Concatenating bipartite graphs

Maria Chudnovsky*
Princeton University, Princeton, NJ 08544, U.S.A.
mchudnov@math.princeton.edu

Patrick Hompe
Princeton University, Princeton, NJ 08544, USA
Current address: University of Waterloo, Waterloo, Ontario N2L3G1, Canada
PatrickHompe@gmail.com

Alex Scott†
Mathematical Institute, University of Oxford, Oxford OX2 6GG, U.K.
scott@maths.ox.ac.uk

Paul Seymour‡
Princeton University, Princeton, NJ 08544, U.S.A.
pds@math.princeton.edu

Sophie Spirkl§
Rutgers University, New Brunswick, NJ 08901, U.S.A.
Current address: University of Waterloo, Waterloo, Ontario N2L3G1, Canada
sspirkl@uwaterloo.ca

Submitted: Jan 15, 2019; Accepted: Apr 27, 2022; Published: Jun 3, 2022

*Supported by NSF grant DMS 1763817 and US Army Research Office Grant W911NF-16-1-0404.
†Supported by a Leverhulme Trust Research Fellowship.
‡Supported by ONR grant N00014-14-1-0084, AFOSR grant A9550-19-1-0187, and NSF grants DMS-1265563 and DMS-1800053.
§This material is based upon work supported by the National Science Foundation under Award No. DMS-1802201.

Abstract

Let $x, y \in (0, 1]$, and let A, B, C be disjoint nonempty stable subsets of a graph G, where every vertex in A has at least $x|B|$ neighbours in B, and every vertex in B has at least $y|C|$ neighbours in C, and there are no edges between A, C. We denote by $\phi(x, y)$ the maximum z such that, in all such graphs G, there is a vertex $v \in C$ that is joined to at least $z|A|$ vertices in A by two-edge paths. This function has some interesting properties: we show, for instance, that $\phi(x, y) = \phi(y, x)$ for all x, y, and there is a discontinuity in $\phi(x, x)$ when $1/x$ is an integer. For $z = 1/2, 2/3, 1/3, 3/4, 2/5, 3/5$, we try to find the (complicated) boundary between the set of pairs (x, y) with $\phi(x, y) \geqslant z$ and the pairs with $\phi(x, y) < z$. We also consider what happens if in addition every vertex in B has at least $x|A|$ neighbours in A, and every vertex in C has at least $y|B|$ neighbours in B.

We raise several questions and conjectures; for instance, it is open whether $\phi(x, x) \geqslant 1/2$ for all $x > 1/3$.

Mathematics Subject Classifications: 05C35, 05C70

1 Introduction

Our interest in the topic of this paper grew mainly from the Caccetta-Häggkvist conjecture [1] from 1978:

1.1. Conjecture: *For every integer $k \geqslant 1$, and all $n > 0$, if G is an n-vertex digraph in which every vertex has out-degree at least n/k, then G has girth at most k.*

(All graphs and digraphs in this paper are finite, and have no loops or parallel edges, though digraphs might have "antiparallel edges", that is, directed cycles of length two. The *girth* of a digraph is the minimum length of a directed cycle.) Conjecture 1.1 is true for $k = 1, 2$ but the case $k = 3$ is still open and is of particular interest. There are many possible extensions and variations (see [8] for a survey), and here are two:

1.2. Conjecture: *If G is a non-null digraph of girth at least three, there is a vertex v such that the number of vertices with (directed) distance exactly two from v is at least the out-degree of v.*

This would imply 1.1 when $k = 3$ for digraphs with all in-degrees and out-degrees at least $n/3$.

1.3. Conjecture: *If G is a non-null digraph with girth at least three, there is a vertex v such that the number of vertices with (directed) distance one or two to v is at least the twice the out-degree of v.*

This would imply 1.1 when $k = 3$. The first is more well-known (the "second neighbourhood conjecture", from 1990), but the second is also interesting.

Sometimes, questions about digraphs can usefully be converted into questions about directed bipartite graphs, by what we call "bipartite expansion": given a digraph G with vertex set $\{v_1, \ldots, v_n\}$, take two disjoint sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$, make a_i adjacent from b_i for $1 \leqslant i \leqslant n$, and for every edge v_iv_j of G, make b_j adjacent from a_i. Applying bipartite expansion to 1.1 led two of us [7] to the following two conjectures:

1.4. Conjecture: *For every integer $k \geqslant 1$, if G is a digraph with a bipartition (A, B) with $|A| = |B| > 0$, and every vertex has out-degree more than $|A|/(k+1)$, then G has a directed cycle of length at most $2k$.*

It is shown in [7] that this is implied by 1.1, and is best possible if true, and is true for $k = 1, 2, 3, 4, 6$ and all $k \geqslant 224{,}539$.

1.5. Conjecture: *For every integer $k \geqslant 1$, and every pair of reals $\alpha, \beta > 0$ with $k\alpha + \beta > 1$, if G is a non-null digraph with bipartition (A, B), and every vertex in A has out-degree at least $\beta|B|$, and every vertex in B has out-degree at least $\alpha|A|$, then G has girth at most $2k$.*

It is shown in [7] that this implies 1.1, and is true for $k = 1, 2$. Let us mention also the following theorem of [7], a bipartite analogue of 1.3:

1.6. *Let G be a directed bipartite graph with no directed cycle of length two, and let (A, B) be a bipartition. Suppose that every vertex in A has at least $\beta|B|$ out-neighbours in B, and every vertex in B has at least $\alpha|A|$ out-neighbours in A, where $\alpha, \beta > 0$. Then there is a vertex $v \in B$ and at least $(\alpha + \beta)|A|$ vertices $u \in A$ such that there is a directed path from u to v of length at most three.*

These results and conjectures for bipartite digraphs are quite pretty, and led us to consider "tripartite digraphs": a digraph with vertex set partitioned into three stable sets A, B, C, where its edges are directed cyclically, that is, from A to B, and from B to C, and from C to A. For instance, start with a digraph G, with vertex set $\{v_1, \ldots, v_n\}$, and make three disjoint sets $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$, and for each edge v_iv_j of G, make b_j adjacent from a_i, and c_j adjacent from b_i, and a_j adjacent from c_i. Let this new digraph be H. Then H has a directed cycle of length three if and only if G has one, and so one might hope to get a version of 1.1 (in the $k = 3$ case) for such tripartite digraphs. Indeed, one might hope that if every vertex of G has outdegree at least $|G|/3$, then H must have a directed triangle; but this is false. The reason this does not work is that there are loopless digraphs G with minimum out-degree at least $|G|/3$ (indeed, at least $|G|/2$) that have directed cycles of length two but not of length three (for instance, replace every edge of a complete bipartite graph $K_{n,n}$ with two antiparallel edges); and directed cycles of length two in G do not translate into directed cycles of H.

Even so, it seems like a good question: if G is a graph, with vertex set partitioned into stable sets A, B, C, and every vertex in A is adjacent to at least $x|B|$ vertices in B, and every vertex in B to at least $y|C|$ vertices in C, and every vertex in C to at least $z|A|$ vertices in A, when must G have a triangle? A good understanding of this might be helpful for the versions of the Caccetta-Häggkvist conjecture given above. This is the topic of this paper.

Another way to state the question is, suppose we have disjoint sets A, B, C, and there are edges between A, B and between B, C such that every vertex in A is adjacent to at least $x|B|$ vertices in B, and every vertex in B to at least $y|C|$ vertices in C; when can we fill in edges between C, A without getting a triangle, such that every vertex in C is adjacent to at least $z|A|$ vertices in A? This is true if and only if no vertex in C can reach more than $(1-z)|A|$ vertices in A by two-edge paths. So for which values of z can we guarantee that some vertex in C can reach more than $(1-z)|A|$ vertices in A by two-edge paths? Or, essentially the same question: which values of z will guarantee that some vertex in C can reach at least $z|A|$ vertices in A by two-edge paths?

Let us make some definitions. A *tripartition* of a graph G is a partition (A, B, C) of $V(G)$ where A, B, C are all nonempty stable sets. We denote the semi-open interval $\{x : 0 < x \leqslant 1\}$ of real numbers by $(0, 1]$. For $x, y \in (0, 1]$, we say a graph G is *(x, y)-constrained, via a tripartition (A, B, C),* if

- every vertex in A has at least $x|B|$ neighbours in B;
- every vertex in B has at least $y|C|$ neighbours in C; and
- there are no edges between A and C.

For $v \in V(G)$, $N(v)$ denotes its set of neighbours, and $N^2(v)$ is the set of vertices with distance exactly two from v. We write $N^2_A(v)$ for $N^2(v) \cap A$, and so on. A first observation:

1.7. *Let $x, y \in (0, 1]$, and let Z be the set of all $z \in (0, 1]$ such that, for every graph G, if G is (x, y)-constrained via (A, B, C) then $|N^2_A(v)| \geqslant z|A|$ for some $v \in C$. Then* $\sup\{z \in Z\}$ *belongs to Z.*

Proof. Let $z' = \sup\{z \in Z\}$, and let G be an (x, y)-constrained graph, via (A, B, C). We must show that $|N^2_A(v)| \geqslant z'|A|$ for some $v \in C$. We may assume that $z' > 0$; so there exists z with $0 < z < z'$, such that $\lceil z|A| \rceil = \lceil z'|A| \rceil$. Since $z' = \sup\{z \in Z\}$ and $z < z'$, and Z is an initial interval of $(0, 1]$, it follows that $z \in Z$, and so $|N^2_A(v)| \geqslant z|A|$ for some $v \in C$. Consequently, $|N^2_A(v)| \geqslant \lceil z|A| \rceil \geqslant z'|A|$, as required. This proves 1.7. ▮

We define $\phi(x, y)$ to be $\sup\{z \in Z\}$, as defined in 1.7. The objective of this paper is to study the properties of the function ϕ. We will show, for instance, that:

- $\phi(x, y) = \phi(y, x)$ for all x, y (proved in 2.3); and
- for each integer $k > 1$, there is a discontinuity in $\phi(x, x)$ when $x = 1/k$: $\phi(x, x) \leqslant 1/k$ when $x \leqslant 1/k$, and $\phi(x, x) \geqslant \frac{2k-3}{2k^2-4k+1}$ when $x > 1/k$.

We have a trivial lower bound, that will be used throughout the paper:

1.8. *$\phi(x, y) \geqslant \max(x, y)$ for all $x, y > 0$.*

Proof. Let G be (x, y)-constrained, via (A, B, C). Since every vertex in A has at least $x|B|$ neighbours in B, and $B \neq \emptyset$, there exists $u \in B$ with at least $x|A|$ neighbours in A; let $v \in C$ be adjacent to u (this is possible since $y > 0$), and then $|N^2_A(v)| \geqslant x|A|$. Consequently, $\phi(x, y) \geqslant x$. Now every vertex in A can reach at least $y|C|$ vertices in C by two-edge paths (since $x > 0$); and so by averaging, some vertex in C can reach at least $y|A|$ vertices in A by two-edge paths. Hence, $\phi(x, y) \geqslant y$. This proves 1.8. ▮

And a trivial upper bound (used to prove 3.2, 6.1 and 6.7):

1.9. *For all* $x, y \in (0,1]$,

$$\phi(x,y) \leqslant \frac{\lceil kx \rceil + \lceil ky \rceil - 1}{k}$$

for every integer $k \geqslant 1$.

Proof. Let $x, y \in (0,1]$, and let $k \geqslant 1$ be an integer. Let A, B, C be three disjoint sets each of cardinality k, where $A = \{a_1, \ldots, a_k\}$, $B = \{b_1, \ldots, b_k\}$ and $C = \{c_1, \ldots, c_k\}$. Make a graph G with vertex set $A \cup B \cup C$ as follows. Let $g = \lceil kx \rceil$, and for $1 \leqslant i \leqslant k$ make a_i adjacent to $b_i, b_{i+1}, \ldots, b_{i+g-1}$ (reading subscripts modulo k). Now let $h = \lceil ky \rceil$, and for $1 \leqslant i \leqslant k$ make b_i adjacent to $c_i, c_{i+1}, \ldots, c_{i+h-1}$ (reading subscripts modulo k). Then G is (x,y)-constrained via (A,B,C); and for $1 \leqslant i \leqslant k$, $N_A^2(c_i) = \{a_i, a_{i-1}, \ldots, a_{i-g-h+2}\}$ (again, reading subscripts modulo k). Consequently, $\phi(x,y) \leqslant (g+h-1)/k$. This proves 1.9. ∎

In particular, we have:

1.10. *For every integer* $k \geqslant 1$, *if* $x, y > 0$ *and* $\max(x,y) = 1/k$ *then* $\phi(x,y) = 1/k$.

Proof. From 1.8, $\phi(x,y) \geqslant 1/k$; and the graph consisting of k disjoint three-vertex paths shows that $\phi(x,y) \leqslant 1/k$. (This also follows from 1.9, since $\lceil kx \rceil, \lceil ky \rceil = 1$.) This proves 1.10. ∎

What makes the function ϕ interesting is that for some values of x, y, 1.8 is far from best possible, and indeed 1.9 seems closer to the truth. Initially we hoped to extend Kneser's theorem from additive group theory [6] to a general graph-theoretic setting, via a corresponding wild conjecture that the bound in 1.9 is always best possible, that is, that for all $x, y \in (0,1]$, there is an integer $k > 0$ with $\phi(x,y) = \frac{\lceil kx \rceil + \lceil ky \rceil - 1}{k}$. This turns out to be false, but perhaps not ridiculously false; maybe something like it is true.

There are two other related problems:

- Let us say G is (x,y)-*biconstrained* (*via* (A,B,C)) if G is (x,y)-constrained via (A,B,C), and in addition
  - every vertex in B has at least $x|A|$ neighbours in A, and
  - every vertex in C has at least $y|B|$ neighbours in B.
- Say G is (x,y)-*exact* (*via* (A,B,C)) if G is (x,y)-constrained via (A,B,C), and in addition there exist $x' \geqslant x$ and $y' \geqslant y$ such that
  - every vertex in A has exactly $x'|B|$ neighbours in B;
  - every vertex in B has exactly $x'|A|$ neighbours in A;
  - every vertex in B has exactly $y'|C|$ neighbours in C; and
  - every vertex in C has exactly $y'|B|$ neighbours in B.

We shall sometime use “mono-constrained” to clarify that we mean the (x, y)-constrained case and not the (x, y)-biconstrained case. Let $\psi(x, y)$ be the analogue of $\phi(x, y)$ for biconstrained graphs; that is, the maximum z such that for all G, if G is (x, y)-biconstrained via (A, B, C), then $|N_A^2(v)| \geqslant z|A|$ for some $v \in C$. (As before, this maximum exists.) Similarly, let $\xi(x, y)$ be the analogue of ϕ and ψ for the exact case. Then we have

1.11. *For all* $x, y \in (0, 1]$,

$$\max(x, y) \leqslant \phi(x, y) \leqslant \psi(x, y) \leqslant \xi(x, y) \leqslant \frac{\lceil kx \rceil + \lceil ky \rceil - 1}{k}$$

for every integer $k \geqslant 1$.

The proof of the non-trivial part of this is the same as the proof of 1.9. One might hope that ψ (and even more ξ) are better-behaved than ϕ, although this seems not to be true. For instance, we proved that $\phi(x, y) = \phi(y, x)$ for all x, y, but were not able to decide whether the same holds for ψ. And we were unable to prove anything whatsoever for the exact case that is not true for the biconstrained case, so the paper will focus on ϕ and ψ.

Let us see an example. Start with the graph of figure 1. Each vertex has a number written next to it in the figure; replace each vertex v by a set X_v of new vertices of the specified cardinality, and for each edge uv of the figure make every vertex in X_u adjacent to every vertex in X_v. This results in a graph with 81 vertices, divided into three sets of 27 corresponding to the three rows of the figure; call these A, B, C (labelled from top to bottom). The graph produced is $(13/27, 1/9)$-biconstrained via (A, B, C), and yet $|N_A^2(v)| = 13$ for every vertex $v \in C$; so this proves that $\psi(13/27, 1/9) \leqslant 13/27$ (and therefore equality holds, by 1.11). This shows that there need not exist an integer k with $\psi(x, y) = \frac{\lceil kx \rceil + \lceil ky \rceil - 1}{k}$. The same graph, used from bottom to top, shows that $\psi(1/9, 13/27) = 13/27$.

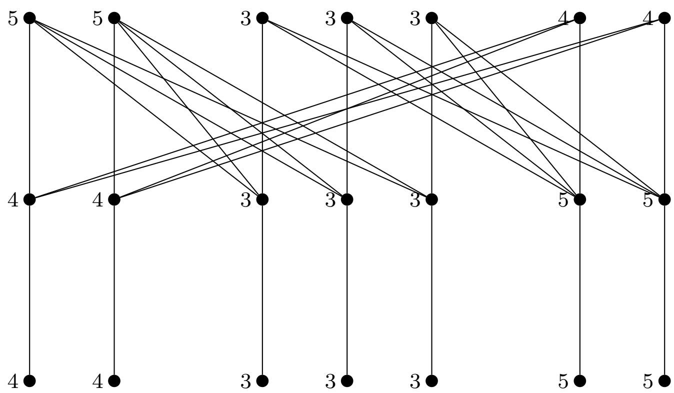

Figure 1: $\psi(13/27, 1/9) = 13/27$

The example is not yet $(13/27, 1/9)$-exact, because some vertices in A have three, four or five neighbours in B, and vice versa. We can make it exact as follows. For each edge

uv of the figure with u in the second row and v in the third, the two sets X_u, X_v have the same cardinality, one of three, four, five. Delete some edges between X_u and X_v such that every vertex in X_u has exactly three neighbours in X_v and vice versa. Then the modified graph is $(13/27, 1/9)$-exact, and shows that $\xi(13/27, 1/9) = 13/27$. Consequently, even for the supposedly nicest function ξ of our three functions, there is not always an integer k with $\xi(x, y) = \frac{\lceil kx\rceil+\lceil ky\rceil-1}{k}$.

So what can we prove about the functions ϕ and ψ? There are many ways to approach this. Some of our results are of the general form "what bounds can we place on $\phi(x, y)$ or $\psi(x, y)$ as a function of x, y?". Others are of the form "for which x, y is $\phi(x, y)$ or $\psi(x, y) \geqslant z$?" where z is some simple rational number, because with z fixed it is easier to see graphically the values of x, y that are far from being decided, and because some useful proof techniques naturally yield results in this form. (For instance, we sometimes look for a small set of vertices that covers one of A, B, C, and this approach leads to results of the form described.) One might also look for results of the form "for which x, y is $\phi(x, y)$ or $\psi(x, y) > z$?" for some fixed z, and this is different; indeed, when $1/z$ is an integer, it has a nice answer, namely if and only if $\max(x, y) > z$ (proved in 6.1). But we did not find any other results of this form that were not consequences of results of the other two forms.

Thus, we focus on seven special cases, $x = y$, and $z = 1/2, 2/3, 1/3, 3/4, 2/5, 3/5$, but in each case the results for ϕ and for ψ are quite different. The paper is organized as follows:

- We begin with a proof that $\phi(x, y) = \phi(y, x)$ for all x, y.
- Then we give some general upper bounds on $\phi(x, y)$ and $\psi(x, y)$, particularly focussing on the case when $x = y$. We determine $\psi(x, x)$ exactly, and show that $\phi(x, x)$ has a discontinuity whenever $1/x$ is an integer.
- Next we consider when $\phi(x, y) \geqslant 1/2$, or $\psi(x, y) \geqslant 1/2$. There are several theorems that this is true for certain pairs (x, y), and their union fills a good part of the (x, y)-square. We also give a number of constructions that shows the statement is *not* true for certain pairs (x, y). Ideally this would fill the complementary part of the square, but there is an "undecided" band of varying width down the middle.
- Then we do the same for $2/3$ instead of $1/2$; and then for $1/3, 3/4, 2/5, 3/5$.
- Finally, we discuss some other questions and approaches.

Some of our results appear in [4] and [5].

Incidentally, we could ask instead, if G is (x, y)-constrained via (A, B, C), which values of z will guarantee that some vertex in A can reach at least $z|C|$ vertices in C by two-edge paths? But this is of no interest; it is true if $z \leqslant y$, and false if $z > y$, since perhaps all the vertices in B have the same neighbours in C.

2 Weighted graphs and some linear programming

In this section we prove that $\phi(x,y) = \phi(y,x)$ for all x, y. The argument uses linear programming, and we need some preparation. We denote the set of real numbers by $\mathbb{R}$, and the non-negative reals numbers by $\mathbb{R}_+$. A *weighted graph* (G, w) consists of a graph G together with a function $w : V(G) \to \mathbb{R}_+$. If $X \subseteq V(G)$, we denote $\sum_{v \in X} w(v)$ by $w(X)$. Let (G, w) be a weighted graph, and (A, B, C) a tripartition of G. If $x, y \in (0, 1]$, a weighted graph (G, w) is (x, y)-*constrained* via (A, B, C), if:

- $\sum_{v \in A} w(v) = \sum_{v \in B} w(v) = \sum_{v \in C} w(v) = 1$;
- for each $v \in A$, $w(N(v) \cap B) \geqslant x$; and
- for each $v \in B$, $w(N(v) \cap C) \geqslant y$.

Similarly, we say (G, w) is (x, y)-*biconstrained* via (A, B, C), if in addition:

- for each $v \in B$, $w(N(v) \cap A) \geqslant x$; and
- for each $v \in C$, $w(N(v) \cap B) \geqslant y$.

To make the graph of figure 1 into an appropriate weighted graph, divide all the numbers by 27.

2.1. *For $x, y, z \in (0, 1]$, the following are equivalent:*

- $\phi(x, y) \geqslant z$*;*
- $w(N_A^2(v)) \geqslant z$ *for some* $v \in C$*, for every weighted graph* (G, w) *that is* (x, y)*-constrained via a tripartition* (A, B, C)*.*

Similarly, the following are equivalent:

- $\psi(x, y) \geqslant z$*;*
- $w(N_A^2(v)) \geqslant z$ *for some* $v \in C$*, for every weighted graph* (G, w) *that is* (x, y)*-biconstrained via a tripartition* (A, B, C)*.*

Proof. To prove the "if" direction of the first statement, let G be (x, y)-constrained via (A, B, C). Define $w(v) = 1/|A|$, for each $v \in A$, and $w(v) = 1/|B|$ for $v \in B$ and similarly for $v \in C$. Then (G, w) is an (x, y)-constrained weighted graph, and the claim follows. The "if" direction of the second statement is proved similarly.

For the "only if" direction, let (G, w) be a weighted graph, (x, y)-constrained via (A, B, C), and suppose such a weighted graph can be chosen with $w(N_A^2(v)) < z$ for each $v \in C$. Consequently, we may choose (G, w) such that in addition, w is rational-valued. Choose an integer $N > 0$ such that $Nw(v)$ is an integer for each $v \in G$. For each $v \in V(G)$, take a set X_v of $Nw(v)$ new vertices; and make a graph G' with vertex set $\bigcup_{v \in V(G)} X_v$, by making every vertex of X_u adjacent to every vertex of X_v for all adjacent $u, v \in V(G)$. Let $A' = \bigcup_{v \in A} X_v$, and define B', C' similarly; then (A', B', C') is a tripartition of G', and G' is (x, y)-constrained via (A', B', C'). Since in G, $w(N_A^2(v)) < z$ for each $v \in C$, it follows that in G', $|N_{A'}^2(v')| < z|A'|$ for each $v' \in C'$, a contradiction. The "only if" direction of the second statement is similar. This proves 2.1. ∎

Let G be a graph with a bipartition (A, B), and let $w : B \to \mathbb{R}_+$ be some function. We define $w(A \to B)$ to mean the minimum, over all $u \in A$, of $w(N(u))$ (taking $w(A \to B) = 0$ if $A = \emptyset$).

2.2. *Let G be a graph with a bipartition (A, B), and let $w : B \to \mathbb{R}_+$ be some function such that $w(B) = 1$. Then either*

- *there is a function $w' : B \to \mathbb{R}_+$, such that $w'(B) = 1$ and $w'(A \to B) \geqslant w(A \to B)$, and such that $w'(v) = 0$ for some $v \in B$; or*
- *there is a function $f : A \to \mathbb{R}_+$, such that $f(A) = 1$ and $f(B \to A) \geqslant w(A \to B)$.*

Proof. We may assume that $A \neq \emptyset$. If some vertex in A has no neighbour in B, then $w(A \to B) = 0$ and the second bullet holds; so we assume that each vertex in A has a neighbour in B.

Let $x = w(A \to B)$. The function w', defined by $w'(v) = 1/|B|$ for each $v \in B$, satisfies $w'(A \to B) > 0$, since every vertex in A has a neighbour in B. Thus, we may assume that $x > 0$, replacing w by w' if necessary.

Let M be the 0/1-matrix $(a_{uv} : u \in A, v \in B)$, where $a_{uv} = 1$ if and only if u, v are adjacent. Let $\mathbf{1}_A \in \mathbb{R}^A$ be the vector of all 1's, and define $\mathbf{1}_B$ similarly. Then $w \in \mathbb{R}^B_+$ satisfies:

- $\mathbf{1}_B^T w = 1$; and
- $Mw \geqslant x\mathbf{1}_A$.

Consequently, $b = w/x$ satisfies $b \in \mathbb{R}^B_+$, and

- $\mathbf{1}_B^T b = 1/x$; and
- $Mb \geqslant \mathbf{1}_A$.

Choose $q \in \mathbb{R}^B_+$ with $Mq \geqslant \mathbf{1}_A$, with $\mathbf{1}_B^T q$ minimum. Thus, $\mathbf{1}_B^T q \leqslant 1/x$. Since $Mq \geqslant \mathbf{1}_A$ and G has an edge, it follows that $\mathbf{1}_B^T q > 0$; let $1/y = \mathbf{1}_B^T q$, and define $w' = yq$. Then $y \geqslant x$, and $\mathbf{1}_B^T w' = 1$ and $Mw' \geqslant y\mathbf{1}_A$, and so we may assume that $w'(v) > 0$ for each $v \in B$, because otherwise the first bullet holds.

Now q minimizes $\mathbf{1}_B^T q$ subject to the linear constraints $q \in \mathbb{R}^B_+$ and $Mq \geqslant \mathbf{1}_A$. From the linear programming duality theorem, there exists $p \in \mathbb{R}^A_+$ such that $p^T M \leqslant \mathbf{1}_B^T$, and $p^T \mathbf{1}_A = \mathbf{1}_B^T q = 1/y$. Define $f = yp$. Then $f : A \to \mathbb{R}_+$ satisfies $f(A) = 1$, and $f(N(v)) \leqslant y$ for each $v \in B$.

Let $v' \in B$; we claim that $f(N(v')) = y$. This follows from the "complementary slackness" principle, but we give the argument in full, as follows. Let $s = w'(v')(y - f(N(v')))$. Thus, $s \geqslant 0$, and we will show $s = 0$. We have

$$y = \sum_{v \in B} yw'(v) \geqslant s + \sum_{v \in B} \sum_{u \in N(v)} w'(v)f(u) = s + \sum_{u \in A} \sum_{v \in N(u)} f(u)w'(v) \geqslant s + \sum_{u \in A} yf(u) = s + y.$$

Consequently, $s = 0$, as claimed. Hence, f satisfies the second bullet. This proves 2.2. ∎

From 2.2 we deduce a very useful result, used throughout the paper

2.3. *If $x, y \in (0,1]$ then $\phi(x,y) = \phi(y,x)$.*

Proof. Let $z = \phi(x,y)$, and choose a weighted graph (G,w) that is (x,y)-constrained via (A,B,C), such that $w(N^2_A(v)) \leqslant z$ for each $z \in C$. Moreover, choose G with $|V(G)|$ minimum. If there is a function $w' : B \to \mathbb{R}_+$, such that $w'(B) = 1$ and $w'(A \to B) \geqslant w(A \to B)$, and such that $w'(v) = 0$ for some $v \in B$, then we may replace w by a new weight function, changing w to w' on B and otherwise keeping w unchanged, and then we may delete the vertex $v \in B$ with $w'(v) = 0$, contrary to the minimality of $|V(G)|$. Thus, there is no such w', and so by 2.2, there is a function $f : A \to \mathbb{R}_+$, such that $f(A) = 1$ and $f(B \to A) \geqslant w(A \to B) \geqslant x$. Similarly, there is a function $g : B \to \mathbb{R}_+$, such that $g(B) = 1$ and $g(C \to B) \geqslant y$. Let H be the graph with bipartition (A,C) in which $u \in A$ and $v \in C$ are adjacent if $u \notin N^2_A(v)$ in G. Thus, in H, $w(C \to A) \geqslant 1 - z$; and so from 2.2 and the minimality of $|V(G)|$, there is a function $h : C \to \mathbb{R}_+$, such that $h(C) = 1$ and (in H) $h(A \to C) \geqslant 1 - z$. Let w' be defined by the union of f, g and h in the natural sense; then (G, w') is a weighted graph and is (y,x)-constrained via (C,B,A), and $w'(N^2_C(v)) \leqslant z$ for each $v \in A$. This proves that $\phi(y,x) \leqslant z$, and so proves 2.3. ▮

We remark that we have not been able to prove an analogue of 2.3 for the biconstrained case, or for the exact case, although we have no counterexample for either one.

There is another useful application of 2.2, the following:

2.4. *Let (G,w) be an (x,y)-constrained weighted graph, via (A,B,C), with the property that $w(N^2_A(v)) \leqslant z$ for each $v \in C$. Suppose that there exists $X \subseteq A$ with $|X| < z^{-1}$ such that $\bigcup_{v \in X} N^2_C(v) = C$. Then there exists $u \in A$ and a weighted graph (G', w') such that*

- *G' is obtained from G by deleting u;*
- *(G', w') is (x,y)-constrained via (A', B, C), where $A' = A \setminus \{u\}$;*
- *in G', $w'(N^2_{A'}(v)) \leqslant z$ for all $v \in C$; and*
- *$w'(u) = w(u)$ for all $u \in B \cup C$.*

Proof. Suppose not. Let H be the graph with bipartition (A,C), in which $u \in A$ and $v \in C$ are adjacent if $u \notin N^2_A(v)$ in G. Then by 2.2, applied to H, there is a function $h : C \to \mathbb{R}_+$, such that $h(C) = 1$ and (in H) $h(A \to C) \geqslant 1 - z$. Consequently, in G, $h(N^2_C(v)) \leqslant z$ for each $v \in A$. In particular, $h(N^2_C(v)) \leqslant z$ for each $v \in X$, and so $h(C) \leqslant z|A| < 1$, a contradiction. This proves 2.4. ▮

Let $x, y, z \in (0,1]$. We say that (x,y,z) is *triangular* if no triangle-free graph G admits a tripartition A, B, C of $V(G)$ with the following properties:

- A, B, C are nonempty stable sets;
- every vertex in A has at least $x|B|$ neighbours in B;

- every vertex in B has at least $y|C|$ neighbours in C; and
- every vertex in C has at least $z|A|$ neighbours in A.

As we mentioned in the introduction, it is possible to reformulate results about $\phi(x, y)$ in terms of triangular triples, because we have:

2.5. *For $x, y, z \in (0, 1]$, $\phi(x, y) > 1 - z$ if and only if (x, y, z) is triangular. Consequently, the three statements $\phi(x, y) \leqslant 1 - z$, $\phi(z, x) \leqslant 1 - y$, and $\phi(y, z) \leqslant 1 - x$ are equivalent.*

Proof. Suppose that (x, y, z) is not triangular. Then there is a triangle-free graph G with a tripartition (A, B, C), satisfying the three bullets in the definition of "triangular". Let H be the subgraph of G with $V(H) = V(G)$, obtained by deleting all edges between A and C. If $v \in C$, then $N_A^2(v)$ (defined with respect to H) contains only vertices in A that are nonadjacent to v in G, since G is triangle-free; and so $|N_A^2(v)| \leqslant |A| - z|A|$, since in G, v has at least $z|A|$ neighbours in A. Consequently, $\phi(x, y) \leqslant 1 - z$.

For the reverse implication, suppose that $\phi(x, y) \leqslant 1 - z$, and let H be (x, y)-constrained via (A, B, C), such that $|N_A^2(v)| \leqslant |A| - z|A|$ for each $v \in C$. Make a graph G by adding certain edges to H, namely for each $v \in C$ and $u \in A$, add an edge uv if $u \notin N_A^2(v)$. Then G is triangle-free, and every vertex $v \in C$ is adjacent in G to at least $|A| - (1 - z)|A| = z|A|$ vertices in A; and so (x, y, z) is not triangular.

In particular, (x, y, z) is triangular if and only if (z, x, y) is triangular; so it follows that $\phi(x, y) \leqslant 1 - z$ if and only if $\phi(z, x) \leqslant 1 - y$, and similarly if and only if $\phi(y, z) \leqslant 1 - x$. This proves 2.5. ∎

We call the equivalence of the second statement of 2.5 "rotating".

It is awkward to express the biconstrained problem in the language of triangular triples, but we can do so as follows. For $x, y, z \in (0, 1]$ we say that (x^*, y, z) is *triangular* if no triangle-free graph G admits a tripartition (A, B, C) that satisfies the three bullets of the previous definition, and in addition satisfies

- every vertex in B has at least $x|A|$ neighbours in A.

Similarly, we say (x^*, y^*, z) is triangular if no triangle-free graph G admits a tripartition (A, B, C) that satisfies the three bullets of the previous definition, and in addition satisfies

- every vertex in B has at least $x|A|$ neighbours in A; and
- every vertex in C has at least $y|B|$ neighbours in B;

and so on. Then, with a proof like that of 2.5, we have:

- For $x, y, z \in (0, 1]$, $\psi(x, y) > 1 - z$ if and only if (x^*, y^*, z) is triangular.

We also need some shorthand for results of the form "if $x' > x$ then (x', y, z) is triangular"; let us say "(x^+, y, z) is triangular" to mean "(x', y, z) is triangular for all $x' > x$", and treat the other two coordinates similarly. We will mix the two systems of notation, in expressions such as "(x^{+*}, y^+, z) is triangular", meaning "(x'^*, y^+, z) is triangular for all $x' > x$".

Thus, in triangular language, and assuming some results from later in the paper, we have the following.

- $(1/2^{+*}, 1/3^*, 1/3^+)$ is triangular: because 7.1 says that $(x^*, 1/3^*, 1/3^+)$ is triangular when $x > 1/2$.
- $(1/2^+, 1/3^+, 1/3^*)$ is triangular, since 7.2 shows that $(1/3^*, 1/2^+, 1/3^+)$ is triangular, and rotating gives that $(1/2^+, 1/3^+, 1/3^*)$ is triangular.
- $(1/2^+, 1/3^{+*}, 1/3^*)$ is triangular; this follows from 4.1 with $k = 2$ and rotating.
- $(1/2^+, 1/3^*, 1/3^{+*})$ is triangular; this also follows from 4.1 with $k = 2$ and rotating.

These four statements are similar, but no two are equivalent, and it would be good to find a common strengthening. Note, however, that $(1/2^{+*}, 1/3^*, 1/3^*)$ is not triangular, and indeed $(2/3^*, 1/3^*, 1/3^*)$ is not triangular. We have not been able to decide whether $(1/2^+, 1/3^+, 1/3)$ and $(1/2^+, 1/3, 1/3^+)$ are triangular, or indeed whether $(1/2^{+*}, 1/3^+, 1/3^+)$ is triangular. This extends to weighted graphs in the natural way. For instance, the weighted graph of figure 2 (identify the vertices on the left with those on the right, in order) shows that $(4/7, 2/7, 3/8)$ is not triangular.

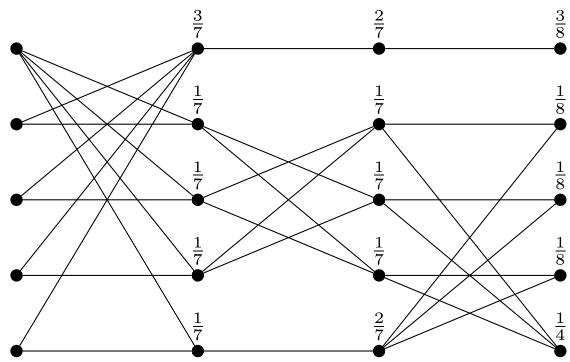

Figure 2: $(4/7, 2/7, 3/8)$ is not triangular

3 Constructions

In this section we construct some graphs to prove upper bounds on $\phi(x, y)$ or $\psi(x, y)$ for certain values of x, y. We begin with a result that will be used several times later in the paper:

3.1. *Let $x, y \in (0, 1]$, and let $z \in (0, 1]$ such that $z/(1-z) = \phi(x/(1-x), y/(1-y))$; then $\phi(x, y) \leqslant z$.*

Proof. Let (G', w') be a weighted graph that is $(x/(1-x), y/(1-y))$-constrained via some tripartition (A', B', C'), such that $w'(N^2_{A'}(v)) \leqslant z/(1-z)$ for each $v \in C'$. Add three new vertices a, b, c to G', and two edges ab and bc, forming G. Define w by

$$\begin{aligned} w(a) &= z \\ w(v) &= (1-z)w'(v) \text{ for each } v \in A' \\ w(b) &= x \\ w(v) &= (1-x)w'(v) \text{ for each } v \in B' \\ w(c) &= y \\ w(v) &= (1-y)w'(v) \text{ for each } v \in C'. \end{aligned}$$

Then G is (x, y)-constrained via $(A' \cup \{a\}, B' \cup \{b\}, C' \cup \{c\})$ and shows that $\phi(x, y) \leqslant z$. This proves 3.1. ∎

The first application of 3.1 is:

3.2. *Let $k \geqslant 0$ be an integer, and let $x, y \in (0, 1]$ with $kx, ky < 1$ and $\frac{x}{1-kx} + \frac{y}{1-ky} \leqslant 1$, with strict inequality if x or y is irrational; then $\phi(x, y) < \frac{1}{k+1}$.*

Proof. By increasing x and y if necessary, we may assume that x, y are rational. Suppose first that $k = 0$; then we may assume that $x + y = 1$. Choose an integer $n \geqslant 1$ such that nx (and hence ny) is an integer. By 1.9,

$$\phi(x, y) \leqslant \frac{\lceil nx \rceil + \lceil ny \rceil - 1}{n} = x + y - 1/n < 1.$$

This completes the proof for $k = 0$. For general k we proceed by induction on k. We may assume that $k > 0$; let $x, y \in (0, 1]$ with $\frac{x}{1-kx} + \frac{y}{1-ky} \leqslant 1$, with strict inequality if x or y is irrational. Let $x' = x/(1-x)$, and $y' = y/(1-y)$. Thus, $x', y' \in (0, 1]$ with

$$\frac{x'}{1-(k-1)x'} + \frac{y'}{1-(k-1)y'} = \frac{x}{1-kx} + \frac{y}{1-ky} \leqslant 1,$$

with strict inequality if x' or y' is irrational. From the inductive hypothesis, $\phi(x', y') < 1/k$. Let z satisfy $z/(1-z) = \phi(x', y')$; then $z/(1-z) < 1/k$, and so $z < 1/(k+1)$. From 3.1, $\phi(x, y) \leqslant z < 1/(k+1)$. This proves 3.2. ∎

The next result is used for $k = 2, 3$, and also to prove 7.8.

3.3. *Let $k \geqslant 0$ be an integer, and let $x, y \in (0, 1]$ with $x + ky \leqslant 1$ and $kx + y \leqslant 1$, with strict inequality in both if x or y is irrational; then $\psi(x, y) < \frac{1}{k}$.*

Proof. Again, we may assume that x, y are rational. Let $s = \max(x, y)$; thus, $s < 1/k$. Choose an integer $N \geqslant 1$ such that $p = xN/(1-(k-1)s)$ and $q = yN/(1-(k-1)s)$ are integers. (It follows that $p+q \leqslant N$, from the hypothesis.) Let G be a graph with vertex set partitioned into three sets A, B, C, with $|A| = N+k$ and $|B| = |C| = N+k-1$; let

$$\begin{aligned} A &= \{a_1, a_2, \dots, a_N, a'_1, \dots, a'_{k-1}, a^*\}, \\ B &= \{b_1, b_2, \dots, b_N, b'_1, \dots, b'_{k-1}\}, \\ C &= \{c_1, c_2, \dots, c_N, c'_1, \dots, c'_{k-1}\}. \end{aligned}$$

Let G have the following edges:

- for $1 \leqslant i \leqslant N$, a_i is adjacent to $b_i, b_{i+1}, \dots, b_{i+p-1}$ reading subscripts modulo N;
- for $1 \leqslant i \leqslant N$, b_i is adjacent to $c_i, c_{i+1}, \dots, c_{i+q-1}$ reading subscripts modulo N;
- for $1 \leqslant i \leqslant k-1$, a'_i is adjacent to b'_i, and b'_i is adjacent to c'_i.
- a^* is adjacent to b_i for $1 \leqslant i \leqslant N$.

(Thus, this is the same as in the graph implicitly used in the proof of 3.2, except for the extra vertex a^*.) Let r satisfy

$$krN = 1/k - x.$$

Thus $r > 0$. For each $v \in V(G)$, define $w(v)$ as follows:

- $w(v) = (k-1)r(N+1)/N$ for $v \in \{a_1, \dots, a_N\}$; $w(v) = 1/k - r$ for $v \in \{a'_1, \dots, a'_{k-1}\}$;
- $w(a^*) = 1/k - N(k-1)r$;
- $w(v) = (1-(k-1)s)/N$ for $v \in \{b_1, \dots, b_N\}$; $w(v) = s$ for $v \in \{b'_1, \dots, b'_{k-1}\}$; and
- $w(v) = (1-(k-1)y)/N$ for $v \in \{c_1, \dots, c_N\}$; $w(v) = y$ for $v \in \{c'_1, \dots, c'_{k-1}\}$.

Then (G, w) is a weighted graph. We claim it is (x, y)-biconstrained via (A, B, C), and $w(N_A^2(v)) < 1/k$ for each $v \in C$. To see this we must verify:

$$\begin{aligned} x &\leqslant p(1-(k-1)s)/N \\ y &\leqslant q(1-(k-1)y)/N \\ y &\leqslant q(1-(k-1)s)/N \\ x &\leqslant 1/k - r \\ x &\leqslant p(k-1)r(N+1)/N + 1/k - N(k-1)r, \text{ and} \\ 1/k &> 1/k - N(k-1)r + (p+q-1)(k-1)r(N+1)/N. \end{aligned}$$

The first and third hold with equality from the definitions of p, q, and the second follows since $y \leqslant s$. The fourth follows from the definition of r. For the fifth, on substituting for p and simplifying, we need to show that $r(k-1)(N - x(N+1)/(1-(k-1)s)) \leqslant 1/k - x$, and this follows from the definition of r. Finally, the sixth simplifies to $(p+q-1)(N+1)/N < N$, and this is true since $p+q \leqslant N$. Consequently, $\psi(x, y) < \frac{1}{k}$, by 2.1. This proves 3.3. ∎

We also need the next three results later in the paper. The first has three applications, in 7.7, 9.9 and 11.3:

3.4. *Let $x', y', z' \in (0,1)$, with $\psi(x', y') \leqslant z'$. If $x, y, z \in (0,1]$ satisfy $x \leqslant 1/(2-x')$, $y \leqslant y'/(1+y')$, $x + (1-x')y/y' \leqslant 1$, $z \geqslant 1/(2-z')$, and*

$$x \leqslant \frac{z - z' + x'(1-z)}{1-z'},$$

then $\psi(x, y) \leqslant z$.

Proof. Since $x' \leqslant z'$ (because $\psi(x', y') \leqslant z'$) it follows that

$$x \leqslant 1/(2-x') \leqslant 1/(2-z') \leqslant z.$$

Let G' be (x, y)-biconstrained via (A, B, C), such that $|N_A^2(u)| \leqslant z'|A|$ for all $u \in C$. Add three vertices a, b, c to the graph, and edges from a to every vertex in B, edges from b to every vertex in A, and an edge between b and c. Let this new graph be G. Assign weights as follows:

$$\begin{aligned}
w(a) &= p \\
w(v) &= (1-p)/|A| \text{ for each } v \in A \\
w(b) &= q \\
w(v) &= (1-q)/|B| \text{ for each } v \in B \\
w(c) &= y \\
w(v) &= (1-y)/|C| \text{ for each } v \in C.
\end{aligned}$$

We will choose p, q such that the weighted graph (G, w) is (x, y)-biconstrained via $(A \cup \{a\}, B \cup \{b\}, C \cup \{c\})$ and $w(N^2_{A\cup\{a\}}(u)) \leqslant z$ for all $u \in C \cup \{c\}$. The conditions are: $1-p \geqslant x$, $1-q \geqslant x$, $q \geqslant y$, $(1-q)x'+q \geqslant x$, $(1-p)x'+p \geqslant x$, $(1-y)y' \geqslant y$, $(1-q)y' \geqslant y$, $1-p \leqslant z$, and $(1-p)z'+p \leqslant z$. These are equivalent to the following:

$$\max\left(1-z, \frac{x-x'}{1-x'}\right) \leqslant p \leqslant \min\left(1-x, \frac{z-z'}{1-z'}\right)$$
$$\max\left(y, \frac{x-x'}{1-x'}\right) \leqslant q \leqslant \min\left(1-x, 1-\frac{y}{y'}\right)$$

Thus, it suffices to show that the lower bound on p is at most the upper bound on p, and the same for q. We obtain eight conditions, which simplify to those given in the theorem statement. This proves 3.4. ∎

The next result is applied in 7.8, 9.9 and 11.5:

3.5. *Let $x', y', z' \in (0,1)$, with $\psi(x', y') \leqslant z'$. If $x, y \in (0,1]$ satisfy $y \leqslant 1/(2-y')$, $x \leqslant x'/(1+x')$, $x \leqslant x'z$, $(1-y')x/x' + y \leqslant 1$, $z \geqslant 1/(2-z')$, and $x \leqslant (z-z')/(1-z')$, then $\psi(x, y) \leqslant z$.*

Proof. Let G' be (x, y)-biconstrained via (A, B, C), such that $|N_A^2(w)| \leqslant z'|A|$ for all $w \in C$. Add three vertices a, b, c to G', with an edge from a to b, edges from b to every vertex in C, and edges from c to every vertex in B. Let this new graph be G. Assign weights as follows:

$$\begin{aligned} w(a) &= p \\ w(v) &= (1-p)/|A| \text{ for each } v \in A \\ w(b) &= q \\ w(v) &= (1-q)/|B| \text{ for each } v \in B \\ w(c) &= 1-y \\ w(v) &= y/|C| \text{ for each } v \in C. \end{aligned}$$

The conditions that the weighted graph (G, w) is (x, y)-biconstrained via $(A \cup \{a\}, B \cup \{b|], C \cup \{c\})$ with $w(N^2_{A\cup\{a\}}(u)) \leqslant z$ for all $u \in C \cup \{c\}$ can be written as follows:

$$\max(x, 1-z) \leqslant p \leqslant \min\left(1 - \frac{x}{x'}, \frac{z-z'}{1-z'}\right)$$
$$\max\left(x, \frac{y-y'}{1-y'}\right) \leqslant q \leqslant \min\left(1-y, 1-\frac{x}{x'}\right).$$

We need to check that the lower bound for p is at most the upper bound for p, and the same for q. This gives eight conditions, which simplify (using that $1 - y' > x'$, since $\psi(x', y') < 1$) to those given in the theorem. This proves 3.5. ∎

The next is used to prove 10.5:

3.6. *Let $s, t \geqslant 1$ be integers with $s/t \leqslant 1/2$. Let $x, y \in (0, 1]$, satisfying $tx/s + y \leqslant 1$, $x + ty/s \leqslant 1$, and either $sy \leqslant x$ or $sx \leqslant y$. Furthermore, if either x or y is irrational, let strict inequality hold in all of these, that is, $tx/s + y < 1$, $x + ty/s < 1$, and either $sy < x$ or $sx < y$. Then $\psi(x, y) < s/t$.*

Proof. By increasing x or y if necessary, we may assume that x, y are both rational. Let $k+1 = \frac{t}{s}$. In terms of k, the hypotheses become $k \geqslant 1$, $(k+1)x + y \leqslant 1$, $x + (k+1)y \leqslant 1$, and either $sy \leqslant x$ or $sx \leqslant y$.

Suppose first that $sy \leqslant x$. Choose an integer $N \geqslant 1$ such that $p = xN/(1-kx)$ and $q = yN/(1-kx)$ are integers, and thus $p+q \leqslant (x+y)N/(x+y) = N$. Let G_1 be the graph with vertices $\{a_1, \ldots, a_N, a^*, b_1, \ldots, b_N, c_1, \ldots, c_N\}$ where (reading subscripts modulo N) each a_i is adjacent to $b_i, \ldots, b_{i+p-1}$, each b_i is adjacent to $c_i, \ldots, c_{i+q-1}$, and a^* is adjacent to all of the b_i.

Let $m = t - s$. Let G_2 be the graph with vertex set $\{a'_1, \ldots, a'_m, b'_1, \ldots, b'_m, c'_1, \ldots, c'_m\}$, where each a'_i is adjacent to $b'_i, \ldots, b'_{i+s-1}$ (reading subscripts modulo m), and each b'_i is adjacent to c'_i. Let G be the disjoint union of G_1 and G_2. Let $A = \{a_1, \ldots, a_N, a^*, a'_1, \ldots, a'_m\}$, and $B = \{b_1, \ldots, b_N, b'_1, \ldots, b'_m\}$ and define C similarly.

Let r satisfy $(k+1)rN = \frac{1}{k+1} - x$. Assign weights as follows:

$$\begin{aligned} w(a_i) &= kr(N+1)/N \\ w(a'_i) &= 1/b - r/s \\ w(a^*) &= 1/(k+1) - Nkr \\ w(b_i) &= (1-kx)/N \\ w(b'_i) &= x/s \\ w(c_i) &= (1-ksy)/N \\ w(c'_i) &= y. \end{aligned}$$

This defines a weighted graph (G, w), (x, y)-biconstrained via (A, B, C), such that

$$w(N_A^2(v)) < s/t = 1/(k+1)$$

for all $v \in C$, and so $\psi(x, y) < s/t$, as desired.

Now suppose $sy > x$, and consequently $sx \leqslant y$. Choose an integer $N \geqslant 1$ such that $p = xN/(1-ky)$, $q = yN/(1-ky)$ are integers, and thus $p+q \leqslant N$. Let G_1 be as before. Let $m = b-a$, and let G_2 be the graph with vertex set $\{a'_1, \ldots, a'_m, b'_1, \ldots, b'_m, c'_1, \ldots, c'_m\}$, where each a'_i is adjacent to b'_i, and each b'_i is adjacent to $c'_i, \ldots, c'_{i+a-1}$, reading subscripts modulo m (thus, this is the earlier graph G_2 flipped). Let G be the disjoint union of G_1 and G_2, and define A, B, C as before. Let $r > 0$ satisfy $(k+1)rN \leqslant 1/(k+1) - x$ and $r \leqslant 1/(k+1) - y$. Assign weights as follows:

$$\begin{aligned} w(a_i) &= kr(N+1)/N \\ w(a'_i) &= 1/t - r/s \\ w(a^*) &= 1/(k+1) - Nkr \\ w(b_i) &= (1-ky)/N \\ w(b'_i) &= y/s \\ w(c_i) &= (1-ky)/N \\ w(c'_i) &= y/s. \end{aligned}$$

Then (G, w) is a weighted graph, (x, y)-biconstrained via (A, B, C), and $w(N_A^2(v)) < s/t$ for all $v \in C$, showing that $\psi(x, y) < s/t$. This proves 3.6. ∎

The next result is used to prove 7.10, 7.11, 9.10, 9.11, 9.12, 9.13, 11.6 and 11.7.

3.7. *Let $x, y, z \in (0, 1]$, with $y \leqslant 1/2 < x$. If $\phi(2 - 1/x, y/(1-y)) \leqslant 2 - 1/z$, then $\phi(x, y) \leqslant z$.*

Proof. Let $x' = (2x-1)/x$, and $y' = y/(1-y)$, and let $z' = \phi(2 - 1/x, y/(1-y))$. Let G' be a graph that is (x', y')-constrained via (A, B, C), such that $|N_A^2(w)| \leqslant z'|A|$ for all $w \in C$. Add three vertices a, b, c to the graph, and edges from a to every vertex in B, edges from b to every vertex in A, and an edge between b and c. Let this new graph be G.

Assign weights $w(v)$ $(v \in V(G))$ as follows:

$$\begin{aligned} w(a) &= 1-z \\ w(v) &= z/|A| \text{ for each } v \in A \\ w(b) &= 1-x \\ w(v) &= x/|B| \text{ for each } v \in B \\ w(c) &= y \\ w(v) &= (1-y)/|C| \text{ for each } v \in C. \end{aligned}$$

Then the weighted graph (G, w) is (x, y)-constrained via $(A \cup \{a\}, B \cup \{b\}, C \cup \{c\})$, since $xx' + (1-x) = x$ and $(1-y)y' = y$. Moreover, $w(N^2_A(v)) \leqslant (1-z) + zz' \leqslant z$ for all $v \in C$; and $w(N^2_{A\cup\{a\}}(c)) = z$. Thus $\phi(x, y) \leqslant z$. This proves 3.7. ∎

4 Biconstrained graphs

In this section we prove some lower bounds on $\psi(x, y)$. On the diagonal $x = y$, $\psi(x, y)$ behaves perfectly; it turns out that for all x, $\psi(x, x) = 1/k$, where k is the largest integer with $1/k \geqslant x$. That follows from the next result via 4.2 and 4.3. The next result will also be used to prove 6.3 and in section 7, and contributes to Figure 7:

4.1. *For all integers $k \geqslant 1$, if $x, y \in (0, 1]$ with $x + ky > 1$ and $kx + \frac{x}{1-(k-1)y} \geqslant 1$, then $\psi(x, y) \geqslant 1/k$.*

Proof. By 1.8 we may assume that $x, y < 1/k$. Let G be (x, y)-biconstrained, via (A, B, C). We must show that $|N^2_A(v)| \geqslant |A|/k$ for some $v \in C$, so we suppose that this is false. Choose $K \subseteq C$ with $|K| \leqslant k$, and, subject to that, with $|K|$ maximum such that the sets $N(v)$ $(v \in K)$ are pairwise disjoint. Let $I \subseteq A$ be the union of the sets $N^2_A(v)$ $(v \in K)$, and let $J \subseteq B$ be the union of the sets $N(v)$ $(v \in K)$. It follows that

(1) *$|A \setminus I| > (1 - |K|/k)|A|$, and $|B \setminus J| \leqslant (1 - |K|y)|B|$.*

If $|K| = k$, then by (1), $|B \setminus J| \leqslant (1 - ky)|B| < x|B|$, and since every vertex in A has $x|B|$ neighbours in B, it follows that every vertex in A has a neighbour in J, that is, $I = A$, contrary to (1). Thus $|K| < k$.

Since each vertex in $A \setminus I$ has at least $x|B|$ neighbours in B, and they all belong to $B \setminus J$, some vertex $t \in B \setminus J$ has at least

$$x|B|\frac{|A \setminus I|}{|B \setminus J|} \geqslant x\frac{1 - |K|/k}{1 - |K|y}|A|$$

neighbours in $A \setminus I$ by (1). Since $|K| \leqslant k-1$ and $ky < 1$, and therefore $|K|y < 1$, it follows that

$$\frac{1-|K|/k}{1-|K|y} \geqslant \frac{1-(k-1)/k}{1-(k-1)y} = \frac{1}{k(1-(k-1)y)},$$

and so t has at least $\frac{x|A|}{k(1-(k-1)y)}$ neighbours in $A \setminus I$. Let $u \in C$ be adjacent to t. From the maximality of K, u has a neighbour $w \in N(v)$ for some $v \in K$. Since w has at least $x|A|$ neighbours in I, it follows that

$$|N_A^2(u)| \geqslant x|A| + \frac{x|A|}{k(1-(k-1)y)} \geqslant |A|/k,$$

a contradiction. This proves 4.1. ∎

We deduce:

4.2. *For all integers $k \geqslant 1$, if $x, y \in (0,1]$ with $x + ky > 1$ and $kx + y \geqslant 1$, then $\psi(x,y) \geqslant 1/k$.*

Proof. If $k = 1$ the result is easy (and follows from 5.2 below), so we assume that $k \geqslant 2$; and hence we may assume that $x, y < 1/k \leqslant 1/2$ by 1.8. By 4.1 we may assume (for a contradiction) that $kx + \frac{x}{1-(k-1)y} < 1$. Consequently, $kx + \frac{x}{1-(k-1)(1-kx)} < 1$. Let $t = 1 - kx$. Then $\frac{1-t}{1-(k-1)t} < kt$, and so $k(k-1)t^2 - (k+1)t + 1 < 0$. This is quadratic in t, with discriminant $(k+1)^2 - 4k(k-1)$, and the latter is negative if $k > 2$; so we may assume that $k = 2$. Then $2t^2 - 3t + 1 < 0$, so $(2t-1)(t-1) < 0$, that is, $1/2 < t < 1$. But $t = 1 - 2x$, so $1/2 < 1 - 2x < 1$, that is, $x < 1/4$. But $2x + y \geqslant 1$ and $y < 1/2$, a contradiction. This proves 4.2. ∎

4.2 implies the result stated earlier, that:

4.3. *For all $x \geqslant 0$, $\psi(x,x) = 1/k$, where k is the largest integer with $1/k \geqslant x$.*

Proof. Certainly $\psi(x,x) \leqslant 1/k$, since by 1.11,

$$\psi(x,x) \leqslant \frac{\lceil kx \rceil + \lceil kx \rceil - 1}{k} = 1/k.$$

Equality holds by 4.2. This proves 4.3. ∎

Next we need a lemma, used for 4.5:

4.4. *Let $k \geqslant 1$ be an integer, let $(k-1)/k^2 \leqslant y \leqslant 1$, and let (A,B,C) be a tripartition of a graph G, such that:*

- *every vertex in B has at least $y|C|$ neighbours in C; and*
- *$|N_A^2(v)| < |A|/k$ for each $v \in C$.*

Then there exist $v_1, \dots, v_k \in A$ such that $N(v_i) \cap N(v_j) = \emptyset$ for $1 \leqslant i < j \leqslant k$.

Proof. If some vertex v in A has degree zero, then we may take $v_1 = \cdots = v_k = v$. So we assume that every vertex in A has a neighbour in B. For each $v \in A$, let $c(v) = |N^2_C(v)|$, and let $A(v) \subseteq A$ be the set of vertices in A that have a neighbour in $N(v)$. Let $|A(v)| = a(v)$.

(1) *For each $v \in A$, $c(v) > kya(v)|C|/|A|$.*

If we choose $u \in N^2_C(v)$ independently at random, then since every vertex in $A(v)$ has at least $y|C|$ second neighbours in $N^2_C(v)$, the probability that a given vertex $w \in A(v)$ belongs to $N^2_A(u)$ is at least $y|C|/c(v)$, and so the expectation of $|N^2_A(u)|$ is at least $(y|C|/c(v))a(v)$. On the other hand, the expectation of $|N^2_A(u)|$ is less than $|A|/k$. This proves (1).

Let H be the graph with vertex set A, in which distinct u, v are adjacent if (in G) u, v have a common neighbour in B. Thus, every vertex v has degree $a(v) - 1$ in H. So $2|E(H)| = \sum_{v \in A}(a(v) - 1)$; but

$$(ky|C|/|A|) \sum_{v \in A} a(v) \leqslant \sum_{v \in A} c(v) = \sum_{v \in A} |N^2_C(v)| = \sum_{u \in C} |N^2_A(u)| < |A| \cdot |C|/k.$$

Consequently,

$$2|E(H)| < (|A| \cdot |C|/k)/(ky|C|/|A|) - |A| = |A|^2/(k^2 y) - |A| \leqslant |A|^2/(k-1) - |A|.$$

By Turán's theorem, H has a stable set of cardinality k. This proves 4.4. ∎

We deduce the next result, which is used to prove 6.3 and contributes to figures 5 and 7:

4.5. *Let $k \geqslant 1$ be an integer, and let $x, y \in (0, 1]$ where $y \geqslant (k-1)/k^2$ and $kx + y > 1$. Let G be (x, y)-constrained via (A, B, C), such that every vertex in C has at least $y|B|$ neighbours in B. Then $|N^2_A(v)| \geqslant |A|/k$ for some $v \in C$. Consequently, $\psi(x, y) \geqslant 1/k$.*

Proof. Suppose not; then there is a weighted graph (G', w), (x, y)-constrained via some tripartition (A', B', C'), such that

- for each $v \in C'$, $w(N(v)) \geqslant y|B'|$; and
- for each $v \in C'$, $w(N^2_{A'}(v)) < 1/k$.

Choose such a weighted graph (G', w) with $|V(G')|$ minimum, and let $z < 1/k$ such that $w(N^2_{A'}(v)) \leqslant z$ for each $v \in C'$. By 4.4, there exist $v_1, \dots, v_k \in A'$ such that $N(v_1), \dots, N(v_k)$ are pairwise disjoint. Consequently, $w(N(v_1) \cup \cdots \cup N(v_k)) \geqslant kx$; and since $w(N(u)) \geqslant y > 1 - kx$ for each $u \in C'$, it follows that $\bigcup_{v \in X} N^2_{C'}(v) = C'$ where $X = \{v_1, \dots, v_k\}$. But $|X| < z^{-1}$, contrary to 2.4 and the minimality of $|V(G')|$. This proves the first claim, and the second follows. This proves 4.5. ∎

5 The mono-constrained case

In this section we are mostly concerned with $\phi(x, y)$ when $x = y$. We know that ψ behaves well on the diagonal $x = y$, because of 4.3, so what about ϕ? More generally, what about an analogue of 4.1 or 4.2 with ψ replaced by ϕ?

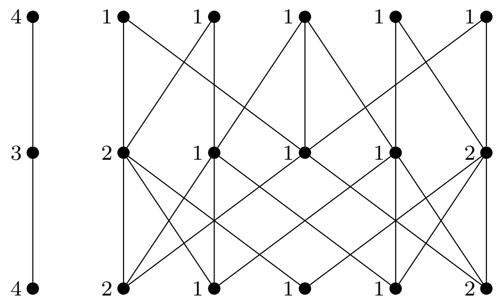

Figure 3: $\phi(3/10, 4/11) \leqslant 4/9$.

If we replace ψ by ϕ in 4.1, it becomes false, even with $k = 2$, because $\phi(3/10, 4/11) \leqslant 4/9$, as the graph of figure 3 shows (the sets A, B, C are the rows, and the numbers on the vertices are used as in figure 1). But as far as we know, 4.2 might hold with ψ replaced by ϕ. Let us state this as a conjecture:

5.1. Conjecture: *For all integers $k \geqslant 1$, if $x, y \in (0, 1]$ with $x + ky > 1$ and $kx + y \geqslant 1$, then $\phi(x, y) \geqslant 1/k$.*

On the other hand, we have not even been able to prove what is presumably the simplest nontrivial case of this, namely that $\phi(x, y) \geqslant 1/2$ for all x, y with $x, y > 1/3$. But we do have several results approaching 5.1. First, it is true with $k = 1$; we have the trivial:

5.2. *For $x, y \in (0, 1]$, if $x + y > 1$, or $x + y = 1$ and x is irrational, then $\phi(x, y) = 1$.*

Proof. Let G be (x, y)-constrained via (A, B, C). Then some vertex $v \in C$ has at least $y|B|$ neighbours in B, and strictly more if y is irrational; and so $N_A^2(v) = A$, as every vertex in A has at least $x|B|$ neighbours in B. This proves 5.2. ∎

This is used for 4.2, 5.6 and 7.9. It is tight, in that if $x + y = 1$ and x, y are rational, then $\phi(x, y) < 1$. We omit the proof, which is easy.

The conjecture 5.1 would imply that $\phi(x, x) \geqslant 1/2$ if $x > 1/3$. We have not been able to prove this, but we can show that $\phi(x, x) > 3/7$ if $x > 1/3$. That is implied by the following:

5.3. *Let $k \geqslant 2$ be an integer; then for $x, y \in (0, 1]$, if $y > 1/k$ then*

$$\phi(x, y) \geqslant \frac{x(2 - 3x)}{kx(1 - x) + x^2 - 3x + 1}.$$

Indeed if $k = 2$, then $\phi(x, y) \geqslant 2x - x^2$ (which is larger).

Proof. Let G be (x, y)-constrained via (A, B, C). If x is irrational then G is (x, y)-constrained via (A, B, C), for some rational $x' > x$; so we may assume that x is rational, by increasing x if necessary. Suppose that $k = 2$, and choose $v_1, v_2 \in B$ independently and uniformly at random. For each $u \in A$, the probability that u is adjacent to at least one of v_1, v_2 is at least $2x - x^2$, since u has at least $x|B|$ neighbours in B; and so we may choose v_1, v_2 such that at least $(2x - x^2)|A|$ vertices in A are adjacent to at least one of them. But v_1, v_2 have a common neighbour in C, since $y > 1/2$, and the claim follows.

Thus, we may assume that $k \geqslant 3$. By 1.8, $\phi(x, y) \geqslant x$, and so we may assume that

$$\frac{x(2-3x)}{kx(1-x)+x^2-3x+1} > x,$$

that is, $x < 1/(k-1)$. Consequently, $x \leqslant (k-2)/(k-1)$ since $k \geqslant 3$. Define

$$\begin{aligned} p &= \frac{x(1-x)}{kx(1-x)+x^2-3x+1}, \\ s &= \frac{x}{(k-2)(1-x)}, \text{ and} \\ m &= \frac{x(2-3x)}{kx(1-x)+x^2-3x+1}. \end{aligned}$$

These are all non-negative, and p is rational with denominator T say; and by replacing each vertex by T copies, we may assume that $p|A|$ is an integer. Since $x \leqslant (k-2)/(k-1)$ it follows that $s \leqslant 1$.

For $1 \leqslant i \leqslant k-1$, we define $v_i \in B$, and a subset P_i of $N_A(v_i)$ with $|P_i| = p|A|$, inductively, as follows. Let $Q = P_1 \cup \cdots \cup P_{i-1}$.

(1) *There exists $v_i \in B$ such that $sa + b \geqslant x(s|Q| + |A| - |Q|)$, where $a = |N_A(v_i) \cap Q|$ and $b = |N_A(v_i) \setminus Q|$.*

Suppose not; then summing over all $v \in B$, we deduce that

$$\sum_{v \in B} s|N_A(v) \cap Q| + \sum_{v \in B} |N_A(v) \setminus Q| < x(s|Q| + |A| - |Q|)|B|.$$

But the first sum is s times the number of edges between Q and B, and so at least $xs|B| \cdot |Q|$; and the second is similarly at least $x|B|(|A| - |Q|)$, a contradiction. This proves (1).

Let v_i be as in (1). Thus, $sa + b \geqslant x(s|Q| + |A| - |Q|) \geqslant x(1 - (1-s)(k-2)p)|A|$. In particular, since

$$a + b \geqslant sa + b \geqslant x(1 - (1-s)(k-2)p)|A| = p|A|,$$

there exists $P_i \subseteq N_A(v_i)$ of cardinality $p|A|$. Also, since $a \leqslant (k-2)p|A|$, and so

$$s(k-2)p|A| + b \geqslant sa + b \geqslant x(1 - (1-s)(k-2)p)|A|,$$

it follows that

$$b \geqslant x(1-(1-s)(k-2)p)|A| - s(k-2)p|A| = (m-p)|A|,$$

and so

$$|N_A(v_h) \cup N_A(v_i)| \geqslant m|A|,$$

for $1 \leqslant h < i$. This completes the inductive definition of $v_1, \dots, v_{k-1}$ and $P_1, \dots, P_{k-1}$.

Let $P = P_1 \cup \cdots \cup P_{k-1}$. Then $|P| \leqslant (k-1)p|A|$. Since every vertex in $A \setminus P$ has at least $x|B|$ neighbours in B, there exists $v_k \in B$ with at least $x(|A|-|P|) \geqslant x(1-(k-1)p)|A|$ neighbours in $A \setminus P$. Let P_k be its set of neighbours in $A \setminus P$. Then for all i with $1 \leqslant i \leqslant k-1$,

$$|P_i| + |P_k| \geqslant (x(1-(k-1)p)+p)|A| = m|A|.$$

Consequently, for all distinct $v, v' \in \{v_1, \dots, v_k\}$, $|N_A(v) \cup N_A(v')| \geqslant m|A|$. But since $y > 1/k$, some two of $v_1, \dots, v_k$ have a common neighbour $u \in C$, and so $|N^2_A(u)| \geqslant m$. This proves 5.3. ∎

We deduce from 5.3 a version of 4.3 for the mono-constrained case:

5.4. *For $y \in (0,1]$, if $y > 1/k$ where $k \geqslant 2$ is an integer, then $\phi(1/k, y) \geqslant \frac{2k-3}{2k^2-4k+1}$.*

Consequently, $\phi(1/k, y) \geqslant 1/k + 1/(2k^2) + \Omega(k^{-3})$.

5.4 tells us in particular that $\phi(x,x) \geqslant \frac{2k-3}{2k^2-4k+1} > 1/k$ when $x > 1/k$ (if $k \geqslant 2$ is an integer), and since $\phi(1/k, 1/k) = 1/k$, there is a discontinuity in $\phi(x,x)$ when $x = 1/k$, and the limit of $\phi(x,x)$ as $x \to 1/k$ from above is different from $\phi(1/k,1/k)$. What happens when $x \to 1/k$ from below? The next results investigate this. We will show that if x is sufficiently close to $1/k$ from below, then $\phi(x,x) = 1/k$.

5.5. *If $k > 0$ is an integer and $x \in (0,1]$ satisfies $(1-x)^k < x$, then $\phi(x,x) \geqslant 1/k$. In particular, if $x > 0.382$ then $\phi(x,x) \geqslant 1/2$, and if $x > 0.318$ then $\phi(x,x) > 1/3$.*

Proof. Let G be (x,x)-constrained via (A,B,C). If we choosing k vertices from C uniformly at random, the number of vertices in B nonadjacent to all of them is at most $(1-x)^k|B|$ in expectation; and so there exist $v_1, \dots, v_k \in C$ such that at most $(1-x)^k|B|$ vertices in B are nonadjacent to all of them. Since $(1-x)^k|B| < x|B|$, it follows that the sets $N^2_A(v_i)$ $(1 \leqslant i \leqslant k)$ have union A, and so one of them has cardinality at least $|A|/k$. This proves 5.5. ∎

The proof of 5.5 is very simple, but the result is not of any value. It is of no use when $k \geqslant 4$ because then $(1-x)^k < x$ implies $x > 1/k$; and we will prove in 6.6 and 8.3 that $\phi(x,x) \geqslant 1/2$ when $x > 0.352202$, and $\phi(x,x) \geqslant 1/3$ when $x \geqslant 0.28231$, which are stronger than 5.5 when $k = 2, 3$. Here is another approach to the same question, more successful for larger values of k (see figure 4):

5.6. *Let $k \geqslant 1$ be an integer, and let $x \geqslant 1/k - \varepsilon$ where $\varepsilon = 1/(13k^3)$. Then $\phi(x,x) \geqslant 1/k$.*

Proof. We may assume that $x = 1/k - \varepsilon$. By 5.2 we may assume that $k \geqslant 2$. We leave the reader to check that

- $1/(2k) - \varepsilon > 6k^2\varepsilon$;
- $x > 1/(k+1)$; and
- $(2kx - 1)/(2k - 1) > (k\varepsilon)/(x + k\varepsilon)$.

(These are inequalities we will need later.) Let G be (x, x)-constrained via (A, B, C), and suppose that $|N_A^2(v)| < |A|/k$ for each $v \in C$. Let P be the set of vertices in B that have at most $(1/k - 2k\varepsilon)|A|$ neighbours in A.

(1) $|P| \leqslant |B|/(2k)$.

Every vertex in B has fewer than $|A|/k$ neighbours in A, and so the number of edges between A and B is at most $|P|(1/k - 2k\varepsilon)|A| + (|B| - |P|)|A|/k$. On the other hand, the number of such edges is at least $(1/k - \varepsilon)|A| \cdot |B|$; and so

$$|P|(1/k - 2k\varepsilon)|A| + (|B| - |P|)|A|/k \geqslant (1/k - \varepsilon)|A| \cdot |B|,$$

which simplifies to $2k|P| \leqslant |B|$. This proves (1).

(2) *If $u, v \in B \setminus P$ have a common neighbour in C, then $|N_A(u) \setminus N_A(v)| \leqslant 2k\varepsilon|A|$.*

Since $u, v \in B \setminus P$ have a common neighbour in C, it follows that $|N_A(u) \cup N_A(v)| \leqslant |A|/k$. But $|N_A(u)| \geqslant (1/k - 2k\varepsilon)|A|$ since $u \in B \setminus P$, and so $|N_A(u) \setminus N_A(v)| \leqslant 2k\varepsilon|A|$. This proves (2).

(3) *There exist $v_1, \ldots, v_k \in B \setminus P$ such that for $1 \leqslant i < j \leqslant k$, there are at least $(1/(2k) - \varepsilon)|A|/k$ vertices in A that are adjacent to v_j and not to v_i.*

Choose $v_1, \ldots, v_k \in B \setminus P$ as follows. Choose $v_1 \in B \setminus P$ arbitrarily. Inductively, suppose we have defined $v_1, \ldots, v_i$ where $i < k$. Each has at most $|A|/k$ neighbours in A, and so the set of vertices in A adjacent to one of $v_1, \ldots, v_i$ has cardinality at most $(i/k)|A| \leqslant (1 - 1/k)|A|$. Let D be the set of vertices in A nonadjacent to each of $v_1, \ldots, v_i$; then $|D| \geqslant |A|/k$. Since, by (1), each vertex in D has at least $x|B| - |P| \geqslant (1/(2k) - \varepsilon)|B|$ neighbours in $B \setminus P$, there exists $v_{i+1} \in B \setminus P$ with at least $(1/(2k) - \varepsilon)|A|/k$ neighbours in D. This completes the inductive definition. We see that for $1 \leqslant i < j \leqslant k$, there are at least $(1/(2k) - \varepsilon)|A|/k$ vertices in A that are adjacent to v_j and not to v_i. This proves (3).

Let H be the bipartite graph $G[(B \setminus P) \cup C]$.

(4) *For $1 \leqslant i < j \leqslant k$, v_i and v_j belong to distinct components of H.*

From (2), the sets $N_C(v_1), \dots, N_C(v_k)$ are pairwise disjoint, because $(1/(2k) - \varepsilon)|A|/k > 2k\varepsilon|A|$. Suppose that there is a path of H joining some two of $v_1, \dots, v_k$, and take the shortest such path Q; between v_i and v_j say, where $j > i$. Let Q have m vertices in B, say $u_1, \dots, u_m$ in order where $u_1 = v_i$. We claim that $m \leqslant 4$. For suppose that $m \geqslant 5$. From the minimality of the length of Q, u_3 has no common neighbour in C with any of $v_1, \dots, v_k$, and so the sets $N_C(u_3), N_C(v_1), \dots, N_C(v_k)$ are pairwise disjoint, which is impossible since $x > 1/(k+1)$. Thus $m \leqslant 4$. By applying (2) to each pair of consecutive members of $V(Q) \cap B$, we deduce that

$$|N_A(v_j) \setminus N_A(v_i)| \leqslant (m-1)2k\varepsilon|A| \leqslant 6k\varepsilon|A|.$$

But $|N_A(v_j) \setminus N_A(v_i)| \geqslant (1/(2k) - \varepsilon)|A|/k$, and so $(1/(2k) - \varepsilon)|A|/k \leqslant 6k\varepsilon|A|$, a contradiction. This proves (4).

For $1 \leqslant i \leqslant k$, let H_i be the component of H containing v_i, and let $V(H_i) \cap B = B_i$ and $V(H_i) \cap C = C_i$. If there exists $v \in B \setminus P$ that does not belong to any of $B_1, \dots, B_k$, then the sets $N_C(v), N_C(v_1), \dots, N_C(v_k)$ are pairwise disjoint, which is impossible since they all have cardinality at least $x|C|$, and $(k+1)x > 1$. Consequently, the sets $B_1, \dots, B_k$ and P form a partition of B.

(5) *For $1 \leqslant i \leqslant k$ there exists $u_i \in C_i$ adjacent to at least $\frac{(1-k\varepsilon)}{1+k(k-1)\varepsilon}|B_i|$ vertices in B_i.*

For $1 \leqslant i \leqslant k$, since v_i has at least $x|C|$ neighbours in C, it follows that $|C_i| \geqslant x|C|$. Let $1 \leqslant i \leqslant k$. Since $C_1, \dots, C_k$ are pairwise disjoint, and the union of the sets C_j $(j \in \{1, \dots, k\} \setminus \{i\})$ has cardinality at least $(k-1)x|C|$, it follows that

$$|C_i| \leqslant |C| - (k-1)x|C| = x|C| + k\varepsilon|C|.$$

There are at least $x|B_i| \cdot |C|$ edges between B_i and C_i, and so some vertex in C_i has at least

$$x(|C|/|C_i|)|B_i| \geqslant x(|C|/(x|C| + k\varepsilon|C|))|B_i| = (x/(x+k\varepsilon))|B_i|$$

neighbours in B_i. By substituting $x = 1/k - \varepsilon$, this proves (5).

For $1 \leqslant i \leqslant k$, let $A_i = N_A^2(u_i)$. Since $|A_i| < |A|/k$ for $1 \leqslant i \leqslant k$, there exists $v \in A$ that is in none of $A_1, \dots, A_k$. Now v has at least $x|B|$ neighbours in B, and they all belong to $B_1 \cup \dots \cup B_k$ except for at most $|P|$ of them. Consequently, there exists $i \in \{1, \dots, k\}$ such that v has at least $(x|B| - |P|)|B_i|/|B \setminus P|$ neighbours in B_i. Since $v \notin A_i$, it follows that

$$(x|B| - |P|)|B_i|/|B \setminus P| + (x/(x+k\varepsilon))|B_i| \leqslant |B_i|.$$

Since $x|B| \leqslant |B|$ and $|P| \leqslant |B|/(2k)$ by (1), it follows that

$$(x|B| - |P|)|B_i|/|B \setminus P| \geqslant (x - 1/(2k))|B_i|/(1 - 1/(2k)) = (2kx-1)|B_i|/(2k-1),$$

and so $(2kx-1)/(2k-1) \leqslant k\varepsilon/(x+k\varepsilon)$, a contradiction. This proves 5.6. ∎

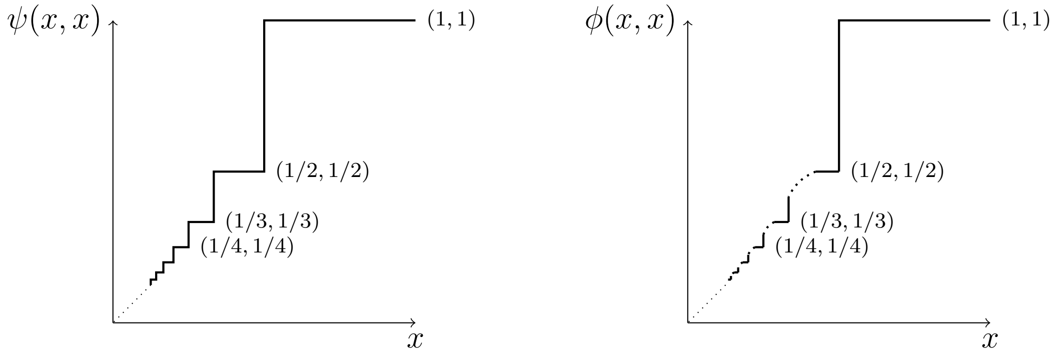

Figure 4: Graphs of $\psi(x, x)$ and $\phi(x, x)$

For comparison, in figure 4 we give graphs of the function $\psi(x, x)$ (which we know completely, because of 4.3), and the function $\phi(x, x)$ (which we only know partially, from 5.6, 5.4 and 1.10.)

The next result is a useful general lower bound on $\phi(x, y)$. It will be used to prove 6.6, 9.7, 10.3 and 11.3, and contributes to figure 6:

5.7. *For $x, y, z \in (0, 1]$, if $y > 1/2$ and $4x^2y(1-z) \geqslant (z-x)^2$ then $\phi(x, y) \geqslant z$. If in addition $4x^2y(1-z) > (z-x)^2$ then $\phi(x, y) > z$.*

Proof. Let G be (x, y)-constrained via (A, B, C), and suppose that $|N_A^2(w)| < z|A|$ for each $w \in C$. There are at least $xy|A| \cdot |B| \cdot |C|$ two-edge paths between A and C, and so there is a vertex $w \in C$ that is an end of at least $xy|A| \cdot |B|$ such paths. Let w be an end of exactly $xq|A| \cdot |B|$ such paths; thus $y \leqslant q$. Let $B_1 = N_B(w)$, and let $t = |B_1|/|B|$. Since $|N_A^2(w)| \leqslant z|A|$, there exists $A_1 \subseteq A$ including $N_A^2(w)$ with $|A_1| = z|A|$ (we may assume the latter is an integer.) For each $u \in A_1$ let u have exactly $d(u)|B|$ neighbours in B_1, and therefore at least $(x - d(u))|B|$ neighbours in $B \setminus B_1$. It follows that

$$\sum_{u \in A_1} d(u) = qx|A|.$$

Let $v_1 \in B_1$ and $v_2 \in B \setminus B_1$, and let $A(v_1, v_2) = N_A(v_1) \cup N_A(v_2)$. For every such choice of v_1, v_2, since $y > 1/2$, there is a vertex w' in C adjacent to both v_1, v_2, and since $|N_A^2(w')| < z|A|$, it follows that $|A(v_1, v_2)| < z|A|$. Let us choose $v_1 \in B_1$ and $v_2 \in B \setminus B_1$, uniformly at random. It follows that the expected value of $|A(v_1, v_2)|$ is less than $z|A|$. The expected value of $|A(v_1, v_2) \cap A_1|$ is at least

$$\sum_{u \in A_1} \left(\frac{d(u)}{t} + \frac{x - d(u)}{1-t} - \frac{d(u)(x - d(u))}{t(1-t)} \right)$$

and the expected value of $|A(v_1, v_2) \setminus A_1|$ is at least

$$\sum_{u \in A \setminus A_1} \frac{x}{1-t}.$$

Consequently, the sum of these two is less than $z|A|$, and so

$$\sum_{u\in A_1}\left(\frac{d(u)}{t}+\frac{x-d(u)}{1-t}-\frac{d(u)(x-d(u))}{t(1-t)}\right)+\sum_{u\in A\setminus A_1}\frac{x}{1-t}<z|A|.$$

Since $\sum_{u\in A_1} d(v)=xq|A|$, this simplifies to

$$xq|A|(1-2t-x)+\sum_{u\in A_1} d(u)^2+xt|A|<zt(1-t)|A|.$$

Now since $\sum_{u\in A_1} d(v)=xq|A|$ and $|A_1|=z|A|$, it follows from the Cauchy-Schwarz inequality that $\sum_{u\in A_1} d(u)^2\geqslant x^2q^2|A|/z$. Consequently,

$$xq|A|(1-2t-x)+x^2q^2|A|/z+xt|A|<zt(1-t)|A|.$$

This can be rewritten as:

$$(zt-xq+x/2-z/2)^2+x^2q(1-z)-(z-x)^2/4<0.$$

Since the first term above is a square, it is nonnegative, and so, since $q\geqslant y$, it follows that

$$x^2y(1-z)-(z-x)^2/4<0,$$

contrary to the hypothesis. This proves the first statement of the theorem, and the second is immediate by slightly increasing z. This proves 5.7. ∎

6 When is $\phi(x,y)$ or $\psi(x,y)\geqslant 1/2$?

Another way to approach the problem of understanding ϕ and ψ is to ask, given some value z, for which $x,y\in(0,1]$ is $\phi(x,y)\geqslant z$? Or we could ask the same question for ψ, or ask when $\phi(x,y)>z$. For instance:

6.1. *If $k\geqslant 1$ is an integer, then for $x,y\in(0,1]$, $\phi(x,y)>1/k$ if and only if $\max(x,y)>1/k$.*

This follows trivially from 1.9 and 1.8. And the same holds with ϕ replaced by ψ. But deciding when $\psi(x,y)\geqslant 1/k$ or $\phi(x,y)\geqslant 1/k$ seems to be much less obvious. In this section we discuss when $\psi(x,y)$ or $\phi(x,y)$ is at least 1/2.

For $x,y\in(0,1]$, we say (temporarily) that (x,y) is *good* if $\psi(x,y)\geqslant 1/2$, and *bad* otherwise. The "map" of good and bad points is shown in the left half of figure 5. The solid black curve borders the known bad points, and the dotted curve borders the good points; between them is undecided. The borders are complicated, and we have indicated in the figure which theorem is responsible for each stretch of border.

Let us explain some of the details. First, if $\max(x,y)\geqslant 1/2$, then (x,y) is good; and all pairs (x,y) with $x+2y,2x+y\leqslant 1$ are bad, by 3.3. We searched by computer to find

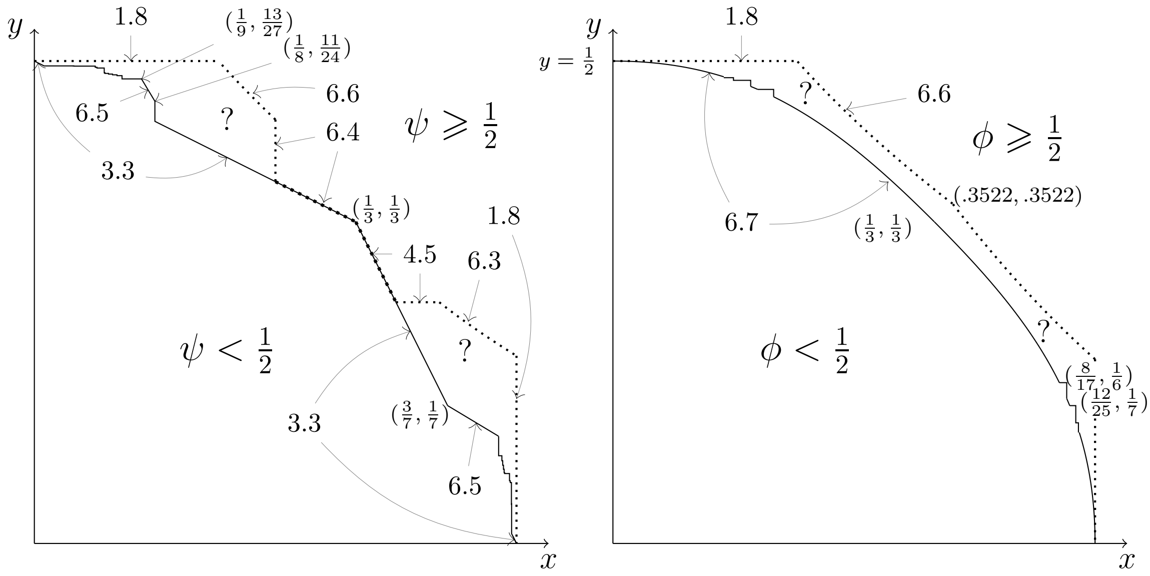

Figure 5: In the left-hand figure, $\psi(x, y) < 1/2$ for pairs (x, y) below the solid line, and $\psi(x, y) \geqslant 1/2$ above the dotted one; between we don't know. The right-hand figure does the same for ϕ.

other examples of bad pairs (x, y), and found about 12 maximal such pairs of rationals, with numerator and denominator at most 100. In fact we only searched for pairs (x, y) where the corresponding (x, y)-biconstrained graph is similar to the graph obtained from figure 1, that is, it is obtained by "blowing up" the vertices of another graph in which the graph between two of the three parts is a matching. All these examples not only show that $\psi(x, y) < 1/2$, but also that $\psi(y, x) < 1/2$, and $\xi(x, y) < 1/2$. In particular, for every bad pair (x, y) we found by computer search, (y, x) is another. This is just an artifact of our method of search, and is not evidence that the set of all bad pairs is closed under switching x and y (though it might be; it is for ϕ, by 2.3). For each bad pair (x, y) the computer found, all pairs (x', y') with $x' \leqslant x$ and $y' \leqslant y$ are also bad, and that gave us a step function bordering the area of the known bad points. We improved on this; we were able to smooth out some of the steps of the step function, by means of 3.3 and 6.5, so the step function the computer found now only survives towards the ends of the solid black curve in the figure. (These "fills" are not invariant under switching x and y.) We give the coordinates of some bad pairs that we find particularly interesting. The apparent asymmetry between x and y in the left half of the figure is just asymmetry among what we have been able to prove; we have no proof of asymmetry. The right half of figure 5 does the same for ϕ. Here there is symmetry exchanging x and y, by 2.3, and so we only "explain" half of the border.

A graph has *radius* r if there is a vertex u such that every vertex has distance at most r from u, and for all $r' < r$ there is no u such that every vertex has distance at most r' from u. We will need the following theorem of Erdős, Saks and Sós [2]:

6.2. *Let G be a connected graph with radius at least r, where $r \geqslant 1$ is an integer. Then G has an induced path with $2r-1$ vertices, and consequently has a stable set of cardinality at least r.*

When we have more than one graph defined using the same vertices, we speak of "H-distance" to mean distance in the graph H, and so on. The next result will be used to contribute to figure 5 when $.4199 < x < .5$ and $.1945 < y < .25$ (see figure 5 for the relevant section of the border).

6.3. *Let $x, y \in (0,1]$, such that*

$$x^2(1+3y) + x(4y^2 - y - 2) + 1 - 2y + 2y^3 < 0.$$

Then $\psi(x,y) \geqslant 1/2$.

Proof. Let G be (x,y)-biconstrained via (A,B,C), and suppose that $|N_A^2(v)| < |A|/2$ for each $v \in C$. Then 1.8 implies that $x, y < 1/2$. Suppose that $y > 1/4$. The given inequality implies that $56x^2 - 64x + 17 < 0$, and so $x > .41$. Since $2x + y > 1$, 4.5 implies that $y \leqslant 1/4$, a contradiction. Thus $y \leqslant 1/4$. We leave the reader to verify that, when $y \leqslant 1/4$, the following are consequences of the given inequality:

- $\frac{x}{3-3y} > 1 - 2x$; and in particular, $\frac{x}{1-y} > 1 - 2x$, so from 4.1 with $k = 2$ it follows that $x + 2y \leqslant 1$;
- $\frac{y}{3-6y} > 1 - 2x$; and so $\frac{y}{2-2y} > 1 - 2x$, since $y \leqslant 1/4$; and
- $x + 3y > 1$.

(We found the easiest way to check these is to have a computer plot the various curves.) Let H be the bipartite graph $G[B \cup C]$.

(1) *If $v, v' \in C$ have H-distance at most $2t$ where $t > 0$ is an integer, then*

$$|N_A^2(v') \setminus N_A^2(v)| < t(1/2 - x)|A|.$$

Take a path P of H joining v and v', of length at most $2t$. Let the vertices of P in C be

$$v = v_0, \dots, v_t = v',$$

in order. For $1 \leqslant i \leqslant t$ let $u_i \in B$ be adjacent to v_{i-1} and v_i. Then for $1 \leqslant i \leqslant t$, $N_A^2(v_{i-1}) \cap N_A^2(v_i)$ includes $N_A(u_i)$ and hence has cardinality at least $x|A|$; and since $|N_A^2(v_i)| < |A|/2$, it follows that $|N_A^2(v_i) \setminus N_A^2(v_{i-1})| < (1/2 - x)|A|$. But the union of the t sets $N_A^2(v_i) \setminus N_A^2(v_{i-1})$ includes $N_A^2(v') \setminus N_A^2(v)$, and so the latter has cardinality less than $t(1/2 - x)|A|$. This proves (1).

(2) *There do not exist $v_1, \dots, v_4 \in C$, pairwise with no common neighbour in B.*

If such $v_1, \dots, v_4$ exist, then every three of $N(v_1), \dots, N(v_4)$ have union of cardinality at least $3y$; and since $3y > 1 - x$, every vertex in A has a neighbour in at least two of $N(v_1), \dots, N(v_4)$. Consequently, every vertex in A belongs to at least two of $N_A^2(v_1), \dots, N_A^2(v_4)$, and so one of $N_A^2(v_1), \dots, N_A^2(v_4)$ has cardinality at least $|A|/2$, a contradiction. This proves (2).

(3) *H has at least two components.*

Suppose not, and let H' be the graph with vertex set C in which v, v' are adjacent if they have a common neighbour in H. By (2), it follows that H' has no stable set of cardinality four, and so has radius at most three by 6.2. Choose $v \in C$ such that every vertex in C has H'-distance at most three from v. Let $B_1 = N_B(v)$ and $A_1 = N_A^2(v)$. Every vertex in $A \setminus A_1$ has at least $x|B|$ neighbours in $B \setminus B_1$, and so some vertex $u \in B \setminus B_1$ has at least $x(|B|/|B \setminus B_1|)|A \setminus A_1|$ neighbours in $A \setminus A_1$. Let A_2 be the set of neighbours of u in $A \setminus A_1$. Since $|B \setminus B_1| \leqslant (1 - y)|B|$ and $|A \setminus A_1| > |A|/2$, and $\frac{x}{3-3y} > 1 - 2x$, it follows that

$$|A_2| \geqslant (x/(1 - y))|A|/2 \geqslant 3(1/2 - x)|A|.$$

Let $v' \in C$ be adjacent to u. Since the H'-distance from v to v' is at most three, the H-distance from v to v' is at most six. By (1), $|N_A^2(v') \setminus N_A^2(v)| < 3(1/2 - x)|A|$, a contradiction. This proves (3).

(4) *If H' is a component of H then $|V(H') \cap B| \leqslant (1 - x)|B|$.*

Suppose that $|V(H') \cap B| > (1 - x)|B|$; then every vertex in A has a neighbour in $V(H')$. By (2), and since H has at least two components, there do not exist three vertices in $C \cap V(H')$ pairwise with no common neighbour, and so by 6.2, it follows that there is a vertex $v \in C \cap V(H')$ with H'-distance at most four from every vertex in $C \cap V(H')$. Let $A' = N_A^2(v)$; then $|A'| < |A|/2$. Since every vertex in $A \setminus A'$ has at least $y|C|$ second neighbours in $C \cap V(H')$, and $|C \cap V(H')| \leqslant (1-y)|C|$, some vertex $v' \in C \cap V(H')$ has at least $(y/(1-y))|A \setminus A'|$ second neighbours in $A \setminus A'$. By (1), $(y/(1-y))|A \setminus A'| < 2(1/2-x)|A|$. But $|A'| < |A|/2$, so $y/(4(1 - y)) \leqslant 1/2 - x$, a contradiction. This proves (4).

(5) *Some component H' of H satisfies $(1 - x)|B| \geqslant |V(H') \cap B| \geqslant x|B|$.*

By (2) and (3), H has either two or three components. If H has only two components, then they both satisfy (5), by (4); so we assume there are three. Let the components of H be H_1, H_2, H_3, and for $1 \leqslant i \leqslant 3$, let $V(H_i) \cap B = B_i$ and $V(H_i) \cap C = C_i$; and let $|B_i|/|B| = b_i$ and $|C_i|/|C| = c_i$. Suppose that $b_1, b_2, b_3 < x$. Consequently, every vertex in A has neighbours in at least two of B_1, B_2, B_3. For $1 \leqslant i \leqslant 3$, let A_i be the set of vertices in A with a neighbour in B_i. Thus, every vertex in A belongs to at least two of A_1, A_2, A_3, so from the symmetry we may assume that $|A_1| \geqslant 2|A|/3$. By (2), every two vertices in C_1 have a common neighbour in B. Choose $v \in C_1$, and let $A' = N_A^2(v)$; then $|A'| \leqslant |A|/2$. Since every vertex in A_1 has at least $y|C|$ second neighbours in C_1, some vertex v' in C_1 has at least $(y/c_1)|A_1 \setminus A'|$ second neighbours in $A_1 \setminus A'$. By (1), $(y/c_1)|A_1 \setminus A'| < (1/2 - x)|A|$. But $|A'| < |A|/2$, so $|A_1 \setminus A'| \geqslant |A|/6$; and $c_1 \leqslant 1 - 2y$, so $y/(6(1-2y)) \leqslant 1/2 - x$, a contradiction. This proves (5).

(6) *Every vertex in C has at most $(1-x-y)|B|$ neighbours in B. Consequently,*

$$\frac{|V(H') \cap B|}{|B|} \leqslant \frac{1-x-y}{y} \frac{|V(H') \cap C|}{|C|}$$

for each component H' of H.

Suppose that $v \in C$ has more than $(1-x-y)|B|$ neighbours in B. Choose $v' \in C$ in a different component of H; so v, v' have no common neighbour in B. Consequently,

$$|N(v) \cup N(v')| > ((1-x-y)+y))|B|,$$

and so every vertex in A has a neighbour in $N(v) \cup N(v')$. But then one of $|N_A^2(v)|, |N_A^2(v')|$ is at least $|A|/2$, a contradiction. This proves the first assertion. Let H' be a component of H. Then H' has at least $y|C| \cdot |V(H') \cap B|$ edges, and at most $(1-x-y)|B| \cdot |V(H') \cap C|$ edges, so the second claim follows. This proves (6).

Let H' be as in (5), and take the union of the other (one or two) components of H. We obtain nonnull subgraphs H_1, H_2 of H, pairwise vertex-disjoint and with union H, such that $|V(H_i) \cap B| \geqslant x|B|$ for $i = 1, 2$. For $i = 1, 2$, let $V(H_i) \cap B = B_i$ and $V(H_i) \cap C = C_i$; and let $|B_i|/|B| = b_i$ and $|C_i|/|C| = c_i$. Thus $b_1, b_2 \geqslant x$. From (6), $b_i \leqslant (1-x-y)c_i/y$ for $i = 1, 2$; and $c_1, c_2 \geqslant y$, since every vertex in B_i has at least $y|C|$ neighbours in C_i. Also $b_1 + b_2 = c_1 + c_2 = 1$.

For $i = 1, 2$ let A_i be the set of vertices $v \in A$ that have more than $(b_i - y)|B|$ neighbours in B_i. Let $A_0 = A \setminus (A_1 \cup A_2)$. Hence, if $u \in A_0$, then since u has at least $x|B|$ neighbours in B, u has at least $(x+y-b_2)|B|$ neighbours in B_1, and at least $(x+y-b_1)|B|$ neighbours in B_2.

Since A_1, A_2 and A_0 have union A, we may assume that $|A_1| + |A_0|/2 \geqslant |A|/2$. Now $A_1 \subseteq N_A^2(v)$ for each $v \in C_1$, since if $u \in A_1$, then u has more than $(b_1 - y)|B|$ neighbours in B_1, and v has at least $y|B|$ neighbours in B. Consequently, $|N_A^2(v) \cap A_0| < |A_0|/2$ for each $v \in C_1$.

Let us choose $v \in C_1$ uniformly at random; then the expected number of second neighbours of v in A_0 is less than $|A_0|/2$, and so for some vertex $u \in A_0$, the probability that $u \in N_A^2(v)$ is less than $1/2$. Let D be the set of neighbours of u in B_1. Then $|D| \geqslant (x+y-b_2)|B|$, and the probability that v has a neighbour in D is less than $1/2$. Thus, more than $|C_1|/2$ vertices in C_1 have no neighbour in D. On the other hand, the expectation of the number of neighbours of v in D is at least $|D|y/c_1$; and so there exists $v \in C_1$ with more than $2|D|y/c_1$ neighbours in D. Also there exists $v' \in C_1$ with no neighbours in D. It follows that

$$|N_B(v) \cup N_B(v')| \geqslant y|B| + 2|D|y/c_1 > (y + 2(x+y-b_2)y/c_1)|B|.$$

Some vertex in A_0 is not a second neighbour of either of v, v', and so

$$|N_B(v) \cup N_B(v')| < (b_1 - (x + y - b_2))|B|.$$

Consequently, $y+2(x+y+b_1-1)y/c_1 \leqslant 1-x-y$. Now $c_1 \leqslant (1-x-2y+yb_1)/(1-x-y)$ since

$$1 - b_1 = b_2 \leqslant (1 - x - y)c_2/y = (1 - x - y)(1 - c_1)/y.$$

So

$$2y(x + y + b_1 - 1)(1 - x - y)/(1 - x - 2y + yb_1) \leqslant 1 - x - 2y,$$

that is,

$$b_1y(1 - x) \leqslant x^2(1 + 2y) + x(-2 + 4y^2) + 1 - 2y + 2y^3.$$

But $b_1 \geqslant x$, contrary to the hypothesis. This proves 6.3. ∎

The next result contributes to figure 5:

6.4. *Let $x, y \in (0, 1]$, such that $x + 2y > 1$, $x \geqslant 1/4$ and $y \geqslant 1/3$. Then $\psi(x, y) \geqslant 1/2$.*

Proof. The only lower bound constraints on y are $y \geqslant 1/2-x/2$ and $y \geqslant 1/3$, and these are both satisfied if $y = 0.38$ since $x \geqslant 1/4$. Hence, we may assume that $y \leqslant 0.38$, by replacing y by $\min(y, 0.38)$. Consequently, $y^2 - 3y + 1 > 0$, and so

$$(1 - y)^3 < 1 - 2y < x.$$

Let G be (x, y)-biconstrained via (A, B, C), and suppose that $|N^2_A(w)| < |A|/2$ for each $w \in C$. Choose $w_1, w_2, w_3 \in C$ uniformly at random. The expected number of vertices in B nonadjacent to all of w_1, w_2, w_3 is at most $(1 - y)^3|B| < x|B|$; so we may choose w_1, w_2, w_3 such that fewer than $x|B|$ vertices in B are nonadjacent to all of w_1, w_2, w_3. For $i = 1, 2, 3$ let $A_i = N^2_A(w_i)$. Thus, $A_1 \cup A_2 \cup A_3 = A$. In particular, one of A_2, A_3, say A_2, includes at least half of $A \setminus A_1$; and since $|A_1| < |A|/2$, it follows that $|A_2 \setminus A_1| > |A|/4$. Since $|A_2| < |A|/2$, it follows that $|A_1 \cap A_2| < |A|/4 < x|A|$; and so $N_B(w_1), N_B(w_2)$ are disjoint (because any common neighbour would have at least $x|A|$ neighbours in A, all belonging to $A_1 \cap A_2$). Hence,

$$|N_B(w_1) \cup N_B(w_2)| \geqslant 2y|B| > (1 - x)|B|,$$

and so $A_1 \cup A_2 = A$, contradicting that $|A_1|, |A_2| < |A|/2$. This proves 6.4. ∎

The next result also contributes to figure 5:

6.5. *Let $x, y \in (0, 1]$, such that $x \leqslant 13/27$ and $y \leqslant 1/7$ and $3x + 5y \leqslant 2$. Then $\psi(x, y) \leqslant 13/27$. If in addition $y \leqslant 1/8$, then $\psi(y, x) \leqslant 1/2$.*

Proof. We claim that, for both statements of the theorem, we may assume that $3x+5y = 2$. By increasing x, we may assume that either $x = 13/27$ or $3x+5y = 2$; and if $x = 13/27$ then $y \leqslant 1/9$, since $3x + 5y \leqslant 2$, and by increasing y we may assume that $3x + 5y = 2$.

This proves our claim. Since $3x + 5y = 2$ and $x \leqslant 13/27$, it follows that $y \geqslant 1/9$; and since $y \leqslant 1/7$ it follows that $x \geqslant 3/7$.

We return to the graph of figure 1. Let A, B, C be the three rows of vertices, in order where A is the top row. We need to adjust the vertex weights. Define $p = 1/2 - x/2 - y$ and $r = (x - y)/2$. With the vertices in the same order as the figure, take vertex weights as follows:

$$\begin{array}{ccc} 5/27, 5/27, & 1/9, 1/9, 1/9, & 4/27, 4/27 \\ p, \quad p, & y, \; y, \; y, & r, \quad r \\ 1/7, \; 1/7, & 1/7, 1/7, 1/7, & 1/7, \; 1/7 \end{array}$$

One can check (it takes some time and we omit the details) that this defines an (x, y)-biconstrained weighted graph showing that $\psi(x, y) \leqslant 13/27$. For the second statement, take the same graph and same vertex weighting, except replace the third row (of all one-sevenths) in the table above, by

$$p', \quad p', \quad y,\ y,\ y, \quad r', \quad r'$$

where $p' = 1/2 - 3y$, and $r' = 3y/2$. This weighted graph is (y, x)-biconstrained via (C, B, A), and shows that $\psi(y, x) \leqslant 1/2$. (Again, we leave the reader to check that this works.) This proves 6.5. ∎

Now the mono-constrained case: for which pairs (x, y) is $\phi(x, y) \geqslant 1/2$? Now we have symmetry between x and y, and we found some examples of pairs (x, y) with $\phi(x, y) < 1/2$ on a computer searching randomly. (Conjecture 5.1 says that all points above both the lines $x + 2y = 1$ and $2x + y = 1$ should be good, and indeed, all the maximal examples the computer found lie in the wedges between the lines.)

The next result strengthens 5.5 when $k = 2$, and contributes to figure 5:

6.6. *Let $x, y \in (0, 1]$, such that $2x^2y \geqslant (1 - x - y)^2$. Then $\phi(x, y) \geqslant 1/2$.*

Proof. Suppose that $\phi(x, y) = 1/2 - \varepsilon$ where $\varepsilon > 0$. Then by 2.3 and 2.5 we have $\phi(x, 1/2 + \varepsilon) \leqslant 1 - y$. Let $y' = 1/2 + \varepsilon$ and $z = 1 - y$. Since $y' > 1/2$ and $\phi(x, y') \leqslant z$, the second statement of 5.7 implies that $4x^2y'(1 - z) \leqslant (z - x)^2$, and so $2x^2y < (1 - x - y)^2$, a contradiction. This proves 6.6. ∎

In particular, 6.6 implies that $\phi(x, x) \geqslant 1/2$ if $x \geqslant 0.352202$, which is stronger than 5.5 when $k = 2$.

The next result is used to prove 7.10, 7.11 and 8.4, and contributes to figure 5:

6.7. *Let $x, y \in (0, 1]$ with $x < 1/3$ and $y < (1 - x)^2/(2 - 4x + 6x^2)$; then $\phi(x, y) < 1/2$.*

Proof. Since $(1 - x)^2/(2 - 4x + 6x^2) < 1/2$ for all $x > 0$, it follows that $y < 1/2$. Since $x < 1/3$ it follows that $(1 - x)^2/(2 - 4x + 6x^2) > 1/3$, and so by increasing y, we may assume that $y > 1/3$. Also, by increasing y slightly if necessary, we may assume that $s = (1/y - 2)^{1/2}$ is rational. Thus, $0 < s < 1 \leqslant 1/y - 1$ and $1 + 2/s \leqslant 1/x$, since

$1/3 < y < 1/2$ and $y \leqslant (1-x)^2/(2-4x+6x^2)$. Choose an integer $N \geqslant 2$ such that sN is an integer.

Choose a graph G_1 that is $(s, 1-s)$-constrained via a tripartition (A_1, B_1, C_1), such that $|A_1| = |B_1| = |C_1| = N$ and $N^2_{A_1}(v) \neq A_1$ for each $v \in C_1$. (It is easy to see that such a graph exists, for instance, one of the graphs used in 1.9.) Let G_2 be isomorphic to G_1, and let (A_2, B_2, C_3) be the corresponding tripartition. Take the disjoint union of G_1 and G_2, and add edges to make every vertex in B_1 adjacent to every vertex in C_2. Add three more vertices a, b, c, where a is adjacent to b, b is adjacent to every vertex in C_1, and c is adjacent to every vertex in B_2, forming G. We define a weighting w of G as follows. Let $p = 1/(2N)$ and $q = 1/2 - 1/(4N^2)$. Define w by:

$$\begin{aligned}
w(a) &= 1-p-q \\
w(v) &= p/N \text{ for each } v \in A_1 \\
w(v) &= q/N \text{ for each } v \in A_2 \\
w(b) &= 1-2x/s \\
w(v) &= x/(Ns) \text{ for each } v \in B_1 \cup B_2 \\
w(c) &= 1-(1+s)y \\
w(v) &= y/N \text{ for each } v \in C_1 \\
w(v) &= sy/N \text{ for each } v \in C_2
\end{aligned}$$

Define $A = A_1 \cup A_2 \cup \{a\}$ and define B, C similarly. Then the weighted graph (G, w) is (x, y)-constrained via (A, B, C), and proves that $\phi(x, y) < 1/2$. (To see the latter, observe that, for instance, if $v \in C_1$ then

$$w(N^2_A(v)) \leqslant 1-p-q+p(1-1/N) = 1-q-p/N = 1-(1/2-1/(4N^2))-1/(2N^2) < 1/2,$$

from the choice of G_1). This proves 6.7. ∎

7 The 2/3 level

When is $\phi(x, y) \geqslant 2/3$; or the same question for ψ? In this section we say what we know about these. The results are shown in figure 6.

The next two results are used for 7.3 and in figure 6:

7.1. *If $x > 1/2$ then $\psi(x, 1/3) \geqslant 2/3$.*

Proof. Let G be $(x, 1/3)$-biconstrained via (A, B, C), and suppose for a contradiction that $|N^2_A(v)| < 2|A|/3$ for all $v \in C$. By averaging, there exists $v_0 \in A$ such that $|N^2_C(v_0)| < 2|C|/3$. Let $B_0 = N(v_0)$ and $C_0 = N^2_C(v_0)$. Hence, $|B_0| \geqslant x|B|$, and $|C_0| < 2|C|/3$, and there are no edges between B_0 and $C \setminus C_0$, and every vertex in C_0 has a neighbour in B_0.

Choose $v_1 \in C_0$. Thus, $N(v_1) \cap B_0 \neq \emptyset$. Let $B_1 = N(v_1)$ and $A_1 = N^2_A(v_1)$. So $|A_1| \geqslant x|A|$, and $|A_1| < 2|A|/3$. Every vertex $v \in A \setminus A_1$ has a neighbour in B_0, since

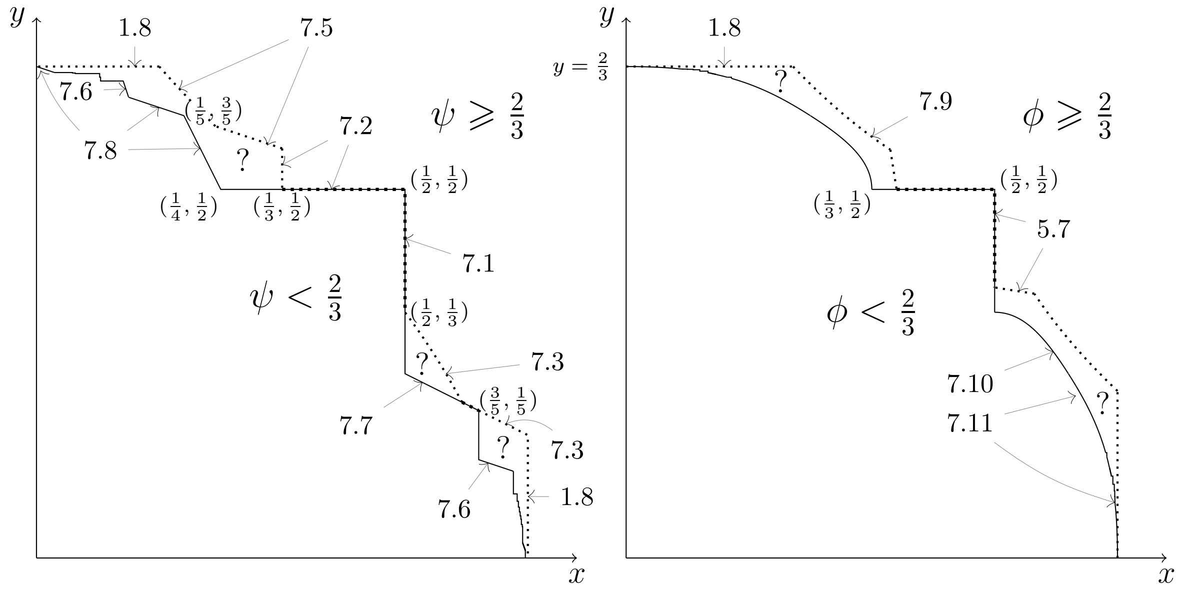

Figure 6: When $\psi(x, y) < 2/3$ and when $\phi(x, y) < 2/3$.

$|B \backslash B_0| < |B|/2 < |N(v)|$. Consequently, every vertex in $A \backslash A_1$ has at least $|C|/3 \geqslant |C_0|/2$ second neighbours in C_0, and by averaging it follows that some vertex $v_2 \in C_0$ has at least $|A \setminus A_1|/2$ second neighbours in $A \setminus A_1$. Let $B_2 = N(v_2)$ and $A_2 = N^2_A(v_2)$. Then $|A_2 \setminus A_1| \geqslant |A \setminus A_1|/2 \geqslant |A|/6$. If there exists $u \in B_1 \cap B_2$, then since u has at least $x|A|$ neighbours in A_1, and they all belong to A_2, it follows that

$$|A_2| = |A_2 \cap A_1| + |A_2 \setminus A_1| \geqslant x|A| + |A|/6 \geqslant 2|A|/3,$$

a contradiction. Consequently, $B_1 \cap B_2 = \emptyset$.

In particular, $|B_1 \cup B_2| \geqslant 2|B|/3$, and so every vertex in A has a neighbour in $B_1 \cup B_2$; and so $A_1 \cup A_2 = A$. Since $|A_1|, |A_2| < 2|A|/3$, it follows that $|A_1 \cap A_2| < |A|/3$. For $i = 1, 2$, choose $b_i \in B_i \cap B_0$. Then $N(b_i) \cap A \subseteq A_i$ for $i = 1, 2$, and so $|N(b_1) \cap N(b_2) \cap A| < |A|/3$. Consequently, $|(N(b_1) \cup N(b_2)) \cap A| > 2|A|/3$. Since $b_1, b_2 \in B_0$ and they each have at least $|C|/3$ neighbours in C_0, and $|C_0| < 2|C|/3$, it follows that they have a common neighbour $v \in C_0$. But then $N(b_1) \cup N(b_2) \cap A \subseteq N^2_A(v)$, and so $|N^2_A(v)| \geqslant 2|A|/3$, a contradiction. This proves 7.1. ∎

7.2. *Let $y > 1/2$, and let G be $(1/3, y)$-constrained via (A, B, C), such that every vertex in B has at least $|A|/3$ neighbours in A. Then there exists $w \in C$ such that $|N^2_A(w)| \geqslant \frac{2}{3}|A|$. Consequently, $\psi(1/3, y) \geqslant 2/3$.*

Proof. By averaging, there exists $v_1 \in C$ with at least $y|B|$ neighbours in B. Let $B_1 = N(v_1)$ and $A_1 = N^2_A(v_1)$. Thus $|B_1| \geqslant y|B|$. Since every vertex in $B \setminus B_1$ has at least $y|C|$ neighbours in C, some vertex $v_2 \in C$ has at least $y|B \setminus B_1|$ neighbours in $B \setminus B_1$. Let $B_2 = N(v_2)$ and $A_2 = N^2_A(v_2)$. Thus, $|B_2 \setminus B_1| \geqslant y|B \setminus B_1|$, and so

$$|B_1 \cup B_2| \geqslant |B_1| + y|B \setminus B_1| = y|B| + (1-y)|B_1| \geqslant y|B| + y(1-y)|B| = (2-y)y|B| > 3|B|/4.$$

In particular, since every vertex in A has at least $|B|/3$ neighbours in B, it follows that $A_1 \cup A_2 = A$. But $B_1 \cap B_2 \neq \emptyset$, since $|B_1|, |B_2| \geqslant y|B| > |B|/2$, and so there exists $b \in B_1 \cap B_2$; and since b has at least $|A|/3$ neighbours in A, and they all belong to $A_1 \cap A_2$, it follows that $|A_1 \cap A_2| \geqslant |A|/3$. Since $|A_1 \cup A_2| = |A|$, it follows that $|A_1| + |A_2| \geqslant 4|A|/3$, and so one of $|A_1|, |A_2|$ is at least $2|A|/3$. This proves 7.2. ∎

The last two results are closely related, via reformulation into triangular language, as we saw in section 2. The graph of figure 2 shows that $\phi(4/7, 2/7) \leqslant 5/8$, and so we studied $\psi(4/7, 2/7)$, and proved the following, which is used in figure 6.

7.3. *If $x, y \in (0, 1]$ such that $\max(x, y) > 1/2$, $x \geqslant 1/3$, $x+2y > 1$, and $3x+y/(1-y) > 2$, then $\psi(x, y) \geqslant 2/3$.*

Proof. Let G be (x, y)-biconstrained, via (A, B, C), and suppose for a contradiction that $|N_A^2(v)| < 2|A|/3$ for each $v \in C$. By 7.2, $y \leqslant 1/2$ since $x \geqslant 1/3$; and so $x > 1/2$ since $\max(x, y) > 1/2$. Hence, $y < 1/3$ by 7.1. Also $x < 2/3$, by 1.8.

(1) *For all $v_1, v_2 \in C$, if $|N(v_1) \cup N(v_2)| > (1 - x)|B|$ then $N(v_1) \cap N(v_2) = \emptyset$, $N_A^2(v_1) \cup N_A^2(v_2) = A$, and $|N_A^2(v_1) \cap N_A^2(v_2)| < |A|/3$.*

Every vertex in A has a neighbour in $N(u) \cup N(v)$, and so $N_A^2(u) \cup N_A^2(v) = A$. Since $|N_A^2(u)| < 2|A|/3$ and $|N_A^2(v)| < 2|A|/3$ it follows that $|N_A^2(u) \cap N_A^2(v)| < |A|/3$, and so there is no vertex in $N(u) \cap N(v)$ (since any such vertex would have at least $x|A|$ neighbours in A, all belonging to $N_A^2(u) \cap N_A^2(v)$). This proves (1).

(2) *There exist $v_1, v_2 \in C$ with $N(v_1) \cap N(v_2) = \emptyset$.*

Choose $v_1 \in C$. Since every vertex in $A \setminus N_A^2(v_1)$ has at least $x|B|$ second neighbours in $B \setminus N(v_1)$, some vertex $u_2 \in B \setminus N(v_1)$ has at least $(x/(3(1 - y)))|A|$ neighbours in $A \setminus N_A^2(v_1)$. Let $v_2 \in C$ be adjacent to u_2. If v_1, v_2 have a common neighbour u_1, then since $N_A(u_1) \subseteq N_A^2(v_2)$, it follows that $|N_A^2(v_2)| \geqslant (x/(3(1 - y)) + x)|A|$, and so $x/(3(1 - y)) + x < 2/3$, that is, $(4 - 3y)x < 2 - 2y < (4 - 3y)/2$, and so $x < 1/2$, a contradiction. This proves (2).

(3) *If $v_1, v_2, v_3 \in C$ and $N(v_1) \cap N(v_2) = \emptyset$ then $N(v_3)$ is disjoint from exactly one of $N(v_1), N(v_2)$.*

If $N(v_3)$ is disjoint from both $N(v_1), N(v_2)$, then every two of $N(v_1), N(v_2), N(v_3)$ have union of cardinality more than $(1 - x)|B|$, and so every vertex in A belongs to at least two of $N_A^2(v_i)$ $(i = 1, 2, 3)$. Consequently, one of $N_A^2(v_i)$ $(i = 1, 2, 3)$ has cardinality at least $a|A|/3$, a contradiction. Now suppose that $N(u_3)$ has nonempty intersection with both $N(v_1), N(v_2)$. Thus, $|N_A^2(v_i) \cap N_A^2(v_3)| \geqslant x|A|$ for $i = 1, 2$, and since $|N_A^2(v_1) \cap N_A^2(v_2)| < |A|/3$, it follows that $|N_A^2(v_3)| \geqslant (2x - 1/3)|A| \geqslant 2|A|/3$ since $x \geqslant 1/2$, a contradiction. This proves (3).

Let H be the bipartite graph $G[B \cup C]$. From (2) and (3), H has exactly two components H_1 and H_2 say. Let $C_i = V(H_i) \cup C$ and $B_i = V(H_i) \cap B$ for $i = 1, 2$. Then from (3), every two vertices in C_i have a common neighbour in B_i, for $i = 1, 2$. Let $c_i = |C_i|/|C|$, for $i = 1, 2$. Thus $c_1 + c_2 = 1$. We may assume that $b_1 \geqslant 1/2$. Choose $v_1 \in C_1$. Since $|A \setminus N_A^2(v_1)| > |A|/3$, and every vertex in $A \setminus N_A^2(v_1)$ has at least $y|C|$ second neighbours in C_1, some vertex $v_2 \in C_1$ has at least $(y/(3c_1))|A|$ second neighbours in $A \setminus N_A^2(v_1)$. But since $v_1, v_2 \in C_1$, they have a common neighbour in B_1; therefore $|N_A^2(v_2)| \geqslant (y/(3c_1)+x)|A|$, and so $y/(3c_1)+x < 2/3$. Now $c_1 \leqslant 1-y$, so $3x+y/(1-y) < 2$, contrary to the hypothesis. This proves 7.3. ∎

If $A \subseteq V(G)$ and $X \subseteq V(G) \setminus A$, $N_A(X)$ denotes the set of vertices in A with a neighbour in X. The next result is a useful lemma which says, roughly speaking, the larger X is, the larger $N_A(X)$ is. In this section we only use it for $k = 1$, but we will use it with $k = 2$ for three results in the section discussing when $\psi(x, y) \geqslant 3/4$.

7.4. *Let $x, y, z \in (0, 1]$, and suppose that G is (x, y)-biconstrained via (A, B, C), and $|N_A^2(w)| < z|A|$ for all $w \in C$. Then for all integers $k \geqslant 1$, if $B' \subseteq B$ with*

$$\frac{|B'|}{|B|} > (k-1)(1-y) + \max(1-y, 1-x/(1-y)),$$

then $|N_A(B')| > (x + k(1-z))|A|$.

Proof. We proceed by induction on k, and so we assume that either $k = 1$ or the result holds for $k-1$. Let $A' = N_A(B')$. Every vertex in $B \setminus B'$ has at least $y|C|$ neighbours in C, so there exists $v \in C$ with at least $y|B \setminus B'|$ neighbours in $B \setminus B'$. By hypothesis, $|B'|/|B| > 1 - x/(1-y)$, that is, $x|B| + y|B \setminus B'| > |B \setminus B'|$. But every vertex in $A \setminus A'$ has at least $x|B|$ neighbours in $B \setminus B'$, and therefore has one such neighbour adjacent to v; and so $A \setminus A' \subseteq N_A^2(v)$. Let $B'' = B' \cap N_B(v)$, and $A'' = N_A(B'')$.

(1) $|A''| \geqslant (x + (k-1)(1-z))|A|$.

Since $|N_B(v)| \geqslant y|B|$, it follows that $|B''| \geqslant y|B| - (|B| - |B'|) = |B'| - (1-y)|B|$. If $k = 1$, then $|B'|/|B| > 1-y$ by hypothesis, and therefore $B'' \neq \emptyset$, and so $|A''| \geqslant x|A|$ as claimed. If $k \geqslant 2$, then since $|B'|/|B| > (k-1)(1-y) + \max(1-y, 1-x/(1-y))$, it follows that

$$\frac{|B''|}{|B|} > (k-2)(1-y) + \max(1-y, 1-x/(1-y)),$$

and so $|A''| \geqslant (x + (k-1)(1-z))|A|$ from the inductive hypothesis. This proves (1).

Since $A'' \subseteq A'$, and $A'' \cup (A \setminus A') \subseteq N_A^2(v)$, it follows that

$$z|A| \geqslant |N_A^2(v)| \geqslant (x + (k-1)(1-z))|A| + |A \setminus A'|$$

and so $|A'| \geqslant (x + k(1-z))|A|$. This proves 7.4. ∎

We apply 7.4 to prove the next result, which is used for figure 6:

7.5. *Let $x, y \in (0,1]$, such that $y > 1/2$, $x + 3y > 2$ and $x > 2(1-y)^2/(2-y)$. Then $\psi(x,y) \geqslant 2/3$.*

Proof. Let G be (x,y)-biconstrained, via (A,B,C), and suppose for a contradiction that $|N_A^2(v)| < 2|A|/3$ for each $v \in C$.

(1) *If $B' \subseteq B$ with $|B'|/|B| > \max(1-y, 1-x/(1-y))$, then there are at least $(x+1/3)|A|$ vertices in A with a neighbour in B'. In particular, if $|B'| \geqslant (x+2y-1)|B|$ then the same conclusion holds.*

The first statement follows from 7.4 with $k = 1$. For the second, $x + 2y - 1 > 1 - y$ since $x + 3y \geqslant 2$; and $x+2y-1 > 1-x/(1-y)$ since $x > 2(1-y)^2/(2-y)$. Consequently, $x + 2y - 1 > \max(1-y, 1-x/(1-y))$. This proves the second statement and so proves (1).

Say $w_1, w_2 \in C$ are *close* if $|N_B(w_1) \cup N_B(w_2)| \leqslant (1-x)|B|$.

(2) *There exists $w_1 \in C$ such that the set of vertices in C that are close to w_1 has cardinality at least $|C|/2$.*

This is trivial if every two vertices in C are close; so we assume there exist $w_1, w_2 \in C$ that are not close. Consequently, every vertex in A has a neighbour in $N_B(w_1) \cup N_B(w_2)$, and so $N_A^2(w_1) \cup N_A^2(w_2) = A$. If there exists $w \in C$ that is not close to either of w_1, w_2 then similarly $N_A^2(w_1) \cup N_A^2(w) = A$ and $N_A^2(w_2) \cup N_A^2(w) = A$; and so every vertex in A belongs to at least two of $N_A^2(w_1), N_A^2(w_2), N_A^2(w)$, and therefore one of these three sets has cardinality at least $2|A|/3$, a contradiction. Thus, exchanging w_1, w_2 if necessary, we may assume that at least half of all vertices in C are close to w_1. This proves (2).

Let C_1 be the set of vertices in C that are close to w_1; thus $|C_1| \geqslant |C|/2$. Let $B_1 = N_B(w_1)$ and $A_1 = N_A^2(w_1)$. Since $|B_1| \geqslant y|B|$ and every vertex in $A \setminus A_1$ has at least $x|B|$ neighbours in $B \setminus B_1$, there exists $v_2 \in B \setminus B_1$ with at least $\frac{x}{1-y}|A \setminus A_1|$ neighbours in $A \setminus A_1$. Since $y > 1/2$ and $|C_1| \geqslant |C|/2$, v_2 has a neighbour $w_2 \in C_1$. Since w_2 is close to w_1 it follows that $|N_B(w_1) \cup N_B(w_2)| \leqslant (1-x)|B|$, and so $|N_B(w_1) \cap N_B(w_2)| \geqslant (x+2y-1)|B|$. From (1), there are at least $(x+1/3)|A|$ vertices in A with a neighbour in $N_B(w_1) \cap N_B(w_2)$. These vertices all belong to A_1, and so

$$|N^2(w_2)| \geqslant \frac{x}{1-y}|A \setminus A_1| + \left(x + \frac{1}{3}\right)|A| \geqslant \left(\frac{x}{3(1-y)} + x + \frac{1}{3}\right)|A|.$$

Consequently, $\frac{x}{3(1-y)} + x + 1/3 < 2/3$, that is, $\frac{x}{1-y} + 3x < 1$. By hypothesis, $x > 2(1-y)^2/(2-y)$, and so substitution for x yields

$$\frac{2(1-y)^2/(2-y)}{1-y} + 6(1-y)^2/(2-y) < 1,$$

which simplifies to $(3y-2)(2y-3) < 0$, contrary to 1.8. This proves 7.5. ∎

The next result is used to prove 9.9, and in figure 6:

7.6. *Let $x, y \in (0,1]$. If either*

- $x \leqslant 11/17$ *and* $y \leqslant 1/7$ *and* $x + 3y \leqslant 1$*; or*
- $x < 1/8$ *and* $y \leqslant 11/17$ *and* $3x + y \leqslant 1$

then $\psi(x, y) < 2/3$.

Proof. Take the graph consisting of seven disjoint copies of a three-vertex path, numbered a_i, b_i, c_i in order ($1 \leqslant i \leqslant 7$). For $1 \leqslant i \leqslant 3$ and $4 \leqslant j \leqslant 7$, make a_i adjacent to b_j and make a_j adjacent to b_i, forming G. Let $A = \{a_i \ : 1 \leqslant i \leqslant 7\}$ and define B, C similarly.

For the first statement, we may assume by increasing x, y that $x + 3y = 1$. It follows that $x \geqslant 4/7$ (because $y \leqslant 1/7$) and similarly $y \geqslant 2/17$, and $4x = 4 - 12y \geqslant 1 + 9y$. For $1 \leqslant i \leqslant 3$, let $w(a_i) = 3/17$, $w(b_i) = (4x - 1)/9$, and $w(c_i) = 1/7$. For $4 \leqslant i \leqslant 7$, let $w(a_i) = 2/17$, $w(b_i) = (1 - x)/3$, and $w(c_i) = 1/7$. Then this weighted graph is (x, y)-biconstrained via (A, B, C) and shows that $\psi(x, y) < 2/3$. This proves the first statement.

For the second statement, we may assume that $3x + y = 1$, and so $x \geqslant 2/17$ and $y > 5/8$, and so $3y = 3 - 9x \leqslant 1 + 8x$. Let us take the same graph and redefine w, as follows. For $1 \leqslant i \leqslant 3$, let $w(a_i) = (1 - y)/2$ and $w(b_i) = w(c_i) = (1 - 4x)/3$. For $4 \leqslant i \leqslant 7$, let $w(a_i) = (3y - 1)/8$ and $w(b_i) = w(c_i) = x$. Then this weighted graph is (x, y)-biconstrained via (C, B, A) and shows that $\psi(x, y) < 2/3$. This proves the second statement, and hence proves 7.6. ∎

The next result is used for figure 6:

7.7. *Let $x', y', z' \in (0,1]$ such that $\psi(x', y') \leqslant z' < 1/2$; and let $x, y \in (0,1]$ satisfy $x \leqslant \frac{1}{2-x'}$, $x < 1 - \frac{1-x'}{3(1-z')}$, $y \leqslant \frac{y'}{1+y'}$ and $x + \frac{1-x'}{y'}y \leqslant 1$. Then $\psi(x, y) < 2/3$. Consequently:*

- $\psi(x, y) < 2/3$ *if* $x \leqslant 3/5$ *and* $y \leqslant 1/4$ *and* $x + 2y \leqslant 1$*;*
- $\psi(x, y) < 2/3$ *if* $x \leqslant 5/8$ *and* $y \leqslant 1/6$ *and* $x + 3y \leqslant 1$.

Proof. The first statement follows from 3.4 taking z slightly less than $2/3$. The two statements in bullets follow by setting $x' = y' = z' = 1/3$, and then $x' = z' = 2/5$ and $y' = 1/5$. This proves 7.7. ∎

The next result is used to prove 9.9, and in figure 6:

7.8. *Let $x', y', z' \in (0,1]$ such that $\psi(x', y') \leqslant z' < 1/2$; and let $x, y \in (0,1]$ satisfy $x < \frac{2x'}{3}$, $y \leqslant \frac{1}{2-y'}$, and $\frac{1-y'}{x'}x + y \leqslant 1$. Then $\psi(x, y) < 2/3$. Consequently:*

- $\psi(x, y) < 2/3$ *if* $y \leqslant 3/5$ *and* $2x + y \leqslant 1$*; and*
- $\psi(x, y) < 2/3$ *if* $y \geqslant 3/5$ *and* $x + 3y \leqslant 2$*, and* $x + 3y < 2$ *if* x *or* y *is irrational.*

Proof. The first statement follows from 3.5, taking z slightly less than $2/3$. To prove the first bullet, let $2x + y \leqslant 1$, and so $x \leqslant 1/2$. If also $y \leqslant 1/2$ then $\psi(x, y) \leqslant 1/2$ by 1.10; so we may assume that $y > 1/2$. We claim there is an integer $k \geqslant 1$ with

$$\frac{3 - 3y}{6y - 2} < k \leqslant \frac{1 - y}{4y - 2}.$$

To see this, if $y > 5/9$ we can take $k = 1$ (because we are given that $y \leqslant 3/5$), and if $y \leqslant 5/9$, then

$$\frac{1 - y}{4y - 2} - \frac{3 - 3y}{6y - 2} \geqslant 1,$$

and so again k exists. Thus, $y \leqslant \frac{2k+1}{4k+1}$, and $x \leqslant \frac{1-y}{2} < \frac{2k}{6k+3}$. Let $x' = z' = k/(2k + 1)$, and $y' = 1/(2k + 1)$. Then the claim follows from the first statement.

For the second bullet, let $x + 3y \leqslant 2$ with $y \geqslant 3/5$, with $x + 3y < 2$ if x or y is irrational. Consequently, we may assume that x, y are rational, by increasing them slightly if necessary. Let $x' = 2/y - 3$, and $y' = 2 - 1/y$; it follows that $x' + 2y' \leqslant 1$ and $2x' + y' \leqslant 1$, and x', y' are rational, and so $\psi(x', y') < 1/2$ by 3.3. The result follows from the first statement. This proves 7.8. ∎

For the mono-constrained question, we have a result used for 10.4, and in figure 6:

7.9. *For $x, y \in (0, 1]$, if $y \leqslant 1/2$ and $x > (1 - y)^2/(1 - 2y^2)$ then $\phi(x, y) \geqslant 2/3$.*

Proof. Let G be (x, y)-constrained via (A, B, C). If $x + y > 1$ the result follows from 5.2, so we may assume that $x + y \leqslant 1$. Since $x > (1 - y)^2/(1 - 2y^2)$, we may also assume that

- x, y are rational; and
- every vertex in A has strictly more than $x|B|$ neighbours in B

by reducing x and y a little if necessary while retaining the property that $x > (1-y)^2/(1-2y^2)$.

Let $p = (1 - x - y)/(1 - 2y)$. Thus, p is rational, so we may assume (by multiplying vertices) that $p|B|$ is an integer. Also $p \leqslant y$, since $x > (1 - y)^2/(1 - 2y^2)$. Let $s = (x - (1 - y)^2)/(y(1 - y))$. It follows that $0 \leqslant s \leqslant 1$, since $x > (1 - y)^2/(1 - 2y^2)$ and $x + y \leqslant 1$.

Choose $v_1 \in C$ with at least $y|B|$ neighbours in B, and let $B_1 \subseteq N(v_1)$ with $|B_1| = y|B|$. Choose $v_2 \in C$ such that $sb_0 + b_2 \geqslant y(sy + (1 - y))$, where $b_0|B| = |N(v_2) \cap B_1|$ and $b_2|B| = |N(v_2) \setminus B_1|$. (Such a vertex exists by averaging.) We claim that $b_0 + b_2 \geqslant p$; for from the definition of s,

$$b_0 + b_2 \geqslant sb_0 + b_2 \geqslant y(sy + (1 - y)) = y((x - (1 - y)^2)/(1 - y) + 1 - y) = xy/(1 - y),$$

and $p = (1 - x - y)/(1 - 2y) \leqslant xy/(1 - y)$ since $x > (1 - y)^2/(1 - 2y^2)$.

Also we claim that $b_2 \geqslant 1 - x - y$; for from the definition of s,

$$sy + 1 - x - y = y(sy + 1 - y) \leqslant sb_0 + b_2 \leqslant sy + b_2.$$

Consequently, $|N(v_1) \cup N(v_2)| \geqslant (1 - x)|B|$, and there exist $P_1 \subseteq N(v_1)$ and $P_2 \subseteq N(v_2)$, both of cardinality $p|B|$. Choose $v_3 \in C$ with at least $y(1 - 2p)|B|$ neighbours in $B \setminus (P_1 \cup P_2)$. Then for $i = 1, 2$,

$$|P_i \cup N(v_3)| \geqslant (y(1 - 2p) + p)|B| \geqslant (1 - x)|B|.$$

Since every vertex in A has strictly more than $x|B|$ neighbours in B, it follows that every vertex in A belongs to at least two of the sets $N_A^2(v_i)$ $(i = 1, 2, 3)$; and so one of these sets has cardinality at least $2|A|/3$. This proves 7.9. ∎

The next result is used to prove 9.10, 9.11, 10.6, and in figure 6:

7.10. *For all $x, y \in (0, 1]$, if $x < 3/5$ and $y \leqslant (x^2 - 2x + 1)/(19x^2 - 22x + 7)$, then $\phi(x, y) < 2/3$.*

Proof. Since $(x^2 - 2x + 1)/(19x^2 - 22x + 7) \leqslant 1/3$ for all x, it follows that $y \leqslant 1/3$, and so we may assume that $x > 1/2$, or else the result is true since $\phi(1/2, 1/2) = 1/2$. Let $x' = 2 - 1/x$ and $y' = y/(1 - y)$. Thus $0 < x' < 1/3$. Moreover,

$$y' \leqslant \frac{(1 - x')^2}{2 - 4x' + 6x'^2},$$

since $y \leqslant (x^2 - 2x + 1)/(19x^2 - 22x + 7)$. By 6.7, $\phi(x', y') < 1/2$. Choose z with $\phi(x', y') < 2 - 1/z < 1/2$. Thus, $z < 2/3$, and by 3.7, it follows that $\phi(x, y) \leqslant z < 2/3$. This proves 7.10. ∎

The next result is used to prove 9.12, 9.13 and 10.7, and in figure 6:

7.11. *For all $x, y \in (0, 1]$, if $y < 1/4$ and $x \leqslant (12y^2 - 8y + 2)/(20y^2 - 12y + 3)$, then $\phi(x, y) < 2/3$.*

Proof. We may assume that $x > 1/2$, since $\phi(1/2, 1/2) = 1/2$. Let $x' = 2 - 1/x$ and $y' = y/(1 - y)$. Thus, $x', y' \in (0, 1]$, and $y' < 1/3$, and $x' \leqslant (1 - y')^2/(2 - 4y' + 6y'^2)$ since $x \leqslant (12y^2 - 8y + 2)/(20y^2 - 12y + 3)$. By 6.7, $\phi(y', x') < 1/2$, and so $\phi(x', y') < 1/2$ by 2.3. By 3.7 (taking z with $\phi(x', y') < 2 - 1/z < 1/2$), it follows that $\phi(x, y) < 2/3$. This proves 7.11. ∎

8 The 1/3 level

Next we do the same for $\psi(x, y) \geqslant 1/3$ and $\phi(x, y) \geqslant 1/3$. Figure 7 summarizes our results.

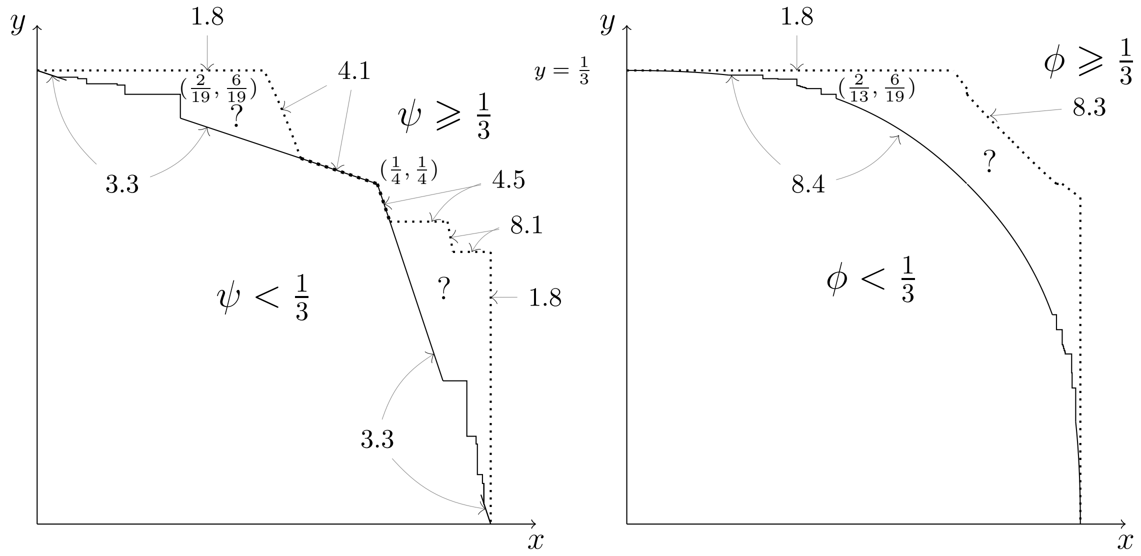

Figure 7: When $\psi(x, y) < 1/3$ and when $\phi(x, y) < 1/3$.

The next result gives part of figure 7:

8.1. *Let $x, y \in (0, 1]$ with $y > \frac{1}{5}$ and $3x + \frac{y}{3(1-y)} \geqslant 1$. Then $\psi(x, y) \geqslant \frac{1}{3}$.*

Proof. We may assume that $y \leqslant 1/3$, by 1.8, and so $y/(3(1-y)) \leqslant 1/6$. Consequently, $x \geqslant 5/18$, and in particular $x > 2y/3$. Also, since $1/5 \leqslant y \leqslant 1/3$, it follows that $3y - y/(3(1-y)) > 1/2$; and so

$$\left(3x + \frac{y}{3(1-y)}\right) + \left(3y - \frac{y}{3(1-y)}\right) > \frac{3}{2},$$

and consequently $x + y > 1/2$. Let G be (x, y)-biconstrained via (A, B, C), and suppose that $|N^2_A(v)| < |A|/3$ for each $v \in C$. Let H be the subgraph induced on $B \cup C$, and let $H_1, \ldots, H_k$ be its components. Let $B_i = V(H_i) \cap B$ and $C_i = V(H_i) \cap C$, and $b_i = |B_i|/|B|$, $c_i = |C_i|/|C|$, for $1 \leqslant i \leqslant k$. Since $y > 0$, B_i, C_i are both nonempty and so $b_i, c_i \geqslant y$ for $1 \leqslant i \leqslant k$. For $1 \leqslant i \leqslant k$, let A_i be the set of vertices in A with a neighbour in B_i, and let A^*_i be the set of vertices in A such that $N(v) \subseteq B_i$.

(1) *$k \geqslant 2$.*

Suppose that $k = 1$, and let H' be the graph with vertex set B in which u, u' are adjacent if u, u' have a common neighbour in H. Then every stable set of H' has cardinality at most 4. By 6.2 there is a vertex $u_1 \in B$ with H'-distance at most four to every other vertex in B; and so the H-distance from u_1 to each vertex in B is at most eight. Let $v_1 \in C$ be adjacent to u_1. Let $A' = A \setminus N^2_A(v_1)$ and $B' = B \setminus N(v_1)$. Hence, $|A'| > 2|A|/3$. Since every vertex in A' has at least $x|B|$ neighbours in B', and $|B'| \leqslant (1-y)|B|$, some vertex $u \in B'$ has at least

$$\frac{x|A'|}{1-y} \geqslant \frac{2x|A|}{3(1-y)}$$

neighbours in A'. Choose a path of H between u_1 and u of length at most eight, and let its vertices be u_1-v_2-u_2-$\cdots$-v_t-$u_t = u$ say, in order. Thus, $t \leqslant 5$, and so there exists i with $1 \leqslant i \leqslant t-1$ such that there are at least $|N_{A'}(u)|/4$ vertices that belong to $N_A(u_{i+1}) \setminus N_A(u_i)$. Since $|N_A(u_i)| \geqslant x|A|$, it follows that

$$|N_A^2(v_{i+1})| \geqslant x|A| + |N_{A'}(u)|/4 \geqslant \frac{x+2x}{12(1-y)}|A| \geqslant |A|/3,$$

a contradiction, since $x \geqslant 2y/3$ and so $x + x/(6(1-y)) \geqslant x + y/(9(1-y)) \geqslant 1/3$. This proves (1).

(2) *$b_i \leqslant 1-x-y < 1/2$ for $1 \leqslant i \leqslant k$, and so $k \geqslant 3$.*

Suppose that $b_1 > 1-x-y$ say. Thus, if $u \in A \setminus A_1$, then $u \in N_A^2(v)$ for every $v \in C \setminus C_1$; and so $|A \setminus A_1| < |A|/3$, and so $|A_1| > 2|A|/3$. Let H' be the graph with vertex set C_1 in which v, v' are adjacent if they have a common H_1-neighbour in B_1. Thus, H' has stability number at most three (by (1)) and so has radius at most three, by 6.2. Choose $v_1 \in C_1$ such that every vertex in C_1 has H_1-distance at most six from v_1. Let $A' = A_1 \setminus N_A^2(v_1)$; thus $|A'| > |A|/3$. Since every vertex in A' has a neighbour in B_1 and hence has at least $y|C|$ second neighbours in C_1, there exists $v \in C_1$ such that

$$|N_{A'}^2(v)| \geqslant \frac{y}{|C_1|}|A'| \geqslant \frac{y}{3(1-y)}|A|,$$

since $|C_1| \leqslant (1-y)|C|$. Choose a path of H_1 between v_1, v of length at most six, with vertices v_1-u_1-v_2-$\cdots$-u_{t-1}-$v_t = v$ say where $t \leqslant 4$. Then for some i with $1 \leqslant i \leqslant t-1$,

$$|N_{A'}^2(v_{i+1}) \setminus N_{A'}^2(v_i)| \geqslant \frac{y}{9(1-y)}|A|,$$

and hence

$$|N_{A'}^2(v_{i+1})| \geqslant \left(x + \frac{y}{9(1-y)}\right)|A|$$

since all vertices of $N_A(u_i)$ belong to $N_A^2(v_{i+1})$ and do not belong to $N_{A'}^2(v_{i+1}) \setminus N_{A'}^2(v_i)$. But $3x + y/(3(1-y)) \geqslant 1$, a contradiction. This proves (2).

By (2), $k \geqslant 3$; and $k \leqslant 4$ since $y > 1/5$. We may assume that $|B_1|, |B_2| \geqslant |B_i|$ for $i \geqslant 3$; let $B_0 = \bigcup_{3 \leqslant i \leqslant k} B_i$, and $C_0 = \bigcup_{3 \leqslant i \leqslant k} C_i$. Hence, $|B_0| \leqslant |B|/2$ since $k \leqslant 4$. Let $b_0 = |B_0|/|B|$ and $c_0 = |C_0|/|C|$; let A_0 be the set of vertices in A with a neighbour in B_0, and let A_0^* be the set of vertices in A such that $N(v) \subseteq B_0$. For $0 \leqslant i < j \leqslant 2$, choose $A_{ij} = A_{ji} \subseteq A_i \cap A_j$ such that the sets $A_{12}, A_{13}, A_{23}, A_0^*, A_1^*, A_2^*$ are pairwise disjoint and have union A. For $0 \leqslant i \leqslant 2$ let $a_i = |A_i|/|A|$ and $a_i^* = |A_i^*|/|A|$, and for $0 \leqslant i, j \leqslant 2$

with $i \neq j$ let $a_{ij} = |A_{ij}|/|A|$. Since $b_1, b_2 \leqslant 1 - x - y < x + y$ and $b_0 \leqslant 1/2 < x + y$, we have $b_i < x + y$ for $i = 0, 1, 2$. Let $0 \leqslant i \leqslant 2$, and choose $v \in C_i$ uniformly at random. Then $A_i^* \subseteq N_A^2(v)$ because $b_i < x + y$, and the expected value of $|N_A^2(v) \cap (A_i \setminus A_i^*)|$ is at least $(y/c_i)|A_i \setminus A_i^*|$; so the expected value of $|N_A^2(v)|$ is at least

$$|A_i^*| + \frac{y}{c_i}|A_i| \setminus |A_i^*| = \left(a_i^* + \frac{y}{c_i}(a_i - a_i^*)\right)|A|.$$

Since $|N_A^2(v)| < |A|/3$, it follows that $a_i^* + (y/c_i)(a_i - a_i^*) < 1/3$. Now A_0^*, A_{01}, A_{02} are pairwise disjoint subsets of A_0, so $a_{01} + a_{02} \leqslant a_0 - a_0^*$; and hence

$$a_0^* + (y/c_0)(a_{01} + a_{02}) \leqslant a_0^* + (y/c_0)(a_0 - a_0^*) < 1/3.$$

Similarly we have $a_1^* + (y/c_1)(a_{01} + a_{12}) < 1/3$ and $a_2^* + (y/c_2)(a_{02} + a_{12}) < 1/3$; and by summing these three inequalities and using the equation

$$a_1^* + a_2^* + a_3^* + a_{12} + a_{13} + a_{23} = 1$$

we obtain

$$a_{12}\left(\frac{y}{c_1} + \frac{y}{c_2} - 1\right) + a_{13}\left(\frac{y}{c_1} + \frac{y}{c_3} - 1\right) + a_{23}\left(\frac{y}{c_2} + \frac{y}{c_3} - 1\right) < 0.$$

Consequently, there exist distinct $i, j \in \{0, 1, 2\}$ with $y/c_i + y/c_j - 1 < 0$. But $1/c_i + 1/c_j \geqslant 4/(c_i + c_j)$, and $c_i + c_j \leqslant 1 - y$, and so $4y/(1 - y) < 1$, a contradiction. This proves 8.1. ∎

For ϕ, we need the following, used to prove 8.3.

8.2. *Let $x, y \in (0, 1]$. Let G be $(x, 2/3)$-constrained via (A, B, C), such that every three vertices in B have a common neighbour in C, and every vertex $w \in C$ satisfies $|N_A^2(w)| < (1 - y)|A|$. If $v_1 \in B$ has $a|A|$ neighbours in A then*

$$a \leqslant 1 - (1 + 2x - 5x^2)y/(1 - x)^2.$$

Proof. Let $v_1 \in B$ have $a|A|$ neighbours in A. Define $A_1 = N_A(v_1)$ and choose $C_1 \subseteq N_C(v_1)$ with $|C_1| = 2|C|/3$ (we may assume this is an integer). Let $A_1' = A \setminus A_1$ and $C_1' = C \setminus C_1$.

(1) *If some vertex $v_2 \in B$ has a set A_2 of $t|B|$ neighbours in A_1', then*

- *every $v \in B$ has at most $(1 - y - a - t)|A|$ neighbours in $A \setminus (A_1 \cup A_2)$; and*
- *the sum over all $v \in B$ of the number of neighbours of v in $A \setminus (A_1 \cup A_2)$ is $(1 - a - t)x|A||B|$.*

The first claim follows since v_1, v_2, v have a common neighbour in C. The second holds since every vertex in $A \setminus (A_1 \cup A_2)$ has $x|B|$ neighbours. The third follows. This proves (1).

The sum over $u \in A_1'$, of $|N^2_{C_1}(u)|$, is at most $\frac{2}{3}(1-y-a)|A||C|$; and for each u, $|N^2_{C_1}(u)| \geqslant \max_{v \in N_B(u)} |N_{C_1}(v)|$. But the latter is at least

$$\sum_{v \in N_B(u)} |N_{C_1}(v)|/(x|B|).$$

It follows that

$$\sum_{u \in A_1'} \sum_{v \in N_B(u)} |N_{C_1}(v)| \leqslant \frac{2}{3} x(1-y-a)|A||B||C|.$$

Consequently,

$$\sum_{v \in B} |N_{A_1'}(v)||N_{C_1}(v)| \leqslant \frac{2}{3} x(1-y-a)|A||B||C|.$$

Moreover, each vertex in C_1' has at least $x|B|$ nonneighbours in B, and so there are at most $(1-x)/3|B||C|$ edges between B and C_1'. Hence, there are at least $(1+x)/3|B||C|$ edges between B and C_1.

For each $v \in B$, let $p(v) = |N_{A_1'}(v)|/|A|$. Thus, $\sum_{v \in B} p(v) = x(1-a)|B|$. By setting $q(v) = 3|N_{C_1}(v)|/|C| - 1$ we deduce: for each $v \in B$ there exists $q(v)$ such that

- for each $v \in B$, $1/3 \leqslant q(v)/3 + 1/3 \leqslant 2/3$, that is, $0 \leqslant q(v) \leqslant 1$;
- $\sum v \in B(q(v)/3 + 1/3) \geqslant (1+x)/3|B|$, that is, $\sum v \in Bq(v) \geqslant x|B|$;
- $\sum_{v \in B} p(v)(q(v)/3 + 1/3) \leqslant \frac{2}{3}x(1-y-a)|B|$, that is, $\sum_{v \in B} p(v)q(v) \leqslant (x - 2xy - xa)|B|$.

Let $Q \subseteq B$ be the $x|B|$ vertices in B (we may assume this is an integer) with $p(v)$ smallest. Then the expression in the last bullet above is minimized by setting $q(v) = 1$ for $v \in Q$, and $q(v) = 0$ for $v \in B \setminus Q$. Consequently, $\sum_{v \in Q} p(v) \leqslant (x - xa - 2xy)|B|$.

Choose $v_2 \in B \setminus Q$ with $|N_{A_1'}(v_2)|$ maximum; A_2 say, where $|A_2| = t|A|$. By (1), every $v \in B$ has at most $(1-y-a-t)|A|$ neighbours in $A_1' \setminus A_2$, and the sum over all $v \in B$ of the number of neighbours of v in $A_1' \setminus A_2$ is $(1-a-t)x|A||B|$. So the number of edges between $A_1' \setminus A_2$ and $B \setminus Q$ is at most $(1-y-a-t)(1-x)|A||B|$; and the number between $A_1' \setminus A_2$ and Q is at most $(x - xa - 2xy)|A||B|$, since $\sum_{v \in Q} p(v) \leqslant (x - xa - 2xy)|B|$. Hence, the number between $A_1' \setminus A_2$ and B is at most $((1-y-a-t)(1-x) + x - xa - 2xy)|A||B|$, and since this number equals $(1-a-t)x|A||B|$, it follows that

$$(1-y-a-t)(1-x) + x - xa - 2xy \geqslant (1-a-t)x,$$

that is,

$$1 - y - a - t - x + xa - xy + 2tx \geqslant 0.$$

Consequently, $t \leqslant (1-x-y-a+xa-xy)/(1-2x)$.

Now $t(|B|-|Q|)|A|+(x-xa-2xy)|A||B| \geqslant x(1-a)|A||B|$, so

$$t(1-x) \geqslant 2xy.$$

Hence,

$$(1-x-y-a+xa-xy)/(1-2x) \geqslant t \geqslant 2xy/(1-x),$$

that is,

$$(1-x)(1-x-y-a+xa-xy) \geqslant (1-2x)2xy.$$

Consequently,

$$(1-x)^2(1-a) \geqslant (1+2x-5x^2)y.$$

This proves 8.2. ∎

The next result is used for figure 7:

8.3. *Let $x, y \in (0,1]$ with $(1-x)^2 > (2x+1)(1+2x-5x^2)y$ and $x \geqslant 1/4$ and $4x^2y^2 \geqslant (1-y)(x-y)^2$. Then $\phi(x,y) \geqslant 1/3$. Consequently, if $x > 0.28231$ then $\phi(x,x) \geqslant 1/3$.*

Proof. Let $\phi(x,y) = z$ and suppose that $z < 1/3$. Then there is a graph G that is $(x, 1-z)$-constrained via A, B, C, such that $|N_A^2(w)| \leqslant (1-y)|A|$ for each $w \in C$, by 2.5 and 2.3. As in 5.7, there exists $w \in C$ such that there are at least $x(1-z)|A| \cdot |B|$ edges between $N_B(w)$ and $N_A^2(w)$. Define $B_1 = N_B(w)$ and $B_2 = B \setminus B_1$; and let $B_2 = t|B|$. For each $u \in A$, let u have $d(u)|B|$ neighbours in B_1. Let $A_1 = N_A^2(w)$ and $A_2 = A \setminus A_1$; and let $|A_2| = s|A|$. Thus, $d(u) = 0$ for each $u \in A_2$.

(1) *$t \geqslant x$, and $s \geqslant y$, and $0 \leqslant d(u) \leqslant \min(x, 1-t)$ for each $u \in A$. Also*

$$x(1-y)|A| \geqslant \sum_{u \in A_1} d(v) \geqslant x(1-z)|A|.$$

We may assume that every vertex in A has degree exactly $x|B|$; so $0 \leqslant d(u) \leqslant \min(x, 1-t)$ for each $u \in A$. Since $|A_1| \leqslant (1-y)|A|$, it follows that $s \geqslant y$. In particular, $A_2 \neq \emptyset$, and so some vertex in A_2 has $x|B|$ neighbours in B_2, and so $t \geqslant x$. Since there are at least $x(1-z)|A| \cdot |B|$ edges between $N_B(w)$ and $N_A^2(w)$, it follows that $\sum_{u \in A_1} d(v) \geqslant x(1-z)|A|$. Since $|A_1| \leqslant (1-y)|A|$ and every vertex in A_1 has degree exactly $x|B|$, it follows that the number of edges between A_1 and B_1 is at most $x(1-y)|A| \cdot |B|$, and so $x(1-y)|A| \geqslant \sum_{u \in A_1} d(v)$. This proves (1).

(2) $1-t \geqslant \frac{2x(1-x)^2/3}{(1-x)^2-(1+2x-5x^2)y}$. *Consequently, $x < 1-3t/2$ and so $x < 1-t$.*

There are at least $2x|A| \cdot |B|/3$ edges between B_1 and A, since $z < 1/3$. But each vertex in B_1 has at most

$$\left(1 - \frac{(1+2x-5x^2)y}{(1-x)^2}\right)|A|$$

neighbours in A, by 8.2, and the first claim follows. To show that $x < 1 - 3t/2$, suppose not; then

$$2x/3 + 1/3 > 1 - t \geqslant \frac{2x(1-x)^2/3}{(1-x)^2 - (1 + 2x - 5x^2)y}$$

and so

$$(2x+1)((1-x)^2 - (1 + 2x - 5x^2)y) > 2x(1-x)^2,$$

that is,

$$(1-x)^2 > (2x+1)(1 + 2x - 5x^2)y$$

contrary to the hypothesis. This proves (2).

Let us choose $v_1, v_1' \in A_1$ uniformly and independently at random, and choose $v_2 \in A_2$ uniformly at random. Then for $u \in A$, the probability that all of v_1, v_1', v_2 are nonadjacent to u is

$$\frac{t - x + d(v)}{t} \left(\frac{1 - t - d(v)}{1-t} \right)^2.$$

Since $1 - y > 2/3$ and so v_1, v_1', v_2 have a common neighbour in C, say w', and $|N_A^2(w')| \leqslant (1-y)|A|$, it follows that

$$\sum_{u \in A} \left(1 - \frac{t - x + d(v)}{t} \left(\frac{1 - t - d(v)}{1-t} \right)^2 \right) \leqslant (1-y)|A|,$$

that is,

$$\sum_{u \in A} \frac{t + d(u) - x}{t} \left(\frac{t + d(u) - 1}{1-t} \right)^2 \geqslant y|A|.$$

This can be rewritten as:

$$\sum_{u \in A} f(d(v)) \geqslant t(1-t)^2(1-y)|A|,$$

where $f(r)$ is the polynomial $(r + t - x)(r + t - 1)^2$. We therefore need to investigate the maximum value of $\sum_{u \in A} f(d(v))$ (which we call "the objective function") over all choices of the numbers $d(u)(u \in A)$ satisfying the various constraints, and verify that this maximum is less than $t(1-t)^2(1-y)|A|$.

The derivative of $f(r)$ is zero when $3r^2 + 2(3t - x - 2)r + (3t - 2x - 1)(t - 1) = 0$, which has roots $r = 1 - t$ and $r = (2x+1)/3 - t$. Let us define $r_0 = (2x+1)/3 - t$. Since $r_0 < 1 - t$, the function $f(r)$ increases for $r < r_0$ and for $r > 1 - t$, and decreases for $r_0 < r < 1 - t$.

The second derivative of $f(r)$ is zero when $3r + 3t - x - 2 = 0$, that is, when $r = r_1$ where $r_1 = 2/3 + x/3 - t$. By (2), $x < r_1$, and we are only concerned $f(r)$ for r in with the range $0 \leqslant r \leqslant x$; so in particular all such r are less than r_1. The function $f(r)$ is concave through the range $0 \leqslant r \leqslant r_1$, since its second derivative is at most zero.

Let us choose real numbers $d(v)(v \in A)$ satisfying the constraints

- $0 \leqslant d(u) \leqslant x$ for each $u \in A$;
- $d(u) = 0$ for at least $y|A|$ vertices $u \in A$;
- $x(1-y)|A| \geqslant \sum_{u \in A_1} d(v) \geqslant 2x|A|/3$

to maximize the function $\sum_{u \in A} f(d(v))$. From the concavity of f, it follows that there exists r^* with $0 < r^* \leqslant x$ such that $d(v) \in \{0, r^*\}$ for all v (because if there were u, v with $d(u), d(v)$ distinct and nonzero, replacing them both by $(d(u)+d(v))/2$ would still satisfy the constraints and increase the objective function). Similarly, if there were more than $y|A|$ vertices v with $d(v) = 0$, then choose some one of them, v say, and choose some u with $d(u) > 0$; then again replacing them both by $(d(u) + d(v))/2$ would still satisfy the constraints and increase the objective function. We deduce that there are exactly $y|A|$ vertices v with $d(v) = 0$.

Now the problem breaks into three cases, depending which of the constraints $x(1 - y)|A| \geqslant \sum_{u \in A_1} d(v) \geqslant 2x|A|/3$ hold with equality.

Suppose first that neither holds with equality. Then from the optimality of the objective function, it follows that $r^* = r_0$, and since

$$x(1-y)|A| \geqslant \sum_{u \in A_1} d(v) \geqslant x(1-z)|A|,$$

it follows that

$$x(1-y) \geqslant (1-y)r_0 \geqslant x(1-z),$$

that is,

$$x \geqslant (2x+1)/3 - t \geqslant 2x/(3-3y).$$

Thus, if t satisfies

$$(1-x)/3 \leqslant t \leqslant (2x+1)/3 - 2x/(3-3y)$$

then there is a possible optimal solution where the objective function has value

$$y|A|f(0) + (1-y)|A|f(r_0).$$

Now $f(0) = (t-x)(t-1)^2$, and

$$f(r_0) = (r_0 + t - x)(r_0 + t - 1)^2 = ((1-x)/3)((2x-2)/3)^2 = 4(1-x)^3/27.$$

We must therefore check that for t in the given range,

$$y|A|(t-x)(1-t)^2 + 4(1-y)|A|(1-x)^3/27 < t(1-t)^2(1-y)|A|,$$

This simplifies to:

$$4(1-x)^3(1-y) < 27(1-t)^2(t - 2ty + xy).$$

Now the function $27(1-t)^2(t - 2ty + xy)$ has no local minimum at t with $t < 1$, and so is minimized at one of the ends of the range. Since $t \geqslant x$, we might as well replace the lower extreme of the range by $t \geqslant x$ (because it makes the arithmetic easier); so to check the lower extreme, we need to check that

$$4(1-x)^3(1-y) < 27x(1-x)^2(1-y),$$

that is, $4(1-x) < 27x$, which is true by hypothesis.

For the upper extreme, $t \leqslant (2x+1)/3 - 2x/(3-3y) < 1/3$; so it suffices to check that

$$4(1-x)^3(1-y) < 27(1-t)^2(t-2ty+xy)$$

when $t = 1/3$, that is, to check $(1-x)^3(1-y) < (1-2y+3xy)$. But $(1-x)^3 < 1/2$ by hypothesis, and $(1-2y+3xy)/(1-y) \geqslant (1-2y)/(1-y) \geqslant 1/2$ since $y < 1/3$. This finished the first of the three cases.

Now let us assume that $x(1-y)|A| = \sum_{u\in A_1} d(v)$. It follows from the optimality of the objective function that $r^* \leqslant r_0$. Moreover, since $x(1-y)|A| = \sum_{u\in A_1} d(v)$, it follows that $x(1-y)|A| = (1-y)|A|r^*$, so $r^* = x$. This is only possible if $x \leqslant r_0$, that is, $t < (1-x)/3$; and this is impossible since $t \geqslant x \geqslant 1/4$. This finishes the second case.

Finally, we assume that $\sum_{u\in A_1} d(v) = 2x|A|/3$. It follows from the optimality of the objective function that $r^* \geqslant r_0$. Moreover, since $\sum_{u\in A_1} d(v) = 2x|A|/3$, it follows that $(1-y)|A|r^* = 2x|A|/3$, that is, $r^* = 2x/(3-3y)$. We must check that

$$y(t-x)(t-1)^2 + (1-y)(r^*+t-x)(r^*+t-1)^2 < t(1-t)^2(1-y).$$

This is cubic in t, and, collecting the various powers of t, it becomes:

$$\begin{gathered} yt^3 + t^2(-xy-2y+(1-y)(r^*-x)+2(1-y)(r^*-1)+2(1-y)) \\ +t(y+2xy+(1-y)(r^*-1)^2+2(1-y)(r^*-x)(r^*-1)-(1-y)) \\ +(-xy+(1-y)(r^*-x)(r^*-1)^2) < 0. \end{gathered}$$

This simplifies to:

$$yt^3+(x-2y)t^2+t(y-2x/3+4x^2y/(3-3y))-xy+x(3y-1)(2x/3-1+y)^2/(3(1-y)^2) < 0.$$

The derivative of the left side with respect to t is

$$3yt^2 + 2(x-2y)t + y - 2x/3 + 4x^2y/(3-3y),$$

which can be rewritten as

$$3y(t+(x-2y)/(3y))^2 - (x-y)^2/(3y) + 4x^2y/(3-3y).$$

Since by hypothesis, $-(x-y)^2/(3y) + 4x^2y/(3-3y) \geqslant 0$, the derivative is nonnegative, at every value of t. Thus, we only need verify the inequality for the maximum value of t that lies in the range.

By (2), $t \leqslant 2(1-x)/3$; so it is enough to verify that

$$y(t-x)(t-1)^2 + (1-y)(r^*+t-x)(r^*+t-1)^2 < t(1-t)^2(1-y)$$

holds when $t = 2(1-x)/3$. Thus, we need to check that

$$y(2(1-x)/3)^3 + (x-2y)(2(1-x)/3)^2 + (2(1-x)/3)(y - 2x/3 + 4x^2y/(3-3y))$$
$$-xy + x(3y-1)(2x/3 - 1 + y)^2/(3(1-y)^2) < 0.$$

We checked on a computer that this is true for all x, y with $1/4 < x < 1/3$ and $0 < y \leqslant x$. This proves 8.3. ∎

The next result is used for 11.6 and 11.7, and in figure 7:

8.4. *Let $x, y \in (0, 1]$ with $x \leqslant \frac{1}{4}$ and $y < \frac{(1-2x)^2}{3-12x+16x^2}$; then $\phi(x, y) < \frac{1}{3}$.*

Proof. Since $\frac{(1-2x)^2}{3-12x+16x^2} \leqslant 1/3$ for all $x \geqslant 0$, it follows that $y \leqslant 1/3$. Let $x' = x/(1-x)$ and $y' = y/(1-y)$; then $x', y' \in (0, 1]$, and $x' < 1/3$ and $y < (1-x)^2/(2-4x+6x^2)$. By 6.7 it follows that $\phi(x', y') < 1/2$. Choose z with $\phi(x', y') < z/(1-z) < 1/2$; then $\phi(x, y) \leqslant z < 1/3$ by 3.1. This proves 8.4. ∎

9 The 3/4 level

In this section we investigate when $\psi(x, y) \geqslant 3/4$ and $\phi(x, y) \geqslant 3/4$. The results are shown in figure 8.

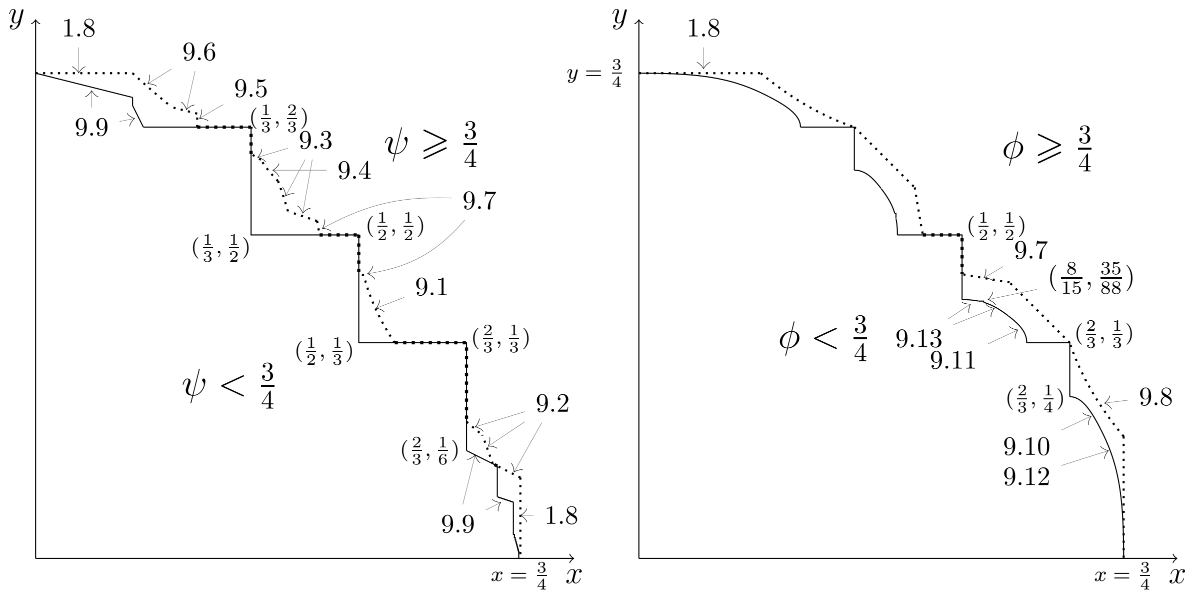

Figure 8: When $\psi(x, y) < 3/4$ and when $\phi(x, y) < 3/4$.

The next thirteen results all contribute to figure 8:

9.1. *Let $x, y \in (0,1]$, such that $y > 1/3$, $x \geqslant 1/2$, and $2y - 2y^2 > 1 - x$. Then $\psi(x,y) \geqslant 3/4$.*

Proof. Let G be a graph that is (x,y)-biconstrained via (A,B,C). We can assume that x, y are rational, and by multiplying vertices if necessary that $y|B| \in \mathbb{Z}$. Let $v_1 \in C$, and let $B_1 \subseteq N(v_1)$ be such that $|B_1| = y|B|$. Choose $v_2 \in C$ with at least $y|B \setminus B_1| = y(1-y)|B|$ neighbours in $B \setminus B_1$, and choose $B_2 \subseteq N(v_2)$ with $|B_2| = y|B|$. Choose $v_3 \in C$ with at least $y(1-2y)$ neighbours in $B \setminus (B_1 \cup B_2)$. Thus, $|N(v_1) \cup N(v_2)| \geqslant y + y(1-y) > 1-x$, and for $i = 1,2$, $|N(v_i) \cup N(v_3)| \geqslant y + y(1-2y) > 1-x$.

For $1 \leqslant i \leqslant 3$, let $A_i = N_A^2(v_i)$. Since $y > 1/3$, it follows that there exist i, j with $1 \leqslant i < j \leqslant 3$ such that $N(v_i) \cap N(v_j) \neq \emptyset$, and so $|A_i \cap A_j| \geqslant x|A| \geqslant |A|/2$. But $A_i \cup A_j = A$ since $|N(v_i) \cup N(v_j)| > 1-x$, and so $|A_i| + |A_j| \geqslant 3|A|/2$, and therefore one of $|A_i|, |A_j| \geqslant 3|A|/4$. This proves 9.1. ∎

9.2. *Let $x, y \in (0,1]$, such that $x > 2/3$, $x + 2y > 1$, and either $4(1-x)(1-y) \leqslant 1$ or $x > 1 - 2y + 2y^2$. Then $\psi(x,y) \geqslant 3/4$.*

Proof. If $x + y > 1$ the result follows from 4.1 with $k = 1$, so we may assume that $y \leqslant 1 - x < 1/3$. Let G be (x,y)-biconstrained via (A,B,C), and suppose that $|N_A^2(v)| < 3|A|/4$ for each $v \in C$. Let H be the graph with $V(H) = V(C)$, where distinct u, v are adjacent in H if and only if u and v have distance two in G (that is, in G they have a common neighbour in B).

(1) *For all $w_1, w_2, w_3 \in C$, if $N(w_1) \cap N(w_2) \neq \emptyset$ and $N(w_2) \cap N(w_3) \neq \emptyset$, then $N(w_1) \cap N(w_3) \neq \emptyset$. Consequently, each component of H is a complete graph.*

Suppose that $w_1, w_2, w_3 \in C$, and $N(w_1) \cap N(w_3) = \emptyset$, and $v_i \in N(w_i) \cap N(w_2)$ for $i = 1,3$. Let $A_i = N_A^2(w_i)$ for $i = 1,2,3$. Since $x + 2y > 1$ we have $A_1 \cup A_3 = A$. Consequently, $|A_1 \cap A_3| < |A|/2$, and since $N_A(v_1) \cap N_A(v_3) \subseteq A_1 \cap A_3$, it follows that $|N_A(v_1) \cap N_A(v_3)| < |A|/2$. Thus,

$$|N_A^2(w_2)| \geqslant |N_A(v_1) \cup N_A(v_3)| > (2x - 1/2)|A| \geqslant 3|A|/4,$$

a contradiction. This proves (1).

Let α be the size of the largest stable set in H, that is, the number of components of H. Let the vertex sets of the components of H be $C_1, \ldots, C_\alpha$, and for $1 \leqslant i \leqslant \alpha$ let B_i be the set of vertices in B with a neighbour in C_i. The sets $B_1, \ldots, B_\alpha$ have union B, and from the definition of H, they are pairwise disjoint. For $1 \leqslant i \leqslant \alpha$, let $w_i \in C_i$ and let $A_i = N_A^2(w_i)$.

(2) *For $1 \leqslant i < j \leqslant \alpha$, $A_i \cup A_j = A$, and so $|A_i \cap A_j| < |A|/2$. Consequently, $\alpha \leqslant 3$.*

Since w_i, w_j have no common neighbour in B, it follows that $|N(w_i) \cup N(w_j)| \geqslant 2y|B| >$

$(1-x)|B|$, and so $A_i \cup A_j = A$. Since $|A_i|, |A_j| < 3|A|/4$, it follows that $|A_i \cap A_j| < |A|/2$. This proves the first assertion. Suppose that $\alpha \geqslant 4$. By the first assertion, every vertex in A belongs to at least three of $A_1, \dots, A_4$. Consequently, some A_i has cardinality at least $3|A|/4$, a contradiction. This proves (2).

(3) $\alpha \neq 1$.

Suppose that $\alpha = 1$. Every vertex in $A \setminus A_1$ has at least $x|B|$ neighbours in $B \setminus N(w_1)$, so we may choose $v \in B$ with at least

$$x|A \setminus A_1|/(1-y) > x|A|/(4-4y) \geqslant |A|/12$$

neighbours in $A \setminus A_1$. Let $w \in C$ be a neighbour of v. Since w_1, w have a common neighbour, it follows that

$$|N_A^2(w)| > (x + 1/12)|A| \geqslant 3|A|/4,$$

a contradiction. This proves (3).

(4) *For* $1 \leqslant i \leqslant \alpha$, *if* $|B_i| > (1-x)|B|$ *then* $|C_i| > \frac{y}{3-4x}|C|$.

Suppose that $|B_i| > (1-x)|B|$ say, and let $|C_i| = c|C|$. Since $x > 2/3$, every vertex $u \in A \setminus A_i$ has a neighbour in B_i, and so $|N_C^2(u) \cap C_i| \geqslant y|C| = (y/c)|C_i|$. Hence, there exists $w \in C_i$ such that

$$|N_A^2(w) \cap (A \setminus A_i)| \geqslant \frac{y}{c}|A \setminus A_i| > \frac{y}{4c}|A|.$$

But w, w_i have a common neighbour, and so $|N_A^2(w) \cap A_i| \geqslant x|A|$, and therefore

$$\frac{3}{4}|A| > |N_A^2(w)| \geqslant (x + \frac{y}{4c})|A|.$$

Consequently, $3/4 > x + y/(4c)$, and so $c > y/(3-4x)$. This proves (4).

(5) $x > 1 - 2y + 2y^2$ *and* $\alpha = 2$.

Since $\alpha \leqslant 3$, we may assume without loss of generality that $|B_1| \geqslant |B|/3 > (1-x)|B|$. Since each $|C_i| \geqslant y|C|$, it follows that $|C_1| \leqslant (1-(\alpha-1)y)|C|$. By (4), $1-(\alpha-1)y > y/(3-4x)$, and since $\alpha \geqslant 2$, it follows that $1-y > y/(3-4x)$, that is, $4(1-x)(1-y) > 1$. From the hypothesis it follows that $x > 1-2y+2y^2$. This proves the first claim. Suppose that $\alpha > 2$; then

$$1 - 2y > \frac{y}{3-4x} > \frac{y}{3-4(1-2y+2y^2)}$$

which simplifies to $(1-y)(1-4y)^2 < 0$, a contradiction. This proves (5).

We may assume without loss of generality that $|C_1| \leqslant |C|/2$. By (4), $|B_1| \leqslant (1-x)|B| < |B|/2$, and so $|B_2| \geqslant |B|/2$.

Every vertex in $B_2 \setminus N(w_2)$ is adjacent to at least a fraction $y/(1-y)$ of the vertices of C_2, and hence there exists $w \in C_2$ with

$$|N(w) \setminus N(w_2)| \geqslant \frac{y}{1-y}|B_2 \setminus N(w_2)|.$$

Thus,

$$|N(w) \cup N(w_2)| \geqslant |N(w_2)| + \frac{y}{1-y}|B_2 \setminus N(w_2)| = \frac{y}{1-y}|B_2| + \frac{1-2y}{1-y}|N(w_2)|.$$

Since $|B_2| \geqslant x|B|$ and $|N(w_2)| \geqslant y|B|$, it follows that

$$|N(w) \cup N(w_2)| \geqslant \left(\frac{xy}{1-y} + \frac{y(1-2y)}{1-y}\right)|B| > (1-x)|B|,$$

(because $x > 1-2y+2y^2$ by (5)). Thus, $N_A^2(w) \cup N_A^2(w_2) = A$, and since w and w_2 have a common neighbour in B_2 it follows that

$$|N_A^2(w)| + |N_A^2(w_2)| \geqslant (x+1)|A| \geqslant 3|A|/2$$

and so one of $|N_A^2(w)|, |N_A^2(w_2)| \geqslant 3|A|/4$, a contradiction. This proves 9.2. ∎

9.3. *Let $x, y \in (0,1]$, with $x > 1/3$, $y > 1/2$, $x+3y > 2$, and either $x \geqslant (5-6y)/(11-12y)$ or $x \geqslant (3-4y)/(4-4y)$. Then $\psi(x,y) \geqslant 3/4$.*

Proof. Let G be (x,y)-biconstrained via (A,B,C), and suppose that $N_A^2(v) < 3|A|/4$ for each $v \in C$. By 4.1 with $k=1$ it follows that $x+y \leqslant 1$. Let H be the graph with $V(H) = V(C)$, in which distinct u, v are adjacent if and only if $|N(u) \cup N(v)| \leqslant (1-x)|B|$.

(1) *For all $u, v \in C$, if u, v are nonadjacent in H then $N_A^2(u) \cup N_A^2(v) = A$. If u, v are adjacent in H then $|N_A^2(u) \cap N_A^2(v)| > (x+1/4)|A|$.*

If u, v are nonadjacent in H, then $|N(u) \cup N(v)| > (1-x)|B|$, and so every vertex in A has a neighbour in $N(u) \cup N(v)$, that is, $N_A^2(u) \cup N_A^2(v) = A$. Now we assume that u, v are adjacent in H. Consequently,

$$|N(v_1) \cap N(v_2)| > 2y - (1-x) > 1-y$$

by hypothesis. Moreover, $x > 1/3$ and $y > 1/2$ imply that

$$|N(u) \cap N(v)| > 2y + x - 1 > 1/3 > 1 - x/(1-y).$$

Thus, 7.4 implies that $|N_A^2(u) \cap N_A^2(v)| > (x+1/4)|A|$. This proves (1).

If there exist $w_1, \dots, w_4 \in C$, pairwise nonadjacent in H, then by (1) each pair of the sets of the sets $N^2_A(w_i)$ $(1 \leqslant i \leqslant 4)$ has union A, and so each vertex in A belongs to at least three of the four sets; and so one of the four sets has cardinality at least $3|A|/4$, a contradiction. Thus, we may choose $w_1, w_2, w_3 \in C$ such that every other vertex in C is adjacent in H to at least one of w_1, w_2, w_3. Choose a partition $C = C_1 \cup C_2 \cup C_3$ such that for $1 \leqslant i \leqslant 3$, every vertex in C_i is equal or adjacent in H to w_i.

(2) $|C_i| \leqslant (1-y)|C|$ *for* $1 \leqslant i \leqslant 3$.

Suppose that $|C_1| > (1-y)|C|$. Define $B_1 = N(w)$ and $A_1 = N^2_A(w)$. Choose $v \in B \setminus N(w)$ with at least $x|A \setminus A_1|/(1-y) > x|A|/(4-4y)$ neighbours in $A \setminus A_1$. Since $|C_1| > (1-y)|C|$, there exists $w \in C_1$ adjacent to v. Then

$$|N^2_A(w)| > x|A|/(4-4y) + (x+1/4)|A| \geqslant 3|A|/4$$

since $x > 1/3$ and $y > 1/2$, a contradiction. This proves (2).

(3) *Every vertex in B has neighbours in exactly two of* C_1, C_2, C_3.

Since each $|C_i| \leqslant (1-y)|C| < y|C|$ by (2), it follows that every vertex in B has neighbours in at least two of C_1, C_2, C_3. Suppose that $v \in B$ has a neighbour $w'_i \in C_i$ for $i = 1, 2, 3$. Let $A_i = N^2_A(w'_i)$ for $i = 1, 2, 3$. For $1 \leqslant i < j \leqslant 3$, $A_i \cup A_j = A$ by (1), and so every vertex of A belongs to at least two of A_1, A_2, A_3, and $N_A(u)$ is a subset of all three of A_1, A_2, A_3. Consequently,

$$|A_1| + |A_2| + |A_3| \geqslant 2|A| + |N_A(w)| \geqslant (2+x)|A| \geqslant 9|A|/4$$

and so some $|A_i| \geqslant 3|A|/4$, a contradiction. This proves (3).

From (3) we may partition B into B_1, B_2, B_3 such that every vertex in B_1 has neighbours in C_2 and in C_3 but not in C_1, and similarly for B_2, B_3. Without loss of generality, we may assume that $|B_1| \leqslant 1/3$. Let $A_1 = N^2_A(w_1)$.

(4) $|N^2_A(w) \setminus A_1| < (2-4x)|A \setminus A_1|$ *for each* $w \in C_1$.

By (1) $|N^2_A(w_1) \cap N^2_A(w)| \geqslant (x+1/4)|A|$, and since $|N^2_A(w)| < 3|A|/4$, it follows that

$$|N^2_A(w) \setminus A_1| < (1/2 - x)|A| \leqslant (2-4x)|A \setminus A_1|.$$

This proves (4).

This has two consequences. The first is that $x < (3-4y)/(4-4y)$. To see this, by (4) we may choose $u \in A \setminus A_1$ such that $|N^2_C(u) \cap C_1| < (2-4x)|C_1|$. Since $|B_1| \leqslant |B|/3$, u has a neighbour $v \in B_2 \cup B_3$, and we may assume that $v \in B_2$ from the symmetry. So at least $y|C|$ neighbours of v belong to $C_1 \cup C_3$, and therefore at least $(y-(1-y))|C|$ neighbours belong to C_1, since $|C_2| \leqslant (1-y)|C|$. So $(2y-1)|C| < (2-4x)|C_1| \leqslant (2-4x)(1-y)|C|$, and hence $x < (3-4y)/(4-4y)$ as claimed.

The second consequence is that $x < (5-6y)/(11-12y)$. To see this, let $S = B \setminus (B_1 \cup N(w_1))$. By (4) and since each vertex in $B_2 \cup B_3$ has a neighbour in C_1, it follows that each vertex $v \in B_2 \cup B_3$ has fewer than $(2-4x)|A \setminus A_1|$ neighbours in $A \setminus A_1$. Since $S \subseteq B_2 \cup B_3$, it follows that some vertex $u \in A \setminus A_1$ has fewer than $(2-4x)|S|$ neighbours in S. But u has no neighbours in $N(w_1)$, and only at most $r|B|$ neighbours in B_1; and since it has at least $x|B|$ neighbours in total, we deduce that

$$x|B| < (2-4x)|S| + r|B| \leqslant (2-4x)(1-r-y)|B| + r|B|$$

(since $B_1 \cap N(w) = \emptyset$ and $|N(w_1)| \geqslant y|B|$ and therefore $|S| \leqslant (1-r-y)|B|$). Consequently,

$$x < (2-4x)(1-r-y) + r = (2-4x)(1-y) + r(4x-1) \leqslant (2-4x)(1-y) + (4x-1)/3$$

and so $x < (5-6y)/(11-12y)$.

We have shown then that $x < (3-4y)/(4-4y)$ and $x < (5-6y)/(11-12y)$; but this contradicts the hypothesis. This proves 9.3. ∎

9.4. *Let $x, y \in (0, 1]$ with $1/2 < y < 2/3$ and $x > 6y^2 - 8y + 3$. Then $\psi(x, y) \geqslant 3/4$.*

Proof. We may assume that $y \in \mathbb{Q}$, by decreasing y if necessary. Let G be (x, y)-biconstrained via (A, B, C), and suppose that $|N^2_A(w)| < 3|A|/4$ for each $w \in C$. By multiplying vertices if necessary, we may assume that $y|B| \in \mathbb{Z}$. Since $x > 6y^2 - 8y + 3 = 6(y-2/3)^2 + 1/3$, it follows that $x > 1/3$. Let H be the graph with $V(H) = C$ in which distinct $u, v \in C$ are adjacent if and only if $|N(u) \cup N(v)| \leqslant (1-x)|B|$. It follows that if u, v are nonadjacent in H, then $N^2_A(u) \cup N^2_A(v) = A$. As in the proof of 9.3, there do not $w_1, \ldots, w_4 \in C$, pairwise nonadjacent in H; and so we may choose $w_1, w_2, w_3 \in C$ and a partition $C = C_1 \cup C_2 \cup C_3$ such that for $1 \leqslant i \leqslant 3$, every vertex in C_i is equal or adjacent in H to w_i. Let $|C_i| = c_i|C|$, and choose $B_i \subseteq N_(w_i)$ with $|B_i| = y|B|$ for $1 \leqslant i \leqslant 3$. Let F be the set of all edges vw of G with $v \in B$ and $w \in C$, such that for $1 \leqslant i \leqslant 3$, not both $v \in B_i$ and $w \in C_i$.

(1) $\frac{|F|}{|B||C|} \leqslant (1-x-y) < (2-3y)(2y-1)$.

Let $w \in C$, with $w \in C_i$ say; then since w, w_i are adjacent in H, it follows that w has at most $(1-x-y)|B|$ neighbours in $B \setminus B_i$. Thus, $|F| \leqslant (1-x-y)|B| \cdot |C|$. But $1-x-y < (2-3y)(2y-1)$ since $x > 6y^2 - 8y + 3$. This proves (1).

Let $p_1 = |B_1 \setminus (B_2 \cup B_3)|/|B|$, and define p_2, p_3 similarly. Let $q_1 = |(B_2 \cup B_3) \setminus B_1|/|B|$, and define q_2, q_3 similarly. Let $p_0 = |B \setminus (B_1 \cup B_2 \cup B_3)|/|B|$, and $q_0 = |B_1 \cup B_2 \cup B_3|/|B|$. Let $q = q_1 + q_2 + q_3$. Thus,

$$\begin{aligned} p_0 + p_1 + p_2 + p_3 + q_0 + q_1 + q_2 + q_3 &= 1 \\ p_1 + q_0 + q_2 + q_3 &= y \\ p_2 + q_0 + q_3 + q_1 &= y \\ p_3 + q_0 + q_1 + q_2 &= y. \end{aligned}$$

By subtracting the last three of these from the first, we obtain

$$p_0 - 2q_0 - (q_1 + q_2 + q_3) = 1 - 3y,$$

and so $p_0 = 2q_0 + q - 3y + 1$.

Every vertex in $B \setminus (B_1 \cup B_2 \cup B_3)$ is incident with at least $y|C|$ edges in F, every vertex in $B_1 \setminus (B_2 \cup B_3)$ is incident with at least $(y - c_1)|C|$ edges in F, and every vertex in $(B_2 \cup B_3) \setminus B_1$ is incident with at least $\max(y - c_2 - c_3, 0) = \max(y + c_1 - 1, 0)$ edges in F (and similar statements hold for c_2, c_3). Summing, we deduce that

$$\frac{|F|}{|B||C|} \geqslant p_0 y + \sum_{1 \leqslant i \leqslant 3} (p_i(y - c_i) + q_i \max(y + c_i - 1, 0)).$$

Since $p_i = y - q_0 - q + q_i$ for $i = 1, 2, 3$, and $c_1 + c_2 + c_3 = 1$, it follows that

$$\sum_{1 \leqslant i \leqslant 3} p_i(y - c_i) = \sum_{1 \leqslant i \leqslant 3} (y - q_0 - q + q_i)(y - c_i) = (y - q_0 - q)(3y - 1) + qy - \sum_{1 \leqslant i \leqslant 3} q_i c_i.$$

Also, $p_0 = 2q_0 + q - 3y + 1$, and so

$$\frac{|F|}{|B||C|} \geqslant (2q_0 + q - 3y + 1)y + (y - q_0 - q)(3y - 1) + qy - \sum_{1 \leqslant i \leqslant 3} q_i c_i + \sum_{1 \leqslant i \leqslant 3} q_i \max(y + c_1 - 1, 0).$$

This simplifies to

$$\frac{|F|}{|B||C|} \geqslant (1 - y)q_0 + \sum_{i \in I} q_i(1 - y - c_i)$$

where I is the set of $i \in \{1, 2, 3\}$ such that $c_i < 1 - y$. From (1) we deduce that

$$(1 - y)q_0 + \sum_{i \in I} q_i(1 - y - c_i) < (2 - 3y)(2y - 1).$$

In particular it follows that $(1 - y)q_0 < (2 - 3y)(2y - 1) \leqslant (1 - y)(2y - 1)$, and so $q_0 < 2y - 1$. Moreover, since $|B_2 \cup B_3| \leqslant |B|$, it follows that $|B_2 \cap B_3| \geqslant (2y - 1)|B|$, and so $q_1 \geqslant 2y - 1 - q_0$, and the same holds for q_2, q_3. Consequently,

$$(1 - y)q_0 + \sum_{i \in I} (2y - 1 - q_0)(1 - y - c_i) < (2 - 3y)(2y - 1),$$

and so

$$(1 - y)q_0 + \sum_{1 \leqslant i \leqslant 3} (2y - 1 - q_0)(1 - y - c_i) < (2 - 3y)(2y - 1),$$

since $2y - 1 - q_0 > 0$. This simplifies to $(2y - 1)q_0 < 0$, a contradiction. This proves 9.4. ∎

9.5. *Let $x, y \in (0,1]$ with $x \geqslant 1/4$ and $y > 2/3$. Then $\psi(x,y) \geqslant 3/4$.*

Proof. Suppose that G is (x,y)-biconstrained via (A,B,C), and $|N_A^2(w)| < 3|A|/4$ for each $w \in C$. Since $x \geqslant 1/4$ and $y > 2/3$, it follows that $x > 2(1-y)^2$, that is, $y > (1-y) + 1 - x/(1-y)$; and also $y > 2 - 2y$. Let $w \in C$. By 7.4 with $k = 2$ and $B' = N(w)$, it follows that

$$|N_A^2(v)| > (x + 1/2)|A| \geqslant 3|A|/4,$$

which is a contradiction. This proves 9.5. ∎

9.6. *Let $x, y \in (0,1]$ with $y > 2/3$, $x + 4y > 3$ and $x > 3(1-y)^2/(2-y)$. Then $\psi(x,y) \geqslant 3/4$.*

Proof. Suppose that G is (x,y)-biconstrained via (A,B,C), and $|N_A^2(w)| < 3|A|/4$ for each $w \in C$. Consequently, $y \leqslant 3/4$, and so $x \geqslant 3/20$ since $x > 3(1-y)^2/(2-y)$. From the hypotheses it follows that

$$2y + x - 1 > 1 - y + \max(1-y, 1 - x/(1-y)).$$

If $w_1, w_2 \in C$ with $|N(w_1) \cup N(w_2)| \leqslant (1-x)|B|$ then $|N(w_1) \cap N(w_2)| \geqslant (2y+x-1)|B|$. Thus, 7.4 applied with $k = 2$ tells us that, for all such $w_1, w_2 \in C$, more than $(x+1/2)|A|$ vertices in A have a neighbour in $N(w_1) \cap N(w_2)$. Let H be the graph with vertex set C, in which w_1, w_2 are adjacent if $|N(w_1) \cup N(w_2)| \leqslant (1-x)|B|$. As in the proof of 9.3, there is no stable set of size at least four in H. It follows that there exist $w_1, w_2, w_3 \in C$ and a partition $C = C_1 \cup C_2 \cup C_3$ such that for $1 \leqslant i \leqslant 3$, every vertex in C_i is equal to or adjacent in H to w_i. We assume without loss of generality that $|C_1| \geqslant 1/3$. Since $y > 2/3$, every vertex in B has a neighbour in C_1. Let $B_1 = N(w_1)$ and $A_1 = N_A^2(w_1)$, and choose $v \in B \setminus B_1$ with more than $x|A|/(4-4y)$ neighbours in $A \setminus A_1$. Since $y > 2/3$, there exists $w \in C_1$ adjacent to v. Then

$$|N_A^2(w)| > (x + 1/2 + x/(4-4y))|A| \geqslant 3|A|/4$$

since $x \geqslant 3/20 \geqslant 1/7$ and $y \geqslant 2/3$, a contradiction. This proves 9.6. ∎

9.7. *Let $x, y \in (0,1]$ with $y > 1/2$ and $x^2y \geqslant (3/4 - x)^2$. Then $\phi(x,y) \geqslant 3/4$.*

Proof. Apply 5.7 with $z = 3/4$. ∎

9.8. *Let $x, y \in (0,1]$ with $y < 1/3$ and $x > \frac{1-y}{1+y-3y^2}$. Then $\phi(x,y) \geqslant 3/4$.*

Proof. Suppose that G is (x,y)-constrained via (A,B,C), and $|N_A^2(w)| < 3|A|/4$ for each $w \in C$. Consequently, $x + y \leqslant 1$. By reducing x or y if necessary, we may assume that every vertex in A has strictly more than $x|B|$ neighbours in B, and that x, y are rational. Let

$$p = \frac{1-x-y}{1-3y}.$$

By multiplying vertices, we may also assume that $y|B|$ and $p|B|$ are integers. Note that the hypotheses imply that $p < xy$.

(1) *There exists $s \in [0,1]$ such that for all b, c, if $0 \leqslant a \leqslant y$ and $sa + b \geqslant y(sy + 1 - y)$, then $a + b \geqslant p$ and $b \geqslant 1 - x - y$.*

We claim first that

$$\max\left(0, \frac{p - y(1-y)}{y^2}\right) \leqslant \min\left(\frac{2y - y^2 + x - 1}{y - y^2}, 1\right).$$

To see this, we need to check that $0 \leqslant \frac{2y-y^2+x-1}{y-y^2}$, and $\frac{p-y(1-y)}{y^2} \leqslant 1$, and $\frac{p-y(1-y)}{y^2} \leqslant \frac{2y-y^2+x-1}{y-y^2}$. The first is true since

$$\frac{x}{1-y} > \frac{1}{1+y-3y^2} = 1 - y + \frac{4y^2 - 3y^3}{1+y-3y^2} \geqslant 1 - y.$$

The second is true since $p \leqslant xy \leqslant y$. The third simplifies to $p/y \leqslant x/(1-y)$, and this is true since $p \leqslant xy$. This proves the claim, and so there exists s such that

$$\max\left(0, \frac{p - y(1-y)}{y^2}\right) \leqslant s \leqslant \min\left(\frac{2y - y^2 + x - 1}{y - y^2}, 1\right).$$

We will show that s satisfies (1). Suppose that $0 \leqslant a \leqslant y$ and $sa + b \geqslant y(sy + 1 - y)$. Then

$$a + b \geqslant sa + b \geqslant y(sy + 1 - y) \geqslant p$$

and

$$sy + b \geqslant sa + b \geqslant y(sy + 1 - y) \geqslant sy + 1 - x - y$$

(and therefore $b \geqslant 1 - x - y$). This proves (1).

(2) *There exists $t \in [0,1]$ such that for all a, b, if $0 \leqslant a \leqslant 2p$ and $ta + b \geqslant y(1 - 2p(1-t))$ then $a + b \geqslant p$ and $p + b \geqslant 1 - x$.*

We claim first that

$$\max\left(0, \frac{2py + p - y}{2py}\right) \leqslant \min\left(\frac{x + y + p - 2py - 1}{2p(1-y)}, 1\right).$$

To see this we must check that $0 \leqslant \frac{x+y+p-2py-1}{2p(1-y)}$, and $\frac{2py+p-y}{2py} \leqslant 1$, and $\frac{2py+p-y}{2py} \leqslant \frac{x+y+p-2py-1}{2p(1-y)}$. The first is true since

$$p - 2py = \frac{(1-x-y)(1-2y)}{1-3y} \geqslant 1 - x - y.$$

The second is true since $p \leqslant xy \leqslant y$; and the third simplifies to $p \leqslant xy$ and so is true. This proves the claim, and so there exists t with

$$\max\left(0, \frac{2py + p - y}{2py}\right) \leqslant t \leqslant \min\left(\frac{x + y + p - 2py - 1}{2p(1-y)}, 0\right).$$

We will show that t satisfies (2). Let a, b satisfy $0 \leqslant a \leqslant 2y$ and $ta + b \geqslant y(1 - 2p(1-t))$. Then

$$a + b \geqslant ta + b \geqslant y(1 - 2p(1-t)) \geqslant p$$

and

$$2tp + b \geqslant ta + b \geqslant y(1 - 2p(1-t)) \geqslant 2tp + 1 - x - p$$

(and so $b \geqslant 1 - x - p$). This proves (2).

Choose $w_1 \in C$ with at least $y|B|$ neighbours in B.

(3) *There exists $w_2 \in C$ such that $|N(w_2)| \geqslant p|B|$ and $|N(w_1) \cup N(w_2)| \geqslant (1-x)|B|$.*

Choose $B_1 \subseteq N(w_1)$ with $|B_1| = y|B|$. Choose s as in (1). Then

$$\begin{aligned}\sum_{w \in C} (s|N(w) \cap B_1| + |N(w) \setminus B_1|) &= \sum_{v \in B_1} s|N(v) \cap C| + \sum_{v \in B \setminus B_1} |N(v) \cap C| \\ &\geqslant (sy^2 + y(1-y))|B| \cdot |C|.\end{aligned}$$

Consequently, we may choose $w_2 \in C$ such that

$$s\frac{|N(w_2) \cap B_1|}{|B|} + \frac{|N(w_2) \setminus B_1|}{|B|} \geqslant y(sy + (1-y)).$$

Since

$$0 \leqslant \frac{|N(w_2) \cap B_1|}{|B|} \leqslant \frac{|B_1|}{|B|} = y,$$

the choice of s implies that $|N(w_2)| \geqslant p|B|$ and $|N(w_2) \setminus B_1| \geqslant (1 - x - y)|B|$, and so $|N(w_1) \cup N(w_2)| \geqslant (1-x)|B|$. This proves (3).

(4) *There exists $w_3 \in C$ such that $|N(w_3)| \geqslant p|B|$, and $|N(w_i) \cup N(w_3)| \geqslant (1-x)|B|$ for $i = 1, 2$.*

Since $|N(w_1)|, |N(w_2)| \geqslant p|B|$ and $p|B|$ is an integer, we may choose $R \subseteq B$ with $|R| = 2p|B|$ such that $|N(w_1) \cap R|, |N(w_2) \cap R| \geqslant p|B|$. Choose t as in (2). As in the proof of (3), there exists $w_3 \in C$ with

$$t\frac{|N(w_3) \cap R|}{|B|} + \frac{|N(w_3) \setminus R|}{|B|} \geqslant y(2pt + (1-2p)) = y(1 - 2y(1-t)).$$

From the choice of t, it follows that $|N(w_3)|/|B| \geqslant p$, and $|N(w_3) \setminus R| \geqslant 1 - x - p$, and consequently $|N(w_1) \cup N(w_3)|, |N(w_2) \cup N(w_3)| \geqslant (1 - x)|B|$. This proves (4).

For $1 \leqslant i \leqslant 3$, choose $B_i \subseteq N(w_i)$ with $|B_i| = p|B|$. Since $|B_1 \cup B_2 \cup B_3| \leqslant 3p$, we may choose $w_4 \in C$ with at least $y(1 - 3p)|B|$ neighbours in $B \setminus (B_1 \cup B_2 \cup B_3)$. Then for all $1 \leqslant i \leqslant 3$ we have:

$$|X_i \cup N(w_4)| \geqslant (p + y(1 - 3p))|B| \geqslant (1 - x)|B|$$

by the definition of p. It follows that for $1 \leqslant i < j \leqslant 4$, $|N(w_i) \cup N(w_j)| \geqslant (1-x)|B|$, and so (since every vertex in A has strictly more than $x|B|$ neighbours in B) it follows that $N^2_A(w_i) \cup N^2_A(w_j) = A$. Thus, every vertex in A belongs to at least three of the four sets $N^2_A(w_i)$ $(1 \leqslant i \leqslant 4)$, and so one of them has cardinality at least $3|A|/4$, a contradiction. This proves 9.8. ▮

9.9. *Let $x, y \in (0, 1]$. Then $\psi(x, y) < 3/4$ if either:*

- *$x \leqslant 1/6$ and $y \leqslant 5/7$ and $2x + y \leqslant 1$; or*
- *$x \leqslant 3/20$ and $x + 4y \leqslant 3$, and $x + 4y < 3$ if x is irrational; or*
- *$x \leqslant 17/23$ and $y \leqslant 1/8$ and $x + 3y \leqslant 1$; or*
- *$x \leqslant 5/7$ and $y \leqslant 1/6$ and $x + 2y \leqslant 1$.*

Proof. If x', y' with $2x' + y' \leqslant 1$ and $y' \leqslant 3/5$, then $\psi(x', y') < 2/3$ by the first bullet of 7.8. Given x, y as in the first bullet, the hypotheses imply that there is a choice of x', y' with $2x' + y' \leqslant 1$ and $y' \leqslant 3/5$, and which also satisfy the hypotheses of 3.5 with $z' = \psi(x', y')$ and z slightly less than $3/4$ (checking this needs some lengthy calculation, which we omit); and so the first statement follows from 3.5. The second statement follows similarly from 3.5 and the second bullet of 7.8. The third statement follows from 3.4 and the first bullet of 7.6; and the fourth follows by applying 3.4 with $z = \max(2/7, x)$, taking $x' = 3/5$, $y' = 1/5$ and $z' = 3/5$. This proves 9.9. ▮

The next two results are both obtained by applying 3.7 to 7.10.

9.10. *If $x, y \in (0, 1]$, with $x < 5/7$ and $y \leqslant \frac{(1-x)^2}{40x^2-56x+20}$, then $\phi(x, y) < 3/4$.*

Proof. Since $\frac{(1-x)^2}{40x^2-56x+20} \leqslant 1/4$ for all $x \geqslant 0$, it follows that $y \leqslant 1/4$, and so we may assume that $x > 1/2$, since $\phi(1/2, 1/2) = 1/2$. Let $x' = 2 - 1/x$ and $y' = y/(1 - y)$. Thus, $x', y' \in (0, 1]$, and $x' < 3/5$ since $x < 5/7$, and $y' \leqslant \frac{x'^2-2x'+1}{19x'^2-22x'+7}$ since $y \leqslant \frac{(1-x)^2}{40x^2-56x+20}$. By 7.10, it follows that $\phi(x', y') < 2/3$. By 3.7 (taking z with $\phi(x', y') < 2 - 1/z < 2/3$), it follows that $\phi(x, y) < 3/4$. This proves 9.10. ▮

9.11. *If $x, y \in (0,1]$, with $y < 3/8$ and $x \leqslant \frac{48y^2-36y+7}{92y^2-68y+13}$, then $\phi(x,y) < 3/4$.*

Proof. We may assume that $x > 1/2$, since $\phi(1/2, 1/2) = 1/2$. Let $x' = 2 - 1/x$ and $y' = y/(1-y)$. Thus, $x', y' \in (0,1]$, and $y' < 3/5$, and $x' \leqslant \frac{y'^2-2y'+1}{19y'^2-22y'+7}$ since $x \leqslant \frac{48y^2-36y+7}{92y^2-68y+13}$. By 7.10 and 2.3, it follows that $\phi(x', y') = \phi(y', x') < 2/3$, and so by 3.7 (taking z with $\phi(x', y') < 2 - 1/z < 2/3$), it follows that $\phi(x,y) < 3/4$. This proves 9.11. ∎

The next two results are proved similarly, using 7.11 instead of 7.10.

9.12. *If $x, y \in (0,1]$, with $y < 1/5$ and $x \leqslant \frac{35y^2-18y+3}{48y^2-24y+4}$, then $\phi(x,y) < 3/4$.*

Proof. We may assume that $x > 1/2$. Let $x' = 2 - 1/x$ and $y' = y/(1-y)$. Thus, $x', y' \in (0,1]$, and $y' < 1/4$, and $x' \leqslant \frac{12y'^2-8y'+2}{20y'^2-12y'+3}$ since $x \leqslant \frac{35y^2-18y+3}{48y^2-24y+4}$. By 7.11, it follows that $\phi(x', y') < 2/3$. By 3.7 (taking z with $\phi(x', y') < 2 - 1/z < 2/3$), it follows that $\phi(x,y) < 3/4$. This proves 9.12. ∎

9.13. *If $x, y \in (0,1]$, with $x < 4/7$ and $y \leqslant \frac{34x^2-40x+12}{93x^2-108x+32}$, then $\phi(x,y) < 3/4$.*

Proof. Since $34x^2 - 40x + 1293x^2 - 108x + 32 \leqslant 2/5$ for all $x \geqslant 0$. it follows that $y \leqslant 2/5$, and so we may assume that $x > 1/2$. Let $x' = 2 - 1/x$ and $y' = y/(1-y)$. Thus, $x', y' \in (0,1]$, and $x' < 1/4$, and $y' \leqslant \frac{12x'^2-8x'+2}{20x'^2-12x'+3}$ since $y \leqslant \frac{34x^2-40x+12}{93x^2-108x+32}$. By 7.11 and 2.3, it follows that $\phi(x', y') = \phi(y', x') < 2/3$, and so by 3.7 (taking z with $\phi(x', y') < 2 - 1/z < 2/3$), it follows that $\phi(x,y) < 3/4$. This proves 9.13. ∎

10 The 2/5 level

Next, we analyze when $\psi, \phi \geqslant 2/5$. The results are shown in figure 9.

The seven results in this section are all motivated as contributions to figure 9.

10.1. *Let $x, y \in (0,1]$ with $x \geqslant 1/5$, $y > 1/3$, and $3y - 2y^2 > 1 - x$; then $\psi(x,y) \geqslant 2/5$.*

Proof. Suppose that G is (x,y)-biconstrained via (A,B,C), and $|N^2_A(w)| < (2/5)|A|$ for each $w \in C$. Let $w_1 \in C$, and let $A_1 = N^2_A(w_1)$. By averaging, there exists $w_2 \in C$ such that

$$|A_2 \setminus A_1| \geqslant y|A \setminus A_1| > (3y/5)|A|.$$

where $A_2 = N^2_A(w_2)$. Since $|A_2| < 2|A|/5$, it follows that

$$|A_2 \cap A_1| < (2/5 - 3y/5)|A| < x|A|,$$

and so $N(w_1) \cap N(w_2) = \emptyset$. Let $B' = N(w_1) \cup N(w_2)$; thus $|B'| \geqslant 2y|B|$. By averaging, there exists $w_3 \in C$ such that $|N(w_3) \setminus B'| \geqslant y|B \setminus B'|$, and so

$$\begin{aligned}|N(w_1) \cup N(w_2) \cup N(w_3)| \geqslant& |B'| + y|B \setminus B'| = y|B| + (1-y)|B'| \\ \geqslant& (y + (1-y)(2y))|B| > (1-x)|B|.\end{aligned}$$

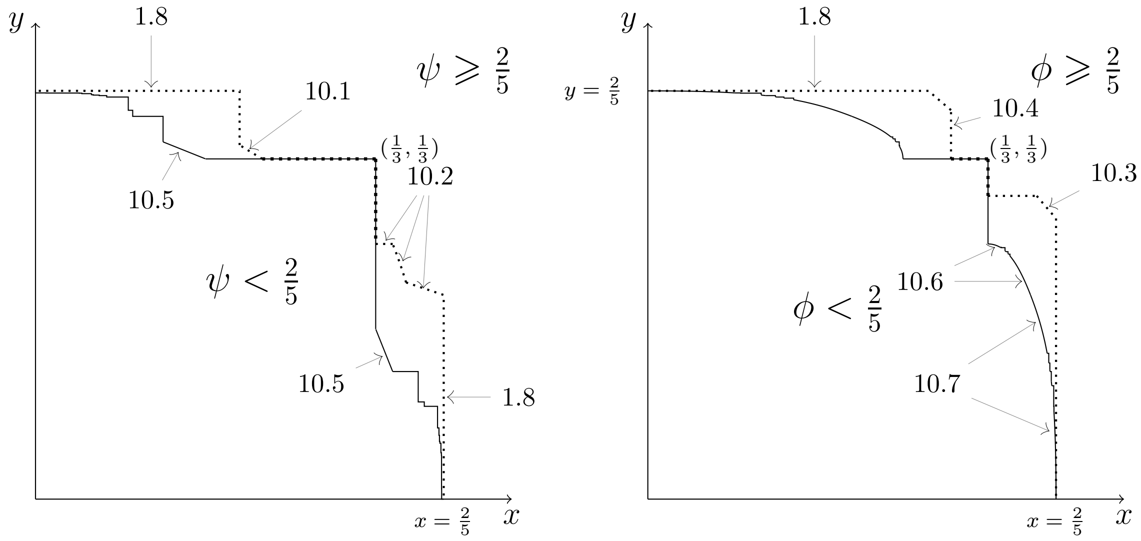

Figure 9: When $\psi(x, y) < 2/5$ and when $\phi(x, y) < 2/5$.

Hence, setting $A_3 = N_A^2(w_3)$, it follows that $A_1 \cup A_2 \cup A_3 = A$.

Since $y > 1/3$, some pair of $N(w_1), N(w_2), N(w_3)$ have nonempty intersection, and so some pair of A_1, A_2, A_3 have intersection of cardinality at least $x|A| \geqslant |A|/5$. But then

$$|A_1| + |A_2| + |A_3| \geqslant |A_1 \cup A_2 \cup A_3| + |A|/5 = (6/5)|A|,$$

which is impossible since $|A_i| < (2/5)|A|$ for $1 \leqslant i \leqslant 3$. This proves 10.1. ∎

10.2. *Let $x, y \in (0, 1]$ with $x > 1/3$, $x+3y > 1$, and either $y \geqslant 1/4$ or $x+y/(10(1-2y)) \geqslant 2/5$; then $\psi(x, y) \geqslant 2/5$.*

Proof. Suppose that G is (x, y)-biconstrained via (A, B, C), and $|N_A^2(w)| < (2/5)|A|$ for each $w \in C$. Choose $w_1, \dots, w_\alpha \in C$ with α maximum such that $N(w_1), \dots, N(w_\alpha)$ are pairwise disjoint.

(1) $\alpha = 3$.

Suppose that $\alpha \leqslant 2$. Let A' be the union of the sets $N_A^2(w_i)$ for $1 \leqslant i \leqslant \alpha - 1$, and let B' be the union of the sets $N(w_i)$ for $1 \leqslant i \leqslant \alpha - 1$. So $|A'| < (2\alpha/5)|A| \leqslant (4/5)|A|$, and $|B'| \geqslant y|B|$. By averaging, there exists $v \in B \setminus B'$ such that

$$|N(v) \cap (A \setminus A')| \geqslant (x/(1-y))|A \setminus A'| \geqslant (3x/(5(1-y)))|A|;$$

let $w \in C$ be adjacent to v. Since $N(w)$ has nonempty intersection with $N(w_i)$ for some $i < \alpha$, it follows that $|N_A^2(w) \cap A'| \geqslant x|A|$. Adding, we deduce that

$$|N_A^2(w)| \geqslant (3x/(5(1-y)))|A| + x|A| \geqslant (2/5)|A|,$$

a contradiction.

Thus, $\alpha \geqslant 3$; suppose that $\alpha \geqslant 4$. Since $x + 3y > 1$ it follows that every vertex in A belongs to at least two of the sets $N_A^2(w_i)$ $(1 \leqslant i \leqslant 4)$, and so one of these four sets has cardinality at least $|A|/2 \geqslant (2/5)|A|$, a contradiction. This proves (1).

(2) *If $w \in C$ then $N(w) \cap N(w_i)$ is nonempty for exactly one value of $i \in \{1, 2, 3\}$.*

Since $\alpha = 3$, it follows that $N(w) \cap N(w_i)$ is nonempty for at least one such i; suppose that $N(w)$ has nonempty intersection with both $N(w_1), N(w_2)$ say. Let $A_i = N_A^2(w_i)$ for $i = 1, 2, 3$, and $A_0 = N_A^2(w)$. Since $A_1 \cup A_2 \cup A_3 = A$, and each $|A_i| < (2/5)|A|$, there are fewer than $|A|/5$ vertices in A that belong to more than one of A_1, A_2, A_3, and in particular $|A_1 \cap A_2| < |A|/5$. But $|A_0 \cap A_i| \geqslant x|A|$ for $i = 1, 2$, and so $|A_0| \geqslant (2x-1/5)|A| \geqslant (2/5)|A|$, a contradiction. This proves (2).

From (2), we can partition $B = B_1 \cup B_2 \cup B_3$, and partition $C = C_1 \cup C_2 \cup C_3$, such that all six of these sets are nonempty, and for all distinct $i, j \in \{1, 2, 3\}$ there is no edge between B_i and C_j, and for all $i \in \{1, 2, 3\}$ and all $w, w' \in C_i$, $N(w) \cap N(w') \neq \emptyset$. Let $|B_i| = b_i|B|$ and $C_i| = c_i|C|$ for $i = 1, 2, 3$. Without loss of generality we may assume that $b_3 \leqslant 1/3 < x$, and so every vertex in A has a neighbour in $B_1 \cup B_2$.

(3) $x + y/(10(1 - 2y)) < 2/5$ *and so* $y \geqslant 1/4$.

Suppose that $x + y/(10(1 - 2y)) \geqslant 2/5$. Without loss of generality, we may assume that at least $|A|/2$ vertices in A have a neighbour in B_1. Choose $w \in C_1$. Since $|N_A^2(w)| < (2/5)|A|$, there are at least $|A|/10$ vertices $u \in A \setminus N_A^2(w)$ that have a neighbour in B_1. For each such u, $|N_C^2(u) \cap C_1| \geqslant y|C|$, and since $|C_1| \leqslant (1 - 2y)|C|$ (because $|C_2|, |C_3| \geqslant y|C|$), it follows that $|N_C^2(u) \cap C_1| \geqslant (y/(1 - 2y))|C_1|$. Consequently, there exists $w' \in C_1$ such that $N_A^2(w')$ contains at least $(y/(10(1-2y)))|A|$ vertices in $A \setminus N_A^2(w)$. Since w, w' have a common neighbour, it follows that $|N_A^2(w) \cap N_A^2(w')| \geqslant x|A|$, and so

$$|N_A^2(w')| \geqslant (x + y/(10(1 - 2y)))|A| \geqslant (2/5)|A|,$$

a contradiction. Thus, $x + y/(10(1 - 2y)) < 2/5$, and so $y \geqslant 1/4$ from the hypothesis. This proves (3).

Since $(b_1 - y) + (b_2 - y) + (b_3 - y) = 1 - 3y < x$, it follows that for every vertex $u \in A$, there exists $i \in \{1, 2, 3\}$ such that $|N(u) \cap B_i| \geqslant (b_i - y)|B|$; and consequently there is a partition $A = A_1 \cup A_2 \cup A_3$ such that for $i = 1, 2, 3$, every vertex in A_i has more than $(b_i - y)|B|$ neighbours in B_i. It follows that $A_i \subseteq N_A^2(w)$ for each $w \in C_i$. Let $|A_i| = a_i|A|$ for $i = 1, 2, 3$.

For $i = 1, 2$, let D_i be the set of vertices in A_3 with a neighbour in B_i, and let $d_i = |D_i|/|A|$. For $i = 1, 2$, if $u \in D_i$ then

$$|N_C^2(u) \cap C_i| \geqslant y|C| \geqslant (y/(1 - 2y))|C_i|,$$

and so there exists $w \in C_i$ such that

$$|N_A^2(w) \cap D_i| \geqslant (y/(1-2y))|A_i| = (y/(1-2y))d_i|A|;$$

and since $A_i \subseteq N_A^2(w)$, it follows that $(y/(1-2y))d_i + a_i < 2/5$. Since $d_1 + d_2 \geqslant a_3$, and $a_1 + a_2 = 1 - a_3$, summing for $i = 1, 2$ yields that

$$4/5 > (y/(1-2y))(d_1+d_2) + (a_1+a_2) \geqslant (y/(1-2y))a_3 + (1-a_3),$$

that is, $(1-3y)a_3/(1-2y) > 1/5$; and since $a_3 < 2/5$, this implies that $y < 1/4$, a contradiction. This proves 10.2. ∎

10.3. *If $x, y \in (0,1]$ and $12x^2y \geqslant 5(1-x-y)^2$, then $\phi(x,y) \geqslant 2/5$.*

Proof. Suppose not. Then $\phi(x,y) = 1-(3/5+\epsilon)$ for some $\epsilon > 0$, so by rotating we have $\phi(x, 3/5+\epsilon) \leqslant 1-y$. But 5.7 gives $\phi(x, 3/5+\epsilon) > 1-y$, a contradiction. This proves 10.3. ∎

10.4. *If $x, y \in (0,1]$ and $x > 1/3$ and $y \geqslant (5-\sqrt{3})/11$, then $\phi(x,y) \geqslant 2/5$.*

Proof. Suppose not. Then $\phi(x,y) = 1-(3/5+\epsilon)$ for some $\epsilon > 0$, so by rotating we have $\phi(3/5+\epsilon, y) \leqslant 1-x < 2/3$. But $3/5 \geqslant (1-y)^2/(1-2y^2)$, since $y \geqslant (5-\sqrt{3})/11$; and so 7.9 gives that $\phi(3/5+\epsilon, y) \geqslant 2/3$, a contradiction. This proves 10.4. ∎

10.5. *If $x, y \in (0,1]$, and either $5x/2 + y \leqslant 1$ and $2y \leqslant x$, or $x + 5y/2 \leqslant 1$ and $2x \leqslant y$, then $\psi(x,y) < 2/5$.*

Proof. Apply 3.6 with $s/t = 2/5$. This proves 10.5. ∎

10.6. *If $x, y \in (0,1]$ with $x \leqslant 3/8$ and $y \leqslant \frac{(2x-1)^2}{52x^2-40x+8}$, then $\phi(x,y) < 2/5$.*

Proof. Define $x' = x/(1-x)$ and $y' = y/(1-y)$. Then $x', y' \in (0,1]$, and $x' < 3/5$, and $y' \leqslant \frac{(1-x')^2}{19x^2-22x+7}$, and so $\phi(x',y') < 2/3$ by 7.10. Choose z with $\phi(x',y') < z/(1-z) < 2/3$; then $\phi(x,y) < z < 2/5$ by 3.1. This proves 10.6. ∎

10.7. *If $x, y \in (0,1]$ with $y \leqslant 1/5$ and $x \leqslant \frac{22y^2-12y+2}{57y^2-30y+5}$, then $\phi(x,y) < 2/5$.*

Proof. Define $x' = x/(1-x)$ and $y' = y/(1-y)$. Then $x', y' \in (0,1]$, and $y' < 1/4$, and $x' \leqslant \frac{12y^2-8y+2}{20y^2-12y+3}$, and and so $\phi(x',y') < 2/3$ by 7.11. Choose z with $\phi(x',y') < z/(1-z) < 2/3$; then $\phi(x,y) < z < 2/5$ by 3.1. This proves 10.7. ∎

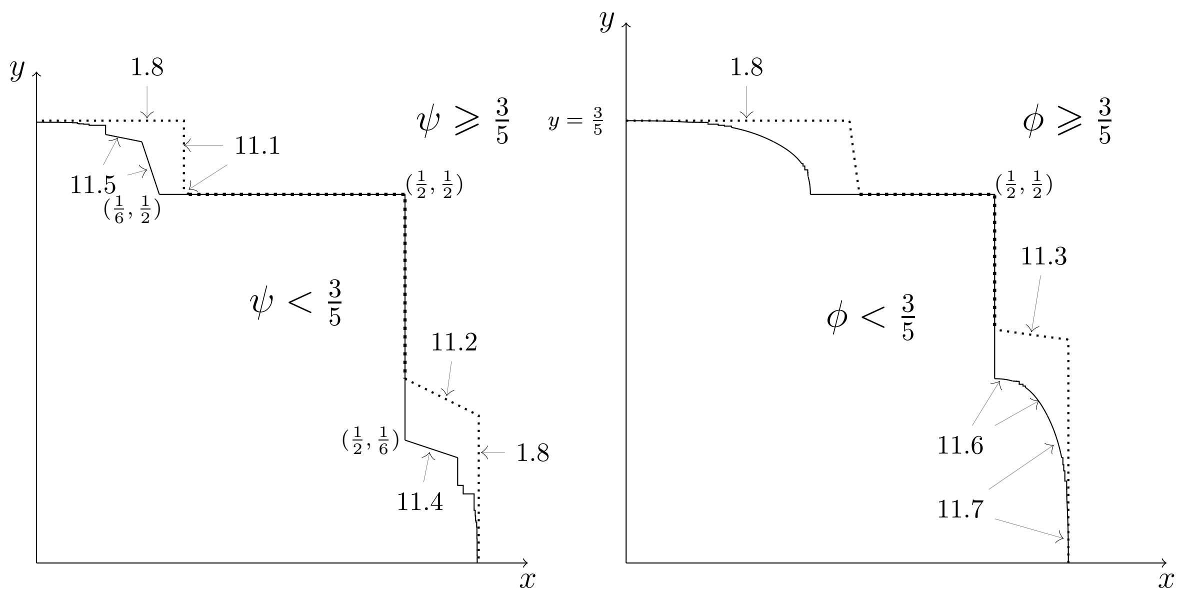

Figure 10: When $\psi(x, y) < 3/5$ and when $\phi(x, y) < 3/5$.

11 The 3/5 level

Next, we analyze when $\psi \geqslant 3/5$, and similarly for ϕ. The results are shown in figure 10.

The seven results in this section all contribute to figure 10.

11.1. *If $x, y \in (0, 1]$ with $y > 1/2$, $x \geqslant 1/5$, and*

$$2y - \frac{3 - 5x}{3 - 3x} y^2 > 1 - x,$$

then $\psi(x, y) \geqslant 3/5$.

Proof. Suppose not, and let G be (x, y)-biconstrained via (A, B, C), such that $|N_A^2(w)| < 3|A|/5$ for all $w \in C$. We can assume that x, y are rational, and by multiplying vertices if necessary, we can assume that both $x|B|$ and $|C|/5$ are integers. By averaging, there exists $u \in A$ such that $|N_C^2(u)| < 3|C|/5$. Choose $B' \subseteq N(u)$ with $|B'| = x|B|$, and choose $C' \subseteq C$ with $N_C^2(u) \subseteq C'$ and $|C'| = 3|C|/5$.

(1) *There exist $w_1 \in C'$ and $w_2 \in C \setminus C'$ such that $|N(w_1) \cup N(w_2)| > (1 - x)|B|$.*

Choose $w_1 \in C'$ and $w_2 \in C \setminus C'$ uniformly and independently at random, and let $S = N(w_1) \cup N(w_2)$. We will compute the expectation of $|S|$. Let $t = 2/5$. For each $v \in B_1$, v has at least $y|C|$ neighbours in C', so the probability it is in S is at least $y/(1 - t)$. For each $v \in B \setminus B'$, define $d(v) = |N(v) \cap (C \setminus C_1)|/|C|$. Then the probability that v is a neighbour of w_2 is $d(v)/t$, and the probability that v is a neighbour of w_1 and not a neighbour of w_2 is at least

$$\left(1 - \frac{d(v)}{t}\right) \frac{y - d(v)}{1 - t}.$$

Thus, the expectation of $|S|$ is at least

$$\begin{aligned}\sum_{v \in B'} \frac{y}{1-t} + \sum_{v \in B \setminus B'} &\left(\frac{d(v)}{t} + \left(1 - \frac{d(v)}{t}\right) \frac{y - d(v)}{1 - t} \right) \\ &= \sum_{v \in B} \frac{y}{1-t} + \sum_{v \in B \setminus B'} \frac{(1 - y - 2t)d(v) + d(v)^2}{t(1-t)}.\end{aligned}$$

Choose q with

$$\sum_{v \in B \setminus B'} d(u) = qt|B|.$$

Each vertex $w \in C \setminus C'$ has at least $y|B|$ neighbours in $B \setminus B'$, so it follows that $q \geqslant y$. Thus, the expectation of $|S|$ is at least

$$\frac{y}{1-t}|B| + \frac{1 - y - 2t}{1-t} q|B| + \sum_{v \in B \setminus B'} \frac{d(v)^2}{t(1-t)}.$$

Since $\sum_{v \in B \setminus B'} d(v) = qt|B|$ and $|B \setminus B'| = (1-x)|B|$, it follows by Cauchy-Schwarz that

$$\sum_{v \in B \setminus B'} d(v)^2 \geqslant \frac{q^2 t^2}{1-x} |B|.$$

Thus, the expectation of $|S|$ is at least

$$\frac{y}{1-t}|B| + \frac{1-y-2t}{1-t} q|B| + \frac{q^2 t}{(1-x)(1-t)}|B| = \left(y + q(1-y-2t) + \frac{q^2 t}{1-x} \right) \frac{|B|}{1-t}.$$

To prove (1), it suffices to show that the expectation of $|S|$ is more than $(1-x)|B|$, and so it suffices to show that

$$y + q(1-y-2t) + \frac{q^2 t}{1-x} > (1-t)(1-x).$$

Remembering that $t = 2/5$, this is

$$2(1-5y)q + \frac{4q^2}{1-x} + 10y > 6(1-x),$$

and the derivative of the left-hand side with respect to q is

$$2(1-5y) + \frac{8q}{1-x} \geqslant 2(1-5y) + \frac{8y}{4/5} = 2 > 0$$

for $q \geqslant y$. It follows that the left-hand side is minimized when $q = y$, so it suffices to show that

$$2(1-5y)y + \frac{4y^2}{1-x} + 10y > 6(1-x),$$

which is equivalent to the hypothesis. This proves (1).

Let w_1, w_2 be as in (1), and let $A_i = N_A^2(w_i)$ for $i = 1, 2$. Then $A = A_1 \cup A_2$, and since $y > 1/2$ we have $N(v_1) \cap N(v_2) \neq \emptyset$, and consequently $|A_1 \cap A_2| \geqslant x|A| \geqslant |A|/5$. Then

$$|A_1| + |A_2| = |A_1 \cup A_2| + |A_1 \cap A_2| \geqslant 6|A|/5$$

and so we have $|A_i| \geqslant 3|A|/5$ for some i, a contradiction. This proves 11.1. ∎

11.2. *If $x, y \in (0, 1]$ with $x > 1/2$ and $x + 2y > 1$, then $\psi(x, y) \geqslant 3/5$.*

Proof. Let G be (x, y)-biconstrained via (A, B, C), and suppose that $|N_A^2(w)| < (3/5)|A|$ for each $w \in C$. Choose α maximum such that there exist $w_1, \ldots, w_\alpha \in C$ where $N(w_i) \cap N(w_j) = \emptyset$ for $1 \leqslant i < j \leqslant \alpha$.

(1) $\alpha = 2$.

Since $x+2y > 1$ it follows that $N_A^2(w_i) \cup N_A^2(w_j) = A$ for all distinct $i, j \in \{1, \ldots, \alpha$, and so if $\alpha \geqslant 3$ then every vertex in A belongs to at least two of the sets $N_A^2(w_1), N_A^2(w_2), N_A^2(w_3)$, which is impossible since they each have cardinality less than $(3/5)|A|$. So $\alpha \leqslant 2$.

Suppose that $\alpha = 1$; then every $w_2 \in C$ satisfies $N(w_1) \cap N(w_2) \neq \emptyset$. Let $B_1 = N(w_1)$ and $A_1 = N_A^2(w_1)$; thus $|A_1| < (3/5)|A|$ and $|B_1| \geqslant y|B|$. Choose $v \in B \setminus B_1$ with v at least $x|A \setminus A_1|/(1-y) > 2x|A|/(5(1-y))$ neighbours in $A \setminus A_1$, and let $w_2 \in C$ be a neighbour of v. Then

$$(3/5)|A| > |N_A^2(w_2)| > x|A| + 2x|A|/(5(1-y)) > \left(x + \frac{4x}{5(x+1)}\right)|A| \geqslant 3|A|/5$$

since $x > 1/2$, a contradiction. This proves (1).

Since $\alpha = 2$, every vertex $w \in C$ shares a neighbour with at least one of w_1, w_2. Let $A_i = N_A^2(w_i)$ for $i = 1, 2$. Since $x+2y > 1$, we have $A = A_1 \cup A_2$, and so $|A_1 \cap A_2| < |A|/5$ because $|A_1|, |A_2| < 3|A|/5$. Then, if some $w \in C$ shares a neighbour with w_1 and shares a neighbour with w_2, it follows that $|N_A^2(w)| > 2x|A| - |A|/5 > 4|A|/5$, a contradiction.

Thus, every vertex in C shares a neighbour with exactly one of w_1 and w_2. Let H be the bipartite graph $G[B \cup C]$. It follows that there are exactly two components of H, say H_1, H_2, where $w_i \in V(H_i)$ for $i = 1, 2$. Let $B_i = B \cap H_i$ and $C_i = C \cap H_i$ for $i = 1, 2$. Without loss of generality we may assume that $|B_1| \geqslant |B|/2$. It follows that for each $u \in A$, u has a neighbour in B_1 and consequently

$$|N_C^2(u) \cap C_1| \geqslant y|C| \leqslant \frac{y}{1-y}|C_1|,$$

because $|C_2| \geqslant y|C|$, and thus $|C_1| \leqslant (1-y)|C|$. Since $|A \setminus A_1| > (2/5)|A|$, there exists $w \in C_1$ with more than $2|A|y/(5(1-y))$ neighbours in $A \setminus A_1$. But w and w_1 share a common neighbour, so

$$|N^2_A(w)| > \frac{2y|A|}{5(1-y)} + x|A| > \left(\frac{2y}{5(1-y)} + (1-2y)\right)|A| \geqslant 3|A|/5$$

since the last inequality is equivalent to $5y^2 - 5y + 1 \geqslant 0$, which is true because $y \leqslant 1/4$ (since $x + 2y > 1$, and $x > 1/2$). This proves 11.2. ∎

11.3. *If $x, y \in (0,1]$ with $y > 1/2$ and $40x^2y \geqslant (3-5x)^2$, then $\phi(x,y) \geqslant 3/5$.*

Proof. Apply 5.7 with $z = 3/5$. This proves 11.3. ∎

11.4. *If $x, y \in (0,1]$ with $x \leqslant 4/7$ and $y \leqslant 1/2$ and $x + 3y \leqslant 1$, then $\psi(x,y) < 3/5$.*

Proof. We may assume that $x > 1/2$ since $\psi(1/2,1/2) < 3/5$; and so $y < 1/6$ since $x + 3y \leqslant 1$. The claim follows from applying 3.4 with z slightly less than 3/5 and $x' = y' = z' = 1/4$. This proves 11.4. ∎

11.5. *If $x, y \in (0,1]$, such that $3x + y \leqslant 1$, and $x + 5y \leqslant 3$, with strict inequality in both if x or y is irrational, then $\psi(x,y) < 3/5$.*

Proof. By increasing x, y if necessary, we may assume that x, y are rational. Suppose that $\psi(x,y) \geqslant 3/5$. We claim first that:

(1) *$x < 1/6$, and $y > 1/2$, and $5xy + 15y < 9$, and $x < (1-y)/(5y)$, and $x \leqslant 3(1-y)^2/(1+5y)$.*

Since $3x + y \leqslant 1$, it follows that $x < 1/3$, and $y > 1/2$ since $\psi(1/2,1/2) = 1/2 < 3/5$. Thus, $x < 1/6$, since $3x + y \leqslant 1$. This proves the first two statements. Since $x + 5y \leqslant 3$, it follows that $y < 3/5$, and so $5xy + 15y < 3x + 15y \leqslant 9$. This proves the third statement. For the fourth, $5x < 3x/y$ (since $y < 3/5$), and $3x \leqslant 1-y$, and so $5x < (1-y)/y$. Finally, for the fifth statement, if $y \leqslant 4/7$, then $1 + 5y \leqslant 9 - 9y$, and so

$$x \leqslant 9x(1-y)/(1+5y) \leqslant 3(1-y)^2/(1+5y);$$

and if $y \geqslant 4/7$, then

$$(3-5y)(1+5y) = 3 + 10y - 25y^2 \leqslant 3 - 6y + 3y^2 = 3(1-y)^2$$

and so

$$x \leqslant 3 - 5y \leqslant 3(1-y)^2/(1+5y).$$

This proves (1).

Since $x < 1/6$, it follows that $x/(1-x) < (5x)/3$ and $(1-y)/(3y) < 1/3$. The hypotheses (via (1)) imply that

$$\frac{2x}{3-3y-x} \leqslant \min\left(\frac{3}{y} - 5, \frac{1-y}{3y}\right),$$

and

$$\frac{5x}{3} < \min\left(\frac{3}{y} - 5, \frac{1-y}{3y}\right).$$

Consequently, there exists a rational x' with $x/(1-x) < 5x/3 < x'$, and

$$\frac{2x}{3-3y-x} \leqslant x' \leqslant \min\left(\frac{3}{y} - 5, \frac{1-y}{3y}, \frac{1}{3}\right).$$

Thus,

$$\max\left(\frac{2y-1}{y}, 1 - \frac{x'(1-y)}{x}\right) \leqslant \min\left(1 - 3x', \frac{1-x'}{3}\right);$$

choose a rational y' between them. Then $x' + 3y' \leqslant 1$ and $3x' + y' \leqslant 1$, and so $\psi(x', y') < 1/3$, by (theorem 3.3 of the paper). Let $\psi(x', y') = z' < 1/3$, and choose $z < 3/5$ with $(1-z)/z \leqslant 1 - z'$, and $(1-z)/(1-x) \leqslant 1 - z'$, and $z \geqslant x/x'$. Then from 3.5, $\psi(x, y) \leqslant z < 3/5$, a contradiction. This proves 11.5. ▮

11.6. *If $x, y \in (0, 1]$ with $x < 4/7$ and $y \leqslant \frac{(3x-2)^2}{52x^2-64x+20}$, then $\phi(x, y) < 3/5$.*

Proof. Since $\frac{(3x-2)^2}{52x^2-64x+20} \leqslant 1/4$ for all $x \geqslant 0$, it follows that $y \leqslant 1/4$. Let $x' = 2 - 1/x$ and $y' = y/(1-y)$. Then $x', y' \in (0, 1]$, and $x' < 1/4$, and $y' \leqslant \frac{(1-2x')^2}{3-12x'+16x'^2}$ since $y \leqslant \frac{(3x-2)^2}{52x^2-64x+20}$. By 8.4, $\phi(x', y') < 1/3$. By 3.7, with $\phi(x', y') < 2 - 1/z < 1/3$, it follows that $\phi(x, y) \leqslant z < 3/5$. This proves 11.6. ▮

11.7. *If $x, y \in (0, 1]$ with $y < 1/5$ and $x \leqslant \frac{31y^2-18y+3}{53y^2-30y+5}$, then $\phi(x, y) < 3/5$.*

Proof. Since $\frac{(3x-2)^2}{52x^2-64x+20} \leqslant 1/4$ for all $x \geqslant 0$, it follows that $y \leqslant 1/4$. Let $x' = 2-1/x$ and $y' = y/(1-y)$. Then $x', y' \in (0, 1]$, and $x' < 1/4$, and $x' \leqslant \frac{(1-2y')^2}{3-12y'+16y'^2}$ since $x \leqslant \frac{31y^2-18y+3}{53y^2-30y+5}$. By 8.4 and 2.3, $\phi(x', y') = \phi(y', x') < 1/3$. By 3.7, with $\phi(x', y') < 2 - 1/z < 1/3$, it follows that $\phi(x, y) \leqslant z < 3/5$. This proves 11.7. ▮

12 Peaceful coexistence

We have not been able to evaluate $\phi(x, y)$ in general, but here is an easier question (that we also cannot do, but it seems to be less far out of reach). It is always true that $\phi(x, y) \geqslant x$, by 1.8, but if y is sufficiently small then equality may hold. For fixed x, what is the largest y such that $\phi(x, y) = x$?

Let (G, w) be a weighted graph. We say it is *x-regular* via a bipartition (A, B) if

- $|A| = |B|$, and $w(v) > 0$ for each $v \in V(G)$;
- the $0,1$-adjacent matrix between A and B is nonsingular;
- $\sum_{u\in A} w(u) = \sum_{v\in B} w(v) = 1$; and
- for each $u \in V(G)$, $\sum_{v\in N(u)} w(v) = x$.

(Note that the fourth bullet is required to hold both for $u \in A$ and for $u \in B$.) Its *order* is $|A|$, and its *min-weight* is $\min_{v\in B} w(v)$. We will show:

12.1. *For $x, y \in (0, 1]$, $\phi(x, y) = x$ if and only if there is an x-regular bipartite weighted graph with order at most $1/y$.*

Proof. If there is a such a weighted graph (G, w), via (A, B), where $|A| = |B| = n$ say, let C be a set of n new vertices, and add a perfect matching between B and C. Extend w to C by defining $w(v) = 1/n$ for each $v \in C$. The weighted graph just made is $(x, 1/n)$-constrained, and shows that $\phi(x, 1/n) \leqslant x$, and consequently $\phi(x, y) \leqslant x$ (and so $\phi(x, y) = x$).

For the converse, suppose that G is (x, y)-constrained via (A, B, C), and $|N^2_A(v)| \leqslant x|A|$ for each $v \in C$.

(1) *Each vertex in A has exactly $x|B|$ neighbours in B, and each vertex in B has exactly $x|A|$ neighbours in A.*

Each vertex $u \in B$ has at most $x|A|$ neighbours in A, since u has a neighbour $v \in C$ and $|N^2_A(v)| \leqslant x|A|$. Since each vertex in A has at least $x|B|$ neighbours in B, averaging shows that equality holds throughout. That proves (1).

Say two vertices in A are *twins* if they have the same neighbour set in B, and two vertices in B are *twins* if they have the same neighbour set in A. This defines equivalence relations of A and B, and we call the equivalence classes *twin classes*.

(2) *For each vertex $v \in C$, all its neighbours in B are twins, and so $N(v)$ is a subset of a twin class of B.*

By (1) each vertex in $N(v)$ has $x|A|$ neighbours in A, and all these vertices belong to $N^2_A(v)$; and since $|N^2_A(v)| = x|A|$, equality holds, and in particular, all vertices in $N(v)$ are twins. This proves (2).

Let $\mathcal{T}$ be the set of all twins classes of B. For each $T \in \mathcal{T}$, let $C(T)$ be the set of all $v \in C$ with $N(v) \subseteq T$. Thus, the sets $C(T)$ $(T \in \mathcal{T})$ are nonempty, pairwise disjoint and have union C. There is one of cardinality at most $|C|/|\mathcal{T}|$, say $C(T)$; and then each vertex in T has only at most $|C|/|\mathcal{T}|$ neighbours in C, and so $y \leqslant 1/|\mathcal{T}|$.

Choose one vertex from each twin class of A and of B, and let H be the subgraph induced on this set. For each vertex v of H, let $w(v) = |T|/|B|$ if $v \in T$ for some twin class T of B, and $w(v) = |T|/|A|$ if $v \in T$ for some twin class T of A. Then we have:

- (H, w) is a bipartite graph, with bipartition (A_0, B_0) say;
- $\sum_{u \in A_0} w(u) = \sum_{v \in B_0} w(v) = 1$;
- for each $u \in V(H)$, $\sum_{v \in N(u)} w(v) = x$; and
- $|B_0| \leqslant 1/y$.

Let us choose a weighted graph (H, w) and bipartition with these properties, with $|V(H)|$ minimum. If there is a function $f : A \to \mathbb{R}$ such that $\sum_{u \in N(v)} f(u) = 0$ for each $v \in B$, not identically zero, then by adding a suitable multiple of f to the restriction of w to A, we can arrange that $w(u) = 0$ for some $u \in A$, and then u can be deleted, contrary to the minimality of $|V(H)|$. Thus, there is no such f, and similarly there is no $f : B \to \mathbb{R}$ such that $\sum_{v \in N(u)} f(v) = 0$ for each $u \in A$, not identically zero. Consequently, $|A_0| = |B_0| = n$ say, and the adjacency matrix between A_0 and B_0 is nonsingular. Moreover $w(v) > 0$ for each $v \in V(H)$, from the minimality of $V(H)$. This proves 12.1. ∎

By 2.3, $\phi(x, y) = x$ if and only $\phi(y, x) = x$, so this also answers the analogous question for $\phi(y, x)$. If x is irrational, there is no x-regular bipartite weighted graph, and so $\phi(x, y) > x$ for all $y > 0$. If $x \in (0, 1]$ is rational, let us define the *order* of $x \in (0, 1]$ to be the minimum order of x-regular bipartite weighted graphs. If $x = p/q$ say where $p, q > 0$ are integers, then the order of x is at most q, because one can construct an appropriate cyclic shift graph. But the order of x can be strictly less than q. For instance, the top part of the graph of figure 1 is 13/27-regular (take as vertex-weights the numbers given, divided by 27), and so the order of 13/27 is at most seven. Figure 11 gives a smaller example, showing that the order of 2/5 is at most four.

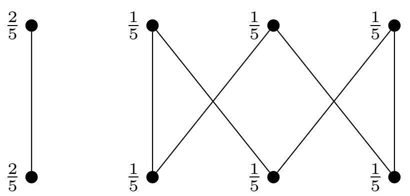

Figure 11: A 2/5-regular weighted bipartite graph of order four.

We can prove that the order is also bounded below by a function of q that goes to infinity with q. More exactly, if G is p/q-regular (in lowest terms) and has order n, then q is at most $(n+1)^{(n+1)/2}$. This follows from a theorem of Hadamard [3], that every $n \times n$ 0,1-matrix has determinant at most $(n+1)^{(n+1)/2} 2^{-n}$. We do not know whether there are weighted bipartite graphs with order n that are p/q-regular (in lowest terms), where q is exponentially large in n. (Hadamard $n \times n$ 0,1-matrices have determinants that achieve Hadamard's bound, and they exist when $n + 1$ is a power of two, but they give weighted bipartite graphs that are vertex-transitive, and which therefore are p/q-regular with $q = n$.)

One could ask the same question for the biconstrained problem: given x, for which values of y is it true that $\psi(x, y) = x$? A similar analysis (we omit the details) shows:

12.2. *For $x, y \in (0, 1]$, the following are equivalent:*

- $\psi(x, y) = x$;
- $\psi(y, x) = x$; *and*
- *there is an x-regular bipartite weighted graph with min-weight at least y.*

References

[1] L. Caccetta and R. Häggkvist, "On minimal digraphs with given girth", *Proc. Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing, Congress. Numer.*, XXI, Utilitas Math., Winnipeg, Man., 1978, 181–187.

[2] P. Erdős, M. Saks, and V. Sós, "Maximum induced trees in graphs", *J. Combinatorial Theory, Ser. B*, 41 (1986), 61–79.

[3] J. Hadamard, "Résolution d'une question relative aux déterminants", *Bulletin des Sciences Mathématiques*, 17 (1893), 240–246.

[4] P. Hompe, *The Girth of Digraphs and Concatenating Bipartite Graphs*, Senior Thesis, Princeton, 2019.

[5] P. Hompe, "Some results on concatenating bipartite graphs", arXiv:1908.07453.

[6] M. Kneser, "Abschätzungen der asymptotischen Dichte von Summenmengen", *Math. Z.* (in German), 58 (1953), 459–484.

[7] P. Seymour and S. Spirkl, "Short directed cycles in bipartite digraphs", *Combinatorica* **40** (2020), 575–599, arXiv:1809.08324.

[8] B. Sullivan, "A summary of problems and results related to the Caccetta-Häggkvist conjecture", arXiv:math/0605646.