Caterpillars in Erdős-Hajnal

Anita Liebenau[1]
School of Mathematics and Statistics, UNSW Sydney, NSW 2052, Australia.
Email: a.liebenau@unsw.edu.au

Marcin Pilipczuk[2]
Institute of Informatics, University of Warsaw, Poland

Paul Seymour[3] and Sophie Spirkl
Princeton University, Princeton, NJ 08544, USA

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Abstract

Let T be a tree such that all its vertices of degree more than two lie on one path; that is, T is a caterpillar subdivision. We prove that there exists $\epsilon > 0$ such that for every graph G with $|V(G)| \geq 2$ not containing T as an induced subgraph, either some vertex has at least $\epsilon|V(G)|$ neighbours, or there are two disjoint sets of vertices A, B, both of cardinality at least $\epsilon|V(G)|$, where there is no edge joining A and B.

A consequence is: for every caterpillar subdivision T, there exists $c > 0$ such that for every graph G containing neither of T and its complement as an induced subgraph, G has a clique or stable set with at least $|V(G)|^c$ vertices. This extends a theorem of Bousquet, Lagoutte and Thomassé [1], who proved the same when T is a path, and a recent theorem of Choromanski, Falik, Liebenau, Patel and Pilipczuk [2], who proved it when T is a "hook".

1 Introduction

The Erdős-Hajnal conjecture [6, 7] asserts:

1.1 Conjecture: *For every graph H, there exists $c > 0$ such that every H-free graph G satisfies*

$$\max(\omega(G), \alpha(G)) \geq |V(G)|^c.$$

(All graphs in this paper are finite and have no loops or parallel edges. A graph G is *H-free* if no induced subgraph of G is isomorphic to H; and $\omega(G), \alpha(G)$ denote the cardinalities of the largest cliques and stable sets in G respectively, and $\omega(G)$ is called the *clique number* of G.) This conjecture has been investigated heavily, and nevertheless has been proved only for very restricted graphs H (see [3] for a survey, and see [9] for progress on the conjecture in a geometric setting). In particular it has not yet been proved when H is a five-vertex path.

On the other hand, a theorem of Bousquet, Lagoutte and Thomassé [1] asserts the following ($\overline{H}$ denotes the complement of a graph H):

1.2 *For every path H, there exists $c > 0$ such that every graph G that is both H-free and $\overline{H}$-free satisfies* $\max(\omega(G), \alpha(G)) \geq |V(G)|^c$.

Let us say H is a *hook* if H is a tree obtained from a path by adding a vertex adjacent to the third vertex of the path. Two of us, with Choromanski, Falik, and Patel [2], extended 1.2, proving:

1.3 *For every hook H, there exists $c > 0$ such that every graph G that is both H-free and $\overline{H}$-free satisfies* $\max(\omega(G), \alpha(G)) \geq |V(G)|^c$.

The main step of the proof of 1.2 is the following:

1.4 *For every path H, there exists $\epsilon > 0$ such that for every H-free graph G with $|V(G)| \geq 2$, either some vertex has at least $\epsilon|V(G)|$ neighbours, or there are two anticomplete sets of vertices A, B, both of cardinality at least $\epsilon|V(G)|$.*

(Two sets $A, B \subseteq V(G)$ are *complete* to each other if $A \cap B = \emptyset$ and every vertex in A is adjacent to every vertex in B; and *anticomplete* to each other if they are complete to each other in $\overline{G}$.)

It is natural to ask, which other graphs H have the property of 1.4? Let us say a graph H has the *sparse strong EH-property* if there exists $\epsilon > 0$ such that for every H-free graph G with $|V(G)| \geq 2$, either some vertex has at least $\epsilon|V(G)|$ neighbours, or there are two anticomplete sets of vertices A, B, both of cardinality at least $\epsilon|V(G)|$. Which graphs have the sparse strong EH-property?

And here is a related question: let us say a graph has the *symmetric strong EH-property* if there exists $\epsilon > 0$ such that for every graph G that is both H-free and $\overline{H}$-free, with $|V(G)| \geq 2$, there are two disjoint sets of vertices, both of cardinality at least $\epsilon|V(G)|$, and either complete or anticomplete to each other. Which graphs have the symmetric strong EH-property?

It follows from a theorem of Rödl [13] (and see [8] for a version with much better constants) that every graph with the sparse property has the symmetric property; and Erdős's construction [5] of a graph with large girth and large chromatic number also shows that every graph with the sparse property is a forest, and every graph with the symmetric property is either a forest or the complement of one. (We omit all these proofs, which are easy; see [2] for more details.) We conjecture the converses, that is:

1.5 Conjectures:

- *A graph H has the sparse strong EH-property if and only if H is a forest.*
- *A graph H has the symmetric strong EH-property if and only if one of H, $\overline{H}$ is a forest.*

The first implies the second, because of the theorem of Rödl [13]. These two conjectures are reminiscent of the Gyárfás-Sumner conjecture, which we discuss later. (Since this paper was submitted for publication, both of these conjectures have been proved, in [4].)

A graph H is a *caterpillar* if H is a tree and some path of H contains all vertices with degree at least two; and a *caterpillar subdivision* if H is a tree and some path of H contains all vertices with degree at least three. (Thus a graph is a caterpillar subdivision if and only if it can be obtained from a caterpillar by subdividing edges.) We will prove:

1.6 *Every caterpillar subdivision has the sparse strong EH-property.*

1.6 implies the next result, which generalizes 1.2 and 1.3. (This theorem was proved independently by the first two authors and by the last two, but since the proofs were virtually identical we have combined the two papers into one. The original paper by the first two authors is available [12].) If $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced on X.

1.7 *Let H, J be caterpillar subdivisions. Then there exists $c > 0$ such that for every graph G, if G is both H-free and $\overline{J}$-free, then* $\max(\omega(G), \alpha(G)) \geq |V(G)|^c$.

Proof of 1.7, assuming 1.6. There is a caterpillar subdivision such that both H, J are induced subgraphs of it, and so, by replacing H, J by this graph, we may assume that $H = J$. Let ϵ satisfy 1.6; so $0 \leq \epsilon \leq 1$. By a theorem of Rödl [13],

(1) *There exists $\delta > 0$ such that for every H-free graph G, there is a subset $X \subseteq V(G)$ with $|X| \geq \delta|V(G)|$ such that one of $G[X], \overline{G}[X]$ has at most $\epsilon|X|^2/4$ edges.*

Choose c such that $2(\epsilon\delta/2)^{2c} = 1$. A graph is *perfect* if chromatic number equals clique number for all its induced subgraphs. For a graph G, let $\pi(G)$ denote the maximum cardinality of a subset X such that $G[X]$ is perfect; we will prove by induction on $|V(G|$ that if G is both H-free and $\overline{H}$-free, then $\pi(G) \geq |V(G)|^{2c}$ (and consequently the theorem will follow, since $\alpha(G)\omega(G) \geq \pi(G)$). If $|V(G)| \leq 1$ the result is trivial, and if $2 \leq |V(G)| \leq 2/\delta$ then $\pi(G) \geq 2 \geq |V(G)|^{2c}$ as required, since $(2/\delta)^{2c} \leq (2/(\epsilon\delta))^{2c} = 2$. Thus we may assume that $|V(G)| > 2/\delta$. By (1) there is a subset $X \subseteq V(G)$ with $|X| \geq \delta|V(G)|$ such that one of $G[X], \overline{G}[X]$ has at most $\epsilon|X|^2/4$ edges; and by replacing G by its complement if necessary, we may assume that $G[X]$ has at most $\epsilon|X|^2/4$ edges. Choose distinct $v_1, \ldots, v_k \in X$, maximal such that for $1 \leq i \leq k$, v_i has at least $\epsilon|X|/2$ neighbours in $X \setminus \{v_1, \ldots, v_i\}$. Let $Y = X \setminus \{v_1, \ldots, v_k\}$. It follows that $k \leq |X|/2$, and every vertex in Y has fewer than $\epsilon|X|/2$ neighbours in Y, from the maximality of k. Thus $|Y| \geq |X|/2 \geq \delta|V(G)|/2$, and $G[Y]$ has maximum degree less than $\epsilon|X|/2 \leq \epsilon|Y|$. Since $|V(G)| > 2/\delta$, it follows that $|X| > 2$ and so $|Y| > 1$. By 1.6 applied to $G[Y]$, there are two anticomplete sets of vertices A, B, both of cardinality at least $\epsilon|Y|$. From the inductive hypothesis $\pi(G[A]) \geq |A|^{2c}$ and $\pi(G[B]) \geq |B|^{2c}$, and so

$$\pi(G) \geq |A|^{2c} + |B|^{2c} \geq 2(\epsilon|Y|)^{2c} \geq 2(\epsilon\delta|V(G)|/2)^{2c} = |V(G)|^{2c}.$$

This proves 1.7. ∎

Let G be a graph and for every subset $X \subseteq V(G)$ let $\mu(X)$ be a real number, satisfying:

- $\mu(\emptyset) = 0$ and $\mu(V(G)) = 1$, and $\mu(X) \le \mu(Y)$ for all X, Y with $X \subseteq Y$; and
- $\mu(X \cup Y) \le \mu(X) + \mu(Y)$ for all disjoint sets X, Y.

We call such a function μ a *mass* on G. For instance, we could take $\mu(X) = |X|/|V(G)|$, or $\mu(X) = \chi(G[X])/\chi(G)$, where χ denotes chromatic number. We denote by $N(v)$ the set of neighbours of v. The result 1.6 can be extended to graphs with masses, in the following way:

1.8 *For every caterpillar subdivision H, there exists $\epsilon > 0$ such that for every H-free graph G, and mass μ on G, either*

- *$\mu(\{v\}) \ge \epsilon$ for some vertex v; or*
- *$\mu(N(v)) \ge \epsilon$ for some vertex v; or*
- *there are two anticomplete sets of vertices A, B, where $\mu(A), \mu(B) \ge \epsilon$.*

We prove this in the next section. It implies 1.6, setting $\mu(X) = |X|/|V(G)|$. To see this, observe that if only the first outcome holds, and $\mu(\{v\}) \ge \epsilon$ for some v, then v has no neighbours (or else the second outcome would hold), and $\mu(V(G) \setminus \{v\}) < \epsilon$ (or else the third outcome would hold); and so $\mu(v) > 1 - \epsilon$. Adding, $2\mu(v) > \epsilon + (1 - \epsilon) = 1$, and so $\mu(v) > 1/2$, and hence $|V(G)| = 1$.

But 1.8 also has an interesting application to the Gyárfás-Sumner conjecture [10, 14], which states that for every tree T and every integer $k \ge 0$, there exists $f(T, k)$ such that every T-free graph with clique number at most k has chromatic number at most $f(T, k)$. This has not been proved in general, and not even for caterpillars; and not even for trees with exactly two vertices of degree more than two (such a tree is a simple kind of caterpillar subdivision). But by induction on k, one could assume that for every vertex v, the chromatic number of the subgraph induced on $N(v)$ is bounded; and so the following consequence of 1.8 might be of interest.

1.9 *Let T be a caterpillar subdivision, and $k \ge 0$ an integer. Let ϵ satisfy 1.8. Suppose that every T-free graph with clique number $< k$ has chromatic number at most $c \ge 1$. Then in every T-free graph with clique number at most k and chromatic number more than c/ϵ, there are two anticomplete sets of vertices A, B, where $\chi(G[A]), \chi(G[B]) \ge \epsilon\chi(G)$.*

Proof. Let G be a T-free graph with $\omega(G) \le k$. Define $\mu(X) = \chi(G[X])/\chi(G)$, for each $X \subseteq V(G)$. Thus one of the three outcomes of 1.8 holds. The first implies that $\chi(G) \le 1/\epsilon$, and the second implies that $\chi(G) \le c/\epsilon$, in both cases a contradiction. So the third holds. This proves 1.9. ∎

Incidentally, perhaps one can unify the Gyárfás-Sumner conjecture and 1.5, in the natural way (using masses).

2 The main proof

In this section we prove 1.8, but before the details of the proof, let us sketch the idea. If X, Y are disjoint subsets of $V(G)$, we say that X *covers* Y if every vertex in Y has a neighbour in X. First let T be a caterpillar, rather than a caterpillar subdivision, and suppose that G is a T-free graph with a mass that does not satisfy the theorem. We choose some large number (depending on T) of disjoint subsets of $V(G)$, each with large mass (let us call them "blocks"). It follows from the falsity of the third bullet of 1.8 that for every two blocks, most of the vertices in one will have neighbours in the other, so we are well-equipped with edges between blocks. Choose a block B_1, and let us grow a subset X of it, one vertex at a time, until there is some other block, say B_2, that is at least half covered by X. We cannot use B_1 as a block any more, and we discard it, retaining only the set X. Also we discard from B_2 the part of B_2 that is not covered by X, and for every other block B_3 say, discard from B_3 the part that is covered by X. We now have many disjoint blocks (one fewer than before), all still with large mass (about half what it was before), together with one more set X that covers one of our blocks and has no edges to the others. Now pick another block (which could be B_2) and do it again, growing a subset of it until it covers half of a different block, and so on. We can construct more complicated patterns of covering, by judiciously choosing which block to grow within next. This will enable us to find a copy of the caterpillar T, with all its vertices in different blocks.

In the case when T is a caterpillar subdivision, we were not able to prove that there is a copy of T with all its vertices in different blocks. But T can be obtained from some caterpillar T' by subdividing some of its leaf edges (not subdividing the spine of T'). We find a copy of T' with all its vertices in different blocks, and grow each leaf of T' to an appropriately long path within the block that contained the leaf, by using "spires", a variant of the proof of Gyárfás [11] showing the χ-boundedness of the graphs not containing a fixed path.

Let us turn to the details. Throughout the remainder of this section, $\epsilon > 0$ is some real number that will be specified later, and G is a graph with a mass μ, satisfying:

(1) $\mu(\{v\}) < \epsilon$ for every vertex v;

(2) $\mu(N(v)) < \epsilon$ for every vertex v; and

(3) there do not exist two subsets A, B of $V(G)$, anticomplete, with $\mu(A), \mu(B) \geq \epsilon$.

We will show that, for every caterpillar subdivision T, if ϵ is sufficiently small, then G contains T as an induced subgraph, which will prove 1.8. We refer to the three statements above as the "axioms".

2.1 *Let $X \subseteq V(G)$. If $\mu(X) \geq 3\epsilon$ then $\mu(X') > \mu(X) - \epsilon$ for the vertex set X' of some component of $G[X]$.*

Proof. Let the vertex sets of the components of $G[X]$ be $X_1, \dots, X_k$ say. Choose $i \geq 1$ minimal such that $\mu(X_1 \cup \dots \cup X_i) \geq \epsilon$. Then from axiom (3), $\mu(X_{i+1} \cup \dots \cup X_n) < \epsilon$; and from the minimality of i, $\mu(X_1 \cup \dots \cup X_{i-1}) < \epsilon$. But

$$\mu(X_1 \cup \dots \cup X_{i-1}) + \mu(X_i) + \mu(X_{i+1} \cup \dots \cup X_n) \geq \mu(X) \geq 3\epsilon,$$

and so $\mu(X_i) \geq \epsilon$. From axiom (3), the union of all other components has mass less than ϵ, and so $\mu(X_i) > \mu(X) - \epsilon$. This proves 2.1. ∎

We observe that since the union of all components of $G[X]$ different from X' has mass less than ϵ, the set X' in 2.1 is unique, and we call it the *big piece* of X.

2.2 *Let $X \subseteq Y \subseteq V(G)$. If $\mu(X) \geq 3\epsilon$ then the big piece of X is a subset of the big piece of Y.*

Proof. The big piece of X has mass at least ϵ, and is a subset of the vertex set of some component of $G[Y]$; and therefore is a subset of the big piece of Y. This proves 2.2. ∎

Let $\tau \geq 3$ be an integer. If $X \subseteq V(G)$, a τ-*spire* in X is a sequence $(x_1, \ldots, x_\tau, Z)$, where

- $x_1, \ldots, x_\tau$ are the vertices in order of an induced τ-vertex path of $G[X]$;
- $Z \subseteq X \setminus \{x_1, \ldots, x_{\tau-1}\}$, and $x_\tau \in Z$;
- $x_1, \ldots, x_{\tau-1}$ have no neighbours in $Z \setminus \{x_\tau\}$; and
- $G[Z]$ is connected.

2.3 *Let $\tau \geq 3$ be an integer, and let $X \subseteq V(G)$ with $\mu(X) \geq (\tau + 2)\epsilon$; then there is a τ-spire $(x_1, \ldots, x_\tau, Z)$ in X where $\mu(Z) \geq \mu(X) - \tau\epsilon$.*

Proof. Let Z_1 be the big piece of X, and choose $x_1 \in Z_1$. Let Z_2 be the big piece of $X \setminus N(x_1)$. Since $\mu(X \setminus N(x_1)) \geq 3\epsilon$, from axiom (2) and since $\tau \geq 3$, 2.2 implies that $Z_2 \subseteq Z_1$. Now x_1 is a one-vertex component of $G[X \setminus N(x_1)]$, and therefore not its big piece, by axiom (1); and since $Z_2 \subseteq Z_1$, some neighbour x_2 of x_1 has a neighbour in Z_2.

Inductively, suppose that $2 \leq i < \tau$, and we have defined $x_1, \ldots, x_i$ and Z_i, where

- $x_1, \ldots, x_i$ are the vertices in order of an induced i-vertex path of $G[X]$;
- Z_i is the big piece of $X \setminus \bigcup_{1 \leq h \leq i-1} N(x_h)$; and
- x_i has a neighbour in Z_i.

Let $Y_{i+1} = X \setminus \bigcup_{1 \leq h \leq i} N(x_h)$. Axiom (2) implies that $\mu(Y_{i+1}) \geq \mu(X) - i\epsilon \geq 3\epsilon$. Let Z_{i+1} be the big piece of Y_{i+1}. By 2.2, $Z_{i+1} \subseteq Z_i$, and so some neighbour x_{i+1} of x_i has a neighbour in Z_{i+1}. This completes the inductive definition.

Then $(x_1, \ldots, x_\tau, Z_\tau \cup \{x_\tau\})$ is a τ-spire in X, and $\mu(Z_\tau) \geq \mu(X) - \tau\epsilon$, by axiom (2) and 2.1. This proves 2.3. ∎

Let H be a caterpillar, and choose a vertex v which is an end of some path P of H that contains all vertices with degree at least two; and call v the *head* of the caterpillar. The *spine* is the minimal path of H with one end v that contains all vertices of degree at least two. The pair (H, v) is thus a rooted tree rather than a tree, but we will normally speak of it as a tree and let the head be implicit.

Again, let $\tau \geq 3$ be an integer. A caterpillar is a τ-*chrysalis* if

- its spine has at most $\tau + 1$ vertices;
- every vertex of the spine different from the head has degree exactly τ; and
- the head has degree at most $\tau - 1$, and the head has degree one if the spine has $\tau + 1$ vertices.

The τ-chrysalis with most vertices therefore has $\tau^2 - \tau + 2$ vertices, and is unique; let us call it the *τ-butterfly.* It is the only τ-chrysalis in which the spine has $\tau + 1$ vertices.

Now let N be a disjoint union of τ-chrysalises $H_1, \dots, H_k$; we call N a *τ-nursery.* We define

$$\phi(N) = \sum_{1 \le i \le k} 2^{|V(H_i)|}.$$

If N, M are τ-nurseries, we say that M is an *improvement* of N if M has fewer components than N and $\phi(M) \ge \phi(N)$.

Returning to the graph G with mass μ, we need to define what it is for a τ-nursery N to be "realizable" in G. Let us direct all the edges of N towards the heads; thus, for every edge uv of N, if v is on the path between u and the head of the component of N containing u, we direct the edge uv from u to v. A vertex v of N is a *leaf* if it has indegree zero and outdegree one in N; that is, if and only if it does not belong to the spine of its component. Let $0 \le \kappa \le 1$, and for each vertex $v \in V(N)$, let $X_v \subseteq V(G)$, satisfying the following conditions:

- the sets X_v $(v \in V(N))$ are pairwise disjoint;
- for each leaf v of N there is a τ-spire $(x_v^1, \dots, x_v^\tau, Z_v)$ in X_v, and $X_v = \{x_v^1, \dots, x_v^\tau\} \cup Z_v$;
- for all distinct $u, v \in V(N)$, if v is a leaf then $\{x_v^1, \dots, x_v^\tau\}$ is anticomplete to X_u;
- for all distinct $u, v \in V(N)$, if there is an edge of G between X_u, X_v then either u, v are adjacent in N or both u, v are heads of components of N;
- for every directed edge $u \to v$ of N, X_u covers X_v;
- for each $v \in V(N)$, if v is the head of a component of N then $\mu(X_v) \ge \kappa$.

If such a function X_v $(v \in V(N))$ exists we call it a *κ-realization* of N in G, and say N is *κ-realizable* in G. We need:

2.4 *Let $\tau \ge 3$ be an integer, and let $0 \le \kappa, \kappa' \le 1$, with $\kappa \ge 2\kappa' + (\tau + 2)\epsilon$. Let N be a τ-nursery with at least two components, and in which no component is the τ-butterfly. If N is κ-realizable in G, there is an improvement N' of N that is κ'-realizable in G.*

Proof. Let the components of N be $H_1, \dots, H_k$, where $|V(H_1)| \le \dots \le |V(H_k)|$, and for $1 \le i \le k$ let h_i be the head of H_i. Let X_v $(v \in V(N))$ be a κ-realization of N in G. If there exists $i \in \{1, \dots, k\}$ such that h_i has degree $\tau - 1$, choose such a value of i, maximum; and otherwise, let $i = 1$. By 2.3, since $\mu(X_{h_i}) \ge \kappa \ge (\tau+2)\epsilon$, there is a τ-spire $(x_1, \dots, x_\tau, Z)$ say in X_{h_i} where $\mu(Z) \ge \mu(X_{h_i}) - \tau\epsilon \ge \epsilon$.

For each $j \in \{1, \dots, k\}$ with $j \ne i$, let $Y_{h_j} \subseteq X_{h_j}$ be the set of vertices in X_{h_j} with no neighbour in $\{x_1, \dots, x_\tau\}$. Thus

$$\mu(Y_{h_j}) \ge \mu(X_{h_j}) - \tau\epsilon \ge \kappa - \tau\epsilon \ge 2(\kappa' + \epsilon)$$

from axiom (2). Since $G[Z]$ is connected and $x_\tau \in Z$, we can number the vertices of Z as $z_1, \dots, z_n$ say, such that $z_1 = x_\tau$ and $G[\{z_1, \dots, z_m\}]$ is connected for $1 \le m \le n$. Since $k \ge 2$, there exists $j \ne i$ with $1 \le j \le k$; but $\mu(Y_{h_j}) \ge 2(\kappa' + \epsilon)$, and by axiom (3), the set of vertices in Y_{h_j} with no neighbour in Z has mass less than ϵ. Consequently we may choose m with $1 \le m \le n$, minimum such that for some $j \in \{1, \dots, k\} \setminus \{i\}$, the set of vertices in Y_{h_j} with no neighbour in $\{z_1, \dots, z_m\}$ has mass less than $\kappa' + \epsilon$. Since no vertex in Y_{h_j} is adjacent to z_1, it follows that $m \ge 2$.

- If $j < i$, it follows that the degree of h_i in N is exactly $\tau - 1$. Let N' be the graph obtained from N by adding the edge h_ih_j, and deleting all vertices in $V(H_j) \setminus \{h_j\}$. Let H'_i be the component of N' that contains the edge h_ih_j, and let us assign its head to be h_j. Thus H'_i is a τ-chrysalis, and so N' is a τ-nursery. Since N' has $k-1$ components and $|V(H_i)| \geq |V(H_j)|$ (because $i > j$) it follows that $\phi(N') \geq \phi(N)$, and N' is an improvement of N.

- If $j > i$, it follows that the degree of h_j in N is at most $\tau - 2$. Let N' be the graph obtained from N by adding the edge h_ih_j, and deleting all vertices in $V(H_i) \setminus \{h_i\}$. Let H'_j be the component of N' that contains the edge h_ih_j, and let us assign its head to be h_j. Thus H'_j is a τ-chrysalis, and again N' is an improvement of N.

For each $v \in V(N')$ define X'_v as follows:

- if $v \neq \{h_1, \ldots, h_k\}$ let $X'_v = X_v$;
- let $X'_{h_i} = \{z_1, \ldots, z_m\} \cup \{x_1, \ldots, x_\tau\}$;
- let X'_{h_j} be the set of vertices in Y_{h_j} with a neighbour in $\{z_1, \ldots, z_m\}$;
- for $1 \leq \ell \leq k$ with $\ell \neq i, j$, let X'_{h_ℓ} be the set of vertices in Y_{h_ℓ} with no neighbour in $\{z_1, \ldots, z_m\}$.

We see that X'_{h_i} covers X'_{h_j}, and has no edges to X'_{h_ℓ} for $1 \leq \ell \leq k$ with $\ell \neq i, j$. Moreover, $\mu(X'_{h_j}) \geq \kappa'$. Let $1 \leq \ell \leq k$ with $\ell \neq i, j$; then, since $m \geq 2$ and from the choice of m, the mass of the set of vertices in Y_{h_ℓ} with no neighbour in $\{z_1, \ldots, z_{m-1}\}$ is at least $\kappa' + \epsilon$. Hence $\mu(X'_{h_\ell}) \geq \kappa'$. It follows that the function X'_v $(v \in V(N'))$ is a κ'-realization of N' in G. This proves 2.4. ∎

Now let T be a caterpillar subdivision. We say that an integer $\tau \geq 3$ *fits* T if

- there is a path of T with at most τ vertices containing all vertices of T of degree more than two;
- T has maximum degree at most τ; and
- every path of T in which every internal vertex has degree two in T has at most τ vertices.

2.5 *Let T be a caterpillar subdivision, and let τ fit T. If G is T-free then for $\kappa > 0$, the τ-butterfly is not κ-realizable in G.*

Proof. Suppose that X_v $(v \in V(N))$ is a κ-realization in G of the τ-butterfly N. Now N is connected, and since $|V(N)| = \tau^2 - \tau + 2$, the spine of N has exactly $\tau + 1$ vertices and they all have degree τ except the head which has degree one. Let the spine of N have vertices $v_0, v_1, \ldots, v_\tau$ in order, where v_0 is the head of N. Since $\mu(X_{v_0}) \geq \kappa > 0$, it follows that $X_{v_0} \neq \emptyset$; choose $p_{v_0} \in X_{v_0}$. For $1 \leq i \leq \tau$, choose $p_{v_i} \in X_{v_i}$ adjacent to $p_{v_{i-1}}$; this is possible since X_{v_i} covers $X_{v_{i-1}}$. Now let u be a leaf of N, with neighbour v say. From the definition of a realization, there is a τ-spire $(x^1_u, \ldots, x^\tau_u, Z_u)$ in X_u, and $X_u = \{x^1_u, \ldots, x^\tau_u\} \cup Z_u$. Since p_v has a neighbour in X_u, and $G[Z_u]$ is connected and contains x^τ_u, and none of $x^1_u, \ldots, x^{\tau-1}_u$ have neighbours in $Z_u \setminus \{x^\tau_u\}$, there is an induced path P_u with τ vertices, with one end p_v and with all other vertices in X_u. Let H be the induced subgraph of G consisting of the union of all these paths P_u (over all leaves u of N) and the path induced on $\{p_{v_0}, \ldots, p_{v_\tau}\}$; then T is isomorphic to an induced subgraph of H, contradicting that G is T-free. This proves 2.5. ∎

Now we can prove the main theorem 1.8, which we restate.

2.6 *For every caterpillar subdivision T, there exists $\epsilon > 0$ such that for every T-free graph G, and mass μ on G, either*

- *$\mu(\{v\}) \geq \epsilon$ for some vertex v; or*
- *$\mu(N(v)) \geq \epsilon$ for some vertex v; or*
- *there are two anticomplete sets of vertices A, B, where $\mu(A), \mu(B) \geq \epsilon$.*

Proof. Choose τ fitting T, and let $p = 2^{\tau^2}$. Define ϵ such that $\epsilon^{-1} = p2^p(\tau + 3)$. We will show that ϵ satisfies the theorem. Suppose not, and choose a T-free graph G, and mass μ on G not satisfying the theorem (and therefore satisfying the axioms). For $0 \leq i \leq p$ define $\kappa_i = 2^{-i}p^{-1} - (\tau + 2)\epsilon$. Thus $0 \leq \kappa_i \leq 1$ for each i. Moreover, $\kappa_p = \epsilon$, and for $1 \leq i \leq p$,

$$\kappa_{i-1} = 2\kappa_i + (\tau + 2)\epsilon.$$

Choose $X_1, \ldots, X_P \subseteq V(G)$, pairwise disjoint, with $\kappa_0 \leq \mu(X_i) < \kappa_0 + \epsilon$ for $1 \leq i \leq P$, with P maximum. We claim that $P \geq p$; for suppose not. Then the union of $X_1, \ldots, X_P$ has mass at most $(p-1)(\kappa_0 + \epsilon)$, and since $(p-1)(\kappa_0 + \epsilon) \leq 1 - \kappa_0$, there exists a set of mass at least κ_0 disjoint from this union. Choose such a set, X_{P+1} say, minimal; then from the minimality of X_{P+1}, and since $\mu(\{v\}) < \epsilon$ for each vertex v, it follows that $\mu(X_{P+1}) < \kappa_0 + \epsilon$, contrary to the maximality of P. This proves that $P \geq p$.

Let N_0 be the τ-nursery with p components, each an isolated vertex. It follows that N_0 is κ_0-realizable in G and $\phi(N_0) = 2p$. Choose a sequence $N_1, \ldots, N_q$ of τ-nurseries, such that for $1 \leq i \leq q$, N_i is an improvement of N_{i-1}, and N_i is κ_i-realizable in G, with q maximum. It follows that $\phi(N_i) \geq \phi(N_{i-1})$ for $1 \leq i \leq q$, from the definition of an improvement, and so $\phi(N_q) \geq 2p$, and in particular, N_q is nonnull. But N_i has at most $p - i$ components for $0 \leq i \leq q$, and so $q \leq p - 1$. Thus κ_{q+1} is defined. By 2.5 no component of N_q is the τ-butterfly, and so N_q has at most one component by 2.4, and therefore has at most $\tau^2 - \tau + 1$ vertices. But $\phi(N_q) \geq 2p$, which is impossible.

Thus there is no such pair G, μ. This proves 2.6. ∎

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