

# On Aharoni's rainbow generalization of the Caccetta-Häggkvist conjecture

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### Abstract

For a digraph  $G$  and  $v \in V(G)$ , let  $\delta^+(v)$  be the number of out-neighbors of  $v$  in  $G$ . The Caccetta-Häggkvist conjecture states that for all  $k \geq 1$ , if  $G$  is a digraph with  $n = |V(G)|$  such that  $\delta^+(v) \geq k$  for all  $v \in V(G)$ , then  $G$  contains a directed cycle of length at most  $\lceil n/k \rceil$ . In [2], Aharoni proposes a generalization of this conjecture, that a simple edge-colored graph on  $n$  vertices with  $n$  color classes, each of size  $k$ , has a rainbow cycle of length at most  $\lceil n/k \rceil$ . In this paper, we prove this conjecture if each color class has size  $\Omega(k \log k)$ .

# 1 Introduction and preliminaries

A graph or digraph is *simple* if there are no loops or parallel edges. For a simple digraph  $G$  and a vertex  $v \in V(G)$ , let  $\delta^+(v)$  denote the number of out-neighbors of  $v$  in  $G$ . A famous conjecture in graph theory is the following, due to Caccetta and Häggkvist [1]:

**Conjecture 1.1 (Caccetta-Häggkvist)** *Suppose  $n, k$  are positive integers, and let  $G$  be a simple digraph on  $n$  vertices with  $\delta^+(v) \geq k$  for all  $v \in V(G)$ ; then  $G$  contains a directed cycle of length at most  $\lceil n/k \rceil$ .*

For a graph  $G$  and a function  $c : E(G) \rightarrow \{1, \dots, |V(G)|\}$ , a *rainbow cycle* (with respect to  $c$ ) is a cycle  $C$  in  $G$  such that for all  $e, f \in E(C)$  with  $e \neq f$ , we have  $c(e) \neq c(f)$ . We will refer to  $c$  as a *coloring* of the edges of  $G$ .<sup>\*</sup> We say that  $c$  has *color classes of size at least  $k$*  for  $k \in \mathbb{N}$  if  $|c^{-1}(i)| \geq k$  for all  $i \in \{1, \dots, |V(G)|\}$ .

In [2], Aharoni proposes a generalization of Conjecture 1.1:

**Conjecture 1.2 ([2])** *Let  $n, k$  be positive integers, and let  $G$  be a simple graph on  $n$  vertices. Let  $c : E(G) \rightarrow \{1, \dots, n\}$  be a coloring of the edges of  $G$  with color classes of size at least  $k$ ; then  $G$  has a rainbow cycle of length at most  $\lceil n/k \rceil$ .*

In a recent paper, Devos et al. [4] prove that Conjecture 1.2 is true for  $k = 2$ :

**Theorem 1.3 ([4])** *Let  $G$  be a simple graph on  $n$  vertices, and let  $c$  be a coloring of the edges of  $G$  with color classes of size at least 2; then there exists a rainbow cycle of length at most  $\lceil n/2 \rceil$ .*

We also make use of the following results due to Bollobás and Szemerédi [3] and Shen [5], respectively. The first deals with the girth of a simple graph, while the second is an approximate result for Conjecture 1.1. In this paper,  $\log$  denotes the logarithm with base 2.

**Theorem 1.4 ([3])** *For all  $n \geq 4$  and  $k \geq 2$ , if  $G$  is a simple graph on  $n$  vertices with  $n + k$  edges, then  $G$  contains a cycle of length at most*

$$\frac{2(n+k)}{3k}(\log k + \log \log k + 4).$$

**Theorem 1.5 ([5])** *Let  $G$  be a simple digraph with  $\delta^+(v) \geq k$  for all  $v \in V(G)$ . Then  $G$  contains a directed cycle of length at most  $\lceil n/k \rceil + 73$ .*

## 2 Main result

Our main result is the following:

**Theorem 2.1** *Let  $k > 1$  be an integer, and let  $G$  be a graph. Let  $c$  be a coloring of the edges of  $G$  with color classes of size at least  $301k \log k$ . Then  $G$  contains a rainbow cycle of length at most  $\lceil n/k \rceil$ .*

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<sup>\*</sup>Note that  $c$  is not required to be a proper edge-coloring.

**Proof.** We proceed by induction on the number of vertices. Let  $f(k) = 7k \log k$ , and let  $G$  be a graph on  $n$  vertices. Let  $c$  be a coloring of the edges of  $G$  with color classes of size at least  $43f(k)$ . Suppose for a contradiction that there is no rainbow cycle of length at most  $\lceil n/k \rceil$ . Note that  $G$  has at least  $43f(k)n$  edges, and therefore,  $n > 43f(k)$ .

For  $v \in V(G)$  and  $i \in \{1, \dots, n\}$ , we say that  $i$  is *dominant at  $v$*  if  $v$  is incident with at least  $7f(k)$  edges  $e$  such that  $c(e) = i$ . We call a vertex  $v \in V(G)$  *color-dominating* if there exists  $i \in \{1, \dots, n\}$  such that  $i$  is dominant at  $v$ . We call a color  $i \in \{1, \dots, n\}$  *vertex-dominating* if there exists a vertex  $v \in V(G)$  such that  $i$  is dominant at  $v$ . Let us say that  $H \subseteq V(G)$  is *nice* if

- for every vertex-dominating color  $i \in \{1, \dots, n\}$ , there is a vertex  $v \in V(G) \setminus H$  such that  $i$  is dominant at  $v$ ; and
- there are at most  $|H|$  colors  $i \in \{1, \dots, n\}$  such that  $i$  is not vertex-dominating and for all  $e \in c^{-1}(i)$ , at least one end of  $e$  is in  $H$ .

(Claim 1) *If there is a nice set  $H \subseteq V(G)$  with  $6f(k) \leq |H| < n$ , then there is a nice set  $H' \subseteq V(G)$  with  $|H'| = \lceil 6f(k) \rceil$ .*

We remove vertices from  $H$  one-by-one such that the remaining set is nice. Suppose that we have removed  $j \geq 0$  vertices from  $H$ , leaving a nice set  $H_j$  with  $|H_j| > \lceil 6f(k) \rceil$ . Let  $C_j$  be the set of colors  $i \in \{1, \dots, n\}$  which are not vertex-dominating and also do not have an edge  $e$  with  $c(e) = i$  such that both ends of  $e$  are in  $V(G) \setminus H_j$ . From the definition of a nice set, we know  $|C_j| \leq |H_j|$ . If  $|C_j| < |H_j|$ , then removing any vertex from  $H_j$  gives a smaller nice set. So, we may assume that  $|C_j| = |H_j|$ . If there is a color  $i$  in  $C_j$  and an edge  $e = uv \in c^{-1}(i)$  with  $v \in H_j$  and  $u \in G \setminus H_j$ , then  $H_j \setminus \{v\}$  is nice. If there is no such  $i \in C_j$ , then for every color  $i \in C_j$ , all edges in  $c^{-1}(i)$  have both their ends in  $H_j$ . Now applying induction to the subgraph of  $G$  with vertex set  $H_j$  and edge set  $c^{-1}(C_j)$  gives a rainbow cycle of length at most  $\lceil n/k \rceil$  in  $G$ , a contradiction. This proves [Claim 1](#).

(Claim 2) *There is a nice set  $H' \subseteq V(G)$  with  $|H'| = \lceil 6f(k) \rceil$ .*

For each vertex-dominating color  $i$ , we pick a vertex  $v_i$  such that  $i$  is dominant at  $v_i$ , and let  $S$  be the set of these vertices  $v_i$ . Let  $H = V(G) \setminus S$ . Note that  $H$  is nice; thus by [Claim 1](#), we may assume that either  $|H| < 6f(k)$  or  $|H| = n$ .

We first consider the case when  $|H| = n$ . Since  $43f(k) \geq 2$ , [Theorem 1.3](#) guarantees the existence of a rainbow cycle  $K$  of length at most  $n/2 + 1$  in  $G$ . Let  $H' = V(G) \setminus V(K)$ . Then  $H'$  is nice, and  $n > |H'| \geq n/2 - 1 \geq 6f(k)$ ; so by [Claim 1](#),  $G$  contains a nice set of size  $\lceil 6f(k) \rceil$ .

Now we may assume that  $|H| < 6f(k)$ . We construct a digraph  $G'$  with  $V(G') = S$ , and for all  $i, j$  with  $v_i, v_j \in S$ , there is an arc  $v_i \rightarrow v_j$  if  $v_i v_j \in E(G)$  and  $c(v_i v_j) = i$ . Every vertex  $v_i$  is incident with at least  $7f(k)$  edges  $e$  with  $c(e) = i$ , and since  $|H| < 6f(k)$ , there are at least  $f(k)$  edges  $e = v_i u$  with  $c(e) = i$  and  $u \in S$ . Therefore,  $\delta^+(G') \geq f(k)$ .

Now, we claim  $n/f(k) + 74 \leq n/k$ , which is equivalent to  $74kf(k) \leq n(f(k) - k)$ . Since  $k \geq 2$ , we have  $\log(k) \geq 117/301$ , and thus  $74kf(k) \leq 43f(k)(f(k) - k) \leq n(f(k) - k)$ , as claimed.

Then, by applying [Theorem 1.5](#) to  $G'$  we obtain a directed cycle  $K$  of length at most  $\lceil n/f(k) \rceil + 73 \leq \lceil n/k \rceil$  in  $G'$ . The edges of  $G$  that correspond to arcs of  $K$  form a rainbow cycle of length at most  $\lceil n/k \rceil$  in  $G$ , a contradiction. This proves [Claim 2](#). ■

Let  $H \subseteq V(G)$  be a nice set with  $|H| = \lceil 6f(k) \rceil$ . Then there exists  $H' \subseteq H$  (Claim 3) such that  $|H'| \geq \lceil 2f(k) \rceil$  and such that for at least  $n - \lceil f(k) \rceil + 1$  colors  $i$ , at least one edge  $e \in c^{-1}(i)$  has both ends in  $V(G) \setminus H'$ .

Let  $C$  be the set of colors  $i$  which are not vertex-dominating and for which no edge of  $c^{-1}(i)$  has both ends in  $V(G) \setminus H$ . Since  $H$  is nice, it follows that  $|C| \leq |H| = \lceil 6f(k) \rceil$ . Let  $D \subseteq C$  be the set of colors  $i \in C$  such that there is a vertex  $v \in H$  which is incident with all edges in  $c^{-1}(i)$  that have one end in  $H$  and the other in  $V(G) \setminus H$ . We claim that  $|D| \leq \lceil f(k) \rceil - 1$ . Indeed, for each color  $i \in D$ , there are at least  $\lceil 36f(k) \rceil$  edges in  $c^{-1}(i)$  with both ends in  $H$  since  $i$  is not vertex-dominating. If  $|D| > \lceil f(k) \rceil - 1$ , then we obtain more than  $(f(k) - 1)(36f(k))$  edges with both ends in  $H$ . Now, since  $k \geq 2$ , we have  $f(k) \geq 72/23$ , and it follows that:

$$(f(k) - 1)(36f(k)) \geq \frac{49f(k)^2}{2} \geq \frac{(6f(k) + 1)^2}{2} \geq \frac{|H|^2}{2}$$

which gives a contradiction. Thus,  $|D| \leq \lceil f(k) \rceil - 1$ .

Next, we claim there exists  $H' \subseteq H$  such that  $|H'| = \lceil 2f(k) \rceil$  and such that for all  $i \in \{1, \dots, n\} \setminus D$ , there is an edge  $e \in c^{-1}(i)$  with both ends in  $V(G) \setminus H'$ . To see this, we construct a graph  $J$  with vertex set  $H$  and the following set of edges. For each  $i \in C \setminus D$ , we choose two vertices  $v_1^i, v_2^i \in H$ , each incident with an edge in  $c^{-1}(i)$  whose other end is in  $V(G) \setminus H$ ; we know from the definition of  $D$  that this is possible. Now, the graph  $J$  has  $|H|$  vertices and at most  $|H|$  edges, and so  $J$  has a stable set  $H' \subseteq V(J)$  of size at least  $|V(J)|/3 \geq 2f(k)$ ; and so  $|H'| \geq \lceil 2f(k) \rceil$ .

Now, for every color  $i \in C \setminus D$ ,  $V(G) \setminus H'$  contains at least one of  $v_1^i, v_2^i$ , and therefore, there is an edge in  $c^{-1}(i)$  with both ends in  $V(G) \setminus H'$ . Moreover, for every  $i \in \{1, \dots, n\} \setminus C$ , either  $i$  dominates a vertex  $v$  in  $V(G) \setminus H \subseteq V(G) \setminus H'$  (and so, since  $|H'| < 7f(k)$ , there is an edge in  $c^{-1}(i)$  incident with  $v$  whose other end is not in  $H'$ ); or there is an edge in  $c^{-1}(i)$  with both ends in  $V(G) \setminus H \subseteq V(G) \setminus H'$ . Thus, for at least  $n - |D| \geq n - \lceil f(k) \rceil + 1$  colors  $i$ , at least one edge in  $c^{-1}(i)$  has both ends in  $V(G) \setminus H'$ . This proves [Claim 3](#).

By combining [Claim 2](#) and [Claim 3](#), we conclude that there exists  $H' \subseteq V(G)$  with  $|H'| \geq \lceil 2f(k) \rceil$ , and such that for at least  $n - \lceil f(k) \rceil + 1$  colors  $i$ , at least one edge in  $c^{-1}(i)$  has both ends in  $V(G) \setminus H'$ . Let  $H''$  be a subgraph of  $G$  with vertex set  $V(G) \setminus H'$ , obtained by taking exactly one edge in  $c^{-1}(i)$  with both ends in  $V(G) \setminus H'$  for all  $i \in \{1, \dots, n\}$  which have such an edge. It follows that  $|E(H'')| \geq |V(H'')| + \lceil f(k) \rceil$ .

Now, we claim that  $\frac{2(n+f(k))}{3\lceil f(k) \rceil}(\log \log(f(k)) + \log(f(k)) + 4) \leq \frac{n}{k}$ . Using  $f(k) < n/43$ , it suffices to show:

$$\frac{88(\log \log(f(k)) + \log(f(k)) + 4)}{129} \leq 7 \log(k)$$

Let  $g(k) = 7 \log(k) - \frac{88}{129}(\log \log(f(k)) + \log(f(k)) + 4)$ . We have that  $g(2) > 0$ , and for  $k \geq 2$  we have:

$$f(k)g'(k) \ln(2) = 49 \log(k) - \frac{88}{129} f'(k) \left( \frac{1}{\log(f(k)) \ln(2)} + 1 \right) > 0$$

since for  $k \geq 2$  we have:

$$f'(k) \left( \frac{1}{\log(f(k)) \ln(2)} + 1 \right) < (7 + 7 \log(k))(3) \leq 49 \log(k)$$

So  $g'(k) > 0$  for  $k \geq 2$ , and it follows that  $g(k) \geq 0$  for  $k \geq 2$ , as desired.

Then, Theorem 1.4 gives a rainbow cycle of length at most  $\frac{2(n+f(k))}{3(f(k))}(\log \log(f(k)) + \log(f(k)) + 4) \leq \lceil \frac{n}{k} \rceil$ , a contradiction. This proves Theorem 2.1. ■

We have an immediate corollary which gives us a result for the case of  $\Omega(n \log n)$  color classes each of size  $k$ :

**Corollary 2.2** *Let  $k$  be a positive integer and let  $G$  be a simple graph on  $n$  vertices. Let  $c : E(G) \rightarrow \{1, \dots, t\}$  with  $t \geq 303n \log n$ , and with  $|c^{-1}(i)| \geq k$  for all  $i \in \{1, \dots, t\}$ . Then  $G$  contains a rainbow cycle in  $G$  of length at most  $\lceil n/k \rceil$ .*

**Proof.** Note that  $t \geq 303n \log n \geq 303n \log k$ . Since  $303n \log k \geq n \lceil 301 \log k \rceil$ , we can partition  $\{1, \dots, t\}$  into  $n$  parts, each of size at least  $\lceil 301 \log k \rceil$ ; that is, there is a function  $f : \{1, \dots, t\} \rightarrow \{1, \dots, n\}$  such that  $|f^{-1}(i)| \geq \lceil 301 \log k \rceil$  for all  $i \in \{1, \dots, n\}$ . By Theorem 2.1, applied to  $G$  and  $f \circ c$ , we obtain a rainbow cycle of length at most  $\lceil n/k \rceil$  in  $G$  with respect to  $f \circ c$ , which is also rainbow with respect to  $c$ . This proves Corollary 2.2. ■

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