Numerical analysis of space-time hybridized discontinuous Galerkin methods for incompressible flows

by

Keegan L. A. Kirk

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Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner: Clint Dawson
Professor, Dept. of Aerospace Engineering,
University of Texas at Austin

Supervisor(s): Sander Rhebergen
Professor, Dept. of Applied Mathematics,
University of Waterloo

Internal Members: Lilia Krivodonova
Professor, Dept. of Applied Mathematics,
University of Waterloo

Kirsten Morris
Professor, Dept. of Applied Mathematics,
University of Waterloo

Internal-External Member: Christopher Batty
Professor, Dept. of Computer Science,
University of Waterloo

Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Statement of Contributions

Keegan L. A. Kirk was the sole author of Chapters 1, 2 and 7, which were written under the supervision of Dr. Sander Rhebergen. Chapters 1, 2 and 7 were not written for publication. The research presented in Chapters 3 and 6 has been published in peer-reviewed journals, and the research presented in Chapters 4 and 5 has been submitted for publication in peer-reviewed journals. Keegan L. A. Kirk's contributions to Chapters 3 to 6 are clarified below.

Chapter 3: The research presented in this chapter is a product of collaboration between Keegan L. A. Kirk and Dr. Sander Rhebergen and has been published in [54]. The project was devised jointly by Keegan L. A. Kirk and Dr. Sander Rhebergen. Keegan L. A. Kirk developed the theoretical formalism and worked out all of the technical details and proofs. Dr. Sander Rhebergen performed the numerical simulations and provided proof verification. The manuscript was jointly written and edited by Keegan L. A. Kirk and Dr. Sander Rhebergen.

Chapter 4: The research presented in this chapter is a product of collaboration between Keegan L. A. Kirk, Dr. Tamás Horváth, and Dr. Sander Rhebergen and has been submitted for publication [53]. The project was devised jointly by Keegan L. A. Kirk and Dr. Sander Rhebergen. Keegan L. A. Kirk developed the theoretical formalism, worked out all of the technical details and proofs, and wrote the first draft of the manuscript. Editing and proof verification were provided by Dr. Sander Rhebergen. Dr. Tamás Horváth performed the numerical simulations.

Chapter 5: The research presented in this chapter is a product of collaboration between Keegan L. A. Kirk, Dr. Ayçıl Çeşmelioğlu, and Dr. Sander Rhebergen and has been submitted for publication [51]. The project was devised by Keegan L. A. Kirk based on the work from Chapter 4. Keegan L. A. Kirk developed the theoretical formalism, worked out all of the technical details and proofs, and wrote the first draft of the manuscript. Editing and proof verification were provided by Dr. Ayçıl Çeşmelioğlu and Dr. Sander Rhebergen.

Chapter 6: The research presented in this chapter is a product of collaboration between Keegan L. A. Kirk, Dr. Tamás Horváth, Dr. Ayçıl Çeşmelioğlu, and Dr. Sander Rhebergen and has been published in [52]. The project was devised by Dr. Sander Rhebergen. Keegan L. A. Kirk developed the theoretical formalism, worked out all of the technical details and proofs, and wrote the first draft of the manuscript. Editing and proof verification were provided by Dr. Sander Rhebergen and Dr. Ayçıl Çeşmelioğlu. Dr. Tamás Horváth performed the numerical simulations.

Abstract

Many industrial problems require the solution of the incompressible Navier–Stokes equations on moving and deforming domains. Notable examples include the simulation of rotating wind turbines in strong air flow, wave impact on offshore structures, and arterial blood flow in the human body. A viable candidate for the numerical solution of the Navier–Stokes equations on time-dependent domains is the space-time discontinuous Galerkin (DG) method, which makes no distinction between spatial and temporal variables. Space-time DG is well suited to handle moving and deforming domains but at a significant increase in computational cost in comparison to traditional time-stepping methods.

Attempts to rectify this situation have led to the pairing of space-time DG with the hybridized discontinuous Galerkin (HDG) method, which was developed to reduce the computational expense of DG. The combination of the two methods results in a scheme that retains the high-order spatial and temporal accuracy and geometric flexibility of space-time DG at a reduced cost. Moreover, the use of hybridization allows for the design of *pressure-robust* space-time methods on time-dependent domains, which is a class of mimetic methods that inherit at the discrete level a fundamental invariance property of the incompressible Navier–Stokes equations.

The space-time HDG method has been successfully applied to incompressible flow problems on time-dependent domains; however, at present, no supporting theoretical analysis can be found in the literature. This thesis is a first step toward such an analysis. In particular, we perform a thorough theoretical convergence analysis of a space-time HDG method for the incompressible Navier–Stokes equations on *fixed domains*, and of a space-time HDG method for the linear advection-diffusion equation on *time-dependent domains*. The former contribution elucidates the difficulties involved in the theoretical analysis of space-time HDG methods for the Navier–Stokes equations, while the latter contribution introduces a framework for the convergence analysis of space-time HDG methods on time-dependent domains.

We begin with an a priori error analysis of a pressure-robust HDG method for the stationary Navier–Stokes equations. Then, we provide an a priori error analysis of a pressure-robust space-time HDG method from which we conclude that the space-time HDG method converges to *strong solutions* of the Navier–Stokes equations. This leaves open the question of convergence to *weak solutions*, which we answer in the affirmative using *compactness* techniques. Finally, we provide an a priori error analysis of a space-time HDG method for the linear advection-diffusion equation on time-dependent domains.

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Dedication

To my family, friends, and teachers.

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Chapter 1

Introduction

1.1 Aims and motivation

The incompressible Navier–Stokes equations are a set of nonlinear partial differential equations (PDEs) that govern the dynamics of viscous incompressible Newtonian fluids. They are central to the mathematical study of fluids, and as such see wide application across a broad spectrum of scientific fields. The solution of the Navier–Stokes equations, or any nonlinear PDE for that matter, is an extremely difficult task in general. In fact, exact solutions can only be found in very idealized scenarios. When faced with modeling complex fluid flows in physically realistic scenarios, we must turn to numerical methods to approximate the solution of the Navier–Stokes equations.

Historically, the mixed finite element method has been a popular candidate for the numerical solution of the incompressible Navier–Stokes equations. Many classic finite elements, such as the Crouzeix–Raviart [25] and Taylor–Hood [42] methods, are still widely used nearly five decades after their inception. However, a notable deficiency of many finite element methods for the incompressible Navier–Stokes equations is that the incompressibility constraint is not satisfied exactly at the discrete level. Instead, the approximate velocity field is only *discretely divergence free* (i.e. in an integral sense when tested against functions from the approximate pressure space). As we shall soon see, the violation of the incompressibility constraint at the discrete level leads to "pollution" in the error in the velocity field by the error in the pressure. This phenomenon has been the topic of intensive research over the past decade, and has given birth to the study of so-called *pressure-robust* finite element methods whose velocity errors are decoupled from the pressure error [47, 62].

To complicate matters further, many important industrial applications of the Navier–Stokes equations require the consideration of moving or deforming domains. Some examples include the modeling of arterial blood flow, free surface waves, and multiphase flows in which the interface between fluids evolves over time. When confronted with such a situation, care must be taken to ensure that the evolution of the domain is captured by the numerical method. A class of methods especially suited for this task is the space-time finite element method which captures the motion of the domain by recasting it as a stationary domain in space-time. While conforming (i.e., continuous) space-time finite element methods have been successfully applied to incompressible flow problems on time-dependent domains [48, 66, 68, 97], they are not locally conservative which can prove problematic for convection dominated flows. From this point of view, space-time discontinuous Galerkin methods are more fitting for the solution of the Navier–Stokes equations on time-dependent domains [74, 76, 99].

Naturally, one may wonder about pressure-robustness in the context of incompressible flows on time-dependent domains. Surprisingly, very few results can be found in this direction. Recently, a class of pressure-robust space-time hybridized discontinuous Galerkin methods for the Navier–Stokes equations on time-dependent domains has been introduced in [43, 44]. While the method performs well in numerical experiments, a theoretical convergence analysis has proven to be highly nontrivial. The purpose of this thesis is to provide a first step toward such an analysis.

In the remainder of this chapter, we introduce the core concepts behind the space-time hybridized discontinuous Galerkin method, discuss the objectives and contributions of this thesis, and provide an overview of the coming chapters.

Discontinuous Galerkin methods. The discontinuous Galerkin (DG) method has seen a substantial increase in popularity amongst computational fluid dynamicists over the course of the past two decades. As the name would seem to suggest, the DG method differs from classical (continuous) finite element methods (henceforth referred to as CG methods) in that it allows for the use of discontinuous basis functions for the trial and test spaces. As outlined below, the increased flexibility afforded by allowing for discontinuities in the finite element spaces is substantial, but this flexibility comes at a significant increase in computational expense.

The fundamental difference between the DG method and the CG method is that basis functions can be localized to single mesh elements. As a result, the discrete equations resulting from the DG method can be posed locally on each mesh element. Of course, there must be *some* communication between local solutions on adjacent elements so that they may be patched together to recover a meaningful global solution. Given the local problem on a single mesh element, we achieve this communication by treating the local solutions on neighbouring elements as boundary data through the use of so-called "numerical fluxes". This yields a more compact "stencil" in comparison to the CG method since the solution on a single mesh element is coupled only to the solutions on neighbouring elements with a shared interface.

By defining basis functions locally on each mesh element, the DG method can easily handle irregular and non-matching meshes which makes it an advantageous method to pair with h-adaptivity. Moreover, p-adaptivity can be achieved naturally by varying the polynomial degree of the local basis functions in each mesh element. However, the trade off for these advantages is increased computational cost in comparison to the CG method. As illustrated in Figure 1.1, the continuity constraint imposed on the basis functions in the CG method permits shared degrees of freedom (DOFs) between adjacent elements. DG methods, by contrast, share no DOFs between elements and this duplication of DOFs ultimately leads to an increase in the size of the global linear system.

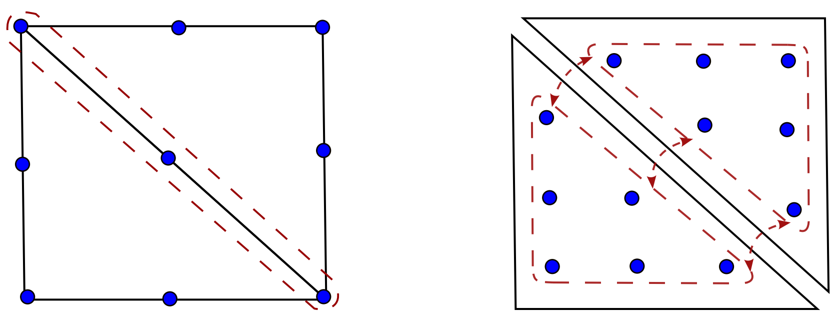

Figure 1.1: The coupling of DOFs on neighbouring elements for the CG method (left) and the DG method (right).

As previously mentioned, the DG method introduces numerical fluxes through which the solution on neighbouring elements is coupled. Given that there is some flexibility in the choice of numerical flux, it seems appropriate to design it to mimic the underlying dynamics of the PDE. This is particularly advantageous when dealing with transport phenomena and conservation laws, as the numerical flux provides a mechanism to introduce upwinding as well as local conservation of appropriate physical quantities into the DG method. Thus, the DG method can be viewed as a synthesis of the CG method and the finite volume method (FVM); it inherits the local conservation properties and other attractive features of the FVM while retaining the potential for higher-order accuracy enjoyed by the CG method. For this reason, the DG method has seen much success when applied to incompressible flow problems; see e.g. the articles [21, 22, 39, 41, 49, 65, 72, 86, 103] and the monographs

[28, 81].

Hybridized discontinuous Galerkin methods. To reduce the size of the global linear system while retaining the advantages of the DG method, one can instead consider the hybridized discontinuous Galerkin (HDG) method [19]. In the HDG method, an additional finite element space is introduced on the interfaces between mesh elements. An approximation of the solution's trace on mesh interfaces is sought in this new finite element space. Rather than coupling the local solutions on neighbouring mesh elements directly, the numerical flux couples them indirectly through their communication with the approximate trace on the shared interface (see Figure 1.2).

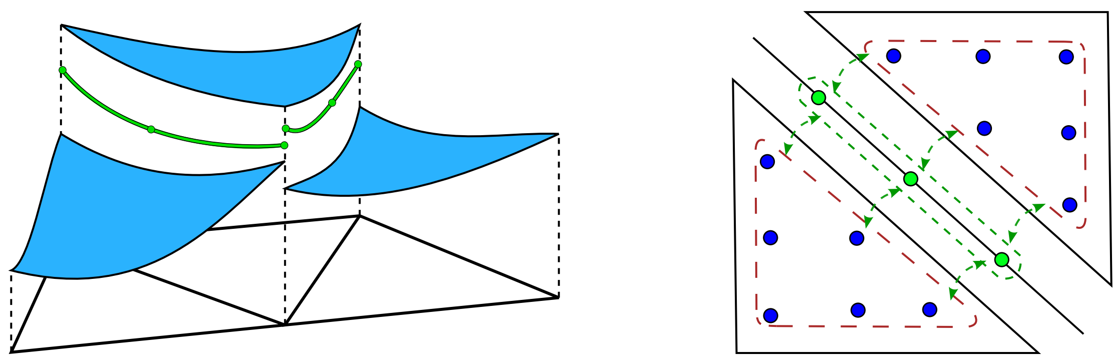

Figure 1.2: Sketch of the HDG solution (left) and the coupling of DOFs on neighbouring elements for the HDG method (right).

At first, the introduction of another unknown into the problem may seem counter intuitive: how can the size of the global system be reduced by introducing *even more* unknowns? The key is in the additional constraint required to close the system. In particular, the weak continuity (in an integral sense) of the normal component of the numerical flux is enforced across the interfaces of mesh elements. This is called the *transmission condition*, and it allows for the elimination of the DOFs on the interior of each element using static condensation resulting in a global system of equations for *only the approximate trace of the solution*. Once the approximate trace has been found, an inexpensive post-processing yields the solution on element interiors.

As the HDG method inherits the favourable properties of the DG method but with a reduction in the total number of globally coupled degrees of freedom, it is an excellent choice for the solution of incompressible flow problems. Some examples of HDG methods for incompressible flows can be found in, e.g., [14, 36, 55, 58, 59, 56, 69, 73, 78].

Space-time finite element methods. Typically, the numerical solution of an evolution equation employs the *method of lines* wherein the problem is first discretized in space using a finite element method to obtain a system of ordinary differential equations (ODEs) in time. The resulting system of ODEs can then be discretized using an appropriately chosen time integration scheme often based on the finite difference method. This approach works well when the domain Ω is *fixed in time*. However, if the domain evolves over time, there is an immediate problem. Consider, for the sake of argument, the backward difference operator: for fixed $x \in \Omega(t)$,

$$\left.\frac{\partial u}{\partial t}\right|_{t=t_{n+1}} \approx \frac{u(x, t_{n+1}) - u(x, t_n)}{\Delta t} \tag{1.1}$$

Since the domain evolves with time, a spatial point $x \in \Omega(t_n)$ at a given time t_n may not remain in the domain at time t_{n+1} (see Figure 1.3). In such a scenario, the difference quotient in eq. (1.1) may fail to be defined.

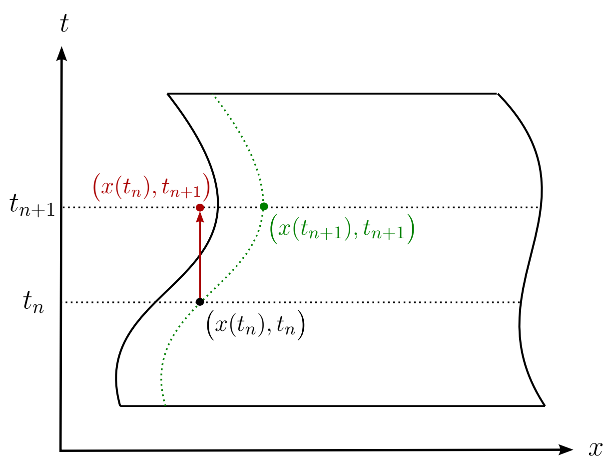

Figure 1.3: The method of lines on a time dependent domain $\Omega(t)$.

A possible way to overcome this problem is by taking a Lagrangian or Arbitrary Lagrangian-Eulerian (ALE) approach, in which a change of coordinates is performed to map the problem onto a fixed reference domain more amenable to the method of lines (see Figure 1.4). This allows the use of relatively inexpensive time integration schemes to solve moving boundary problems. However, a careful choice of time integrator is required to ensure satisfaction of the geometric conservation law (GCL), which can be characterized as a numerical method's ability to preserve uniform (constant) flow solutions under mesh movement. While the exact role that the GCL plays in the stability and convergence of numerical methods for moving boundary problems is controversial [7], it has been observed

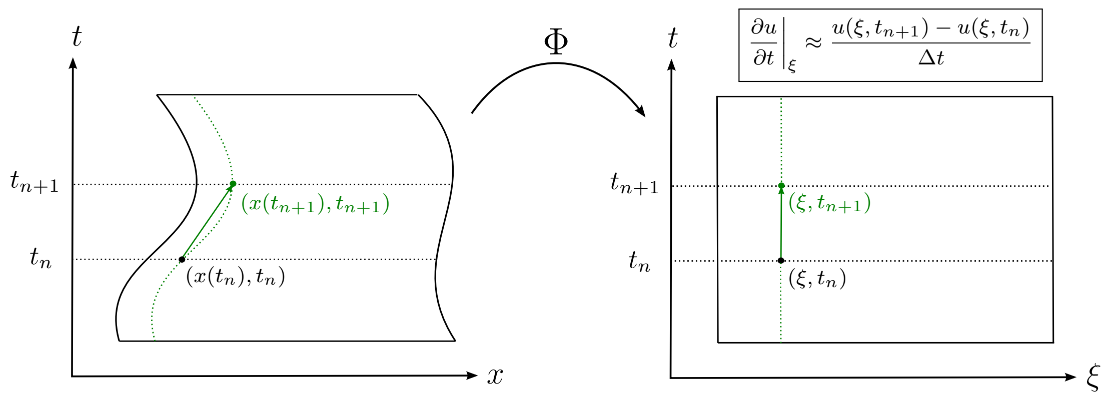

Figure 1.4: Combining a Lagrangian/ALE approach with the method of lines.

that failure to satisfy the GCL can result in a loss of accuracy in numerical solutions on moving meshes [71].

A second approach, at the heart of this thesis, is to employ a *space-time* finite element method. In space-time methods, a time-dependent problem on a moving d-dimensional domain is recast as a stationary problem on a fixed $(d+1)$-dimensional space-time domain by formally removing the distinction between spatial and temporal variables. This space-time domain is partitioned into *space-time elements* (see Figure 1.5), and the finite element method is used in both space and time. As a result, the movement of the domain is naturally captured by the numerical discretization. Moreover, space–time methods are known to automatically satisfy the GCL [60].

Of course, a subclass of the space-time finite element method is the space-time DG method. As the name suggests, the space-time DG method employs DG in both space and time. Thus, the attractive properties of the DG method (local conservation, *hp*-adaptivity, etc.) hold *both* in space and time. Furthermore, by using a discontinuous-in-time trial space, the problem can be localized to a single time interval or *space-time slab* as illustrated in red in Figure 1.5. This allows for a sequential approach where the solution is computed in a single space-time slab and then used as an initial condition for the next space-time slab. This significantly reduces the amount of memory required as only a single space-time slab is ever stored at once. Alternative to the slab-by-slab approach is the all-at-once approach where the entire space-time mesh is stored at once. While the all-at-once approach is much more memory intensive, it is more amenable to parallelization than the slab-by-slab

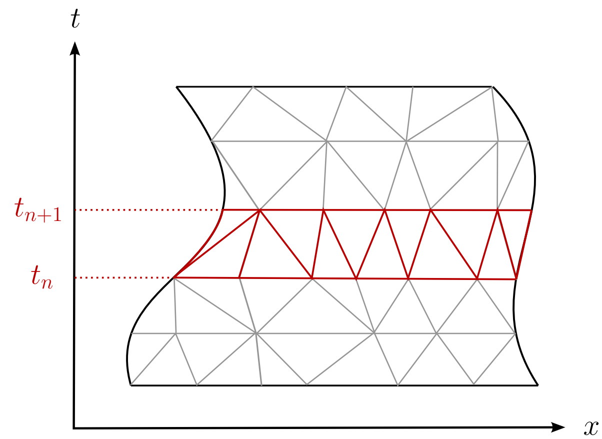

Figure 1.5: A tessellation of the space-time domain with space-time elements.

approach [92]. Examples of space-time finite element methods for the incompressible flows on time-dependent domains can be found in e.g. [48, 66, 68, 76, 97, 99].

Though our primary motivation for the use of space-time finite element methods is the ease in which they handle time-dependent domains, much of the analysis in this thesis considers *fixed domains*. As such, it is only appropriate to point out some of the benefits that the space-time DG method has even on fixed domains. The most obvious is that higher-order accuracy in time can be achieved simply by increasing the degree of the polynomial solution in time. Other benefits include local temporal refinement, p-adaptivity in time, and the possibility of using differing meshes at each time level. The catch, of course, is a significant increase in the computational cost over more traditional time integration methods. Examples of space-time finite element methods for incompressible flows on fixed domains can be found in, e.g., [1, 17].

Space-time hybridized discontinuous Galerkin methods. As we have seen, the use of the DG method comes at a significant computational expense, especially so when DG is used in both space and time! To offset the computational cost of solving a $(d+1)$-dimensional problem with DG, the space-time HDG method was introduced in the articles [74, 75]. The combination of space-time DG and HDG allows for the use of static condensation to reduce the problem to one of merely finding the approximate traces on the facets of space-time elements. Roughly speaking, this reduces the size of the global system to one of a d-dimensional problem since the approximate trace is defined on a d-dimensional surface in $\mathbb{R}^{d+1}$. More recent examples of space-time HDG methods for the incompressible Navier–Stokes equations can be found in [43, 44].

Pressure-robust finite element methods. For simplicity, we shall restrict our discussion of pressure-robustness to the steady incompressible Navier–Stokes system on an open bounded domain $\Omega \subset \mathbb{R}^d$: given an external body force f and kinematic viscosity ν, find a velocity field u and pressure p satisfying

$$-\nu\Delta u + \nabla \cdot (u \otimes u) + \nabla p = f, \quad \text{in } \Omega, \tag{1.2a}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega, \tag{1.2b}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.2c}$$

One peculiarity with many finite element methods for eq. (1.2) is that the error in the velocity field is polluted by the error in the pressure. More precisely, given finite element spaces V_h for the velocity and Q_h for the pressure,

$$\|u - u_h\|_{V_h} \leq C \left(\inf_{v_h \in V_h} \|u - v_h\|_{V_h} + \frac{1}{\nu} \inf_{q_h \in Q_h} \|p - q_h\|_{Q_h} \right). \tag{1.3}$$

Classic examples of finite element methods satisfying eq. (1.3) include the Taylor–Hood element [42], the Crouzeix–Raviart element [25], the MINI element [3], and discontinuous Galerkin methods [20, 23, 28]. At first glance, the additional dependence on the pressure in the error bound eq. (1.3) may not appear significant; however, the appearance of the inverse of the kinematic viscosity is problematic if one wishes to consider convection dominated flows ($\nu \ll 1$).

From the physical point of view, the problem stems from the following observation [47, 62]: if the external force is perturbed by an irrotational field, i.e. $f \to f + \nabla\varphi$, for some scalar potential φ, the solution becomes

$$(u, p) \to (u, p + \varphi).$$

In other words, the additional irrotational force field $\nabla\varphi$ is balanced by the pressure gradient, leaving the velocity field unchanged. Viewed from this lens, eq. (1.3) would seem to indicate that this fundamental invariance property of incompressible flows is not inherited at the discrete level.

From the mathematical point of view, the source of this peculiarity is related to the *divergence constraint* eq. (1.2b). Many finite element methods for incompressible flow problems satisfy this constraint only approximately. This constitutes a variational crime in the sense that if K denotes the kernel of the distributional divergence operator and K_h denotes the kernel of the discrete divergence operator arising from the finite element discretization, then $K_h \not\subset K$. However, if indeed $K_h \subset K$, it holds that [6, Theorem 5.25]

$$\|u - u_h\|_{V_h} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{V_h}. \tag{1.4}$$

We call finite element methods that satisfy eq. (1.4) *pressure-robust*. As simple a task as it may seem, it is not immediately obvious how one can design a finite element method to ensure that $K_h \subset K$. Fortunately, it has been recognized that pressure-robust finite element methods can be equivalently characterized as $H(\text{div};\Omega)$-conforming methods that produce pointwise solenoidal velocity approximations [47]. Examples of pressure-robust finite element methods can be found in, e.g., [36, 47, 56, 57, 59, 62, 89]. Note that it is not our purpose to investigate the advantages of pressure-robust discretizations. We instead refer the reader to the thesis [88], which gives an excellent overview of the advantages of pressure-robust discretizations for both stationary and transient incompressible flows.

Objectives and contributions. The present thesis is concerned with the numerical analysis of a class of pressure-robust space-time hybridized discontinuous Galerkin methods for the incompressible Navier–Stokes equations. While our main motivation for the use of space-time methods is the ease in which they handle time-dependent domains, no theoretical analysis of a space-time HDG method for the Navier–Stokes equations has appeared in the literature, even on fixed domains. Our purpose is to fill this gap as a first step toward a theoretical analysis valid on time-dependent domains. The main contributions of this thesis are fourfold:

1. We provide an a priori error analysis of the pressure-robust HDG method of Rhebergen and Wells [78] in the case of the stationary Navier–Stokes equations, which was previously missing from the literature.

2. We provide the first a priori error analysis of a space-time HDG method for the transient Navier–Stokes equations on *fixed* domains. Our analysis requires the spatial domain $\Omega \subset \mathbb{R}^d$ to be a convex polygon ($d = 2$) or polyhedron ($d = 3$) as well as a restriction on the size of the problem data to ensure the existence of a *strong solution* to the Navier–Stokes equations (see Theorem 2.3.2 below).

3. We further show, using a compactness argument, that this space-time HDG method converges to a weak solution to the Navier–Stokes equations (in the sense of Leray–Hopf) even in the absence of additional regularity. This fills the gap left by our a priori analysis, as we no longer require convexity of the spatial domain nor a restriction on the size of the problem data. However, this approach does not yield rates of convergence.

4. Finally, we provide an a priori error analysis of a space-time HDG method for the advection-diffusion equation on *time-dependent* domains. To the author's knowledge, this constitutes the first error analysis of a space-time HDG method on time-dependent domains in general.

1.2 Thesis outline

The remaining chapters of this thesis are structured as follows:

Chapter 2. First, we rapidly recall many of the tools from functional analysis that we will require in subsequent chapters. Then, we introduce the basic mathematical theory of the incompressible Navier–Stokes equations.

Chapter 3. We study the pointwise solenoidal HDG method proposed by Rhebergen and Wells in [78] in the case of the stationary Navier–Stokes equations. We show that under mild assumptions on the size of the problem data, the resulting nonlinear algebraic system arising from the HDG discretization is uniquely solvable using a fixed-point argument à la Brouwer. Then, we perform an a priori error analysis of the method and prove that the velocity and pressure approximations converge at the optimal rate in suitable norms. Most importantly, we show that the error in the velocity approximation is independent of the pressure, and thus the method is pressure-robust. The contents of this chapter have been taken, with slight modification, from the article:

> K. L. A. Kirk and S. Rhebergen, *Analysis of a pressure-robust hybridized discontinuous Galerkin method for the stationary Navier–Stokes equations*, Journal of Scientific Computing, 81 (2019), pp. 881–897. https://doi.org/10.1007/s10915-019-01040-y

Chapter 4. We study a space-time HDG method for the incompressible Navier–Stokes equations on fixed domains based on the HDG method analyzed in Chapter 3. Using a topological degree argument, we show that there exists a solution to the nonlinear algebraic system arising from the space-time HDG discretization in both two and three spatial dimensions. Then, using a novel discrete Ladyzhenskaya inequality and fine properties of polynomials, we obtain a uniform-in-time bound on the discrete velocity in two spatial dimensions. This uniform bound allows us to show that the discrete velocity solution is unique in two dimensions under a restriction on the size of the problem data. Next, we prove a space-time inf-sup condition and conclude from the Ladyzhenskaya-Babuška-Brezzi theorem that to each discrete velocity solution there exists a corresponding unique discrete pressure solution.

Further, we derive optimal a priori error bounds for the velocity under the assumption that the Navier–Stokes system admits a strong solution (see Theorem 2.3.2). This places a restriction on the size of the problem data as well as the shape of the spatial domain $\Omega \subset \mathbb{R}^d$. Notably, the error bounds that we obtain for the velocity are independent of the pressure, thus proving that the method is pressure-robust. Finally, we obtain sub-optimal a priori error bounds for the pressure. The contents of this chapter have been taken, with slight modification, from the article:

> K. L. A Kirk, T. L. Horváth, and S. Rhebergen, *Analysis of an exactly mass conserving space-time hybridized discontinuous Galerkin method for the time-dependent Navier–Stokes equations*, (To appear in Mathematics of Computation) https://arxiv.org/abs/2103.13492

Chapter 5. Since our a priori analysis in Chapter 4 requires additional regularity of the exact solution, we cannot use it to conclude that the space-time HDG method converges to a *weak solution* of the Navier–Stokes equations as the time step and mesh size tend to zero. In this chapter, we prove that this is indeed the case, and moreover the weak solution is one in the sense of Leray–Hopf (i.e. it satisfies an appropriate energy inequality), filling the gap left by the previous chapter. Our analysis hinges on the introduction of discrete differential operators as well as a discrete version of the Aubin–Lions–Simon compactness theorem. The contents of this chapter have been taken, with slight modification, from the article:

> K. L. A Kirk, A. Çeşmelioğlu, and S. Rhebergen, *Convergence to weak solutions of a space-time hybridized discontinuous Galerkin method for the incompressible Navier–Stokes equations*, Mathematics of Computation. https://doi.org/10.1090/mcom/3780

Chapter 6. In this penultimate chapter, we analyze a space-time HDG method for a linear advection-diffusion equation on time-dependent domains. Following [38, 95, 99], we make use of *anisotropic* Sobolev spaces which provide a suitable alternative to Bochner spaces when the underlying spatial domain is evolving in time. Using novel anisotropic (in space and time) inverse and trace inequalities, we prove that the discrete bilinear form is coercive and satisfies an inf-sup condition with respect to a "streamline diffusion"-like norm that controls the time derivative of the discrete solution. Finally, we derive anisotropic (in space and time) a priori error estimates. The contents of this chapter have been taken, with slight modification, from the article:

> K. L. A Kirk, T. L. Horváth, A. Çeşmelioğlu, and S. Rhebergen, *Analysis of a space-time hybridizable discontinuous Galerkin method for the advection-diffusion problem on time-dependent domains*, SIAM Journal on Numerical Analysis, 57 (2019), pp. 1677–1696. https://doi.org/10.1137/18M1202049

Chapter 7. Finally, we conclude the thesis by discussing possible future avenues of research based on the work of Chapters 3 through 6. In particular, we discuss the extension of our analysis to space-time HDG methods for the Navier–Stokes equations on *time-dependent domains*.

Chapter 2

Preliminaries

The finite element method can be seen as a way to approximate the underlying *function spaces* (typically, Banach or Hilbert spaces) associated with the solution of a PDE rather than the differential operators involved. It should come at no surprise then that the framework for the theoretical analysis of the finite element method is that of functional analysis. In this chapter, we rapidly recall some of the concepts and tools from the theory of functional analysis and function spaces that we will require throughout the remainder of the thesis.

2.1 Functional analytic tools

In this section, we introduce some of the tools from functional analysis that we require in the sequel. Note that throughout this thesis, all vector spaces are assumed real. It will be assumed throughout that the reader is familiar with the basics of Hilbert space and Banach space theory. For more information, the interested reader may consult any standard text on functional analysis, e.g., [10, 18, 84, 105].

2.1.1 Topological preliminaries

We begin by introducing two lemmas that provide us with powerful tools for the study of nonlinear systems in finite-dimensional vector spaces:

Lemma 2.1.1 (Corollary to Brouwer's fixed point theorem [18, Theorem 9.9-3])**.** *Let $(X, (\cdot,\cdot)_X)$ be a finite-dimensional Hilbert space and let $f : X \to X$ be a continuous mapping with the following property: there exists $M > 0$ such that*

$$(f(v), v)_X \geq 0 \quad \text{for all } v \in X \text{ such that } \|v\|_X = M.$$

Then, there exists a $v_0 \in X$ such that $\|v_0\|_X \leq M$ and $f(v_0) = 0$.

Lemma 2.1.2 (Topological degree argument [28, Lemma 6.42])**.** *Let $(X, \|\cdot\|_X)$ be a finite-dimensional Banach space. Let $M > 0$ and let $\Psi : X \times [0,1] \to X$ satisfy*

1. *Ψ is continuous.*
2. *$\Psi(\cdot, 0)$ is an affine function and the equation $\Psi(v, 0) = 0$ has a solution $v \in X$ such that $\|v\|_X < M$.*
3. *For any $(v, \rho) \in X \times [0,1]$, $\Psi(v, \rho) = 0$ implies $\|v\|_X < M$.*

Then, there exists $v \in X$ such that $\Psi(v, 1) = 0$ and $\|v\|_X < M$.

2.1.2 Elementary results

Next, we recall a number of elementary results from linear functional analysis. To set notation, let X be a Banach space with topological dual X'. Throughout this thesis, we will denote the value of a continuous linear functional $f \in X'$ at $v \in X$ by the *duality* pairing

$$f(v) = \langle f, v \rangle_{X' \times X}. \tag{2.1}$$

Example 2.1.1. *To get a feel for what we mean by the duality pairing eq. (2.1), consider the Sobolev space $X = H^1(-1,1)$ (see Definition 2.2.4 below) and let f be the Dirac delta "function" $\delta_0 : H^1(-1,1) \to \mathbb{R}$ defined by*

$$v \mapsto \delta_0(v) = v(0), \quad \forall v \in H^1(-1,1).$$

Observe that $\delta_0 \in X' = \left(H^1(-1,1)\right)'$, since by the Sobolev embedding theorem (Theorem 2.2.6 below),

$$|\delta_0(v)| = |v(0)| \leq C\|v\|_{H^1(-1,1)}, \quad \forall v \in H^1(-1,1).$$

Following the convention eq. (2.1), we write

$$\langle \delta_0, v \rangle_{X' \times X} = v(0).$$

Theorem 2.1.1 (Riesz Representation Theorem [18, Theorem 4.6-1])**.** *Let $(V, (\cdot, \cdot)_V)$ be a Hilbert space. Given any continuous linear functional $f \in V'$, there exists a unique element $z_f \in V$ such that*

$$\langle f, v\rangle_{V' \times V} = (z_f, v)_V.$$

This defines a linear isomorphism $R : V \to V'$, $R : z_f \mapsto f$ by

$$\langle f, v\rangle_{V' \times V} = \langle R z_f, v\rangle_{V' \times V} = (z_f, v)_H, \quad \forall v \in V.$$

Theorem 2.1.2 (Lax–Milgram [18, Theorem 6.2-1])**.** *Let V be a Hilbert space, let $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ be a bounded and coercive bilinear form, and let $f : V \to \mathbb{R}$ be a continuous linear functional. In other words, there exist constants $C_1, C_2, C_3 > 0$ such that for all $u, v \in V$,*

$$|a(u, v)| \leq C_1 \|u\|_V \|v\|_V\,, \quad a(u, u) \geq C_2 \|u\|_V^2\,, \quad |\langle f, v\rangle_{V' \times V}| \leq C_3 \|v\|_V\,.$$

Then, there exists a unique solution to the variational problem: find $u \in V$ such that

$$a(u, v) = \langle f, v\rangle_{V' \times V}, \quad \forall v \in V.$$

Theorem 2.1.3 (Banach–Nečas–Babuška [28, Theorem 1.1])**.** *Let X be a Banach space and let Y be a reflexive Banach space. Suppose that $a(\cdot, \cdot) : X \times Y \to \mathbb{R}$ is a continuous bilinear form: there exists $C_1 > 0$ such that*

$$|a(v, w)| \leq C_1 \|v\|_X \|w\|_Y\,, \quad \forall v \in X,\ w \in Y.$$

Let $f \in Y'$ and consider the problem: find $u \in X$ such that for all $w \in Y$, it holds that

$$a(u, w) = \langle f, w\rangle_{Y' \times Y}. \tag{2.2}$$

Then, problem eq. (2.2) has a unique solution if and only if the following two conditions hold:

(i) There is a $C_2 > 0$ such that for all $v \in X$,

$$C_2 \|v\|_X \leq \sup_{0 \neq w \in Y} \frac{a(v, w)}{\|w\|_Y}, \tag{2.3}$$

(ii) For all $w \in Y$,

$$(\forall v \in X,\ a(v, w) = 0) \Rightarrow (w = 0).$$

Note that condition eq. (2.3) is equivalent to the following *inf-sup condition*:

$$C_2 \leq \inf_{0 \neq v \in X} \sup_{0 \neq w \in Y} \frac{a(v,w)}{\|v\|_X \|w\|_Y}.$$

Remark 2.1.1. *If in Theorem 2.1.3 the Banach spaces X and Y are finite-dimensional, condition (ii) is superfluous.*

Theorem 2.1.4 (Hahn–Banach [18, Theorem 5.9-1])**.** *Let X be a normed vector space, let Y be a subset of X, and let $f : Y \to \mathbb{R}$ be a continuous linear functional. Then, there exists a continuous linear functional $\tilde{f} : X \to \mathbb{R}$ satisfying*

$$\langle \tilde{f}, y \rangle_{X' \times X} = \langle f, y \rangle_{Y' \times Y}, \text{ for all } y \in Y, \quad \text{ and } \quad \|\tilde{f}\|_{X'} = \|f\|_{Y'}.$$

Corollary 2.1.1 ([18, Theorem 5.9-7])**.** *Let Y be a subspace of a normed vector space X. Then, Y is dense in X if and only if*

$$\langle f, y \rangle_{X' \times X} = 0 \text{ for all } y \in Y$$

implies that f is the zero functional.

2.1.3 Modes of convergence in Banach spaces

To prove convergence of the space-time HDG scheme to weak solutions of the Navier–Stokes equations in Chapter 5, we require various types of convergence in Banach spaces, which we summarize below.

Definition 2.1.1 (Strong convergence)**.** *Let X be a Banach space. A sequence of elements $(x_n)_{n \in \mathbb{N}}$ in X is said to converge strongly to a function $x \in X$ if*

$$\lim_{n \to \infty} \|x_n - x\|_X = 0.$$

We denote strong convergence by an arrow: $x_n \to x$ as $n \to \infty$.

Definition 2.1.2 (Weak convergence)**.** *Let X be a Banach space and let X' denote its topological dual space. A sequence of elements $(x_n)_{n \in \mathbb{N}}$ in X is said to converge weakly to a function $x \in X$ if for each $f \in X'$, it holds that*

$$\lim_{n \to \infty} f(x_n) = f(x).$$

We denote weak convergence by a half-arrow: $x_n \rightharpoonup x$ as $n \to \infty$.

Definition 2.1.3 (Weak-$\star$ convergence)**.** *Let X be a Banach space and let X' denote its topological dual space. A sequence of elements $(f_n)_{n\in\mathbb{N}}$ in X' is said to converge weakly-$\star$ to an element $f \in X'$ if for each $x \in X$, it holds that*

$$\lim_{n\to\infty} f_n(x) = f(x).$$

We denote weak-$\star$ convergence by a half-arrow with a star on top: $f_n \overset{\star}{\rightharpoonup} f$ as $n \to \infty$.

2.1.4 Gelfand triples $V \subset H \subset V'$

We end off this section by discussing Gelfand triples (also known as evolution triples due to their importance to the study of abstract evolution equations), which will arise in Chapter 5. Our discussion will follow [10, Section 5.2]. Let V be a separable and reflexive Banach space, and let H be a separable Hilbert space. Assume that V is dense in H, and that the embedding $V \subset H$ is continuous: there exists a $C > 0$ such that

$$\|v\|_H \le C\|v\|_V\,, \quad \forall v \in V.$$

Then, we can identify H with a dense subspace of V' and we write

$$V \subset H \subset V', \tag{2.4}$$

where both embeddings are continuous and dense. We call the triplet (V, H, V') a Gelfand triple, and we call H a pivot space. The importance of the Gelfand triple lies in the fact that the duality pairing on V, when restricted to H, coincides with the inner-product on H:

$$\langle u, v\rangle_{V'\times V} = (u, v)_H, \quad \forall u \in H,\ \forall v \in V. \tag{2.5}$$

Below, we justify that such an identification can be made. The following discussion is technical, and can be skipped by the reader willing to take eq. (2.4) and eq. (2.5) at face value.

Justification of the identification $H \subset V'$

Define a mapping $T : H' \to V'$ by

$$\langle Tf, v\rangle_{V'\times V} = \langle f, v\rangle_{H'\times H}, \tag{2.6}$$

that is, Tf is the restriction of $f \in H'$ to V. Note that the linearity of T follows from the linearity of the duality product on H. By Theorem 2.1.1, we can identify $H \cong H'$ using the Riesz isomorphism $R : H \to H'$. This, combined with the definition of the operator T, shows that if $u \in H$ and $v \in V$, we can identify the duality product on V with the inner product on H as follows:

$$\langle (T \circ R)u, v\rangle_{V' \times V} = \langle Ru, v\rangle_{H' \times H} = (u, v)_H.$$

To justify the identification of H with a subspace of V', we need to show that to each element $u \in H$ there corresponds a unique element $(T \circ R)u \in V'$. We first show that the linear operator T is injective. To this end, note that if $Tf = 0$, then $\langle f, v\rangle_{H' \times H} = 0$ for all $v \in V$. Since V is dense in H, this is equivalent to $f = 0$ by Corollary 2.1.1. Thus, both $R : H \to H'$ and $T : H' \to V'$ are injective, and so is their composition $(T \circ R) : H \to V'$. Therefore, to each $u \in H$, we can associate a unique element $(T \circ R)u$ of V'.

It remains to show that the embedding $TH' \subset V'$ is continuous and dense. We begin by showing the former. The fact that $V \subset H$ with continuous embedding ensures that T is continuous. Indeed, for all $v \in V$, we have

$$\frac{\langle f, v\rangle_{H' \times H}}{\|v\|_V} \le \frac{\|f\|_{H'}\|v\|_H}{\|v\|_V} \le C\|f\|_{H'},$$

and thus, by the definition of T, for any $f \in H'$,

$$\|Tf\|_{V'} = \sup_{0 \ne v \in V} \frac{\langle Tf, v\rangle_{V' \times V}}{\|v\|_V} = \sup_{0 \ne v \in V} \frac{\langle f, v\rangle_{H' \times H}}{\|v\|_V} \le C\|f\|_{H'}.$$

Therefore, TH' embeds continuously into V.

Finally, we show the image of T is dense in V'. By Corollary 2.1.1, it suffices to show for $\varphi \in V''$

$$\langle \varphi, Tf\rangle_{V'' \times V'} = 0, \ \forall f \in H', \quad \Rightarrow \quad \varphi = 0.$$

Let $f \in H'$ be arbitrary. Recall [10, Section 3.5] that since V is reflexive, we can identify $V \cong JV = V''$ via the canonical injection $J : V \to V''$ satisfying for all $v \in V$ and $\psi \in V'$,

$$\langle Jv, \psi\rangle_{V'' \times V'} = \langle \psi, v\rangle_{V' \times V}.$$

Suppose now that $\langle \varphi, Tf\rangle_{V'' \times V'} = 0$ for all $f \in H'$. By the reflexity of V, we can find a $v \in V$ such that $\varphi = Jv$, and

$$0 = \langle \varphi, Tf\rangle_{V'' \times V'} = \langle Jv, Tf\rangle_{V'' \times V'} = \langle Tf, v\rangle_{V' \times V}, \quad \forall f \in H'.$$

By the definition of the linear operator T, this is equivalent to

$$0 = \langle f, v\rangle_{H', H}, \quad \forall f \in H',$$

but then v must be the zero element of H (thus also V) (again, thanks to Corollary 2.1.1). By the linearity of J, we have $\psi = Jv = 0$, and thus TH' is dense in V'. Consequently, we can identify H' with a dense subspace of V' via the mapping $T : H' \ni f \mapsto Tf \in V'$: $H' \cong TH' \subset V'$, and the embedding is continuous and dense. Moreover, by Theorem 2.1.1, we can identify $H \cong H'$ using the Riesz isomorphism $R : H \to H'$.

Remark 2.1.2. *By the triplet eq.* (2.4)*, we really mean*

$$V \subset H \cong RH = H' \cong TH' \subset V'.$$

2.2 Function spaces

The purpose of this subsection is to introduce various function spaces that will be used extensively in the sequel. We have made no attempt to be exhaustive in our treatment. For more information, we refer the interested reader to e.g., [8, 18].

2.2.1 The Lebesgue spaces $L^p(\Omega)$

In what follows, it will be assumed that the reader is familiar with the Lebesgue theory of integration, but we will briefly review some of the basics. In particular, we will be concerned with measure spaces $(\Omega, \mathcal{M}, \mu)$, where $\Omega \subset \mathbb{R}^d$ is an open, bounded, and connected subset of $\mathbb{R}^d$, $\mathcal{M}$ is the σ-algebra of Lebesgue measurable subsets of Ω, and $\mu : \mathcal{M} \to [0, \infty]$ is the Lebesgue measure on $\mathbb{R}^d$. For a construction of the Lebesgue measure on $\mathbb{R}^d$, we refer to, e.g., [35, 83]. Given a measurable set $E \subset \Omega$ with $\mu(E) = 0$, we will say that a property holds *almost everywhere* (a.e.) provided it holds on $\Omega \setminus E$.

We define the vector space of summable functions on Ω:

$$\mathcal{L}^1(\Omega) = \left\{ f : \Omega \to \mathbb{R} \ : \ f \text{ is measurable and } \int_\Omega |f| \, \mathrm{d}x < \infty \right\}.$$

As it stands, the mapping $f \mapsto \int_\Omega |f| \, \mathrm{d}x$ defines a semi-norm on the space $\mathcal{L}^1(\Omega)$, since $\int_\Omega |f| \, \mathrm{d}x = 0$ need not imply that $f = 0$ (consider a measurable function taking nonzero values only on a set of measure zero). However, we wish to leverage the tools from Banach space theory introduced in Section 2.1. To this end, we define an equivalence relation on $\mathcal{L}^1(\Omega)$: $f \sim g$ if $f = g$ a.e. in Ω, and consider the quotient space $L^1(\Omega) = \mathcal{L}^1(\Omega)/\sim$. We will abuse notation by identifying functions in $\mathcal{L}^1(\Omega)$ with their equivalence classes in $L^1(\Omega)$. The mapping $f \mapsto \int_\Omega |f| \, \mathrm{d}x$ then defines a norm on $L^1(\Omega)$.

Next, we recall a number of basic results concerning Lebesgue integration.

Theorem 2.2.1 (Fatou's Lemma [18, Theorem 1.15-2]). *Let* $\{f_n\}_{n\in\mathbb{N}} \subset L^1(\Omega)$ *be a sequence of non-negative functions. Then,*

$$\int_\Omega (\liminf_{n\to\infty} f_n(x))\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega f_n(x)\,\mathrm{d}x.$$

Theorem 2.2.2 (Dominated Convergence Theorem [18, Theorem 1.15-3]). *Let* $\{f_n\}_{n\in\mathbb{N}} \subset L^1(\Omega)$ *be a sequence converging pointwise a.e. to* f*, and suppose* $|f_n(x)| \le g(x)$ *for some* $g \in L^1(\Omega)$*. Then,* $f \in L^1(\Omega)$ *and it holds that*

$$\lim_{n\to\infty} \int_\Omega f_n\,\mathrm{d}x = \int_\Omega f\,\mathrm{d}x.$$

Theorem 2.2.3 (Fubini's Theorem [18, Theorem 1.15-5]). *Given two Lebesgue measurable sets* $\Omega_1 \subset \mathbb{R}^n$*,* $\Omega_2 \subset \mathbb{R}^m$*, and a function* $f \in L^1(\Omega_1 \times \Omega_2)$*, it holds that*

$$\int_{\Omega_1\times\Omega_2} f(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_{\Omega_1}\left(\int_{\Omega_2} f(x_1,x_2)\,\mathrm{d}x_2\right)\mathrm{d}x_1 = \int_{\Omega_2}\left(\int_{\Omega_1} f(x_1,x_2)\,\mathrm{d}x_1\right)\mathrm{d}x_2.$$

For more information on the Lebesgue theory of integration, or abstract measure theory in general, the interested reader may consult [35, 83].

Definition 2.2.1 (Lebesgue spaces). *Let* $p \in \mathbb{R}$ *be such that* $1 \le p \le \infty$*. If* $1 \le p < \infty$*, we define*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;:\; f \text{ is measurable and } \|f\|_{L^p(\Omega)} < \infty \right\},$$

where

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p\,\mathrm{d}x\right)^{1/p}.$$

In the case $p = \infty$*, we define*

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;:\; f \text{ is measurable and } \|f\|_{L^\infty(\Omega)} < \infty \right\},$$

where

$$\|f\|_{L^\infty(\Omega)} := \inf\left\{ M \;:\; |f(x)| \le M \text{ a.e. on } \Omega \right\}.$$

As before, we abuse notation by identifying functions with their equivalence classes. The space $L^p(\Omega)$, equipped with the norm $\|\cdot\|_{L^p(\Omega)}$, is a Banach space for $1 \le p \le \infty$; if $1 < p < \infty$, it is reflexive, and if $1 \le p < \infty$ it is separable. For $p = 2$, the space $L^2(\Omega)$ becomes a Hilbert space when equipped with the inner product

$$(f,g)_{L^2(\Omega)} = \int_\Omega fg\,\mathrm{d}x.$$

Often, we will write $(\cdot,\cdot)_D$ for the L^2-inner product over a measurable set D for brevity.

2.2.2 Test functions and distributions

We now give a brief overview of the theory of distributions. For more information, the interested reader may consult, e.g., [18, 35, 84]. Let $\Omega \subset \mathbb{R}^d$ be a domain and consider the corresponding vector space of compactly supported smooth functions $C_c^\infty(\Omega)$. We equip this space with the topology described in, e.g., [84, Chapter 6.3] and denote the resulting topological vector space by $\mathcal{D}(\Omega)$. We call $\mathcal{D}(\Omega)$ the space of *test functions*.

Definition 2.2.2 (Sequential convergence in $\mathcal{D}(\Omega)$). *A sequence of test functions $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ is said to converge towards a test function $\phi \in \mathcal{D}(\Omega)$ if $\{\phi_n\}_{n\in\mathbb{N}} \subset C_c^\infty(K)$ for some compact set $K \subset \Omega$ and $\partial^\alpha \phi_n \to \partial^\alpha \phi$ as $n \to \infty$ uniformly for any multi-index $\alpha \in \mathbb{N}^d$.*

We now seek to characterize the topological dual space of $\mathcal{D}(\Omega)$. We say a linear map $T : \mathcal{D}(\Omega) \to \mathbb{R}$ is *continuous* if, given any sequence of test functions $(\phi_n)_{n\in\mathbb{N}}$ converging to ϕ in $\mathcal{D}(\Omega)$, it holds that

$$T(\phi_n) \to T(\phi) \text{ as } n \to \infty.$$

We call such a map a *distribution*, and we denote the space of distributions by $\mathcal{D}'(\Omega)$. We equip this space with the weak-$\star$ topology: a sequence of distributions $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{D}'(\Omega)$ is said to converge towards a distribution $T \in \mathcal{D}'(\Omega)$ if, for all $\phi \in \mathcal{D}(\Omega)$,

$$T_n(\phi) \to T(\phi) \text{ as } n \to \infty.$$

We refer to T as the *distributional limit* of $\{T_n\}_{n\in\mathbb{N}}$, and it is unique.

Next, we define the notion of a distributional derivative, motivated by the following integration by parts formula: given a multi-index $\alpha \in \mathbb{N}^d$, a function $u \in C^{|\alpha|}(\Omega)$, and a test function $\phi \in \mathcal{D}(\Omega)$, it holds that:

$$\int_\Omega (\partial^\alpha u)\phi \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u(\partial^\alpha \phi)\,\mathrm{d}x. \tag{2.7}$$

Definition 2.2.3 (Distributional derivatives). *Let $T \in \mathcal{D}'(\Omega)$ be a distribution and $\alpha \in \mathbb{N}^d$ be a multi-index. The derivative of T in the sense of distributions is the unique distribution $\partial^\alpha T \in \mathcal{D}'(\Omega)$ defined by the formula*

$$\partial^\alpha T(\phi) = (-1)^{|\alpha|} T(\partial^\alpha \phi), \quad \forall \phi \in \mathcal{D}(\Omega). \tag{2.8}$$

In general, an object of the space $\mathcal{D}'(\Omega)$ need not be a function (see, e.g., Example 2.1.1). Conversely, any *locally integrable* function f can be associated with a distribution $T_f \in$

$\mathcal{D}'(\Omega)$. By locally integrable, we mean f is measurable and defined a.e. on Ω and $f \in L^1(K)$ for every compact set $K \subset \Omega$. We denote the space of locally integrable functions on Ω by $L^1_{\text{loc}}(\Omega)$. Given $f \in L^1_{\text{loc}}(\Omega)$, we define the distribution $T_f \in \mathcal{D}'(\Omega)$ by

$$T_f(\phi) = \int_\Omega f\phi \, \mathrm{d}x, \quad \phi \in \mathcal{D}(\Omega).$$

We will abuse notation by identifying f with T_f and thus $L^1_{\text{loc}}(\Omega)$ with a subspace of $\mathcal{D}'(\Omega)$. Similarly, if a distribution $T \in \mathcal{D}'(\Omega)$ is such that $T = T_f$ for $f \in L^1_{\text{loc}}(\Omega)$, we write $T \in L^1_{\text{loc}}(\Omega)$.

Proposition 2.2.1 ([8, Proposition II.2.43])**.** *Let $1 \leq p < \infty$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(\Omega)$ which converges weakly towards $f \in L^p(\Omega)$. Then it holds that*

$$f_n \to f, \quad \text{in } \mathcal{D}'(\Omega) \text{ as } n \to \infty.$$

Moreover, if $p = \infty$ and $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^\infty(\Omega)$ which converges weakly-$\star$ to $f \in L^\infty(\Omega)$, then we have

$$f_n \to f, \quad \text{in } \mathcal{D}'(\Omega) \text{ as } n \to \infty.$$

2.2.3 Sobolev spaces

If the distributional derivative $\partial^\alpha u$ of a locally integrable function $u \in L^1_{\text{loc}}(\Omega)$ can itself be identified with a locally integrable function $g_\alpha \in L^1_{\text{loc}}(\Omega)$, then we say that g_α is the *weak derivative* of u. More precisely, $\partial^\alpha u = g_\alpha$ in the weak sense if

$$\int_\Omega u\partial^\alpha \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g_\alpha \phi \, \mathrm{d}x, \quad \forall \phi \in \mathcal{D}(\Omega).$$

If u is sufficiently smooth (e.g., $u \in C^{|\alpha|}(\Omega)$), the weak and classical derivatives of u coincide. Among the set of all weakly differentiable functions, we will give special attention to those whose weak derivatives are elements of the Lebesgue spaces $L^p(\Omega)$. These functions furnish the *Sobolev spaces*.

Definition 2.2.4 (Sobolev spaces)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open set, $k \geq 1$ be an integer, and $1 \leq p \leq \infty$. We define the Sobolev space*

$$W^{k,p}(\Omega) = \left\{ v \in L^p(\Omega) \, : \, \partial^\alpha v \in L^p(\Omega), 1 \leq |\alpha| \leq k \right\}.$$

If $p = 2$, we write $W^{k,2}(\Omega) = H^k(\Omega)$.

The space $W^{k,p}(\Omega)$ is a Banach space when equipped with the norms

$$\|v\|_{W^{k,p}(\Omega)} := \left(\sum_{0\le|\alpha|\le k} \|\partial^\alpha v\|^p_{L^p(\Omega)} \right)^{1/p}, \quad \text{if } 1 \le p < \infty,$$
$$\|v\|_{W^{k,\infty}(\Omega)} := \max_{0\le|\alpha|\le k} \|\partial^\alpha v\|_{L^\infty(\Omega)}, \quad \text{if } p = \infty.$$

We define also the semi-norms

$$|v|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha|=k} \|\partial^\alpha v\|^p_{L^p(\Omega)} \right)^{1/p}, \quad \text{if } 1 \le p < \infty,$$
$$|v|_{W^{k,\infty}(\Omega)} := \max_{|\alpha|=k} \|\partial^\alpha v\|_{L^\infty(\Omega)}, \quad \text{if } p = \infty.$$

The space $W^{k,p}(\Omega)$ is reflexive if $1 < p < \infty$, and separable if $1 \le p < \infty$. The space $H^k(\Omega)$ is a Hilbert space. Alternatively, one may view $W^{k,p}(\Omega)$ as the closure of $C^k(\overline{\Omega})$ with respect to the topology induced by the $W^{k,p}(\Omega)$ norm.

For many of the results we list below, additional geometrical assumptions on the open, bounded, and connected set $\Omega \subset \mathbb{R}^d$ are required. Our presentation will closely follow [18, Section 1.18]. For simplicity, we shall view the boundary $\partial\Omega$ of Ω as being locally the graph of a Lipschitz continuous function φ. More precisely, we assume there exists constants $\alpha > 0$ and $L > 0$, a finite number of local orthogonal coordinate systems $(y_1^r, \dots, y_{d-1}^r, y_d^r) = (\boldsymbol{y}_r, y_d^r) \in \mathbb{R}^{d-1} \times \mathbb{R}$, and corresponding functions $\varphi_r : \omega_r := \left\{ \boldsymbol{y}_r \in \mathbb{R}^{d-1} \,:\, |\boldsymbol{y}_r| < \alpha \right\} \to \mathbb{R}$, $1 \le r \le R$ such that

$$\text{(i)} \quad \partial\Omega = \bigcup_{r=1}^{R} \left\{ (\boldsymbol{y}_r, y_r) \,:\, y_r = \varphi_r(\boldsymbol{y}_r);\ |\boldsymbol{y}_r| < \alpha \right\},$$
$$\text{(ii)} \quad |\varphi_r(\boldsymbol{y}_r) - \varphi_r(\boldsymbol{z}_r)| \le L|\boldsymbol{y}_r - \boldsymbol{z}_r|, \quad \forall \boldsymbol{y}_r, \boldsymbol{z}_r \in \omega_r,\ 1 \le r \le R.$$

Moreover, we assume that Ω is *locally on the same side of its boundary*; that is, there exists a constant $\beta > 0$ such that

$$\text{(iii)} \quad \left\{ (\boldsymbol{y}_r, y_r) \,:\, \boldsymbol{y}_r \in \omega_r \text{ and } \varphi_r(\boldsymbol{y}_r) < y_r^d < \varphi_r(\boldsymbol{y}_r) + \beta \right\} \subset \Omega,\ 1 \le r \le R,$$
$$\text{(iv)} \quad \left\{ (\boldsymbol{y}_r, y_r) \,:\, \boldsymbol{y}_r \in \omega_r \text{ and } \varphi_r(\boldsymbol{y}_r) - \beta < y_r^d < \varphi_r(\boldsymbol{y}_r) \right\} \subset \mathbb{R}^d \setminus \overline{\Omega},\ 1 \le r \le R.$$

In the sequel, we refer to an open, connected set satisfying conditions (i)–(iv) as a *Lipschitz domain* (see Figure 2.1 for the case $d = 2$).

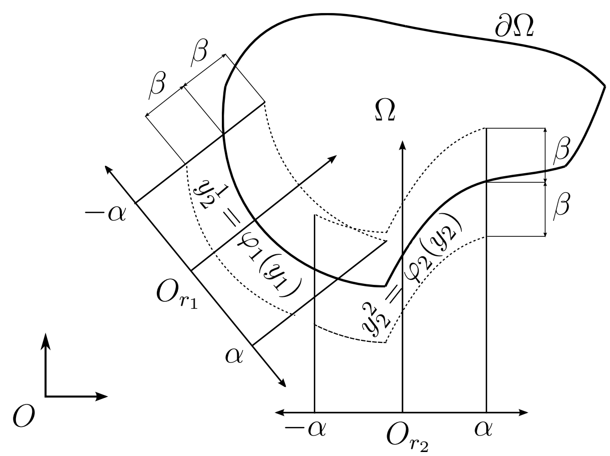

Figure 2.1: A Lipschitz domain in $\mathbb{R}^2$.

Since elements of $L^p(\Omega)$ are equivalence classes of functions agreeing up to a set of (d-dimensional) Lebesgue measure zero, the question arises whether we can assign boundary values to elements of $W^{k,p}(\Omega)$. The problem lies in the fact that the boundary $\partial\Omega$ is a $(d-1)$-dimensional surface and thus $|\partial\Omega|_d = 0$, with $|\cdot|_d$ denoting the d-dimensional Lebesgue measure. Generally, we cannot assign boundary values in a pointwise sense. However, using the Hahn–Banach theorem (Theorem 2.1.4), we can define an operator $\gamma \in \mathcal{L}(W^{1,p}(\Omega); L^{p^\#}(\Omega))$ (with $p^\#$ defined in Theorem 2.2.4 below) such that $\gamma : u \mapsto u|_{\partial\Omega}$ for all $u \in C^1(\overline{\Omega})$. More precisely, we have the following result that allows us to consider traces of functions in $W^{1,p}(\Omega)$ as elements of $L^p(\partial\Omega)$:

Theorem 2.2.4 (Trace theorem [18, Section 6.6])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $1 \le p < \infty$. There exists a trace operator $\gamma \in \mathcal{L}(W^{1,p}(\Omega), L^{p^\#}(\partial\Omega))$, where $1 \le p^\# < \infty$ if $p = d$, and*

$$\frac{1}{p^\#} = \frac{1}{p} - \frac{p-1}{p(d-1)}, \quad \text{if } 1 \le p < d.$$

Moreover, there exists a constant $C > 0$ such that for all $v \in W^{1,p}(\Omega)$,

$$\|\gamma(v)\|_{L^{p\#}(\partial\Omega)} \le C\|v\|_{L^p(\Omega)}^{1-1/p}\|v\|_{W^{1,p}(\Omega)}^{1/p}. \tag{2.9}$$

Where no confusion may arise, we will abuse notation by writing $\gamma(u) = u|_{\partial\Omega}$. Note that the operator γ is not surjective onto the space $L^{p^\#}(\Omega)$ and thus a general element of

$L^{p^\#}(\Omega)$ need not have a lifting in $W^{1,p}(\Omega)$. For this reason, we define the following *trace spaces* which are simply the image of $W^{1,p}(\Omega)$ under γ:

$$W^{1-\frac{1}{p},p}(\partial\Omega) := \left\{\gamma(v) \in L^{p^\#}(\partial\Omega) \,:\, v \in W^{1,p}(\Omega)\right\}, \quad \text{for } 1 \le p < d,$$
$$H^{1/2}(\partial\Omega) := \left\{\gamma(v) \in L^2(\partial\Omega) \,:\, v \in H^1(\Omega)\right\}, \quad \text{if } p = 2.$$

We will also need to consider functions with vanishing trace. In this case, we define

$$W_0^{1,p}(\Omega) := \left\{v \in W^{1,p}(\Omega) \,:\, \gamma(v) = 0\right\}, \quad \text{for } 1 \le p < d,$$
$$H_0^1(\Omega) := \left\{v \in H^1(\Omega) \,:\, \gamma(v) = 0\right\}, \quad \text{if } p = 2.$$

Theorem 2.2.5 (Poincaré–Friedrichs [18, Section 6.6])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $1 \le p < \infty$. There exists a constant $C_1 > 0$ such that*

$$\int_\Omega |v|^p \,\mathrm{d}x \le C_1 \left(|v|^p_{W^{1,p}(\Omega)} + \left| \int_\Omega v \,\mathrm{d}x \right|^p \right), \quad \forall\, v \in W^{1,p}(\Omega). \tag{2.10}$$

Moreover, if $v \in W^{1,p}(\Omega)$ satisfies $\gamma(v) = 0$ on $\partial\Omega$, then there exists a constant $C_2 > 0$ such that

$$\|v\|_{L^p(\Omega)} \le C_2 |v|_{W^{1,p}(\Omega)}. \tag{2.11}$$

Theorem 2.2.6 (Sobolev–Rellich–Kondrachov [8, Theorem III.2.34])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Define the critical Sobolev exponent $p^\star$ associated with p by:*

$$\begin{cases} \frac{1}{p^\star} = \frac{1}{p} - \frac{1}{d}, & \text{for } p < d, \\ 1 \le p^\star < \infty, & \text{for } p = d. \end{cases}$$

For $1 \le p < \infty$ and $1 \le q \le p^\star$ we have the continuous embedding

$$W^{1,p}(\Omega) \subset L^q(\Omega), \tag{2.12}$$

and the embedding is compact for $1 \le q < p^\star$. Furthermore, for $d < p \le \infty$ and $0 \le a \le 1 - d/p$, we have the continuous embedding

$$W^{1,p}(\Omega) \subset C^{0,a}(\Omega), \tag{2.13}$$

and the embedding is compact for $0 \le a < 1 - d/p$.

By eq. (2.13), we mean that given $u \in W^{1,p}(\Omega)$, there exists a Hölder continuous representative belonging to the same equivalence class as u.

Theorem 2.2.7 (Gagliardo–Nirenberg [8, Proposition III.2.35])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $1 \leq p \leq \infty$ and $p \leq q \leq p^\star$. There is a constant $C > 0$ such that*

$$\|v\|_{L^q(\Omega)} \leq C\|v\|_{L^p(\Omega)}^{1+d/q-d/p}\|v\|_{W^{1,p}(\Omega)}^{d/p-d/q}, \quad \forall\, v \in W^{1,p}(\Omega). \tag{2.14}$$

Furthermore, if $\gamma(v) = 0$, we have

$$\|v\|_{L^q(\Omega)} \leq C\|v\|_{L^p(\Omega)}^{1+d/q-d/p}\|\nabla v\|_{L^p(\Omega)}^{d/p-d/q}, \quad \forall\, v \in W_0^{1,p}(\Omega). \tag{2.15}$$

Choosing $d \in \{2, 3\}$, $q = 4$, and $p = 2$ in eq. (2.15), we recover the classic Ladyzhenskaya inequality [37, pp. 55]: for all $v \in H_0^1(\Omega)$,

$$\|v\|_{L^4(\Omega)} \leq \begin{cases} C\|v\|_{L^2(\Omega)}^{1/2}\|\nabla v\|_{L^2(\Omega)}^{1/2}, & \text{if } d = 2, \\ C\|v\|_{L^2(\Omega)}^{1/4}\|\nabla v\|_{L^2(\Omega)}^{3/4}, & \text{if } d = 3. \end{cases} \tag{2.16}$$

2.2.4 The Sobolev space $H(\mathrm{div}; \Omega)$

We begin by defining the weak divergence operator of a function $u \in L^1_{\mathrm{loc}}(\Omega)^d$. If there exists a function $g \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega g\phi \,\mathrm{d}x = -\int_\Omega u \cdot \nabla\phi \,\mathrm{d}x, \quad \forall \phi \in \mathcal{D}(\Omega),$$

then we say that g is the *weak divergence* of u and we write $\nabla \cdot u = g$.

Definition 2.2.5 (The space $H(\mathrm{div}; \Omega)$)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded Lipschitz domain. We define*

$$H(\mathrm{div}; \Omega) := \left\{ u \in L^2(\Omega)^d \,:\, \nabla \cdot u \in L^2(\Omega) \right\}. \tag{2.17}$$

$H(\mathrm{div}; \Omega)$ is a Hilbert space when equipped with the inner product:

$$(u, v)_{H(\mathrm{div};\Omega)} = \int_\Omega u \cdot v \,\mathrm{d}x + \int_\Omega \nabla \cdot u \, \nabla \cdot v \,\mathrm{d}x. \tag{2.18}$$

A normal trace operator for fields in $H(\mathrm{div}; \Omega)$ can be defined using duality:

Lemma 2.2.1. *Let $u \in H(\mathrm{div}; \Omega)$. There exists a surjective normal trace operator $\gamma_n \in \mathcal{L}(H(\mathrm{div}; \Omega); H^{-1/2}(\partial\Omega))$ satisfying the Green's formula*

$$\int_\Omega v \nabla \cdot u \,\mathrm{d}x + \int_\Omega u \cdot \nabla v \,\mathrm{d}x = \langle \gamma_n(u), \gamma(v) \rangle_{H^{-1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}, \quad \forall v \in H^1(\Omega). \tag{2.19}$$

Where no confusion may arise, we will abuse notation by writing $\gamma_n(u) = u \cdot n|_{\partial\Omega}$.

2.2.5 Bochner–Sobolev spaces

Next, we consider vector-valued functions defined on an interval $I \subset \mathbb{R}$ taking values in a Banach space X. We say a function $s : I \to X$ is a *simple function* if its range is a finite set of values $\{v_1, \dots, v_n\}$ where $v_i \in X$ and the sets $A_i = s^{-1}(v_i)$ are Lebesgue measurable. We then say that a function $u : I \to X$ is *Bochner measurable* if there exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ of simple functions such that

$$\lim_{k\to\infty} u_k(t) = u(t) \text{ for a.e. } t \in I.$$

Definition 2.2.6 (The Bochner space $L^p(I;X)$)**.** *Let X be a Banach space and let $I \subset \mathbb{R}$. If $1 \le p < \infty$, we define*

$$L^p(I;X) = \left\{ f : I \to X \,:\, f \text{ is Bochner measurable and } \|f\|_{L^p(I;X)} < \infty \right\},$$

where

$$\|u\|_{L^p(I;X)} := \left(\int_I \|u\|_X^p \, \mathrm{d}t \right)^{1/p}.$$

In the case $p = \infty$, we define

$$L^\infty(I;X) = \left\{ f : I \to X \,:\, f \text{ is Bochner measurable and } \|f\|_{L^\infty(I;X)} < \infty \right\},$$

where

$$\|f\|_{L^\infty(I;X)} := \inf \left\{ M \,:\, \|f(t)\|_X \le M \text{ a.e. on } I \right\}.$$

Theorem 2.2.8 (Dual space of $L^p(I;X)$ [82, Proposition 1.38])**.** *Let X be a Banach space, let $1 \le p < \infty$ and let q be the Hölder conjugate of p. Then, $L^q(I;X') \subset \big(L^p(I,X)\big)'$. Furthermore, if X' is separable, then*

$$\big(L^p(I,X)\big)' \cong L^q(I;X'),$$

with duality pairing

$$\langle f, v \rangle_{L^q(I,X')\times L^p(I,X)} := \int_I \langle f(t), v(t) \rangle_{X'\times X} \, \mathrm{d}t, \quad f \in L^p(I,X), v \in L^q(I,X'). \tag{2.20}$$

Next, we introduce Bochner–Sobolev spaces. Given an interval $I \subset \mathbb{R}$, Banach spaces X and Y with $X \subset Y$, and $1 \le p, q \le \infty$, we say that a function $u \in L^p(I;X)$ has a distributional (time) derivative in $L^q(I;Y)$ if there exists a function $g \in L^q(I;Y)$ such that

$$\int_I u(t)\phi'(t) \, \mathrm{d}t = -\int_I g(t)\phi(t) \, \mathrm{d}t, \quad \forall \phi \in \mathcal{D}(I).$$

If such a function g exists, it is unique and we write $\frac{\mathrm{d}u}{\mathrm{d}t} = g$. Higher order derivatives are defined analogously.

Definition 2.2.7 (Bochner–Sobolev spaces)**.** *Let X and Y be Banach spaces with $X \subset Y$. We define the Bochner–Sobolev space*

$$W^{1,p,q}(I; X, Y) = \left\{ u \in L^p(I; X) \,:\, \tfrac{\mathrm{d}u}{\mathrm{d}t} \in L^q(I; Y) \right\},$$

which is a Banach space when equipped with the norm

$$\|u\|_{W^{1,p,q}(I;X,Y)} := \|u\|_{L^p(I;X)} + \left\| \tfrac{\mathrm{d}u}{\mathrm{d}t} \right\|_{L^q(I;Y)}.$$

In the Hilbertian case when $p = q = 2$ and $X = Y$, we will simply write

$$W^{1,2,2}(I; X, X) := H^1(I; X).$$

Analogously, for higher order derivatives, we define

$$H^k(I; X) = \left\{ u \in L^2(I; X) \,:\, \tfrac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} \in L^2(I; X), 1 \le \alpha \le k \right\},$$

which we equip with the norm

$$\|u\|_{H^k(I;X)} := \left(\sum_{\alpha=0}^{k} \left\| \tfrac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha} \right\|^2_{L^2(I;X)} \right)^{1/2}.$$

Lemma 2.2.2 ([8, Proposition II.5.11])**.** *Let $1 \le p, q \le \infty$, and suppose X and Y are two Banach spaces such that $X \subset Y$ with continuous and dense embedding. Then, $W^{1,p,q}(I; X, Y) \subset C(I; Y)$ with continuous embedding.*

Theorem 2.2.9 (Lions–Magenes [8, Theorem II.5.12])**.** *Let V and H be separable Hilbert spaces, and suppose the triplet (V, H, V') is a Gelfand triple (Section 2.1.4). Let $1 \le p, q \le \infty$ with Hölder conjugates p' and q' and let $u \in W^{1,p,q'}(I; V, V')$ and $v \in W^{1,q,p'}(I; V, V')$. Then, the function $t \mapsto (u(t), v(t))_H$ has a continuous representative on I and for all $t_1, t_2 \in I$, the following integration by parts formula holds:*

$$(u(t_2), v(t_2))_H - (u(t_1), v(t_1))_H = \int_{t_1}^{t_2} \left\langle \tfrac{\mathrm{d}u}{\mathrm{d}t}, v \right\rangle_{V' \times V} \mathrm{d}t + \int_{t_1}^{t_2} \left\langle \tfrac{\mathrm{d}v}{\mathrm{d}t}, u \right\rangle_{V' \times V} \mathrm{d}t.$$

Theorem 2.2.10 (Arzelà–Ascoli [91, Lemma 1])**.** *Let B be Banach space. A set $\mathcal{F}$ of $C(0, T; B)$ is relatively compact if and only if:*

(i) The set $\mathcal{F}(t) = \{f(t) \mid f \in \mathcal{F}\}$ *is relatively compact in* B *for all* $0 < t < T$*, and*

(ii) $\mathcal{F}$ *is uniformly equicontinuous:* $\forall \epsilon > 0$*, there exists a* $\delta > 0$ *such that*

$$\big\|f(t_2) - f(t_1)\big\|_B < \epsilon$$

for all $f \in \mathcal{F}$ *and all* $t_1, t_2 \in (0, T)$ *such that* $|t_2 - t_1| < \delta$*.*

Theorem 2.2.11 (Simon [91])**.** *Let* $1 \le p < \infty$*, let* B *be a Banach space, and let* $\mathcal{F} \subset L^p(0, T; B)$*. The set* $\mathcal{F}$ *is relatively compact in* $L^p(0, T; B)$ *if and only if:*

(i) The set $\left\{\int_{t_1}^{t_2} f(t)\,\mathrm{d}t \mid f \in \mathcal{F}\right\}$ *is relatively compact in* B *for all* $0 < t_1 < t_2 < T$*, and*

(ii) $\|\tau_h f - f\|_{L^p(0,T-h;B)} \to 0$ *as* $h \to 0$ *uniformly for all* $f \in \mathcal{F}$*, where* $\tau_h f = f(t + h)$ *for* $h > 0$*. Equivalently,* $\forall \epsilon > 0$ *there exists a* $\delta > 0$ *such that for all* $f \in \mathcal{F}$ *and for all* $h < \delta$*, we have*

$$\int_0^{T-h} \big\|f(t + h) - f(t)\big\|_B^p \,\mathrm{d}t < \epsilon.$$

Theorem 2.2.12 (Aubin–Lions–Simon [8, Theorem II.5.16])**.** *Let* $B_0 \subset B_1 \subset B_2$ *be three Banach spaces such that* $B_1 \subset B_2$ *with continuous embedding and* $B_0 \subset B_1$ *with compact embedding. Let* p, r *be such that* $1 \le p, r < \infty$*. Then, the embedding of* $W^{1,p,r}(0, T; B_0, B_2)$ *into* $L^p(0, T; B_1)$ *is compact.*

2.3 The incompressible Navier–Stokes equations

As this thesis is concerned with the numerical analysis of the incompressible Navier–Stokes equations, we briefly review some aspects of the basic theory of weak and strong solutions that will be used extensively throughout Chapter 4 and Chapter 5. Consider the transient Navier–Stokes system posed on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$: given a suitably chosen body force f, kinematic viscosity $0 < \nu \le 1$, and initial data u_0, find (u, p) such that

$$\partial_t u - \nu \Delta u + \nabla \cdot (u \otimes u) + \nabla p = f, \qquad \text{in } \Omega \times (0, T], \tag{2.21a}$$
$$\nabla \cdot u = 0, \qquad \text{in } \Omega \times (0, T], \tag{2.21b}$$
$$u = 0, \qquad \text{on } \partial\Omega \times (0, T], \tag{2.21c}$$
$$u(x, 0) = u_0(x), \qquad \text{in } \Omega. \tag{2.21d}$$

We will begin our discussion of the Navier–Stokes system with the theory of weak solutions of Leray–Hopf type [96]. The starting point is the following space of solenoidal smooth vector fields:

$$\mathscr{V} = \left\{ u \in C_0^\infty(\Omega)^d \mid \nabla \cdot u = 0 \right\}.$$

We define two function spaces, H and V, as the closures of $\mathscr{V}$ in the norm topologies of $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively. For an open, bounded Lipschitz set Ω, we have the following characterizations of H and V [96, Theorems I.1.4 and I.1.6]:

$$H = \left\{ u \in L^2(\Omega)^d \mid \nabla \cdot u = 0 \text{ and } u \cdot n = 0 \right\}, \tag{2.22}$$

$$V = \left\{ u \in H_0^1(\Omega)^d \mid \nabla \cdot u = 0 \right\}. \tag{2.23}$$

We note that $V \subset H$ with dense and continuous embedding and thus the triplet (V, H, V') is a Gelfand triple (Section 2.1.4). The natural setting for weak velocity solutions of eq. (2.21) is the class $L^2(0,T;V) \cap L^\infty(0,T;H)$. By testing eq. (2.21a) with test functions from V and integrating by parts in space, we have the following abstract ODE for the velocity field u: for a.e. $t \in (0,T]$,

$$\langle \tfrac{\mathrm{d}u}{\mathrm{d}t}, v\rangle_{V'\times V} + \nu(\nabla u, \nabla v) \; + ((u\cdot\nabla)u, v) = \langle f, v\rangle_{V'\times V}, \quad \forall v \in V, \tag{2.24a}$$

$$u(0) = u_0, \tag{2.24b}$$

in the sense of distributions. We recall the following classical result concerning solutions to eq. (2.24):

Theorem 2.3.1. *[96, Theorems III.3.1 and III.3.2] Given $f \in L^2(0,T;V')$ and $u_0 \in H$, there exists at least one function $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ satisfying the weak formulation eq. (2.24). Moreover, u is weakly continuous from $[0,T]$ in the sense that for all $v \in H$, $t \to (u(t), v)$ is a continuous function. If $d = 2$, it is well known that this solution is unique and furthermore $u \in C(0,T;H)$. Uniqueness in three dimensions remains an open problem.*

Remark 2.3.1 (The energy inequality)**.** *In two dimensions, the weak solution to the Navier–Stokes equations satisfies the following* energy equality*: for all $s \in (0,T)$,*

$$\left\|u(s)\right\|_{L^2(\Omega)}^2 + 2\nu \int_0^s \|u\|_V^2 \,\mathrm{d}t \; = \|u_0\|_{L^2(\Omega)}^2 + 2\int_0^s \langle f, u\rangle_{H^{-1}\times H_0^1} \,\mathrm{d}t. \tag{2.25}$$

In three dimensions, we say that a weak solution is of Leray–Hopf *type if it satisfies the* energy inequality*: for a.e. $s \in (0,T)$,*

$$\left\|u(s)\right\|_{L^2(\Omega)}^2 + 2\nu \int_0^s \|u\|_V^2 \,\mathrm{d}t \; \le \|u_0\|_{L^2(\Omega)}^2 + 2\int_0^s \langle f, u\rangle_{H^{-1}\times H_0^1} \,\mathrm{d}t. \tag{2.26}$$

Remark 2.3.2 (On an equivalent formulation)**.** *It is possible to define an equivalent formulation to eq.* (2.24) *using time-dependent test functions* $\varphi \in C_c(0,T;V)$ *(see e.g. [8, Section V.1.2.2]): find* $u \in L^2(0,T;V)$ *such that* $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^1(0,T;V')$ *satisfying for all* $\varphi \in C_c(0,T;V)$*,*

$$\int_0^T \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \varphi \right\rangle_{V'\times V} \mathrm{d}t + \int_0^T ((u\cdot\nabla)u, \varphi)\,\mathrm{d}t + \nu \int_0^T (\nabla u, \nabla \varphi)\,\mathrm{d}t = \int_0^T \langle f, \varphi\rangle_{V'\times V}\,\mathrm{d}t. \tag{2.27}$$

Remark 2.3.3 (On recovering the pressure solution)**.** *In general, a pressure field cannot be associated to the weak velocity solution of the Navier–Stokes equations if* $f \in L^2(0,T;V')$*; see [90]. However, if instead* $f \in L^2(0,T;H^{-1}(\Omega)^d)$*, there is an associated pressure field* $p \in W^{-1,\infty}(0,T;L^2_0(\Omega))$ *satisfying the Navier–Stokes equations in the distributional sense (see e.g. [90, Proposition 5] or [8, Theorem V.1.4]).*

Remark 2.3.4 (On the regularity of weak solutions and consistency)**.** *For the space-time HDG scheme studied in this thesis, we will require a stronger notion of a solution to the Navier–Stokes problem. Indeed, the discrete space we will define for the approximate velocity field is non-conforming in* V*. Consequently, we cannot consider at the discrete level the duality pairing* $\langle f, v_h\rangle_{V'\times V}$ *without modifying the test function* v_h *with an appropriate smoothing operator, which would introduce a consistency error; see e.g. [4]. Moreover, it will become evident in Chapter 4 we require at least* $(u,p) \in H^1(0,T;L^2(\Omega)^d) \cap L^2(0,T;H^{\frac{3}{2}+\epsilon}(\Omega)^d) \times L^2(0,T;H^1(\Omega)) \cap L^2(0,T;L^2_0(\Omega))$*,* $\epsilon > 0$*, for the consistency of our numerical scheme. The question of whether the proposed numerical scheme converges under the minimal regularity assumptions given in Theorem 2.3.1 is the subject of Chapter 5.*

We next seek conditions on the existence of a stronger solution to the Navier–Stokes system. We now assume that (at least) $f \in L^2(0,T;H)$ and $u_0 \in V$. In two dimensions, if Ω is C^2 (or convex), this is enough to ensure the existence of a unique solution $(u,p) \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)^d \cap V) \times L^2(0,T;H^1(\Omega)) \cap L^2(0,T;L^2_0(\Omega))$, and furthermore, $\partial_t u \in L^2(0,T;H)$ [96, Theorem III.3.10].

However, in three dimensions the situation is more complicated. It is possible to prove the existence of a unique strong solution to the Navier–Stokes system with an important caveat: loss of globality of time. In other words, the solution can be shown to exist on some time interval $(0,T^*]$ with T^* depending on the problem data. There are two possible cases:

1. Short lifetime and large data.

2. Long lifetime and small data.

In Chapter 4, we will focus on the latter case which we summarize in the following theorem:

Theorem 2.3.2 ([24, Theorem 9.3], [17, Theorem 5.4])**.** *Let $\Omega \subset \mathbb{R}^3$ be a convex polyhedral domain. There exists a $C > 0$, dependent on the final time T, such that if $u_0 \in V$ and $f \in L^2(0,T;H)$ satisfy*

$$\|u_0\|_V^2 + \frac{1}{\nu}\|f\|^2_{L^2(0,T;L^2(\Omega))} \leq C\nu^2, \tag{2.28}$$

then there exists a unique strong solution *with $(u,p) \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)^3 \cap V) \times L^2(0,T;H^1(\Omega)) \cap L^2(0,T;L_0^2(\Omega))$ and $\partial_t u \in L^2(0,T;H)$ such that*

$$\|u\|^2_{L^\infty(0,T;V)} + \nu\|u\|^2_{L^2(0,T;H^2(\Omega))} \leq C\nu^2, \quad \|\partial_t u\|^2_{L^2(0,T;L^2(\Omega))} \leq C\nu^3. \tag{2.29}$$

We note that $u \in L^2(0,T;H^2(\Omega)^d \cap V) \subset L^2(0,T;H)$ and $\partial_t u \in L^2(0,T;H)$ ensures that $u \in H^1(0,T;H)$. The assumption on the problem data eq. (2.28) can be interpreted as small initial data and body force, or large viscosity and arbitrary data.

Remark 2.3.5. *Recall that the Stokes operator $A : H^2(\Omega)^d \cap V \to H$ is defined as the Helmholtz projection of the vector Laplace operator; see e.g. [24, Chapter 4]. Inspecting the proof of [24, Theorem 9.3], the assumption on the smoothness of the domain in Theorem 2.3.2 is required to ensure that $\|Au\|_{L^2(\Omega)}$ is a norm on $H^2(\Omega)^d \cap V$ equivalent to the $H^2(\Omega)^d$ norm, which is in turn a consequence of the regularity theory of the linear, stationary Stokes problem. Therefore, the conclusion of Theorem 2.3.2 remains valid if $\Omega \subset \mathbb{R}^3$ is a convex polyhedron [26].*

Chapter 3

Pressure-robust HDG for the steady problem

In [78], a simple class of HDG methods that produce a pointwise solenoidal discrete velocity field belonging to $H(\text{div};\Omega)$ is introduced. It was shown therein that the method is exactly mass conserving, locally momentum conserving, and energy stable. Moreover, through a series of numerical experiments, it was observed that the velocity error is independent of the pressure (hence, pressure-robust). However, no accompanying error analysis was provided to theoretically confirm that the method is pressure-robust. As the HDG method in [78] forms the basis of the space-time HDG method considered in Chapter 4 and Chapter 5, the purpose of this chapter is to fill this gap.

We begin by showing that the nonlinear algebraic system of equations arising from the HDG discretization is well-posed under a restriction on the size of the problem data using a fixed point argument. Then, we derive optimal error estimates in the velocity which are independent of the pressure in a discrete analogue of the H^1-norm typical in the analysis of HDG methods, followed by optimal error estimates in the pressure in the L^2-norm. This confirms that the method is pressure-robust. Lastly, under the assumption that the domain Ω is convex, we derive optimal error estimates for the velocity in the $L^2(\Omega)$-norm.

This chapter is organized as follows: we present the steady Navier–Stokes problem in Section 3.1 and the HDG method is introduced in Section 3.2. Notation and properties of the multilinear forms involved are discussed in Section 3.3. Existence and uniqueness of the discrete solution are shown in Section 3.4. We derive optimal pressure-robust error estimates for the velocity in a mesh dependent energy norm, the pressure in the L^2-norm and optimal L^2-error estimates for the velocity in Section 3.5. Finally, numerical examples are presented in Section 3.6 to confirm the theory.

This chapter is reprinted, with slight modification, from the following article:

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3.1 The steady Navier–Stokes equations

Let $\Omega \subset \mathbb{R}^d$ be a polygonal ($d = 2$) or polyhedral ($d = 3$) domain with boundary Γ. We consider the Navier–Stokes equations: given a body force $f : \Omega \to \mathbb{R}^d$ and kinematic viscosity $\nu \in \mathbb{R}^+$, find the velocity $u : \Omega \to \mathbb{R}^d$ and pressure $p : \Omega \to \mathbb{R}$ such that

$$-\nu\nabla^2 u + \nabla\cdot(u\otimes u) + \nabla p = f \qquad \text{in } \Omega, \tag{3.1a}$$

$$\nabla\cdot u = 0 \qquad \text{in } \Omega, \tag{3.1b}$$

$$u = 0 \qquad \text{on } \Gamma. \tag{3.1c}$$

It is well known, e.g., [96], that given a body force $f \in \left[L^2(\Omega)\right]^d$, the variational formulation of the Navier–Stokes problem eq. (3.1): find $(u, p) \in \left[H_0^1(\Omega)\right]^d \times L_0^2(\Omega)$ such that

$$\int_\Omega \nu\nabla u : \nabla v \,\mathrm{d}x + \int_\Omega (u\cdot\nabla u)\cdot v \,\mathrm{d}x - \int_\Omega p\nabla\cdot v \,\mathrm{d}x = \int_\Omega f\cdot v \,\mathrm{d}x \quad \forall v \in \left[H_0^1(\Omega)\right]^d \tag{3.2a}$$

$$\int_\Omega q\nabla\cdot u \,\mathrm{d}x = 0 \qquad \forall q \in L_0^2(\Omega), \tag{3.2b}$$

admits a unique solution provided

$$\|f\|_{L^2(\Omega)} \le \nu^2 (C_o C_p)^{-1}, \tag{3.3}$$

where C_p is the Poincaré constant (Theorem 2.2.5) and C_o is a constant depending only on Ω and d. In addition, the velocity satisfies the stability estimate

$$\|u\|_{H^1(\Omega)} \le C_p \nu^{-1} \|f\|_{L^2(\Omega)}, \tag{3.4}$$

3.2 The HDG method

Let $\mathcal{T} = \{K\}$ denote the triangulation of the domain Ω into simplices K. Furthermore, let $\mathcal{F}$ and Γ^0 denote, respectively, the set and union of all edges of $\mathcal{T}$. We denote the characteristic length of a cell K by h_K and we denote the outward unit normal vector on the boundary of a cell, ∂K, by n. We introduce discontinuous finite element approximation spaces for the velocity and pressure:

$$V_h := \left\{ v_h \in \left[L^2(\Omega)\right]^d,\ v_h \in \left[P_k(K)\right]^d\ \forall K \in \mathcal{T} \right\}, \tag{3.5a}$$

$$Q_h := \left\{ q_h \in L^2(\Omega),\ q_h \in P_{k-1}(K)\ \forall K \in \mathcal{T} \right\}. \tag{3.5b}$$

In addition, we introduce also discontinuous finite element approximation spaces for the approximate traces of the velocity and pressure:

$$\bar{V}_h := \left\{ \bar{v}_h \in \left[L^2(\mathcal{F})\right]^d,\ \bar{v}_h \in \left[P_k(F)\right]^d\ \forall F \in \mathcal{F},\ \bar{v}_h = 0 \text{ on } \Gamma \right\}, \tag{3.6a}$$

$$\bar{Q}_h := \left\{ \bar{q}_h \in L^2(\mathcal{F}),\ \bar{q}_h \in P_k(F)\ \forall F \in \mathcal{F} \right\}, \tag{3.6b}$$

For notational convenience, we denote function pairs in $\boldsymbol{V}_h$ and $\boldsymbol{Q}_h$ by boldface, e.g., $\boldsymbol{v}_h = (v_h, \bar{v}_h) \in \boldsymbol{V}_h$ and $\boldsymbol{q}_h = (q_h, \bar{q}_h) \in \boldsymbol{Q}_h$.

The HDG formulation for the Navier–Stokes problem eq. (3.1) is given by [78]: given $f \in \left[L^2(\Omega)\right]^d$, find $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in \boldsymbol{X}_h$ such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h, v_h) = \sum_{K \in \mathcal{T}} \int_K f \cdot v_h \,\mathrm{d}x \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.7a}$$

$$b_h(\boldsymbol{q}_h, u_h) = 0 \qquad \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h. \tag{3.7b}$$

The discrete forms $a_h(\cdot,\cdot) : \boldsymbol{V}_h \times \boldsymbol{V}_h \to \mathbb{R}$, $b_h(\cdot,\cdot) : \boldsymbol{Q}_h \times V_h \to \mathbb{R}$, and $o_h(\cdot;\cdot,\cdot) : V_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h \to \mathbb{R}$ appearing in eq. (3.7) serve as approximations to the viscous, pressure-velocity coupling, and convection terms, respectively. We define them as in [78]:

$$
\begin{aligned}
a_h(\boldsymbol{u}, \boldsymbol{v}) &:= \sum_{K \in \mathcal{T}_h} \int_K \nabla u : \nabla v \,\mathrm{d}x + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\alpha}{h_K} (u - \bar{u}) \cdot (v - \bar{v}) \,\mathrm{d}s && \text{(3.8a)} \\
&\quad - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left[(u - \bar{u}) \cdot \partial_n v + \partial_n u \cdot (v - \bar{v}) \right] \mathrm{d}s, \\
o_h(w; \boldsymbol{u}, \boldsymbol{v}) &:= - \sum_{K \in \mathcal{T}_h} \int_K u \otimes w : \nabla v \,\mathrm{d}x + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tfrac{1}{2} w \cdot n \, (u + \bar{u}) \cdot (v - \bar{v}) \,\mathrm{d}s && \text{(3.8b)} \\
&\quad + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \tfrac{1}{2} |w \cdot n| \, (u - \bar{u}) \cdot (v - \bar{v}) \,\mathrm{d}s, \\
b_h(\boldsymbol{p}, v) &:= - \sum_{K \in \mathcal{T}_h} \int_K p \nabla \cdot v \,\mathrm{d}x + \sum_{K \in \mathcal{T}_h} \int_{\partial K} v \cdot n \, \bar{p} \,\mathrm{d}s. && \text{(3.8c)}
\end{aligned}
$$

The parameter $\alpha > 0$ appearing in the bilinear form $a_h(\cdot, \cdot)$ is a penalty parameter typical of interior penalty type discretizations, which must be chosen sufficiently large to ensure stability [77]. It was shown in [77] for the Stokes problem and [78] for the Navier–Stokes problem that the approximate velocity $u_h \in V_h$ obtained from the hybridized discontinuous Galerkin discretization eq. (3.7) possesses two appealing properties, namely, $\nabla \cdot u_h = 0$ pointwise and $u_h \in H(\mathrm{div}; \Omega)$. These properties are key to proving a pressure-robust error estimate for the velocity field in Section 3.5.

3.3 Preliminaries

In this section we present some stability and boundedness results of the hybridized discontinuous Galerkin method eq. (3.7) and some other preliminaries. To set notation, let

$$
\begin{aligned}
V(h) &:= V_h + \left[H_0^1(\Omega)\right]^d \cap \left[H^2(\Omega)\right]^d, \qquad & Q(h) &:= Q_h + L_0^2(\Omega) \cap H^1(\Omega), && \text{(3.9a)} \\
\bar{V}(h) &:= \bar{V}_h + \left[H_0^{3/2}(\Gamma^0)\right]^d, & \bar{Q}(h) &:= \bar{Q}_h + H_0^{1/2}(\Gamma^0), && \text{(3.9b)}
\end{aligned}
$$

and $\boldsymbol{V}(h) := V(h) \times \bar{V}(h)$, $\boldsymbol{Q}(h) := Q(h) \times \bar{Q}(h)$ and $\boldsymbol{X}(h) := \boldsymbol{V}(h) \times \boldsymbol{Q}(h)$. Frequent use will also be made of functions in the following space:

$$
V_h^{\mathrm{div}} := \left\{ v_h \in V_h : \; b_h(\boldsymbol{q}_h, v_h) = 0 \;\forall \boldsymbol{q}_h \in \boldsymbol{Q}_h \right\}. \tag{3.10}
$$

We denote the trace operator by $\gamma : H^k(\Omega) \to H^{k-1/2}(\Gamma^0)$ to restrict functions in $H^s(\Omega)$ to Γ^0. The trace operator is applied component-wise for functions in $\left[H^s(\Omega)\right]^d$. Given D an open subset of $\mathbb{R}^d$ we denote for scalar-valued functions $p, q \in L^2(D)$ the standard inner-product by $(p,q)_D := \int_D pq \,\mathrm{d}x$ and its corresponding norm $\|p\|_D := \sqrt{(p,p)_D}$. Furthermore, we define $(p,q)_{\mathcal{T}} := \sum_{K\in\mathcal{T}}(p,q)_K$ and denote the usual L^2-norm on Ω by $\|p\| := \sqrt{(p,p)_{\mathcal{T}}}$. For scalar-valued functions $p, q \in L^2(F)$, where $F \subset \mathbb{R}^{d-1}$, we define the inner-product $\langle p, q\rangle_F := \int_F pq \,\mathrm{d}s$ with norm $\|p\|_F = \sqrt{\langle p,p\rangle_F}$. Similar definitions hold for vector-valued functions.

We introduce the following mesh-dependent inner-product and norms:

$$(\boldsymbol{u},\boldsymbol{v})_v := (\nabla u, \nabla v)_{\mathcal{T}} + \sum_{K\in\mathcal{T}} \alpha h_K^{-1} \langle \bar{u} - u, \bar{v} - v\rangle_{\partial K} \qquad \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}(h), \tag{3.11a}$$

$$|||\boldsymbol{v}|||_v^2 := \sum_{K\in\mathcal{T}} \|\nabla v\|_K^2 + \sum_{K\in\mathcal{T}} \alpha h_K^{-1} \|\bar{v} - v\|_{\partial K}^2 \qquad \boldsymbol{v} \in \boldsymbol{V}(h), \tag{3.11b}$$

$$|||\boldsymbol{v}|||_{v'}^2 := |||\boldsymbol{v}|||_v^2 + \sum_{K\in\mathcal{T}} \frac{h_K}{\alpha} \left\| \frac{\partial v}{\partial n} \right\|_{\partial K}^2 \qquad \boldsymbol{v} \in \boldsymbol{V}(h), \tag{3.11c}$$

$$|||\boldsymbol{q}|||_p^2 := \|q\|^2 + \sum_{K\in\mathcal{T}} h_K \|\bar{q}\|_{\partial K}^2 \qquad \boldsymbol{q} \in \boldsymbol{Q}(h), \tag{3.11d}$$

where we note that $|||\cdot|||_v$ and $|||\cdot|||_{v'}$ are equivalent on $\boldsymbol{V}_h$, see [77]. We define also

$$|||(\boldsymbol{v}_h, \boldsymbol{q}_h)|||_{v,p}^2 := \nu |||\boldsymbol{v}_h|||_v^2 + \nu^{-1} |||\boldsymbol{q}_h|||_p^2 \qquad (\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h, \tag{3.12a}$$

$$\begin{aligned} |||(\boldsymbol{v}, \boldsymbol{q})|||_{v',p'}^2 &:= |||(\boldsymbol{v}, \boldsymbol{q})|||_{v,p}^2 + \sum_{K\in\mathcal{T}} \frac{\nu h_K}{\alpha} \left\| \frac{\partial v}{\partial n} \right\|_{\partial K}^2 \\ &= \nu |||\boldsymbol{v}|||_{v'}^2 + \nu^{-1} |||\boldsymbol{q}|||_p^2 \qquad (\boldsymbol{v}, \boldsymbol{q}) \in \boldsymbol{X}(h). \end{aligned} \tag{3.12b}$$

The standard discrete H^1-norm for $v \in V(h)$ is defined as $\|v\|_{1,h} := |||(v_h, \{\!\{v_h\}\!\})|||_v$, where $\{\!\{v\}\!\} := \frac{1}{2}(v^+ + v^-)$ is the average operator and $v^\pm$ denote the trace of v from the interior of $K^\pm$. Furthermore, use will be made of the following discrete Poincaré inequality:

$$\|v_h\| \le c_p \|v_h\|_{1,h} \le c_p |||\boldsymbol{v}_h|||_v \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.13}$$

where c_p is a constant independent of h_K [28, Theorem 5.3].

Previously it was shown [77, Lemmas 4.2 and 4.3] that for sufficiently large α, the bilinear form $a_h(\cdot,\cdot)$ is coercive and bounded, i.e., there exist constants $c_a^s > 0$ and $c_a^b > 0$, independent of h, such that for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(h)$

$$a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \ge \nu c_a^s |||\boldsymbol{v}_h|||_v^2 \quad \text{and} \quad \left|a_h(\boldsymbol{u}, \boldsymbol{v})\right| \le \nu c_a^b |||\boldsymbol{u}|||_{v'} |||\boldsymbol{v}|||_{v'}. \tag{3.14}$$

The boundedness of $b_h(\cdot,\cdot)$ was shown in the proof of [77, Lemma 4.8], i.e., there exists a constant $c_b^b > 0$, independent of h, such that for all $\boldsymbol{v} \in \boldsymbol{V}(h)$ and $\boldsymbol{q} \in \boldsymbol{Q}(h)$,

$$\left|b_h(\boldsymbol{q}, v)\right| \leq c_b^b \|\!|\boldsymbol{v}|\!\|_v \|\!|\boldsymbol{q}|\!\|_p, \tag{3.15}$$

while the stability of $b_h(\cdot,\cdot)$ was proven in [79, Lemma 1]: there exists a constant $\beta_p > 0$, independent of h, such that for all $\boldsymbol{q}_h \in \boldsymbol{Q}_h$,

$$\beta_p \|\!|\boldsymbol{q}_h|\!\|_p \leq \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b_h(\boldsymbol{q}_h, v_h)}{\|\!|\boldsymbol{v}_h|\!\|_v}. \tag{3.16}$$

Discrete inf-sup stability follows from coercivity of $a_h(\cdot,\cdot)$ eq. (3.14) and the stability of $b_h(\cdot,\cdot)$ eq. (3.16), e.g. [28, Lemma 6.13]: there exists a constant $\sigma > 0$, independent of h and ν, such that for all $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h$

$$\sigma \left|\!\left|\!\left|(\boldsymbol{v}_h, \boldsymbol{q}_h)\right|\!\right|\!\right|_{v,p} \leq \sup_{(\boldsymbol{w}_h, \boldsymbol{r}_h) \in \boldsymbol{X}_h} \frac{a_h(\boldsymbol{v}_h, \boldsymbol{w}_h) + b_h(\boldsymbol{q}_h, w_h) - b_h(\boldsymbol{r}_h, v_h)}{\left|\!\left|\!\left|(\boldsymbol{w}_h, \boldsymbol{r}_h)\right|\!\right|\!\right|_{v,p}}. \tag{3.17}$$

For the form $o_h(\cdot;\cdot,\cdot)$ it was shown [14, Proposition 3.6] that for $w_h \in V_h^{\text{div}}$

$$o_h(w_h; \boldsymbol{v}_h, \boldsymbol{v}_h) = \frac{1}{2} \sum_{K \in \mathcal{T}} \int_{\partial K} |w_h \cdot n| |v_h - \bar{v}_h|^2 \,\mathrm{d}s \geq 0 \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.18}$$

It was also shown [14, Proposition 3.4] that for $w_1, w_2 \in V(h)$, $\boldsymbol{u} \in \boldsymbol{V}(h)$ and $\boldsymbol{v} \in \boldsymbol{V}(h)$ that

$$\left|o_h(w_1; \boldsymbol{u}, \boldsymbol{v}) - o_h(w_2; \boldsymbol{u}, \boldsymbol{v})\right| \leq c_o \|w_1 - w_2\|_{1,h} \|\!|\boldsymbol{u}|\!\|_v \|\!|\boldsymbol{v}|\!\|_v. \tag{3.19}$$

Finally, we note that if $(u, p) \in \left(\left[H_0^1(\Omega)\right]^d \cap \left[H^2(\Omega)\right]^d\right) \times \left(L_0^2(\Omega) \cap H^1(\Omega)\right)$, letting $\boldsymbol{u} = (u, \gamma(u))$ and $\boldsymbol{p} = (p, \gamma(p))$, then

$$a_h(\boldsymbol{u}, \boldsymbol{v}_h) + o_h(u; \boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{p}, v_h) + b_h(\boldsymbol{q}_h, u) = \int_\Omega f \cdot v_h \,\mathrm{d}x \quad \forall (\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h. \tag{3.20}$$

This consistency result follows immediately from [77, Lemma 4.1] and noting that, after integration by parts, using that u and $\bar{v}_h$ are single-valued on cell boundaries, and that $\bar{v}_h = 0$ on Γ,

$$o_h(u; \boldsymbol{u}, \boldsymbol{v}_h) = \sum_{K \in \mathcal{T}} \int_K \nabla \cdot (u \otimes u) \cdot v_h \,\mathrm{d}x. \tag{3.21}$$

3.4 Existence and uniqueness

The hybridized discontinuous Galerkin method for the Navier–Stokes problem eq. (3.7) results in a system of nonlinear algebraic equations. To show existence and uniqueness of this nonlinear system, we use the classic Brouwer's fixed point theorem Lemma 2.1.1.

Lemma 3.4.1 (Existence and uniqueness)**.** *Assuming*

$$\|f\|_{L^2(\Omega)} < \frac{(\nu c_a^s)^2}{c_o c_p}, \tag{3.22}$$

where c_p is the constant from eq. (3.13), c_a^s is the constant from eq. (3.14), and c_o is the constant from eq. (3.19), there exists a unique solution $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in \boldsymbol{X}_h$ to the hybridizable discontinuous Galerkin method for the Navier–Stokes problem eq. (3.7). Furthermore,

$$|\!|\!| \boldsymbol{u}_h |\!|\!|_v \le c_p (c_a^s \nu)^{-1} \|f\|_{L^2(\Omega)} \quad \textit{and} \quad \sigma |\!|\!| (\boldsymbol{u}_h, \boldsymbol{p}_h) |\!|\!|_{v,p} \le c_p \|f\|_{L^2(\Omega)} + \frac{c_o c_p^2}{(c_a^s \nu)^2} \|f\|^2_{L^2(\Omega)}, \tag{3.23}$$

where σ is the discrete inf-sup constant eq. (3.17).

Proof. We prove first existence of a solution $\boldsymbol{u}_h \in V_h^{\text{div}} \times \bar{V}_h$ to eq. (3.7). We start by defining a mapping $\Psi : V_h^{\text{div}} \times \bar{V}_h \to V_h^{\text{div}} \times \bar{V}_h$ by

$$\forall \boldsymbol{w}_h, \boldsymbol{v}_h \in V_h^{\text{div}} \times \bar{V}_h, \quad \big(\Psi(\boldsymbol{w}_h), \boldsymbol{v}_h\big)_v = a_h(\boldsymbol{w}_h, \boldsymbol{v}_h) + o_h(w_h; \boldsymbol{w}_h, \boldsymbol{v}_h) - (f, v_h)_{\mathcal{T}}. \tag{3.24}$$

Taking $\boldsymbol{v}_h = \boldsymbol{w}_h$ in eq. (3.24) we find by coercivity of $a_h(\cdot,\cdot)$ eq. (3.14), positivity of $o_h(\cdot;\cdot,\cdot)$ eq. (3.18), Cauchy–Schwarz and eq. (3.13),

$$\big(\Psi(\boldsymbol{w}_h), \boldsymbol{w}_h\big)_v \ge \Big(\nu c_a^s |\!|\!| \boldsymbol{w}_h |\!|\!|_v - c_p \|f\|_{L^2(\Omega)}\Big) |\!|\!| \boldsymbol{w}_h |\!|\!|_v. \tag{3.25}$$

For all $\boldsymbol{w}_h \in V_h^{\text{div}} \times \bar{V}_h$ that satisfy $|\!|\!| \boldsymbol{w}_h |\!|\!|_v = c_p (c_a^s \nu)^{-1} \|f\|_{L^2(\Omega)}$ we therefore find that $(\Psi(\boldsymbol{w}_h), \boldsymbol{w}_h)_v \ge 0$. A corollary to Brouwer's fixed point theorem (Lemma 2.1.1) implies the existence of $\boldsymbol{u}_h \in \mathcal{B}_h := \{\boldsymbol{v}_h \in V_h^{\text{div}} \times \bar{V}_h \ : \ |\!|\!| \boldsymbol{v}_h |\!|\!|_v \le c_p (c_a^s \nu)^{-1} \|f\|_{L^2(\Omega)}\}$ such that $\Psi(\boldsymbol{u}_h) = 0$. Equivalently, there exists $\boldsymbol{u}_h \in V_h^{\text{div}} \times \bar{V}_h$ satisfying the first estimate in eq. (3.23) and

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h) = (f, v_h)_{\mathcal{T}} \quad \forall \boldsymbol{v}_h \in V_h^{\text{div}} \times \bar{V}_h, \tag{3.26}$$

proving the existence of a solution $\boldsymbol{u}_h$ to eq. (3.7) restricted to $V_h^{\text{div}} \times \bar{V}_h$.

Next we prove uniqueness of $\boldsymbol{u}_h \in V_h^{\text{div}} \times \bar{V}_h$ to eq. (3.7). For this, assume two solutions $\boldsymbol{u}_{h,1} \in V_h^{\text{div}} \times \bar{V}_h$ and $\boldsymbol{u}_{h,2} \in V_h^{\text{div}} \times \bar{V}_h$ that both solve eq. (3.7). We will show that $\boldsymbol{u}_{h,1} = \boldsymbol{u}_{h,2}$ under the smallness assumption eq. (3.22). We first note that coercivity of $a_h(\cdot,\cdot)$ eq. (3.14) implies

$$\nu c_a^s \left\| \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2} \right\|_v^2 \le a_h(\boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}, \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}). \tag{3.27}$$

Furthermore, note that for any $\boldsymbol{v}_h \in V_h^{\text{div}} \times \bar{V}_h$,

$$a_h(\boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}, \boldsymbol{v}_h) + o_h(u_{h,1}; \boldsymbol{u}_{h,1}, \boldsymbol{v}_h) - o_h(u_{h,2}; \boldsymbol{u}_{h,2}, \boldsymbol{v}_h) = 0. \tag{3.28}$$

Combining eq. (3.27) and eq. (3.28),

$$\begin{aligned} \nu c_a^s \left\| \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2} \right\|_v^2 \le & o_h(u_{h,2}; \boldsymbol{u}_{h,1}, \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}) - o_h(u_{h,1}; \boldsymbol{u}_{h,1}, \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}) \\ & - o_h(u_{h,2}; \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}, \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}) \\ \le & o_h(u_{h,2} - u_{h,1}; \boldsymbol{u}_{h,1}, \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}), \end{aligned} \tag{3.29}$$

since $o_h(u_{h,2}; \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}, \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2}) \ge 0$ by eq. (3.18). Next, by eq. (3.19) and eq. (3.23)

$$\begin{aligned} \nu c_a^s \left\| \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2} \right\|_v^2 &\le c_o \left\| u_{h,2} - u_{h,1} \right\|_{1,h} \left\| \boldsymbol{u}_{h,1} \right\|_v \left\| \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2} \right\|_v \\ &\le c_o \left\| \boldsymbol{u}_{h,1} \right\|_v \left\| \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2} \right\|_v^2 \\ &\le c_o c_p (c_a^s \nu)^{-1} \|f\|_{L^2(\Omega)} \left\| \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2} \right\|_v^2, \end{aligned} \tag{3.30}$$

implying

$$\left((\nu c_a^s)^2 - c_o c_p \|f\|_{L^2(\Omega)} \right) \left\| \boldsymbol{u}_{h,1} - \boldsymbol{u}_{h,2} \right\|_v^2 \le 0. \tag{3.31}$$

By eq. (3.22) it follows that $\boldsymbol{u}_{h,1} = \boldsymbol{u}_{h,2}$, proving uniqueness of $\boldsymbol{u}_h \in V_h^{\text{div}} \times \bar{V}_h$.

We next prove the existence and uniqueness of $\boldsymbol{p}_h$. Given the solution $\boldsymbol{u}_h \in V_h^{\text{div}} \times \bar{V}_h$, the pressure $\boldsymbol{p}_h \in \boldsymbol{Q}_h$ is the solution to

$$b_h(\boldsymbol{p}_h, v_h) = (f, v_h)_{\mathcal{T}} - a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{3.32}$$

Since $a_h(\boldsymbol{u}_h, \cdot)$ and $o_h(u_h; \boldsymbol{u}_h, \cdot)$ are bounded linear functionals on $\boldsymbol{V}_h$ by, respectively eq. (3.14) and eq. (3.19), the right-hand side itself is a bounded linear functional on $\boldsymbol{V}_h$. Existence of a unique solution $\boldsymbol{p}_h \in \boldsymbol{Q}_h$ to eq. (3.32) is now guaranteed by the inf-sup condition eq. (3.16) thanks to Theorem 2.1.3.

Lastly, we prove the second estimate of eq. (3.23). By eq. (3.7) $a_h(\boldsymbol{u}_h, \boldsymbol{w}_h) + b_h(\boldsymbol{p}_h, w_h) - b_h(\boldsymbol{r}_h, u_h) = (f, w_h)_{\mathcal{T}} - o_h(u_h; \boldsymbol{u}_h, \boldsymbol{w}_h)$. Discrete inf-sup stability eq. (3.17), and boundedness of $o_h(\cdot;\cdot,\cdot)$ eq. (3.19) therefore result in

$$\sigma \left|\!\left|\!\left| (\boldsymbol{u}_h, \boldsymbol{p}_h) \right|\!\right|\!\right|_{v,p} \le \sup_{(\boldsymbol{w}_h, \boldsymbol{r}_h) \in \boldsymbol{X}_h} \frac{(f, w_h)_{\mathcal{T}} - o_h(u_h; \boldsymbol{u}_h, \boldsymbol{w}_h)}{\left|\!\left|\!\left| (\boldsymbol{w}_h, \boldsymbol{r}_h) \right|\!\right|\!\right|_{v,p}} \le c_p \|f\|_{L^2(\Omega)} + c_o \left|\!\left|\!\left| \boldsymbol{u}_h \right|\!\right|\!\right|_v^2. \tag{3.33}$$

The result follows from the first estimate in eq. (3.23). □

3.5 Error analysis

In this section we prove that the HDG method eq. (3.7) for the Navier–Stokes problem is pressure-robust, i.e., the velocity error is pressure-independent. Let $\Pi_{\text{BDM}} : \left[H^1(\Omega)\right]^d \to V_h$ be the usual Brezzi–Douglas–Marini (BDM) interpolation operator as given in the following lemma [40, Lemma 7].

Lemma 3.5.1. *If the mesh consists of triangles in two dimensions or tetrahedra in three dimensions there is an interpolation operator $\Pi_{\text{BDM}} : [H^1(\Omega)]^d \to V_h$ with the following properties for all $u \in [H^{k+1}(K)]^d$:*

(i) $[\![n \cdot \Pi_{\text{BDM}} u]\!] = 0$, *where* $[\![a]\!] = a^+ + a^-$ *and* $[\![a]\!] = a$ *on, respectively, interior and boundary faces is the usual jump operator.*

(ii) $\|u - \Pi_{\text{BDM}} u\|_{H^m(K)} \le c h_K^{l-m} \|u\|_{H^l(K)}$ *with* $m = 0, 1, 2$ *and* $\min(1, m) \le l \le k + 1$.

(iii) $\int_K q(\nabla \cdot u - \nabla \cdot \Pi_{\text{BDM}} u) \, \mathrm{d}x = 0$ *for all* $q \in P_{k-1}(K)$.

(iv) $\int_F \bar{q}(n \cdot u - n \cdot \Pi_{\text{BDM}} u) \, \mathrm{d}s = 0$ *for all* $\bar{q} \in P_k(F)$, *where* F *is a face on* ∂K.

Furthermore, let $\bar{\Pi}_V$, Π_Q and $\bar{\Pi}_Q$ be the standard L^2-projection operators onto $\bar{V}_h$, Q_h and $\bar{Q}_h$, respectively. We then introduce the approximation and interpolation errors

$$\begin{aligned} &\xi_u = u - \Pi_{\text{BDM}} u, && \zeta_u = u_h - \Pi_{\text{BDM}} u, && \bar{\xi}_u = \gamma(u) - \bar{\Pi}_V u, && \bar{\zeta}_u = \bar{u}_h - \bar{\Pi}_V u, \\ &\xi_p = p - \Pi_Q p, && \zeta_p = p_h - \Pi_Q p, && \bar{\xi}_p = \gamma(p) - \bar{\Pi}_Q p, && \bar{\zeta}_p = \bar{p}_h - \bar{\Pi}_Q p, \end{aligned}$$

and, for notational convenience, $\boldsymbol{\xi}_u = (\xi_u, \bar{\xi}_u)$, $\boldsymbol{\zeta}_u = (\zeta_u, \bar{\zeta}_u)$, $\boldsymbol{\xi}_p = (\xi_p, \bar{\xi}_p)$ and $\boldsymbol{\zeta}_p = (\zeta_p, \bar{\zeta}_p)$. Subtracting now the HDG method eq. (3.7) from the consistency result eq. (3.20) and splitting the errors, we obtain the following error equation:

$$\begin{aligned} a_h(\boldsymbol{\zeta}_u, \boldsymbol{v}_h) + b_h(\boldsymbol{\zeta}_p, v_h) + b_h(\boldsymbol{q}_h, \zeta_u) =& a_h(\boldsymbol{\xi}_u, \boldsymbol{v}_h) + b_h(\boldsymbol{\xi}_p, v_h) + b_h(\boldsymbol{q}_h, \xi_u) \\ &- o_h(u; \boldsymbol{\zeta}_u, \boldsymbol{v}_h) - o_h(\zeta_u; \boldsymbol{u}_h, \boldsymbol{v}_h) \\ &+ o_h(u; \boldsymbol{\xi}_u, \boldsymbol{v}_h) + o_h(\xi_u; \boldsymbol{u}_h, \boldsymbol{v}_h). \end{aligned} \tag{3.34}$$

In the following lemma we will find an energy estimate for the velocity error.

Theorem 3.5.1 (Pressure robust velocity error estimate)**.** *Let C_p and C_o be the constants in eq. (3.3). Furthermore let c_p be the discrete Poincaré constant of eq. (3.13), c_o the constant in eq. (3.19) and c_a^s the constant in eq. (3.14). Let $u \in \left[H^{k+1}(\Omega)\right]^d$ be the velocity solution to the Navier–Stokes problem eq. (3.1), $\boldsymbol{u} = (u, \gamma(u))$, and $\boldsymbol{u}_h \in \boldsymbol{V}_h$ the velocity solution of the HDG discretization eq. (3.7). Then assuming the smallness condition*

$$c_o' c_p' \|f\|_{L^2(\Omega)} \le \tfrac{1}{2}\nu^2 (c_a')^2, \tag{3.35}$$

where $c_p' = \max\left\{C_p, c_p\right\}$, $c_o' = \max\left\{C_o, c_o\right\}$ and $c_a' = \min\left\{1, c_a^s\right\}$ we obtain the pressure-robust velocity error estimate

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_v \le c h^k \|u\|_{H^{k+1}(\Omega)}, \tag{3.36}$$

where $c > 0$ a constant independent of h and ν.

Proof. In the error equation eq. (3.34) take $(\boldsymbol{v}_h, \boldsymbol{q}_h) = (\boldsymbol{\zeta}_u, -\boldsymbol{\zeta}_p)$. Then, by coercivity of $a_h(\cdot,\cdot)$ eq. (3.14)

$$\begin{aligned} \nu c_a^s |||\boldsymbol{\zeta}_u|||_v^2 \le a_h(\boldsymbol{\zeta}_u, \boldsymbol{\zeta}_u) =& a_h(\boldsymbol{\xi}_u, \boldsymbol{\zeta}_u) + b_h(\boldsymbol{\xi}_p, \zeta_u) - b_h(\boldsymbol{\zeta}_p, \xi_u) \\ &- o_h(u; \boldsymbol{\zeta}_u, \boldsymbol{\zeta}_u) - o_h(\zeta_u; \boldsymbol{u}_h, \boldsymbol{\zeta}_u) \\ &+ o_h(u; \boldsymbol{\xi}_u, \boldsymbol{\zeta}_u) + o_h(\xi_u; \boldsymbol{u}_h, \boldsymbol{\zeta}_u). \end{aligned} \tag{3.37}$$

By properties of the BDM interpolation operator and using that u_h is pointwise divergence-free and divergence-conforming, we note that $b_h(\boldsymbol{\xi}_p, \zeta_u) = 0$ and $b_h(\boldsymbol{\zeta}_p, \xi_u) = 0$. Furthermore, $o_h(u; \boldsymbol{\zeta}_u, \boldsymbol{\zeta}_u) \ge 0$ so that

$$\nu c_a^s |||\boldsymbol{\zeta}_u|||_v^2 \le a_h(\boldsymbol{\xi}_u, \boldsymbol{\zeta}_u) - o_h(\zeta_u; \boldsymbol{u}_h, \boldsymbol{\zeta}_u) + o_h(u; \boldsymbol{\xi}_u, \boldsymbol{\zeta}_u) + o_h(\xi_u; \boldsymbol{u}_h, \boldsymbol{\zeta}_u). \tag{3.38}$$

We next bound each term on the right-hand side separately.

By boundedness of $a_h(\cdot,\cdot)$ eq. (3.14),

$$a_h(\boldsymbol{\xi}_u, \boldsymbol{\zeta}_u) \le \nu c_a^b |||\boldsymbol{\xi}_u|||_{v'} |||\boldsymbol{\zeta}_u|||_{v'} \le \nu c c_a^b |||\boldsymbol{\xi}_u|||_{v'} |||\boldsymbol{\zeta}_u|||_v, \tag{3.39}$$

where the second inequality is by equivalence of $|||\cdot|||_{v'}$ and $|||\cdot|||_v$ on $\boldsymbol{V}_h$.

From eq. (3.4) and eq. (3.35) it follows that $\|u\|_{H^1(\Omega)} \le \frac{1}{2}c_a' c_o^{-1}\nu$. Furthermore, from eq. (3.23) and eq. (3.35) it follows that $|||\boldsymbol{u}_h|||_v \le \frac{1}{2}c_a' c_o^{-1}\nu$. Then, by eq. (3.19),

$$
\begin{aligned}
o_h(u;\boldsymbol{\xi}_u,\boldsymbol{\zeta}_u) &\le c_o\|u\|_{H^1(\Omega)}\,|||\boldsymbol{\xi}_u|||_v|||\boldsymbol{\zeta}_u|||_v \le \tfrac{1}{2}c_a'\nu|||\boldsymbol{\xi}_u|||_v|||\boldsymbol{\zeta}_u|||_v, && (3.40a)\\
o_h(\xi_u;\boldsymbol{u}_h,\boldsymbol{\zeta}_u) &\le c_o\|\xi_u\|_{1,h}\,|||\boldsymbol{u}_h|||_v|||\boldsymbol{\zeta}_u|||_v \le \tfrac{1}{2}c_a'\nu|||\boldsymbol{\xi}_u|||_v|||\boldsymbol{\zeta}_u|||_v, && (3.40b)\\
o_h(\zeta_u;\boldsymbol{u}_h,\boldsymbol{\zeta}_u) &\le c_o\|\zeta_u\|_{1,h}\,|||\boldsymbol{u}_h|||_v|||\boldsymbol{\zeta}_u|||_v \le \tfrac{1}{2}c_a'\nu|||\boldsymbol{\zeta}_u|||_v^2. && (3.40c)
\end{aligned}
$$

Combining eq. (3.38)–eq. (3.40) and dividing by $|||\boldsymbol{\zeta}_u|||_v$,

$$
\tfrac{1}{2}c_a'\nu|||\boldsymbol{\zeta}_u|||_v \le (\nu c_a^s - \tfrac{1}{2}c_a'\nu)|||\boldsymbol{\zeta}_u|||_v \le \nu(c_a' + c c_a^b)|||\boldsymbol{\xi}_u|||_{v'}. \tag{3.41}
$$

The result follows by a triangle inequality and the interpolation estimates of the BDM interpolation operator defined in Lemma 3.5.1 and the L^2-projection operator. □

Given the velocity error estimate of the previous theorem we can now state an error estimate for the pressure in the L^2-norm.

Lemma 3.5.2 (Pressure error estimate in the L^2-norm)**.** *Let $(u,p) \in \left[H^{k+1}(\Omega)\right]^d \times H^k(\Omega)$ be the solution to the Navier–Stokes problem eq. (3.1) and $\boldsymbol{u} = (u,\gamma(u))$ and $\boldsymbol{p} = (p,\gamma(p))$. Let $(\boldsymbol{u}_h,\boldsymbol{p}_h) \in \boldsymbol{X}_h$ solve eq. (3.7), then*

$$
\|p - p_h\|_{L^2(\Omega)} \le c\left(h^k\|p\|_{H^k(\Omega)} + h^k\|u\|_{H^{k+1}(\Omega)}\right), \tag{3.42}
$$

with $c > 0$ a constant independent of h and ν.

Proof. By the triangle inequality and the inf-sup condition eq. (3.16),

$$
\begin{aligned}
\|p - p_h\|_{L^2(\Omega)} &\le |||\boldsymbol{p} - \boldsymbol{q}_h|||_p + |||\boldsymbol{p}_h - \boldsymbol{q}_h|||_p \\
&\le |||\boldsymbol{p} - \boldsymbol{q}_h|||_p + \beta_p^{-1}\sup_{\boldsymbol{v}\in\boldsymbol{V}_h}\frac{b_h(\boldsymbol{p}-\boldsymbol{p}_h, v_h)}{|||\boldsymbol{v}_h|||_v} + \beta_p^{-1}\sup_{\boldsymbol{v}\in\boldsymbol{V}_h}\frac{b_h(\boldsymbol{p}-\boldsymbol{q}_h, v_h)}{|||\boldsymbol{v}_h|||_v}.
\end{aligned} \tag{3.43}
$$

Bounding the third term on the right-hand side using the boundedness of $b_h(\cdot,\cdot)$ eq. (3.15),

$$
\|p - p_h\|_{L^2(\Omega)} \le \left(1 + \beta_p^{-1}c_b^b\right)|||\boldsymbol{p} - \boldsymbol{q}_h|||_p + \beta_p^{-1}\sup_{\boldsymbol{v}\in\boldsymbol{V}_h}\frac{b_h(\boldsymbol{p}-\boldsymbol{p}_h, v_h)}{|||\boldsymbol{v}_h|||_v}. \tag{3.44}
$$

Proceeding as in the velocity error estimate,

$$\begin{aligned} b_h(\boldsymbol{p}-\boldsymbol{p}_h, v_h) &= a_h(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h) + o_h(\boldsymbol{u}-\boldsymbol{u}_h; \boldsymbol{u}, \boldsymbol{v}_h) - o_h(\boldsymbol{u}_h; \boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{v}_h) \\ &\leq \left(c_a^b + c_a'\right) \nu \| \boldsymbol{u}-\boldsymbol{u}_h \|_{v'} \| \boldsymbol{v}_h \|_v . \end{aligned} \tag{3.45}$$

Combining eq. (3.44) and eq. (3.45), and since $\boldsymbol{q}_h \in \boldsymbol{Q}_h$ is arbitrary,

$$\|p - p_h\|_{L^2(\Omega)} \leq \left(1 + \beta_p^{-1} c_b^b\right) \inf_{\boldsymbol{q}_h \in \boldsymbol{Q}_h} \|\boldsymbol{p} - \boldsymbol{q}_h\|_p + \beta_p^{-1}\left(c_a^b + c_\alpha\right) \nu \|\boldsymbol{u} - \boldsymbol{u}_h\|_{v'} . \tag{3.46}$$

Standard interpolation estimates for the L^2-projection can be used to show that

$$\inf_{\boldsymbol{q}_h \in \boldsymbol{Q}_h} \|\boldsymbol{p} - \boldsymbol{q}_h\|_p \leq c h^k \|p\|_{H^k(\Omega)} , \tag{3.47}$$

where c is a constant independent of h. To bound the second term on the right-hand side of eq. (3.46), note that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{v'} \leq \|\boldsymbol{\xi}_u\|_{v'} + \|\boldsymbol{\zeta}_u\|_{v'} \leq \|\boldsymbol{\xi}_u\|_{v'} + c\|\boldsymbol{\zeta}_u\|_v \leq c\|\boldsymbol{\xi}_u\|_{v'}, \tag{3.48}$$

where the last inequality is by eq. (3.41). The result follows from eq. (3.46), eq. (3.47), eq. (3.48) and the interpolation estimates of the BDM interpolation operator defined in lemma 3.5.1 and the L^2-projection operator. □

We end this section by showing the velocity error estimate in the L^2-norm. For this we require the solution (ϕ, ψ) to the following dual problem [46, Chapter 6]:

$$-\nu \nabla^2 \phi - \nabla \cdot (u \otimes \phi) - \nabla \psi - (\nabla \phi)^T u = g \qquad \text{in } \Omega, \tag{3.49a}$$

$$\nabla \cdot \phi = 0 \qquad \text{in } \Omega, \tag{3.49b}$$

$$\phi = 0 \qquad \text{on } \Gamma. \tag{3.49c}$$

We assume the following regularity estimate:

$$\|\phi\|_{H^2(\Omega)} + \|\psi\|_{H^1(\Omega)} \leq c_r \|g\|_{L^2(\Omega)} , \tag{3.50}$$

with $c_r > 0$ a constant independent of h. This regularity estimate holds for a convex polyhedron Ω assuming $u \in \left[L^\infty(\Omega)\right]^d$ [14]. It will be convenient to introduce the interpolation errors

$$\xi_\phi = \phi - \Pi_{\mathrm{BDM}}\phi, \qquad \bar{\xi}_\phi = \gamma(\phi) - \bar{\Pi}_V \phi,$$

$$\xi_\psi = \psi - \Pi_Q \psi, \qquad \bar{\xi}_\psi = \gamma(\psi) - \bar{\Pi}_Q \psi.$$

and $\boldsymbol{\xi}_\phi = (\xi_\phi, \bar{\xi}_\phi)$, $\boldsymbol{\xi}_\psi = (\xi_\psi, \bar{\xi}_\psi)$.

Lemma 3.5.3 (Velocity error estimate in the L^2-norm)**.** *Let* $u \in \left[H^{k+1}(\Omega)\right]^d \cap \left[L^\infty(\Omega)\right]^d$ *be the velocity solution to the Navier–Stokes problem eq.* (3.1)*,* $\boldsymbol{u} = (u, \gamma(u))$*, and* $\boldsymbol{u}_h \in \boldsymbol{V}_h$ *the velocity solution of the HDG discretization eq.* (3.7)*. Subject to the regularity condition eq.* (3.50)*, there exists a constant* $C > 0$*, independent of* h*, such that*

$$\|u - u_h\|_{L^2(\Omega)} \leq C h^{k+1} \|u\|_{H^{k+1}(\Omega)} . \tag{3.51}$$

Proof. By definition of $a_h(\cdot,\cdot)$ eq. (3.8a), integration by parts, using the single-valuedness of u, $\partial_n \phi$ and $\bar{u}_h$ across cell boundaries, and that $u = \bar{u}_h = 0$ on Γ, we note that

$$a_h(\boldsymbol{u} - \boldsymbol{u}_h, (\phi, \gamma(\phi))) = -\sum_{K \in \mathcal{T}_h} \int_K \nu (u - u_h) \cdot \nabla^2 \phi \, \mathrm{d}x. \tag{3.52}$$

Furthermore, by definition of eq. (3.8b), using that $\phi = \gamma(\phi)$ on cell boundaries and the identity $(a \otimes b) : C = b \cdot C^T a$ for vectors $a, b \in \mathbb{R}^m$ and tensor $C \in \mathbb{R}^{n \times n}$

$$o_h(u - u_h; (u, \gamma(u)), (\phi, \gamma(\phi))) = -\sum_{K \in \mathcal{T}_h} \int_K (u - u_h) \cdot (\nabla \phi)^T u \, \mathrm{d}x. \tag{3.53}$$

Similarly, using again the identity $(a \otimes b) : C = b \cdot C^T a$,

$$o_h(u; \boldsymbol{u} - \boldsymbol{u}_h, (\phi, \gamma(\phi))) = -\sum_{K \in \mathcal{T}_h} \int_K (u - u_h) \cdot \nabla \cdot (u \otimes \phi) \, \mathrm{d}x, \tag{3.54}$$

where we used also that $(\nabla \phi) u = (u \cdot \nabla)\phi$ and, for divergence-free u, $\nabla \cdot (u \otimes \phi) = (u \cdot \nabla)\phi$.

Next, by definition of b_h eq. (3.8c), integration by parts, using that u, $u_h \cdot n$ and ψ are single-valued across cell boundaries, and that $u = u_h = 0$ on Γ,

$$-\, b_h((\psi, \gamma(\psi)), u - u_h) = -\sum_{K \in \mathcal{T}} \int_K \nabla \psi \cdot (u - u_h) \, \mathrm{d}x. \tag{3.55}$$

Once again from the definition of b_h eq. (3.8c),

$$b_h(\boldsymbol{p} - \boldsymbol{p}_h, \phi) = -\int_\Omega (p - p_h) \nabla \cdot \phi \, \mathrm{d}x = 0. \tag{3.56}$$

since $\nabla \cdot \phi = 0$.

Adding eq. (3.52)–eq. (3.56),

$$
\begin{aligned}
a_h(\boldsymbol{u}-\boldsymbol{u}_h,(\phi,\gamma(\phi)))&+o_h(u-u_h;(u,\gamma(u)),(\phi,\gamma(\phi)))\\
&+o_h(u;\boldsymbol{u}-\boldsymbol{u}_h,(\phi,\gamma(\phi)))+b_h(\boldsymbol{p}-\boldsymbol{p}_h,\phi)-b_h((\psi,\gamma(\psi)),u-u_h)\\
&=\sum_{K\in\mathcal{T}}\int_K(u-u_h)\cdot\Big(-\nu\nabla^2\phi-\nabla\cdot(u\otimes\phi)-\nabla\psi-(\nabla\phi)^Tu\Big)\,\mathrm{d}x.
\end{aligned}\tag{3.57}
$$

Taking $g=u-u_h$ in eq. (3.49) we therefore find that

$$
\begin{aligned}
\|u-u_h\|^2_{L^2(\Omega)}=a_h(\boldsymbol{u}-\boldsymbol{u}_h,(\phi,\gamma(\phi)))&+b_h(\boldsymbol{p}-\boldsymbol{p}_h,\phi)+o_h(u-u_h;(u,\gamma(u)),(\phi,\gamma(\phi)))\\
&+o_h(u;\boldsymbol{u}-\boldsymbol{u}_h,(\phi,\gamma(\phi)))-b_h((\psi,\gamma(\psi)),u-u_h).
\end{aligned}\tag{3.58}
$$

Next, from the consistency of the scheme eq. (3.20),

$$
a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h)+b_h(\boldsymbol{p}-\boldsymbol{p}_h,v_h)-o_h(u_h;\boldsymbol{u}_h,\boldsymbol{v}_h)+o_h(u;\boldsymbol{u},\boldsymbol{v}_h)-b_h(\boldsymbol{q}_h,u-u_h)=0.\tag{3.59}
$$

Subtract now eq. (3.59) from eq. (3.58) and choose $\boldsymbol{v}_h=(\Pi_{\text{BDM}}\phi,\bar{\Pi}_V\phi)$ and $\boldsymbol{q}_h=(\Pi_Q\psi,\bar{\Pi}_Q\psi)$. Algebraic manipulation then results in

$$
\begin{aligned}
\|u-u_h\|^2_{L^2(\Omega)}=&a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\xi}_\phi)+b_h(\boldsymbol{p}-\boldsymbol{p}_h,\xi_\phi)+o_h(u-u_h;(u,\gamma(u)),\boldsymbol{\xi}_\phi)\\
&-o_h(u-u_h;\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\xi}_\phi)+o_h(u;\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\xi}_\phi)\\
&+o_h(u-u_h;\boldsymbol{u}-\boldsymbol{u}_h,(\phi,\gamma(\phi)))-b_h(\boldsymbol{\xi}_\psi,u-u_h)\\
=&T_1+T_2+T_3+T_4+T_5+T_6+T_7.
\end{aligned}\tag{3.60}
$$

Note first that

$$
\begin{aligned}
T_2=b_h(\boldsymbol{p}-\boldsymbol{p}_h,\xi_\phi)=-\sum_{K\in\mathcal{T}_h}\int_K(p-p_h)\nabla\cdot(\phi-\Pi_{\text{BDM}}\phi)\,\mathrm{d}x&\\
+\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\phi-\Pi_{\text{BDM}}\phi)\cdot n\bar{p}_h\,\mathrm{d}s=0,&
\end{aligned}\tag{3.61}
$$

by properties of the BDM interpolation operator and the L^2-projection operator Π_Q. We next bound the remaining terms in eq. (3.60). By boundedness of $a_h(\cdot,\cdot)$ eq. (3.14),

$$
T_1\le c_a^b\nu|\!|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!|\!|_{v'}\Big|\!\Big|\!\Big|\boldsymbol{\xi}_\phi\Big|\!\Big|\!\Big|_{v'}.\tag{3.62}
$$

Next, the interpolation property (ii) of the BDM projection in lemma 3.5.1 and the interpolation properties of the L^2-projection $\bar{\Pi}_V$ followed by assumption eq. (3.50), yield

$$\left\|\!\left|\boldsymbol{\xi}_\phi\right|\!\right\|_{v'} \leq h\|\phi\|_{H^2(\Omega)} \leq h\|u - u_h\|_{L^2(\Omega)}\,, \tag{3.63}$$

so that

$$T_1 \leq c\nu h|\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!|_{v'}\|u - u_h\|_{L^2(\Omega)}\,. \tag{3.64}$$

From boundedness of the trilinear form $o_h(\cdot;\cdot,\cdot)$ eq. (3.19), the smallness assumption eq. (3.35), and eq. (3.63)

$$T_3 \leq \tfrac{1}{2}c_\alpha \nu|\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!|_{v'}\left\|\!\left|\boldsymbol{\xi}_\phi\right|\!\right\|_{v'} \leq c\nu h|\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!|_{v'}\|u - u_h\|_{L^2(\Omega)}\,, \tag{3.65}$$

and, similarly,

$$T_4 + T_5 \leq c\nu h|\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!|_{v'}\|u - u_h\|_{L^2(\Omega)}\,. \tag{3.66}$$

For T_6, using the boundedness of the trilinear form $o_h(\cdot;\cdot,\cdot)$ eq. (3.19), the fact that $\|\phi\|_{H^1(\Omega)} \leq \|\phi\|_{H^2(\Omega)}$, and eq. (3.36),

$$\begin{aligned} T_6 &\leq C_O|\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!|_{v'}^2\|\phi\|_1 \\ &\leq ch|\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!|_{v'}\|u\|_2\|\phi\|_2 \\ &\leq ch|\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!|_{v'}\|u\|_2\|u - u_h\|_{L^2(\Omega)}\,. \end{aligned} \tag{3.67}$$

To bound T_7, we use the boundedness of $b_h(\cdot,\cdot)$ eq. (3.15), standard interpolation estimates for the L^2-projections Π_Q and $\bar{\Pi}_Q$, and the regularity assumption eq. (3.50) to find

$$T_7 \leq c_b^b\left\|\!\left|\boldsymbol{\xi}_\psi\right|\!\right\|_p|\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!|_{v'} \leq ch\|\psi\|_{H^1(\Omega)}\,|\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!|_{v'} \leq ch|\!|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!|\!|_{v'}\|u - u_h\|_{L^2(\Omega)}\,. \tag{3.68}$$

The result follows after collecting eq. (3.64)–eq. (3.68), dividing both sides by $\|u - u_h\|_{L^2(\Omega)}$ and applying the interpolation estimates of the BDM interpolant defined in Lemma 3.5.1 and the L^2-projection operator. □

3.6 Numerical examples

In this section we present numerical examples that demonstrate optimality and pressure-robustness of the scheme. All numerical examples have been implemented with the penalty parameter $\alpha = 10k^2$ using the high order finite element library NGSolve [85]. In all test cases below, we compare the HDG method analyzed in this paper to the HDG method proposed in [55]. The method proposed in [55] considers a smaller pressure trace function space in that $\bar{Q}_h$ eq. (3.6b) is replaced by

$$\tilde{Q}_h := \left\{ \bar{q}_h \in L^2(\mathcal{F}),\ \bar{q}_h \in P_{k-1}(F)\ \forall F \in \mathcal{F} \right\}.$$

The velocity and pressure estimates of this scheme are optimal, see [77] for the analysis of the Stokes problem. Despite the velocity field obtained by the discretization in [55] being pointwise divergence free, the method is not pressure robust. This can be attributed to the fact that the approximate velocity field is not divergence-conforming.

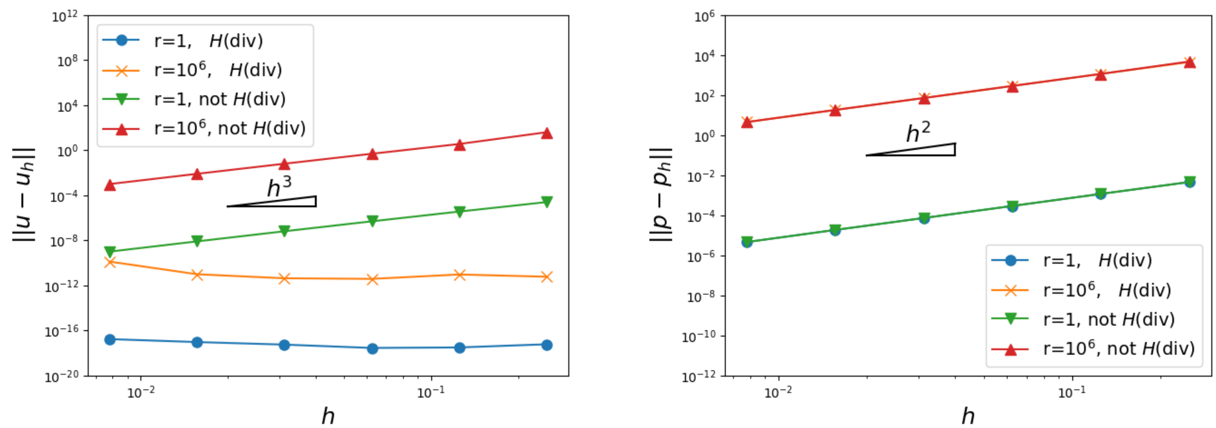
3.6.1 No flow problem

In this first example we consider the no flow problem from [47, Example 1.1] adapted to the stationary Navier–Stokes problem. For this we take $\Omega = (0,1)^2$, set $\nu = 1$, and apply homogeneous Dirichlet boundary conditions. The source term is taken to be $f = (0, r(1-y+3y^2))$, where $r > 0$ is a parameter. The exact solution to this problem is $u = 0$ and $p = r(y^3 - \frac{1}{2}y^2 + y - \frac{7}{12})$. Changing the parameter r should affect only the pressure solution. This example tests whether the numerical scheme mimics this property.

In Figure 3.1 we plot the velocity and pressure errors using a polynomial approximation with $k = 2$ for $r = 1$ and $r = 10^6$. We observe in Figure 3.1a that the velocity error using the HDG method that is not divergence-conforming is, as expected, not pressure-robust. Although the velocity converges optimally, increasing the parameter r increases the error in the velocity. Conversely, the error in the velocity of the divergence-conforming method is of machine-precision, no matter the grid size. Although the error in the velocity increases as r increases, this can be attributed to an increase in the condition number of the matrix that needs to be inverted at each Picard iteration. The pressure-errors are identical for both HDG methods, see Figure 3.1b. The errors in the pressure converge optimally and increase as r increases.

3.6.2 Potential flow problem

We next consider the potential flow problem from [63, Example 4]. Setting $f = 0$, this test case was constructed such that pressure is balanced by the nonlinear convection terms, and serves to show that nonlinear convection terms can also induce a lack of pressure-robustness [47]. On the domain $\Omega = (-\frac{1}{2}, \frac{1}{2})^2$, we consider the steady Navier–Stokes



(a) Velocity error.

(b) Pressure error.

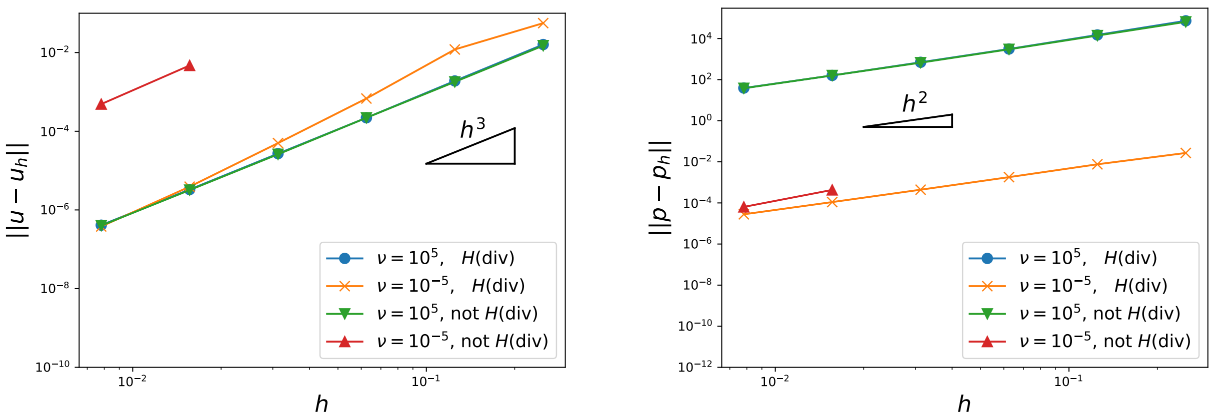
Figure 3.1: Results for the no flow problem in Section 3.6.1 using polynomial degree $k = 2$. Observe that the pressure errors are identical for both HDG methods.

equations

$$
\begin{aligned}
-\nu \Delta u + \nabla \cdot (u \otimes u) + \nabla p &= 0, \\
\nabla \cdot u &= 0,
\end{aligned}
$$

and the boundary conditions are chosen such that the exact solution is given by $u = \nabla \phi$ and $p = -\frac{1}{2}|u|^2$, with the harmonic function $\phi = y^5 + 5x^4 y - 10x^2 y^3$. In Figure 3.2 we plot the velocity and pressure errors using a polynomial approximation with $k = 2$ for $\nu = 10^5$ and $\nu = 10^{-5}$. We observe optimal rates of convergence for both methods for velocity and pressure.

For the HDG method that is not divergence-conforming, however, the errors in the velocity and pressure increase significantly as the viscosity is decreased. Furthermore, there is no convergence of the non-linear solvers for large h for the case that $\nu = 10^{-5}$. This was observed also in [47, 63] for schemes that are not pressure-robust. For the divergence-conforming method, the errors in velocity and pressure are unaffected by the decrease in viscosity and there are no problems associated with the convergence of the non-linear solver.



(a) Example 2, Velocity error.

(b) Example 2, Pressure error.

Figure 3.2: Results for the potential flow problem in Section 3.6.2 using polynomial degree $k = 2$. The pressure errors and velocity errors are identical for both HDG methods in the case $\nu = 10^5$, while the HDG method that is not divergence-conforming fails to converge for large h in the case $\nu = 10^{-5}$.

Chapter 4

Pressure-robust space-time HDG for the time-dependent problem on fixed domains: Convergence to strong solutions

In this chapter, we present the first analysis of a space-time HDG method for the incompressible Navier–Stokes equations. The main result is an optimal error estimate for the velocity which is independent of the pressure, thus proving that the method is pressure-robust. The key to our error analysis is a "parabolic Stokes projection" introduced by Chrysafinos and Walkington [17] suitably modified to accomodate the space-time HDG setting. The projection is defined as the space-time HDG solution of a linear Stokes problem, and thus the projection is pointwise divergence free and belongs to $H(\text{div}; \Omega)$.

Along the way, we investigate the well-posedness of the nonlinear algebraic system arising from the space-time HDG discretization. To our knowledge, this is the first work to consider the well-posedness of a space-time finite element method for the incompressible Navier–Stokes equations. This is made complicated by the *fully discrete* nature of the scheme, as we cannot leverage tools from ODE theory (e.g., Carathéodory's theorem) as is done in the semidiscrete case. Instead, we will use a topological degree argument based on Lemma 2.1.2 to prove that there exists a discrete solution in both two and three spatial dimensions, and that this discrete solution satisfies a suitable energy inequality.

The uniqueness of the discrete velocity solution is subtle. As the discrete problem eq. (4.4) must be considered in an integral sense over each space-time slab (and not pointwise in time), we will require sharper bounds on the nonlinear convection term than eq. (4.10). This is made possible by a novel discrete version of the Ladyzhenskaya inequality eq. (2.16) valid for broken polynomial spaces. Much like the continuous problem, we can only prove that the solution to the nonlinear algebraic system is unique in two spatial dimensions due to the difference in the scaling of the exponents in this Ladyzhenskaya inequality with respect to the spatial dimension.

Additionally, our proof of uniqueness in two dimensions requires a bound on the discrete velocity solution u_h in $L^\infty(0,T;L^2(\Omega)^d)$ that is uniform with respect to the time step Δt and mesh size h. We remark that for the low order scheme in time ($k=1$), this uniform bound is furnished by the energy estimate Lemma 4.1.1. However, for higher order schemes in time ($k \geq 2$), this energy bound is insufficient. The point of failure is that, in general, $\operatorname{ess\,sup}_{0<t\leq T}\; u_h(t)$ is not attained at the partition points of the time-interval for higher order schemes in time. Consequently, the energy bound in Lemma 4.1.1 does not bound the discrete velocity solution u_h in $L^\infty(0,T;L^2(\Omega)^d)$.

Let us briefly recall how one obtains such a bound for the continuous problem. In two dimensions, bounds on the continuous solution u in $L^\infty(0,T;L^2(\Omega)^d)$ are obtained by testing eq. (2.24) with u and integrating to an arbitrary time $s \in (0,T]$. In the equivalent space-time variational formulation of the Navier–Stokes equations given in eq. (2.27), this amounts to choosing $\chi_{[0,s)}u$ as a test function. In three dimensions, weak solutions satisfying eq. (2.24) are no longer regular enough to be used as test functions since $\partial_t u \in L^{4/3}(0,T;V')$, and energy bounds must be obtained through other means; see e.g. [46, Lemma 7.21]. However, in principle, the strong solutions in Theorem 2.3.2 satisfying $\partial_t u \in L^2(0,T;H)$ possess enough regularity to proceed as in the case of two dimensions. The difficulty at the discrete level is that, in constrast to the continuous problem, $\chi_{(0,s]}u_h$ is not, in general, an element of the velocity finite element space and hence cannot be used as a test function. To circumvent this problem, we will make use of tools introduced by Chrysafinos and Walkington [15, 16, 17] which exploit fine properties of polynomials to provide discrete approximations to the characteristic function.

The final piece of the puzzle for our analysis is a uniform-in-time bound on the parabolic Stokes projection. Our plan is to follow the proof of [17, Theorem 4.10]. Therein, an essential ingredient is a discrete (spatial) Stokes operator. Unfortunately, as the usual L^2-inner product offers no control over the discrete facet solution, it is *not* an inner-product on the HDG space. Therefore, we cannot rely on the Riesz representation theorem (Theorem 2.1.1) in the HDG setting as is usually done to infer the existence of a discrete Stokes operator. Instead, we introduce a novel discrete Stokes operator by mimicking the static condensation that occurs for the HDG method at the algebraic level following ideas from [12]. Lastly, we derive an error estimate for the pressure which is unfortunately sub-optimal due to the use of an inverse inequality to control the time derivative of the velocity error.

This chapter is organized as follows: in Section 4.1, we introduce the numerical method and present the main results of this chapter. In Section 4.2, we study the conservation properties of the numerical scheme and introduce analysis tools that required for our analysis. We consider the well-posedness of the nonlinear algebraic system arising from the numerical scheme in Section 4.3. Section 4.4 and Section 4.5 are dedicated to the error analysis for the velocity and pressure, respectively. Finally, we present a numerical test case with a manufactured solution in Section 4.6 to verify the convergence rates predicted by the theory both when the spatial error dominates and when the temporal error dominates.

The contents of this chapter have been taken, with modification, from the article:

> K. L. A Kirk, T. L. Horváth, and S. Rhebergen, *Analysis of an exactly mass conserving space-time hybridized discontinuous Galerkin method for the time-dependent Navier–Stokes equations*, (To appear in Mathematics of Computation) https://arxiv.org/abs/2103.13492,

with permission from the American Mathematical Society (AMS).

4.1 The space-time HDG method and main results

4.1.1 Notation

We use standard notation for Lebesgue and Sobolev spaces: given a bounded measurable set D, we denote by $L^p(D)$ the space of p-integrable functions. When $p = 2$, we denote the $L^2(D)$ inner product by $(\cdot,\cdot)_D$. We denote by $W^{k,p}(D)$ the Sobolev space of functions whose k^{th} distributional derivative is p-integrable. When $p = 2$, we write $W^{k,p}(D) = H^k(D)$. Provided the boundary of D is smooth enough to admit a continuous trace operator, we define $H_0^k(D)$ to be the subspace of $H^k(D)$ of functions with vanishing trace on the boundary of D. We denote the space of polynomials of degree $k \geq 0$ on D by $P_k(D)$.

Next, for any Banach space U and for $1 \leq p < \infty$, we let $L^p(0,T;U)$ denote the space of p-integrable functions defined on $[0,T]$ taking values in U. This is a Banach space equipped with the norm

$$\|u\|_{L^p(0,T;U)} = \left(\int_0^T \|u\|_U^p \,\mathrm{d}t \right)^{1/p}.$$

When $p = \infty$, we denote by $L^\infty(0,T;U)$ the Banach space of essentially bounded functions

$$L^\infty(0,T;U) := \left\{ u : [0,T] \to U \mid \text{ess sup}_{0\le t\le T} \left\| u(t) \right\|_U < \infty \right\},$$

where ess sup denotes the essential supremum. By $H^k(0,T;U)$ we denote the Bochner-Sobolev space for $k \ge 1$:

$$H^k(0,T;U) := \left\{ u \in L^2(0,T;U) \mid \tfrac{\mathrm{d}^j u}{\mathrm{d}t^j} \in L^2(0,T;U),\ j = 1,\dots,k \right\},$$

endowed with the norm

$$\|u\|_{H^k(0,T;U)} = \left(\sum_{j=0}^{k} \left\| \tfrac{\mathrm{d}^j u}{\mathrm{d}t^j} \right\|^2_{L^2(0,T;L^2(\Omega))} \right)^{1/2}.$$

Let $C(0,T;U)$ denote the Banach space of (time) continuous functions equipped with the norm

$$\|u\|_{C(0,T;U)} = \sup_{0\le t\le T} \left\| u(t) \right\|_U .$$

By $C_c(0,T;U)$ we denote the space of (time) continuous functions with compact support in the interval $(0,T)$. Lastly, given a Banach space U, we let $P_k(0,T;U)$ denote the space of polynomials of degree $k \ge 0$ in time taking values in U.

4.1.2 The continuous problem

In this chapter, we are concerned with the numerical solution of the transient Navier–Stokes system posed on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$: given a suitably chosen body force f, kinematic viscosity $0 < \nu \le 1$, and initial data u_0, find (u,p) such that

$$\partial_t u - \nu \Delta u + \nabla \cdot (u \otimes u) + \nabla p = f, \qquad \text{in } \Omega \times (0,T], \tag{4.1a}$$
$$\nabla \cdot u = 0, \qquad \text{in } \Omega \times (0,T], \tag{4.1b}$$
$$u = 0, \qquad \text{on } \partial\Omega \times (0,T], \tag{4.1c}$$
$$u(x,0) = u_0(x), \qquad \text{in } \Omega. \tag{4.1d}$$

Recall from Section 2.3 that the natural setting for weak velocity solutions of eq. (4.1) is the class $L^2(0,T;V) \cap L^\infty(0,T;H)$, with H and V defined in eq. (2.22) and eq. (2.23), respectively. However, as a discontinuous method, the HDG method introduces additional stabilization which is a potential source of consistency error if the exact solution is not sufficiently regular. For this reason, we make the following assumption to ensure the existence of a strong solution in the sense of Theorem 2.3.2:

Assumption 1. Let $C > 0$ be the constant from Theorem 2.3.2. We assume that eq. (2.28) holds. Note that, since $u_0 \in V \subset H_0^1(\Omega)^d$, eq. (2.28) and the Poincaré inequality (Theorem 2.2.5) imply the existence of another constant $C_\star > 0$ such that

$$\|u_0\|_{L^2(\Omega)}^2 + \frac{1}{\nu}\|f\|_{L^2(0,T;L^2(\Omega))}^2 \leq C_\star \nu^2. \tag{4.2}$$

Remark 4.1.1. *If $\Omega \subset \mathbb{R}^2$ is a convex polygon, the existence of a global unique strong solution (u, p) can be shown without any restriction on the problem data (see e.g. [96]). However, we will later require a similar restriction on the data to prove the uniqueness of the discrete solution in two dimensions. We therefore assume eq. (2.28) even in the two dimensional case.*

Therefore, given $f \in L^2(0,T;H)$ and $u_0 \in V$ satisfying the small data assumption eq. (4.2), we consider the following space-time formulation for the strong solution to the Navier–Stokes system: for all $(v, q) \in L^2(0,T;H_0^1(\Omega)^d) \cap H^1(0,T;L^2(\Omega)^d) \times L^2(0,T;L_0^2(\Omega) \cap H^1(\Omega))$, find $(u,p) \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)^d \cap V) \cap H^1(0,T;H) \times L^2(0,T;L_0^2(\Omega) \cap H^1(\Omega))$ satisfying

$$\begin{aligned} -\int_0^T (u, \partial_t v)\,\mathrm{d}t + \int_0^T ((u\cdot\nabla)u, v)\,\mathrm{d}t + \nu\int_0^T (\nabla u, \nabla v)\,\mathrm{d}t + \int_0^T (\nabla p, v)\,\mathrm{d}t \\ + (u(T), v(T)) - \int_0^T (q, \nabla\cdot u)\,\mathrm{d}t = (u_0, v(0)) + \int_0^T (f, v)\,\mathrm{d}t. \end{aligned} \tag{4.3}$$

4.1.3 The numerical method

To obtain a triangulation of the space-time domain $\Omega \times (0,T)$, we first tessellate the spatial domain $\Omega \subset \mathbb{R}^d$, $d = \{2,3\}$ with simplicial elements (if $d = 2$), or tetrahedral elements (if $d = 3$). We denote the resulting tessellation by $\mathcal{T}_h = \{K\}$. Furthermore, we let $\mathcal{F}_h$ and $\partial\mathcal{T}_h$ denote, respectively, the set and union of all edges of $\mathcal{T}_h$. By h_K, we denote the diameter of the element $K \in \mathcal{T}_h$, and we let $h = \max_{K\in\mathcal{T}_h} h_K$. We make the following assumptions on the spatial mesh:

(i) For each $h \in \mathcal{H}$, $\mathcal{T}_h$ is conforming in the sense that given two elements $K_1, K_2 \in \mathcal{T}_h$, either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is a common vertex ($d = 2$) or edge ($d = 3$), or a common face of K_1 and K_2.

(ii) For each $h \in \mathcal{H}$, $\mathcal{T}_h$ is quasi-uniform; i.e., there exists a $C_U > 0$ such that $h \le C_U h_K$ for all $K \in \mathcal{T}_h$.

(iii) For each $h \in \mathcal{H}$, each face $F \in \mathcal{F}_h$ satisfies an *equivalence condition*: that is, given $h_F = \text{diam}(F)$, there exist constants $C_e, C^e > 0$ such that $C_e h_K \le h_F \le C^e h_K$ for all $K \in \mathcal{T}_h$ and for all $F \in \mathcal{F}_h$ where $F \subset \partial K$.

Next, we partition the time interval $[0, T]$ into a series of $N + 1$ time-levels $0 = t_0 < t_1 < \cdots < t_N = T$ of length $\Delta t_n = t_{n+1} - t_n$. For simplicity of presentation, we assume a uniform time step size $\Delta t_n = \Delta t$ for $0 \le n \le N$. We remark, however, that a variable time step size poses no additional difficulty in the application nor the analysis of the method. A space-time slab is then defined as $\mathcal{E}^n = \Omega \times I_n$, with $I_n = (t_n, t_{n+1})$. We then tessellate the space-time slab $\mathcal{E}^n$ with space-time prisms $K \times I_n$, i.e. $\mathcal{E}^n = \bigcup_{K \in \mathcal{T}_h} K \times I_n$. We denote this tessellation by $\mathscr{T}_h^n$. Combining each space-time slab $n = 0, \dots, N-1$, we obtain a tessellation of the space-time domain $\mathscr{T}_h = \bigcup_{n=0}^{N-1} \mathscr{T}_h^n$.

4.1.3.1 The space-time hybridized DG method

We discretize the Navier–Stokes problem eq. (4.1) using the exactly mass conserving hybridized discontinuous Galerkin method developed in [78] combined with a high-order discontinuous Galerkin time stepping scheme. We first introduce the following discontinuous finite element spaces on $\mathcal{T}_h$:

$$V_h := \left\{ v_h \in L^2(\Omega)^d \mid v_h|_K \in P_{k_s}(K)^d \ \forall K \in \mathcal{T}_h \right\},$$
$$Q_h := \left\{ q_h \in L^2_0(\Omega) \mid q_h|_K \in P_{k_s-1}(K) \ \forall K \in \mathcal{T}_h \right\}.$$

On $\partial\mathcal{T}_h$, we introduce the following facet finite element spaces:

$$\bar{V}_h := \left\{ \bar{v}_h \in L^2(\partial\mathcal{T}_h) \mid \bar{v}_h|_F \in P_{k_s}(F)^d \ \forall F \in \mathcal{F}_h,\ \bar{v}_h|_{\partial\Omega} = 0 \right\},$$
$$\bar{Q}_h := \left\{ \bar{q}_h \in L^2(\partial\mathcal{T}_h) \mid \bar{q}_h|_F \in P_{k_s}(F) \ \forall F \in \mathcal{F}_h \right\}.$$

From these spaces, we construct the following space-time finite element spaces in which we will seek our approximation on each space-time slab $\mathcal{E}^n$:

$$\mathcal{V}_h := \left\{ v_h \in L^2(0, T; L^2(\Omega)^d) \mid v_h|_{(t_n, t_{n+1}]} \in P_{k_t}(t_n, t_{n+1}; V_h),\ \forall n = 0, \dots, N-1 \right\},$$
$$\mathcal{Q}_h := \left\{ q_h \in L^2(0, T; L^2_0(\Omega)) \mid q_h|_{(t_n, t_{n+1}]} \in P_{k_t}(t_n, t_{n+1}; Q_h),\ \forall n = 0, \dots, N-1 \right\},$$
$$\bar{\mathcal{V}}_h := \left\{ \bar{v}_h \in L^2(0, T; L^2(\partial\mathcal{T}_h)^d) \mid \bar{v}_h|_{(t_n, t_{n+1}]} \in P_{k_t}(t_n, t_{n+1}; \bar{V}_h),\ \forall n = 0, \dots, N-1 \right\},$$
$$\bar{\mathcal{Q}}_h := \left\{ \bar{q}_h \in L^2(0, T; L^2(\partial\mathcal{T}_h)) \mid \bar{q}_h|_{(t_n, t_{n+1}]} \in P_{k_t}(t_n, t_{n+1}; \bar{Q}_h),\ \forall n = 0, \dots, N-1 \right\}.$$

We note that, in general, the polynomial degree in time k_t can be chosen independently of the polynomial degree in space k_s, but for ease of presentation we choose $k_t = k_s = k$. This choice forces us to consider $k_t \geq 1$, but the analysis herein is valid also for the case $k_t = 0$ (corresponding to a modified backward Euler scheme). We adopt the following notation for various product spaces of interest in this work:

$$\boldsymbol{V}_h = V_h \times \bar{V}_h, \quad \boldsymbol{Q}_h = Q_h \times \bar{Q}_h, \quad \boldsymbol{\mathcal{V}}_h = \mathcal{V}_h \times \bar{\mathcal{V}}_h, \quad \boldsymbol{\mathcal{Q}}_h = \mathcal{Q}_h \times \bar{\mathcal{Q}}_h.$$

Pairs in these product spaces will be denoted using boldface; for example, $\boldsymbol{v}_h := (v_h, \bar{v}_h) \in \boldsymbol{\mathcal{V}}_h$. On each space-time slab $\mathcal{E}^n$, the space-time HDG method for the Navier–Stokes problem reads: find $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in \boldsymbol{\mathcal{V}}_h \times \boldsymbol{\mathcal{Q}}_h$ satisfying for all $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{\mathcal{V}}_h \times \boldsymbol{\mathcal{Q}}_h$,

$$\begin{aligned} -\int_{I_n} (u_h, \partial_t v_h)_{\mathcal{T}_h} \,\mathrm{d}t + \int_{I_n} \big(\nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h)\big) \,\mathrm{d}t \qquad \text{(4.4a)} \\ + (u_{n+1}^-, v_{n+1}^-)_{\mathcal{T}_h} + \int_{I_n} b_h(\boldsymbol{p}_h, v_h) \,\mathrm{d}t = (u_n^-, v_n^+)_{\mathcal{T}_h} + \int_{I_n} (f, v_h)_{\mathcal{T}_h} \,\mathrm{d}t, \\ \int_{I_n} b_h(\boldsymbol{q}_h, u_h) \,\mathrm{d}t = 0, \qquad \text{(4.4b)} \end{aligned}$$

where $(u, v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \int_K uv \,\mathrm{d}x$. We initialize the numerical scheme by choosing $u_0^- = P_h u_0$ on the first space-time slab $\mathcal{E}_h^0$, where $P_h : L^2(\Omega) \to V_h^{\mathrm{div}}$ is the L^2-projection onto $V_h^{\mathrm{div}} := \big\{u_h \in V_h \,:\, b_h(\boldsymbol{q}_h, u_h) = 0, \ \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h\big\}$, the discretely divergence free subspace of V_h. Here, we denote by $u_n^{\pm}$ the traces at time level t_n from above and below, i.e. $u_n^{\pm} = \lim_{\epsilon \to 0} u_h(t^n \pm \epsilon)$. We define the time jump operator at time t_n by $[u_h]_n = u_n^+ - u_n^-$.

The discrete forms $a_h(\cdot, \cdot) : \boldsymbol{V}_h \times \boldsymbol{V}_h \to \mathbb{R}$, $b_h(\cdot, \cdot) : V_h \times \boldsymbol{Q}_h \to \mathbb{R}$, and $o_h(\cdot; \cdot, \cdot) : V_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h \to \mathbb{R}$ appearing in eq. (4.4) serve as approximations to the viscous, pressure-velocity coupling, and convection terms, respectively. We define them as in Chapter 3:

$$a_h(\boldsymbol{u},\boldsymbol{v}) := \sum_{K\in\mathcal{T}_h}\int_K \nabla u : \nabla v \,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\frac{\alpha}{h_K}(u-\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s, \tag{4.5a}$$

$$- \sum_{K\in\mathcal{T}_h}\int_{\partial K}\big[(u-\bar{u})\cdot\partial_n v_h + \partial_n u\cdot(v-\bar{v})\big]\,\mathrm{d}s,$$

$$o_h(w;\boldsymbol{u},\boldsymbol{v}) := -\sum_{K\in\mathcal{T}_h}\int_K u\otimes w : \nabla v\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\tfrac{1}{2}w\cdot n(u+\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s \tag{4.5b}$$

$$+ \sum_{K\in\mathcal{T}_h}\int_{\partial K}\tfrac{1}{2}|w\cdot n|\,(u-\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s,$$

$$b_h(\boldsymbol{p},v) := -\sum_{K\in\mathcal{T}_h}\int_K p\nabla\cdot v\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K} v\cdot n\bar{p}\,\mathrm{d}s. \tag{4.5c}$$

Here, we slightly abuse notation by using n to denote the outward unit normal n_K to the element K for brevity. To ensure stability of the numerical scheme, $\alpha > 0$ must be chosen sufficiently large [77].

4.1.3.2 Preliminaries

In this subsection, we present some preliminaries and rapidly recall the main properties of the forms eq. (4.5) discussed in the previous chapter. Throughout this section and the rest of the chapter, we denote by $C > 0$ a generic constant independent of the mesh parameters h and Δt and the viscosity ν, but possibly dependent on the domain Ω, the polynomial degree k, and the spatial dimension d. At times we also use the notation $a \lesssim b$ to denote $a \leq Cb$. To set notation, let

$$V(h) := V_h + V\cap H^2(\Omega)^d, \quad \bar{V}(h) := \bar{V}_h + H^{3/2}(\partial\mathcal{T}_h)^d$$

and define the product space $\boldsymbol{V}(h) := V(h) \times \bar{V}(h)$. We introduce the following mesh-dependent inner-products and norms:

$$
\begin{aligned}
(\boldsymbol{u}, \boldsymbol{v})_{0,h} &:= (u, v)_{\mathcal{T}_h} + \sum_{K \in \mathcal{T}_h} h_K (u - \bar{u}, v - \bar{v})_{\partial K}, && \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(h), \\
\|v\|_{1,p,h}^p &:= \sum_{K \in \mathcal{T}_h} \|\nabla v\|_{L^p(K)}^p + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F^{p-1}} \big\| [\![v]\!] \big\|_{L^p(F)}^p, && \forall v \in V(h), \\
|\!|\!| \boldsymbol{v} |\!|\!|_v^2 &:= \sum_{K \in \mathcal{T}_h} \|\nabla v\|_{L^2(K)}^2 + \sum_{K \in \mathcal{T}_h} h_K^{-1} \|\bar{v} - v\|_{L^2(\partial K)}^2, && \forall \boldsymbol{v} \in \boldsymbol{V}(h), \\
|\!|\!| \boldsymbol{v} |\!|\!|_{v'}^2 &:= |\!|\!| \boldsymbol{v} |\!|\!|_v^2 + \sum_{K \in \mathcal{T}_h} h_K \big\| (\nabla v) n \big\|_{L^2(\partial K)}^2, && \forall \boldsymbol{v} \in \boldsymbol{V}(h), \\
|\!|\!| \boldsymbol{q} |\!|\!|_p^2 &:= \|q_h\|_{L^2(\Omega)}^2 + \sum_{K \in \mathcal{T}_h} h_K \|\bar{q}_h\|_{L^2(\partial K)}^2, && \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h,
\end{aligned}
$$

where we note that the equivalence constants of $|\!|\!| \cdot |\!|\!|_v$ and $|\!|\!| \cdot |\!|\!|_{v'}$ on the finite-dimensional space $\boldsymbol{V}_h$ are independent of the mesh size; see [77]. The bilinear form $a_h(\cdot, \cdot)$ is continuous and for sufficiently large α enjoys discrete coercivity [77, Lemmas 4.2 and 4.3], i.e. for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(h)$

$$
a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq C |\!|\!| \boldsymbol{v}_h |\!|\!|_v^2 \quad \text{and} \quad \big| a_h(\boldsymbol{u}, \boldsymbol{v}) \big| \leq C |\!|\!| \boldsymbol{u} |\!|\!|_{v'} |\!|\!| \boldsymbol{v} |\!|\!|_{v'}. \tag{4.8}
$$

The trilinear form $o_h(\cdot; \cdot, \cdot)$ satisfies [14, Proposition 3.6]

$$
o_h(w_h; \boldsymbol{v}_h, \boldsymbol{v}_h) = \frac{1}{2} \sum_{K \in \mathcal{T}} \int_{\partial K} |w_h \cdot n| |v_h - \bar{v}_h|^2 \, \mathrm{d}s \geq 0 \qquad w_h \in V_h^{\mathrm{div}}, \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{4.9}
$$

Further, the trilinear form $o_h(\cdot; \cdot, \cdot)$ is Lipschitz continuous in its first argument [14, Proposition 3.4]: for all $w_1, w_2 \in V(h)$, $\boldsymbol{u} \in \boldsymbol{V}(h)$ and $\boldsymbol{v} \in \boldsymbol{V}(h)$ it holds that

$$
\big| o_h(w_1; \boldsymbol{u}, \boldsymbol{v}) - o_h(w_2; \boldsymbol{u}, \boldsymbol{v}) \big| \leq C \|w_1 - w_2\|_{1,h} \, |\!|\!| \boldsymbol{u} |\!|\!|_v |\!|\!| \boldsymbol{v} |\!|\!|_v. \tag{4.10}
$$

4.1.4 Well-posedness and stability

To the best of the authors' knowledge, a rigourous study of well-posedness for higher-order space-time Galerkin schemes applied to the Navier–Stokes equations has yet to appear in the literature. We remark that for the low order scheme ($k = 1$), uniqueness of the discrete solution is a consequence of the following energy estimate which we will derive in Section 4.3:

Lemma 4.1.1. *Let $d = 2$ or 3, $k \geq 1$, and suppose that $\boldsymbol{u}_h \in \boldsymbol{\mathcal{V}}_h$ is an approximate velocity solution of the Navier–Stokes equations computed using the space-time HDG scheme eq. (4.4) for $n = 0, \ldots, N-1$. There exists a constant $C > 0$, independent of the mesh parameters Δt and h and the viscosity ν but dependent on the domain Ω and polynomial degree k, such that*

$$\left\|u_N^-\right\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1}\left\|[u_h]_n\right\|_{L^2(\Omega)}^2 + \nu\int_0^T |||\boldsymbol{u}_h|||_v^2 \,\mathrm{d}t \leq C\left(\frac{1}{\nu}\int_0^T \|f\|_{L^2(\Omega)}^2 \,\mathrm{d}t \; + \left\|u_0\right\|_{L^2(\Omega)}^2\right),$$

where we define the time jump operator at time t_n as $[u_h]_n = u_n^+ - u_n^-$.

However, for higher order schemes in time ($k \geq 2$), this energy bound is insufficient to prove the uniqueness of the discrete solution since $\operatorname{ess\,sup}_{0<t\leq T} u_h(t)$ need not be not attained at the partition points of the time-interval for higher order schemes in time. Consequently, the energy bound in Lemma 4.1.1 does not bound the discrete velocity solution u_h in $L^\infty(0,T;L^2(\Omega)^d)$. To overcome this challenge, we will make use of tools introduced by Chrysafinos and Walkington [15, 16, 17]. We begin by introducing the exponential interpolant from [17]:

Definition 4.1.1 (Exponential interpolant [17])**.** *Let V be a linear space and $\lambda > 0$ be given. If $v = \sum_{i=0}^k \phi_i(t)v_i \in P_k(I_n;V)$ where $\phi_i(t) \in P_k(I_n)$, $v_i \in V$, the exponential interpolant of v is defined by*

$$\tilde{v} = \sum_{i=0}^k \tilde{\phi}_i(t)v_i,$$

where $\tilde{r}_i(t) \in P_k(I_n)$ is an approximation of $r_i(t)e^{-\lambda(t-t_n)}$ satisfying $\tilde{r}_i(t_n^+) = r_i(t_n^+)$ and

$$\int_{I_n} \tilde{r}_i(t)q(t)\,\mathrm{d}t = \int_{I_n} r_i(t)q(t)e^{-\lambda(t-t_n)}\,\mathrm{d}t, \quad \forall q \in P_{k-1}(I_n). \tag{4.11}$$

Next, we summarize the important properties of the exponential interpolant from [17, Lemma 3.4 and Lemma 3.6]:

Lemma 4.1.2. *Let V and Q be linear spaces and $v \mapsto \tilde{v}$ the exponential interpolant constructed in Definition 4.1.1. If $L(\cdot,\cdot) : V \times Q \to \mathbb{R}$ is a bilinear mapping and $v \in P_k(I_n, V)$, then*

$$\int_{I_n} L(\tilde{v}(t), q(t))\,\mathrm{d}t = \int_{I_n} L(v(t), q(t))e^{-\lambda(t-t_n)}\,\mathrm{d}t, \quad \forall q \in P_{k-1}(I_n, Q). \tag{4.12}$$

If $(\cdot,\cdot)_V$ is a semi-inner product on V, then there exists a constant $C > 0$, dependent only on the polynomial degree k, such that for all $v \in P_k(I_n; V)$ and $1 \le p \le \infty$,

$$\|\tilde{v}\|_{L^p(I_n;V)} \le C\|v\|_{L^p(I_n;V)} . \tag{4.13}$$

We now discuss the concept of a discrete characteristic function, introduced by Chrysafinos and Walkington [15, 16, 17] for discontinuous Galerkin time stepping schemes combined with conforming spatial discretizations, and extended to full space-time discontinuous Galerkin discretizations in e.g. [29, 100]. As we have previously noted, given a polynomial $p \in P_k(I_n)$ and a fixed time $s \in (t_n, t_{n+1})$, the function $\chi_{(t_n,s]}p$ no longer lies in $P_k(I_n)$ in general and hence is not an admissable test function in our discrete scheme. In essence, the discrete characteristic function provides us with a discrete approximation of $\chi_{(t_n,s]}p$.

The discrete characteristic function is constructed in two steps. First, given a fixed time $s \in (t_n, t_{n+1})$, we define $p_\chi \in P_k(I_n)$ as the unique polynomial satisfying $p_\chi(t_n^+) = p(t_n^+)$, and

$$\int_{t_n}^{t_{n+1}} p_\chi q \,\mathrm{d}t = \int_{t_n}^{s} pq \,\mathrm{d}t, \quad \forall q \in P_{k-1}(I_n). \tag{4.14}$$

This induces a continuous mapping $p \mapsto \tilde{p}$. Next, given a semi-inner product space V, this construction is extended to approximate functions of the form $\chi_{(t_n,s]}v$ where $v \in P_k(I_n; V)$:

Definition 4.1.2 (Discrete characteristic function [17])**.** *Let V be a semi-inner product space and fix $s \in (t_n, t_{n+1})$. The discrete characteristic function of $v \in P_k(I_n; V)$ is defined as the function $v_\chi \in P_k(I_n; V)$ satisfying $v_\chi(t_n^+) = v(t_n^+)$ and*

$$\int_{t_n}^{t_{n+1}} (v_\chi, w)_V \,\mathrm{d}t = \int_{t_n}^{s} (v, w)_V \,\mathrm{d}t, \quad \forall w \in P_{k-1}(I_n; V).$$

In the following lemma, we summarize the important properties of the discrete characteristic function from [17, Lemma 3.1 and Lemma 3.2]:

Lemma 4.1.3. *Let V be a semi-inner product space. The mapping*

$$v = \sum_{i=0}^{k} \phi_i(t) v_i \mapsto v_\chi = \sum_{i=0}^{k} (\phi_i)_\chi(t) v_i$$

on $P_k(I_n; V)$ is continuous in the sense that there exists a constant $C > 0$, depending only upon the polynomial degree k, such that

$$\|v_\chi\|_{L^2(I_n,V)} \le C \|v\|_{L^2(I_n,V)} . \tag{4.15}$$

Moreover, in the case where $v(t) = z$ is constant in time, we can characterize its discrete characteristic function as $v_\chi = p(t)z$ for $p \in P_k(I_n)$ satisfying $p(t_n^+) = 1$ and

$$\int_{t_n}^{t_{n+1}} pq \,\mathrm{d}t = \int_{t_n}^{s} q \,\mathrm{d}t, \quad \forall q \in P_{k-1}(I_n). \tag{4.16}$$

Further, we have the following bound in $L^\infty(I_n)$ on p:

$$\|p\|_{L^\infty(I_n)} \leq C, \tag{4.17}$$

where the constant C depends only on the polynomial degree k.

With the help of these tools, it is possible to bound the discrete solution u_h in $L^\infty(0, T; L^2(\Omega)^d)$ in two spatial dimensions:

Lemma 4.1.4. *Let $d = 2$, $k \geq 1$, and suppose $\boldsymbol{u}_h \in \boldsymbol{\mathcal{V}}_h$ is an approximate velocity solution of the Navier–Stokes equations computed using the space-time HDG scheme eq. (4.4) for $n = 0, \ldots, N-1$. There exists a constant $C > 0$, independent of the mesh parameters Δt and h and the viscosity ν but dependent on the domain Ω and polynomial degree k, such that*

$$\begin{aligned}
&\|u_h\|^2_{L^\infty(0,T;L^2(\Omega))} \\
&\leq C\left(\frac{1}{\nu}\int_0^T \|f\|^2_{L^2(\Omega)} \,\mathrm{d}t \;+\; \left\|u_0\right\|^2_{L^2(\Omega)}\right) + \frac{C}{\nu^2}\left(\frac{1}{\nu}\int_0^T \|f\|^2_{L^2(\Omega)} \,\mathrm{d}t \;+\; \left\|u_0\right\|^2_{L^2(\Omega)}\right)^2.
\end{aligned}$$

With this bound in hand, we can prove the following uniqueness result in two dimensions for the solution of the nonlinear system of algebraic equations arising from the discrete scheme eq. (4.4):

Theorem 4.1.1 (Uniqueness in two dimensions)**.** *Let $\boldsymbol{u}_h \in \boldsymbol{\mathcal{V}}_h$ be an approximate velocity solution of the Navier–Stokes equations computed using the space-time HDG scheme eq. (4.4) for $n = 0, \ldots, N-1$. If $d = 2$, there exists a constant $C > 0$, independent of the mesh parameters Δt and h and the viscosity ν but dependent on the domain Ω and polynomial degree k, such that if the problem data satisfies eq. (4.2) then $\boldsymbol{u}_h$ is the unique velocity solution to eq. (4.4).*

We defer the proofs of Lemma 4.1.4 and Theorem 4.1.1 to Section 4.3. In addition to the bound on u_h in $L^\infty(0, T; L^2(\Omega)^d)$, the other key ingredient for proving Theorem 4.1.1 is a novel discrete version of the classic Ladyzhenskaya inequality eq. (2.16) valid for broken polynomial spaces. We will discuss this further in Section 4.2. Note that, similar to the continuous theory, the scaling of the exponents in the discrete Ladyzhenskaya inequality in three spatial dimensions prevents us from extending the proof of uniqueness to $d = 3$.

4.1.5 Error analysis

Our main result is a *pressure-robust* error estimate for the approximate velocity arising from the numerical scheme eq. (4.4) under the assumption that the problem data satisfies eq. (4.2):

Theorem 4.1.2 (Velocity error)**.** *Let u be the strong velocity solution to the Navier–Stokes system eq.* (4.1) *guaranteed by Theorem 2.3.2 and assume it further satisfies*

$$u \in H^{k+1}(0,T; V \cap H^2(\Omega)^d) \cap H^1(0,T; H^{k+1}(\Omega)^d),$$

with initial data $u_0 \in H^{k+1}(\Omega)^d$. Let $(u_h, \bar{u}_h) \in \boldsymbol{\mathcal{V}}_h$ be an approximate velocity solution to the Navier–Stokes system computed using the space-time HDG scheme eq. (4.4) *for $n = 0, \dots, N-1$, Then, there exists a constant $C > 0$ such that the error $\boldsymbol{e}_h = (u-u_h, \gamma(u)-\bar{u}_h)$ satisfies*

$$\left\|e_N^-\right\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1}\left\|[e_h]_n\right\|_{L^2(\Omega)}^2 + \nu\int_0^T \|\!|\boldsymbol{e}_h|\!\|_{v'}^2 \,\mathrm{d}t \le \exp\Big(CT\Big)\Big(h^{2k} + \Delta t^{2k+2}\Big)\,C(u),$$

provided the time step satisfies $\Delta t \lesssim \nu$. Here, $C(u)$ depends on Sobolev–Bochner norms of the velocity u, but is independent of the pressure p.

The proof of Theorem 4.1.2 is deferred to Section 4.4. We remark that the time step restriction $\Delta t \lesssim \nu$ in Theorem 4.1.2 is necessary in the proof of this theorem to use a discrete Grönwall inequality; it is not necessary for the stability of the space-time HDG method eq. (4.4), but rather to *quantify the asymptotic rates of convergence.*

Theorem 4.1.3 (Pressure error)**.** *Let $(u_h, \bar{u}_h, p_h, \bar{p}_h) \in \boldsymbol{\mathcal{V}}_h \times \boldsymbol{\mathcal{Q}}_h$ be the approximate solution to the Navier–Stokes system computed using the space-time HDG scheme eq.* (4.4) *for $n = 0, \dots N-1$ and let the solution (u,p) to the Navier–Stokes system satisfy*

$$u \in H^{k+1}(0,T; V \cap H^2(\Omega)^d) \cap H^1(0,T; H^{k+1}(\Omega)^d), \quad p \in H^{k+1}(0,T; H^{k+1}(\Omega) \cap L_0^2(\Omega)),$$

with initial data satisfying $u_0 \in H^{k+1}(\Omega)^d$.

There exists a constant $C > 0$, independent of the mesh parameters Δt and h and the viscosity ν but dependent on the domain Ω and polynomial degree k, such that the error

$$\int_0^T \|p - p_h\|_{L^2(\Omega)}^2 \,\mathrm{d}t \le \exp(CT)\nu^{-1}\left(\Delta t^{2k} + \frac{h^{2k}}{\Delta t^{3/2}}\right) C(u,p),$$

provided the time step satisfies $\Delta t \lesssim \nu$, where the hidden constant is independent of the mesh parameters Δt and h and the viscosity ν. Here, $C(u,p)$ is a constant dependent on Sobolev–Bochner norms of the continuous velocity and pressure (u,p).

The proofs of these main results are deferred to Sections 4.4 and 4.5 after we introduce some of the essential tools for our analysis in the next section.

4.2 Preliminary results

4.2.1 Properties of the numerical scheme

Let $\mathcal{V}_h^{\text{div}}$ denote the subspace of $\mathcal{V}_h$ of discrete divergence free velocity fields:

$$\mathcal{V}_h^{\text{div}} = \left\{ u_h \in \mathcal{V}_h \ : \ \int_{I_n} b_h(\boldsymbol{q}_h, u_h)\,\mathrm{d}t = 0,\ \forall \boldsymbol{q}_h \in \boldsymbol{\mathcal{Q}}_h,\ \forall n = 0, \dots, N-1 \right\}.$$

The following result motivates the use of equal order polynomial degrees in time for both the velocity and pressure approximation spaces:

Lemma 4.2.1. $\mathcal{V}_h^{div} = \left\{ v_h \in \mathcal{V}_h \mid v_h|_{\mathcal{E}^n} \in P_k(I_n; V_h^{div}),\ \forall n = 0, \dots, N-1 \right\}$.

Proof. The proof is very similar to that of [17, Lemma 2.3] with minor modifications and is therefore omitted. □

An immediate consequence of Lemma 4.2.1 is that $u_h(t) \in H$ a.e. $t \in (0, T)$ where H is defined in eq. (2.22). To see this we first expand u_h in terms of an orthonormal basis $\{\phi_i\}_{i=0}^k$ of $P_k(I_n)$ with respect to the $L^2(I_n)$ inner-product:

$$u_h = \sum_{i=0}^{k} \phi_i(t) u_i(x), \quad u_i \in V_h. \tag{4.18}$$

By Lemma 4.2.1, $u_h \in P_k(I_n; V_h^{\text{div}})$, so $u_i \in V_h^{\text{div}}$ for each $i = 0, \dots, k$. Thus,

$$0 = b_h(\boldsymbol{q}_h, u_i) = -\sum_{K \in \mathcal{T}_h} \int_K q_h \nabla \cdot u_i \,\mathrm{d}x + \sum_{K \in \mathcal{T}_h} \int_{\partial K} u_i \cdot n \bar{q}_h \,\mathrm{d}s, \quad \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h.$$

Following the same arguments as [78, Proposition 1] it follows that $\nabla \cdot u_i = 0$ for all $x \in K$, $[\![u_i \cdot n]\!] = 0$ on all $F \in \mathcal{F}_h^{\text{int}}$, and $u_i \cdot n = 0$ on $\partial\Omega$ for $i = 0, \dots, k$. By eq. (4.18) and since H is a linear space the result follows.

Lemma 4.2.2 (Consistency)**.** *Let (u,p) be the strong solution to the Navier–Stokes system eq. (4.1) guaranteed by Theorem 2.3.2. Define $\boldsymbol{u} = (u, \gamma(u))$ and $\boldsymbol{p} = (p, \gamma(p))$. Then, it holds that*

$$-\int_{I_n} (u, \partial_t v_h)_{\mathcal{T}_h} \,\mathrm{d}t + (u(t_{n+1}), v_{n+1}^-)_{\mathcal{T}_h} + \int_{I_n} (\nu a_h(\boldsymbol{u}, \boldsymbol{v}_h) + o_h(u; \boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{p}, v_h)) \,\mathrm{d}t \\ - \int_{I_n} b_h(\boldsymbol{q}_h, u) \,\mathrm{d}t = (u(t_n), v_n^+)_{\mathcal{T}_h} + \int_{I_n} (f, v_h)_{\mathcal{T}_h} \,\mathrm{d}t, \quad \forall (\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{\mathcal{V}}_h \times \boldsymbol{\mathcal{Q}}_h,$$

where $a_h(\cdot,\cdot)$, $o_h(\cdot;\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ are defined in eq. (3.8a), eq. (3.8b), and eq. (3.8c), respectively.

4.2.2 Scalings and embeddings

We begin by recalling a number of results for piece-wise polynomials. First, for polynomials in time, let $(V, (\cdot,\cdot)_V)$ be an inner product space. Then, there exists $C > 0$ such that for all $v \in P_k(I_n, V)$ (see e.g. [17, Lemma 3.5]):

$$\|v\|_{L^p(I_n,V)} \le C\Delta t^{1/p-1/2} \|v\|_{L^2(I_n,V)}, \qquad 1 \le p \le \infty, \tag{4.19a}$$

$$\|\partial_t v\|_{L^2(I_n,V)} \le C\Delta t^{-1} \|v\|_{L^2(I_n,V)}. \tag{4.19b}$$

Next, we recall the following discrete version of the Sobolev embedding theorem valid for broken polynomial spaces $P_r(\mathcal{T}_h) = \left\{ f \in L^2(\Omega) \mid f|_K \in P_r(K),\ \forall K \in \mathcal{T}_h \right\}$ where $r \ge 0$. Let $1 \le p < \infty$, then for all q satisfying $1 \le q \le pd/(d-p)$ if $1 \le p < d$, or $1 \le q < \infty$ if $d \le p < \infty$, there exists a constant $C > 0$ such that [28, Theorem 5.3]:

$$\|v_h\|_{L^q(\Omega)} \le C \|v_h\|_{1,p,h}, \quad \forall v_h \in P_r(\mathcal{T}_h). \tag{4.20}$$

In the case $p = 2$, we write $\|\cdot\|_{1,2,h} = \|\cdot\|_{1,h}$. Note that choosing $p = q = 2$ in eq. (4.20) yields the discrete Poincaré inequality: $\|v_h\|_{L^2(\Omega)} \le C_P \|v_h\|_{1,h}$ for all $v_h \in V_h$. By the triangle inequality, $\|v_h\|_{1,h} \le |||\boldsymbol{v}_h|||_v$, so that

$$\|v_h\|_{L^2(\Omega)} \le C_P |||\boldsymbol{v}_h|||_\nu, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{4.21}$$

We now prove a discrete version of the Ladyzhenskaya inequalities valid for broken polynomial spaces. While the analogue of these inequalities are well known in the context of H^1-conforming finite element methods [32], to our knowledge they have yet to be extended to *non-conforming* finite element spaces.

Lemma 4.2.3 (Ladyzhenskaya inequality for broken polynomial spaces)**.** *There exists a constant $C > 0$ such that for $d \in \{2, 3\}$:*

$$\|v_h\|_{L^4(\Omega)} \le C\|v_h\|_{L^2(\Omega)}^{1/2(d-1)} \|v_h\|_{1,h}^{3/2(5-d)}, \quad \forall v_h \in V_h. \tag{4.22}$$

Proof. It suffices to consider the scalar case. We focus first on the case $d = 2$. Inserting v_h^2 into eq. (4.20) with $r = 2k$, $p = 1$, and $q = 2$ yields $\|v_h\|_{L^4(\Omega)}^2 \le C\|v_h^2\|_{1,1,h}$. The result follows after noting that the right-hand side can be bounded by applying the Cauchy–Schwarz inequality and a local discrete trace inequality $\|v_h\|_{L^2(F)} \le Ch_K^{-1/2}\|v_h\|_{L^2(K)}$ [28, Lemma 1.46]:

$$\frac{1}{2}\|v_h^2\|_{1,1,h} = \sum_{K\in\mathcal{T}_h}\int_K \left|(\nabla v_h)v_h\right| \mathrm{d}x + \sum_{F\in\mathcal{F}_h}\int_F \left|[\![v_h]\!] \cdot \{\!\{v_h\}\!\}\right| \mathrm{d}s \le C\|v_h\|_{L^2(\Omega)}\|v_h\|_{1,h}.$$

For the case $d = 3$, the result follows from the Cauchy–Schwarz inequality and eq. (4.20) with $q = 6$ and $p = 2$. □

For $d = 3$, interpolating between $L^2(\Omega)^d$ and $L^4(\Omega)^d$ and using eq. (4.22) yields:

$$\|v_h\|_{L^3(\Omega)} \le C\|v_h\|_{L^2(\Omega)}^{1/2}\|v_h\|_{1,h}^{1/2}, \quad \forall v_h \in V_h. \tag{4.23}$$

4.3 Well-posedness of the discrete problem

4.3.1 Existence of a discrete solution

We will begin by showing the existence of a solution to the nonlinear system of algebraic equations arising from eq. (4.4) using a topological degree argument (Lemma 2.1.2). We first require the proof of Lemma 4.1.1 and well-posedness of the space-time HDG discretization of a linear time-dependent Stokes problem as discussed next.

4.3.1.1 Proof of Lemma 4.1.1.

Proof. Testing eq. (4.4) with $(v_h, \bar{v}_h, q_h, \bar{q}_h) = (u_h, \bar{u}_h, p_h, \bar{p}_h)$, using the coercivity of $a_h(\cdot,\cdot)$ and the fact that $o_h(\cdot; \boldsymbol{u}_h, \boldsymbol{u}_h) \ge 0$, and integrating by parts in time, we find that there exists a constant $C_1 > 0$ such that

$$\frac{1}{2}\|u_{n+1}^-\|_{L^2(\Omega)}^2 + \frac{1}{2}\|[u_h]_n\|_{L^2(\Omega)}^2 - \frac{1}{2}\|u_n^-\|_{L^2(\Omega)}^2 + C_1\nu\int_{I_n} |\!|\!|\boldsymbol{u}_h|\!|\!|_v^2 \,\mathrm{d}t \le \int_{I_n} (f, u_h)_{\mathcal{T}_h} \,\mathrm{d}t.$$

To bound the right-hand side, we apply the Cauchy–Schwarz inequality, the discrete Poincaré inequality eq. (4.21), and Young's inequality with $\epsilon = C_1/(C_P\nu) > 0$. Rearranging, we see there is a constant $C_2 > 0$ such that

$$\left\|u_{n+1}^-\right\|_{L^2(\Omega)}^2 + \left\|[u_h]_n\right\|_{L^2(\Omega)}^2 - \left\|u_n^-\right\|_{L^2(\Omega)}^2 + \nu C_1 \int_{I_n} \|\!|\boldsymbol{u}_h|\!\|_v^2 \,\mathrm{d}t \le \frac{C_2}{\nu} \int_{I_n} \|f\|_{L^2(\Omega)}^2 \,\mathrm{d}t.$$

The result follows after summing over all space-time slabs. □

4.3.1.2 A linearized problem

Before we can apply the topological degree argument, we will need to study the space-time HDG solution of the linear time-dependent Stokes problem:

$$\partial_t u - \nu\Delta u + \nabla p = f, \quad \text{in } \Omega \times (0,T], \tag{4.24a}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega \times (0,T], \tag{4.24b}$$

$$u = 0, \quad \text{on } \partial\Omega \times (0,T], \tag{4.24c}$$

$$u(x,0) = u_0(x), \quad \text{in } \Omega. \tag{4.24d}$$

Lemma 4.3.1. *There exists a unique pair $\boldsymbol{u}_h \in \mathcal{V}_h^{div} \times \bar{\mathcal{V}}_h$ such that for all $\boldsymbol{v}_h \in \mathcal{V}_h^{div} \times \bar{\mathcal{V}}_h$:*

$$-\int_{I_n} (u_h, \partial_t v_h)_{\mathcal{T}_h} \,\mathrm{d}t + (u_{n+1}^-, v_{n+1}^-)_{\mathcal{T}_h} + \int_{I_n} \nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \,\mathrm{d}t = (u_n^-, v_n^+)_{\mathcal{T}_h} + \int_{I_n} (f, v_h)_{\mathcal{T}_h} \,\mathrm{d}t. \tag{4.25}$$

Note that this is simply the space-time HDG scheme applied to the time-dependent Stokes problem eq. (4.24).

Proof. The result follows from the Lax–Milgram theorem (Theorem 2.1.2) □

4.3.1.3 The topological degree argument

Theorem 4.3.1. *Let $d \in \{2,3\}$ and $k \ge 1$. There exists at least one discrete velocity solution $\boldsymbol{u}_h \in \mathcal{V}_h^{div} \times \bar{\mathcal{V}}_h$ to eq. (4.4) for $n = 0,\dots,N-1$ satisfying the energy estimate Lemma 4.1.1.*

Proof. We set $X = \mathcal{V}_h^{\mathrm{div}} \times \bar{\mathcal{V}}_h$ and equip it with the norm

$$\|\boldsymbol{u}_h\|_X^2 := \left\|u_N^-\right\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \left\|[u_h]_n\right\|_{L^2(\Omega)}^2 + \nu \int_0^T \|\!|\boldsymbol{u}_h|\!\|_v^2 \,\mathrm{d}t.$$

Define the continuous mapping $\Psi : X \times [0,1] \to X$ for $n = 0, \ldots, N-1$, by

$$\begin{aligned}\int_{I_n} (\Psi(\boldsymbol{u}_h, \rho), \boldsymbol{v}_h)_{0,h} \,\mathrm{d}t = &- \int_{I_n} (u_h, \partial_t v_h)_{\mathcal{T}_h} \,\mathrm{d}t + (u_{n+1}^-, v_{n+1}^-)_{\mathcal{T}_h} + \int_{I_n} \nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \,\mathrm{d}t \\ &+ \int_{I_n} \rho o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \,\mathrm{d}t - (u_n^-, v_n^+)_{\mathcal{T}_h} - \int_{I_n} (f, v_h)_{\mathcal{T}_h} \,\mathrm{d}t.\end{aligned}$$

Ψ is well-defined by the Riesz representation theorem (Theorem 2.1.1), verifying item (1) in Lemma 2.1.2. Next, we choose $\boldsymbol{u}_h \in X$ such that $\Psi(\boldsymbol{u}_h, \rho) = 0$ for some $\rho \in [0,1]$. Since $o_h(u_h; \boldsymbol{u}_h, \boldsymbol{u}_h) \geq 0$, we can repeat the proof of Lemma 4.1.1 to bound $\boldsymbol{u}_h$ uniformly with respect to ρ:

$$\left\|u_N^-\right\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \left\|[u_h]_n\right\|_{L^2(\Omega)}^2 + \nu \int_0^T |\!|\!| \boldsymbol{u}_h |\!|\!|_v^2 \,\mathrm{d}t \leq C \left(\frac{1}{\nu} \int_0^T \|f\|_{L^2(\Omega)}^2 \,\mathrm{d}t \; + \left\|u_0\right\|_{L^2(\Omega)}^2 \right),$$

which verifies item (3) in Lemma 2.1.2 with

$$M^2 = C \left(\frac{1}{\nu} \int_0^T \|f\|_{L^2(\Omega)}^2 \,\mathrm{d}t \; + \left\|u_0\right\|_{L^2(\Omega)}^2 \right) + \epsilon,$$

for any $\epsilon > 0$. Finally, note that $\Psi(\cdot, 0) : X \to X$ is an affine function since the nonlinear convection term disappears for $\rho = 0$. By Lemma 4.3.1, there exists a solution to $\Psi(\boldsymbol{u}_h, 0) = 0$, verifying item (2) in Lemma 2.1.2. Therefore, there exists a solution $\boldsymbol{u}_h$ to $\Psi(\boldsymbol{u}_h, 1) = 0$ satisfying $\|\boldsymbol{u}_h\|_X < M$. Equivalently, $\boldsymbol{u}_h \in \mathcal{V}_h^{\mathrm{div}} \times \bar{\mathcal{V}}_h$ solves eq. (4.4) for all $\boldsymbol{v}_h \in \mathcal{V}_h^{\mathrm{div}} \times \bar{\mathcal{V}}_h$ and satisfies the energy bound in Lemma 4.1.1. □

4.3.2 Uniqueness of the discrete velocity in two dimensions

4.3.2.1 Bounds on the convection term

In the analysis that follows, we will require tighter bounds on the trilinear convection form than is provided by eq. (4.10). For this, we will make extensive use of the results of Section 4.2.2. We remark that, although we focus on $d = 2$ for the proof of uniqueness, the bound eq. (4.27) will be essential for the error analysis in both two and three spatial dimensions in Section 4.4.

Lemma 4.3.2. *If $d = 2$, there exists a $C > 0$ such that for all $\boldsymbol{w}_h, \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h$,*

$$\begin{aligned}&|o_h(w_h; \boldsymbol{u}_h, \boldsymbol{v}_h)| \\ &\leq C \|w_h\|_{L^2(\Omega)}^{1/2} |\!|\!| \boldsymbol{u}_h |\!|\!|_v^{1/2} |\!|\!| \boldsymbol{v}_h |\!|\!|_v \left(\|w_h\|_{L^2(\Omega)}^{1/2} |\!|\!| \boldsymbol{u}_h |\!|\!|_v^{1/2} + \|u_h\|_{L^2(\Omega)}^{1/2} |\!|\!| \boldsymbol{w}_h |\!|\!|_v^{1/2} \right).\end{aligned} \tag{4.26}$$

Moreover, if $d \in \{2, 3\}$, there exists a $C > 0$ such that for all $\boldsymbol{w}_h, \boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$|o_h(w_h; \boldsymbol{u}_h, \boldsymbol{v}_h)| \le C\|w_h\|_{L^2(\Omega)}^{1/2} \|\!|\boldsymbol{w}_h|\!\|_v^{1/2} \|\!|\boldsymbol{u}_h|\!\|_v \|\!|\boldsymbol{v}_h|\!\|_v. \tag{4.27}$$

Proof. This proof relies on the following scaling identity for

$$\mu \in R_k(\partial K) := \left\{\mu \in L^2(\partial K) \mid \mu|_F \in P_k(F),\ \forall F \subset \partial K\right\}$$

between L^p and L^2 norms on element boundaries which can be obtained using standard arguments:

$$\|\mu\|_{L^p(\partial K)} \le Ch^{(d-1)(1/p-1/2)}\|\mu\|_{L^2(\partial K)}, \qquad 2 \le p < \infty. \tag{4.28}$$

Now, split $o_h(w_h, \boldsymbol{u}_h, \boldsymbol{v}_h)$ into three terms and bound each separately. Using that $u_h + \bar{u}_h = 2u_h + (\bar{u}_h - u_h)$, we find:

$$\begin{aligned}
&|o_h(w_h, \boldsymbol{u}_h, \boldsymbol{v}_h)| \\
&\le \sum_{K\in\mathcal{T}_h} \int_K |(u_h \otimes w_h) : \nabla v_h|\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h} \int_{\partial K} |w_h \cdot n\,(u_h) \cdot (v_h - \bar{v}_h)|\,\mathrm{d}s \\
&\quad + \sum_{K\in\mathcal{T}_h} \int_{\partial K} |w_h \cdot n||(u_h - \bar{u}_h) \cdot (v_h - \bar{v}_h)|\,\mathrm{d}s = T_1 + T_2 + T_3.
\end{aligned} \tag{4.29}$$

To show eq. (4.26), we first apply the generalized Hölder inequality to T_1 with $p = q = 4$ and $r = 2$, the Cauchy–Schwarz inequality, and eq. (4.22) to find:

$$|T_1| \le C\|u_h\|_{L^2(\Omega)}^{1/2} \|\!|\boldsymbol{u}_h|\!\|_v^{1/2} \|w_h\|_{L^2(\Omega)}^{1/2} \|\!|\boldsymbol{w}_h|\!\|_v^{1/2} \|\!|\boldsymbol{v}_h|\!\|_v.$$

To bound T_2, we apply the generalized Hölder inequality with $p = q = 4$ and $r = 2$ and use the local discrete trace inequality $\|v_h\|_{L^p(\partial K)} \le Ch_K^{-1/p}\|v_h\|_{L^p(K)}$ (see e.g. [28, Lemma 1.52]) for $p = 4$, we have

$$|T_2| \le C \sum_{K\in\mathcal{T}_h} \|w_h\|_{L^4(K)} \|u_h\|_{L^4(K)}\, h_K^{-1/2} \|v_h - \bar{v}_h\|_{L^2(\partial K)}.$$

Applying the Cauchy–Schwarz inequality and eq. (4.22) we find

$$|T_2| \le C\|u_h\|_{L^2(\Omega)}^{1/2} \|\!|\boldsymbol{u}_h|\!\|_v^{1/2} \|w_h\|_{L^2(\Omega)}^{1/2} \|\!|\boldsymbol{w}_h|\!\|_v^{1/2} \|\!|\boldsymbol{v}_h|\!\|_v.$$

To bound T_3, we again apply the generalized Hölder inequality with $p = q = 4$ and $r = 2$, the local discrete trace inequality $\|v_h \cdot n\|_{L^2(\partial K)} \le Ch_K^{-1/2}\|v_h\|_{L^2(K)}$, eq. (4.28) with $d = 2$ and $p = 4$, and the Cauchy–Schwarz inequality to find

$$|T_3| \le C\|w_h\|_{L^2(\Omega)} \|\!|\boldsymbol{u}_h|\!\|_v \|\!|\boldsymbol{v}_h|\!\|_v.$$

Summing the bounds on T_1, T_2, and T_3 yields the result.

The proof of eq. (4.27) differs in the cases of $d = 2$ and $d = 3$. We begin with $d = 3$. To bound T_1, we first apply the generalized Hölder inequality with $p = 3$, $q = 6$ and $r = 2$ followed by the Cauchy–Schwarz inequality to find:

$$|T_1| \le \|w_h\|_{L^3(\Omega)} \|u_h\|_{L^6(\Omega)} \, |\!|\!|\boldsymbol{v}_h|\!|\!|_v .$$

Now applying eq. (4.23) and eq. (4.20) with $q = 6$, we have

$$|T_1| \le C \|w_h\|_{L^2(\Omega)}^{1/2} \, |\!|\!|\boldsymbol{w}_h|\!|\!|_v^{1/2} |\!|\!|\boldsymbol{u}_h|\!|\!|_v |\!|\!|\boldsymbol{v}_h|\!|\!|_v .$$

To bound T_2, we apply the generalized Hölder inequality with $p = 3$, $q = 6$ and $r = 2$ to find

$$|T_2| \le \sum_{K \in \mathcal{T}_h} \|w_h \cdot n\|_{L^3(\partial K)} \|u_h\|_{L^6(\partial K)} \|v_h - \bar{v}_h\|_{L^2(\partial K)} .$$

Next, using the local discrete trace inequality $\|v_h\|_{L^p(\partial K)} \le C h_K^{-1/p} \|v_h\|_{L^p(K)}$ (see e.g. [28, Lemma 1.52]) for $p = 3$ and $p = 6$, the Cauchy–Schwarz inequality, eq. (4.23), and eq. (4.20) with $q = 6$, we have

$$|T_2| \le C \|w_h\|_{L^2(\Omega)}^{1/2} \, |\!|\!|\boldsymbol{w}_h|\!|\!|_v^{1/2} |\!|\!|\boldsymbol{u}_h|\!|\!|_v |\!|\!|\boldsymbol{v}_h|\!|\!|_v .$$

To bound T_3, we again apply the generalized Hölder inequality with $p = 3$, $q = 6$ and $r = 2$:

$$|T_3| \le \sum_{K \in \mathcal{T}_h} \|w_h \cdot n\|_{L^3(\partial K)} \|u_h - \bar{u}_h\|_{L^6(\partial K)} \|v_h - \bar{v}_h\|_{L^2(\partial K)} .$$

Now, applying the local discrete trace inequality $\|v_h \cdot n\|_{L^p(\partial K)} \le C h_K^{-1/p} \|v_h\|_{L^p(K)}$ with $p = 3$, eq. (4.28) with $d = 3$ and $p = 6$, the discrete Cauchy–Schwarz inequality, and eq. (4.23), we have

$$|T_3| \le C \|w_h\|_{L^2(\Omega)}^{1/2} \, |\!|\!|\boldsymbol{w}_h|\!|\!|_v^{1/2} |\!|\!|\boldsymbol{u}_h|\!|\!|_v |\!|\!|\boldsymbol{v}_h|\!|\!|_v .$$

Summing the bounds on T_1, T_2, and T_3 yields the result. The case for $d = 2$ follows similarly, instead using $p = q = 4$ and $r = 2$ in the generalized Hölder inequality, eq. (4.20) with $q = 4$, and eq. (4.28) with $d = 2$ and $p = 4$. □

4.3.2.2 Proof of Lemma 4.1.4

Proof. Let $\tilde{\boldsymbol{u}}_h$ be the exponential interpolant of $\boldsymbol{u}_h$ as defined in eq. (4.11). Testing eq. (4.4) with $\tilde{\boldsymbol{u}}_h$, integrating by parts in time, and using the defining properties of the exponential interpolant, we have

$$\frac{1}{2}\int_{I_n} \tfrac{\mathrm{d}}{\mathrm{d}t}\left\|u_h(t)\right\|^2_{L^2(\Omega)} e^{-\lambda(t-t_n)}\,\mathrm{d}t + \left\|u_n^+\right\|^2_{L^2(\Omega)} + \int_{I_n} (\nu a_h(\boldsymbol{u}_h, \tilde{\boldsymbol{u}}_h) + o_h(u_h; \boldsymbol{u}_h, \tilde{\boldsymbol{u}}_h))\,\mathrm{d}t \\ = (u_n^-, u_n^+)_{\mathcal{T}_h} + \int_{I_n} (f, \tilde{\boldsymbol{u}}_h)_{\mathcal{T}_h}\,\mathrm{d}t.$$

Integrating by parts again in time and applying the Cauchy–Schwarz inequality and Young's inequality to the first term on the right hand side, we have

$$\frac{\lambda}{2}\int_{I_n} \left\|u_h(t)\right\|^2_{L^2(\Omega)} e^{-\lambda(t-t_n)}\,\mathrm{d}t + \frac{1}{2}\left\|u_{n+1}^-\right\|^2_{L^2(\Omega)} e^{-\lambda\Delta t} \\ \le \frac{1}{2}\left\|u_n^-\right\|^2_{L^2(\Omega)} + \int_{I_n} (f, \tilde{u}_h)_{\mathcal{T}_h}\,\mathrm{d}t - \int_{I_n} (\nu a_h(\boldsymbol{u}_h, \tilde{\boldsymbol{u}}_h) + o_h(u_h; \boldsymbol{u}_h, \tilde{\boldsymbol{u}}_h))\,\mathrm{d}t.$$

We now focus on bounding the right-hand side. By the boundedness of $a_h(\cdot,\cdot)$ eq. (4.8) and eq. (4.13), there exists a constant $C_1 > 0$ such that

$$\int_{I_n} |a_h(\boldsymbol{u}_h, \tilde{\boldsymbol{u}}_h)|\,\mathrm{d}t \le C_1 \int_{I_n} |||\boldsymbol{u}_h|||_v^2\,\mathrm{d}t.$$

In two spatial dimensions, we can use Lemma 4.3.2, eq. (4.13), and hence Young's inequality with some $\epsilon_1 > 0$ to find there exists a constant $C_2 > 0$ such that

$$\int_{I_n} o_h(u_h; \boldsymbol{u}_h, \tilde{\boldsymbol{u}}_h)\,\mathrm{d}t \le \frac{\epsilon_1}{2}\|u_h\|^2_{L^\infty(I_n;L^2(\Omega))} + \frac{C_2}{2\epsilon_1}\left(\int_{I_n} |||\boldsymbol{u}_h|||_v^2\,\mathrm{d}t\right)^2.$$

Next by the Cauchy–Schwarz inequality, Young's inequality, the discrete Poincaré inequality, and eq. (4.13), there exists a constant $C_3 > 0$ such that for some $\epsilon_2 > 0$,

$$\int_{I_n} (f, \tilde{u}_h)_{\mathcal{T}_h}\,\mathrm{d}t \le \frac{1}{2\epsilon_2}\int_{I_n} \|f\|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{C_3\epsilon_2}{2}\int_{I_n} |||\boldsymbol{u}_h|||_v^2\,\mathrm{d}t.$$

Thus,

$$\frac{\lambda}{2}\int_{I_n} \|u_h\|^2_{L^2(\Omega)} e^{-\lambda(t-t_n)}\,\mathrm{d}t + \frac{1}{2}\left\|u_{n+1}^-\right\|_{L^2(\Omega)} e^{-\lambda\Delta t} - \frac{\epsilon_1}{2}\|u_h\|^2_{L^\infty(I_n;L^2(\Omega))} \\ \le \frac{1}{2}\left\|u_n^-\right\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon_2}\int_{I_n} \|f\|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{C_3\epsilon_2}{2}\int_{I_n} |||\boldsymbol{u}_h|||_v^2\,\mathrm{d}t \\ + C_1\nu\int_{I_n} |||\boldsymbol{u}_h|||_v^2\,\mathrm{d}t + \frac{C_2}{2\epsilon_1}\left(\int_{I_n} |||\boldsymbol{u}_h|||_v^2\,\mathrm{d}t\right)^2.$$

Choosing $\lambda = 1/\Delta t$ and applying the scaling identity in eq. (4.19a) with $p = \infty$, we find there exists a constant $C_4 > 0$ such that

$$\left(\frac{C_4^{-1}e^{-1}}{2} - \frac{\epsilon_1}{2}\right)\|u_h\|^2_{L^\infty(I_n;L^2(\Omega))} \\ \le \frac{1}{2}\big\|u_n^-\big\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon_2}\int_{I_n}\|f\|^2_{L^2(\Omega)}\,\mathrm{d}t + \frac{C_3\epsilon_2}{2}\int_{I_n}|\!|\!|\boldsymbol{u}_h|\!|\!|_v^2\,\mathrm{d}t \\ + C_1\nu\int_{I_n}|\!|\!|\boldsymbol{u}_h|\!|\!|_v^2\,\mathrm{d}t + \frac{C_2}{2\epsilon_1}\left(\int_{I_n}|\!|\!|\boldsymbol{u}_h|\!|\!|_v^2\,\mathrm{d}t\right)^2.$$

Choosing $\epsilon_1 = C_4^{-1}e^{-1}/2$, $\epsilon_2 = 2\nu$, using the a priori estimates on $\boldsymbol{u}_h$ in Lemma 4.1.1, and rearranging, we see there exists a $C_5 > 0$ such that

$$\|u_h\|^2_{L^\infty(I_n;L^2(\Omega))} \\ \le C_5\left(\frac{1}{\nu}\int_0^T\|f\|^2_{L^2(\Omega)}\,\mathrm{d}t + \big\|u_0\big\|^2_{L^2(\Omega)}\right) + \frac{C_5}{\nu^2}\left(\frac{1}{\nu}\int_0^T\|f\|^2_{L^2(\Omega)}\,\mathrm{d}t + \big\|u_0\big\|^2_{L^2(\Omega)}\right)^2.$$

This bound holds uniformly for every space-time slab, so the result follows.

□

4.3.2.3 Proof of Theorem 4.1.1.

Proof. Consider an arbitrary space-time slab $\mathcal{E}^m$. Suppose $(u_1, \bar{u}_1) \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h$ and $(u_2, \bar{u}_2) \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h$ are two solutions to eq. (4.4) corresponding to the same problem data f and u_0, and set $\boldsymbol{w}_h = \boldsymbol{u}_1 - \boldsymbol{u}_2$. Then, for all $\boldsymbol{v}_h \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h$, it holds that

$$-\int_{I_m}(w_h, \partial_t v_h)_{\mathcal{T}_h}\,\mathrm{d}t + (w_{m+1}^-, v_{m+1}^-)_{\mathcal{T}_h} + \int_{I_m}\nu a_h(\boldsymbol{w}_h, \boldsymbol{v}_h)\,\mathrm{d}t \\ + \int_{I_m}\big(o_h(u_1, \boldsymbol{u}_1, \boldsymbol{v}_h) - o_h(u_2, \boldsymbol{u}_2, \boldsymbol{v}_h)\big)\,\mathrm{d}t = (w_m^-, v_m^+)_{\mathcal{T}_h}. \tag{4.30}$$

Step one: Testing eq. (4.30) with $\boldsymbol{v}_h = \boldsymbol{w}_h$, integrating by parts in time, using the coercivity of $a_h(\cdot,\cdot)$, and noting that $o_h(u_2, \boldsymbol{u}_2, \boldsymbol{w}_h) - o_h(u_1, \boldsymbol{u}_1, \boldsymbol{w}_h) \le -o_h(w_h, \boldsymbol{u}_2, \boldsymbol{w}_h)$ by eq. (4.9), we find

$$\frac{1}{2}\big\|w_{m+1}^-\big\|^2_{L^2(\Omega)} + \frac{1}{2}\big\|[w_h]_m\big\|^2_{L^2(\Omega)} - \frac{1}{2}\big\|w_m^-\big\|^2_{L^2(\Omega)} \\ + C\nu\int_{I_m}|\!|\!|\boldsymbol{w}_h|\!|\!|_v^2\,\mathrm{d}t \le \int_{I_m}|o_h(w_h, \boldsymbol{u}_2, \boldsymbol{w}_h)|\,\mathrm{d}t.$$

Summing over all space-time slabs $n = 0, \ldots, N-1$, rearranging, and noting that $w_0^- = 0$, we see that there exists a $C_1 > 0$ such that

$$\left\|w_N^-\right\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \left\|[w_h]_n\right\|_{L^2(\Omega)}^2 + \nu \int_0^T |||\boldsymbol{w}_h|||_v^2 \,\mathrm{d}t \leq C_1 \int_0^T |o_h(w_h, \boldsymbol{u}_2, \boldsymbol{w}_h)| \,\mathrm{d}t. \tag{4.31}$$

Step two: Fix an integer m such that $0 \leq m \leq N-1$. Testing eq. (4.30) with the discrete characteristic function $\boldsymbol{v}_h = \boldsymbol{w}_\chi$ where $s = \arg\sup_{t \in I_m} \|u_h(t)\|_{L^2(\Omega)}$, integrating by parts in time, using Young's inequality, we have after rearranging

$$\begin{aligned}
&\frac{1}{2} \sup_{t \in I_m} \left\|w_h(t)\right\|_{L^2(\Omega)}^2 - \frac{1}{2} \sup_{t \in I_{m-1}} \left\|w_h(t)\right\|_{L^2(\Omega)}^2 \\
&\leq -\int_{I_m} \big(\nu a_h(\boldsymbol{w}_h, \boldsymbol{w}_\chi) + o_h(u_1, \boldsymbol{u}_1, \boldsymbol{w}_\chi) - o_h(u_2, \boldsymbol{u}_2, \boldsymbol{w}_\chi)\big) \,\mathrm{d}t,
\end{aligned}$$

where we have used that $\sup_{t \in I_{m-1}} \|w(s)\|_{L^2(\Omega)} \geq \|w_m^-\|_{L^2(\Omega)}$. Setting $I_{-1} = \{t_0\} = \{0\}$, we can sum over the space-time slabs $n = 0, \ldots, m$ and use the boundedness of $a_h(\cdot, \cdot)$ and the bound eq. (4.15) to find there exists a constant $C_2 > 0$ such that

$$\frac{1}{2} \sup_{t \in I_m} \left\|w_h(t)\right\|_{L^2(\Omega)}^2 \leq C_2 \nu \int_0^T |||\boldsymbol{w}_h|||_v^2 \,\mathrm{d}t + \int_0^T \left|o_h(u_2, \boldsymbol{u}_2, \boldsymbol{w}_\chi) - o_h(u_1, \boldsymbol{u}_1, \boldsymbol{w}_\chi)\right| \mathrm{d}t, \tag{4.32}$$

where we have used that $\sup_{t \in I_{-1}} \|w_h(t)\|_{L^2(\Omega)} = \|w_0^-\|_{L^2(\Omega)} = 0$. This bound holds uniformly for all space-time slabs, and thus we can replace the supremum over I_n in eq. (4.32) with the supremum over $[0, T]$. Doing so, and adding $2\nu \int_0^T |||\boldsymbol{w}_h|||_v^2 \,\mathrm{d}t$ to both sides we see there exists a constant $C_3 > 0$ such that

$$\begin{aligned}
&\frac{1}{2} \|w_h\|_{L^\infty(0,T;L^2(\Omega))}^2 + 2\nu \int_0^T |||\boldsymbol{w}_h|||_v^2 \,\mathrm{d}t \\
&\quad \leq C_3 \int_0^T \left|o_h(w_h, \boldsymbol{u}_2, \boldsymbol{w}_h)\right| \mathrm{d}t + \int_0^T \left|o_h(w_h, \boldsymbol{u}_2, \boldsymbol{w}_\chi)\right| \mathrm{d}t + \int_0^T \left|o_h(u_1, \boldsymbol{w}_h, \boldsymbol{w}_\chi)\right| \mathrm{d}t.
\end{aligned}$$

Here, we have used the bound eq. (4.31) from step one, that $o_h(u_2, \boldsymbol{u}_2, \boldsymbol{w}_\chi) - o_h(u_1, \boldsymbol{u}_1, \boldsymbol{w}_\chi) = -o_h(w_h, \boldsymbol{u}_2, \boldsymbol{w}_\chi) - o_h(u_1, \boldsymbol{w}_h, \boldsymbol{w}_\chi)$, and the triangle inequality. From Lemma 4.3.2 and two applications of Young's inequality, first with $p = q = 2$ and second with $p = 4, q = 4/3$, we find

$$|o_h(w_h, \boldsymbol{u}_2, \boldsymbol{w}_h)| \leq C_4 \left(\frac{1}{2\epsilon} \|w_h\|_{L^2(\Omega)}^2 |||\boldsymbol{u}_2|||_v^2 + \frac{1}{4\epsilon^3} \|w_h\|_{L^2(\Omega)}^2 \|u_2\|_{L^2(\Omega)}^2 |||\boldsymbol{u}_2|||_v^2 + \frac{5\epsilon}{4} |||\boldsymbol{w}_h|||_v^2 \right).$$

Similarly, from Lemma 4.3.2 and eq. (4.15), we have

$$|o_h(w_h, \boldsymbol{u}_2, \boldsymbol{w}_\chi)| \le C_5 \left(\frac{1}{2\epsilon} \|w_h\|^2_{L^2(\Omega)} \, \|\!|\boldsymbol{u}_2|\!\|^2_v + \frac{1}{4\epsilon^3} \|w_h\|^2_{L^2(\Omega)} \|u_2\|^2_{L^2(\Omega)} \, \|\!|\boldsymbol{u}_2|\!\|^2_v + \frac{5\epsilon}{4} \|\!|\boldsymbol{w}_h|\!\|^2_v \right),$$

and finally,

$$\left| o_h(u_1, \boldsymbol{w}_h, \boldsymbol{w}_\chi) \right| \le C_6 \left(\|u_1\|_{L^2(\Omega)} \, \|\!|\boldsymbol{w}_h|\!\|^2_v + \frac{1}{4\epsilon^3} \|u_1\|^2_{L^2(\Omega)} \|w_h\|^2_{L^2(\Omega)} \, \|\!|\boldsymbol{u}_1|\!\|^2_v + \frac{3\epsilon}{4} \|\!|\boldsymbol{w}_h|\!\|^2_v \right),$$

where $\epsilon > 0$. Collecting the above bounds, choosing $\epsilon = O(\nu)$ sufficiently small and rearranging, we can find a $C_7 > 0$ such that

$$\begin{aligned} &\|w_h\|^2_{L^\infty(0,T;L^2(\Omega))} + \nu \int_0^T \|\!|\boldsymbol{w}_h|\!\|^2_v \,\mathrm{d}t \\ &\le \frac{C_7}{\nu^4} \Bigg(\nu^3 \|u_1\|_{L^\infty(0,T;L^2(\Omega))} + \nu \|u_1\|^2_{L^\infty(0,T;L^2(\Omega))} \int_0^T \|\!|\boldsymbol{u}_1|\!\|^2_v \,\mathrm{d}t \\ &\qquad + \left(\nu^3 + \nu \|u_2\|^2_{L^\infty(0,T;L^2(\Omega))} \right) \int_0^T \|\!|\boldsymbol{u}_2|\!\|^2_v \,\mathrm{d}t \Bigg) \times \left(\|w_h\|^2_{L^\infty(0,T;L^2(\Omega))} + \nu \int_0^T \|\!|\boldsymbol{w}_h|\!\|^2_v \,\mathrm{d}t \right). \end{aligned} \tag{4.33}$$

Step three: For notational convenience, let $\Xi = \nu^{-1} \int_0^T \|f\|^2_{L^2(\Omega)} \,\mathrm{d}t + \|u_0\|^2_{L^2(\Omega)}$. Applying the bounds in Lemma 4.1.1 and Lemma 4.1.4 to eq. (4.33), we find there exists a $C_8 > 0$ such that

$$\begin{aligned} &\|w_h\|^2_{L^\infty(0,T;L^2(\Omega))} + \nu \int_0^T \|\!|\boldsymbol{w}_h|\!\|^2_v \,\mathrm{d}t \\ &\le \frac{C_8}{\nu^4} \left(\nu^3 \Xi^{1/2} + \nu^2 \Xi + \Xi^2 + \nu^{-2} \Xi^3 \right) \left(\|w_h\|^2_{L^\infty(0,T;L^2(\Omega))} + \nu \int_0^T \|\!|\boldsymbol{w}_h|\!\|^2_v \,\mathrm{d}t \right). \end{aligned}$$

The result follows if $\nu < 1$ and $\Xi \le \frac{1}{4} \min \{ C_8^{-1/3}, C_8^{-1/2}, C_8^{-1}, C_8^{-2} \} \nu^2$. □

4.3.3 Recovering the pressure

Existence of the pressure pair $(p_h, \bar{p}_h) \in \boldsymbol{Q}_h$ satisfying eq. (4.4) will require the following inf-sup condition:

Theorem 4.3.2 (Inf-sup condition)**.** *Suppose that the spatial mesh $\mathcal{T}_h$ is conforming and quasi-uniform. There exists a constant $\beta > 0$, independent of h and Δt, such that for all $\boldsymbol{q}_h \in \boldsymbol{\mathcal{Q}}_h$,*

$$\sup_{0\neq \boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h} \frac{\int_{I_n} b_h(\boldsymbol{q}_h, v_h)\,\mathrm{d}t}{\left(\int_{I_n} |||\boldsymbol{v}_h|||_v^2\,\mathrm{d}t\right)^{1/2}} \geq \beta \left(\int_{I_n} |||\boldsymbol{q}_h|||_p^2\,\mathrm{d}t\right)^{1/2}. \tag{4.34}$$

The proof, which exploits the tensor-product structure of the finite element spaces in an essential way, is an extension of the proof of [79, Lemma 1] to the space-time setting.

4.3.3.1 Proof of Theorem 4.3.2.

By [45, Theorem 3.1], the inf-sup condition eq. (4.34) is satisfied if we can decompose $b_h(\cdot,\cdot)$ into $b_1(\cdot,\cdot) : V_h \times Q_h \to \mathbb{R}$ and $b_2(\cdot,\cdot) : V_h \times \bar{Q}_h \to \mathbb{R}$ such that, for some constants $\alpha_1, \alpha_2 > 0$, it holds that

$$\sup_{(v_h,\bar{v}_h)\in \mathcal{Z}_{b_2}\times \bar{\mathcal{V}}_h} \frac{\int_{I_n} b_1(q_h, v_h)\,\mathrm{d}t}{\left(\int_{I_n} |||\boldsymbol{v}_h|||_v^2\,\mathrm{d}t\right)^{1/2}} \geq \alpha_1 \left(\int_{I_n} \|q_h\|_{L^2(\Omega)}^2\,\mathrm{d}t\right)^{1/2}, \tag{4.35a}$$

and

$$\sup_{\boldsymbol{v}_h\in \boldsymbol{\mathcal{V}}_h} \frac{\int_{I_n} b_2(\bar{q}_h, v_h)\,\mathrm{d}t}{\left(\int_{I_n} |||\boldsymbol{v}_h|||_v^2\,\mathrm{d}t\right)^{1/2}} \geq \alpha_2 \left(\sum_{K\in\mathcal{T}_h}\int_{I_n} h_K \|\bar{q}\|_{\partial K}^2\,\mathrm{d}t\right)^{1/2}, \tag{4.35b}$$

where

$$\mathcal{Z}_{b_2} = \left\{ v_h \in \mathcal{V}_h \ : \ \int_{I_n} b_2(v_h, \bar{q}_h)\,\mathrm{d}t = 0, \quad \forall \bar{q}_h \in \bar{\mathcal{Q}}_h \right\}.$$

We thus define

$$b_1(q_h, v_h) = -\sum_{K\in\mathcal{T}_h}\int_K q_h \nabla\cdot v_h\,\mathrm{d}x \quad \text{and} \quad b_2(\bar{q}_h, v_h) = \sum_{K\in\mathcal{T}_h}\int_{\partial K} v_h\cdot n\bar{q}_h\,\mathrm{d}s.$$

We begin by proving eq. (4.35a). The tensor-product structure of the space $\mathcal{Q}_h$ ensures that we can expand any $q_h \in \mathcal{Q}_h$ in terms of an orthonormal basis of $P_k(I_n)$ with respect to the $L^2(I_n)$ inner product:

$$q_h = \sum_{i=0}^{k} \phi_i(t) q_i(x), \quad \phi_i \in P_k(I_n),\ q_i \in Q_h. \tag{4.36}$$

Since each $q_i \in L^2_0(\Omega)$, there exist $z_i \in H^1_0(\Omega)^d$, $0 \le i \le k$ and constants β_i, $0 \le i \le k$ such that $\nabla \cdot z_i = -q_i$ and $\beta_i \|z_i\|_{H^1(\Omega)} \le \|q_i\|_{L^2(\Omega)}$ (see e.g. [28, Theorem 6.5]). We construct the desired $\boldsymbol{\psi}_h = (\psi_h, \bar{\psi}_h) \in \mathcal{Z}_{b_2} \times \bar{\mathcal{V}}_h$ by choosing

$$\psi_h = \sum_{i=0}^{k} \phi_i(t) \Pi_{\mathrm{BDM}} z_i, \quad \text{and} \quad \bar{\psi}_h = \sum_{i=0}^{k} \phi_i(t) \bar{\Pi}_V z_i,$$

where $\Pi_{\mathrm{BDM}} : \left[H^1(\Omega)\right]^d \to V_h$ is the BDM projection (see Lemma 3.5.1) and $\bar{\Pi}_V$ is the L^2 projection onto the space $\bar{V}_h$. By the orthonormality of the basis $\{\phi_i\}_{i=0}^k$, property (v) of Lemma 3.5.1, the single-valuedness of $z_i \cdot n$ and $\bar{q}_i$ across element faces, and the fact that $z_i \in H^1_0(\Omega)$, we have

$$b_2(\bar{q}_h, \psi_h) = \sum_{K \in \mathcal{T}_h} \sum_{i=0}^{k} \int_{\partial K} z_i \cdot n \bar{q}_i \,\mathrm{d}s = 0,$$

and thus $\psi_h \in \mathcal{Z}_{b_2}$. We now show that $\boldsymbol{\psi}_h$ satisfies the inequality in eq. (4.35a) with some $\alpha_1 > 0$ independent of the mesh parameters h and Δt. Given $q_h \in \mathcal{Q}_h$, we can use the expansion eq. (4.36), the definition of z_i, $0 \le i \le k$, and the commuting diagram property of the BDM projection (property (iv) of Lemma 3.5.1) to find

$$\int_{I_n} \|q_h\|^2_{L^2(\Omega)} \,\mathrm{d}t = \int_{I_n} b_1(q_h, \psi_h) \,\mathrm{d}t. \tag{4.37}$$

Next, we need to show existence of a constant $\alpha_1 > 0$, independent of the mesh parameters h and Δt, such that

$$\alpha_1^2 \int_{I_n} \|\!|\!| \boldsymbol{\psi}_h |\!|\!\|^2_v \,\mathrm{d}t \le \int_{I_n} \|q_h\|^2_{L^2(\Omega)} \,\mathrm{d}t. \tag{4.38}$$

But, this can easily be reduced to the proof of [77, Lemma 4.5] by expanding ψ_h in terms of an orthonormal basis of $P_k(I_n)$ with respect to the $L^2(I_n)$ inner-product. Combining eq. (4.37) and eq. (4.38), we have

$$\frac{\int_{I_n} b_1(q_h, \psi_h) \,\mathrm{d}t}{\left(\int_{I_n} \|\!|\!| \boldsymbol{\psi}_h |\!|\!\|^2_v \,\mathrm{d}t\right)^{1/2}} = \frac{\int_{I_n} \|q_h\|^2_{L^2(\Omega)} \,\mathrm{d}t}{\left(\int_{I_n} \|\!|\!| \boldsymbol{\psi}_h |\!|\!\|^2_v \,\mathrm{d}t\right)^{1/2}} \ge \alpha_1^2 \left(\int_{I_n} \|q_h\|^2_{L^2(\Omega)} \,\mathrm{d}t\right)^{1/2},$$

where $\alpha_1 > 0$ depends on the constants β_i, $0 \le i \le k$.

What we have left to show is eq. (4.35b). It suffices to construct an $\boldsymbol{\omega}_h \in \boldsymbol{\mathcal{V}}_h$ such that for some $\alpha_2 > 0$ it holds that

$$\frac{\int_{I_n} b_2(\bar{q}_h, \omega_h) \,\mathrm{d}t}{\left(\int_{I_n} \|\!|\!| \boldsymbol{\omega}_h |\!|\!\|^2_v \,\mathrm{d}t\right)^{1/2}} \ge \alpha_2 \left(\sum_{K \in \mathcal{T}_h} \int_{I_n} h_K \|\bar{q}_h\|^2_{\partial K} \,\mathrm{d}t\right)^{1/2}, \quad \forall \bar{q}_h \in \bar{\mathcal{Q}}_h. \tag{4.39}$$

The tensor-product structure of $\bar{\mathcal{Q}}_h$ ensures that we can expand any $\bar{q}_h \in \bar{\mathcal{Q}}_h$ in terms of an orthonormal basis of $P_k(I_n)$ with respect to the $L^2(I_n)$ inner-product:

$$\bar{q}_h = \sum_{i=0}^{k} \phi_i(t)\bar{q}_i(x), \quad \phi_i \in P_k(I_n),\ \bar{q}_i \in \bar{Q}_h. \tag{4.40}$$

Given $\bar{q}_h \in \bar{\mathcal{Q}}_h$, we construct the required $\boldsymbol{\omega}_h$ by choosing $\bar{\omega}_h = 0$, and defining $\omega_h \in \mathcal{V}_h$ element-wise by:

$$\omega_h|_{K\times I_n} = \sum_{i=0}^{k} \phi_i(t) L^{\mathrm{BDM}}\left(\bar{q}_i|_{\partial K}\right),$$

with $\bar{q}_i \in \bar{Q}_h$ defined as in eq. (4.40). Here, $L^{\mathrm{BDM}} : P_k(\partial K) \to \left(P_k(K)\right)^d$ is the local BDM lifting satisfying for all $\bar{q}_h \in P_k(\partial K)$ (see e.g. [31, Proposition 2.10]):

$$(L^{\mathrm{BDM}}\bar{q}_h)\cdot n = \bar{q}_h, \quad \text{and} \quad \left\|L^{\mathrm{BDM}}\bar{q}_h\right\|_{L^2(K)} \le C h_K^{1/2} \|\bar{q}\|_{L^2(\partial K)}, \quad \forall \bar{q}_h \in P_k(\partial K), \tag{4.41}$$

where n is the unit outward normal to ∂K. Using the first property in eq. (4.41), it can be shown that

$$\int_{I_n} b_2(\bar{q}_h, \omega_h)\,\mathrm{d}t = \sum_{K\in\mathcal{T}_h} \int_{I_n} \|\bar{q}_h\|^2_{L^2(\partial K)}\,\mathrm{d}t. \tag{4.42}$$

The remainder of the proof of eq. (4.35b) can easily be reduced to the proof of [79, Lemma 3] by expanding ω_h in terms of an orthonormal basis of $P_k(I_n)$ with respect to the $L^2(I_n)$ inner-product. In particular, it can be shown that $\alpha_2 = Ch_{\min}/h_{\max}$, which remains uniformly bounded below provided we assume quasi-uniformity of $\mathcal{T}_h$.

Theorem 2.1.3 yields the following corollary:

Corollary 4.3.1. *To each discrete velocity solution pair* $(u_h, \bar{u}_h) \in \boldsymbol{\mathcal{V}}_h$ *guaranteed by Theorem 4.3.1, there exists a unique discrete pressure pair* $(p_h, \bar{p}_h) \in \boldsymbol{\mathcal{Q}}_h$ *satisfying eq.* (4.4).

4.4 Error analysis for the velocity

4.4.1 Space-time projection operators

Let $P_h : L^2(\Omega)^d \to V_h^{\mathrm{div}}$ and $\bar{P}_h : L^2(\Gamma)^d \to \bar{V}_h$ denote the orthogonal L^2-projections onto, respectively, the spaces V_h^{div} and $\bar{V}_h$. The approximation properties of $\bar{P}_h$ are well-known while the approximation properties of P_h rely critically on the fact that $V_h^{\text{div}} \subset H$. In particular, we can exploit the best approximation property of the orthogonal projection along with the approximation properties of the BDM projection to prove:

Lemma 4.4.1. *Let $k \geq 1$, $0 \leq m \leq 2$, and $u \in H^{k+1}(\Omega)^d$. If the spatial mesh $\mathcal{T}_h$ is quasi-uniform and consists triangles in two dimensions or tetrahedras in three dimensions, then the following estimates hold:*

$$\sum_{K \in \mathcal{T}_h} \|u - P_h u\|_{H^m(K)}^2 \lesssim h^{2(k-m+1)} |u|_{H^{k+1}(\Omega)}^2, \tag{4.43}$$

$$\sum_{K \in \mathcal{T}_h} h_K^{-1} \|u - P_h u\|_{L^2(\partial K)}^2 \lesssim h^{2k} |u|_{H^{k+1}(\Omega)}^2. \tag{4.44}$$

Proof. We begin by proving eq. (4.43). For $m = 0$, we have by the best approximation property of the orthogonal L^2-projection onto V_h^{div}:

$$\|u - P_h u\|_{L^2(\Omega)} = \min_{v_h \in V_h^{\text{div}}} \|u - v_h\|_{L^2(\Omega)}.$$

Since $\Pi_{\text{BDM}} u \in V_h^{\text{div}}$, eq. (4.43) follows from standard approximation properties of the BDM projection (Lemma 3.5.1). The proof for $m = 1$ follows by noting that, by triangle inequality,

$$\|u - P_h u\|_{H^m(K)} \leq \|u - \Pi_V u\|_{H^m(K)} + \|\Pi_V u - P_h u\|_{H^m(K)},$$

where Π_V is the orthogonal L^2-projection onto V_h. Using the local inverse inequality $\|u_h\|_{H^1(K)} \leq C h_K^{-1} \|u_h\|_{L^2(K)}$, the quasi-uniformity of the spatial mesh $\mathcal{T}_h$, eq. (4.43), and the approximation properties of Π_V (see e.g. [28, Lemma 1.58]), the result follows. The bound for $m = 2$ follows similarly. To prove eq. (4.44), we note that by the local trace inequality for functions in $H^1(K)$, we have

$$\|u - P_h u\|_{L^2(\partial K)} \leq C \left(h_K^{-1/2} \|u - P_h u\|_{L^2(K)} + \|u - P_h u\|_{L^2(K)}^{1/2} |u - P_h u|_{H^1(K)}^{1/2} \right).$$

The result now follows from the quasi-uniformity of the mesh, the Cauchy–Schwarz inequality, and eq. (4.43). □

Following [17, Definition 4.2], we introduce a space-time projection operator much in the same spirit as the temporal "DG-projection" defined in [98, Eq. (12.9)] or [29, Section 6.1.4], but appropriately modified for divergence free fields. Additionally, we will need an analogue of this temporal DG-projection onto the facet space $\bar{\mathcal{V}}_h$:

Definition 4.4.1. *1.* $\mathcal{P}_h : C(I_n; L^2(\Omega)) \to \mathcal{V}_h$ *satisfying* $(\mathcal{P}_h u)(t_{n+1}^-) = (P_h u)(t_{n+1}^-)$, *with* $(\mathcal{P}_h u)(t_0^-) = P_h u(t_0)$, *and*

$$\int_{I_n} (u - \mathcal{P}_h u, v_h)_{\mathcal{T}_h} \,\mathrm{d}t = 0 \quad \forall v_h \in P_{k-1}(I_n, V_h^{div}). \tag{4.45}$$

2. $\bar{\mathcal{P}}_h : C(I_n; L^2(\Gamma)) \to P_k(I_n; \bar{V}_h)$ *satisfying* $(\bar{\mathcal{P}}_h u)(t_{n+1}^-) = (\bar{P}_h u)(t_{n+1}^-)$

$$\int_{I_n} (u - \bar{\mathcal{P}}_h u, \bar{v}_h)_{\partial\mathcal{T}_h} \,\mathrm{d}t = 0, \quad \forall \bar{v}_h \in P_{k-1}(I_n; \bar{V}_h). \tag{4.46}$$

We summarize the approximation properties of $\mathcal{P}_h$ and $\bar{\mathcal{P}}_h$ in Appendix A.

4.4.2 Parabolic Stokes projection

Motivated by [17, Definition 4.2], we introduce a parabolic Stokes projection which will be crucial to our error analysis in Section 4.4:

Definition 4.4.2 (Parabolic Stokes projection)**.** *Let* u *be the strong velocity solution to the Navier–Stokes system eq. (4.1) guaranteed by Theorem 2.3.2. We define the parabolic Stokes projection* $(\Pi_h u, \bar{\Pi}_h u, \Pi_h p, \bar{\Pi}_h p) \in \mathcal{V}_h^{div} \times \bar{\mathcal{V}}_h \times \boldsymbol{\mathcal{Q}}_h$ *to be the solution to the following space-time HDG scheme:*

$$\begin{aligned} -\int_{I_n} (\Pi_h u, \partial_t v_h)_{\mathcal{T}_h} \,\mathrm{d}t + ((\Pi_h u)_{n+1}^-, v_{n+1}^-)_{\mathcal{T}_h} + \int_{I_n} (\nu a_h(\boldsymbol{\Pi}_h u, \boldsymbol{v}_h) + b_h(\boldsymbol{\Pi}_h p, v_h)) \,\mathrm{d}t \\ = ((\Pi_h u)_n^-, v_n^+)_{\mathcal{T}_h} + \int_{I_n} (\partial_t u, v_h)_{\mathcal{T}_h} \,\mathrm{d}t + \int_{I_n} \nu a_h(\boldsymbol{u}, \boldsymbol{v}_h) \,\mathrm{d}t \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h, \\ \int_{I_n} b_h(\boldsymbol{q}_h, \Pi_h u) \,\mathrm{d}t = 0 \quad \forall \boldsymbol{q}_h \in \boldsymbol{\mathcal{Q}}_h, \end{aligned} \tag{4.47}$$

where $(\Pi_h u)_0^- = P_h u(t_0)$ *and* $(\bar{\Pi}_h u)_0^-$ *may be arbitrarily chosen. Here, we have denoted* $\boldsymbol{\Pi}_h u = (\Pi_h u, \bar{\Pi}_h u)$ *and* $\boldsymbol{\Pi}_h p = (\Pi_h p, \bar{\Pi}_h p)$.

Remark 4.4.1. *We remark that eq. (4.47) is simply a space-time HDG scheme for the evolutionary Stokes problem eq. (4.24) with* $f = u_t - \nu\Delta u$ *and* $u_0 = u(0)$*. Consequently,* $\Pi_h u \in \mathcal{V}_h^{div}$ *and thus* $\Pi_h u \in H$.

4.4.3 Uniform bounds on the parabolic Stokes projection

To perform our error analysis in Section 4.4, we will require a uniform bound on the Stokes projection:

$$\text{ess sup}_{0\le t\le T} |||\boldsymbol{\Pi}_h u|||_\nu \le C(u, u_0, \nu).$$

Our plan is to follow the proof of [17, Theorem 4.10]. Therein, an essential ingredient is a discrete Stokes operator. Unfortunately, as $(\cdot,\cdot)_{\mathcal{T}_h}$ is *not* an inner-product on $\boldsymbol{V}_h$, we cannot leverage the Riesz representation theorem (Theorem 2.1.1) to infer the existence of a discrete Stokes operator in the HDG setting. Instead, we introduce a novel discrete Stokes operator by mimicking the static condensation that occurs for the HDG method at the algebraic level following ideas from [12].

4.4.3.1 Discrete Stokes operator

Consider the variational problem: find $\boldsymbol{\phi}_h \in V_h^{\text{div}} \times \bar{V}_h$ such that

$$a_h(\boldsymbol{\phi}_h, \boldsymbol{w}_h) = (u_h, w_h)_{\mathcal{T}_h}, \quad \forall \boldsymbol{w}_h \in V_h^{\text{div}} \times \bar{V}_h.$$

This problem is well-posed by the Lax–Milgram theorem (Theorem 2.1.2), implying the existence of a well-defined solution operator $S_h : V_h \to V_h^{\text{div}} \times \bar{V}_h$ such that $\boldsymbol{\phi}_h = S_h(u_h)$. Note that S_h *need not be surjective* onto the product space $V_h^{\text{div}} \times \bar{V}_h$. However, as in [12], we can split the solution operator S_h into "element" and "facet" solution operators $S_{\mathcal{K}}$ and $S_{\mathcal{F}}$. We will show that $S_{\mathcal{K}}$ is invertible.

Define the *facet solution operator* $S_{\mathcal{F}} : V_h^{\text{div}} \to \bar{V}_h$ as the unique solution of

$$a_h((v_h, S_{\mathcal{F}}(v_h)), (0, \bar{w}_h)) = 0, \quad \forall \bar{w}_h \in \bar{V}_h. \tag{4.48}$$

Since $a_h(\cdot,\cdot)$ is symmetric, $S_{\mathcal{F}}$ is self-adjoint. Next, we introduce a new bilinear form on $V_h^{\text{div}} \times V_h^{\text{div}}$:

$$a_h^\star(v_h, w_h) = a_h((v_h, S_{\mathcal{F}}(v_h)), (w_h, S_{\mathcal{F}}(w_h))), \tag{4.49}$$

for which we introduce the *element solution operator* $S_{\mathcal{K}} : V_h^{\text{div}} \to V_h^{\text{div}}$ satisfying

$$a_h^\star(S_{\mathcal{K}}(u_h), w_h) = (u_h, w_h)_{\mathcal{T}_h}.$$

It can be shown that $S_h(u_h) = (S_{\mathcal{K}}(u_h), (S_{\mathcal{F}} \circ S_{\mathcal{K}})(u_h))$ (see [12, Lemma 3.1]). We observe that $S_{\mathcal{K}} : V_h^{\text{div}} \to V_h^{\text{div}}$ is injective. By the Rank-Nullity theorem, $S_{\mathcal{K}}$ is bijective. Therefore, we can define an inverse operator $A_h = S_{\mathcal{K}}^{-1}$ satisfying

$$a_h^\star(u_h, w_h) = (A_h u_h, w_h)_{\mathcal{T}_h}, \quad \forall w_h \in V_h^{\text{div}}.$$

By eq. (4.49), we have equivalently that

$$a_h((u_h, S_{\mathcal{F}}(u_h)), (w_h, S_{\mathcal{F}}(w_h))) = (A_h u_h, w_h)_{\mathcal{T}_h}, \quad \forall w_h \in V_h^{\mathrm{div}}. \tag{4.50}$$

Lemma 4.4.2. *Fix a space-time slab $\mathcal{E}^n$. Let $(\Pi_h u, \bar{\Pi}_h u)$ be the velocity components of the parabolic Stokes projection solving eq. (4.47), and let $S_{\mathcal{F}} : V_h^{div} \to \bar{V}_h$ be the facet solution operator introduced in eq. (4.48). Then, it holds that*

$$\bar{\Pi}_h u = S_{\mathcal{F}}(\Pi_h u). \tag{4.51}$$

Proof. Set $(v_h, \bar{v}_h, q_h, \bar{q}_h) = (0, \bar{v}_h, 0, 0)$ in eq. (4.47) and expand $\Pi_h u$, $\bar{\Pi}_h u$, and $\bar{v}_h$ in terms of an orthonormal basis $\{\phi_i(t)\}_{i=0}^k$ of $P_k(I_n)$ with respect to the $L^2(I_n)$ inner-product to find

$$\sum_{i=0}^{k} \sum_{K\in\mathcal{T}_h} \int_{\partial K} \frac{\alpha}{h_K} (\bar{\Pi}_h u)_i \cdot \bar{v}_i \,\mathrm{d}s = \sum_{i=0}^{k} \sum_{K\in\mathcal{T}_h} \int_{\partial K} \left(\frac{\alpha}{h_K} (\Pi_h u)_i - \frac{\partial (\Pi_h u)_i}{\partial n} \right) \cdot \bar{v}_i \,\mathrm{d}s.$$

By the definition of the operator $S_{\mathcal{F}}$, we have for each $i = 0, \ldots, k$,

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \left(\frac{\alpha}{h_K} (\Pi_h u)_i - \frac{\partial (\Pi_h u)_i}{\partial n} \right) \cdot \bar{v}_i \,\mathrm{d}s = \sum_{K\in\mathcal{T}_h} \int_{\partial K} \frac{\alpha}{h_K} S_{\mathcal{F}}((\Pi_h u)_i) \cdot \bar{v}_i \,\mathrm{d}s,$$

and moreover, each $S_{\mathcal{F}}((\Pi_h u)_i)$ is unique. Choosing $\bar{v}_i = (\bar{\Pi}_h u)_i - S_{\mathcal{F}}((\Pi_h u)_i) \in \bar{V}_h$ and rearranging allows us to conclude $(\bar{\Pi}_h u)_i = S_{\mathcal{F}}((\Pi_h u)_i)$ for each $i = 0, \ldots, k$. The result follows by uniqueness of the expansions of u_h and $\bar{u}_h$ with respect to the chosen basis of $P_k(I_n)$ and the linearity of $S_{\mathcal{F}}$. □

Lemma 4.4.3. *Fix a space-time slab $\mathcal{E}^n$. Let $(\Pi_h u, \bar{\Pi}_h u) \in \mathcal{V}_h^{div} \times \bar{\mathcal{V}}_h$ be the velocity components of the parabolic Stokes projection solving eq. (4.47) and let $A_h : V_h^{div} \to V_h^{div}$ be the discrete Stokes operator satisfying eq. (4.50). For notational convenience, we denote $A_h \boldsymbol{\Pi}_h u = (A_h \Pi_h u, A_h \bar{\Pi}_h u)$. Then, for all $t \in I_n$, it holds that:*

$$a_h(\boldsymbol{\Pi}_h u, A_h \boldsymbol{\Pi}_h u) = \|A_h \Pi_h u\|_{L^2(\Omega)}^2, \tag{4.52}$$

$$(\partial_t \Pi_h u, A_h \Pi_h u)_{\mathcal{T}_h} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} a_h(\boldsymbol{\Pi}_h u, \boldsymbol{\Pi}_h u). \tag{4.53}$$

Proof. By Lemma 4.4.2 and the linearity of A_h and $S_{\mathcal{F}}$, we find

$$S_{\mathcal{F}}(A_h \Pi_h u) = A_h (S_{\mathcal{F}} \Pi_h u) = A_h \bar{\Pi}_h u, \quad S_{\mathcal{F}}(\partial_t \Pi_h u) = \partial_t (S_{\mathcal{F}} \Pi_h u) = \partial_t \bar{\Pi}_h u.$$

The conclusion follows from eq. (4.50) after some basic calculations. □

4.4.3.2 Bounding the Stokes projection

Lemma 4.4.4 (Uniform bound on the Stokes projection)**.** *Let u be the strong velocity solution to the Navier–Stokes system eq. (4.1) guaranteed by Theorem 2.3.2 and let $(\Pi_h u, \bar{\Pi}_h u, \Pi_h p, \bar{\Pi}_h p) \in \mathcal{V}_h^{div} \times \bar{\mathcal{V}}_h \times \boldsymbol{\mathcal{Q}}_h$ be the solution to eq. (4.47), where we set $\bar{u}_0^- = \bar{P}_h u_0$. Then, it holds that*

$$\|\boldsymbol{\Pi}_h u\|^2_{L^\infty(0,T;\boldsymbol{\mathcal{V}}_h)} \le C\left(\frac{1}{\nu}\int_0^T \|\partial_t u - \nu\Delta u\|^2_{L^2(\Omega)}\,\mathrm{d}t + \|u_0\|^2_{H^1(\Omega)}\right).$$

Here, we define $\|\boldsymbol{\Pi}_h u\|_{L^\infty(0,T;\boldsymbol{\mathcal{V}}_h)} \equiv$ *ess sup*$_{0\le t\le T}|||\boldsymbol{\Pi}_h u|||_\nu$.

Proof. The proof will proceed in two steps. In the first step, we bound the Stokes projection at the partition points of the time-intervals. In the second step, we use the exponential interpolant, combined with the results of the first step, to obtain a uniform bound on the Stokes projection over $(0,T)$.

Step one: Integrating by parts in time in the term containing the temporal derivative in eq. (4.47), testing with $\boldsymbol{v}_h = A_h\boldsymbol{\Pi}_h u$ and using eq. (4.52) and eq. (4.53) in Lemma 4.4.3, we have

$$\begin{aligned}\frac{1}{2}\int_{I_n}\tfrac{\mathrm{d}}{\mathrm{d}t}a_h(\boldsymbol{\Pi}_h u, \boldsymbol{\Pi}_h u)\,\mathrm{d}t + a_h((\boldsymbol{\Pi}_h u)_n^+, (\boldsymbol{\Pi}_h u)_n^+) + \nu\int_{I_n}\|A_h\Pi_h u\|^2_{L^2(\Omega)}\\ = \int_{I_n}(\partial_t u - \nu\Delta u, A\Pi_h u)_{\mathcal{T}_h}\,\mathrm{d}t + a_h((\boldsymbol{\Pi}_h u)_n^-, (\boldsymbol{\Pi}_h u)_n^+).\end{aligned}$$

Using the coercivity of $a_h(\cdot,\cdot)$, the Cauchy–Schwarz inequality, Young's inequality, and summing over all space-time slabs, we find

$$\begin{aligned}|||(\boldsymbol{\Pi}_h u)_N^-|||_v^2 + \sum_{n=0}^{N-1}|||[\boldsymbol{\Pi}_h u]_n|||_v^2 + \nu\int_0^T\|A_h\Pi_h u\|^2_{L^2(\Omega)}\,\mathrm{d}t\\ \le C\left(\frac{1}{\nu}\int_0^T\|\partial_t u - \nu\Delta u\|^2_{L^2(\Omega)}\,\mathrm{d}t + |||(\boldsymbol{\Pi}_h u)_0^-|||_v^2\right).\end{aligned}$$

As $(\Pi_h u_0)^- = P_h u_0$ and $(\bar{\Pi}_h u)_0^- = \bar{P}_h u_0$, we have from Lemma 4.4.1 and the approximation properties of $\bar{P}_h$ (see e.g. [80]) that

$$
\begin{aligned}
&\left\lvert\!\left\lvert\!\left\lvert (\boldsymbol{\Pi}_h u)_N^- \right\rvert\!\right\rvert\!\right\rvert_v^2 + \sum_{n=0}^{N-1} \left\lvert\!\left\lvert\!\left\lvert [\boldsymbol{\Pi}_h u]_n \right\rvert\!\right\rvert\!\right\rvert_v^2 + \nu \int_0^T \|A\Pi_h u\|_{L^2(\Omega)}^2 \,\mathrm{d}t \\
&\qquad \leq C \left(\frac{1}{\nu} \int_0^T \|\partial_t u - \nu \Delta u\|_{L^2(\Omega)}^2 \,\mathrm{d}t + \|u_0\|_{H^1(\Omega)}^2 \right).
\end{aligned}
\tag{4.54}
$$

Note that for the lowest order scheme ($k = 1$), we can already infer the result.

Step two: It remains to obtain a bound for higher order polynomials in time. For this, we use the exponential interpolant of the pair $A_h \boldsymbol{\Pi}_h u = (A_h \Pi_h u, A_h \bar{\Pi}_h u)$, which we denote by $\tilde{A}_h \boldsymbol{\Pi}_h u = (\tilde{A}_h \Pi_h u, \tilde{A}_h \bar{\Pi}_h u)$. Integrating the first term on the left hand side of eq. (4.47) by parts in time, choosing $\boldsymbol{v}_h = \tilde{A}_h \boldsymbol{\Pi}_h u$, and using eq. (4.11), eq. (4.53), and that $\tilde{A}_h \Pi_h u \in \mathcal{V}_h^{\mathrm{div}}$, we have

$$
\begin{aligned}
&\frac{1}{2} \int_{I_n} e^{-\lambda(t-t_n)} \tfrac{\mathrm{d}}{\mathrm{d}t} a_h(\boldsymbol{\Pi}_h u, \boldsymbol{\Pi}_h u) \,\mathrm{d}t + ((\Pi_h u)_n^+, (A_h \Pi_h u)_n^+)_{\mathcal{T}_h} + \nu \int_{I_n} a_h(\boldsymbol{\Pi}_h u, \tilde{A}_h \boldsymbol{\Pi}_h u) \,\mathrm{d}t \\
&\qquad = \int_{I_n} (\partial_t u - \nu \Delta u, \tilde{A}_h \Pi_h u)_{\mathcal{T}_h} \,\mathrm{d}t + ((\Pi_h u)_n^-, (\tilde{A}_h \Pi_h u)_n^+)_{\mathcal{T}_h}.
\end{aligned}
$$

Proceeding in an identical fashion as in the proof of Lemma 4.1.4, and using eq. (4.13), we obtain

$$
\begin{aligned}
&\frac{e^{-1} C}{2} \|\boldsymbol{\Pi}_h u\|_{L^\infty(I_n; \boldsymbol{\mathcal{V}}_h)}^2 + \frac{e^{-1} C}{2} \left\lvert\!\left\lvert\!\left\lvert (\boldsymbol{\Pi}_h u)_{n+1}^- \right\rvert\!\right\rvert\!\right\rvert_v^2 \\
&\qquad \leq C \left(\frac{1}{\nu} \int_{I_n} \|\partial_t u - \nu \Delta u\|_{L^2(\Omega)}^2 \,\mathrm{d}t + \nu \int_{I_n} \|A\Pi_h u\|_{L^2(\Omega)}^2 \,\mathrm{d}t + \left\lvert\!\left\lvert\!\left\lvert (\boldsymbol{\Pi}_h u)_n^- \right\rvert\!\right\rvert\!\right\rvert_v^2 \right).
\end{aligned}
\tag{4.55}
$$

Bounding the last two terms on the right-hand side of eq. (4.55) using eq. (4.54) and omitting the second (positive) term on the left hand side, we see that there exists a constant $C > 0$ such that

$$
\|\boldsymbol{\Pi}_h u\|_{L^\infty(I_n; \boldsymbol{\mathcal{V}}_h)}^2 \leq C \left(\frac{1}{\nu} \int_0^T \|\partial_t u - \nu \Delta u\|_{L^2(\Omega)}^2 \,\mathrm{d}t + \|u_0\|_{H^1(\Omega)}^2 \right).
$$

This bound holds uniformly for every space-time slab, so the result follows. □

4.4.3.3 Approximation properties of the parabolic Stokes projection

Lemma 4.4.5. *Let u be the strong velocity solution to the Navier–Stokes system eq. (4.1) guaranteed by Theorem 2.3.2, let $(\Pi_h u, \bar{\Pi}_h u) \in \mathcal{V}_h^{div} \times \bar{\mathcal{V}}_h$ be the velocity pair of the Stokes projection eq. (4.47) for $n = 0, \ldots, N-1$, and let $\mathcal{P}_h$ and $\bar{\mathcal{P}}_h$ denote the projections introduced in Definition 4.4.1. Let $\zeta_h = \mathcal{P}_h u - \Pi_h u$, $\xi_h = u - \mathcal{P}_h u$, $\bar{\zeta}_h = \bar{\mathcal{P}}_h u - \bar{\Pi}_h u$ and $\bar{\xi}_h = u - \bar{\mathcal{P}}_h u$. There is a constant $C > 0$ such that*

$$\left\|\zeta_h(t_{n+1}^-)\right\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \left\|[\zeta_h]_i\right\|_{L^2(\Omega)}^2 + \nu \int_0^T |||\boldsymbol{\zeta}_h|||_v^2 \,\mathrm{d}t \leq C\nu \int_0^T |||\boldsymbol{\xi}_h|||_{v'}^2 \,\mathrm{d}t.$$

Proof. Our starting point will be the definition of the parabolic Stokes projection eq. (4.47). We will introduce the splitting $\boldsymbol{u} - \boldsymbol{\Pi}_h u = \boldsymbol{\xi}_h + \boldsymbol{\zeta}_h$, where $\boldsymbol{\xi}_h = (\xi_h, \bar{\xi}_h)$ and $\boldsymbol{\zeta}_h = (\zeta_h, \bar{\zeta}_h)$. Testing eq. (4.47) with $\boldsymbol{v}_h = \boldsymbol{\zeta}_h \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h$, integrating by parts in time, using the defining properties of the projection P_h Definition 4.4.1, the coercivity and boundedness of $a_h(\cdot, \cdot)$ eq. (4.8), the Cauchy–Schwarz inequality and Young's inequality with some sufficiently small $\epsilon > 0$, we have

$$\left\|\zeta_h(t_{n+1}^-)\right\|_{L^2(\Omega)}^2 + \left\|[\zeta_h]_n\right\|_{L^2(\Omega)}^2 - \left\|\zeta_h(t_n^-)\right\|_{L^2(\Omega)}^2 + C\nu \int_{I_n} |||\boldsymbol{\zeta}_h|||_v^2 \,\mathrm{d}t \leq C\nu \int_{I_n} |||\boldsymbol{\xi}_h|||_{v'}^2 \,\mathrm{d}t.$$

We conclude by summing over all space-time slabs and noting that $\zeta_h(t_0^-) = 0$. □

4.4.4 Error analysis for the velocity

4.4.4.1 The error equation

We introduce the notation $\boldsymbol{e}_h = (e_h, \bar{e}_h) = (u - u_h, \gamma(u) - \bar{u}_h)$. From Lemma 4.2.2, we have the following Galerkin orthogonality result:

$$\begin{aligned}
&- \int_{I_n} (e_h, \partial_t v_h)_{\mathcal{T}_h} \,\mathrm{d}t + (e_{n+1}^-, v_{n+1}^-)_{\mathcal{T}_h} + \nu \int_{I_n} a_h(\boldsymbol{e}_h, \boldsymbol{v}_h) \,\mathrm{d}t + \int_{I_n} b_h(\boldsymbol{p} - \boldsymbol{p}_h, v_h) \,\mathrm{d}t \\
&+ \int_{I_n} \big(o_h(u; \boldsymbol{u}, \boldsymbol{v}_h) - o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h)\big) \,\mathrm{d}t - (e_n^-, v_n^+)_{\mathcal{T}_h} = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{V}}_h.
\end{aligned} \tag{4.56}$$

Introducing the splitting $\boldsymbol{e}_h = (\boldsymbol{u} - \boldsymbol{\Pi}_h u) + (\boldsymbol{\Pi}_h u - \boldsymbol{u}_h) = \boldsymbol{\eta}_h + \boldsymbol{\theta}_h$, integrating by parts in the first term on the left hand side, using the definition of the parabolic Stokes projection eq. (4.47), and choosing $\boldsymbol{v}_h = \boldsymbol{\theta}_h \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h$, eq. (4.56) reduces to

$$\begin{aligned}\int_{I_n} (\partial_t \theta_h, \theta_h)_{\mathcal{T}_h} \,\mathrm{d}t + \nu \int_{I_n} a_h(\boldsymbol{\theta}_h, \boldsymbol{\theta}_h) \,\mathrm{d}t + \left(\theta_n^+ - \theta_n^-, \theta_n^+\right)_{\mathcal{T}_h} \\ = - \int_{I_n} \left(o_h(u; \boldsymbol{u}, \boldsymbol{\theta}_h) - o_h(u_h; \boldsymbol{u}_h, \boldsymbol{\theta}_h)\right) \mathrm{d}t,\end{aligned} \tag{4.57}$$

where we have used that $u_h, \Pi_h u \in P_k(I_n, H)$.

Lemma 4.4.6. *Let $(\Pi_h u, \bar{\Pi}_h u) \in \mathcal{V}_h^{div} \times \bar{\mathcal{V}}_h$ be the velocity pair of the Stokes projection eq. (4.47) and let $(u_h, \bar{u}_h) \in \boldsymbol{\mathcal{V}}_h$ be an approximate velocity solution to the Navier–Stokes system computed using the space-time HDG scheme eq. (4.4) for $n = 0, \ldots, N-1$. Let $\theta_h = \Pi_h u - u_h$, $\eta_h = u - \Pi_h u$, $\bar{\theta}_h = \bar{\Pi}_h u - u_h$ and $\bar{\eta}_h = u - \bar{\Pi}_h u$. There exists a constant $C > 0$ such that*

$$\int_{I_n} \|\theta_h\|_{L^2(\Omega)}^2 \,\mathrm{d}t \le C \left(\nu^{1/2} \Delta t^{1/2} \int_{I_n} |\!|\!|\boldsymbol{\theta}_h|\!|\!|_v^2 \,\mathrm{d}t + \nu \Delta t \int_{I_n} |\!|\!|\boldsymbol{\eta}_h|\!|\!|_v^2 \,\mathrm{d}t\right) + \Delta t \left\|\theta_n^-\right\|_{L^2(\Omega)}^2.$$

Proof. We will proceed as in the proof of [17, Theorem 5.2]. Choose $\boldsymbol{z}_h \in V_h^{\text{div}} \times \bar{V}_h$ independent of time. We test eq. (4.56) with the discrete characteristic function $\boldsymbol{z}_\chi \in V_h^{\text{div}} \times \bar{V}_h$ of $\boldsymbol{z}_h$. Recall from Equation (4.15) that we can write $\boldsymbol{z}_\chi = \varphi(t)\boldsymbol{z}_h$, with $\varphi(t)$ satisfying $\varphi(t_n^+) = 1$ as well as eq. (4.16) and eq. (4.17). Then, we have

$$(\theta_h(s), z_h)_{\mathcal{T}_h} = - \int_{I_n} \left(o_h(u; \boldsymbol{u}, \boldsymbol{z}_\chi) - o_h(u_h; \boldsymbol{u}_h, \boldsymbol{z}_\chi) + \nu a_h(\boldsymbol{\theta}_h, \boldsymbol{z}_\chi)\right) \mathrm{d}t + \left(\theta_n^-, z_h\right)_{\mathcal{T}_h}. \tag{4.58}$$

By the boundedness of $a_h(\cdot,\cdot)$ eq. (4.8), the bound on φ eq. (4.17), and the Cauchy–Schwarz inequality,

$$\int_{I_n} |a_h(\boldsymbol{\theta}_h, \boldsymbol{z}_\chi)| \,\mathrm{d}t \le C \Delta t^{1/2} |\!|\!|\boldsymbol{z}_h|\!|\!|_v \left(\int_{I_n} |\!|\!|\boldsymbol{\theta}_h|\!|\!|_v^2 \,\mathrm{d}t\right)^{1/2}. \tag{4.59}$$

After a few algebraic manipulations, we apply eq. (4.10), followed by eq. (4.17), the energy estimate Lemma 4.1.1, the assumption eq. (4.2) on the problem data, and the Cauchy–

Schwarz inequality, to find

$$
\begin{aligned}
&\int_{I_n} |o_h(u;\boldsymbol{u},\boldsymbol{z}_\chi) - o_h(u_h;\boldsymbol{u}_h,\boldsymbol{z}_\chi)| \,\mathrm{d}t \\
&= \int_{I_n} |o_h(u;\boldsymbol{\eta}_h,\boldsymbol{z}_\chi) + o_h(\eta_h;\boldsymbol{\Pi}_h u,\boldsymbol{z}_\chi) + o_h(u_h;\boldsymbol{\theta}_h,\boldsymbol{z}_\chi) - o_h(\theta_h;\boldsymbol{\Pi}_h u,\boldsymbol{z}_\chi)| \,\mathrm{d}t \\
&\le C|\!|\!|\boldsymbol{z}_h|\!|\!|_v \int_{I_n} \Big(\|u\|_{H^1(\Omega)} |\!|\!|\boldsymbol{\eta}_h|\!|\!|_v + |\!|\!|\boldsymbol{\eta}_h|\!|\!|_v |\!|\!|\boldsymbol{\Pi}_h u|\!|\!|_v + |\!|\!|\boldsymbol{u}_h|\!|\!|_v |\!|\!|\boldsymbol{\theta}_h|\!|\!|_v + |\!|\!|\boldsymbol{\Pi}_h u|\!|\!|_v |\!|\!|\boldsymbol{\theta}_h|\!|\!|_v\Big) \,\mathrm{d}t \\
&\le C\nu^{1/2}|\!|\!|\boldsymbol{z}_h|\!|\!|_v \left(\nu^{1/2}\Delta t^{1/2}\left(\int_{I_n} |\!|\!|\boldsymbol{\eta}_h|\!|\!|_v^2 \,\mathrm{d}t\right)^{1/2} + (\nu^{1/2}\Delta t^{1/2}+1)\left(\int_{I_n} |\!|\!|\boldsymbol{\theta}_h|\!|\!|_v^2 \,\mathrm{d}t\right)^{1/2}\right).
\end{aligned}
\tag{4.60}
$$

Combining eq. (4.58), eq. (4.59), and eq. (4.60),

$$
\begin{aligned}
(\theta_h(s), z_h)_{\mathcal{T}_h} \le C|\!|\!|\boldsymbol{z}_h|\!|\!|_v \left(\nu\Delta t^{1/2} + \nu^{1/2}\right)\left(\int_{I_n} |\!|\!|\boldsymbol{\theta}_h|\!|\!|_v^2 \,\mathrm{d}t\right)^{1/2} \\
+ C|\!|\!|\boldsymbol{z}_h|\!|\!|_v \nu\Delta t^{1/2}\left(\int_{I_n} |\!|\!|\boldsymbol{\eta}_h|\!|\!|_v^2 \,\mathrm{d}t\right)^{1/2} + \left(\theta_n^-, z_h\right)_{\mathcal{T}_h}.
\end{aligned}
\tag{4.61}
$$

This holds for any $\boldsymbol{z}_h \in V_h^{\mathrm{div}} \times \bar{V}_h$, so fix $s \in I_n$ and select $\boldsymbol{z}_h = (\theta_h(s), \bar{\theta}_h(s)) \in \boldsymbol{V}_h$ to find

$$
\begin{aligned}
\left\|\theta_h(s)\right\|_{L^2(\Omega)}^2 \le C\left(\nu\Delta t^{1/2} + \nu^{1/2}\right)|\!|\!|\boldsymbol{\theta}_h(s)|\!|\!|_v \left(\int_{I_n} |\!|\!|\boldsymbol{\theta}_h|\!|\!|_v^2 \,\mathrm{d}t\right)^{1/2} + \\
C\nu\Delta t^{1/2}|\!|\!|\boldsymbol{\theta}_h(s)|\!|\!|_v \left(\int_{I_n} |\!|\!|\boldsymbol{\eta}_h|\!|\!|_v^2 \,\mathrm{d}t\right)^{1/2} + \left(\theta_n^-, \theta_h(s)\right)_{\mathcal{T}_h}.
\end{aligned}
\tag{4.62}
$$

This holds for all $s \in I_n$, so the result follows after integrating both sides over I_n and applying the Cauchy–Schwarz inequality and Young's inequality. □

Lemma 4.4.7. *Let $u \in L^\infty(0,T;V) \cap L^2(0,T;V \cap H^2(\Omega)^d) \cap H^1(0,T;H)$ be the strong solution to the continuous Navier–Stokes problem, let $(\Pi_h u, \bar{\Pi}_h u) \in \mathcal{V}_h^{div} \times \bar{\mathcal{V}}_h$ be the velocity pair of the Stokes projection eq. (4.47), and let $(u_h, \bar{u}_h) \in \boldsymbol{\mathcal{V}}_h$ be an approximate velocity solution to the Navier–Stokes system computed using the space-time HDG scheme eq. (4.4) for $n = 0, \dots, N-1$. Let $\theta_h = \Pi_h u - u_h$, $\eta_h = u - \Pi_h u$, $\bar{\theta}_h = \bar{\Pi}_h u - \bar{u}_h$ and $\bar{\eta}_h = u - \bar{\Pi}_h u$. There exists a constant $C > 0$ such that*

$$
\left\|\theta_N^-\right\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1}\left\|[\theta_h]_n\right\|_{L^2(\Omega)}^2 + \nu\int_0^T |\!|\!|\boldsymbol{\theta}_h|\!|\!|_v^2 \,\mathrm{d}t \le C\exp\left(CT\right)\nu\Delta t\int_0^T |\!|\!|\boldsymbol{\eta}_h|\!|\!|_v^2 \,\mathrm{d}t,
$$

provided the time step satisfies $\Delta t \le C\nu$.

Proof. Our starting point for deriving an error estimate for the velocity will be the error equation eq. (4.57). We begin by bounding the nonlinear convection terms. A few algebraic manipulations yield

$$
\begin{aligned}
&-\int_{I_n} (o_h(u;\boldsymbol{u},\boldsymbol{\theta}_h) - o_h(u_h;\boldsymbol{u}_h,\boldsymbol{\theta}_h))\,\mathrm{d}t \\
&\le \int_{I_n} |o_h(u;\boldsymbol{\eta}_h,\boldsymbol{\theta}_h)|\,\mathrm{d}t + \int_{I_n} |o_h(\eta_h;\boldsymbol{\Pi}_h u,\boldsymbol{\theta}_h)|\,\mathrm{d}t + \int_{I_n} |o_h(\theta_h;\boldsymbol{\Pi}_h u,\boldsymbol{\theta}_h)|\,\mathrm{d}t = T_1 + T_2 + T_3,
\end{aligned}
$$

where we have used that $o_h(u_h;\boldsymbol{\theta}_h,\boldsymbol{\theta}_h) \ge 0$. We now bound T_1 and T_2. By eq. (4.10), the assumption eq. (4.2) on the problem data, and Young's inequality with some $\epsilon_1 > 0$, we find

$$
\int_{I_n} |o_h(u;\boldsymbol{\eta}_h,\boldsymbol{\theta}_h)|\,\mathrm{d}t \le \frac{C\nu}{2\epsilon_1}\int_{I_n} |||\boldsymbol{\eta}_h|||_v^2\,\mathrm{d}t + \frac{\nu\epsilon_1}{2}\int_{I_n} |||\boldsymbol{\theta}_h|||_v^2\,\mathrm{d}t, \tag{4.63}
$$

and similarly,

$$
\int_{I_n} |o_h(\eta_h;\boldsymbol{\Pi}_h\boldsymbol{u},\boldsymbol{\theta}_h)|\,\mathrm{d}t \le \frac{C\nu}{2\epsilon_1}\int_{I_n} |||\boldsymbol{\eta}_h|||_v^2\,\mathrm{d}t + \frac{\nu\epsilon_1}{2}\int_{I_n} |||\boldsymbol{\theta}_h|||_v^2\,\mathrm{d}t. \tag{4.64}
$$

The bound on T_3 is more complicated. To begin, we use Lemma 4.3.2 and Hölder's inequality with $p = 4$ and $q = 4/3$ to find

$$
\int_{I_n} |o_h(\theta_h;\boldsymbol{\Pi}_h u,\boldsymbol{\theta}_h)|\,\mathrm{d}t \le C\left(\int_{I_n} \|\theta_h\|_{L^2(\Omega)}^2\,|||\boldsymbol{\Pi}_h u|||_v^4\,\mathrm{d}t\right)^{1/4}\left(\int_{I_n} |||\boldsymbol{\theta}_h|||_v^2\,\mathrm{d}t\right)^{3/4}.
$$

Recall Young's inequality in the form $ab \le \epsilon_2^{p/q}a^p/p + b^q/(q\epsilon_2)$ where $1/p + 1/q = 1$, $1 < q,p < \infty$, $a,b > 0$, and $\epsilon_2 > 0$ (see e.g. [46, Appendix A]). Choosing $p = 4$ and $q = 4/3$ we find

$$
\int_{I_n} |o_h(\theta_h;\boldsymbol{\Pi}_h u,\boldsymbol{\theta}_h)|\,\mathrm{d}t \le C\left(\frac{\epsilon_2^3}{4}\nu^4\int_{I_n} \|\theta_h\|_{L^2(\Omega)}^2\,\mathrm{d}t + \frac{3}{4\epsilon_2}\int_{I_n} |||\boldsymbol{\theta}_h|||_v^2\,\mathrm{d}t\right). \tag{4.65}
$$

Here, we have used the uniform bound on the Stokes projection in Lemma 4.4.4 and the assumption eq. (2.28) on the problem data. Next, we consider the error equation eq. (4.57). Integrating by parts in time on the left hand side of eq. (4.57), combining the result with eq. (4.63), eq. (4.64), eq. (4.65), and using the coercivity of $a_h(\cdot,\cdot)$ eq. (4.8), we have for some constants $C_1, C_2 > 0$:

$$
\begin{aligned}
&\left\|\theta_{n+1}^-\right\|_{L^2(\Omega)}^2 + \left\|[\theta_h]_n\right\|_{L^2(\Omega)}^2 - \left\|\theta_n^-\right\|_{L^2(\Omega)}^2 + C_1\nu\int_{I_n} |||\boldsymbol{\theta}_h|||_v^2\,\mathrm{d}t \\
&\le C_2\left(\nu\epsilon_1^{-1}\int_{I_n} |||\boldsymbol{\eta}_h|||_v^2\,\mathrm{d}t + \epsilon_1\nu\int_{I_n} |||\boldsymbol{\theta}_h|||_v^2\,\mathrm{d}t + \epsilon_2^3\nu^4\int_{I_n} \|\theta_h\|_{L^2(\Omega)}^2\,\mathrm{d}t + \epsilon_2^{-1}\int_{I_n} |||\boldsymbol{\theta}_h|||_v^2\,\mathrm{d}t\right).
\end{aligned} \tag{4.66}
$$

Choosing $\epsilon_1 = C_1/(2C_2)$ and $\epsilon_2 = C_3\nu^{-1}$ where $C_3 > 2C_2/C_1$ in eq. (4.66), letting $C_4 = C_1/2 - C_2/C_3 > 0$, and using Lemma 4.4.6, we have upon rearranging that

$$\left\|\theta_{n+1}^-\right\|_{L^2(\Omega)}^2 + \left\|[\theta_h]_n\right\|_{L^2(\Omega)}^2 + \left(C_4\nu - C_5\nu^{1/2}\Delta t^{1/2}\right)\int_{I_n} \|\!|\boldsymbol{\theta}_h|\!\|_v^2 \,\mathrm{d}t \\ \leq (1 + C_5\Delta t)\left\|\theta_n^-\right\|_{L^2(\Omega)}^2 + C_5\nu\Delta t\int_{I_n} \|\!|\boldsymbol{\eta}_h|\!\|_v^2 \,\mathrm{d}t.$$

Summing over all space-time slabs and noting that $\theta_0^- = 0$, we have

$$\left\|\theta_N^-\right\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1}\left\|[\theta_h]_n\right\|_{L^2(\Omega)}^2 + \left(C_4\nu - C_5\nu^{1/2}\Delta t^{1/2}\right)\int_0^T \|\!|\boldsymbol{\theta}_h|\!\|_v^2 \,\mathrm{d}t \\ \leq C_5\Delta t\left(\sum_{n=0}^{N-1}\left\|\theta_i^-\right\|_{L^2(\Omega)}^2 + \nu\int_0^T \|\!|\boldsymbol{\eta}_h|\!\|_v^2 \,\mathrm{d}t\right).$$

The result follows by a discrete Grönwall inequality [29, Lemma 1.11] for $\Delta t < C_4\nu/(2C_5)$ and using that $\prod_{j=0}^{N-1}(1 + C\Delta t) \leq \exp\left(C\sum_{j=0}^{N-1}\Delta t\right) \leq \exp\left(CT\right)$. □

4.4.5 Proof of Theorem 4.1.2

Proof. Let $\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{u}_h$. We introduce the splitting $\boldsymbol{e}_h = \boldsymbol{\xi}_h + \boldsymbol{\zeta}_h + \boldsymbol{\theta}_h$, where $\boldsymbol{\theta}_h = \boldsymbol{\Pi}_h u - \boldsymbol{u}_h$, $\boldsymbol{\zeta}_h = \boldsymbol{\mathcal{P}}_h u - \boldsymbol{\Pi}_h u$, and $\boldsymbol{\xi}_h = \boldsymbol{u} - \boldsymbol{\mathcal{P}}_h u$. Using the triangle inequality, Lemma 4.4.7, Lemma 4.4.5, and noting that $[\xi_h]_n = 0$ for $n = 0, \ldots, N-1$, we find there exists a constant $C > 0$ such that

$$\left\|e_N^-\right\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1}\left\|[e_h]_n\right\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1}\nu\int_0^T \|\!|\boldsymbol{e}_h|\!\|_{v'}^2 \,\mathrm{d}t \\ \leq \exp\left(CT\right)\left(\left\|\xi_N^-\right\|_{L^2(\Omega)}^2 + \nu\int_0^T \|\!|\boldsymbol{\xi}_h|\!\|_{v'}^2 \,\mathrm{d}t\right). \quad (4.67)$$

To bound the last term on the right-hand side of eq. (4.67), we employ Theorem A.1.2 to find

$$\int_0^T \|\!|\boldsymbol{\xi}_h|\!\|_{v'}^2 \,\mathrm{d}t \lesssim h^{2k}\|u\|_{L^2(0,T,H^{k+1}(\Omega))}^2 + \Delta t^{2k+2}\|u\|_{H^{k+1}(0,T,H^2(\Omega))}^2. \quad (4.68)$$

The result will follow after bounding $\|\xi_N^-\|_{L^2(\Omega)}$. By Lemma 4.4.1, there exists a constant $C > 0$ such that

$$\|\xi_N^-\|^2_{L^2(\Omega)} = \|u(T) - (P_h u)(T)\|^2_{L^2(\Omega)} \lesssim h^{2k+2}\|u\|^2_{C(0,T;H^{k+1}(\Omega))}. \tag{4.69}$$

□

4.5 Error analysis for the pressure

4.5.1 Bounds on temporal derivative of the error

The error analysis for the pressure will require a bound on the temporal derivative of $u - u_h$:

$$\int_{I_n} \|\partial_t(u - u_h)\|^2_{L^2(\Omega)} \,\mathrm{d}t.$$

Lemma 4.5.1. *Let $(\Pi_h u, \bar{\Pi}_h u) \in \mathcal{V}_h^{div} \times \bar{\mathcal{V}}_h$ denote the element and facet velocity components of the Stokes projection eq. (4.47) for $n = 0, \dots, N-1$, and let $\mathcal{P}_h$ denote the projection introduced in Definition 4.4.1. Let $\zeta_h = \mathcal{P}_h u - \Pi_h u$, $\xi_h = u - \mathcal{P}_h u$, $\bar{\zeta}_h = \bar{\mathcal{P}}_h u - \bar{\Pi}_h u$ and $\bar{\xi}_h = u - \bar{\mathcal{P}}_h u$. There exists a constant $C > 0$, independent of the mesh parameters Δt and h and the viscosity ν but dependent on the domain Ω and polynomial degree k, such that*

$$\sum_{n=0}^{N-1} \int_{I_n} \|\partial_t \zeta_h\|^2_{L^2(\Omega)} \,\mathrm{d}t \le C\Delta t^{-1}\nu \int_0^T \|\!|\boldsymbol{\xi}_h|\!\|^2_{v'} \,\mathrm{d}t.$$

Proof. Our starting point will be the definition of the parabolic Stokes projection eq. (4.47). Introducing the splitting $\boldsymbol{u} - \boldsymbol{\Pi}_h \boldsymbol{u} = \boldsymbol{\xi}_h + \boldsymbol{\zeta}_h$ and testing with $v_h = (t - t_n)\partial_t \boldsymbol{\zeta_h} \in P_k(I_n; V_h^{\text{div}}) \times P_k(I_n; \bar{V}_h)$ we find

$$\begin{aligned}
&\int_{I_n} (t - t_n)\|\partial_t \zeta_h\|^2_{L^2(\Omega)} \,\mathrm{d}t + \nu \int_{I_n} (t - t_n) a_h(\boldsymbol{\zeta}_h, \partial_t \boldsymbol{\zeta}_h) \,\mathrm{d}t \\
&= \int_{I_n} (\xi_h, \partial_t \zeta_h)_{\mathcal{T}_h} \,\mathrm{d}t + \int_{I_n} (t - t_n)(\xi_h, \partial_t^2 \zeta_h)_{\mathcal{T}_h} \,\mathrm{d}t \\
&\quad - \Delta t(\xi_h(t_{n+1}^-), \partial_t \zeta_{n+1}^-)_{\mathcal{T}_h} - \nu \int_{I_n} (t - t_n) a_h(\boldsymbol{\xi}_h, \partial_t \boldsymbol{\zeta}_h) \,\mathrm{d}t.
\end{aligned}$$

Now, since $\partial_t \zeta_h, (t-t_n)\partial_t^2 \zeta_h \in P_{k-1}(I_n, V_h^{\text{div}})$,

$$\int_{I_n} (\xi_h, \partial_t \zeta_h)_{\mathcal{T}_h} \,\mathrm{d}t = 0 = \int_{I_n} (t-t_n)(\xi_h, \partial_t^2 \zeta_h)_{\mathcal{T}_h} \,\mathrm{d}t.$$

Therefore,

$$\begin{aligned}
&\int_{I_n} (t-t_n)\|\partial_t \zeta_h\|_{L^2(\Omega)}^2 \,\mathrm{d}t + \nu \int_{I_n} (t-t_n) a_h(\boldsymbol{\zeta}_h, \partial_t \boldsymbol{\zeta}_h) \,\mathrm{d}t \\
&= -\Delta t (\xi_h(t_{n+1}^-), \partial_t \zeta_{n+1}^-)_{\mathcal{T}_h} - \nu \int_{I_n} (t-t_n) a_h(\boldsymbol{\xi}_h, \partial_t \boldsymbol{\zeta}_h) \,\mathrm{d}t.
\end{aligned}$$

Now,

$$\int_{I_n} (t-t_n) a_h(\boldsymbol{\zeta}_h, \partial_t \boldsymbol{\zeta}_h) \,\mathrm{d}t = \frac{1}{2}\int_{I_n} (t-t_n) \frac{\mathrm{d}}{\mathrm{d}t} a_h(\boldsymbol{\zeta}_h, \boldsymbol{\zeta}_h) \,\mathrm{d}t,$$

and integration by parts yields

$$\frac{1}{2}\int_{I_n} (t-t_n) \frac{\mathrm{d}}{\mathrm{d}t} a_h(\boldsymbol{\zeta}_h, \boldsymbol{\zeta}_h) \,\mathrm{d}t = \frac{1}{2}\Delta t a_h(\boldsymbol{\zeta}_h(t_{n+1}^-), \boldsymbol{\zeta}_h(t_{n+1}^-)) - \frac{1}{2}\int_{I_n} a_h(\boldsymbol{\zeta}_h, \boldsymbol{\zeta}_h) \,\mathrm{d}t.$$

By the coercivity of $a_h(\cdot,\cdot)$, $a_h(\boldsymbol{\zeta}_h(t_{n+1}^-), \boldsymbol{\zeta}_h(t_{n+1}^-)) \geq 0$, so

$$\begin{aligned}
&\int_{I_n} (t-t_n)\|\partial_t \zeta_h\|_{L^2(\Omega)}^2 \,\mathrm{d}t \\
&\leq \frac{C_b}{2}\nu \int_{I_n} \|\!|\!|\boldsymbol{\zeta}_h|\!|\!|_v^2 \,\mathrm{d}t - \Delta t (\xi_h(t_{n+1}^-), \partial_t \zeta_{n+1}^-)_{\mathcal{T}_h} - \nu \int_{I_n} (t-t_n) a_h(\boldsymbol{\xi}_h, \partial_t \boldsymbol{\zeta}_h) \,\mathrm{d}t.
\end{aligned}$$

Since $\partial_t \zeta_{n+1}^- \in V_h^{\text{div}}$, it holds that $(\xi_h(t_{n+1}^-), \partial_t \zeta_{n+1}^-)_{\mathcal{T}_h} = 0$. Now, by the boundedness of $a_h(\cdot,\cdot)$, the inverse inequality Equation (4.19b), and Young's inequality, we have

$$\int_{I_n} (t-t_n)|a_h(\boldsymbol{\xi}_h, \partial_t \boldsymbol{\zeta}_h)| \,\mathrm{d}t \leq \frac{C}{2}\int_{I_n} \|\!|\!|\boldsymbol{\xi}_h|\!|\!|_{v'}^2 \,\mathrm{d}t + \frac{C}{2}\int_{I_n} \|\!|\!|\boldsymbol{\zeta}_h|\!|\!|_v^2 \,\mathrm{d}t.$$

Thus,

$$\int_{I_n} (t-t_n)\|\partial_t \zeta_h\|_{L^2(\Omega)}^2 \,\mathrm{d}t \leq C\left(\nu \int_{I_n} \|\!|\!|\boldsymbol{\xi}_h|\!|\!|_{v'}^2 \,\mathrm{d}t + \nu \int_{I_n} \|\!|\!|\boldsymbol{\zeta}_h|\!|\!|_v^2 \,\mathrm{d}t\right).$$

Applying a finite-dimensional scaling argument as in [98, Eq. (12.18)],

$$C\Delta t \int_{I_n} \|\partial_t \zeta_h\|_{L^2(\Omega)}^2 \,\mathrm{d}t \leq \int_{I_n} (t-t_n)\|\partial_t \zeta_h\|_{L^2(\Omega)}^2 \,\mathrm{d}t,$$

so

$$\int_{I_n} \|\partial_t \zeta_h\|^2_{L^2(\Omega)} \,\mathrm{d}t \le C\Delta t^{-1} \left(\nu \int_{I_n} \|\!|\boldsymbol{\xi}_h|\!\|^2_{v'} \,\mathrm{d}t + \nu \int_{I_n} \|\!|\boldsymbol{\zeta}_h|\!\|^2_{v} \,\mathrm{d}t \right).$$

Summing over all space-time slabs and applying Lemma 4.4.5 yields the result. □

Lemma 4.5.2. *Let $(u_h, \bar{u}_h) \in \boldsymbol{\mathcal{V}}_h$ be the approximate velocity solution to the Navier–Stokes system computed using the space-time HDG scheme eq. (4.4) for $n = 0, \dots N-1$ and let u be the velocity solution to the Navier–Stokes system eq. (2.24). Let $\theta_h = u_h - \Pi_h u$, $\xi_h = u - \mathcal{P}_h u$, $\bar{\theta}_h = u_h - \bar{\Pi}_h u$ and $\bar{\xi}_h = u - \bar{\mathcal{P}}_h u$. There exists a constant $C > 0$, independent of the mesh parameters Δt and h and the viscosity ν but dependent on the domain Ω and polynomial degree k, such that*

$$\sum_{n=0}^{N-1} \int_{I_n} \|\partial_t \theta_h\|^2_{L^2(\Omega)} \,\mathrm{d}t \le C\Delta t^{-3/2} \exp\Big(CT\Big) \left(\nu \int_0^T \|\!|\boldsymbol{\xi}_h|\!\|^2_{v'} \,\mathrm{d}t \; + \; \nu \int_0^T \|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|^2_{v} \,\mathrm{d}t \right).$$

Proof. Our starting point is the error equation eq. (4.56). This time, we test with $\boldsymbol{v}_h = (t - t_n)\partial_t \boldsymbol{\theta}_h \in P_{k-1}(I_n; V_h^{\mathrm{div}}) \times P_{k-1}(I_n, V_h^k)$. Since $u_h \in P_k(I_n, H(\mathrm{div}))$, $\nabla \cdot u_h = 0$, $\Pi_h u \in P_k(I_n, H(\mathrm{div}))$, and $\nabla \cdot \Pi_h u = 0$, we have that $\partial_t u_h \in P_{k-1}(I_n, H(\mathrm{div}))$, $\partial_t(\nabla \cdot u_h) = \nabla \cdot \partial_t u_h = 0$, $\partial_t \Pi_h u \in P_{k-1}(I_n, H(\mathrm{div}))$, and $\partial_t(\nabla \cdot \Pi_h u_h) = \nabla \cdot \partial_t \Pi_h u_h = 0$. Therefore, we have

$$\begin{aligned} \int_{I_n} (t - t_n) \|\partial_t \theta_h\|^2_{L^2(\Omega)} \,\mathrm{d}t + \nu \int_{I_n} (t - t_n) a_h(\boldsymbol{\theta}_h, \partial_t \boldsymbol{\theta}_h) \,\mathrm{d}t \\ = - \int_{I_n} (t - t_n) \big(o_h(u; \boldsymbol{u}, \partial_t \boldsymbol{\theta}_h) - o_h(u_h; \boldsymbol{u}_h, \partial_t \boldsymbol{\theta}_h) \big) \,\mathrm{d}t. \end{aligned} \tag{4.70}$$

Now,

$$\begin{aligned} \nu \int_{I_n} (t - t_n) a_h(\boldsymbol{\theta}_h, \partial_t \boldsymbol{\theta}_h) \,\mathrm{d}t &= \frac{1}{2}\nu \int_{I_n} (t - t_n) \tfrac{\mathrm{d}}{\mathrm{d}t} a_h(\boldsymbol{\theta}_h, \boldsymbol{\theta}_h) \,\mathrm{d}t \\ &= \frac{1}{2}\nu \Delta t a_h(\boldsymbol{\theta}^-_{n+1}, \boldsymbol{\theta}^-_{n+1}) - \frac{1}{2}\nu \int_{I_n} a_h(\boldsymbol{\theta}_h, \boldsymbol{\theta}_h) \,\mathrm{d}t \\ &\ge -\frac{1}{2}\nu \int_{I_n} a_h(\boldsymbol{\theta}_h, \boldsymbol{\theta}_h) \,\mathrm{d}t, \end{aligned} \tag{4.71}$$

so by the boundedness of $a_h(\cdot, \cdot)$,

$$\begin{aligned} &\int_{I_n} (t - t_n) \|\partial_t \theta_h\|^2_{L^2(\Omega)} \,\mathrm{d}t \\ &\qquad \le C\nu \int_{I_n} \|\!|\boldsymbol{\theta}_h|\!\|^2_{v} \,\mathrm{d}t \; + \; \Delta t \int_{I_n} \big| \big(o_h(u; \boldsymbol{u}, \partial_t \boldsymbol{\theta}_h) - o_h(u_h; \boldsymbol{u}_h, \partial_t \boldsymbol{\theta}_h) \big) \big| \,\mathrm{d}t. \end{aligned}$$

Now,

$$o_h(u;\boldsymbol{u},\partial_t\boldsymbol{\theta}_h) - o_h(u_h;\boldsymbol{u}_h,\partial_t\boldsymbol{\theta}_h) = o_h(u-u_h;\boldsymbol{u},\partial_t\boldsymbol{\theta}_h) + o_h(u_h;\boldsymbol{u}-\boldsymbol{u}_h,\partial_t\boldsymbol{\theta}_h),$$

so by eq. (4.10) we have

$$\begin{aligned}\Delta t\int_{I_n} &|\left(o_h(u;\boldsymbol{u},\partial_t\boldsymbol{\theta}_h) - o_h(u_h;\boldsymbol{u}_h,\partial_t\boldsymbol{\theta}_h)\right)|\,\mathrm{d}t\\ &\le C\Delta t\int_{I_n}\left(\|u\|_{H^1(\Omega)} + \|\!|\boldsymbol{u}_h|\!\|_v\right)\|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!\|_v\|\!|\partial_t\boldsymbol{\theta}_h|\!\|_v\,\mathrm{d}t\\ &\le C\Delta t\left(\|u\|_{L^\infty(0,T;H^1(\Omega))} + \|\boldsymbol{u}_h\|_{L^\infty(0,T;\boldsymbol{\mathcal{V}}_h)}\right)\int_{I_n}\|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!\|_v\|\!|\partial_t\boldsymbol{\theta}_h|\!\|_v\,\mathrm{d}t.\end{aligned}$$

By Equation (4.19a), the energy estimate Lemma 4.1.1, and the assumption eq. (4.2) on the problem data,

$$\|\boldsymbol{u}_h\|_{L^\infty(0,T;\boldsymbol{\mathcal{V}}_h)} \le C\Delta t^{-1/2}\nu^{1/2}. \tag{4.72}$$

Thus, by the assumption eq. (4.2) on the problem data, Theorem 2.3.2, eq. (2.29), and Equation (4.19b),

$$\begin{aligned}\Delta t\int_{I_n} &|\left(o_h(u;\boldsymbol{u},\partial_t\boldsymbol{\theta}_h) - o_h(u_h;\boldsymbol{u}_h,\partial_t\boldsymbol{\theta}_h)\right)|\,\mathrm{d}t\\ &\le \frac{C\epsilon}{2}\left(\nu^2 + \frac{\nu}{\Delta t}\right)\int_{I_n}\|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!\|_v^2\,\mathrm{d}t + \frac{1}{2\epsilon}\Delta t^2\int_{I_n}\|\!|\partial_t\boldsymbol{\theta}_h|\!\|_v^2\,\mathrm{d}t\\ &\le \frac{C\epsilon}{2}\left(\nu^2 + \frac{\nu}{\Delta t}\right)\int_{I_n}\|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!\|_v^2\,\mathrm{d}t + \frac{C}{2\epsilon}\int_{I_n}\|\!|\boldsymbol{\theta}_h|\!\|_v^2\,\mathrm{d}t.\end{aligned}$$

Choosing $\epsilon = O(\Delta t^{1/2})$, we have

$$\begin{aligned}\Delta t\int_{I_n} &|\left(o_h(u;\boldsymbol{u},\partial_t\boldsymbol{\theta}_h) - o_h(u_h;\boldsymbol{u}_h,\partial_t\boldsymbol{\theta}_h)\right)|\,\mathrm{d}t\\ &\le C\left(\nu^2\Delta t^{1/2} + \nu\Delta t^{-1/2}\right)\int_{I_n}\|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!\|_v^2\,\mathrm{d}t + C\Delta t^{-1/2}\int_{I_n}\|\!|\boldsymbol{\theta}_h|\!\|_v^2\,\mathrm{d}t\\ &\le C\nu\Delta t^{-1/2}\int_{I_n}\|\!|\boldsymbol{u}-\boldsymbol{u}_h|\!\|_v^2\,\mathrm{d}t + C\Delta t^{-1/2}\int_{I_n}\|\!|\boldsymbol{\theta}_h|\!\|_v^2\,\mathrm{d}t.\end{aligned}$$

A finite-dimensional scaling argument [98, Eq. (12.18)] yields

$$C\Delta t\int_{I_n}\|\partial_t\theta_h\|_{L^2(\Omega)}^2\,\mathrm{d}t \le \int_{I_n}(t-t_n)\|\partial_t\theta_h\|_{L^2(\Omega)}^2\,\mathrm{d}t.$$

Therefore, we have

$$\int_{I_n} \|\partial_t \theta_h\|^2_{L^2(\Omega)} \,\mathrm{d}t \le C\nu^{-1}\Delta t^{-3/2} \left(\nu \int_{I_n} |||\boldsymbol{\theta}_h|||_v^2 \,\mathrm{d}t \;+\; \nu \int_{I_n} |||\boldsymbol{u} - \boldsymbol{u}_h|||_v^2 \,\mathrm{d}t \right).$$

Summing over all space-time slabs and applying Lemma 4.4.7, using the splitting $\boldsymbol{\eta}_h = \boldsymbol{\xi}_h + \boldsymbol{\zeta}_h$, and applying Lemma 4.4.5, we have

$$\begin{aligned}
&\sum_{n=0}^{N-1} \int_{I_n} \|\partial_t \theta_h\|^2_{L^2(\Omega)} \,\mathrm{d}t \\
&\le C\nu^{-1}\Delta t^{-3/2} \exp\Big(CT\Big) \left(\nu \int_0^T |||\boldsymbol{\xi}_h|||_{v'}^2 \,\mathrm{d}t \;+\; \nu \int_0^T |||\boldsymbol{u} - \boldsymbol{u}_h|||_v^2 \,\mathrm{d}t \right).
\end{aligned}$$

□

Lemma 4.5.3. *Let $(u_h, \bar{u}_h) \in \boldsymbol{\mathcal{V}}_h$ be the approximate velocity solution to the Navier–Stokes system computed using the space-time HDG scheme eq. (4.4) for $n = 0, \dots N-1$ and let the velocity solution u to the Navier–Stokes system eq. (2.24) satisfy*

$$u \in H^{k+1}(0,T; V \cap H^2(\Omega)^d) \cap H^1(0,T; H^{k+1}(\Omega)^d),$$

with initial data satisfying $u_0 \in H^{k+1}(\Omega)^d$. There exists a constant $C > 0$, independent of the mesh parameters Δt and h and the viscosity ν but dependent on the domain Ω and polynomial degree k, such that

$$\begin{aligned}
&\sum_{n=0}^{N-1} \int_{I_n} \big\|\partial_t (u - u_h)\big\|^2_{L^2(\Omega)} \\
&\le C\nu^{-1} \exp\Big(CT\Big) \left(\Delta t^{2k} \|u\|^2_{H^{k+1}(0,T;H^2(\Omega))} + \frac{h^{2k}}{\Delta t^{3/2}} \|u\|^2_{L^2(0,T;H^{k+1}(\Omega))} \right).
\end{aligned}$$

Proof. First, we introduce the splitting $\boldsymbol{e}_h = \boldsymbol{\xi}_h + \boldsymbol{\zeta}_h + \boldsymbol{\theta}_h$:

$$\sum_{n=0}^{N-1} \int_{I_n} \big\|\partial_t (u - u_h)\big\|^2_{L^2(\Omega)} \le \sum_{n=0}^{N-1} \int_{I_n} \|\partial_t \xi_h\|^2_{L^2(\Omega)} + \sum_{n=0}^{N-1} \int_{I_n} \|\partial_t \zeta_h\|^2_{L^2(\Omega)} + \sum_{n=0}^{N-1} \int_{I_n} \|\partial_t \theta_h\|^2_{L^2(\Omega)}.$$

From Lemma 4.5.1, Lemma 4.5.2, and the assumption that $\Delta t \le 1$, we have:

$$\begin{aligned}
&\sum_{n=0}^{N-1} \int_{I_n} \big\|\partial_t (u - u_h)\big\|^2_{L^2(\Omega)} \\
&\le \sum_{n=0}^{N-1} \int_{I_n} \|\partial_t \xi_h\|^2_{L^2(\Omega)} + C\nu^{-1}\Delta t^{-3/2} \exp\Big(CT\Big) \left(\nu \int_0^T |||\boldsymbol{\xi}_h|||_{v'}^2 \,\mathrm{d}t \;+\; \nu \int_0^T |||\boldsymbol{u} - \boldsymbol{u}_h|||_v^2 \,\mathrm{d}t \right).
\end{aligned}$$

Applying eq. (A.3e), the projection estimates in Theorem A.1.2, and Theorem 4.1.2 (see in particular eq. (4.68) and eq. (4.69)), we have:

$$\begin{aligned}
&\sum_{n=0}^{N-1} \int_{I_n} \big\|\partial_t(u - u_h)\big\|_{L^2(\Omega)}^2 \\
&\leq C\nu^{-1} \exp\Big(CT\Big) \Big(\Delta t^{2k} \|u\|_{H^{k+1}(0,T;L^2(\Omega))}^2 + h^{2k+2} \|u\|_{H^1(0,T;H^{k+1}(\Omega))}^2 \\
&+ \Delta t^{2k+1/2} \|u\|_{H^{k+1}(0,T,H^2(\Omega))}^2 + \Delta t^{-3/2} h^{2k} \Big(h^2 \|u\|_{C(0,T;H^{k+1}(\Omega))}^2 + \|u\|_{L^2(0,T;H^{k+1}(\Omega))}^2\Big)\Big).
\end{aligned}$$

The result follows after collecting leading order terms. □

4.5.2 Proof of Theorem 4.1.3.

Proof. Recall that given $p_h - q_h \in \mathcal{Q}_h$, we can expand it in terms of an orthonormal basis of $P_k(I_n)$:

$$p_h - q_h = \sum_{i=0}^{k} \phi_i(t)(p_i - q_i),$$

where $p_i, q_i \in Q_h$. Mimicking the proof of the inf-sup condition Theorem 4.3.2, given $\boldsymbol{p}_h - \boldsymbol{q}_h \in \boldsymbol{\mathcal{Q}}_h$, we construct $\boldsymbol{\psi}_h \in \boldsymbol{\mathcal{V}}_h$ by choosing

$$\psi_h = \sum_{i=0}^{k} \phi_i(t) \Pi_{\text{BDM}} z_i, \quad \text{and} \quad \bar{\psi}_h = \sum_{i=0}^{k} \phi_i(t) \bar{\Pi}_V z_i, \tag{4.73}$$

where Π_{BDM} is the BDM projection, $\bar{\Pi}_V$ is the L^2-projection onto the space $\bar{V}_h$, and $z_i \in H_0^1(\Omega)^d$ satisfies

$$\nabla \cdot z_i = p_i - q_i, \quad \|z_i\|_{H^1(\Omega)} \leq \beta \|p_i - q_i\|_{L^2(\Omega)},$$

for some $\beta > 0$. Note that it holds that

$$\int_{I_n} \|p_h - q_h\|_{L^2(\Omega)}^2 \,\mathrm{d}t = \int_{I_n} b_h(\boldsymbol{p} - \boldsymbol{q}_h, \psi_h) \,\mathrm{d}t - \int_{I_n} b_h(\boldsymbol{p} - \boldsymbol{p}_h, \psi_h) \,\mathrm{d}t. \tag{4.74}$$

Testing eq. (4.4a) with $\boldsymbol{v}_h = \boldsymbol{\psi}_h$, using Lemma 4.2.2, integrating by parts in time and rearranging, and applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\int_{I_n} b_h(\boldsymbol{p} - \boldsymbol{p}_h, \psi_h) \,\mathrm{d}t \leq& \int_{I_n} \big\|\partial_t(u - u_h)\big\|_{L^2(\Omega)} \|\psi_h\|_{L^2(\Omega)} \,\mathrm{d}t + \big\|[u - u_h]_n\big\|_{L^2(\Omega)} \big\|\psi_n^+\big\|_{L^2(\Omega)} \\
&+ \nu \int_{I_n} \|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|_{v'} \|\!|\boldsymbol{\psi}_h|\!\|_v + \int_{I_n} (o_h(u; \boldsymbol{u}, \boldsymbol{\psi}_h) - o_h(u_h; \boldsymbol{u}_h, \boldsymbol{\psi}_h)) \,\mathrm{d}t.
\end{aligned}$$

Note that there exists a constant $C > 0$ such that $|||\boldsymbol{\psi}_i|||_v \le C\|p_i - q_i\|_{L^2(\Omega)}$ (see [77, eq. (75)]) and thus

$$\int_{I_n} |||\boldsymbol{\psi}_h|||_v^2 \le C \int_{I_n} \|p_h - q_h\|_{L^2(\Omega)}^2 \,\mathrm{d}t. \tag{4.75}$$

Since $o_h(u;\boldsymbol{u},\boldsymbol{\psi}_h) - o_h(u_h;\boldsymbol{u}_h,\boldsymbol{\psi}_h) = o_h(u-u_h;\boldsymbol{u},\boldsymbol{\psi}_h) + o_h(u_h;\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\psi}_h)$, we can apply eq. (4.10) to find

$$\begin{aligned}
&\int_{I_n} | \left(o_h(u;\boldsymbol{u},\boldsymbol{\psi}_h) - o_h(u_h;\boldsymbol{u}_h,\boldsymbol{\psi}_h)\right) | \,\mathrm{d}t \\
&\qquad \le C\left(\|u\|_{L^\infty(0,T;H^1(\Omega))} + \|\boldsymbol{u}_h\|_{L^\infty(0,T;\boldsymbol{\mathcal{V}}_h)}\right) \int_{I_n} |||\boldsymbol{u}-\boldsymbol{u}_h|||_v |||\boldsymbol{\psi}_h|||_v \,\mathrm{d}t.
\end{aligned}$$

Now, by Equation (4.19a), Lemma 4.1.1, Theorem 2.3.2 eq. (2.29), assumption eq. (4.2) on the problem data, and eq. (4.75), there exists a constant $C > 0$ such that

$$\begin{aligned}
&\int_{I_n} | \left(o_h(u;\boldsymbol{u},\boldsymbol{\psi}_h) - o_h(u_h;\boldsymbol{u}_h,\boldsymbol{\psi}_h)\right) | \,\mathrm{d}t \\
&\qquad \le C\Delta t^{-1/2}\left(\nu\int_{I_n} |||\boldsymbol{u}-\boldsymbol{u}_h|||_{v'}^2 \,\mathrm{d}t\right)^{1/2} \left(\int_{I_n} \|p_h - q_h\|_{L^2(\Omega)}^2 \,\mathrm{d}t\right)^{1/2}.
\end{aligned} \tag{4.76}$$

Collecting eq. (4.75) and eq. (4.76) and applying the following discrete trace inequality valid for polynomials in time (see e.g. [29, Lemma 6.42])

$$\left\|\psi_n^+\right\|_{L^2(\Omega)} \le C\Delta t^{-1/2}\left(\int_{I_n} \|\psi_h\|_{L^2(\Omega)}^2 \,\mathrm{d}t\right)^{1/2},$$

we see that there exists a constant $C > 0$ such that

$$\begin{aligned}
&\int_{I_n} |b_h(\boldsymbol{p}-\boldsymbol{p}_h,\psi_h)| \,\mathrm{d}t \\
&\qquad \le C\Delta t^{-1/2}\left(\int_{I_n} \|p_h - q_h\|_{L^2(\Omega)}^2 \,\mathrm{d}t\right)^{1/2} \left(\Delta t^{1/2}\left(\int_{I_n} \left\|\partial_t(u-u_h)\right\|_{L^2(\Omega)}^2 \,\mathrm{d}t\right)^{1/2}\right. \\
&\qquad \left. + \left\|[u-u_h]_n\right\|_{L^2(\Omega)}^2 + \left(\nu\int_{I_n} |||\boldsymbol{u}-\boldsymbol{u}_h|||_{v'}^2 \,\mathrm{d}t\right)^{1/2}\right).
\end{aligned} \tag{4.77}$$

Summing over all space-time slabs and using Lemma 4.5.3 and Theorem 4.1.2 (see in particular eq. (4.68) and eq. (4.69)), we have the following leading order terms

$$\begin{aligned} &\int_0^T |b_h(\boldsymbol{p} - \boldsymbol{p}_h, \psi_h)| \,\mathrm{d}t \\ &\leq C\nu^{-1} \exp\Big(CT\Big) \Big(\int_{I_n} \|p_h - q_h\|^2_{L^2(\Omega)} \,\mathrm{d}t\Big)^{1/2} \\ &\quad \times \bigg(\Delta t^k \|u\|_{H^{k+1}(0,T;H^2(\Omega))} + \frac{h^k}{\Delta t^{3/4}} \|u\|_{L^2(0,T;H^{k+1}(\Omega))}\bigg). \end{aligned} \tag{4.78}$$

It remains to bound

$$\int_{I_n} b_h(\boldsymbol{p} - \boldsymbol{q}_h, \psi_h) \,\mathrm{d}t.$$

Note that since $\psi_h \cdot n|_{\partial K} \in P_k(I_n; P_k(\partial K))$, we have

$$\int_{I_n} b_h(\boldsymbol{p} - \boldsymbol{q}_h, \psi_h) \,\mathrm{d}t = -\sum_{K \in \mathcal{T}_h} \int_{I_n} \bigg(\int_K (p - q_h) \nabla \cdot \psi_h \,\mathrm{d}x \,\mathrm{d}t + \int_{\partial K} \psi_h \cdot n(\Pi_{\bar{Q}} p - \bar{q}_h) \,\mathrm{d}s\bigg) \,\mathrm{d}t,$$

where $\Pi_{\bar{Q}}$ is the L^2-projection onto $P_k(I_n; P_k(\partial K))$. Recalling the definition of ψ_h eq. (4.73), we see that $\psi_h \in P_k(I_n; H(\mathrm{div}))$, and hence $\psi_h \cdot n$ is single-valued across cell facets, as is $\Pi_{\bar{Q}} p - \bar{q}_h$. Thus,

$$\sum_{K \in \mathcal{T}_h} \int_{I_n} \int_{\partial K} \psi_h \cdot n(\Pi_{\bar{Q}} p - \bar{q}_h) \,\mathrm{d}s \,\mathrm{d}t = 0,$$

and therefore we are left with

$$\int_{I_n} b_h(\boldsymbol{p} - \boldsymbol{q}_h, \psi_h) \,\mathrm{d}t = -\sum_{K \in \mathcal{T}_h} \int_{I_n} \int_K (p - q_h) \nabla \cdot \psi_h \,\mathrm{d}t.$$

Noting that $(\nabla \cdot \psi_h)|_K \in P_k(I_n; P_{k-1}(K))$, we can write

$$\int_{I_n} b_h(\boldsymbol{p} - \boldsymbol{q}_h, \psi_h) \,\mathrm{d}t = -\sum_{K \in \mathcal{T}_h} \int_{I_n} \int_K (\Pi_Q p - q_h) \nabla \cdot \psi_h \,\mathrm{d}t,$$

where Π_Q is the L^2-projection onto $P_k(I_n; P_{k-1}(K))$. Furthermore, it can be shown via an expansion in terms of the basis $\{\phi_i(t)\}_{i=0}^k$ of $P_k(I_n)$ and the commuting diagram property of the BDM projection ((iii) in Lemma 3.5.1) that

$$-\sum_{K \in \mathcal{T}_h} \int_{I_n} \int_K (\Pi_Q p - q_h) \nabla \cdot \psi_h \,\mathrm{d}t = -\sum_{K \in \mathcal{T}_h} \int_{I_n} \int_K (\Pi_Q p - q_h)(p_h - q_h) \,\mathrm{d}t.$$

Lastly, we can again use the definition of Π_Q to conclude

$$-\sum_{K\in\mathcal{T}_h}\int_{I_n}\int_K (\Pi_Q p - q_h)\nabla\cdot\psi_h \,\mathrm{d}t = -\sum_{K\in\mathcal{T}_h}\int_{I_n}\int_K (p-q_h)(p_h-q_h)\,\mathrm{d}t.$$

Thus,

$$\int_{I_n} b_h(\boldsymbol{p}-\boldsymbol{q}_h,\psi_h)\,\mathrm{d}t = -\int_{I_n}\int_\Omega (p-q_h)(p_h-q_h)\,\mathrm{d}t. \tag{4.79}$$

With eq. (4.78) and eq. (4.79) in hand, we can return to eq. (4.74), apply the Cauchy–Schwarz inequality, and sum over all space-time slabs to find

$$\left(\int_0^T \|p_h-q_h\|^2_{L^2(\Omega)}\,\mathrm{d}t\right)^{1/2} \le \left(\int_0^T \|p-q_h\|^2_{L^2(\Omega)}\,\mathrm{d}t\right)^{1/2} + E, \tag{4.80}$$

where

$$E = C\nu^{-1}\exp\Big(CT\Big)\left(\Delta t^k\|u\|_{H^{k+1}(0,T;H^2(\Omega))} + \frac{h^k}{\Delta t^{3/4}}\|u\|_{L^2(0,T;H^{k+1}(\Omega))}\right).$$

From the triangle inequality,

$$\left(\int_0^T \|p-p_h\|^2_{L^2(\Omega)}\,\mathrm{d}t\right)^{1/2} \le 2\left(\int_0^T \|p-q_h\|^2_{L^2(\Omega)}\,\mathrm{d}t\right)^{1/2} + E,$$

and we can bound the first term on the right-hand side using the approximation properties of the L^2-projection onto $\mathcal{Q}_h$ (see e.g. [93, Theorem 3]):

$$\int_0^T \|p-q_h\|^2_{L^2(\Omega)}\,\mathrm{d}t \le C\left(h^{2k}\|p\|^2_{L^2(0,T;H^{k+1}(\Omega))} + \Delta t^{2k+2}\|p\|^2_{H^{k+1}(0,T;L^2(\Omega))}\right).$$

The result follows after collecting the leading order terms. □

4.6 Numerical results

In this section, we consider a simple test case with a manufactured solution to verify the theoretical results of the previous sections. We solve the Navier–Stokes equations on the space-time domain $\Omega\times[0,T]=[0,1]^3$. We impose Dirichlet boundary conditions along the boundaries $x = 0$, $x = 1$, $y = 0$, and Neumann boundary conditions along $y = 1$. We choose the problem data such that the exact solution is given by

$$u = \begin{bmatrix} 2 + \sin(2\pi(x-t))\sin(2\pi(y-t)) \\ 2 + \cos(2\pi(x-t))\cos(2\pi(y-t)) \end{bmatrix}, \quad p = \sin(2\pi(x-t))\cos(2\pi(y-t)).$$

This example was implemented using the Modular Finite Element Methods (MFEM) library [2, 67] on prismatic space-time meshes.

We present the velocity and pressure errors, measured in the mesh-dependent $|||\cdot|||_{v'}$-norm and $\|\cdot\|_{L^2(0,T;L^2(\Omega))}$-norm, respectively, and rates of convergence for different levels of space-time refinement with polynomial degrees $k = 2$ and $k = 3$ in Table 4.1. Due to the dominance of the spatial error, we observe that $(\int_0^T |||\boldsymbol{e}_h|||_{v'}^2 \,\mathrm{d}t)^{1/2} = \mathcal{O}(h^k)$, as expected from Theorem 4.1.2. We furthermore observe optimal rates of convergence for the pressure in the $L^2(0,T;L^2(\Omega))$-norm.

Cells per slab	Nr. of slabs	$(\int_0^T |||\boldsymbol{e}_h|||_{v'}^2 \,\mathrm{d}t)^{1/2}$	Rate	$\|p - p_h\|_{L^2(\Omega\times[0,T])}$	Rate
128	20	8.6e-01	-	7.9e-03	-
512	40	2.1e-01	2.0	2.6e-03	1.6
2048	80	5.2e-02	2.0	6.7e-04	1.9
8192	160	1.3e-02	2.0	1.7e-04	2.0
128	20	2.0e-01	-	6.9e-04	-
512	40	2.7e-02	2.9	5.2e-05	3.7
2048	80	3.5e-03	3.0	4.7e-06	3.5
8192	160	4.3e-04	3.0	5.1e-07	3.2

Table 4.1: Rates of convergence when solving eq. (4.1) with $\nu = 10^{-4}$. Note that $\Delta t = 1/(\text{Nr. of slabs})$. Top: using polynomials of degree $k = 2$, bottom: using polynomials of degree $k = 3$.

To explicitly show $k + 1$ rates of convergence in time (as also predicted by Theorem 4.1.2), we now consider a fine enough fixed spatial mesh to ensure that the temporal error dominates over the spatial error. In particular, we choose a mesh consisting of 57800 elements when $k = 2$ and 8192 elements when $k = 3$. In Table 4.2 we observe that $(\int_0^T |||\boldsymbol{e}_h|||_v^2 \,\mathrm{d}t)^{1/2} = \mathcal{O}(\Delta t^{k+1})$ as expected.

Cells per slab	Nr. of slabs	$(\int_0^T \lVert\!\lvert \boldsymbol{e}_h \rVert\!\rvert_v^2 \,\mathrm{d}t)^{1/2}$	Rate
	2	4.4e+00	-
57800	4	9.9e-01	2.1
	8	1.6e-01	2.6
	16	2.3e-02	2.8
	32	3.1e-03	2.9
	2	1.8e+00	-
8192	4	2.1e-01	3.1
	8	1.4e-02	3.9
	16	1.0e-03	3.8

Table 4.2: Time rates of convergence when solving eq. (4.1) with $\nu = 10^{-4}$. Note that $\Delta t = 1/(\text{Nr. of slabs})$. Top: using polynomials of degree $k = 2$, bottom: using polynomials of degree $k = 3$.

Chapter 5

Pressure-robust space-time HDG for the time-dependent problem on fixed domains: Convergence to weak solutions

In this chapter, we continue our study of the space-time hybridized discontinuous Galerkin (HDG) method for the evolutionary incompressible Navier–Stokes equations analyzed in Chapter 4. Therein, we proved that the method is *pressure-robust* and derived optimal rates of convergence in space and time for the velocity field assuming that the Navier–Stokes problem admits a strong solution in the sense of Theorem 2.3.2. However, as a discontinuous method, the HDG method introduces additional stabilization which is a potential source of consistency error if the exact solution is not sufficiently regular. Consequently, the convergence results deduced from the standard a priori analysis of DG and HDG methods often exclude the case of non-smooth solutions which may be present in physically realistic scenarios. For this reason, our analysis in Chapter 4 considered *strong solutions* of the Navier–Stokes system, and cannot be used to deduce convergence to *weak solutions* in the absence of additional regularity.

The purpose of this chapter is to fill this gap by proving that the discrete solution of the space-time HDG scheme analyzed in Chapter 4 for strong solutions converges to a Leray–Hopf weak solution of the evolutionary Navier–Stokes equations. To our knowledge, this is one of the few minimal regularity convergence results available for HDG discretizations in general, and it is the first for a space-time HDG discretization. As a byproduct, we obtain a new proof of the existence of weak solutions of the Navier–Stokes equations. To circumvent the problems posed by the lack of consistency in our numerical scheme, we instead consider the concept of *asymptotic consistency* introduced in [28, Section 5.2]. That is, we aim to show the discrete weak formulation resulting from our space-time HDG discretization converges to the exact weak formulation of the Navier–Stokes equations in a suitable sense as the time step and mesh size tends to zero. Let us briefly discuss the challenges involved.

Consider a countable set of mesh sizes $\mathcal{H}$ whose unique accumulation point is zero, and consider a sequence of discrete velocity solutions $\{u_h\}_{h\in\mathcal{H}}$ satisfying eq. (5.8) below computed on a sequence of space-time meshes such that the time step Δt vanishes along with the mesh size h (but no explicit relation between the two is assumed). We aim to pass to the limit as $h \to 0$ (and thus $\Delta t \to 0$) in eq. (5.8), which requires *compactness*. To this end, the energy bounds obtained on the discrete solution in Chapter 4 allow us to conclude that the sequence of discrete velocities is compact in the *weak topology* of $L^2(0,T;H)$ and the *weak-$\star$ topology* of $L^\infty(0,T;H)$. If the problem were linear, these results would suffice. However, it is well known that nonlinear functions need not be weakly continuous, and thus the nonlinear convection term poses a problem.

To overcome this barrier, we will need to additionally show that $\{u_h\}_{h\in\mathcal{H}}$ is compact in the *strong topology* of $L^2(0,T;L^2(\Omega)^d)$. This is made challenging by the discontinuous nature of our numerical method, as standard compactness results like the Rellich–Kondrachov theorem (Theorem 2.2.6) and the Aubin–Lions–Simon theorem (Theorem 2.2.12) routinely employed at the continuous level are lost and appropriate discrete analogues must be derived. Fortunately, discrete compactness for DG schemes is, at this point, well studied. We mention in particular the works of Buffa and Ortner [11], Di Pietro and Ern [72], and Kikuchi [50], wherein discrete versions of the Rellich–Kondrachov theorem are proven for broken Sobolev and broken polynomial spaces. A common theme among these works is the introduction of a discrete analogue of the gradient operator that incorporates information from the *jumps* of the discrete solution across its discontinuities.

As for a discrete analogue of the Aubin–Lions–Simon theorem in the time-dependent setting, we mention the work of Walkington [102] where it is shown that DG time stepping methods enjoy similar compactness properties to the evolutionary equations they are used to approximate. This is made possible by Simon's characterization of compact sets in $L^p(0,T;B)$ (Theorem 2.2.11) which, unlike Theorem 2.2.12, does not require additional regularity in time. Unfortunately, the results of [102] are valid only for conforming spatial discretizations. This was remedied in [61], wherein a generalization of the work of Walkington valid for broken Sobolev spaces (and thus, for a broad class of non-conforming discretizations) is obtained.

In this chapter, we adapt some of the available discrete functional analysis tools [72] to the HDG setting (see also [50] for similar efforts). We also prove a variation of the discrete Aubin–Lions–Simon theorem in [102] valid for our non-conforming discretization. Our result differs slightly from that of [61] in that we stay entirely within the framework of broken polynomial spaces. In an effort to unify the available discrete functional analysis tools for spatial DG discretizations and DG time stepping, we introduce a discrete time derivative operator in analogy with the aforementioned discrete gradient operator using the time lifting operator in [64, 87], and we show that some of the assumptions required in [61, 102] for compactness can be interpreted using this discrete time derivative.

The remainder of the chapter is organized as follows: In Section 5.1, we introduce notation, recall the space-time HDG method under consideration and some of the key results obtained in Chapter 4. In Section 5.2, we introduce discrete analogues of the gradient operator and time derivative, and recast the numerical scheme in terms of these discrete operators. In Section 5.3, we prove that these discrete operators are bounded uniformly with respect to the mesh size and time step, and as a consequence we obtain convergence of the sequence of discrete velocity solutions as the mesh size and time step tend to zero. In Section 5.4, we show that the limit of this sequence of discrete solutions is a weak solution to the Navier–Stokes equations.

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5.1 Preliminaries

In this section, we discuss the weak formulation for the continuous Navier–Stokes problem eq. (5.1), introduce the space-time HDG method that we will use to approximate solutions of eq. (5.1), and collect a number of useful results for our analysis.

5.1.1 Notation

We use standard notation for Lebesgue and Sobolev spaces: given a bounded measurable set D, we denote by $L^p(D)$ the space of p-integrable functions. When $p = 2$, we denote the

$L^2(D)$ inner product by $(\cdot,\cdot)_D$. We denote by $W^{k,p}(D)$ the Sobolev space of p-integrable functions whose distributional derivatives up to order k are p-integrable. When $p = 2$, we write $W^{k,p}(D) = H^k(D)$. We define $H^1_0(D)$ to be the subspace of $H^1(D)$ of functions with vanishing trace on the boundary of D. We denote the space of polynomials of degree $k \geq 0$ on D by $P_k(D)$. We use standard notation for spaces of vector valued functions with d components, e.g. $L^2(D)^d$, $H^k(D)^d$, $P^k(D)^d$, etc. At times we drop the superscript for convenience, e.g. we denote by $\|\cdot\|_{L^2(\Omega)}$ the norm on both $L^2(\Omega)$ and $L^2(\Omega)^d$.

Next, let U be a Banach space, $I = [a, b]$ an interval in $\mathbb{R}$, and $1 \leq p < \infty$. We denote by $L^p(I;U)$ the Bochner space of p-integrable functions defined on I taking values in U. When $p = \infty$, we denote by $L^\infty(I;U)$ the Bochner space of essentially bounded functions taking values in U and by $C(I;U)$ the space of (time) continuous functions taking values in U. Finally, we let $P_k(I;U)$ denote the space of polynomials of degree $k \geq 0$ in time taking values in U.

5.1.2 The continuous problem

Given a suitably chosen body force f, kinematic viscosity $\nu \in \mathbb{R}^+$, and initial data u_0, we consider the transient Navier–Stokes system posed on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$:

$$\partial_t u - \nu\Delta u + \nabla\cdot(u\otimes u) + \nabla p = f, \qquad \text{in } \Omega\times(0,T], \tag{5.1a}$$

$$\nabla\cdot u = 0, \qquad \text{in } \Omega\times(0,T], \tag{5.1b}$$

$$u = 0, \qquad \text{on } \partial\Omega\times(0,T], \tag{5.1c}$$

$$u(x,0) = u_0(x), \qquad \text{in } \Omega. \tag{5.1d}$$

To avoid any complications arising from curved boundaries, we will assume further that in two spatial dimensions Ω is a polygon and in three spatial dimensions Ω is a polyhedron. As we are interested in weak solutions, we require no assumption that Ω is convex.

Definition 5.1.1 (Weak solution)**.** *Given a body force $f \in L^2(0,T;H^{-1}(\Omega)^d)$ and an initial condition $u_0 \in H$, a function $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ with $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^1(0,T;V')$ is said to be a weak solution of the Navier–Stokes equations eq. (5.1) provided it satisfies for all $\varphi \in C_c(0,T;V)$,*

$$\int_0^T \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \varphi \right\rangle_{V'\times V} \mathrm{d}t + \int_0^T ((u\cdot\nabla)u, \varphi)\,\mathrm{d}t + \nu\int_0^T (\nabla u, \nabla\varphi)\,\mathrm{d}t = \int_0^T \langle f, \varphi\rangle_{H^{-1}\times H^1_0}\,\mathrm{d}t, \tag{5.2}$$

and $u(0) = u_0$ in V' (see e.g. [8, Section V.1.2.2]).

It is well known that weak solutions in the sense of Definition 5.1.1 are weakly continuous from $[0, T]$ into H, and their distributional time derivative possess the further regularity $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^{4/d}(0, T; V')$ (see e.g. [8]). If $d = 2$, this solution is unique and furthermore $u \in C(0, T; H)$. Uniqueness in three dimensions remains an open problem.

Remark 5.1.1 (On the regularity of the body force)**.** *Our main result (Theorem 5.4.3) should remain valid for $f \in L^2(0, T; H^{-1}(\Omega)^d)$ provided there is an appropriate smoothing operator $E_h : V_h \to H_0^1(\Omega)^d$; see e.g. [4]. In particular, if $v_h \to v$ strongly in $L^2(\Omega)^d$, we require also that $E_h v_h \to v$ strongly in $L^2(\Omega)^d$ as $h \to 0$. For simplicity, we focus on body forces $f \in L^2(0, T; L^2(\Omega)^d)$.*

Remark 5.1.2 (The energy inequality)**.** *In two dimensions, the weak solution to the Navier–Stokes equations satisfies the following* energy equality*: for all $s \in (0, T)$,*

$$\left\|u(s)\right\|_{L^2(\Omega)}^2 + 2\nu \int_0^s \|u\|_V^2 \,\mathrm{d}t \;=\; \|u_0\|_{L^2(\Omega)}^2 + 2\int_0^s \langle f, u\rangle_{H^{-1}\times H_0^1} \,\mathrm{d}t. \tag{5.3}$$

In three dimensions, we say that a weak solution is of Leray–Hopf *type if it satisfies the* energy inequality*: for a.e. $s \in (0, T)$,*

$$\left\|u(s)\right\|_{L^2(\Omega)}^2 + 2\nu \int_0^s \|u\|_V^2 \,\mathrm{d}t \;\le\; \|u_0\|_{L^2(\Omega)}^2 + 2\int_0^s \langle f, u\rangle_{H^{-1}\times H_0^1} \,\mathrm{d}t. \tag{5.4}$$

5.1.3 Space-time setting and finite element spaces

In this subsection, we will introduce the space-time slabs, elements, faces, and finite element spaces required for the space-time HDG discretization. We follow some of the definitions introduced in [27]. We define a simplicial mesh of Ω to be a couple $(\mathcal{T}_h, \mathcal{F}_h)$ where the set of mesh elements $\mathcal{T}_h$ is a finite collection of nonempty, disjoint simplices K with boundary ∂K and diameter h_K such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} \overline{K}$. We define the *mesh size* h of $\mathcal{T}_h$ to be $h = \max_{K\in\mathcal{T}_h} h_K$.

The set of mesh faces $\mathcal{F}_h$ is a finite collection of nonempty, disjoint subsets of $\overline{\Omega}$ such that, for any $F \in \mathcal{F}_h$, F is a non-empty, connected subset of a hyperplane in $\mathbb{R}^d$. We assume further that for each $F \in \mathcal{F}_h$, either there exist distinct mesh elements $K_1, K_2 \in \mathcal{T}_h$ such that $F = \partial K_1 \cap \partial K_2$, in which case we call F an interior face, or there exists one mesh element $K \in \mathcal{T}_h$ such that $F = \partial K \cap \partial\Omega$ and we call F a boundary face. Moreover, we assume that the set of mesh faces forms a partition of the mesh skeleton; that is, $\partial\mathcal{T}_h = \bigcup_{K\in\mathcal{T}_h} \partial K = \bigcup_{F\in\mathcal{F}_h} F$. We collect interior faces in the set $\mathcal{F}_h^i$ and boundary faces in the set $\mathcal{F}_h^b$. Note that $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$.

To perform our analysis, we will make the following assumptions on the family of spatial meshes $\{(\mathcal{T}_h, \mathcal{F}_h)\}_{h\in\mathcal{H}}$:

(i) For each $h \in \mathcal{H}$, $\mathcal{T}_h$ is conforming in the sense that given two elements $K_1, K_2 \in \mathcal{T}_h$, either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is a common vertex ($d = 2$) or edge ($d = 3$), or a common face of K_1 and K_2.

(ii) For each $h \in \mathcal{H}$, $\mathcal{T}_h$ is quasi-uniform; i.e., there exists a $C_U > 0$ such that $h \leq C_U h_K$ for all $K \in \mathcal{T}_h$.

(iii) For each $h \in \mathcal{H}$, each face $F \in \mathcal{F}_h$ satisfies an *equivalence condition*: that is, given $h_F = \mathrm{diam}(F)$, there exist constants $C_e, C^e > 0$ such that $C_e h_K \leq h_F \leq C^e h_K$ for all $K \in \mathcal{T}_h$ and for all $F \in \mathcal{F}_h$ where $F \subset \partial K$.

Let $k_s \geq 1$ be a fixed integer. We introduce a pair of discontinuous finite element spaces on $\mathcal{T}_h$:

$$
\begin{aligned}
V_h &:= \left\{ v_h \in L^2(\Omega)^d \mid v_h|_K \in P_{k_s}(K)^d \; \forall K \in \mathcal{T}_h \right\}, \\
Q_h &:= \left\{ q_h \in L^2_0(\Omega) \mid q_h|_K \in P_{k_s-1}(K) \; \forall K \in \mathcal{T}_h \right\},
\end{aligned}
$$

and on $\partial\mathcal{T}_h$, we introduce a pair of discontinuous facet finite element spaces:

$$
\begin{aligned}
\bar{V}_h &:= \left\{ \bar{v}_h \in L^2(\partial\mathcal{T}_h) \mid \bar{v}_h|_F \in P_{k_s}(F)^d \; \forall F \in \mathcal{F}_h, \; \bar{v}_h|_{\partial\Omega} = 0 \right\}, \\
\bar{Q}_h &:= \left\{ \bar{q}_h \in L^2(\partial\mathcal{T}_h) \mid \bar{q}_h|_F \in P_{k_s}(F) \; \forall F \in \mathcal{F}_h \right\}.
\end{aligned}
$$

Next, we partition the time interval $(0, T)$ into a series of $N + 1$ time-levels $0 = t_0 < t_1 < \cdots < t_N = T$ of length $\Delta t_n = t_{n+1} - t_n$, and we define $\tau = \max_{0\leq n\leq N-1} \Delta t_n$. For the compactness result in Theorem B.1.1 to hold, we require this time partition to be quasi-uniform, i.e. there exists a $C_{U'} > 0$ such that $\tau \leq C_{U'} \Delta t_n$ for all $n = 0, \ldots, N - 1$ (see [102] for details). A space-time slab is then defined as $\mathcal{E}^n = \Omega \times I_n$, with $I_n = (t_n, t_{n+1})$. Let $k_t \geq 0$ be a fixed integer (not necessarily chosen to be equal to k_s). We consider the following tensor-product space-time finite element spaces in which we will seek our approximation on each space-time slab $\mathcal{E}^n$:

$$
\begin{aligned}
\mathcal{V}_h &:= \left\{ v_h \in L^2(0, T; L^2(\Omega)^d) \mid v_h|_{\mathcal{E}^n} \in P_{k_t}(I_n; V_h), \; \forall n = 0, \ldots, N - 1 \right\}, \\
\mathcal{Q}_h &:= \left\{ q_h \in L^2(0, T; L^2_0(\Omega)) \mid q_h|_{\mathcal{E}^n} \in P_{k_t}(I_n; Q_h), \; \forall n = 0, \ldots, N - 1 \right\}, \\
\bar{\mathcal{V}}_h &:= \left\{ \bar{v}_h \in L^2(0, T; L^2(\partial\mathcal{T}_h)^d) \mid \bar{v}_h|_{\mathcal{E}^n} \in P_{k_t}(I_n; \bar{V}_h), \; \forall n = 0, \ldots, N - 1 \right\}, \\
\bar{\mathcal{Q}}_h &:= \left\{ \bar{q}_h \in L^2(0, T; L^2(\partial\mathcal{T}_h)) \mid \bar{q}_h|_{\mathcal{E}^n} \in P_{k_t}(I_n; \bar{Q}_h), \; \forall n = 0, \ldots, N - 1 \right\}.
\end{aligned}
$$

As the spaces $\mathcal{V}_h$ and $\mathcal{Q}_h$ are non-conforming, we make use of *broken differential operators*. For $v_h \in \mathcal{V}_h$, we introduce the *broken gradient operator* $\nabla_h v_h$ by the restriction $(\nabla_h v_h)|_K = \nabla(v_h|_K)$ and the *broken time derivative* $\partial_\tau v_h$ by the restriction $(\partial_\tau v_h)|_{I_n} = \partial_t(v_h|_{I_n})$. Moreover, the trace of a function $v_h \in \mathcal{V}_h$ may be double-valued on interior faces $F \in \mathcal{F}_h^i$ as well as across two space-time slabs $\mathcal{E}^n$ and $\mathcal{E}^{n+1}$. For fixed n, on an interior face $F \in \mathcal{F}_h^i$ shared by two elements K^L and K^R, we denote the traces of $v_h \in V_h$ on F by $v_h^L =$ trace of $v_h|_{K^L}$ on F and $v_h^R =$ trace of $v_h|_{K^R}$ on F. We denote by $u_n^\pm$ the traces at time level t_n from above and below, i.e. $u_n^\pm = \lim_{\epsilon \searrow 0} u_h(t_n \pm \epsilon)$.

We introduce the *jump* $[\![\cdot]\!]$ and *average* $\{\!\{\cdot\}\!\}$ of $v_h \in V_h$ across an interior face $F \in \mathcal{F}_h^i$ component-wise: let $[\![v_{h,i}]\!] = v_{h,i}^L - v_{h,i}^R$ and $\{\!\{v_{h,i}\}\!\} = (v_{h,i}^L + v_{h,i}^R)/2$ with $v_{h,i}$ denoting the ith Cartesian component of v_h. The quantities $[\![v_h]\!]$ and $\{\!\{v_h\}\!\}$ are then the vectors with ith Cartesian component $[\![v_{h,i}]\!]$ and $\{\!\{v_{h,i}\}\!\}$, respectively. On boundary faces $F \in \mathcal{F}_h^b$, we set $[\![v_h]\!] = \{\!\{v_h\}\!\} =$ trace of $v_h|_K$ on F, where K is the element such that $F \subset \partial K \cap \partial\Omega$. Lastly, we define the *time jump* of $v_h \in \mathcal{V}_h$ across the space-time slab $\mathcal{E}^n$ by $[v_h]_n = v_n^+ - v_n^-$.

We adopt the following notation for various product spaces of interest in this work: $\boldsymbol{V}_h = V_h \times \bar{V}_h$, $\boldsymbol{Q}_h = Q_h \times \bar{Q}_h$, $\boldsymbol{\mathcal{V}}_h = \mathcal{V}_h \times \bar{\mathcal{V}}_h$, and $\boldsymbol{\mathcal{Q}}_h = \mathcal{Q}_h \times \bar{\mathcal{Q}}_h$. Pairs in these product spaces will be denoted using boldface; for example, $\boldsymbol{v}_h := (v_h, \bar{v}_h) \in \boldsymbol{\mathcal{V}}_h$. Lastly, we introduce two mesh-dependent norms on the spaces V_h and $\boldsymbol{V}_h$, both of which are standard in the study of interior penalty methods:

$$\|v_h\|_{1,h}^2 := \sum_{K \in \mathcal{T}_h} \|\nabla v_h\|_K^2 + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \|[\![v_h]\!]\|_{L^2(F)}^2, \qquad \forall v_h \in V_h,$$

$$|\!|\!|\boldsymbol{v}|\!|\!|_v^2 := \sum_{K \in \mathcal{T}_h} \|\nabla v_h\|_K^2 + \sum_{K \in \mathcal{T}_h} \frac{1}{h_K} \|v_h - \bar{v}_h\|_{\partial K}^2, \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Throughout we use the notation $a \lesssim b$ to denote $a \le Cb$ where C is a constant independent of the mesh parameters h and τ, the viscosity ν, but possibly dependent on the polynomial degrees k_t and k_s, the spatial dimension d, and the domain Ω.

Thanks to the equivalence condition on faces, we have

$$\|v_h\|_{1,h} \lesssim |\!|\!|\boldsymbol{v}_h|\!|\!|_v, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{5.6}$$

and hence we can conclude the following discrete Poincaré inequality holds [28, Corollary 5.4]: for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$\|v_h\|_{L^2(\Omega)} \lesssim |\!|\!|\boldsymbol{v}_h|\!|\!|_v. \tag{5.7}$$

5.1.4 The space-time HDG method

We discretize the Navier–Stokes problem eq. (5.1) using the exactly mass conserving space-time HDG method studied in Chapter 4. This method combines the point-wise divergence free and $H(\mathrm{div};\Omega)$-conforming HDG method studied in Chapter 3 with a discontinuous Galerkin time stepping scheme; see also [43, 44] for related discretizations on space-time tetrahedral meshes on time-dependent domains.

Due to the use of discontinuous-in-time finite element spaces, the discrete space-time HDG formulation can be localized to a single space-time slab; see e.g. [98, Chapter 12]. We first consider the discrete formulation on a single space-time slab, and in Section 5.2.3 we will introduce the equivalent discrete formulation obtained by summing over all space-time slabs to aid us in our analysis. For $n = 0, \dots, N-1$, the space-time HDG method for the Navier–Stokes problem in each space-time slab $\mathcal{E}^n$ reads: find $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in \boldsymbol{\mathcal{V}}_h \times \boldsymbol{\mathcal{Q}}_h$ such that for all test functions $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{\mathcal{V}}_h \times \boldsymbol{\mathcal{Q}}_h$:

$$
\begin{aligned}
-\int_{I_n} (u_h, \partial_t v_h)_{\mathcal{T}_h} \,\mathrm{d}t + (u_{n+1}^-, v_{n+1}^-)_{\mathcal{T}_h} + \int_{I_n} \big(\nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h)\big) \,\mathrm{d}t \\
+ \int_{I_n} b_h(\boldsymbol{p}_h, v_h) \,\mathrm{d}t - \int_{I_n} b_h(\boldsymbol{q}_h, u_h) \,\mathrm{d}t = (u_n^-, v_n^+)_{\mathcal{T}_h} + \int_{I_n} (f, v_h)_{\mathcal{T}_h} \,\mathrm{d}t, \quad (5.8)
\end{aligned}
$$

where $(u, v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (u, v)_K$. Once we have solved eq. (5.8) for u_h in the space-time slab $\mathcal{E}^n$, the trace u_{n+1}^- serves as an initial condition when solving eq. (5.8) on the next space-time slab $\mathcal{E}^{n+1}$. The process is initiated by choosing $u_0^- = \Pi_h^{\mathrm{div}} u_0$ in the first space-time slab $\mathcal{E}^0$, where $u_0 \in H$ is the prescribed initial condition to the continuous problem eq. (5.1), and $\Pi_h^{\mathrm{div}} : L^2(\Omega) \to V_h^{\mathrm{div}}$ is the L^2-projection onto the discretely divergence free subspace $V_h^{\mathrm{div}} \subset V_h$; see eq. (5.13) below and the discussion following.

The discrete forms $a_h(\cdot,\cdot) : \boldsymbol{V}_h \times \boldsymbol{V}_h \to \mathbb{R}$, $b_h(\cdot,\cdot) : V_h \times \boldsymbol{Q}_h \to \mathbb{R}$, and $o_h(\cdot;\cdot,\cdot) : V_h \times \boldsymbol{V}_h \times \boldsymbol{V}_h \to \mathbb{R}$ appearing in eq. (5.8) serve as approximations to the viscous, pressure-velocity coupling, and convection terms, respectively. We define them as in Chapter 3:

$$a_h(\boldsymbol{u},\boldsymbol{v}) := \sum_{K\in\mathcal{T}_h}\int_K \nabla u : \nabla v \,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K} \frac{\alpha}{h_K}(u-\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s \tag{5.9a}$$
$$- \sum_{K\in\mathcal{T}_h}\int_{\partial K}\left[(u-\bar{u})\cdot\partial_n v + \partial_n u\cdot(v-\bar{v})\right]\mathrm{d}s,$$
$$o_h(w;\boldsymbol{u},\boldsymbol{v}) := -\sum_{K\in\mathcal{T}_h}\int_K u\otimes w : \nabla v\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\tfrac{1}{2} w\cdot n\,(u+\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s \tag{5.9b}$$
$$+ \sum_{K\in\mathcal{T}_h}\int_{\partial K}\tfrac{1}{2}|w\cdot n|\,(u-\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s,$$
$$b_h(\boldsymbol{p},v) := -\sum_{K\in\mathcal{T}_h}\int_K p\nabla\cdot v\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K} v\cdot n\,\bar{p}\,\mathrm{d}s. \tag{5.9c}$$

The parameter $\alpha > 0$ appearing in the bilinear form $a_h(\cdot,\cdot)$ is a penalty parameter typical of interior penalty type discretizations. The bilinear form $a_h(\cdot,\cdot)$ is continuous and for sufficiently large α enjoys discrete coercivity [77, Lemmas 4.2 and 4.3], i.e. for all $\boldsymbol{u}_h,\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$|\!|\!|\boldsymbol{v}_h|\!|\!|_v^2 \lesssim a_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \quad \text{and} \quad \left|a_h(\boldsymbol{u}_h,\boldsymbol{v}_h)\right| \lesssim |\!|\!|\boldsymbol{u}_h|\!|\!|_v |\!|\!|\boldsymbol{v}_h|\!|\!|_v. \tag{5.10}$$

The form $o_h(\cdot;\cdot,\cdot)$ satisfies [14, Proposition 3.6]

$$o_h(w_h;\boldsymbol{v}_h,\boldsymbol{v}_h) = \frac{1}{2}\sum_{K\in\mathcal{T}}\int_{\partial K}|w_h\cdot n||v_h-\bar{v}_h|^2\,\mathrm{d}s \geq 0, \qquad w_h\in V_h^{\text{div}},\ \forall \boldsymbol{v}_h\in\boldsymbol{V}_h. \tag{5.11}$$

The form $o_h(\cdot;\cdot,\cdot)$ also satisfies for all $\boldsymbol{u}_h,\boldsymbol{v}_h\in\boldsymbol{V}_h$ and $d\in\{2,3\}$ (Lemma 4.3.2),

$$|o_h(u_h;\boldsymbol{u}_h,\boldsymbol{v}_h)| \lesssim \|u_h\|_{L^2(\Omega)}^{1/(d-1)} |\!|\!|\boldsymbol{u}_h|\!|\!|_v^{d/2} |\!|\!|\boldsymbol{v}_h|\!|\!|_v. \tag{5.12}$$

Frequent use will also be made of functions in the subspace of discretely divergence free velocity fields:

$$\begin{aligned} V_h^{\text{div}} &:= \left\{v_h\in V_h \;:\; b_h(v_h,\boldsymbol{q}_h)=0,\ \forall \boldsymbol{q}_h\in\boldsymbol{Q}_h\right\}, \\ \mathcal{V}_h^{\text{div}} &:= \left\{v_h\in\mathcal{V}_h \;:\; \int_0^T b_h(v_h,\boldsymbol{q}_h)\,\mathrm{d}t = 0,\ \forall\boldsymbol{q}_h\in\boldsymbol{\mathcal{Q}}_h,\ \forall n=0,\dots,N-1\right\}. \end{aligned} \tag{5.13}$$

We note that $V_h^{\text{div}}\subset H(\text{div};\Omega)$, and further $\nabla\cdot v_h = 0$ and $v_h\cdot n|_{\partial\Omega}=0$ for all $v_h\in V_h^{\text{div}}$ (see e.g. [78, Proposition 1]). As Ω is assumed to have a Lipschitz boundary, we therefore have $V_h^{\text{div}}\subset H$. In fact, it can be shown that $V_h^{\text{div}} = V_h\cap H$.

5.1.5 Properties of the space-time HDG scheme

Here, we collect a number of useful results concerning the solution of the space-time HDG scheme eq. (5.8). The existence of solutions to the nonlinear algebraic system arising from eq. (5.8) was shown in Theorem 4.3.1. It was also shown in Chapter 4 that the discrete velocity u_h computed using the space-time HDG scheme is *conforming* in $L^2(0,T;H)$, i.e., if $u_h \in \mathcal{V}_h$ is the element velocity solution of eq. (5.8), then $\nabla \cdot u_h = 0$, $u_h|_{\mathcal{E}^n} \in P_k(I_n; H(\text{div};\Omega))$, and the normal trace of u_h vanishes on the spatial boundary $\partial\Omega$.

Next, we recall an energy estimate that will allow us to conclude that the discrete velocity pair $\boldsymbol{u}_h \in \boldsymbol{\mathcal{V}}_h$ computed using eq. (5.8) is bounded uniformly with respect to the mesh parameters h and τ:

Lemma 5.1.1. *Let $d \in \{2,3\}$, $k_s \geq 1$ and $k_t \geq 0$, and suppose that $\boldsymbol{u}_h \in \boldsymbol{\mathcal{V}}_h$ is solution of space-time HDG scheme eq. (5.8) for $n = 0, \dots, N-1$. For all $0 \leq m \leq N-1$,*

$$\left\|u_{m+1}^-\right\|_{L^2(\Omega)}^2 + \sum_{n=0}^{m} \left\|[u_h]_n\right\|_{L^2(\Omega)}^2 + \nu \int_0^{t_{m+1}} \left\|\!\left|\boldsymbol{u}_h\right|\!\right\|_v^2 \,\mathrm{d}t \leq C(f, u_0, \nu). \tag{5.14}$$

Furthermore, if $k_t \geq 0$ when $d = 2$ and $k_t \in \{0,1\}$ when $d = 3$, it holds that

$$\|u_h\|_{L^\infty(0,T;L^2(\Omega)^d)} \leq C(f, u_0, \nu). \tag{5.15}$$

Here, $C(f, u_0, \nu)$ denotes a constant that depends on the data f, u_0, and ν.

The bounds in Lemma 5.1.1 were proven in Chapter 4 under the assumption that $k_t = k_s \geq 1$ for simplicity of presentation; we remark that the proofs are equally valid for the general case $k_s \geq 1$ and $k_t \geq 0$. Note that for the lower order schemes $k_t \in \{0,1\}$, eq. (5.15) follows directly from eq. (5.14). This can be seen immediately when considering constant polynomials in time ($k_t = 0$). For linear polynomials in time ($k_t = 1$), this follows from the bound (see [101, Section 3]): $\|u_h\|_{L^\infty(0,T;L^2(\Omega)^d)} \leq \max_{0\leq m\leq N-1} \left\|u_{m+1}^-\right\|_{L^2(\Omega)} + \max_{0\leq m\leq N-1} \left\|[u_h]_m\right\|_{L^2(\Omega)}$.

5.2 Lifting operators and discrete differential operators

In this section, we introduce two discrete differential operators that serve as natural approximations to the distributional gradient and distributional time derivative in the space-time HDG setting. These discrete operators enjoy convergence to their continuous counterparts in the weak topologies of appropriate Bochner spaces.

5.2.1 Discrete gradient

First, we introduce a discrete gradient operator that will serve as an approximation of the distributional gradient operator following ideas in [11, 72, 50]. The basic building block of the discrete gradient operator is the following observation [11]: as functions $v_h \in V_h$ are discontinuous, their distributional gradient has a contribution from the jumps of u_h across element interfaces. Therefore, an appropriate approximation of the distributional gradient in the HDG setting must incorporate the contribution from the jumps between the element solution and the facet solution across element boundaries. We do so by constructing an HDG lifting operator following ideas in [50, 70]. For this, we need to introduce the scalar broken polynomial spaces $W_h := \left\{w_h \in L^2(\Omega) \mid w_h|_K \in P_{k_s}(K)\right\}$ and $\bar{W}_h := \left\{w_h \in L^2(\partial\mathcal{T}_h) \mid w_h|_{\partial K} \in R_{k_s}(\partial K)\right\}$, with $R_{k_s}(\partial K)$ defined in Section 4.3.2.1. We first define a *local lifting* $R_h^{\partial K} : L^2(\partial K) \to P_{k_s}(K)^d$ satisfying

$$\int_K R_h^{\partial K}(\mu) \cdot w_h \,\mathrm{d}x = \int_{\partial K} \mu w_h \cdot n \,\mathrm{d}s, \quad \forall w_h \in P_{k_s}(K)^d. \tag{5.16}$$

We then define the *global lifting* $R_h^{k_s} : L^2(\partial\mathcal{T}_h) \to V_h$ by the restriction $R_h^{k_s}(\mu)|_K = R_h^{\partial K}(\mu|_{\partial K})$ for all $K \in \mathcal{T}_h$. Note that $R_h^{k_s}$ satisfies for all $w_h \in V_h$,

$$\sum_{K\in\mathcal{T}_h} \int_K R_h^{k_s}(\mu) \cdot w_h \,\mathrm{d}x = \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu w_h \cdot n \,\mathrm{d}s, \tag{5.17}$$

and it can be shown using the Cauchy–Schwarz inequality and a standard local discrete trace inequality that

$$\|R_h^{k_s}(w_h - \bar{w}_h)\|^2_{L^2(\Omega)} \lesssim \sum_{K\in\mathcal{T}_h} \frac{1}{h_K} \|w_h - \bar{w}_h\|^2_{L^2(\partial K)}, \quad \forall \boldsymbol{w}_h \in W_h \times \bar{W}_h. \tag{5.18}$$

Using the global HDG lifting, we introduce the discrete gradient operator $G_h^{k_s} : W_h \times \bar{W}_h \to V_h$ in the same spirit as in [72, 50]: given $(v, \bar{v}) \in W_h \times \bar{W}_h$, we set

$$G_h^{k_s}(v, \bar{v}) = \nabla_h v - R_h^{k_s}(v - \bar{v}), \tag{5.19}$$

where ∇_h is the broken gradient operator. Crucially, this operator satisfies for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in V_h$ the identity

$$\int_\Omega G_h^{k_s}(\boldsymbol{v}_{h,i}) \cdot w_h \,\mathrm{d}x = \int_\Omega \nabla_h v_{h,i} \cdot w_h \,\mathrm{d}x - \sum_{K\in\mathcal{T}_h} \int_{\partial K} (v_{h,i} - \bar{v}_{h,i}) w_h \cdot n \,\mathrm{d}s,$$

where $v_{h,i}$ and $\bar{v}_{h,i}$ denote the ith Cartesian components of v_h and $\bar{v}_h$, respectively.

5.2.2 Discrete time derivative

To define a discrete time derivative operator that serves as an appropriate approximation of the distributional time derivative, we proceed by analogy with the discrete gradient constructed in the previous section. We follow [64, 87] by introducing a *local time lifting operator* $R^{k_t}_{loc,n} : V_h \to P_{k_t}(I_n; V_h)$ satisfying

$$\int_{I_n} (R^{k_t}_{loc,n}(u_h), v_h)_{\mathcal{T}_h} \, \mathrm{d}t = ([u_h]_n, v_n^+)_{\mathcal{T}_h}, \quad \forall v_h \in \; P_{k_t}(I_n; V_h), \tag{5.20}$$

$$R^{k_t}_{loc,n}(u_h) = \frac{(u_n^+ - u_n^-)}{2} \sum_{m=0}^{k_t} (-1)^m (2m+1) L_m^n(t), \tag{5.21}$$

where the latter representation formula follows from [87, Lemma 6]. Here $L_m^n(t)$ are *mapped Legendre polynomials*; see [87, Section 3]. We then define a *global time lifting* $R^{k_t} : V_h \to \mathcal{V}_h$ by the restriction $R^{k_t}|_{I_n} = R^{k_t}_{loc,n}$. This lifting satisfies:

$$\int_0^T (R^{k_t}(u_h), v_h)_{\mathcal{T}_h} \, \mathrm{d}t = \sum_{n=0}^{N-1} ([u_h]_n, v_n^+)_{\mathcal{T}_h}, \quad \forall v_h \in \mathcal{V}_h. \tag{5.22}$$

With the global time lifting in hand, we define the discrete time derivative $\mathcal{D}_t^{k_t} : \mathcal{V}_h \to \mathcal{V}_h$ of $v_h \in \mathcal{V}_h$ by setting

$$\mathcal{D}_t^{k_t}(v_h) = \partial_\tau v_h + R^{k_t}(v_h). \tag{5.23}$$

Lemma 5.2.1. *Suppose that $u_h \in \mathcal{V}_h^{div}$. Then, it holds that $\mathcal{D}_t^{k_t}(u_h)|_{\mathcal{E}^n} \in P_{k_t}(I_n; H)$ for all $0 \le n \le N-1$.*

Proof. That $\mathcal{D}_t^{k_t}(u_h)$ is divergence free and $H(\mathrm{div};\Omega)$-conforming follows from the fact that the broken time derivative commutes with the divergence operator and the representation formula eq. (5.21). It remains to show that $\mathcal{D}_t^{k_t}(u_h) \cdot n|_{\partial\Omega} = 0$. Note that $u_h \cdot n|_{\partial\Omega} = 0$ implies $(\partial_\tau u_h) \cdot n|_{\partial\Omega} = 0$. This can be seen by considering a single space-time slab $\mathcal{E}^n$ and expanding u_h in terms of a basis $\{\psi_i\}_{i=0}^m$ of $P_{k_t}(I_n)$ to find $(\partial_\tau u_h)|_{I_n} = \sum_{i=0}^{k_t} \partial_t \psi_i u_i$, where $u_i \in V_h$ is such that $u_i \cdot n|_{\partial\Omega} = 0$. Lastly, since $u_n^+ \cdot n|_{\partial\Omega} = u_n^- \cdot n|_{\partial\Omega} = 0$, the representation formula eq. (5.21) shows that indeed $R^{k_t}_{loc,n}(u_h) \cdot n|_{\partial\Omega} = 0$. □

5.2.3 Rewriting the HDG scheme

We now recast the space-time HDG scheme into a form more amenable to the convergence analysis in Section 5.4 using the discrete differential operators introduced above. In what follows, $(u_{h,i})_{1\le i\le d}$, $(\bar{u}_{h,i})_{1\le i\le d}$, $(v_{h,i})_{1\le i\le d}$ and $(\bar{v}_{h,i})_{1\le i\le d}$ will denote the Cartesian components of u_h, $\bar{u}_h$, v_h and $\bar{v}_h$, respectively. We will adopt the convention of summation over repeated indices. Restricting our attention to test functions $\boldsymbol{v}_h \in \mathcal{V}_h^{\mathrm{div}} \times \bar{\mathcal{V}}_h$ in eq. (5.8) to remove the contribution from the pressure-velocity coupling term, integrating by parts in time, summing over all space-time slabs $\mathcal{E}^n$, and using the definitions of the lifting operators, we arrive at the problem: find $\boldsymbol{u}_h \in \mathcal{V}_h^{\mathrm{div}} \times \bar{\mathcal{V}}_h$ satisfying for all $\boldsymbol{v}_h \in \mathcal{V}_h^{\mathrm{div}} \times \bar{\mathcal{V}}_h$,

$$\int_0^T (\mathcal{D}_t^{k_t}(u_h), v_h)_{\mathcal{T}_h}\,\mathrm{d}t + \int_0^T \big(\nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h)\big)\,\mathrm{d}t = \int_0^T (f, v_h)_{\mathcal{T}_h}\,\mathrm{d}t, \tag{5.24}$$

where

$$\begin{aligned} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = &\int_\Omega G_h^{k_s}(\boldsymbol{u}_{h,i}) \cdot G_h^{k_s}(\boldsymbol{v}_{h,i})\,\mathrm{d}x - \int_\Omega R_h^{k_s}(u_{h,i} - \bar{u}_{h,i}) \cdot R_h^{k_s}(v_{h,i} - \bar{v}_{h,i})\,\mathrm{d}x \\ &+ \sum_{K\in\mathcal{T}_h} \int_{\partial K} \frac{\alpha}{h_K}(u_{h,i} - \bar{u}_{h,i})(v_{h,i} - \bar{v}_{h,i})\,\mathrm{d}s, \end{aligned} \tag{5.25}$$

$$\begin{aligned} o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h)\,\mathrm{d}t = &\int_\Omega u_h \cdot G_h^{2k_s}(\boldsymbol{u}_{h,i}) v_{h,i}\,\mathrm{d}x \\ &+ \sum_{K\in\mathcal{T}_h} \int_{\partial K} \tfrac{1}{2}\big(u_h \cdot n + |u_h \cdot n|\big)(u_h - \bar{u}_h)\cdot(v_h - \bar{v}_h)\,\mathrm{d}s. \end{aligned} \tag{5.26}$$

5.3 Uniform bounds on the discrete differential operators

In this section, we derive uniform bounds on the discrete differential operators of the discrete velocity solution introduced in the previous section. In what follows, we suppose that $\boldsymbol{u}_h \in \mathcal{V}_h^{\mathrm{div}} \times \bar{\mathcal{V}}_h$ is a discrete velocity pair solving the space-time HDG formulation eq. (5.8) for $n = 0, \dots, N-1$. We then show that subsequences of the discrete derivatives converge weakly to their continuous counterparts.

5.3.1 Bounding the discrete gradient

Before bounding the discrete gradient of u_h, we pause to mention an immediate consequence of the energy bound Lemma 5.1.1. From the discrete Sobolev embeddings for broken polynomial spaces [72, Theorem 6.1], we can infer using eq. (5.6) that

$$\int_0^T \|u_h\|_{L^q(\Omega)}^2 \,\mathrm{d}t \leq C(f, u_0, \nu), \tag{5.27}$$

where $1 \leq q < \infty$ if $d = 2$ and $1 \leq q \leq 6$ if $d = 3$. Thus, $(u_h)_{h\in\mathcal{H}}$ is bounded in $L^2(0, T; L^q(\Omega)^d)$ for $1 \leq q \leq 6$ and in particular in $L^2(0, T; H)$.

Theorem 5.3.1. *Let $d \in \{2, 3\}$ and suppose $k_t \geq 0$ if $d = 2$ and $k_t \in \{0, 1\}$ if $d = 3$. Let $\boldsymbol{u}_h$ be the solution of the space-time HDG scheme eq. (5.8). Then, provided the penalty parameter $\alpha > 0$ is chosen sufficiently large, it holds that*

$$\int_0^T \|G_h^k(\boldsymbol{u}_{h,i})\|_{L^2(\Omega)}^2 \,\mathrm{d}t \leq C(f, u_0, \nu). \tag{5.28}$$

Proof. The result follows from eq. (5.25) and the energy bound in Lemma 5.1.1, provided $\alpha > 0$ is chosen sufficiently large, since for all $\boldsymbol{u}_h \in \boldsymbol{V}_h$ we have by eq. (5.18) for $i = 1, \dots, d$ that

$$-\left\|R_h^{k_s}(u_{h,i} - \bar{u}_{h,i})\right\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{T}_h} \frac{\alpha}{h_K}\left\|u_{h,i} - \bar{u}_{h,i}\right\|_{L^2(\partial K)}^2$$
$$\geq (\alpha - C) \sum_{K\in\mathcal{T}_h} \frac{1}{h_K}\left\|u_{h,i} - \bar{u}_{h,i}\right\|_{L^2(\partial K)}^2,$$

and therefore,

$$a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq \sum_{i=1}^d \|G_h^{k_s}(\boldsymbol{u}_{h,i})\|_{L^2(\Omega)}^2 . \tag{5.29}$$

Consequently, the sequence $G_h^k(\boldsymbol{u}_{h,i})$ is bounded in $L^2(0, T; L^2(\Omega)^d)$. □

5.3.2 Bounding the discrete time derivative

We now turn our focus to bounding the discrete time derivative of $u_h \in \mathcal{V}_h^{\text{div}}$ uniformly, first in the dual space of $\mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h$, and second in $L^{4/d}(0, T; V')$. The former is required to obtain a *strong compactness* result needed for passage to the limit as $h \to 0$ in the nonlinear convection term, and the second is essential to ensure the distributional time derivatives of accumulation points of the sequence $\{u_h\}_{h\in\mathcal{H}}$ are sufficiently regular to satisfy Definition 5.1.1. That $\mathcal{D}_t^{k_t}(u_h)$ can be identified with an element of $L^{4/d}(0, T; V')$ follows from Lemma 5.2.1 since (V, H, V') form a Gelfand triple (Section 2.1.4).

5.3.2.1 Uniform bound in the dual space of $\mathcal{V}_h^{\mathbf{div}} \times \bar{\mathcal{V}}_h$

To apply the compactness theorem Theorem B.1.1 later on to prove Theorem 5.3.3, we will require $F_h : \boldsymbol{v}_h \mapsto (\mathcal{D}_t^{k_t}(u_h), v_h)_{\mathcal{T}_h}$ to be uniformly bounded $L^{4/d}(0,T;(V_h^{\text{div}} \times \bar{V}_h)')$, with $(V_h^{\text{div}} \times \bar{V}_h)'$ the dual space of $V_h^{\text{div}} \times \bar{V}_h$. We shall see that it suffices to bound $F_h(\boldsymbol{v}_h)$ in the dual space of the fully discrete space $\mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h$, which we equip with the norm

$$\|F_h\|_{(\mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h)'} = \sup_{0 \neq \boldsymbol{v}_h \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h} \frac{\left| \int_0^T F_h(\boldsymbol{v}_h) \,\mathrm{d}t \right|}{\left(\int_0^T |||\boldsymbol{v}_h|||_v^{4/(4-d)} \,\mathrm{d}t \right)^{(4-d)/4}}.$$

This motivates the following result (where we choose $F_h : \boldsymbol{v}_h \mapsto (\mathcal{D}_t^{k_t}(u_h), v_h)_{\mathcal{T}_h}$):

Lemma 5.3.1. *Let $d \in \{2,3\}$ and suppose $k_t \geq 0$ if $d = 2$ and $k_t \in \{0,1\}$ if $d = 3$. Let $\boldsymbol{u}_h$ be the discrete velocity pair arising from the solution of the space-time HDG scheme eq. (5.8). It holds for all $\boldsymbol{v}_h \in \mathcal{V}_h^{div} \times \bar{\mathcal{V}}_h$ that*

$$\left| \int_0^T (\mathcal{D}_t^{k_t}(u_h), v_h)_{\mathcal{T}_h} \,\mathrm{d}t \right| \leq C(u_0, f, \nu, T) \left(\int_0^T |||\boldsymbol{v}_h|||_v^{4/(4-d)} \,\mathrm{d}t \right)^{(4-d)/4}.$$

Proof. Let $\boldsymbol{v}_h \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h$, and use eq. (5.24) to write

$$\int_0^T (\mathcal{D}_t^{k_t} u_h, v_h)_{\mathcal{T}_h} \,\mathrm{d}t = \int_0^T \big((f, v_h)_{\mathcal{T}_h} - \nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) - o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h) \big) \,\mathrm{d}t. \tag{5.30}$$

We now bound each of the three terms on the right-hand side of eq. (5.30), beginning with the first term on the right-hand side. The Cauchy-Schwarz inequality, Hölder's inequality, and the discrete Poincaré inequality eq. (5.7) yield

$$\int_0^T |(f, v_h)_{\mathcal{T}_h}| \,\mathrm{d}t \leq C(f, T) \left(\int_0^T |||\boldsymbol{v}_h|||_v^{4/(4-d)} \,\mathrm{d}t \right)^{(4-d)/4}. \tag{5.31}$$

To bound the linear viscous term on the right-hand side of eq. (5.30), we begin by using the boundedness of $a_h(\cdot,\cdot)$ eq. (5.10) and Hölder's inequality with $p = 4/d$ and $q = 4/(4-d)$ to find

$$\int_0^T |a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)| \,\mathrm{d}t \leq C \left(\int_0^T |||\boldsymbol{u}_h|||_v^{4/d} \,\mathrm{d}t \right)^{d/4} \left(\int_0^T |||\boldsymbol{v}_h|||_v^{4/(4-d)} \,\mathrm{d}t \right)^{(4-d)/4}. \tag{5.32}$$

If $d = 2$, directly using the uniform bound in Lemma 5.1.1, and if $d = 3$, applying Hölder's inequality to the first integral on the right-hand side of eq. (5.32) with $p = 3$ and $q = 3/2$, followed by the uniform bound in Lemma 5.1.1, we find

$$\int_0^T |a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)| \,\mathrm{d}t \leq C(f, u_0, \nu, T) \left(\int_0^T \|\!|\boldsymbol{v}_h|\!\|_v^{4/(4-d)} \,\mathrm{d}t \right)^{(4-d)/4}. \tag{5.33}$$

Lastly, we must bound the nonlinear convection term on the right-hand side of eq. (5.30). For this, we use the bound eq. (5.12), apply the generalized Hölder's inequality with $p = \infty$, $q = 4/d$, and $r = 4/(4-d)$, and use Lemma 5.1.1, to find

$$\int_0^T |o_h(u_h; \boldsymbol{u}_h, \boldsymbol{v}_h)| \,\mathrm{d}t \leq C(f, u_0, \nu) \left(\int_0^T \|\!|\boldsymbol{v}_h|\!\|_v^{4/(4-d)} \,\mathrm{d}t \right)^{(4-d)/4}. \tag{5.34}$$

Collecting eq. (5.31), eq. (5.33), and eq. (5.34) yields the result. □

5.3.2.2 Construction of suitable test functions

To prove a uniform bound on the discrete time derivative in $L^{4/d}(0, T; V')$ (see Theorem 5.3.2), we will need to construct a suitable set of test functions in the discrete space $\mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h$. This will require two preparatory results. The first is a density result for functions of tensor-product type in $C_c(0, T; V)$ taken from [8, Lemma V.1.2] with minor modification:

Lemma 5.3.2. *The set $\mathscr{M}$ of functions φ of the form*

$$\varphi(t, x) = \sum_{k=1}^{M} \eta_k(t) \psi_k(x), \tag{5.35}$$

where $M \geq 1$ is an integer, $\eta_k \in C_c^\infty(0, T)$, and $\psi_k \in \mathscr{V}$, is dense in $C_c(0, T; V)$.

Denote by $\Pi_n^t : L^2(I_n) \to P_{k_t}(I_n)$, $\Pi_h^{\text{div}} : L^2(\Omega) \to V_h^{\text{div}}$, and $\bar{\Pi}_h : H^1(\Omega)^d \to \bar{V}_h$ the orthogonal L^2-projections onto the discrete spaces $P_{k_t}(I_n)$, V_h^{div}, and $\bar{V}_h$, respectively. We define the *global* L^2-projection Π^t in time by the restriction $\Pi^t|_{I_n} = \Pi_n^t$. Given a function $\varphi \in \mathscr{M}$, consider for all $n = 0, \ldots, N-1$,

$$\Pi\varphi|_{\mathcal{E}^n} = \sum_{k=1}^{M} \Pi_n^t \eta_k(t) \Pi_h^{\text{div}} \psi_k(x) \quad \text{and} \quad \bar{\Pi}\varphi|_{\mathcal{E}^n} = \sum_{k=1}^{M} \Pi_n^t \eta_k(t) \bar{\Pi}_h \psi_k(x). \tag{5.36}$$

By construction, $(\Pi v, \bar{\Pi} v) \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h$. We remark that the approximation properties of Π_h^{div} obtained in Lemma 4.4.1 and listed in Lemma B.2.1 require quasi-uniformity of the underlying spatial mesh $\mathcal{T}_h$.

Proposition 5.3.1. *Suppose $d \in \{2, 3\}$. Let $\varphi \in \mathscr{M}$ and let $(\Pi\varphi, \bar{\Pi}\varphi) \in \mathcal{V}_h^{div} \times \bar{\mathcal{V}}_h$ be the discrete test functions constructed in eq. (5.36). Then, the following stability property holds:*

$$\int_0^T |||(\Pi\varphi, \bar{\Pi}\varphi)|||_v^{4/(4-d)} \, \mathrm{d}t \lesssim \int_0^T \|\varphi\|_V^{4/(4-d)} \, \mathrm{d}t, \quad \forall v \in \mathscr{M}. \tag{5.37}$$

Proof. See Appendix B.2.2. □

5.3.2.3 Uniform bound in $L^{4/d}(0, T; V')$

With Lemmas 5.3.1 and 5.3.2, and Proposition 5.3.1 in hand, we can now prove the main result of this subsection. Since V' is separable, we can identify $L^{4/d}(0, T; V') \cong L^{4/(4-d)}(0, T; V)'$ (see e.g. [82, Proposition 1.38]), and since (V, H, V') form a Gelfand triple, we have

$$\|\mathcal{D}_t^{k_t}(u_h)\|_{L^{4/d}(0,T;V')} = \sup_{0 \neq v \in L^{4/(4-d)}(0,T;V)} \frac{\left| \int_0^T (\mathcal{D}_t^{k_t}(u_h), v)_{\mathcal{T}_h} \, \mathrm{d}t \right|}{\|v\|_{L^{4/(4-d)}(0,T;V)}}. \tag{5.38}$$

Theorem 5.3.2 (Uniform bound on the discrete time derivative)**.** *Let $d \in \{2, 3\}$ and suppose $k_t \geq 0$ if $d = 2$ and $k_t \in \{0, 1\}$ if $d = 3$. Let $\boldsymbol{u}_h$ be the discrete velocity pair arising from the solution of the space-time HDG scheme eq. (5.8). Then $\left\|\mathcal{D}_t^{k_t}(u_h)\right\|_{L^{4/d}(0,T;V')} \leq C(f, u_0, \nu, T)$.*

Proof. We follow the strategy used in the proof of [61, Theorem 3.2]. The density of $C_c(0, T; V)$ in $L^p(0, T; V)$ for $1 \leq p < \infty$ gives us also the density of $\mathscr{M}$ in $L^{4/(4-d)}(0; T, V)$. We therefore replace the supremum over $v \in L^{4/(4-d)}(0; T, V)$ in eq. (5.38) with the supremum over $\varphi \in \mathscr{M}$. Now let $\varphi \in \mathscr{M}$ be arbitrary. Using the expansion of φ eq. (5.35), the definitions of the L^2-projections Π^t and Π_h, Proposition 5.3.1, and Lemma 5.3.1, we have

$$
\begin{aligned}
&\|\mathcal{D}_t^{k_t}(u_h)\|_{L^{4/d}(0,T;V')}\\
&\quad= \sup_{0\neq\varphi\in\mathscr{M}} \frac{\left|\int_0^T (\mathcal{D}_t^{k_t}(u_h),\varphi)_{\mathcal{T}_h}\,\mathrm{d}t\right|}{\|\varphi\|_{L^{4/(4-d)}(0,T;V)}}\\
&\quad= \sup_{0\neq\varphi\in\mathscr{M}} \frac{\left|\int_0^T (\mathcal{D}_t^{k_t}(u_h),\Pi\varphi)_{\mathcal{T}_h}\,\mathrm{d}t\right|}{\left(\int_0^T |\!|\!|(\Pi\varphi,\bar{\Pi}\varphi)|\!|\!|_v^{4/(4-d)}\,\mathrm{d}t\right)^{(4-d)/4}} \left(\frac{\int_0^T |\!|\!|(\Pi\varphi,\bar{\Pi}\varphi)|\!|\!|_v^{4/(4-d)}\,\mathrm{d}t}{\int_0^T \|\varphi\|_V^{4/(4-d)}\,\mathrm{d}t}\right)^{(4-d)/4}\\
&\quad\lesssim \sup_{0\neq\varphi\in\mathscr{M}} \frac{\left|\int_0^T (\mathcal{D}_t^{k_t}(u_h),\Pi\varphi)_{\mathcal{T}_h}\,\mathrm{d}t\right|}{\left(\int_0^T |\!|\!|(\Pi\varphi,\bar{\Pi}\varphi)|\!|\!|_v^{4/(4-d)}\,\mathrm{d}t\right)^{(4-d)/4}}\\
&\quad\lesssim \sup_{0\neq\boldsymbol{v}_h\in\mathcal{V}_h^{\mathrm{div}}\times\bar{\mathcal{V}}_h} \frac{\left|\int_0^T (\mathcal{D}_t^{k_t}(u_h),v_h)_{\mathcal{T}_h}\,\mathrm{d}t\right|}{\left(\int_0^T |\!|\!|\boldsymbol{v}_h|\!|\!|_v^{4/(4-d)}\,\mathrm{d}t\right)^{(4-d)/4}} \leq C(f,u_0,\nu,T).
\end{aligned}
$$

□

5.3.3 Compactness

We end this section by summarizing the significance of the uniform bounds on the discrete velocity collected in Lemma 5.1.1, Theorem 5.3.1, and Theorem 5.3.2. In particular, we conclude that subsequences of the discrete velocity solution computed by the space-time HDG scheme eq. (5.8) converge to a limit function u in suitable topologies. The goal of Section 5.4 will be to show that u is in fact a weak solution to the Navier–Stokes problem in the sense of Definition 5.1.1.

Theorem 5.3.3. *Let $\mathcal{H}$ be a countable set of mesh sizes whose unique accumulation point is 0. Let $k_s \geq 1$ and $k_t \geq 0$ if $d = 2$ and $k_t \in \{0,1\}$ if $d = 3$ and suppose that $\{\boldsymbol{u}_h\}_{h\in\mathcal{H}}$ is a sequence of solutions of eq. (5.24) such that $\tau \to 0$ as $h \to 0$. Then, there exists a function $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ with $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^{4/d}(0,T;V')$ such that, up to a (not relabeled)* *subsequence:*

$$
\begin{aligned}
&\text{(i) } u_h \overset{\star}{\rightharpoonup} u && \textit{in } L^\infty(0,T;H),\\
&\text{(ii) } u_h \to u && \textit{in } L^2(0,T;L^2(\Omega)^d),\\
&\text{(iii) } G_h^{k_s}(\boldsymbol{u}_{h,i}) \rightharpoonup \nabla u_i && \textit{in } L^2(0,T;L^2(\Omega)^d),\\
&\text{(iv) } \mathcal{D}_t^{k_t}(u_h) \rightharpoonup \frac{\mathrm{d}u}{\mathrm{d}t} && \textit{in } L^{4/d}(0,T;V').
\end{aligned}
$$

Proof. **(i)** Weak$-\star$ convergence. The existence of a u satisfying (i) is a direct consequence of the uniform $L^\infty(0,T;L^2(\Omega)^d)$ bound in Lemma 5.1.1 and the Banach–Alaoglu theorem [10, Corollary 3.30].

(ii) Strong convergence. This follows from Theorem B.1.1 due to the uniform energy bound in Lemma 5.1.1, the uniform bound on $\mathcal{D}_t^{k_t}(u_h)$ in Lemma 5.3.1 (see also Remark B.1.1 and [61, Theorem 3.2]), and the uniqueness of distributional limits.

(iii) Weak convergence of the discrete gradient. By Theorem 5.3.1 there exists $w \in L^2(0,T;L^2(\Omega)^d)$ such that, upon passage to a subsequence, $G_h^k(\boldsymbol{u}_{h,i}) \rightharpoonup w$ in $L^2(0,T;L^2(\Omega)^d)$ as $h \to 0$. Let $\phi \in C_c^\infty(\mathbb{R}^d)^d$ and $\eta \in C_c^\infty(0,T)$ be arbitrary and let Π_h be the orthogonal L^2-projection onto V_h. Extending $u_{h,i}$, $G_h^{k_s}(\boldsymbol{u}_{h,i})$, $R_h^{k_s}(u_{h,i} - \bar{u}_{h,i})$, u, and w by zero outside of Ω, and integrating by parts element-wise in space, we have for all $\eta \in C_c^\infty(0,T)$ and $\phi \in C_c^\infty(\mathbb{R}^d)^d$ that

$$
\begin{aligned}
&\int_0^T \Big(\int_{\mathbb{R}^d} G_h^{k_s}(\boldsymbol{u}_{h,i}) \cdot \phi \,\mathrm{d}x\Big)\,\eta \,\mathrm{d}t \\
&\qquad = \int_0^T \Big(-\int_{\mathbb{R}^d} u_{h,i} \nabla \cdot \phi \,\mathrm{d}x + \sum_{K \in \mathcal{T}_h} \int_{\partial K} (u_{h,i} - \bar{u}_{h,i})(\phi - \Pi_h \phi) \cdot n \,\mathrm{d}s\Big)\,\eta \,\mathrm{d}t,
\end{aligned}
\tag{5.39}
$$

where we have used eqs. (5.17) and (5.19), that ϕ and $\bar{u}_{h,i}$ are single-valued on element boundaries, and that $\bar{u}_{h,i}|_{\partial\Omega} = 0$. Moreover,

$$
\int_0^T \big\|\eta(\phi - \Pi_h \phi) \cdot n\big\|^2_{L^2(\partial K)} \,\mathrm{d}t \lesssim h^{2\ell+1} \int_0^T \eta^2 \|\phi\|^2_{H^{\ell+1}(\Omega)} \,\mathrm{d}t. \tag{5.40}
$$

As a consequence of eqs. (5.39) and (5.40) and the strong convergence in $L^2(0,T;L^2(\Omega)^d)$ of u_h to u, it holds for all $\eta \in C_c^\infty(0,T)$ that

$$
\int_0^T \Big(\int_{\mathbb{R}^d} w \cdot \phi \,\mathrm{d}x\Big)\,\eta \,\mathrm{d}t = \lim_{h \to 0} \int_0^T \Big(\int_{\mathbb{R}^d} G_h^{k_s}(\boldsymbol{u}_{h,i}) \cdot \phi \,\mathrm{d}x\Big)\,\eta \,\mathrm{d}t = \int_0^T \Big(-\int_{\mathbb{R}^d} u_i \nabla \cdot \phi \,\mathrm{d}x\Big)\,\eta \,\mathrm{d}t. \tag{5.41}
$$

Hence $w = \nabla u_i$ as elements of $L^2(0,T;L^2(\mathbb{R}^d)^d)$, so $u_i \in L^2(0,T;H^1(\mathbb{R}^d))$. As u_i vanishes outside of Ω, the $H^1(\mathbb{R}^d)$-regularity ensures that u_i vanishes on the boundary. As $u \in H$, its distributional divergence vanishes, and thus $u \in L^2(0,T;V)$.

(iv) Weak convergence of the discrete time derivative. By Theorem 5.3.2, there exists a $z \in L^{4/d}(0,T;V')$ such that, upon passage to a subsequence , $\mathcal{D}_t^{k_t}(u_h) \rightharpoonup z$ in $L^{4/d}(0,T;V')$. For arbitrary $v \in V$ and $\eta \in C_c^\infty(0,T)$, we use the definition of $\mathcal{D}_t^{k_t}(u_h)$ eq. (5.23) and integrate by parts in time to find

$$\begin{aligned}\int_0^T &\langle \mathcal{D}_t^{k_t}(u_h), v\eta\rangle_{V'\times V}\,\mathrm{d}t \\ &= -\int_0^T (u_h,v)_{\mathcal{T}_h}\partial_t\eta\,\mathrm{d}t + \sum_{n=0}^{N-1}\left((u_{n+1}^-,v)_{\mathcal{T}_h}\eta(t_{n+1}) - (u_n^-,v)_{\mathcal{T}_h}\eta(t_n)\right).\end{aligned} \tag{5.42}$$

The telescoping sum on the right-hand side of eq. (5.42) vanishes since $\eta(0) = \eta(T) = 0$. Thus, we can take the limit as $h \to 0$ to find that for all $\eta \in C_c^\infty(0,T)$,

$$\int_0^T \eta\langle z,v\rangle_{V'\times V}\,\mathrm{d}t = \lim_{h\to 0}\int_0^T \langle \mathcal{D}_t^{k_t}(u_h), v\eta\rangle_{V'\times V}\,\mathrm{d}t = -\int_0^T \partial_t\eta(u,v)_{\mathcal{T}_h}\,\mathrm{d}t,$$

since $\mathcal{D}_t^{k_t}(u_h) \rightharpoonup z$ in $L^{4/d}(0,T;V')$ and $u_h \to u$ in $L^2(0,T;L^2(\Omega)^d)$ as $h \to 0$. Therefore, $z = \frac{du}{dt}$. □

5.4 Convergence to weak solutions

The remainder of this chapter is dedicated to showing that the limiting function $u \in L^\infty(0;T,H) \cap L^2(0,T;V)$ guaranteed by Theorem 5.3.3 is actually a weak solution of the Navier–Stokes problem in the sense of Definition 5.1.1. The plan is as follows: we first construct a set of test functions in the discrete space that will allow us to conclude upon passage to the limit that u solves eq. (5.2). We will then show that the viscous term $a_h(\cdot,\cdot)$ and the nonlinear convection term $o_h(\cdot;\cdot,\cdot)$ enjoy *asymptotic consistency* in the sense of [28, Definition 5.9], and use this to pass to the limit in eq. (5.24). Finally, we discuss the energy (in)equality and conclude that the constructed weak solution $u \in L^\infty(0,T;H)\cap L^2(0,T;V)$ is a solution in the sense of Leray–Hopf.

5.4.1 Strong convergence of test functions

Passing to the limit in eq. (5.24) will require a suitable set of discrete test functions. We will again use the set $\mathscr{M}$ of functions defined in Lemma 5.3.2 as our basic building block, as it is sufficiently rich to ensure density in $C_c(0,T;V)$ while its tensor-product structure allows us to easily combine spatial and temporal projections onto the discrete spaces. In particular, given $\varphi \in \mathscr{M}$, we will work with the discrete functions $\Pi\varphi$ and $\bar{\Pi}\varphi$, constructed in eq. (5.36). To set notation, we denote by $\Pi\varphi_i$ and $\bar{\Pi}\varphi_i$ the ith Cartesian component of the vector functions $\Pi\varphi$ and $\bar{\Pi}\varphi$, respectively. We first show a strong convergence result for the sequence of discrete test functions $\{(\Pi\varphi, \bar{\Pi}\varphi)\}_{h\in\mathcal{H}}$:

Proposition 5.4.1. *Let $k_s \geq 1$, $k_t \geq 0$ if $d = 2$, and $k_t \in \{0, 1\}$ if $d = 3$ and suppose that $\tau \to 0$ as $h \to 0$. Let $\varphi \in \mathscr{M}$ and consider the sequence of discrete test functions $\{(\Pi\varphi, \bar{\Pi}\varphi)\}_{h\in\mathcal{H}}$ defined in eq. (5.36). Then, it holds that $\Pi\varphi \to \varphi$ strongly in $L^\infty(0,T;L^\infty(\Omega)^d)$ and $G_h^{k_s}((\Pi\varphi_i, \bar{\Pi}\varphi_i)) \to \nabla\varphi_i$ strongly in $L^2(0,T;L^2(\Omega)^d)$ as $h \to 0$.*

Proof. We first record the following consequences of Lemma B.2.1:

$$\|\Pi^t\eta_k - \eta_k\|^2_{L^\infty(0,T)} \lesssim \tau^2\|\eta_k\|^2_{W^{1,\infty}(0,T)}, \tag{5.43a}$$

$$\sum_{K\in\mathcal{T}_h} \|\nabla(\Pi_h^{\text{div}}\psi_k - \psi_k)\|^2_{L^2(K)} \lesssim h^2\|\psi_k\|^2_{H^2(\Omega)}, \tag{5.43b}$$

$$\|\psi_k - \Pi_h^{\text{div}}\psi_k\|_{L^\infty(\Omega)} \lesssim h^{1/2}|\psi_k|_{H^2(\Omega)}. \tag{5.43c}$$

That $\Pi\varphi \to \varphi$ strongly in $L^\infty(0,T;L^\infty(\Omega)^d)$ follows from eq. (5.43), since

$$\begin{aligned} &\|\varphi - \Pi\varphi\|_{L^\infty(0,T;L^\infty(\Omega)^d)} \\ &\lesssim \sum_{k=1}^{m} \Big(\|\eta_k\|_{L^\infty(0,T)}\,\|\psi_k - \Pi_h^{\text{div}}\psi_k\|_{L^\infty(\Omega)} + \|\eta_k - \Pi^t\eta_k\|_{L^\infty(0,T)}\,\|\psi_k\|_{H^2(\Omega)}\Big). \end{aligned} \tag{5.44}$$

We now prove the strong convergence of $G_h^{k_s}((\Pi\varphi_i, \bar{\Pi}\varphi_i))$ to $\nabla\varphi_i$ in $L^2(0,T;L^2(\Omega)^d)$. Using the definition of the discrete gradient eq. (5.19), the triangle inequality, and eq. (5.18), we find

$$\begin{aligned} &\int_0^T \|G_h^{k_s}((\Pi\varphi_i, \bar{\Pi}\varphi_i)) - \nabla\varphi_i\|^2_{L^2(\Omega)}\,\mathrm{d}t \\ &\quad\leq \sum_{K\in\mathcal{T}_h}\int_0^T \|\nabla\Pi_h\varphi - \nabla\varphi\|^2_{L^2(K)}\,\mathrm{d}t + \sum_{K\in\mathcal{T}_h}\int_0^T h_K^{-1}\big\|\Pi\varphi - \bar{\Pi}\varphi\big\|^2_{L^2(\partial K)}\,\mathrm{d}t. \end{aligned} \tag{5.45}$$

We start with the first term on the right-hand side of eq. (5.45). By the definition of $\Pi\varphi$, the triangle inequality, and eq. (5.43), we can write

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\int_0^T \|\nabla\Pi\varphi-\nabla\varphi\|_{L^2(K)}^2\,\mathrm{d}t \\
&\quad\lesssim \sum_{k=1}^{M}\sum_{K\in\mathcal{T}_h}\left(\int_0^T (\Pi^t\eta_k)^2\,\|\nabla(\Pi_h^{\mathrm{div}}\psi_k-\psi_k)\|_{L^2(K)}^2\,\mathrm{d}t+\int_0^T(\Pi^t\eta_k-\eta_k)^2\,\|\nabla\psi_k\|_{L^2(K)}^2\,\mathrm{d}t\right)\\
&\quad\lesssim \sum_{k=1}^{M}\|\eta_k\|_{W^{1,\infty}(0,T)}^2\left(h^2\int_0^T\|\psi_k\|_{H^2(\Omega)}^2\,\mathrm{d}t+\tau^2\int_0^T\|\psi_k\|_{L^2(\Omega)}^2\,\mathrm{d}t\right),
\end{aligned}
$$

which can be seen to vanish as $h\to 0$. Turning now to the second term on the right-hand side of eq. (5.45), we find

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\int_0^T h_K^{-1}\left\|\Pi\varphi-\bar{\Pi}\varphi\right\|_{L^2(\partial K)}^2\,\mathrm{d}t\\
&\quad\lesssim\sum_{k=1}^{M}\left\|\Pi^t\eta_k\right\|_{W^{1,\infty}(0,T)}^2\sum_{K\in\mathcal{T}_h}\int_0^T h_K^{-1}\,\|\Pi_h^{\mathrm{div}}\psi_k-\bar{\Pi}_h\psi_k\|_{L^2(\partial K)}^2\,\mathrm{d}t.
\end{aligned}
\tag{5.46}
$$

Using a discrete local trace inequality, the assumed quasi-uniformity of the spatial mesh, and the approximation properties of Π_h^{div} and $\bar{\Pi}_h$, we find

$$
h_K^{-1}\,\|\Pi_h^{\mathrm{div}}\psi_k-\bar{\Pi}_h\psi_k\|_{L^2(\partial K)}^2\lesssim h^2\|\psi_k\|_{H^2(\Omega)}^2\,, \tag{5.47}
$$

and thus the right-hand side of eq. (5.46) vanishes as $h\to 0$. The result follows. □

5.4.2 Asymptotic consistency of the linear viscous term

We are now in a position to show that the linear viscous term is asymptotically consistent in the sense of [28, Definition 5.9]:

Theorem 5.4.1. *Let $k_s\geq 1$ and $k_t\geq 0$ if $d=2$ and $k_t\in\{0,1\}$ and suppose that $\{\boldsymbol{u}_h\}_{h\in\mathcal{H}}$ is a sequence of solutions of eq. (5.24) such that $\tau\to 0$ as $h\to 0$. Let $\varphi\in\mathscr{M}$, denote by $(\Pi\varphi,\bar{\Pi}\varphi)$ the discrete test functions constructed as in eq. (5.36), and let $u\in L^\infty(0,T;H)\cap L^2(0,T;V)$ be the limit (up to a subsequence) of $\{u_h\}_{h\in\mathcal{H}}$ guaranteed by Theorem 5.3.3. Then, the following asymptotic consistency result holds for the linear viscous term:*

$$
\lim_{h\to 0}\int_0^T a_h(\boldsymbol{u}_h,(\Pi\varphi,\bar{\Pi}\varphi))\,\mathrm{d}t=\int_0^T\int_\Omega\nabla u:\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t.
$$

Proof. Since $G_h^{k_s}((\Pi\varphi_i, \bar{\Pi}\varphi_i)) \to \nabla\varphi_i$ strongly in $L^2(0,T;L^2(\Omega)^d)$ by Proposition 5.4.1 and by Theorem 5.3.3 (iii), $G_h^{k_s}(\boldsymbol{u}_{h,i}) \rightharpoonup \nabla u_i$ weakly in $L^2(0,T;L^2(\Omega)^d)$ as $(\tau, h) \to 0$, we can pass to the limit in the first term of eq. (5.25) to find that

$$\lim_{h\to 0} \int_0^T \int_\Omega G_h^{k_s}(\boldsymbol{u}_{h,i}) \cdot G_h^{k_s}((\Pi\varphi_i, \bar{\Pi}\varphi_i)) \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega \nabla u_i \cdot \nabla\varphi_i \,\mathrm{d}x\,\mathrm{d}t.$$

Turning to the second term of eq. (5.25), we have by the Cauchy-Schwarz inequality, the bound on the global spatial lifting operator eq. (5.18), the definition of $\Pi\varphi$ and $\bar{\Pi}\varphi$, and uniform bound Lemma 5.1.1,

$$\begin{aligned}
&\int_0^T \int_\Omega R_h^{k_s}(u_{h,i} - \bar{u}_{h,i}) \cdot R_h^{k_s}(\Pi\varphi_i - \bar{\Pi}\varphi_i) \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le C(f, u_0, \nu) \left(\sum_{k=1}^{M} \sum_{K\in\mathcal{T}_h} \int_0^T h_K^{-1} \Pi^t \eta_k \left\| \Pi_h^{\mathrm{div}} \psi_k - \bar{\Pi}_h \psi_k \right\|_{L^2(\partial K)}^2 \right)^{1/2},
\end{aligned}$$

which can be seen to vanish as $h \to 0$ by eq. (5.43a) and eq. (5.47). In an identical fashion, we find

$$\lim_{h\to 0} \sum_{K\in\mathcal{T}_h} \int_0^T \int_{\partial K} \frac{\alpha}{h_K} (u_h - \bar{u}_h) \cdot (\Pi\varphi_i - \bar{\Pi}\varphi_i) \,\mathrm{d}s\,\mathrm{d}t = 0.$$

The result follows. □

5.4.3 Asymptotic consistency of the nonlinear convection term

The goal of this subsection is to prove that the nonlinear convection term is asymptotically consistent in the sense of [28, Definition 5.9].

Theorem 5.4.2. *Let $k_s \ge 1$ and $k_t \ge 0$ if $d = 2$ and $k_t \in \{0, 1\}$ and suppose that $\{\boldsymbol{u}_h\}_{h\in\mathcal{H}}$ is a sequence of solutions of eq. (5.24) such that $\tau \to 0$ as $h \to 0$. Let $\varphi \in \mathscr{M}$, denote by $(\Pi\varphi, \bar{\Pi}\varphi)$ the discrete test functions constructed as in eq. (5.36), and let $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ be an accumulation point of $\{u_h\}_{h\in\mathcal{H}}$ guaranteed by Theorem 5.3.3. Then, the following* asymptotic consistency *result holds for the nonlinear convection term:*

$$\lim_{h\to 0} \int_0^T o_h(u_h; \boldsymbol{u}_h, (\Pi\varphi, \bar{\Pi}\varphi)) \,\mathrm{d}t = \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t.$$

Proof. We start with the first term on the right-hand side of eq. (5.26). By Hölder's inequality, we have

$$\begin{aligned}&\int_0^T \|u\varphi_i - u_h \Pi\varphi_i\|^2_{L^2(\Omega)} \, \mathrm{d}t \\ &\lesssim \|\varphi - \Pi\varphi\|^2_{L^\infty(0,T;L^\infty(\Omega)^d)} \int_0^T \|u\|^2_{L^2(\Omega)} \, \mathrm{d}t + \|\Pi\varphi\|^2_{L^\infty(0,T;L^\infty(\Omega)^d)} \int_0^T \|u - u_h\|^2_{L^2(\Omega)} \, \mathrm{d}t,\end{aligned}$$

which can be seen to vanish as $h \to 0$ by Proposition 5.4.1 and Theorem 5.3.3. Therefore, $u_h \Pi\varphi_i \to u\varphi_i$ strongly in $L^2(0, T; L^2(\Omega)^d)$ as $h \to 0$, and this combined with the fact that $G_h^{2k_s}(\boldsymbol{u}_{h,i}) \rightharpoonup \nabla u$ yields

$$\lim_{h\to 0} \int_0^T \int_\Omega u_h \cdot G_h^{2k_s}(\boldsymbol{u}_{h,i}) \Pi\varphi_i \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t.$$

It remains to show that the facet term appearing in eq. (5.26) converges to 0 as $h \to 0$. By the definitions of $\Pi\varphi$ and $\bar{\Pi}\varphi$, proceeding as in the proof of [14, Proposition 3.4], and using the fact that $\int_0^T |\!|\!| \boldsymbol{u}_h |\!|\!|_v^2 \, \mathrm{d}t$ is uniformly bounded by Lemma 5.1.1,

$$\begin{aligned}&\left| \sum_{K\in\mathcal{T}_h} \int_0^T \int_{\partial K} \tfrac{1}{2} \left(u_h \cdot n + |u_h \cdot n| \right) (u_h - \bar{u}_h) \cdot (\Pi\varphi - \bar{\Pi}\varphi) \, \mathrm{d}s \, \mathrm{d}t \right| \\ &\qquad \le C(f, u_0, \nu) \sum_{k=1}^M \|\Pi^t \eta_k\|_{L^\infty(0,T)} \left(\sum_{K\in\mathcal{T}_h} h_K^{-1} \|\Pi_h^{\mathrm{div}} \psi_k - \bar{\Pi}_h \psi_k\|^2_{L^2(\partial K)} \right)^{1/2},\end{aligned}$$

which can be seen to vanish as $h \to 0$ by using the second bound in eq. (5.43a) and eq. (5.47). □

5.4.4 Passing to the limit

With the asymptotic consistency of the linear viscous term (Theorem 5.4.1) and the nonlinear convection term (Theorem 5.4.2), we are ready to pass to the limit in eq. (5.24). Suppose that $\tau \to 0$ as $h \to 0$. Extract from $\{u_h\}_{h\in\mathcal{H}}$ the subsequence satisfying the convergence results listed in Theorem 5.3.3. Let $\varphi \in \mathscr{M}$ and choose $\boldsymbol{v}_h = (\Pi\varphi, \bar{\Pi}\varphi) \in \mathcal{V}_h^{\mathrm{div}} \times \bar{\mathcal{V}}_h$ as a test function in eq. (5.24):

$$\begin{aligned}\int_0^T (\mathcal{D}_t^{k_t}(u_h), \Pi\varphi)_{\mathcal{T}_h} \, \mathrm{d}t + \int_0^T \big(\nu a_h(\boldsymbol{u}_h, (\Pi\varphi, \bar{\Pi}\varphi)) + o_h(u_h; \boldsymbol{u}_h, (\Pi\varphi, \bar{\Pi}\varphi)) \big) \, \mathrm{d}t \\ = \int_0^T (f, \Pi\varphi)_{\mathcal{T}_h} \, \mathrm{d}t.\end{aligned} \tag{5.48}$$

By the definition of $\Pi\varphi$, we have

$$\int_0^T (\mathcal{D}_t^{k_t}(u_h), \Pi\varphi)_{\mathcal{T}_h}\,\mathrm{d}t = \int_0^T (\mathcal{D}_t^{k_t}(u_h), \varphi)_{\mathcal{T}_h}\,\mathrm{d}t = \int_0^T \langle \mathcal{D}_t^{k_t}(u_h), \varphi\rangle_{V'\times V}\,\mathrm{d}t. \tag{5.49}$$

Thus, the weak convergence of $\mathcal{D}_t^{k_t}(u_h)$ to $\frac{\mathrm{d}u}{\mathrm{d}t}$ in $L^{4/d}(0,T;V')$ yields

$$\lim_{h\to 0}\int_0^T (\mathcal{D}_t^{k_t}(u_h), \Pi\varphi)_{\mathcal{T}_h}\,\mathrm{d}t = \int_0^T \big\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \varphi\big\rangle_{V'\times V}\,\mathrm{d}t. \tag{5.50}$$

This, in combination with Theorem 5.4.1 and Theorem 5.4.2, shows that upon passage to the limit as $h \to 0$ in eq. (5.48) that the limit $u \in L^\infty(0,T;H)\cap L^2(0,T;V)$ of the subsequence $\{u_h\}_{h\in\mathcal{H}}$ given by Theorem 5.3.3 satisfies for all $\varphi \in \mathscr{M}$,

$$\int_0^T \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \varphi\right\rangle_{V'\times V}\mathrm{d}t + \int_0^T ((u\cdot\nabla)u, \varphi)\,\mathrm{d}t + \nu\int_0^T (\nabla u, \nabla\varphi)\,\mathrm{d}t = \int_0^T (f,\varphi)_{\mathcal{T}_h}\,\mathrm{d}t. \tag{5.51}$$

By the density of the set $\mathscr{M}$ in $C_c(0,T;V)$, eq. (5.51) holds also for all $\varphi \in C_c(0,T;V)$.

We now show that $u \in L^\infty(0,T;H)\cap L^2(0,T;V)$ satisfies the initial condition in the sense that $u(0) = u_0$ in V'. Our starting point is the definition of the discrete time derivative eq. (5.23) in a single space-time slab $\mathcal{E}^n$:

$$\int_{I_n} (\partial_t u_h, v_h)_{\mathcal{T}_h}\,\mathrm{d}t + (u_n^+ - u_n^-, v_n^+)_{\mathcal{T}_h} = \int_{I_n} (\mathcal{D}_t^{k_t}(u_h), v_h)_{\mathcal{T}_h}\,\mathrm{d}t, \quad \forall v_h \in \mathcal{V}_h. \tag{5.52}$$

Let $\psi \in V$ and $\eta \in C^\infty(0,T)$ such that $\eta(T) = 0$. Define a function $w_h \in \mathcal{V}_h^{\mathrm{div}}$ by setting $w_h|_{\mathcal{E}^n} = \Pi_n^{k_t}\eta\Pi_h^{\mathrm{div}}\psi$ with $\Pi_n^{k_t} : H^1(I_n) \to P_{k_t}(I_n)$ the DG time projection defined as in [34, Section 69.3.2]. Define also the global projection $\Pi^{k_t}|_{I_n} = \Pi_n^{k_t}$. We note that by definition, $(\psi, v_h)_{\mathcal{T}_h} = (\Pi_h^{\mathrm{div}}\psi, v_h)_{\mathcal{T}_h}$ for all $v_h \in V_h^{\mathrm{div}}$, and by the defining properties of the projection $\Pi_n^{k_t}$ (see [34, Eq. (69.26)]),

$$\int_{I_n} (\partial_t u_h, \Pi_h^{\mathrm{div}}\psi)_{\mathcal{T}_h}\Pi_n^{k_t}\eta\,\mathrm{d}t = \int_{I_n} (\partial_t u_h, \Pi_h^{\mathrm{div}}\psi)_{\mathcal{T}_h}\eta\,\mathrm{d}t \quad \text{and} \quad (\Pi_n^{k_t}\eta)(t_n^+) = \eta(t_n),$$

since $\partial_t u_h \in P_{k_t-1}(I_n)$ (with the convention that $P_{-1}(I_n) = \{0\}$). Choosing $v_h = w_h$ in eq. (5.52), we find

$$\int_{I_n} (\partial_t u_h, \psi)_{\mathcal{T}_h}\eta\,\mathrm{d}t + (u_n^+ - u_n^-, \psi)_{\mathcal{T}_h}\eta(t_n) = \int_{I_n} (\mathcal{D}_t^{k_t}(u_h), \psi)_{\mathcal{T}_h}\Pi^{k_t}\eta\,\mathrm{d}t. \tag{5.53}$$

Integrating by parts in time on the left hand side of eq. (5.53), summing over all space-time slabs, and using that $\eta(T) = 0$, we have

$$-\int_0^T (u_h, \psi)_{\mathcal{T}_h} \partial_t \eta \, \mathrm{d}t - (u_0^-, \psi)_{\mathcal{T}_h} \eta(0) = \int_0^T (\mathcal{D}_t^{k_t}(u_h), \psi)_{\mathcal{T}_h} \Pi^{k_t} \eta \, \mathrm{d}t. \tag{5.54}$$

From Theorem 5.3.3 (i) and (iii), and since $u_0^- = \Pi_h^{\mathrm{div}} u_0 \to u_0$ strongly in H, and $\Pi^{k_t}\eta \to \eta$ strongly in $L^{4/(4-d)}(0, T)$ by eq. (5.43a), we can pass to the limit as $h \to 0$ in eq. (5.54) to find

$$-\int_0^T (u, \psi)_{\mathcal{T}_h} \partial_t \eta \, \mathrm{d}t - (u_0, \psi)_{\mathcal{T}_h} \eta(0) = \int_0^T \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \psi \right\rangle_{V' \times V} \eta \, \mathrm{d}t. \tag{5.55}$$

Comparing eq. (5.55) with Theorem 2.2.9, we find that

$$0 = (u(0) - u_0, \psi)_{\mathcal{T}_h} = \langle u(0) - u_0, \psi \rangle_{V' \times V}, \ \forall \psi \in V \quad \Rightarrow \quad u(0) = u_0 \text{ in } V'.$$

Therefore, we have proven:

Theorem 5.4.3. *Let $u_0 \in H$ and $f \in L^2(0, T; L^2(\Omega)^d)$ be given and let $\mathcal{H}$ be a countable set of mesh sizes whose unique accumulation point is 0. Let $k_s \geq 1$ and $k_t \geq 0$ if $d = 2$ and $k_t \in \{0, 1\}$ if $d = 3$ and suppose that $\{\boldsymbol{u}_h\}_{h \in \mathcal{H}}$ is a sequence of solutions of eq. (5.24) such that $\tau \to 0$ as $h \to 0$. Then, upon passage to a subsequence, $\{u_h\}_{h \in \mathcal{H}}$ converges (in the sense of Theorem 5.3.3) to a weak solution of the Navier–Stokes problem eq. (5.2) $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ with $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^{4/d}(0, T; V')$.*

5.4.5 The energy inequality

In three dimensions, we are not guaranteed uniqueness of weak solutions and cannot conclude a priori that the weak solution obtained from Theorem 5.4.3 satisfies the energy inequality eq. (5.4). We show below that the weak solution in fact does satisfy eq. (5.4).

Lemma 5.4.1. *Let $d = 3$, $k_s \geq 1$, and $k_t \in \{0, 1\}$ and suppose that $\tau \to 0$ as $h \to 0$. The weak solution $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ given by Theorem 5.4.3 satisfies the energy inequality for a.e. $s \in (0, T]$:*

$$\|u(s)\|_{L^2(\Omega)}^2 + 2\nu \int_0^s \|u\|_V^2 \, \mathrm{d}t \ \leq \|u_0\|_{L^2(\Omega)}^2 + 2 \int_0^s (f, u)_{L^2(\Omega)} \, \mathrm{d}t. \tag{5.56}$$

Proof. Let $s \in (0,T)$ be fixed and choose $n_s \in \{0,1,\dots,N-1\}$ such that $t_{n_s} \le s \le t_{n_s+1}$. Testing eq. (5.8) with $\boldsymbol{v}_h = \boldsymbol{u}_h \in \mathcal{V}_h^{\mathrm{div}} \times \bar{\mathcal{V}}_h$, using eq. (5.29) and the stabillity of Π_h^{div} in $L^2(\Omega)^d$, and summing from $n = 0$ to $n = n_s$, we have

$$\left\|u_{n_s+1}^-\right\|_{L^2(\Omega)}^2 + 2\nu \sum_{i=1}^{3} \int_0^s \|G_h^{k_s}(\boldsymbol{u}_{h,i})\|_{L^2(\Omega)}^2 \,\mathrm{d}t \le \|u_0\|_{L^2(\Omega)}^2 + 2\int_0^{t_{n_s+1}} (f, u_h)_{\mathcal{T}_h} \,\mathrm{d}t. \quad (5.57)$$

Let us first suppose that $k_t = 0$. Since $u_{t_{n_s+1}}^- = u_h(s)$ for $s \in (t_{n_s}, t_{n_s+1})$ in this case, eq. (5.57) yields

$$\left\|u_h(s)\right\|_{L^2(\Omega)}^2 + 2\nu \sum_{i=1}^{3} \int_0^s \|G_h^{k_s}(\boldsymbol{u}_{h,i})\|_{L^2(\Omega)}^2 \,\mathrm{d}t \le \|u_0\|_{L^2(\Omega)}^2 + 2\int_0^{t_{n_s+1}} (f, u_h)_{\mathcal{T}_h} \,\mathrm{d}t. \quad (5.58)$$

Our goal is to justify passage to the limit in eq. (5.58).

To this end, we multiply both sides of eq. (5.58) by an arbitrary $\phi \in C_c^\infty(\mathbb{R})$ satisfying $\phi \ge 0$ and integrate from $s = 0$ to $s = T$:

$$\begin{aligned} \int_0^T \left(\left\|u_h(s)\right\|_{L^2(\Omega)}^2 + 2\nu \sum_{i=1}^{3} \int_0^s \|G_h^{k_s}(\boldsymbol{u}_{h,i})\|_{L^2(\Omega)}^2 \,\mathrm{d}t \right) \phi(s) \,\mathrm{d}s \\ \le \int_0^T \left(\|u_0\|_{L^2(\Omega)}^2 + 2\int_0^{t_{n_s+1}} (f, u_h)_{\mathcal{T}_h} \,\mathrm{d}t \right) \phi(s) \,\mathrm{d}s. \end{aligned} \quad (5.59)$$

We first consider the integral involving the body force f. By the triangle inequality, the Cauchy-Schwarz inequality, discrete Poincaré inequality eq. (5.7), and the uniform energy bound in Lemma 5.1.1, we obtain

$$\left| \int_0^{t_{n_s+1}} (f, u_h)_{\mathcal{T}_h} \,\mathrm{d}t - \int_0^s (f, u)_{\mathcal{T}_h} \,\mathrm{d}t \right| \lesssim \left(\int_s^{s+\tau} \|f\|_{L^2(\Omega)}^2 \,\mathrm{d}t \right)^{1/2} + \int_0^s |(f, u_h - u)_{\mathcal{T}_h}| \,\mathrm{d}t. \quad (5.60)$$

Since $f \in L^2(0,T; L^2(\Omega)^d)$, the primitive $F(\tau) = \int_s^{s+\tau} \|f\|_{L^2(\Omega)}^2 \,\mathrm{d}t$ is absolutely continuous. This, combined with the fact $u_h \to u$ strongly in $L^2(0,T; L^2(\Omega)^d)$, shows that the right-hand side of eq. (5.60) vanishes as $h \to 0$, and so

$$\lim_{h\to 0} \int_0^{t_{n_s+1}} (f, u_h)_{\mathcal{T}_h} \,\mathrm{d}t = \int_0^s (f, u)_{\mathcal{T}_h} \,\mathrm{d}t.$$

Thus, we can apply Lebesgue's dominated convergence theorem (Theorem 2.2.2) to find

$$\lim_{h\to 0} \int_0^T \left(\int_0^{t_{n_s+1}} (f, u_h)_{\mathcal{T}_h} \,\mathrm{d}t \right) \phi(s) \,\mathrm{d}s = \int_0^T \left(\int_0^s (f, u)_{\mathcal{T}_h} \,\mathrm{d}t \right) \phi(s) \,\mathrm{d}s. \quad (5.61)$$

With eq. (5.61) in hand, we pass to the lower limit as $h \to 0$ in eq. (5.59) and use Fatou's lemma (Theorem 2.2.1), the weak lower semicontinuity of norms, and Theorem 5.3.3:

$$\int_0^T \left(\left\| u(s) \right\|_{L^2(\Omega)}^2 + 2\nu \int_0^s \|u\|_V^2 \, \mathrm{d}t \right) \phi(s) \, \mathrm{d}s \le \int_0^T \left(\|u_0\|_{L^2(\Omega)}^2 + 2 \int_0^s (f, u)_{\mathcal{T}_h} \, \mathrm{d}t \right) \phi(s) \, \mathrm{d}s.$$

As this holds for all $\phi \in C_c^\infty(\mathbb{R})$ satisfying $\phi \ge 0$, we have for a.e. $s \in [0, T]$ [96, pp. 291],

$$\left\| u(s) \right\|_{L^2(\Omega)}^2 + 2\nu \int_0^s \|u\|_V^2 \, \mathrm{d}t \le \|u_0\|_{L^2(\Omega)}^2 + 2 \int_0^s (f, u)_{\mathcal{T}_h} \, \mathrm{d}t.$$

Next, suppose that $k_t = 1$. In this case, eq. (5.57) does not offer direct control over $\|u_h(s)\|_{L^2(\Omega)}$ for $s \in (t_{n_s}, t_{n_s+1})$. Instead, we define $\tilde{u}_h$ to be piecewise constant (in time) function satisfying $\tilde{u}_h|_{\mathcal{E}^m} = u_{m+1}^- = u_h(t_{m+1}^-)$, so that eq. (5.57) yields

$$\|\tilde{u}_h(s)\|_{L^2(\Omega)}^2 + 2\nu \sum_{i=1}^3 \int_0^s \left\| G_h^{k_s}(\boldsymbol{u}_h) \right\|_{L^2(\Omega)}^2 \mathrm{d}t \le \|u_0^-\|_{L^2(\Omega)}^2 + 2 \int_0^{t_{n_s+1}} (f, u_h)_{\mathcal{T}_h} \, \mathrm{d}t.$$

Note that if $u_h \rightharpoonup u$ in $L^2(0, T; L^2(\Omega)^d)$ as $h \to 0$, then also $\tilde{u}_h \rightharpoonup u$ in $L^2(0, T; L^2(\Omega)^d)$ as $h \to 0$ [101, Corollary 3.2]. The weak lower semi-continuity of the norm in $L^2(0, T; L^2(\Omega)^d)$ yields

$$\int_0^T \|u(s)\|_{L^2(\Omega)}^2 \, \phi(s) \, \mathrm{d}s \le \liminf_{h \to 0} \int_0^T \|\tilde{u}_h(s)\|_{L^2(\Omega)}^2 \, \phi(s) \, \mathrm{d}s.$$

The remainder of the proof is identical to the case $k_t = 0$. □

Chapter 6

Space-time HDG for the advection-diffusion problem on moving domains

Many important physical processes are governed by the solution of time-dependent partial differential equations on moving and deforming domains. Of particular importance are advection dominated transport problems, with applications ranging from multi-phase flows separated by evolving interfaces to incompressible flow problems arising from fluid-structure interaction. Since our motivation to study space-time HDG methods is the numerical solution of the incompressible Navier–Stokes equations on time-dependent domains, an appropriate first step is to analyze a space-time HDG scheme for a simpler linear advection–diffusion model.

In this chapter, we present (to our knowledge) the first analysis of a space-time HDG method on time-dependent domains. The first error analysis of a space-time DG method on moving and deforming domains for the linear advection–diffusion equation, however, was performed in [94, 95], and for the Oseen equations in [99], laying the groundwork for the error estimates in Section 6.4. The consideration of moving domains significantly alters the analysis of the method compared to analysis on fixed domains considered in the previous chapters. In particular, moving meshes lack the tensor product structure necessary to use the space-time projections or the temporal inverse and trace inequalities used in Chapter 4, without modification. Moreover, the Bochner–Sobolev spaces used in the previous chapters are no longer the appropriate functional setting as the function spaces defined over a time-dependent domain are themselves time-dependent. Instead, we follow [95, 99] by introducing *anisotropic* Sobolev spaces (first considered in the thesis [38]).

This chapter is organized as follows: In Section 6.1 we discuss the scalar advection–diffusion problem in a space-time setting. Next, in Section 6.2, we discuss the finite element spaces necessary to obtain the weak formulation of the advection–diffusion problem, which we subsequently introduce. Section 6.3 deals with the consistency and stability of the space-time HDG method. Theoretical rates of convergence of the space-time HDG formulation in a mesh-dependent norm on moving grids are derived in Section 6.4. Finally, Section 6.5 presents the results of a numerical example to support the theoretical analysis.

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6.1 The advection–diffusion problem

Let $\Omega(t) \subset \mathbb{R}^d$ be a time-dependent polygonal ($d = 2$) or polyhedral ($d = 3$) domain whose evolution depends continuously on time $t \in [t_0, t_N]$. Let $x = (x_1, \cdots, x_d)$ be the spatial variables and denote the spatial gradient operator by $\overline{\nabla} = (\partial_{x_1}, \cdots, \partial_{x_d})$. We consider the time-dependent advection–diffusion problem

$$\partial_t u + \overline{\nabla} \cdot (\bar{\beta} u) - \nu \overline{\nabla}^2 u = f \qquad \text{in } \Omega(t),\ t_0 < t \le t_N, \tag{6.1}$$

with given advective velocity $\bar{\beta}$, forcing term f and constant and positive diffusion coefficient ν.

Before introducing the space-time HDG method in Section 6.2, we first present the space-time formulation of the advection–diffusion problem eq. (6.1). Let $\mathcal{E} := \{(t, x) : x \in \Omega(t),\ t_0 < t < t_N\} \subset \mathbb{R}^{d+1}$ be a $(d + 1)$-dimensional polyhedral space-time domain. We denote the boundary of $\mathcal{E}$ by $\partial\mathcal{E}$, and note that it is comprised of the hyper-surfaces $\Omega(t_0) := \{(t, x) \in \partial\mathcal{E} :\ t = t_0\}$, $\Omega(t_N) := \{(t, x) \in \partial\mathcal{E} :\ t = t_N\}$, and $\mathcal{Q}_{\mathcal{E}} := \{(t, x) \in \partial\mathcal{E} : t_0 < t < t_N\}$. The outward space-time normal vector to $\partial\mathcal{E}$ is denoted by $n := (n_t, \bar{n})$, where n_t and $\bar{n}$ are the temporal and spatial parts of the space-time normal vector, respectively.

To recast the advection–diffusion problem in the space-time setting we introduce the space-time velocity field $\beta := (1, \bar{\beta})$ and the operator $\nabla := (\partial_t, \overline{\nabla})$. The space-time formulation of eq. (6.1) is then given by

$$\nabla \cdot (\beta u) - \nu \overline{\nabla}^2 u = f \quad \text{in } \mathcal{E}, \tag{6.2}$$

where $f \in L^2(\mathcal{E})$ and where $\beta, \nabla \cdot \beta \in L^\infty(\mathcal{E})$.

We partition the boundary of $\Omega(t)$ such that $\partial\Omega(t) = \Gamma_D(t) \cup \Gamma_N(t)$ and $\Gamma_D(t) \cap \Gamma_N(t) = \emptyset$ and we partition the space-time boundary into $\partial\mathcal{E} = \partial\mathcal{E}_D \cup \partial\mathcal{E}_N$, where $\partial\mathcal{E}_D := \big\{(t,x) : x \in \Gamma_D(t), t_0 < t \le t_N\big\}$ and $\partial\mathcal{E}_N := \big\{(t,x) : x \in \Gamma_N(t) \cup \Omega(t_0), t_0 < t \le t_N\big\}$. Given a suitably smooth function $g : \partial\mathcal{E}_N \to \mathbb{R}$, we prescribe the initial and boundary conditions

$$-\zeta u \beta \cdot n + \nu \overline{\nabla} u \cdot \bar{n} = g \quad \text{on } \partial\mathcal{E}_N, \tag{6.3a}$$

$$u = 0 \quad \text{on } \partial\mathcal{E}_D, \tag{6.3b}$$

where ζ is an indicator function for the inflow boundary of $\mathcal{E}$, i.e., the portions of the boundary where $\beta \cdot n < 0$. Note that eq. (6.3a) imposes the initial condition $u(x,t_0) = g(x)$ on $\Omega(t_0)$.

6.2 The space-time hybridizable discontinuous Galerkin method

In this section we introduce the space-time mesh, the space-time approximation spaces and the space-time HDG formulation for the advection–diffusion problem eq. (6.2)–eq. (6.3).

6.2.1 Description of space-time slabs, faces and elements

We begin this section with a description of the discretization of the space-time domain. First, the time interval $[t_0, t_N]$ is partitioned into the time levels $t_0 < t_1 < \cdots < t_N$, where the n-th time interval is defined as $I_n = (t_n, t_{n+1})$ with length $\Delta t_n = t_{n+1} - t_n$. For simplicity we will assume a fixed time interval length, i.e., $\Delta t_n = \Delta t$ for $n = 0, 1, \cdots, N-1$. For ease of notation, we will denote $\Omega(t_n) = \Omega_n$ in the sequel. The space-time domain is then divided into space-time slabs $\mathcal{E}^n = \mathcal{E} \cap \big(I_n \times \mathbb{R}^d\big)$. Each space-time slab $\mathcal{E}^n$ is bounded by Ω_n, Ω_{n+1}, and $\mathcal{Q}_{\mathcal{E}}^n = \partial\mathcal{E}^n \setminus (\Omega_n \cup \Omega_{n+1})$.

We further divide each space-time slab into space-time elements, $\mathcal{E}^n = \bigcup_j \mathcal{K}_j^n$. To construct the space-time element $\mathcal{K}_j^n$, we divide the domain Ω_n into non-overlapping spatial elements K_j^n, so that $\Omega_n = \bigcup_j K_j^n$. Then, at t_{n+1} the spatial elements K_j^{n+1} are obtained by mapping the nodes of the elements K_j^n into their new position via the transformation describing the deformation of the domain. Each space-time element $\mathcal{K}_j^n$ is obtained by connecting the elements K_j^n and K_j^{n+1} via linear interpolation in time.

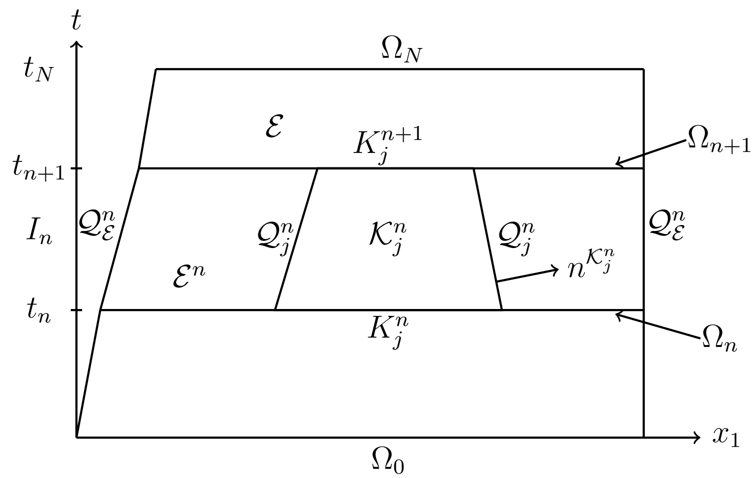

Figure 6.1: An example of a space-time slab in a polyhedral $(1+1)$-dimensional space-time domain.

The boundary of the space-time element $\mathcal{K}_j^n$ consists of K_j^n, K_j^{n+1}, and $\mathcal{Q}_j^n = \partial\mathcal{K}_j^n \setminus (K_j^n \cup K_j^{n+1})$. On $\partial\mathcal{K}_j^n$, the outward unit space-time normal vector is denoted by $n^{\mathcal{K}_j^n} = (n_t^{\mathcal{K}_j^n}, \bar{n}^{\mathcal{K}_j^n})$, where $n_t^{\mathcal{K}_j^n}$ and $\bar{n}^{\mathcal{K}_j^n}$ are, respectively, the temporal and spatial parts of the space-time normal vector. On K_j^n, $n^{\mathcal{K}_j^n} = (-1, 0)$, while on K_j^{n+1}, $n^{\mathcal{K}_j^n} = (1, 0)$. In the remainder of the article, we will drop the subscripts and superscripts when referring to space-time elements, their boundaries and outward normal vectors wherever no confusion will occur.

We complete the description of the space-time domain with the tessellation $\mathcal{T}_h^n$ consisting of all space-time elements in $\mathcal{E}^n$, and $\mathcal{T}_h = \bigcup_n \mathcal{T}_h^n$ consisting of all space-time elements in $\mathcal{E}$. An illustration of a space-time domain is shown in the case of one spatial dimension in Section 6.2.1.

Finally, an interior space-time facet $\mathcal{S}$ is shared by two adjacent elements $\mathcal{K}^L$ and $\mathcal{K}^R$, $\mathcal{S} = \partial\mathcal{K}^L \cap \partial\mathcal{K}^R$, while a boundary facet is a face of $\partial\mathcal{K}$ that lies on $\partial\mathcal{E}$. The set of all facets will be denoted by $\mathcal{F}$, and the union of all facets by Γ.

6.2.2 Approximation spaces

We define the Sobolev space $H^s(\Omega) = \{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega) \text{ for } |\alpha| \leq s\}$, where $D^\alpha v$ denotes the weak derivative of v, α is the multi-index symbol, s a non-negative integer, and

$\Omega \subset \mathbb{R}^n$ is an open domain (see e.g. [9]). The space $H^s(\Omega)$ is equipped with the following norm and semi-norm:

$$\|v\|^2_{H^s(\Omega)} = \sum_{|\alpha|\le s} \|D^\alpha v\|^2_{L^2(\Omega)} \quad \text{and} \quad |v|^2_{H^s(\Omega)} = \sum_{|\alpha|=s} \|D^\alpha v\|^2_{L^2(\Omega)}, \tag{6.4}$$

where $\|\cdot\|_{L^2(\Omega)}$ is the standard L^2-norm on Ω. In the sequel, we will simply write $\|v\|_\Omega = \|v\|_{L^2(\Omega)}$.

Next, we introduce anisotropic Sobolev spaces on an open domain $\Omega \subset \mathbb{R}^{d+1}$ [38]. For simplicity, we follow [95, 99] by restricting the anisotropy to the case where the Sobolev index can differ only between spatial and temporal variables. All spatial variables will have the same index. Let (s_t, s_s) be a pair of non-negative integers, with s_t, s_s corresponding to the spatial and temporal Sobolev indices. For $\alpha_t, \alpha_{s_i} \ge 0$, $i = 1, \dots, d$, we define the anisotropic Sobolev space of order (s_t, s_s) on $\Omega \subset \mathbb{R}^{d+1}$ by

$$H^{(s_t,s_s)}(\Omega) = \{v \in L^2(\Omega) : D^{\alpha_t} D^{\alpha_s} v \in L^2(\Omega) \text{ for } \alpha_t \le s_t, |\alpha_s| \le s_s\}, \tag{6.5}$$

where $\alpha_s = \left(\alpha_{s_1}, \cdots, \alpha_{s_d}\right)$. The anisotropic Sobolev norm and semi-norm are given by, respectively,

$$\|v\|^2_{H^{(s_t,s_s)}(\Omega)} = \sum_{\substack{\alpha_t \le s_t \\ |\alpha_s| \le s_s}} \|D^{\alpha_t} D^{\alpha_s} v\|^2_\Omega \quad \text{and} \quad |v|^2_{H^{(s_t,s_s)}(\Omega)} = \sum_{\substack{\alpha_t = s_t \\ |\alpha_s| = s_s}} \|D^{\alpha_t} D^{\alpha_s} v\|^2_\Omega .$$

We assume that each space-time element $\mathcal{K}$ is the image of a fixed master element $\widehat{\mathcal{K}} = (-1,1)^{d+1}$ under two mappings. First, we construct an intermediate tensor-product element $\widetilde{\mathcal{K}}$ from an affine mapping $F_\mathcal{K} : \widehat{\mathcal{K}} \to \widetilde{\mathcal{K}}$ of the form $F_\mathcal{K}(\hat{x}) = A_\mathcal{K}\hat{x} + b$, where $A_\mathcal{K} = \operatorname{diag}\left(\frac{\Delta t}{2}, \frac{h_1}{2}, \dots, \frac{h_d}{2}\right)$. Here h_i is the edge length in the i-th coordinate direction, Δt the time-step, and $b \in \mathbb{R}^{d+1}$ is a constant vector.

Next, the space-time element $\mathcal{K}$ is obtained from $\widetilde{\mathcal{K}}$ via the suitably regular diffeomorphism $\phi_\mathcal{K} : \widetilde{\mathcal{K}} \to \mathcal{K}$. The mapping $\phi_\mathcal{K}$ determines the shape of the space-time element after the size of the element has been specified by $F_\mathcal{K}$. Following [38], we will assume that the Jacobian of the diffeomorphism $\phi_\mathcal{K}$ satisfies:

$$C_1^{-1} \le |\det J_{\phi_\mathcal{K}}| \le C_1, \qquad \left\|\det J_{\phi_\mathcal{K}\backslash mn}\right\|_{L^\infty(\widetilde{\mathcal{K}})} \le C_2, \quad m, n = 0, \dots, d, \quad \forall \mathcal{K} \in \mathcal{T}_h,$$

where C_1 and C_2 are constants independent of the edge lengths h_i and the time-step Δt, and where $\det J_{\phi_\mathcal{K}\backslash mn}$ denotes the (m,n) minor of $J_{\phi_\mathcal{K}}$.

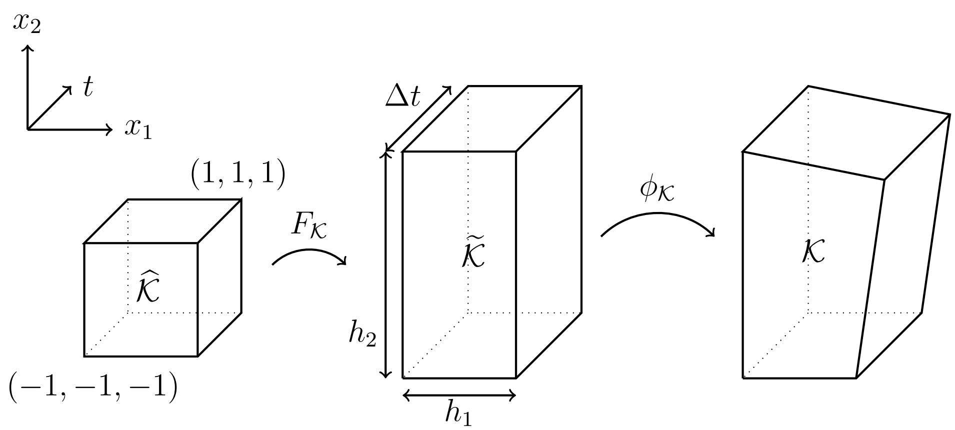

Figure 6.2: Construction of the space-time element $\mathcal{K}$ through an affine mapping $F_{\mathcal{K}} : \widehat{\mathcal{K}} \to \widetilde{\mathcal{K}}$ and a diffeomorphism $\phi_{\mathcal{K}} : \widetilde{\mathcal{K}} \to \mathcal{K}$ [95].

Following [95], we define the Sobolev space $H^{(s_t,s_s)}(\widetilde{\mathcal{K}})$ as

$$H^{(s_t,s_s)}(\widetilde{\mathcal{K}}) = \{v \in L^2(\widetilde{K}) : D^{\alpha_t} D^{\alpha_s} v \in L^2(\widetilde{K}) \text{ for } \alpha_t \le s_t, |\alpha_s| \le s_s\}. \tag{6.6}$$

Furthermore, the Sobolev space $H^{(s_t,s_s)}(\mathcal{K})$ is defined as

$$H^{(s_t,s_s)}(\mathcal{K}) = \{v \in L^2(\mathcal{K}) : v \circ \phi_{\mathcal{K}} \in H^{(s_t,s_s)}(\widetilde{\mathcal{K}})\}, \tag{6.7}$$

see [38, Definition 2.9].

For the analysis in Section 6.3 we require the concept of a broken anisotropic Sobolev space. We assign to $\mathcal{T}_h$ the broken Sobolev space

$$H^{(s_t,s_s)}(\mathcal{T}_h) = \{v \in L^2(\mathcal{E}) : v|_{\mathcal{K}} \in H^{(s_s,s_t)}(\mathcal{K}), \forall \mathcal{K} \in \mathcal{T}_h\}, \tag{6.8}$$

which we equip with the broken anisotropic Sobolev norm and semi-norm, respectively,

$$\|v\|^2_{H^{(s_t,s_s)}(\mathcal{T}_h)} = \sum_{\mathcal{K} \in \mathcal{T}_h} \|v\|^2_{H^{(s_t,s_s)}(\mathcal{K})} \quad \text{and} \quad |v|^2_{H^{(s_t,s_s)}(\mathcal{T}_h)} = \sum_{\mathcal{K} \in \mathcal{T}_h} |v|^2_{H^{(s_t,s_s)}(\mathcal{K})}. \tag{6.9}$$

For $v \in H^{(1,1)}(\mathcal{T}_h)$, we define the broken (space-time) gradient $\nabla_h v$ by $(\nabla_h v)|_{\mathcal{K}} = \nabla(v|_{\mathcal{K}})$, $\forall \mathcal{K} \in \mathcal{T}_h$.

Additionally, we will make use of the following (spatial) shape regularity assumption. Suppose $\mathcal{K} \in \mathcal{T}_h$ is constructed from the fixed reference element $\widehat{\mathcal{K}}$ via the mappings

$F_{\mathcal{K}} : \widehat{K} \to \widetilde{\mathcal{K}}$ and $\phi_{\mathcal{K}} : \widetilde{\mathcal{K}} \to \mathcal{K}$. Let h_K and ρ_K denote the radii of the d-dimensional circumsphere and inscribed sphere of the brick $h_1 \times \cdots \times h_d$, respectively. We assume the existence of a constant $c_r > 0$ such that

$$\frac{h_K}{\rho_K} \le c_r, \quad \forall \mathcal{K} \in \mathcal{T}_h. \tag{6.10}$$

For the HDG method, we require the finite element spaces

$$V_h^{(p_t,p_s)} = \left\{ v_h \in L^2(\mathcal{E}) : v_h|_{\mathcal{K}} \circ \phi_{\mathcal{K}} \circ F_{\mathcal{K}} \in Q_{(p_t,p_s)}(\widehat{\mathcal{K}}), \forall \mathcal{K} \in \mathcal{T}_h \right\}, \tag{6.11}$$

$$\begin{aligned} M_h^{(p_t,p_s)} = \{ \mu_h \in L^2(\Gamma) : \mu_h|_{\mathcal{S}} \circ \phi_{\mathcal{K}} \circ F_{\mathcal{K}} \in Q_{(p_t,p_s)}(\widehat{\mathcal{S}}), \forall \mathcal{S} \in \mathcal{F}, \\ \mu_h = 0 \text{ on } \partial\mathcal{E}_D \}, \end{aligned} \tag{6.12}$$

where $Q_{(p_t,p_s)}(D)$ denotes the set of all tensor-product polynomials of degree p_t in the temporal direction and p_s in each spatial direction on a domain D. Furthermore, we define $\boldsymbol{V}_h = V_h^{(p_t,p_s)} \times M_h^{(p_t,p_s)}$.

6.2.3 Weak formulation

It will be convenient to introduce the bilinear forms

$$\begin{aligned} a_h^a\big((u,\lambda),(v,\mu)\big) = &- \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \beta u \cdot \nabla_h v \,\mathrm{d}x + \int_{\partial\mathcal{E}_N} \frac{1}{2}\big(\beta\cdot n + |\beta\cdot n|\big)\lambda\mu \,\mathrm{d}s \\ &+ \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \frac{1}{2}\big(\beta\cdot n(u+\lambda) + |\beta\cdot n|(u-\lambda)\big)(v-\mu)\,\mathrm{d}s, \end{aligned} \tag{6.13a}$$

$$\begin{aligned} a_h^d\big((u,\lambda),(v,\mu)\big) = &\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nu\overline{\nabla}_h u \cdot \overline{\nabla}_h v \,\mathrm{d}x + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{Q}} \frac{\nu\alpha}{h_K}(u-\lambda)(v-\mu)\,\mathrm{d}s \\ &- \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{Q}} \Big[\nu(u-\lambda)\overline{\nabla}_h v\cdot\bar{n} + \nu\overline{\nabla}_h u\cdot\bar{n}\,(v-\mu)\Big]\,\mathrm{d}s, \end{aligned} \tag{6.13b}$$

where $\alpha > 0$ is a penalty parameter. The space-time HDG method for eq. (6.2)–eq. (6.3) is then given by: find $(u_h, \lambda_h) \in \boldsymbol{V}_h$ such that

$$a_h\big((u_h,\lambda_h),(v_h,\mu_h)\big) = \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} f v_h \,\mathrm{d}x + \int_{\partial\mathcal{E}_N} g\mu_h \,\mathrm{d}s \qquad \forall (v_h,\mu_h) \in \boldsymbol{V}_h, \tag{6.14}$$

where $a_h\big((u,\lambda),(v,\mu)\big) = a_h^a\big((u,\lambda),(v,\mu)\big) + a_h^d\big((u,\lambda),(v,\mu)\big)$.

6.3 Stability and boundedness

In this section we prove stability and boundedness of the space-time HDG method eq. (6.14). Our analysis will make repeated use of local trace and inverse inequalities valid on the finite element space $V_h^{(p_t,p_s)}$. Using ideas from [38], the dependence on the spatial mesh size h_K and time-step Δt is made explicit in these inequalities. Motivated by the fact that these two parameters differ in general, this will allow us to derive error bounds in section 6.4 that are anisotropic in h_K and Δt as in [95, 99]. The local trace and inverse inequalities are summarized in the following lemma.

Lemma 6.3.1. *Assume that $\mathcal{K}$ is a space-time element in $\mathbb{R}^{d+1}$ constructed via the mappings $\phi_{\mathcal{K}} : \widetilde{\mathcal{K}} \to \mathcal{K}$ and $F_{\mathcal{K}} : \widehat{K} \to \widetilde{\mathcal{K}}$ as defined in section 6.2.2. Assume further that the spatial shape regularity condition eq. (6.10) holds. Then, for all $v_h \in V_h^{(p_t,p_s)}$, the following local inverse and trace inequalities hold:*

$$\|\partial_t v_h\|_{\mathcal{K}} \le c_{I,t}\left(\Delta t^{-1} + h_K^{-1}\right)\|v_h\|_{\mathcal{K}}\,, \tag{6.15a}$$

$$\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}} \le c_{I,s} h_K^{-1}\|v_h\|_{\mathcal{K}}\,, \tag{6.15b}$$

$$\|v_h\|_{\mathcal{Q}} \le c_{T,\mathcal{Q}} h_K^{-\frac{1}{2}}\|v_h\|_{\mathcal{K}}\,, \tag{6.15c}$$

$$\|v_h\|_{\partial\mathcal{K}} \le c_{T,\partial\mathcal{K}}\left(\Delta t^{-\frac{1}{2}} + h_K^{-\frac{1}{2}}\right)\|v_h\|_{\mathcal{K}}\,, \tag{6.15d}$$

where $c_{I,s}$, $c_{I,t}$, $c_{T,\mathcal{Q}}$, and $c_{T,\partial\mathcal{K}}$ are constants depending on the polynomial degrees p_t and p_s, the spatial shape-regularity constant c_r, and the Jacobian of the mapping $\phi_{\mathcal{K}}$, but independent of the spatial mesh size h_K and the time step Δt.

Proof. Inequalities eq. (6.15a)–eq. (6.15d) are space-time variants of those found in [38, Corollary 3.54, Corollary 3.59]. □

Additionally, we will require the following discrete Poincaré inequality valid for $(v_h, \mu_h) \in V_h^\star$ [95],

$$\|v_h\|_{\mathcal{E}}^2 \le c_p^2\left(\sum_{\mathcal{K}\in\mathcal{T}_h}\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h}\frac{1}{h_K}\|v_h-\mu_h\|_{\mathcal{Q}}^2\right), \tag{6.16}$$

where $c_p > 0$ is a constant independent of the spatial mesh size h_K and time-step Δt.

Consider the following extended function spaces on $\mathcal{E}$ and Γ:

$$V(h) = V_h^{(p_t,p_s)} + H^2(\mathcal{E}), \qquad M(h) = M_h^{(p_t,p_s)} + H^{3/2}(\Gamma), \tag{6.17}$$

where $H^{3/2}(\Gamma)$ is the trace space of $H^2(\mathcal{E})$. For notational purposes we also introduce $V^\star(h) = V(h) \times M(h)$. We define three norms on $V^\star(h)$. First, the "stability" norm is defined as

$$\left\|\!\left|(v,\mu)\right|\!\right\|_v^2 = \|v\|_{\mathcal{E}}^2 + \left\|\beta_n^{1/2}\mu\right\|_{\partial\mathcal{E}_N}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h}\left\|\beta_n^{1/2}(v-\mu)\right\|_{\partial\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h}\nu\left\|\overline{\nabla}_h v\right\|_{\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\nu}{h_K}\|v-\mu\|_{\mathcal{Q}}^2, \quad (6.18)$$

where for ease of notation we have defined $\beta_n = |\beta\cdot n|$. Additionally, we introduce a stronger norm obtained by endowing the "stability" norm with an additional term controlling the L^2-norm of time derivatives:

$$\left\|\!\left|(v,\mu)\right|\!\right\|_s^2 = \left\|\!\left|(v,\mu)\right|\!\right\|_v^2 + \sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\Delta t h_K^2}{\Delta t + h_K}\|\partial_t v\|_{\mathcal{K}}^2. \quad (6.19)$$

To prove boundedness of the bilinear form in section 6.3.1 we introduce the following norm:

$$\begin{aligned}\left\|\!\left|(v,\mu)\right|\!\right\|_{s,\star}^2 = &\left\|\!\left|(v,\mu)\right|\!\right\|_s^2 + \sum_{\mathcal{K}\in\mathcal{T}_h}\left\|\beta_n^{1/2}v\right\|_{\partial\mathcal{K}^+}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h}\left\|\beta_n^{1/2}\mu\right\|_{\partial\mathcal{K}^-}^2 \\ &+ \sum_{\mathcal{K}\in\mathcal{T}_h} h_K\nu\left\|\overline{\nabla}_h v\cdot\bar{n}\right\|_{\mathcal{Q}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\Delta t + h_K}{\Delta t h_K^2}\|v\|_{\mathcal{K}}^2,\end{aligned} \quad (6.20)$$

where $\partial\mathcal{K}^+$ denotes the outflow part of the boundary (where $\beta\cdot n > 0$) and where $\partial\mathcal{K}^-$ denotes the inflow part of the boundary (where $\beta\cdot n \le 0$). The additional terms are required since the inequalities in Lemma 6.3.1 are valid only on the discrete space $V_h^{(p_t,p_s)}$.

Let $u \in H^2(\mathcal{E})$ solve the advection–diffusion problem eq. (6.2). Defining the trace operator $\gamma : H^2(\mathcal{E}) \to H^{3/2}(\Gamma)$, restricting functions in $H^2(\mathcal{E})$ to Γ, and letting $\boldsymbol{u} = (u, \gamma(u))$, we have

$$a_h(\boldsymbol{u},(v_h,\mu_h)) = \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} f v_h \,\mathrm{d}x + \int_{\partial\mathcal{E}_N} g\mu_h \,\mathrm{d}s \qquad \forall (v_h,\mu_h)\in \boldsymbol{V}_h. \quad (6.21)$$

This consistency result follows by noting that $u = \gamma(u)$ on element boundaries, integration by parts in space-time, single-valuedness of $\beta\cdot n$, $\overline{\nabla}_h u\cdot\bar{n}$, u and μ_h on element boundaries, the fact that $\mu_h = 0$ on $\partial\mathcal{E}_D$, and that u solves eq. (6.2)–eq. (6.3). An immediate consequence of consistency is Galerkin orthogonality: Let $(u_h,\lambda_h)\in\boldsymbol{V}_h$ solve eq. (6.14), then

$$a_h\big((u,\gamma(u)) - (u_h,\lambda_h),(v_h,\mu_h)\big) = 0, \quad \forall (v_h,\mu_h)\in\boldsymbol{V}_h. \quad (6.22)$$

6.3.1 Boundedness

We now turn to the boundedness of the bilinear form.

Lemma 6.3.2 (Boundedness)**.** *There exists a $c_B > 0$, independent of h_K and Δt, such that for all $\boldsymbol{u} = (u, \gamma(u)) \in V^\star(h)$ and all $(v_h, \mu_h) \in \boldsymbol{V}_h$,*

$$\left| a_h(\boldsymbol{u}, (v_h, \mu_h)) \right| \le c_B \|\!|\boldsymbol{u}|\!\|_{s,\star} \|\!|(v_h, \mu_h)|\!\|_{s}. \tag{6.23}$$

Proof. We will begin by bounding each term of the advective part of the bilinear form, $a_h^a(\boldsymbol{u}, \boldsymbol{v}_h)$. We note that

$$\begin{aligned} |a_h^a(\boldsymbol{u}, (v_h, \mu_h))| \le & \left| \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} \beta u \cdot \nabla_h v_h \,\mathrm{d}x \right| + \left| \int_{\partial \mathcal{E}_N} \tfrac{1}{2} \left(\beta \cdot n + |\beta \cdot n| \right) \gamma(u) \mu_h \,\mathrm{d}s \right| \\ & + \left| \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\partial \mathcal{K}} \tfrac{1}{2} \left(\beta \cdot n (u + \gamma(u)) + |\beta \cdot n| (u - \gamma(u)) \right) (v_h - \mu_h) \,\mathrm{d}s \right|. \end{aligned} \tag{6.24}$$

To obtain a bound for the first term on the right-hand side of eq. (6.24), we first recall $\beta \cdot \nabla_h v_h = \partial_t v_h + \bar{\beta} \cdot \overline{\nabla}_h v_h$, so that

$$\left| \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} \beta u \cdot \nabla_h v_h \,\mathrm{d}x \right| \le \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} |u \partial_t v_h| \,\mathrm{d}x + \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} \left| \bar{\beta} u \cdot \overline{\nabla}_h v_h \right| \mathrm{d}x. \tag{6.25}$$

Both terms on the right-hand side may be bounded using the Cauchy–Schwarz inequality:

$$\begin{aligned} \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} |u \partial_t v_h| \,\mathrm{d}x &\le \sum_{\mathcal{K} \in \mathcal{T}_h} \left(\frac{\Delta t + h_K}{\Delta t h_K^2} \right)^{\frac{1}{2}} \|u\|_{\mathcal{K}} \left(\frac{\Delta t h_K^2}{\Delta t + h_K} \right)^{\frac{1}{2}} \|\partial_t v_h\|_{\mathcal{K}} \qquad (6.26) \\ &\le \|\!|\boldsymbol{u}|\!\|_{s,\star} \|\!|(v_h, \mu_h)|\!\|_{s}, \\ \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} \left| \bar{\beta} u \cdot \overline{\nabla}_h v_h \right| \mathrm{d}x &\le \left\| \bar{\beta} \right\|_{L^\infty(\mathcal{E})} \sum_{\mathcal{K} \in \mathcal{T}_h} \nu^{-1/2} \|u\|_{\mathcal{K}} \, \nu^{1/2} \left\| \overline{\nabla}_h v_h \right\|_{\mathcal{K}} \qquad (6.27) \\ &\le \left\| \bar{\beta} \right\|_{L^\infty(\mathcal{E})} \nu^{-1/2} \|\!|\boldsymbol{u}|\!\|_{s,\star} \|\!|(v_h, \mu_h)|\!\|_{s}. \end{aligned}$$

The integral over the mixed boundary $\partial \mathcal{E}_N$ in eq. (6.24) may also be bounded via the Cauchy–Schwarz inequality:

$$\begin{aligned} \left| \int_{\partial \mathcal{E}_N} \tfrac{1}{2} \left(\beta \cdot n + |\beta \cdot n| \right) \gamma(u) \mu_h \,\mathrm{d}s \right| &\le \left\| \beta_n^{1/2} \gamma(u) \right\|_{\partial \mathcal{E}_N} \left\| \beta_n^{1/2} \mu_h \right\|_{\partial \mathcal{E}_N} \\ &\le \|\!|\boldsymbol{u}|\!\|_{s,\star} \|\!|(v_h, \mu_h)|\!\|_{s}. \end{aligned}$$

For the final term appearing on the right-hand side of eq. (6.24), we have the bound

$$
\begin{aligned}
&\left| \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \tfrac{1}{2} \big(\beta\cdot n(u+\gamma(u)) + |\beta\cdot n|(u-\gamma(u))\big)\,(v_h-\mu_h)\,\mathrm{d}s \right| \qquad (6.28)\\
&\quad\le \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}^+} \big|\beta\cdot n\,(v_h-\mu_h)\,u\big|\,\mathrm{d}s + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}^-} \big|\beta\cdot n\,(v_h-\mu_h)\,\gamma(u)\big|\,\mathrm{d}s\\
&\quad\le \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\left\|\beta_n^{1/2} u\right\|_{\partial\mathcal{K}^+} + \left\|\beta_n^{1/2}\gamma(u)\right\|_{\partial\mathcal{K}^-} \right) \left\|\beta_n^{1/2}(v_h-\mu_h)\right\|_{\partial\mathcal{K}}\\
&\quad\le \sqrt{2}\,|||\boldsymbol{u}|||_{s,\star}\,|||(v_h,\mu_h)|||_s,
\end{aligned}
$$

where we used the triangle inequality for the first inequality, the Cauchy–Schwarz inequality for the second inequality, and finally combined the discrete Cauchy–Schwarz inequality with the fact that $(a+b)^2 \le 2(a^2+b^2)$. Collecting the above bounds we obtain, for all $\boldsymbol{u}\in V^\star(h)$ and $(v_h,\mu_h)\in \boldsymbol{V}_h$,

$$
\left|a_h^a(\boldsymbol{u},(v_h,\mu_h))\right| \le c_{B,a}|||\boldsymbol{u}|||_{s,\star}|||(v_h,\mu_h)|||_s, \qquad (6.29)
$$

where $c_{B,a} = 2+\sqrt{2}+\left\|\bar{\beta}\right\|_{L^\infty(\mathcal{E})}\nu^{-1/2}$.

We now shift our focus to the diffusive part of the bilinear form, $a_h^d(\boldsymbol{u},(v_h,\mu_h))$. We note that

$$
\begin{aligned}
|a_h^d(\boldsymbol{u},(v_h,\mu_h))| \le{}& \left| \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nu\overline{\nabla}_h u\cdot\overline{\nabla}_h v_h\,\mathrm{d}x \right| + \left| \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{Q}} \frac{\nu\alpha}{h_K}(u-\gamma(u))\,(v_h-\mu_h)\,\mathrm{d}s \right|\\
&+ \left| \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{Q}} \left[\nu(u-\gamma(u))\overline{\nabla}_h v_h\cdot\bar{n} + \nu\overline{\nabla}_h u\cdot\bar{n}\,(v_h-\mu_h)\right]\mathrm{d}s \right|. \qquad (6.30)
\end{aligned}
$$

By the Cauchy–Schwarz inequality, the first two terms on the right hand side of eq. (6.30) can be bounded by $(1+\alpha)|||\boldsymbol{u}|||_{s,\star}|||(v_h,\mu_h)|||_s$. To bound the remaining term of $a_h^d(\boldsymbol{u},(v_h,\mu_h))$, we note that

$$
\begin{aligned}
&\left| \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{Q}} \left[\nu(u-\gamma(u))\overline{\nabla}_h v_h\cdot\bar{n} + \nu\overline{\nabla}_h u\cdot\bar{n}\,(v_h-\mu_h)\right]\mathrm{d}s \right| \qquad (6.31)\\
&\quad\le \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{Q}} \left|\nu(u-\gamma(u))\overline{\nabla}_h v_h\cdot\bar{n}\right|\mathrm{d}s + \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{Q}} \left|\nu\overline{\nabla}_h u\cdot\bar{n}\,(v_h-\mu_h)\right|\mathrm{d}s.
\end{aligned}
$$

Application of the Cauchy–Schwarz inequality to the first term on the right-hand side of eq. (6.31), followed by the trace inequality eq. (6.15c), yields

$$\begin{aligned}\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\left|\nu(u-\gamma(u))\overline{\nabla}_h v_h\cdot\bar{n}\right|\,\mathrm{d}s \\ &\le c_{T,\mathcal{Q}}\left(\sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\nu}{h_K}\left\|u-\gamma(u)\right\|_{\mathcal{Q}}^2\right)^{\frac{1}{2}}\left(\sum_{\mathcal{K}\in\mathcal{T}_h}\nu\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}}^2\right)^{\frac{1}{2}} \\ &\le c_{T,\mathcal{Q}}\lvert\lvert\lvert\boldsymbol{u}\rvert\rvert\rvert_{s,\star}\lvert\lvert\lvert(v_h,\mu_h)\rvert\rvert\rvert_s.\end{aligned} \tag{6.32}$$

Finally, to bound the second term on the right-hand side of eq. (6.31), we apply the Cauchy–Schwarz inequality:

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\left|\nu\overline{\nabla}_h u\cdot\bar{n}\,(v_h-\mu_h)\right|\,\mathrm{d}s\le\lvert\lvert\lvert\boldsymbol{u}\rvert\rvert\rvert_{s,\star}\lvert\lvert\lvert(v_h,\mu_h)\rvert\rvert\rvert_s. \tag{6.33}$$

Therefore, for all $\boldsymbol{u}\in V^\star(h)$ and $(v_h,\mu_h)\in\boldsymbol{V}_h$,

$$\left|a_h^d(\boldsymbol{u},(v_h,\mu_h))\right|\le c_{B,d}\lvert\lvert\lvert\boldsymbol{u}\rvert\rvert\rvert_{s,\star}\lvert\lvert\lvert(v_h,\mu_h)\rvert\rvert\rvert_s, \tag{6.34}$$

where $c_{B,d}=2+\alpha+c_{T,\mathcal{Q}}$. Combining eq. (6.29) with eq. (6.34) yields the assertion with $c_B=c_{B,a}+c_{B,d}$. □

In the sequel, we will also make use of the following bound valid for all $(u_h,\lambda_h),(v_h,\mu_h)\in\boldsymbol{V}_h$:

$$|a_h^d((u_h,\lambda_h),(v_h,\mu_h))|\le c_d\lvert\lvert\lvert(u_h,\lambda_h)\rvert\rvert\rvert_v\lvert\lvert\lvert(v_h,\mu_h)\rvert\rvert\rvert_v, \tag{6.35}$$

which follows immediately from eq. (6.34) using the equivalence of norms on finite-dimensional spaces. However, to quantify the constant c_d to ensure its independence of h_K and Δt, we proceed as follows: note that

$$\begin{aligned}|a_h^d((u_h,\lambda_h),(v_h,\mu_h))|\le&\left|\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\nu\overline{\nabla}_h u_h\cdot\overline{\nabla}_h v_h\,\mathrm{d}x\right|+\left|\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\frac{\nu\alpha}{h_K}(u_h-\lambda_h)\,(v_h-\mu_h)\,\mathrm{d}s\right| \\ &+\left|\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\left[\nu(u_h-\lambda_h)\overline{\nabla}_h v_h\cdot\bar{n}+\nu\overline{\nabla}_h u\cdot\bar{n}\,(v_h-\mu_h)\right]\mathrm{d}s\right|.\end{aligned} \tag{6.36}$$

By the Cauchy–Schwarz inequality, the first two terms on the right-hand side of eq. (6.36) can be bounded by $(1+\alpha)\|\!|(u_h,\lambda_h)|\!\|_v |\!\|(v_h,\mu_h)|\!\|_v$. To bound the remaining term on the right-hand side of eq. (6.36), we note that

$$\begin{aligned}\Big|\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\Big[\nu(u_h-\lambda_h)\overline{\nabla}_h v_h\cdot\bar{n}+\nu\overline{\nabla}_h u_h\cdot\bar{n}\,(v_h-\mu_h)\Big]\,\mathrm{d}s\Big| \\ \le \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\Big|\nu(u_h-\lambda_h)\overline{\nabla}_h v_h\cdot\bar{n}\Big|\,\mathrm{d}s+\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\Big|\nu\overline{\nabla}_h u_h\cdot\bar{n}\,(v_h-\mu_h)\Big|\,\mathrm{d}s.\end{aligned} \tag{6.37}$$

Application of the Cauchy–Schwarz inequality to the first term on the right-hand side of eq. (6.37), followed by the trace inequality eq. (6.15c), yields

$$\begin{aligned}\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\Big|\nu(u_h-\lambda_h)\overline{\nabla}_h v_h\cdot\bar{n}\Big|\,\mathrm{d}s \\ \le c_{T,\mathcal{Q}}\left(\sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\nu}{h_K}\|u_h-\lambda_h\|_{\mathcal{Q}}^2\right)^{\frac{1}{2}}\left(\sum_{\mathcal{K}\in\mathcal{T}_h}\nu\Big\|\overline{\nabla}_h v_h\Big\|_{\mathcal{K}}^2\right)^{\frac{1}{2}} \\ \le c_{T,\mathcal{Q}}|\!\|(u_h,\lambda_h)|\!\|_v|\!\|(v_h,\mu_h)|\!\|_v.\end{aligned} \tag{6.38}$$

Finally, to bound the second term on the right-hand side of eq. (6.37), we apply the Cauchy–Schwarz inequality followed by the trace inequality eq. (6.15c) to find

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\Big|\nu\overline{\nabla}_h u_h\cdot\bar{n}\,(v_h-\mu_h)\Big|\,\mathrm{d}s\le c_{T,\mathcal{Q}}|\!\|(u_h,\lambda_h)|\!\|_v|\!\|(v_h,\mu_h)|\!\|_v. \tag{6.39}$$

The result follows with $c_d=1+\alpha+2c_{T,\mathcal{Q}}$.

6.3.2 Stability

Next we demonstrate that the method is stable in the norm eq. (6.18) over the space $V_h^\star$:

Lemma 6.3.3 (Stability)**.** *Let α be the penalty parameter appearing in eq. (6.13b) which is such that $\alpha > c_{T,\mathcal{Q}}^2$ where $c_{T,\mathcal{Q}}$ is the constant from the local trace inequality eq. (6.15c). Further, let $c_\alpha=(\alpha-c_{T,\mathcal{Q}}^2)/(1+\alpha)$ and suppose there exists a constant $\beta_0>0$ such that*

$$\frac{c_\alpha\nu}{c_p^2}+\inf_{x\in\mathcal{E}}\overline{\nabla}_h\cdot\bar{\beta}\ge\beta_0>0, \tag{6.40}$$

where c_p is the constant from the discrete Poincaré inequality eq. (6.16). Then there exists a constant c_c, independent of h_K and Δt, such that

$$a_h((v_h,\mu_h),(v_h,\mu_h)) \ge c_c \left\|\!\left|(v_h,\mu_h)\right|\!\right\|_v^2, \qquad \forall (v_h,\mu_h)\in \boldsymbol{V}_h. \tag{6.41}$$

Proof. By definition of the bilinear form $a_h^a(\cdot,\cdot)$ in eq. (6.13a),

$$\begin{aligned} a_h^a((v_h,\mu_h),(v_h,\mu_h)) = {}& \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} v_h^2 \nabla_h\cdot\beta \,\mathrm{d}x - \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}} \frac{1}{2}\beta\cdot n v_h^2 \,\mathrm{d}s \\ &+ \int_{\partial\mathcal{E}_N}\frac{1}{2}\left(\beta\cdot n + |\beta\cdot n|\right)\mu_h^2\,\mathrm{d}s + \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}}\frac{1}{2}\beta\cdot n(v_h+\mu_h)(v_h-\mu_h)\,\mathrm{d}s \\ &+ \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}}\frac{1}{2}|\beta\cdot n|(v_h-\mu_h)^2\,\mathrm{d}s, \end{aligned} \tag{6.42}$$

where we used that $-2v_h\beta\cdot\nabla_h v_h = -\nabla_h\cdot(\beta v_h^2) + v_h^2\nabla_h\cdot\beta$ and applied Gauss' Theorem. Expanding the fourth integral on the right-hand side and using the fact that $\beta\cdot n$ and μ_h are single-valued on element boundaries, and that $\mu_h = 0$ on $\partial\mathcal{E}_D$, eq. (6.42) reduces to

$$\begin{aligned} a_h^a((v_h,\mu_h),(v_h,\mu_h)) = {}& \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}} v_h^2\nabla_h\cdot\beta\,\mathrm{d}x + \int_{\partial\mathcal{E}_N}\frac{1}{2}|\beta\cdot n|\mu_h^2\,\mathrm{d}s \\ &+ \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}}\frac{1}{2}|\beta\cdot n|(v_h-\mu_h)^2\,\mathrm{d}s. \end{aligned} \tag{6.43}$$

Next, by definition of the bilinear form $a_h^d(\cdot,\cdot)$ in eq. (6.13b),

$$\begin{aligned} a_h^d((v_h,\mu_h),(v_h,\mu_h)) = {}& \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\nu\left|\overline{\nabla}_h v_h\right|^2\mathrm{d}x + \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\frac{\nu\alpha}{h_K}(v_h-\mu_h)^2\,\mathrm{d}s \\ &- \sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}} 2\nu\overline{\nabla}_h v_h\cdot\bar{n}\,(v_h-\mu_h)\,\mathrm{d}s. \end{aligned} \tag{6.44}$$

Applying the Cauchy–Schwarz inequality and the trace inequality eq. (6.15c) to the third term on the right-hand side of eq. (6.44),

$$\left|2\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{Q}}\nu\overline{\nabla}_h v_h\cdot\bar{n}(v_h-\mu_h)\,\mathrm{d}s\right| \le 2\nu^{1/2}c_{T,\mathcal{Q}}\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}}\nu^{1/2}h_K^{-1/2}\|v_h-\mu_h\|_{\mathcal{Q}}. \tag{6.45}$$

Combining eq. (6.44) and eq. (6.45), and choosing $\alpha > c_{T,\mathcal{Q}}^2$,

$$\begin{aligned} a_h^d((v_h,\mu_h),(v_h,\mu_h)) \\ &\ge \sum_{\mathcal{K}\in\mathcal{T}_h}\left(\nu\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}}^2 - 2c_{T,\mathcal{Q}}\nu\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}} h_K^{-1/2}\|v_h-\mu_h\|_{\mathcal{Q}} + \frac{\nu\alpha}{h_K}\|v_h-\mu_h\|_{\mathcal{Q}}^2\right) \\ &\ge \sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\alpha - c_{T,\mathcal{Q}}^2}{1+\alpha}\left(\nu\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}}^2 + \frac{\nu}{h_K}\|v_h-\mu_h\|_{\mathcal{Q}}^2\right). \end{aligned} \tag{6.46}$$

The second inequality follows from noting that for $\alpha > \psi^2$, with ψ a positive real number, it holds that $x^2 - 2\psi xy + \alpha y^2 \ge (\alpha - \psi^2)(x^2+y^2)/(1+\alpha)$, for $x, y \in \mathbb{R}$ [28], and taking $x = \nu^{1/2}\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}}$, $y = \nu^{1/2}h_K^{-1/2}\|v_h-\mu_h\|_{\mathcal{Q}}$ and $\psi = c_{T,\mathcal{Q}}$. Combining eq. (6.43) and eq. (6.46), and using that $\nabla_h \cdot \beta = \overline{\nabla}_h \cdot \bar{\beta}$,

$$\begin{aligned} a_h((v_h,\mu_h),(v_h,\mu_h)) \ge \sum_{\mathcal{K}\in\mathcal{T}_h}\frac{1}{2}\int_{\mathcal{K}} v_h^2\overline{\nabla}_h\cdot\bar{\beta}\,\mathrm{d}x + \frac{1}{2}\left\|\beta_n^{1/2}\mu_h\right\|_{\partial\mathcal{E}_N}^2 \\ + \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\left\|\beta_n^{1/2}(v_h-\mu_h)\right\|_{\partial\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h}c_\alpha\nu\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h}c_\alpha\frac{\nu}{h_K}\|v_h-\mu_h\|_{\mathcal{Q}}^2. \end{aligned} \tag{6.47}$$

Using the discrete Poincaré inequality eq. (6.16) and eq. (6.40), we obtain from eq. (6.47):

$$\begin{aligned} a_h((v_h,\mu_h),(v_h,\mu_h)) \ge \frac{1}{2}\beta_0\|v_h\|_{\mathcal{E}}^2 + \frac{1}{2}\left\|\beta_n^{1/2}\mu_h\right\|_{\partial\mathcal{E}_N}^2 \\ + \frac{1}{2}\sum_{\mathcal{K}\in\mathcal{T}_h}\left\|\beta_n^{1/2}(v_h-\mu_h)\right\|_{\partial\mathcal{K}}^2 + \frac{1}{2}c_\alpha\sum_{\mathcal{K}\in\mathcal{T}_h}\nu\left\|\overline{\nabla}_h v_h\right\|_{\mathcal{K}}^2 + \frac{1}{2}c_\alpha\sum_{\mathcal{K}\in\mathcal{T}_h}\frac{\nu}{h_K}\|v_h-\mu_h\|_{\mathcal{Q}}^2. \end{aligned} \tag{6.48}$$

The result follows with $c_c = \min(\beta_0, c_\alpha)/2$. □

6.3.3 The inf-sup condition

Stability was proven in Section 6.3.2 with respect to the norm $\left|\!\left|\!\left|(\cdot,\cdot)\right|\!\right|\!\right|_v$. To obtain the error estimates in Section 6.4, we instead consider a norm with additional control over the time derivatives of the solution. For this we prove an inf-sup condition with respect to the stronger norm Equation (6.19) following ideas in [13, 28, 104]. We first state the inf-sup condition.

Theorem 6.3.1 (The inf-sup condition)**.** *There exists $c_i > 0$, independent of h_K and Δt, such that for all $(w_h, \lambda_h) \in \boldsymbol{V}_h$*

$$c_i \left|\!\left|\!\left| (w_h, \lambda_h) \right|\!\right|\!\right|_s \le \sup_{(v_h,\mu_h)\in \boldsymbol{V}_h} \frac{a_h((w_h, \lambda_h), (v_h, \mu_h))}{\left|\!\left|\!\left| (v_h, \mu_h) \right|\!\right|\!\right|_s} . \tag{6.49}$$

The proof of the inf-sup condition follows after the following two intermediate results.

Lemma 6.3.4. *Let $(w_h, \lambda_h) \in \boldsymbol{V}_h$ and let $z_h = \frac{\Delta t h_K^2}{\Delta t + h_K} \partial_t w_h$. There exists a $c_1 > 0$, independent of h_K and Δt, such that*

$$\left|\!\left|\!\left| (z_h, 0) \right|\!\right|\!\right|_s \le c_1 \left|\!\left|\!\left| (w_h, \lambda_h) \right|\!\right|\!\right|_s .$$

Proof. We bound each component of $\left|\!\left|\!\left| (z_h, 0) \right|\!\right|\!\right|_s$ term-by-term. Using the inverse inequality eq. (6.15a) and that $h_K < 1$, we have

$$\|z_h\|_{\mathcal{E}}^2 = \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\frac{\Delta t h_K^2}{\Delta t + h_K} \right)^2 \|\partial_t w_h\|_{\mathcal{K}}^2 \le c_{I,t}^2 \|w_h\|_{\mathcal{E}}^2 .$$

Similarly, the inverse inequality eq. (6.15a) and $h_K < 1$ yields

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \nu \left\| \overline{\nabla}_h z_h \right\|_{\mathcal{K}}^2 = \sum_{\mathcal{K}\in\mathcal{T}_h} \nu \left(\frac{\Delta t h_K^2}{\Delta t + h_K} \right)^2 \left\| \partial_t (\overline{\nabla}_h w_h) \right\|_{\mathcal{K}}^2 \le c_{I,t}^2 \sum_{\mathcal{K}\in\mathcal{T}_h} \nu \left\| \overline{\nabla}_h w_h \right\|_{\mathcal{K}}^2 .$$

Next, the facet term arising from the advective portion of the norm may be bounded using the trace inequality eq. (6.15d):

$$\begin{aligned} \sum_{\mathcal{K}\in\mathcal{T}_h} \left\| \beta_n^{1/2} z_h \right\|_{\partial\mathcal{K}}^2 &\le \|\beta\|_{L^\infty(\mathcal{E})} \sum_{\mathcal{K}\in\mathcal{T}_h} \|z_h\|_{\partial\mathcal{K}}^2 \\ &\le c_{T,\partial\mathcal{K}}^2 \|\beta\|_{L^\infty(\mathcal{E})} \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\frac{\Delta t h_K^2}{\Delta t + h_K} \right) \|\partial_t w_h\|_{\mathcal{K}}^2 . \end{aligned}$$

The facet term diffusive portion of the norm may be bounded with an application of eq. (6.15c) and eq. (6.15a):

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\nu}{h_K} \|z_h\|_{\mathcal{Q}}^2 = \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\nu}{h_K} \left(\frac{\Delta t h_K^2}{\Delta t + h_K} \right)^2 \|\partial_t w_h\|_{\mathcal{Q}}^2 \le c_{T,\mathcal{Q}}^2 c_{I,t}^2 \sum_{\mathcal{K}\in\mathcal{T}_h} \nu \|w_h\|_{\mathcal{K}}^2 .$$

For the remaining term, eq. (6.15a) yields

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \|\partial_t z_h\|_{\mathcal{K}}^2 \leq c_{I,t}^2 \sum_{\mathcal{K}\in\mathcal{T}_h} \left(\frac{\Delta t h_K^2}{\Delta t + h_K} \|\partial_t w_h\|_{\mathcal{K}}^2 \right).$$

Collecting the above bounds, we obtain lemma 6.3.4, with $c_1 = 3c_{I,t}^2 + c_{T,\partial\mathcal{K}}^2 \|\beta\|_{L^\infty(\mathcal{E})} + c_{T,\mathcal{Q}}^2 c_{I,t}^2$. □

Lemma 6.3.5. *Let $(w_h, \lambda_h) \in \boldsymbol{V}_h$ and let $z_h = \frac{\Delta t h_K^2}{\Delta t + h_K}\partial_t w_h$. There exists a $c_2 > 0$, independent of h_K and Δt, such that if $(v_h, \mu_h) = c_2(w_h, \lambda_h) + (z_h, 0) \in \boldsymbol{V}_h$, then*

$$\frac{1}{2}\left|\!\left|\!\left|(w_h, \lambda_h)\right|\!\right|\!\right|_s^2 \leq a_h((w_h, \lambda_h), (v_h, \mu_h)). \tag{6.50}$$

Proof. Note that $a_h((w_h, \lambda_h), (z_h, 0)) = a_h^a((w_h, \lambda_h), (z_h, 0)) + a_h^d((w_h, \lambda_h), (z_h, 0))$. Integrating by parts the volume integral of $a_h^a(\cdot, \cdot)$ we have the following decomposition:

$$\begin{aligned}
\sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \|\partial_t w_h\|_{\mathcal{K}}^2 &= a_h((w_h, \lambda_h), (z_h, 0)) - a_h^d((w_h, \lambda_h), (z_h, 0)) \\
&- \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \int_{\mathcal{K}} w_h \overline{\nabla}_h \cdot \bar{\beta} \partial_t w_h \,\mathrm{d}x - \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \int_{\mathcal{K}} \bar{\beta} \cdot \overline{\nabla}_h w_h \partial_t w_h \,\mathrm{d}x \\
&+ \frac{1}{2} \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \int_{\partial\mathcal{K}} \left(\beta \cdot n - |\beta \cdot n|\right) (w_h - \lambda_h)\, \partial_t w_h \,\mathrm{d}s.
\end{aligned} \tag{6.51}$$

From the boundedness of the diffusive part of the bilinear form eq. (6.35), and application of Young's inequality, with $\epsilon_1 > 0$, we obtain the following bound for the second term on the right-hand side of eq. (6.51):

$$\begin{aligned}
|a_h^d((w_h, \lambda_h), (z_h, 0))| &\leq \frac{c_d}{2\epsilon_1} \left|\!\left|\!\left|(z_h, 0)\right|\!\right|\!\right|_v^2 + \frac{c_d \epsilon_1}{2} \left|\!\left|\!\left|(w_h, \lambda_h)\right|\!\right|\!\right|_v^2 \\
&\leq \frac{c_d c_1^2}{2\epsilon_1} \left|\!\left|\!\left|(w_h, \lambda_h)\right|\!\right|\!\right|_s^2 + \frac{c_d \epsilon_1}{2} \left|\!\left|\!\left|(w_h, \lambda_h)\right|\!\right|\!\right|_v^2 \\
&\leq \frac{c_d c_1^2}{2\epsilon_1} \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \|\partial_t w_h\|_{\mathcal{K}}^2 + \left(\frac{c_d c_1^2}{2\epsilon_1} + \frac{c_d \epsilon_1}{2} \right) \left|\!\left|\!\left|(w_h, \lambda_h)\right|\!\right|\!\right|_v^2,
\end{aligned}$$

where we have used the fact that $\left|\!\left|\!\left|\cdot\right|\!\right|\!\right|_v \leq \left|\!\left|\!\left|\cdot\right|\!\right|\!\right|_s$ and applied Lemma 6.3.4 in the second inequality, and the definition of $\left|\!\left|\!\left|\cdot\right|\!\right|\!\right|_s$ in the third inequality. Next, to bound the third term on the right-hand side of eq. (6.51) we apply the Cauchy–Schwarz inequality and eq. (6.15a) to obtain

$$\left| \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \int_{\mathcal{K}} w_h \overline{\nabla}_h \cdot \bar{\beta} \partial_t w_h \,\mathrm{d}x \right| \le c_{I,t} \left\| \overline{\nabla}_h \cdot \bar{\beta} \right\|_{L^\infty(\mathcal{E})} \sum_{\mathcal{K}\in\mathcal{T}_h} h_K \|w_h\|_{\mathcal{K}}^2 .$$

As for the fourth term on the right-hand side of eq. (6.51), we first apply the Cauchy–Schwarz inequality, Young's inequality with some $\epsilon_2 > 0$ and eq. (6.15b),

$$\begin{aligned} \left| \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \int_{\mathcal{K}} \bar{\beta} \cdot \overline{\nabla}_h w_h \partial_t w_h \,\mathrm{d}x \right| &\le \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \|\bar{\beta}\|_{L^\infty(\mathcal{K})} \|\partial_t w_h\|_{\mathcal{K}} \left\| \overline{\nabla}_h w_h \right\|_{\mathcal{K}} \\ &\le \|\bar{\beta}\|_{L^\infty(\mathcal{E})} \left[\sum_{\mathcal{K}\in\mathcal{T}_h} \frac{c_{I,s}^2}{2\epsilon_2} \|w_h\|_{\mathcal{K}}^2 + \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\epsilon_2}{2} \left(\frac{\Delta t h_K^2}{\Delta t + h_K} \right) \|\partial_t w_h\|_{\mathcal{K}}^2 \right] . \end{aligned}$$

For the remaining term on the right-hand side of eq. (6.51), we use the Cauchy–Schwarz inequality, Young's inequality with some $\epsilon_3 > 0$, and apply the trace inequality eq. (6.15d) to find:

$$\begin{aligned} &\left| \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \int_{\partial\mathcal{K}} \frac{1}{2} \left(\beta \cdot n - |\beta \cdot n| \right) \partial_t w_h \left(w_h - \lambda_h \right) \mathrm{d}s \right| \\ &\qquad \le \frac{c_{T,\partial\mathcal{K}}^2}{2\epsilon_3} \sum_{\mathcal{K}\in\mathcal{T}_h} h_K \frac{\Delta t h_K^2}{\Delta t + h_K} \|\partial_t w_h\|_{\mathcal{K}}^2 + \frac{\epsilon_3}{2} \|\beta\|_{L^\infty(\mathcal{E})} \sum_{\mathcal{K}\in\mathcal{T}_h} \left\| \beta_n^{1/2} (w_h - \lambda_h) \right\|_{\partial\mathcal{K}}^2 . \end{aligned} \tag{6.52}$$

Combining all of the above estimates,

$$\begin{aligned} \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \|\partial_t w_h\|_{\mathcal{K}}^2 &\le a_h((w_h, \lambda_h), (z_h, 0)) \\ &\quad + \left(\frac{c_d c_1^2}{2\epsilon_1} + \frac{\epsilon_2}{2} \|\bar{\beta}\|_{L^\infty(\mathcal{E})} + \frac{c_{T,\partial\mathcal{K}}^2}{2\epsilon_3} \right) \sum_{\mathcal{K}\in\mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \|\partial_t w_h\|_{\mathcal{K}}^2 \\ &\quad + \left(\frac{c_d c_1^2}{2\epsilon_1} + \frac{c_d \epsilon_1}{2} + c_{I,t} \left\| \overline{\nabla}_h \cdot \bar{\beta} \right\|_{L^\infty(\mathcal{E})} + \frac{c_{I,s}^2}{2\epsilon_2} \|\bar{\beta}\|_{L^\infty(\mathcal{E})} \right. \\ &\qquad \left. + \frac{\epsilon_3}{2} \|\beta\|_{L^\infty(\mathcal{E})} \right) |||(w_h, \lambda_h)|||_v^2 . \end{aligned} \tag{6.53}$$

Choosing $\epsilon_1 = 2c_d c_1^2$, $\epsilon_2 = 1/(4\|\bar{\beta}\|_{L^\infty(\mathcal{E})})$, and $\epsilon_3 = 4c_{T,\partial\mathcal{K}}^2$, adding $\frac{1}{2}\|\!|(w_h, \lambda_h)|\!\|_v^2$ to both sides, and rearranging yields

$$\frac{1}{2}\|\!|(w_h, \lambda_h)|\!\|_s^2 \le a_h((w_h, \lambda_h), (z_h, 0)) \\ + \left(\frac{3}{4} + c_d^2 c_1^2 + c_{I,t}\left\|\overline{\nabla}_h \cdot \bar{\beta}\right\|_{L^\infty(\mathcal{E})} + 2c_{I,s}^2\|\bar{\beta}\|_{L^\infty(\mathcal{E})}^2 + 2c_{T,\partial\mathcal{K}}^2\|\beta\|_{L^\infty(\mathcal{E})}\right)\|\!|(w_h, \lambda_h)|\!\|_v^2. \quad (6.54)$$

From the stability of $a_h(\cdot, \cdot)$, Lemma 6.3.3, we have the bound

$$\frac{1}{2}\|\!|(w_h, \lambda_h)|\!\|_s^2 \le a_h((w_h, \lambda_h), (z_h, 0)) + c_2 a_h((w_h, \lambda_h), (w_h, \lambda_h)) \quad (6.55)$$

where $c_2 = c_c^{-1}(\frac{3}{4} + c_d^2 c_1^2 + c_{I,t}\left\|\overline{\nabla}_h \cdot \bar{\beta}\right\|_{L^\infty(\mathcal{E})} + 2c_{I,s}^2\|\bar{\beta}\|_{L^\infty(\mathcal{E})}^2 + 2c_{T,\partial\mathcal{K}}^2\|\beta\|_{L^\infty(\mathcal{E})})$. The result follows. □

Combining Lemma 6.3.4 and Lemma 6.3.5 now yields the proof for the inf-sup condition stated in Theorem 6.3.1.

Proof of Theorem 6.3.1. Given any $(w_h, \lambda_h) \in \boldsymbol{V}_h$, consider the linear combination $(v_h, \mu_h) = c_2(w_h, \lambda_h) + (z_h, 0)$, with $z_h = \frac{\Delta t h_K^2}{\Delta t + h_K}\partial_t w_h$ and c_2 the constant from Lemma 6.3.5. An application of the triangle inequality and the combination of Lemma 6.3.4 and Lemma 6.3.5 yields

$$\begin{aligned} \|\!|(v_h, \mu_h)|\!\|_s \|\!|(w_h, \lambda_h)|\!\|_s &\le \|\!|(z_h, 0)|\!\|_s \|\!|(w_h, \lambda_h)|\!\|_s + c_2\|\!|(w_h, \lambda_h)|\!\|_s^2 \\ &\le (c_1 + c_2)\|\!|(w_h, \lambda_h)|\!\|_s^2 \\ &\le 2(c_1 + c_2)a_h((w_h, \lambda_h), (v_h, \mu_h)), \end{aligned}$$

which implies the inf-sup condition with $c_i = \frac{1}{2}(c_1 + c_2)^{-1}$. □

6.4 Error analysis

We now turn to the error analysis of the space-time HDG method. The following Céa-like lemma will prove useful in obtaining the global error estimate in Theorem 6.4.1.

Lemma 6.4.1 (Convergence)**.** *If* $\boldsymbol{u} = (u, \gamma(u)) \in H^2(\mathcal{E}) \times H^{3/2}(\Gamma)$, *where* u *solves eq.* (6.1), *and* $(u_h, \lambda_h) \in \boldsymbol{V}_h$ *is the solution to the discrete problem eq.* (6.14), *then*

$$\|\!|\boldsymbol{u} - (u_h, \lambda_h)|\!\|_s \le \left(1 + \frac{c_B}{c_i}\right) \inf_{(v_h, \mu_h) \in \boldsymbol{V}_h} \|\!|\boldsymbol{u} - (v_h, \mu_h)|\!\|_{s,\star}. \quad (6.56)$$

Proof. From inf-sup stability (Theorem 6.3.1), Galerkin orthogonality eq. (6.22), and boundedness (Lemma 6.3.2), we have for any $(w_h, \omega_h) \in \boldsymbol{V}_h$

$$
\begin{aligned}
c_i \left|\!\left|\!\left| (u_h, \lambda_h) - (w_h, \omega_h) \right|\!\right|\!\right|_s &\le \sup_{(v_h,\mu_h)\in \boldsymbol{V}_h} \frac{a_h((u_h, \lambda_h) - (w_h, \omega_h), (v_h, \mu_h))}{\left|\!\left|\!\left| (v_h, \mu_h) \right|\!\right|\!\right|_s} \\
&= \sup_{(v_h,\mu_h)\in \boldsymbol{V}_h} \frac{a_h(\boldsymbol{u} - (w_h, \omega_h), (v_h, \mu_h))}{\left|\!\left|\!\left| (v_h, \mu_h) \right|\!\right|\!\right|_s} \\
&\le c_B \sup_{(v_h,\mu_h)\in \boldsymbol{V}_h} \frac{\left|\!\left|\!\left| \boldsymbol{u} - (w_h, \omega_h) \right|\!\right|\!\right|_{s,\star} \left|\!\left|\!\left| (v_h, \mu_h) \right|\!\right|\!\right|_s}{\left|\!\left|\!\left| (v_h, \mu_h) \right|\!\right|\!\right|_s} \\
&= c_B \left|\!\left|\!\left| \boldsymbol{u} - (w_h, \omega_h) \right|\!\right|\!\right|_{s,\star}.
\end{aligned}
$$

The result follows after application of the triangle inequality to $\left|\!\left|\!\left| \boldsymbol{u} - (u_h, \lambda_h) \right|\!\right|\!\right|_s$. □

We next define the projections $\mathcal{P} : L^2(\mathcal{E}) \to V_h^{(p_t,p_s)}$ and $\mathcal{P}^\partial : L^2(\Gamma) \to M_h^{(p_t,p_s)}$ which satisfy

$$
\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} (w - \mathcal{P}w)\, v_h \,\mathrm{d}x = 0, \qquad \forall\, v_h \in V_h^{(p_t,p_s)}, \tag{6.57}
$$

$$
\sum_{\mathcal{S}\in\mathcal{F}} \int_{\mathcal{S}} \left(\lambda - \mathcal{P}^\partial \lambda\right) \mu_h \,\mathrm{d}s = 0, \qquad \forall\, \mu_h \in M_h^{(p_t,p_s)}. \tag{6.58}
$$

These projections will be used to obtain interpolation estimates.

Lemma 6.4.2 (Interpolation estimates)**.** *Assume that $\mathcal{K}$ is a space-time element in $\mathbb{R}^{d+1}$ constructed via two mappings $\phi_{\mathcal{K}}$ and $F_{\mathcal{K}}$, with $F_{\mathcal{K}} : \widehat{\mathcal{K}} \to \widetilde{\mathcal{K}}$ and $\phi_{\mathcal{K}} : \widetilde{\mathcal{K}} \to \mathcal{K}$. Assume that the spatial shape-regularity condition eq. (6.10) holds. Suppose $u|_{\mathcal{K}} \in H^{(p_t+1,p_s+1)}(\mathcal{K})$ solves eq. (6.2)–eq. (6.3). Then, the error $u - \mathcal{P}u$, its trace at the boundary $\partial\mathcal{K}$, and the* *error $u - \mathcal{P}^{\partial} u$ on $\partial\mathcal{K}$ satisfy the following error bounds:*

$$\|u - \mathcal{P}u\|_{\mathcal{K}}^2 \leq c\left(h_K^{2p_s+2} + \Delta t^{2p_t+2}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}, \tag{6.59}$$

$$\left\|\overline{\nabla}_h(u - \mathcal{P}u)\right\|_{\mathcal{K}}^2 \leq c\left(h_K^{2p_s} + \Delta t^{2p_t+2}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}, \tag{6.60}$$

$$\left\|\partial_t(u - \mathcal{P}u)\right\|_{\mathcal{K}}^2 \leq c\left(h_K^{2p_s} + \Delta t^{2p_t}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}, \tag{6.61}$$

$$\left\|\overline{\nabla}_h(u - \mathcal{P}u)\cdot \bar{n}\right\|_{\mathcal{Q}}^2 \leq c\left(h_K^{2p_s-1} + h_K^{-1}\Delta t^{2p_t+2}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}, \tag{6.62}$$

$$\|u - \mathcal{P}u\|_{\partial\mathcal{K}}^2 \leq c\left(h_K^{2p_s+1} + \Delta t^{2p_t+1}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}, \tag{6.63}$$

$$\left\|u - \mathcal{P}^{\partial}\gamma(u)\right\|_{\partial\mathcal{K}} \leq c\left(h_K^{2p_s+1} + \Delta t^{2p_t+1}\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}, \tag{6.64}$$

where c depends only on the spatial dimension d, the polynomial degrees p_t and p_s, the spatial shape-regularity constant c_r, and the Jacobian of the mapping $\phi_{\mathcal{K}}$.

Proof. The bounds eq. (6.59), eq. (6.60) and eq. (6.63) have been obtained previously in [95, Lemma 6.1 and Remark 6.2] by generalizing [38, Lemmas 3.13 and 3.17] to higher dimensions. We relax the assumption in [95, Remark 6.2] that all spatial edge lengths are equal through the spatial shape-regularity assumption eq. (6.10). In doing so, the bound eq. (6.61) may be obtained in an identical fashion to eq. (6.60). The bound eq. (6.62) is obtained as follows: we derive a bound for the spatial derivative of the interpolation error over each face $\partial\mathcal{K}_i$, where $i = 1, \ldots, d$, generalizing [38, Lemma 3.20] to the space-time setting. Then, summing over the faces $i = 1, \ldots, d$ we obtain a bound of the spatial derivatives of the interpolation error over $\mathcal{Q} = \partial\mathcal{K}\backslash(K^n \cup K^{n+1})$, and sum over all of the spatial derivatives to obtain the result. Lastly, the bound eq. (6.64) may be inferred from the bound eq. (6.63) by the optimality of the L^2-projection $\mathcal{P}^{\partial}$ on facets. □

With the interpolation estimates in place, we can now derive an error bound in the $|||\cdot|||_s$ norm:

Theorem 6.4.1 (Global error estimate)**.** *Suppose that $\mathcal{K}$ is a space-time element in $\mathbb{R}^{d+1}$ constructed via two mappings $\phi_{\mathcal{K}}$ and $F_{\mathcal{K}}$, with $F_{\mathcal{K}} : \widehat{\mathcal{K}} \to \widetilde{\mathcal{K}}$ and $\phi_{\mathcal{K}} : \widetilde{\mathcal{K}} \to \mathcal{K}$, and that the spatial shape-regularity condition eq. (6.10) holds. Let $\boldsymbol{u} = (u, \gamma(u))$, where $u|_{\mathcal{K}} \in H^{(p_t+1,p_s+1)}(\mathcal{K})$ solves the advection–diffusion problem eq. (6.2), and where $\gamma(u)$ denotes the trace of u on $\partial\mathcal{K}$. Furthermore, let $(u_h, \lambda_h) \in \boldsymbol{V}_h$ be the solution to the discrete problem eq. (6.14). Then, the following error bound holds:*

$$|||\boldsymbol{u} - (u_h, \lambda_h)|||_s^2 \;\leq\; C\left(h^{2p_s} + \Delta t^{2p_t+1} + \nu\left(h^{2p_s} + h^{-1}\Delta t^{2p_t+1}\right)\right)\|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{E})}, \tag{6.65}$$

where $h = max_{\mathcal{K}\in\mathcal{T}_h} h_K$ is the spatial mesh size Δt is the time-step and $C > 0$ a constant.

Proof. By Lemma 6.4.1, we may bound the discretization error $\boldsymbol{u} - (u_h, \lambda_h)$ in the $|||\cdot|||_s$ norm by the interpolation error $\boldsymbol{u} - (\mathcal{P}u, \mathcal{P}^\partial\gamma(u))$ in the $|||\cdot|||_{s,\star}$ norm:

$$\left|\left|\left|\boldsymbol{u} - (u_h, \lambda_h)\right|\right|\right|_s \leq \left(1 + \frac{c_B}{c_i}\right) \left|\left|\left|\boldsymbol{u} - (\mathcal{P}u, \mathcal{P}^\partial\gamma(u))\right|\right|\right|_{s,\star}. \tag{6.66}$$

Thus, it suffices to bound each term of $\left|\left|\left|\boldsymbol{u} - (\mathcal{P}u, \mathcal{P}^\partial\gamma(u))\right|\right|\right|_{s,\star}$ using the interpolation estimates in Lemma 6.4.2.

First, combining the terms involving $\|u - \mathcal{P}u\|_{\mathcal{K}}$, applying eq. (6.59), and collecting the leading order terms,

$$\left(1 + \frac{\Delta t + h_K}{\Delta t h_K^2}\right) \|u - \mathcal{P}u\|_{\mathcal{K}}^2 \leq c\left(h_K^{2p_s} + \Delta t^{2p_t+2} h_K^{-2}\right) \|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}. \tag{6.67}$$

Using the fact that $\frac{\Delta t h_K^2}{\Delta t + h_K} \leq \Delta t h_K$ and applying the estimate eq. (6.61), we have

$$\frac{\Delta t h_K^2}{\Delta t + h_K} \left\|\partial_t(u - \mathcal{P}u)\right\|_{\mathcal{K}}^2 \leq c\left(h_K^{2p_s+1}\Delta t + h_K \Delta t^{2p_t+1}\right) \|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}. \tag{6.68}$$

Next, an application of eq. (6.60) yields

$$\nu \left\|\overline{\nabla}_h(u - \mathcal{P}u)\right\|_{\mathcal{K}}^2 \leq c\nu\left(h_K^{2p_s} + \Delta t^{2p_t+2}\right) \|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}. \tag{6.69}$$

Using the triangle inequality, eq. (6.63), and eq. (6.64), all of the advective facet terms may be bounded as follows:

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \left(\left\|\beta_n^{1/2}(u - \mathcal{P}u)\right\|_{\partial\mathcal{K}}^2 + \left\|\beta_n^{1/2}(u - \mathcal{P}^\partial u)\right\|_{\partial\mathcal{K}}^2\right) \leq$$
$$c\|\beta\|_{L^\infty(\mathcal{E})} \sum_{\mathcal{K}\in\mathcal{T}_h} \left(h_K^{2p_s+1} + \Delta t^{2p_t+1}\right) \|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}. \tag{6.70}$$

For the diffusive facet term, we again apply the triangle inequality, eq. (6.63), and eq. (6.64) to obtain

$$\frac{\nu}{h_K}\|u - \mathcal{P}u\|_{\partial\mathcal{K}}^2 + \frac{\nu}{h_K}\left\|u - \mathcal{P}^\partial u\right\|_{\partial\mathcal{K}}^2 \leq c\nu\left(h_K^{2p_s} + h_K^{-1}\Delta t^{2p_t+1}\right) \|u\|^2_{H^{(p_t+1,p_s+1)}(\mathcal{K})}. \tag{6.71}$$

Lastly, applying eq. (6.62),

$$h_K \nu \left\| \overline{\nabla}_h (u - \mathcal{P}u) \cdot \bar{n} \right\|_{\mathcal{Q}}^2 \leq c\nu \left(h_K^{2p_s} + \Delta t^{2p_t+2} \right) \|u\|_{H^{(p_t+1,p_s+1)}(\mathcal{K})}^2 . \qquad (6.72)$$

Summing over all $\mathcal{K} \in \mathcal{T}_h$, collecting all of the above estimates, and returning to eq. (6.66) yields the assertion. □

6.5 Numerical example

In this section we validate the results of the previous sections. For this we consider the rotating Gaussian pulse test case on a time-dependent domain as introduced in [75, Section 4.3]. We solve eq. (6.2)–eq. (6.3) with $\bar{\beta} = (-4x_2, 4x_1)^T$ and $f = 0$. The boundary and initial conditions are set such that the exact solution is given by

$$u(t, x_1, x_2) = \frac{\sigma^2}{\sigma^2 + 2\nu t} \exp\left(-\frac{(\tilde{x}_1 - x_{1c})^2 + (\tilde{x}_2 - x_{2c})^2}{2\sigma^2 + 4\nu t} \right), \qquad (6.73)$$

where $\tilde{x}_1 = x_1 \cos(4t) + x_2 \sin(4t)$, $\tilde{x}_2 = -x_1 \sin(4t) + x_2 \cos(4t)$, $(x_{1c}, x_{2c}) = (-0.2, 0.1)$. Furthermore, we set $\sigma = 0.1$. The advection–diffusion problem is solved on a time-dependent domain. The deformation is based on a transformation of a uniform space-time mesh $(t, x_1^0, x_2^0) \in [0, t_N] \times [-0.5, 0.5]^2$ given by

$$x_i = x_i^0 + A\left(\tfrac{1}{2} - x_i^0\right) \sin\left(2\pi\left(\tfrac{1}{2} - x_i^* + t\right)\right) \qquad i = 1, 2, \qquad (6.74)$$

where and $(x_1^*, x_2^*) = (x_2, x_1)$ and $A = 0.1$. We take $t_N = 1$.

This example was implemented using the Modular Finite Element Methods (MFEM) library [2, 67] on unstructured hexahedral space-time meshes. The solution on the time-dependent domain is shown at different points in time in Figure 6.3. In Table 6.1 we compute the rates of convergence in the $|||(\cdot,\cdot)|||_s$ norm using polynomial degree $p = p_t = p_s = 1, 2, 3$. We consider both $\nu = 10^{-2}$ and $\nu = 10^{-6}$. Mesh refinement is done simultaneously in space and time. For the case that $\nu = 10^{-2}$ we obtain rates of convergence of approximately p, as expected from Theorem 6.4.1, while for $\nu = 10^{-6}$ we obtain slightly better rates of convergence, namely $p + 1/2$.

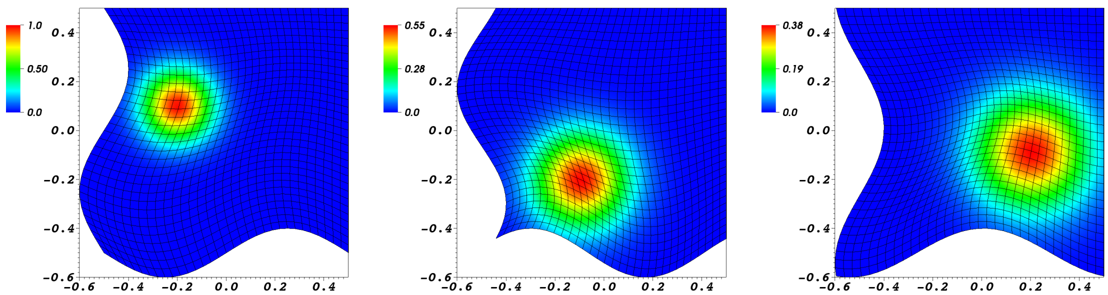

Figure 6.3: The mesh and solution for $\nu = 10^{-2}$ at time levels $t = 0, 0.4, 0.8$ (left to right).

Cells per slab	Nr. of slabs	$p = 1$	rates	$p = 2$	rates	$p = 3$	rates
64	8	8.00e-2	-	1.52e-2	-	2.87e-3	-
256	16	3.15e-2	1.3	3.24e-3	2.2	2.92e-4	3.3
1024	32	1.30e-2	1.3	7.03e-4	2.2	3.21e-5	3.2
4096	64	5.95e-3	1.1	1.64e-4	2.1	3.80e-6	3.1
64	8	1.75e-1	-	3.71e-2	-	6.67e-3	-
256	16	7.78e-2	1.2	6.23e-3	2.6	5.60e-4	3.6
1024	32	2.51e-2	1.6	1.03e-3	2.6	4.64e-5	3.6
4096	64	7.60e-3	1.7	1.76e-4	2.5	3.88e-6	3.6

Table 6.1: Rates of convergence when solving the advection–diffusion problem eq. (6.2)–eq. (6.3) on a time-dependent domain with mesh deformation satisfying eq. (6.74) with $\nu = 10^{-2}$ (top) and $\nu = 10^{-6}$ (bottom).

Chapter 7

Conclusions

In this thesis, we presented our work on the theoretical analysis of space-time HDG schemes for incompressible flow problems. Throughout, we have extended many of the technical tools used in the analysis of finite element methods to the HDG setting in a novel way. In this final chapter, we briefly summarize what we have achieved and discuss possible avenues of future research based on this work.

7.1 Summary

In Chapter 3, we presented a theoretical analysis of the HDG method of Rhebergen and Wells [78] for the steady Navier–Stokes equations. We showed that there exists a unique solution to the nonlinear algebraic system that arises from the HDG discretization under a small data condition. Furthermore, we proved optimal a priori error estimates for both the velocity and pressure. In particular, the error in the velocity is shown to be independent of the error in the pressure, confirming theoretically what was observed in the numerical experiments in [78].

In Chapter 4, we analyzed an exactly mass conserving space-time HDG method for the time-dependent incompressible Navier-Stokes equations on fixed domains. We proved that the method is energy stable, that there exists a solution to the nonlinear algebraic system arising from the discretization in both two and three spatial dimensions, and that this solution is unique in two spatial dimensions under a small data condition. We then derived optimal a priori error estimates for the velocity which are independent of the pressure, proving that the method is indeed pressure-robust. This required us to consider *strong* *solutions* of the Navier–Stokes equations, which placed a restriction on the shape of the spatial domain and the size of the problem data. Finally, we derived (sub-optimal) error estimates for the pressure, and presented a numerical experiment to verify the theoretically predicted rates of convergence.

In Chapter 5, we filled the gap left by the analysis of the previous chapter by proving that the space-time HDG method converges to a weak solution of the Navier–Stokes equations in the absence of additional regularity as the mesh size and time step tend to zero. This removes the restrictions placed on the shape of the spatial domain and problem data in Chapter 4. Our analysis required the introduction of *discrete differential operators*, which we extended to the HDG setting. Moreover, a discrete compactness result reminiscent of the Aubin–Lions–Simon theorem is proven in order to pass to the limit in the nonlinear convection term. Finally, we showed that this weak solution satisfies a suitable energy inequality in three spatial dimensions, and thus is a solution in the sense of Leray–Hopf.

Finally, in Chapter 6, we provide a first step toward the analysis of space-time HDG schemes for the incompressible Navier–Stokes equations on time-dependent domains by considering the simpler linear advection-diffusion equation. Our analysis made extensive use of anisotropic interpolant estimates and novel anisotropic inverse and trace inequalities. We proved that the resulting algebraic system from the space-time HDG discretization is well-posed, and we derived error estimates that are anisotropic in the time step and spatial mesh size which were then verified with a numerical experiment.

7.2 Future directions

The first and most immediate recommendation for future work is to remedy the sub-optimal error estimates for the pressure obtained in Chapter 4. The source of the problem seems to be that the time derivative of the error in the velocity is estimated in the L^2-norm, which introduces a negative power of the time-step through the use of an inverse inequality. A more appropriate approach would be to estimate the time derivative of the error in a discrete analogue of a negative Sobolev norm. This would be done using techniques similar to those in Chapter 5 when bounding the discrete time derivative in the dual space V'. Continuing on this point, it would also be interesting to see if optimal rates of convergence for the velocity can be obtained in the $L^2(0,T;L^2(\Omega)^d)$-norm. A possible approach would involve the consideration of a backward-in-time linearized dual problem satisfying suitable parabolic regularity properties, analogous to the technique used to prove the L^2-estimates for the velocity in Chapter 3 in the elliptic setting.

Next, as mentioned in the thesis introduction, our primary motivation for the use of the space-time HDG method is for the numerical solution of the Navier–Stokes equations on time dependent domains. Since the space-time HDG method introduced in [43, 44] uses space-time tetrahedral elements, the techniques that we have presented in this thesis are not applicable. Instead, we propose the use of mapped *space-time prismatic* elements, so that the tensor-product structure can be exploited to combine the techniques used in Chapter 4 with the analysis framework used in Chapter 6 for time-dependent domains. This would require the use of the *piola transformation* to ensure that the discrete velocity solution remains pointwise divergence free and divergence conforming on moving meshes.

However, if one insists on analyzing the space-time HDG method in [43, 44] on space-time tetrahedral elements, a possible approach would be to consider the following *elliptic regularization* of the momentum equation:

$$\epsilon \partial_{tt} u + \partial_t u - \nu \Delta u + \nabla \cdot (u \otimes u) + \nabla p = f, \qquad \text{on } \Omega(t),$$

complemented with suitable initial and boundary conditions. Here $\epsilon > 0$ is a small parameter. This regularized problem is now an *elliptic* problem in $\mathbb{R}^{d+1}$, and thus the techniques used for the steady elliptic problem in Chapter 4 are applicable. The difficulty in this approach is justifying that the exact solution to the regularized system converges as $\epsilon \to 0$ to a solution of the Navier–Stokes equations on time-dependent domains, which is made especially challenging by the fact that the error analysis requires $\Omega(t)$ to be polygonal or polyhedral (and thus, not smooth).

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APPENDICES

Appendix A

Appendix to Chapter 4

A.1 Approximation properties of $\mathcal{P}_h$ and $\bar{\mathcal{P}}_h$

Here, we briefly outline the approximation properties of the projections $\mathcal{P}_h$ and $\bar{\mathcal{P}}_h$ introduced in Definition 4.4.1. We require some results from [29] adapted to the present setting:

Theorem A.1.1 ([29, Theorem 6.9]). *The projections defined in Definition 4.4.1 exist and are unique. Furthermore, for all $n = 0, \ldots, N-1$,*

$$(\mathcal{P}_h v)|_{I_n} = \mathcal{P}_h(P_h v)|_{I_n} = P_h(\mathcal{P}_h v|_{I_n}), \quad (\bar{\mathcal{P}}_h v)|_{I_n} = \bar{\mathcal{P}}_h(\bar{P}_h v)|_{I_n} = \bar{P}_h(\bar{\mathcal{P}}_h v|_{I_n}).$$

We first introduce a temporal "DG-projection" $P^t : C(I_n) \to P_k(I_n)$ satisfying

$$\int_{I_n} \left(P^t w(t) - w(t)\right) v \,\mathrm{d}t = 0,$$

for all $v \in P_{k-1}(I_n)$ and $P^t w(t_{n+1}^-) = w(t_{n+1}^-)$ (see, e.g., [98, Chapter 12] or [29, Lemma 6.11]). For $u \in H^{k+1}(I_n)$ this projection satisfies

$$\|u - P^t u\|_{H^s(I_n)} \lesssim \Delta t^{r-s} |u|_{H^r(I_n)}, \quad 0 \le s \le 1 \le r \le k. \tag{A.1}$$

Lemma A.1.1. *Let $\varphi \in C(I_n; V_h^{div})$ and $\bar{\psi} \in C(I_n; \bar{V}_h)$. Then*

$$\mathcal{P}_h \varphi(x,t) = P^t \varphi(x,t) \qquad \forall x \in \{K, F\}, \ \forall \{K \in \mathcal{T}_h, F \in \mathcal{F}_h\}, \tag{A.2a}$$

$$\bar{\mathcal{P}}_h \bar{\psi}(x,t) = P^t \bar{\psi}(x,t) \qquad \forall x \in F, \ \forall F \in \mathcal{F}_h. \tag{A.2b}$$

Proof. The proofs of eq. (A.2a) and eq. (A.2b) follow the proof of [29, Lemma 6.11] with minor modifications. □

With Lemma A.1.1 in hand, we can prove the following results:

Theorem A.1.2. *Let $k \geq 1$, $m \in \{0,1\}$, and*

$$u \in H^{k+1}(I_n; H_0^1(\Omega)^d \cap H^2(\Omega)^d) \cap C(I_n; H^{k+1}(\Omega)^d).$$

Let $\mathcal{P}_h$ and $\bar{\mathcal{P}}_h$ be the projections defined in Definition 4.4.1. Let $l = 0$ if $m \leq 1$ and $l = m$ if $m = 2$. Then, the following estimates hold:

$$\sum_{K\in\mathcal{T}_h} \int_{I_n} |u - \mathcal{P}_h u|^2_{H^m(K)} \,\mathrm{d}t \tag{A.3a}$$

$$\lesssim h^{2(k-m+1)} \|u\|^2_{L^2(I_n;H^{k+1}(\Omega))} + h^{-l}\Delta t^{2k+2}\|u\|^2_{H^{k+1}(I_n;H^m(\Omega))},$$

$$\sum_{K\in\mathcal{T}_h} h_K^{-1} \int_{I_n} \left\|\mathcal{P}_h u - \bar{\mathcal{P}}_h u\right\|^2_{L^2(\partial K)} \mathrm{d}t \lesssim h^{2k}\|u\|^2_{L^2(I_n;H^{k+1}(\Omega))}, \tag{A.3b}$$

$$\sum_{K\in\mathcal{T}_h} h_K \int_{I_n} \left\|\nabla(u - \mathcal{P}_h u)n\right\|^2_{L^2(\partial K)} \mathrm{d}t \tag{A.3c}$$

$$\lesssim h^{2k}\|u\|^2_{L^2(I_n;H^{k+1}(\Omega))} + \Delta t^{2k+2}\|u\|^2_{H^{k+1}(I_n;H^2(\Omega))},$$

$$\int_{I_n} \|\partial_t(u - \mathcal{P}_h u)\|^2_{L^2(\Omega))} \,\mathrm{d}t \tag{A.3d}$$

$$\lesssim \left(\Delta t^{2k}\|u\|^2_{H^{k+1}(I_n;L^2(\Omega))} + h^{2k+2}\left\|u\right\|^2_{H^1(I_n;H^{k+1}(\Omega))}\right). \tag{A.3e}$$

Proof. First, eq. (A.3a) can be shown in a similar fashion to [29, Lemmas 6.17 and 6.18] using the approximation properties of the spatial projection P_h given in Lemma 4.4.1 and the approximation properties of P^t given in eq. (A.1). For eq. (A.3b) we recall that $(\mathcal{P}_h v)|_{I_n} = \mathcal{P}_h(P_h v)|_{I_n}$ and $(\bar{\mathcal{P}}_h v)|_{I_n} = \bar{\mathcal{P}}_h(\bar{P}_h v)|_{I_n}$ by Theorem A.1.1, so by Lemma A.1.1, Fubini's theorem, and the stability of P^t in the $L^2(I_n)$ norm, we have

$$\sum_{K\in\mathcal{T}_h} \frac{1}{h_K} \int_{I_n} \|\mathcal{P}_h u - \bar{\mathcal{P}}_h u\|^2_{L^2(\partial K)} \,\mathrm{d}t \leq C \sum_{K\in\mathcal{T}_h} \frac{1}{h_K} \int_{I_n} \|P_h u - \bar{P}_h u\|^2_{L^2(\partial K)} \,\mathrm{d}t.$$

We conclude using the triangle inequality and the approximation properties of the spatial projections P_h and $\bar{P}_h$. Finally, eq. (A.3c) follows from eq. (A.3a) after noting that a local trace inequality yields

$$h_K \left\|\nabla(u - \mathcal{P}_h u)n\right\|^2_{L^2(\partial K)} \leq |u - \mathcal{P}_h u|^2_{H^1(K)} + h_K^2 |u - \mathcal{P}_h u|^2_{H^2(K)}.$$

Finally, eq. (A.3e) follows from Theorem A.1.1, Lemma A.1.1, Lemma 4.4.1, and eq. (A.1) since

$$\begin{aligned}
\|\partial_t(u - \mathcal{P}_h u)\|_{L^2(I_n;L^2(\Omega))} &= \|\partial_t(u - \mathcal{P}_h P_h u)\|_{L^2(I_n;L^2(\Omega))} \\
&= \|\partial_t(u - P^t P_h u)\|_{L^2(I_n;L^2(\Omega))} \\
&\leq \|\partial_t u - P_h \partial_t u\|_{L^2(I_n;L^2(\Omega))} + \|\partial_t(P_h u - P^t P_h u)\|_{L^2(I_n;L^2(\Omega))} \,,
\end{aligned}$$

where we have used the fact that the spatial projection P_h commutes with differentiation in time. □

Appendix B

Appendix to Chapter 5

B.1 Discrete compactness of the velocity

We recall the discrete compactness theory for DG time stepping developed by Walkington in [102] with a minor modification to fit the current non-conforming spatial discretization.

Remark B.1.1. *To our knowledge, the compactness theorem in [102, Theorem 3.1] for DG-in-time discretizations was first extended to non-conforming spatial approximations in [61]. Note that we can apply [61, Theorem 3.2] in our setting to conclude that the sequence* $\{u_h\}_{h\in\mathcal{H}}$ *is relatively compact in* $L^2(0,T;L^2(\Omega)^d)$ *by selecting (using the notation of [61] with* Y *and* X *replacing* V *and* H *therein to avoid confusion):*

$$W = H_0^1(\Omega)^d, \quad W(\mathcal{T}_h) = H^1(\mathcal{T}_h)^d, \quad Y = [BV(\Omega)^d \cap L^4(\Omega)^d; L^4(\Omega)^d]_{1/2},$$
$$X = L^2(\Omega)^d, \quad W' = H^{-1}(\Omega)^d, \quad W_h = V_h,$$

where $H^1(\mathcal{T}_h)^d$ *is the broken* H^1 *space equipped with the* $\|\cdot\|_{1,h}$*-norm [11, 28, 29],* $BV(\Omega)^d$ *is the space of functions of bounded variation [11], and* $[Y_0,Y_1]_\theta$ *denotes the complex interpolation between Banach spaces* Y_0, Y_1 *with exponent* $\theta \in (0,1)$ *[5].*

We present below a simple proof of a special case of [61, Theorem 3.2] that stays directly within the framework of broken polynomial spaces and their discrete functional analysis tools. This avoids the need to construct a non-conforming space that embeds compactly into $L^2(\Omega)^d$ and is made possible by the following discrete Rellich–Kondrachov theorem valid for broken polynomial spaces [28, Theorem 5.6]:

Lemma B.1.1 (Discrete Rellich–Kondrachov)**.** *Let $\mathcal{H}$ be a countable set of mesh sizes whose unique accumulation point is* 0*. Assume $\{(\mathcal{T}_h, \mathcal{F}_h)\}_{h\in\mathcal{H}}$ is a sequence of conforming and shape-regular simplicial meshes. Let $\{v_h\}_{h\in\mathcal{H}}$ be a sequence in $\{V_h\}_{h\in\mathcal{H}}$ bounded in the $\|\cdot\|_{1,h}$-norm. Then, for all $1 \le q < \infty$ if $d = 2$ and $1 \le q \le 6$ if $d = 3$, the sequence $\{v_h\}_{h\in\mathcal{H}}$ is relatively compact in $L^q(\Omega)^d$.*

Theorem B.1.1 (Compactness)**.** *Let $q = 4/d$. Let the sequence $\{\boldsymbol{u}_h\}_{h\in\mathcal{H}}$ be such that for each $h \in \mathcal{H}$, $\boldsymbol{u}_h \in \mathcal{V}_h^{div} \times \bar{\mathcal{V}}_h$. Then, $\{u_h\}_{h\in\mathcal{H}}$ is relatively compact in $L^2(0, T; L^2(\Omega)^d)$ if:*

- *(i) $\{\boldsymbol{u}_h\}_{h\in\mathcal{H}}$ is uniformly bounded in the sense that $\int_0^T \|\!|\boldsymbol{u}_h|\!\|_v^2 \,\mathrm{d}t \le M$ for some $M > 0$ independent of the mesh parameters h and τ.*
- *(ii) For each $h \in \mathcal{H}$, the following bound on the discrete time derivative of u_h holds uniformly for $q' = 4/(4 - d)$:*

$$\left| \int_0^T (\mathcal{D}_t^{k_t}(u_h), v_h)_{\mathcal{T}_h} \,\mathrm{d}t \right| \lesssim \left(\int_0^T \|\!|\boldsymbol{v}_h|\!\|_v^{q'} \,\mathrm{d}t \right)^{1/q'}, \quad \forall \boldsymbol{v}_h \in \mathcal{V}_h^{\mathrm{div}} \times \bar{\mathcal{V}}_h.$$

Proof. The proof, which proceeds in three steps, follows closely the proof of [102, Theorem 3.1] with minor modifications.

Step one (equicontinuity): Step one follows exactly the proof of [102, Lemma 3.3]; here we show that the assumptions therein can be interpreted as a uniform bound on the discrete time derivative eq. (5.23). By definition, it holds that

$$\int_{I_n} (\partial_t u_h, v_h)_{\mathcal{T}_h} + ([u_h]_n, v_n^+)_{\mathcal{T}_h} = \int_{I_n} (\mathcal{D}_t^{k_t}(u_h), v_h)_{\mathcal{T}_h} \,\mathrm{d}t, \quad \forall v_h \in \mathcal{V}_h. \tag{B.1}$$

Comparing with [102, Lemma 3.3], we require $F_h : \boldsymbol{v}_h \mapsto (\mathcal{D}_t^{k_t}(u_h), v_h)_{\mathcal{T}_h}$ to be uniformly bounded $L^q(0, T; (V_h^{\mathrm{div}} \times \bar{V}_h)')$, where $(V_h^{\mathrm{div}} \times \bar{V}_h)'$ is the dual space of $V_h^{\mathrm{div}} \times \bar{V}_h$. We show that assumption (ii) suffices. As the space $V_h^{\mathrm{div}} \times \bar{V}_h$ and its dual are finite-dimensional (hence separable), we make the identification $L^q(0, T; (V_h^{\mathrm{div}} \times \bar{V}_h)') \cong L^{q'}(0, T; V_h^{\mathrm{div}} \times \bar{V}_h)$, and we have

$$\|F_h\|_{L^q(0,T;(V_h^{\mathrm{div}} \times \bar{V}_h)')} = \sup_{0 \ne \boldsymbol{v} \in L^{q'}(0,T;V_h^{\mathrm{div}} \times \bar{V}_h)} \frac{\left| \int_0^T F_h(\boldsymbol{v}) \,\mathrm{d}t \right|}{\left(\int_0^T \|\!|\boldsymbol{v}|\!\|_v^{q'} \,\mathrm{d}t \right)^{1/q'}}. \tag{B.2}$$

Choose F_h in eq. (B.2) to be the functional that maps for each $t \in [0, T]$,

$$L^{q'}(0, T; V_h^{\mathrm{div}} \times \bar{V}_h) \ni (v, \bar{v}) =: \boldsymbol{v} \mapsto (\mathcal{D}_t^{k_t}(u_h), v)_{\mathcal{T}_h} \in \mathbb{R}.$$

Since $\boldsymbol{v} \in L^{q'}(0,T;V_h^{\text{div}} \times \bar{V}_h)$, we have $\Pi^t \boldsymbol{v} \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h$, and the stability of the L^2-projection Π^t in $L^{q'}(I_n)$ [30] yields

$$\|F_h\|_{L^q(0,T;(V_h^{\text{div}} \times \bar{V}_h)')} \lesssim \sup_{\boldsymbol{v} \in L^{q'}(0,T;V_h^{\text{div}} \times \bar{V}_h)} \frac{\left| \int_0^T (\mathcal{D}_t^{k_t}(u_h), \Pi^t v)_{\mathcal{T}_h} \, \mathrm{d}t \right|}{\left(\int_0^T \|\!|\Pi^t \boldsymbol{v}|\!\|_v^{q'} \, \mathrm{d}t \right)^{1/q'}}$$
$$\lesssim \sup_{\boldsymbol{v}_h \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h} \frac{\left| \int_0^T (\mathcal{D}_t^{k_t}(u_h), v_h)_{\mathcal{T}_h} \, \mathrm{d}t \right|}{\left(\int_0^T \|\!|\boldsymbol{v}_h|\!\|_v^{q'} \, \mathrm{d}t \right)^{1/q'}},$$

which is uniformly bounded by assumption (ii) of the theorem. Proceeding in an identical fashion to [102, Lemma 3.3], we find that

$$\int_\delta^T \left\| u_h(t) - u_h(t-\delta) \right\|_{L^2(\Omega)}^2 \mathrm{d}t \le C \max(\tau, \delta)^{1/q'} \delta^{1/2}. \tag{B.3}$$

Step two (relative compactness in $L^2(\theta, T-\theta; L^2(\Omega)^d)$): We aim to show that for all $0 < \theta < T/2$, the set $\left\{ u_h|_{\theta, T-\theta} \mid h \in \mathcal{H} \right\}$ is relatively compact in $L^2(\theta, T-\theta; L^2(\Omega)^d)$. The proof is a minor modification of [102, Theorem 3.2]. To this end, we construct a sequence of regularized functions so that we may leverage the classical Arzelà–Ascoli theorem (Theorem 2.2.10). Let $\phi \in C_c^\infty(-1,1)$ be nonnegative with unit integral. For $\delta > 0$, set $\phi_\delta(s) = (1/\delta)\phi(s/\delta)$. Extend u_h by zero outside of $[0,T]$ and consider the sequence of mollified functions $\left\{ u_h^\delta \right\}_{h \in \mathcal{H}}$, where $u_h^\delta(t) = \phi_\delta * u_h(t)$.

Since $\int_0^T \|u_h\|_{1,h}^2 \, \mathrm{d}t$ is uniformly bounded by assumption, we have

$$\|u_h^\delta(t)\|_{1,h}^2 \le \delta \sup_{|s|<\delta} |\phi_\delta(t-s)|^2 \int_0^\delta \left\| u_h(s) \right\|_{1,h}^2 \mathrm{d}s \le M^\star,$$

with $M^\star$ a constant independent of h and τ. Thus, by Lemma B.1.1, the sequence $\{u_h^\delta(t)\}_{h \in \mathcal{H}}$ is relatively compact in $L^2(\Omega)^d$ for each $t \in [0,T]$. Furthermore, the uniform Lipschitz continuity of the mollifiers $\phi_\delta(s)$ ensures the sequence $\left\{ u_h^\delta(t) \right\}_{h \in \mathcal{H}}$ is equicontinuous on $[0,T]$. By the Arzelà–Ascoli theorem, the sequence $\left\{ u_h^\delta \right\}_{h \in \mathcal{H}}$ is relatively compact in $C(0,T;L^2(\Omega)^d)$ and thus also $L^2(0,T;L^2(\Omega)^d)$ as the former embeds continuously into the latter. As relatively compact sets are totally bounded,

$$\forall \epsilon > 0, \ \exists h_1, \dots, h_M \subset \mathcal{H} \ s.t. \ \{u_h^\delta\}_{h \in \mathcal{H}} \subset \bigcup_{i=1}^M B_\epsilon(u_{h_i}),$$

where B_ϵ is an ϵ-ball in the metric induced by the $L^2(0,T;L^2(\Omega)^d)$-norm. The remainder of the proof that the set $\{u_h|_{[\theta,T-\theta]} \mid h \in \mathcal{H}\}$ is relatively compact in $L^2(\theta, T-\theta; L^2(\Omega)^d)$ for all $0 < \theta < T/2$ is identical to that of [102, Theorem 3.2].

Step three (finishing up): The equicontinuity eq. (B.3) and [102, Lemma 3.4] ensure that the sequence $\{u_h\}_{h\in\mathcal{H}}$ is bounded uniformly in $L^r(0,T;L^2(\Omega)^d)$ for $1 \le r < 4$. Consequently, for all $\epsilon > 0$ we can find $\theta > 0$ such that

$$\int_0^\theta \|u_h(t)\|_{L^2(\Omega)}^2 \,\mathrm{d}t + \int_{T-\theta}^T \|u_h(t)\|_{L^2(\Omega)}^2 \,\mathrm{d}t \le \epsilon.$$

It follows that $\{u_h \mid h \in \mathcal{H}\}$ is the uniform limit of relatively compact sets in $L^2(0,T;L^2(\Omega)^d)$ [91, Section 2]. Thus, $\{u_h \mid h \in \mathcal{H}\}$ is relatively compact in $L^2(0,T;L^2(\Omega)^d)$.

B.2 Properties of the projections Π and $\bar{\Pi}$

B.2.1 Approximation properties of Π^t and Π_h^{div}

Lemma B.2.1 (Approximation properties of Π_h^{div} and Π^t). *Let $\ell \ge 0$ and suppose that $\eta \in W^{\ell+1,\infty}(0,T)$ and $\psi \in H^{\ell+1}(\Omega)^d$. Then, for all $n = 0, \dots, N-1$,*

$$\|\eta - \Pi^t\eta\|_{L^\infty(I_n)} \lesssim \tau^{\ell+1}|\eta|_{W^{\ell+1,\infty}(I_n)}, \tag{B.4}$$

and if $\mathcal{T}_h$ is conforming and quasi-uniform, we have for $0 \le m \le 2$ and $K \in \mathcal{T}_h$,

$$\sum_{K\in\mathcal{T}_h} \|\psi - \Pi_h^{\mathrm{div}}\psi\|_{H^m(K)}^2 \lesssim h^{2(\ell-m+1)}|\psi|_{H^{\ell+1}(\Omega)}^2, \tag{B.5}$$

$$\|\psi - \Pi_h^{\mathrm{div}}\psi\|_{L^\infty(\Omega)} \lesssim h^{1/2}|\psi|_{H^2(\Omega)}, \tag{B.6}$$

the latter requiring $\ell \ge 1$.

Proof. Estimate eq. (B.4) is standard, see e.g. [33, Lemma 11.18]. The proof of eq. (B.5) is given in Lemma 4.4.1; we note that therein it is assumed that $\ell \ge 1$ but the proof easily extends to the case $\ell = 0$. We now show eq. (B.6). Let $\widehat{K}$ be the reference simplex in $\mathbb{R}^d$ and suppose that $F_K : \widehat{K} \to K$ is an affine mapping; denote its Jacobian matrix by J_K.

As $H^2(\widehat{K}) \subset L^\infty(\widehat{K})$ with continuous embedding for $d \le 3$, we have by repeated use of [33, Lemma 11.7]:

$$\|\psi - \Pi_h^{\text{div}}\psi\|_{L^\infty(K)} \lesssim \|J_K\|_{\ell^2}^2 \, |\det J_K|^{-1/2} \, \|\psi - \Pi_h^{\text{div}}\psi\|_{H^2(K)} .$$

Since $\mathcal{T}_h$ is assumed quasi-uniform and hence shape-regular, we have $\|J_K\|_{\ell^2}^2 \lesssim h_K^2$ and $|\det J_K|^{-1/2} \lesssim h_K^{-d/2}$ (see e.g. [33, Lemma 11.1], [31, Chapter 1.2]). Thus, for $d \le 3$,

$$\|\psi - \Pi_h^{\text{div}}\psi\|_{L^\infty(K)} \lesssim h^{1/2} \, \|\psi\|_{H^2(\Omega)} .$$

The result follows by noting that this bound holds uniformly for all $K \in \mathcal{T}_h$. □

B.2.2 Proof of Proposition 5.3.1

It suffices to show the inequality in eq. (5.37) on a single space-time slab $\mathcal{E}^n$; the result then follows by summing over all space-time slabs. Let $\varphi \in \mathscr{M}$. By the definitions of the norm $|||\cdot|||_v$ and the projections $\Pi\varphi$ and $\bar{\Pi}\varphi$ given in eq. (5.36), we have

$$\begin{aligned} &|||(\Pi\varphi, \bar{\Pi}\varphi)|||_v^{4/(4-d)} \\ &= \left(\sum_{K\in\mathcal{T}_h} \int_K |\nabla \Pi_h^{\text{div}} \Pi^t \sum_{k=1}^M \eta_k \psi_k|^2 \, \mathrm{d}x + \sum_{K\in\mathcal{T}_h} h_K^{-1} \int_{\partial K} |(\Pi_h^{\text{div}} - \bar{\Pi}_h)\Pi^t \sum_{k=1}^M \eta_k \psi_k|^2 \, \mathrm{d}x \right)^{2/(4-d)} . \end{aligned}$$

Available approximation results for the projection $\bar{\Pi}_h$ and eq. (B.5) yield for $\Psi \in H^1(\Omega)^d$,

$$\sum_{K\in\mathcal{T}_h} \left(\|\nabla \Pi_h^{\text{div}} \Psi\|_{L^2(K)}^2 + h_K^{-1} \, \|(\Pi_h^{\text{div}} - \bar{\Pi}_h)\Psi\|_{L^2(\partial K)}^2 \right) \lesssim \|\nabla \Psi\|_{L^2(\Omega)}^2 .$$

Therefore, we have

$$|||(\Pi\varphi, \bar{\Pi}\varphi)|||_v^{4/(4-d)} \lesssim \left(\int_\Omega |\Pi^t \sum_{k=1}^M \eta_k \nabla \psi_k|^2 \, \mathrm{d}x \right)^{2/(4-d)}, \quad \forall \varphi \in \mathscr{M}. \tag{B.7}$$

If $d = 2$, we can integrate eq. (B.7) over I_n and use Fubini's theorem and the stability of the projection Π^t in $L^2(I_n)$ to find

$$\int_{I_n} |||(\Pi\varphi, \bar{\Pi}\varphi)|||_v^2 \, \mathrm{d}t \lesssim \int_{I_n} \|\varphi\|_V^2 \, \mathrm{d}t, \quad \forall \varphi \in \mathscr{M},$$

as required. On the other hand, if $d = 3$, we integrate eq. (B.7) over I_n, and apply a finite-dimensional scaling argument between norms in $L^2(I_n)$ and $L^1(I_n)$ (see e.g. [28, Lemma 1.50]) to find:

$$\int_{I_n} \left|\!\left|\!\left| (\Pi\varphi, \bar{\Pi}\varphi) \right|\!\right|\!\right|_v^4 \mathrm{d}t \lesssim C\tau^{-1} \left(\int_{I_n} \int_{\Omega} |\Pi^t \sum_{k=1}^{M} \eta_k \nabla \psi_k|^2 \,\mathrm{d}x \,\mathrm{d}t \right)^2 . \tag{B.8}$$

Using Fubini's theorem to interchange the temporal and spatial integrals in eq. (B.8) as necessary, we can apply the stability of the projection Π^t in the $L^2(I_n)$ norm followed by the Cauchy–Schwarz inequality applied to the temporal integral to find

$$\int_{I_n} \left|\!\left|\!\left| (\Pi\varphi, \bar{\Pi}\varphi) \right|\!\right|\!\right|_v^4 \mathrm{d}t \lesssim \tau^{-1} \left(\int_{I_n} \|\varphi\|_V^2 \,\mathrm{d}t \right)^2 \lesssim \int_{I_n} \|\varphi\|_V^4 \,\mathrm{d}t, \quad \forall \varphi \in \mathscr{M}.$$

□