Even Pairs and Prism Corners in Square-Free Berge Graphs

Maria Chudnovsky [*]
Princeton University, Princeton, NJ 08544

Frédéric Maffray [†]
CNRS, Laboratoire G-SCOP, University of Grenoble-Alpes, France

Paul Seymour [‡]
Princeton University, Princeton, NJ 08544

Sophie Spirkl
Princeton University, Princeton, NJ 08544

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Abstract

Let G be a Berge graph such that no induced subgraph is a 4-cycle or a line-graph of a bipartite subdivision of K_4. We show that every such graph G either is a complete graph or has an even pair.

1 Introduction

All graphs in this paper are finite and simple. For a graph G and $X \subseteq V(G)$, $G|X$ denotes the induced subgraph of G with vertex set X. Two disjoint sets X, Y of vertices in a graph are *complete* to each other if every vertex in X is adjacent to every vertex in Y, and *anticomplete* to each other if no vertex in X is adjacent to a vertex in Y. We say that v is *complete* to $X \subseteq V(G)$ if $\{v\}$ is complete to X, and v is *anticomplete* to $X \subseteq V(G)$ if $\{v\}$ is anticomplete to X. For $X, Y \subseteq V(G)$, we say that X *touches* Y if either $X \cap Y \neq \emptyset$ or there exists $x \in X, y \in Y$ so that $xy \in E(G)$. Let $v \in V(G)$; we say that v *touches* $X \subseteq V(G)$ if $\{v\}$ touches X; and we say that v is a *neighbor of* X if $v \notin X$, but v touches X.

For a vertex $v \in V(G)$, we let $N_G(v) = N(v)$ denote the set of neighbors of v in G. A *clique* in a graph is a set of pairwise adjacent vertices, and for a graph G, $\omega(G)$ denotes the size of the largest clique in G. By a *path* in a graph we mean an induced path, and the *length* of a path is the number of edges in it. A path is *odd* if its length is odd, and *even* otherwise.

Let $k \geq 4$ be an integer. A *hole of length* k in a graph is an induced subgraph isomorphic to the k-vertex cycle C_k, and an *antihole of length* k is an induced subgraph isomorphic to C_k^c (here G^c denotes the complement of G). A hole (or antihole) is *odd* if its length is odd. A graph is called *Berge* if it has no holes of odd length, and no antiholes of odd length. A hole of length four is called a *square*, and a graph is *square-free* if it does not contain a square.

An *even pair* in a graph is a pair of vertices $\{u, v\}$ such that every path from u to v is even, and in particular, u and v are non-adjacent. (We remind the reader that by a path we always mean an

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induced path.) The *contraction* operation for even pairs is defined as follows. The graph G' is obtained from G by *contracting* $\{u, v\}$ if

- $V(G') = (V(G) \setminus \{u, v\}) \cup \{w\}$;
- $G' \setminus \{w\} = G \setminus \{u, v\}$; and
- $N_{G'}(w) = N_G(u) \cup N_G(v)$.

It is not difficult to see that if G is Berge and G' is obtained from G by contracting an even pair $\{u, v\}$, then G' is Berge, and that $\omega(G') = \omega(G)$ [3]. Moreover, given a coloring of G' with $\omega(G)$ colors, one can obtain a coloring of G with $\omega(G)$ colors by assigning u and v the color of w, and keeping the colors of the remaining vertices unchanged. A graph G is called *even contractile* if there is a sequence of graphs $G_1, \ldots, G_t$ where $G_1 = G$, G_t is the complete graph with $\omega(G)$ vertices, and for $i \in \{1, \ldots, t-1\}$, G_{i+1} is obtained from G_i by contracting an even pair of G_i. If the even pair to be contracted at every stage can be found algorithmically, as for instance in Theorem 1.3, this leads to a polynomial time coloring algorithm. For this reason, and because of their role in the understanding of the structure of Berge graphs, much attention has been devoted to determining which Berge graphs are even contractile, or have even pairs.

A *prism* K in a graph G is an induced subgraph consisting of two disjoint *triangles* $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ and three disjoint paths P_1, P_2, P_3, where P_i has ends a_i and b_i, and for $1 \leq i < j \leq 3$ the only edges between $V(P_i)$ and $V(P_j)$ are a_ia_j and b_ib_j. The vertices in $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ are called *corners* of the prism. A vertex $v \in V(G) \setminus V(K)$ is *major* for K if v has at least two neighbors in each of the triangles of K. A vertex $v \in V(G)$ is *rough* for the prism K if there exist $\{i, j, k\} = \{1, 2, 3\}$ such that either v is an end of the path P_k, or v has a neighbor in $\{a_i, b_i\}$ and in $\{a_j, b_j\}$ and either

- there is a path Q, called the *normal path*, from v to an interior vertex of P_k such that the set of interior vertices of Q is anticomplete to $\{a_i, a_j, b_i, b_j\}$; or
- P_k has length one and v is adjacent to both a_k and b_k;

P_k is called a *base path* for v in K.

We say that a vertex v is a *corner* (*major, rough*) in G if there is a prism K in G such that v is a corner (major, rough) for K. A vertex is *smooth* if it is not rough.

A prism is *odd* if P_1, P_2, P_3 are all odd, and *even* if they are all even. It is easy to see that if G is Berge, then every prism is either even or odd. Everett and Reed made the following conjecture:

It is not true that every major vertex is a rough vertex, because major vertices need not have neighbors in the interior of paths, but in the case of odd prisms, the following holds:

Lemma 1.1. *If K is an odd prism in a Berge graph G, and v is major for K, then v is rough for K.*

Proof. If there is an i, say $i = 1$, so that v does not have a neighbor in P_i, then v is adjacent to a_2, a_3, b_2, b_3 because v is major, and thus v-a_2-a_1-P_1-b_1-b_3-v is an odd hole in G, a contradiction. Thus, v has a neighbor in every path P_i of K.

Because v is major, if there is some k such that v is adjacent to neither a_k nor b_k, then v is adjacent to some vertex in the interior of P_k, and to a_i, a_j, b_i, b_j; hence it is a rough vertex with base path P_k.

Otherwise, v is adjacent to at least one of $\{a_i, b_i\}$ for each i. There is at least one path P_k of K such that v is adjacent to both its ends. But P_k is odd, and so v-a_k-P_k-b_k-v is an odd hole unless v has a neighbor in the interior of P_k or P_k has length one. Thus, v is rough for K with base path P_k. $\square$

For a prism K, a subset of its vertex set is *local* if it is contained in one path or one triangle of K. A vertex $v \in V(G) \setminus V(K)$ is a *local neighbor of* K if v has a neighbor in K and the set of neighbors of v in $V(K)$ is local.

Conjecture 1.2 ([3])**.** *If a Berge graph has no odd prism and no antihole of length at least six, then it is even contractile.*

Conjecture 1.2 is still open, but the following weaker statements have been proved:

Theorem 1.3 ([7])**.** *If a Berge graph has no prism and no antihole of length at least six, then it is even contractile.*

Theorem 1.4 ([5])**.** *If a square-free Berge graph has no odd prism, then either it is a complete graph or it has an even pair.*

A graph H is a *subdivision* of a graph G if H is obtained from G by repeatedly subdividing edges. H is a *bipartite subdivision* of G if H is a subdivision of G, and H is bipartite. The line-graph $L(G)$ of G is the graph with vertex set $E(G)$, and such that $e, f \in E(G)$ are adjacent in $L(G)$ if and only if e and f share an end in G. A graph G is called *flat* if it is Berge and it contains no induced subgraph isomorphic to the line-graph of a bipartite subdivision of K_4.

Hougardy [4] made the following related conjecture:

Conjecture 1.5 ([4])**.** *If G is a minimal Berge graph with no even pair, then G is either an even antihole of length at least six, or the line-graph of a bipartite graph.*

Here we prove the following result, in the spirit of Theorem 1.4:

Theorem 1.6. *If G is a square-free flat graph, then either G is a complete graph or G has an even pair.*

In view of Conjecture 1.2 and Theorem 1.6 one might hope that the common generalization holds, i.e. that every flat graph with no antihole of length at least six is either a complete graph or has an even pair. This is not the case, as the graph on the left in Figure 1 shows. It is the line-graph of the bipartite series-parallel graph on the right; thus it is flat, and it contains no antihole of length at least six, but it does not have an even pair.

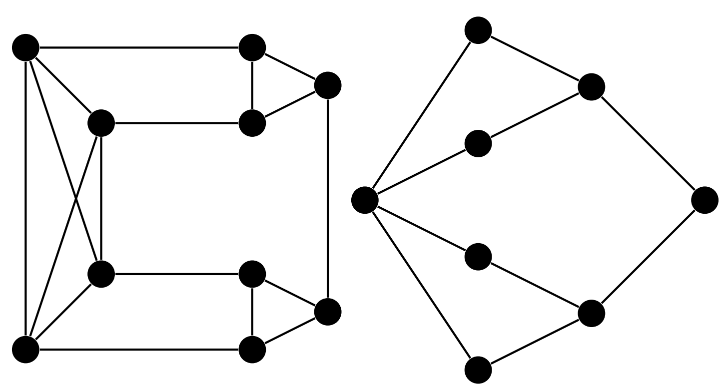

Figure 1: G and a bipartite series-parallel graph H with $G = L(H)$

To prove Theorem 1.6 we use an idea first suggested by the second author [6] to approach Conjecture 1.2. A vertex is *simplicial* if its neighbor set is a clique. Note that a graph that is the disjoint union of cliques has no non-simplicial vertices. The second author conjectured that

Conjecture 1.7 ([6])**.** *Every Berge graph with no odd prism and no antihole of length at least six either is a disjoint union of cliques or has a vertex that is not a corner and not simplicial.*

He further suggested that

Conjecture 1.8 ([6]). *If G is a Berge graph with no antihole of length at least six, and $v \in V(G)$ is not a corner and not simplicial, then the neighbor set of v includes an even pair of G.*

Our first result is a variant of Conjecture 1.7:

Theorem 1.9. *Let G be a square-free flat graph. Then either G is a disjoint union of cliques, or some $v \in V(G)$ is smooth and not simplicial.*

We then closely follow the outline of the proof of Theorem 1.3 and show the following variant of Conjecture 1.8.

Theorem 1.10. *Let G be a Berge graph with no antihole of length at least six, such that every proper induced subgraph of G either is a complete graph or has an even pair. Let v be a vertex of G that is smooth and not simplicial. Then the neighborhood of v includes an even pair of G.*

Proof of Theorem 1.6, assuming Theorems 1.9 and 1.10. We prove this by induction on $|V(G)|$. We may assume that G is not complete. If G is the union of at least two disjoint cliques, then two vertices in different connected components form an even pair; thus we may assume that G is not a disjoint union of cliques. By Theorem 1.9, there is a smooth, non-simplicial vertex. Since G is square-free, it contains no antihole of length at least six, so by Theorem 1.10 there is an even pair in G. This proves Theorem 1.6. □

The proof of Theorem 1.9 appears in Sections 2 and 3, and Section 4 is devoted to the proof of Theorem 1.10.

2 Prism Corners

Let G be a non-null square-free flat graph with no clique cutset. We want to show that there is a vertex in G that is smooth.

A *megaprism* in G is an induced subgraph P such that $V(P)$ admits a partition into twelve sets $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4$ with the following properties:

- $A_1, A_2, A_3, B_1, B_2, B_3 \neq \emptyset$;
- A_i is complete to A_j, and B_i is complete to B_j for all distinct $i, j \in \{1, 2, 3\}$; the sets $A = A_1 \cup A_2 \cup A_3$ and $B = B_1 \cup B_2 \cup B_3$ are called *potatoes* of P;
- for $i = 1, 2, 3$, the vertex set of every component of $G|C_i$ touches A_i and B_i, every vertex in A_i has a neighbor in $B_i \cup C_i$, and every vertex in B_i has a neighbor in $A_i \cup C_i$;
- the vertex set of every component of $G|A_4$ touches A, and the vertex set of every component of $G|B_4$ touches B;
- for every edge uv of P, $\{u, v\}$ is a subset of one of the sets $A \cup A_4$, $B \cup B_4$, C_4, S_1, S_2, S_3, where $S_i = A_i \cup B_i \cup C_i$ for $i = 1, 2, 3$; and
- every vertex in $M = V(G) \setminus V(P)$ is *major* for P, that is, it is complete to two of A_1, A_2, A_3 and to two of B_1, B_2, B_3.

Thus, every component of $G|C_4$ is a component of P, and we shall soon show that $C_4 = \emptyset$. Components of $G|A_4$ and $G|B_4$ are called *side components*. For $i = 1, 2, 3$, the set $S_i = A_i \cup C_i \cup B_i$ is called a *strip* of P with *interior* C_i. A path from A_i to B_i with interior in C_i is called a *rung* of the strip S_i. We call the sets $A_i, B_i (i = 1, 2, 3)$ *P-ends*.

The following is proved (with different terminology) in the first paragraph of the proofs of Theorems 4.2 and 5.2 of [1].

Theorem 2.1. *Let G be a square-free flat graph that contains a prism with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$. Then there is a megaprism P such that the sets A, B as defined above satisfy that $\{a_1, a_2, a_3\} \subseteq A$ and $\{b_1, b_2, b_3\} \subseteq B$.*

In Lemmas 2.2-2.6, we always assume that G is a non-null square-free flat graph with no clique cutset, and that P is a megaprism in G with notation as above.

The following was proved in [1], Theorem 5.2 (4). We include a short proof for completeness.

Lemma 2.2. *M is a clique and for each potato of P, at most one of the P-ends included in this potato is not complete to M. Also, $C_4 = \emptyset$.*

Proof. Suppose that M is not a clique. Then, M contains two non-adjacent vertices x, y, each complete to two P-ends in each potato. Hence, they have common neighbors $a \in A$ and $b \in B$, and if a and b can be chosen to be not adjacent, this forms a square, a contradiction. Therefore, a and b are adjacent; thus both are in the same strip, say S_1, and x, y have no common neighbors in $A_2 \cup A_3$. Up to symmetry, this implies that since x, y are major, x is complete to A_2 and has no neighbors in A_3 and y is complete to A_3 and has no neighbors in A_2. Let $v \in A_2, u \in A_3$; then x-b-y-u-v-x is a hole of length five, which cannot happen in a Berge graph. This proves that M is a clique.

Next, we suppose that two P-ends in the same potato contain vertices x, y that are not complete to M. Then there are major vertices u, v such that x is adjacent to u but not v, and y is adjacent to v but not u. Since x is adjacent to y and u is adjacent to v, they form a square.

Finally, since M is a clique and not a clique cutset, it follows that $C_4 = \emptyset$. □

We say that a P-end is *good* if it is a clique and all its vertices are complete to M. Otherwise, it is called a *bad* P-end.

Lemma 2.3. *For each potato, at most one of the P-ends it includes is a bad P-end.*

Proof. By Lemma 2.2, at most one P-end is not complete to M. If there are two P-ends that are not cliques, there is a square formed by two pairs of non-adjacent vertices that are complete to each other. If u, v are in the same P-end in a potato p and non-adjacent, and x is in a different P-end in p and non-adjacent to some $y \in M$, then x-u-y-v-x is a square. □

Let S_i be strip of P. We define $\widetilde{S_i}$ as follows.

- if both A_i and B_i are good P-ends, then $\widetilde{S_i} = C_i$;
- if A_i is a good P-end and B_i is a bad P-end, then $\widetilde{S_i} = C_i \cup B_i \cup B_4$;
- if B_i is a good P-end and A_i is a bad P-end, then $\widetilde{S_i} = C_i \cup A_i \cup A_4$; and
- if both A_i and B_i are bad P-ends, then $\widetilde{S_i} = C_i \cup A_i \cup A_4 \cup B_i \cup B_4$.

In Lemma 2.7, we will show that if there is a prism, then some strip of some megaprism has no prism corners in its interior. Our strategy to prove this is, we can assume there is a prism and hence a megaprism; choose a megaprism with a strip "minimal" in some sense, and prove that no vertex in the interior of the strip is a corner. The intuition behind this is, if a vertex v in the interior of this minimal strip (S say) is a corner, then grow the corresponding prism to a megaprism; it is difficult for the strips of the new megaprism to "escape" from S – more or less, only one can escape through each end of S – so one will be trapped inside of S, and this will contradict the minimality of S. But there are difficulties. First, we need to make sure that there are vertices in the interior of S; so let us choose S to be a strip with no rung of length one, and subject to that with something minimal. Second, too many strips of the new megaprism can sometimes "escape" from S into side components, when the corresponding end of S is not a clique, so we would like to consider this as not really escaping, which means we sometimes need to include the side components as part of the strip. This led us to try choosing a megaprism and a strip S with no rungs of length one and then with $\widetilde{S}$ minimal, and this is an approach that works. It may not be the simplest method, but the example of Figure 2 shows that several simpler methods do not work.

In the graph of Figure 2, the sets $A_1 = \{v_1\}, A_2 = \{v_2\}, A_3 = \{v_3\}, C_1 = \{v_{15}, v_{16}, v_{17}\}, C_2 = \{v_4\}, C_3 = \{v_5\}, B_1 = \{v_8, v_9, v_{18}\}, B_2 = \{v_6\}, B_3 = \{v_7\}, B_4 = \{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$ define a megaprism, and there is another from the left-right symmetry. Those are the only two; and an internal vertex of S_1 is a corner vertex, so whatever we minimize must not be minimized by this strip. The advantage of minimizing $\widetilde{S}$ is that $\widetilde{S_1}$ includes the corresponding set of a strip of the other megaprism, and so is not minimal. (A more obvious fix for this example is, choose S with as few vertices as possible. That works for this example, but we could subdivide the edge v_2v_4 a large even number of times, and the same for $v_3v_5, v_{10}v_{12}, v_{11}v_{13}$, and then simply counting vertices no longer works.)

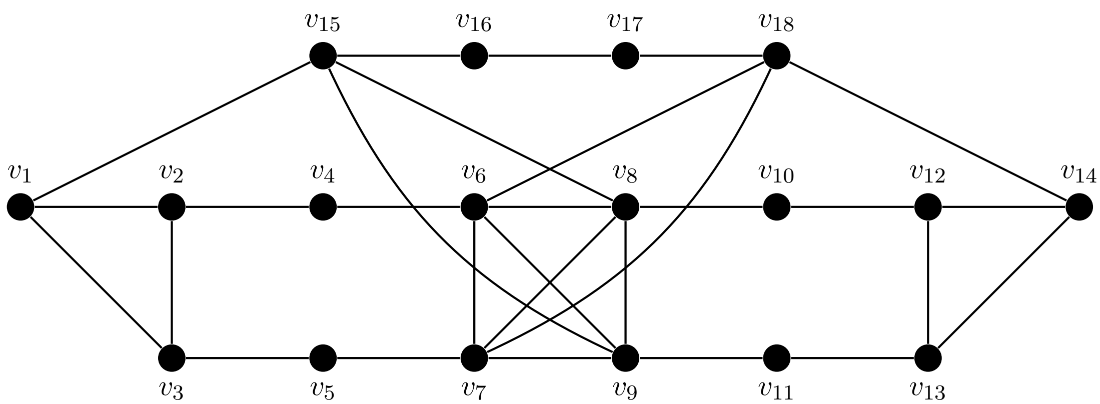

Figure 2: A hard example

Lemma 2.4. *Let S_i be a strip with a bad P-end A_i; then every component of A_4 has a neighbor in A_i. The same holds for B_i and B_4.*

Proof. This follows because the union of M and good P-ends in A is not a clique cutset. □

Lemma 2.5. *For $i \in \{1, 2, 3\}$, every vertex with distance one from $\widetilde{S_i}$ is either in M or in a good P-end, and therefore complete to M. Consequently, every path from a vertex in $\widetilde{S_i}$ to a vertex not in $\widetilde{S_i}$ contains a vertex that is either in M or in a good P-end. Moreover, if $X \subseteq V(P)$ is connected, X touches $\widetilde{S_i}$, and no vertex $v \in X$ is complete to $M \setminus \{v\}$, then $X \subseteq \widetilde{S_i}$.*

Proof. Let $i = 1$, say. First, note that vertices with distance one from $\widetilde{S_1}$ are either in M or A or B. Moreover, if A_1 is a good P-end, then vertices in A with distance one from $\widetilde{S_1}$ are in A_1. If A_1 is a bad P-end, then vertices in A with distance one from $\widetilde{S_1}$ are in $A_2 \cup A_3$, and since A_1 is a bad P-end by Lemmma 2.3, both A_2 and A_3 are good P-ends. The last statement of the lemma follows since every vertex v in a good P-end is complete to M by definition, and every vertex v in M is complete to $M \setminus \{v\}$ by Lemma 2.2. □

Lemma 2.6. *For all distinct $i, j \in \{1,2,3\}$, $\widetilde{S_i}$ and $\widetilde{S_j}$ do not touch. Let R be a path with one end in $\widetilde{S_i}$ and the other end in $\widetilde{S_j}$ for some $i \neq j$. Then the interior of R contains either a vertex in M, or a vertex in a good P-end. Moreover, $\widetilde{S_i}$ and $\widetilde{S_j}$ are anticomplete to each other for all $i \neq j$.*

Proof. Since by Lemma 2.3 at most one P-end in each potato is bad, it follows that $\widetilde{S_i}$ and $\widetilde{S_j}$ do not touch, and in particular they are anticomplete to each other. But $V(R) \not\subseteq \widetilde{S_i}$, and R is connected, so it contains a vertex with distance one from $\widetilde{S_i}$, and the result follows from Lemma 2.5. $\square$

Since G is square-free, for every megaprism, at most one of its strips has a rung of length one.

Lemma 2.7. *Let G be a non-null square-free flat graph with no clique cutset, and P, S_1 be chosen such that P is a megaprism in G with partition $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4$ (where $C_4 = \emptyset$), and S_1 is a strip of P with no rung of length one, and among all such choices of P, S_1, the set $\widetilde{S_1}$ is minimal with respect to inclusion.*

Let P' be another megaprism in G, and let $A'_1, A'_2, A'_3, A'_4, B'_1, B'_2, B'_3, B'_4, C'_1, C'_2, C'_3, C'_4$ (where $C'_4 = \emptyset$) be the sets of the partition of P', and let $A' = A'_1 \cup A'_2 \cup A'_3$ and $B' = B'_1 \cup B'_2 \cup B'_3$ be the potatoes of P'. Then $(A' \cup B') \cap C_1 = \emptyset$. In particular, no vertex in C_1 is a corner.

Proof. Assume for a contradiction that $(A' \cup B') \cap C_1 \neq \emptyset$. For $i = 1,2,3$, let $S'_i = A'_i \cup C'_i \cup B'_i$ denote the strips of P', and let $M' = V(G) \setminus V(P')$ denote the major vertices for P'. By Lemma 2.2, both M and M' are cliques. We distinguish several cases depending on which vertices in $V(P')$ are also in M.

(1) *$M \cap C'_i = \emptyset$ for $i = 1,2,3$. Moreover, for $i = 1,2,3$, either $M \cap A'_i = \emptyset$ or $M \cap B'_i = \emptyset$.*

Let $\{i,j,k\} = \{1,2,3\}$, and suppose that either $M \cap C'_i \neq \emptyset$ or $M \cap A'_i, M \cap B'_i \neq \emptyset$. Define $X = S'_j \cup S'_k \cup \widetilde{S'_j} \cup \widetilde{S'_k}$; then $G|X$ is connected by Lemma 2.4 and C_1 touches X, but X contains no vertex complete to $M \cap S'_i$. Thus, by Lemma 2.5, $\widetilde{S'_j}, \widetilde{S'_k} \subseteq \widetilde{S_1}$. Since one of S'_j, S'_k has no rung of length one, this contradicts the minimality of $\widetilde{S_1}$. This proves (1).

(2) *For some $i \in \{1,2,3\}$, either $M \cap V(P') \subseteq A'_i$ or $M \cap V(P') \subseteq B'_i$.*

Suppose not; then by (1), we may assume that either $M \cap A'_4 \neq \emptyset$ or M contains vertices from at least two different strips of P'. In the latter case, since M is a clique, we may assume that M contains vertices in two P'-ends in A'. Let $X = V(P') \setminus (A' \cup A'_4)$; then no vertex in X is complete to $M \cap (A' \cup A'_4)$, but $G|X$ is connected and C_1 touches X. By Lemma 2.5, $X \subseteq \widetilde{S_1}$. Let S'_i be a strip of P' with a good P'-end in A' and no rung of length one; then $\widetilde{S'_i} \subset \widetilde{S_1}$, a contradiction. This proves (2).

(3) *Let $X = C'_2 \cup C'_3 \cup B'_2 \cup B'_3$ together with the components of $G|B'_4$ that have neighbors in $B'_2 \cup B'_3$. If $M \cap A'_1 \neq \emptyset$, then $X \subseteq \widetilde{S_1}$.*

Suppose not. Then we may assume that $M \cap V(P') \subseteq A'_1$ by (2). Let $w \in M \cap A'_1$. Since X is connected and does not touch $\{w\} \subseteq M$, it suffices to show that $\widetilde{S_1}$ touches X by Lemma 2.5.

Since $C_1 \subseteq \widetilde{S_1}$, we may assume that C_1 does not touch X. By our assumption, $C_1 \cap (A' \cup B') \neq \emptyset$. Since every vertex in $B' \cup A'_2 \cup A'_3$ touches X, it follows that $\widetilde{S_1} \cap (B' \cup A'_2 \cup A'_3) = \emptyset$. Consequently, $C_1 \cap A'_1 \neq \emptyset$. Since $C_1 \cap A'_1$ is complete to $A'_2 \cup A'_3$, and since $(A'_2 \cup A'_3) \cap M = \emptyset$, it follows that $A'_2 \cup A'_3 \subseteq (A_1 \cup B_1) \setminus \widetilde{S_1}$.

By symmetry, we may assume that $(A'_2 \cup A'_3) \cap A_1 \neq \emptyset$. It follows that $A_1 \not\subseteq \widetilde{S_1}$, and so A_1 is a good P-end, and in particular, A_1 is a clique and every vertex in A_1 touches $A'_2 \cup A'_3$. Let R be a rung of S'_1 from a vertex in $C_1 \cap A'_1$ to a vertex $b \in B'_1$. If $b \in \widetilde{S_1}$, then X touches $\widetilde{S_1}$. Therefore, we may assume that $b \notin \widetilde{S_1}$, and consequently, some vertex z in $C'_1 \cup B'_1$ has distance one from $\widetilde{S_1}$. But z does not touch $A'_2 \cup A'_3$, so z is not in A_1. Since $M \cap (C'_1 \cup B'_1) = \emptyset$, it follows that $z \notin M$. Therefore, z is in

B. Since z has distance one from $\widetilde{S_1}$, z is in a good P-end. It follows that z is touches every vertex of B, and since no vertex in $A_2' \cup A_3'$ touches z, it follows that $(A_2' \cup A_3') \cap B = \emptyset$. Since $A_2' \cup A_3' \subseteq A_1 \cup B_1$, it follows that $A_2' \cup A_3' \subseteq A_1$.

Let $Y = V(R) \cup X$. Since neither X nor $V(R) \setminus A_1'$ contains a vertex complete to $A_2' \cup A_3'$, and since $V(R) \cap A_1' \subseteq C_1$, it follows that $Y \cap A = \emptyset$. Since A is a cutset separating $A_1' \cap C_1 \subseteq Y$ from A_4, it follows that $Y \cap A_4 = \emptyset$, and consequently, $X \cap (A \cup A_4) = \emptyset$. But the set X has a neighbor in $A_2' \cup A_3' \subseteq A_1$, and since $X \cap M = \emptyset$, it follows that X contains a vertex in $\widetilde{S_1}$. Hence $X \subseteq \widetilde{S_1}$, which proves (3).

(4) $\quad M \cap V(P') = \emptyset$.

Suppose not, and by (1), let $w \in M \cap A_1'$, say, and let $Z = A_2' \cup A_3'$. Let X be as in (3); and so $X \subseteq \widetilde{S_1}$. Let us pick a strip S_i' of P' such that its P'-end A_i' is good, and it has no rung of length one. Consequently $\widetilde{S_i'} \subseteq C_i' \cup B_i' \cup B_4'$. By the minimality of $\widetilde{S_1}$, it follows that $\widetilde{S_i'} \not\subset \widetilde{S_1}$. Therefore, $i = 1$ by (3), since $\widetilde{S_2'}, \widetilde{S_3'} \subset X$ by Lemma 2.4 if their P'-end in A' is good. Consequently, A_1' is a good P'-end of S_1' and S_1' has no rung of length one. Moreover, since at most one of S_2', S_3' has a rung of length one, it follows that one of A_2', A_3' is a bad P'-end, and so Z is either not a clique or not complete to M'. Also, vertices in $B_1' \cup B_4'$ are not adjacent to w, and $B_1' \cup B_4' \cup X$ is connected, so by Lemma 2.5, $B_1' \cup B_4' \subseteq \widetilde{S_1}$. Since $\widetilde{S_1'} \not\subset \widetilde{S_1}$, it follows that $C_1' \setminus \widetilde{S_1} \neq \emptyset$. Since $C_1' \cup B_1' \cup X$ is connected and $B_1' \cup X \subseteq \widetilde{S_1}$, Lemma 2.5 implies that there is a vertex $a \in C_1'$ that is in a good P-end. Without loss of generality, we assume that $a \in A$; then a is complete to $A \setminus \{a\}$.

Since at most one strip has a rung of length one, there exists $j \in \{2, 3\}$ such that S_j' has no rung of length one. Then $\widetilde{S_j'} \not\subset \widetilde{S_1}$; and since $B_1' \subseteq \widetilde{S_1} \setminus \widetilde{S_j'}$, it follows that $\emptyset \neq \widetilde{S_j'} \setminus \widetilde{S_1} \subseteq \widetilde{S_j'} \setminus X$. The set $X \cup \widetilde{S_j'}$ is connected, so $\widetilde{S_j'} \setminus \widetilde{S_1}$ contains a vertex b in a good P-end by Lemma 2.5. It follows that $b \in A_2' \cup A_3' \cup A_4'$, because $b \notin X$. Since b is not adjacent to a, it follows that $b \notin A$ and thus, $b \in B$ and b is complete to $B \setminus \{b\}$.

We claim that $Z \cap (A_4 \cup A) = \emptyset$, and that $M' \cap (A_2 \cup A_3 \cup A_4) = \emptyset$. Let $Y = S_2' \cup S_3' \cup \widetilde{S_2'} \cup \widetilde{S_3'}$; then Y contains no neighbor of a and hence Y is disjoint from A. The set Y is connected and contains b, which implies that $Y \cap A_4 = \emptyset$. Since $Z \subset Y$, it follows that $Z \cap (A_4 \cup A) = \emptyset$. Moreover, $B_2' \cup B_3' \subseteq X \cap Y$, and so $B_2' \cup B_3' \subseteq C_1 \cup B_1 \cup B_4$. Since every vertex in M' has a neighbor in $B_2' \cup B_3'$, it follows that $M' \cap (A_2 \cup A_3 \cup A_4) = \emptyset$.

We choose a path R as follows. Let R_1 be a one- or two-vertex path from b to a vertex in $B_2 \cup B_3$, depending whether $b \in B_2 \cup B_3$ or $b \in B_1$. Let R_2 be a rung of P starting at the end of R_1 in $B_2 \cup B_3$ and ending at some vertex $r \in A_2 \cup A_3$. Finally, let $R = R_1 \cup R_2$.

By construction, since $b \notin \widetilde{S_1}$, every interior vertex of R has distance at least two from $\widetilde{S_1}$. Therefore, no interior vertex of R is in X or M'. Moreover, $r \notin X$ because $r \notin \widetilde{S_1}$; and $r \notin M'$, because $r \in A_2 \cup A_3$, and we proved that $(A_2 \cup A_3) \cap M' = \emptyset$.

Let $R^* = R$ if $r = a$, and $R^* = a$-r-R-b otherwise. Then R^* has ends $a \in C_1'$ and $b \in A_2' \cup A_3' \cup A_4'$. Since $A_1' \cup M' \cup X$ is a cutset separating C_1' from $A_2' \cup A_3' \cup A_4'$, and $(\{a\} \cup V(R)) \cap (X \cup M') = \emptyset$, and $a, b \notin A_1'$, it follows that $(V(R) \setminus \{b\}) \cap A_1' \neq \emptyset$. Let $y \in V(R) \cap A_1'$.

Since $N(r) \cap \widetilde{S_1} \subseteq A \cup A_4$, and no internal vertex of R has a neighbor in $\widetilde{S_1}$, it follows that $N(y) \cap \widetilde{S_1} \subseteq A \cup A_4$. But y is complete to Z, and $Z \cap (A_4 \cup A) = \emptyset$, so $Z \cap \widetilde{S_1} = \emptyset$. Every vertex in Z touches $X \subseteq \widetilde{S_1}$, and since $Z \cap (A_4 \cup A \cup \widetilde{S_1}) = \emptyset$, it follows that $Z \subseteq B \cup B_4$. All vertices in Z have distance one from $\widetilde{S_1}$, and $Z \cap M = \emptyset$, so all vertices of Z are in good P-ends; it follows that Z is included in the union of all good P-ends in B. Thus, Z is a clique and complete to M.

We showed earlier that Z is either not a clique or not complete to M', and so there is a vertex $m \in M' \setminus M$ such that m is not complete to Z. It follows that $m \notin B$, and we already proved that $M' \cap (A_2 \cup A_3 \cup A_4) = \emptyset$, so $m \notin A_2 \cup A_3 \cup A_4$. Since m has a neighbor in $Z \subseteq B$, and S_1 has no rung of length one, $m \notin A_1$. Since m has a neighbor in $B'_2 \cup B'_3 \subseteq X \subseteq \widetilde{S_1}$, $m \notin C_2 \cup C_3$. Since $y \in A'_1$ and A'_1 is a good P'-end, m is adjacent to y and so $m \notin C_1$. It follows that $m \in B_4$, and so B_1 is a bad P-end, since m touches $\widetilde{S_1}$. But then $V(R) \cap B = \{b\}$, and so m is not adjacent to y, a contradiction; (4) follows.

By (4) we have $M \subseteq M'$, and so $V(P') \subseteq V(P)$.

(5) *If A_1 is a good P-end, and there exist distinct $i, j \in \{1, 2, 3\}$ such that $A_1 \cap (A'_i \cup A'_4), A_1 \cap (A'_j \cup A'_4) \neq \emptyset$, then $V(P') \setminus (A' \cup A'_4) \subseteq V(P) \setminus (A \cup A_4)$.*

Let $a_i \in A_1 \cap (A'_i \cup A'_4), a_j \in A_1 \cap (A'_j \cup A'_4)$, and let X be as in (3). Then no vertex in X touches both a_i and a_j. Since A_1 is a good P-end, every vertex in A touches a_i and a_j, and therefore, $X \cap A = \emptyset$. Moreover, $X \subset V(P')$, so $X \cap M = \emptyset$. Since $A \cup M$ is a cutset separating A_4 from $V(P) \setminus (A \cup A_4)$ and by (3), X touches C_1, it follows that $X \cap A_4 = \emptyset$. This proves (5).

A potato p of P' is *sweet* if $p \subseteq S_1 \cup \widetilde{S_1}$, and the good P'-ends in p are included in S_1.

(6) *There is a sweet potato of P'.*

There exists i such that $(A'_i \cup B'_i) \cap C_1 \neq \emptyset$; choose a value of i with this property such that S'_i has a rung of length one if possible. We may assume from the symmetry that $i = 1$ and $A'_1 \cap C_1 \neq \emptyset$; and, since at most one strip of P' has a rung of length one, it follows that for $j = 2, 3$, if S'_j has a rung of length one then $(A'_j \cup B'_j) \cap C_1 = \emptyset$. Let $v \in A'_1 \cap C_1$.

Every vertex in $A'_2 \cup A'_3$ has a neighbor in C_1, and it follows that $A'_2 \cup A'_3 \subseteq S_1$. Suppose that (6) is false. In particular there exists $w \in A'_1 \setminus S_1$, for otherwise both statements of (6) are true. Then w is non-adjacent to v. Consequently, A'_1 is not a clique, and so A'_2 and A'_3 are good P'-ends by Lemma 2.3. It follows that $A'_1 \not\subseteq S_1 \cup \widetilde{S_1}$, since (6) is false, and therefore we may assume that $w \notin \widetilde{S_1}$. Since v, w have a common neighbor in $V(P') \subseteq V(P)$, it follows that $w \notin C_2 \cup C_3$, and we may assume without loss of generality that $w \in A_2 \cup A_3 \cup A_4$, and thus $A'_2 \cup A'_3 \subseteq A_1$.

We claim that if A_1 is a good P-end, then $A_4 \subseteq A'_4$ and $A \cup A_4 \subseteq A' \cup A'_4 \cup M'$. For let $X = V(P') \setminus (A' \cup A'_4)$. Since A_1 is a good P-end and $A'_2, A'_3 \subseteq A_1$, (5) implies that $X \cap (A \cup A_4) = \emptyset$. Thus $A \cup A_4 \subseteq A' \cup A'_4 \cup M'$. If $u \in A_4$, then u has no neighbor in $V(P) \setminus (A \cup A_4)$, and so u has no neighbor in X, and hence $u \notin A' \cup M'$; and therefore $u \in A'_4$. The claim follows.

Suppose that $w \in A_4$. Since $w \notin \widetilde{S_1}$, it follows that A_1 is a good P-end, and yet $w \in A_4 \setminus A'_4$, a contradiction. This proves that $w \in A_2 \cup A_3$.

Let $X = \widetilde{S'_2} \cup \widetilde{S'_3} \cup B'_2 \cup B'_3$. Note that since A'_2 and A'_3 are good P'-ends, it follows that $X \cap (A'_2 \cup A'_3) = \emptyset$. Since no vertex in X is complete to $A'_2 \cup A'_3$ or adjacent to w, it follows that $X \cap A = \emptyset$.

We claim that S'_2, S'_3 have no rungs of length one. For suppose that $a \in A'_2$ is adjacent to $b \in B'_2$ say; so $a \in S_1$. From the initial choice of S'_1, it follows that $(A'_2 \cup B'_2) \cap C_1 = \emptyset$, and in particular $b \notin C_1$. Since S_1 has no rung of length one, $b \notin B_1$; and since $b \in X$, it follows that $b \notin A$, and so $b \in A_4$. Since X is connected and $A \cap X = \emptyset$, we deduce that $X \subseteq A_4$. Since $X \not\subseteq A'_4$, A_1 is a bad P-end, and so

$$\widetilde{S'_3} \subseteq X \subseteq A_4 \subseteq \widetilde{S_1} \setminus \{v\},$$

contrary to the minimality of $\widetilde{S_1}$, since S'_3 has no rung of length one. This proves that S'_2, S'_3 have no rungs of length one.

For $i = 2, 3$, $\widetilde{S'_i}$ is not a subset of $\widetilde{S_1}$ from the minimality of $\widetilde{S_1}$, since $v \in \widetilde{S_1} \setminus \widetilde{S'_i}$; and so there is a path R_i of $G|(A'_i \cup \widetilde{S'_i})$ from A'_i to some vertex in $\widetilde{S'_i} \setminus \widetilde{S_1}$, such that all its vertices except the first belong to $\widetilde{S'_i}$. Choose such a path R_i, of minimum length, and let its ends be $y_i \in A'_i$ and $z_i \in \widetilde{S'_i}$. Certainly R_i has length at least one, since $A'_i \cap \widetilde{S'_i} = \emptyset$. Let the neighbor of z_i in R_i be z'_i.

Now z_2, z_3 are non-adjacent, since $\widetilde{S'_2}$ is anticomplete to $\widetilde{S'_3}$ by Lemma 2.6. Consequently they do not both belong to good P-ends in B; and neither of them is in a good P-end in A, since neither z_2 nor z_3 is complete to $A'_2 \cup A'_3$. Thus one of z_2, z_3 is not in a good P-end, say z_2, and so by Lemma 2.5, $z'_2 \notin \widetilde{S_1}$. From the minimality of R_2, $z'_2 \notin \widetilde{S'_2}$, and so $z'_2 = y_2$ and $y_2 \notin \widetilde{S_1}$. Since $y_2 \in A_1$ it follows that A_1 is a good P-end, and hence $A \cup A_4 \subseteq A' \cup A'_4 \cup M'$; and so $z_2 \notin A \cup A_4$. Since z_2, y_2 are adjacent, and S_1 has no rung of length one, it follows that $z_2 \in C_1$, contradicting that $z_2 \notin \widetilde{S_1}$. This proves (6).

For the remainder of the proof, we will always assume that A' is a sweet potato of P'.

(7) *For $i = 1, 2, 3$, if $\widetilde{S'_i} \not\subseteq \widetilde{S_1}$, then there exists $z_i \in \widetilde{S'_i} \cup A'_i$ touching $\widetilde{S'_i} \setminus \widetilde{S_1}$ with distance one from $\widetilde{S_1}$. Consequently, z_i is in a good P-end, and either $z_i \in \widetilde{S'_i}$ or z_i is in a good P-end of S_1.*

Let $x_i \in \widetilde{S'_i} \setminus \widetilde{S_1}$, and let R_i be a path from x_i to a vertex $y_i \in A'_i$ such that $V(R_i) \setminus \{y_i\} \subseteq \widetilde{S'_i}$. By Lemma 2.4, such a path exists. Since $y_i \in A'$ and A' is sweet, so y_i touches $\widetilde{S_1}$; let z_i be the first vertex of R_i (starting at x_i) that touches $\widetilde{S_1}$. It follows that $z_i \notin \widetilde{S_1}$, and either $z_i \in \widetilde{S'_i}$, or $z_i = y_i$ and R_i has at least two vertices and a neighbor of z_i in $V(R_i)$ is in $\widetilde{S'_i} \setminus \widetilde{S_1}$. Consequently, z_i touches $\widetilde{S'_i} \setminus \widetilde{S_1}$. Since $x_i \notin \widetilde{S_1}$, we know that z_i has distance one from $\widetilde{S_1}$. Moreover, $V(P') \cap M = \emptyset$, so by Lemma 2.5, z_i is in a good P-end. We may therefore assume that $z_i \notin \widetilde{S'_i}$; consequently, A'_i is a good P'-end, and $z_i \in A'_i$. Since A' is sweet, it follows that $A'_i \subseteq S_1$, and so $z_i \in S_1$. Thus, $z_i \in A_1 \cup B_1$, and the P-end of S_1 containing z_i is good. This proves (7).

(8) *If z_i, z_j as in (7) exist for $i \neq j$ with $i, j \in \{1, 2, 3\}$, then z_i and z_j are in different potatoes of P. Consequently, one of z_1, z_2, z_3 does not exist.*

Suppose that z_1 and z_2 exist and belong to the same potato A, say. Since z_1, z_2 are in good P-ends in A, it follows that z_1 is adjacent to z_2, which means that either $z_1 \notin \widetilde{S'_1}$ or $z_2 \notin \widetilde{S'_2}$; and without loss of generality, let $z_1 \notin \widetilde{S'_1}$. Therefore, $z_1 \in A'_1$ and A'_1 is a good P'-end; and so z_1 is in a good P-end of S_1 by (7). Hence A_1 is a good P-end, and since $z_1, z_2 \in A$ have distance one from $\widetilde{S_1}$, it follows that $z_1, z_2 \in A_1$. Also, $z_2 \in N(z_1) \cap (\widetilde{S'_2} \cup A'_2)$, so $z_2 \in A'_2 \cup A'_4$. Let $X = V(P') \setminus (A' \cup A'_4)$; then $X \cap (A \cup A_4) = \emptyset$ by (5), since $z_1 \in A_1 \cap A'_1$ and $z_2 \in A_1 \cap (A'_2 \cup A'_4)$. Since z_1 touches $\widetilde{S'_1} \setminus \widetilde{S_1}$ and $z_1 \notin \widetilde{S'_1}$, there is a neighbor x of z_1 with $x \in \widetilde{S'_1} \setminus \widetilde{S_1}$. Since A'_1 is a good P'-end, it follows that $x \in X$, so $x \notin A \cup A_4$. But x is adjacent to $z_1 \in A_1$, and consequently, $x \in C_1 \subseteq \widetilde{S_1}$, since S_1 has no rung of length one. This is a contradiction, because $x \notin \widetilde{S_1}$; (8) follows.

(9) *For $i = 1, 2, 3$, if $\widetilde{S'_i} \subseteq \widetilde{S_1}$, then $\widetilde{S'_i} \subset \widetilde{S_1}$, and so S'_i has a rung of length one.*

Let Y be the union of all good P'-ends in A'; because A' is sweet, we know that $Y \subseteq S_1$. Since Y only includes good P'-ends, it follows that $Y \cap \widetilde{S'_i} = \emptyset$ for $i = 1, 2, 3$. We may assume that $Y \cap \widetilde{S_1} = \emptyset$, for otherwise (9) holds. Therefore, $Y \subseteq S_1 \setminus \widetilde{S_1}$. Since Y is a clique and S_1 has no rung of length one, we may assume that $Y \subseteq A_1$, and A_1 is a good P-end of S_1. Let $X = V(P') \setminus (A' \cup A'_4)$; then $X \cap (A \cup A_4 \cup M) = \emptyset$ by (5), since $Y \subseteq A_1$. Let $j \neq i$ such that $A'_j \subset Y$, and let $v_j \in C'_j \cup B'_j$ with a neighbor in A'_j. Then $v_j \in X$, so $v_j \notin A \cup A_4 \cup M$, but v_j has a neighbor in $Y \subseteq A_1$, and hence $v_j \in C_1$. Therefore, $v_j \in \widetilde{S_1} \setminus \widetilde{S'_i}$. This proves (9).

By (7) and (8), we may assume that $\widetilde{S'_1} \subseteq \widetilde{S_1}$. By (9), we know S'_1 has a rung ab of length one with $a \in A'_1$ and $b \in B'_1$. Since G is square-free, and so S'_2, S'_3 have no rungs of length one, we deduce that $\widetilde{S'_2} \not\subseteq \widetilde{S_1}$ and $\widetilde{S'_3} \not\subseteq \widetilde{S_1}$. So there exist z_2 and z_3 as in (7), and by (8), z_2 and z_3 are in different potatoes of P, so without loss of generality, let $z_2 \in A$, $z_3 \in B$. We know that $a \in A' \subseteq S_1 \cup \widetilde{S_1}$, so both a

and b touch $S_1 \cup \widetilde{S_1}$. Also, z_2 and z_3 are each adjacent to at most one of a, b. Since z_2 is complete to $A \setminus \{z_2\}$, it follows that $\{a, b\} \not\subseteq A$, and similarly, $\{a, b\} \not\subseteq B$. Since $a \in \widetilde{S_1} \cup S_1$ and S_1 has no rung of length one, it follows that $\{a, b\} \not\subseteq A \cup B$. For the remainder of the proof, we fix a, b, z_2 and z_3.

(10) *Every vertex in M' is adjacent to either a or b.*

Suppose that $m \in M'$ is non-adjacent to both a and b. Then by Lemma 2.3 m is complete to A'_2 and B'_3. Let $a' \in A'_2$ and $b' \in B'_3$; then m-a'-a-b-b'-m is a hole of length five. This is a contradiction, because G is Berge. This proves (10).

(11) *The vertices z_2 and z_3 are not adjacent.*

Since $\widetilde{S'_2}$ and $\widetilde{S'_3}$ do not touch by Lemma 2.6, we may assume that one of z_2, z_3 is in a good P'-end in A'; without loss of generality, say $z_2 \in A'_2$, and A'_2 is a good P'-end. By (7), $z_2 \in A_1$. Since S_1 has no rung of length one, z_2 is non-adjacent to every vertex in B, and in particular, z_2 is not adjacent to z_3, which proves (11).

(12) $(A_4 \cup B_4) \cap \{a, b\} = \emptyset$.

Assume for a contradiction that $a \in A_4$. Since $a \in A' \subseteq S_1 \cup \widetilde{S_1}$, A_1 is a bad P-end of S_1. Since z_2 is in a good P-end in A, it follows that $z_2 \in A_2 \cup A_3$, and since A_1 is bad, it follows from (7) that $z_2 \in \widetilde{S'_2}$. Also, A'_1 is a bad P'-end, because A' is sweet, and therefore, since a is complete to $A'_2 \cup A'_3$, again because A' is sweet, $A'_2 \cup A'_3 \subseteq A_1$. Since $z_2 \in A_2 \cup A_3$, this implies that $z_2 \notin A'_2 \cup A'_3$, but z_2 is complete to $A'_2 \cup A'_3$, and thus $z_2 \in A'_4$. But A'_2 is a good P'-end of S'_2, since A'_1 is a bad P'-end, so $A'_4 \cap \widetilde{S'_2} \cup A'_2 = \emptyset$, a contradiction to the fact that $z \in \widetilde{S'_2} \cup A'_2$. Therefore, $a \notin A_4$, and by symmetry, $a \notin B_4$, and similarly, $b \notin A_4 \cup B_4$. This proves (12).

(13) *If $\{a, b\} \cap A_1 \neq \emptyset$, then $z_2 \in \widetilde{S'_2}$. If $\{a, b\} \cap B_1 \neq \emptyset$, then $z_3 \in \widetilde{S'_3}$.*

Suppose not. By symmetry, we may assume that $\{a, b\} \cap A_1 \neq \emptyset$, and that z_2 is in A'_2 and A'_2 is a good P'-end. In particular, z_2 does not touch b. Since z_2 touches every vertex of A, it follows that $b \notin A$, and since $\{a, b\} \cap A_1 \neq \emptyset$, it follows that $\{a, b\} \cap A_1 = \{a\}$. We let $X = \widetilde{S'_2} \cup B'_1 \cup B'_2$.

Since z_2 is in a good P'-end A'_2, by (7), it follows that $z_2 \in A_1$ and A_1 is a good P-end. Hence (5) implies that $X \subseteq V(P) \setminus (A \cup A_4)$, because $a, z_2 \in A_1$. Thus $\widetilde{S'_2} \subset X$ is disjoint from $A \cup A_4$. Since $A'_2 \subseteq S_1$ (as A' is sweet) and a is complete to A'_2, we know that $A'_2 \subseteq A_1 \cup C_1$. Let Z be a component of $\widetilde{S'_2}$. Then Z has a neighbor in A'_2, and so Z touches C_1. By (11), $z_3 \notin A'_3$, and so $z_3 \in \widetilde{S'_3}$. Since z_3 touches every vertex in B, but z_3 touches no vertex in $\widetilde{S'_2}$, it follows that $Z \cap B = \emptyset$. Finally, since $Z \cap (A \cup B \cup M) = \emptyset$, and Z touches C_1, it follows that $Z \subseteq C_1$. Consequently, $\widetilde{S'_2} \subseteq C_1 \subseteq \widetilde{S_1}$, a contradiction. This proves (13).

(14) *If $a \in A_1 \cup B_1$, then $b \in C_1$.*

By symmetry, we may assume that $a \in A_1$. By (13), we know that $z_2 \in \widetilde{S'_2}$, and since z_2 is adjacent to a (since z_2 is in a good P-end of A), it follows that $z_2 \in A'_2 \cup A'_4$. Hence A'_2 is a bad P'-end, so A'_3 is a good P'-end. Let $Y = \widetilde{S'_3} \cup B'$. Then $Y \cap (A \cup M) = \emptyset$, since Y does not touch $z_2 \in (A'_2 \cup A'_4) \cap \widetilde{S'_2}$, but z_2 touches every vertex in A, since z_2 is in a good P-end in A. Since Y is connected and touches $z_3 \in B$, it follows that $Y \cap A_4 = \emptyset$. But $b \in Y \cap N(a)$, and consequently, $b \notin A \cup A_4 \cup M$, and hence $b \in C_1$. This proves (14).

(15) *If $a \in C_1$, then $b \in A_1 \cup B_1$.*

Suppose not. Since $a \in C_1$, it follows that $b \in S_1$, and so $b \in C_1$. Let $Z = S_2 \cup S_3$. Every vertex in Z is non-adjacent to both a and b. By (10), it follows that $M' \cap Z = \emptyset$. Also, no vertex in Z is in a good P'-end, because no vertex in Z is adjacent to a or b.

Let R be a path from z_2 to z_3 containing a rung of S_2 as a subpath (exists by (11)). Since $z_2, z_3 \in V(P')$ and every internal vertex of R is in Z, it follows that $V(R) \cap M' = \emptyset$. We claim that some vertex of R belongs to a good P'-end; for if z_2 or z_3 is in a good P'-end then this is true, and otherwise R is a path from $\widetilde{S_2'}$ to $\widetilde{S_3'}$, and the claim follows from Lemma 2.6. Let $r \in V(R)$ be in a good P'-end. From what we proved above it follows that $r \notin Z$, and so r is one of z_2, z_3, and we may assume from the symmetry that $r = z_2$. Since $z_2 \in A_2' \cup \widetilde{S_2'}$, it is not the case that B_2' is a good P'-end and $z_2 \in B_2'$; so A_2' is a good P'-end and $z_2 \in A_2'$.

It follows that $z_2 \in A_1$, and thus $N(z_2) \cap Z = A_2 \cup A_3$. Since z_2, z_3 are not adjacent by (11), it follows that $z_3 \notin A_3'$, and so $z_3 \in \widetilde{S_3'}$. Thus, Z touches $z_3 \in \widetilde{S_3'}$ (because $z_3 \in B$), Z is connected, and Z contains no vertex in M' or in a good P'-end, so by Lemma 2.5, $Z \subseteq \widetilde{S_3'}$. Hence $A_2 \cup A_3 \subseteq N(z_2) \cap Z \subseteq A_3' \cup A_4'$. Since a is complete to A_3' and anticomplete to $A_2 \cup A_3$, it follows that $A_2 \cup A_3 \subseteq A_4'$, and so $Z \cap A_4' \neq \emptyset$. Since $z_2 \in A_2'$, it follows that $Z \cap A_3' \subseteq N(z_2) \cap Z \subseteq A_2 \cup A_3 \subseteq A_4'$. Since $A_3' \cap A_4' = \emptyset$, this implies that $Z \cap A_3' = \emptyset$. Since Z is connected, Z is included in $\widetilde{S_3'} \setminus A_3'$, and Z includes the non-empty set $A_2 \cup A_3 \subseteq A_4'$, it follows that $Z \subseteq A_4'$. Consequently, every vertex in $A \cup B \cup M$ touches A_4'. Let $X = V(P') \setminus (A' \cup A_4')$; then X is disjoint from $A \cup B \cup M$, because X does not touch A_4'. But X touches C_1, and so $X \subseteq C_1$. Since A_2' is a good P'-end, it follows that $\widetilde{S_2'} \subset X \subseteq C_1 \subseteq \widetilde{S_1}$. This is a contradiction, and (15) follows.

Together, (4), (12), (14), and (15) imply that either $a \in A_1 \cup B_1$ and $b \in C_1$, or $a \in C_1$ and $b \in A_1 \cup B_1$. By symmetry, we assume from now on that $\{a, b\} \cap A_1 \neq \emptyset$ and $\{a, b\} \cap C_1 \neq \emptyset$. By (13), it follows that $z_2 \in \widetilde{S_2'}$.

(16) $M' \cap (S_2 \cup S_3) = \emptyset$.

Every vertex in $B_2 \cup B_3 \cup C_2 \cup C_3$ is non-adjacent to both a and b. Thus, (10) implies that $M' \cap (S_2 \cup S_3) \subseteq A_2 \cup A_3$. Therefore, we may assume that some vertex m in $A_2 \cup A_3$ is in M'. Then m is not adjacent to the vertex in $c \in \{a, b\} \cap C_1$, and hence c is in a bad P'-end of S_1' in a potato p of P'. Since every vertex in $S_3' \cap p$ is adjacent to both m and c, it follows that $S_3' \cap p \subseteq A_1$. Consequently, z_2 has a neighbor in $S_3' \cap p$. But S_1' has a bad P'-end in p, and so $\widetilde{S_2'}$ does not touch $S_3' \cap p$, a contradiction. Thus, $M' \cap (S_2 \cup S_3) = \emptyset$, and we have proved (16).

(17) *If $z_2 \in A_1$, then no vertex in $A_2 \cup A_3$ is in a good P'-end in B'.*

Suppose that some vertex r is in $A_2 \cup A_3$ and in a good P'-end in B', and that $z_2 \in A_1$. Then r and b are adjacent, and so $b \in A_1$. Since z_2 is in a good P-end by (7), it follows that z_2 is adjacent to b and r. Thus $z_2 \in B_2' \cup B_4'$, so B_2' is a bad P'-end, and $r \in B_1' \cup B_3'$. Moreover, $A' \subseteq S_1 \cup \widetilde{S_1}$ since A' is a sweet potato; and $B' \subseteq A \cup A_4$, because every vertex in $B' \setminus \{r, b\}$ is adjacent to both r and b. Therefore, and by (16), $Z = B_2 \cup B_3 \cup C_2 \cup C_3$ is a connected set disjoint from $A' \cup B' \cup M'$. Furthermore, $z_3 \in \widetilde{S_3'} \cup A_3' \subseteq A_4' \cup A_3' \cup C_3'$ touches Z, so $Z \subseteq A_4'$ or $Z \subseteq C_3'$. But r has a neighbor in Z, so $Z \not\subseteq A_4'$, and thus $Z \subseteq C_3'$. Every vertex in $A_2 \cup A_3$ is adjacent to $b \in B_1'$ and has a neighbor in $Z \subseteq C_3'$, so $A_2 \cup A_3 \subseteq B_3'$. Since $B' \subseteq A \cup A_4$ and $A_2 \cup A_3 \subseteq B_3'$, it follows that $B_2' \cap (A_2 \cup A_3) = \emptyset$, and thus $B_2' \subseteq A_1 \cup A_4$.

Since $z_2 \in A_1$ and z_2 is in a good P-end it follows that A_1 is a good P-end, and so every vertex in A touches b. Let $Y = A_2' \cup C_2' \cup \{a\}$; then Y contains no neighbor of b except a, so $Y \cap A = \emptyset$. Since $a \in Y$ and Y is connected, it follows that $Y \subseteq V(P) \setminus (A \cup A_4)$, and in particular Y is anticomplete to A_4. Since every vertex in B_2' touches Y, we deduce that $A_4 \cap B_2' = \emptyset$, and so $B_2' \subseteq A_1$.

Since $B'_2 \subseteq A_1$, it follows that B'_2 is a clique and M is complete to B'_2. Since B'_2 is a bad P'-end, there exists a vertex $m \in M' \setminus M$ such that m is not complete to B'_2; and therefore $m \notin A$. Since m is complete to $B'_3 \supseteq A_2 \cup A_3$, it follows that $m \in A_4$. Then the good P'-ends in A' are complete to m and contained in S_1, and thus the good P'-ends in A' are contained in A_1, and hence complete to z_2. This is a contradiction, because $z_2 \in B'_2 \cup B'_4$. This proves (17).

(18) *Neither z_2 nor z_3 is in a good P'-end in A'.*

We already proved that z_2 is not in a good P'-end in A' by (13). Therefore, we suppose that z_3 is in a good P'-end A'_3, and since $z_3 \in B$ is adjacent to a and S_1 has no rung of length one, it follows that $a \in C_1$, $b \in A_1$, and $z_3 \in B_1$. Since z_2 is adjacent to b (since z_2 is in a good P-end), it follows that $z_2 \in B'_2 \cup B'_4$, and B'_2 is a bad P'-end. Let $Y = S_2 \cup S_3$; then Y contains no vertex in $M' \cup A'$ by (16) and because $Y \cap (S_1 \cup \widetilde{S_1}) = \emptyset$, but A' is sweet and so $A' \subseteq S_1 \cup \widetilde{S_1}$. Let $Z = C_2 \cup C_3 \cup B_2 \cup B_3$; then $Z \subset Y$. Since B'_2 is a bad P'-end, every vertex in B' touches $b \in B'_1$. No vertex in Z touches $b \in A_1$, so $Z \cap B' = \emptyset$.

We claim that $Y \subseteq \widetilde{S'_2} \cup B'$. Suppose that $z_2 \in A_1$; then by (17), no vertex in $A_2 \cup A_3$ is in a good P'-end in B'. Thus, Y does not contain a vertex in a good P'-end of B', and since Y is disjoint from $A' \cup M'$ and Y touches $z_2 \in \widetilde{S'_2}$, it follows that $Y \subseteq \widetilde{S'_2}$. Therefore, we may assume that $z_2 \notin A_1$; and so $z_2 \in A_2 \cup A_3$, and therefore z_2 touches Z, so we have $Z \subseteq \widetilde{S'_2}$. Every vertex in $A_2 \cup A_3$ is adjacent to $b \in B'_1$, and touches $z_2 \in (B'_2 \cup B'_4) \cap \widetilde{S'_2}$; so $A_2 \cup A_3 \subseteq B' \cup (B'_4 \cap \widetilde{S'_2})$, and thus $Y \subseteq \widetilde{S'_2} \cup B'$. This proves our claim.

But Y touches $z_3 \in A'_3$. Since S'_3 has no rung of length one, Y contains a vertex in $A'_2 \cup A'_4$. Since $Y \cap (A' \cup M') = \emptyset$, it follows that $Y \subseteq A'_4$. But $z_2 \in B'_2 \cup B'_4$ touches Y, a contradiction. This proves (18).

Let R be a path from z_2 to z_3 containing a rung of either S_2 or S_3 as a subpath; we choose R so that if $z_2 \in A_2 \cup A_3$, then the rung starts at z_2. By (18), R is a path from $\widetilde{S'_2}$ to $\widetilde{S'_3}$. By Lemma 2.6, $V(R) \setminus \{z_2, z_3\}$ contains a vertex in M' or in a good P'-end. By (16), $V(R) \cap M' = \emptyset$. No vertex of $V(R) \setminus \{z_2, z_3\}$ is in a good P'-end in A', because all good P'-ends in A' are included in S_1 as A' is a sweet potato of P'. So there exists $r \in V(R) \setminus \{z_2, z_3\}$ such that r is in a good P'-end in B'. Since r is adjacent to b, it follows that $r \in A_2 \cup A_3$, and since $r \neq z_2$, it follows that $z_2 \in A_1$, which contradicts (17). Thus, our initial assumption that $(A' \cup B') \cap C_1 \neq \emptyset$ is false. Now, by Theorem 2.1, it follows that no vertex in C_1 is a corner. This concludes the proof. □

3 Rough vertices

To prove that there is a vertex in C_1 that is not rough, we first recall some results and definitions from [2].

A *pyramid* with *triangle* $\{x_1, x_2, x_3\}$ and *apex* x (where x, x_1, x_2, x_3 are four distinct vertices) is a graph containing three paths P_1, P_2, P_3 such that for each $i = 1, 2, 3$, P_i is an (x_i, x)-path, $\{x_1, x_2, x_3\}$ is a clique, at least two of the paths have length at least two, and there are no other vertices or edges. A *pyramid in a graph G* is an induced subgraph of G that is a pyramid.

Lemma 3.1 (2.4 in [2])**.** *If a graph G contains no odd hole, then G contains no pyramid.*

Theorem 3.2 (10.1 in [2])**.** *In a Berge graph G let R_1, R_2, R_3 be three paths that form a prism K, with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, where each R_i has ends a_i and b_i. Let $F \subset V(G) \setminus V(K)$ be connected, such that its set of neighbors in K is not local, but some vertex in F has a neighbor in K. Assume that no vertex in F is major with respect to K. Then there is a path f_1-$\cdots$-f_n in F with $n \geq 1$, such that (up to symmetry) either:*

1. *f_1 has two adjacent neighbors in R_1, and f_n has two adjacent neighbors in R_2, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K)$, or*

2. *$n \geq 2$, f_1 is adjacent to a_1, a_2, a_3, and f_n is adjacent to b_1, b_2, b_3, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K)$, or*

3. *$n \geq 2$, f_1 is adjacent to a_1, a_2, and f_n is adjacent to b_1, b_2, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K)$, or*

4. *f_1 is adjacent to a_1, a_2, and there is at least one edge between f_n and $V(R_3) \setminus \{a_3\}$, and there are no other edges between $\{f_1, \dots, f_n\}$ and $V(K) \setminus \{a_3\}$.*

Note that in item 1 the set $V(K) \cup \{f_1, \dots, f_n\}$ induces the line-graph of a bipartite subdivision of K_4, and a subset of that set induces a prism of which f_1 and f_n are corners. In items 2 and 3, G has a prism with corners $a_1, a_2, f_1, b_1, b_2, f_n$. In item 4, G has a prism with corners $a_1, a_2, f_1, b_1, b_2, b_3$. Hence in all items f_1 is the corner of a prism. In particular, when F consists of only one vertex we obtain the following using Lemma 1.1.

Theorem 3.3. *In a Berge graph G, let K be an odd prism and $x \in V(G) \setminus V(K)$ be a smooth vertex. If x has a neighbor in K, then x is a local neighbor of K.*

Lemma 3.4. *Let G be a flat graph. Let K be a prism with paths P_1, P_2, P_3, and let v be rough for K such that v has a normal path Q from v to a base path P_1. Then no vertex in the interior of Q has a neighbor in P_2 or P_3.*

Proof. Let F be the set of interior vertices of Q. We may assume that F is non-empty, and therefore, some vertex in F has a neighbor in the interior of P_1; we may assume that some vertex in F has a neighbor in $V(P_2) \cup V(P_3)$, and thus F does not attach locally to K. Then, by Theorem 3.2, since G is flat, F contains a vertex f_1 with two neighbors in either $\{a_1, a_2, a_3\}$ or $\{b_1, b_2, b_3\}$; but every vertex in F is non-adjacent to every vertex in $\{a_2, a_3, b_2, b_3\}$, a contradiction. $\square$

Theorem 3.2 implies that in a flat graph, every vertex which is rough for a prism K is either a corner or major for K. Here we use a related statement:

Lemma 3.5. *Let K be a prism in a flat graph G and let v be a vertex that is rough for K; then either v is a corner in G, possibly for another prism, or there exist $\{i, j, k\} = \{1, 2, 3\}$ such that v is adjacent to a_i and b_j, and P_k is the base path for v in K. In particular, if v is not a corner, then v is major for K.*

Proof. Let the prism consist of triangles $\{a_i, a_j, a_k\}$ and $\{b_i, b_j, b_k\}$ as well as paths P_i, P_j, P_k, where P_k is the base path for v in K. If v is complete to $\{a_i, b_j\}$ or $\{a_j, b_i\}$, then the result follows from Theorem 3.2 with $F = \{v\}$; thus we may assume that this is not the case. We may assume that v is not an end of P_k, for otherwise v is a corner. By definition of a rough vertex, v has a neighbor in $\{a_i, b_i\}$ and in $\{a_j, b_j\}$. Thus, we may assume that v is adjacent to a_i and a_j, and anticomplete to $\{b_i, b_j\}$. Let P'_k denote the path from v to b_k obtained as vb_k, or as the path from v to b_k in $G|(V(Q) \cup V(P_k))$, where Q is a normal path. By Lemma 3.4, no vertex in the interior of Q has a neighbor in $V(P_i) \cup V(P_j)$. If v has no neighbors in the interior of P_i, P_j, then $\{a_i, a_j, v\}$, $\{b_i, b_j, b_k\}$, P_i, P_j, P'_k forms a prism and v is a corner, and the statement of the lemma follows.

Thus, we may assume that v has a neighbor in $(V(P_i) \cup V(P_j)) \setminus \{a_i, a_j\}$. Let u_l be the neighbor of v closest to b_l on P_l for $l = i, j$. It follows that $u_l \neq b_l$ for $l = 1, 2$. Let P'_l denote the path in $G|(\{v\} \cup V(P_l))$ from v to b_l. Then both P'_i and P'_j have an interior vertex, and thus $v, P'_i, P'_j, P'_k, \{b_i, b_j, b_k\}$ form a pyramid, contrary to Lemma 3.1. This concludes the proof. $\square$

Lemma 3.6. *Let G be a non-null square-free flat graph with no clique cutset, and P, S_1 be chosen such that P is a megaprism in G with with partition $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4$ (where $C_4 = \emptyset$), and S_1 is a strip of P with no rung of length one, and among all such choices of P, S_1, the set $\widetilde{S_1}$ is minimal with respect to inclusion. There is a smooth vertex v in the interior of the strip S_1.*

Proof. Let v be a vertex in C_1 such that $N(v) \cap (A_1 \cup B_1)$ is minimal with respect to inclusion.

(1) *$N(v) \cap (M \cup A_1)$ and $N(v) \cap (M \cup B_1)$ are cliques, and $N(v) \cap A_1$ is anticomplete to $N(v) \cap B_1$. In particular, $G|(N(v) \cap (A_1 \cup M \cup B_1))$ is the union of two complete graphs.*

Suppose that v has two non-adjacent neighbors $x, y \in M \cup A_1$. Since M is a clique by Lemma 2.2, we may assume that $x \in A_1$. Let $a \in A_2$; then y is adjacent to a, because either $y \in A_1$ or $y \in M$, and y is not complete to A_1, so y is complete to A_2. Therefore, v-x-a-y-v is a square, a contradiction. Since S_1 has no rung of length one, this implies (1).

We may assume that v is rough for some prism K; v is not a corner by Lemma 2.7 and Theorem 2.1. By Lemma 3.5, there exist $\{i, j, k\} = \{1, 2, 3\}$ such that v is adjacent to a_i and b_j and P_k is the base path for v in K.

Suppose that v has a normal path Q from v to the interior of a base path P_k in K; let w be the neighbor of v in $V(Q)$. We know that v is adjacent to b_j and a_i by Lemma 3.5. The set $\{w, a_i, b_j\}$ is a stable set, and $a_i, b_j \notin C_1$ by Lemma 2.7, so $a_i, b_j \in A_1 \cup M \cup B_1$. By (1), we may assume that $a_i \in A_1$, $b_j \in B_1$, and consequently, $w \in C_1$. Now w is not adjacent to either of a_i, b_j, and by the choice of v with $N(v) \cap (A_1 \cup B_1)$ minimal, w has a neighbor x in A_1 or B_1, say A_1, such that x is not a neighbor of v. We know that x is not adjacent to b_j, because S_1 has no rung of length one. But then x-w-v-b_j and a_i-v-b_j are two rungs in S_1 of different parity, so adding a rung from A_2 to B_2 will complete one of them to an odd hole, a contradiction.

Therefore, we may assume by the definition of a rough vertex that v is adjacent to a_i, b_j, a_k, b_k and $P_k = a_k$-b_k. Then $\{a_i, b_j, a_k, b_k\} \subseteq N(v) \cap (A_1 \cup M \cup B_1)$. But $G|\{a_i, b_j, a_k, b_k\}$ is a four-vertex path, and thus not the union of two complete graphs; this contradicts (1). ☐

Lemma 3.7. *Let G be a graph, and let C be a clique cutset in G such that there exist $A, B \neq \emptyset$ with $A \cup B \cup C = V(G)$, $A \cap B = \emptyset$, and such that there are no edges between A and B in G. Let $v \in A \cup C$ be a smooth vertex in $G|(A \cup C)$, and either $G|(B \cup C)$ contains no prism or $v \in A$. Then v is smooth in G.*

Proof. We may assume that v is rough in G. By Lemma 3.5, v is either a corner of a prism, or there is a prism K so that v is adjacent to two corners a_i and b_j for some $i \neq j$, $i, j \in \{1, 2, 3\}$. Consequently, there is a prism K in G so that v has two non-adjacent neighbors in $V(K)$, and v is rough for K in G. Let Q be a normal path for v and K if it exists, and $Q = \emptyset$ otherwise. Let $H = G|(V(Q) \cup V(K) \cup \{v\})$; it follows that v is rough for K in H. Consequently, $V(H) \cap B \neq \emptyset$, because v is smooth in $G|(A \cup C)$. Since H has no clique cutset, this implies that $V(H) \subseteq B \cup C$, and therefore $G|(B \cup C)$ contains a prism, and hence $v \in A$. This is a contradiction, since $V(H) \subseteq B \cup C$. ☐

We are now ready to prove Theorem 1.9, which we restate.

Theorem 3.8. *Let G be a square-free flat graph. Then either G is a disjoint union of cliques or some $v \in V(G)$ is smooth and not simplicial.*

Proof. We may assume that G is not a disjoint union of cliques. If G does not contain a prism, then every vertex of G is smooth, and since G is not a disjoint union of cliques, G has a non-simplicial vertex.

From now on, we may assume that G contains a prism. We prove by induction on $|V(G)|$ that if G contains a prism, G contains two distinct non-adjacent smooth and non-simplicial vertices.

Suppose that G has no clique cutset. Then, no vertex of G is simplicial, because otherwise $N(v)$ is a clique cutset in G. Since G contains a prism, we can find a megaprism P by Theorem 2.1. Let S_1, S_2 be strips of P with no rung of length one. For $i = 1, 2$, let P'^i be a megaprism with a strip $S_1'^i$ with no rung of length one and such that $\widetilde{S_1'^i} \subseteq \widetilde{S_i}$, and subject to that $\widetilde{S_1'^i}$ is minimal with respect to inclusion. P'^i exists, since $P'^i = P$ (after possibly relabelling strips) is such a megaprism. By Lemma 3.6, it follows that for $i = 1, 2$, there is a smooth vertex $v_i \in \widetilde{S_1'^i} \subseteq \widetilde{S_i}$. It follows that v_1 and v_2 are distinct and non-adjacent, since $\widetilde{S_1}$ and $\widetilde{S_2}$ do not touch by Lemma 2.6, and hence v_1 and v_2 are the desired vertices.

Therefore, we may assume that G has a clique cutset $C \subseteq V(G)$ such that there exist $A, B \neq \emptyset$ with $A \cup B \cup C = V(G)$, $A \cap B = \emptyset$, and such that there are no edges between A and B in G. Since prisms do not have a clique cutset, and since G contains a prism, at least one of $G|(A \cup C)$ and $G|(B \cup C)$ contains a prism as well. Suppose that both $G|(A \cup C)$ and $G|(B \cup C)$ contain a prism. By induction, $G|(A \cup C)$ contains two distinct non-adjacent smooth non-simplicial vertices u_A, v_A, and $G|(B \cup C)$ contains two distinct non-adjacent smooth non-simplicial vertices u_B, v_B. Since C is a clique, at most one of u_A, v_A is contained in C; the same holds for u_B, v_B. Thus, we may assume that $u_A, u_B \notin C$. Then, u_A, u_B are distinct, non-adjacent, not simplicial, and smooth by Lemma 3.7. Therefore, we may assume that $G|(A \cup C)$ contains a prism, but $G|(B \cup C)$ does not. Then, by induction, $G|(A \cup C)$ contains two distinct non-adjacent smooth non-simplicial vertices u, v. By Lemma 3.7, u, v are smooth in G as well, which concludes the proof. □

4 Even pairs

In this section we give the proof of Theorem 1.10. This proof closely follows the proof of the main results in [5] and [7].

Let $P = x\text{-}x'\text{-}\cdots\text{-}y'\text{-}y$ be a path of length at least three in G. Following [2], we say that a pair $\{u, v\}$ of non-adjacent vertices of $V(G) \setminus P$ is a *leap* for P if $N(u) \cap P = \{x, x', y\}$ and $N(v) \cap P = \{x, y', y\}$. Note that in that case $P \cup \{u, v\}$ induces a prism, whose corners are u, v, x, x', y, y'.

Let $T \subseteq V(G)$. The set T is called *anticonnected* if $G^c|T$ is connected, where G^c denotes the complement of G. A vertex is called *T-complete* if it is complete to T, and $C(T)$ denotes the set of all T-complete vertices. An edge is a *T-edge* if both its ends are T-complete. An induced subgraph Q of G is an *antipath* in G if $G^c|V(Q)$ is a path in G^c.

Lemma 4.1 ([2, 8])**.** *In a Berge graph G, let P be a path and $T \subset V(G)$ be an anticonnected set such that $V(P) \cap T = \emptyset$ and the ends of P are T-complete. Then either:*

1. *P has even length and has an even number of T-edges;*
2. *P has odd length and has an odd number of T-edges;*
3. *P has odd length at least three and there is a leap for P in T;*
4. *P has length three and its two interior vertices are the ends of an odd antipath Q whose interior is in T (and consequently $V(P) \cup V(Q)$ induces an antihole in G).*

Lemma 4.2 (2.3 in [2])**.** *In a Berge graph G, let H be a hole and $T \subset V(G)$ be an anticonnected set such that $V(H) \cap T = \emptyset$. Then either the number of T-edges in H is even, or H has exactly two T-complete vertices and they are adjacent.*

Lemma 4.3. *Let G be a Berge graph that contains no antihole of length at least six. Let P be a path in G and $T \subset V(G)$ be an anticonnected set such that $V(P) \cap T = \emptyset$, the ends of P are T-complete, and some vertex in T is smooth. Then the number of T-edges in P has the same parity as the length of P. In particular if P has odd length at least three, then some interior vertex of P is T-complete.*

Proof. If P has length one or two the lemma holds trivially, so assume that P has length at least three. Let $P = u\text{-}u'\text{-}\cdots\text{-}v'\text{-}v$. We apply Lemma 4.1 to P and T. If we have outcome 1 or 2 of Lemma 4.1, then the lemma holds. We know that outcome 4 does not hold since G contains no antihole of length at least six. Hence suppose that outcome 3 of Lemma 4.1 holds, so T contains a leap $\{a, b\}$ for P. Then $P \cup \{a, b\}$ induces a prism K, whose triangles are $\{a, u, u'\}$ and $\{b, v, v'\}$, and a-v, b-u are two paths of K, so K is an odd prism. Let x be a smooth vertex in T; so $x \notin \{a, b\}$. Since x is adjacent to u and v, it follows that it is not a local neighbor of K, but x has neighbors in K; but this contradicts Theorem 3.3. □

Lemma 4.4. *Let G be a Berge graph that contains no antihole of length at least six. Let H be a hole in G, let $P = x\text{-}\cdots\text{-}y$ be a path in G, and let $T \subset V(G)$ be an anticonnected set that contains a smooth vertex σ, such that $V(H)$, $V(P)$ and T are pairwise disjoint. Assume that there are disjoint edges ab, cd of H such that the edges between H and P are ax and bx, and c, d, y are T-complete. Then one of a, b is T-complete.*

Proof. We may assume that a, c, d, b lie in this order along H. We call P_1 the (a, c)-path contained in $H \setminus \{b, d\}$ and P_2 the (b, d)-path contained in $H \setminus \{a, c\}$. Let $t \in T$. Since t has a neighbor in $V(P)$, there is a path from t to x with interior in $V(P)$; denote this path by $S(t)$. We may assume that neither a nor b is complete to T.

(1) *There exists $q \in T$ with q non-adjacent to both a and b.*

Since T is anticonnected, there is an antipath Q from a to b with interior in T. We claim that Q has length two. Suppose not. Since a-Q-b-z-a is not an antihole (of length at least five) for any $z \in \{c, d, y\}$, it follows that $ac, bd \in E(G)$ and $x = y$. But now a-Q-b-c-y-d-a is an antihole of length at least five, a contradiction. This proves that Q has length two, and the interior vertex q of Q satisfies (1).

(2) *Let $t \in T$ be anticomplete to $\{a, b\}$. Then t is anticomplete to $V(H) \setminus \{c, d\}$, and $K(t) = G|(V(H) \cup V(S(t)))$ is a prism with triangles $\{a, b, x\}$ and $\{c, d, t\}$.*

Suppose that t has a neighbor in $V(H) \setminus \{c, d\}$. We may assume that t has a neighbor in $V(P_1) \setminus \{c\}$. Now there is a path R_1 from t to a with interior in $V(P_1) \setminus \{c\}$, a path R_2 from t to b with interior in $V(P_2)$, and $G|(V(R_1) \cup V(R_2) \cup V(S(t)))$ is a pyramid with triangle $\{a, b, x\}$ and apex t, contrary to Lemma 3.1. This proves that t is anticomplete to $V(H) \setminus \{c, d\}$. But now $K(t) = G|(V(H) \cup V(S(t)))$ is a prism with triangles $\{a, b, x\}$ and $\{c, d, t\}$. This proves (2).

Let q be as in (1). It follows from (2) that σ is adjacent to at least one of a, b (for σ is not a corner vertex), and in particular $\sigma \neq q$. Since σ is complete to $\{c, d\}$, it follows that σ is not a local neighbor of $K(q)$, but σ has a neighbor in $K(q)$, and so, since σ is smooth, Theorem 3.3 implies that $K(q)$ is an even prism. In particular, q is non-adjacent to x, and $x \neq y$. Let x' be the neighbor of x in P. Since σ has a neighbor in $V(P) \setminus \{x\}$, namely y, there is a path from σ to x' with interior in $V(P) \setminus \{x\}$; let $R(\sigma)$ be this path. Now $R(\sigma)$ contains a path from σ to the interior of $S(q)$, and every vertex in the interior of $R(\sigma)$ is non-adjacent to a, b, c and d. Since σ is adjacent to both c and d, it follows that σ is rough for $K(q)$ with base path $S(q)$, a contradiction. □

Lemma 4.5. *Let G be a Berge graph that contains no antihole of length at least six. Let H be a hole in G, let $P = x\text{-}\cdots\text{-}y$ be a path in G, and let $T \subset V(G)$ be an anticonnected set that contains a smooth vertex, such that $V(H)$, $V(P)$ and T are pairwise disjoint. Assume that $V(H) \cup V(P)$ is connected, and that there are adjacent vertices $u, v \in H$ such that u, v and x are T-complete. Then either some vertex of P is adjacent to one of u, v, or some vertex of $H \setminus \{u, v\}$ is T-complete.*

Proof. Suppose that the lemma does not hold and choose a counterexample such that $|V(H) \cup V(P)|$ is minimal. Hence y is the only vertex of P that has a neighbor in H, and no vertex of $V(P) \setminus \{x\}$ is complete to T. Let u' (resp. v') be the neighbor of y closest to u along $H \setminus \{v\}$ (resp. to v along $H \setminus \{u\}$). By the assumption, $u' \neq u$ and $v' \neq v$. Call H_u the path from u to u' in $H \setminus \{v\}$, and call H_v the path from v to v' in $H \setminus \{u\}$.

Suppose that $u' = v'$. Then one of the paths $P \cup H_u$ and $P \cup H_v$ is odd (note that this odd path is of length at least 3), and has no T-complete vertex in its interior, contrary to Lemma 4.3. Therefore $u' \neq v'$.

Suppose that $u'v' \in E$. Then we can apply Lemma 4.4 to the hole H, the path P, and the set T, and we obtain that one of u', v' is T-complete, a contradiction. Therefore $u'v' \notin E$. Consider the hole H' induced by $V(H_u) \cup V(H_v) \cup \{y\}$. If $x \neq y$, then H', $P \setminus \{y\}$, and T form a counterexample to the lemma with $|V(H') \cup V(P \setminus \{y\})| < |V(H) \cup V(P)|$, a contradiction. Therefore, $x = y$, but then H' has exactly one T-edge, but also contains the T-complete vertex x. This contradicts Lemma 4.2. □

Now we can give the proof of Theorem 1.10, which we restate here for clarity.

Theorem 4.6. *Let G be a Berge graph with no antihole of length at least six. Assume that every proper induced subgraph of G either is a complete graph or has an even pair. Let σ be a vertex of G that is smooth and not simplicial. Then the neighborhood of σ includes an even pair of G.*

Proof. For $X \subseteq V(G)$, we let $C(X)$ denote the set of all common neighbors of X in G. For a path R and $x, y \in V(R)$, $R[x, y]$ denotes the subpath of R with ends x and y.

There is a set $T \subset V(G)$ such that T is anticonnected, $\sigma \in T$, and $C(T)$ is not a clique (because $\{\sigma\}$ itself has these properties), and we choose T maximal with these properties. Let $Z = V(G) \setminus (T \cup C(T))$. An *outer path* is a path of length at least two whose ends are T-complete and whose interior vertices are in Z.

(1) *Every outer path has length ≥ 4 and even.*

Indeed, Lemma 4.3 implies that there is no outer path of odd length. Moreover, suppose that P is an outer path of length two. Let z be the interior vertex of P. The set of $T \cup \{z\}$-complete vertices is equal to $C(T) \cap N(z)$, which is not a clique because it contains the ends of P, and is anti-connected because $z \notin C(T)$, so $T \cup \{z\}$ contradicts the maximality of T. Thus (1) holds.

(2) *We may assume that there is an outer path.*

Suppose that there is no outer path. By the hypothesis, the subgraph $G|C(T)$ has an even pair $\{a, b\}$. Consider an (a, b)-path P in G. If P has a vertex $t \in T$ then $P = a\text{-}t\text{-}b$, so P has length two. Now suppose that $V(P) \cap T = \emptyset$. Then P lies entirely in $C(T)$, for otherwise P would contain an outer path. Hence P has even length. This means that $\{a, b\}$ is an even pair of G and the theorem is proved. Thus (2) holds.

By (2), we can choose an outer path $\alpha\text{-}z_1\text{-}\cdots\text{-}z_n\text{-}\beta$, with $\alpha, \beta \in C(T)$ and $z_1, \ldots, z_n \in Z$, such that n is minimal. By (1), n is odd and $n \geq 3$. Let $R = z_1\text{-}\cdots\text{-}z_n$. Define:

$$\begin{aligned} A &= \{v \in C(T) \mid v \textit{ is adjacent to } z_1 \textit{ and has no neighbor in } \{z_2, \ldots, z_n\}\}, \\ B &= \{v \in C(T) \mid v \textit{ is adjacent to } z_n \textit{ and has no neighbor in } \{z_1, \ldots, z_{n-1}\}\}. \end{aligned}$$

Note that A is not empty, because $\alpha \in A$, and that A is a clique, for otherwise there is an outer path $a\text{-}z_1\text{-}a'$ for every two non-adjacent vertices $a, a' \in A$, contradicting (1). Likewise B is a non-empty clique. Clearly, $A \cap B = \emptyset$. Moreover, there is no edge ab with $a \in A$ and $b \in B$, for otherwise

$\{a, z_1, \ldots, z_n, b\}$ induces an odd hole. We will show that some well-chosen vertices $a \in A$ and $b \in B$ form an even pair of G.

(3) *Every T-complete vertex that has a neighbor in R is either in $A \cup B$ or complete to $A \cup B$.*

Pick a T-complete vertex w that has a neighbor $z_i \in R$ $(1 \le i \le n)$. Suppose that w is not complete to $A \cup B$; so, up to symmetry, w is not adjacent to a vertex $u \in A$. Let i be the smallest integer such that w is adjacent to z_i. Then u-z_1-$\cdots$-z_i-w is an outer path. The minimality of n implies $i = n$, and so $w \in B$. Thus (3) holds.

We say an *R-segment* is a path in G of length at least one whose ends have a neighbor in R and whose interior vertices have no neighbor in R.

(4) *Let Q be an R-segment that contains an odd number of T-edges. Assume that $V(Q) \cap (B \cup C(A \cup B)) = \emptyset$ and $V(Q) \not\subseteq A$. Then Q has length at least two, $V(Q) \cap V(R) = \emptyset$, there are exactly two T-complete vertices in Q and they are adjacent, and there are vertices $z_i, z_j \in V(R)$ such that $V(Q) \cup V(R[z_i, z_j])$ induces a hole H_Q.*

Note that since Q is a path and contains a T-edge, it follows that $V(Q) \cap T = \emptyset$, for otherwise $G|V(Q)$ would contain a triangle.

Let x, y be the ends of Q, and let x' (resp. y') be the T-complete vertex in Q closest to x (resp. closest to y). So $x' \neq y'$. Suppose that Q has length one. Then x, y are T-complete, so $x, y \notin V(R)$. Moreover each of x, y has a neighbor in R, by the definition of an R-segment. By (3) and since $V(Q) \cap (B \cap C(A \cap B)) = \emptyset$, we have that $x, y \in A$, a contradiction since $V(Q) \not\subset A$. So Q has length at least two. It follows, by the definition of an R-segment, that $V(Q) \cap V(R) = \emptyset$. Moreover, x has a neighbor $z_i \in V(R)$ and y has a neighbor $z_j \in V(R)$. We choose z_i and z_j such that the path $R[z_i, z_j]$ is minimal; so every interior vertex of that path has no neighbor in Q. Hence $V(Q) \cup V(R[z_i, z_j])$ induces a hole H_Q. The hole H_Q has an odd number of T-edges (the same as Q), so Lemma 4.2 implies that x' and y' are the only T-complete vertices in H_Q and they are adjacent. Thus (4) holds.

Let $P = u$-u'-$\cdots$-v'-v an odd path with $u \in A$ and $v \in B$. Then P has length at least three, since there is no edge between A and B; also P contains no vertex of T and no $(A \cup B)$-complete vertex.

(5) *Let $P = u$-u'-$\cdots$-v'-v be an odd path with $u \in A$ and $v \in B$, and with $u', v' \notin A \cup B$. The only edges between R and $V(P) \cap C(T)$ are $z_1 u$ and $z_n v$.*

Suppose that zw is an edge with $z \in R$ and $w \in V(P) \cap C(T)$. As observed above w is not complete to $A \cup B$, so, by (3), we have $w \in A \cup B$. If $w \in A$, then, since A is a clique, it follows that $w = u$, because $u' \notin A$. The case $w \in B$ is similar. Thus (5) holds.

(6) *If $P = u$-u'-$\cdots$-v'-v is an odd path with $u \in A$ and $v \in B$, then exactly one of $u' \in A$ or $v' \in B$ holds.*

We prove (6) by induction on the length of P. First, suppose that both $u' \in A$ and $v' \in B$ hold. The path $P' = P[u', v']$ has odd length and, since there is no edge between A and B, this length is at least three. Put $P' = u'$-u''-$\cdots$-v''-v'. By the induction hypothesis applied to P', one of $u'' \in A$ or $v'' \in B$ holds; but this contradicts the fact that A and B are cliques. So at most one of $u' \in A$ and $v' \in B$ holds.

We may assume that $u' \notin A$ and $v' \notin B$. We will show that this leads to a contradiction. Some interior vertex of P has a neighbor in R, for otherwise $V(R) \cap V(P) = \emptyset$ and $V(R) \cup V(P)$ induces an odd hole. Since u and v also have a neighbor in R, we deduce that P has at least two R-segments. On the other hand, Lemma 4.3 implies that P has an odd number of T-edges. It follows that there is an

R-segment Q of P that contains an odd number of T-edges, and that Q does not contain both u and v, say Q does not contain v. Clearly, $V(Q) \cap (B \cup C(A \cup B)) = \emptyset$ and $V(Q) \not\subseteq A$. By (4), Q contains exactly two T-complete vertices x', y' and they are adjacent, and we use the notation H_Q, z_i, z_j as in (4). Call x and y the ends of Q, and assume that u, x, x', y', y, v lie in that order along P. By (5) we have $y' \neq y$ since $v \notin Q$.

Let $k = \max\{i, j\}$. Define a path S by setting $S = z_{k+1}$-R-z_n-v if $k < n$ and $S = v$ if $k = n$. Note that $G|(V(H_Q) \cup V(S))$ is connected since z_k is a vertex of H_Q adjacent to S. We can apply Lemma 4.5 to the triple (H_Q, S, T); so some vertex $z \in S$ has a neighbor in $\{x', y'\}$. However, v itself has no neighbor in $\{x', y'\}$ because x', y', y, v are four distinct vertices in that order along P. So $z \in \{z_{k+1}, \dots, z_n\}$. But z is not adjacent to y' because y' has no neighbor in R since y' is in the interior of Q; so z is adjacent to x'. Then x' has a neighbor in R, so $x' = x = u$ by (5), so $x' \in A$, but then the edge zx' contradicts (5) because $z \neq z_1$. This completes the proof that either $u' \in A$ or $v' \in B$ holds, and (6) follows.

For each vertex $b \in B$ we define a binary relation $<_b$ on A by setting $a <_b a'$ if there exists an odd path a-a'-$\cdots$-b. For each vertex $a \in A$ we define a binary relation $<_a$ on B similarly.

(7) *For each $b \in B$ the relation $<_b$ is antisymmetric.*

Suppose not; then there exist $b \in B$ and odd paths $P_u = u_0$-u_1-$\cdots$-u_p with $p \geq 3$, p odd, $P_v = v_0$-v_1-$\cdots$-v_q with $q \geq 3$, q odd such that $u_0 = v_1$ and $v_0 = u_1$, $u_0, v_0 \in A$ and $u_p = v_q = b$. By (6) we know that $P_u \setminus \{u_0, u_1, b\}$ and $P_v \setminus \{v_0, v_1, b\}$ contain no vertex from $A \cup B$.

Let r be the smallest integer such that a vertex $u_r \in P_u \setminus \{u_0, u_1\}$ has a neighbor in $P_v \setminus \{v_0, v_1\}$, and let s be the smallest integer such that $u_r v_s$ is an edge, with $2 \leq s \leq q$. Such integers exist since u_{p-1} is adjacent to v_q. Now $\{u_1, \dots, u_r, v_1, \dots, v_s\}$ induces a hole H, so r and s have the same parity, and u_r and v_s are different and adjacent. Hence $2 \leq r < p$ and $2 \leq s < q$.

We claim that $u_r v_{s+1}$ is an edge, and it is the only edge between $P_u[u_1, u_r]$ and $P_v[v_{s+1}, v_q]$. By the choice of r, $\{u_1, \dots, u_{r-1}\}$ is anticomplete to $\{v_{s+1}, \dots, v_q\}$. Let t be the largest integer such that there is an edge $u_r v_t$ with $2 \leq s \leq t \leq q$. Suppose that $t-s$ is even. Then $V(P_u[u_1, u_r]) \cup V(P_v[v_t, v_q])$ induces an odd path from A to B. Its second vertex is u_2, and its penultimate vertex w is either v_{q-1} (if $t < q$) or u_r (if $t = q$). By (6) applied to that path, we should have either $u_2 \in A$ or $w \in B$, but by (6), neither P_u nor P_v contains a vertex $(A \cup B) \setminus \{u_1, v_1, b\}$; this is a contradiction. Therefore, $t-s$ is odd. Suppose that $t \geq s+3$. Then $V(P_v[v_1, v_s]) \cup \{u_r\} \cup V(P_v[v_t, v_q])$ induces an odd path from A to B. Its second vertex is v_2, and its penultimate vertex y is either v_{q-1} (if $t < q$) or u_r (if $t = q$). By (6) applied to that path, we should have either $v_2 \in A$ or $y \in B$, but by (6), neither P_u nor P_v contains a vertex $(A \cup B) \setminus \{u_1, v_1, b\}$; this is a contradiction. Hence $t = s+1$, which proves our claim.

We continue with the proof of (7). Consider the T-complete vertices in $H \setminus \{u_1, v_1\}$. We can apply Lemma 4.4 to the hole H, the path $P_v[v_{s+1}, v_q]$ and the set T, with respect to the edges $u_1 v_1$ and $u_r v_s$, and we obtain that one of u_r, v_s is T-complete, so H contains at least three T-complete vertices. By Lemma 4.2, H has an even number of T-edges.

We claim that some vertex of $H \setminus \{u_1, v_1\}$ has a neighbor in R. Suppose the contrary. In particular $V(H) \cap V(R) = \emptyset$. There is a path S from z_1 to v_{s+1} in $V(R) \cup \{v_{s+1}, \dots, v_q\}$, with vertex set $\{z_1, \dots, z_k, v_{s+1}, \dots, v_l\}$, say; in particular z_k is adjacent to v_l.

Then, by our previous claim, the three paths $P_u[u_1, u_r]$, $P_v[v_1, v_s]$ and S form a prism K, whose corners are $u_1, v_1, z_1, u_r, v_s, v_{s+1}$, and T is complete to u_1, v_1 and one of u_r, v_s. If K is an odd prism, then this contradicts Theorem 3.3 because σ is a smooth vertex in T, so K is even, and in particular either $k \neq 1$ or $l \neq s+1$. Also, since K is even, $q \neq s+1$. Since σ is adjacent to $b = v_q$, there is a path Q_1 from σ to z_k with interior in $\{z_{k+1}, \dots, z_n, b\}$, and a path Q_2 from σ to v_l with interior in $\{v_{l+1}, \dots, v_q\}$. Since either $k \neq 1$ or $l \neq s+1$, at least one of Q_1, Q_2 is a path from σ to the interior of S. Moreover, $\{u_1, v_1, u_r, v_s\}$ is anticomplete to $(V(Q_1) \cup V(Q_2)) \setminus \{\sigma, z_k, v_l\}$, and so σ is rough for K, a contradiction. This proves our claim that some vertex of $H \setminus \{u_1, v_1\}$ has a neighbor in R.

It follows that H has at least three R-segments. Observe that u_1-v_1 is an R-segment that contains one T-edge. Since H has an even number of T-edges, there exists an R-segment Q of H, different from u_1-v_1, that contains an odd number of T-edges. Call x and y the ends of Q, and call x' and y' respectively the first and last T-complete vertices of Q, so that u_1, x, x', y', y, v_1 lie in this order along H. Clearly, $V(Q) \cap (B \cup C(A \cup B)) = \emptyset$ and $V(Q) \not\subseteq A$. By (4) x' and y' are the only T-complete vertices in Q and they are adjacent. We use the same notation H_Q, z_i, z_j as in (4). Since H has at least three R-segments, we have either $x \neq u_1$ or $y \neq v_1$, so let us assume up to symmetry that $y \neq v_1$.

Suppose that $x \neq u_1$. So $x', y' \notin A$. By (3), we have $x \neq x'$. Let $h = \min\{i, j\}$. Define a path $P_1 = u_1$-z_1-R-z_{h-1} if $h \geq 2$ and $P_1 = u_1$ if $h = 1$. We can apply Lemma 4.5 to the triple (H_Q, P_1, T), which implies that a vertex of P_1 is adjacent to one of x', y'. However, x' and y' have no neighbor in R, by (3); and so, one of x', y' is adjacent to u_1. Since $x \neq x'$, and $x \neq u_1$, it follows that neither x' nor y' are adjacent to u_1, a contradiction. Hence $x = u_1$, which implies $x' = u_1$, so $i = 1$, and $y' = u_2$, so $y \neq y'$ since Q has length at least two. If y is adjacent to v_1, then $V(H_Q) \cup \{v_1\}$ induces a pyramid (with triangle $\{u_1, v_1, z_1\}$ and apex y), a contradiction to Lemma 3.1. So y is not adjacent to v_1. It follows that $\{v_1\} \cup V(R[z_1, z_j]) \cup V(Q[y, y'])$ induces a path. This path is odd (because $H_Q \setminus \{u_1\}$ is even, and y' is adjacent to u_1), of length at least three, and its ends are T-complete and its interior vertices are not T-complete, which contradicts Lemma 4.3. Thus (7) holds.

(8) *For each $b \in B$, the relation $<_b$ is transitive.*

Let u, v, w be three vertices of A such that $u <_b v <_b w$. Since $v <_b w$, there exists an odd path $P = v_0$-v_1-$\cdots$-v_q with $v_0 = v$, $v_1 = w$, $v_q = b$, q odd, $q \geq 3$. By (6) we have $v_{q-1} \notin B$. If u has no neighbor in $P[v_2, v_q]$ then $\{u\} \cup V(P[v_1, v_q])$ induces an odd path to b, implying $u <_b w$ as desired. Hence we may assume that u has a neighbor v_i in $P[v_2, v_q]$, and let i be the largest such integer. We have $i < q$ as there is no edge between A and B. If i is odd (so $3 \leq i \leq q-2$), then $\{u\} \cup V(P[v_i, v_q])$ induces an odd path with $u \in A$ and $v_q \in B$; applying (6) to this path, we have either $v_i \in A$ or $v_{q-1} \in B$. The former is impossible because A is a clique, and we saw above that $v_{q-1} \notin B$. Hence i is even (with $2 \leq i \leq q-1$). Then $\{v_0, u\} \cup V(P[v_i, v_q])$ induces an odd path to b, implying $v <_b u$ and contradicting (7). Thus (8) holds.

Facts (7) and (8) mean that $<_b$ is a strict order relation for each $b \in B$. Let $\mathrm{Max}(b)$ denote the set of vertices of A that are maximal for $<_b$. Similarly, for each vertex $a \in A$ the relation $<_a$ is a strict order. Let $\mathrm{Max}(a)$ denote the set of vertices of B that are maximal for $<_a$.

(9) *There exist $a \in A$ and $b \in B$ such that $a \in Max(b)$ and $b \in Max(a)$.*

For two vertices $a \in A$ and $b \in B$, let $D_b(a) = \{a' \in A \mid a' <_b a\}$, and let $D_a(b)$ be defined similarly. Choose a and b such that $D_b(a)$ is maximized and, subject to this first criterion, such that $D_a(b)$ is maximized. The first criterion implies that $a \in \mathrm{Max}(b)$. Hence let us assume that $b \notin \mathrm{Max}(a)$. So there exists $v \in \mathrm{Max}(a)$ such that $b <_a v$. We claim that $D_b(a) \subseteq D_v(a)$. To prove this, let $a' \in D_b(a)$. So there is an odd path $P = a'$-u_1-$\cdots$-u_k with $u_1 = a$ and $u_k = b$. Let j be the smallest integer such that v is adjacent to u_j. If $j = k$, then a-P-b-v is an odd path, implying $v <_a b$, a contradiction. So $j < k$. Hence a-P-u_j-v is a path, and (6) implies that it is even, so a'-a-P-u_j-v is an odd path, and so $a' <_v a$. This proves that $D_v(a) \supseteq D_b(a)$. On the other hand we have $D_a(v) \supseteq D_a(b) \cup \{b\}$ since $b <_a v$. So the pair $\{a, v\}$ contradicts the choice of $\{a, b\}$. Thus (9) holds.

Finally, we observe that the pair $\{a, b\}$ given by (9) is an even pair of G. Indeed, if there is an odd path a-a'-$\cdots$-b'-b, then (6) implies either $a' \in A$ and $a <_b a'$, or $b' \in B$ and $b <_a b'$, so either $a \notin \mathrm{Max}(b)$ or $b \notin \mathrm{Max}(a)$, a contradiction. This completes the proof of Theorem 4.6. $\square$

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References

[1] M. Chudnovsky, I. Lo, F. Maffray, N. Trotignon, K. Vušković, "Coloring square-free Berge graphs", *arXiv preprint*, arXiv:1509.09195 (2015).

[2] M. Chudnovsky, N. Robertson, P. Seymour, R.Thomas, "The strong perfect graph theorem", *Annals of Mathematics* 164 (2006), 51–229.

[3] H. Everett, C.M.H. de Figueiredo, C. Linhares Sales, F. Maffray, O. Porto, B.A. Reed, "Even pairs", in: *Perfect Graphs*, J.L. Ramírez-Alfonsín and B.A. Reed eds., Wiley Interscience, 2001, 67–92.

[4] S. Hougardy, "Even and odd pairs in line-graphs of bipartite graphs", *Eur. J. Comb.* 16 (1995), 17–21.

[5] F. Maffray, "Coloring square-free Berge graphs with no odd prism", *arXiv preprint*, arXiv:1502.03695 (2015).

[6] F. Maffray, *private communication.*

[7] F. Maffray, N. Trotignon, "Algorithms for perfectly contractile graphs", *SIAM Journal on Discrete Mathematics* 19 (2005), 553–574.

[8] F. Roussel, P. Rubio, "About skew partitions in minimal imperfect graphs", *J. Comb. Theory, Ser. B* 83 (2001), 171–190.