Induced subgraphs of graphs with large chromatic number. VIII. Longer odd holes

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Abstract

In an earlier paper, two of us proved that for all κ, every graph with clique number at most κ and sufficiently large chromatic number has an odd hole (a "hole" is an induced cycle of length at least four). In this paper we prove a strengthening; for all κ, ℓ, every graph with clique number at most κ and sufficiently large chromatic number has either a hole of length five or an odd hole of length more than ℓ. This approaches a well-known conjecture of András Gyárfás that for all integers κ, ℓ, every graph with clique number at most κ and sufficiently large chromatic number has an odd hole of length more than ℓ.

1 Introduction

All graphs in this paper are finite and have no loops or parallel edges. We denote the chromatic number of a graph G by $\chi(G)$, and its clique number (the cardinality of its largest clique) by $\omega(G)$. A *hole* in G means an induced subgraph which is a cycle of length at least four, and an *odd* hole is one with odd length. A 5-*hole* means a hole of length five. Two of us proved in [6] a conjecture of András Gyárfás [4], that:

1.1 *For all $\kappa \geq 0$ there exists $c \geq 0$ such that for every graph G, if $\omega(G) \leq \kappa$ and $\chi(G) > c$ then G has an odd hole.*

The same paper of Gyárfás gives a stronger conjecture that has remained open:

1.2 Conjecture: *For all $\kappa, \ell \geq 0$ there exists c such that for every graph G, if $\omega(G) \leq \kappa$ and $\chi(G) > c$ then G has an odd hole of length more than ℓ.*

In this paper we give a result strengthening 1.1 but still weaker than 1.2, the following:

1.3 *For all $\kappa, \ell \geq 0$, there exists $c \geq 0$ such that for every graph G, if $\omega(G) \leq \kappa$ and $\chi(G) > c$ then G has either a 5-hole or an odd hole of length more than ℓ.*

In particular, this gives another proof of 1.1, slightly easier than the original. We remark that there have been several other partial results approaching the conjecture 1.2, in [1, 2, 5, 7]. The two strongest of these, implying the others, are the results of [2, 7] respectively, namely:

1.4 *For all $\kappa, \ell \geq 0$, there exists $c \geq 0$ such that for every graph G, if $\omega(G) \leq \kappa$ and $\chi(G) > c$ then G has a hole of length more than ℓ.*

1.5 *For all $\ell \geq 0$, there exists $c \geq 0$ such that for every graph G, if $\omega(G) \leq 2$ and $\chi(G) > c$ then G has holes of ℓ consecutive lengths (and in particular has an odd hole of length more than ℓ).*

2 Some preliminaries

If G is a graph and $X, Y \subseteq V(G)$, we say that Y *covers* X if $X \cap Y = \emptyset$ and every vertex in X has a neighbour in Y. We will frequently need the following:

2.1 *Let B_1, B_2, C be subsets of $V(G)$, such that B_1, B_2 both cover C (possibly $B_1 = B_2$). Let $X \subseteq C$ be a clique with cardinality $\omega(G)$. Then there exist $b_1 \in B_1$ and $b_2 \in B_2$, distinct, and both with neighbours in X, such that either b_1, b_2 are adjacent, or there is an induced path of length three between them with interior in X.*

Proof. Choose $b_1 \in B_1 \cup B_2$ with as many neighbours in X as possible. From the symmetry we may assume that $b_1 \in B_1$. Since $|X| = \omega(G)$, there exists $x \in X$ nonadjacent to b_1. Choose $b_2 \in B_2$ adjacent to x. If b_1, b_2 are adjacent then the theorem holds, so we assume not. From the choice of b_1, there exists $y \in X$ adjacent to b_1 and not to b_2. But then b_1-y-x-b_2 is an induced path of length three. This proves 2.1. ∎

If $X \subseteq V(G)$, the subgraph of G induced on X is denoted by $G[X]$, and we often write $\chi(X)$ for $\chi(G[X])$. The *distance* between two vertices u, v of G is the length of a shortest path between u, v, or ∞ if there is no such path. If $v \in V(G)$ and $\rho \geq 0$ is an integer, $N_G^\rho(v)$ or $N^\rho(v)$ denotes the set of all vertices u with distance exactly ρ from v, and $N_G^\rho[v]$ or $N^\rho[v]$ denotes the set of all v with distance at most ρ from v. If G is a nonnull graph and $\rho \geq 1$, we define $\chi^\rho(G)$ to be the maximum of $\chi(N^\rho[v])$ taken over all vertices v of G. (For the null graph G we define $\chi^\rho(G) = 0$.)

We might as well assume that $\ell \geq 5$ in 1.3; and the proof of 1.3 will be induction on κ, with ℓ fixed, so we may assume that $\kappa \geq 2$ and the result holds for all smaller κ. In particular there exists τ such that for every graph G, if $\omega(G) \leq \kappa - 1$ and G has no 5-hole and no odd hole of length more than ℓ then $\chi(G) < \tau$. We fix such κ, ℓ, τ, thoughout the paper. Let us say a graph G is a *candidate* if $\omega(G) \leq \kappa$, and G has no 5-hole and no odd hole of length more than ℓ. We must show that there exists c such that every candidate has chromatic number at most c.

We observe first (a result that has been proved many times before):

2.2 *Let G be a candidate, and let $v \in V(G)$. Then $\chi(N^1[v]) \leq \tau$, and $\chi(N^2[v]) \leq \tau^2$.*

Proof. Since $\omega(G) \leq \kappa$, it follows that $\omega(G[N^1(v)]) \leq \kappa - 1$, and so $\chi(N^1(v)) < \tau$. Consequently $\chi(N^1[v]) \leq \tau$. Take a partition $X_1, \dots, X_\tau$ of $N^1(v)$ into stable sets, and let $Y_1, \dots, Y_\tau$ be a partition of $N^2(v)$ such that for $1 \leq i \leq \tau$, every vertex in Y_i has a neighbour in X_i. Suppose that for some i, there is a clique $Z \subseteq Y_i$ with $|Z| = \kappa$. By 2.1, there exist $b, b' \in X_i$ joined by an induced path of length one or three with interior in Y_i. Length one is impossible since X_i is stable; and length three is impossible since adding v would give a 5-hole. This proves that $\omega(G[Y_i]) < \kappa$, and hence $\chi(Y_i) \leq \tau - 1$, for $1 \leq i \leq \tau$. Consequently $\chi(Y_1 \cup \dots \cup Y_\tau) \leq \tau(\tau - 1)$. Since $N^2(v) = Y_1 \cup \dots \cup Y_\tau$, and $N^2[v] = N^2(v) \cup N^1[v]$, it follows that $\chi(N^2[v]) \leq \tau(\tau - 1) + \tau = \tau^2$. This proves 2.2. ▮

We remark that the proof just given uses that G has no 5-hole, and with a view to 1.2, it would be sensible to use this hypothesis as little as possible. Its use here can be avoided by the method sketched at the end of section 2 of [2], at the cost of a much longer proof (and a worse bound on $\chi(N^2(v))$).

As in several other papers of this series, the proof of 1.3 breaks into cases depending whether there is an induced subgraph of large chromatic number such that every ball of small radius in it has bounded chromatic number, or not. Let us make this more precise.

Let $\mathbb{N}$ denote the set of nonnegative integers, and let $\phi : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. For $\rho \geq 1$, let us say a graph G is *(ρ, ϕ)-controlled* if $\chi(H) \leq \phi(\chi^\rho(H))$ for every induced subgraph H of G. Let us say a class of graphs $\mathcal{C}$ is *ρ-controlled* if there is a nondecreasing function $\phi : \mathbb{N} \to \mathbb{N}$ such that every graph in the class is (ρ, ϕ)-controlled.

Thus 2.2 implies that for every 2-controlled class $\mathcal{C}$ of candidates, there exists c such that every graph in $\mathcal{C}$ has chromatic number at most c. To see this, choose ϕ such that every graph in $\mathcal{C}$ is $(2, \phi)$-controlled. Let $G \in \mathcal{C}$; then 2.2 implies that $\chi^2(G) \leq \tau^2$, and so $\chi(G) \leq \phi(\tau^2)$. Consequently setting $c = \phi(\tau^2)$ satisfies the requirement.

Our first major goal is to extend this to larger values of ρ, that is:

2.3 *Let $\rho \geq 2$, and let $\mathcal{C}$ be a ρ-controlled class of candidates. Then there exists c such that every member of $\mathcal{C}$ has chromatic number at most c.*

The proof will take several steps, spread over the next two sections. We will need the following (its proof is an argument of Gyárfás [4]):

2.4 *Let G be a graph, let $k \geq 0$, let $C \subseteq V(G)$, and let $x_0 \in V(G) \setminus C$, such that*

- *$G[C]$ is connected;*
- *x_0 has a neighbour in C; and*
- *$\chi(C) > k\chi^1(G)$.*

Then there is an induced path $x_0\text{-}\cdots\text{-}x_k$ of G where $x_1, \ldots, x_k \in C$, and a subset C' of C, with the following properties:

- *$x_0, \ldots, x_k \notin C'$;*
- *$G[C']$ is connected;*
- *x_k has a neighbour in C', and $x_0, \ldots, x_{k-1}$ have no neighbours in C'; and*
- *$\chi(C') \geq \chi(C) - k\chi^1(G)$.*

Proof. We proceed by induction on k; the result holds if $k = 0$, so we assume that $k > 0$ and the result holds for $k-1$. Consequently there is an induced path $x_0\text{-}\cdots\text{-}x_{k-1}$ of G where $x_1, \ldots, x_{k-1} \in C$, and a subset C'' of C, such that

- $x_0, \ldots, x_{k-1} \notin C''$;
- $G[C'']$ is connected;
- x_{k-1} has a neighbour in C'', and $x_0, \ldots, x_{k-2}$ have no neighbours in C''; and
- $\chi(C'') \geq \chi(C) - (k-1)\chi^1(G)$.

Let N be the set of neighbours of x_{k-1}, and let C' be the vertex set of a component of $G[C'' \setminus N]$, chosen with $\chi(C')$ maximum (there is such a component since $\chi(C'') > \chi^1(G) \geq \chi(N)$). Let x_k be a neighbour of x_{k-1} with a neighbour in C''. Then $x_0\text{-}\cdots\text{-}x_k$ and C' satisfy the theorem. This proves 2.4. ▮

3 Levellings and multicoverings

If $X, Y \subseteq V(G)$, we say X, Y are *anticomplete* if $X \cap Y = \emptyset$ and there are no edges between X and Y. A *levelling* in a graph G is a sequence of pairwise disjoint subsets $(L_0, L_1, \ldots, L_k)$ of $V(G)$ such that

- $|L_0| = 1$;
- for $1 \leq i \leq k$, L_{i-1} covers L_i; and
- for $0 \leq i < j \leq k$, if $j > i + 1$ then L_i is anticomplete to L_j.

If $\mathcal{L} = (L_0, L_1, \dots, L_k)$ is a levelling, L_k is called the *base* of $\mathcal{L}$, and the vertex in L_0 is the *apex* of $\mathcal{L}$, and $L_0 \cup \dots \cup L_k$ is the *vertex set* of $\mathcal{L}$, denoted by $V(\mathcal{L})$.

For $1 \le i \le n$ let $\mathcal{L}_i$ be a levelling in G with vertex set V_i, and let $C \subseteq V(G)$. We say that $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ is a *multicovering* of C if

- $V_1, \dots, V_n, C$ are pairwise disjoint;
- $1 \le i < j \le n$, every vertex in V_i with a neighbour in V_j belongs to the base of $\mathcal{L}_i$;
- for $1 \le i \le n$, every vertex in V_i with a neighbour in C belong to the base of $\mathcal{L}_i$; and
- for $1 \le i \le n$, the base of $\mathcal{L}_i$ covers C.

We call n the *length* of the multicovering $(\mathcal{L}_1, \dots, \mathcal{L}_n)$. A multicovering $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ is *independent* if for $1 \le i < j \le n$, every vertex in $V(\mathcal{L}_j)$ with a neighbour in $V(\mathcal{L}_i)$ belongs to the base of $\mathcal{L}_j$.

Next we need an object rather like a multicovering but different. For $1 \le i \le n$ let $\mathcal{L}_i$ be a levelling in G with vertex set V_i, and let $B, C \subseteq V(G)$. We say that $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ is a *polycovering* of (B, C) if

- the sets $V_1, \dots, V_n, B, C$ are pairwise disjoint;
- the sets $V_1, \dots, V_n, C$ are pairwise anticomplete;
- B covers C, and V_i covers B for $1 \le i \le n$;
- let $\mathcal{L}_1$ be $(L_0, \dots, L_k)$; then $(L_0, \dots, L_k, B)$ is a levelling.

Again, we call n its *length*.

A levelling $(L_0, \dots, L_k)$ has *height* k, and if $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ is a multicovering of C and each $\mathcal{L}_i$ has height k we call it a *k-multicovering* of C. If $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ is a polycovering of (B, C) and each $\mathcal{L}_i$ has height k, and in addition for $1 \le i \le n$, every vertex in $V(\mathcal{L}_i)$ with a neighbour in B belongs to the base of $\mathcal{L}_i$, we call $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ a *k-polycovering* of (B, C).

A levelling $(L_0, \dots, L_k)$ is *stable* if each of the sets $L_0, \dots, L_k$ is stable. A multicovering $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ of C is *stable* if each $\mathcal{L}_i$ is stable; and a polycovering $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ of (B, C) is *stable* if B is stable and each $\mathcal{L}_i$ is stable.

3.1 *Let $\rho \ge 2$, let $\mathcal{C}$ be a ρ-controlled class of graphs, and let $\tau_{\rho-1}$ be such that $\chi^{\rho-1}(G) \le \tau_{\rho-1}$ for each $G \in \mathcal{C}$. For all $c \ge 0$ and $n \ge 0$, there exists c' such that if $G \in \mathcal{C}$ is a graph with chromatic number more than c', then there is a stable $(\rho-1)$-multicovering $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ in G of a set C with $\chi(C) > c$.*

Proof. Choose ϕ such that every graph in $\mathcal{C}$ is (ρ, ϕ)-controlled. We proceed by induction on n. The claim holds if $n = 0$, so we assume that $n > 0$ and that the theorem holds with n replaced by $n - 1$ and c' replaced by c''. Let $c_2 = c'' + \tau_{\rho-1}$, let $c_1 = \tau_{\rho-1}^{\rho} c_2$, and let $c' = \phi(c_1)$; we claim that c' satisfies the theorem. For let $G \in \mathcal{C}$ with $\chi(G) > c'$. Since G is (ρ, ϕ)-controlled, it follows that $\phi(\chi^{\rho}(G)) > c'$, and since ϕ is nondecreasing and $c' = \phi(c_1)$, we deduce that $\chi^{\rho}(G) > c_1$. Consequently there is a vertex v of G such that $\chi(N^{\rho}[v]) > c_1$. Now $\chi^{\rho-1}(G) \le \tau_{\rho-1}$, and in particular $\chi(N^{\rho-1}[v]) \le \tau_{\rho-1}$, and so $\chi(N^{\rho}(v)) > c_1 - \tau_{\rho-1} = c''$. Since $\chi^{\rho-1}(G) \le \tau_{\rho-1}$, there is a $\tau_{\rho-1}$-colouring of $G[N^{\rho-1}[v]]$, say ψ. For each $v \in N^\rho(u)$, take a path P_u between v, u of length ρ; each of its vertices except u is assigned a colour by ψ. Let f_u be the sequence of the colours of the vertices of $P_u \setminus \{u\}$, in order starting from v. There are only $\tau_{\rho-1}^\rho$ possibilities for f_u; so there exists $C_2 \subseteq C_1$ with $\chi(C_2) > c_2$, such that all the sequences f_u are the same for all $u \in C_2$. For $0 \le i \le \rho$ let L_i be the set of vertices w such that for some $u \in C$, w is the ith vertex of P_u. It follows that $L_0, \dots, L_{\rho-1}$ are all stable.

Let $\mathcal{L}_1 = (L_0, \dots, L_{\rho-1})$; then $\mathcal{L}_1$ is a stable levelling. From the inductive hypothesis applied to $G[L_\rho]$, there is a stable $(\rho-1)$-multicovering $(\mathcal{L}_2, \dots, \mathcal{L}_n)$ in G of a set C with $\chi(C) > c$; and then $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ satisfies the theorem. This proves 3.1. ■

If $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ is an independent multicovering in G of C, we say it is *starred* if there exist b_i in the base of $\mathcal{L}_i$ for each $i \in \{1, \dots, n\}$, and $z \in C$, such that each b_i is adjacent to z, and the vertices $b_i (1 \le i \le n)$ are pairwise nonadjacent. We observe:

3.2 *For all $n \ge 0$ there exists $n' \ge 0$ with the following property. Let G be a candidate, and let $(\mathcal{L}_1, \dots, \mathcal{L}_{n'})$ be a multicovering in G of some set $C \neq \emptyset$. Then some n-term subsequence of the sequence $(\mathcal{L}_1, \dots, \mathcal{L}_{n'})$ is starred.*

Proof. Choose n' such that every graph with at least n' vertices has either a stable set of size n or a clique of size κ. We claim that n' satisfies the theorem. For let G, C and $(\mathcal{L}_1, \dots, \mathcal{L}_{n'})$ be as in the theorem, and let $z \in C$. For $1 \le i \le n'$ choose a neighbour b_i of z in the base of $\mathcal{L}_i$. Since $\omega(G) \le \kappa$, the subgraph induced on the vertices $\{b_i : 1 \le i \le n'\}$ has no clique of size κ; so it has an stable set of size n. The corresponding subsequence of the multicovering is starred. This proves 3.2. ■

With this we can polish 3.1 a little, as follows:

3.3 *Let $\rho \ge 2$, let $\mathcal{C}$ be a ρ-controlled class of candidates, and let $\tau_{\rho-1}$ be such that $\chi^{\rho-1}(G) \le \tau_{\rho-1}$ for each $G \in \mathcal{C}$. For all $c, n \ge 0$, there exists c' with the following properties. Let $G \in \mathcal{C}$ such that $\chi(G) > c'$. Then there exists $C \subseteq V(G)$ with $\chi(C) > c$, and either*

- *there is a starred independent stable $(\rho-1)$-multicovering $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ in G of C; or*
- *there is a stable $(\rho-2)$-polycovering $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ in G of (B, C) for some B.*

Proof. Let n' be as in 3.2. Let $c_2 = 2^{n^2 n'} c$, let $c_1 = c_2 + nn'\tau_{\rho-1}$, and let c' satisfy 3.1 with n replaced by nn' and c by c_1; we claim that c' satisfies the theorem. For let $G \in \mathcal{C}$ such that $\chi(G) > c'$. By 3.1, there is a stable $(\rho-1)$-multicovering $(\mathcal{L}_1, \dots, \mathcal{L}_{nn'})$ in G of a set C_1 with $\chi(C_1) > c_1$. Let X be the set of vertices $v \in C_1$ such that for some $i \in \{1, \dots, nn'\}$, the distance in G between v and the apex of $\mathcal{L}_i$ is less than ρ. Thus $\chi(X) \le nn'\tau_{\rho-1}$, and so $C_1 \setminus X$ has chromatic number more than c_2. For $1 \le i \le nn'$, let B_i be the base of $\mathcal{L}_i$.

For each $v \in C_1 \setminus X$ and $1 \le i \le nn'$, choose a neighbour $b(v,i)$ of v in B_i. For each $v \in C_1 \setminus X$, and $1 \le i < j \le nn'$, let $f_{ij}(v) = 1$ if $b(v,i)$ has a neighbour in $V_j \setminus B_j$, and $f_{ij}(v) = 0$ otherwise. There are at most $2^{n^2 n'^2}$ possibilities for the matrix of numbers $f_{ij}(v)$ $(1 \le i < j \le nn')$, so there exist a subset C of $C_1 \setminus X$ with $\chi(C) > c$ and $f_{ij} \in \{0,1\}$ for all i, j with $1 \le i < j \le nn'$, such that $f_{ij}(v) = f_{ij}$ for each $v \in C$.

Suppose first that for some i there are at least $n-1$ values of j with $i < j \le nn'$ such that $f_{ij} = 1$, say $j_1, \dots, j_{n-1}$. Let B be the set of vertices in B_i with a neighbour in C and with a neighbour in

$V_j \setminus B_j$ for each $j \in \{j_1, \ldots, j_{n-1}\}$. Since $b(v, i) \in B$ for each $v \in C$, it follows that B covers C. For each $j \in \{i, j_1, \ldots, j_{n-1}\}$, let $\mathcal{L}'_j$ be obtained from $\mathcal{L}_j$ by removing its final term (that is, its base). Since every vertex $v \in B$ has a neighbour in C, and this neighbour has distance at least ρ from the apex of $\mathcal{L}_j$, it follows that v has distance at least $\rho - 1$ from this apex; and hence every neighbour of v in V_j belongs to one of the last two terms of the sequence $\mathcal{L}_j$. Since v has a neighbour in $V_j \setminus B_j$, it follows that v has a neighbour in the base of $\mathcal{L}'_j$, and all its neighbours in $V(\mathcal{L}'_j)$ belong to the base of $\mathcal{L}'_j$. Consequently $(\mathcal{L}'_i, \mathcal{L}'_{j_1}, \ldots, \mathcal{L}'_{j_{n-1}})$ is a stable $(\rho - 2)$-polycovering of (B, C) and the second bullet of the theorem holds.

We may therefore assume that for each i, there are fewer than n choices of j with $i < j \leq nn'$ such that $f_{ij} = 1$. The graph with vertex set $\{1, \ldots, nn'\}$ in which i, j are adjacent (for $i < j$) if $f_{ij} = 1$ is therefore $(n - 1)$-degenerate and so n-colourable, and since it has nn' vertices, it has a stable set of cardinality n'. Hence there are n' numbers $i_1 < \cdots < i_{n'}$ such that $f_{ij} = 0$ for all $i, j \in \{i_1, \ldots, i_{n'}\}$ with $i < j$. For each $i \in \{i_1, \ldots, i_{n'}\}$, let B'_i be the set $\{b(v, i) : v \in C\}$, and let $\mathcal{L}'_i$ be obtained from $\mathcal{L}_i$ by replacing its final term with B'_i. Then $(\mathcal{L}'_{i_1}, \ldots, \mathcal{L}'_{i_{n'}})$ is an independent stable multicovering of C, and by 3.2 and the choice of n', it has a subsequence of length n which is starred; and hence the first bullet of the theorem holds. This proves 3.3. ▮

4 $(\rho - 1)$-multicoverings

Some notation: let $\mathcal{L}$ be a levelling $(L_0, \ldots, L_k)$ say, and let $p, q \in L_k$. Then there is an induced path P joining p, q with $V(P) \subseteq V(\mathcal{L})$, using at most two vertices of L_i for $0 \leq i \leq k$. Moreover, if the levelling is stable, this path has even length. We denote some such path by $\mathcal{L}(p, q)$.

Let us return to the proof of 2.3. By 3.3, we may assume that we have one of the two outcomes of 3.3, and first we handle the first case, by the following theorem.

4.1 *For all $\rho \geq 3$ there exists c with the following property. Let G be a candidate, and let $(\mathcal{L}_1, \mathcal{L}_2)$ be a starred independent stable $(\rho - 1)$-multicovering in G of a set C. Then $\chi(C) \leq c$.*

Proof. Let $c = (\ell + 8)\tau^2 + \ell\tau$; and let G, C and $(\mathcal{L}_1, \mathcal{L}_2)$ be as in the theorem. Suppose that $\chi(C) > c$. For $i = 1, 2$ let B_i be the base of $\mathcal{L}_i$.

Since the multicovering is starred, there exists $z \in C$, with neighbours $b_1 \in B_1$ and $b_2 \in B_2$ that are not adjacent. Since $\chi(C) > c \geq \tau$, there is a clique $X \subseteq C$ of cardinality κ. By 2.1 there exist $b'_1 \in B_1$ and $b'_2 \in B_2$ joined by an induced path Q of length one or three with interior in X.

Let $Y = \{b_1, b_2, z\} \cup V(Q)$, and let Z be the set of vertices in C with distance at least 3 from every vertex in Y. Since $|Y| \leq 7$, it follows that

$$\chi(Z) > c - 7\tau^2 = (\ell + 1)\tau^2 + \ell\tau.$$

Let W be a component of $G[Z]$ with maximum chromatic number, and choose $x_0 \in B_1$ with a neighbour in W. By 2.4, since $\chi(W) = \chi(Z) > \ell\tau$, there is an induced path x_0-$\cdots$-x_ℓ of G where $x_1, \ldots, x_\ell \in W$, and a subset C' of W, such that:

- $x_1, \ldots, x_\ell \notin C'$;
- $G[C']$ is connected;

- x_ℓ has a neighbour in C', and $x_0, \dots, x_{\ell-1}$ have no neighbours in C'; and
- $\chi(C') \ge \chi(W) - \ell\tau > (\ell+1)\tau^2$.

From the last bullet above, there is a vertex in C' with distance at least 3 from each of $x_0, \dots, x_\ell$. Choose a neighbour of this vertex in B_2, say y, and let R be an induced path between x_0, y with interior in W such that $x_1, \dots, x_\ell$ are all vertices of R. This exists since y is nonadjacent to $x_0, \dots, x_\ell$ (because it has a neighbour with distance at least 3 from each of them). In summary then, R is a path of length at least $\ell+1$, between x_0, y, and $V(R)$ is anticomplete to Y.

Suppose first that R has odd length. Then the union of R, $\mathcal{L}_1(x_0, b_1)$, $\mathcal{L}_2(y, b_2)$ and the path b_1-z-b_2 is an odd hole of length more than ℓ, a contradiction. If R has even length, then the union of R, $\mathcal{L}_1(x_0, b'_1)$, $\mathcal{L}_2(y, b'_2)$ and Q is an odd hole of length more than ℓ, again a contradiction. This proves 4.1. ▮

Now we handle the second outcome of 3.3.

4.2 *For all $\rho \ge 3$ there exist c with the following property. Let G be a candidate, and let $(\mathcal{L}_1, \mathcal{L}_2)$ be a stable $(\rho-2)$-polycovering in G of a pair (B, C). Then $\chi(C) \le c$.*

Proof. Let $c = 2(\ell-1)\tau + 2(\ell+7)\tau^2$, let $G, \mathcal{L}_1, \mathcal{L}_2, B, C$ be as in the theorem, and suppose that $\chi(C) > c$. Let A_1, A_2 be the bases of $\mathcal{L}_1, \mathcal{L}_2$ respectively. Thus every vertex in $V(\mathcal{L}_i)$ with a neighbour in B belongs to A_i, for $i = 1, 2$. By 2.1, since $c > \tau$, there is an induced path p_2-$\cdots$-p_5 of length three where $p_2, p_5 \in B$ and $p_3, p_4 \in C$. Among all such choices of $p_2, \dots, p_5$, choose p_2-$\cdots$-p_5 such that the set of vertices in A_2 with a neighbour in $\{p_2, p_5\}$ is minimal.

Let us say $a \in A_1 \cup A_2$ is a *grandparent* of $z \in C$ if there exists $b \in B$ such that a-b-z is an induced path. Every vertex of C has a grandparent in A_1, and every vertex in A_1 is nonadjacent to at least one of p_2, p_5. By reversing the path p_2-$\cdots$-p_5 if necessary, we may assume that the set Z_1 of vertices in C that have a grandparent in A_1 that is nonadjacent to p_5 has chromatic number at least $\chi(C)/2$, and hence more than $(\ell-1)\tau + (\ell+7)\tau^2$.

Let Y_1 be the set of vertices in B that have a neighbour in A_1 nonadjacent to p_5. It follows that Z_1 is the set of vertices in C that have a neighbour in Y_1. Let Y_2 be the set of all vertices in Y_1 with a neighbour in A_2 nonadjacent to p_2. For every vertex in $Y_1 \setminus Y_2$, all its neighbours in A_2 are adjacent to p_2; and so, from the minimality of the set of neighbours in A_2 of p_2, p_5 (and since p_5 has a neighbour in A_1 that is not adjacent to p_2), there is no induced path p'_2-$\cdots$-p'_5 of length three where $p'_2, p'_5 \in Y_1 \setminus Y_2$ and $p'_3, p'_4 \in C$. By 2.1, the set of vertices in C with a neighbour in $Y_1 \setminus Y_2$ has chromatic number at most τ; and so the set Z_2 of vertices in Z_1 with a neighbour in Y_2 has chromatic number at least $\chi(Z_1) - \tau > (\ell-2)\tau + (\ell+7)\tau^2$.

Choose $p_1 \in A_1$ adjacent to p_2, and $p_6 \in A_2$ adjacent to p_5. Since p_2, p_5 have no common neighbour in $A_1 \cup A_2$ (because G has no 5-hole), it follows that p_1-$\cdots$-p_6 is an induced path. Let $Z_3 \subseteq Z_2$ be the set of vertices in Z_2 with distance at least three from each of $p_1, \dots, p_6$. Consequently $\chi(Z_3) \ge \chi(Z_2) - 6\tau^2 > (\ell-2)\tau + (\ell+1)\tau^2$. Let C' be the vertex set of a component of $G[Z_3]$, chosen with maximum chromatic number. Let Y_3 be the set of vertices in Y_2 with a neighbour in C'. Since Y_2 covers Z_2 it follows that Y_3 covers C'.

Now $Y_3 \subseteq Y_1$, so every vertex in Y_3 has a neighbour in A_1 nonadjacent to p_5; choose $A'_1 \subseteq A_1$ minimal such that no vertex in A'_1 is adjacent to p_5, and A'_1 covers Y_3. Since $Y_3 \neq \emptyset$, there exists $q'_1 \in A'_1$. From the minimality of A'_1, there exists $q_2 \in Y_3$ such that q'_1 is its only neighbour in A'_1.

Since $q_2 \in Y_3 \subseteq Y_2$, there exists $q_1 \in A_2$ adjacent to q_2 and nonadjacent to p_2. Since $q_2 \in Y_3$, q_2 has a neighbour in C'. Since $\chi(C') > (\ell-2)\tau$, by 2.4 there is an induced path $q_2\text{-}\cdots\text{-}q_\ell$ where $q_3, \ldots, q_\ell \in C'$, and a subset C'' of C', such that

- $q_2, \ldots, q_\ell \notin C''$;
- $G[C'']$ is connected;
- q_ℓ has a neighbour in C'', and $q_2, \ldots, q_{\ell-1}$ have no neighbours in C''; and
- $\chi(C'') \geq \chi(C') - (\ell-2)\tau > (\ell+1)\tau^2$.

Since $\chi(C'') > (\ell+1)\tau^2$, there exists $z \in C''$ with distance at least three from each of $q'_1, q_1, q_2, \ldots, q_\ell$. Since $z \in C' \subseteq Z_2$, z has a neighbour $y \in Y_2$. Since $G[C'']$ is connected, the path $q_2\text{-}\cdots\text{-}q_\ell$ is a subpath of an induced path $q_2\text{-}\cdots\text{-}q_{n-1}$ where $q_{n-1} = y$ and $q_{\ell+1}, \ldots, q_{n-2} \in C''$. Now q_1 is nonadjacent to q_{n-1}, since q_{n-1} is adjacent to z and z has distance at least three from q_1. Consequently $q_1\text{-}\cdots\text{-}q_{n-1}$ is an induced path. Let $q_n \in A'_1$ be adjacent to q_{n-1}. Since $q_n \neq q'_1$ (because z has distance at least three from q'_1) and q'_1 is the only neighbour of q_2 in A'_1, it follows that q_2, q_n are nonadjacent; and so $q_1\text{-}\cdots\text{-}q_n$ is an induced path Q say.

Let P be the path $p_1\text{-}\cdots\text{-}p_6$. Since every vertex of Q has distance at most two from some vertex in C', and every vertex in C' has distance at least three from every vertex of P, it follows that $V(P) \cap V(Q) = \emptyset$. We need to investigate edges between $V(P)$ and $V(Q)$; suppose then that p_i is adjacent to q_j where $1 \leq i \leq 6$ and $1 \leq j \leq n$. Since the distance between p_i and C' is at least three, it follows that the distance between q_j and C' is at least two, and so $j \in \{1, n\}$. Since there are no edges between $\{q_1, q_n\}$ and $\{p_1, p_3, p_4, p_6\}$, it follows that $i \in \{2, 5\}$.

Now q_1 is not adjacent to p_2, from the choice of q_1; and q_n is not adjacent to p_5, since p_5 has no neighbour in A'_1. Thus the only possibilities for edges between P and Q are p_2q_n and p_5q_1. If p_2 is adjacent to q_n let R be the path $p_2\text{-}q_n$. If p_2, q_n are not adjacent, let R be the path obtained by adding the edge p_1p_2 to the even path $\mathcal{L}_1(p_1, q_n)$. In either case R has odd length. Similarly, there is an induced path S of odd length between p_5, q_1, with $V(S) \setminus \{p_5, p_6, q_1\} \subseteq V(\mathcal{L}_2) \setminus A_2$. But then the union of P, Q, R, S is an odd hole of length more than ℓ, a contradiction. This proves 4.2. ▌

We deduce 2.3, which we restate:

4.3 *Let $\rho \geq 2$, and let $\mathcal{C}$ be a ρ-controlled class of candidates. Then there exists c such that every graph in $\mathcal{C}$ has chromatic number at most c.*

Proof. We proceed by induction on ρ. As was noted after 2.2, the claim holds for $\rho = 2$, so we assume that $\rho > 2$ and the claim holds for $\rho - 1$.

Choose c_1 such that 4.1 is satisfied with c replaced by c_1, choose c_2 such that 4.2 is satisfied with c replaced by c_2, and let $c = \max(c_1, c_2)$. For each integer $x \geq 0$, let $\phi(x) \geq \phi(x-1)$ (or $\phi(x) \geq 0$ if $x = 0$) be such that 3.3 is satisfied with $\tau_{\rho-1}$ replaced by x, n replaced by 2, and c' replaced by $\phi(x)$. Let $\mathcal{C}_x$ be the class of all induced subgraphs H of members of $\mathcal{C}$ with $\chi^{\rho-1}(H) \leq x$.

Suppose that for some x, there exists $G \in \mathcal{C}_x$ with $\chi(G) > \phi(x)$. By 3.3, there exists $C \subseteq V(G)$ with $\chi(C) > c$, and either

- there is a starred independent stable $(\rho-1)$-multicovering $(\mathcal{L}_1, \mathcal{L}_2)$ in G of C; or

- there is a stable $(\rho-2)$-polycovering $(\mathcal{L}_1, \mathcal{L}_2)$ in G of (B, C) for some B.

By 4.1 the first is impossible; and by 4.2 the second is impossible.

Thus there is no such G; that is, for every induced subgraph H of a member of $\mathcal{C}$, $\chi(H) \leq \phi(x)$ for all $x \geq \chi^{\rho-1}(H)$, and in particular, $\chi(H) \leq \phi(\chi^{\rho-1}(H))$. Consequently every graph in $\mathcal{C}$ is $(\rho-1, \phi)$-controlled, and so $\mathcal{C}$ is $(\rho-1)$-controlled, and the result follows from the inductive hypothesis. This proves 4.3. ▮

5 Multicoverings in the uncontrolled case

To show 1.3, we need to show that every candidate has bounded chromatic number. In view of 4.3, it suffices to show that the class of all candidates is ℓ-controlled. Suppose not; then as in the proof of 4.3, there exist x and a class of candidates G with $\chi^{\ell}(G) \leq x$ and with unbounded chromatic number. Thus it suffices to prove the following:

5.1 *For all τ_ℓ there exists c such that if G is a candidate with $\chi^{\ell}(G) \leq \tau_\ell$ then $\chi(G) \leq c$.*

Proving this is the goal of the remainder of the paper.

In 3.1 we could obtain stable multicoverings, but this depended on ρ-control, and no longer works. But we can at least arrange that the bases of our levellings are stable. Let us say a levelling is *stable-based* if its base is stable, and a multicovering is *stable-based* if each term is stable-based. We begin with:

5.2 *For all $\tau_\ell, c \geq 0$ there exists $c' \geq 0$ such that if G is a candidate with $\chi^{\ell}(G) \leq \tau_\ell$ and $\chi(G) > c'$ then there is a levelling $(L_0, \ldots, L_k)$ in G with $\chi(L_k) > c$ and L_{k-1} stable.*

Proof. Let $c_2 = \tau_\ell + 2\tau$, let $c_1 = c^2 c_2$, and let $c' = 2c_1$. Let G be a candidate with $\chi^{\ell}(G) \leq \tau_\ell$ and $\chi(G) > c'$.

(1) *There is a levelling $(L_0, \ldots, L_k)$ in G such that $\chi(L_k) > c_1$.*

For let G_1 be a component of G with maximum chromatic number, and let $z_0 \in V(G_1)$. For all $i \geq 0$ let L_i be the set of vertices with distance i from z_0. Then there exists k such that $\chi(L_k) \geq \chi(G)/2 > c'/2 = c_1$. This proves (1).

(2) *There is a levelling $(L_0, \ldots, L_k)$ in G with the following properties:*

- *$\chi(L_k) > c_1$;*
- *$G[L_k]$ is connected; and*
- *for $0 \leq i < k$ and for every vertex $v \in L_i$, there exists $u \in L_{i+1}$ such that v is the unique neighbour of u in L_i.*

For choose $L_0, \dots, L_k$ as in (1) with $L_0 \cup \dots \cup L_k$ minimal. Consequently deleting any vertex of L_k reduces the chromatic number of $G[L_k]$, and hence $G[L_k]$ is connected. Also, for $0 \le i < k$ and $v \in L_i$,

$$(L_0, \dots, L_{i-1}, L_i \setminus \{v\}, L_{i+1}, \dots, L_k)$$

is not a levelling, and so v is the unique neighbour in L_i of some vertex in L_{i+1}. This proves (2).

Since $c_1 \ge \tau_\ell$, it follows that $k > \ell \ge 5$. Choose $z \in L_{k-2}$. Let X be the set of vertices in L_{k-2} with distance at least $\ell + 1$ from z in G. Hence $\chi(X) \ge \chi(L_{k-2}) - \tau_\ell$. Let $A = L_0 \cup \dots \cup L_{k-3}$ and $B = L_{k-1} \cup L_k$.

(3) *For all $v \in X$, either every induced path between v, z with interior in A is even and every induced path between v, z with interior in B is even, or every induced path between v, z with interior in A is odd and every induced path between v, z with interior in B is odd.*

For there is an induced path between v, z with interior in A, from the definition of a levelling; and there is one with interior in B, by (2). Each such path has length more than ℓ, and the union of a path of the first type and a path of the second is a hole of length more than ℓ and is consequently even. This proves (3).

Let X_0 be the set of all vertices $v \in X$ such that every induced path between v, z with interior in A is even, and $X_1 = X \setminus X_0$.

(4) *Let $j \in \{0, 1\}$, and let $u, v \in X_j$ be adjacent; then every neighbour of u in L_{k-3} is also adjacent to v, and vice versa.*

For suppose that $w \in L_{k-3}$ is adjacent to u and not to v. By (2), there is an induced path between v, L_k containing no neighbour of u except v, and also there is an induced path between z, L_k containing no neighbour of u; and since $G[L_k]$ is connected and contains no neighbours of u, it follows that there is an induced path Q between v, z containing no neighbours of u except v, with interior in B. Choose an induced path P between w and some neighbour z' of z in L_{k-3}, with interior in $L_0 \cup \dots \cup L_{k-4}$. But then adding the edges uw and zz' to P gives an induced path P' between v, z with interior in A, which therefore has the same parity as Q, by (3); and so adding the edge uv to the union of P' and Q gives an odd hole of length more than ℓ, which is impossible. This proves (4).

It follows that for $i = 0, 1$, $\omega(G[X_i]) < \kappa$, since every connected subgraph of $G[X_i]$ has a common neighbour in L_{k-3} by (4). Hence $\chi(X_i) \le \tau$ for $i = 0, 1$. We deduce that $\chi(L_{k-2}) \le \tau_\ell + 2\tau = c_2$. Take a partition of $G[L_{k-2}]$ into c_2 stable sets, say $Y_1, \dots, Y_{c_2}$. Every vertex in L_{k-1} has a neighbour in at least one of these sets, so there is a partition $Y'_1, \dots, Y'_{c_2}$ of L_{k-1} such that for $1 \le i \le c_2$, Y_i covers Y'_i. If some Y'_i has chromatic number more than c, then $(L_0, \dots, L_{k-3}, Y_i, Y'_i)$ satisfies the theorem, so we assume not. Hence $\chi(L_{k-1}) \le cc_2$. Take a partition of L_{k-1} into cc_2 stable sets $Z_1, \dots, Z_{cc_2}$, and take a partition $Z'_1, \dots, Z'_{cc_2}$ of L_k such that each Z_i covers Z'_i. Since $\chi(L_k) > c^2 c_2$, there exists i with $\chi(Z'_i) > c$; and so $(L_0, \dots, L_{k-2}, Z_i, Z'_i)$ satisfies the theorem. This proves 5.2. ▮

We deduce:

5.3 *For all $\tau_\ell, c, n \ge 0$, there exists c' such that if G is a candidate with chromatic number more than c' and with $\chi^\ell(G) \le \tau_\ell$, then there is a stable-based multicovering $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ in G of a set C with $\chi(C) > c$.*

Proof. We proceed by induction on n; the result holds for $n = 0$, so we assume that $n > 0$ and the result holds for $n - 1$. Choose c'' such that the theorem is satisfied with n replaced by $n - 1$ and c' replaced by c''. Choose c' such that 5.2 is satisfied with c replaced by c''. We claim that c' satisfies the theorem.

For let G be a candidate with chromatic number more than c' and with $\chi^\ell(G) \le \tau_\ell$. By 5.2 there is a levelling $(L_0, \dots, L_k)$ in G with $\chi(L_k) > c''$ and L_{k-1} stable. Let $\mathcal{L}_1$ be the levelling $(L_0, \dots, L_{k-1})$; then it is stable-based. From the inductive hypothesis, there is a stable-based multicovering $(\mathcal{L}_2, \dots, \mathcal{L}_n)$ in $G[L_k]$ of some set $C \subseteq L_k$ with $\chi(C) > c$. But then $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ satisfies the theorem. This proves 5.3. ▌

This can be polished just as we polished 3.1 in 3.3, to give the following (the proof is exactly analogous to that of 3.3 and we omit it).

5.4 *For all $c, n, \tau_\ell \ge 0$, there exists c' with the following properties. Let G be a candidate such that $\chi^\ell(G) \le \tau_\ell$ and $\chi(G) > c'$. Then there exists $C \subseteq V(G)$ with $\chi(C) > c$, and either*

- *a starred independent stable-based multicovering $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ in G of C, or*
- *a polycovering $(\mathcal{L}_1, \dots, \mathcal{L}_n)$ in G of (B, C), for some stable set B.*

6 Finishing the uncontrolled case

We may assume that one of the two outcomes of 5.4 holds, and we handle them separately. The first case will be handled by the following:

6.1 *For all $\tau_\ell \ge 0$ there exists c with the following property. Let G be a candidate with $\chi^\ell(G) \le \tau_\ell$, and let $(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ be a starred independent multicovering in G of a set C. Then $\chi(C) \le c$.*

Proof. For $i = 1, 2, 3$, let B_i be the base of $\mathcal{L}_i$. Since the multicovering is starred, there exist $z \in C$ and $b_i \in B_i$ for $i = 1, 2, 3$, such that z is adjacent to b_1, b_2, b_3, and b_1, b_2, b_3 are pairwise nonadjacent.

Let Z be the set of all $v \in C$ with distance at least $\ell + 1$ from z. Consequently $\chi(Z) \ge \chi(C) - \tau_\ell$. For each $v \in B_i$, let $P_i(v)$ be some path $\mathcal{L}_i(v, b_i)$. Each vertex in W has neighbours in B_1, B_2, B_3, and the corresponding paths $P_i(v) (i = 1, 2, 3)$ may be even or odd, a total of eight possibilities. Thus there exists $W \subseteq Z$ with $\chi(W) \ge \chi(Z)/8$ and $f_1, f_2, f_3 \in \{0, 1\}$, such that for all $w \in W$ and $i = 1, 2, 3$, w has a neighbour v in B_i such that the path $P_i(v)$ has even length if $f_i = 0$ and odd length if $f_i = 1$.

Now two of f_1, f_2, f_3 are equal, say f_1, f_2 (without loss of generality, since reordering the levellings in an independent multicovering gives another). For $i = 1, 2$, let B'_i be the set of $v \in B_i$ such that $P_i(v)$ has length of parity f_i. It follows that B'_1, B'_2 each cover W.

Since $\chi(W) > \tau$, by 2.1 there exist $b'_1 \in B'_1$ and $b'_2 \in B'_2$, joined by an induced path Q of length one or three with interior in W, and b'_1, b'_2 both have neighbours in W. In particular the distance between b'_i and a_0 is at least ℓ, and so $P_i(b'_i)$ has length at least ℓ for $i = 1, 2$. The sets

$$\{z\},\ V(P_1(b'_1)) \setminus V(Q),\ V(P_2(b'_2)) \setminus V(Q),\ V(Q)$$

are pairwise disjoint, and we claim that the only edges between these sets are the edges zb_1, zb_2, and edges of $P_1(b_1'), P_2(b_2')$. To see this, note that there are no edges between z and $V(Q)$, since every vertex of Q has distance at least ℓ from z. Moreover, every vertex of $P_i(b_i') \setminus V(Q)$ belongs to $V(\mathcal{L}_i) \setminus B_i$ except for b_i, and so since the multicovering is independent, the only edges between $P_1(b_1') \setminus V(Q)$ and $P_2(b_2') \setminus V(Q)$ are between b_1, b_2, and hence there are no such edges since b_1, b_2 are nonadjacent. Hence the union of Q, $P_1(b_1)$ and $P_2(b_2)$ is an odd hole of length more than ℓ, a contradiction. This proves 6.1. ▮

For the second case of 5.4 we use the following:

6.2 *For all $\tau_\ell \geq 0$ there exists c with the following property. Let G be a candidate with $\chi^\ell(G) \leq \tau_\ell$, and let $(\mathcal{L}_1, \mathcal{L}_2)$ be a polycovering in G of a pair (B, C), where B is stable. Then $\chi(C) \leq c$.*

Proof. For $i = 1, 2$, let $V_i = V(\mathcal{L}_i)$. Choose $b \in B$, and let Z be the set of vertices in C with distance at least $\ell + 1$ from b. Consequently $\chi(Z) > \chi(C) - \tau_\ell$. Let $\mathcal{L}_1 = (L_0, \dots, L_k)$; then from the definition of polycovering, it follows that $(L_0, \dots, L_k, B)$ is a levelling $\mathcal{L}$ say. For each $v \in B$ with a neighbour in Z, let $P_1(v)$ be some path $\mathcal{L}(v, b)$. For each vertex $v \in B$ with a neighbour in Z, choose an induced path $P_2(v)$ between v, b with interior in V_2, of minimum length.

If $P_1(v)$ is even and $P_2(v)$ is odd, or vice versa, then $P_1(v) \cup P_2(v)$ is an odd hole of length more than ℓ, a contradiction. Thus either $P_1(v), P_2(v)$ are both even or they are both odd. Let B_0' be the set of $v \in B$ with a neighbour in W such that $P_1(v), P_2(v)$ are both even, and B_1' the set with $P_1(v), P_2(v)$ both odd. Every vertex in W has a neighbour in one of B_0', B_1', so there exists $W' \subseteq W$ with $\chi(W') > \chi(W)/2$, and $B' \subseteq B$, and $f \in \{0, 1\}$, such that B' covers W', and for each $v \in B'$, $P_1(v)$ and $P_2(v)$ both have parity f. Since $\chi(W') > \tau$, and B is stable, by 2.1 there exist $b_1, b_2 \in B'$ joined by an induced path of length three with interior in W'.

By exchanging b_1, b_2 if necessary, we may assume that $P_2(b_1)$ has length at least that of $P_2(b_2)$. Now b_1 has no neighbour in $P_1(b_2)$, since b_1 has no neighbours in $L_0 \cup \dots \cup L_{k-1}$, and b_1, b_2 have no common neighbour in the base of $\mathcal{L}_1$ (since G has no 5-hole), and b_1 has distance at least ℓ from b. Now b_1, b_2 also have no common neighbour in the base of $\mathcal{L}_2$ (since G has no 5-hole); and so b_1 has no neighbour in $P_2(b_2)$, since there is no path between b_1, b with interior in V_2 of length less than that of $P_2(b_2)$. But then the union of $P_1(b_1), P_2(b_2)$ and Q is an odd hole of length more than ℓ, a contradiction. This proves 6.2. ▮

We deduce 5.1, which we restate:

6.3 *For all τ_ℓ there exists c such that if G is a candidate with $\chi^\ell(G) \leq \tau_\ell$ then $\chi(G) \leq c$.*

Proof. Let 6.1 be satisfied with c replaced by c_1, and let 6.2 be satisfied with c replaced by c_2. Let $c_0 = \max(c_1, c_2)$, and let 5.4 be satisfied with c', c, n replaced by $c, c_0, 3$ respectively. Suppose that G is a candidate with $\chi^\ell(G) \leq \tau_\ell$ and $\chi(G) > c$. By 5.4 there exists $C \subseteq V(G)$ with $\chi(C) > c_0$, and either

- an independent stable-based multicovering $(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ in G of C, or
- a polycovering $(\mathcal{L}_1, \mathcal{L}_2)$ in G of (B, C), for some stable set B.

But the first contradicts 6.1 and the second contradicts 6.2. This proves 6.3, and hence completes the proof of 1.3. ▮

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