod of T; since x_2 is a tripod of T_1; and since x_3 is a tripod of T_2, it follows that $\text{def}(T) = \text{def}(T_1) = \text{def}(T_2) = \text{def}(T_3)$. Moreover, by Claim 9, $v(T_3) \geq 5$. Thus by the minimality of T, we have that $d(T_3) \geq 3 - \gamma$. In addition, note that $v(T_3) = v(T_2) - 1$, that $v(T_2) = v(T_1) - 1$, and that $v(T_1) = v(T) - 1$. Thus, letting $T_0 := T$, we have that

$$d(T_{i-1}) = d(T_i) - \varepsilon + \alpha\left(b(T_i) - b(T_{i-1}) + q(T_i) - q(T_{i-1})\right) \tag{3.5.2}$$

for each $i \in \{1, 2, 3\}$.

Let $z_3, x_2, u_1, u_2, q_1, q_2, \ldots$ be the neighbours of z_3 listed in their cyclic order around z_3, where $\{u_1, u_2\} \subseteq V(C)$. Let $R = N(x_3) \setminus \{z_3, x_2, u_1, u_2, q_1\}$. We claim no vertex $r \in R$ is in the quasiboundary of T_2; otherwise, given the existence of z_3 and q_1, we have that r is a strong dividing vertex of T_3, contradicting Claim 18.

Note that $R \subseteq B(T_3) \subseteq Q(T_3)$. Thus since $x_3 \in Q(T_2) \setminus Q(T_3)$ and $R \subseteq Q(T_3) \setminus Q(T_2)$, it follows from Equation 3.5.2 that

$$d(T_2) \geq d(T_3) - \varepsilon + 2\alpha(|R| - 1).$$

By Claims 33 and 34, $d(T) \geq d(T_2) - 2\varepsilon$.

Combining these results, we have that

$$\begin{aligned} d(T) &\geq d(T_2) - 2\varepsilon \\ &\geq (d(T_3) - \varepsilon + 2\alpha(|R| - 1)) - 2\varepsilon. \end{aligned}$$

Suppose that $\deg(x_3) \geq 7$, and so that $|R| \geq 2$. Then $d(T) \geq d(T_3) - 3\varepsilon + 2\alpha$. Since $d(T_3) \geq 3 - \gamma$, this implies $d(T) \geq 3 - \gamma - 3\varepsilon + 2\alpha$. This is a contradiction, since $3\varepsilon \leq 2\alpha$ by (I1). Thus $|R| = 1$, and so $\deg(x_3) = 6$ and $d(T) \geq d(T_3) - 3\varepsilon$, as desired.

Suppose now that $\deg(x_3) < 6$, and so that $\deg(x_3) = 5$ by Proposition 3.2.11 (2). Recall that by Claim 22 (1), x_1 is a tripod of T. It follows from Claim 24 that $|S(x_1)| = 2$; and moreover since x_2 is a tripod of T_1 by Claim 22 (1), we have further that $|S(x_2)| = 3$. Thus there exists a colour $c_2 \in S(x_2)$ such that $x_1[x_2, c_2] \notin S(x_1)$. Let $\phi(x_2) = c_2$. Note that $N(z_2) \cap V(C) = \emptyset$ by Claim 35. Thus by Claim 24, $|S(x_2)| = 5$, and so there exists two distinct colours d_1 and d_2 in $S(z_2)$ such that $x_1[z_2, d_i] \notin S(x_1)$ and $x_2[z_2, d_i] \neq c_2$ for $i \in \{1, 2\}$.

Furthermore, $N(z_3) \cap V(C) = \emptyset$ by Claim 35. Thus by Claim 24, $|S(z_3)| = 5$. It follows that there exists an $i \in \{1, 2\}$ such that $S(z_3) \setminus (\{z_3[z_2, d_i], z_3[x_2, c_2]\} \cup \{z_3[x_3, c] : c \in S(x_3)\})$ is non-empty, since $|S(z_3)| = 5$ and $z_3[z_2, d_1] \neq z_3[z_2, d_2]$. Without loss of generality, suppose that $i = 1$. Let $d_3 \in S(z_3) \setminus (\{z_3[z_2, d_i], z_3[x_2, c_2]\} \cup \{z_3(x_3, c) : c \in S(x_3)\})$. Finally, let $\phi(z_2) = d_1$ and $\phi(z_3) = d_3$.

Let C' be obtained from C_3 by deleting x_2 and adding the vertices z_2, z_3 as well as edges x_1z_2, z_2z_3, and z_3x_3. Let $T' = T\langle C'\rangle = (G', C', (L, M))$. We claim that $G' - \{x_3q_1, x_1z_1\}$ has a C'-critical subgraph. To see this, note that if ϕ extends to $G' - \{x_3q_1, x_1z_1\}$, then ϕ extends to an (L, M)-colouring of G by redefining $\phi(x_1)$ as a colour in $S(x_1) \setminus x_1[z_1, \phi(z_1)]$ and $\phi(x_3)$ as a colour in $S(x_3) \setminus \{x_3[q_1, \phi(q_1)], x_3[x_2, c_2]\}$. Such choices exist, since $|S(x_1)| = 2$ and $|S(x_3)| = 3$ by Claim 24 since x_3 has exactly two neighbours in $V(C)$ by Claims 27-31. This contradicts the fact that ϕ does not extend to G. Thus ϕ does not extend to an (L, M)-colouring of $G' - \{x_3q_1, x_1z_1\}$, and so by Proposition 3.2.9 we have that $G' - \{x_3q_1, x_1z_1\}$ has a C'-critical subgraph G''. Note that $v(T') \geq 2$ by Claim 9, and $|E(G') \setminus E(G'')| \geq 2$. Moreover, we claim that C' is chordless: this follows easily from the facts that C is chordless by Claim 13; that neither z_2 nor z_3 have a neighbour in $V(C)$ by Claim 35; that $x_1z_3 \notin E(G)$ since G is planar; and that $z_2x_3 \notin E(G)$ since z_3 has degree at least five by Proposition 3.2.11 (2). Thus $|E(G'') \setminus E(C')| \geq 2$.

By Claim 10 (3) applied to T' and G'', we find that $d(T') \geq 5 - (2\alpha + \varepsilon) - \gamma$. Moreover, $v(T') = v(T_2) + 3$, $b(T') \geq b(T_2) - 3$, and similarly $q(T') \geq q(T_2) - 3$. Thus $s(T') \geq s(T_2) - (3\varepsilon + 6\alpha)$. Furthermore, $\mathrm{def}(T') = \mathrm{def}(T_2) + 1$. Putting all of this together,

$$\begin{aligned} 5 - (2\alpha + \varepsilon) - \gamma &\leq d(T') \\ &\leq \mathrm{def}(T') - s(T') \\ &\leq (\mathrm{def}(T_2) + 1) - (s(T_2) - (3\varepsilon + 6\alpha)) \\ &\leq d(T_2) + 1 + (3\varepsilon + 6\alpha), \end{aligned}$$

which implies that $d(T_2) \geq 4 - \gamma - (8\alpha + 4\varepsilon)$. By Claim 33, $d(T) \geq d(T_2) - 2\varepsilon$, and so $d(T) \geq 4 - \gamma - (8\alpha + 6\varepsilon)$. This is a contradiction, since $8\alpha + 6\varepsilon \leq 1$ by (I2) and (I3). □

We will complete the proof of Theorem 3.4.7 by showing that x_3 is not of type (2,0,0), thereby arriving at a contradiction. Before we do this, we need the following key claim which establishes some of the correspondence assignment in the graph near x_3.

Claim 37. *The following both hold.*

(i) The vertex z_1 has exactly one neighbour in $V(C)$, and there do not exist colours $d_1 \in S(x_1)$ and $d_2 \in S(x_2)$ such that $z_2[x_1, d_1] = z_2[x_2, d_2]$.

(ii) Let $T_1 = (G_1, C_1, (L, M))$. Let $\phi(x_1) \in S(x_1)$, let $S'(v) = S(v)$ for all $v \in V(G') \setminus N(x_1)$, let $S'(v) = S(v) \setminus v[x_1, \phi(x_1)]$ for each $v \in N(x_1)$. The vertex z_2 has exactly one neighbour in $V(C')$. Moreover, there do not exist colours $d_2 \in S'(x_2)$ and $d_3 \in S'(x_3)$ such that $z_3[x_2, d_2] = z_3[x_3, d_3]$.

Proof. We begin by proving (i). First we will show that z_1 has exactly one neighbour in $V(C)$. To see this, suppose not. Since z_1 is in the boundary of T by Claim 25, it follows that z_1 has at least one neighbour in $V(C)$. Thus z_1 has at least two neighbours in $V(C)$. Since x_1 is adjacent to z_1 and there are no edges in $E(G)$ with both endpoints in X_1 by Claim 21 (3), it follows that that $z_1 \notin X_1$ and so that z_1 has exactly two neighbours in $V(C)$. Thus by Claim 24 we have that $|S(z_1)| = 3$. Similarly, since x_2 is a tripod of T_1 by Claim 22 (1), by Claim 24, $|S(x_2)| = 3$. Since x_1 is a tripod of T by Claim 21 (1), we have further from Claim 24 that $|S(x_1)| = 2$. Thus there exists a colour $c_1 \in S(z_1)$ and a colour $c_2 \in S(x_2)$ such that $x_1[z_1, c_1] \notin S(x_1)$ and $x_1[x_2, c_2] \notin S(x_1)$. Let $C' = C \oplus x_1 \oplus z_1 \oplus x_2$, and let $\phi(z_1) = c_1$ and $\phi(x_2) = c_2$. Let $T' = (G', C', (L, M)) = T\langle C'\rangle$. We claim $G' - x_1z_2$ has a C'-critical subgraph. To see this, note that if ϕ extends to an (L, M)-colouring of $G' - x_1z_2$, then by redefining $\phi(x_1)$ to be a colour in $S(x_1) \setminus x_1[z_2, \phi(z_2)]$ we obtain an extension of ϕ to an (L, M)-colouring of G, a contradiction. Thus ϕ does not extend to $G' - x_1z_2$, and so by Proposition 3.2.9, we have that $G' - x_1z_2$ contains a C'-critical subgraph G''. But G'' is a proper subgraph of G', and T' is a 3-relaxation of T. This contradicts Claim 20.

Thus z_1 has exactly one neighbour in $V(C)$, and so by Claim 24 we have that $|S(z_1)| = 4$. Since $|S(x_1)| = 2$, there exist two distinct colours c_1 and c_2 in $S(z_1)$ with $x_1[z_1, c_1] \notin S(x_1)$ and $x_1[z_1, c_2] \notin S(x_1)$.

We now proceed with the rest of the claim. Suppose for a contradiction that there exist $d_1 \in S(x_1)$ and $d_2 \in S(x_2)$ such that $z_2[x_1, d_1] = z_2[x_2, d_2]$. Since z_2 has no neighbours in $V(C)$ by Claim 35, by Claim 24 we have that $|S(z_2)| = 5$, and so there exists a colour $c_3 \in S(z_2)$ and an $i \in \{1, 2\}$ such that $c_3 \neq z_2[z_1, c_i]$, such that $x_1[z_2, c_3] \notin S(x_1)$, and such that $x_2[z_2, c_3] \notin S(x_2)$: that is, there is a colour choice c_i for z_1 that avoids $S(x_1)$, and a colour choice $c_3 \in S(z_2)$ that avoids $c_i \in S(z_1)$ as well as $S(x_1)$ and $S(x_2)$. See Figure 3.3 for an illustration of the matchings described. Let C'' be obtained from $(C \oplus x_1 \oplus x_2) \setminus \{x_1\}$ by adding the vertices z_1 and z_2 as well as edges yz_1, z_1z_2, z_2x_2, where $y \in N(z_1) \cap V(C)$. Let $T'' = (G'', C'', (L, M)) = T\langle C''\rangle$. Recall that $\deg(x_2) = 6$ by Claim 22 (2); and $N(x_2) \setminus (V(C'') \cup \{x_1\}) = \{z_3, x_3\}$.

We claim that $G'' \setminus \{x_2z_3, x_2x_3\}$ has a C''-critical subgraph. To see this, choose $\phi(z_1) = c_i$, and $\phi(z_2) = c_3$. If ϕ extends to an (L, M)-colouring of $G'' \setminus \{x_1, x_2\}$, then ϕ extends to an (L, M)-colouring of G by first choosing $\phi(x_2) \in S(x_2) \setminus \{x_2[z_3, \phi(z_3)], x_2[x_3, \phi(x_3)]\}$, and then choosing $\phi(x_1) \in S(x_1) \setminus \{x_1[x_2, \phi(x_2)]\}$. Note this is possible, since $|S(x_1)| = 2$ and $|S(x_2)| = 3$. This is a contradiction, since ϕ does not extend to G by assumption. Thus ϕ does not extend to an (L, M)-colouring of $G'' - \{x_1, x_2\}$, and so by Proposition 3.2.9 we have that $G'' \setminus \{x_2z_3, x_2x_3\}$ has a C''-critical subgraph. But then G'' contains a proper C''-critical subgraph G^*. Note that $|E(G'') \setminus E(G^*)| \geq 2$, and by Claim 9, $v(T'') \geq 3$. Finally, we claim $|E(G^*) \setminus E(C'')| \geq 2$. To see this, note that since C is chordless by Claim

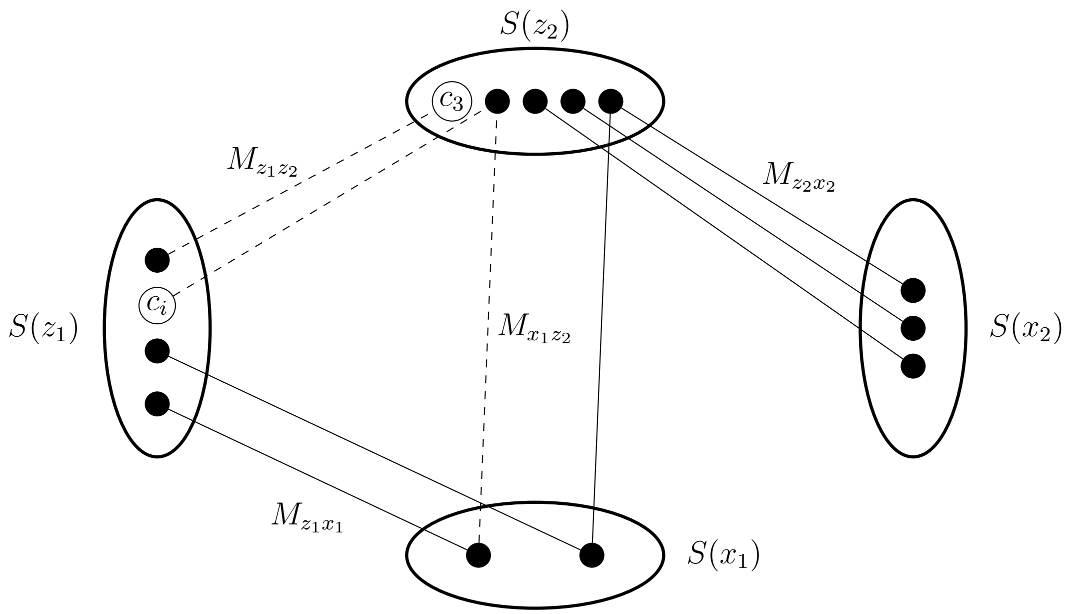

Figure 3.3: The matchings between $S(x_1)$, $S(x_2)$, $S(z_1)$, and $S(z_2)$, as described in Claim 37. The matching $M_{x_2x_1}$ is omitted for clarity. We assume there exists a colour $d_1 \in S(x_1)$ and $d_2 \in S(x_2)$ such that $z_2[x_1, d_1] = z_2[x_2, d_2]$. Without loss of generality, we may assume that the solid edges in the matchings are as shown. No matter the remainder of the edges in $M_{z_1z_2}$ and $M_{x_1z_2}$, there exist colours $c_i \in S(z_1)$ and $c_3 \in S(z_2)$ such that c_i is unmatched in $M_{x_1z_1}$, such that c_3 is unmatched in $M_{x_1z_2}$ and $M_{x_2z_2}$, and such that $c_3 \neq z_2[z_1, c_i]$.

13; since z_1 has exactly one neighbour y in $V(C)$ as shown above; since neither z_2 nor z_3 have a neighbour in $V(C)$ by Claim 35; and since $z_1x_2 \notin E(G)$ since G is planar, it follows that C'' is chordless. Thus $|E(G^*) \setminus E(C'')| \geq 2$. By Claim 10 (3) applied to T'' and G^*, we find that $d(T'') \geq 5 - (2\alpha + \varepsilon) - \gamma$. In addition, $s(T_2) \leq s(T'') + 2(2\alpha + \varepsilon)$, $\text{def}(T_2) \geq \text{def}(T'') - 1$, and so $d(T_2) \geq d(T'') - 1 - 2(2\alpha + \varepsilon)$. Thus $d(T_2) \geq 4 - \gamma - 3(2\alpha + \varepsilon)$. By Claim 33, $d(T) \geq d(T_2) - 2\varepsilon$ and so $d(T) \geq d(T_2) - 2\varepsilon \geq 4 - \gamma - (6\alpha + 5\varepsilon)$. But then $d(T) \geq 3 - \gamma$ since by (I2) and (I3) we have that $6\alpha + 5\varepsilon \leq 1$. This contradicts the fact that T is a counterexample.

The proof of (ii) is nearly identical. For each $uv \in E(G')$, let M'_{uv} be the restriction of M_{uv} to $S'(u)$ and $S'(v)$. Recall that by Claim 12, we have that $|M_{z_2x_1}| = 5$. Note that z_2 is not adjacent to a vertex in $V(C)$ by Claim 35. Thus as z_2 is adjacent to x_1, we have that z_2 has exactly one neighbour in $V(C) \oplus x_1$, and so by Claim 12, we have that $|S'(z_2)| = 4$.

Similarly, $|S'(x_2) = 2|$. Thus there exist two colours $c_1, c_2 \in S'(z_2)$ such that for $i \in \{1,2\}$, we have that $x_2[x_2, c_i] \notin S'(x_2)$. Moreover, since $|S'(x_2)| = 2$ and $|S'(z_3)| = 5$, by Claim 12 we have that that $|M'_{x_2z_3}| = 2$. Recall that $|S(x_3)| = 2$ by Claim 24 and that $|S(z_3)| = 5$ by Claims 35 and 24. Since G is planar, neither x_3 nor z_3 is adjacent to x_1. Thus $|M'_{x_3z_2}| = 3$. Suppose for a contradiction that there exist colours $d_2 \in S'(x_2)$ and $d_3 \in S'(x_3)$ such that $z_3[x_2, d_2] = z_3[x_3, d_3]$. Then there exists a colour $c_3 \in S'(z_3)$ and an $i \in \{1,2\}$ such that $c_3 \neq z_3[z_2, c_i]$, such that $x_2[z_3, c_3] \notin S'(x_2)$, and such that $x_3[z_3, c_3] \notin S'(x_3)$: that is, there is a colour choice c_i for z_2 that avoids $S'(x_2)$, and a colour choice $c_3 \in S'(z_3)$ that avoids $c_i \in S'(z_2)$ as well as $S'(x_2)$ and $S'(x_3)$. See Figure 3.4 for an illustration of the matchings involved. Let C' be obtained from $(C_1 \oplus x_2 \oplus x_3) \setminus \{x_2\}$ by adding the vertices z_2 and z_3 as well as edges x_1z_2, z_2z_3, z_3x_3. Let $T' = T\langle C'' \rangle = (G', C', (L, M))$. Recall that $\deg(x_3) = 6$ by Claim 36. Let $N(x_3) \setminus (V(C) \cup \{x_2, z_3\}) = \{z_4, z_5\}$.

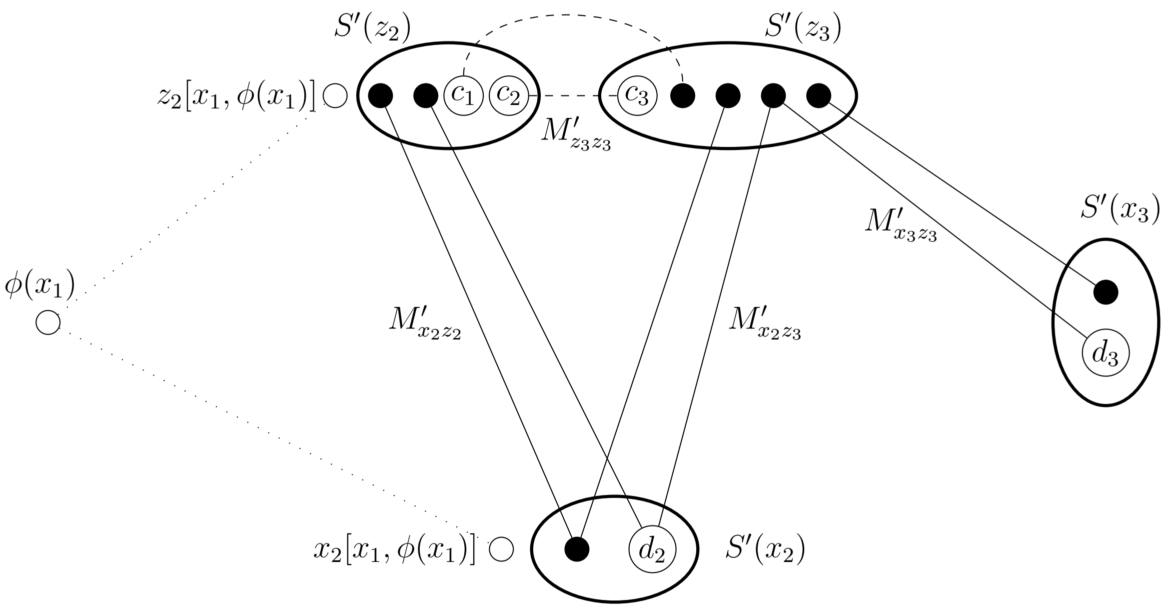

Figure 3.4: The matchings between $S'(x_2)$, $S'(x_3)$, $S'(z_2)$, and $S'(z_3)$, as described in the proof of the second statement in Claim 37. The matching $M'_{x_2x_3}$ is omitted for clarity. We assume there exists a colour $d_2 \in S'(x_2)$ and $d_3 \in S'(x_3)$ such that $z_2[x_2, d_2] = z_2[x_3, d_3]$. Without loss of generality, we may assume that the solid edges in the matchings are as shown. No matter the matching $M'_{z_2z_3}$, there exist colours $c_i \in S'(z_2)$ and $c_3 \in S'(z_2)$ such that c_i is unmatched in $M'_{x_2z_2}$, such that c_3 is unmatched in $M'_{x_2z_3}$ and $M'_{x_3z_3}$, and such that $c_3 \neq z_3[z_2, c_i]$.

We claim that $G' \backslash \{x_3z_4, x_3z_5\}$ has a C'-critical subgraph. To see this, choose $\phi(z_2) = c_i$, and $\phi(z_3) = c_3$. If ϕ extends to an (L, M)-colouring of $G' \setminus \{x_2, x_3\}$, then ϕ extends to an (L, M)-colouring of G by first choosing $\phi(x_3) \in S'(x_3) \setminus \{x_3[z_4, \phi(z_4)], x_3[z_5, \phi(z_5)]\}$, and then choosing $\phi(x_2) \in S'(x_2) \setminus \{x_2[x_3, \phi(x_3)]\}$. Note this is possible, since $|S'(x_2)| = 2$ and $|S'(x_3)| = 3$. This is a contradiction, since ϕ does not extend to G by assumption. Thus ϕ does not extend to an (L, M)-colouring of $G' - \{x_2, x_3\}$, and so by Proposition 3.2.9 we have that $G' \setminus \{x_3z_4, x_3z_5\}$ has a C'-critical subgraph. But then G' contains a proper C'-critical subgraph G^*. Note that $|E(G') \setminus E(G^*)| \geq 2$, and by Claim 9, $v(T') \geq 3$. Finally, we claim that C' is chordless: this follows from the facts that C is chordless by Claim 13; that neither z_2 nor z_3 have neighbours in $V(C)$ by Claim 35; and that $z_3z_1 \notin V(C)$ since G is planar. Thus $|E(G^*) \setminus E(C')| \geq 2$. By Claim 10 (3) applied to T' and G^*, we find that $d(T') \geq 5 - (2\alpha + \varepsilon) - \gamma$.

In addition, $s(T_3) \leq s(T') + 2(2\alpha + \varepsilon)$, and $\text{def}(T_3) \geq \text{def}(T') - 1$. Thus $d(T_3) \geq d(T') - 1 - 2(2\alpha + \varepsilon)$, and so since $d(T') \geq 5 - (2\alpha + \varepsilon) - \gamma$, we have that $d(T_3) \geq 4 - \gamma - 3(2\alpha + \varepsilon)$. By Claim 36, $d(T) \geq d(T_3) - 3\varepsilon$. Thus $d(T) \geq 4 - \gamma - 6\alpha - 6\varepsilon$. But this is a contradiction, since by (I2) and (I3) we have that $6\alpha + 6\varepsilon \leq 1$. □

We now show x_3 is not of type (2,0,0), thus completing the proof of Theorem 3.4.7.

Claim 38. *x_3 is not of type (2,0,0).*

Proof. Suppose not. By Claim 21 (1), we have that x_1 is a tripod of T. By Claim 24, it follows that $|S(x_1)| = 2$. Let $|S(x_1)| = \{c_1, c_2\}$, and for each $i \in \{1, 2\}$, let ϕ_i be an extension of ϕ to x_1 with $\phi_i(x_1) = c_i$; let $S_i(x_2) = S(x_2) \setminus x_2[x_1, c_i]$; and similarly let $S_i(x_3) = S(x_3) \setminus x_3[x_1, c_i]$. Note that $S_1(x_2) \neq S_2(x_2)$ since c_1 and c_2 are distinct. Furthermore, note that x_3 is not adjacent to x_1 since x_3 is a tripod of T_2 of type (2,0,0): thus $S_i(x_3) = S(x_3)$ is a fixed set that does not depend on i. Let $M^i_{z_3x_2}$ be the restriction of $M_{z_3x_2}$ to $S_i(z_3)$ and $S_i(x_2)$. Finally, let $S = S(z_3) \setminus \{z_3[x_3, d] : d \in S_i(x_3)\}$. Since $S_i(x_3)$ is fixed, so too is S.

Note that by Claim 12, we have that $|M^i_{z_3x_2}| = 2$ for each $i \in \{1, 2\}$, and moreover by Claim 37 (2) there does not exist a colour $d_3 \in S_i(x_3)$ and a colour $d_2 \in S_i(x_2)$ such that $z_3[x_2, d_2] = z_3[x_3, d_3]$. Since S is fixed, this implies that $M^1(z_3x_2) = M^2(z_3x_2)$. This is a contradiction, since $S_1(x_2)$ and $S_2(x_2)$ are distinct sets of size two. □

3.6 Hyperbolicity and Implications

In this section, we prove Theorem 1.2.23, and discuss the implications of this result. We will need the following equivalent definition for the deficiency of a graph.

Lemma 3.6.1 (Proof taken from Lemma 3.3, [35])**.** *If G is a 2-connected plane graph with outer cycle C, then*

$$\text{def}(G) = |V(C)| - 3 - \sum_{f \in \mathcal{F}(G)} (|f| - 3).$$

Proof. Using Euler's formula for planar graphs, we have that $|V(G)| - |E(G)| + (|\mathcal{F}(G)| + 1) = 2$; equivalently, $|V(C)| + v(G) + |\mathcal{F}(G)| + 1 = |E(G)| + 2$. Hence

$$\begin{aligned} |V(C)| - 3 - \sum_{f \in \mathcal{F}(G)} (|f| - 3) &= |V(C)| - 3 - (2|E(G)| - |E(C)|) + 3|\mathcal{F}(G)|) \\ &= 2|V(C)| - 2|E(G)| - 3 + 3|E(G)| - 3|V(C)| - 3v(G) + 3 \\ &= |E(G) \setminus E(C)| - 3v(G) \quad \text{since } |E(C)| = |V(C)| \\ &= \text{def}(G), \quad \text{by definition.} \end{aligned}$$

□

Theorem 3.4.7 implies the following.

Theorem 3.6.2. *If $T = (G, C, (L, M))$ is a critical canvas, then*

$$\varepsilon |V(G) \setminus V(C)| + \sum_{f \in \mathcal{F}(G)} (|f| - 3) \leq |V(C)| - 4.$$

Proof. If $T = (G, C, (L, M))$ is a critical canvas, then it follows from Lemma 3.6.1 that $d(T) = |V(C)| - 3 - \sum_{f \in \mathcal{F}(G)} (|f| - 3) - \varepsilon v(G) - \alpha(b(T) + q(T))$. Thus $d(T) \leq |V(C)| - 3 - \sum_{f \in \mathcal{F}(G)} (|f| - 3) - \varepsilon v(G)$. By Theorem 3.4.7, if $v(T) \geq 2$ then $3 - \gamma \leq d(T)$. If $v(T) \leq 1$, then by Proposition 3.4.6 $d(T) \geq 1$. Thus $1 \leq |V(C)| - 3 - \sum_{f \in \mathcal{F}(G)} (|f| - 3) - \varepsilon v(G)$, and so

$$\varepsilon |V(G) \setminus V(C)| + \sum_{f \in \mathcal{F}(G)} (|f| - 3) \leq |V(C)| - 4,$$

as desired. □

Omitting the $\varepsilon |V(G) \setminus V(C)|$ term, we obtain the following.

Corollary 3.6.3. *If $(G, C, (L, M))$ is a critical canvas, then every internal face f of G satisfies $|f| < |C| - 1$.*

Omitting instead the face sizes from Theorem 3.6.2 gives the theorem below.

Theorem 3.6.4. *If $(G, C, (L, M))$ is a critical canvas and ε is as in Theorem 3.4.7, then $|V(G)| \leq \frac{1+\varepsilon}{\varepsilon}|V(C)|$.*

Proof. By Theorem 3.6.2, we have that $|V(G) \setminus V(C)| \leq \frac{1}{\varepsilon}|V(C)|$. Thus $|V(G)| \leq \frac{1}{\varepsilon}|V(C)| + |V(C)|$, and so the result follows. □

To obtain the best possible bound in Theorem 3.6.4, we wish to maximize ε subject to inequalities (I1-I3). To that end, we choose gives $\alpha = \frac{1}{25}$, $\varepsilon = \frac{1}{50}$, and $\gamma = \frac{7}{10}$, giving $V(G) \leq 51|V(C)|$ in Theorem 3.6.4.

Theorem 1.2.23 follows from Theorem 3.6.4 as shown below.

Proof of Theorem 1.2.23. Let $G, C, (L, M)$, and H be as in Theorem 1.2.23. We claim that H is C-critical. Suppose not. Then there exists a proper subgraph H' of H such that every (L, M)-colouring of C that extends to H' also extends to H. But since every (L, M)-colouring C that extends to H also extends to G, we have that H' contradicts the minimality of H. Thus H is C-critical, and so by Theorem 3.6.4 we have $|V(H)| \leq 51|V(C)|$, as desired. □

We now discuss the implications of Theorem 3.6.4. To begin, we show how Theorem 3.4.7 implies that the family of graphs which are critical for 5-correspondence colouring forms a hyperbolic family. Note the theorem below is merely a more explicit version of Theorem 1.2.19. Following this, we discuss the implications of the hyperbolicity of a family of graphs as described by Postle and Thomas in [37].

Theorem 3.6.5. *The family $\mathcal{F}$ of embedded graphs that are critical for 5-correspondence colouring is hyperbolic with Cheeger constant* 50.

Proof. Let G be a graph that is (L, M)-critical, where (L, M) is a 5-correspondence colouring. Note that G is connected, as otherwise since every subgraph of G admits an (L, M)-colouring, it follows that each component of G admits an (L, M)-colouring and thus so does G itself, contradicting the definition of (L, M)-critical. Suppose that G is embedded in a surface Σ, and let $\lambda : \mathbf{S}^1 \to \Sigma$ be a closed curve intersecting G in only its vertices and bounding an open disk Δ. Let Y be the set of vertices of G that are intersected by

λ, and let X be the set of vertices in Δ. The theorem follows from showing that if X is non-empty, then $|X| \leq 50(|Y| - 1)$. Let $G_1 := G[X \cup Y]$, and let $G_2 := G \setminus G_1$. Since G is critical for 5-correspondence colouring, there exists a colouring of G_2 that extends every proper subgraph of G containing G_2 but not to G itself. Since $X \neq \emptyset$, it follows that G_1 is $G[Y]$-critical. By Theorem 1.2.7, it follows that $|Y| \geq 3$. Let $v_0, v_1, v_2, \cdots, v_k$ be the vertices of Y appearing in a cyclic order along λ. Let C be the cycle $v_0 v_1 \cdots v_k v_0$. Since G_1 is $G[Y]$-critical, it follows that $G_1 \cup C$ is C-critical. By Lemma 3.2.5, $G_1 \cup C$ is 2-connected, and hence $(G_1, C, (L, M))$ is a canvas. By Theorem 3.6.2 with $\varepsilon = \frac{1}{50}$, we have that $|V(G_1) \setminus V(C)| \leq 50(|V(S)| - 4)$. The result follows. □

Showing that such a family of critical graphs is hyperbolic has many interesting implications, as described in [37]. We highlight a few in particular below, following a definition.

Definition 3.6.6. A *non-contractible cycle* in a surface is a cycle that cannot be continuously deformed to a single point. An embedded graph is *ρ-locally planar* if every cycle (in the graph) that is non-contractible (in the surface) has length at least ρ.

In [37], Postle and Thomas show the following.

Theorem 3.6.7 (Postle & Thomas, [37])**.** *For every hyperbolic family $\mathcal{F}$ of embedded graphs that is closed under curve cutting there exists a constant $k > 0$ such that every graph $G \in \mathcal{F}$ embedded in a surface of Euler genus g has a non-contractible cycle of length at most $k \log(g + 1)$.*

Using this, we obtain the following theorem as a corollary to Theorem 3.6.5.

Theorem 1.2.20. *For every surface Σ, there exists a constant $\rho > 0$ such that every ρ-locally planar graph that embeds in Σ is 5-correspondence-colourable.*

Proof. Let Σ be a fixed surface, and let g be the Euler genus of Σ. Let $\mathcal{F}$ be the family of embedded graphs that are critical for 5-correspondence colouring. By Theorem 1.2.19, $\mathcal{F}$ is a hyperbolic family. By Theorem 3.6.7, there exists a constant $k > 0$ such that every graph $G \in \mathcal{F}$ embedded in Σ has a non-contractible cycle of length at most $k \log(g + 1)$.

Let $\rho > k \log(g + 1)$, and let G be a ρ-locally planar graph that embeds in Σ. Suppose for a contradiction that G is not 5-correspondence-colourable. Then G contains a subgraph H in $\mathcal{F}$. But then H contains a non-contractible cycle of length at most $k \log(g + 1)$. This is a contradiction, since $H \subseteq G$ and G is ρ-locally planar. □

In addition, following the work of Dvořák and Kawarabayashi in [12], Theorem 1.2.23 implies the following. Note that by *linear-time algorithms*, we mean algorithms whose run-time is linear in the number of vertices in the graph.

Theorem 1.2.24. *Let Σ be a fixed surface. There exists a linear-time algorithm that takes as input an embedded graph (G, Σ) and 5-correspondence assignment (L, M) for G with lists of bounded size and determines whether or not G is (L, M)-colourable.*

Theorem 1.2.25. *Let Σ be a fixed surface. There exists a linear-time algorithm that takes as input an embedded graph (G, Σ) and determines whether or not G is 5-correspondence-colourable.*

We note that the correspondence assignment in Theorem 1.2.24 is fixed, whereas in Theorem 1.2.25 it is not.

The algorithms in the theorems above are the same as those given by Dvořák and Kawarabayashi in [12]. We refer to [12] for a complete description of the algorithms and the proof of correctness. Following a necessary definition, a brief overview is given below to give the reader an overall sense of the algorithms.

Definition 3.6.8. Given a graph G, a *tree decomposition* is a tree T where $V(T)$ is a set of subsets of $V(G)$ and T satisfies the following properties:

- $\cup_{U \in V(T)} U = V(G)$,
- for each $uv \in E(G)$, there exists some $W \in V(T)$ such that $\{u, v\} \subseteq W$, and
- for each $V, U \in V(T)$, if there exists a vertex $w \in V(G)$ with $w \in V \cap U$, then $w \in Y$ for each vertex $Y \in V(T)$ in the unique path from V to U in T.

The *width* of T is defined as $\min_{V \in V(T)} |V| - 1$. The *tree-width* of G is the minimum width across all tree decompositions of G.

The idea behind the algorithm in Theorem 1.2.24 is the following: given an embedded graph (G, Σ), we first find a subgraph H of G of tree-width linear in the genus of Σ such that each component of $G - H$ is connected to H via a cut of bounded size. This subgraph H has specific structure in G: namely, H has a bounded number of components, each component of $G - H$ is locally planar, and H is formed by the union of short, non-contractible cycles connected by paths of bounded distance. The graph H can be found via a linear-time algorithm of Dvořák, Král', and Thomas [13]. Using Theorem 1.2.23, one can show that H-critical graphs in the components of $G - H$ have logarithmic distance to H. If a component of $G - H$ contains an H-critical subgraph G', we claim G' has tree-width logarithmic in $|V(G)|$. To see this, it suffices to add a new vertex v adjacent to all vertices in $H \cap G'$. Note this increases the genus of $(H \cap G') \cup \{v\}$ by at most $2k$, where k is the number of components of $H \cap G'$. We then use the following result of Eppstein.

Theorem 3.6.9 (Eppstein, [16])**.** *There exists a constant c such that every graph G of Euler genus g and radius r has tree-width at most $c(g+1)r$. Furthermore, a tree decomposition of this width can be found in time $O((g+1)r|V(G)|)$.*

Thus G' has tree-width $O(r)$, and since r is logarithmic in $|V(H)|$ and $|V(H)| \leq |V(G)|$, it follows that G' has tree-width logarithmic in $|V(G)|$. We find a tree decomposition of this width, and use a standard dynamic programming algorithm (see for instance [22]) to colour H, and extend this colouring to the components of $G' - H$. As G' has bounded tree-width, the algorithm runs in $O(|V(G')|)$ time.

The algorithm in Theorem 1.2.25 is similar in spirit, but requires checking not only whether an (L, M)-colouring extends, but whether (L, M')-colourings extend for all possible sets of matchings M'. Note that a graph G is 5-correspondence colourable if and only if it is 5-correspondence colourable for every 5-correspondence assignment (L, M) with $|L(v)| = 5$ for all $v \in V(G)$ and $|M_{uv}| = 5$ for all $uv \in E(G)$. It thus suffices to check matchings M' satisfying these requirements. Since the lists are of bounded size, the dynamic programming algorithm mentioned above runs in $O(|V(G)|)$ time. Note further that in correspondence colouring (as opposed to list colouring) we may assume that the list assignment is the same set of five colours for each vertex: the matchings between adjacent lists determine the meaning of these colours.

3.7 The girth at least five case

For this final section of Chapter 3 (and indeed for the remainder of the thesis), we expand the notion of canvas as follows.

Definition 3.7.1. We say the triple $(G, S, (L, M))$ is a *canvas* if G plane graph, S is any connected subgraph of G, and (L, M) is a correspondence assignment for the vertices of G such that there exists an (L, M)-colouring of S and

- either $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(S)$, or

- G has girth at least five and $|L(v)| \geq 3$ for all $v \in V(G) \setminus V(S)$.

To close this chapter, we observe the following.

Observation 3.7.2. *Let $\varepsilon, \alpha > 0$ satisfy the following:*

(I1) $9\varepsilon \leq \alpha$

(I2) $2.5\alpha + 5.5\varepsilon \leq 1$

(I3) $11\varepsilon + 1 \leq 3\alpha$.

If $T = (G, S, L)$ is a critical canvas where:

- *G has girth at least five,*
- *G is not composed of exactly S and one edge not in S,*
- *G is not composed of exactly S together with one vertex of degree 3, then*

then $3e(T) - (5 + \varepsilon)v(T) - \alpha q(T) \geq 3$.

This is the correspondence colouring analogue of a nearly identical theorem for list colouring of Postle: Theorem 3.9, [34]. Beyond the change from list colouring to correspondence colouring, the key difference between the statements of Theorem 3.7.2 and Theorem 3.9 in [34] is that S is connected (as opposed to having at most two components). This change allows us to use Theorem 2.11 in [34] (which describes structures arising from critical canvases $(G, S, (L, M))$ where S is connected, and which holds for correspondence colouring) in lieu of Theorem 2.12 (which allows S to have two components, and which is not currently known to hold for correspondence colouring). Otherwise, the proof of Theorem 3.9 in [34] carries over to the correspondence colouring framework with only standard, minor changes: namely, when we perform reductions (colouring a strict subgraph of a minimum counterexample, deleting this subgraph, and removing vertices' colours from neighbours' lists), we delete *corresponding* colours from neighbouring lists, rather than identical colours.

The proof is similar in spirit to that of Theorem 3.4.7. However, as noted in Section 1.4, Postle and Thomas' list colouring theorem in the 5-choosability case does *not* carry over to correspondence colouring. The colouring arguments in Postle and Thomas' theorem for 5-choosability rely on the fact that for a triangle ux_2z_2u in a minimum counterexample with list assignment S, if $S(u) \subseteq S(x_2)$, then $S(z_2) \setminus (S(x_2) \cup S(u)) = S(z_2) \setminus S(x_2)$. This implies that it is possible to colour z_2 from $S(z_2)$ while avoiding the lists of both x_2 and u. This argument crucially does not hold for correspondence colouring: an analogous argument to that in shows merely that for a correspondence assignment (S, M), we have $|M_{x_2u}| = |S(u)|$, which of course implies nothing about M_{z_2u}. Crucially, the proof in the 5-choosability case involves keeping track of lists along a cycle. This is not the case in Postle's proof for 3-choosability: the colouring arguments involve only deleting vertices and removing their colours (or in the correspondence framework, their *corresponding* colours) from the lists of neighbours, and do not keep track of what these colours correspond to. Moreover, no arguments rely on keeping track of what colours are or are not available in a cycle: the colouring arguments involve only trees branching from vertices in S in the minimum counterexample.

Theorem 3.7.2 implies the following, which is the correspondence colouring analogue of Theorem 1.8 in [34].

Observation 3.7.3. *Let G be a plane graph of girth at least five, let (L, M) be a 3-correspondence assignment for G, and let C be a facial cycle of G. If G is C-critical with respect to (L, M), then $|V(G)| \leq 89|V(C)|$.*

Observation 3.7.3 in turn implies the corollary below.

Corollary 3.7.4. *The graphs of girth at least five that are critical for 3-correspondence colouring form a hyperbolic family.*

Proof. Let (G, Σ) be an embedded graph of girth at least five that is (L, M)-critical, where (L, M) is a 3-correspondence colouring. Note that G is connected, and by Theorem 4.3.4, Σ is not the plane. Let $\lambda : \mathbb{S}^1 \to \Sigma$ be a closed curve intersecting G in only its vertices and bounding an open disk Δ. Let Y be the set of vertices of G that are intersected by λ, and let X be the set of vertices in Δ. The theorem follows by showing that if X is non-empty, then $|X| \leq 395(|Y|-1)$. Let $G_1 := G[X \cup Y]$, and let $G_2 := G \setminus G_1$. Since G is critical for 3-correspondence colouring, there exists a colouring of G_2 that extends to every proper subgraph of G containing G_2 but not to G itself. Since $G_1 \setminus Y \neq \emptyset$, it follows that G_1 is $G[Y]$-critical; and by Theorem 4.3.4, $|Y| \geq 3$. Let $v_0, v_1, \ldots, v_k$ be the vertices in Y appearing in a cyclic order along λ. Working modulo $k+1$, for each $i \in \{0, \ldots, k\}$ and each non-adjacent pair v_i, v_{i+1}, let P_i be the path $v_i u_i u_{i+1} v_{i+1}$. For each adjacent pair v_i, v_{i+1}, let $P_i = v_i v_{i+1}$. Let $C = \cup_{i=0}^{k} P_i$. Since G_1 is $G[Y]$-critical, it follows that $G_1 \cup C$ is C-critical. By Observation 3.7.3, $|V(G_1 \cup C)| \leq 89|V(C)|$; or equivalently, that $|V(G_1)| \leq 88|V(C)|$. Since $|V(C)| \leq 3|Y|$, it follows that $|V(G_1)| \leq 264|V(Y)|$; and since $G_1 = G[X \cup Y]$, we have further that $|X| \leq 263|Y|$. Since $|Y| \geq 3$, we have that $263 \leq 132(|Y|-1)$, and so $|X| \leq 263(|Y|-1) + 132(|Y|-1)$. Thus $|X| \leq 395(|Y|-1)$, as desired. $\square$

Corollary 3.7.4 gives the theorem below. The proof is identical to that of Theorem 1.2.20.

Theorem 1.2.28. *For every surface Σ, there exists a constant $\rho > 0$ such that every ρ-locally planar graph of girth at least five that embeds in Σ is 3-correspondence-colourable.*

Per the work of Dvořák and Kawarabayashi in [12], Observation 3.7.3 implies the following. Note that by *linear-time algorithms*, we mean algorithms whose run-time is linear in the number of vertices in the graph.

Theorem 1.2.26. *Let Σ be a fixed surface. There exists a linear-time algorithm that takes as input an embedded graph of girth at least five (G, Σ) and a 3-correspondence assignment (L, M) for G with lists of bounded size and determines whether or not G is (L, M)-colourable.*

Theorem 1.2.27. *Let Σ be a fixed surface. There exists a linear-time algorithm that takes as input an embedded graph of girth at least five (G, Σ) and determines whether or not G is 3-correspondence-colourable.*

We note that the correspondence assignment in Theorem 1.2.26 is fixed, whereas in Theorem 1.2.27 it is not.

Chapter 4

Counting Correspondence Colourings

4.1 Introduction

4.1.1 Results

Using the results of the previous section as well as several key lemmas from [33], we will show that if G is a planar graph and (L, M) is a 5-correspondence assignment for G, then G has exponentially many (L, M)-colourings. This proves a conjecture of Langhede and Thomassen [29].

In particular, we will show the following.

Theorem 1.2.36. *If G is a planar graph with at least three vertices and (L, M) is a 5-correspondence assignment for G, then G has at least $2^{\frac{|V(G)|+306}{67}}$ distinct (L, M)-colourings.*

An analogous result (with better exponent) for list colouring was proved by Thomassen [44].

Theorem 4.1.1 (Thomassen, [44])**.** *If G is a planar graph and L is a 5-list-assignment for G, then G has at least $2^{\frac{|V(G)|}{9}}$ distinct L-colourings.*

As correspondence colouring generalizes list colouring, our theorem also implies that every planar graph with 5-list assignment L has exponentially many L-colourings. Though the bound in Theorem 1.2.36 is worse than that given by Thomassen in Theorem 4.1.1, our proof has the advantage of being shorter and less technical than that of Thomassen's.

Our proof uses Theorem 3.6.4; by using a theorem of Postle and Thomas [35] instead of Theorem 3.6.4, we obtain a better constant than that in Theorem 1.2.36 for list colouring —though still not matching the constant in Theorem 4.1.1.

Theorem 4.1.2. *If G is a planar graph with at least three vertices and L is a 5-list assignment for G, then G has at least $2^{\frac{|V(G)|+114}{25}}$ distinct L-colourings.*

In Section 4.3, we prove the following theorem for planar graphs of girth at least five.

Theorem 1.2.37. *If G is a planar graph with at least two vertices and girth at least five and (L, M) is a 3-correspondence assignment for G, then G has at least $2^{\frac{|V(G)|+890}{292}}$ distinct (L, M)-colourings.*

Theorem 1.2.37 strengthens the following list colouring result of Thomassen in both that it holds for correspondence colouring, and in that the exponent in the theorem statement is better.

Theorem 4.1.3 (Thomassen, [45])**.** *If G is a planar graph of girth at least five, L is a 3-list assignment for G, and $|V(G)| \geq 2$, then G has at least $2^{\frac{|V(G)|}{10000}}$ distinct L-colourings.*

(The above is directly implied by Theorem 4.3 in [45].)

4.1.2 Outline of Chapter

Subsection 4.1.3 contains useful definitions and key results from [33] which will be used in proving the main results of this chapter.

Theorems 1.2.36 and 4.1.2 are corollaries to a more technical theorem (Theorem 4.2.6) involving counting the number of extensions of a precoloured subgraph. The bulk of Section 4.2 is dedicated to the proof of Theorem 4.2.6. We will delay the proof of Theorems 1.2.36 and 4.1.2 until the end of Section 4.2, following the proof of Theorem 4.2.6.

Similarly, Theorem 1.2.37 is a corollary of a more technical theorem (Theorem 4.3.5), the proof of which constitutes the bulk of Section 4.3. Consequently, the proof of Theorem 1.2.37 is found at the end of Section 4.3.

4.1.3 Tools and Nomenclature

We begin with a few necessary definitions. In this chapter, we will extend the notion of *canvas* to include triples $(G, S, (L, M))$ as described below.

Definition 4.1.4. We say the triple $(G, S, (L, M))$ is a *canvas* if G is a plane graph, S is any connected subgraph of G, and (L, M) is a correspondence assignment for the vertices of G such that there exists an (L, M)-colouring of S and

- either $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(S)$, or
- G has girth at least five and $|L(v)| \geq 3$ for all $v \in V(G) \setminus V(S)$.

Definition 4.1.5. Let G be a graph, and let $H \subseteq G$. We define $v(G|H) := |V(G) \setminus V(H)|$, and $e(G|H) := |E(G) \setminus E(H)|$.

Definition 4.1.6. Let G be a graph of girth at least $g \in \{3, 5\}$, and let $H \subseteq G$. We define $\text{def}_g(G|H) := (g-2)e(G|H) - g \cdot v(G|H)$.

Note that when $g = 3$, this matches the definition of deficiency given in Chapter 3. In addition to deficiency, we will need the following quantity.

Definition 4.1.7. Let G be a graph, and H a subgraph of G. For $g \in \{3, 5\}$ and $\varepsilon > 0$, we define $d_{g,\varepsilon}(G|H) := (g-2)e(G|H) - (g+\varepsilon)v(G|H)$. Equivalently, $d_{g,\varepsilon}(G|H) := \text{def}_g(G|H) - \varepsilon v(G|H)$.

When $g = 3$, this nearly matches the definition of $d(\cdot)$ given in the previous chapter, ignoring the boundary and quasiboundary terms. We will also need the following notion, defined for list colouring in [33].

Definition 4.1.8. Let G be a graph, and let H be an induced subgraph of G. We say H is *r-deletable* if for every correspondence assignment (L, M) of H such that $|L(v)| \geq r - (\deg_G(v) - \deg_H(v))$ for each $v \in V(G)$, the graph H has an (L, M)-colouring no matter the correspondence assignment (L, M).

If H is an r-deletable subgraph of G, then every (L, M)-colouring of $G \setminus V(H)$ extends to an (L, M)-colouring of H. In fact, the definition above captures an even stronger notion: for an r-deletable subgraph $H \subseteq G$, an (L, M)-colouring of $G \setminus V(H)$ extends to an (L, M)-colouring of H no matter the correspondence assignment (L, M) and no matter the structure of $G \setminus V(H)$.

4.2 The girth three case

In this section, we prove Theorem 4.2.6 which we will show afterwards implies Theorems 1.2.36 and 4.1.2.

We will need the following three results, which are the correspondence colouring analogues of Theorem 5.20, Proposition 5.21, and Lemma 5.22 in [33].

Lemma 4.2.1. *There exists $\varepsilon > 0$ such that following holds: If G is a plane graph and H is a connected subgraph of G such that G is H-critical with respect to a 5-correspondence colouring, then $d_{3,\varepsilon}(G|H) \geq 0$.*

This is implied by Theorem 3.4.6 in the case where $v(G|H) \leq 1$. For list colouring, the $v(G|H) \geq 2$ case is implied by Theorem 4.6 in [35] with $\varepsilon = \frac{1}{18}$. For correspondence colouring, the $v(G|H) \geq 2$ case is implied by Theorem 3.4.7, by ignoring the boundary and quasiboundary terms in the expression for $d(G)$. For correspondence colouring, $\varepsilon = \frac{1}{50}$.

The proof of the following proposition is nearly identical to that of Proposition 5.21 in [33]; we include the proof for the purposes of cohesion.

Proposition 4.2.2. *Let G be a graph and H a proper subgraph of G such that $V(H) \neq V(G)$. If $G - V(H)$ is not r-deletable in G, then there exists a subgraph G_0 of G containing H such that G_0 is H-critical for r-correspondence colouring.*

Proof (Adapted from Proposition 5.21, [33]). Since $G - V(H)$ is not r-deletable in G, there exists a correspondence assignment (L_0, M_0) such that $|L_0(v)| \geq r - \deg_H(v)$ for each $v \in V(G) \setminus V(H)$ and $G - V(H)$ is not (L_0, M_0)-colourable. Define a new correspondence assignment (L, M) of G as follows. For each $v \in V(H)$, define c_v to be a new colour not appearing in any other list. Define R to be a set of $r - 1$ new colours distinct not appearing in any other list or in $\cup_{v \in V(H)} \{c_v\}$. Set $L(v) = \{c_v\} \cup R$ for each $v \in V(H)$. For each $u \in V(G) \setminus V(H)$, let $L(u) = L_0(u) \cup \{c_v : v \in N(u) \cap V(H)\}$. For each $uv \notin E(H)$, set $M_{uv} = (M_0)_{uv}$. For each $uv \in E(H)$, set $M_{uv} = \emptyset$. Finally, for each uv with $u \in V(G) \setminus V(H)$ and $v \in V(H)$, set $M_{uv} = \{(u, c_v)(v, c_v)\}$. Now (L, M) is an r-correspondence assignment of G. Let ϕ be the colouring of H given by $\phi(v) = c_v$ for every $v \in V(H)$. Since $G - V(H)$ is not (L_0, M_0)-colourable, it follows that ϕ does not extend to an (L, M)-colouring of G. Let G' be an inclusion-wise minimal subgraph of G containing H such that ϕ does not extend to an (L, M)-colouring of G'. By the minimality of G', we have that ϕ extends to an (L, M)-colouring of every proper subgraph of G' containing H. Thus G' is H-critical with respect to (L, M). Hence by definition, $G'$is H-critical for r-correspondence colouring, as desired. □

The lemma below is nearly identical to that of Proposition 5.22 in [33], where it was originally written for list colouring rather than correspondence colouring. We include the proof for the purposes of cohesion.

Lemma 4.2.3 (Lemma 5.22, [33]). *Let ε be as in Lemma 4.2.1. If G is a plane graph with girth at least $g \in \{3, 5\}$ and H is a connected subgraph of G such that there does not exist $X \subseteq V(G) \setminus V(H)$ such that $G[X]$ is $(8-g)$-deletable in G, then $d_{g,\varepsilon}(G|H) \geq 0$.*

Proof. We proceed by induction on $v(G|H)+e(G|H)$. If $V(H) = V(G)$, then $d_{g,\varepsilon}(G|H) \geq 0$ as desired. So we may assume that $V(H) \neq V(G)$. By assumption, $G - V(H)$ is not $(8-g)$-deletable in G. By Proposition 4.2.2, it follows that there exists a subgraph G_0 of G containing H such that G_0 is H-critical for $(8-g)$-correspondence colouring. Note that H is a proper subgraph of G_0 by definition of H-critical. By Theorem 4.2.1, we have that $d_{g,\varepsilon}(G_0|H) \geq 0$. By definition of critical, since H is connected it follows that G_0 is connected. Note that $v(G|G_0) + e(G|G_0) < v(G|H) + e(G|H)$. Hence by induction, $d_{g,\varepsilon}(G|G_0) \geq 0$. By definition of $d_{g,\varepsilon}$, we have that $d_{g,\varepsilon}(G|H) = d_{g,\varepsilon}(G|G_0) + d_{g,\varepsilon}(G_0|H) \geq 0 + 0 = 0$, as desired. □

We make the following easy observation which follows directly from the definitions of def_3 and $d_{g,\varepsilon}$.

Observation 4.2.4. *Let G be a graph, and H a subgraph of G. If $d_{3,\varepsilon}(G|H) \geq 0$, then $v(G|H) \leq \varepsilon^{-1} \cdot \text{def}_3(G|H)$.*

Proof. By definition, $d_{3,\varepsilon}(G|H) = \text{def}_3(G|H) - \varepsilon v(G|H)$. Since $d_{3,\varepsilon}(G|H) \geq 0$, it follows that $0 \leq \text{def}_3(G|H) - \varepsilon v(G|H)$. Note that $\varepsilon > 0$; by isolating $v(G|H)$, we obtain the desired result. □

Finally, we will need the following theorem, due to Thomassen. This theorem was originally written in the language of list colouring; however, as pointed out by Dvořák and Postle in [15], the proof also carries over to the realm of correspondence colouring.

Theorem 4.2.5 (Thomassen, [40]). *Let G be a plane graph. Let C be the subgraph of G whose edge- and vertex-set are precisely those of the outer face boundary walk of G. Let (L, M) be a correspondence assignment for G where $|L(v)| \geq 1$ for a path $S \subseteq C$ with $|V(S)| \leq 2$; where $|L(v)| \geq 3$ for all $v \in V(C) \setminus V(S)$; and where $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$. Then every (L, M)-colouring of S extends to an (L, M)-colouring of G.*

The proof of Theorem 4.2.6 follows. The reader may find it helpful to consult Figure 4.1 while reading for a depiction of the cases considered in the proof.

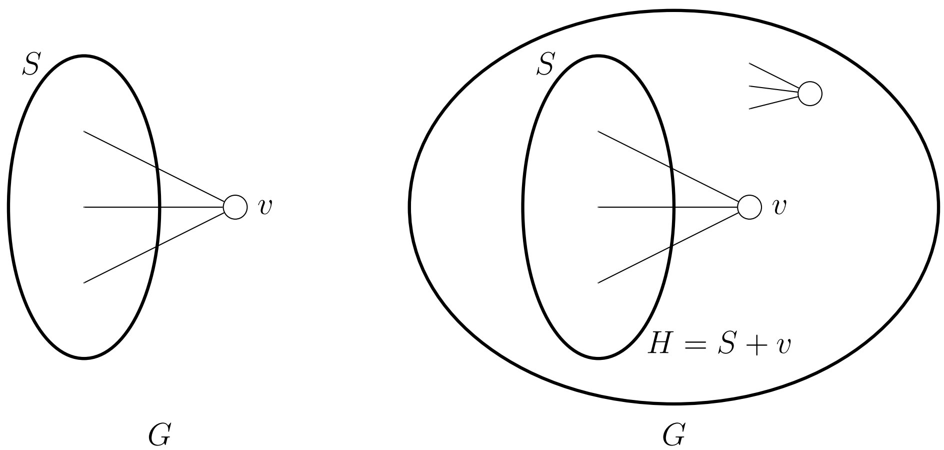

Figure 4.1: The cases to consider for Theorem 4.2.6. First, the case where $V(G) = V(S) \cup \{v\}$; here, we consider each possible value of $\deg(v)$. For the case where $|V(G)| \geq |V(S)| + 2$, we let $H = S + v$. Note then that $|V(G)| - |V(S)| > |V(G)| - |V(H)|$ and $|V(G)| - |V(S)| > |V(H)| - |V(S)|$.

Theorem 4.2.6. *Let ε be as in Lemma 4.2.3. Let G be a plane graph, let S be a connected subgraph of G, and let (L, M) be a 5-correspondence assignment for G. If ϕ is an (L, M)-colouring of S that extends to an (L, M)-colouring of G, then*

$$\log_2 E(\phi) \geq \frac{v(G|S) - (\varepsilon^{-1} + 1)\mathrm{def}_3(G|S)}{67},$$

where $E(\phi)$ is the number of extensions of ϕ to G.

Proof. We proceed by induction on $v(G|S)$. We may assume that $v(G|S) \neq 0$, as otherwise there is nothing to prove. First suppose that $v(G|S) = 1$, and let $v \in V(G) \setminus V(S)$. Then

$$\frac{v(G|S) - (\varepsilon^{-1} + 1)\mathrm{def}_3(G|S)}{67} = \frac{1 - (\varepsilon^{-1} + 1)(\deg(v) - 3)}{67}.$$

Note that when $\deg(v) \geq 4$, the right hand side is negative since $\varepsilon^{-1} > 0$. Since ϕ extends to an (L, M) colouring of G by assumption, it follows that $\log_2 E(\phi) \geq 0$, and so $\log_2 E(\phi) \geq \frac{v(G|S) - (\varepsilon^{-1}+1)\mathrm{def}_3(G|S)}{67}$ holds as desired. We may therefore assume that $\deg(v) \leq 3$. Since $|L(v)| \geq 5$ it follows that $E(\phi) \geq 5 - \deg(v)$. Therefore it suffices to show that $\log_2(5-\deg(v)) \geq \frac{1-(\varepsilon^{-1}+1)(\deg(v)-3)}{67}$, or equivalently, that $67 \geq \frac{1-(\varepsilon^{-1}+1)(\deg(v)-3)}{\log_2(5-\deg(v))}$. The right hand side is maximized when $\deg(v) = 0$, in which case it is easy to verify that $67 > \frac{154}{\log_2(5)}$.

We may therefore assume that $v(G|S) \geq 2$.

Before proceeding with the remainder of the case analysis, we will need the following claim.

Claim 39. *There does not exist a graph $H \subsetneq G$ with $S \subsetneq H$ and $|V(S)| < |V(H)| < |V(G)|$ such that $G - V(H)$ is a 5-deletable subgraph of G.*

Proof. Suppose not. Let $G' = G - V(H)$. Note that G' is an induced subgraph of G. Since $|V(S)| < |V(H)| < |V(G)$, we have that $v(H|S) < v(G|S)$, and so it follows by induction that there are at least $2^{\frac{v(H|S)-(\varepsilon^{-1}+1)\text{def}_3(H|S)}{67}}$ extensions of ϕ to H. By definition of 5-deletable subgraph, since G' is 5-deletable each of these extensions of ϕ to an (L, M)-colouring of H extends further to an (L, M) colouring of G', and thus to G. Since $|V(S)| < |V(H)| < |V(G)$, it follows that $v(G|H) < v(G|S)$, and so by induction for each extension of ϕ to an (L, M) colouring ϕ' of H there are at least $2^{\frac{v(G|H)-(\varepsilon^{-1}+1)\text{def}_3(G|H)}{67}}$ extensions of ϕ' to G. Therefore

$$\begin{aligned}\log_2 E(\phi) &\geq \frac{v(H|S) - (\varepsilon^{-1}+1)\text{def}_3(H|S)}{67} + \frac{v(G|H) - (\varepsilon^{-1}+1)\text{def}_3(G|H)}{67} \\ &= \frac{v(H|S) + v(G|H) - (\varepsilon^{-1}+1)(\text{def}_3(H|S) + \text{def}_3(G|H))}{67} \\ &= \frac{v(G|S) - (\varepsilon^{-1}+1)\text{def}_3(G|S)}{67},\end{aligned}$$

as desired. □

Among all vertices in $V(G) \setminus V(S)$ that have a neighbour in S, choose a vertex v that maximizes $|N(v) \cap V(S)|$. Let $H = S + v$. Since ϕ extends to an (L, M)-colouring of G, there is at least $1 = 2^0$ extension ϕ' of ϕ to H where ϕ' extends further to an (L, M)-colouring of G. Since ϕ' extends to G and $v(G|H) < v(G|S)$, by induction there are at least $2^{\frac{v(G|H)-(\varepsilon^{-1}+1)\text{def}_3(G|H)}{67}}$ extensions of ϕ' to G. Therefore

$$\log_2 E(\phi) \geq 0 + \frac{v(G|H) - (\varepsilon^{-1}+1)\text{def}_3(G|H)}{67}.$$

Moreover, since $v(G|H) = v(G|S) - 1$ and $\text{def}_3(G|H) = \text{def}_3(G|S) - \deg_H(v) + 3$, it follows that

$$\log_2 E(\phi) \geq \frac{v(G|S) - 1 - (\varepsilon^{-1} + 1)(\text{def}_3(G|S) - \deg_H(v) + 3)}{67}.$$

If $\deg_H(v) \geq 4$, the desired result immediately holds since $\varepsilon > 0$. Thus we may assume $\deg_H(v) \leq 3$. First suppose $\deg_H(v) \leq 2$. Since $|L(v)| \geq 5$, there are at least $5 - \deg_H(v)$ extensions of ϕ to an (L, M) colouring of H. We claim each of these extensions extends further to an (L, M)-colouring of G. To see this, let ϕ' be an extension of ϕ to an (L, M)-colouring of H. Let $G' = G - V(H)$. Note that G' is induced. Let (L', M) be a correspondence assignment for $G' + v$ where for each $u \in V(G')$, we set $L'(u) = L(u) \setminus \{u[w, \phi'(w)] : w \in V(S) \cap N(u)\}$. Let C be the graph whose vertex- and edge-set are precisely those of the outer face boundary walk of $G' + v$. Note that since we chose v to maximize $|N(v) \cap V(S)|$ and $\deg_H(v) \leq 2$ by assumption, every vertex u in $V(C)$ has $|L(u)| \geq 3$. Moreover, every vertex u in $V(G') \setminus V(C)$ has $|L'(u)| \geq 5$ since L is a 5-list assignment for G and no vertex in $V(G') \setminus V(C)$ is adjacent to a vertex in S since G is planar. By Theorem 4.2.5, it follows that ϕ' extends to $G' + v$, and thus to G. Since ϕ' was an arbitrary extension of ϕ to H, it follows that $G' = G - V(H)$ is a 5-deletable subgraph of G; and since $|V(S)| < |V(H)| < |V(G)|$, this contradicts Claim 39.

We may therefore assume that $\deg_H(v) = 3$.

By Claim 39, there does not exist $X \subseteq V(G) \setminus V(H)$ such that $G[X]$ is 5-deletable in G. Thus by Lemma 4.2.3, we have that

$$d_{3,\varepsilon}(G|H) \geq 0. \tag{4.2.1}$$

It follows from Observation 4.2.4 that

$$v(G|H) \leq \varepsilon^{-1}\text{def}_3(G|H). \tag{4.2.2}$$

Moreover, since $v(G|S) = 1 + v(G|H)$ and $\text{def}_3(G|S) = \text{def}_3(G|H) + \text{def}_3(H|S)$, it follows that

$$\frac{v(G|S) - (\varepsilon^{-1} + 1)\text{def}_3(G|S)}{67} = \frac{1 + v(G|H) - (\varepsilon^{-1} + 1)(\text{def}_3(G|H) + \text{def}_3(H|S))}{67}.$$

By definition of $\text{def}_3(G|S)$, we have that $\text{def}_3(H|S) = \deg_H(v) - 3$, and so

$$\begin{aligned}
\frac{v(G|S) - (\varepsilon^{-1}+1)\text{def}_3(G|S)}{67} &= \frac{1 + v(G|H) - (\varepsilon^{-1}+1)(\text{def}_3(G|H) + \deg_H(v) - 3)}{67} \\
&\frac{1 + v(G|H) - (\varepsilon^{-1}+1)\text{def}_3(G|H)}{67} \quad \text{since } \deg_H(v) = 3 \\
&\leq \frac{1 + \varepsilon^{-1}\text{def}_3(G|H) - (\varepsilon^{-1}+1)\text{def}_3(G|H)}{67} \quad \text{by Obs. 4.2.4} \\
&\leq \frac{1 - \text{def}_3(G|H)}{67}.
\end{aligned}$$

By Lemma 4.2.3, we have that $\text{def}_3(G|H) \geq v(G|H)\varepsilon$, and since $v(G|S) \geq 2$, it follows that $v(G|H) \geq 1$ and so $\text{def}_3(G|H) \geq \varepsilon$. Since $\text{def}_3(G|H)$ is integral, $\text{def}_3(G|H) \geq 1$, and thus the right hand side above is at most 0. It follows that

$$\frac{v(G|S) - (\varepsilon^{-1}+1)\text{def}_3(G|S)}{67} \leq 0.$$

But since ϕ extends to an (L, M)-colouring of G, $\log_2(E(\phi)) \geq 0$. Thus

$$\log_2 E(\phi) \geq \frac{v(G|S) - (\varepsilon^{-1}+1)\text{def}_3(G|S)}{67},$$

as desired. □

Using essentially the same proof technique as Theorem 4.2.6 (but with $\varepsilon = \frac{1}{18}$) we obtain the following result.

Theorem 4.2.7. *Let G be a planar graph, S a connected subgraph of G, and L a 5-list assignment for G. If ϕ is an L-colouring of S that extends to an (L, M)-colouring of G, then*

$$\log_2 E(\phi) \geq \frac{v(G|S) - 19\text{def}_3(G|S)}{25},$$

where $E(\phi)$ is the number of extensions of ϕ to G.

As a corollary to Theorem 4.2.6, we obtain Theorem 1.2.36, restated and proved below. Theorem 1.2.36 proves a conjecture of Langhede and Thomassen [29].

Theorem 1.2.36. *If G is a planar graph with at least three vertices and (L, M) is a 5-correspondence assignment for G, then G has at least $2^{\frac{|V(G)|+306}{67}}$ distinct (L, M)-colourings.*

Proof. Let S be the empty graph, and ϕ a trivial colouring of S. Let $E(\phi)$ be the number of extensions of ϕ to G. Since G is planar, we have that $|E(G)| \leq 3|V(G)-6$. By Theorem 4.2.6,

$$\begin{aligned}\log_2 E(\phi) &\geq \frac{|V(G)| - (50+1)(|E(G)| - 3|V(G)|)}{67} \\ &\geq \frac{|V(G)| - 51(3|V(G)| - 6 - 3|V(G)|}{67} \quad \text{since } e(G) \leq 3v(G) - 6 \\ &= \frac{|V(G)| + 306}{67},\end{aligned}$$

as desired. □

Similarly, as a corollary to Theorem 4.2.7 we obtain Theorem 4.1.2, restated below.

Theorem 4.1.2. *If G is a planar graph with at least three vertices and L is a 5-list assignment for G, then G has at least $2^{\frac{|V(G)|+114}{25}}$ distinct L-colourings.*

(As mentioned in the introduction, Theorem 4.1.2 is not the strongest such result: Thomassen proved a similar theorem with a bound of $2^{\frac{|V(G)|}{9}}$ [44].)

4.3 The girth at least five case

We now prove an analogous result to Theorem 4.2.6 in the case of planar graphs of girth at least five. The proof is similar in spirit to that of the girth three case, but requires a stronger version of Lemma 4.2.1. In particular, we will need the following lemma.

Lemma 4.3.1. *Let G be a plane graph with girth at least five and let H be a connected subgraph of G such that G is H-critical for 3-correspondence colouring. Setting $\varepsilon = \frac{1}{88}$, we have that $d_{5,\varepsilon}(G|H) \geq 3$.*

Proof. For $v(G|H) \geq 2$ or $v(G|H) = 1$ and $e(G|H) \neq 3$, this is directly implied by Observation 3.7.2, ignoring the quasiboundary term. Suppose now that $v(G|H) = 0$. Since G is H-critical, it follows that H is a proper subgraph of G and so that $e(G|H) \geq 1$. Thus $d_{3,\varepsilon} \geq 3 - 0 = 3$. If $v(G|H) = 1$ and $e(G|H) = 3$, then $d_{5,\varepsilon} = 3 \cdot 3 - 5 \cdot 1 - \varepsilon \cdot 1 = 4 - \varepsilon$. This is at least 3, since $\varepsilon = \frac{1}{88}$. □

Using this, we now establish a stronger version of Lemma 4.2.3. The proof is identical to that of Lemma 4.2.3.

Lemma 4.3.2. *Let ε be as in Lemma 4.3.1. If G is a plane graph with girth at least 5 and H is a connected subgraph of G such that there does not exist $X \subseteq V(G) \setminus V(H)$ such that $G[X]$ is 3-deletable in G, then $d_{5,\varepsilon}(G|H) \geq 3$.*

Similar to the case of Observation 4.2.4, this trivially implies the following.

Observation 4.3.3. *Let G be a graph of girth at least 5, and H a subgraph of G. If $d_{5,\varepsilon}(G|H) \geq 0$, then $v(G|H) \leq \varepsilon^{-1} \cdot (\mathrm{def}_5(G|H) - 3)$.*

Finally, we will need the following theorem, due to Thomassen. As in the girth 3 case, this theorem was originally written in the language of list colouring; however, as pointed out by Dvořák and Postle in [15], the proof also carries over to the realm of correspondence colouring.

Theorem 4.3.4 (Thomassen, [43])**.** *Let G be a plane graph of girth at least five. Let C be the subgraph of G whose edge- and vertex-set are precisely those of the outer face boundary walk of G. Let (L, M) be a correspondence assignment for G where $|L(v)| \geq 1$ for each vertex v in a path or cycle $S \subseteq C$ with $|V(S)| \leq 6$; where $|L(v)| = 2$ for each vertex v in an independent set A of vertices in $|V(C) \setminus V(S)|$; where $|L(v)| \geq 3$ for all $v \in V(G) \setminus (A \cup V(S))$; and where there is no edge between vertices in A and vertices in S. Then every (L, M)-colouring of S extends to an (L, M)-colouring of G.*

We now prove the following theorem, which is the girth at least five analogue to Theorem 4.2.6. The reader may find it helpful to consult Figure 4.2 while reading for a depiction of the cases considered in the proof.

Theorem 4.3.5. *Let G be a plane graph of girth at least five, let S be a connected subgraph of G, and let (L, M) be a 3-correspondence assignment for G. If ϕ is an (L, M)-colouring of S that extends to an (L, M)-colouring of G, then*

$$\log_2 E(\phi) \geq \frac{v(G|S) - 89\mathrm{def}_5(G|S)}{282}.$$

Proof. We proceed by induction on $v(G|S)$. We may assume that $v(G|S) \neq 0$, as otherwise there is nothing to prove. First suppose that $v(G|S) = 1$, and let $v \in V(G) \setminus V(S)$. Then

$$\frac{v(G|S) - 89\mathrm{def}_5(G|S)}{282} = \frac{1 - 89(3\deg(v) - 5)}{282}.$$

Note that when $\deg(v) \geq 2$, the right hand side is negative. Since ϕ extends to an (L, M)-colouring of G by assumption, it follows that $\log_2 E(\phi) \geq 0$, and so

$$\log_2 E(\phi) \geq \frac{v(G|S) - 89\text{def}_5(G|S)}{282},$$

as desired. Thus we may assume that $\deg(v) \leq 1$. Since $|L(v)| \geq 3$, we have that $E(\phi) \geq 3 - \deg(v)$. Therefore it suffices to show that $\log_2(3 - \deg(v)) \geq \frac{1-89(3\deg(v)-5)}{282}$; or, equivalently, that

$$282 \geq \frac{1 - 89(3\deg(v) - 5)}{\log_2(3 - \deg(v))}.$$

When $\deg(v) = 0$, the right hand side equals $\frac{451}{\log_2(3)} < 282$. When $\deg(v) = 1$, the right hand side equals 179, which again is less than 282. Thus the inequality above holds.

We may therefore assume that $v(G|S) \geq 2$.

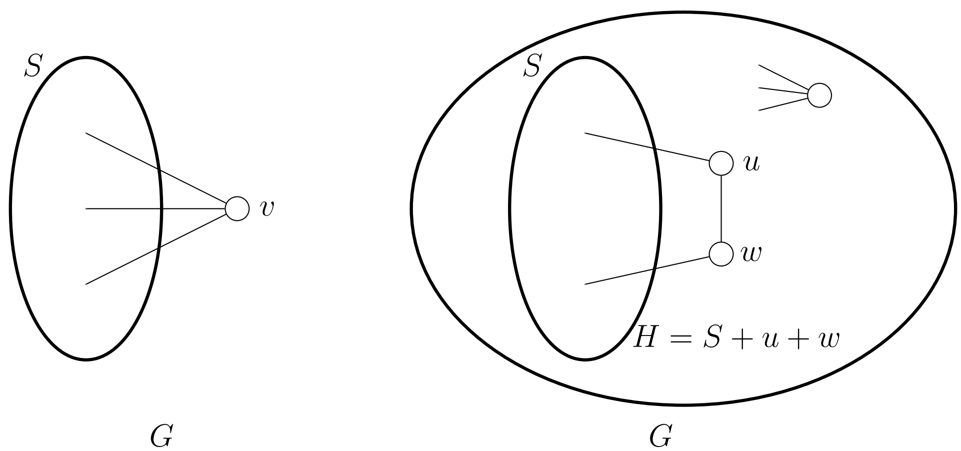

Figure 4.2: Two of the cases to consider for Theorem 4.3.5. First, the case where $V(G) = V(S) \cup \{v\}$; here, we consider each possible value of $\deg(v)$. The case where $|V(G)| = |V(S)| + 2$ is easily dealt with, and is not pictured. For the case where $|V(G)| \geq |V(S)| + 3$, we let $H = S + u + w$. Note then that $|V(G)| - |V(S)| > |V(G)| - |V(H)|$ and $|V(G)| - |V(S)| > |V(H)| - |V(S)|$.

Before proceeding with the remainder of the case analysis, we will need the following claim.

Claim 40. *There does not exist a graph $H \subsetneq G$ with $S \subsetneq H$ and $|V(S)| < |V(H)| < |V(G)|$ such that $G - V(H)$ is a 3-deletable subgraph of G.*

Proof. Suppose not. Let $G' = G - V(H)$ be a 3-deletable subgraph. Note that G' is induced. Since $S \subsetneq H$ and $H \subsetneq G$ and $|V(S)| < |V(H)| < |V(G)|$, it follows that $v(H|S) < v(G|S)$. By induction, there are at least $2^{\frac{v(H|S)-89\text{def}_5(H|S)}{282}}$ extensions of ϕ to H. Since G' is a 3-deletable subgraph of G, by definition each of these extensions of ϕ to an (L,M)-colouring of H extends further to an (L,M) colouring of G', and therefore to G.

Since $H \subsetneq G$ and $S \subsetneq H$ and $|V(S)| < |V(H)| < |V(G)|$, we have that $v(G|H) < v(G|S)$, and so by induction for each extension of ϕ to an (L,M) colouring ϕ' of H there are at least $2^{\frac{v(G|H)-89\text{def}_5(G|H)}{282}}$ extensions of ϕ' to G. Therefore

$$\begin{aligned}\log_2 E(\phi) &\geq \frac{v(H|S) - 89\text{def}_5(H|S)}{282} + \frac{v(G|H) - 89\text{def}_5(G|H)}{282} \\ &= \frac{v(H|S) + v(G|H) - 89(\text{def}_5(H|S) + \text{def}_5(G|H))}{282} \\ &= \frac{v(G|S) - 89\text{def}_5(G|S)}{282},\end{aligned}$$

as desired. □

Among all vertices in $V(G) \setminus V(S)$ that have a neighbour in S, choose a vertex v that maximizes $|N(v) \cap V(S)|$. Let $H = S + v$. Since ϕ extends to an (L,M)-colouring of G, there is at least $1 = 2^0$ extension ϕ' of ϕ to H where ϕ' extends further to an (L,M)-colouring of G. Since ϕ' extends to G and $v(G|H) < v(G|S)$, by induction there are at least $2^{\frac{v(G|H)-89\text{def}_5(G|H)}{282}}$ extensions of ϕ' to G. Therefore

$$\log_2 E(\phi) \geq 0 + \frac{v(G|H) - 89\text{def}_5(G|H)}{282}.$$

Since $\text{def}_5(G|H) = \text{def}_5(G|S) - 3\deg_H(v) + 5$, it follows that

$$\begin{aligned}\log_2 E(\phi) \geq &= \frac{v(G|S) - 1 - 89(\text{def}_5(G|S) - 3\deg_H(v) + 5)}{282} \\ &= \frac{v(G|S) - 1 - 89\text{def}_5(G|S)}{282} - \frac{89(5 - 3\deg_H(v))}{282}.\end{aligned}$$

If $\deg_H(v) \geq 2$, then $\log_2 E(\phi) \geq \frac{v(G|S)-1-89\text{def}_5(G|S)}{282}$, as desired. Thus we may assume $\deg_H(v) \leq 1$. If $\deg_H(v) = 0$, we claim $G - V(H)$ is 3-deletable. To see this, let ϕ' be an extension of ϕ to an (L,M)-colouring of H. By Theorem 4.3.4, G' admits an (L',M)-colouring ϕ''. Note that since v was chosen to maximize $N(v) \cap V(S)$ and $\deg_H(v) = 0$, it follows that every vertex in $V(G) \setminus V(H)$ has no neighbours in $V(S)$, and so that $\phi'' \cup \phi'$ is an extension of ϕ' to G. Thus $G - V(H)$ is 3-deletable, contradicting Claim 40. We may therefore assume that $\deg_H(v) = 1$.

Suppose now that $v(G|S) = 2$, and let $\{u\} = V(G) \setminus V(H)$. Since $v \in V(G) \setminus V(S)$ was chosen to maximize $|N(v) \cap V(S)|$, it follows that $|N(u) \cap V(S)| \leq 1$. But then $|N(u) \cap V(H)| \leq 2$; and since $|L(u)| \geq 3$, it follows that $G \setminus V(H) = u$ is 3-deletable, contradicting Claim 40.

Thus $v(G|S) \geq 3$. Let X be the set of vertices in $V(G) \setminus V(S)$ with at least one neighbour in S. Note that $v \in X$. By our choice of v, since $\deg_H(v) = 1$ it follows that every vertex in U has exactly one neighbour in S. If X is an independent set, we claim that $G - V(H)$ is a 3-deletable subgraph of G. To see this, let ϕ' be an extension of ϕ to an (L, M)-colouring of H. Let (L', M) be a list assignment for $G - V(H)$, where $L'(u) = L(u) \setminus \{u[w, \phi'(w)] : w \in V(H) \cap N(u)\}$. By Theorem 4.3.4, $G - V(H)$ is (L', M)-colourable; and so $G - V(H)$ is 3-deletable, contradicting Claim 40.

Thus X is not an independent set, and so there exist vertices $u, w \in X$ such that $uw \in E(G)$. Let $H' = S + u + w$. Since every vertex in X has exactly one neighbour in S, it follows that $e(H'|S) = 3$. Thus $\text{def}_5(G|H') = \text{def}_5(G|S) + 1$; and by Claim 40, there does not exist $X \subseteq V(G) \setminus V(H')$ such that $G[X]$ is 3-deletable in G. Thus by Lemma 4.3.2, $d_{5,\varepsilon}(G|H') \geq 3$, and so by Observation 4.3.3, $\text{def}_5(G|H') \geq \varepsilon \cdot v(G|H') + 3$. Thus $\text{def}_5(G|S) \geq 2 + \varepsilon \cdot v(G|H')$. It follows that

$$\begin{aligned} v(G|S) - 89\text{def}_5(G|S) &\leq v(G|S) - 89(2 + \varepsilon v(G|H)) \\ &= v(G|S) - 89(2 + \varepsilon(v(G|S) - 2)) \\ &= v(G|S) - 89\varepsilon \cdot v(G|S) - 89(2 - 2\varepsilon). \end{aligned}$$

As $\varepsilon = \frac{1}{88}$, the above is negative. Thus

$$0 > \frac{v(G|S) - 89\text{def}_5(G|S)}{282}.$$

Since ϕ extends to an (L, M)-colouring of G, it follows that $\log_2 E(\phi) \geq 0$, and so $\log_2 E(\phi) > \frac{v(G|S) - 89\text{def}_5(G|S)}{282}$, as desired. □

As an easy corollary, we obtain Theorem 1.2.37.

Proof of Theorem 1.2.37. Let S be the empty graph, and ϕ a trivial colouring of S. Let $E(\phi)$ be the number of extensions of ϕ to G. If G has two vertices, then G has at least 5×4 colourings, and so $E(\phi) \geq 20 > 2^{\frac{892}{292}}$, as desired. If G has three vertices, then since G has girth at least five $|E(G)| \leq 2$ and so $E(\phi) \geq 5 \times 4 \times 4 > 2^{\frac{893}{292}}$. Thus we may assume G has at least four vertices. Since G is planar and has girth at least five, $3|E(G)| \leq 5|V(G)| - 10$. By Theorem 4.3.5,

$$\begin{aligned}
\log_2 E(\phi) &\geq \frac{|V(G)| - 89(3|E(G)| - 5|V(G))}{282} \\
&\geq \frac{|V(G)| - 89(-10)}{282} \quad \text{since } 3|E(G)| - 5|V(G)| \leq -10 \\
&= \frac{|V(G)| + 890}{292},
\end{aligned}$$

as desired. □

Chapter 5

Conclusion and Open Problems

Many of the results in this thesis suggest that what determines the difficulty of colouring problems in planar graphs is really only local structure, rather than the global structure of the graph. For instance, our local choosability result (Theorem 1.2.11) shows that, given a planar graph, *if even a single vertex is not contained in a short cycle*, then already the colouring problem becomes easier. Indeed, it is not colouring planar graphs with short cycles that is difficult (in the sense that it requires lists of larger size): rather it seems to be colouring the short cycles themselves within the planar graphs. Theorem 3.6.4 suggests that, in order to determine whether a (5-correspondence) colouring of a subgraph C extends to a colouring of the whole graph, it is enough to check whether it extends to a relatively small neighbourhood of C. In a similar vein, our results on locally planar graphs (Theorems 1.2.20 and 1.2.28) suggests that even in the case of certain classes of non-planar graphs, if the graphs' local structure is planar, then that is enough to ensure planar-like colouring properties.

As we demonstrate in Chapter 3 (and as is shown by Postle and Thomas in [37]), showing a family of graphs is hyperbolic is a clear avenue to pursue to obtain results on locally planar graphs (among other things). Following the work of Dvořák and Kawarabayashi [11], hyperbolicity theorems can also be used to prove algorithmic results: indeed, for the decidability problem of colouring graphs embedded on fixed surfaces we obtain algorithms that are not only poly-time, but *linear-time*. These questions —whether locally planar graphs are colourable, as well as the existence of the algorithms described —are very natural and interesting in their own right, but barring hyperbolicity, there seems to be no general method or obvious strategy for tackling them. This is in part the reason for proving hyperbolicity theorems like Theorems 3.4.7 and Observation 3.7.2: they provide a clear path for studying these natural questions.

In Chapter 4, we demonstrate that these stronger hyperbolicity theorems have yet another fascinating implication: they can be used to show there exist exponentially many colourings of planar graphs. This too is an interesting and natural question, and the fact that it too falls under the umbrella of implications of hyperbolicity further motivates the study of hyperbolicity theorems like Theorem 3.4.7.

This thesis answers many interesting questions, and raises many others. For instance, we note the proof of Theorem 1.2.11 does not hold for correspondence colouring. In particular, Lemma 2.2.14 relies on a colouring argument that does not translate to the correspondence framework, and it is not immediately obvious how to get around this.

Question 5.0.1. *Let G be a planar graph, and let (L, M) be an arbitrary correspondence assignment for G where L is a local girth list assignment. Is G (L, M)-colourable?*

It would also be interesting to investigate whether Theorem 1.2.11 extends to graph classes beyond planar graphs. In particular, we ask the following.

Question 5.0.2. *For every surface Σ, does there exist a constant $\rho > 0$ such that every ρ-locally planar graph that embeds in Σ is local girth choosable?*

It would also be interesting to investigate whether the analogous correspondence colouring result holds. As correspondence colouring generalizes list colouring, this would of course imply a positive answer to Question 5.0.2.

We are also interested in algorithmic questions.

Question 5.0.3. *For graphs embedded in a fixed surface, does there exist a poly-time algorithm for the decidability problem of local girth choosability?*

Again, it would be interesting to investigate the answer to the analogous question for correspondence colouring.

In light of Theorem 1.2.11, we raise the question below.

Question 5.0.4. *Given a planar graph G with at least three vertices and a local girth list assignment L for G, do there exist exponentially many distinct L-colourings of G?*

Though each of the questions above is interesting in its own right, perhaps most interesting of all are the following two.

Question 5.0.5. *Does there exist a theorem analogous to Theorem 3.4.7 for local girth choosability?*

Question 5.0.6. *Does there exist a theorem analogous to Theorem 3.4.7 for local girth correspondence colouring?*

As mentioned in the introduction, a positive answer to this last question would simultaneously imply all the results in this thesis. In addition, it would imply a positive answer to all of the questions raised thus far (with the possible exception of Question 5.0.4; however, it seems very likely that our method described in Chapter 4 could be used to to answer Question 5.0.4 assuming a theorem like that alluded to in Question 5.0.6.

In addition to hyperbolicity, Postle and Thomas also introduce the notion of *strong hyperbolicity.* While hyperbolicity implies a bound on the number of vertices in an open disk, strong hyperbolicity bounds the number of vertices in an open annulus. In addition to all the implications of hyperbolicity, strong hyperbolicity has further interesting implications: we refer the reader to [37] for details. We leave the reader with the following family of problems: it would be interesting to investigate whether there exist strong hyperbolicity theorems for the families of correspondence-critical graphs described in this thesis. Given the intricacies involved in the proofs of our hyperbolicity results, these questions seem especially daunting.

References

[1] Noga Alon and Michael Tarsi. Colorings and orientations of graphs. *Combinatorica*, 12(2):125–134, 1992.

[2] Kenneth Appel and Wolfgang Haken. Every planar map is four colorable. Part I: Discharging. *Illinois Journal of Mathematics*, 21:429–490, 1977.

[3] Kenneth Appel, Wolfgang Haken, and John Koch. Every planar map is four colorable. Part II: Reducibility. *Illinois Journal of Mathematics*, 21(3):491–567, 1977.

[4] Anton Bernshteyn and Alexandr Kostochka. On differences between DP-coloring and list coloring. *Matematicheskie Trudy*, 21(2):61–71, 2018.

[5] Marthe Bonamy, Michelle Delcourt, Richard Lang, and Luke Postle. Edge-colouring graphs with local list sizes. *arXiv preprint arXiv:2007.14944*, 2020.

[6] Oleg Borodin, Alexandr Kostochka, and Douglas Woodall. List edge and list total colourings of multigraphs. *Journal of Combinatorial Theory, Series B*, 71(2):184–204, 1997.

[7] Ewan Davies, Rémi de Joannis de Verclos, Ross J Kang, and François Pirot. Coloring triangle-free graphs with local list sizes. *Random Structures & Algorithms*, 57(3):730–744, 2020.

[8] Matt DeVos, Ken-ichi Kawarabayashi, and Bojan Mohar. *Locally planar graphs are 5-choosable*. University of Ljubljana, Inst. of Mathematics, Physics and Mechanics, 2006.

[9] R. Diestel. *Graph Theory: 5th edition*. Springer Graduate Texts in Mathematics. Springer-Verlag, © Reinhard Diestel, 2017.

[10] G.A. Dirac. Note on the colouring of graphs. *Mathematische Zeitschrift*, 54(4):347–353, 1951.

[11] Zdenek Dvorák and Ken-ichi Kawarabayashi. Choosability of planar graphs of girth 5. *arXiv preprint arXiv:1109.2976*, 2011.

[12] Zdeněk Dvořák and Ken-ichi Kawarabayashi. List-coloring embedded graphs. In *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1004–1012. SIAM, 2013.

[13] Zdeněk Dvořák, Daniel Král', and Robin Thomas. Coloring triangle-free graphs on surfaces. In *Proceedings of the twentieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 120–129. SIAM, 2009.

[14] Zdeněk Dvořák, Bernard Lidickỳ, Bojan Mohar, and Luke Postle. 5-list-coloring planar graphs with distant precolored vertices. *Journal of Combinatorial Theory, Series B*, 122:311–352, 2017.

[15] Zdeněk Dvořák and Luke Postle. Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. *Journal of Combinatorial Theory, Series B*, 129:38–54, 2018.

[16] David Eppstein. Diameter and treewidth in minor-closed graph families. *Algorithmica*, 27(3):275–291, 2000.

[17] Paul Erdős, Arthur L Rubin, and Herbert Taylor. Choosability in graphs. In *Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium*, volume 26, pages 125–157, 1979.

[18] Tibor Gallai. Kritische Graphen I. *Magyar Tud. Akad. Mat. Kutato Int. Kozl.*, 8:165–192, 1963.

[19] Tibor Gallai. Kritische Graphen II. *Magyar Tud. Akad. Mat. Kutato Int. Kozl.*, 8:373–395, 1963.

[20] Fred Galvin. The list chromatic index of a bipartite multigraph. *Journal of Combinatorial Theory, Series B*, 63(1):153–158, 1995.

[21] H Grötzsch. Ein Dreifarbensatz für Dreikreisfreie Netze auf der Kugel. *Wiss. Z. Martin Luther Univ. Halle-Wittenberg, Math. Nat. Reihe*, 8:109–120, 1959.

[22] Klaus Jansen and Petra Scheffler. Generalized coloring for tree-like graphs. *Discrete Applied Mathematics*, 75(2):135–155, 1997.

[23] Tommy R Jensen and Bjarne Toft. *Graph coloring problems*. John Wiley & Sons, 1994.

[24] Jeff Kahn. Asymptotically good list-colorings. *Journal of Combinatorial Theory, Series A*, 73(1):1–59, 1996.

[25] Tom Kelly and Luke Postle. Exponentially many 4-list-colorings of triangle-free graphs on surfaces. *Journal of Graph Theory*, 87(2):230–238, 2018.

[26] Tom Kelly and Luke Postle. A local epsilon version of Reed's conjecture. *Journal of Combinatorial Theory, Series B*, 141:181–222, 2020.

[27] Alexandr Kostochka and Michael Stiebitz. Excess in colour-critical graphs. In *Graph theory and combinatorial biology (Balatonlelle, 1996)*, pages 87–99. János Bolyai Math. Soc., Budapest, 1999.

[28] Alexandr Kostochka and Matthew Yancey. Ore's conjecture on color-critical graphs is almost true. *Journal of Combinatorial Theory, Series B*, 109:73–101, 2014.

[29] Rikke Langhede and Carsten Thomassen. Exponentially many $\mathbf{Z}_5$-colorings in simple planar graphs. *Discrete Mathematics*, 344(9):112474, 2021.

[30] Bojan Mohar and Carsten Thomassen. *Graphs on surfaces*, volume 10. JHU press, 2001.

[31] Oystein Ore. *The four-color problem*. Academic Press, 1967.

[32] Luke Postle. *5-list-coloring graphs on surfaces*. Georgia Institute of Technology, 2012.

[33] Luke Postle. Linear-time and efficient distributed algorithms for list coloring graphs on surfaces. In *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 929–941. IEEE, 2019.

[34] Luke Postle. 3-list-coloring graphs of girth at least five on surfaces. *Journal of Combinatorial Theory, Series B*, 147:1–36, 2021.

[35] Luke Postle and Robin Thomas. Five-list-coloring graphs on surfaces II. A linear bound for critical graphs in a disk. *Journal of Combinatorial Theory, Series B*, 119:42–65, 2016.

[36] Luke Postle and Robin Thomas. Five-list coloring graphs on surfaces III. One list of size one and one list of size two. *Journal of Combinatorial Theory, Series B*, 128:1–16, 2018.

[37] Luke Postle and Robin Thomas. Hyperbolic families and coloring graphs on surfaces. *Transactions of the American Mathematical Society, Series B*, 5(7):167–221, 2018.

[38] Bruce Reed. ω, δ, and χ. *Journal of Graph Theory*, 27(4):177–212, 1998.

[39] Carsten Thomassen. Five-coloring maps on surfaces. *Journal of Combinatorial Theory, Series B*, 59(1):89–105, 1993.

[40] Carsten Thomassen. Every planar graph is 5-choosable. *Journal of Combinatorial Theory, Series B*, 62(1):180–181, 1994.

[41] Carsten Thomassen. 3-list-coloring planar graphs of girth 5. *Journal of Combinatorial Theory, Series B*, 64(1):101–107, 1995.

[42] Carsten Thomassen. Color-critical graphs on a fixed surface. *Journal of Combinatorial Theory, Series B*, 70(1):67–100, 1997.

[43] Carsten Thomassen. A short list color proof of Grötzsch's theorem. *Journal of Combinatorial Theory, Series B*, 88(1):189–192, 2003.

[44] Carsten Thomassen. Exponentially many 5-list-colorings of planar graphs. *Journal of Combinatorial Theory, Series B*, 97(4):571–583, 2007.

[45] Carsten Thomassen. Many 3-colorings of triangle-free planar graphs. *Journal of Combinatorial Theory, Series B*, 97(3):334–349, 2007.

[46] V.G. Vizing. Vertex colourings with given colours. *Metody Diskret. Analiz.*, pages 3–10, 1976.

[47] Margit Voigt. List colourings of planar graphs. *Discrete Mathematics*, 120(1-3):215–219, 1993.

[48] Margit Voigt. A not 3-choosable planar graph without 3-cycles. *Discrete Mathematics*, 146(1-3):325–328, 1995.

[49] Carl Roger Yerger Jr. *Color-critical graphs on surfaces*. Georgia Institute of Technology, 2010.