Multiple Criteria Subset Selection Under Interdependence of Actions

by

Siamak Rajabi

A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Systems Design Engineering

Waterloo, Ontario, Canada, 1997

0-612-22230-6

The University of Waterloo requires the signatures of all persons using or photocopying this thesis. Please sign below, and give address and date.

To my parents for their loving encouragement,

and to my wife, Vida, and my daughter, Tannaz,
who are always with me.

Abstract

Multiple Criteria Decison Making (MCDM) is a problem that has been studied extensively. Most pitfalls are by now well-known, and many proven algorithms permit choices to be made efficiently. But, when the problem is a multiple criteria subset selection, new difficulties appear, and most algorithms of MCDM are either inapplicable or impractical.

Even when actions are *independent*, so their cumulative effects are additive, multiple criteria subset selection is a challenging problem. Moreover, applicable multiple criteria subset selection approaches suffer from large computational requirements. To deal with these difficulties, techniques are introduced for screening individual actions when a subset of a large discrete set of independent actions is to be selected, both when the number of actions to be selected is given a priori, and when the subset to be selected must satisfy several constraints.

When actions are *interdependent* the subset selection problem becomes even harder. A novel definition and characterization of the interdependence of actions in the context of multiple criteria subset selection problems are presented. Most of the interdependence discussion can be generalized to sets of actions rather than individual actions. Exploration of the main relationships of set-independence and action-independence produces several different methods for evaluating a set of interdependent actions. A general approach to evaluate a combination of interdependent actions is proposed and applied to the multiple criteria structure. The effects of interdependence of actions on the modeling and resolution of a subset choice problem are illustrated, and the importance of taking interdependence of actions into account is demonstrated.

The subset selection problem under interdependence of actions is formulated as a multiple criteria integer program and two solution methodologies are proposed. The advantages of these approaches in comparison to others are discussed. These methodologies and associated analytical techniques are applied to an on-going water supply planning problem in the Regional Municipality of Waterloo. The results indicate both the importance of interdependence of actions and the effectiveness of the proposed methodologies.

ACKNOWLEDGMENTS

This thesis would not have been possible without the guidance and support of my supervisors Professor Keith W. Hipel and Professor D. Marc Kilgour. I would like to express my deepest gratitude to them for their inspiration, encouragement and many hours of discussion during the four year development of this thesis. It is my greatest honor and distinct pleasure to acknowledge their support.

I would also like to thank my committee members Professor M. Magazine, Professor N. Fraser, Professor L.Tuncel and Professor P. Calamai for their constructive comments on my thesis. Special thanks are due to my external examiner Professor B. Hobbs from the Johns Hopkins University for his valuable suggestions to improve my thesis.

I am also grateful to my friends Dr. A. Seifi for reading parts of my thesis and Dr. Y. Wang for his assistance during my work in PAMI Lab.

I would also like to express my sincere appreciation to my dear wife, Vida, for her love, support and sacrifice throughout my years at the University of Waterloo. The accomplishment of writing this thesis belongs equally to Vida.

Finally, I would also like to thank the Iranian Ministry of Higher Education for providing financial support for this research.

Contents

List of Tables

List of Figures

Glossary

MCDM	Multiple Criteria Decision Making,
DM	Decision Maker,
MADM	Multi-Attribute Decision Making,
MOMP	Multiple Objective Mathematical Programming,
MCZO	Multiple Criteria Zero-One,
GP	Goal programming,
WDL	Waste Disposal Location,
WWSPP	Waterloo Water Supply Planning Problem,
$\mathbf{A}$	Set of available actions,
$\mathbf{Z}$	Feasible region in criterion space,
$\mathbf{X}$	Feasible region in decision space,
$\mathbf{P}$	Set of criteria,
$Eff(\mathbf{A})$	Set of efficient actions in set $\mathbf{A}$,
$d(a)$	Set of efficient actions dominating action a,
$dom^{-1}(a)$	Set of actions dominating action a,
$Dom(\mathbf{A})$	Set of dominated actions in $\mathbf{A}$,
$Dom^{-1}(\mathbf{S})$	Set of actions that dominate some actions in $\mathbf{S}$,
$PO(\mathbf{A})$	Set of potentially optimal actions,
$DPO_m(\mathbf{A})$	Set of dominated potentially optimal actions for

	m-best actions problem,
$Eff_T(\mathbf{A})$	Set of Totally-efficient actions in set $\mathbf{A}$,
$d_T(a)$	Set of T-efficient actions that totally dominate a,
$dom_T^{-1}(a)$	Set of actions that totally dominate a,
z^{**}	Ideal point solution in criterion space,
Λ	Set of all strictly positive weighting vectors where $\Lambda = \{\lambda \in \mathbf{R}^{\lvert\mathbf{P}\rvert} \lvert \lambda_p > 0, \sum_{p=1}^{\lvert\mathbf{P}\rvert} \lambda_p = 1\}$,
$Qe(\mathbf{A})$	Set of weakly-efficient actions in set $\mathbf{A}$,
G_p	Amount of goal on criterion p,
d_p^+	Positive deviation from goal on criterion p,
d_p^-	Negative deviation from goal on criterion p,
$c_p(a_l) = c_p^l$	Consequence of action a_l on criterion p,
$v_p(a_l) = v_p^l$	Value of action a_l on criterion p,
$v(a_l)$	Overall value of action a_l.
$\mathbf{V}$	Set of monotonic value functions,
$\mathbf{V}_L \in \mathbf{V}$	Set of linear value functions,
$\succ$	Dominance relation,
$\mathbf{A}_{(n)}$	Collection of subsets of $\mathbf{A}$ containing n actions,
$\phi_p(\mathbf{A}_1, \mathbf{A}_2 \lvert \mathbf{A}^0)$	Amount of interdependence of sets $\mathbf{A}_1$ and $\mathbf{A}_2$ on p when set $\mathbf{A}^0$ has already been selected,
$\mathbf{I}_p$	Independence relation on p,
$\sim \mathbf{I}_p$	Interdependence relation on p,
$\gamma_p(\mathbf{A}_1, \mathbf{A}_2 \lvert \mathbf{A}^0)$	Synergy of $\mathbf{A}_1$ and $\mathbf{A}_2$ given $\mathbf{A}^0$ on criterion p,
$c_p(\mathbf{A}_1 \lvert \mathbf{A}^0)$	Consequence of $\mathbf{A}_1$ given $\mathbf{A}^0$ on criterion p,
$\Delta_p(\mathbf{S})$	Amount of simple dependence within set $\mathbf{S}$ on criterion p,
$O_p(\mathbf{S})$	Order of dependence within set $\mathbf{S}$ on criterion p,

O_p	Overall order of dependence on criterion p,
$\phi_p(\mathbf{A}_1, \mathbf{A}_2)$	Amount of simple interdependence of $\mathbf{A}_1$ and $\mathbf{A}_2$ on p,
$\Theta(\mathbf{S})$	Class of all covers of set $\mathbf{S}$,
$\theta \in \Theta(\mathbf{S})$	A particular cover of set $\mathbf{S}$,
$\Psi(\mathbf{S})$	Class of all partitions of set $\mathbf{S}$,
$\psi \in \Psi(\mathbf{S})$	A particular partition of set $\mathbf{S}$,
$\hat{\Psi}_p(\mathbf{S}) \subseteq \Psi(\mathbf{S})$	Partitions of $\mathbf{S}$ that satisfy additivity condition of $\mathbf{S}$ on p,
$D_p(\mathbf{S}\|\psi)$	Degree of additivity of $\mathbf{S}$ with respect to ψ on p,
$D_p(\mathbf{S})$	Overall degree of additivity of $\mathbf{S}$ on p,
$\bar{D}_p(\mathbf{S})$	Degree of separability of $\mathbf{S}$ on p,
$\psi_p^0(\mathbf{S})$	Maximal partition of $\mathbf{S}$ on p,
$\mathbf{L}_p^k$	Subsets of actions with order of dependence of k on p.

Chapter 1

Motivation and Objectives

1.1 Motivation

MCDM problems naturally arise in many situations, both strategic and routine. For instance, a typical multiple criteria problem takes place when a family uses criteria such as price, size, distance from shopping and schools, and aesthetics in order to decide which house to buy from a wide selection. Given the expectation of the family and availability of houses, often there does not exist a house that satisfies all the necessities. Hence, the family must trade-off among different criteria to select the house that gives maximum satisfaction. Another example is a government's decision about which combination of alternative sources of energy generation to select in order to meet long-term energy demand, while considering cost and environmental impacts criteria. Many methods and theories have been developed during the past two decades in both continuous and discrete problems for solving a wide range of multiple criteria decision situations.

The definition and generation of actions [1] is an important step in the process of employing MCDM, but one to which little research effort has been devoted [63, 129, 120]. For most MCDM tools, it is assumed that the decision maker deals with a predefined and clearly specified set of actions, perhaps defined by a set of decision variables and constraints, from which a preferred action(s) is to be selected.

In many real world decision problems, the decision maker(s) is interested in selecting a combination of actions rather than an individual action. For example, the manager of a company might like to select a *set* of products to manufacture. Research and development departments often consider a *set* of projects for analysis. Also, a government which is responsible for developing the long term water supply for a region may employ a combination of sources, such as ground-water, lakes, and river water, to satisfy future demand.

Situations in which a subset of actions is to be selected from a discrete set of actions have not received much attention in the multiple criteria literature [127]. Moving from single action selection to multiple criteria subset selection increases the complexity of the decision problem. In fact, most available multiple criteria subset selection methods for finding the set of non-dominated solutions are applicable only to small problems [127]. Hence, it is useful to develop techniques for removing inferior actions before attempting to solve the problem through formal methods. Several screening approaches have been suggested in the literature. But, most of them are only suitable for single action selection. Therefore, there is a need to develop procedures for adapting these screening approaches for multiple criteria subset selection problems.

Most multiple criteria models assume strict independence of actions. Yet inter-

[1] *In this research we discriminate between* action *and* alternative. *An action is assumed to be an individual object and by alternative we mean a combination of actions.*

dependence of actions can be found in many real-world subset selection problems. Consider, for instance, the pressing problem of disposing solid wastes. Possible actions include using one or more of a number of potential dumping sites, incineration at one or more locations, introducing by-laws to reduce the amount of waste generated in the first place, plus a range of recycling measures. Criteria may include cost, infrastructure requirements, environmental risk, potential acceptability, and aesthetics. An optimal solution may consist of a set of actions that, typically are interdependent for one or more of the criteria on which they are to be evaluated. Other examples may include the selection of different products to be produced in a firm, research and development or investment projects, transportation routes, and computer systems.

Often, interdependence of actions is overlooked or it is treated in some unnatural way which leads to the solutions which are not the best choices [81]. This is due partly to the ill-structure of the interdependence relation, difficulty in formulating them, and trouble in measuring the amount of interdependence [112, 31]. Recently, however, as the importance of interdependence in some applications was recognized, techniques were developed for estimating the amount of interdependence of actions [112].

Interdependence of actions is more crucial in MCDM problems, because different types of interdependencies may occur across several conflicting criteria and change the solutions of the problem. Moreover, the set of non-dominated solutions may be changed extensively in the presence of interdependent actions. Even though several formulations of interdependence appear in the literature, all restrict the type or extent of interdependence in some ways.

In this thesis, we present a novel definition of interdependence of actions and sets of actions in MCDM and assess the main properties of interdependence using these definitions. The effects of interdependence on subset selection, especially in the multiple criteria framework, are examined and techniques for evaluating subsets of actions that are interdependent according to specific criteria are presented.

1.2 Objectives

The main objective of this thesis is to develop models and associated analytical techniques for multiple criteria subset selection problems under interdependence of actions. Figure 1.1 depicts the main focus of this thesis. The following are some specific goals:

1. To introduce effective techniques for screening actions when a subset of a large discrete set of actions is to be selected. This includes:
   - identifying conditions under which individually dominated actions can be screened out from the set of feasible actions,
   - developing techniques to remove inferior actions when the number of actions to be selected is given a priori, and
   - proposing techniques for screening individually dominated actions when the possible subset to be selected is defined by a set of constraints.
2. To present a general framework for independence and interdependence of sets of actions in MCDM problems. This includes:
   - analyzing the effects of interdependence of actions in multiple criteria subset selection problems,

- presenting a foundation for the definition and characterization of interdependence of actions, especially in the presence of multiple criteria,
- assessing the relationship between the independence of two sets and independence of their proper subsets, and
- proposing different techniques to facilitate the evaluation of sets of interdependent actions.

3. To propose a solution methodology to solve a multiple criteria subset selection problem under interdependence of actions. This requires:
   - formulating a discrete multiple criteria subset selection problem under interdependence of actions,
   - exploiting the structure of the formulated problem to propose an improved solution methodology, and
   - developing a methodology that generates a representative subset of nondominated solutions, and overcomes some of the difficulties existing in current satisficing approaches.

4. To apply the suggested solution methodology to a real-world water supply planning problem in the Regional Municipality of Waterloo, located in southern Ontario, Canada. This includes:
   - identifying the criteria, available actions and interdependence among actions,
   - constructing a mathematical model that represents all criteria, as well as resource and technological constraints, and
   - demonstrating the importance of interdependence of actions in the Waterloo Water Supply Planning Problem.

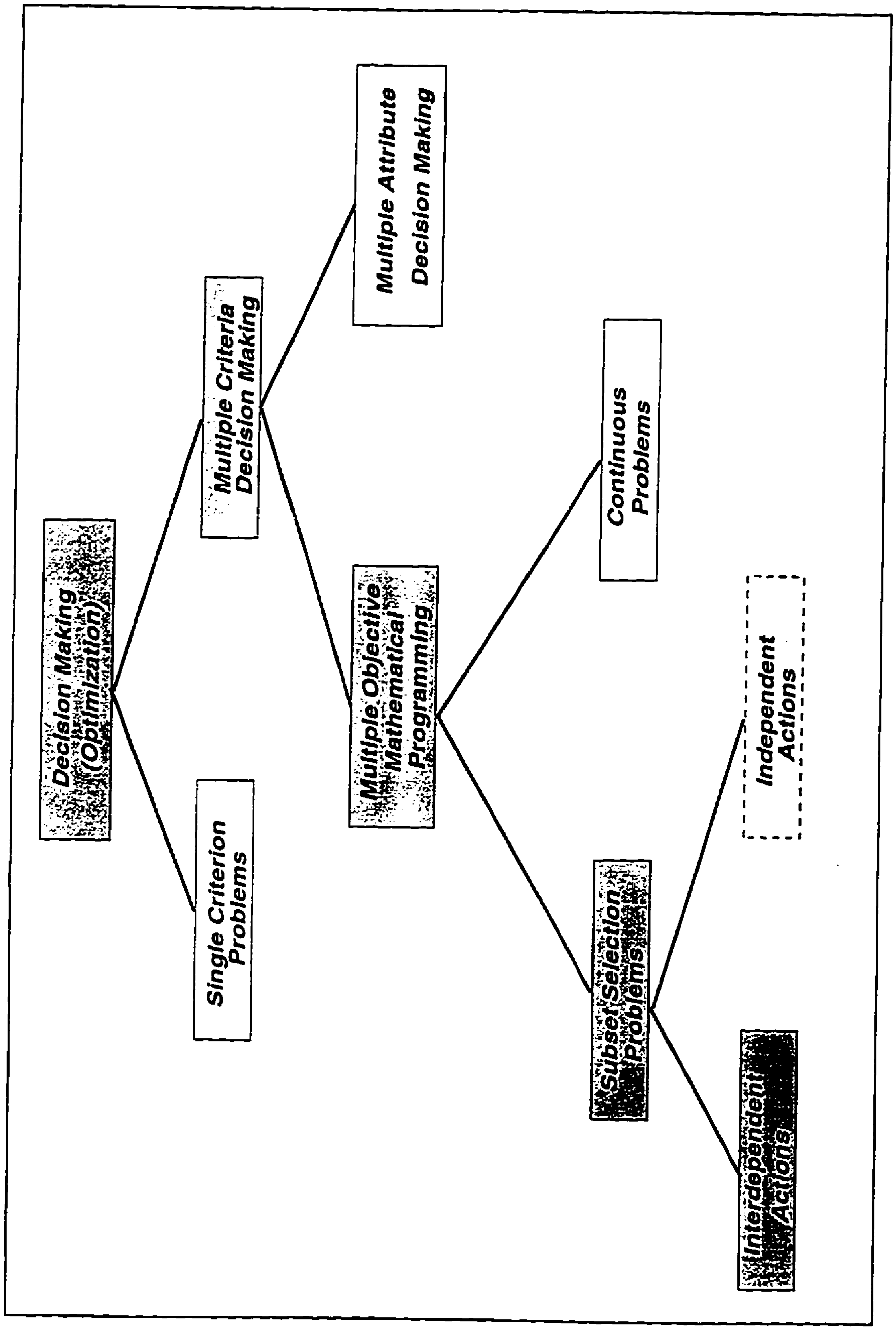


Figure 1.1: The Main Areas of Study in the Thesis

1.3 Overview of the Thesis

This chapter presents the motivation and the main objectives of the thesis. Chapter 2 presents a brief overview of MCDM concepts and reviews some of the multiple criteria subset selection problem approaches. Additionally, the reference programming methods are discussed in detail. Next, Chapter 3 introduces new techniques for screening individual actions in multiple criteria subset selection problem. In this chapter it is assumed that there is not any interdependence among actions. Furthermore, it is shown that usage of conventional dominance procedures for screening may eliminate some good actions from the set of feasible actions. Subsequently, techniques are proposed to examine individually dominated actions for multiple criteria subset selection problems with respect to two specific cases: when the number of actions to be selected is specified a priori, and when a set of constraints specifies the number of actions to be chosen.

Chapter 4 focuses on modelling interdependence of actions in MCDM. The importance of interdependence of actions in multiple criteria subset selection is shown and a general framework for interdependence of sets of actions is presented. Different types of interdependence and important special cases are also discussed. Then, the main differences of our definition with conventional definitions of interdependence are explained. Following the introduction of interdependence, Chapter 5 discusses the evaluation of interdependence of actions. It begins with presenting a general formulation for evaluating the consequences of a set of interdependent actions. Next, several techniques are presented to examine the independence of two sets of actions and to evaluate the sets of interdependent actions. Furthermore, in this chapter useful connections between independence of two sets and independence of their proper subsets are explored. Finally, this chapter ends with presenting a new definition of additivity of a set of interdependent actions. Using this definition it is shown how to decompose a set of interdependent actions into some subsets such that minimum number of interdependence evaluations is necessary.

Chapter 6 provides a formulation for subset selection problems under interdependence of actions. Then, two solution methodologies are presented to solve the formulated problem. Finally, the main advantages of the proposed methods in comparison with other available approaches are discussed.

To demonstrate the effectiveness of the proposed methods and to show the effects of interdependence of actions on solutions. a real world water resources planning problem in the Regional Municipality of Waterloo is studied in Chapter 7. The experience gained and lessons learned in applying the proposed approaches to the Waterloo Water Supply Strategy, are discussed. Subsequently, a summary of the accomplishments and main contributions of the thesis are given in Chapter 8. Finally, this thesis ends by presenting some possible directions for future research.

Chapter 2

Background and Literature Review of MCDM

2.1 A Brief Historical Perspective of MCDM

Today, it is well understood that most decision problems inherently involve choices that ought to be judged according to more than one criterion. In fact, MCDM problems arise naturally in many situations, both strategic and tactical, and MCDM methods have been widely applied in public policy, engineering, and design. For example, in the selection of plans for a road, construction costs, usage, and expected rate and severity of accidents are some of the main criteria. In water resources planning, criteria such as power generation capacity, flood control capability, and environmental impacts may be essential. In designing a gear-box, several criteria, including volume of material, maximal peripheral velocity between gears, width of the gear-box, and distance between axes of input and output shafts, should be minimized simultaneously [88]. Increasing the output quality level and reducing the overall inspection cost are two conflicting criteria applicable to the design of quality control policies in a production line.

MCDM dates back to the late 19th century, when the concept of equilibrium in consumer economics was introduced by Edgeworth and Pareto [115, 116]. However, MCDM became a useful decision technology in the early 1970s, when the applications of operations research extended to strategic levels of decision making. Specifically, after the first conference on MCDM, held at the University of South Carolina in 1973, the field has been one of the fastest growing areas in operations research, as evidenced by the enormous number of books, journal articles, and congresses in both the theory and application of MCDM methods [129, 118].

2.2 MCDM: Concepts and Definitions

MCDM consists of a set of tools to help a Decision Maker (DM) or a group of DMs to make a decision by *finding*, *selecting*, *sorting*, or *ranking* a set of actions according to two or more criteria, which are usually conflicting. A possible set of actions, A, may be specified explicitly by listing its member, or implicitly by identifying a set of decision variables and the constraints they must satisfy. The definition and generation of *actions* is an important step in the process of MCDM but one to which relatively little research effort has been devoted [129, 63, 120]. For most real-world problems there is no pre-existing set of well-defined actions. Most often, before, any formal decision analysis can be undertaken, some preliminary work to define, combine, expand, or reduce the set of feasible actions is necessary. The set of feasible actions can be reduced by removing some inferior actions, identifying those that do not meet some level of acceptability, or that do not meet key performance standards on criteria. Despite many potential applications, situations in which a subset of actions (alternative) is to be selected from a discrete set of actions have not received much attention within the multiple criteria literature [127]. Moreover, in many real-world multiple criteria subset selection problems, there are some kinds of interdependence among actions. Yet, most multiple criteria models assume strict independence of actions.

The set of *criteria*, **P**, by which actions are to be compared, is another element of MCDM. There is no consistent definition of a criterion by researchers. Vincke [129] defines a criterion as a function f, defined on set of actions, taking its values in a totally ordered set. Bouyssou [13] defines a criterion as a tool for comparing actions according to a particular significance axis. Finally, Yu [137] expresses criteria as a set of functions that are relevant to making a decision.[1]

The criteria are usually in conflict with each other, especially if each criterion represents the interest of a specific group of DMs. For example, building a factory in a region may generate job opportunities, but on the other hand may introduce adverse environmental impacts. Increasing the frequency of inspection in a production line decreases the number of defects but, on the other hand, increases the cost of quality control. Thus, it is rare to find an action that is best according to all criteria, and asking for an optimal solution to an MCDM problem does not make sense. Rather, one must search for a *compromise* solution that appropriately reconciles the different criteria. To find this compromise solution, it is necessary to learn something about the DM's preferences over the criteria. Hence, the role of the DM in MCDM is more explicit, and more crucial than in single objective optimization. There are many difficulties in introducing value judgments of the DM(s) into MCDM problems. One should keep in mind as well that due to the behavioral influence of tradeoffs across criteria, in many situations it is impossible

[1] *In this thesis, the terms criterion and objective are used interchangeably.*

to find a solution simply by implementing a mathematical model [105].

Once the actions and the criteria are constructed, one must measure or evaluate each actions according to each criterion. Most optimization procedures are based on the assumption that one can assign a real number to represent the consequences of an action according to a criterion. However, in many real-world applications this is often a very difficult task. This issue is more important in MCDM, because in many MCDM applications some criteria are not quantitative. Often, the natural way to express the consequences of actions is by using *ordinal* information, whereby the actions are ranked according to each non-quantitative criterion.

MCDM can be classified into two main branches, Multiple Attribute Decision Making (MADM) and Multiple Objective Mathematical Programming (MOMP). The former applies mainly when the set of actions, **A**, is defined explicitly by listing its finite members; the latter when **A** is defined implicitly by a set of constraints to be satisfied. Usually, in MADM the size of **A** is small and in MOMP the number of actions is large. Even though both MADM and MOMP have been used for solving multiple criteria subset selection problems, a natural way to tackle these problems is through MOMP approaches[2]. Figures 2.1 and 2.2 shows some of different types of MADM and MOMP approaches, respectively. This thesis mainly concerned with MOMP problems.

Without loss of generality, assume that all criteria are to be maximized. Then, an MOMP problem can be expressed as follows:

[2] *AHP (Analytic Hierarchy Process) [108] and PROMETHE V [15] are among MADM approaches that have been used for subset selection problems (see [2] and [108]).*

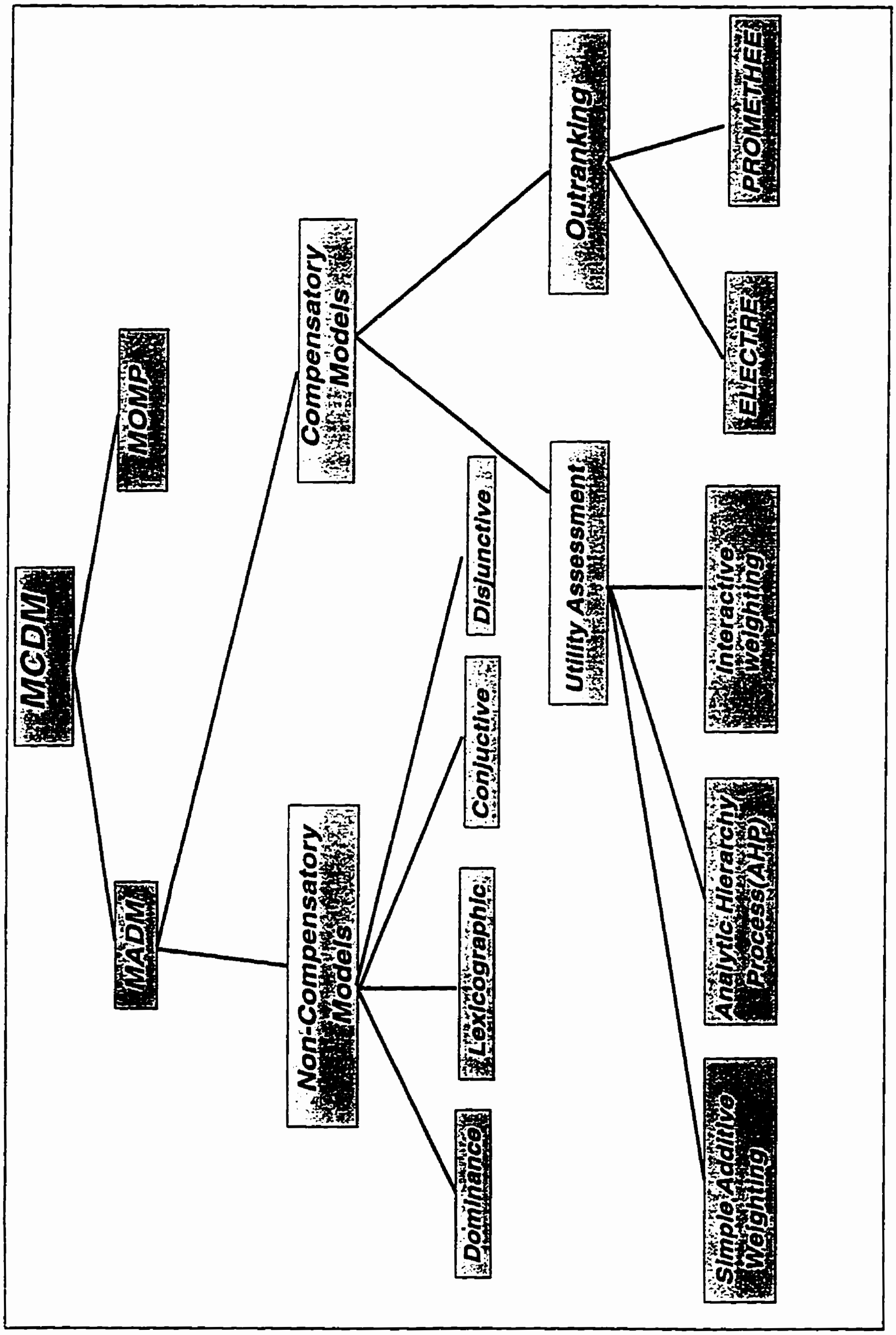


Figure 2.1: Some of the Different Types of MADM Approaches

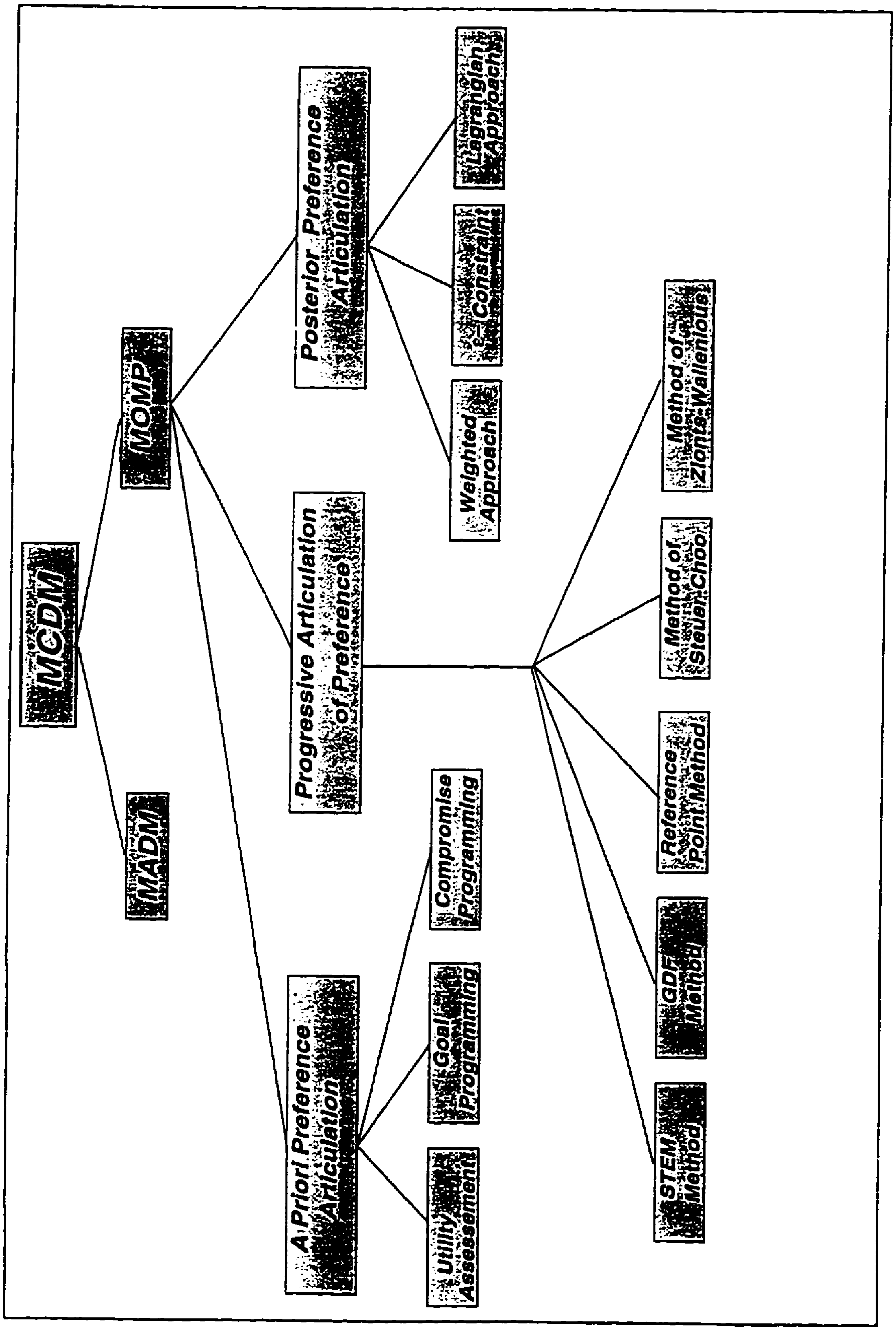


Figure 2.2: Some of the Different Types of MOMP Approaches

$$\textbf{(B1)} \qquad \textbf{Maximize} \qquad \{f_1(x), \ldots, f_p(x), \ldots, f_{|\mathbf{P}|}(x)\}$$
$$\textbf{Subject to}:$$
$$x \in \mathbf{X},$$

where, $f_p(x)$ is the pth objective function, x is the vector of decision variables and $\mathbf{X}$ is the feasible space. The main characteristics of a multiple criteria problem which distinguishes it from single criterion problems is that there generally does not exist a solution that simultaneously maximizes all of the objectives. A solution which is best according to all objectives is called the *ideal point* and is denoted

$$z^{**} = (z_1^*, z_2^*, \cdots, z_p^*, \cdots, z_{|\mathbf{P}|}^*),$$

where $z_p^* = max\{f_p(x)\}$ and $x \in \mathbf{X}$. Some MCDM approaches use this ideal point for assessing other solutions. Most theories of MCDM can be characterized according to the non-dominated (efficient or Pareto optimal) solution concept.[3]

Definition 2.1 *A solution x in* **(B1)** *is defined to be non-dominated if there is no other solution x^0 such that:*

$$f_p(x^0) \geq f_p(x) \qquad \forall p = 1, \ldots, |\mathbf{P}|,$$
$$\text{and} \qquad f_p(x^0) > f_p(x) \qquad \text{for at least one } p.$$

The set of efficient solutions of $\mathbf{X}$ is denoted by $Eff(\mathbf{X})$. Geoffrion [35] introduces the concept of *properly efficient* solutions. He argues that all members of $Eff(\mathbf{X})$

[3] *In this thesis, efficient solutions and non-dominated solutions will be used interchangeably.*

may not be considered as reasonable solutions and a rational DM would always choose an action which is properly efficient.

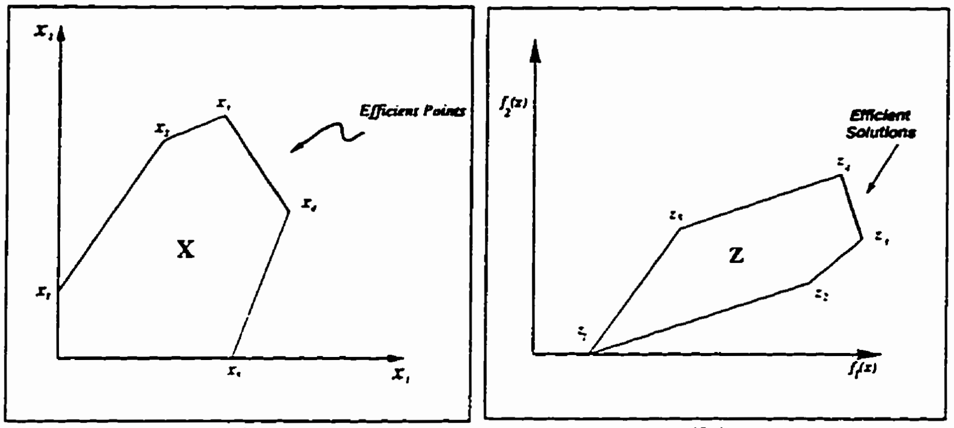
Definition 2.2 *Solution x is properly efficient, if it is efficient and for any $y \in \mathbf{X}$ and any criterion p; the following ratio is bounded above for some k[35]:*

$$\frac{f_p(y) - f_p(x)}{f_k(x) - f_k(y)}. \tag{2.1}$$

The above ratio shows the improvement in criterion p divided by the decrement in criterion k with changing a decision from x to y. Hence, if this ratio is not bounded from above, a very small decrement in the criterion k leads to a very large improvement in criterion p and a rational DM usually prefers this exchange [102]. Nevertheless, in most practical problems, the concepts of efficient and properly efficient solutions are not very different. The *improper efficient solutions* can only occur in some specific types of nonlinear multi-objective and multi-criteria discrete optimization problems with an infinite number of actions. For more description and examples see [118].

Basically, in single objective problems, the study is conducted in a decision space. However, in multiple objective programming, it is more convenient to transform the feasible region in the decision space $\mathbf{X}$ into a feasible region in *criterion space* [4], $\mathbf{Z}$. Hence, a feasible region in criterion space is the image of $\mathbf{X}$ under $f_p(\cdot)$ for all p. The following example illustrates this concept.

[4] *Note that, for the single objective problem, the feasible region in criterion space is a straight line segment.*



(a) (b)

Figure 2.3: The Set of Feasible Actions in (a)Decision Space, (b)Criterion Space

Example 2.1 Consider the following 2-criterion problem.

$$\begin{aligned} &\textbf{Maximize} && f_1(x) = x_1 + x_2, \\ &\textbf{Maximize} && f_2(x) = x_1, \\ &\textbf{Subject to}: && \\ & && -3x_1 + 2x_2 \leq 2, \\ & && -x_1 + 2x_2 \leq 5, \\ & && 4x_1 + 3x_2 \leq 20, \\ & && 3x_1 - x_2 \leq 8, \\ & && x_1 \geq 0, \quad x_2 \geq 0. \end{aligned}$$

Figure 2.3a shows the set of feasible points in the decision space and Figure 2.3b shows the set of feasible solutions in the criterion space. Every extreme point in **X** corresponds to an extreme solution in **Z**. Any solution that lies on the line $z_3 - z_4$

is an efficient solution. Likewise, all points on line $x_3 - x_4$ are efficient points.

Much research has been devoted to finding effective ways for obtaining the set of efficient solutions [102]. Geoffrion's theorem provide some basic research ideas. Consider Problem **B1**.

Theorem 2.1 *If $x \in \mathbf{X}$ maximizes*

$$\sum_{p=1}^{|\mathbf{P}|} \lambda_p . f_p(x), \tag{2.2}$$

for some λ, where $\lambda_p \in \Lambda$ and

$$\Lambda = \left\{ \lambda \in \mathbf{R}^{|\mathbf{P}|} \mid \lambda_p > 0, \sum_{p=1}^{|\mathbf{P}|} \lambda_p = 1 \right\}, \tag{2.3}$$

then x is an efficient solution [35].

The above theorem indicates that regardless of the shape of the feasible space and $f_p(.)$, if x is the maximal solution of the convex combination of all criteria, then x is efficient. However, this theorem provides only a necessary condition. To be sufficient more conditions are required.

Theorem 2.2 *If $\mathbf{X}$ is convex, let $x \in \mathbf{X}$ be efficient. Then there exists $\lambda \in \Lambda$ in which action x is a maximum solution of (2.2) [35].*

Due to the convexity assumption of Theorem 2.2, Geoffrion's theorem cannot explore all extreme efficient points in non-convex discrete and nonlinear programming.

For the linear case. the solution of (2.2) is basically the set of non-dominated *extreme points*, and if the DM's true utility function is nonlinear (which is the case in most situations). then it is possible that his best solution will not lie on the extreme points. To handle this, methods have been developed for generating other non-extreme efficient points. These methods are mainly based on the convex combinations of the extreme efficient points [118, 102].

Another concept of efficiency is *weak-efficiency* or *quasi-efficiency*, which is denoted by $Qe(\mathbf{X})$. Consider Problem **B1**,

Definition 2.3 *Solution x is called quasi-efficient if there does not exist another solution x^0 such that [118]:*

$$f_p(x) < f_p(x^0) \qquad \forall p. \tag{2.4}$$

By setting $\lambda \geq 0$ in (2.3). Theorem 2.1 produces quasi-efficient solutions. Quasi-efficiency is a more relaxed definition of efficiency and one can observe that:

$$Eff(\mathbf{X}) \subseteq Qe(\mathbf{X}). \tag{2.5}$$

The notion of efficiency is especially important in studying deterministic problems, although selected concepts such as *stochastic dominance, mean variance dominance, probability* and *utility dominance* have also been defined in the MCDM literature [137]. Note that efficiency is weaker than optimality in the sense that in most cases there exist many efficient points but no ideal point. Hence, after finding the set of efficient solutions the DM still must choose one member of this set.

2.3 Overview of Multiple Objective Mathematical Programming

One important feature of MCDM is the great diversity of developed procedures. In fact, the wide variety of methods has encouraged some researchers to build models for selecting the best MCDM approach in some general and specific application areas [53, 49]. In this section, some of the well known MOMP approaches that will be addressed in upcoming sections are discussed. One can classify MOMP approaches according to how and when preference information is articulated:

1. A priori preference information.
2. Progressive articulation of preference information, and
3. Posterior preference information.

Below each of these approaches are briefly explained.

2.3.1 A priori preference information

A priori preference information methods of MOMP begin with an exploration of the value function of the DM. Once the preference structure of the DM has been assessed, all objectives are aggregated into one, thereby changing the problem to a single objective optimization problem. In most cases, assessment of the DM's value function is quite difficult and involves a great deal of subjectivity. Many theories and procedures have been developed for determination and characterization of the DM's preference structure for both deterministic and probabilistic cases. The capability of these procedures in conditions of uncertainty, and their usefulness for sensitivity analyses are among their advantages [60]. In addition, the theoretical foundations of these methods add to their attractiveness. However, they generally require the assumption that a value function exists, and often that it is additive; even if one is prepared to make these assumptions, it may be extremely difficult to construct a value function in practice. It is worth mentioning that these approaches have been mainly used in MADM. In addition to the *additive* form of the value function, others such as *multiplicative*, *polynomial*, *partially additive*, and so on, have been proposed in the literature.

2.3.2 Progressive articulation of preference information

Due to the great difficulty of determining a DM's preference explicitly, many procedures try to elicit them progressively. Methods that alternate between computation and interaction with the DM are called *interactive*. The process starts with little or no preference information. At each iteration, a set of solutions (usually, non-dominated solutions) is presented to the DM. As each solution is examined, the DM decides upon the updated preference information and inputs it into the model. The process terminates when the DM is satisfied with the solution currently proposed by the model.

The first interactive method called STEM, was proposed by Benonyan *et al.* in 1971 [8]. Although originally proposed for solving linear problems, its structure permits it to be applied to integer and nonlinear problems. This procedure is based on reducing the feasible space by adding more constraints, obtained through interaction with the DM. The augmented weighted Chebyshev method is used for assessing the compromise solution at each step.

The GDF method of Geoffrion *et al.* [36] is another interactive approach to

MOMP. Using an implicit utility function, this method attempts to find the best solution using the Frank-Wolf gradient algorithm.

The method of Zionts and Wallenious [142] is applicable to linear problems. Relying on the assumption that the DM's utility function is pseudo-concave, this method generates extreme efficient points at each iteration. Adjacent extreme points are compared by the DM and this information is added to the model for the next iteration.

The method of Steuer and Choo [119] generates samples of the efficient points by using the augmented weighted Chebyshev norm. Using a filtering algorithm, this method gives a pre-specified number of efficient points, that are dispersed throughout the space of efficient solutions, and can be considered representative. For further information about other types of interactive method see Steuer [118].

2.3.3 Reference Programming

One of the main class of approaches which are used both as a priori preference information as well as in progressive articulation of preference is reference programming.

The main concept of reference programming is based on a rationality framework, called *satisficing* (a combination of satisfactory and sufficient) decision making proposed by Simon [114]. For many organizations, a solution that is as close as possible to a goal is more acceptable than an optimal solution. This reflects the fact that usually real-world problems are dynamic, prone to error in measurement, under time pressure, complex, and ill-defined. The *goals* or *aspiration levels* to be satisfied may be based on past performance, the DM's intuition, the level of competition, etc. Moreover, these goals are not fixed, and can be changed to reflect a circumstance such as the difficulty of reaching previous goals. Eilon [26] states that "optimizing is the science of the ultimate and satisficing is the art of feasible". In fact, the idea of using *heuristics* to address a problem is inspired by the idea of satisficing, an acceptable and usually good solution in hand is better than an optimal solution in the bush. The satisficing approach is sometimes called *bounded rationality*. Below, three main methods of reference programming are discussed.

Goal Programming

One popular method which is designed primarily for use with a priori preference information is Goal Programming. Goal Programming (GP) is perhaps the first formal technique of MCDM. The term GP was first used by Charnes and Cooper in 1961 [19]. This method has been recognized as the most popular and most accepted method in MOMP [134, 57]. Different versions of GP have been proposed in the literature. The fundamental idea is that the best solution is as close as possible to some predefined goals. Therefore, one must implement the following two steps before solving the problem:

- specify the level of goals for all criteria,
- define the distance metric, to measure the closeness of feasible solutions to the target. This distance metric is usually called the achievement function.

It is assumed that the DM can specify the desired goal (G_p) for each objective. Therefore, problem **B1** can be written as follows:

$$
\begin{array}{lll}
\textbf{(B2)} & goal\ \{Z_1 = f_1(x)\} & (Z_1 \geq G_1), \\
 & goal\ \{Z_2 = f_2(x)\} & (Z_2 \geq G_2), \\
 & \quad\vdots \qquad \vdots & \\
 & goal\ \{Z_{|\mathbf{P}|} = f_{|\mathbf{P}|}(x)\} & (Z_{|\mathbf{P}|} \geq G_{|\mathbf{P}|}),
\end{array}
$$

Subject to :

$$x \in \mathbf{X}.$$

The information in parentheses on the left shows the goals specified for different objective functions. The achievement function in GP is to minimize the deviations (positive and/or negative) from DM's aspiration level, according to a distance function. The achievement function in GP is based on the following general distance metric:

$$\textbf{Minimize} \quad z = \left[\sum_{p=1}^{|\mathbf{P}|} w_p^{\alpha} \left| \frac{G_p - f_p(x)}{k_p} \right|^{\alpha} \right]^{1/\alpha}, \ \alpha \in \{1, 2, 3, \ldots\} \cup \{\infty\}, \qquad (2.6)$$

where w_p is the importance of the deviation from goal on pth criterion, $k_p > 0$ is the normalizing constant for pth criterion, and G_p is the aspiration level on criterion p.

Note that the above expression does not differentiate between positive and negative deviations. In more general case, one can assign different penalties for over-achievement and under-achievement. Charnes and Cooper [19] define the following change of variables:

$$d_p^+ = 1/2[\mid G_p - f_p(x) \mid + (G_p - f_p(x))], \qquad (2.7)$$

$$d_p^- = 1/2[\mid G_p - f_p(x) \mid - (G_p - f_p(x))], \qquad (2.8)$$

where d_p^+ and d_p^- represent negative and positive deviations from a specified goal on criterion p. Adding (2.7) and (2.8) gives,

$$d_p^+ + d_p^- = |G_p - f_p(x)|. \qquad (2.9)$$

Substituting this in the achievement function (2.6).

$$\textbf{Minimize } z = \left[\sum_{p=1}^{|P|} w_p^\alpha \left(\frac{d_p^+ + d_p^-}{k_p} \right)^\alpha \right]^{1/\alpha} \quad \alpha \in \{1, 2, 3, \ldots\} \cup \{\infty\}. \tag{2.10}$$

But. $d_p^+ . d_p^- = 0$. because both positive and negative deviations cannot be simultaneously nonzero. Hence. (2.10) can be written as:

$$\textbf{Minimize } z = \left[\sum_{p=1}^{|P|} w_p^\alpha \left(\frac{d_p^+}{k_p} \right)^\alpha + \sum_{p=1}^{|P|} w_p^\alpha \left(\frac{d_p^-}{k_p} \right)^\alpha \right]^{1/\alpha} . \ \alpha \in \{1, 2, 3, \ldots\} \cup \{\infty\}. \tag{2.11}$$

On the other hand. subtracting (2.8) from (2.7) gives:

$$d_p^+ - d_p^- = G_p - f_p(x). \tag{2.12}$$

which can be served as a goal restriction for the pth objective function. It follows that taking into account both (2.7) and (2.8), the achievement function is equivalent to:

$$\begin{aligned} \textbf{Minimize } z \ &= \ \left[\sum_{p=1}^{|P|} w_p^\alpha \left(\frac{d_p^+}{k_p} \right)^\alpha + \sum_{p=1}^{|P|} w_p^\alpha \left(\frac{d_p^-}{k_p} \right)^\alpha \right]^{1/\alpha} . \ \alpha \in \{1, 2, 3, \ldots\} \cup \{\infty\} \\ \textbf{(B3)} \quad & \quad \textbf{Subject to:} \\ & \quad f_p(x) + d_p^- - d_p^+ = G_p, \qquad \forall p \in \mathbf{P}. \end{aligned}$$

Program **B3** along with the original constraints of (**B2**) constitute the general structure of a GP problem. Using different values of α for measuring the overall distance of the objectives from goals leads to different types of GP techniques. By increasing the value of α. more emphasis is given to the largest deviations from goals in (**B3**). With $\alpha = 1$ the achievement function is a simple additive function.

In this case, the GP problem is called Archemedian (weighted) GP.

$$\textbf{Minimize} \quad z = \sum_{p=1}^{|\mathbf{P}|} w_p \left(\frac{d_p^+}{k_p}\right) + \sum_{p=1}^{|\mathbf{P}|} w_p \left(\frac{d_p^-}{k_p}\right),$$

Subject to :

$$f_p(x) + d_p^- - d_p^+ = G_p,$$

$$x \in \mathbf{X}.$$

Also, when $\alpha = 2$, the achievement function is an Euclidean function given as follows:

$$\textbf{Minimize} \quad z = \left[\sum_{p=1}^{|\mathbf{P}|} w_p^2 \left(\frac{d_p^+}{k_p}\right)^2 + \sum_{p=1}^{|\mathbf{P}|} w_p^2 \left(\frac{d_p^-}{k_p}\right)^2\right]^{1/2}. \tag{2.13}$$

There has been little usage of the Euclidean achievement function in GP applications. Setting $\alpha = \infty$ leads to minimax or *Chebyshev* GP. In this case, only the largest deviation from the goal is taken into account.

Therefore, selecting $\alpha = 1$ and assuming that objective functions have been normalized, the GP formulation of (**B2**) can be represented as follows:

$$(\mathbf{B4}) \qquad \textbf{Minimize} \quad \sum_{p=1}^{|\mathbf{P}|} w_p^- d_p^-.$$

Subject to :

$$f_1(x) + d_1^- - d_1^+ = G_1,$$

$$f_2(x) + d_2^- - d_2^+ = G_2,$$

$$\vdots \qquad \vdots$$

$$f_{|\mathbf{P}|}(x) + d_{|\mathbf{P}|}^- - d_{|\mathbf{P}|}^+ = G_{|\mathbf{P}|},$$

$$d_p^-, d_p^+ \geq 0,$$

$$x \in \mathbf{X}.$$

Note that because the objective function in the original program is maximization, the positive deviations do not have any penalty and therefore have been omitted in (**B4**). The GP problems can be solved with most standard mathematical programming procedure and software. Hence, it can be supported by strong sensitivity analysis capabilities of these procedures such as shadow prices and analyses of ranges.

In spite of the above mentioned advantages, one should be very cautious when applying this approach. Except to the Chebychev type, the same assumptions as multi-attribute value theory are necessary in GP such as additive independence of attributes, ratio scaled weights, and interval scaled attribute value function [52]. Moreover, if the goals are assigned at or greater than ideal point, the Archemedian GP chosses the same solution as linear additive function.

The most critical problem in GP is that sometimes the optimal solution may be dominated. This phenomenon was first noted by Zeleny and Cochrane [141] and Cohon and Marks [22], and afterwards discussed by other researchers. The following example demonstrates this issue for a simple integer problem:

Example 2.2 Consider the following multiple criteria integer program:

$$\begin{aligned}
\textbf{Maximize} \quad & f_1(x) = x_1, \\
\textbf{Maximize} \quad & f_2(x) = x_2, \\
\textbf{Subject to}: \quad & \\
& x_1 + x_2 \leq 7 \\
& x_1 \leq 5, \qquad x_2 \leq 5, \\
& x_1, x_2 \quad integer.
\end{aligned}$$

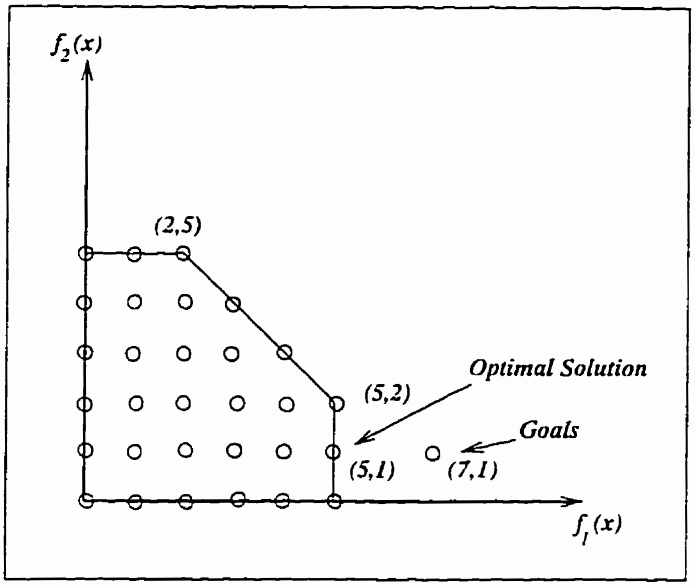


Figure 2.4: The Dominated Solution in Goal Programming

Figure 2.4 shows the feasible criterion space. Assume that the DM assigns (7.1) as the goals for first and second criteria, respectively. with equal importance for deviations from the goal on each criterion. The Archemedian GP formulation is.

$$\textbf{Minimize} \quad d_1^- + d_2^-$$

Subject to :

$$x_1 + d_1^- - d_1^+ = 7.$$

$$x_2 + d_2^- - d_2^+ = 1.$$

$$x_1 + x_2 \leq 7.$$

$$x_1 \leq 5,$$

$$x_2 \leq 5,$$

$$x_1, x_2 \ \textit{integer}.$$

The optimal solution is $f_1^*(x) = 5$ and $f_2^*(x) = 1$. This solution is obviously dominated by $(f_1(x) = 5, f_2(x) = 2)$, as shown in Figure 2.4. The solution is also dominated for the corresponding Chebyshev GP.

Another difficulty in GP arises when the levels of goals are significantly different. In this case, regardless of the importance of each criterion, the model may yield a solution in favor of the criteria with large levels.

Compromise Programming

To be as close as possible to an *ideal point*, that is, a point which is the best from all points of view, is a rational approach [139]. Based on this idea, Zeleny proposes the compromise programming method. He believes that the DM makes tradeoffs among actions with respect to their distance from an ideal point and he selects the closest point. Therefore, similar to GP, compromise programming is based on a reference point and a measure of distance in which the reference point is the ideal point and the measure of distance is the family of weighted L_α metrics as follows:

$$L_\alpha = \left(\sum_{p=1}^{|\mathbf{P}|} w_p^\alpha (z_p^* - f_p(x))^\alpha \right)^{1/\alpha}, \tag{2.14}$$

where z_p^* is the optimal solution of objective function p. A point which minimizes this metric is considered as a solution. By changing the value of parameter α, $(1 \leq \alpha \leq \infty)$, the *compromise set* $(C(\mathbf{A}))$ will be constructed. This set, for $w_p \geq 0$, is always efficient, given that at least one solution for $\alpha = \infty$ is efficient [102]. For this case, one can observe that:

$$C(\mathbf{A}) \subseteq Eff(\mathbf{A}). \tag{2.15}$$

Furthermore, Zeleny argues that humans strive to be as far as possible from an *anti-ideal* (nadir) point. Therefore, one could also build up a compromise set based on the anti-ideal point and select the points which are in both sets as a best representative of the preferred solution [139].

Reference Point Method

As explained before, one of the main criticism of GP is that an optimal solution of this approach may be dominated. This weakness of GP has led to the development of an approach, called the reference point method. The *reference point method* of Wierzbicki [135] is a multiple criteria approach based on the satisficing concept.[5] In the reference point method, the aspiration level specified by the DM is projected onto the non-dominated space. Hence, even if a DM underestimates the aspiration level, the model does not generate dominated solutions.

The reference point method uses a *scalarizing function* similar to the Chebyshev norm, to find a solution close to an aspiration level. Sawaragi *et al.* [110] and Wierzbicki [136] have shown that the Chebyshev norm type is the only scalariziation function that produces non-dominated solutions, regardless of the structure of the problem. However, a scalariziation function of Chebyshev norm may also produce quasi-efficient solutions. To exclude quasi-efficient solutions, an augmented Chebyshev norm is used in this method. Hence, the reference point method produces efficient solutions in nonlinear as well as discrete multiple objective problems [77].

Let $\bar{q} \in \mathbf{R}^{|P|}$ be a reference point in the criterion space,[6]. Then, a typical

[5] *Wierzbicki calls his approach a* quasi-satisficing *method.*

[6] *Note that the reference point does not need to be in the feasible criterion space,* $\mathbf{Z}$.

scalarizing function used in the reference point method is [7]

$$S(\overline{q}, z, w) = \max_{1 \leq p \leq |\mathbf{P}|} \left\{ w_p(\overline{q}_p - f_p(x)) \right\} + \epsilon \sum_{p=1}^{|\mathbf{P}|} w_p f_p(x), \tag{2.16}$$

where ϵ is a sufficiently small positive number. The amount of w_p is usually calculated according to the values of ideal and nadir points of the problem (see, for example, [84]). A quasi-satisficing solution is obtained by solving the following program:

$$\textbf{(B5)} \qquad \textbf{Minimize} \qquad S(\overline{q}, z, w)$$
$$\textbf{Subject to:} \qquad x \in \mathbf{X}.$$

The main difference between the function in (2.16) and other types of functions used in goal or compromise programming is that the aspiration level $\overline{q}$ does not need to be unattainable in order for the program to achieve efficient solutions, because this function remains monotone, even if the reference point is located inside of $\mathbf{Z}$. In other words, depending upon the location of the reference point, this function switches from minimization to maximization [77]. Note that the reference point method is usually used in an interactive manner. Hence, the DM can changes his aspiration level at each iteration.

2.3.4 Posterior preference information

Posterior preference information methods start by solving the problem without articulating the preference structure. Then a compromise solution is obtained by assessing the preference structure. Usually, the first step is carried out by *vec-*

[7] *Many different kinds of scalarizing functions have been proposed for the reference point method.*

tor optimization, by which a set of efficient solutions, or a subset, is generated. Three main approaches for generating efficient solutions are the *weighted approach* which uses the Geoffrion's theorem, *kth objective ϵ-constraint*, and *k-th objective Lagrangian approach*. Finding the set of efficient solutions in situations for which the decision space is not convex (such as in subset selection problems) is a difficult problem. The next section reviews the main techniques available to deal with multiple objective integer problems.

Discrete Multiple Objective Mathematical Programming

Combinatorial optimization is a powerful tool for many real-world applications. Despite recent improvements in both combinatorial and multiple criteria optimization, there have been few advances in multiple criteria combinatorial optimization. In his bibliography on the applications of multiple criteria optimization, White [134] listed more than 500 papers, including only a few on multiple criteria combinatorial problems. This is due partly to the inherent difficulty of combinatorial optimization, which becomes more difficult by introducing multiple objectives. Hence, because of the many applications using this method and its current theoretical weakness, this field is considered an important challenge for future research [127].

Geoffrion's theorems, which provide the fundamental procedure for finding efficient points, cannot find all efficient solutions of discrete problems. Thus, obtaining the set of efficient points in multiple criteria discrete problems is generally quite difficult. Consider the following problem:

$$\textbf{(B6)} \qquad \textbf{Maximize} \quad f_p(x) = C_p x; \qquad \forall p \in \mid \mathbf{P} \mid$$

$$\textbf{Subject to :}$$

$$x \in \mathbf{X}$$

$$x \in \{0, 1\},$$

where C_p is a vector of size $\mid \mathbf{A} \mid$. Due to the non-convexity of **(B6)**, Geoffrion's theorem can only find the *Supported Efficient* solutions. Unsupported efficient solutions are efficient points which are dominated by some convex combination of other efficient points. Therefore, those efficient solutions which are unsupported (*convex dominated*) remain hidden. In Figure 2.5, $\mathbf{z}^5$ and $\mathbf{z}^6$ are unsupported efficient points which are dominated by the convex combination of $\mathbf{z}^4$ and $\mathbf{z}^7$. Due to this fact and the inherent differences between discrete and continuous problems, most of the theories and procedures developed for multiple criteria continuous (and especially linear) problems are not applicable to discrete cases. In some applications such as shortest-path problems, using Geoffrion's theorem as an approximation method to find a subset of efficient solutions may omit large portions of efficient solutions [130].

Teghem and Kunsch [125] review interactive models in *multiple objective integer linear programming.* They state that because of the great difficulty of finding all efficient points in integer cases, interactive methods are quite useful. However, they criticize most interactive procedures for asking the DM too many questions. Rasmussen [99] reviews multiple criteria zero-one programming and concludes that the study of this area of research has not advanced very far; most methods can be used only for small problems. One procedure for solving multiple criteria zero-one problems is proposed by Pasternak and Passy [90], who use a variant of Balas' filter method.

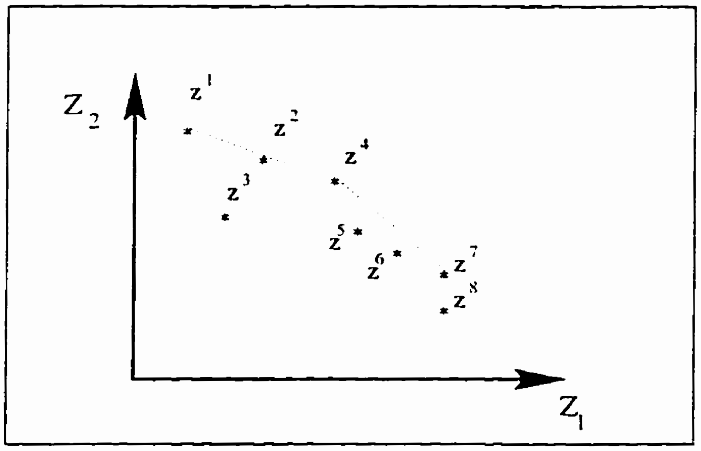


Figure 2.5: The Unsupported Efficient Solutions

Bitran [9] proposes an implicit enumeration method that generates all efficient points for multiple criteria zero-one problems. He introduces a relaxation of (**B6**) as:

$$\textbf{(B7)} \qquad \textbf{Maximize} \quad f_p(x) = C_p x: \quad \forall p \in \mid \mathbf{P} \mid$$
$$\textbf{Subject to}:$$
$$x \in \{0.1\}.$$

He argues that all efficient points of (**B7**) that are feasible in the original Problem are also efficient in (**B6**). To determine the other efficient points of (**B6**), one should identify those points that are dominated by the points in (**B7**) which are not feasible in (**B6**). Thus, Bitran's procedure has two main steps:

1. detecting the efficient points of the relaxed problem (**B7**), and
2. examining all non-efficient points and obtaining those which are dominated by infeasible points in (**B7**).

The main shortcoming of this approach is the required large computation time. which makes it suitable only for small problems.

Later, in 1979. Bitran [10] improved his algorithm. This revised algorithm has a great computational advantage over the first one. Bitran and Lawrance [11] applied this Bitran's new procedure to a service office location problem. Bitran and Rivera [12] propose an implicit enumeration method for solving multiple criteria zero-one problems. They also tailor their algorithm for use with a particular class of facility location problems.

Villarreal and Karwan [128] introduce a combined dynamic programming approach for solving linear multiple criteria integer programming which could also be used for zero-one programming.

Deckro and Winkofsky [24] propose an implicit enumeration method for linear multiple criteria zero-one problems. This approach is based on bounding and direction of preferences and compares favorably with Bitran's second method. They conclude that their approach may be useful for large sparse problems.

Steuer and Choo [118] suggest an *interactive weighted Chebyshev* method for multiple criteria linear optimization that can also be used for linear integer programming. This method can find both supported and unsupported efficient solutions.

Gabbani and Magazine [34] propose an interactive heuristic procedure for multiple criteria integer programming. Their method has two main parts. First. an algorithm searches for a set of criterion weights that would produce the most preferred solution of a linear utility function. Second. the search space is narrowed down using interactions with the DM. This approach can find only the supported efficient points.

Korhonen *et al.* [74] present a procedure in which the DM's utility function is assumed to be *quasi-concave*. Using the DM's responses, convex cones, that are used for eliminating inferior solutions, are generated.

Lee and Luebbe [76] propose a method for zero-one goal programming problems. Their algorithm is based on finding non-zero variables that satisfy each constraint and each priority level, and the partitioning of the problem into subproblems according to priority levels.

Ramesh *et al.* [98] provide an interactive branch and bound method for multiple criteria integer problems, similar to the procedure of Zionts and Walleninus [142, 143] for multicriteria linear programming. The DM's utility function is assumed to be *pseudoconcave*. This method uses the notions of *convex cones* and *λ-constraints* for removing undesirable points.

Karaivanova *et al.* [61] present an interactive heuristic approach for application to linear multiple criteria integer problems. Their method is based on the *augmented weighted Chebyshev metric* for generating supported and unsupported solutions. To solve the single objective integer problem obtained from the augmented weighted Chebyshev metric, a heuristic approach is used. This method requires less computational time for large problems than Steuer and Choo's approach [119].

Ulungu and Teghem [127], present a comprehensive survey on multiple criteria combinatorial optimization, reviewing multiple objective *transshipment, network flow, location, traveling salesman, set covering* and *knapsack* problems. This study shows that most multiple criteria combinatorial procedures are applicable only for small problems and the lack of good heuristics is also quite obvious in this field.

2.4 Summary

In this chapter, some MCDM concepts that are related to the discussion in the forthcoming chapters, were reviewed. We focused on well-known approaches of MOMP problems. In particular, different methodologies of reference programming were reviewed. Difficulties of finding non-dominated solutions in multiple criteria subset selection problems were discussed and available techniques to solve multiple criteria integer problems were studied.

Chapter 3

Screening in Multiple Criteria Subset Choice

3.1 Introduction

In many MCDM problems, DMs are interested in selecting a combination of actions rather than one individual action. Moreover, in practical decision problems the set of feasible actions is often very large, making it worthwhile to identify the most promising actions for more detailed investigation. This is particularly important when a subset of actions is to be selected, since the number of available *alternatives* (combinations of actions) can be enormous due to the combinatorial nature of the problem. Hence, in the early stages of the decision process, it is generally very useful to distinguish those actions (or subsets of actions) that seem reasonable from those that seem inferior. If this phase of the selection process, called screening, is carried out effectively, then the prior phase - generating new actions - will be facilitated and encouraged.

The main objective of screening is to remove inferior actions from the set of potential actions, so that those remaining can be subsequently investigated in more detail, perhaps using more accurate information or more refined assessment criteria. Several approaches for screening actions have been addressed in the literature, among them *feasibility testing*, the *dominance relation*, *elementary methods*, *successive elimination*, and *bounding the performance level*. These methods have been used in the context of certainty as well as uncertainty, in single and multiple criteria problems, and for qualitative as well as quantitative criteria [80, 55]. Most of them, however, are not suitable for subset selection problems: as Example 3.2 (below) will demonstrate, they should be used only with extreme caution.

One popular technique for screening actions in MCDM is the dominance relation. An action is dominated when there exists another action that scores at least as well on all criteria and strictly better on at least one criterion. In many cases, the set of non-dominated actions is very large, so that even after dominated actions are removed from consideration, the DM may still face a difficult task. Hence, significant effort has been devoted to enriching and extending the dominance relationship to screen out even more actions. Most of these approaches are based on including information about the DM's preference structure, and thereby reducing the decision space by screening out actions that are un-dominated but dispreferred [6, 83, 70, 74].

This chapter addresses procedures for screening actions when a subset of a large *discrete set* of actions is to be selected. We show some difficulties that may be encountered in screening actions in this context. In particular, we demonstrate that not all individually dominated actions can be safely removed from consideration. Subsequently, we give conditions under which an individually dominated action cannot possibly belong to an optimal subset, and so can be safely screened from

Table 3.1: The Effect of Screening Actions in Subset Selection

Number of Actions in **A**	*% of Reduction of* **A**	*% of Reduction of Feasible Alternatives*			
		$m = 1$	$m = 2$	$m = \mid \mathbf{A} \mid /5$	m *Unrestricted*
10	10	10	20	20	50
	20	20	38	38	75
	30	30	53	53	87
20	10	10	19	37	75
	20	20	37	62	93
	30	30	52	79	98
50	10	10	19	69	96
	20	20	36	92	99.9
	30	30	51	98	99.9
100	10	10	19	90	99.9
	20	20	36	99	99.9
	30	30	51	99.9	99.9

the set of feasible actions.

Note that it is very useful to have reliable procedures to screen out individually inferior actions in subset selection problems. Table 3.1 demonstrates numerically how screening actions can reduce the size of a subset selection problem. In this table **A** is the set of actions and m is the number of actions to be selected. For instance. if **A** contains 20 actions of which four are to be selected. reducing **A** by 30% (6 actions) reduces the number of feasible alternatives by 79%. Screening individual actions can make many subset selection methodologies applicable by dramatically reducing the size of the problem.

We address the problem of screening actions in subset selection problems for two cases:

1. The m-best actions problem: the number of actions to be selected. m. is given

a priori.

2. The j-constraints problem: the possible subsets to be selected satisfy j constraints.

Note that the j-constraints problem is a generalization of the m-best actions problem.

3.2 The m-best Actions Problem

In this section, we consider a subset choice problem in which a pre-specified number of actions, $m \geq 2$, is to be selected. Such problems may arise in practical cases such as selecting sites for m new facilities or choosing candidates for m open positions. As will be seen below, the particular features of screening appear in even the simplest special case, $m = 2$.

Several procedures have been proposed for the m-best action problem in the MCDM literature. The method of Sage and White [109] obtains a preferred subset of actions by reducing the number of feasible actions. Dominance relations, restrictions on criterion weights, and available information about the DM's utility function are used for this purpose. Korhonen *et al.* [74] provide a procedure which requires the DM to compare pairs of actions. Using *cones of inferior solutions*, this procedure makes pairwise comparisons to eliminate inferior actions. Koksalan *et al.* [70] present some variations of the cones of inferior solutions to reduce the number of required comparisons. Assuming the DM's utility to be quasiconcave, Koksalan [69] presents a method similar to the methods of Korhonen *et al.* [74] and Koksalan *et al.* [70] for subset selection.

First we explain the terminology and notation. Throughout this thesis, the word *consequence* (outcome) will be used to refer to an objective measure of an action or alternative according to a criterion, and the word *value* will be used to refer to DM's subjective evaluation of a consequence. For example, a criterion for potential waste disposal sites is capacity. Capacity may be measured objectively as a consequence (such as millions of cubic meters) or subjectively, as the contribution of that additional capacity to a community's welfare.

It has been observed in practice that the consequence on any criterion is usually additive when more than one action is selected. Hence, the consequence of a subset of actions can be obtained by summing the consequences of each action in the subset. The value of a subset, however, cannot usually be obtained by summation of the individual values, often because of a saturation, or diminishing marginal value effect. Figure 3.1 shows a typical value function on a single consequence. Note that usually value is considered as a function of the consequences of all criteria, and may not be additive across criteria.

Let **A** be the set of actions, and **P** the set of criteria. We assume that **A** and **P** are both finite. Denote the consequence of action $a_l \in \mathbf{A}$ on criterion $p \in \mathbf{P}$ by $c_p(a_l) = c_p^l$. Thus, action a_l is described by its consequences,

$$c(a_l) = (c_1^l, \ldots, c_p^l, \ldots, c_{|\mathbf{P}|}^l).$$

The DM's overall value (utility) for action $a_l \in \mathbf{A}$ is a function, $v(.)$, of $|\mathbf{P}|$ variables, $c_1^l, \ldots, c_p^l, \ldots, c_{|\mathbf{P}|}^l$. Thus, we write

$$v(a_l) = v(c_1^l, \ldots, c_p^l, \ldots, c_{|\mathbf{P}|}^l).$$

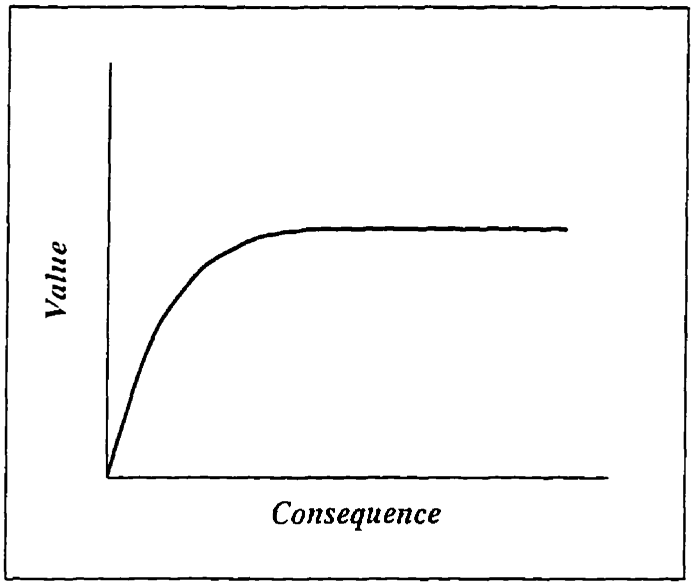


Figure 3.1: A Typical Monotonic Value Function.

Throughout, we assume that $v(.)$ is a strictly monotonically increasing function in each of its $|\mathbf{P}|$ arguments. For instance, the value function used in Example 3.2 (below) is a linear function of consequences, and hence:

$$v(a) = \sum_{p \in \mathbf{P}} w_p c_p(a). \tag{3.1}$$

where $w_p > 0$ is the global importance of criterion p. Note that the value function (3.1) is very simple and does not reflect any saturation effects.

Let $\mathbf{V}$ denote the set of all strictly monotonically increasing value functions. Thus, for each $v \in \mathbf{V}$, $v : \mathbf{R}^{|\mathbf{P}|} \rightarrow \mathbf{R}$ by $v(a_l) = v(c_1^l, \ldots, c_p^l, \ldots, c_{|\mathbf{P}|}^l)$, and v is strictly monotonically increasing in each argument. Note that the assumption of value as an increasing function of consequences is for exposition only, and can be easily expanded to include values that are decreasing in consequences such as cost or damage to the environment. Define $\mathbf{V}_L \subset \mathbf{V}$ to be the set of all linear value functions, as in (3.1).

In general, when subsets of actions are to be evaluated, it will be assumed that consequences are additive over subsets, but values need not be. Thus, if $\mathbf{S} \subseteq \mathbf{A}$, then

$$c_p(\mathbf{S}) = \sum_{a_l \in \mathbf{S}} c_p^l,$$
$$v(\mathbf{S}) = v(c_1(\mathbf{S}), \ldots, c_p(\mathbf{S}), \ldots, c_{|\mathbf{P}|}(\mathbf{S})).$$

An action $a_l \in \mathbf{A}$ *dominates* action $a_k \in \mathbf{A}$ iff $c_p^l \geq c_p^k \; \forall p \in \mathbf{P}$ and $\exists p \in \mathbf{P}$ such that $c_p^l > c_p^k$. In this case, we write $a_l \succ a_k$. An action that is not dominated by any other action is called *efficient*.

For any set of actions, $\mathbf{S} \subseteq \mathbf{A}$, let $Dom(\mathbf{S}) = \{a \in \mathbf{A} : a^0 \succ a \;\; \exists a^0 \in \mathbf{S}\}$. Thus, $Dom(\mathbf{S})$ is the set of all actions dominated by some action in $\mathbf{S}$. In particular, $Dom(\mathbf{A})$ is the set of all dominated actions, and

$$Eff(\mathbf{A}) = \mathbf{A} - Dom(\mathbf{A}),$$

is the set of all efficient actions.

For any action $a \in \mathbf{A}$, let $dom^{-1}(a) = \{a^0 \in \mathbf{A} : a^0 \succ a\}$. For any set of actions $\mathbf{S} \subseteq \mathbf{A}$, let $Dom^{-1}(\mathbf{S}) = \{a^0 \in \mathbf{A} : a^0 \succ a, \; \exists a \in \mathbf{S}\}$. Thus, $dom^{-1}(a)$ is the set of actions that dominate a, and $Dom^{-1}(\mathbf{S})$ is the set of actions that dominate some action in $\mathbf{S}$. Clearly,

$$Dom^{-1}(\mathbf{S}) = \bigcup\{dom^{-1}(a) : \; a \in \mathbf{S}\}.$$

Table 3.2: Consequences of Seven Feasible Actions

Criteria	*Actions*						
	a_1	a_2	a_3	a_4	a_5	a_6	a_7
p_1	4	7	7	9	5	7	8
p_2	3	2	2	1	3	4	3
p_3	7	8	5	4	6	9	6

For any action $a \in \mathbf{A}$ define

$$d(a) = dom^{-1}(a) \cap Eff(\mathbf{A}).$$

Then $d(a)$ is the set of efficient actions that dominate a. The following example illustrates the above definitions:

Example 3.1 Let $\mathbf{A} = \{a_1, \ldots, a_7\}$, $|\mathbf{P}| = 3$, and $\mathbf{S} = \{a_3, a_4, a_5\} \subseteq \mathbf{A}$. The consequences of the actions in $\mathbf{A}$ according to each criterion are shown in Table 3.2. For these actions,

$$Eff(\mathbf{A}) = \{a_4, a_6, a_7\}, \text{ and } Dom(\mathbf{A}) = \{a_1, a_2, a_3, a_5\},$$

$$Dom(\mathbf{S}) = \emptyset, \; dom^{-1}(a_5) = \{a_6, a_7\}, \; dom^{-1}(a_3) = \{a_2, a_6, a_7\}$$

$$Dom^{-1}(\mathbf{S}) = \{a_2, a_6, a_7\}, \text{ and}$$

$$d(a_5) = dom^{-1}(a_5) \cap Eff(\mathbf{A}) = \{a_6, a_7\}, \text{and}$$

$$d(a_3) = dom^{-1}(a_3) \cap Eff(\mathbf{A}) = \{a_6, a_7\}.$$

It is immediate that if $a \in Eff(\mathbf{A})$, then $d(a) = \emptyset$. The converse of this observation is also true.

Theorem 3.1 *If $a \in Dom(\mathbf{A})$, then $d(a) \neq \emptyset$.*

Proof: Suppose that $a_1 \in Dom(\mathbf{A})$. Then there exists $a_2 \neq a_1$ such that $a_2 \succ a_1$. If $a_2 \in Eff(\mathbf{A})$, the proof is complete. If not, $a_2 \in Dom(\mathbf{A})$, and there exists a_3 such that $a_3 \succ a_2$. Because domination is a transitive relation, $a_3 \succ a_1$ also. If $a_3 \in Eff(\mathbf{A})$, the proof is complete. Otherwise, note that a_1, a_2 and a_3 must be distinct, because domination is anti-reflexive. Continue in this way, choosing $a_4 \succ a_3$, *etc.* Since $\mathbf{A}$, and therefore $Dom(\mathbf{A})$, is finite, eventually $a_n \in Eff(\mathbf{A})$ will be found such that $a_n \succ a_{n-1} \succ \ldots \succ a_1$. □

As already pointed out, a widely used screening method for the best action selection problem is removal of dominated actions. The following simple example shows that in subset selection, good subsets may be lost when individually dominated actions are removed.

Example 3.2 Assume that a pair of actions is to be selected. The consequences of actions a_1, a_2, a_3, a_4, and a_5 according to criteria, 1,2, and 3, are given in Table 3.3. Observe that action a_1 is dominated by action a_5. Following the standard screening procedure, action a_1 is removed from further examination; it is assumed that the best pair of actions is to be found among the remaining non-dominated actions. Suppose that the DM's value function is linear additive with global weights (0.2, 0.6, 0.2) for the three criteria. Table 3.4 shows the scores of all possible pairs of actions on each criterion and their overall values. Clearly, the combination of actions a_1 and a_5 is better than any other combination of two actions. Hence, it would be a mistake to eliminate a_1 from the set of feasible actions–the optimal combination of two actions would become unavailable.

As Example 3.2 shows, when a subset of actions is to be selected, the dominated set $Dom(\mathbf{A})$ should not be removed without further consideration. Yet, attempting to include all feasible actions in the MCDM process may result in a problem of

Table 3.3: Consequences of Five Feasible Actions

Criteria	weight	*Actions*				
		a_1	a_2	a_3	a_4	a_5
p_1	0.2	6	10	6	8	7
p_2	0.6	7	3	4	6	9
p_3	0.2	4	2	7	3	5

Table 3.4: Consequences of Possible Pairs of Actions

Criteria	weight	*Pair of Actions*				
		a_1, a_2	a_1, a_3	a_1, a_4	a_1, a_5	a_2, a_3
p_1	0.2	16	12	14	13	16
p_2	0.6	10	11	13	16	7
p_3	0.2	6	11	7	9	9
Overall Value	1.0	10.4	11.2	12	14	9.2
Criteria	weight	*Pair of Actions*				
		a_2, a_4	a_2, a_5	a_3, a_4	a_3, a_5	a_4, a_5
p_1	0.2	18	17	14	13	15
p_2	0.6	9	12	14	13	15
p_3	0.2	5	7	10	12	8
Overall Value	1.0	10	12	10.8	12.8	13.6

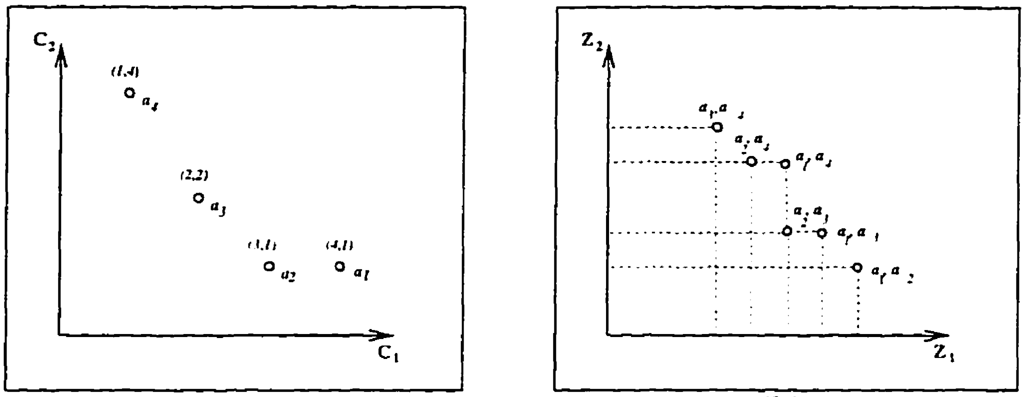


Figure 3.2: (a)Individual Actions in Criterion Space(b)Pairs of Actions in Criterion Space

unmanageable size. Therefore, it would be very useful to have exact (or heuristic) methods to eliminate those dominated actions that cannot possibly be in a best subset, no matter what the DM's value function.

Clearly, in an m-best actions problem, an action which is individually dominated can be included in an efficient subset. A geometric illustration is given by Figure 3.2. Action a_2 is dominated by action a_1 as shown in Figure 3.2(a), but the pair $\{a_1, a_2\}$ is efficient in the 2-best actions problem (Figure 3.2 (b)).

In what follows, we discuss conditions under which a dominated action can be safely removed in a subset selection problem. The following theorem demonstrates that a dominated action can be included in an optimal subset, $\mathbf{A}^*$, only if all actions that dominate it are also included in $\mathbf{A}^*$.

Theorem 3.2 *Suppose that* $\mathbf{A}^*$ *is an m-best subset for some* $m \geq 2$. *Let* $a_l \in Dom(\mathbf{A})$. *If* $dom^{-1}(a_l) \not\subset \mathbf{A}^*$, *then* $a_l \notin \mathbf{A}^*$.

Proof: Assume that $a_l \in \mathbf{A}^*$ but $dom^{-1}(a_l) \not\subset \mathbf{A}^*$. Then there exists $a_k \in Eff(\mathbf{A})$

such that $a_k \succ a_l$ and $a_k \notin \mathbf{A}^*$. Consider $\mathbf{A}^0 = \mathbf{A}^* \cup \{a_k\} - \{a_l\}$. Clearly, $|\mathbf{A}^0| = m$, and for any criterion p, $c_p(\mathbf{A}^0) = c_p(\mathbf{A}^*) + c_p^k - c_p^l$. It follows that, for every criterion p, $c_p(\mathbf{A}^0) \geq c_p(\mathbf{A}^*)$ and, for some criterion, $p' \in \mathbf{P}$, $c_{p'}(\mathbf{A}^0) > c_{p'}(\mathbf{A}^*)$. It follows from monotonicity that $v(\mathbf{A}^0) > v(\mathbf{A}^*)$, contradicting the hypothesis that the value of $\mathbf{A}^*$ is a maximum. Thus, $a_l \notin \mathbf{A}^*$, completing the proof. □

According to Theorem 3.2, if the best set of actions $\mathbf{A}^*$ does not contain $dom^{-1}(a_l)$, then a_l can be eelimminated in the selection process. On the other hand, it follows that

$$a_l \in \mathbf{A}^* \implies dom^{-1}(a_l) \subset \mathbf{A}^*.$$

so that, when a dominated action is in the optimal subset, then every action that dominates it (and in particular every efficient actions that dominates it) must also be in the optimal subset.

To apply Theorem 3.2 to screening, one must examine whether $dom^{-1}(a_l) \subset \mathbf{A}^*$. For this purpose, one often can use an easily evaluated approximation to $v(\cdot)$ to demonstrate that $dom^{-1}(a_l) \not\subset \mathbf{A}^*$, which implies that a_l need not be considered. See [7, 83, 68], for example, for procedures to approximate the value function.

In an m-best actions problem, suppose that action $a \in Dom(\mathbf{A})$ satisfies $|dom^{-1}(a)| \geq m$. Then Theorem 2 implies that a can be screened out, because $|dom^{-1}(a) \cup \{a\}| > m + 1$.

For action $a \in \mathbf{A}$ satisfying $|dom^{-1}(a)| < m$, we propose below a procedure to determine whether a can possibly be included in the optimal subset under any monotonic value function, *i.e.* whether a can be screened out. For this purpose, we introduce the concept of *Dominated Potentially Optimal* (DPO). The idea is that among dominated actions only a DPO_m action can be included among the m best actions. We first define the concept of *Potentially Optimal* (PO):

Definition 3.1 *Action* $a_i \in \mathbf{A}$ *is* potentially optimal *(PO) if there exists at least one* $v \in \mathbf{V}$ *such that* $v(a_i) \geq v(a_l)$ *for all* $a_l \in \mathbf{A}$.

The set of potentially optimal actions in $\mathbf{A}$ is denoted $PO(\mathbf{A})$. The following mathematical program can be used to determine whether action a_i is potentially optimal:

$$
\begin{aligned}
(\mathbf{D1}(a_i)) \qquad & \textbf{Minimize} \quad \delta \\
& \textbf{Subject to:} \\
& \qquad v(a_i) - v(a_l) + \delta \geq 0, \qquad a_l \in \mathbf{A} \setminus \{a_i\}, \\
& \qquad v \in \mathbf{V}.
\end{aligned}
$$

The above program seeks a value function v that minimizes δ. For instance, if $v \in \mathbf{V}_L$ as in (3.1), the program $(\mathbf{D1}(a_i))$ determines the criterion weights, w_p, that minimize δ . If the optimal value of this problem is non-positive, then $a_i \in PO(\mathbf{A})$, because $\delta^* \leq 0$ implies that there is a value function that makes a_i at least as preferable as all other actions.

Example 3.3 Consider Table 3.3. To examine if action a_2 is potentially optimal within $\{a_1, \ldots, a_5\}$ based on linear value functions $(\mathbf{V}_L)$ only, the following program is constructed:

$$
\begin{aligned}
(\mathbf{D1}(a_2)) \qquad & \textbf{Minimize} \quad \delta \\
& \textbf{Subject to:} \\
& \qquad 4w_1 - 4w_2 - 2w_3 + \delta \geq 0,
\end{aligned}
$$

$$4w_1 - 1w_2 - 5w_3 + \delta \geq 0,$$
$$2w_1 - 3w_2 - 1w_3 + \delta \geq 0,$$
$$3w_1 - 6w_2 - 3w_3 + \delta \geq 0,$$
$$w_1 + w_2 + w_3 = 1,$$
$$w_p \geq \epsilon, \qquad p = 1, 2, 3,$$

where ϵ in the last set of constraints is a small positive number and ensures that the value function is strictly increasing. The optimal solution of this program is $\delta^* = -1.92$ with $w_1 = 0.98, w_2 = 0.01$, and $w_3 = 0.01$. Hence, a_2 is potentially optimal. However, if we construct a similar program to determine whether a_4 is PO, we obtain $\delta^* > 0$. Hence, a_4 is not potentially optimal.

The concept of potentially optimal actions has been addressed by some researchers in the context of multi-attribute decision theory (see for example, [47], [60], [5], [133]). This notion has been especially useful in situations where some partial information on the DM's preferences is available [5]. White [133] proves that under a strictly monotonic function a potentially optimal action is always non-dominated. However, there may be some non-dominated actions which are not potentially optimal. Therefore,

$$PO(\mathbf{A}) \subseteq Eff(\mathbf{A}). \tag{3.2}$$

Hence, when the value function is strictly monotonic, a dominated action in **A** cannot be potentially optimal in **A**. In general, however, there is no such relation between potentially optimal and efficient actions. Consider a 2-criterion problem with the actions shown in Figure 3.3. Assume that $v(a) = w_1c_1(a) + w_2c_2(a)$, such that $w_1 \geq 0, w_2 \geq 0$. Then, action a_1 is dominated but potentially optimal, and

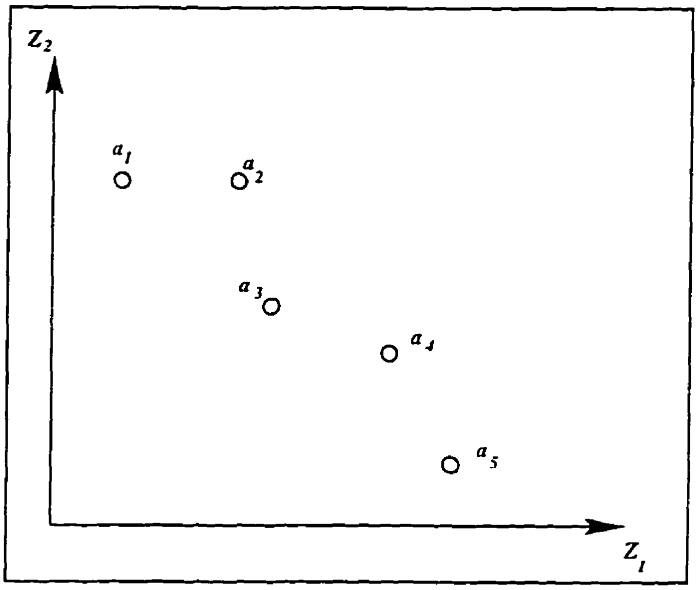


Figure 3.3: Dominated and Potentially Optimal Actions.

action a_3 is efficient but not potentially optimal.

Now, consider Example 3.2 (see Table 3.3) and assume that $v \in \mathbf{V}_L$. Then action a_4 is non-dominated, but it is not potentially optimal for $v \in \mathbf{V}_L$, because no linear value function can make it better than both a_5 and a_2 simultaneously.[1] In fact, Hazen [47] proves that for an additive value function, if an action is non-dominated but not potentially optimal, then it is dominated by a convex combination of other actions. In Example 3.2, action a_4 is dominated by a convex combination of actions a_5 and a_2.

It is noteworthy that if the DM is confident of some relations on the parameters of v, such as upper and lower bounds on criterion weights, or if the DM can holistically state some relations among actions, then this information may be added to the set of constraints of $(\mathrm{D1}(a_i))$. In this way, more specific solutions may be

[1] *Note that if the value function is not restricted to* $\mathbf{V}_L$, a_4 *is potentially optimal. For instance,* a_4 *maximizes the monotonic value function* $v(c_1, c_2, c_3) = c_1 + c_2 + c_3 + 20c_1.min\{c_1, c_2, c_3\}$.

obtained.

Example 3.4 Consider Example 3.3 in which it was shown that action a_2 is potentially optimal. Now assume that the DM specifies the following information for criterion importance:

$$0.1 \leq w_1 \leq 0.3; \quad 0.3 \leq w_2 \leq 0.6; \quad 0.2 \leq w_3 \leq 0.5.$$

Under this partial information, $\delta^* > 0$ in $(\mathbf{D}_1(a_2))$. Hence, action a_2 is no longer potentially optimal.

We now define the notion of dominated potentially optimal. The concept of PO was defined with respect to the standard problem of selecting the best action. But membership of an action in the set of DPO actions depends on the number of actions to be selected. Let $\mathbf{A}_{(n)}$ denote the collection of all subsets of $\mathbf{A}$ that contain n actions. For instance, $\mathbf{A}_{(2)}$ is the collection of all pairs of actions in $\mathbf{A}$. Hence, similar to $PO(\mathbf{A})$ which denotes individual potentially optimal actions, $PO(\mathbf{A}_{(n)})$ is the set of potentially optimal subsets with cardinality n within set $\mathbf{A}_{(n)}$. Note that $\mathbf{A}^i_{(n)} \in PO(\mathbf{A}_{(n)})$ implies that there exists a value function such that $\mathbf{A}^i_{(n)}$ is as good as any other subset with cardinality n. In the m-best actions problem, the concept of " DPO " is defined as follows:

Definition 3.2 *Action $a_k \in \mathbf{A}$ belongs to the Dominated Potentially Optimal set of the m-best actions problems, $DPO_m(\mathbf{A})$, if*
1) $a_k \in Dom(\mathbf{A})$, and
2) there exists $\mathbf{A}^i_{(m)} \in \mathbf{A}_{(m)}$ such that $a_k \in \mathbf{A}^i_{(m)}$, and $\mathbf{A}^i_{(m)} \in PO(\mathbf{A}_{(m)})$.

According to Definition 3.2 an action is $DPO_m(\mathbf{A})$ if it is dominated, yet is also a member of at least one potentially optimal subset with cardinality m. Moreover, Definition 3.2 states that if a dominated action, a_k, is not in $DPO_m(\mathbf{A})$, then there is no $\mathbf{A}^i_{(m)} \in \mathbf{A}_{(m)}$ that includes a_k such that $\mathbf{A}^i_{(m)} \in PO(\mathbf{A}_{(m)})$. Further, according to the definition of PO actions, a_k cannot be a member of any m-best subset of action. Hence, a_k can be removed from the set of actions. In other words,

$$a_k \notin DPO_m(\mathbf{A}) \Longrightarrow a_k \notin \mathbf{A}^* \tag{3.3}$$

To use Definition 3.2 to determine whether $a_k \in DPO_m(\mathbf{A})$, the sets $\mathbf{A}_{(m)}$ and $PO(\mathbf{A}_{(m)})$ must be known. However, as pointed out above, $\mathbf{A}_{(m)}$ may be quite large: generating $PO(\mathbf{A}_{(m)})$ may be cumbersome and excessively demanding in time and computation. Hence, it would be quite useful to determine membership of an action in $DPO_m(\mathbf{A})$ by examining individual actions rather than subsets in $\mathbf{A}_{(m)}$. The following theorem provides an alternative way to determine if a_k is a DPO_m action. Note that if $|dom^{-1}(a_k)| \geq m$, then $a_k \notin DPO_m(\mathbf{A})$.

Theorem 3.3 *Let $a_k \in Dom(\mathbf{A})$, and $m > |dom^{-1}(a_k)|$. Then $a_k \in DPO_m(\mathbf{A})$ iff for some $q \geq |\mathbf{A}| - m$, $\exists \mathbf{S} \in \mathbf{A}_{(q+1)}$, such that $a_k \in \mathbf{S}$ and $a_k \in PO(\mathbf{S})$.*

Proof:

1) Suppose that q and $\mathbf{S}$ are as stated in the hypothesis and that $a_k \in PO(\mathbf{S})$. We prove that $a_k \in \mathbf{T}$ for some $\mathbf{T} \in \mathbf{A}_{(m)}$ such that $\mathbf{T} \in PO(\mathbf{A}_{(m)})$. By assumption, there exists $v_0 \in \mathbf{V}$ such that $v_0(a_k) \geq v_0(a_l)$ $\forall a_l \in \mathbf{S}$. Note that $|\mathbf{S}| = q+1$, or $|\mathbf{S}| \geq |\mathbf{A}| - m + 1$.

Under value function v_0, a_k scores at least as high as $|\mathbf{A}| - m + 1$ actions in $\mathbf{A}$. Hence, there must exist $\mathbf{T} \in \mathbf{A}_{(m)}$ containing a_k such that $\mathbf{T}$ is optimal under v_0. In other words,

$$a_k \in \mathbf{T} \in PO(\mathbf{A}_{(m)}).$$

2) Suppose that $a_k \in \mathbf{T} \in \mathbf{A}_{(m)}$, and $\mathbf{T} \in PO(\mathbf{A}_{(m)})$. We prove that there exists $q \geq |\mathbf{A}| - m$ and $\mathbf{S} \in \mathbf{A}_{(q+1)}$, such that $a_k \in \mathbf{S}$ and $a_k \in PO(\mathbf{S})$.

Assume that $\mathbf{T}$ is optimal under $v_0 \in \mathbf{V}$. Under v_0, a_k is no less preferred than at least $|\mathbf{A}| - m$ actions, because otherwise an action not in $\mathbf{T}$ would have been included in it. Hence, $\mathbf{S} = \mathbf{A} - \mathbf{T} \cup \{a_k\}$ has the property that $|\mathbf{S}| =| \mathbf{A} | -m + 1$ and $v_0(a_k) \geq v_0(a_l)$, $\forall a_l \in \mathbf{S}$. Thus $a_k \in PO(\mathbf{S})$. □

In Theorem 3.3, the set $\mathbf{S}$ cannot contain $dom^{-1}(a_k)$, because, according to relation (3.2), a_k cannot be PO in a set that contains actions that dominate it. Hence, in Theorem 3.3, $\mathbf{S} \subseteq \mathbf{A} - dom^{-1}(a_k)$.

Example 3.5 *Consider Table 3.2 in Example 3.1 with* $m = 4$. *Note that* $a_5 \in Dom(\mathbf{A})$, *and* $dom^{-1}(a_5) = \{a_6, a_7\}$. *Because* $m > |dom^{-1}(a_5)|$, *Theorem 3.3 requires that* $q \geq 3$. *To determine whether* $a_5 \in DPO(\mathbf{A})$, *it is sufficient to determine whether there exists a set* $\mathbf{S} \subseteq \mathbf{A} - dom^{-1}(a_5)$, *such that* $|\mathbf{S}| = 4, a_5 \in \mathbf{S}$, *and* $a_5 \in PO(\mathbf{S})$.

To clarify Theorem 3.3, consider the problem of selecting a pair of actions ($m = 2$), where $| dom^{-1}(a_k) |= 1$. The following corollary is the immediate result of Theorem 3.3 and Definition 3.2:

Corollary 3.1 *Suppose* $a_k \in Dom(\mathbf{A})$, *and* $|dom^{-1}(a_k)| = m - 1$. *Then* $a_k \in DPO_m(\mathbf{A})$ *iff* $a_k \in PO(\mathbf{A} - dom^{-1}(a_k))$.

We utilize the following mathematical program to show whether $a_k \in DPO(\mathbf{A})$ in an m-best actions problem:

(**D2**(a_k)) **Minimize** δ

Subject to :

$$\begin{array}{lll}
(a) & v(a_k) - v(a_l) + \delta \geq -M(1 - \alpha_l) & a_l \in \mathbf{A} \setminus \{dom^{-1}(a_k) \cup \{a_k\}\}. \\
(b) & \sum_{l \in \mathbf{A} \setminus \{dom^{-1}(a_k) \cup a_k\}} \alpha_l = q. & \\
(c) & \alpha_l \in \{0.1\}. & l \in \mathbf{A} \setminus \{dom^{-1}(a_k) \cup \{a_k\}\}. \\
(d) & v \in \mathbf{V}. &
\end{array}$$

where M is a sufficiently large number and $q \geq |\mathbf{A}| - m$. The following theorem shows how Program $\mathbf{D2}(a_k)$ determines whether action a_k is DPO:

Theorem 3.4 *Let $a_k \in Dom(\mathbf{A})$ and $m > |dom^{-1}(a_k)|$. Then $a_k \in DPO_m(\mathbf{A})$ iff the optimal solution of Problem $\mathbf{D2}(a_k)$ is non-positive ($\delta^* \leq 0$).*

Proof: First, we show that when $\delta^* \leq 0$, then there exists a $v \in \mathbf{V}$ and a set $\mathbf{S} \subseteq \mathbf{A} \setminus (dom^{-1}(a_k) \cup \{a_k\})$ such that

$$v(a_k) \geq v(a_l) \quad \forall a_l \in \mathbf{S}.$$

This implies that $a_k \in DPO_m(\mathbf{A})$, according to Definition 3.2 and Theorem 3.3.

Note that if $\alpha_l = 0$ then constraint l in the constraint set (a) becomes

$$v(a_k) - v(a_l) + \delta \geq -M. \tag{3.4}$$

Because M is a large number, (3.4) is always true, and will not affect the solution of ($\mathbf{D2}(a_k)$). On the other hand, when $\alpha_l = 1$, constraint l in constraint set (a)

becomes:

$$v(a_k) - v(a_l) + \delta \geq 0. \tag{3.5}$$

Let $N^1 = |\{l : \alpha_l = 1\}|$ at the optimal solution of Problem $\mathbf{D2}(a_k)$. To satisfy constraint (b), N^1 must equal q. In this case, there are q constraints similar to (3.5) for different values of l such that $a_l \in \mathbf{A} \setminus (dom^{-1}(a_k) \cup \{a_k\})$. Therefore, $\delta^* \leq 0$ means that there exists a value function $v_0 \in \mathbf{V}$ such that $v_0(a_k) \geq v_0(a_l)$ for at least q actions, a_l. Hence, a_k is PO in a set of $q+1$ actions within $\mathbf{A} \setminus dom^{-1}(a_k)$ and, according to Theorem 3.3, $a_k \in DPO_m(\mathbf{A})$.

Now we show that if a_k is in $DPO_m(\mathbf{A})$, the optimal solution of $\mathbf{D2}(a_k)$ is non-positive. Suppose that $\delta^* > 0$. Hence, Program $\mathbf{D2}(a_k)$ cannot find a set $\mathbf{S} \subseteq \mathbf{A}_{(q)}$ such that $v_0(a_k) \geq v_0(a_l)$ for a $v_0 \in \mathbf{V}$. From Theorem 3.3, a_k is not in $DPO_m(\mathbf{A})$. $\square$

For the special case when $m = 2$ and $|\ dom^{-1}(a_k)\ | = 1$, Theorem 3.3 requires that $q \geq |\ \mathbf{A}\ | -1$. Hence, action a_k is to be compared with all actions in $\mathbf{A} - dom^{-1}(a_k)$. In Program $\mathbf{D2}(a_k)$, constraint (b) becomes

$$\alpha_l = 1, \quad \forall a_l \in \mathbf{A} \setminus \{dom^{-1}(a_k) \cup \{a_k\}\}.$$

Then Program $(\mathbf{D2}(a_k))$ reduces to the following:

$(\mathbf{D3}(a_k))$ **Minimize** δ

Subject to :

$$v(a_k) - v(a_l) + \delta \geq 0, \quad a_l \in \mathbf{A} \setminus (dom^{-1}(a_k) \cup \{a_k\}),$$

$$v \in \mathbf{V}.$$

Example 3.6 Consider Table 3.3, and assume that a pair of actions is to be selected. To determine if action a_1, which is dominated by a_5, is $DPO_2(\mathbf{A})$ under *linear* value functions. we note that $dom^{-1}(a_1) = \{a_5\}$. Consequently, we construct the following program:

$$
\begin{array}{lll}
(\mathbf{D3}(a_1)) & \textbf{Minimize} & \delta \\
 & \textbf{Subject to}: & \\
 & & -4w_1 + 4w_2 + 2w_3 + \delta \geq 0, \\
 & & 3w_2 - 3w_3 + \delta \geq 0, \\
 & & -2w_1 + w_2 + w_3 + \delta \geq 0, \\
 & & w_1 + w_2 + w_3 = 1, \\
 & & w_p \geq \epsilon, \quad p = 1, 2, 3,
\end{array}
$$

where. $w_p > 0$ is the weight of criterion p in a linear value function in $\mathbf{V}_L$ as in (3.1). The optimal solution of the above program is $\delta^* = -0.97$. Hence, a_1 is in $DPO_2(\mathbf{A})$. On the other hand, if the score of action a_1 on the second criterion were changed to 5. then the optimal value of Problem $\mathbf{D3}(a_1)$ would be $\delta^* = 0.33$, implying that a_1 is not in $DPO_2(\mathbf{A})$.

The following theorem shows that to determine whether an action is DPO. it is only necessary to compare it with actions in $Eff(\mathbf{A}) \setminus (dom^{-1}(a_k) \cup \{a_k\})$ rather than $\mathbf{A} \setminus (dom^{-1}(a_k) \cup \{a_k\})$. Clearly. this reduces the number of actions that a_k should be compared with.

Theorem 3.5 *Let $a_k \in Dom(\mathbf{A})$ and $m >| dom^{-1}(a_k) |$ in an m-best actions problem. Then,*

$$a_k \in DPO_m(\mathbf{A}) \Longleftrightarrow a_k \in DPO_m(Eff(\mathbf{A}) \cup \{a_k\}).$$

Proof: Clearly, if $a_k \in DPO_m(\mathbf{A})$, then $a_k \in DPO_m(Eff(\mathbf{A}) \cup \{a_k\})$, because $a_k \in Dom(\mathbf{A})$ and the former set contains the latter: if a_k is DPO in a set, it is also DPO in any subset.

Now, we show that

$$a_k \in DPO_m(Eff(\mathbf{A}) \cup \{a_k\}) \Longrightarrow a_k \in DPO_m(\mathbf{A})$$

According to Theorem 3.1, any dominated action is dominated by at least one efficient action. On the other hand, according to Theorem 3.4, a dominated action a_k is DPO iff the optimal solution of Problem $\mathbf{D2}(a_k)$ is $\delta^* \leq 0$. Now, suppose $a_i \in Dom(\mathbf{A})$ and $a_j \in dom^{-1}(a_i)$.

Consider the two constraints in Problem $\mathbf{D2}(a_k)$ in which action a_k is compared with actions a_i and a_j. First, suppose that α_i and α_j have been set to be one in the optimal solution of $(\mathbf{D2}(a_k))$, so that

$$(1) \qquad v(a_k) - v(a_j) + \delta^* \geq 0$$

$$(2) \qquad v(a_k) - v(a_i) + \delta^* \geq 0.$$

at the optimal solution. Since v is an increasing monotonic function, $a_j \succ a_i$ implies $v(a_j) > v(a_i)$. Therefore, if $\delta^* \leq 0$ in (1), for some $v \in \mathbf{V}$, then this v also satisfies

(2) with $\delta^* \leq 0$. Hence, including constraint (2) in Problem $\mathbf{D2}(a_k)$ does not change the value of the optimal solution, permitting us to remove all constraints corresponding to dominated actions in $(\mathbf{D2}(a_k))$. Now suppose that in the optimal solution of $(\mathbf{D2}(a_k))$, $\alpha_i = 1$ and $\alpha_j = 0$, so that

$$(1) \qquad v(a_k) - v(a_j) + \delta^* \geq -M$$

$$(2) \qquad v(a_k) - v(a_i) + \delta^* \geq 0.$$

In this case, removal of a_i may change the optimal solution of $(\mathbf{D2}(a_k))$. But, note that removing a_i, decreases the number of α that has to be one to ensure $a_k \in DPO_m(\mathbf{A})$, because, in this case, $q = |\mathbf{A} \setminus \{a_i\}| - m$.

Clearly, when $\alpha_i = 0$ and $\alpha_j = 1$ or when both α_i and α_j are zero, removing a_i does not change the optimal solution of $(\mathbf{D2}(a_k))$. This completes the proof that any action which is in $DPO_m(Eff(\mathbf{A}) \cup \{a_k\})$ is also in $DPO_m(\mathbf{A})$, and vice versa. □

Note that when one applies the result of Theorem 3.5 to Program $\mathbf{D2}(a_k)$, $q = | Eff(\mathbf{A}) | - m$. To summarize the discussion in this section, we first demonstrated that in a subset selection problem, an individually dominated action should not be removed from the set of actions without further examination. Then we defined the concept of a DPO_m action in m-best actions problem (Definition 3.2). Subsequently, a method to identify a DPO_m action by inspecting individual actions rather than subsets of actions (Theorem 3.3) was exhibited. Program $\mathbf{D2}(a_k)$ was introduced as a practical way to identify DPO_m actions. Figure 3.4 demonstrates the relationship among some of the screening concepts introduced in this section. The following example illustrates our results:

Example 3.7 Consider Table 3.5 which shows the consequences of eight actions

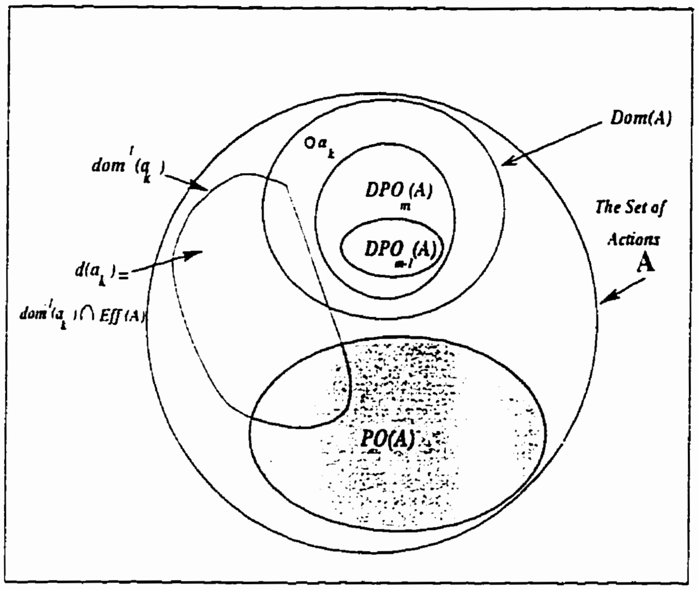


Figure 3.4: Relationship among Screening Concepts.

according to three criteria. Suppose that $m = 4$ actions are to be selected. The dominance relations are

$$a_5 \succ a_4, \quad a_7 \succ a_1, \quad a_8 \succ a_1, \quad \text{and} \quad a_6 \succ a_3.$$

To examine whether a_1, which is dominated by a_7 and a_8, can be removed from the set of actions, we can do the following:

According to Definition 3.1, $a_k \in \mathbf{A}^*$ only if $a_k \in DPO_m(\mathbf{A})$. Hence, we must determine whether $a_k \in DPO_m(\mathbf{A})$. Because, $dom^{-1}(a_1) = \{a_7, a_8\}$, the number of actions that would be selected in addition to $dom^{-1}(a_1)$ is

$$m- \mid dom^{-1}(a_1) \mid = 2.$$

Hence, the minimum number of actions to which a_1 must be preferred to be eligible for inclusion in $\mathbf{A}^*$, is

$$|\mathbf{A} \setminus (dom^{-1}(a_1) \cup \{a_1\})| - 2 + 1 = 4.$$

Note that as introduced in Theorem 3.3, $q = |\mathbf{A}| - m = 4$, which equals the above expression. According to Theorem 3.4, $a_1 \in DPO_m(\mathbf{A})$ *iff* the optimal solution of the following mathematical program is non-positive:

$$(\mathbf{D2}(a_1)) \qquad \textbf{Minimize} \quad \delta$$

Subject to :

$$v(a_1) - v(a_l) + \delta \geq -M(1 - \alpha_l) \qquad a_l \in \mathbf{A} \setminus \{a_1, a_7, a_8\}$$

$$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 4,$$

$$\alpha_l \in \{0, 1\} \qquad l \in \mathbf{A} \setminus \{a_1, a_7, a_8\},$$

$$v \in \mathbf{V}.$$

For linear value functions, of the form (3.1), the Program $\mathbf{D2}(a_1)$ becomes:

$$(\mathbf{D}'\mathbf{2}(a_1)) \qquad \textbf{Minimize} \quad \delta$$

Subject to :

$$-5w_1 + w_2 + w_3 + \delta \geq -M(1 - \alpha_2)$$

$$4w_1 - 3w_2 - 2w_3 + \delta \geq -M(1 - \alpha_3)$$

$$2w_1 + w_2 - 5w_3 + \delta \geq -M(1 - \alpha_4)$$

$$w_1 - w_2 - 6w_3 + \delta \geq -M(1 - \alpha_5)$$

$$3w_1 - 4w_2 - 3w_3 + \delta \geq -M(1 - \alpha_6)$$

$$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 4$$

$$w_1 + w_2 + w_3 = 1;$$

$$w_p \geq \epsilon, \quad p = 1, 2, 3.$$

$$\alpha_l \in \{0, 1\} \qquad l \in \mathbf{A} \setminus \{a_1, a_7, a_8\}.$$

The optimal solution of $(\mathbf{D}'\mathbf{2}(a_1))$ is $\delta^* = -0.82$ with $w_1 = 0.96, w_2 = 0.02, w_3 = 0.02$. The program assigns the value 1 to the binary variables $\alpha_3, \alpha_4, \alpha_5, \alpha_6$, and selects $\mathbf{A}_{(q)} = \{a_3, a_4, a_5, a_6\}$. Therefore, action a_1 is DPO_4 and should not be removed from the set of actions. If the DM specifies $w_1 \leq w_2 \leq w_3$, then action a_1 is not potentially optimal. Note that we can utilize Theorem 3.5 to simplify the above program. Since $\{a_3, a_4\} \in Dom(\mathbf{A})$, it is not necessary to compare action a_1 with actions a_3 and a_4; one can use the same procedure with $q = \mid Eff(\mathbf{A}) \mid - m = 2$. In this case, Program $\mathbf{D}'2(a_1)$ reduces to the following program:

$(\mathbf{D}''2(a_1))$ **Minimize** δ

Subject to :

$$-5w_1 + w_2 + w_3 + \delta \geq -M(1 - \alpha_2)$$

$$w_1 - w_2 - 6w_3 + \delta \geq -M(1 - \alpha_5)$$

$$3w_1 - 4w_2 - 3w_3 + \delta \geq -M(1 - \alpha_6)$$

$$\alpha_2 + \alpha_5 + \alpha_6 = 2$$

$$w_1 + w_2 + w_3 = 1;$$

$$w_p \geq \epsilon, \quad p = 1, 2, 3,$$

$$\alpha_l \in \{0, 1\} \qquad l \in \mathbf{A} \setminus \{a_1, a_3, a_4, a_7, a_8\}.$$

Table 3.5: Consequences of Actions According to Three Criteria

Criteria	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
p_1	5	10	1	3	4	2	8	6
p_2	4	3	7	3	5	8	5	6
p_3	3	2	5	8	9	6	4	5

The optimal solution of $(\mathbf{D}''\mathbf{2}(a_1))$ is $\delta^* = -0.82$. indicating that a_1 is DPO_4. Note that if $m = 3$ actions (rather than four) are to be selected. the optimal solution of the Program $\mathbf{D}'\mathbf{2}(a_1)$ or $\mathbf{D}''\mathbf{2}(a_1)$ is $\delta^* = 1.219$. Hence. in this case action a_1 is not in DPO_3 and can be eliminated from the set of feasible actions.

3.3 The j-Constraints Problem

We now address the same problem as in previous section. except that the number of actions to be selected is not pre-specified. Instead. the subsets that may be selected are defined by a set of constraints. In fact. the j-constraints problem is a generalization of the m-best actions problem. because m-best actions problem is a 1-constraints problem in which the only constraint specifies that m actions are to be selected.

Here. we consider only the *binary multidimensional knapsack* problem with multiple objectives. which commonly arises in project selection. Without loss of generality. assume that all criteria are to be maximized. The problem under consideration is as follows:

$$\textbf{(D4)} \qquad \textbf{Maximize} \quad z_p = \sum_{i \in \mathbf{A}} c_p^i x_i, \qquad p = 1, \ldots, |\, \mathbf{P} \,|$$

Subject to :

$$\sum_{i \in \mathbf{A}} b_{ij} x_i \leq B_j, \qquad j = 1, \ldots, |\, \mathbf{J} \,|,$$

$$x_i \in \{0, 1\}.$$

where $\mathbf{P}$ is the set of criteria. J is the set of constraints. $c_p^i \geq 0$ is the consequence of action a_i on criterion p. $b_{ij} \geq 0$ is the rate of consumption of the jth resource by action a_i. and B_j is the total amount of the jth resource available for consumption. The binary variable x_i is defined as follows:

$$x_i = \begin{cases} 1 & \text{if } a_i \text{ is selected;} \\ 0 & \text{if } a_i \text{ is not selected.} \end{cases}$$

Note that the main difference between the j-constraints problem and the m-best actions problem is that here a dominated action may be included in $\mathbf{A}^*$. because of its low rate of resource consumption. In the context of Program **D4** we define the notion of *T-efficiency*[2] of an action as follows:

Definition 3.3 *Action a_i is T-efficient (T-nondominated) if there does not exist another action a_h such that*
(1) $c_p^i \leq c_p^h \ \forall p \in \mathbf{P}$, and
(2) $b_{ij} \geq b_{hj} \ \forall j \in \mathbf{J}$.
with one of the inequalities in (1) being strict.

When an action is T-dominated. there is another action that would improve any selection of actions. We denote the set of T-dominated actions by $Dom_T(\mathbf{A})$.

[2] *T-efficiency stands for Total-efficiency.*

and its complement by $Eff_T(\mathbf{A}) = \mathbf{A} - Dom_T(\mathbf{A})$. For each $a_k \in Dom_T(\mathbf{A})$, the set of actions dominating a_k is denoted by $dom_T^{-1}(a_k)$. Moreover, $d_T(a_k) = dom_T^{-1}(a_k) \cap Eff_T(\mathbf{A})$. Clearly, any action in $dom_T^{-1}(a_k)$ is preferred to a_k for inclusion in the best set of actions, because each objective will be increased by an equal or greater amount with equal or less consumption of resources.

As discussed in Chapter 2, it is difficult and time-consuming to solve a multiple criteria integer problem, such as Program **D4** [127]. Hence, it is useful to remove any T-dominated actions and thereby reduce the size of the problem. However, an individually T-dominated action may be in the best set of actions, unless some specific conditions hold.

The following theorem, which is similar to the Theorem 3.2, shows that a T-dominated action, a_k, cannot be included in the optimal solution of (**D4**) when $dom_T^{-1}(a_k) \not\subset \mathbf{A}^*$.

Theorem 3.6 *Suppose* $\mathbf{A}^*$ *is the optimal solution of Program* **D4**. *Let* $a_k \in Dom_T(\mathbf{A})$. *Then*

$$dom_T^{-1}(a_k) \not\subset \mathbf{A}^* \Longrightarrow a_k \notin \mathbf{A}^*.$$

Proof: Suppose $a_k \in \mathbf{A}^*$ and $dom_T^{-1}(a_k) \not\subset \mathbf{A}^*$. Consider an action $a_e \in dom_T^{-1}(a_k)$. Construct another set $\mathbf{A}^0 \neq \mathbf{A}^*$ by dropping a_k and including a_e.

$$\mathbf{A}^0 = \mathbf{A}^* \cup \{a_e\} - \{a_k\}$$

Since $a_e \in dom_T^{-1}(a_k)$, we have $a_e \succ a_k$. First, we notice that $\mathbf{A}^0$ is feasible, because according to the definition of T-efficiency, $b_{ej} \leq b_{kj} \;\; \forall j$, and therefore,

$$\sum_{a_i \in \mathbf{A}^0} b_{ij} = \sum_{a_i \in \mathbf{A}^*} b_{ij} - b_{kj} + b_{ej} \leq \sum_{a_i \in \mathbf{A}^*} b_{ij} \leq b_j, \qquad \forall j. \tag{3.6}$$

Also, $a_e \succ a_k$ implies $c_p^e \geq c_p^k$ $\forall p \in \mathbf{P}$ and $c_p^e > c_p^k$ for some p. Hence,

$$\sum_{a_i \in \mathbf{A}^0} c_P^i = \sum_{a_i \in \mathbf{A}^*} c_p^i - c_p^k + c_p^e \geq \sum_{a_i \in \mathbf{A}^*} c_p^i, \tag{3.7}$$

for all p, with strict inequality for al least one $p \in \mathbf{P}$. Hence, $\mathbf{A}^*$ is not optimal, contradicting the assumption and completing the proof. □

To determine whether a T-dominated action a_k can be removed from the list of feasible actions in Program **D4**, one can employ a concept similar to DPO_m in the previous section. However, as explained previously, one of the requirements to examine whether $a_k \in DPO_m(\mathbf{A})$ is knowing, a priori, the number of actions to be selected, m. In the j-constraints problem, the number of actions to be selected is not given a priori. Let $a_k \in Dom_T(\mathbf{A})$ and m and m' be any two positive integers such that $m < m'$. Then, according to the definition of PO and DPO in the previous section,

$$a_k \notin DPO_{m'}(\mathbf{A}) \Longrightarrow a_k \notin DPO_m(\mathbf{A}).$$

Therefore, if m is the biggest number of actions that can be selected in Program **D4**, all the procedures for the m-best actions problem can be used in the j-constraints problem.

Hence, we first construct a program to obtain the *maximum* number of actions that can possibly be included in a feasible solution of (**D4**), and then we utilize the results of the m-best actions problem for this case. Let action a_k be T-dominated. Then the following program determines the maximum number of actions that can be included in a feasible solution of (**D4**), in addition to $dom_T^{-1}(a_k)$:

$$(\mathbf{D5}(a_k)) \qquad \textbf{Maximize} \quad \hat{m} = \sum_{a_i \in \mathbf{A} \setminus dom_T^{-1}(a_k)} x_i$$

Subject to :

$$\sum_{a_i \in \mathbf{A} \setminus dom_T^{-1}(a_k)} b_{ij} x_i \leq \mathbf{B}_j - \sum_{a_i \in dom_T^{-1}(a_k)} b_{ij} \qquad j = 1, \ldots, |\mathbf{J}|$$

$$x_i \in \{0, 1\}.$$

The Program $\mathbf{D5}(a_k)$ is a zero-one program that. because of the structure of its objective function. can be solved rather easily. Suppose $\hat{m}^*$ is the optimal solution of $(\mathbf{D5}(a_k))$. When $\hat{m}^*$ is known. the j-constraints problem can be reduced to an $\hat{m}^*$-best actions problem. Hence. let $PO(\mathbf{A}_{(\hat{m}^*)})$ denote the set of potentially optimal subsets with cardinality $\hat{m}^*$. as defined according to Definition 3.1. Then. the concept of *T-Dominated Potentially Optimal* is defined as follows:

Definition 3.4 *Action $a_k \in \mathbf{A}$ is T-dominated potentially optimal ($a_k \in DPO_T$) if*

1) $a_k \in Dom_T(\mathbf{A})$, and

2) there exists $\mathbf{A}^i_{(\hat{m}^*)} \in \mathbf{A}_{(\hat{m}^*)}$ such that $a_k \in \mathbf{A}^i_{(\hat{m}^*)}$. and $\mathbf{A}^i_{(\hat{m}^*)} \in PO(\mathbf{A}_{(\hat{m}^*)})$.

The following theorem shows that if Problem $\mathbf{D5}(a_k)$ is infeasible or its optimal solution is non-positive. then a_k cannot be in the optimal subset:

Theorem 3.7 *Consider the j-constraints problem $\mathbf{D4}$, and let $a_k \in Dom_T(\mathbf{A})$. If Program $\mathbf{D5}(a_k)$ is infeasible or its optimal solution, $\hat{m}^*$, is non-positive, then $a_k \notin \mathbf{A}^*$.*

Proof: The optimal solution of the Program $\mathbf{D5}$ is the maximum number of actions that can be selected if set $dom_T^{-1}(a_k)$ has already been selected. The infeasibility of Program **D5** indicates that $dom_T^{-1}(a_k) \not\subset \mathbf{A}^*$. But, according to Theorem 3.6,

$$dom_T^{-1}(a_k) \not\subset \mathbf{A}^* \Longrightarrow a_k \notin \mathbf{A}^*.$$

Thus, a_k cannot be in $\mathbf{A}^*$. Similarly, $\hat{m}^* = 0$ indicates that if $dom_T^{-1}(a_k) \subset \mathbf{A}^*$ no more actions can be included in $\mathbf{A}^*$. Thus, $a_k \notin \mathbf{A}^*$. □

Now, suppose that the optimal solution of $(\mathbf{D5}(a_k))$, $\hat{m}^* > 0$, *i.e.* $|\mathbf{A}^*| - |dom_T^{-1}(a_k)| \geq 1$.

Since $\hat{m}^*$ is an upper bound for the size of solutions in **D4**, all the procedures for m-best actions are applicable in this case. One can utilize Program $\mathbf{D2}(a_k)$ from the last section with $q = |\mathbf{A}| - \hat{m}^*$ to determine whether $a_k \in DPO_T(\mathbf{A})$. Then $\delta^* \leq 0$ indicates that there exists a value function under which a_k is better than at least q other actions in $\mathbf{A} \setminus dom_T^{-1}(a_k)$. Therefore, a_k may belong to the best set of actions. Otherwise, $\delta^* > 0$, and a_k can be removed from the set of feasible actions, as illustrated next.

Example 3.8 Consider a subset selection problem in which $|\mathbf{A}| = 6$, $|\mathbf{P}| = 3$, and $|\mathbf{J}| = 2$:

$$\begin{array}{ll}
\textbf{Maximize} & 5x_1 + 7x_2 + 4x_3 + 3x_4 + 6x_5 + 7x_6; \\
\textbf{Maximize} & 2x_1 + 2x_2 + 2x_3 + 2x_4 + 3x_5 + 7x_6; \\
\textbf{Maximize} & 7x_1 + 2x_2 + 6x_3 + x_4 + 2x_5 + 3x_6; \\
\textbf{Subject to}: & \\
 & 2x_1 + 4x_2 + 5x_3 + x_4 + 3x_5 + 8x_6 \leq 8; \\
 & 5x_1 + 6x_2 + 7x_3 + 5x_4 + 8x_5 + 7x_6 \leq 18.
\end{array}$$

In this problem, $Dom_T(\mathbf{A}) = \{a_3\}$, $dom_T^{-1}(a_3) = \{a_1\}$, and $Eff_T(\mathbf{A}) = \{a_1, a_2, a_4, a_5, a_6\}$. To determine whether action a_3 can be removed from the set of feasible actions, we first solve the following program to find the maximum number of actions that can be selected, assuming a_1 has already been selected.

$$\textbf{Maximize} \quad Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6;$$

Subject to :

$$4x_2 + 5x_3 + x_4 + 3x_5 + 8x_6 \leq 6;$$

$$6x_2 + 7x_3 + 5x_4 + 8x_5 + 7x_6 \leq 13;$$

The solution of this problem is $Z^* = 2$. Hence, at most two more actions can be selected in addition to a_1. Thus, $m = 2$ and $q = 2$. We can construct Problem $\mathbf{D2}(a_3)$ as follows:

$$(\mathbf{D2}(a_3)) \qquad \textbf{Minimize} \quad \delta$$

Subject to :

$$v(a_3) - v(a_l) + \delta \geq -M(1 - \alpha_l), \qquad l \in \mathbf{A} \setminus (dom_T^{-1}(a_3) \cup \{a_3\});$$

$$\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 2;$$

$$\alpha_l \in \{0, 1\}, \qquad l \in \mathbf{A} \setminus (dom_T^{-1}(a_3) \cup \{a_3\}),$$

or, assuming the DM's value function is linear,

$$(\mathbf{D2}(a_3)) \qquad \textbf{Minimize} \quad \delta$$

Subject to :

$$-3w_1 + 4w_3 \geq -M(1 - \alpha_2);$$

$$w_1 + 5w_3 \geq -M(1 - \alpha_4);$$

$$-2w_1 - w_2 + 4w_3 \geq -M(1 - \alpha_5);$$
$$-3w_1 - 5w_2 \geq -M(1 - \alpha_6);$$
$$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 = 2;$$
$$w_1 + w_2 + w_3 = 1;$$
$$w_p \geq \epsilon, \quad p = 1, 2, 3,$$
$$\alpha_l \in \{0, 1\} \qquad l \in \mathbf{A} \setminus (dom_T^{-1}(a_3) \cup \{a_3\}).$$

The solution of the above problem is ($\delta^* = -3.78$) with $w_1 = 0.02, w_2 = 0.02, w_3 = 0.098$ as optimal parameters. Hence, $a_3 \in DPO_T(\mathbf{A})$ and a_3 may be included in the best set of actions. Note that if the consequence of action a_6 on the last criterion changes to 8, then δ^* is positive and $a_3 \notin DPO_T(\mathbf{A})$.

If the DM can assign some partial information to the value function, the result may change. For example, assume that the following information on parameters of the value function is given:

$$0.1 \leq w_1 \leq 0.3, \quad 0.2 \leq w_2 \leq 0.4, \quad 0.3 \leq w_3 \leq 0.5.$$

Under this information, $\delta^* = 0.7$ and $a_3 \notin DPO_T(\mathbf{A})$.

3.4 Conclusions

This chapter addresses the problem of screening individual actions in an MCDM subset selection problem. It is shown that removing individually dominated actions, even though it may create a problem that is considerably easier to solve, may be unsatisfactory in that optimal subsets might become inaccessible. Subsequently, the concept of DPO actions is defined for both m-best actions and j-constraints problems and it is shown that those individually dominated actions that are not DPO cannot be in the best subset of actions. Moreover, conditions are given for which a dominated action is DPO are explored and a method is proposed to recognize the DPO actions for both the m-best actions and j-constraints problems.

Chapter 4

Modeling Action-Interdependence in MCDM

4.1 Introduction and Literature Review

In the previous chapter, it was pointed out that in many MCDM problems, a given decision maker must select a subset of actions rather than a single action. There are two main approaches to analyzing a subset selection problem. The first is to enumerate all subsets, remove those that are infeasible, and then select the best using conventional MADM methods. This process might begin, for example, with the identification of dominated subsets.

The second approach is to find the best subset of actions directly from the set of all available actions. The main advantage of the first approach is that one can employ directly any MADM procedure to select the best subset. Moreover, since only one subset is to be chosen, interdependence of actions is not relevant; one incorporates the interdependence of actions in their evaluations under the relevant criteria, and not in the actual selection procedure. However, when the number of actions is large, enumerating and evaluating all feasible subsets can be extremely time consuming. For example, Rajabi *et al.* [93] model a large-scale water resources planning problem in the Regional Municipality of Waterloo, Ontario, Canada. Even though many actions were recognized to be infeasible and removed in the preliminary phase of the study, more than $2^{50} \simeq 10^{15}$ combinations of actions remained.

In most situations, evaluation according to a criterion may be readily available for individual actions but not for sets of actions, because these values are typically obtained from experts in different fields who prefer to evaluate each individual action on its own. In reality, time considerations, diversity of fields, and lack of established procedures for eliciting information about interdependence mean that often knowledge about interdependence is sketchy. As a result, a great deal of subjectivity may be involved in aggregating values of actions into values of subsets of actions [2]. Therefore, in most real-world applications, especially when the number of actions is large, it is preferable to tackle a multiple criteria subset selection problem directly through the underlying individual actions. Even though this chapter and the next deal mainly with the second approach, most of the discussion in these two chapters also applies to the first approach.

Some standard MCDM procedures are applicable to subset selection, but they generally rely on the assumption of independence of actions. Unfortunately, actions are clearly *inter*dependent in many real-world subset selection problems. For instance, in decisions about how to dispose of solid wastes from a metropolitan area, possible actions include using one or more of a number of potential dumping sites, incineration at one or more locations, introducing by-laws to reduce the amount of waste generated in the first place, plus a range of recycling measures. Criteria for evaluating each action may include cost, infrastructure requirements, environmental risk, political acceptability, and aesthetics. A satisficing or optimal solution may consist of a set of actions that, typically, are interdependent for one or more of the criteria on which they are evaluated.

Other examples in which interdependence of actions may occur include selection of research and development or investment projects, transportation routes, computer systems, and time stream decisions (see also [31] and [112] for more examples).

This chapter is mainly concerned with exploring the notion of interdependence of actions and the adaptation of MCDM methods to subset selection in the presence of interdependence. Following a brief literature review in this section, Section 4.2 puts forward a small case study to introduce the problem. Subsequently, Section 4.3 presents some definitions of independence and interdependence of sets of actions, discusses their main features, and explains some special cases. General approaches for evaluating a set of interdependent actions and the relationship between independence (interdependence) of sets and independence (interdependence) of actions are discussed in the next chapter.

Various formulations of interdependence appear in the literature. However, most restrict the type or extent of interdependence in some way. One common approach is to consider only interdependence between two actions, or binary interdependence. Fishburn and LaValle [31] give a thorough discussion of the evaluation of subsets of actions when interactions are binary. They identify necessary conditions on preferences in order that the value of any subset equals the sum of individual action values plus binary interaction terms. Additionally, they characterize preferences between two subsets when the only available values are those of the individual actions. Fishburn and LaValle also provide the first ordinal characterization of interdependence. They list transportation-route selection, household and corporate budgets, student admissions, restaurant-menu composition, and optional accessories for a new car as representative decision problems in which interdependence of actions should be considered.

Often, the existence of interdependence among actions has been overlooked or ignored. For instance, an analysis by Keeney and colleagues to select three out of five sites for nuclear waste disposal employed multi-attribute utility theory [81]. However, after reviewing their proposal, the U.S. Department of Energy selected a different subset from the one they proposed. In assessing this disparity, Keeney [64] argued that the logic of the analysis involved evaluating individually each of the five sites. However, the selection of the sites should have been based on portfolio selection principles, recognizing that individual performance is not as important as the performance of the whole. He concluded that the individual examination of the sites did not address some important considerations that affected all sites.

Rajabi *et al.* [92] model a multiple criteria subset selection problem with any number of interdependent actions as a non-linear multiple criteria integer programming problem. They suggest a variant of goal programming for solving the problem. Elsewhere, they apply their approach to a long term water supply planning problem [93, 95].

Santhanam and Kyparisis [112] present a model for the selection of information system projects in the presence of interdependence. They formulate the subset selection problem as an integer programming problem with some nonlinear terms to reflect interdependence. They apply their procedure to choosing management information system projects with any number of actions.

Gomes [42] introduces the concept of interdependence between two actions in an urban transportation system. Having assumed the probability of choosing highway project j to be p_j, he defines the value of another project, i, that is interdependent with j according to some criterion, as

$$v_i = (v_{i|j})p_j + (v_{i|\sim j})(1 - p_j),$$

where $v_{i|j}$ is the value of action i when action j has been selected and $v_{i|\sim j}$ is the value of action i when j has not been selected. Through an example he shows that the ranking of actions may be changed as a result of this kind of interdependence. Tzeng and Teng [126] use a fuzzy multi-objective model to select a subset of interdependent transportation projects. They classify such projects as *independent*, *complementary*, or *substitutive*. For independent projects, the objective value of the combination is equal to the sum of individual performances according to each criterion. Two projects are called complementary if the result of investment of their combination is greater than the sum of the individual results, and substitutive if the amount of their combination is less than their sum. Further, Tzeng and Teng measure performance and interdependence using fuzzy numbers.

Aaker and Tyebjee [1] describe interdependence among R&D projects in a single objective framework. They address three different types of interdependence: *overlap* in project resource utilization, in which the projects use a common budget, facilities, or manpower; *technical* interdependence, in which the success or failure of a project influences the progress of other projects, and *effect* interdependence, in which there is synergy among projects. In the latter case utilities are not additive where subset of projects is selected.

Evans and Fairbairn [29] note that many NASA mission projects are highly interdependent. But their primary concern is how this interdependence should affect the project implementation sequence.

Interdependence of *criteria* has been also addressed in the MCDM literature (see, for example, [48]). Criteria may be interdependent due to correlations among the elements of the evaluation matrix (the values of the actions according to the criteria); this is called *statistical interdependence.* Alternatively, criteria can be interdependent in the framework of multi-attribute theory. When the directions of increase or decrease on two or more criteria are the same, then these criteria are correlated. Generally, statistical interdependence of two criteria suggests the existence of factors that affect both criteria in the same direction or in opposite directions.

Much research has been devoted to the concept of interdependence of criteria in multi-attribute theory (see, for example, [65]). Criterion p_1 is preferentially independent of p_2 if the relative preference of actions that differ only in their evaluation according to p_1 does not depend on their evaluation on criterion p_2. Let (x, y) denote the evaluation of an action according to criteria p_1 and p_2. Then p_1 and p_2 are preferentially independent *iff*

$$(x_1, \alpha) \succeq (x_2, \alpha) \quad for\ some\ \alpha \Longrightarrow (x_1, \beta) \succeq (x_2, \beta) \quad for \quad all \quad \beta. \tag{4.1}$$

Hence, when the criteria are preferentially independent, one can rank a set of actions according to a criterion without considering the rest of the criteria. Preferential independence among n criteria can be defined similarly [65, 32].

The primary purpose of this chapter is to present new and general definitions of interdependence of actions and of sets of actions, and to assess the main properties of interdependence using the definitions. The effects of interdependence on subset choice, especially in a multiple objective framework, are examined, and several techniques for evaluating subsets of actions that are interdependent according to a specific criterion are presented.

4.2 Interdependence in Multiple Criteria Subset Selection

Recall that $\mathbf{A}$ is the set of possible actions, and $\mathbf{P}$ is the set of criteria on which they are to be evaluated. Denote the value of $\mathbf{A}^0 \subseteq \mathbf{A}$ with respect to criterion $p \in \mathbf{P}$ by $c_p(\mathbf{A}^0)$. If $\mathbf{A}^0 = \{a_{i_1}, a_{i_2}, \cdots, a_{i_n}\}$, write $c_p(\mathbf{A}^0) = c_p(i_1, i_2, \cdots, i_n)$. Without loss of generality, set $c_p(\emptyset) = 0$. Note that this notation implies that consequence is a subset property, *i.e.* the consequence of $\mathbf{A}^0 \subseteq \mathbf{A}$ never depends on the order of selection of the actions within $\mathbf{A}^0$.

Before introducing a formal definition of interdependence, an example is presented to demonstrate the importance of interdependence in multiple criteria subset selection problems. This example shows that a naive application of conventional MCDM methods can produce serious errors in subset selection problems, especially when actions are interdependent.

The example will be called the Waste Disposal Location (WDL) problem. The objective is to identify the two best among five potential sites of equal capacity. The criteria are *proximity to population*, *infrastructure requirements* (such as need for roads, water, and electricity supply), and *environmental risk*. All criteria are measured so that higher values are preferred.

The WDL problem is characterized by certain interdependencies. Building a new road near sites 4 and 5 could serve both sites; if sites 4 and 5 are both selected, a saving in infrastructure investment will be obtained. In this case, the total value of sites 4 and 5 is increased by 10% if both are selected. Hence, sites 4 and 5 have

Table 4.1: WDL Example: Normalized Consequences of Five Feasible Sites

Criteria	*Actions*					*Weights*
	a_1	a_2	a_3	a_4	a_5	
(1)Population	0.45	0.45	1	0.55	0.84	0.23
(2)Infrastructure	0.8	0.7	0.75	0.83	0.83	0.39
(3)Environmental Risk	0.6	0.87	0.5	0.75	0.6	0.38
Additive Value	0.644	0.707	0.713	0.735	0.745	1

a *positive synergy* of 10% on the infrastructure criterion. As well. if sites 1 and 2 together are selected. then a single power plant facility may be built for both. taking advantage of economies of scale. Thus. actions 1 and 2 have a positive synergy of 30%. Finally. the evaluation of site 4 or 5 on the environmental risk criterion depends on whether the other site is selected. If either of the sites is selected. then selecting the other aggravates the risk of environmental damage in the region and. hence. these two actions have a *negative synergy* of 30% on the environmental risk criterion. Figure 4.1 illustrates the interdependence of the five actions on the infrastructure and environmental risk criteria.

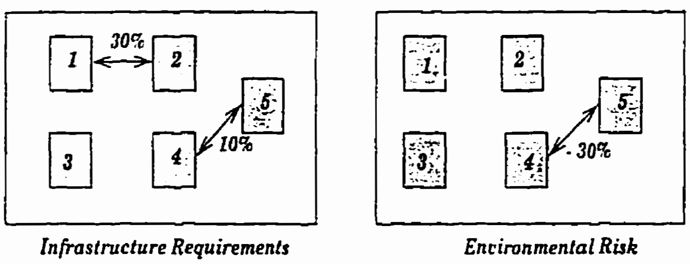


Figure 4.1: WDL Example: Interdependent Actions and Synergy Levels

Table 4.1 shows the individual performance of each action according to each criterion. When applying a *linear value function* with the weights shown in Table 4.1. action a_5 is best. and action a_4 second best. Note that in this example the additive value of an action a. $v(a)$, is calculated as follows:

$$v(a) = \sum_{p=1}^{3} w_p c_p(a).$$

where w_p is the weight of criterion p. and $c_p(a)$ is the evaluation of action a on criterion p. Therefore. a naive conclusion. which would apply if there were no interdependencies. is that the optimal subset is $\mathbf{A}_1^* = \{a_4. a_5\}$. Note. however. that although $v(4) = 0.735$ and $v(5) = 0.745$. $v(4.5) = 1.384$ because of interdependence.

A less naive method of subset selection is to select the single best action. then re-evaluate each action according to the amount that it would increase or decrease the value on each criterion if selected. then choose the best of these. etc. This is called the *greedy algorithm*. If it is applied after a_5 is selected. a_3 is found to add the most value. To reiterate. the overall value increment of action a_3 when a_5 is selected is greater than the increment of any other action. Because of interdependencies. $c_2(4.5) - c_2(5) = 0.996$ and $c_3(4.5) - c_3(5) = 0.34$. Therefore. $\mathbf{A}_2^* = \{a_5. a_3\}$. with $v(3.5) = 1.457$. is certainly an improvement over $\mathbf{A}_1^* = \{a_4. a_5\}$. Table 4.2 shows the reevaluated consequences of acting when action a_5 is selected.

However. by exhaustive examination one finds that the best subset of actions is $\mathbf{A}^* = \{a_1. a_2\}$. with $v(1.2) = 1.526$. Note that action a_2 is in the fourth position in Table 4.1. and action a_1 is not only in fifth position. but it is also dominated by actions a_5 and a_4. Table 4.3 depicts the reevaluated consequences of actions when action a_2 is selected.

One should keep in mind that the presence of several criteria is crucial to draw-

Table 4.2: WDL Example: Reevaluated Site Consequences after Selecting Site 5

Criteria	*Actions*				*Weight*
	a_1	a_2	a_3	a_4	
(1)Population	0.45	0.45	1	0.55	0.23
(2)Infrastructure	0.8	0.7	0.75	0.99	0.39
(3)Environmental Risk	0.6	0.87	0.5	0.34	0.38
Increment to Value	0.644	0.707	0.713	0.503	1

Table 4.3: WDL Example: Reevaluated Site Consequences after Selecting Site 2

Criteria	*Actions*				*Weight*
	a_1	a_3	a_4	a_5	
Population	0.45	1	0.55	0.84	0.23
Infrastructure	1.25	0.75	0.83	0.83	0.39
Environmental Risk	0.6	0.5	0.75	0.6	0.38
Increment to Value	0.819	0.713	0.735	0.745	1

ing conclusions about the WDL example. Suppose, for example, that only the environmental risk criterion (including interdependence) were taken into account. Then the best feasible subset of actions is $\{a_2, a_4\}$. This choice remains the same using either of the naive choice methods (isolated values or greedy algorithm) or using exhaustive examination of all feasible subsets. Hence, subset selection can be sensitive to interdependence of actions in multiple criteria decision making.

Another use of MCDM methods is to identify all efficient (non-dominated) subsets. Recall that in the WDL example, only subsets with two actions are feasible. Ignoring interdependence, the efficient feasible subsets are

$$\{\{a_2, a_3\}, \{a_2, a_4\}, \{a_2, a_5\}, \{a_3, a_4\}, \{a_3, a_5\}, \{a_4, a_5\}\}.$$

However, taking into account the interdependence of actions, the efficient feasible subsets become

$$\{\{a_1, a_2\}, \{a_1, a_5\}, \{a_2, a_3\}, \{a_2, a_4\}, \{a_2, a_5\}, \{a_3, a_4\}, \{a_3, a_5\}\}.$$

The subset $\{a_4, a_5\}$, which dominates several other pairs in the first case, is now itself dominated. Also, note that even though there is no interdependence between a_1 and a_5, the pair $\{a_1, a_5\}$ is included as an efficient pair in the second case. Thus, a feasible subset can be efficient when interdependence is taken into account, but not when it is ignored, even though the members of that subset exhibit no interdependence on any criterion. In general, the set of efficient solutions may change extensively in the presence of interdependence of actions. Furthermore, when an aggregate evaluation function is given, an example can be found with the property that the optimal solution remains the same when interdependence is considered, but the set of efficient solutions changes significantly. Hence, interdependence of actions can be important in virtually any MCDM procedure.

The WDL example, although very simple, demonstrates that when there are interdependent actions, all subsets must be examined in order to find the best one, or to find the set of efficient solutions; selection according to the ranked list of individual actions may not yield the optimal solution. Moreover, this example illustrates that the selection of a subset of actions based on the greedy algorithm can be quite misleading. In the next section, a formal definition and characterization of interdependence of sets of actions and interdependence of actions are presented. Subsequently, important special cases are discussed.

4.3 Interdependence of Sets of Actions

Below, a formal definition of the interdependence of actions is presented. However, first the interdependence of sets of actions is considered; then interdependence of actions is treated as a special case of set interdependence. In fact, it is more useful to define *dependence* as the amount of interdependence, and *independence* as the absence of interdependence.

Roughly, two sets of actions, $\mathbf{A}_1$ and $\mathbf{A}_2$ are independent on criterion p if the selection of $\mathbf{A}_1$ has no effect on the evaluation of $\mathbf{A}_2$, and vice versa, no matter what other actions have already been selected. If $\mathbf{A}_1$ and $\mathbf{A}_2$ are not independent, then they are interdependent. Recall that for any $\mathbf{S} \subseteq \mathbf{A}$, $c_p(\mathbf{S})$ is the evaluation (or consequence) of $\mathbf{S}$ on criterion $p \in \mathbf{P}$.

Definition 4.1 *Let* $\mathbf{A}_1, \mathbf{A}_2 \subseteq \mathbf{A}$, $\mathbf{A}_1 \cap \mathbf{A}_2 = \emptyset$, $\mathbf{A}_1 \neq \emptyset$, $\mathbf{A}_2 \neq \emptyset$, *and let* $\mathbf{A}^0 \subseteq \mathbf{A} \setminus (\mathbf{A}_1 \cup \mathbf{A}_2)$. *Then the amount of interdependence of* $\mathbf{A}_1$ *on* $\mathbf{A}_2$, *given* $\mathbf{A}^0$, *on* *criterion* $p \in \mathbf{P}$ *is*

$$\phi_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0) = c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}^0) - c_p(\mathbf{A}_1 \cup \mathbf{A}^0) - c_p(\mathbf{A}_2 \cup \mathbf{A}^0) + c_p(\mathbf{A}^0). \quad (4.2)$$

Note that the amount of interdependence of $\mathbf{A}_2$ on $\mathbf{A}_1$ equals the amount of interdependence of $\mathbf{A}_1$ on $\mathbf{A}_2$, because it follows from (4.2) that

$$\phi_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0) = \phi_p(\mathbf{A}_2, \mathbf{A}_1 \mid \mathbf{A}^0).$$

It is noteworthy that if $\mathbf{A}_1 \cap \mathbf{A}_2 \neq \emptyset$, one can include common actions in $\mathbf{A}^0$ and use (4.2), or alternatively define the amount of interdependence as

$$\phi_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0) =$$
$$c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}^0) - c_p(\mathbf{A}_1 \cup \mathbf{A}^0) - c_p(\mathbf{A}_2 \cup \mathbf{A}^0) + c_p[(\mathbf{A}_1 \cap \mathbf{A}_2) \cup \mathbf{A}^0]. \quad (4.3)$$

We will not pursue this definition here, but will always assume that $\mathbf{A}_1$ and $\mathbf{A}_2$ are disjoint.

Definition 4.2 *Let* $\mathbf{A}_1, \mathbf{A}_2 \subseteq \mathbf{A}$, $\mathbf{A}_1 \cap \mathbf{A}_2 = \emptyset, \mathbf{A}_1 \neq \emptyset, \mathbf{A}_2 \neq \emptyset$, *and let* $\mathbf{A}^0 \subseteq \mathbf{A} \setminus (\mathbf{A}_1 \cup \mathbf{A}_2)$. *Then* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are independent given* $\mathbf{A}^0$, *according to criterion* $p \in \mathbf{P}$, *if*

$$\phi_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0) = 0.$$

In this case, we write $\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{A}_2$. Independence of $\mathbf{A}_1$ and $\mathbf{A}_2$ implies that the amount by which the selection of set $\mathbf{A}_1$ increases the consequence on criterion p does not depend on whether $\mathbf{A}_2$ is also selected.

Note that independence given $\mathbf{A}^0$ on criterion p, $(\mathbf{I}_p \mid \mathbf{A}^0)$, is a relation on the disjoint subsets of $\mathbf{A} \setminus \mathbf{A}^0$. Moreover, since $\phi_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0)$ is symmetric in $\mathbf{A}_1$ and $\mathbf{A}_2$, $(\mathbf{I}_p \mid \mathbf{A}^0)$ is a symmetric relation, *i.e.*

$$\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{A}_2 \iff \mathbf{A}_2(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{A}_1. \tag{4.4}$$

However, independence is not necessarily transitive. In particular, if $\mathbf{A}_1, \mathbf{A}_2$, and $\mathbf{A}_3$ are nonempty and pairwise disjoint, then it can happen that

$$\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{A}_2 \quad and \quad \mathbf{A}_2(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{A}_3 \quad but \quad \sim [\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{A}_3], \tag{4.5}$$

where $\sim [\cdot]$ symbolizes "Not" of $[\cdot]$. To illustrate how non-transitivity of independence can occur in reality, assume that $\mathbf{A}_1, \mathbf{A}_2$, and $\mathbf{A}_3$ are three ground-water sources in different regions. If $\mathbf{A}_2$ is located far from both $\mathbf{A}_1$ and $\mathbf{A}_3$, then the amount of water extraction from $\mathbf{A}_1$ or $\mathbf{A}_3$ would typically not affect the amount of water extraction from $\mathbf{A}_2$ and vice versa. However, there may be a close relationship between water extraction from $\mathbf{A}_1$ and $\mathbf{A}_3$, if they are adjacent. Thus $\mathbf{A}_1$ and $\mathbf{A}_2$, and $\mathbf{A}_2$ and $\mathbf{A}_3$, may be independent, while $\mathbf{A}_1$ and $\mathbf{A}_3$ are highly interdependent.

To appreciate the meaning of independence, note from (4.2) that $\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{A}_2$ if

$$c_p(\mathbf{A}^0 \cup \mathbf{A}_1) - c_p(\mathbf{A}^0) = c_p(\mathbf{A}^0 \cup \mathbf{A}_1 \cup \mathbf{A}_2) - c_p(\mathbf{A}^0 \cup \mathbf{A}_2). \tag{4.6}$$

This relation indicates that the increase in consequence on criterion p following the selection of $\mathbf{A}_1$ does not depend on whether $\mathbf{A}_2$ has already been selected. In other words, if $\mathbf{A}_1$ and $\mathbf{A}_2$ are independent, the increase in consequence following the selection of $\mathbf{A}_1$ is $c_p(\mathbf{A}^0 \cup \mathbf{A}_1) - c_p(\mathbf{A}^0)$, whether or not $\mathbf{A}_2$ is also selected.

Likewise, the increase in consequence after choosing $\mathbf{A}_2$ is $c_p(\mathbf{A}^0 \cup \mathbf{A}_2) - c_p(\mathbf{A}^0)$, whether or not $\mathbf{A}_1$ is also selected. It follows that, if $\mathbf{A}_1$ and $\mathbf{A}_2$ are independent, then the increase in consequence subsequent to the selection of both of them is $c_p(\mathbf{A}^0 \cup \mathbf{A}_1) + c_p(\mathbf{A}^0 \cup \mathbf{A}_2) - 2c_p(\mathbf{A}^0)$.

Hence, if (4.6) does not hold, there is a synergistic relation between $\mathbf{A}_1$ and $\mathbf{A}_2$ on criterion p. Define the *synergy* of $\mathbf{A}_1$ and $\mathbf{A}_2$, given $\mathbf{A}^0$, on criterion p as

$$\begin{aligned}\gamma_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0) &= \\ &\frac{[c_p(\mathbf{A}^0 \cup \mathbf{A}_1 \cup \mathbf{A}_2) - c_p(\mathbf{A}^0)] - [c_p(\mathbf{A}^0 \cup \mathbf{A}_1) + c_p(\mathbf{A}^0 \cup \mathbf{A}_2) - 2c_p(\mathbf{A}^0)]}{[c_p(\mathbf{A}^0 \cup \mathbf{A}_1) + c_p(\mathbf{A}^0 \cup \mathbf{A}_2) - 2c_p(\mathbf{A}^0)]} \\ &= \frac{\phi_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0)}{c_p(\mathbf{A}^0 \cup \mathbf{A}_1) + c_p(\mathbf{A}^0 \cup \mathbf{A}_2) - 2c_p(\mathbf{A}^0)}. \end{aligned} \tag{4.7}$$

Note that

$$\gamma_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0) = \frac{\text{Actual increase in consequence} - \text{Independent increase in consequence}}{\text{Independent increase in consequence}}. \tag{4.8}$$

Hence, the synergy of two sets $\mathbf{A}_1$ and $\mathbf{A}_2$ is the amount of their interdependence divided by the "independent" increase in consequence following the selection of both sets. [1] For instance, in the WDL example, if $\mathbf{A}^0 = \emptyset$, then the synergy of actions a_4 and a_5 on the infrastructure criterion is

$$\gamma_2(a_1, a_2 \mid \emptyset) = \frac{\phi_p(a_1, a_2 \mid \emptyset)}{c_p(a_1) + c_p(a_2)} = 10\%.$$

[1] *Note that γ_p can take any numerical value-positive or negative.*

By substituting (4.7) into (4.2), the consequence of the combination of $\mathbf{A}_1$ and $\mathbf{A}_2$ given $\mathbf{A}^0$ equals

$$\begin{aligned} c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}^0) &= [c_p(\mathbf{A}_1 \cup \mathbf{A}^0) + c_p(\mathbf{A}_2 \cup \mathbf{A}^0)][1 + \gamma_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0)] \\ &\quad - c_p(\mathbf{A}^0)[1 + 2\gamma_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0)]. \end{aligned} \tag{4.9}$$

Expression (4.9) shows how synergy, γ_p, can be interpreted as an increase, or a decrease, in the consequence of joint selection of two actions. Note that γ_p can take both positive and negative values. If $\gamma_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0) > 0$, we say that $\mathbf{A}_1$ and $\mathbf{A}_2$ have positive synergy given $\mathbf{A}^0$. Similarly, if $\gamma_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0) < 0$, $\mathbf{A}_1$ and $\mathbf{A}_2$ have negative synergy given $\mathbf{A}^0$. As an example of negative synergy, suppose that a_i and a_j are two sites that can utilize some common facilities, thereby reducing infrastructure costs in the WDL example. Hence, with the selection of both a_i and a_j, the total infrastructure cost will be decreased. On the other hand, assume that the selection of both a_i and a_j aggravates the environmental situation of the region. Accordingly, the synergy of these two actions on environmental risks would be positive.

As is evident from (4.7), the synergy of two sets of actions depends on the set $\mathbf{A}^0$ that has already been selected. In fact, it is possible for the synergy of two sets to be positive when $\mathbf{A}_1^0$ is selected and negative when $\mathbf{A}_2^0 \neq \mathbf{A}_1^0$ is selected.

Example 4.1 *Let* $c_p(1) = c_p(2) = c_p(3) = c_p(4) = 2$, $c_p(1,2) = c_p(2,3) = c_p(4,3) = 3, c_p(1,3) = c_p(1,4) = 4$, $c_p(1,2,3) = 5$, $c_p(1,4,3) = 4$. *Then the synergy levels of two actions* a_1 *and* a_3, *given* a_2 *and given* a_4, *are*

$$\gamma_p(a_1, a_3 \mid \{a_2\}) = 50\%$$

$$\gamma_p(a_1, a_3 \mid \{a_4\}) = -33\%.$$

Suppose that the value according to criterion $p \in \mathbf{P}$ is to be maximized. Then, if $\gamma_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0)$ is a large positive number, $\mathbf{A}_1$ and $\mathbf{A}_2$ approach a *complementary* relation. On the other hand, if $\gamma_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0)$ is a large negative number, $\mathbf{A}_1$ and $\mathbf{A}_2$ are close to being mutually exclusive. Concepts similar to complementary and mutually exclusive have been widely used in project selection decision problems.

Definition 4.3 *Let* $\mathbf{A}_1, \mathbf{A}_2 \subseteq \mathbf{A}$, $\mathbf{A}_1 \cap \mathbf{A}_2 = \emptyset$, $\mathbf{A}_1 \neq \emptyset$, $\mathbf{A}_2 \neq \emptyset$, *and let* $\mathbf{A}^0 \subseteq \mathbf{A} \setminus (\mathbf{A}_1 \cup \mathbf{A}_2)$. *Then* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are independent on criterion* p, *if*

$$\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{A}_2 \qquad \forall \mathbf{A}^0 \subseteq \mathbf{A} \setminus \mathbf{A}_1 \cup \mathbf{A}_2. \tag{4.10}$$

Thus, two sets of actions $\mathbf{A}_1$ and $\mathbf{A}_2$ are independent if they are independent given any $\mathbf{A}^0 \subseteq \mathbf{A} \setminus \mathbf{A}_1 \cup \mathbf{A}_2$. If $\mathbf{A}_1$ and $\mathbf{A}_2$ are not independent on criterion p, they are *interdependent* on criterion p. It follows from the above definition that $\mathbf{A}_1$ and $\mathbf{A}_2$ are interdependent on criterion p if

$$\exists\ \mathbf{A}^0 \subseteq \mathbf{A} \setminus (\mathbf{A}_1 \cup \mathbf{A}_2) \quad \text{such that} \quad \sim [\mathbf{A}_1\ (\mathbf{I}_p \mid \mathbf{A}^0)\ \mathbf{A}_2]. \tag{4.11}$$

Hence, $\mathbf{A}_1$ and $\mathbf{A}_2$ are interdependent on criterion p if there exists $\mathbf{A}^0$ such that (4.6) does not hold. We call two sets *unconditionally interdependent* if they are interdependent (*i.e.* (4.6) fails) for all $\mathbf{A}^0$, and *conditionally interdependent* if they are interdependent given some $\mathbf{A}^0$. In particular, $\mathbf{A}_1$ and $\mathbf{A}_2$ are conditionally interdependent if they are independent given some $\mathbf{A}^0_1$, but not given some $\mathbf{A}^0_2 \neq \mathbf{A}^0_1$. Another possibility, shown in Example 4.1, is that $\mathbf{A}_1$ and $\mathbf{A}_2$ have positive synergy given $\mathbf{A}^0_1$, but negative synergy given $\mathbf{A}^0_2 \neq \mathbf{A}^0_1$.

Since independence is a symmetric relation, interdependence (which is the negation of independence) is also symmetric. Moreover, interdependence is not necessarily transitive. Consider the following example:

Example 4.2 *Assume* $\mathbf{A}^0 = \emptyset$. *Consider three sets* $\mathbf{A}_1$, $\mathbf{A}_2$, *and* $\mathbf{A}_3$, *such that*

$$c_p(\mathbf{A}_1) = c_p(\mathbf{A}_2) = c_p(\mathbf{A}_3) = 3,$$
$$c_p(\mathbf{A}_1 \cup \mathbf{A}_2) = c_p(\mathbf{A}_2 \cup \mathbf{A}_3) = 7, \quad \text{and} \quad c_p(\mathbf{A}_1 \cup \mathbf{A}_3) = 6.$$

Then it follows that

$$\phi_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0) = \phi_p(\mathbf{A}_2, \mathbf{A}_3 \mid \mathbf{A}^0) = 1, \;\; \textit{but} \;\; \phi_p(\mathbf{A}_1, \mathbf{A}_3 \mid \mathbf{A}^0) = 0.$$

Thus,

$$\sim [\mathbf{A}_1 \;\; (\mathbf{I}_p \mid \mathbf{A}^0) \;\; \mathbf{A}_2], \qquad \sim [\mathbf{A}_2 \;\; (\mathbf{I}_p \mid \mathbf{A}^0) \;\; \mathbf{A}_3], \qquad \text{and} \qquad \mathbf{A}_1 \;\; (\mathbf{I}_p \mid \mathbf{A}^0) \;\; \mathbf{A}_3.$$

It is easy to overlook conditional interdependence, although it can be as important as unconditional interdependence in applications. Consider two sets of ground-water sources $\mathbf{A}_1$ and $\mathbf{A}_2$ that are independent. If a third set of ground-water sources close to both sources is selected, the total water extraction of the two sources, $\mathbf{A}_1$ and $\mathbf{A}_2$, may be reduced, making $\mathbf{A}_1$ and $\mathbf{A}_2$ interdependent.

As another example of conditional interdependence, consider the selection of two software systems that work independently. The total benefit of their usage thus equals the sum of the individual benefits. However, suppose that with the selection of appropriate hardware, these two software systems produce positive synergy. Thus, the systems are independent if the hardware is not selected, and interdependent if it is selected. Accordingly, we call these two software programs conditionally interdependent. An important special case of conditional interdependence is defined next.

Definition 4.4 *Let* $\mathbf{A}_1, \mathbf{A}_2 \subseteq \mathbf{A}$, $\mathbf{A}_1 \cap \mathbf{A}_2 = \emptyset$, $\mathbf{A}_1 \neq \emptyset, \mathbf{A}_2 \neq \emptyset$. *Then* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are simply independent if* $\mathbf{A}_1(\mathbf{I}_p \mid \emptyset)\mathbf{A}_2$*; otherwise,* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are simply interdependent.*

Note that by (4.2), $\mathbf{A}_1$ and $\mathbf{A}_2$ are simply independent if $\phi_p(\mathbf{A}_1, \mathbf{A}_2 \mid \emptyset) = 0$. According to (4.6), this is equivalent to

$$c_p(\mathbf{A}_1 \cup \mathbf{A}_2) = c_p(\mathbf{A}_1) + c_p(\mathbf{A}_2), \tag{4.12}$$

because $c_p(\emptyset) = 0$. In other words, two sets are simply independent if the value of their combination equals the sum of their individual values. This concept is similar to the idea of conventional independence of actions. All past research on interdependence of actions is restricted to the concepts of simple independence and interdependence.

Note that two sets of actions may be simply independent, yet interdependent (when some non-empty set of actions has already been selected), as shown by the following example:

Example 4.3 *Consider three actions* a_1, a_2 *and* a_3 *such that* $c_p(1) = c_p(2) = c_p(3) = 2$ *and* $c_p(1,2,3) = 5$, $c_p(1,2) = 3$, $c_p(2,3) = 3$, *and* $c_p(1,3) = 4$. *Actions* a_1 *and* a_3 *are simply independent on criterion* p, *because*

$$c_p(1,3) - c_p(1) - c_p(3) = 0.$$

But when action a_2 has already been selected, a_1 and a_3 are interdependent, because

$$c_p(1,2,3) - c_p(1,2) - c_p(2,3) + c_p(2) = 1,$$

so that

$$c_p(1,2) - c_p(2) \neq c_p(1,2,3) - c_p(2,3).$$

Hence

$$a_1 \ (\mathbf{I}_p \mid \emptyset) \ a_3 \qquad \textit{and} \qquad \sim [a_1(\mathbf{I}_p \mid \{a_2\})a_3].$$

If two sets are conditionally independent given $\mathbf{A}^0$, one may need to know whether these two sets are also independent given a subset or superset of $\mathbf{A}^0$. In what follows, we explore conditions under which the independence of two sets given $\mathbf{A}^0$ implies their independence on any subset or superset of $\mathbf{A}^0$. Let $\mathbf{A}^0$ and $\mathbf{A}^{0'}$ be two nonempty sets of actions such that $\mathbf{A}^{0'} \subseteq \mathbf{A}^0$. The following theorem shows a relationship between $\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{A}_2$ and $\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}^{0'})\mathbf{A}_2$.

Theorem 4.1 *Let $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}^0, \mathbf{A}^{0'}$ be nonempty sets such that $\mathbf{A}_1 \cap \mathbf{A}_2 = \emptyset$, $\mathbf{A}^{0'} \subseteq \mathbf{A}^0$, $\mathbf{A}^0 \subseteq \mathbf{A} \setminus \mathbf{A}_1 \cup \mathbf{A}_2$. Define $\mathbf{B} = \mathbf{A}^0 \setminus \mathbf{A}^{0'}$. Then*

$$\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{A}_2 \iff \mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}^{0'})\mathbf{A}_2, \ \textit{when} \quad \begin{cases} 1)\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}^{0'})\mathbf{B}, \\ 2)\mathbf{A}_2(\mathbf{I}_p \mid \mathbf{A}^{0'})\mathbf{B}, \textit{and} \\ 3)(\mathbf{A}_1 \cup \mathbf{A}_2)(\mathbf{I}_p \mid \mathbf{A}^{0'})\mathbf{B}. \end{cases}$$

Proof: see Appendix A.1.

The above theorem shows that independence of two sets given $\mathbf{A}^0$ does not imply their independence on subset or superset of $\mathbf{A}^0$ unless all three indicated conditions in this theorem hold. The following corollary is a special case of Theorem 4.1 when $\mathbf{A}^{0'} = \emptyset$. This corollary shows a useful relationship between simple independence and conditional independence of two sets.

Corollary 4.1 *Let* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}^0$ *be nonempty sets such that* $\mathbf{A}_1 \cap \mathbf{A}_2 = \emptyset$. *Then*

$$\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{A}_2 \iff \mathbf{A}_1(\mathbf{I}_p \mid \emptyset)\mathbf{A}_2, \textit{ when } \begin{cases} 1)\mathbf{A}_1(\mathbf{I}_p \mid \emptyset)\mathbf{A}^0, \\ 2)\mathbf{A}_2(\mathbf{I}_p \mid \emptyset)\mathbf{A}^0, \textit{and} \\ 3)(\mathbf{A}_1 \cup \mathbf{A}_2)(\mathbf{I}_p \mid \emptyset)\mathbf{A}^0. \end{cases}$$

The above corollary indicates that simple independence of two sets implies their conditional independence given $\mathbf{A}^0$, when both sets and their union are simply independent with $\mathbf{A}^0$. Theorem 4.1 and Corollary 4.1 can also be used for examining interdependence of two sets. For instance,

$$\sim [\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{A}_2] \iff \sim [\mathbf{A}_1(\mathbf{I}_p \mid \emptyset)\mathbf{A}_2], \textit{ when } \begin{cases} 1)\mathbf{A}_1(\mathbf{I}_p \mid \emptyset)\mathbf{A}^0, \\ 2)\mathbf{A}_2(\mathbf{I}_p \mid \emptyset)\mathbf{A}^0, \textit{and} \\ 3)(\mathbf{A}_1 \cup \mathbf{A}_2)(\mathbf{I}_p \mid \emptyset)\mathbf{A}^0. \end{cases}$$

The following theorem shows another relationship between simple and conditional independence.

Theorem 4.2 *Let* $\mathbf{A}_1, \mathbf{A}_2$, *and* $\mathbf{A}_3$ *be three nonempty and disjoint sets of actions, and let* $\mathbf{A}_1(\mathbf{I}_p \mid \emptyset)\mathbf{A}_2$. *Then* $\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}_2)\mathbf{A}_3$ *iff* $\mathbf{A}_1(\mathbf{I}_p \mid \emptyset)(\mathbf{A}_2 \cup \mathbf{A}_3)$.

Proof: in Appendix A.2.

The following corollary, an immediate consequence of Theorem 4.2, shows a relationship between simple and conditional interdependence.

Corollary 4.2 *Let* $\mathbf{A}_1, \mathbf{A}_2$, *and* $\mathbf{A}_3$ *be three nonempty and disjoint sets of actions, and let* $\mathbf{A}_1(\mathbf{I}_p \mid \emptyset)\mathbf{A}_2$. *Then* $\sim [\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}_2)\mathbf{A}_3]$ *iff* $\sim [\mathbf{A}_1(\mathbf{I}_p \mid \emptyset)(\mathbf{A}_2 \cup \mathbf{A}_3)]$.

The following theorem is a generalization of Theorem 4.2.

Theorem 4.3 *Let* $\mathbf{A}_1, \mathbf{A}_1, \ldots, \mathbf{A}_n$ *be nonempty and disjoint sets of actions. Then* $\mathbf{A}_1(\mathbf{I}_p \mid \emptyset)\mathbf{A}_2 \cup \mathbf{A}_3 \ldots \cup \mathbf{A}_{n-1} \cup \mathbf{A}_n$, *provided that* $\mathbf{A}_1(\mathbf{I}_p \mid \emptyset)\mathbf{A}_2$, $\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}_2)\mathbf{A}_3$, $\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}_2 \cup \mathbf{A}_3)\mathbf{A}_4$, ..., *and* $\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}_2 \cup \mathbf{A}_3 \ldots \cup \mathbf{A}_{n-1})\mathbf{A}_n$.

Proof: in Appendix A.3

For two sets to be interdependent, interdependence on one criterion suffices.

Definition 4.5 *Let* $\mathbf{A}_1, \mathbf{A}_2 \subseteq \mathbf{A}$, $\mathbf{A}_1 \cap \mathbf{A}_2 = \emptyset, \mathbf{A}_1 \neq \emptyset, \mathbf{A}_2 \neq \emptyset$, *and let* $\mathbf{A}^0 \subseteq \mathbf{A} \setminus (\mathbf{A}_1 \cup \mathbf{A}_2)$. *Then*

$$\sim [\mathbf{A}_1\ (\mathbf{I} \mid \mathbf{A}^0)\ \mathbf{A}_2] \Longrightarrow \exists p \in \mathbf{P} \quad \text{such that} \quad \sim [\mathbf{A}_1\ (\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{A}_2], \tag{4.13}$$

so that $(\mathbf{I} \mid \mathbf{A}^0)$ expresses independence on every criterion. Hence, two sets of actions are interdependent if there is at least one criterion under which they are interdependent.

We now define two special cases of set-independence, namely interdependence of a set and an action, and interdependence of two individual actions.

Definition 4.6 *Let* $a_i \in \mathbf{A}$, $\mathbf{S} \subseteq \mathbf{A} \setminus \{a_i\}$, $\mathbf{S} \neq \emptyset$, *and let* $\mathbf{A}^0 \subseteq \mathbf{A} \setminus (\mathbf{S} \cup \{a_i\})$. *Then* a_i *and* $\mathbf{S}$ *are independent if*

$$\phi_p(a_i, \mathbf{S} \mid \mathbf{A}^0) = 0.$$

Note that $\phi_p(a_i, \mathbf{S} \mid \mathbf{A}^0)$ is defined by (4.2).

The next example shows that even though independence of an action and a set is a symmetric relation. *i.e.*

$$a_i\ (\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S} \Longleftrightarrow \mathbf{S}(\mathbf{I}_p \mid \mathbf{A}^0)a_i.$$

the independence of a_i and $\mathbf{S}$ is a property of the two sets $\{a_i\}$ and $\mathbf{S}$ and not of the set $\{a_i\} \cup \mathbf{S}$.

Example 4.4 *Consider three actions a_1, a_2 and a_3 such that $c_p(1) = c_p(2) = c_p(3) = 2$ and $c_p(1,2) = 4$. $c_p(1,3) = 4$. $c_p(2,3) = 5$. and $c_p(1,2,3) = 7$. Then $a_1\ (\mathbf{I}_p \mid \emptyset)\{a_2, a_3\}$ because*

$$c_p(1) - c_p(1,2,3) + c_p(2,3) = 0.$$

On the other hand:

$$c_p(1,2) - c_p(1,2,3) + c_p(3) \neq 0.$$

Hence.

$$a_1(\mathbf{I}_p \mid \emptyset)\{a_2, a_3\} \quad but \quad \sim [\{a_1, a_2\}(\mathbf{I}_p \mid \emptyset)\ a_3].$$

In general. suppose $\mathbf{S}_1, \mathbf{S}_2 \subseteq \mathbf{S}$, $\mathbf{S}_1 \neq \mathbf{S}_2$ and $\mathbf{A}^0 \cap \mathbf{S} = \emptyset$. Then, the independence of $\mathbf{S}_1$ and $\mathbf{S} \setminus \mathbf{S}_1$ on criterion p. $\mathbf{S}_1(\mathbf{I}_p \mid \mathbf{A}^0)(\mathbf{S} \setminus \mathbf{S}_1)$. does not imply the independence of $\mathbf{S}_2$ and $\mathbf{S} \setminus \mathbf{S}_2$ under criterion p; as the example shows. $\sim [\mathbf{S}_2(\mathbf{I}_p \mid \mathbf{A}^0)(\mathbf{S} \setminus \mathbf{S}_2)]$ is possible.

One important case of interdependence is interdependence of two actions. All previous research of interdependence has been limited to the concept of interdependence of actions (see. for example. [31], and [1]). The importance of interdependence of actions stems from the fact that often a set of actions is selected by choosing individual actions, one at a time. Moreover, carefully designed procedures for dealing with interdependence of actions may be expandable to interdependence of sets.

Two actions a_i and a_j are independent given $\mathbf{A}^0$ if the amount by which action a_i increases the evaluation on criterion p does not depend on whether action a_j is also selected. In other words, for $p \in \mathbf{P}$, $a_i, a_j \in \mathbf{A}$, and any $\mathbf{A}^0 \subseteq \mathbf{A} \setminus \{a_i, a_j\}$,

$$a_i \; (\mathbf{I}_p \mid \mathbf{A}^0) \; a_j \;\; if \;\; \phi_p(a_i, a_j \mid \mathbf{A}^0) = 0. \tag{4.14}$$

where $\phi_p(\ldots \mid .)$ is as in (4.2). Note that $a_i \; (\mathbf{I}_p \mid \mathbf{A}^0) \; a_j$ denotes the independence of actions a_i and a_j on criterion p given $\mathbf{A}^0$. Similar to the interdependence of sets, we denote the conditional interdependence of actions a_i and a_j by $\sim [a_i \; (\mathbf{I}_p \mid \mathbf{A}^0) a_j]$. For instance, $\sim [a_i(\mathbf{I}_p \mid \{a_k\}) a_j]$ indicates that $\phi_p(a_i, a_j \mid a_k) \neq 0$.

4.4 Conclusions

This chapter introduces a new definition and characterization of interdependence of actions for subset selection problems. It is shown that ignoring interdependence of actions in multiple criteria decision problems is riskier than in single criterion decision problems. Through a simple example it is demonstrated that using the greedy algorithm to choose a small subset of actions in the presence of interdependence can be quite misleading.

The interdependence of actions is generalized to set interdependence. In fact, interdependence of actions is treated as an special case of set interdependence. Interdependence is characterized as conditional and unconditional, and the main differences of conditional and unconditional interdependence, compared to conventional approaches, are explained. The next chapter is mainly concerned with exploring the relationships between set-interdependence and action-interdependence.

Chapter 5

Interdependence Evaluation

5.1 Introduction

In the previous chapter the notion of interdependence of sets of actions was introduced. Additionally, as special cases, the interdependence of two actions and interdependence of an action and a set were discussed. This chapter presents a general framework for evaluating the consequence of a set of interdependent actions. Moreover, the relationship between set-interdependence and action-interdependence is explored. Section 5.2 presents a general methodology to evaluate the consequence of a set in the presence of interdependence. It also puts forward a new and general definition for order of interdependence. Next, Section 5.3 discusses the evaluation of interdependence of two sets according to the amount of interdependence of their subsets. A thorough analysis of the relationship between interdependence of sets and interdependence of actions is addressed in Section 5.4. Subsequently, Section 5.5 presents a new definition of additivity of a set of interdependent actions. Finally, conclusions are drawn in Section 5.6.

5.2 Evaluating a Set of Interdependent Actions on a Criterion

In this section, we present a general framework to evaluate a subset of interdependent actions. It has been observed in practice that, often, the evaluation on any criterion is additive when more than one action is selected. For instance, the overall cost of a set of independent projects is the sum of all individual project costs. However, interdependence of actions does occur; our objective is to measure and account for its effects.

Define $c_p(\mathbf{A}_1 \mid \mathbf{A}^0) = c_p(\mathbf{A}^0 \cup \mathbf{A}_1) - c_p(\mathbf{A}^0)$, where $p \in \mathbf{P}$ and $\mathbf{A}_1 \cap \mathbf{A}^0 = \emptyset$. We call $c_p(\mathbf{A}_1 \mid \mathbf{A}^0)$ the *consequence increment* of $\mathbf{A}_1$ given $\mathbf{A}^0$ on criterion p. From (4.2)

$$\phi_p(\mathbf{A}_1, \mathbf{A}_2 \mid \mathbf{A}^0) = c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \mid \mathbf{A}^0) - c_p(\mathbf{A}_1 \mid \mathbf{A}^0) - c_p(\mathbf{A}_2 \mid \mathbf{A}^0). \qquad (5.1)$$

Now, if $\mathbf{A}_1(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{A}_2$, (5.1) reduces to

$$c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \mid \mathbf{A}^0) = c_p(\mathbf{A}_1 \mid \mathbf{A}^0) + c_p(\mathbf{A}_2 \mid \mathbf{A}^0). \qquad (5.2)$$

Equation (5.2) shows that independence of two sets is equivalent to the additivity of their consequence increments. Moreover, (5.1) indicates how the *existence* and *amount* of interdependence between two sets depends upon the set of actions $\mathbf{A}^0$ that has already been selected. To simplify the discussion in the rest of this section, we consider only the concepts of simple independence and simple interdependence of actions, and hence we assume that $\mathbf{A}^0 = \emptyset$. The results are expandable to the general case, unless otherwise specified.

The symbol $\Delta_p(\mathbf{S})$ will denote the *amount of simple dependence within set* $\mathbf{S} \subseteq \mathbf{A}$ on criterion p. We call $\Delta_p(\mathbf{S})$ the dependence of set $\mathbf{S}$. Following Fishburn and LaValle [30], we define $\Delta_p(\mathbf{S})$, for any set $\mathbf{S} \subseteq \mathbf{A}$ and $p \in \mathbf{P}$, as follows:

$$\Delta_p(\mathbf{S}) = \Delta_p(a_1, \ldots, a_i, \ldots, a_n) = c_p(\mathbf{S}) - \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}| = n-1}} c_p(\mathbf{T})$$

$$+ \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}| = n-2}} c_p(\mathbf{T}) + \cdots + (-1)^{n+1} \sum_{i \in \mathbf{S}} c_p(i). \tag{5.3}$$

Note that the value of $\Delta_p(\mathbf{S})$ can always be calculated, as it depends only on the value of $c_p(.)$. For a set of actions with two or three elements

$$\Delta_p(a_i, a_j) = c_p(i, j) - c_p(i) - c_p(j) \tag{5.4}$$

$$\Delta_p(a_i, a_j, a_k) = c_p(i, j, k) - c_p(i, j) - c_p(i, k) - c_p(j, k) + c_p(i) + c_p(j) + c_p(k). \tag{5.5}$$

Based on the above definition, Fishburn and LaValle prove that the value of any set of actions, S, $|\mathbf{S}| \geq 2$, can be calculated as follows:

$$c_p(\mathbf{S}) = \sum_{i \in \mathbf{S}} c_p(i) + \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}| \geq 2}} \Delta_p(\mathbf{T}). \tag{5.6}$$

Models (5.3) and (5.6) include dependencies within any number of actions. In many practical cases, there are a priori restrictions on dependence. Define the *order of dependence* of set S on criterion $p \in \mathbf{P}$, $O_p(\mathbf{S})$, as the cardinality of largest subset of S, $\mathbf{T}$, such that $\Delta_p(\mathbf{T}) \neq 0$. Hence, when $O_p(\mathbf{S}) = k$, then the value of $\mathbf{S}$ can be calculated as

$$c_p(\mathbf{S}) = \sum_{i \in \mathbf{S}} c_p(i) + \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}| \leq k}} \Delta_p(\mathbf{T}), \tag{5.7}$$

because $\Delta_p(\mathbf{T}) = 0$ for any $|\mathbf{T}| > k$. In general, define the order of dependence on criterion p, O_p, as the cardinality of the largest subset, $\mathbf{T} \subseteq \mathbf{A}$, such that $\Delta_p(\mathbf{T}) \neq 0$. Hence, when $O_p = k$, the value of any set of actions in $\mathbf{A}$ can be calculated using (5.7). Note that $O_p(\mathbf{S})$ is a property of set $\mathbf{S}$, and O_p is a property of the set of all actions, $\mathbf{A}$. In practical cases, it is useful to find the order of dependence within a set of actions. The following theorem shows how to find this quantity:

Theorem 5.1 *Let* $\mathbf{S} \subseteq \mathbf{A}$, $|\mathbf{S}| \geq 2$. *Then the order of dependence of* $\mathbf{S}$, $O_p(\mathbf{S})$, *on criterion* p *is* k, *where* $k \leq |\mathbf{S}|$, *iff the consequence of* $\mathbf{S}$ *can be written as:*

$$\begin{aligned} c_p(\mathbf{S}) &= \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}| = k}} c_p(\mathbf{T}) - (|\mathbf{S}| - k) \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}| = k-1}} c_p(\mathbf{T}) \\ &+ \left(\frac{1}{2!}\right) [(|\mathbf{S}| - k)(|\mathbf{S}| - (k-1))] \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}| = k-2}} c_p(\mathbf{T}) + \cdots \\ &\pm \left(\frac{1}{(k-1)!}\right) [(|\mathbf{S}| - k)(|\mathbf{S}| - (k-1)) \cdots (|\mathbf{S}| - 2)] \sum_{a_i \in \mathbf{S}} c_p(i), \quad k \leq |\mathbf{S}|. \end{aligned} \tag{5.8}$$

Proof: See Appendix A.4

Note that Expression 5.8 depends only on $c_p(\cdot)$. Hence, it can be used to determine the order of interdependence of a set.

Example 5.1 Using Theorem 5.1 one can show that $O_p(\mathbf{S}) = 2$, and $O_p(\mathbf{S}) = 3$ if

$c_p(\mathbf{S})$ can be represented as (5.9), and (5.10), respectively,

$$c_p(\mathbf{S}) = \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}|=2}} c_p(\mathbf{T}) - (|\mathbf{S}|-2) \sum_{a_i \in \mathbf{S}} c_p(i), \qquad |\mathbf{S}| \geq 2, \tag{5.9}$$

$$\begin{aligned} c_p(\mathbf{S}) &= \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}|=3}} c_p(\mathbf{T}) - (|\mathbf{S}|-3) \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}|=2}} c_p(\mathbf{T}) \\ &+ \frac{1}{2}((|\mathbf{S}|-3)(|\mathbf{S}|-2)) \sum_{a_i \in \mathbf{S}} c_p(i), \qquad |\mathbf{S}| \geq 3. \end{aligned} \tag{5.10}$$

Clearly, when $O_p(\mathbf{S}) = 1$, the consequence of set $\mathbf{S}$ can be written as the summation of the consequences of its individual actions. By convention, $O_p(\mathbf{S}) = 0$ means that there is no other way to express $c_p(\mathbf{S})$. The following corollary is the immediate result of the definitions of set-interdependence and order of dependence:

Corollary 5.1 *Let* $\emptyset \neq \mathbf{S}_1 \subseteq \mathbf{A}$, $\emptyset \neq \mathbf{S}_2 \subseteq \mathbf{A}$, *and* $\mathbf{S}_1 \cap \mathbf{S}_2 = \emptyset$. *Moreover, let* $O_p(\mathbf{S}_1) = k_1$ *and* $O_p(\mathbf{S}_2) = k_2$. *Then*

$$Max\{k_1, k_2\} \leq O_p(\mathbf{S}_1 \cup \mathbf{S}_2) \leq k_1 + k_2. \tag{5.11}$$

In some cases, it may be difficult to estimate the amount of dependence within large sets of actions. Moreover, the computational requirements to evaluate a set increase rapidly as the order of dependence increases. Hence, in some situations it is beneficial to ignore higher order dependence. For example, Fishburn and LaValle [31] restrict their model to dependencies within pairs of actions only. As is shown in Sections 5.4 and 5.5, restricting the order of dependence can produce useful connections between interdependence of sets and interdependence of actions.

5.3 Evaluating Interdependence of Sets on a Criterion

In the previous section, the amount of dependence within a set of actions was defined. We now derive useful expressions to evaluate the interdependence of an action and a set, and subsequently, generalize it for interdependence of one set and another. In particular, we demonstrate how the amount of interdependence of two sets can be expressed in terms of the amount of interdependence of their proper subsets. Here, we consider only simple independence and interdependence, and hence we assume that $\mathbf{A}^0 = \emptyset$. However, the results are applicable to more general cases, unless otherwise specified. For the sake of simplicity in notation, we denote $\phi_p(\mathrm{S}_1, \mathrm{S}_2 \mid \emptyset)$ and $\mathrm{S}_1(\mathbf{I}_p|\emptyset)\mathrm{S}_2$ by $\phi_p(\mathrm{S}_1, \mathrm{S}_2)$ and $\mathrm{S}_1\ \mathbf{I}_p\ \mathrm{S}_2$, respectively.

The following theorem gives an expression for the amount of interdependence of an action and a set of actions in terms of dependence within individual actions. Note that $\phi_p(\mathrm{S}_1, \mathrm{S}_2)$ is the amount of interdependence of S_1 and S_2 and is defined according to (4.2), while $\Delta_p(\mathrm{S})$ is dependence within the set S and is defined according to (5.3).

Theorem 5.2 *Let* $\mathrm{S} = \{a_1, \cdots, a_j, \cdots, a_n\}$, $\mathrm{S} \subseteq \mathbf{A} \setminus \{a_i\}$, *and* $\phi_p(a_i, \mathrm{S})$ *denote the amount of interdependence of action* a_i *and set* S. *Then the dependence of* a_i *and* S *can be represented as follows:*

$$\phi_p(a_i, \mathrm{S}) = \sum_{\emptyset \neq \mathrm{T} \subseteq \mathrm{S}} \Delta_p(\{a_i\} \cup \mathbf{T}), \tag{5.12}$$

where $\Delta_p(\{a_i\} \cup \mathbf{T})$ *is defined according to* (5.3).

Proof: According to 4.2. when $\mathbf{A}^0 = \emptyset$

$$\phi_p(a_i, \mathbf{S}) = c_p(a_i \cup \mathbf{S}) - c_p(\mathbf{S}) - c_p(i). \tag{5.13}$$

But, based on (5.6)

$$c_p(\mathbf{S}) = \sum_{j \in \mathbf{S}} c_p(j) + \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}| \geq 2}} \Delta_p(\mathbf{T}). \tag{5.14}$$

Substituting (5.14) into (5.13) gives:

$$\phi_p(a_i, \mathbf{S}) = \sum_{j \in \mathbf{S}} c_p(j) + c_p(i) + \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}| \geq 2}} \Delta_p(\mathbf{T})$$

$$+ \sum_{\emptyset \neq \mathbf{T} \subseteq \mathbf{S}} \Delta_p(\{a_i\} \cup \mathbf{T}) - \sum_{j \in \mathbf{S}} c_p(j) - \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}| \geq 2}} \Delta_p(\mathbf{T}) - c_p(i).$$

Hence,

$$\phi_p(a_i, \mathbf{S}) = \sum_{\emptyset \neq \mathbf{T} \subseteq \mathbf{S}} \Delta_p(\{a_i\} \cup \mathbf{T}). \square \tag{5.15}$$

A recursive expression for the amount of interdependence of an action and a set can be useful in estimating the consequence of a set of interdependent actions. The following theorem establishes a relation between the amount of interdependence of a_i and $\mathbf{S}_n$, and of a_i and $\mathbf{S}_{n+1}$.

Theorem 5.3 *Let* $a_i \in \mathbf{A}$ *and* $\mathbf{S}_n, \mathbf{S}_{n+1} \subseteq \mathbf{A} \setminus \{a_i\}$, $\mathbf{S}_n = \{a_1, a_2, \cdots, a_j, \cdots, a_n\}$ *and* $\mathbf{S}_{n+1} = \mathbf{S}_n \cup \{a_{n+1}\}$. *Then*

$$\phi_p(a_i, \mathbf{S}_{n+1}) = \phi_p(a_i, \mathbf{S}_n) + \sum_{\mathbf{T} \subseteq \mathbf{S}_n} \Delta_p\left(\{a_i, a_{n+1}\} \cup \mathbf{T}\right). \tag{5.16}$$

(Note that the summation includes a $\mathbf{T} = \emptyset$ *term.)*

Proof: According to 4.2. when $\mathbf{A}^0 = \emptyset$

$$\phi_p(a_i, \mathbf{S}_{n+1}) = c_p(\mathbf{S}_n \cup \{a_i, a_{n+1}\}) - c_p(\mathbf{S}_n \cup \{a_{n+1}\}) - c_p(i). \quad (5.17)$$

But, using (5.6):

$$\begin{aligned} c_p(\mathbf{S}_n \cup \{a_i, a_{n+1}\}) = & \sum_{j \in S_n} c_p(j) + c_p(i) + c_p(n+1) + \sum_{\substack{\mathbf{T} \subseteq \mathbf{S}_n \\ |\mathbf{T}| \geq 2}} \Delta_p(\mathbf{T}) \\ & + \sum_{\emptyset \neq \mathbf{T} \subseteq \mathbf{S}_n} \Delta_p(\{a_i\} \cup \mathbf{T}) + \sum_{\emptyset \neq \mathbf{T} \subseteq \mathbf{S}_n} \Delta_p(\{a_{n+1}\} \cup \mathbf{T}) \\ & + \sum_{\mathbf{T} \subseteq \mathbf{S}_n} \Delta_p(\{a_i, a_{n+1}\} \cup \mathbf{T}). \end{aligned}$$

and

$$\begin{aligned} c_p(\mathbf{S}_n \cup \{a_{n+1}\}) &= \sum_{j \in S_n} c_p(j) + c_p(n+1) + \sum_{\substack{\mathbf{T} \subseteq \mathbf{S}_n \\ |\mathbf{T}| \geq 2}} \Delta_p(\mathbf{T}) \\ &+ \sum_{\emptyset \neq \mathbf{T} \subseteq \mathbf{S}_n} \Delta_p(\{a_{n+1}\} \cup \mathbf{T}). \end{aligned}$$

Substituting into (5.17) gives:

$$\begin{aligned} \phi_p(a_i, \mathbf{S}_{n+1}) &= \sum_{\emptyset \neq \mathbf{T} \subseteq \mathbf{S}_n} \Delta_p(\{a_i\} \cup \mathbf{T}) \\ &+ \sum_{\mathbf{T} \subseteq \mathbf{S}_n} \Delta_p(\{a_i, a_{n+1}\} \cup \mathbf{T}), \end{aligned}$$

or

$$\phi_p(a_i, \mathbf{S}_{n+1}) = \phi_p(a_i, \mathbf{S}_n) + \sum_{\mathbf{T} \subseteq \mathbf{S}_n} \Delta_p(\{a_i, a_{n+1}\} \cup \mathbf{T}). \square$$

With sequential substitution of interdependence terms as in Theorem 5.3, one can find a recursive expression for the interdependence of a set and an action. Let

$\mathbf{S}_k = \{a_1, \cdots, a_k\} \setminus \{a_i\}$, $\mathbf{S}_{k-t} = \{a_1, \cdots, a_{k-t}\} \setminus \{a_i\}$, and $\mathbf{S}_t = \{a_{k-t+1}, \cdots, a_k\} \setminus \{a_i\}$. Then, for every $t < k$ (see Appendix A.5),

$$\phi_p(a_i, \mathbf{S}_k) = \phi_p(a_i, \mathbf{S}_{k-t}) + \sum_{\mathbf{T}_1 \subseteq \mathbf{S}_{k-t}} \sum_{\emptyset \neq \mathbf{T}_2 \subseteq \mathbf{S}_t} \Delta_p(\{a_i\} \cup \mathbf{T}_1 \cup \mathbf{T}_2). \tag{5.18}$$

For example, consider $\mathbf{S}_5 = \{a_1, \cdots, a_5\}$. If $\phi_p(a_i, \mathbf{S}_3)$ is known, where $\mathbf{S}_3 = \{a_1, a_2, a_3\}$, then

$$\begin{aligned} \phi_p(a_i, \mathbf{S}_5) &= \phi_p(a_i, \mathbf{S}_3) + \sum_{\substack{\mathbf{T} \subseteq \mathbf{S}_3 \\ |\mathbf{T}| \leq |\mathbf{S}_3|}} \Delta_p(\{a_i, a_4\} \cup \mathbf{T}) \\ &+ \sum_{\substack{\mathbf{T} \subseteq \mathbf{S}_3 \\ |\mathbf{T}| \leq |\mathbf{S}_3|}} \Delta_p(\{a_i, a_5\} \cup \mathbf{T}) + \sum_{\substack{\mathbf{T} \subseteq \mathbf{S}_3 \\ |\mathbf{T}| \leq |\mathbf{S}_3|}} \Delta_p(\{a_i, a_4, a_5\} \cup \mathbf{T}). \end{aligned}$$

The amount of interdependence of two sets can be obtained similarly.

Theorem 5.4 *Assume* $\mathbf{S}_1, \mathbf{S}_2 \subseteq \mathbf{A}$, *and* $\mathbf{S}_1 \cap \mathbf{S}_2 = \emptyset$. *Then the amount of interdependence of* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *is as follows:*

$$\phi_p(\mathbf{S}_1, \mathbf{S}_2) = \sum_{\emptyset \neq \mathbf{T}_1 \subseteq \mathbf{S}_1} \sum_{\emptyset \neq \mathbf{T}_2 \subseteq \mathbf{S}_2} \Delta_p(\mathbf{T}_1 \cup \mathbf{T}_2). \tag{5.19}$$

Proof: See Appendix A.6.

For instance, the amount of interdependence of sets $\{a_1, a_2\}$ and $\{a_3, a_4\}$ equals

$$\begin{aligned} \phi_p(\{a_1, a_2\}, \{a_3, a_4\}) &= \Delta_p(a_1, a_3) + \Delta_p(a_1, a_4) \\ &+ \Delta_p(a_2, a_3) + \Delta_p(a_2, a_4) \\ &+ \Delta_p(a_1, a_2, a_3) + \Delta_p(a_1, a_2, a_4) \end{aligned}$$

$$+ \quad \Delta_p(a_1, a_3, a_4) + \Delta_p(a_2, a_3, a_4)$$
$$+ \quad \Delta_p(a_1, a_2, a_3, a_4).$$

Similar to (5.18), there is a recursive expression for interdependence of two sets in terms of interdependence of their subsets. Let $\mathbf{S}_1, \mathbf{S}_2 \subseteq \mathbf{A}, \mathbf{S}_1 \neq \emptyset, \mathbf{S}_2 \neq \emptyset, \mathbf{S}_1 \cap \mathbf{S}_2 = \emptyset, \mathbf{S}_1' \subseteq \mathbf{S}_1$, and $\mathbf{S}_2' \subseteq \mathbf{S}_2$. Then,

$$\begin{aligned}\phi_p(\mathbf{S}_1, \mathbf{S}_2) &= \phi_p(\mathbf{S}_1', \mathbf{S}_2') + \sum_{\emptyset \neq \mathbf{T}_1 \subseteq \mathbf{S}_1 \backslash \mathbf{S}_1'} \sum_{\emptyset \neq \mathbf{T}_2 \subseteq \mathbf{S}_2} \Delta_p(\mathbf{T}_1 \cup \mathbf{T}_2) \\ &+ \sum_{\emptyset \neq \mathbf{T}_1 \subseteq \mathbf{S}_1} \sum_{\emptyset \neq \mathbf{T}_2 \subseteq \mathbf{S}_2 \backslash \mathbf{S}_2'} \Delta_p(\mathbf{T}_1 \cup \mathbf{T}_2), \qquad (5.20)\end{aligned}$$

where $\Delta_p(.)$ is defined according to (5.3).

5.4 Relationship Between Interdependence of *Sets* and Interdependence of *Actions*

One of the main difficulties in evaluating the consequence of a set of actions is measuring the interdependence among its components. This issue has been addressed by many researchers (for example, [31], [112], [93], [94]). In fact, decision makers and analysts often ignore interdependence and use additive models to evaluate the consequence of a set of actions because of the difficulty of measuring interdependence. Moreover, studies that do consider interdependence impose limitations on its type or structure. Recently, however, as the importance of interdependence in some applications was recognized, techniques were developed for estimating the amount of interdependence among two or more actions [112].

In this section, we propose several different approaches to testing whether actions, or sets of actions, are independent, and to evaluating sets of interdependent actions. Furthermore, we establish useful connections among these approaches. The relationships between independence of sets and independence of actions provide a basis for these procedures. For instance, to estimate the consequence of a set, one can partition it into independent subsets such that the consequence of the set can be represented as an additive function of the consequences of its subsets. One can also decompose the set so as to minimize the number of interdependence terms in the evaluation.

On the other hand, in some real-world applications only partial information concerning the consequences of individual actions and subsets of actions is available. For instance, it is possible that only the consequences of individual actions, and of interactions for a few subsets, are available. Nonetheless, one can estimate the dependence of actions using this information. The discussion in this section sheds some light on these issues.

Recall that independence of an action a_i with individual actions does not imply independence of a_i with the set consisting of their union. For example, it may be that

$$a_i(\mathbf{I}_p \mid \mathbf{A}^0)a_j \text{ and } a_i(\mathbf{I}_p \mid \mathbf{A}^0)a_k, \quad \text{but} \quad \sim [a_i(\mathbf{I}_p \mid \mathbf{A}^0)\{a_j, a_k\}].$$

This can occur when $\{a_j, a_k\}$ has properties not shared by any individual action. For instance, in the context of the WDL example, it could be that sites a_1, a_2 and a_3 are pairwise independent on the infrastructure criterion, but if all three sites were selected together, a large common facility could be built for all sites to take the advantage of economies of scale.

We address the interrelationship between set-independence and action-independence in two cases:

1. There is no restriction on the sign of synergy on the criterion.

2. Synergy on the criterion is either always non-negative, or always non-positive.

5.4.1 General Case: Interdependence Unrestricted

First, we make no assumptions about the sign of synergy of interdependent sets of actions. The amount of interdependence may be zero, positive, or negative. Using some strong conditions, useful relations between the interdependence of sets of actions and interdependence between pairs of actions contained in these sets can be established.

Interdependence can be a difficult property to understand as it does not pass directly from sets to their subsets or supersets. For example, it is possible for an action to be interdependent with some or all actions in a set, yet to be independent of that set considered as a whole. This occurs when the values of the interdependence of an action with different subsets of a set "cancel". For example, assume that $\Delta_p(a_i, a_j) = -5$, $\Delta_p(a_i, a_k) = 3$, and $\Delta_p(a_i, a_j, a_k) = 2$. Then

$$\phi_p(a_i, \{a_j, a_k\}) = \Delta_p(a_i, a_j) + \Delta_p(a_i, a_k) + \Delta_p(a_i, a_j, a_k) = 0$$

Hence, $a_i \mathbf{I}_p \{a_j, a_k\}$, but $\sim [a_i \ \mathbf{I}_p \ a_j]$ and $\sim [a_i \ \mathbf{I}_p \ a_k]$.

The following theorem establishes the relation between independence of a set and an action, and independence of two actions. In particular, it is shown that to adjoin a set of actions to either of a pair of independent sets such that independence is preserved, the augmenting subset should be independent of

1. the set it is added to, and
2. the union of the original sets.

Theorem 5.5 *Let* $\mathbf{S}, \mathbf{S}' \subseteq \mathbf{A}, \mathbf{S} \cap \mathbf{S}' = \emptyset$. *Assume* $\mathbf{S}_1'$ *and* $\mathbf{S}_2'$ *partition* $\mathbf{S}'$. *Then, for any* $\mathbf{A}^0 \subseteq \mathbf{A} \setminus (\mathbf{S} \cup \mathbf{S}')$, *if* $\mathbf{S}_1'(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}_2'$ *and* $\mathbf{S}_2'(\mathbf{I}_p \mid \mathbf{A}^0)(\mathbf{S}_1' \cup \mathbf{S})$, *then* $\mathbf{S}(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}'$ *iff* $\mathbf{S}(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}_1'$.

Proof: Assume that

$$\mathbf{S}_1'(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}_2' \text{ and } \mathbf{S}_2'(\mathbf{I}_p \mid \mathbf{A}^0)(\mathbf{S}_1' \cup \mathbf{S}).$$

(1) Proof of $\mathbf{S}(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}' \Longrightarrow \mathbf{S}(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}_1'$.
Since $\mathbf{S}(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}'$,

$$c_p(\mathbf{S} \cup \mathbf{A}^0) - c_p(\mathbf{A}^0) = c_p(\mathbf{S} \cup \mathbf{S}' \cup \mathbf{A}^0) - c_p(\mathbf{S}' \cup \mathbf{A}^0).$$

Because, $\mathbf{S}_1'(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}_2'$ it now follows that

$$c_p(\mathbf{S} \cup \mathbf{S}' \cup \mathbf{A}^0) = c_p(\mathbf{S} \cup \mathbf{A}^0) - c_p(\mathbf{A}^0) + c_p(\mathbf{S}_1' \cup \mathbf{A}^0) + c_p(\mathbf{S}_2' \cup \mathbf{A}^0) - c_p(\mathbf{A}^0). \tag{5.21}$$

On the other hand, because $\mathbf{S}_2'(\mathbf{I}_p \mid \mathbf{A}^0)(\mathbf{S}_1' \cup \mathbf{S})$, we have

$$c_p(\mathbf{S}_2' \cup \mathbf{A}^0) - c_p(\mathbf{A}^0) = c_p(\mathbf{S}_1' \cup \mathbf{S}_2' \cup \mathbf{S} \cup \mathbf{A}^0) - c_p(\mathbf{S}_1' \cup \mathbf{S} \cup \mathbf{A}^0). \tag{5.22}$$

Substituting (5.21) into (5.22) yields

$$c_p(\mathbf{S} \cup \mathbf{A}^0) - c_p(\mathbf{A}^0) = c_p(\mathbf{S}_1' \cup \mathbf{S} \cup \mathbf{A}^0) - c_p(\mathbf{S}_1' \cup \mathbf{A}^0). \tag{5.23}$$

which means, by definition.

$$\mathbf{S}(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}_1'.$$

(2) Proof that $\mathbf{S}(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}_1' \Longrightarrow \mathbf{S}(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}'$.

Since $\mathbf{S}_2'(\mathbf{I}_p \mid \mathbf{A}^0)(\mathbf{S}_1' \cup \mathbf{S})$. (5.22) holds. Because $\mathbf{S}_1'(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}_2'$.

$$c_p(\mathbf{S}_1' \cup \mathbf{A}^0) - c_p(\mathbf{A}^0) = c_p(\mathbf{S}_1' \cup \mathbf{S}_2' \cup \mathbf{A}^0) - c_p(\mathbf{S}_2' \cup \mathbf{A}^0). \tag{5.24}$$

Substituting (5.22) into (5.24) yields

$$c_p(\mathbf{S} \cup \mathbf{S}_1' \cup \mathbf{S}_2' \cup \mathbf{A}^0) - c_p(\mathbf{S} \cup \mathbf{S}_1' \cup \mathbf{A}^0) = c_p(\mathbf{S}_1' \cup \mathbf{S}_2' \cup \mathbf{A}^0) - c_p(\mathbf{S}_1' \cup \mathbf{A}^0). \tag{5.25}$$

On the other hand. $\mathbf{S}(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}_1'$ implies

$$c_p(\mathbf{S} \cup \mathbf{A}^0) - c_p(\mathbf{A}^0) = c_p(\mathbf{S} \cup \mathbf{S}_1' \cup \mathbf{A}^0) - c_p(\mathbf{S}_1' \cup \mathbf{A}^0). \tag{5.26}$$

Substituting (5.26) into (5.25) gives

$$c_p(\mathbf{S} \cup \mathbf{A}^0) - c_p(\mathbf{A}^0) = c_p(\mathbf{S} \cup \mathbf{S}_1' \cup \mathbf{S}_2' \cup \mathbf{A}^0) - c_p(\mathbf{S}_1' \cup \mathbf{S}_2' \cup \mathbf{A}^0). \tag{5.27}$$

which implies by definition that

$$\mathbf{S}(\mathbf{I}_p \mid \mathbf{A}^0)(\mathbf{S}_1' \cup \mathbf{S}').\square$$

Theorem 5.5 can be interpreted as providing sufficient conditions for the independence of two sets to be implied by the independence of one of them with a proper subset of the other, and vice versa.

Now, we state a special case of Theorem 5.5 which shows what conditions are required to preserve independence of two actions when additional actions are joined to one of them.

Corollary 5.2 *Assume actions a_i, a_j, and $a_k \in \mathbf{A}$, and $\mathbf{A}^0 \subseteq \mathbf{A} \setminus \{a_i, a_j, a_k\}$. If* $a_j(\mathbf{I}_p \mid \mathbf{A}^0)a_k$ and $a_k(\mathbf{I}_p \mid \mathbf{A}^0)(a_i, a_j)$ *, then* $a_i(\mathbf{I}_p \mid \mathbf{A}^0)(a_j, a_k)$ *iff* $a_i(\mathbf{I}_p \mid \mathbf{A}^0)a_j$.

Proof: Similar to Theorem 5.5.

The relation between *interdependence* with a set and with one of its subsets follows from the theorem for independence.

Corollary 5.3 *Let* $\mathbf{S}, \mathbf{S}' \subseteq \mathbf{A}, \mathbf{S} \cap \mathbf{S}' = \emptyset$ *and* $\mathbf{A}^0 \subseteq \mathbf{A} \setminus (\mathbf{S} \cup \mathbf{S}')$. *Assume that* $\mathbf{S}'_1$ *and* $\mathbf{S}'_2$ *partition* $\mathbf{S}'$. *Then* $\mathbf{S}$ *and* $\mathbf{S}'_1$ *are interdependent given* $\mathbf{A}^0$ *iff* $\mathbf{S}$ *and* $\mathbf{S}'$ *are interdependent given* $\mathbf{A}^0$ *provided that*

$$\mathbf{S}'_1(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}'_2 \text{ and } \mathbf{S}'_2(\mathbf{I}_p \mid \mathbf{A}^0)\{\mathbf{S}'_1 \cup \mathbf{S}\}$$

Proof: The proof is immediate from Theorem 5.5.□

In summary, unless more conditions hold, interdependence (independence) of one set with another does not imply interdependence (independence) of the set with any proper subset, and vice versa.

Note that dependence within a set of actions does not imply the dependence within any of its proper subsets. For instance, for three actions it is possible that there is no interdependence within pairs of actions. yet dependence does exist when three actions are selected. Figure 5.2 shows three different situations which may arise in dependence of three actions and below is a numerical example illustrating Figure 5.2.

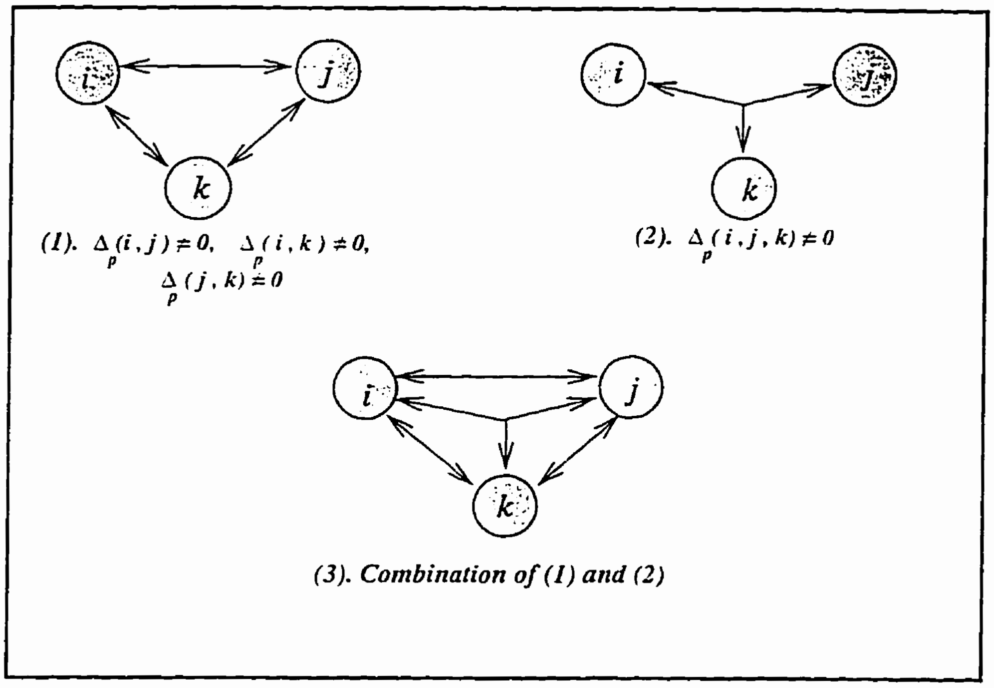


Figure 5.1: Kinds of Interdependence among Three Actions.

Example 5.2 Assume $c_p(i) = c_p(j) = c_p(k) = 2$.

(1) If $c_p(i,j) = 5, c_p(i,k) = 3, c_p(j,k) = 6$ and $c_p(i,j,k) = 8$, then

$$\Delta_p(a_i, a_j) = 1, \Delta_p(a_i, a_k) = -1, \Delta_p(a_j, a_k) = 2, \ and \ \Delta_p(a_i, a_j, a_k) = 0$$

which corresponds to case (1) in Figure .

(2) If $c_p(i,j) = c_p(i,k) = c_p(j,k) = 4$ and $c_p(i,j,k) = 8$, then

$$\Delta_p(a_i, a_j) = \Delta_p(a_i, a_k) = \Delta_p(a_j, a_k) = 0 \;\; and \; \Delta_p(a_i, a_j, a_k) = 2$$

which corresponds to case (2) in Figure 2.

(3) Finally, if $c_p(i,j) = 5$, $c_p(i,k) = 3$, $c_p(j,k) = 6$ and $c_p(i,j,k) = 7$, then

$$\Delta_p(a_i, a_j) = 1, \Delta_p(a_i, a_k) = -1, \Delta_p(a_j, a_k) = 2, \;\; and \; \Delta_p(a_i, a_j, a_k) = -1$$

which corresponds to case (3) in figure 5.2.

Another consequence of Theorem 5.5 is a corollary that provides an alternative way to assess independence or interdependence.

Corollary 5.4 *Sets* $\mathbf{S}$ *and* $\mathbf{S}'$ *are independent given* $\mathbf{A}^0 \subseteq \mathbf{A} \setminus (\mathbf{S} \cup \mathbf{S}')$ *if* $\mathbf{S}'$ *can be partitioned into two subsets,* $\mathbf{S}_1'$ *and* $\mathbf{S}_2'$, *such that*

$$\mathbf{S}(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}_1', \;\; \mathbf{S}_1'(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}_2' \;\; \text{, and } \mathbf{S}_2'(\mathbf{I}_p \mid \mathbf{A}^0)\{\mathbf{S} \cup \mathbf{S}_1'\}. \qquad (5.28)$$

According to Corollary 5.4, to demonstrate the independence of two sets, $\mathbf{S}$ and $\mathbf{S}'$, it is sufficient to find a partition of $\mathbf{S}'$ into two subsets such that (5.28) holds. On the other hand, according to Corollary 5.3, $\mathbf{S}$ and $\mathbf{S}'$ are interdependent if any part of (5.28) fails.

The following example demonstrates a possible use of Theorem 5.5 and its corollaries. In particular, it shows how Theorem 5.5 can help to find the amount of dependence within all pairs of actions with partial information on dependence of some pairs of actions.

Example 5.3 Assume $\mathbf{S} = \{a_1, a_2\}$, $\mathbf{S}' = \{a_3, a_4, a_5\}$, $\mathbf{A}^0 = \emptyset$, and $\mathbf{S}(\mathbf{I}_p \mid \mathbf{A}^0)\mathbf{S}'$. Let $\Delta_p(a_2, a_3) = \Delta_p(a_2, a_5) = 2$, $\Delta_p(a_3, a_5) = 0$, and suppose that dependences among more than two actions are all negligible (i.e. $O_p = 2$). Find the dependence within all pairs of actions.

According to Theorem 5.5 and Corollary 5.3, since $\mathbf{S}\ \mathbf{I}_p\ \mathbf{S}'$, one can partition $\mathbf{S}'$ into $\mathbf{S}'_1 = \{a_3\}$ and $\mathbf{S}'_2 = \{a_4, a_5\}$ such that

$$\{a_1, a_2\}\ \mathbf{I}_p\ \{a_3\}; \qquad \{a_3\}\ \mathbf{I}_p\ \{a_4, a_5\}; \text{ and } \quad \{a_4, a_5\}\ \mathbf{I}_p\ \{a_1, a_2, a_3\}.$$

Using the definition of set interdependence, and the assumption that dependence of more than two actions is zero, we have

$$\phi_p(\{a_1, a_2\}, \{a_3\}) = 0 \Longrightarrow \Delta_p(\{a_1, a_3\}) + \Delta_p(\{a_2, a_3\}) = 0. \qquad (5.29)$$

$$\phi_p(\{a_3\}, \{a_4, a_5\}) = 0 \Longrightarrow \Delta_p(\{a_3, a_4\} + \Delta_p(\{a_3, a_5\} = 0. \qquad (5.30)$$

$$\begin{aligned} &\phi_p(\{a_4, a_5\}, \{a_1, a_2, a_3\}) = 0 \Longrightarrow \\ &\Delta_p(\{a_1, a_4\}) + \Delta_p(\{a_2, a_4\}) + \Delta_p(\{a_3, a_4\}) \\ &+\Delta_p(\{a_1, a_5\}) + \Delta_p(\{a_2, a_5\}) + \Delta_p(\{a_3, a_5\}) = 0. \end{aligned} \qquad (5.31)$$

Using (5.29), (5.30), and (5.31),

$$\begin{aligned} &\Delta_p(\{a_1, a_3\}) = -2, \quad \Delta_p(\{a_3, a_4\}) = 0, \\ &\Delta_p(\{a_1, a_4\}) + \Delta_p(\{a_2, a_4\}) + \Delta_p(\{a_1, a_5\}) = -2. \end{aligned}$$

Using the same procedure and partitioning $\mathbf{S}'$ into $\{a_3, a_4\}$ and $\{a_5\}$, one can find that $\Delta_p(\{a_1, a_5\}) = -2$ and $\Delta_p(\{a_1, a_4\}) = -\Delta_p(\{a_2, a_4\})$.

5.4.2 Case 2: Unicity of Sign of Synergy on a Criterion

In real-world problems, criteria often have the property that actions, or sets of actions, are either independent or always have synergy of the same sign. These are called *positive synergy* or *negative synergy* criteria. Moreover, when there exist both positive and negative synergies on a criterion, one can often decompose the criterion into positive and negative sub-criteria; then evaluation of actions can be carried out using each sub-criterion.

Building the set of criteria in such a way that all criteria are either positive or negative makes it easier to derive useful connections between action and set interdependence. Hence, utilization of the multiple criteria structure eliminates some of the difficulties of evaluating a set of interdependent actions. This, in turn, helps in implementing some efficient procedures to estimate the amount of interdependence. It is noteworthy that, in general, unicity of sign cannot be used with utility or aggregated value functions. Throughout this section, we assume that a criterion is either positive or negative. The following theorem shows that in the case of unicity, independence of two sets implies independence of all their subsets; in other words, in this case independence is hereditary.

Theorem 5.6 *Assume that $p \in \mathbf{P}$ is positive or negative. Let* $\mathsf{S} \subseteq \mathbf{A}, \mathsf{S}' \subseteq \mathbf{A}$, $\mathsf{S} \neq \emptyset, \mathsf{S}' \neq \emptyset$ *and* $\mathsf{S} \cap \mathsf{S}' = \emptyset$. *Then* $\mathsf{S}\ \mathbf{I}_p\ \mathsf{S}'$ *implies* $\mathsf{S}_i\ \mathbf{I}_p\ \mathsf{S}'_j \quad \forall\ \mathsf{S}_i \subseteq \mathsf{S}$, $\forall\, \mathsf{S}'_j \subseteq \mathsf{S}'$.

Proof: Without loss of generality assume that $p \in \mathbf{P}$ is a positive criterion. Assume that $\exists\ S_1 \subseteq S$ and $\exists\ S_2 \subseteq S'$, such that $\phi_p(S_1, S_2) > 0$. According to (5.20)

$$\begin{aligned}\phi_p(S, S') &= \phi_p(S_1, S_2) + \sum_{\emptyset \neq T_1 \subseteq S \setminus S_1} \sum_{\emptyset \neq T_2 \subseteq S'} \Delta_p(T_1 \cup T_2)\\ &+ \sum_{\emptyset \neq T_1 \subseteq S} \sum_{\emptyset \neq T_2 \subseteq S' \setminus S_2} \Delta_p(T_1 \cup T_2),\end{aligned}$$

or

$$\phi_p(S, S') = \phi_p(S_1, S_2) + \underbrace{\phi_p(S \setminus S_1, S')}_{a} + \underbrace{\phi_p(S, S' \setminus S_2)}_{b}. \tag{5.32}$$

Since $S\ I_p\ S'$, $\phi_p(S, S') = 0$. Thus, for vanishing the right hand side of (5.32), at least one of the terms a or b should be negative. But, this is against the assumption that p is a positive criterion. Hence, S_1 and S_2 must be independent on criterion p. □

According to Theorem 5.6 under unicity of synergy, if two sets are independent, then all their subsets must be independent.

It follows from Theorem 5.6 that for every $S_1 \subseteq A$ and $S_2 \subseteq A$ such that $S_1 \cap S_2 = \emptyset$,

$$\sim [S_1\ I_p\ S_2] \Longrightarrow \sim [S_1'\ I_p\ S_2'] \quad \forall S_1' \supseteq S_1, \text{ and } \forall S_2' \supseteq S_2. \tag{5.33}$$

In other words, interdependence of two sets implies interdependence of their supersets. The next corollary is an immediate consequence of Theorem 5.6.

Corollary 5.5 *Let $a_i \in A$ and $S \subseteq A \setminus \{a_i\}$. Then $a_i\ I_p\ S$ implies that $a_i\ I_p\ S'$ $\forall S' \subseteq S$.*

In many decision problems, one wants to examine the independence of two sets when the relations among some of their subsets are known. Let $S_{(k)}$ denote the collection of all subsets of $\mathbf{S}$ with cardinality k. The following theorem indicates necessary and sufficient conditions for independence of two sets to be implied by independence of their proper subsets. under the unicity of the synergy condition.

Theorem 5.7 *Let* $m.n > 0$ *and fix* $\mathbf{S} \in \mathbf{A}_{(n)}$ *and* $\mathbf{T} \in \mathbf{A}_{(m)}$ *such that* $\mathbf{S} \cap \mathbf{T} = \emptyset$. *Then* $\mathbf{S}\ \mathbf{I}_p\ \mathbf{T}$ *iff* $\Delta_p(\mathbf{S} \cup \mathbf{T}) = 0$ *and* $\forall \mathbf{S}_1 \in \mathbf{S}_{(n-1)}, \mathbf{T}_1 \in \mathbf{T}_{(m-1)}, \mathbf{S}_1\ \mathbf{I}_p\ \mathbf{T}_1$.

Proof: The condition is necessary according to Theorem 5.6. which states that

$$\mathbf{S}\ \mathbf{I}_p\ \mathbf{T} \Longrightarrow \mathbf{S}_1\ \mathbf{I}_p\ \mathbf{T}_1 \quad \forall \mathbf{S}_1 \in \mathbf{S}_{(n-1)} \text{ and } \forall \mathbf{T}_1 \in \mathbf{T}_{(m-1)}.$$

To prove that it is sufficient, we must show that $\phi_p(\mathbf{S}, \mathbf{T}) = 0$. According to (5.19).

$$\phi_p(\mathbf{S}, \mathbf{T}) = \sum_{\emptyset \neq \mathbf{S}_1 \subseteq \mathbf{S}} \sum_{\emptyset \neq \mathbf{T}_1 \subseteq \mathbf{T}} \Delta_p(\mathbf{S}_1 \cup \mathbf{T}_1).$$

Because $\mathbf{S}_1\ \mathbf{I}_p\ \mathbf{T}_1 \ \ \forall\ \mathbf{S}_1 \in \mathbf{S}_{(n-1)}$ and $\mathbf{T}_1 \in \mathbf{T}_{(m-1)}$. Theorem 5.6 implies

$$\mathbf{S}_2\ \mathbf{I}_p\ \mathbf{T}_2, \quad \forall \mathbf{S}_2 \in \mathbf{S}_{(i)}, \mathbf{T}_2 \in \mathbf{T}_{(j)}.$$

where $i \leq n-1, j \leq m-1$. Therefore. $\phi_p(\mathbf{S}_1, \mathbf{T}_1) = 0$ whenever $\mathbf{S}_1 \subseteq \mathbf{S}, |\ \mathbf{S}_1\ | \leq n-1, \mathbf{T}_1 \subseteq \mathbf{T}, |\ \mathbf{T}_1\ | \leq m-1$. But, because of the unicity of sign of synergy,

$$\phi_p(\mathbf{S}_1, \mathbf{T}_1) = 0 \Longrightarrow \Delta_p(\mathbf{S}_1 \cup \mathbf{T}_1) = 0.$$

for any such $\mathbf{S}_1$ and $\mathbf{T}_1$. Hence, (5.19) reduces to:

$$\phi_p(\mathbf{S}, \mathbf{T}) = \Delta_p(\mathbf{S} \cup \mathbf{T}).$$

But, by assumption, $\Delta_p(\mathbf{S} \cup \mathbf{T}) = 0$. Hence, $\phi_p(\mathbf{S}, \mathbf{T}) = 0$ and $\mathbf{S}\ \mathbf{I}_p\ \mathbf{T}$. □

In summary, Theorem 5.7 shows that, in order to prove independence of two sets $\mathbf{S}$ and $\mathbf{T}$, $\mid \mathbf{S} \mid = n$, $\mid \mathbf{T} \mid = m$, one has to examine $n \times m$ independence relations among the n subsets of $\mathbf{S}_{(n-1)}$ and the m subsets of $\mathbf{T}_{(m-1)}$, and also show that $\Delta_p(\mathbf{S} \cup \mathbf{T}) = 0$. For instance, to use this method to show the independence of $\mathbf{S} = \{a_1, a_2, a_3, a_4\}$, and $\mathbf{T} = \{a_5, a_6, a_7\}$, twelve independence relations have to be proven, as well as $\Delta_p(\mathbf{S} \cup \mathbf{T}) = 0$. The following corollary is the immediate result of Theorem 5.7.

Corollary 5.6 *Let* $a_i \in \mathbf{A}$ *and* $\mathbf{S} \subseteq \mathbf{A} \setminus a_i$. *Then* $a_i\ \mathbf{I}_p\ \mathbf{S}$ *if and only if* $a_i\ \mathbf{I}_p\ \mathbf{S}_1$ $\forall \mathbf{S}_1 \in \mathbf{S}_{(n-1)}$, *and* $\Delta_p(\{a_i\} \cup \mathbf{S}) = 0$.

For instance, to show the independence of a_1 and $\mathbf{S} = \{a_2, a_3, a_4\}$, the following relations must be proven:

$$a_1\ \mathbf{I}_p\ \{a_2, a_3\}, \quad a_1\ \mathbf{I}_p\ \{a_2, a_4\}, \quad a_1\ \mathbf{I}_p\ \{a_3, a_4\},$$
$$\Delta_p(\{a_1, a_2, a_3, a_4\}) = 0.$$

Recall from (5.2) that the order of dependence on criterion p, O_p, is the cardinality of the largest subset $\mathbf{T} \subseteq \mathbf{A}$ such that $\Delta_p(\mathbf{T}) \neq 0$. As explained previously, in some cases it is more convenient to restrict the order of dependence of a set. The following theorem is useful for proving independence of two sets according to independence of their proper subsets under a restriction on the order of dependence.

Theorem 5.8 *Assume that the order of dependence on a positive or negative criterion* $p \in \mathbf{P}$ *is* $O_p = k$. *Let* $\mathbf{S}_i \subset \mathbf{A}$, $\mathbf{S}_i \neq \emptyset$, $i = 1, \cdots, n$, *and* $\mathbf{T}_j \subset \mathbf{A}$, $\mathbf{T}_j \neq \emptyset$.

$j = 1, \cdots, m$, and $\mathbf{S}_i \cap \mathbf{T}_j = \emptyset$. Moreover, let $| \mathbf{S}_i \cup \mathbf{T}_j | \geq k$. Then the following statements are equivalent:

(1). $(\bigcup_i \mathbf{S}_i)\ \mathbf{I}_p\ (\bigcup_j \mathbf{T}_j)$

(2). $\mathbf{S}_i\ \mathbf{I}_p\ \mathbf{T}_j, \quad \forall i, j.$

Proof: **(1)** implies **(2)** because, according to Theorem 5.6, if two sets are independent then any pair of their subsets is also independent.

To show that **(2)** implies **(1)**, assume that $\mathbf{S}_i\ \mathbf{I}_p\ \mathbf{T}_j$, $\forall i, j$. Let $\mathbf{S} = \bigcup_i \mathbf{S}_i$ and $\mathbf{T} = \bigcup_j \mathbf{T}_j$. We have to show that $\phi_p(\mathbf{S}, \mathbf{T}) = 0$. According to (5.19)

$$\phi_p(\mathbf{S}, \mathbf{T}) = \sum_{\emptyset \neq \mathbf{S}_i \subseteq \mathbf{S}} \ \sum_{\emptyset \neq \mathbf{T}_j \subseteq \mathbf{T}} \Delta_p(\mathbf{S}_i \cup \mathbf{T}_j).$$

To demonstrate that this expression vanishes, we can show that all the interdependent terms $\Delta_p(.)$ are zero. Since $\mathbf{S}_i\ \mathbf{I}_p\ \mathbf{T}_j$, $\forall i, j$, then according to Theorem 5.6 under the assumption of unicity of sign $\mathbf{S}_{i'}\ \mathbf{I}_p\ \mathbf{T}_{j'}$ for any $\mathbf{S}_{i'} \subseteq \mathbf{S}_i$ and any $\mathbf{T}_{j'} \subseteq \mathbf{T}_j$. Therefore,

$$\sum_{\emptyset \neq \mathbf{S}_{i'} \subseteq \mathbf{S}_i} \ \sum_{\emptyset \neq \mathbf{T}_{j'} \subseteq \mathbf{T}_j} \Delta_p(\mathbf{S}_{i'} \cup \mathbf{T}_{j'}) = 0; \qquad \forall i, j.$$

On the other hand, the order of interdependence is k, and $| \mathbf{S}_i \cup \mathbf{T}_j | \geq k$. Hence

$$\Delta_p(\mathbf{S}_i \cup \mathbf{T}_j) = 0 \qquad \textit{when} \quad | \mathbf{S}_i \cup \mathbf{T}_j | \geq k.$$

Thus, all interdependence terms in (5-7) are zero and hence $(\bigcup_i \mathbf{S}_i)\ \mathbf{I}_p\ (\bigcup_j \mathbf{T}_j)$. ◻

The following theorem is a useful way of recognizing independence or interdependence of two sets. Recall that for any set $\mathbf{S} \subseteq \mathbf{A}$, the collection of subset, $\theta^0(\mathbf{S}) = \{\mathbf{S}_1, \ldots, \mathbf{S}_i, \ldots \mathbf{S}_n\}$, is a *cover* of $\mathbf{S}$ *iff* $\bigcup_i \mathbf{S}_i = \mathbf{S}$. Let $\Theta(\mathbf{S})$ denote the class of all covers of set $\mathbf{S}$, and $\theta^0(\mathbf{S})$ represent an element of $\Theta(\mathbf{S})$.

Theorem 5.9 *Assume that order of dependence on a positive or negative criterion, $p \in \mathbf{P}$, is $O_p = k$. Let $\mathbf{S} \subseteq \mathbf{A}, \mathbf{T} \subseteq \mathbf{A}$, and $\mathbf{S} \cap \mathbf{T} = \emptyset$. Then $\mathbf{S}\ \mathbf{I}_p\ \mathbf{T}$ iff $\exists \theta_i \in \Theta(\mathbf{S})$, and $\exists \theta_j' \in \Theta(\mathbf{T})$, such that for every $\mathbf{S}_i \in \theta_i$, $\mathbf{T}_j \in \theta_j'$, $\mid \mathbf{S}_i \cup \mathbf{T}_j \mid \geq k$, and $\mathbf{S}_i\ \mathbf{I}_p\ \mathbf{T}_j$.*

Proof: Immediate from Theorem 5.8. □

In summary, to show the independence of two sets it is sufficient to find two covers for each set such that all pairs of sets, one from each cover, are independent.

Note that Theorem 5.9 implies that when $O_p = 2$, then two sets are independent *iff* all pairs of actions, one from each set, are independent. The following example shows a simple application of Theorems 5.8 and 5.9.

Example 5.4 Let $\mathbf{S} = \{a_1, a_2, a_3, a_4\}$ and $\mathbf{T} = \{a_5, a_6, a_7\}$. Suppose that the following information is available concerning the relations among different subsets of $\mathbf{S}$ and $\mathbf{T}$:

$$(1) \quad \{a_2, a_3, a_4\}\ \mathbf{I}_p\ \{a_7\} \quad \text{and} \quad \{a_1, a_4\}\ \mathbf{I}_p\ \{a_7\},$$

$$(2) \quad \{a_1, a_2\}\ \mathbf{I}_p\ \{a_5\} \quad \text{and} \quad \{a_3, a_4\}\ \mathbf{I}_p\ \{a_5\},$$

$$(3) \quad \{a_1, a_4\}\ \mathbf{I}_p\ \{a_5, a_6\} \quad \text{and} \quad \{a_3, a_2\}\ \mathbf{I}_p\ \{a_6\}.$$

Suppose that $O_p = 3$. Then according to Theorem 5.8, **(1)**, **(2)**, and **(3)** are, respectively, equivalent to

$$(4) \quad \{a_1, a_2, a_3, a_4\}\ \mathbf{I}_p\ \{a_7\},$$

(5) $\{a_1, a_2, a_3, a_4\}$ $\mathbf{I}_p$ $\{a_5\}$,

(6) $\{a_1, a_2, a_3, a_4\}$ $\mathbf{I}_p$ $\{a_6\}$.

But (4),(5), and (6) imply that

$$\{a_1, a_2, a_3, a_4\} \ \mathbf{I}_p \ \{a_5, a_6, a_7\}.$$

Hence, when $O_p = 3$, $c_p(\mathbf{S} \cup \mathbf{T}) = c_p(\mathbf{S}) + c_p(\mathbf{T})$.

5.5 Additivity of an Interdependent Set

In this section, properties of a set of actions are explored to show how the consequence of a set of interdependent actions can be represented in terms of the consequences of its proper subsets. Here, the main objective is to propose an alternative approach for evaluating the consequence of a set of actions. Specifically, we show how to decompose a set so that the consequence of the set can be evaluated additively, as the sum of the consequences of the subsets in the partition, or such that the number of interdependence terms is minimized. Recall from Section 5.2 that in general the consequence of a set of interdependent actions can be evaluated using the following expression:

$$c_p(\mathbf{S}) = \sum_{i \in \mathbf{S}} c_p(i) + \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}| \geq 2}} \Delta_p(\mathbf{T}). \tag{5.34}$$

As pointed out in previous sections, estimating $\Delta_p(.)$ is the difficult part of evaluating the consequence of a set. Here we introduce alternative ways to represent an interdependent set and to evaluate its consequence in a way that avoids the need to estimate all interdependencies in that set.

For sake of simplicity in discussion, we assume throughout this section the following, unless explicitly stated, otherwise.

1. Actions are either independent or have *simple* interdependence. Recall that two actions a_i and a_j are simply interdependent *iff* $\mathbf{A}^0 = \emptyset$.
2. The sign of interdependence of actions on each criterion is either positive or negative, *i.e.* the unicity of sign of interdependence holds for every criterion.

Later, we will show that these assumptions can be relaxed, with some modifications.

5.5.1 Additivity of An Interdependent Set: Definitions and Concepts

As explained before, one of the basic assumptions in the conventional subset selection problem is that the consequence of a set of actions can be represented as the sum of the consequences of individual actions. In other words, it is assumed that for any set $\mathbf{S} = \{a_1, \ldots, a_i, \ldots, a_m\} \subseteq \mathbf{A}$,

$$c_p(\mathbf{S}) = \sum_{a_i \in \mathbf{S}} c_p(a_i). \tag{5.35}$$

Usually, a set that satisfies (5.35) is said to be additive. In Section 5.2 we observed that this assumption is not true for a set of actions with interdependence; one has to use (5.34) to find the consequence of the set. In this section, we introduce new definitions and techniques that generalize the notion of additivity of a set of actions.

We call a set of actions, $\mathbf{S}$, additive if its consequence equals the sum of the consequences of the subset in the partition of $\mathbf{S}$. Recall that $\psi = \{S_1, \ldots, S_j, \ldots, S_q\}$

is a partition of $\mathbf{S}$ *iff* $S_j \neq \emptyset\ \forall j$, $\bigcap_{j\in\psi} S_j = \emptyset$ and $\bigcup_{j\in\psi} S_j = \mathbf{S}$. Let $\Psi(\mathbf{S})$ be the class of all partitions of set $\mathbf{S}$. Then, the notion of additivity of a set $\mathbf{S}$ is defined as follows:

Definition 5.1 *Let $\emptyset \neq \mathbf{S} \subseteq \mathbf{A}$. Then $\mathbf{S}$ is additive on p with respect to $\psi \in \Psi(\mathbf{S})$, $\psi = \{S_1, \ldots, S_j, \ldots, S_q\}$ if*

$$c_p(\mathbf{S}) = \sum_{j=1}^{q} c_p(S_j). \tag{5.36}$$

When $\mathbf{S}$ is additive in respect to partition ψ, then the cardinality of the largest subset in ψ is called the *degree of additivity* of $\mathbf{S}$ with respect to ψ, and is denoted by $D_p(\mathbf{S} \mid \psi)$. More specifically,

Definition 5.2 *Let $\emptyset \neq \mathbf{S} \subseteq \mathbf{A}$, and $\psi = \{S_1, \ldots, S_j, \ldots, S_q\}$ be a partition of $\mathbf{S}$ such that* (5.36) *holds. Then the degree of additivity of $\mathbf{S}$ with respect to ψ on criterion p is*

$$D_p(\mathbf{S} \mid \psi) = \max_{j=1,\cdots,q} \mid S_j \mid . \tag{5.37}$$

Note that $D_p(\mathbf{S} \mid \psi)$ is a property of a specific partition of $\mathbf{S}$, ψ. Hence, set $\mathbf{S}$ may have different degrees of additivity for various choices of ψ satisfying (5.36). Moreover, additivity of a set $\mathbf{S}$ is defined according to a specific criterion p; the degree of additivity of a set may be different for various criteria.

Clearly, when the consequence of $\mathbf{S}$ can be represented as the sum of consequences of individual actions in $\mathbf{S}$, then $\psi = \{a_1, \ldots, a_i, \ldots, a_m\}$ satisfies (5.36), and therefore, the degree of additivity of $\mathbf{S}$ on p in respect to ψ is one and vice versa. In other words, $D_p(\mathbf{S} \mid \psi) = 1$ implies conventional additivity of $\mathbf{S}$, on criterion p.

The following theorem shows the necessary and sufficient conditions for a set of actions to be additive in respect to a partition:

Theorem 5.10 *Let $\emptyset \neq \mathbf{S} \subseteq \mathbf{A}$, $\psi \in \Psi(\mathbf{S})$, $\psi = \{S_1, \dots, S_j, \dots, S_q\}$. Then* $\mathbf{S}$ *is additive with respect to ψ iff*

$$S_i \, \mathbf{I} \, S_j \qquad for \quad S_i \neq S_j \in \psi.$$

Proof: The condition is necessary according to the definition of additivity of set S. Hence,

$$S_j \, \mathbf{I}_p \, S_{j'} \qquad for \quad j \neq j' \in \psi \Longrightarrow \ \mathbf{S} \text{ is additive with respect to } \psi.$$

To show that it is also sufficient, note that if $\mathbf{S}$ is additive with respect to ψ then

$$c_p(\mathbf{S}) = \sum_{j=1}^{q} c_p(\mathbf{S}_j).$$

Now assume that for some S_k and S_l in ψ, S_k and S_l are not independent. Then,

$$c_p(S_k \cup S_l) \neq c_p(S_k) + c_p(S_l).$$

It follows that because of unicity of sign

$$c_p(\mathbf{S}) = c_p\left(\bigcup_{j\setminus k,l} S_j \cup S_k \cup S_l\right) \neq \sum_{j\setminus k,l} c_p(S_j) + c_p(S_k) + c_p(S_l) \neq \sum_{j=1}^{q} c_p(S_j),$$

which contradicts the assumption. Thus,

$$\mathbf{S} \text{ is additive with respect to } \psi \implies S_j \, \mathbf{I}_p \, S_{j'} \qquad for \quad j \neq j' \in \psi. \ \square$$

According to Theorem 5.10, to represent the consequence of a set of actions as sum of the consequences of its disjoint subsets, one has to find a partition with independent elements. In the previous sections, we showed the required conditions for independence of two sets of actions based on independence of their proper subsets. Here we assume that the sign of interdependence on each criterion is the same and there is only simple interdependence among actions, enabling us to examine the independence of two sets, using Theorems 5.7 and 5.8. It is noteworthy that with these theorems one can determine the independence of two sets in more general cases.

Clearly, there may be several partitions that satisfy the conditions in Theorem 5.10. Let $\hat{\Psi}_p(\mathrm{S}) \subseteq \Psi(\mathrm{S})$ be the set of all partitions with respect to which S is additive on criterion p. We want to find a partition in $\hat{\Psi}_p(\mathrm{S})$ that has the maximum number of elements. In other words, we seek a partition in $\hat{\Psi}_p(\mathrm{S})$ such that the cardinalities of its elements is minimized. Our interest stems from the fact that most often it is easier to measure the consequences of smaller subsets of actions. This is even more important in multiple criteria situations, when there are several consequences on different criteria to be evaluated. Let $\psi_p^0(\mathrm{S}) \in \hat{\Psi}_p(\mathrm{S})$ be a partition of S that satisfies (5.36) and has the maximum number of elements. We call $\psi_p^0(\mathrm{S})$ the *maximal partition* of set S on criterion p.

Later, we will show that for every set of actions there is one and only one maximal partition on each criterion. The cardinality of that maximal partition, $\psi_p^0(\mathrm{S})$, determines how much set S can be decomposed without violating (5.36). Hence, it is called the *degree of separability* of S on criterion p. Since the degree of additivity and the degree of separability have inverse properties, we denote the degree of separability by $\bar{D}_p(\mathrm{S})$. Hence,

$$\bar{D}_p(\mathrm{S}) = \max_{\psi \in \hat{\Psi}_p(\mathrm{S})} |\psi|.$$

Clearly, a set may have different degrees of separability on different criteria.

The following corollary is the immediate result of Theorem 5.10 and the definition of degree of separability.

Corollary 5.7 *Let $\emptyset \neq \mathbf{S} \subseteq \mathbf{A}$. Then the degree of separability of $\mathbf{S}$ is the largest integer k such that actions in $\mathbf{S}$ can be partitioned into k independent sets.*

Recall that we have defined the degree of additivity of a set on a partition, ψ, as the cardinality of the largest subset in ψ. The *overall* degree of additivity of a set , $\mathbf{S}$, on criterion p is defined in respect to its maximal partition, and is denoted by $D_p(\mathbf{S})$.

Definition 5.3 *The overall degree of additivity of a set on a criterion p, $D_p(\mathbf{S})$, is the cardinality of the largest subset of its maximal partition, ψ^0. In other words, if $\psi^0(\mathbf{S}) = \{S_1^0, \ldots, S_j^0, \ldots, S_q^0\}$ is the maximal partition of $\mathbf{S}$, then*

$$D_p(\mathbf{S}) = \max_{j=1,\ldots,q} \mid S_j^0 \mid . \tag{5.38}$$

Definition 5.3 indicates that $D_p(\mathbf{S})$ is the smallest degree of additivity on all partitions of $\mathbf{S}$ satisfying (5.36). Hence,

$$D_p(\mathbf{S}) = \min_{\psi \in \Psi_p(\mathbf{S})} D_p(\mathbf{S} \mid \psi). \tag{5.39}$$

Note that the degree of additivity, $D_p(\mathbf{S})$, is different from the order of dependence, $O_p(\mathbf{S})$, that was defined in Section 5.2. The former is the property of the maximal partition of $\mathbf{S}$ on criterion p and is defined as the cardinality of the largest subset in the maximal partition. The latter is the largest cardinality of any $\mathbf{T} \subset \mathbf{S}$ with non-zero amount of simple dependence. Nevertheless, it is clear that when

$O_p(\mathbf{S}) = 1$, then $D_p(\mathbf{S}) = 1$. Also, it can be shown that for any nonempty set of actions $D_p(\mathbf{S}) \geq O_p(\mathbf{S})$.

As pointed out, the degree of separability and degree of additivity have inverse properties. When $\bar{D}_p(\mathbf{S})$ increases, $D_p(\mathbf{S})$ decreases and vice versa. Two special cases require more explanation. When $\bar{D}_p(\mathbf{S}) = |\mathbf{S}|$, then $D_p(\mathbf{S}) = 1$. In this case, according to Theorem 5.10 all actions in **S** are independent and the consequence of **S** can be written as the sum of the consequences of its individual actions. On the other hand, $\bar{D}_p(\mathbf{S}) = 1$ or $D_p(\mathbf{S}) = | \mathbf{S} |$ implies that the maximal partition of **S** has only one element, which is the set **S** itself. In this case, set **S** is called a *completely interdependent* set. The concept of a completely interdependent set is an important notion that we will use later in this section.

Definition 5.4 *Let* $\mathbf{S} \subseteq \mathbf{A}, | \mathbf{S} | \geq 2$. *Then,* **S** *is completely interdependent if* **S** *cannot be partitioned into two subsets such that (5.36) holds.*

We now propose the following theorem the proof of which follows directly from the above definition and result of Theorem 5.9.

Theorem 5.11 *Let* $\mathbf{S} \subseteq \mathbf{A}, | \mathbf{S} | \geq 2$, *and* $O_p(\mathbf{S}) = k$. *Then* **S** *is completely interdependent on p iff for every two disjoint subsets of* **S**, S_1 *and* S_2 *such that* $S_1 \cup S_2 = \mathbf{S}$, $\exists S_1' \subset S_1$ *and* $\exists S_2' \subset S_2$, *and* $|S_1' \cup S_2'| \leq k$, $\sim [S_1' \ \mathbf{I}_p \ S_2']$.

The following corollary is the special case of Theorem 5.11 when, $O_p(\mathbf{S}) = 2$.

Corollary 5.8 *Let* $\mathbf{S} \subseteq \mathbf{A}, | \mathbf{S} | \geq 2$, *and* $O_p(\mathbf{S}) = 2$. *Then* **S** *is completely interdependent iff for every two nonempty disjoint subsets,* S_1 *and* S_2, *that cover* **S**, *there exist two interdependent actions, one in* S_1 *and one in* S_2.

The following example summarizes the above discussion:

Example 5.5 Let set S $= \{a_1, \ldots, a_8\}$, and $O_p(\mathrm{S}) = 2$, and suppose the set of interdependent pairs of actions on criterion p is

$$\{(a_1, a_2), (a_3, a_8), (a_4, a_5), (a_5, a_6)\}.$$

Consider an arbitrary partition of S as follows:

$$\psi_1 = \{\{a_1, a_2\}, \{a_4, a_5, a_6, a_3, a_8\}, \{a_7\}\}.$$

All elements of partition ψ_1 are independent, because there are no interdependent pairs of actions, from different subsets in the partition. Hence, according to Theorem 5.10, S is additive with respect to ψ_1.

The degree of additivity of S in respect to ψ_1 is calculated as follows:

$$D_p(\mathrm{S} \mid \psi_1) = Max\{\mid \{a_1, a_2\} \mid, \mid \{a_4, a_5, a_6, a_3, a_8\} \mid, \mid \{a_7\} \mid\} = 5.$$

However, ψ_1 is not the maximal partition of S because its second element (*i.e* $\{a_4, a_5, a_6, a_3, a_8\}$) can be decomposed into $\{a_4, a_5, a_6\}$ and $\{a_3, a_8\}$ without violating the additivity of S. Moreover, this element is not completely interdependent. The maximal partition of S is

$$\psi_p^0 = \{\{a_1, a_2\}, \{a_4, a_5, a_6\}, \{a_3, a_8\}, \{a_7\}\}.$$

Therefore, $\bar{D}_p(\mathrm{S}) = \mid \psi^0 \mid = 4$ and $D_p(\mathrm{S}) = 3$.

Even though $O_p(\mathrm{S}) = 2$ in the above example, the degree of additivity of S equals 3. In most cases, evaluating the consequence of a set of actions through its maximal partition is easiest, because this partition decomposes the set into its smallest possible subsets such that (5.36) holds, and evaluating smaller subsets is easier. The following theorem shows that a partition is maximal when it satisfies (5.36), and all of its elements are completely interdependent.

Theorem 5.12 *Let $\emptyset \neq \mathbf{S} \subseteq \mathbf{A}$. Then $\psi_p^0(\mathbf{S})$ is the maximal partition of* $\mathbf{S}$ *on criterion p iff $\psi_p^0(\mathbf{S}) \in \hat{\Psi}_p$ and all elements of $\psi_p^0(\mathbf{S})$ are completely interdependent.*

Proof: We first prove that if $\psi_p^0(\mathbf{S})$ is maximal, then all elements of $\psi_p^0(\mathbf{S})$ are completely interdependent.

Assume that $\psi_p^0(\mathbf{S})$ is maximal. Suppose that $\exists S_e \in \psi_p^0(\mathbf{S})$ such that S_e is not completely interdependent. Then according to Definition 5.4 and Theorem 5.11, S_e can be partitioned into two subsets, S_{e_1} and S_{e_2} such that

$$c_p(S_e) = c_p(S_{e_1}) + c_p(S_{e_2}).$$

Hence,

$$\begin{aligned} c_p(\mathbf{S}) &= \sum_{j \neq e} c_p(S_j) + c_p(S_e). \\ &= \sum_{j \neq e} c_p(S_j) + c_p(S_{e_1}) + c_p(S_{e_2}) \end{aligned}$$

Hence, the additivity of S is maintained, implying that $\psi_p^0(\mathbf{S})$ is not maximal, which contradicts the initial assumption.

Now, we prove that if all elements of $\psi_p^0(\mathbf{S})$ are completely interdependent then $\psi_p^0(\mathbf{S})$ is maximal. When every element of $\psi_p^0(\mathbf{S})$ is completely interdependent, then according to Definition 5.4, none of them can be partitioned into any two independent subsets. Hence, $\psi_p^0(\mathbf{S})$ is the largest partition that satisfies (5.36). Therefore, $\psi_p^0(\mathbf{S})$ is maximal, completing the proof. □

Note that in Example 5.5 all the elements of ψ_p^0 are completely interdependent. Hence, this partition is maximal. On the other hand, in this example, the second element of partition ψ_1 is not completely interdependent and can be partitioned into two independent subsets. Therefore, this partition is not maximal. Obviously, there may exist several completely interdependent subsets for a set of actions. However, we are interested in finding the biggest completely interdependent subset of a set of interdependent actions, because, according to Theorem 5.12, the cardinality of the biggest completely interdependent subset indicates the overall degree of additivity of a set. The following theorem shows that for every set of actions, there is only one maximal partition on each criterion.

Theorem 5.13 *Let $\emptyset \neq \mathbf{S} \subseteq \mathbf{A}$. Then there is one and only one maximal partition of set $\mathbf{S}$ on each criterion.*

Proof: Let ψ_p^0 be the maximal partition of $\mathbf{S}$. Assume that ψ_p^0 is not unique and there exists another partition $\hat{\psi}_p^0$ that satisfy (5.36) and $\mid \hat{\psi}_p^0 \mid = \mid \psi_p^0 \mid$. We prove that $\hat{\psi}_p^0$ and ψ_p^0 are identical by showing that all elements of these two partitions are identical.

Consider a subset $S^0 \in \psi_p^0$. We show that if ψ_p^0 and $\hat{\psi}_p^0$ both satisfy (5.36) and $\mid \hat{\psi}_p^0 \mid = \mid \psi_p^0 \mid$, then there exists a subset $\hat{S}^0 \in \hat{\psi}_p^0$, identical to S^0.

According to the necessary condition of Theorem 5.12, if ψ_p^0 is maximal, all the elements of ψ_p^0 should be completely interdependent. On the other hand, if $|\hat{\psi}_p^0| = |\psi_p^0|$, then all the elements of $\hat{\psi}_p^0$ must be completely interdependent. Because, otherwise, if one element of $\hat{\psi}_p^0$ is not completely interdependent, then according to the sufficient condition of Theorem 5.12 this element can be partitioned into two subsets without violating (5.36), and hence,

$$| \hat{\psi}_p^0 |>| \psi_p^0 |,$$

which contradicts the assumption. Hence, all the elements in ψ_p^0 and $\hat{\psi}_p^0$ are completely interdependent.

Moreover, since ψ_p^0 and $\hat{\psi}_p^0$ must satisfy (5.36), according to Theorem 5.10, all the elements in ψ_p^0 and all the elements in $\hat{\psi}_p^0$ are independent of each other. Note that because of the unicity of sign, the independence of two sets S_1 and S_2 indicates that

$$\nexists\, a_i \in S_1 \;\nexists\, a_j \in S_2 \;\; such \;\; that \;\; \sim (a_i \, \mathbf{I} \, a_j).$$

Now, we show that for every $S_1 \in \psi_p^0$ there exists $S_2 \in \hat{\psi}_p^0$ identical to S_1. For this purpose, we show that all the elements of S_1 and S_2 are the same. Consider $a_1 \in S_1$, find a set in $\hat{\psi}_p^0$ that contains a_1. Call this set S_2. Because, S_1 is completely interdependent, there is another action a_2 in S_1 that is interdependent with a_1. Action a_2 should also be in S_2 because according to the above discussion, all the elements of $\hat{\psi}^0$ should be independent of each other. In the same way, one can show that every action that is included in S_1 should also be in set S_2. Hence, S_1 and S_2 are identical.

Similarly, it can be proven that for any set in ψ_p^0 there is an identical set in $\hat{\psi}_p^0$. Hence, ψ^0 and $\hat{\psi}^0$ are identical and the proof is complete. □

According to Theorem 5.13, the maximal partition of every nonempty set of actions is unique. For any real number x, let $\lfloor x \rfloor$ denote the greatest integer which is less than or equal to x. The following theorem establishes a lower bound for the overall degree of additivity of a set.

Theorem 5.14 *Let $\emptyset \neq \mathbf{S} \subseteq \mathbf{A}$ and ψ_p^0 be the maximal partition. Then,*

$$D_p(\mathbf{S}) \geq \left\lfloor \frac{|\mathbf{S}|}{|\psi_p^0|} \right\rfloor .$$

Proof: Let $\psi_p^0 = \{S_1, \ldots, S_j, \ldots, S_q\}$. Since $D_p(\mathbf{S})$ is the largest cardinality of all sets in partition ψ^0, then:

$$\sum_j | S_j | \leq D_p(\mathbf{S}) \cdot | \psi_p^0 |.$$

$$| \mathbf{S} | \leq D_p(\mathbf{S}) \cdot | \psi_p^0 |.$$

$$D_p(\mathbf{S}) \geq \left\lfloor \frac{|\mathbf{S}|}{|\psi_p^0|} \right\rfloor . \square$$

The following algorithm determines a maximal partition of a set of interdependent actions. In this algorithm $F_k, k = 1, \ldots, q$ denotes the set of actions interdependent with k. For instance, in the Example 5.5, $F_5 = \{a_4, a_6\}$.

Begin Procedure

Find the maximal partition of $\mathbf{S} = \{a_1, \ldots, a_i, \ldots, a_m\}$

  $j := 1, \psi_j := \emptyset, \mathbf{D} := \mathbf{S}$

  WHILE $i \leq | \mathbf{D} |$

    CONSTRUCT F_i for $a_i \in \mathbf{D}$

    LET $\psi_j = F_i$

      FOR $k = i + 1$ to $| \mathbf{D} |$

IF $\psi_j \cap F_k = \emptyset$ **then end**

ELSE

LET $\psi_j = \psi_j \cup F_k \cup a_i \cup a_k$

END FOR

IF $\psi_j = F_i$, **then** $\psi_j = a_i \cup F_i$

LET $j = j + 1, \mathbf{D} := \mathbf{D} - \psi_j$

END

The collection of ψ_j is the maximal partition of S.

End Procedure

5.5.2 Additivity of a Set and Graph Theory

In this section, we shed some light on the relationships between concepts introduced in this section and similar concepts in the theory of graphs. The concept of additivity of a set of actions is similar to the notion of a connected graph and stable set (independent set) in graph theory. For a graph $G(V, E)$, where V denotes the set of vertices and E the set of edges, a *connected* subgraph is a set of vertices such that there is a *path* between any pair of vertices, and a *stable set* of vertices is a subset W of V such that no two elements of W are connected.[1] A set W with maximum number of elements is called the *maximum stable set* and the cardinality of the maximum stable set is called the *order of stability* of graph G. A standard problem in graph theory and combinatorics is to find the maximal stable set of vertices for a given graph. In general, this problem is difficult to solve [16].

A set of actions with binary interdependence, *i.e.* with $O_p = 2$,[2] can be repre-

[1] *For a detailed description of graph theory concepts, refer to [16].*

[2] *Recall that when the order of dependence is two,* $\Delta_p(\mathrm{S}) = 0$ *for* $|\mathrm{S}| \geq 3$.

sented as a graph in which nodes represent the actions and existence of edges between nodes represents the binary interdependence relation between actions. Hence. for every set of interdependent actions there exists a corresponding graph. Finding the maximal partition of a set is equivalent to partitioning the nodes of a corresponding graph such that no edge joins two different elements of the partition. and every node in each element of that partition is connected to at least one other node. Moreover. every completely interdependent subset of a set corresponds to a connected subgraph of the corresponding graph and vice versa. For example Figure 5.2 shows the corresponding graph of set S in Example 5.5.

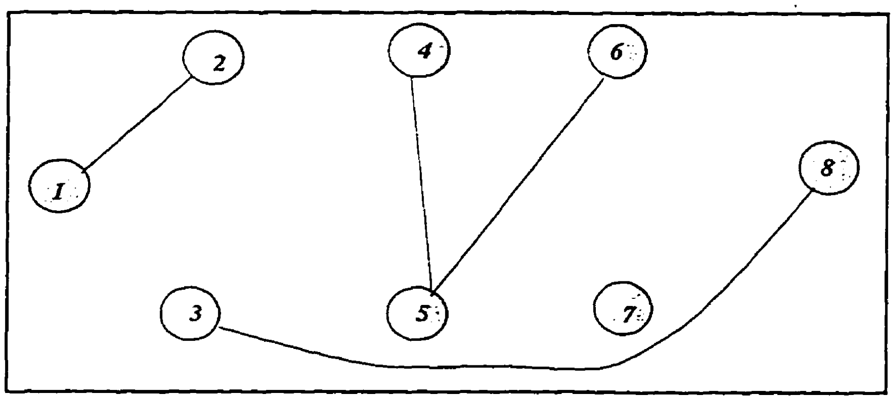


Figure 5.2: Graph Corresponding to Example 5.5

The concept of hyper-graph [3] can be employed to represent a set of actions with higher order of dependence. For instance. if in Example 5.5 actions in sets $\{a_2, a_3, a_5\}$ and actions in set $\{a_4, a_5, a_6\}$ are interdependent with order of dependence 3. then the corresponding graph is shown in 5.3.

The following theorem establishes a relation between order of separability of a set of actions and order of stability of its corresponding graph.

[3] *For a set* $S = \{a_1, \ldots, a_m\}$ *and a family* $E = \{E_1, \ldots, E_j, \ldots, E_e\}$ *of subsets of* S. $H = (S, E)$ *is a hyper-graph if* $E_j \neq \emptyset$ *for* $j = 1, \ldots, e$ *and* $\bigcup_j E_j = S$.

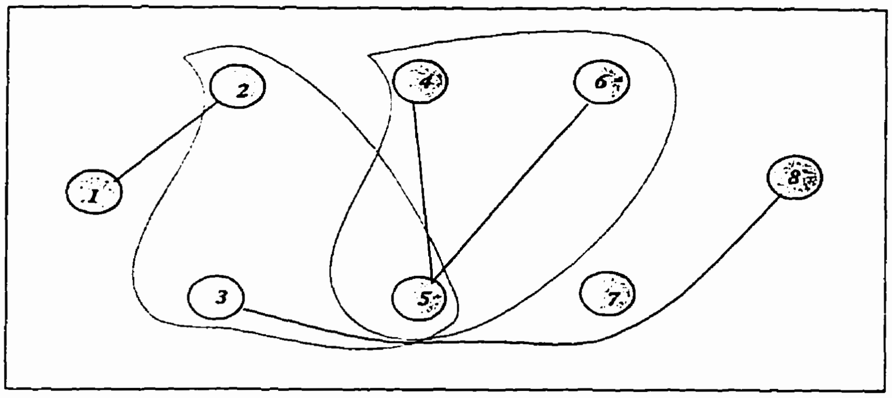


Figure 5.3: Hypergraph Corresponding to Example 5.5

Theorem 5.15 *Let* $\emptyset \neq \mathbf{S} \subseteq \mathbf{A}$, $O_p(\mathbf{S}) = 2$, *and suppose the interdependence relation on actions in* $\mathbf{S}$ *is transitive. Then the order of separability of* $\mathbf{S}$ *equals the order of stability of the corresponding graph of* $\mathbf{S}$.

Proof:
Let $\bar{D}_p(\mathbf{S})$ and $\alpha(\mathbf{S})$ denote the degree of separability of $\mathbf{S}$ and order of stability of its corresponding graph, respectively. Recall that according to the definition $\alpha(\mathbf{S})$ is the maximum number of nodes (actions) such that no two of them are connected (interdependent). On the other hand, $\bar{D}_p(\mathbf{S})$ is the cardinality of the maximal partition of $\mathbf{S}$.

Since, all the elements in the maximal partition, $\psi_p^0(\mathbf{S})$, must be completely interdependent, and since according to the assumption all interdependence relations in S are transitive, from each element of $\psi_p^0(\mathbf{S})$, only one node can be selected for inclusion in the corresponding $\alpha(\mathbf{S})$. On the other hand, since all the elements of $\psi_p^0(\mathbf{S})$ are independent, from every element of $\psi_p^0(\mathbf{S})$ one node can be selected in $\alpha(\mathbf{S})$. Therefore, because $\alpha(\mathbf{S})$ is the maximum stable set, for every element of $\psi_p^0(\mathbf{S})$ there is one and only one node in the corresponding stable set. Therefore, $|\bar{D}_p(\mathbf{S})| = \alpha(\mathbf{S})$, completing the proof. □

Note that in general when interdependence is not transitive, the above theorem does not necessarily hold. For instance, in Figure 5.2, $\alpha(S) = 5$, and $| \psi_p^0(S) | = 4$.

5.6 Summary and Conclusions

This chapter addressed general procedures to evaluate the consequence of an interdependent set of actions. Different approaches were introduced to measure the interdependence of two sets of actions according to the amount of interdependence of their proper subsets. The order of dependence of a set of actions was formally defined and a practical method was presented to determine the order of dependence of a set of interdependent action. Furthermore, several different approaches were proposed to determine whether actions, or sets of actions, are independent. Accordingly, various techniques were proposed to evaluate the sets of interdependent actions.

Throughout this chapter, useful relationships between independence of sets and independence of actions are developed. Using the concept of additivity of a set, an approach was introduced to evaluate the consequence of a set according to its independent partitions.

The theory of interdependence introduced in this chapter, and the preceding, can be used in subset selection problems for MADM, when there are small number of actions, as well as in MOMP, when there are a large number of actions. In the next chapter we propose a general approach in the MOMP framework to solve a multiple criteria subset selection problem under interdependence of actions.

Chapter 6

Formulation and Solution Methodologies

6.1 Introduction

In the previous chapters, several techniques were proposed to evaluate the consequence of a set of interdependent actions. This chapter is mainly concerned with the formulation of a subset selection problem under interdependence of actions and solution methods for this problem. Throughout this chapter, we assume that one can examine and estimate the interdependence of actions on each criterion, using the techniques presented in Chapter 5.

The organization of this chapter is as follows. Section 6.2 deals with the statement and formulation of the problem of selecting a subset of actions from a large discrete set of actions. Section 6.3 discusses existing approaches to solve this problem and explains the main advantages and disadvantages of each approach. Then, Section 6.4 discusses the problem of dominated solutions in Goal Programming

(GP) models. Subsequently, two solution methodologies are proposed in Sections 6.5 and 6.6 to overcome some of the shortcomings of current GP approaches. Finally, appropriate conclusions are drawn in Section 6.7.

6.2 Problem Formulation

Recall that **A** is the set of actions, **P** is the set of criteria, and c_p^i is the consequence of action a_i according to criterion p. Define the binary variable x_i by

$$x_i = \begin{cases} 1 & \text{if } a_i \text{ is selected;} \\ 0 & \text{if } a_i \text{ is not selected.} \end{cases}$$

Without loss of generality, assume that all criteria are to be maximized. A multiple criteria subset selection problem is expressed as follows:

$$\textbf{Maximize} \quad f_p(x) = \sum_{p=1}^{|\mathbf{P}|} c_p^i x_i, \qquad \forall\, p \in \mathbf{P}.$$

Subject to :

$$x \in \mathbf{X},$$
$$x_i \in \{0, 1\}, \qquad i = 1, \ldots, |\mathbf{A}|,$$

where **X** denotes the feasible decision set. For example, in the m-best action problem addressed in Chapter 3,

$$\mathbf{X} = \left\{ \left\{ (x_1, \ldots, x_{|\mathbf{A}|}) \right\} : x_i \in \{0, 1\}, \sum_i x_i = m \right\}.$$

Now, let $\mathbf{L}_p^k$ denote the set of subsets of actions with order of dependence k

according to criterion p.[1] For instance, in the WDL example in Chapter 4, we have

$$\mathbf{L}_1^k = \emptyset \ \ \forall k = 1, 2, 3;$$
$$\mathbf{L}_2^2 = \{\{a_4, a_5\}, \{a_1, a_2\}\}; \ \mathbf{L}_3^2 = \{\{a_4, a_5\}\}; \quad \text{and}$$
$$\mathbf{L}_p^k = \emptyset \ \forall p \in \mathbf{P} \quad \mathit{for} \quad k \geq 3.$$

According to our definition of interdependence, a general subset selection problem under interdependence of actions with resource constraints, when $O_p = K$, is formulated as a Multiple Criteria Zero-One (MCZO) problem as follows:

$$\begin{aligned}
\textbf{(Q1)} \quad \textbf{Maximize} \ f_p(x) = \sum_{i=1}^{|\mathbf{A}|} c_p^i x_i \quad & + \sum_{S \in \mathbf{L}_p^2} \Delta_p(S) \cdot \left(\prod_{a_i \in S} x_i \right) \\
& + \sum_{S \in \mathbf{L}_p^3} \Delta_p(S) \cdot \left(\prod_{a_i \in S} x_i \right) \\
& + \ \cdots \\
& + \sum_{S \in \mathbf{L}_p^K} \Delta_p(S) \cdot \left(\prod_{a_i \in S} x_i \right), \qquad \forall p \in \mathbf{P}, \\
\textbf{Subject to}: \quad & \\
& x \in \mathbf{X}, \\
& x_i \in \{0, 1\}, \quad i = 1, 2, \ldots, |\mathbf{A}|,
\end{aligned}$$

where c_p^i is the consequence of action i according to criterion p, and $\Delta_p(S)$, defined according to (5.3), denotes the amount of simple dependence within set S. Problem **Q1** is a general mathematical form in which if all actions in S are selected, then $\prod_{a_i \in S} x_i = 1$ and $\Delta_p(S)$ will be added to the overall consequence of criterion p,

[1] *For definition of order of dependence refer to Section 5.2.*

$f_p(x)$. Program **Q1** can be rewritten as,

$$(\mathbf{Q2}) \quad \textbf{Maximize} \quad f_p(x) = \sum_{i=1}^{|\mathbf{A}|} c_p^i x_i \ + \sum_{k=2}^{K} \sum_{S \in \mathbf{L}_p^k} \Delta_p(S) \cdot \left(\prod_{a_i \in S} x_i \right) ; \ \forall p \in \mathbf{P}$$

Subject to :

$$x \in \mathbf{X},$$

$$x_i \in \{0, 1\} \qquad i = 1, 2, \ldots, |\mathbf{A}|.$$

For example, when $O_p = 3$, the objective functions of (**Q2**) can be stated as

$$\textbf{Maximize } f_p(x) = \sum_{i=1}^{|\mathbf{A}|} c_p^i x_i + \sum_{\{a_i, a_j\} \in \mathbf{L}_p^2} \Delta_p(i, j) x_i x_j + \sum_{\{a_i, a_j, a_k\} \in \mathbf{L}_p^3} \Delta_p(i, j, k) x_i x_j x_k; \ \forall p \in \mathbf{P}. \tag{6.1}$$

where $\Delta_p(i, j)$ and $\Delta_p(i, j, k)$ is the amount of simple dependence within a_i, a_j, and within a_i, a_j, and a_k, respectively. Thus, a multiple criteria subset selection problem under interdependence of actions is formulated as an MCZO problem with some nonlinear terms. Nonlinearities in (**Q2**) may cause some difficulties. The next subsection deals with removing the polynomial terms in (**Q2**).

6.2.1 Removing Nonlinearity

Most theory and procedures of integer programming have been developed for the case of linear objectives and constraints. Hence, it is useful to find an equivalent linear problem for (**Q2**). Since all x_i in (**Q2**)are restricted to be zero or one, the nonlinear terms can be removed easily.

For each $S = \{a_{i_1}, \ldots, a_{i_j}, \ldots, a_{i_k}\} \in \mathbf{L}_p^k$ for any k and p, define

$$Y_S = x_{i_1} . x_{i_2} \cdots x_{i_j} \cdots x_{i_k}. \tag{6.2}$$

and add the two following constraints:

$$x_{i_1} + \cdots + x_{i_j} + \cdots + x_{i_k} - Y_S \leq k-1, \quad (6.3)$$

$$-x_{i_1} - \cdots - x_{i_j} - \cdots - x_{i_k} + kY_S \leq 0, \quad (6.4)$$

where $Y_S \in \{0,1\}$. These two constraints ensure that Y_S takes the value of one *iff* all actions in S are selected. Hence, to convert (**Q2**) to a linear zero-one problem, one new binary variable and two constraints must be added for each interdependence term.

Glover and Woolsey [38] proposed to substitute polynomial cross-product terms by a continuous variables rather than by integer variables. They show that these continuous variables automatically take zero or one values.

Similarly, we can change the variable Y_S to a continuous variable by replacing (6.4) with the following set of constraints[2]:

$$x_{i_j} \geq Y_S, \quad \forall\, a_{i_j} \in S, \quad (6.5)$$

where $Y_S \geq 0$. To demonstrate this idea, consider the following example:

Example 6.1 Suppose that a subset selection problem is given as follows:

$$\textbf{Maximize} \quad f_1(x) = \sum_{i=1}^{|\mathbf{A}|} c_1^i x_i + \Delta_1(1,2)x_1x_2 + \Delta_1(2,3,4)x_2x_3x_4,$$

$$\textbf{Maximize} \quad f_2(x) = \sum_{i=1}^{|\mathbf{A}|} c_2^i x_i + \Delta_2(2,3)x_2x_3 + \Delta_2(2,3,4)x_2x_3x_4,$$

Subject to :

[2] *Note that Glover and Woolsey's [38] procedure is independent of the number of objectives. Hence, their procedure can be applied to a multiple objective case.*

$$x \in \mathbf{X},$$
$$x_i \in \{0,1\}, \quad i = 1, 2, \ldots, |\mathbf{A}|.$$

This problem can be converted into a linear integer program as follows:

$$
\begin{aligned}
\textbf{Maximize} \quad f_1(x) &= \sum_{i=1}^{|\mathbf{A}|} c_1^i x_i + \Delta_1(1,2) \underbrace{Y_{1,2}}_{x_1 x_2} + \Delta_1(2,3,4) \underbrace{Y_{2,3,4}}_{x_2 x_3 x_4}. \\
\textbf{Maximize} \quad f_2(x) &= \sum_{i=1}^{|\mathbf{A}|} c_2^i x_i + \Delta_2(2,3) \underbrace{Y_{2,3}}_{x_2 x_3} + \Delta_2(2,3,4) \underbrace{Y_{2,3,4}}_{x_2 x_3 x_4}.
\end{aligned}
$$

Subject to :

$$
\begin{aligned}
& x_1 + x_2 - Y_{1,2} \le 1, \\
& x_2 + x_3 - Y_{2,3} \le 1, \\
& x_2 + x_3 + x_4 - Y_{2,3,4} \le 2, \\
& x_1 \ge Y_{1,2}, \\
& x_2 \ge Y_{1,2}, \\
& x_2 \ge Y_{2,3}, \\
& x_3 \ge Y_{2,3}, \\
& x_2 \ge Y_{2,3,4}, \\
& x_3 \ge Y_{2,3,4}, \\
& x_4 \ge Y_{2,3,4}, \\
& Y_{1,2}, Y_{2,3,4} \ge 0, \\
& x \in \mathbf{X}, \\
& x_i \in \{0,1\}, \quad \forall a_i \in \mathbf{A}.
\end{aligned}
$$

In general, a subset selection problem in MCDM under interdependence of actions can be formulated as a *linear* mixed integer program,

$$
\begin{aligned}
(\mathbf{Q3}) \quad \textbf{Maximize } f_p(x) &= \sum_{i=1}^{|\mathbf{A}|} c_p^i x_i + \sum_{k=2}^{K} \sum_{S \in \mathbf{L}_p^k} \Delta_p(S) \cdot Y_S; \ \forall p \in \mathbf{P} \\
\textbf{Subject to:} & \\
& \sum_{a_i \in S} x_i - Y_S \leq |S| - 1, \qquad \forall S \in \mathbf{L}_p^k, \forall p, k, \\
& x_i \geq Y_S, \qquad \forall x_i \in S, \ \forall S \in \mathbf{L}_p^k, \ \forall p, k, \\
& x \in \mathbf{X}, \\
& x_i \in \{0, 1\}, \quad \forall a_i \in \mathbf{A}, \\
& Y_S \geq 0, \quad \forall S \in \mathbf{L}_p^k, \forall p, k,
\end{aligned}
$$

where Y_S is defined according to (6.2). The following sections review some of the solution approaches available to solve (**Q3**) and explain their main advantages and shortcomings. Subsequently, two solution methodologies are proposed.

6.3 Solution Approaches

One could choose, for instance, one of the following three general approaches to solve Problem **Q3** (see also Chapter 2 for a review of solution techniques):

1. Assess the utility function of the DM to aggregate all objectives into one; then solve the single objective problem.

2. Solve a vector optimization problem to find the set of efficient solutions.

3. Use a GP approach.

Each of the above approaches has its own strengths and weaknesses. Assessing the DM's value function is quite difficult and may involve a great deal of subjectivity. this is especiaaly critical when the number of criteria is large. In vector optimization there are two main difficulties. First, the set of efficient solutions is usually large so, after using this method, the DM still faces a difficult problem of selecting the best solution. For example, Ruhe [107] shows that for a particular class of bi-criteria transshipment problem, there are 2^n supported efficient solutions, where n is the number of nodes. Second, due to the non-convexity of the decision space in **(Q3)**, the set of unsupported efficient solutions in a MCZO problem may be quite difficult to obtain (see Chapter 2). Even for class P of the combinatorial problems such as assignment problem in which the *unimodularity property* holds and the integer solutions can be found by solving linear programming problem, the Geoffrion's theorem cannot find all efficient solutions. In fact, most MCZO methods are applicable only to small problems [127]. Moreover, as pointed out in Chapter 3, when one wishes to select a subset of actions, the individually dominated actions should not be removed first, since there is a substantial possibility that, under some value functions, a subset including some dominated actions may be the best alternative (subset of actions). In the presence of interdependence, this kind of occurrence becomes even more likely, leading to a large number of decision variables in the MCZO problem.

GP is the most popular method in MOMP because of its combination of validity and acceptance by decision makers. It has been used widely in many different areas of application. White [134] surveyed multiple criteria optimization publications and found that 280 out of 400 papers involve variations on GP techniques. The main strengths of GP were described in Chapter 2.

Hence, GP can be considered as a suitable approach to solve **(Q3)**. However, conventional GP is not without difficulties. GP is often used to select the best alternative according to the aspiration levels and priority of objectives specified by the DM. However, in many situations the DM wants only to find some good solutions to choose among, possibly using qualitative criteria [86]. Moreover, the optimal solution of a GP problem may be dominated. The following section describes this in more detail.

6.4 Dominated Solutions in GP Problems

In spite of the popularity and the many recognized advantages of GP, this methodology has been criticized by researchers from several aspects. As explained in Chapter 2, one of the difficulties in GP is that the resulting optimal solution may be dominated. In what follows, we discuss the issue of non-dominated solutions in GP, in more detail. First, we present alternative GP formulations. Suppose that $\mathbf{P} = \{p_1, \ldots, p_j, \ldots, p_{|\mathbf{P}|}\}$ is the set of criteria. For sake of simplicity in notation, assume that the importance of the criteria decreases according to the order of their subscripts. Then the multiple objective problem **Q3** can be formulated as a lexicographic GP problem as follows:

$$(\mathbf{Q4}) \quad \textbf{Lex. Minimize} \quad \hat{\mathbf{d}} = (d_1^-, \ldots, d_p^-, \ldots, d_{|\mathbf{P}|}^-)$$

Subject to :

$$\sum_{i=1}^{|\mathbf{A}|} c_p^i x_i + \sum_{k=2}^{K} \sum_{S \in \mathbf{L}_p^k} \Delta_p(S) Y_S + d_p^- - d_p^+ = G_p; \qquad \forall p \in \mathbf{P},$$

$$\sum_{a_i \in S} x_i - Y_S \leq |S| - 1, \qquad \forall S \in \mathbf{L}_p^k, \forall p, k,$$

$$x_i \geq Y_S, \qquad \forall x_i \in S, \ \forall S \in \mathbf{L}_p^k, \ \forall p, k,$$

$$x \in X,$$
$$x_i \in \{0, 1\}, \qquad Y_S \geq 0,$$

where d_p^- and d_p^+ are negative and positive deviations of solutions from the goal on criterion p, G_p, and the rest of the notation is the same as in **Q3**. Similarly, the Archemedian GP model of (**Q3**) is given by,

$$(\mathbf{Q5}) \qquad \textbf{Minimize} \ \sum_{p=1}^{|\mathbf{P}|} w_p d_p^-,$$

subject to the same set of constraints as in (**Q4**). In this problem, w_p is the weight of negative deviation from G_p. Similarly, the associated Chebyshev GP formulation of (**Q3**) is,

$$(\mathbf{Q6}) \qquad \textbf{Minimize} \ \left(max\{w_1 d_1^-, \ldots, w_p d_p^-, \ldots, w_{|\mathbf{P}|} d_{|\mathbf{P}|}^-\}\right),$$

subject to the same set of constraints as in (**Q4**). Now, we define the concept of *GP-efficient solution* as follows:

Definition 6.1 *Let S be any solution to a GP problem formulated as* (**Q4**), (**Q5**), or (**Q6**), *and let $d_p^-(S)$ be its associated negative deviation from the goal on criterion p. Then S is a GP-efficient solution if there does not exist another feasible solution, S^0, such that*

$$d_p^-(S^0) \leq d_p^-(S), \qquad \forall p \in \mathbf{P},$$

with at least one of the inequalities strict.

Clearly, when a solution is GP-efficient, decreasing its deviation from goal on one criterion leads to an increase in the deviation from goal on at least one other criterion. The following theorem shows that the optimal solution of a GP problem is GP-efficient.

Theorem 6.1 *Let S^* be an optimal solution to GP problem* **Q4** *or* **Q5**. *Then S^* is a GP-efficient solution. Moreover, let* **H** *be the set of optimal solutions to* **(Q6)**. *Then there exists a solution $S^* \in \mathbf{H}$ such that S^* is GP-efficient.*

Proof:

We first prove that if S^* is an optimal solution of **(Q4)**, then S^* is GP-efficient. Let S^* be an optimal solution of **(Q4)** and let S' be any feasible solution to the vector maximization Problem **Q3**. Suppose that the negative deviations of solutions S^* and S' from the aspiration levels are $(d_1^*, \ldots, d_p^*, \ldots, d_{|p|}^*)$ and $(d_1', \ldots, d_p', \ldots, d_{|p|}')$, respectively. For simplicity in notation, assume that the criteria are listed in decreasing order of importance in the lexicographic GP problem.

In lexicographic GP, first the deviation from the most important criterion is minimized; if there are multiple optimal solutions, then in the second stage the deviation on the second most important criterion is minimized (without increasing the first deviation), and so on. Hence, d_1^* is the minimum deviation from the most important goal which is obtained by solving the corresponding single objective problem. Therefore, $d_1^* \leq d_1'$. If $d_1^* < d_1'$, then S' cannot dominate S^*. Thus, S^* is GP-efficient. If $d_1^* = d_1'$, we move to the second level of priority. Since d_2^* is the minimum deviation from the goal without worsening the first priority, thus $d_2^* \leq d_2'$. Similar to the first step, if $d_2^* < d_2'$, the proof is complete. Otherwise, we move to the third priority. The rest of proof is similar to the first and second steps.

Note that if there are no optimal solutions at one stage, p, then the values of the remaining deviations, $d_{p+1}, d_{p+2}, \ldots, d_{|p|}$ can be fixed and the proof for this case is complete. Hence, an optimal solution to **(Q4)** is GP-efficient.[3]

Now suppose that S^* is an optimal solution of Problem **Q5** and let its negative deviations from the aspiration levels be $(d_1^*, \ldots, d_p^*, \ldots, d_{|p|}^*)$. Suppose S^* is not GP-efficient. Since the set of feasible solutions is finite, there exists a solution for **(Q3)** , S', with negative deviation $(d_1', \ldots, d_p', \ldots, d_{|p|}')$ such that,

$$d_p' \leq d_p^*, \quad \forall p \in \mathbf{P}. \tag{6.6}$$

with at least one inequality being strict. Since in **(Q5)**, $w_p > 0$, (6.6) implies that

$$\sum_{p=1}^{|\mathbf{P}|} w_p d_p^* > \sum_{p=1}^{|\mathbf{P}|} w_p d_p'.$$

contradicting the assumption that S^* is optimal. Hence, the optimal solution of **(Q5)** is GP-efficient.[4]

Finally suppose that $\mathbf{H}$ is the set of optimal solutions of **(Q6)**. We show that there exists a solution in $\mathbf{H}$, called S^*, which is GP-efficient. Let h^* be the value of the optimal solution of **(Q6)**, *i.e.*,

$$h^* = min\left(max\{w_1 d_1, \ldots, w_p d_p, \ldots, w_{|\mathbf{P}|} d_{|\mathbf{P}|}\}\right). \tag{6.7}$$

Suppose that there does not exist any optimal solution in $\mathbf{H}$, which is GP-efficient. Let $\hat{S} \in \mathbf{H}$ be an optimal solution of **(Q6)** that is not dominated by another solution in $\mathbf{H}$.

Since the feasible space is finite, the set of GP-efficient solution is not empty. Hence, if $\hat{S}$ is not GP-efficient, then there exists a GP-efficient solution, $\bar{S} \notin \mathbf{H}$

[3] *This part of proof is similar to the proof of theorem 1 in [58].*

[4] *This part of proof is similar to the proof of generating efficient solutions in weighted approach.*

whose negative deviations are $\{\bar{d}_1, \ldots, \bar{d}_p, \ldots, \bar{d}_{|p|}\}$ such that

$$\bar{d}_p \leq \hat{d}_p \quad \forall p \in \mathbf{P}. \tag{6.8}$$

with at least one inequality strict. But, because h^* is optimal value of **(Q6)**, (6.8) implies that $\bar{S}$ is an optimal solution of **(Q6)** (*i.e.* $\bar{S} \in \mathbf{H}$), contradicting the assumption. Hence, there exists an optimal solution of **(Q6)** that is GP-efficient.[5] □

Therefore, the optimal solutions of (**Q4**) and (**Q5**) are GP-efficient and at least one of the optimal solutions of (**Q6**) is GP-efficient. However, a GP-efficient solution is not necessarily an efficient solution of the original multiple objective Problem **Q3**. Nevertheless, it is useful to examine the conditions for which a GP-efficient solution is an efficient solution of the original problem.

Let $f_p^*(x)$ denote the optimal solution of multiple objective problem (**Q3**) on criterion p and let S^* be the optimal solution of corresponding GP problem with d_p^{-*} and d_p^{+*} as negative and positive deviations on criterion p. Clearly, when all negative deviations of the GP-efficient solution are nonzero (*i.e.* when $d_p^{*-} > 0 \;\; \forall p \in \mathbf{P}$), then the GP-efficient solution is also an efficient solution of (**Q3**), because, in this case, the specified aspiration levels are not attainable and $f_p^*(x) < G_p$. Therefore,

$$f_p^*(x) = G_p - d_p^{-*}, \qquad \forall p \in \mathbf{P}, \tag{6.9}$$

which indicates that the optimal solution of the GP problem, S^*, is an efficient solution of (**Q3**). Otherwise, if $d_p^{-*} = 0$ for some p, then $f_p^*(x) \geq G_p - d_p^{-*} + d_p^{+*}$ or $f_p^*(x) \geq G_p + d_p^{+*}$, implying that S^* may be dominated.

[5] *This part of proof is similar to the proof of Theorem 14.15 in [118].*

Hence, a GP-efficient solution of **(Q4)**, **(Q5)** or **(Q6)** is an efficient solution of the original problem, **(Q3)**, if the optimal solution of the following problem is zero.

(Q7) **Maximize** $q = \sum_{p=1}^{|\mathbf{P}|} \delta_p$

Subject to :

$$\sum_{i=1}^{|\mathbf{A}|} c_p^i x_i + \sum_{k=2}^{K} \sum_{S \in \mathbf{L}_p^k} \Delta_p(S) \cdot \left(\prod_{a_i \in S} x_i \right) - \delta_p = G_p - d_p^{-*} + d_p^{+*}, \quad \forall\, p \in \mathbf{P}.$$

$$\delta_p \geq 0.$$

with the rest of constraints as in **(Q3)**. If the optimal solution of the above program is positive, then there exists one criterion p such that

$$\sum_{i=1}^{|\mathbf{A}|} c_p^i x_i + \sum_{k=2}^{K} \sum_{S \in \mathbf{L}_p^k} \Delta_p(S) \cdot \left(\prod_{a_i \in S} x_i \right) > G_p - d_p^{-*} + d_p^{+*},$$

indicating that objective function p can make its current solution on criterion p better without worsening any other criteria. On the other hand, if the optimal solution of **(Q7)** is zero, then the GP solution is non-dominated because, $f_p^*(x) = G_p - d_p^* \;\; \forall\, p \in \mathbf{P}$.

In the next two sections, two variations of zero-one GP method suitable for problems with interdependence of actions are proposed to overcome the aforementioned difficulties of GP. The next section presents a method combining lexicographic GP and vector optimization that generates a subset of non-dominated alternatives (subsets of actions). Then, Section 6.6 presents a modified GP method based on a combination of Archemedian and Chebyshev GP techniques.

6.5 A Modified Lexicographic GP Method

As described in Chapter 2 one type of GP is lexicographic (preemptive) GP in which the criteria are in priority order. In lexicographic GP, satisfying the first priority goal is considered to be much more important than satisfying the second one, and so on. This section proposes a modified GP method that generates a subset of efficient solutions. In the proposed method, first a lexicographic GP is solved, and then its solutions are used in a vector optimization problem to find a subset of efficient solutions.

In general, the solution of a lexicographic GP Problem may require as many as $|\mathbf{P}|$ stages, in which one goal is tried at each stage. Hence, the most important criterion is minimized first; if it has multiple optimal solutions, the second step is started in which d_2 will be minimized without worsening achievement on the first goal, and so forth. The sequential solution is complete as soon as the optimal solution of any stage is unique. Therefore, in lexicographic GP it is possible that some lower priority goals will never be taken into account.

One of the advantages of the lexicographic zero-one approach is that the problem can be solved using a sequence of zero-one GP problems. This allows one to use any zero-one programming routine so that models of the same size as single objective zero-one problems can be solved (see [57] and [59] for details).

Solving (**Q4**) often leads to a unique solution that depends on the aspiration levels and goal priorities. This solution may be dominated [140]. As explained earlier, when a DM faces a complex problem such as subset selection under interdependence of actions, she or he may be willing to consider more than one attractive solution in order to reexamine and select among them. This way other criteria that are difficult to state as mathematical functions can be included, enabling the

DM to select the best alternative according to both qualitative and quantitative measures [92]. On the other hand, presenting all efficient alternatives to the DM through vector optimization may not be useful. In fact, due to the huge number of non-dominated solutions in practical problems, only those techniques that generate a small and representative portion of non-dominated solutions can be considered successful [59, 144].

A lexicographic GP technique can be modified to obtain some efficient solutions that in some sense represent the set of all efficient solutions. We present the proposed method for two cases: 1) when the optimal solution of **(Q7)**, q^*, is zero, and 2) when $q^* > 0$.

6.5.1 Case 1: $q^* = 0$

As described in the previous section, when the optimal solution of **(Q7)**, is zero, the GP-efficient solution is non-dominated. Thus, if $q^* = 0$, we need only to find some other representative efficient solutions.

To obtain other efficient solutions, when $q^* = 0$, the DM is asked first to change the levels of goals on all or some of the criteria. Suppose the DM decomposes the set of criteria into two disjoint subsets $\mathbf{P}_1$, and $\mathbf{P}_2$. The subset $\mathbf{P}_1$ is the set of criteria that can be degraded or acceptable as they are, subject to improvement on one or more criteria in $\mathbf{P}_2$. [6] Moreover, suppose that $\underline{g_p}$, $0 \leq \underline{g_p} \leq 1$, is the level of degradation on criterion $p \in \mathbf{P}_1$, and let $\overline{g_p}$, $0 \leq \overline{g_p} \leq 1$, be the level of improvement on criterion $p \in \mathbf{P}_2$ which have been specified by the DM. Consider Problem $\mathbf{Q'4}$

[6] *Note that since the solution is already efficient, it is impossible to improve all the criteria.*

below.

$$(\mathbf{Q'4}) \quad \textbf{Lex. Minimize} \quad \hat{\mathrm{d}} = (d_1^-, \ldots, d_p^-, \ldots, d_{|\mathbf{P}|}^-)$$

Subject to :

$$\sum_{i=1}^{|\mathbf{A}|} c_p^i x_i + \sum_{k=2}^{K} \sum_{S \in \mathbf{L}_p^k} \Delta_p(S) \cdot Y_S + d_p^- - d_p^+ = G_p(1 - \underline{g_p}); \qquad \forall p \in \mathbf{P}_1$$

$$\sum_{i=1}^{|\mathbf{A}|} c_p^i x_i + \sum_{k=2}^{K} \sum_{S \in \mathbf{L}_p^k} \Delta_p(S) \cdot Y_S + d_p^- - d_p^+ = G_p(1 + \overline{g_p}); \qquad \forall p \in \mathbf{P}_2$$

with the rest of constraints similar to **(Q3)**. Recall that in the above program. G_p is the goal on the pth criterion for the initial lexicographic problem. **Q4**. As discussed in the previous section, solving $(\mathbf{Q'}4)$ gives an efficient solution when

$$G_p(1 - \underline{g_p}) > f_p^*(x) \qquad \forall p \in \mathbf{P}_1, \text{ and}$$

$$G_p(1 + \overline{g_p}) > f_p^*(x) \qquad \forall p \in \mathbf{P}_2.$$

But the optimal solution of **(Q4)** is efficient. Thus. $G_p > f_p^*(x)$ $\forall p \in \mathbf{P}$, which implies that $G_p(1 + \overline{g_p}) > f_p^*(x)$ $\forall p \in \mathbf{P}_2$. Therefore. to ensure that the optimal solution of $(\mathbf{Q'}4)$ is efficient. it is required that $G_p \underline{g_p} < G_p - f_p^*(x)$, or $\underline{g_p} < \frac{d_p^{-*} + d_p^{+**}}{G_p}$ $\forall p \in \mathbf{P}_1$, where d_p^{-*} is the negative deviation of optimal solution of Problem **Q4** on criterion p. Hence, in the second step, the DM sets the changes on the goals in order to find another efficient solutions.

Since the efficient solutions obtained from solving **(Q4)** and $(\mathbf{Q'}4)$ are based on the order of importance of criteria and the aspirations levels, one of these solutions is either optimal or near-optimal based on the DM's value function. Therefore, other efficient solutions close to these two efficient solutions can be considered as an attractive subset of efficient solutions, which most likely includes the optimal solution.

In what follows we show a simple procedure for finding some attractive efficient solutions close to the optimal solutions of **(Q4)** and **(Q$'$4)**, according to a given distance metric. These are called *adjacent* efficient solutions.

According to Geoffrion's Theorem [35], every optimal solution to the following program for different value of λ, $\lambda_p \in \Lambda = \{\lambda \in \mathbf{R}^{|\mathbf{P}|} | \lambda_p > 0, \sum_{p=1}^{|\mathbf{P}|} \lambda_p = 1\}$ is efficient:

$$\textbf{Maximize} \sum_{p \in \mathbf{P}} \lambda_p f_p(x), \tag{6.10}$$

subject to the same set of constraints as in **(Q3)**. Let $\mathbf{z}^1 = \{z_1^1, \ldots, z_p^1, \ldots, z_{|\mathbf{P}|}^1\}$ and $\mathbf{z}^2 = \{z_1^2, \ldots, z_p^2, \ldots, z_{|\mathbf{P}|}^2\}$ be the criterion vector of optimal solutions of **(Q4)** and **(Q$'$4)**, respectively. Among all efficient solutions of program **Q3**, one would like to find some adjacent efficient solutions to $\mathbf{z}^1$. For this purpose, one could use the following program:

$$(\mathbf{Q}''4) \qquad \textbf{Maximize} \quad \sum_{p \in \mathbf{P}} \lambda_p f_p(x),$$

$$\textbf{Subject to:}$$

$$\left[\sum_{p=1}^{|\mathbf{P}|} \left(\pi_p \left| z_p^1 - f_p(x) \right| \right)^{\alpha} \right]^{\frac{1}{\alpha}} \leq \beta, \tag{6.11}$$

with the rest of constraints as in **(Q3)**. Constraint 6.11 ensures that every efficient solution obtained is in the neighborhood of the first generated efficient solution, $\mathbf{z}^1$, according to an L_α distance metric. In this constraint, π_p is a factor for equalization of ranges on different criteria and is calculated by:

$$\pi_p = \frac{1}{R_p} \left[\sum_{p=1}^{|\mathbf{P}|} \frac{1}{R_p} \right]^{-1}.$$

where $R_p = \left| z_p^1 - z_p^2 \right|$. There is no specific rule to determine the amount of distance parameter, β. One suggestion is the following:

$$\beta = \frac{\left[\sum_{p=1}^{|\mathbf{P}|} (\pi_p R_p)^\alpha\right]^{\frac{1}{\alpha}}}{h} \quad \alpha \in \{1, 2, \ldots,\} \cup \{\infty\}, \tag{6.12}$$

where R_p and π_p are defined as above, and h is an arbitrary number that indicates the largeness of the neighborhood [7]. Increasing the value of h decreases the number of adjacent efficient solutions to be generated, and decreasing it enlarges the neighborhood definitions and allows more solutions to be generated.

One may add the following two sets of constraints to Problem $\mathbf{Q}''\mathbf{4}$ to shrink the feasible space and thereby facilitate solving the problem. These constraints specify that the values of the new efficient solutions on each criterion, p, are bounded by z_p^1, and z_p^2.

$$f_p(x) \geq \min\{z_p^1, z_p^2\} \qquad \forall p \in \mathbf{P},$$
$$f_p(x) \leq \max\{z_p^1, z_p^2\} \qquad \forall p \in \mathbf{P}.$$

Setting $\alpha = 1$ (*i.e.* for the L_1 norm), the constraint (1) in ($\mathbf{Q}''4$) becomes

$$\sum_{p \in \mathbf{P}} \pi_p \left| z_p^1 - f_p(x) \right| \leq \beta. \tag{6.13}$$

Now define $u_p - u_p' = z_p^1 - f_p(x) \quad \forall p \in \mathbf{P}$. Then (6.13) can be replaced with the following set of constraints:

$$\sum_{p \in \mathbf{P}} \pi_p \left(u_p - u_p' \right) \leq \beta,$$

[7] *Steuer [118], in his forward filtering approach, suggests a similar distance parameter for initialization of his screening approach. He sets $h = 4$ as a good starting point.*

$$z_p^1 - f_p(x) - u_p + u_p^{'} = 0. \qquad \forall p \in \mathbf{P}.$$

Hence, for the L_1 norm, $(\mathbf{Q}''4)$ becomes

$$\begin{aligned} &\textbf{Maximize} \quad \sum_{p\in\mathbf{P}} \lambda_p f_p(x), \\ &\textbf{Subject to}: \\ &\qquad \sum_{p\in\mathbf{P}} \pi_p \left(u_p - u_p^{'}\right) \le \beta, \\ &\qquad z_p^1 - f_p(x) - u_p + u_p^{'} = 0. \qquad \forall p \in \mathbf{P}. \end{aligned}$$

where $\beta = \frac{1}{h}\sum_{p=1}^{|\mathbf{P}|} \pi_p R_p$, $\lambda \in \Lambda$, and the rest of constraints are as in $(\mathbf{Q3})$. Similarly, for L_∞, constraint 6.11 in $(\mathbf{Q}''4)$ changes to the following:

$$\begin{aligned} &\pi_p \left(u_p - u_p^{'}\right) \le \beta, \qquad \forall p \in \mathbf{P} \\ &z_p^1 - f_p(x) - u_p + u_p^{'} = 0. \qquad \forall p \in \mathbf{P}. \end{aligned}$$

where $\beta = \frac{1}{h}\left[\sum_{p=1}^{|\mathbf{P}|}(\pi_p R_p)^\infty\right]^{\frac{1}{\infty}} = \frac{\max_p R_p}{h}$. Hence, using either the L_1 or L_∞ distance metric makes $(\mathbf{Q}''\mathbf{4})$ a linear integer problem. Other L_α metrics correspond to nonlinear, and hence more difficult, problems, but an L_1, or L_∞ approximation, or a combination of these two, may find a subset of efficient solutions close to $\mathbf{z}_p^1$ for any L_α, when the number of criteria is small. Note that as the number of criteria increases, depending on the shape of the non-dominated solutions z_p may be very different from the convex combination of z_1 and z_∞.

6.5.2 Case 2: $q^* > 0$

Now, suppose that in **(Q7)**, $q^* > 0$. Hence, the optimal solution of **(Q4)** is not efficient. In this case, one way to find a subset of efficient solutions, is to use Hannan's formulation [46]. In the context of linear GP, he proves that for a dominated solution, either an alternative optimal solution that is not dominated can be found, or at least one of the objective functions is unbounded. If all goals are bounded, then the following vector maximization program provides a set of non-dominated solutions:

$$
\begin{aligned}
\textbf{(Q8)} \qquad & \textbf{Maximize} \quad f_p(x), \qquad\qquad p = 1, \ldots, |\, \mathbf{P} \,|, \\
& \textbf{Subject to:} \\
& \sum_{i=1}^{|\mathbf{A}|} c_p^i x_i + \sum_{k=2}^{K} \sum_{S \in \mathbf{L}_p^k} \Delta_p(S) \cdot Y_S \geq G_p - d_p^{-*} + d_p^{+*}, \qquad \forall p \in \mathbf{P},
\end{aligned}
\tag{6.14}
$$

subject to the same set of constraints as in **(Q3)**. Note that because of constraints (6.14), Problem **Q8** is easier to solve than the original vector optimization **(Q3)**. In this formulation, d_p^{-*} and d_p^{+*} are the optimal solutions to **(Q4)** for specified aspiration levels, and Y_S is the binary variable substituted for $(\prod_{a_i \in S} x_i)$. Note that if the DM aspires to a difficult-to-achieve target, there may be no new solution for the above problem: the only optimal solution to **(Q8)** would be the efficient solution of **(Q4)**.

Problem **Q8** is a vector optimization problem in which the objectives are the original objective functions in **(Q3)** plus additional constraints to ensure that the objective functions are not less than $G_p - d_p^{-*} + d_p^{+*}$. It is noteworthy that in linear GP, it is not necessary to include d_p^{+*} for the constraints in **(Q8)**, because for an

Table 6.1: The Consequence of Five Actions in WDL example

Criteria	*Actions*				
	a_1	a_2	a_3	a_4	a_5
p_1 ↑	0.45	0.45	1	0.55	0.84
p_2 ↑	0.8	0.7	0.75	0.83	0.83
p_3 ↓	0.6	0.87	0.5	0.75	0.6

optimal solution of a linear GP problem, $d_p^{+*} = 0$, for any maximization criterion. The following simple example illustrates the above discussion:

Example 6.2 Consider Table 6.1 which shows the normalized consequences of five actions according to three criteria. Arrows show the direction of preference for each criterion. Suppose that a pair of actions are to be selected. The interdependent actions and their corresponding dependence values are,

$$\mathbf{L}_1^k = \emptyset,\ \forall k,\ \mathbf{L}_2^2 = \{(a_1, a_2), (a_4, a_5)\}, \text{and } \mathbf{L}_3^2 = \{(a_4, a_5)\},$$
$$\Delta_2(1,2) = 0.3, \quad \Delta_2(4,5) = -0.25, \quad \Delta_3(4,5) = -0.2.$$

Assume that (1.0, 1.5, 1.3) are the aspiration levels for the first, second and third criteria, respectively with the priority of (d_2^-, d_3^+, d_1^-). The lexicographic GP for this problem is

$$\textbf{Lex Minimize} \quad \dot{\mathrm{d}} = (d_2^-, d_3^+, d_1^-)$$

Subject to :

$$.45x_1 + .45x_2 + x_3 + .55x_4 + .84x_5 + d_1^- - d_1^+ = 1.0,$$
$$.8x_1 + .7x_2 + .75x_3 + .83x_4 + .83x_5 + .3Y_{1,2} - .25Y_{4,5} + d_2^- - d_2^+ = 1.5$$
$$.6x_1 + .87x_2 + .5x_3 + .75x_4 + .6x_5 - 0.2Y_{4,5} + d_3^- - d_3^+ = 1.3,$$

$$\left.\begin{array}{l}\sum_{i=1}^{5} x_i = 2, \\ x_1 + x_2 - Y_{1,2} \leq 1, \\ x_4 + x_5 - Y_{4,5} \leq 1, \\ x_1 \geq Y_{1,2},\ x_2 \geq Y_{1,2},\ x_4 \geq Y_{4,5},\ x_5 \geq Y_{4,5}, \\ x_i \in \{0,1\},\ Y_{1,2} \geq 0,\ Y_{4,5} \geq 0. \end{array}\right\} \quad (6.15)$$

The optimal solution of this program using a sequential zero-one approach is (x_3, x_4) with (1.55, 1.58, 1.25) for the first, second, and third criteria, respectively. This solution is dominated. Now, we construct the associated vector maximization problem:

$$\textbf{Maximize} \quad (f_1(x), f_2(x), f_3(x))$$

Subject to :

$$.45x_1 + .45x_2 + x_3 + .55x_4 + .84x_5 \geq 1.0 - 0 + .55,$$

$$.8x_1 + .7x_2 + .75x_3 + .83x_4 + .83x_5 + 0.3Y_{1,2} - 0.25Y_{4,5} \geq 1.5 - 0 + .08,$$

$$.6x_1 + .87x_2 + .5x_3 + .75x_4 + .6x_5 - 0.2Y_{4,5} \leq 1.3 - .05 + 0,$$

the set of constraints (6.15).

Solving this problem using the convex combination of the criteria for different values of $\lambda \in \Lambda$ (see Geoffrion's Theorem in Chapter 2) gives several GP-efficient solutions as follows:

$$(x_1, x_5),\ (x_3, x_5),\ (x_1, x_2).$$

Note that (x_3, x_5) dominates (x_3, x_4). Now suppose that the DM specifies (2,2,1) as the goals. In this case the solution of the GP problem is $z^1 = (x_1, x_2)$ with deviations of (1.1, .2, .47) from goals. This solution is efficient. Hence, we seek other efficient solutions using the DM's preference information.

Suppose that the DM specifies (2.5,1.5,1.3) as new goals. Note that the goal for the first criterion has been improved while the second and third ones have been degraded. Solving the lexicographic GP for these goals give $z^2 = (x_3, x_5)$ as another efficient solution. Now we search for other efficient solutions in the neighborhood of z^1.

Let the L_1 metric be chosen as distance function. Hence,

$$R_1 = 0.94,\ R_2 = 0.28,\ R_3 = 0.48,\ \pi_1 = .158,\ \pi_2 = .576,\ \pi_3 = .31,\ \text{and}\ \beta = \frac{.519}{h}.$$

If $h = 1$ is chosen for the distance parameter, the following program finds some other efficient solutions close to z^1:

$$\begin{aligned}\textbf{Maximize}\quad f(x) = {} & \lambda_1\,(.45x_1 + .45x_2 + x_3 + .55x_4 + .84x_5) + \\ & \lambda_2\,(.8x_1 + .7x_2 + .75x_3 + .83x_4 + .83x_5 + 0.3Y_{1,2} - 0.25Y_{4,5}) - \\ & \lambda_3\,(.6x_1 + .87x_2 + .5x_3 + .75x_4 + .6x_5 - 0.2Y_{4,5}),\end{aligned}$$

Subject to :

$$\left.\begin{aligned}&.45x_1 + .45x_2 + x_3 + .55x_4 + .84x_5 \geq .9,\\ &.45x_1 + .45x_2 + x_3 + .55x_4 + .84x_5 \leq 1.84,\\ &.8x_1 + .7x_2 + .75x_3 + .83x_4 + .83x_5 + 0.3Y_{1,2} - 0.25Y_{4,5} \geq 1.58,\\ &.8x_1 + .7x_2 + .75x_3 + .83x_4 + .83x_5 + 0.3Y_{1,2} - 0.25Y_{4,5} \leq 1.8,\\ &.6x_1 + .87x_2 + .5x_3 + .75x_4 + .6x_5 - 0.2Y_{4,5} \geq .99,\\ &.6x_1 + .87x_2 + .5x_3 + .75x_4 + .6x_5 - 0.2Y_{4,5} \leq 1.47,\end{aligned}\right\} \quad (6.16)$$

$$.184\left(u_1 - u_1'\right) \leq .516,$$

$$.456\left(u_2 - u_2'\right) \leq .516,$$

$$1.36\left(u_3 - u_3'\right) \leq .516,$$

$$.45x_1 + .45x_2 + x_3 + .55x_4 + .84x_5 - .9 + u_1 - u_1' = 0,$$

$.8x_1 + .7x_2 + .75x_3 + .83x_4 + .83x_5 + 0.3Y_{1,2} - 0.25Y_{4,5} - 1.8 + u_2 - u_2' = 0,$

$.6x_1 + .87x_2 + .5x_3 + .75x_4 + .6x_5 - 0.2Y_{4,5} - 1.47 + u_3 - u_3' = 0,$

the set of constraints (6.15).

Solving the above problem by selecting $\lambda_1 = 0.1, \lambda_2 = 0.8, \lambda_3 = 0.1$ gives (x_1, x_5) as an another efficient solution in the neighborhood of z^1. The optional set of constraints (6.16) ensures that the criterion values of the solutions of the above problem on each criterion, p, are bounded by z_p^1 and z_p^2. Removing this set of constraints may lead to producing more non-dominated solutions.

Note that if more than one DM is involved in the process of decision making, each DM can assign his or her own aspiration levels and the above model can be used to identify desirable alternatives for each DM. In this way, a compromise solution involving all DMs may be obtained. The next section discusses another GP approach to solve Problem **Q3**.

6.6 Combined Chebyshev-Archemedian Goal Programming

As explained in Chapter 2, Chebyshev and Archemedian GP are two widely used techniques to solve multiple criteria problems. Despite the many advantages, and many successful applications, they have been criticized by some researchers, often after thorough experiments [120]. In this section, we propose a new GP technique that removes some of the shortcomings discussed in the previous chapter, while maintaining the original GP structure.

Some difficulties that have been mentioned in the literature in regard to Archemedian GP are as follows:

- Archemedian GP may generate solutions which are far from some criterion goals. In other words, even though the weighted sum of the deviations is minimized, the solution may be far from the goal on some criterion. For example, consider Figure 6.1 in which **Z** is the feasible criterion space, and **G** is the goal specified by the DM. Both criteria are to be maximized. The Archemedian GP solution is located at point (a). This solution has zero deviation from the goal on first criterion and a relatively large deviation on the second criterion.

- In most practical MCDM problems, large deviations from a specified goal are more likely to be of importance than small deviations. But Archemedian GP does not take this issue into account, because, the unit cost for deviations of any distance is constant. Stewart [120] suggests that using the L_2 norm, instead of Archemedian GP, may alleviate this difficulty. However, the problem is more difficult to solve using the L_2 norm. Moreover, there is no special justification for using this norm to reflect the DM's behavior, rather than other distance functions.

In Chebyshev GP the most critical criterion always receives the most attention, the solution is not so sensitive to the choice of weights, and aggregation of deviations is avoided. However, Chebyshev GP may produce a solution with a high weighted sum of deviations. It has also been shown that Chebyshev GP solutions may reject some reasonable solutions in favor of others that are more balanced [59]. In Figure 6.1 the point (b) is the solution of the Chebyshev GP problem.

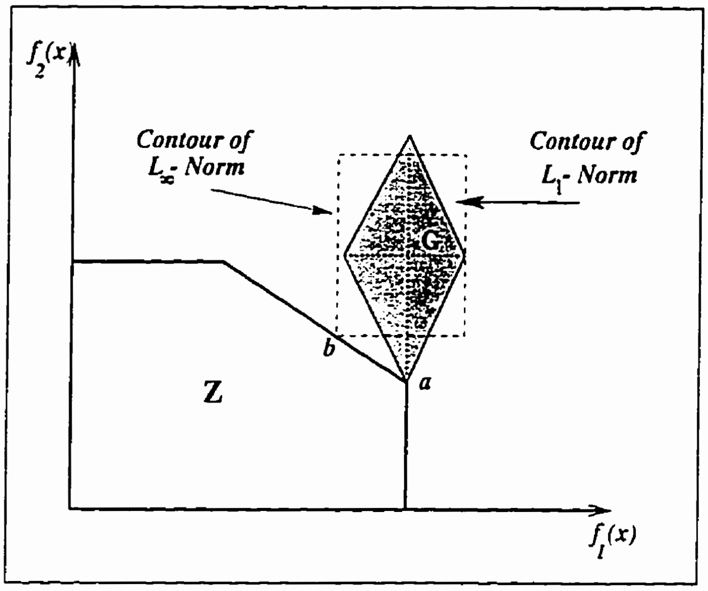


Figure 6.1: The Chebyshev and Archemedian GP Solutions

Both Chebyshev and Archemedian GP are designed to obtain one solution. However, in some cases DMs prefer more than one good solution. In what follows we propose a new GP approach which may overcome the above-mentioned shortcomings of Chebyshev and Archemedian GP techniques.

Consider Problems **Q5** and **Q6** presented in the previous section. We construct the following two-objective problem:

$$(\mathbf{Q9}) \qquad \textbf{Minimize} \quad \sum_{p=1}^{|P|} w_p d_p^-,$$

$$\textbf{Minimize} \quad \left(max\{w_1 d_1^-, \ldots, w_p d_p^-, \ldots, w_{|P|} d_{|P|}^-\}\right),$$

subject to the same set of constraints as in (**Q5**) or (**Q6**). We call (**Q9**) a *multiple objective GP* problem in which both the weighted sum of deviations and the maximum deviations from goals are to be minimized simultaneously. Now we define the concept of *combined GP-efficient* solution as follows:

Definition 6.2 *Let S be a solution to* (**Q9**) *and $d_p^-(S)$ be its corresponding deviation from goal on criterion p. Without loss of generality, assume that all criteria are to be maximized. Then S is a combined GP-efficient solution if there is no other solution S^0 such that*

$$\begin{aligned} \sum_{p=1}^{|\mathbf{P}|} w_p d_p^-(S) &\geq \sum_{p=1}^{|\mathbf{P}|} w_p d_p^-(S^0), \\ \max_{1\leq p\leq |\mathbf{P}|} \left(w_p d_p^-(S)\right) &\geq \max_{1\leq p\leq |\mathbf{P}|} \left(w_p d_p^-(S^0)\right). \end{aligned}$$

with at least one of the inequalities strict.

We now propose the following theorem, the proof of which follows directly from Definition 6.2.

Theorem 6.2 *Every solution to* (**Q9**) *is a combined GP-efficient solution.*

Definition 6.2 is indeed a description of the efficient solutions of (**Q9**). Solving (**Q9**) usually gives several combined GP-efficient solutions which are more balanced than solutions of (**Q5**) and (**Q6**). Clearly, the two solutions to (**Q9**) are the optimal solutions of (**Q5**) and (**Q6**), because they yield the best values on the first and second criteria, respectively.

One can use any of the multi-criteria integer programming techniques, presented in Chapter 2, to solve Problem **Q9**. However, given the difficulty of finding unsupported efficient solutions in multi-criteria integer problems and the fact that only some representative combined GP-efficient solutions are needed, we use the weighted approach (Geoffrion's Theorem) to find a portion of the efficient solutions of (**Q9**). Hence, a convex combination of the first and second criteria is suggested to solve (**Q9**) as follows:

(d_p

subject to ... $\leq$ ... w_p an
d_p^- are defi ... 0) is a
efficient sc ... lecisio
space. (Q9 ... (Q10
namely un ... e othe
efficient so. ... 10) i
the Archer ... tion.

Clearly. ... then ... 0) ha
one solutio ... This
solution is

The ob ... the ... nd L_∞
norms. wh ... text o
locational ... the ... a cost o
serving cus ... away
from a faci ... median
and *p-cent* ... k. ... ed by
the fact th ... ture ... aim of
most locati ... m and
minimax) ... timal
solutions o

Note th ... then

[8] *For defin*

$$\textbf{(Q10)} \qquad \textbf{Minimize} \quad \left[(1-\lambda)\sum_{p=1}^{|\mathbf{P}|} w_p d_p^- + \lambda \max_{1\le p\le |\mathbf{P}|}(w_p d_p^-)\right],$$

subject to the same set of constraints as in (**Q4**). where $0 \le \lambda \le 1$. and w_p and d_p^- are defined as before. According to Theorem 2.2 every solution to (**Q10**) is an efficient solution to (**Q9**). However. because of the non-convexity of the decision space. (**Q9**) may have efficient solutions that cannot be found by solving (**Q10**). namely unsupported efficient solutions that are convex dominated by some other efficient solutions. In extreme cases. when $\lambda = 0$. the optimal solution of (**Q10**) is the Archemedian GP solution. and when $\lambda = 1$. it is the Chebyshev GP solution.

Clearly. if (**Q5**) and (**Q6**) have an identical optimal solution. then (**Q10**) has one solution identical to the optimal solutions of (**Q5**) and (**Q6**) for all λ. This solution is also the *ideal solution* of (**Q10**) [8].

The objective function of (**Q10**) can be viewed as a hybrid of the L_1 and L_∞ norms. which can be denoted as $L_{1,\infty}$. A similar norm can be found in the context of locational decision problems. where it is important to consider both the total cost of serving customers as well as the service for those customers who are located far away from a facility [17]. Halpern [44. 45] considers a convex combination of *p-median* and *p-center* problems to locate a facility on an undirected network. Motivated by the fact that neither the minisum nor the minimax criterion can capture the aim of most locational problems. Burkard *et. al* [17] propose a two-criterion (minisum and minimax) zero-one problem and explore the properties of the efficient and optimal solutions of this problem.

Note that when there are only two criteria and all constraints are linear. then

[8] *For definition of ideal solution refer to Chapter 2.*

the combined Chebyshev-Archemedian GP is similar to the concept of compromise set discussed in Chapter 2. In fact, when the solutions of the L_1 and L_∞ problem, lie on the same edge of the feasible region, the solution sets of these two problems are identical.

As shown in Figure 6.1, the contours of Archemedian and Chebyshev GP for a 2-criterion problem are diamond and rectangular, respectively. It is useful to observe the contour of combined Chebyshev-Archemedian GP.

Let (G_1, G_2) be the goals specified for criterion $f_1(x)$ and $f_2(x)$, respectively. The convex combination of L_1 and L_∞ can be written as follows:

$$\begin{aligned} L_{1,\infty} &= (1-\lambda)L_1 + \lambda L_\infty, \\ &= (1-\lambda)\left(w_1|f_1(x) - G_1| + w_2|f_2(x) - G_2|\right) + \\ &\quad \lambda\left(\max\{w_1|f_1(x) - G_1|, w_2|f_2(x) - G_2|\}\right). \end{aligned}$$

For simplicity, assume that the axes in criterion space are shifted such that $G_1 = 0$ and $G_2 = 0$. Hence,

$$\begin{aligned} L_{1,\infty} &= (1-\lambda)\underbrace{\left(w_1|f_1(x)| + w_2|f_2(x)|\right)}_{a} + \lambda\left(\max\{w_1|f_1(x)|, w_2|f_2(x)|\}\right), \\ &= (1-\lambda)\underbrace{\left(max\{w_1|f_1(x)|, w_2|f_2(x)|\} + \min\{w_1|f_1(x)|, w_2|f_2(x)|\}\right)}_{a} \\ &\quad + \lambda\left(\max\{w_1|f_1(x)|, w_2|f_2(x)|\}\right), \\ &= \max\{w_1|f_1(x)|, w_2|f_2(x)|\} + (1-\lambda)\min\{w_1|f_1(x)|, w_2|f_2(x)|\}. \end{aligned}$$

Hence, the contour of $L_{1,\infty}$ for constant C is,

$$\max\{w_1|f_1(x)|, w_2|f_2(x)|\} + (1-\lambda)\min\{w_1|f_1(x)|, w_2|f_2(x)|\} = C,$$

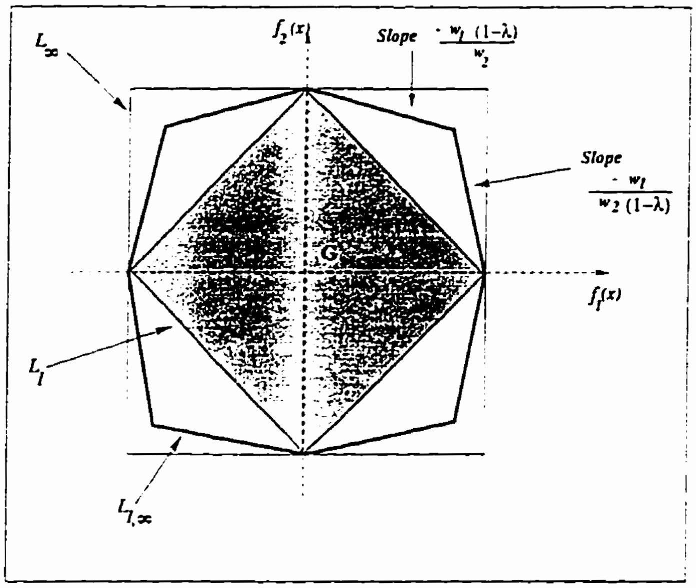


Figure 6.2: Combined Chebyshev-Archemedian Contour for a Two-Criterion Problem

which can be represented as

$$\begin{cases} w_1|f_1(x)| + w_2(1-\lambda)|f_2(x)| = C & if \;\; w_1|f_1(x)| > w_2|f_2(x)|; \\ w_1|f_2(x)| + w_1(1-\lambda)|f_1(x)| = C & if \;\; w_2|f_2(x)| > w_1|f_1(x)|. \end{cases}$$

Both cases together produce the octagon shown in Figure 6.2.

Solving (**Q10**) for different values of λ gives different combined GP-efficient solutions for the DM to choose among. Since the objective function in (**Q10**) is not smooth, we transform it into the following equivalent GP problem:

$$\textbf{Minimize} \qquad (1-\lambda)\sum_{p=1}^{|\mathbf{P}|} w_p d_p^- + \alpha,$$

(**Q11**) **Subject to :**

$$\lambda w_p d_p^- \leq \alpha; \qquad \forall p \in \mathbf{P}.$$

$$\sum_{i=1}^{|\mathbf{A}|} c_p^i x_i + \sum_{k=2}^{K} \sum_{S \in \mathbf{L}_p^k} \Delta_p(S) Y_S + d_p^- - d_p^+ = G_p, \quad \forall p \in \mathbf{P}$$

$$\sum_{a_i \in S} x_i - Y_S \leq |S| - 1. \qquad \forall S \in \mathbf{L}_p^k, \forall p, k,$$

$$x_i \geq Y_S, \qquad \forall x_i \in S, \ \forall S \in \mathbf{L}_p^k, \ \forall p, k,$$

$$x \in \mathbf{X}, \ x_i \in \{0, 1\}, \ Y_S \geq 0.$$

Problem **Q11** can be solved by any single-objective integer programming technique for different values of λ. Moreover, (**Q11**) does not destroy the GP structure and can be solved as efficiently as any weighted or Chebyshev integer GP problem.

Example 6.3 Consider the following multiple objective linear problem whose corresponding feasible set in decision space and criterion space is shown in Figure 6.3:

$$\textbf{Maximize}: \quad f_1(x) = 2x_1 + x_2,$$

$$\textbf{Maximize}: \quad f_2(x) = -x_1 + 2x_2,$$

Subject to :

$$-x_1 + 3x_2 \leq 21,$$

$$x_1 + 3x_2 \leq 27,$$

$$4x_1 + 3x_2 \leq 45,$$

$$3x_1 + x_2 \leq 30.$$

Suppose that the DM specifies $G_1 = 40$, and $G_2 = 20$ as goals, with equal weight. The optimal solution of the Archemedian GP problem is $x_1 = 3, x_2 = 8$ with $d_1^- = 26, d_2^- = 7$. The optimal solution of Chebyshev GP problem is $x_1 = 8.07, x_2 = 4.23$ with $d_1^- = 19.6, d_2^- = 19.6$. Note that the solution of Archemedian

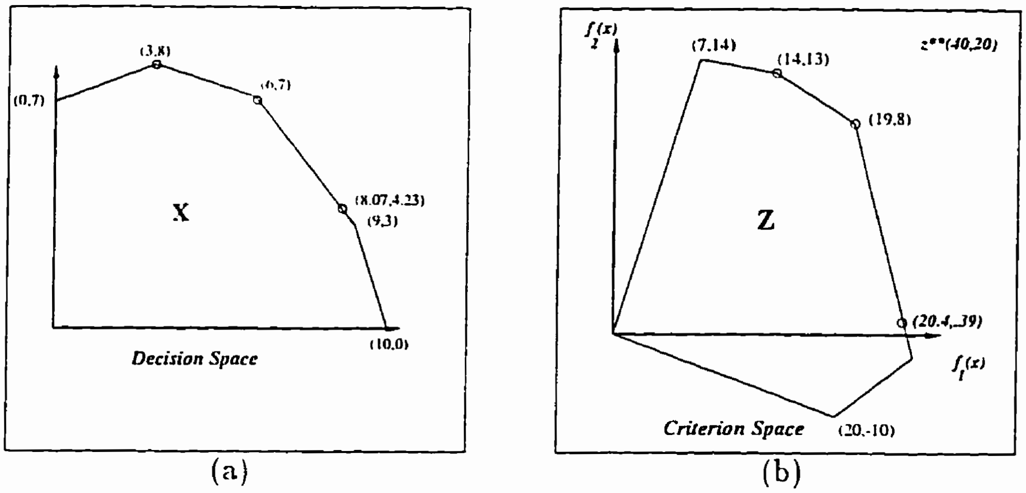


Figure 6.3: The Set of Feasible Actions in (a)Decision Space. (b)Criterion Space (Example 6.3)

GP problem has a relatively large deviation from the first criterion. and solution of Chebyshev GP problem has a large weighted average deviation.

Using the hybrid Chebyshev-Archemedian GP technique. one can find another solution $x_1 = 6. x_2 = 7$ with negative deviations as $d_1^- = 21, d_2^- = 12$. Hence. this program has three supported combined GP-efficient solutions.

$$(3, 8), (8.07, 4.23), (6.7).$$

The DM can select among these three GP-efficient solutions, perhaps using qualitative criteria.

The above discussion demonstrated that a multiple objective GP approach in which both weighted deviations and maximum deviation from goals are simultaneously minimized. overcomes some of the difficulties of Archemedian or Chebyshev GP techniques that mentioned earlier. Nevertheless. this approach is not guaranteed to find efficient solutions.

Example 6.4 Example 2.2 showed that the solutions of the Archemedian and Chebyshev GP can be dominated. It can be shown that the solution of the combined Archemedian-Chebyshev GP is $x_1 = 5. x_2 = 1$ which is also dominated.

The method presented in Section 6.5 can be used to improve on a dominated solution of the combined Archemedian-Chebyshev GP until it is efficient. An alternative way to provide efficient solutions in GP is to change the conventional structure of GP. as explained below.

Section 6.4 shows that when a goal is attainable. then the optimum solution of the GP problem may be dominated. The structure of GP problem is such that only negative deviations. d_p^-. are penalized for a maximization criterion p. and only positive deviations. d_p^+. are penalized for a minimization criterion. Without loss of generality. assume that all criteria are to be maximized. Since the only aim of this GP methodology is to minimize the negative deviations. if the goals are attainable. then all optimal solutions are equivalent. provided that $d_p^- = 0$ for all p.

An analogy concept to GP can be found in *Enforcement of Environmental Laws and Regulations*. Often a firm that does not meet the standards specified by environmental agency pays a penalty. Hence. if the firm complies. there is no penalty but if the firm violates. a penalty will be imposed. On the other hand, some laws also offer economic incentives to firms that exceed the standards. These laws reward firms for over-achievement on environmental standards [54]. They give the firm an incentive to exceed the standard.

The same idea can be used in GP problem. The GP structure can be changed such that the DM is allowed to assign both a penalty for under-achievement and reward for over-achievement. Clearly, the absolute rate of penalty and reward need not be necessarily the same. Furthermore, if the DM does not specify any reward for exceeding the goal, in other words, if he or she assigns the value of zero for over-achievement, then there is no benefit for exceeding the goals in DM's point of view. Moreover, in some cases the DM may believe that over-achieving a goal may degrade some other intangible and qualitative goals. In these cases, a dominated solution generated by GP is called an *acceptable dominated solution* because it is in accordance to the DM's values. It can be shown that the new GP structure does not generate dominated solutions; under it, the procedure seeks better solutions when the goals are attainable.

6.7 Summary and Conclusions

In this chapter, the problem of multiple criteria subset selection was formulated as a non-linear MCZO programming model. Techniques were presented to remove the non-linearity in the model. The main difficulties of available approaches to solve the model were discussed and two modified GP solution methodologies were proposed to overcome these problems. The first approach used the lexicographic GP and vector optimization, in sequence, to obtain a subset of non-dominated solutions. The second approach was based on a two-criterion GP problem in which a combination of Archemedian and Chebyshev GP was employed. It was shown that the combined Chebyshev-Archemedian GP generates a subset of balanced solutions. The next chapter employs these solution methodologies in an on-going water supply planning project. In the GP models discussed in this chapter, a "priori" prefrence is assumed. One could use these models as interactive methods.

Chapter 7

Case Study: Waterloo Water Supply Planning

7.1 Introduction

In this chapter. the solution methodologies that proposed in the previous chapter are applied to a long-term water supply planning problem in the Regional Municipality of Waterloo. Ontario. Canada. to select the best combination of water supply actions. Decisions about water resources have been widely recognized as being multiple objective in nature. In fact. many theories and concepts of MCDM have been inspired by water resources planning problems (see. for example. [120]). Usually. water supply planning has diverse economic. social. environmental. and political objectives. During the past two decades. many MCDM techniques have been developed for use in water resources planning problems (see. for example. [21. 22. 41. 43. 49. 93. 95]. and [122]).

As an example of the use of MCDM in water resources planning. consider the work of Roy *et al.* [106], which uses ELECTRE III to program a water supply system in Poland by setting up a priority order of water users based on socio-economic criteria and then selecting the best water supply sources. Abu-Taleb and Mareschal [2] employ PROMETHEE V to select a set of technical, managerial, pricing, and regulatory water resources options for Jordan. First, all options are evaluated according to an aggregated criterion. Then, a zero-one programming approach with budget, geographic dispersion, and compatibility constraints, is used to select the best combination of options.

Stewart and Scott [121] develop a scenario-based procedure to select a subset of water policies in South Africa. They use a statistical experimental design technique to generate a set of scenarios called the *background set*. The *reference point method* is then used to select scenarios from the background set to form the *foreground set*. The weights for the "scalarizing function" are generated randomly and those scenarios that most frequently minimize this function are selected for the foreground set. A foreground set is generated for every group of DMs. Then the procedure searches for consensus among parties.

Netto *et al.* [85] employ a two-stage procedure for evaluating long-term water supply systems in Southern France. In the first stage, ELECTRE III is used to reduce the number of feasible alternatives. Subsequently, in the second stage, an extension of ELECTRE III is employed to carry out multi-actor, multi-criteria selection among the remaining alternatives.

In many water resources planning problems, the definition and generation of actions is crucial to effective decision making [66]. Yet little research effort has been devoted to this step. Moreover, the optimal choice of a subset of (discrete) water supply actions has not received much attention in the literature. Even though *inter*dependence of water supply strategies occurs commonly in real-world problems, multiple objective methods have assumed strict *in*dependence of individual actions.

The main objective of this chapter is to implement the proposed models and associated analytical techniques to select, within a multiple objective framework, the best combination of long-term water supply strategies for the Regional Municipality of Waterloo, Ontario, Canada. The problem is formulated as a multiple criteria integer program with interdependent actions. Different types of interdependencies in the problem are shown to be essential features. Due to the large number of potential actions and the non-convexity of the decision space, it is quite difficult to identify the non-dominated alternatives. Instead, the combined Chebyshev-Archemedian GP and the modified lexicographic GP techniques are suggested to obtain a subset of non-dominated combinations of actions. The experience gained and lessons learned in applying the proposed approach to the Waterloo water supply strategy. are discussed.

The organization of this chapter is as follows. Section 7.2 briefly describes the background and characteristics of the Waterloo Water Supply Planning Problem (WWSPP). Then, Section 7.3 explains different kinds of interdependencies that exist in the WWSPP. The general mathematical model and the combined Chebyshev-Archemedian model of WWSPP are presented Sections 7.4 and 7.5. Subsequently. Section 7.5 presents a solution methodology to solve the model. Next Section 7.6 discusses the input data. Subsequently, Sections 7.7 and 7.8 present a brief discussion of the solutions. Finally, a range of conclusions are drawn in Section 7.9.

7.2 Problem Definition

7.2.1 Background

The Regional Municipality of Waterloo is located in the southwestern part of Ontario, Canada and comprises the three cities of Kitchener, Waterloo and Cambridge, as well as several rural areas. Figure 7.1 shows where Waterloo is located within the Great Lake Drainage Basin situated in the center of North America and Canada and overlapping parts of Canada and the United States. In fact, the Canadian province of Ontario is an Indian word which means *sparkling water*. Figure 7.2 displays the Grand river basin within which the Regional Municipality of Waterloo is located. The Waterloo region has an area of almost 1350 km^2 and is one of the most prosperous and industrialized areas in Canada, with population of almost 0.5 million people. At present, the Waterloo region is one of the largest communities in North America to rely almost entirely on ground water. More than 90% of Waterloo's potable water is provided by some 126 wells; the remainder is drawn from the Grand River which flows through the region. Due to increases in residential, industrial, and commercial demand, and decreases in the reliability of ground water sources, the Regional Government is currently developing a Long Term Water Strategy to the year 2041 (Associated Engineering [4]).

7.2.2 Problem Characteristics

Most water resources planning problems share features such as conflicting objectives, concerned parties with different points of view, and uncertainty over demand and availability of resources. In the WWSPP, conventional multiple objective procedures are especially difficult to apply for the following reasons:

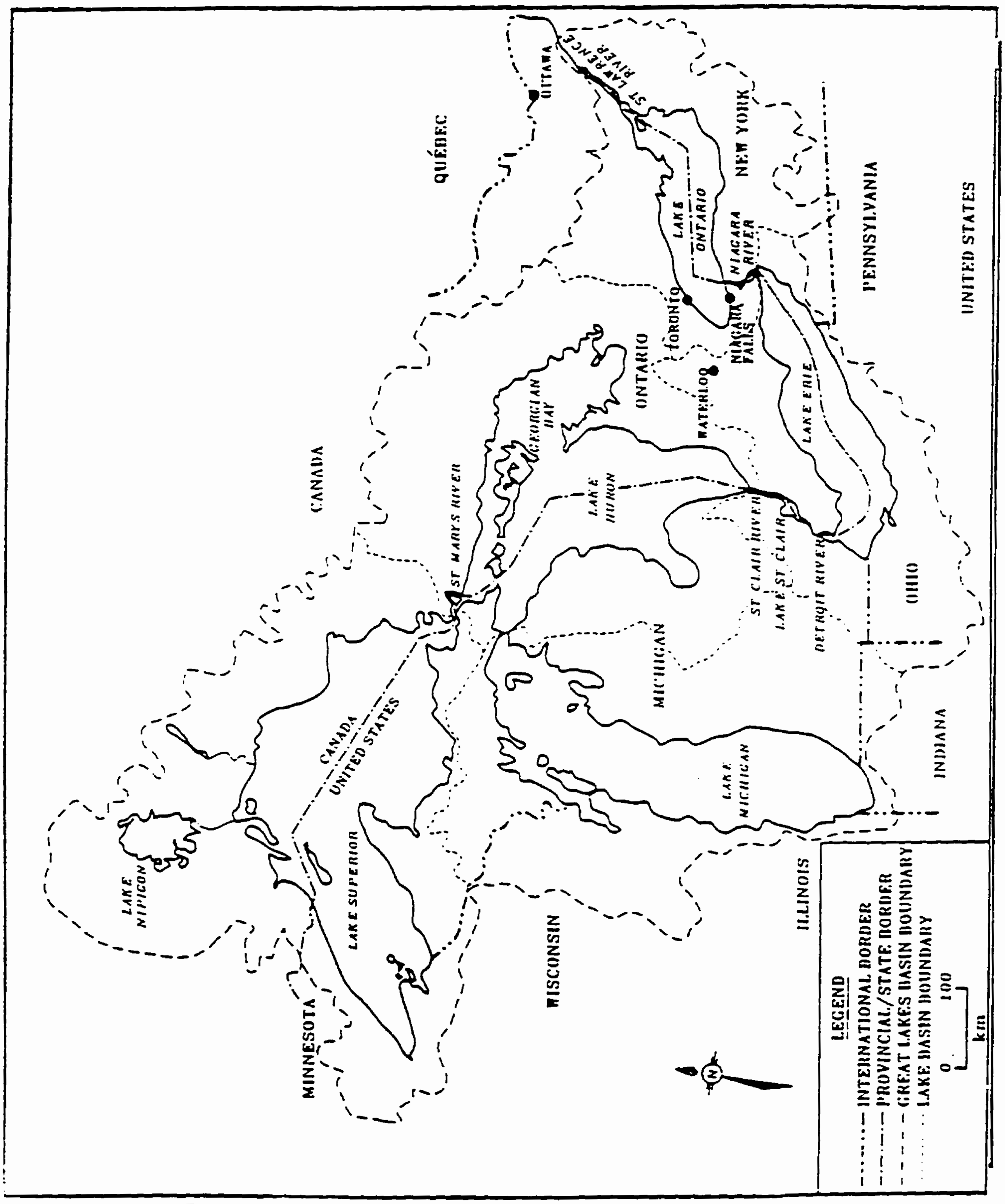


Figure 7.1: The Location of Regional Municipality of Waterloo

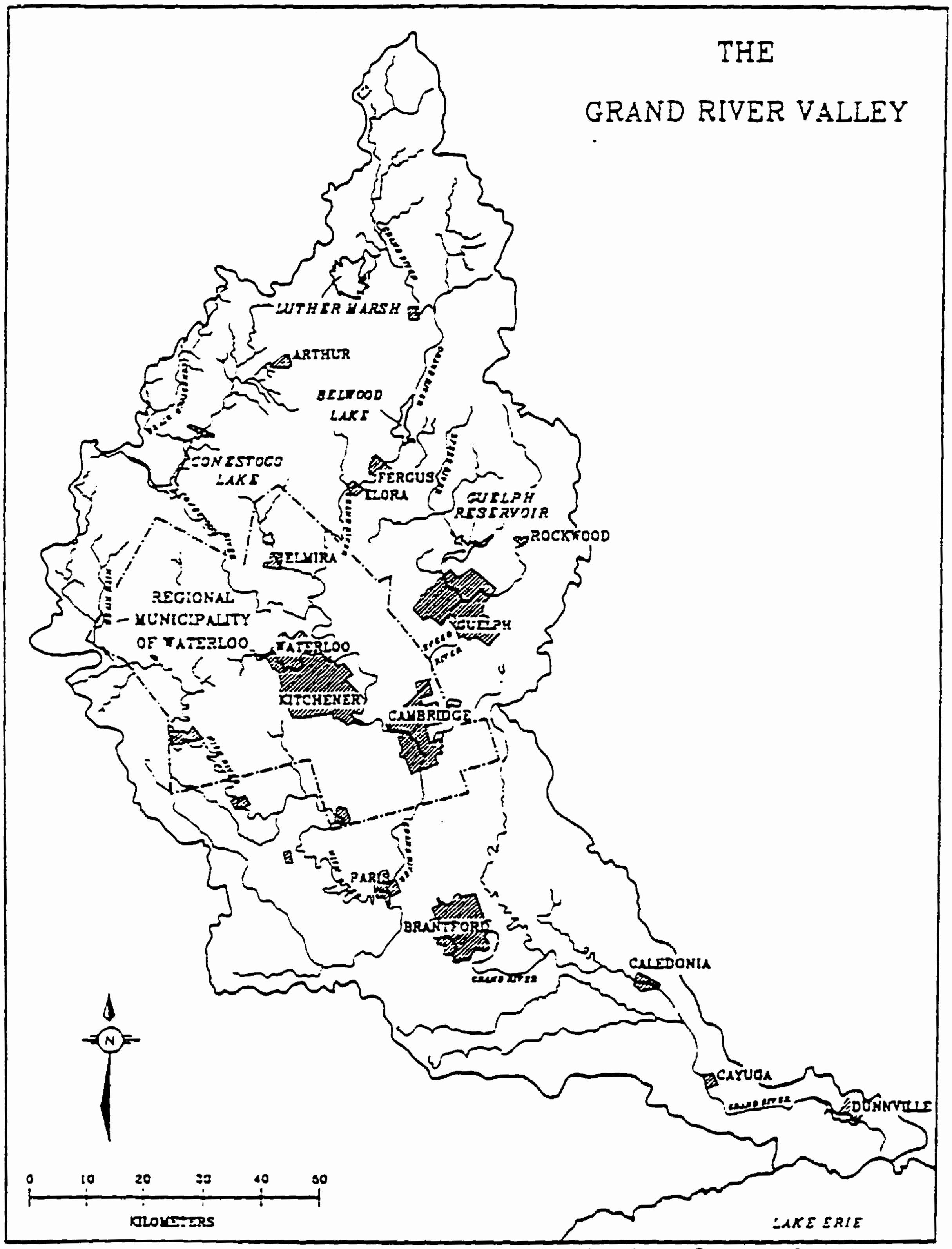


Figure 7.2: The Grand River Basin Located in Southern Ontario, Canada.

- Available actions have never been stated precisely in a form suitable for comparison. Most multiple criteria decision procedures begin with a predefined set of actions. However, in the WWSPP, the relationships among actions require careful study, especially for reconsidering those actions that had been screened out earlier. The problem of generating and defining actions in water resources problems was recently addressed by Stewart and Scott [121], and Keeney *et al.* [66].

- Most MCDM methods assume strict independence of actions. However, it is evident that there are significant positive and negative synergies among actions. Section 7.3 addresses this issue in detail.

- Both tangible and intangible criteria must be taken into account. Moreover, subjectivities underlie some of the tangible criteria, making the evaluation of actions more complicated. For instance, the costs of projects are affected by various funding details, such as who will pay and what assistance will be received from higher levels of government.

- Objectives are not clearly defined. Objectives introduced by various DMs and discussed in technical documents, are interdependent and correlated. Hence, before modeling the problem, the criteria must be refined. The next subsection addresses the main criteria for the WWSPP.

- As in many water resources problems, each criterion reflects the primary interest of one group of DMs. Different parties are involved in WWSPP including among others: Chambers of Commerce, environmental groups, homebuilders associations, service clubs, academia, and agriculture groups [4]. Trade-offs among criteria can therefore be difficult. However, the main DM is the Regional Municipality of Waterloo which is in charge of regional water development. It is noteworthy that some DMs have conflicting interpretations of their interests. For example, public perception of water quality differs especially between urban and rural areas.

- Three strategies have been promoted by different parties and are referred to as tradition, security, and displacement. Each strategy is based on a specific philosophy as follows:
  - *Tradition* means to not expand sources of water until demand exceeds supply. This strategy runs a high risk of shortages due to unexpected events. The Waterloo region has occasionally experienced contamination of some wells, leading to short-term water shortages. For example, the wells supplying the town of Elmira were closed down due to pollution of the underground aquifer. Elmira now receives its water via a pipeline from the city of Waterloo.
  - According to the *security* strategy, additional capacity should be developed to secure the region from any potential loss of water resources. This strategy increases confidence that water demand will be met, but also increases investment and operating costs.
  - The *displacement* strategy emphasizes the replacement of current sources of water. This would have several advantages. For instance, water from alternative sources such as one of the Great Lakes would not require domestic softening, and supplies would become more reliable and secure.

It is clear that each strategy implies a specific volume of water for the region. Hence, the best subset of water supply actions may be different for each strategy.

- Demand varies continuously and implementation of each project creates a step-size increase in water supply. Intuitively, a water strategy that has a low gap between demand and implementation is preferred.

7.2.3 Criterion Identification

The overall purpose of WWSPP is to design and implement the best water resources plan to satisfy long-term demand. In light of this purpose, more specific objectives such as *low cost*, *good water quality*, *few infrastructure impacts*, *minimum environmental impacts*, *high security* and *reliability* in order to have low *risk*, and *sufficient supply capability* have been proposed for measuring the effectiveness of possible actions. Below is a brief description of each criterion.

- *Cost* - This criterion measures the cost of water to the year 2041, including investment cost, operations and maintenance costs, cost of purchasing water from other regions if required, and cost of standard treatment.

- *Water quality* - The *Ontario Water Resources Act*, implemented in 1972, is the main legislative instrument of the *Ministry of Environment and Energy (MOEE)* for regulating water quality in the province. The *Ontario Clean Water Agency (OCWA)* mentioned at the start of the Ontario Water Resource Act is part of the MOEE and its mission is to oversee the development of municipal water and wastewater infrastructure. All water sources must meet the *Ontario Water Standards* for now and the future. The level of treatment depends on the water supply action. This criterion also reflects public opinion on the aesthetic aspects of water quality. For some actions, it is difficult to judge water quality. For instance, the physical, chemical, and bacteriological characteristics of ground water from different fields may vary considerably.

- *Infrastructure Impacts* - Each action requires certain modifications to the existing water supply system, such as expansion of water mains, and construction of reservoirs, pumping stations. The main part of this criterion is quantitative, but some intangible effects must also be taken into account.

- *Environmental Impacts* - This criterion refers to the long-term and short-term environmental impacts of actions on environment. Effects of implementing each action on agriculture and farm well, fisheries, wetlands, recreation, and surface water are considered in this criterion. These impacts are more important for actions involving new construction such as, pipelines. Extensive monitoring is required for ground water sources; its cost can be considered as a tangible portion of the environmental impacts.

- *Risk* - Maximizing the security and reliability that adequate supplies of high quality water are provided, is a major concern for all DMs. Supplementary water resources that can be used in emergencies decrease these risks. Selecting actions that increase the flexibility of water supply leads to a low risk plan. A project is flexible if it is multi-purpose, quick to implement, easy to expand, and easy to modify in the case of unpredicted changes.

- *Supply Capability* - Most water resources planning research considers supply capability to be a set of constraints to be satisfied. However, in WWSPP different strategies (*i.e.* traditional, security, and displacement) may lead to various policies for satisfying water demand. Therefore, supply capability is included as an objective that should be maximized in the model. According to this objective those actions that provide large supply capability in future are preferred. Clearly, larger supply capability imposes more cost. Note that, meeting minimum demand based on traditional strategy in each region is considered as a hard constraints in the model.

7.2.4 Available Actions

The definition and generation of actions is an important step in the process of multi-objective water resources planning, but one to which little research effort has been devoted. Characteristically, water resources planning problems present a wide variety of possible actions. Most often. actions are not predefined clearly; in some cases. it is hard to determine when actions are feasible [66].

Usually, the number of actions is reduced to a manageable size using screening procedures (see. for example. [55]) or intuitive techniques. However. most of these procedures are ad-hoc: using them may eliminate some potentially good alternatives. This problem is often serious for situations in which a subset of actions is to be selected. because as is discussed in Chapter 3. the best subset of actions may contain dominated actions [91]. Figure 7.3 categorizes the set of main actions and their sub-actions for the WWSPP. (Capacities are measured in Million Imperial Gallons per Day. or $MIGD$.)

In the following, each main actions is briefly described.

1. *Ground water* (GW) - Currently. almost all water of the region is provided by ground water from wells in different fields. This main action is concerned with more development of ground water supplies.

2. *Aquifer Recharge* (AQ) - This set of actions is based on the storage of treated drinking water in a suitable aquifer during periods of water surplus for using in seasonal peaks. emergency water demand. or for subsequent years as short-term and long-term water supply.

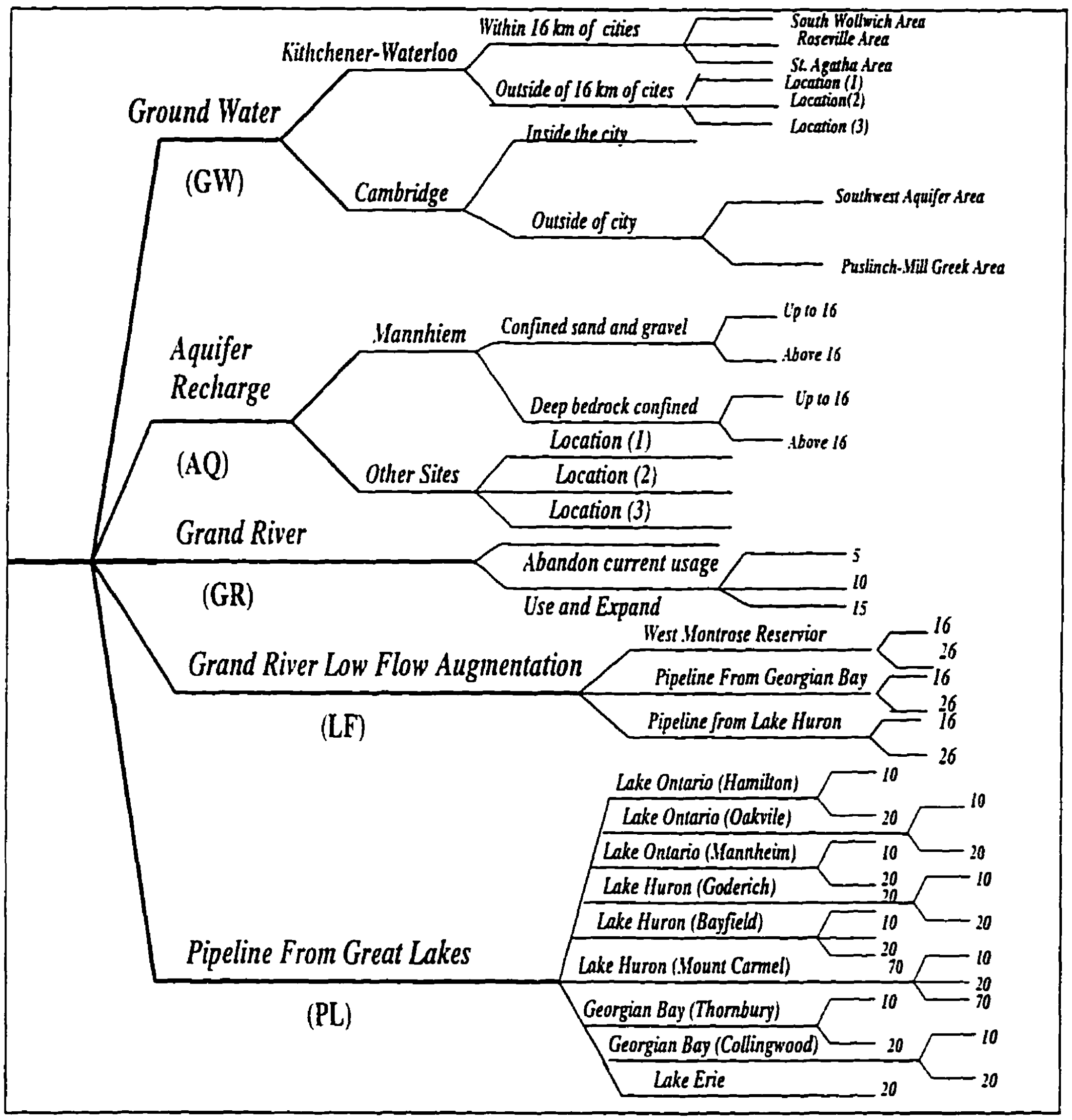


Figure 7.3: Main Actions and Sub-actions for the Waterloo Water Supply Planning Problem and Their Supply Capacities.

3. *Grand River* (GR) - Currently, a small portion of the region's water is provided by the Grand River. This action suggests higher abstraction from this river.

4. *Grand River low flow augmentation* (LF) - To provide the opportunity for additional summer abstraction, one set of proposed actions is augmentation of the Grand River in low flow periods by using some reservoirs or pipelines from one of the Great Lakes.

5. *Pipeline from Great Lakes* (PL) - This set of actions includes constructing pipelines from one of the Great Lakes through different routes.

In addition to the above actions, there are many suggested *managerial* (such as incentive policies for recycling water), *pricing*, and *regulatory* policies that can be implemented along with any solution to the Waterloo water supply problem. Even though choice of these policies may affect the selection of the best subset, we do not include these policies in the subset selection problem, for several reasons. Firstly, including all these policies makes the problem very large and unmanageable, and secondly, these policies in some cases can be implemented independently from selection of water actions. Moreover, the effects of some policies can be examined by considering different scenarios.

7.3 Interdependence of Actions

Most systematic approaches to water resources planning have assumed independence of actions, even though actions are clearly interdependent in many real-world water resources problems. Interdependence of actions is more common in the multiple objective context since the combinations of actions may be interdependent according to different objectives. There has been little research exploring the concepts and characteristics of interdependence of actions in the water resources planning problem.

In the WWSPP, there are different kinds of interdependencies among actions that cannot be overlooked. Table 7.1 describes some groups of interdependent actions and the criteria under which they are interdependent, categorizes the interdependencies, and indicates whether the interdependence is positive (+) or negative (-). Note that in addition to those given in Table 7.1, there are some conventional interdependencies that affect the implementation of actions. For example, two options of aquifer Recharge cannot be implemented simultaneously.

As described in Chapter 4, actions can be interdependent either conditionally or unconditionally. Recall that when two actions affect each other (on a criterion) no matter what other actions are selected, they are unconditionally interdependent, but if the connection holds only when specific other action(s) are selected, they are conditionally interdependent. As an example of conditional interdependence, suppose that a_1 and a_3 are two independent wells. However, if well a_2 is close to both a_1 and a_3, then when a_2 is selected the amount of water extraction from either a_1 or a_3 affects the amount that can be extracted from the other. Hence, a_1 and a_3 are conditionally interdependent.

7.4 Model Building

This section explains the main elements of the mathematical model developed to select the best combination of actions for the WWSPP. The problem is formulated as a multiple-objective mixed-integer programming problem with some non-linear terms that arise due to the interdependence of actions.

Table 7.1: Examples of Interdependence of Actions in the Waterloo Water Supply Planning Problem

Groups of Actions	**Criterion**	**Type**	**Description**
Different ground water fields	Supply capability	Direct (-)	Water extraction from one field decreases the water extraction from other fields.
Wells in one subregion	Supply capability	Direct and indirect (-)	Water extraction from one well affects other wells.
Aquifer Recharge and Grand River low-flow augmentation	Cost	Direct (+)	Aquifer Recharge and low flow augmentation could not be accomplished without a new treatment facility and/or a reservoir.
Ground water	Water quality	Direct and indirect (-)	The cost of monitoring each well decreases when more wells are selected.
Ground water	Environmental impacts	Direct (+)	Additional wells aggravate the effects on agriculture, farm wells, and wetlands.
Grand River and ground water	Risk	Direct (+)	Risk increases with the selection of these actions.
Grand River and low-flow augmentation	Risk	Direct (+)	Risk increases with the selection of these actions.
Pipeline and ground water	Infrastructure Cost	Direct (+)	The infrastructure increases due to the major differences between these two actions.
Pipeline and ground water	Risk	Direct (-)	Because they rely on two completely different water sources.

7.4.1 Notation

The notation used for formulating a model for the WWSPP is as follows:

T = number of planning periods = horizon. The planning horizon (1996-2041) has been divided into five periods.

t = index corresponding to planning period; $t = 1, \ldots, T$.

$\mathbf{R}$ = the set of subregions, $\mathbf{R} = \{r_1, \cdots, r_r, \cdots, r_{|\mathbf{R}|}\}$.

r = index corresponding to subregion: Kitchener-Waterloo=1, Cambridge=2, and rural areas=3.

$\mathbf{A}$ = the set of actions, $\mathbf{A} = \{a_1, \ldots, a_i, \ldots, a_{|\mathbf{A}|}\}$.

$\mathbf{A}$ = $\mathbf{A}_{GW} \cup \mathbf{A}_{AQ} \cup \mathbf{A}_{GR} \cup \mathbf{A}_{LF} \cup \mathbf{A}_{PL}$, indicating the union of sets of actions in ground water, Aquifer Recharge, Grand River, low flow augmentation, and pipeline.

i = index corresponding to a proposed action.

x^t_{ir} = the *fraction* of water from action i assigned to subregion r in period t.

Z^t_i = binary variable corresponding to action i at time t such that

$$Z^t_i = \begin{cases} 1 & \text{if action } i \text{ is used in time } t, \\ 0 & \text{otherwise.} \end{cases}$$

C_i = the supply capability of ith action. Hence, $C_i x^t_{ir}$ is the amount of water of ith action assigned to region r in period t.

j = index corresponding to the actions in use in 1996.

$\mathbf{A}'$ = the set of actions in use in 1996.

D^r_t = the demand in period t for subregion r according to traditional supply strategy.

y_{jr}^t = the *fraction* of water from initial action j to be used in subregion r in period t.

C_j' = the supply capability of the jth action currently being used. Hence, $C_j' y_{ir}^t$ is amount of water from the jth old action assigned to region r in period t.

$\mathbf{P}$ = the set of criteria: {cost, infrastructure impacts, water quality, environmental impacts, risk, supply capability}

p = index corresponding to the set of criteria, $p = 1, \ldots, |\mathbf{P}|$.

$(FS)_{pi}$ = the fixed score of the ith action according to criterion p.

$(VS)_{pi}^t$ = the variable score of the action i for period t according to criterion p.

$\mathbf{L}_p$ = collection of all sets of interdependent actions according to criterion p.

$\mathbf{L}_p^k$ = collection of interdependencies among k actions on criterion p. For example, $\mathbf{L}_p^2$ is the set of all pairs of interdependent actions on criterion p.

$\Delta_p(S)$ = the amount of simple dependence within actions in set S on criterion p. For example, $\Delta_p(i, j)$ is the amount of simple dependence within actions a_i and a_j according to criterion p, and $\Delta_p(i, j, k)$ the amount of dependence among actions a_i, a_j, and a_k according to criterion p.

7.4.2 Problem Formulation

In this section, the objective functions and the main constraints of the WWSPP are explained. The solutions of the presented model address the following questions:

1. Which actions are to be implemented?

2. What level of capacity from each action is to be selected?

3. At which time should these actions be implemented?

4. What percentage of each selected actions should be assigned to each subregion?

- Objective functions

For each criterion p: $p = 1, \ldots, |\mathbf{P}|$ the following functions are to be maximized or minimized:

$$\text{Max(Min)} \quad Z_p = \underbrace{\sum_{i=1}^{|\mathbf{A}|}(FS)^t_{pi}\delta_i}_{a} + \underbrace{\sum_{i=1}^{|\mathbf{A}|}\sum_{t=1}^{T}(VS)^t_{pi}\sum_{r=1}^{|\mathbf{R}|}x^t_{ir}}_{b} + \underbrace{\sum_{j=1}^{|\mathbf{A}'|}\sum_{t=1}^{T}(VS)^t_{pj}\sum_{r=1}^{|\mathbf{R}|}y^t_{jr}}_{c}$$
$$+ \underbrace{\sum_{k=2}^{|\mathbf{A}|}\sum_{S\in\mathbf{L}^k_p}\Delta_p(S)\cdot\left(\prod_{a_i\in S}\delta_i\right)}_{d} \qquad for \;\; p = 1, \ldots, |\mathbf{P}|.$$

In the above set of objective functions, the term a considers the sum of fixed scores of all selected actions. For instance, when $p = 1$, it shows the investment cost of the selected actions. Term b is the sum of variable scores[1] of selected actions on each criterion over the planning horizon. Term c indicates the sum of variable scores of actions in use. The last term of the objective function, term (d), represents the amount of interdependence of actions. For instance, when actions a_i and a_j are

[1] *Implementation of an action involves both fixed and variable scores. For example, the fixed cost of an action is mainly the construction cost which is fixed regardless of the number of periods that this action is to be used, while the variable cost is mainly maintenance and operation, which depend on the number of periods the action is used and the amount of water extracted from it.*

interdependent on criterion p the quadratic term $(\Delta_p(i,j)\delta_i\delta_j)$ appears in objective function p. The binary variable, δ_i in the set of objective functions is defined as follows:

$$\sum_{t=1}^{T} Z_i^t - M_1\delta_i \leq 0; \qquad for \;\; a_i \in \mathbf{A}. \tag{7.1}$$

$$\sum_{t=1}^{T} Z_i^t \geq \delta_i; \qquad for \;\; a_i \in \mathbf{A}. \tag{7.2}$$

where M_1 is a sufficiently large number. Expressions 7.1 and 7.2 ensure that δ_i takes the value 1 if and only if action a_i is used at least once during the time horizon. Introducing variables δ_i, significantly decreases the number of non-linear terms arising from interdependent actions.

- **Constraints**

- *Demand:*

In accordance to the traditional supply strategy, one must satisfy the average demand for each subregion in each period. This set of constraints ensures that for all region and all periods, enough water supply is assigned.

$$\sum_{i=1}^{|\mathbf{A}|} C_i x_{ir}^t + \sum_{j=1}^{|\mathbf{A}'|} C'_{jr} y_{jr}^t \geq D_r^t, \;\; for \;\; t = 1, 2, \ldots, T, \; and \, r = 1, \cdots, |\, \mathbf{R} \,|, \tag{7.3}$$

- *Budget*

The set of constraints in Expression 7.4 specifies that the total investment, maintenance and operating costs (first criterion) should not exceed available funds for each period.

$$\sum_{i=1}^{|\mathbf{A}|}(FS)_{1i}^t\delta_i + \sum_{i=1}^{|\mathbf{A}|}(VS)_{1i}^t\sum_{r=1}^{|\mathbf{R}|}x_{ir}^t + \sum_{j=1}^{|\mathbf{A}'|}(VS)_{1j}^t\sum_{r=1}^{|\mathbf{R}|}y_{jr}^t$$
$$+ \sum_{k=2}^{|\mathbf{A}|}\sum_{S\in \mathbf{L}_1^k}\Delta_1(S)\cdot\left(\prod_{a_i\in S}\delta_i\right) \le B_t, \qquad for\ t = 1,\cdots,T, \tag{7.4}$$

- *Technological constraints:*

Constraints 7.5 and 7.6 force variable Z_i^t to take the value 1, if and only if action a_i is used at least once in a subregion. Also, (7.7) and (7.8) ensure that the total usage of each new and old actions do not exceed their capacities.

$$\sum_{r=1}^{|\mathbf{R}|}x_{ir}^t \le M_2 Z_i^t, \qquad i = 1,\cdots,|\,\mathbf{A}\,|, \quad t = 1,\cdots,T, \tag{7.5}$$

$$M_3\sum_{r=1}^{|\mathbf{R}|}x_{ir}^t \ge Z_i^t, \qquad i = 1,\cdots,|\,\mathbf{A}\,|, \quad t = 1,\cdots,T, \tag{7.6}$$

$$\sum_{r=1}^{|\mathbf{R}|}x_{ir}^t \le 1, \qquad i = 1,\cdots,|\,\mathbf{A}\,|, \quad t = 1,\cdots,T, \tag{7.7}$$

$$\sum_{r=1}^{|\mathbf{R}|}y_{jr}^t \le 1, \qquad j = 1,\cdots,|\,\mathbf{A}'\,|, \quad t = 1,\cdots,T, \tag{7.8}$$

where M_2 and M_3 are sufficiently large numbers. For some main actions, only one sub-action can be selected. For example,

$$\sum_{a_i\in \mathbf{A}_{lf}} Z_i^t \le 1, \qquad for\ \ t = 1,\ldots,T, \tag{7.9}$$

$$\sum_{a_i\in \mathbf{A}_{pl}} Z_i^t \le 1, \qquad for\ \ t = 1,\ldots,T. \tag{7.10}$$

The set of constraints in (7.9) and (7.10) ensure that at most one action from each of the variables of low flow augmentation and pipeline actions can be selected.

- *Variable types:*

$$x_{ir}^t \geq 0; y_{jr}^t \geq 0: \quad \text{for all } a_i \in \mathbf{A}, \quad r = 1 \cdots, \mid R \mid, \text{ and } t = 1, \cdots T,$$
$$Z_i^t, \delta_i \in \{0, 1\}: \quad \text{for all } a_i \in \mathbf{A} \quad t = 1, \cdots T. \tag{7.11}$$

7.5 Solution Methodology

The WWSPP, which is formulated in the previous section, is a nonlinear multiple criteria mixed integer programming. Since most of the theories of integer programming are developed in the framework of linear cases[2], it is more convenient to convert the above nonlinear program to a linear one. This can be accomplished by using the techniques presented in Chapter 6. For each $S = \{i_1, i_2, \ldots, i_k\} \in \mathbf{L}_p^k$ (given k and p), define $Q_S = \delta_{i_1}.\delta_{i_2} \cdots \delta_{i_k}$ and add the two following constraints:

$$\delta_{i_1} + \delta_{i_2} + \cdots + \delta_{i_k} - Q_S \leq k - 1, \tag{7.12}$$
$$-\delta_{i_1} - \delta_{i_2} - \cdots - \delta_{i_k} + kQ_S \leq 0. \tag{7.13}$$

In this way, a multiple objective subset selection problem under interdependence of actions can be formulated as a *linear* multiple objective mixed integer problem. The difficulty of solving a mixed integer problem is highly dependent upon the number of integer variables. Hence, following Glover and Woolsey (1974), one can

[2] *Most integer programming approaches are based on solving a sequence of linear problems.*

change each cross-product variable (Q_S) to a continuous variable by replacing (7.13) with the following set of inequalities:

$$\delta_{i_j} \geq Q_S \quad \forall\, j \in \mathbf{S}. \tag{7.14}$$

For detailed discussion refer to Section 6.2.1 in the previous chapter. Let d_p^+ and d_p^- denote positive and negative deviations from goal on criterion p. Then, the WWSPP can be reformulated based on a hybrid $L_{1,\infty}$ norm described in previous chapter as follows:

$$\textbf{Minimize} \quad \left\{ \underbrace{[(1-\lambda)\sum_{p} w_p(d_p^+ + d_p^-)]}_{e} + \lambda Max_p \underbrace{(w_p(d_p^+ + d_p^-))}_{f} \right\}$$

$$\textbf{Subject to:}$$

$$\sum_{i=1}^{|\mathbf{A}|}(FS)_{pi}^t \delta_i + \sum_{i=1}^{|\mathbf{A}|}\sum_{t=1}^{T}(VS)_{pi}^t \sum_{r=1}^{|\mathbf{R}|} x_{ir}^t + \sum_{j=1}^{|\mathbf{A}'|}\sum_{t=1}^{T}(VS)_{pj}^t \sum_{r=1}^{|\mathbf{R}|} y_{jr}^t$$

$$+ \sum_{k=2}^{|\mathbf{A}|}\sum_{S \in \mathbf{L}_p^k} \Delta_p(S) \cdot \left(\prod_{a_i \in S} \delta_i \right) + d_p^- - d_p^+ = G_p,$$

$$for \;\; p = 1, \ldots, |\,\mathbf{P}\,|\,. \tag{7.15}$$

The Set of Constraints 7.1 to 7.11.

In the above formulation, the term shown by e is the Archemedian part of the goal programming problem and f is Chebyshev part. Also, G_p is the aspiration level assigned for criterion p. w_p is the amount of penalty for unit deviation of objective function p from the specified goal, and λ is the coefficient of tendency towards Chebyshev or weighted average norm. The above model can be reformulated as follows:

$$\textbf{Minimize} \quad \left\{(1-\lambda)\left(\sum_{p} w_p(d_p^+ + d_p^-)\right) + \lambda\beta\right\},$$

Subject to :

$$w_p(d_p^+ + d_p^-) \leq \beta \quad \forall p \tag{7.16}$$

$$\text{Constraints 7.1 to 7.11 and (7.15).} \tag{7.17}$$

The next section provides some numerical information for the WWSPP.

7.6 Input Data

Population growth is the main cause of increase in water demand in the Waterloo region. Table 7.2 shows the predicted water demand for each subregion, in terms of $MIGD$ to the year 2041.

Table 7.2: Water Demand in Three Main Areas to the Year 2041

Regions	1996	2001	2006	2011	2016	2021	2026	2031	2036	2041
Kit/Wat	30.1	31.8	33.9	36	38.2	40	41.6	43.3	44.7	46
Cambridge	15.1	16	17	18.1	19.2	20	20.8	21.6	22.3	23
Rural	4	4.7	5.3	6.1	6.9	7.4	8.1	8.5	8.9	9.3

Note that for some of the WWSPP actions, there exist different variations with different capacities and various specifications. To reduce the number of discrete variables, in this model, we do not take into account those actions that have been removed from the set of available actions in the screening process addressed by

Associate Engineers [4] or those actions that are very similar and cannot be selected together. Actions that are considered in the mathematical modeling include:

- Ground water, option 1, (GW1)- Developing additional ground water sources in the vicinity of Kitchener-Waterloo.
- Ground water, option 2, (GW2) - Developing more ground water sources in *new* fields. The new sources of ground water are located in the South Woolwich Area, the Reseville Area, and St. Agatha Area.
- Aquifer Recharge, option 1, (AQ1) - Constructing dual purpose Recharge and recover wells in Mannheim site with capacity of 10 $MIGD$.
- Aquifer Recharge, option 2, (AQ2) - Constructing dual purpose Recharge and recover wells in Mannheim site with capacity of 20 $MIGD$.
- Grand River, (GR) - water extraction from Grand River during times of peak demand.
- Grand River Low Flow Augmentation (LF1)- Augmentation of Grand River water flow by implementing West Montrose Dam.
- Grand River Low Flow Augmentation (LF2)- Augmentation of Grand River water flow by constructing a pipeline from Georgian Bay.
- Grand River Low Flow Augmentation (LF3) - Augmentation of Grand River water flow by using a pipeline from Lake Huron.
- Pipeline (PL1) - Transporting water to the region via a high pressure pipeline from Lake Ontario.

- Pipeline (PL2) - Transporting water to the region via a high pressure pipeline from Lake Erie using Manticoke water treatment facility.
- Pipeline (PL3) - Transporting water to the region via a high pressure pipeline from Lake Huron in Goderich connection.
- Pipeline (PL4) - Transporting water to the region via a high pressure pipeline from Georgian Bay in Thornbury location.

Tables 7.3 provides the actual values of the water actions according to the main criteria. The scores for water quality. environmental impacts and risk criteria are estimated according to the preliminary evaluation obtained by Associated Engineering [4]. Arrows show the direction of preference for each criterion where an upward arrow means that a higher value is more preferred and downward arrow indicates a lower value is more preferred. Here. it is assumed that the preference of the DM is monotonically increasing or decreasing on each criterion.

Table 7.4 presents the set of all suggested interdependencies and the estimated values of *synergy* between each pair of actions in WWSPP. Note that synergy between two actions is defined according to (4.7). For sake of simplicity. only binary interdependencies are considered. here. Furthermore. since interdependence is a symmetric relation. only one side of the interdependence between two actions are shown in this table.

Since, there is no explicit information on the DMs' goal for each criterion. the *ideal point* of the problem is used as the initial target of the problem. Recall that the ideal point is a solution which is best according to all criteria. In other words. we solve the model separately for each criterion to find the optimal solution for that criterion. The collection of these optimum solutions for all criteria constitute the

Table 7.3: Scores of Actions According to Criteria

Actions	Inves. Cost ↓	Oper. Cost ↓	Water quality ↑	Inf. Impact ↓	Env. Impact ↓	Risk ↓	Supply Capability ↑
GW1	\$100	\$4	50	30	60	80	29
GW2	\$61	\$2.4	50	30	60	80	20
AQ1	\$8.6	\$5.9	70	40	45	50	40
AQ2	\$17	\$8.8	70	50	45	50	40
GR	\$5	\$2	30	30	40	80	5
LF1	\$112	\$6.2	60	60	50	60	50
LF2	\$123.6	\$6.6	60	60	40	70	unlimited
LF3	\$111.25	\$6.7	60	60	90	70	unlimited
PL1	\$120.4	\$4.2	70	60	80	30	unlimited
PL2	\$126	\$3.4	70	65	80	30	unlimited
PL3	\$181	\$2.3	80	60	80	30	unlimited
PL4	\$222	\$2.5	70	60	80	30	unlimited

Table 7.4: Interdependent Actions and Their Estimated Values in WWSPP

Actions	Ground Water	Low Flow	Grand River	Aquifer Recharge
Ground Water	– –	– –	– –	– –
Low Flow	–	–	–	–
Grand River	Risk (+0.2) Water Quality (-0.1)	Risk (+0.2) Infra.(-0.1)	– –	– –
Aquifer Recharge	Risk (-0.1) Infra. (+0.15)	Cost (+0.2) Risk (+0.15)	Risk (-0.1) Infra. (+0.2)	– –
Pipeline	Risk (-0.1) Infra (+0.2)	Risk (+0.1) Env. (+0.1)	Risk (-0.1) Infra. (+0.2)	Env. (+0.1) Risk (+0.2)

ideal point of the WWSPP problem. Solving the overall goal programming problem with the ideal point as the target provides some initial solutions to the DM. If the DM is not satisfied by this set of solutions, or if he or she wants to examine the robustness of the solutions, the second step is started.

In the second step, the DM specifies the percentage of the ideal point for each criterion that can be downgraded without penalty. Then, the model is solved for this new target. The decision process is terminated when the DM is satisfied with the solution. The model is built such that the DM can easily enter these percentages.

The importance of each criterion is reflected as the rate of penalty for unit deviation from the goal of each criterion in the model. These rates are estimated according to the preliminary study by Associated Engineering [4] and interviews with personnel in the Regional Municipality of Waterloo and are given as follows:

$$w_{cost} = 0.3, w_{water\ quality} = 0.1, w_{infrast.\ impacts} = 0.1,$$

$$w_{environ.\ impacts} = 0.1, w_{risk} = 0.2, w_{supply\ capability} = 0.2.$$

7.7 Discussion of Results

The WWSPP is modeled using GAMS (General Algebraic Modeling System) and solved with LAMPS (Linear and Mixed-Integer Programming). Different logical constraints and *special ordered sets* are added to the set of constraints to reduce the computational time. Also, the planning horizon is divided into 5, rather than 10 periods to reduce the number of integer variables. The combined Chebyshev-Archemedian model is then employed to solve the model.

As pointed out, the main objective of studying WWSPP is to select a set of promising subsets of water supply actions. Moreover, in this section it is demonstrated that:

1. Interdependence of actions is important and should not be ignored. In other words, it is shown that solutions of the model with and without interdependence are quite different.

2. The convex combination of weighted and Chebyshev GP produces different GP-nondominated solutions. Hence. the DM has the opportunity to compare these different solutions by perhaps considering the criteria that could not be stated as a mathematical formula.

To achieve the above mentioned objectives. several versions of the WWSPP were solved:

1. Different values of $0 \leq \lambda \leq 1$,

2. With and without interdependence,

3. Weighted and unweighted Chebyshev norms.

4. Combinations of the above.

Solving the model for different values of λ provides some combined-GP nondominated solutions to the problem (see Chapter 6). All these solutions are potentially good decisions that the DM can choose among them according to his or her preferred GP objective function.

Even though the model presented in this chapter is inspired by a real-world water resources problem. the following simplifications are considered in the model:

-*No explicit uncertainty* - Some of the elements of the WWSPP involve uncertainty and inaccuracy. For example, actual water demand may not be as accurate as its forecast, the estimated capacities and reliability of some of the proposed actions are not accurate, and the cost of implementing actions are not precise. In this model we do not consider any explicit uncertainty for the above mentioned parameters. Nonetheless, one can use sensitivity analyses on different uncertain parameters to assess the effect of uncertainty on problem solutions (see for example, [79]).

-*Longer time period* - To reduce the discrete variables and hence to decrease the computational requirement. we divide the planning horizon into five periods with 10 years length. Clearly. using shorter time periods provide more accurate solutions.

Table 7.5 shows the set of selected actions in the case of weighted Chebyshev. for different values of λ as well as the deviations of the solutions from goals for two different cases: when interdependence of actions is taken into account and in the case of ignorance of interdependence. The third and sixth column of this table shows the deviation of the solutions from goals for cost, water quality, infrastructure impacts, environmental impacts, risk and supply capability criteria, respectively. Also, Figure 7.4 depicts the information in Table 7.5 in a schematic form.

As Table 7.5 and Figure 7.4 show. the sets of selected actions are different for the two cases of considering and ignoring the interdependence of actions. When $0.4 \leq \lambda < 0.8$ the best solution for the case of interdependence is AQ2 and GW1, and for the case of no interdependence is AQ2 and PL2. The reason is that the desirable synergetic effects of AQ2 and GW1 is more than desirable synergies between AQ2 and PL2. As Table 7.4 indicates AQ2 and GW1 hold a desirable synergy on risk criterion. But, PL2 and AQ2 have undesirable synergy on both environmental effects impacts and risk criterion. Hence, when interdependence of actions is taken into account, the combination of AQ2 and GW1 is better than combination of AQ2

Table 7.5: WWSPP: Sets of Actions Selected and Deviations from Goal (Weighted Chebyshev)

λ	With Interdependence	Deviations	Without Interdependence	Deviations from goals
$0.0 \leq \lambda < 0.4$	AQ2, PL2	(248,33,111, 110,60,5)	AQ2, PL2	(241,33,100, 120,50,5)
$0.4 \leq \lambda < 0.8$	AQ2, GW1	(236, 47,75, 70,65,60)	AQ2, PL2	(241,33,100, 120,50,5)
$0.8 \leq \lambda < 0.95$	AQ2, GW1	(236, 47,75, 70,65,60)	AQ2, GW1	(236, 47,65, 70,65,60)
$\lambda = 1$	GW1, AQ1, GR1	(234,96,105, 130,101,170)	GW1, AQ1, GR1	(234,87,105, 140,90,170)

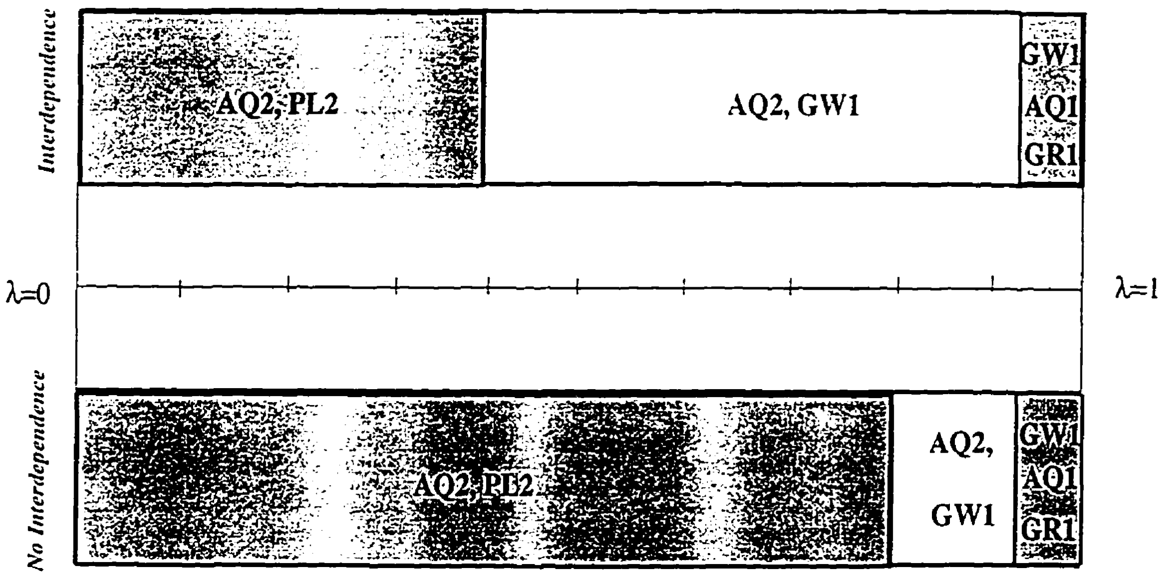


Figure 7.4: Weighted Case, the Selected Actions for Different Values of λ

and PL2.

Note that even though AQ2 is selected for both cases, the usage of this action is quite different. For the case of interdependence 17 and 41.5 percent of AQ2 capacity is used by rural area, in forth and fifth periods, respectively. However, in case of no interdependence the same amount of water in the same periods is utilized by Kitchener and Waterloo.

Moreover, Table 7.5 shows that Combined Archemedian - Chebyshev model produces several different GP non-dominated solutions with different properties. Aquifer Recharge is the only action which is recommended in all of the cases. The main reason for selecting Aquifer Recharge is that the investment cost of this action in comparison with other actions is quite low (see Table 7.3); implementation of this action and using a portion of its capacity is justifiable. An important practical observation of the solution is that if the planning horizon is extended, then other actions may be selected instead of Aquifer Recharge.

Note that as the value of λ increases and hence the objective function of GP model approaches a Chebyshev norm, the maximum deviation from the goals over all criteria is minimized. However, at the same time the sum of weighted deviations is increased substantially. The cost criterion has the maximum deviation in all of the situations. Table 7.5 also shows that moving from a pure weighted GP to a pure Chebyshev GP does not make substantial change in the maximum weighted deviations.

Table 7.6 and Figure 7.5 show the same information as in Table 7.5 and Figure 7.4 when an unweighted Chebyshev is used in the objective function. The solutions of the model when $0.15 \leq \lambda < 0.6$ are different for the cases of interdependence and no interdependence.

Table 7.6: WWSPP: Sets of Actions Selected and Deviations from Goal. (Unweighted Chebyshev)

λ	With Interdependence	Deviations	Without Interdependence	Deviations
$0.0 \leq \lambda < 0.15$	AQ2, PL2	(248,33,111, 110,60,5)	AQ2, PL2	(241,33,100, 120, 50, 5)
$0.15 \leq \lambda < 0.6$	AQ2, GW1	(236, 47,65, 70,65,60)	AQ2, PL2	(241,33,100, 120,50,5)
$0.6 \leq \lambda < 0.95$	AQ2, GW1	(236, 47,65, 80,65,60)	AQ2, GW1	(236, 47,65, 80,65,60)
$\lambda = 1$	GW1, AQ1, GR1	(234,96,105, 130,101,170)	GW1, AQ1, GR1	(234,87,105, 140,90,170)

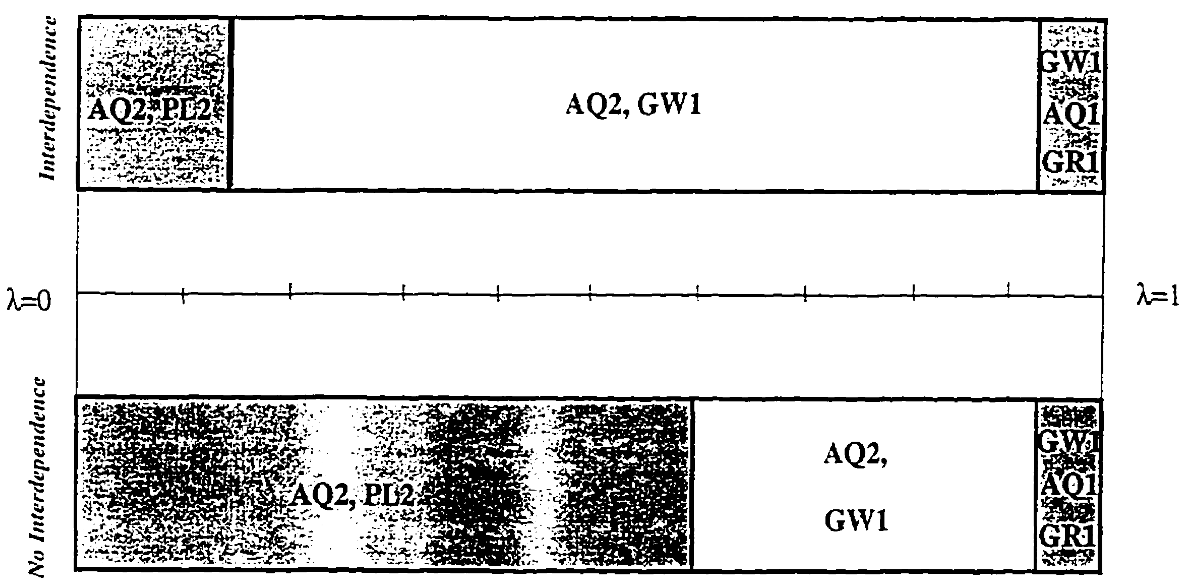


Figure 7.5: Unweighted case, the Selected Actions for Different Values of λ

Table 7.7 shows the percentages of water utilization of each selected action by each subregion, when $\lambda = .5$, interdependence is taking into account, and a weighted Chebyshev norm is used in the objective function. For this situation, new ground water sources have to be implemented from the early stage of planning. At the beginning, only a small portion of the capacity of this resource is used (12.5%), and gradually the usage is increased such that in the fourth and fifth period all the capacity of this action (10 $MIGD$) will be used. On the other hand, another selected action, AQ2, is only needed in the last two periods, and only 41% of its capacity will be utilized at the end of the planning horizon. As pointed out earlier, AQ2 is selected because of its low investment cost, even though the operating cost of this action is relatively high. Additionally, in this case, the analysis recommends that the whole capacity of old ground water and Grand River should be utilized: replacing them with new water sources is not justifiable. This is mainly because cost criterion has the highest priority over other criteria.

Table 7.8 shows information similar to that in Table 7.7 except that interdependence of actions is ignored. As it is shown in this table, the solution is quite different in comparison with the case in which interdependencies are taken into account. For this situation, a second pipeline option (PL2) is chosen instead of ground water. In the first period, only 12% of its capacity is utilized and gradually the usage is increased. In the fourth period, the entire capacity of action PL2 is used for Kitchener/Waterloo while in the fifth period it is assigned to Cambridge. Again, AQ is used partially for only the last two periods. Additionally, current water supply actions (ground water and Grand River) are completely being used, in all periods. Therefore, if the interdependence of actions is ignored, the solution changes dramatically.

Table 7.9 shows the solution of WWSPP, when the GP objective function is

Table 7.7: Optimal Water Supply Assignment for each Subregion (Interdependence Case, ($\lambda = .5$))

Periods	Regions	*Selected Actions*			
		AQ2	GW1	OGW	OGR
1996-2001	KW			0.693	0.217
	Cambridge		0.125	0.307	
	Rural				0.783
2002-2011	KW		0.51	0.586	
	Cambridge			0.411	
	Rural			0.002	1
2012-2021	KW			0.773	1
	Cambridge		0.87	0.059	
	Rural			0.168	
2022-2031	KW			0.848	1
	Cambridge		1	0.036	
	Rural	0.17		0.116	
2031-2041	KW			0.909	1
	Cambridge		1	0.068	
	Rural	0.415		0.023	

Table 7.8: Optimal Water Supply Assignment for each Subregion (No Interdependence Case, ($\lambda = .5$))

Periods	regions	*Selected Actions*			
		AQ2	PL2	OGW	OGR
1996-2001	KW			0.693	0.127
	Cambridge		0.125	0.307	
	Rural				0.783
2002-2011	KW			0.818	
	Cambridge		0.505	0.182	
	Rural		0.005		1
2012-2021	KW		0.87	0.377	1
	Cambridge			0.455	
	Rural			0.168	
2022-2031	KW	0.17	1	0.316	1
	Cambridge			0.491	
	Rural			0.193	
2032-2041	KW	0.415		0.726	1
	Cambridge		1	0.068	
	Rural			0.211	

Table 7.9: Optimal Water Supply Assignment for each Subregion (Interdependence Case, ($\lambda = 1$))

Periods	regions	*Selected Actions* GW1	AQ1	GR1	OGW	OGR
1996-2001	KW				0.586	1
	Cambridge				0.364	
	Rural	0.125			0.05	
2002-2011	KW	0.51			0.586	
	Cambridge				0.275	1
	Rural				0.139	
2012-2021	KW				0.773	1
	Cambridge	0.87			0.059	
	Rural				0.168	
2022-2031	KW				0.848	1
	Cambridge	1			0.036	
	Rural			0.68	0.139	
2032-2041	KW	0.51			0.909	1
	Cambridge	1			0.068	
	Rural		0.33	1	0.023	

purely a Chebyshev norm (*i.e.* $\lambda = 1$) and interdependence of actions is considered. In this case, three new actions are selected for the region. GW1, AQ1, and GR1. These new actions along with OGW and OGR provide a solution such that its maximum deviation from the target over all criteria is less than any other feasible solutions, even though the deviations from other criterion goals increase. Note that for this situation, GR1 is utilized only in last two periods and AQ1 is only needed partially in the last period.

Moreover, the GP solution is equivalent to the optimal solution of the single objective problem when minimizing cost is the only objective.

Table 7.10: Optimal Water Supply Assignment for each Subregion (No Interdependence Case, ($\lambda = 1$))

Periods	regions	*Selected Actions*				
		GW1	AQ1	GR1	OGW	OGR
1996-2001	KW	0.125			0.636	0.127
	Cambridge				0.364	
	Rural					0.783
2002-2011	KW				0.818	
	Cambridge	0.505			0.182	
	Rural	0.0055				1
2012-2021	KW				0.773	1
	Cambridge	0.5			0.227	
	Rural	0.37				
2022-2031	KW	1		0.68	0.316	1
	Cambridge				0.491	
	Rural				0.193	
2032-2041	KW	1	0.33	1	0.266	1
	Cambridge				0.523	
	Rural				0.211	

7.8 Applying the Modified Lexicographic GP to WWSPP

In this section we briefly explain the main results of applying the modified lexicographic approach to WWSPP. The priority and the level of goals for criteria are specified in the first and second steps as shown in Table 7.11. Given this information, we apply a sequential integer program to WWSPP and obtain (GW1,AQ1,GR1) as the best solution. This solution is efficient. Hence, according to the modified lexicographic approach, we must find another efficient solution by trading-off on the level of goals for different criteria.

Suppose the revised level of goals are as shown in row 5 in Table 7.11. Solving the problem using the new goals with the same priority on criteria gives (AQ2,PL3) as another efficient solution. Now, we must find other efficient solutions close to (GW1,AQ1,GR1) according to a distance metric. Assume that L_1 has been selected as an appropriate distance metric for measuring closeness. Hence,

$$\pi_1 = .05,\ \ \pi_2 = .03,\ \ \pi_3 = .55,\ \ \pi_4 = .08,\ \ \pi_5 = .043,\ \ \pi_6 = .015,\ \ \text{and } \beta = \frac{14.53}{h}.$$

Now the following program is solved for different value of λ to find some other efficient solutions:

$$\textbf{Maximize} \quad \sum_{p=1}^{|P|} \lambda_p \left(\sum_{i=1}^{|\mathbf{A}|} (FS)^t_{pi}\delta_i + \sum_{i=1}^{|\mathbf{A}|}\sum_{t=1}^{T} (VS)^t_{pi} \sum_{r=1}^{|\mathbf{R}|} x^t_{ir} + \sum_{j=1}^{|\mathbf{A}'|}\sum_{t=1}^{T} (VS)^t_{pj} \sum_{r=1}^{|\mathbf{R}|} y^t_{jr} + \sum_{k=2}^{|\mathbf{A}|} \sum_{S\in \mathbf{L}^k_p} \Delta_p(S)\cdot\left(\prod_{a_i\in S}\delta_i\right)\right),$$

Subject to :

$$\sum_{i=1}^{|\mathbf{A}|}(FS)^t_{pi}\delta_i + \sum_{i=1}^{|\mathbf{A}|}\sum_{t=1}^{T}(VS)^t_{pi}\sum_{r=1}^{|\mathbf{R}|}x^t_{ir} + \sum_{j=1}^{|\mathbf{A}|}\sum_{t=1}^{T}(VS)^t_{pj}\sum_{r=1}^{|\mathbf{R}|}y^t_{jr}$$

$$+\sum_{k=2}^{|\mathbf{A}|}\sum_{S\in \mathbf{L}^k_p}\Delta_p(S)\cdot\left(\prod_{a_i\in S}\delta_i\right) - \mathbf{z}^1_\mathbf{p} + u_p - u'_p = 0,$$

$$for \;\; p = 1, \ldots, |\, \mathbf{P}\,|,$$

$$\sum_{p=1}^{|\mathbf{P}|}\pi_p\,(u_p - v_p) \le \frac{14.53}{h},$$

The Set of Constraints 7.1 to 7.11.

In the above program $\mathbf{z}^1_\mathbf{p}$ is the criterion value of the first solution and u_p and u'_p are the auxiliary variables that defined in Chapter 6. Solving the above problem for $h = 1$ gives (AQ1. PL3) and (AQ1. PL2) as new supported efficient solutions.

Table 7.11: Main Steps of Applying the Modified Lexicographic GP to WWSPP (Interdependence Case. L_1 Norm)

Steps	*Tasks*	*Results*
1	Priority of Criteria	Inves. Cost. Oper. Cost, Risk, Water Quality. Inf. Impact, Env. Impact, Supply Capability
2	Initial Aspiration levels	(300. 40, 30, 40,30,40)
3	The Solution of the Modified Lexicographic GP	(GW1, AQ1, GR1)
4	Type of Solution	Efficient
5	The revised Aspiration levels	(400. 100, 100, 40, 20, 20)
6	The Efficient solution with the Revised Aspiration Levels	(AQ2, PL3)
7	The New Solutions of the Vector Optimization Problem	(AQ1, PL3) (AQ1, PL2)

When one selects L_∞ as distance metrics, the only efficient solution generated by solving the vector optimization problem is $(GW1, AQ2, GR1)$. This solution is similar to $(GW1. AQ1. GR1)$ which is generated in the second step.

Moreover, Table 7.12 shows the same information as in Table 7.11 when interdependence of actions is ignored.

Table 7.12: Main Steps of Applying the Modified Lexicographic GP to WWSPP (No Interdependence. L_1 Norm)

Steps	*Tasks*	*Results*
1	Priority of Criteria	Inves. Cost. Oper. Cost. Risk. Water Quality. Inf. Impact. Env. Impact. Supply Capability
2	Initial Aspiration levels	(300. 40. 30. 40.30.40)
3	The Solution of the Modified Lexicographic GP	(GW1. AQ1. GR1)
4	Type of Solution	Efficient
5	The revised Aspiration levels	(400. 100. 100. 40. 20. 20)
6	The Efficient solution with the Revised Aspiration Levels	(AQ1, GR1, PL3)
7	The New Solutions of the Vector Optimization Problem	(AQ1. PL3); (AQ2.PL2) (AQ2, PL3)

7.9 Conclusions

In this chapter. a real-world water supply planning problem was modeled as a multiple objective mixed integer programming problem. The main features of the problem, especially interdependence of actions, were discussed. It was shown that due to this characteristics conventional multiple criteria procedures would be difficult to apply. This case study showed the importance of interdependence of actions, even under moderate amount of interdependence. It was also demonstrated that the combined Chebyshev-Archemedian GP and the modified lexicographic approaches are useful to generate different attractive solutions. Finally, the solutions of the model in different cases were discussed and it was shown that the interdependence of actions should not be ignored.

Chapter 8

Contributions and Future Research

8.1 Main Contributions of the Thesis

The main contribution of this thesis is the development of novel definitions and characterizations of interdependence of actions in multiple criteria subset selection problems. Interdependence of actions was generalized to any number of actions and extended into set-interdependence. In fact, most conventional interdependence formulations can be defined as special cases of our definition. Furthermore, the subset selection problem under interdependence of actions was formulated as a multiple criteria zero-one problem, and two modified GP methodologies were proposed to find a representative subset of solutions. Motivated by the fact that most multiple criteria integer problems are difficult to solve, an approach was proposed to screen out those actions that cannot possibly be in the best set of actions. An application to the choice of future sources of municipal water supply for the Regional Municipality of Waterloo, Canada, shows the effectiveness of the proposed methodologies and associated analytical techniques. In summary, the main contributions of this thesis are as follow:

1. A new screening approach was developed for multiple criteria subset selection problems. Specifically, techniques were proposed to find and remove those individually dominated actions that cannot possibly be in the best set of actions. The techniques were utilized for both m-best actions and j-constraints problems.

2. Novel definitions and characterizations of interdependence of actions in multiple criteria subset selection problems were introduced. The interdependence was identified as conditional and unconditional, and the main differences of conditional and unconditional interdependence in comparison with conventional approaches were discussed.

3. The concept of interdependence of actions was generalized to interdependence of sets of actions and useful relationships between set-interdependence and action-interdependence were established.

4. Using relationships between interdependence of sets and interdependence of their subsets, several different approaches were proposed for evaluating the amount of interdependence among actions.

5. The concept of order of dependence was defined and a technique to distinguish the order of dependence of a set of interdependent actions was proposed. Moreover, the importance of identifying the order of dependence of a set of actions to establish useful relationships between independence of two sets and independence of their proper subsets were shown.

6. The concept of additivity of a set was introduced and it was shown that how to decompose a set such that the its consequence can be evaluated as an additive function of the consequences of the subsets in the partition, or such that the number of interdependence terms is minimized. Moreover, the main relationships between the notion of additivity of a set and independence of its subsets were explored.

7. A general procedure was introduced to formulate a subset selection problem under interdependence of actions. Subsequently, the difficulties of obtaining the non-dominated solutions of the formulated problem were put forward and the following two modified GP methodologies were developed to find an attractive subset of efficient solutions:

    - Modified lexicographic GP - The lexicographic GP and vector optimization approaches were employed in sequence to find a portion of efficient solutions that most likely include the best solution according to the DM's value function.

    - Hybrid Chebyshev-Archemedian GP - Difficulties of Archemedian and Chebyshev GP were discussed and a GP approach according to the convex combination of Archemedian and Chebyshev GP was proposed. It was shown that this approach overcomes some of the shortcomings of the conventional GP.

8. The proposed concepts and methodologies were applied to a real-world water supply planning problem. The problem was formulated as a mixed integer programming model. The solutions of the problem demonstrated the importance of considering interdependence of actions in subset selection problem under moderate amount of interdependence. Moreover, the effectiveness of

the combined Chebyshev-Archemedian GP and the modified lexicographic GP approaches in generating different GP-efficient solutions for the WWSPP were demonstrated.

8.2 Suggestions for Future Research

The research contained in this thesis opens up a range of new avenues for future productive research. The following are some suggested areas of research:

1. To expand the screening approach for subset selection problems under interdependence of actions. The screening approach presented in Chapter 3 was restricted to the case when all actions are independent. The theory and methodology of screening actions in subset selection problems can be appropriately developed for handling decision situations for which there exist interdependencies among actions.

2. To modify and utilize other screening approaches for subset selection problems. In this thesis, we analyzed the dominance method for screening actions. One can adapt other screening approaches which were originally designed for multiple criteria single action selection, such as successive elimination and bounding the performance for subset choice problems. This should be accomplished for both m-best actions and j-constraints problems. Due to the inherent difficulty of combinatorial problems, it is quite useful to find methods for screening actions and removing those that cannot be included in the best or efficient subsets of actions.

3. To apply the screening approaches to other MCZO problems. In this thesis we restricted the implementation of our screening approach to a multidimensional

knapsack problem. The proposed concepts and techniques can be applied to other types of MCZO problems.

4. To investigate the computational complexity of the screening approach. The screening approach presented in chapter 3 is considered as a pre-processing stage in solving a multiple criteria subset selection problem. Hence, it is useful to discuss the complexity of the screening approach to justify its utilization as a pre-processing stage.

5. To integrate the screening approach with some well-known MCDM interactive approach. In an MCDM interactive approach, the DM's preference information is used to remove actions that cannot be optimal. This is usually referred to as dominance in the *reduced decision space*, because the decision space is made smaller using the DM's preference information. Most of these approaches are suitable for single action selection. The methods of Koksalan *et al.* [71] and Korhenon *et al.* [74] are among interactive MCDM approaches, that can be integrated into the concept of screening presented in Chapter 3. More specifically, the *cone of inferior solutions* [71, 74] can be built such that every action in the cone cannot possibly be in the best subset of actions.

6. To explore further the notion of interdependence in MCDM subset selection. Establishing more useful relationships between interdependence of sets and interdependence of their proper subsets can be quite useful to evaluate interdependent sets.

7. To investigate the notion of interdependence of actions for ordinal preference information. The theory of interdependence introduced in this thesis is confined to the situations where there exists cardinal information on consequence

and amount of interdependence of actions. However, in many real-world problems only ordinal information on interdependence is available. The definition and characterization of interdependence can be expanded to take into account ordinal and qualitative information, as well as a combination of qualitative and quantitative preferences.

8. To explore possible domain of applications of the theory of interdependence in real-world decision making problems.

9. To employ more concepts of graph theory for exhibiting other properties of interdependence of actions. In Chapter 5, some elementary concepts of graph theory were used to explore features of interdependence of actions. More concepts from the theory of graph can be used to capture its other properties. For example, the hypergraph notion can be utilized to characterize the interdependence of the order of three or more.

10. To modify the conventional GP methodology so that in addition to entering a penalty for under-achievement, one can input rewards for over-achievement of the solution. This eliminates some of the difficulties in GP such as generation of dominated solutions.

11. To integrate screening approaches with the interdependence notion within the framework of the GP methodology. In other words, screening actions can be accomplished in reference to aspiration levels and in the presence of interdependence of actions.

12. To construct a more detailed model for WWSPP such that other technical information along with the stochastic nature of the problem, can be incorporated into the model.

13. To implement other techniques presented in this thesis into WWSPP problem. In Chapter 7 we showed the results of implementation of the elementary concept of interdependence, the combined Chebyshev-Archemedian method and the modified lexicographic method into the WWSPP problem. It is useful to apply other proposed concepts and techniques in this thesis, such as screening approach into WWSPP problem.

14. To extend WWSPP problem such that both qualitative and quantitative information can be taken into account. Like other real-world multiple criteria problems. WWSPP contains both qualitative and quantitative information. Developing a model for WWSPP so that both kinds of information can be handled would be very useful.

15. To implement a comprehensive sensitivity analysis for WWSPP problem. A comprehensive sensitivity analysis is required to investigate how the solutions of WWSPP are influenced due to changes in model parameter. Designing a systematic sensitivity analysis is especially important in WWSPP model, because in this research we have considered only a deterministic model. Moreover, due to the subjectivities involved in assigning weights and aspiration levels in GP. it is quite important to find sensitivity of a solution when the GP parameters are changed.

Appendix A

Proofs of Theorems

A.1 Proof of Theorem 4.1

1)Proof that

$$\mathbf{A}_1(\mathbf{I} \mid \mathbf{A}^0)\mathbf{A}_2 \quad \Longrightarrow \quad \mathbf{A}_1(\mathbf{I} \mid \mathbf{A}^{0'})\mathbf{A}_2$$

Since $\mathbf{A}_1(\mathbf{I} \mid \mathbf{A}^0)\mathbf{A}_2$, we have

$$c_p(\mathbf{A}_1 \cup \mathbf{A}^0) - c_p(\mathbf{A}^0) = c_p(\mathbf{A}_1 \cup \mathbf{A}^0 \cup \mathbf{A}_2) - c_p(\mathbf{A}^0 \cup \mathbf{A}_2). \tag{A.1}$$

But $\mathbf{A}_1(\mathbf{I} \mid \mathbf{A}^{0'})\mathbf{B}$ implies that

$$c_p(\mathbf{A}_1 \cup \mathbf{A}^{0'} \cup \mathbf{B}) - c_p(\mathbf{A}^{0'} \cup \mathbf{B}) = c_p(\mathbf{A}_1 \cup \mathbf{A}^{0'}) - c_p(\mathbf{A}^{0'}). \tag{A.2}$$

However, $\mathbf{A}^{0'} \cup \mathbf{B} = \mathbf{A}^0$, because $\mathbf{B} = \mathbf{A}^0 \setminus \mathbf{A}^{0'}$. Thus A.2 is equivalent to

$$c_p(\mathbf{A}_1 \cup \mathbf{A}^0) - c_p(\mathbf{A}^0) = c_p(\mathbf{A}_1 \cup \mathbf{A}^{0'}) - c_p(\mathbf{A}^{0'}). \tag{A.3}$$

On the other hand, $\mathbf{A}_2(\mathbf{I} \mid \mathbf{A}^{0'})\mathbf{B}$ gives

$$c_p(\mathbf{A}_2 \cup \mathbf{A}^{0'}) - c_p(\mathbf{A}^{0'}) = c_p(\mathbf{A}_2 \cup \mathbf{A}^{0'} \cup \mathbf{B}) - c_p(\mathbf{A}^{0'} \cup \mathbf{B}), \tag{A.4}$$

or

$$c_p(\mathbf{A}_2 \cup \mathbf{A}^0) = c_p(\mathbf{A}_2 \cup \mathbf{A}^{0'}) - c_p(\mathbf{A}^{0'}) + c_p(\mathbf{A}^0). \tag{A.5}$$

In addition. $(\mathbf{A}_1 \cup \mathbf{A}_2)(\mathbf{I} \mid \mathbf{A}^{0'})\mathbf{B}$ implies that

$$c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}^{0'}) - c_p(\mathbf{A}^{0'}) = c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}^{0'} \cup \mathbf{B}) - c_p(\mathbf{A}^{0'} \cup \mathbf{B}), \tag{A.6}$$

or

$$c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}^0) = c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}^{0'}) - c_p(\mathbf{A}^{0'}) + c_p(\mathbf{A}^0). \tag{A.7}$$

Substituting the right hand sides of (A.2), (A.3).(A.5). into (A.1).

$$\begin{aligned} c_p(\mathbf{A}_1 \cup \mathbf{A}^{0'}) - c_p(\mathbf{A}^{0'}) &= c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}^{0'}) - c_p(\mathbf{A}^{0'}) \\ &\quad c_p(\mathbf{A}^0) - c_p(\mathbf{A}_2 \cup \mathbf{A}^{0'}) + c_p(\mathbf{A}^{0'}) - c_p(\mathbf{A}^0). \end{aligned} \tag{A.8}$$

Simplifying and rewriting A.8 gives,

$$c_p(\mathbf{A}_1 \cup \mathbf{A}^{0'}) - c_p(\mathbf{A}^{0'}) = c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}^{0'}) - c_p(\mathbf{A}_2 \cup \mathbf{A}^{0'}), \tag{A.9}$$

or

$$\mathbf{A}_1(\mathbf{I} \mid \mathbf{A}^{0'})\mathbf{A}_2.$$

2)Proof that

$$\mathbf{A}_1(\mathbf{I} \mid \mathbf{A}^{0'})\mathbf{A}_2 \Longrightarrow \mathbf{A}_1(\mathbf{I} \mid \mathbf{A}^0)\mathbf{A}_2.$$

The proof is similar to the necessary condition and can be obtained by rearranging (A.3),(A.5), (A.7), and substituting into (A.9). □

A.2 Proof of Theorem 4.2:

1)Proof that

$$\mathbf{A}_1(\mathbf{I} \mid \emptyset)(\mathbf{A}_2 \cup \mathbf{A}_3) \Longrightarrow \mathbf{A}_1(\mathbf{I} \mid \mathbf{A}_2)\mathbf{A}_3.$$

Since $\mathbf{A}_1(\mathbf{I} \mid \emptyset)(\mathbf{A}_2 \cup \mathbf{A}_3)$. we have

$$c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}_3) = c_p(\mathbf{A}_1) + c_p(\mathbf{A}_2 \cup \mathbf{A}_3). \tag{A.10}$$

Since $\mathbf{A}_1(\mathbf{I} \mid \emptyset)\mathbf{A}_2$,

$$c_p(\mathbf{A}_1) = c_p(\mathbf{A}_1 \cup \mathbf{A}_2) - c_p(\mathbf{A}_2). \tag{A.11}$$

Substituting (A.11) into (A.10) and rearranging,

$$c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}_3) - c_p(\mathbf{A}_2 \cup \mathbf{A}_3) = c_p(\mathbf{A}_1 \cup \mathbf{A}_2) - c_p(\mathbf{A}_2). \tag{A.12}$$

Hence,

$$\mathbf{A}_1(\mathbf{I} \mid \mathbf{A}_2)\mathbf{A}_3.$$

2) Proof of sufficient condition is similar to the necessary condition. □

A.3 Proof of Theorem 4.3:

Since $\mathbf{A}_1(\mathbf{I} \mid \mathbf{A}_2 \cup \mathbf{A}_3 \ldots \cup \mathbf{A}_{n-1})\mathbf{A}_n$,

$$c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \ldots \mathbf{A}_{n-1}) - c_p(\mathbf{A}_2 \cup \ldots \mathbf{A}_{n-1}) = c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \ldots \mathbf{A}_{n-1} \cup \mathbf{A}_n) - c_p(\mathbf{A}_2 \cup \ldots \mathbf{A}_{n-1} \cup \mathbf{A}_n). \quad \text{(A.13)}$$

Define

$$\mathbf{Y} = c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \ldots \mathbf{A}_{n-1}) - c_p(\mathbf{A}_2 \cup \ldots \mathbf{A}_{n-1}).$$

Since $\mathbf{A}_1(\mathbf{I} \mid \mathbf{A}_2 \cup \mathbf{A}_3 \ldots \cup \mathbf{A}_{n-2})\mathbf{A}_{n-1}$,

$$\mathbf{Y} = c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \ldots \mathbf{A}_{n-2}) - c_p(\mathbf{A}_2 \cup \ldots \mathbf{A}_{n-2}).$$

Similarly, becuase $\mathbf{A}_1(\mathbf{I} \mid \mathbf{A}_2 \cup \mathbf{A}_3 \ldots \cup \mathbf{A}_{n-3})\mathbf{A}_{n-2}$,

$$\mathbf{Y} = c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \ldots \mathbf{A}_{n-3}) - c_p(\mathbf{A}_2 \cup \ldots \mathbf{A}_{n-3}).$$

With successive substitution in the same fashion,

$$\mathbf{Y} = c_p(\mathbf{A}_1 \cup \mathbf{A}_2) - c_p(\mathbf{A}_2),$$

and finally since $\mathbf{A}_1(\mathbf{I} \mid \emptyset)\mathbf{A}_2$,

$$\mathbf{Y} = c_p(\mathbf{A}_1).$$

Substituting $\mathbf{Y}$ into the left hand side of (A.13),

$$c_p(\mathbf{A}_1) = c_p(\mathbf{A}_1 \cup \mathbf{A}_2 \cup \ldots \mathbf{A}_{n-1} \cup \mathbf{A}_n) - c_p(\mathbf{A}_2 \cup \ldots \mathbf{A}_{n-1} \cup \mathbf{A}_n).$$

Hence,

$$\mathbf{A}_1(\mathbf{I} \mid \emptyset)\mathbf{A}_2 \cup \mathbf{A}_3 \ldots \cup \mathbf{A}_{n-1} \cup \mathbf{A}_n . \Box$$

A.4 Proof of Theorem 5.1

We prove the theorem by induction. We first show that Expression 5.8 is true for $O_p(\mathbf{S}) \leq 2$.

According to (5.7), when $O_p = 2$, we have

$$c_p(\mathbf{S}) = \sum_{i \in \mathbf{S}} c_p(i) + \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}|=2}} \Delta_p(\mathbf{T}). \tag{A.14}$$

Finding the value of $\Delta_p(\mathbf{T})$ from (5.4) and substituting it into (A.14) gives:

$$\begin{aligned} c_p(\mathbf{S}) &= \sum_{i \in \mathbf{S}} c_p(i) + \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}|=2}} \left(c_p(\mathbf{T}) - \sum_{a_i \in \mathbf{T}} c_p(i) \right), \\ &= \sum_{i \in \mathbf{S}} c_p(i) + \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}|=2}} c_p(\mathbf{T}) - (|\,\mathbf{S}\,| - 1) \sum_{a_i \in \mathbf{S}} c_p(i), \\ &= \sum_{\substack{\mathbf{T} \subseteq \mathbf{S} \\ |\mathbf{T}|=2}} c_p(\mathbf{T}) - (|\,\mathbf{S}\,| - 2) \sum_{a_i \in \mathbf{S}} c_p(i). \end{aligned}$$

Hence, the theorem is true for $O_p(\mathbf{S}) = 2$. Now we show that if the theorem is true for k, it is also true for $k+1$. For this purpose, first we replace all k in Expression 5.8 by $k+1$, and then we prove that the resulting expression is true. Replacing k by $k+1$ in (5.8) gives.

$$\begin{aligned}
c_p(\mathbf{S}) &= \sum_{\substack{\mathbf{T}\subseteq \mathbf{S}\\ |\mathbf{T}|=k+1}} c_p(\mathbf{T}) - (|\,\mathbf{S}\,| -(k+1)) \sum_{\substack{\mathbf{T}\subseteq \mathbf{S}\\ |\mathbf{T}|=k}} c_p(\mathbf{T}) \\
&+ \left(\frac{1}{2!}\right)[(|\,\mathbf{S}\,| -(k+1))(|\,\mathbf{S}\,| -k)] \sum_{\substack{\mathbf{T}\subseteq \mathbf{S}\\ |\mathbf{T}|=k-1}} c_p(\mathbf{T}) + \cdots \\
&\pm \left(\frac{1}{(k-1)!}\right)[(|\,\mathbf{S}\,| -(k+1))(|\,\mathbf{S}\,| -k)\cdots(|\,\mathbf{S}\,| -3)] \sum_{\substack{\mathbf{T}\subseteq \mathbf{S}\\ |\mathbf{T}|=2}} c_p(\mathbf{T}) \\
&\pm \left(\frac{1}{k!}\right)[(|\,\mathbf{S}\,| -(k+1))(|\,\mathbf{S}\,| -k)\cdots(|\,\mathbf{S}\,| -2)] \sum_{a_i\in \mathbf{S}} c_p(i), \qquad k \leq |\,\mathbf{S}\,| .
\end{aligned} \tag{A.15}$$

On the other hand according to (5.7), if $O_p(\mathbf{S}) = k+1$, then

$$\begin{aligned}
c_p(\mathbf{S}) &= \sum_{i\in \mathbf{S}} c_p(i) + \sum_{\substack{\mathbf{T}\subseteq \mathbf{S}\\ |\mathbf{T}|\leq k+1}} \Delta_p(\mathbf{T}). \\
&= \sum_{i\in \mathbf{S}} c_p(i) + \sum_{\substack{\mathbf{T}\subseteq \mathbf{S}\\ |\mathbf{T}|\leq k+1}} \Delta_p(\mathbf{T}) + \sum_{\substack{\mathbf{T}\subseteq \mathbf{S}\\ |\mathbf{T}|= k+1}} \Delta_p(\mathbf{T}).
\end{aligned} \tag{A.16}$$

But, according to (5.3). the last term in right hand side of (A.16) can be written as follows:

$$\sum_{\substack{\mathbf{T}\subseteq \mathbf{S}\\ |\mathbf{T}|=k+1}} \Delta_p(\mathbf{T}) = \sum_{\substack{\mathbf{T}\subseteq \mathbf{S}\\ |\mathbf{T}|=k+1}} \left(c_p(\mathbf{T}) - \sum_{\substack{\mathbf{T}\subseteq \mathbf{S}\\ |\mathbf{T}|=k}} c_p(\mathbf{T}) + \sum_{\substack{\mathbf{T}\subseteq \mathbf{S}\\ |\mathbf{T}|=k-1}} c_p(\mathbf{T}) \right.$$

$$+\cdots+(-1)^k \sum_{a_i \in S} c_p(i) \Bigg) .$$

The above expression can be rewritten as,

$$\sum_{\substack{T \subseteq S \\ |T|=k+1}} \Delta_p(T) = \sum_{\substack{T \subseteq S \\ |T|=k+1}} c_p(T) - (|S| - k) \sum_{\substack{T \subseteq S \\ |T|=k}} c_p(T)$$

$$+\binom{|S|-(k-1)}{2} \sum_{\substack{T \subseteq S \\ |T|=k-1}} c_p(T) - \binom{|S|-(k-2)}{3} \sum_{\substack{T \subseteq S \\ |T|=k-2}} c_p(T)$$

$$+\cdots+(-1)^k \binom{|S|-1}{k-1} \sum_{a_i \in S} c_p(i). \tag{A.17}$$

Now, substituting the amount of $\sum_{\substack{T \subseteq S \\ |T|=k+1}} \Delta_p(T)$ from (A.17), and amount of $\sum_{\substack{T \subseteq S \\ |T| \le k}} \Delta_p(T)$ from (5.8) into (A.16), and rearranging produces,

$$\begin{aligned} c_p(S) &= \sum_{\substack{T \subseteq S \\ |T|=k}} c_p(T) - (|S| - k + 1) \sum_{\substack{T \subseteq S \\ |T|=k}} c_p(T) \\ &+ \left(-|S| + k + \binom{|S|-(k-1)}{2}\right) \sum_{\substack{T \subseteq S \\ |T|=k-1}} c_p(T) \\ &- \left(\binom{|S|-(k-2)}{3} + \frac{1}{2!}(|S| - k)(|S| - (k-1))\right) \sum_{\substack{T \subseteq S \\ |T|=k-2}} c_p(T) + \dots \end{aligned}$$

$$\pm \left((|\mathbf{S}| - k)(|\mathbf{S}| - (k-1)) \cdots (|\mathbf{S}| - 2) + \binom{|\mathbf{S}| - 1}{k-1} \right) \sum_{a_i \in \mathbf{S}} c_p(i). \quad (A.18)$$

By simplifying the coefficient in (A.18), one can show that (A.18) and (A.15) are equivalent and this complete the proof. □

A.5 Proof of Expression 5.18

According to (4.2), when $\mathbf{A}^0 = \emptyset$

$$\phi_p(a_i, \mathbf{S}_k) = c_p(a_i \cup \mathbf{S}_{k-t} \cup \mathbf{S}_t) - c_p(i) - c_p(\mathbf{S}_{k-t} \cup \mathbf{S}_t). \quad (A.19)$$

But, using (5.6) gives,

$$\begin{aligned} c_p(a_i \cup \mathbf{S}_{k-t} \cup \mathbf{S}_t) &= c_p(i) + \sum_{j \in S_{k-t}} c_p(j) + \sum_{m \in S_t} c_p(m) + \\ &\sum_{\substack{\mathbf{T}_1 \subseteq \mathbf{S}_{k-t} \\ |\mathbf{T}| \geq 2}} \Delta_p(\mathbf{T}_1) + \sum_{\substack{\mathbf{T}_2 \subseteq \mathbf{S}_t \\ |\mathbf{T}_2| \geq 2}} \Delta_p(\mathbf{T}_2) \\ &+ \sum_{\emptyset \neq \mathbf{T}_1 \subseteq \mathbf{S}_{k-t}} \Delta_p(\{a_i\} \cup \mathbf{T}_1\}) + \sum_{\emptyset \neq \mathbf{T}_2 \subseteq \mathbf{S}_t} \Delta_p(\{a_i\} \cup \mathbf{T}_2) \\ &+ \sum_{\emptyset \neq \mathbf{T}_1 \subseteq \mathbf{S}_{k-t}} \sum_{\emptyset \neq \mathbf{T}_2 \subseteq \mathbf{S}_t} \Delta_p(\{a_i\} \cup \mathbf{T}_1 \cup \mathbf{T}_2\}) + \sum_{\emptyset \neq \mathbf{T}_1 \subseteq \mathbf{S}_{k-t}} \sum_{\emptyset \neq \mathbf{T}_2 \subseteq \mathbf{S}_t} \Delta_p(\mathbf{T}_1 \cup \mathbf{T}_2\}), \end{aligned} \quad (A.20)$$

and

$$c_p(\mathbf{S}_{k-t} \cup \mathbf{S}_t) = \sum_{j \in S_{k-t}} c_p(j) + \sum_{m \in S_t} c_p(m) + \sum_{\substack{\mathbf{T}_1 \subseteq \mathbf{S}_{k-t} \\ |\mathbf{T}| \geq 2}} \Delta_p(\mathbf{T}_1)$$

$$+ \sum_{\substack{T_2 \subseteq S_t \\ |T| \geq 2}} \Delta_p(\mathbf{T}_2) + \sum_{\emptyset \neq \mathbf{T}_1 \subseteq \mathbf{S}_{k-t}} \sum_{\emptyset \neq \mathbf{T}_2 \subseteq \mathbf{S}_t} \Delta_p(\mathbf{T}_1 \cup \mathbf{T}_2). \tag{A.21}$$

Substituting (A.20) and (A.21) into (A.19),

$$\phi_p(a_i, \mathrm{S}_k) = \sum_{\emptyset \neq \mathbf{T}_1 \subseteq \mathbf{S}_{k-t}} \Delta_p(\{a_i\} \cup \mathbf{T}_1\}) + \sum_{\emptyset \neq \mathbf{T}_2 \subseteq \mathbf{S}_t} \Delta_p(\{a_i\} \cup \mathbf{T}_2)$$
$$+ \sum_{\emptyset \neq \mathbf{T}_1 \subseteq \mathbf{S}_{k-t}} \sum_{\emptyset \neq \mathbf{T}_2 \subseteq \mathbf{S}_t} \Delta_p(\{a_i\} \cup \mathbf{T}_1 \cup \mathbf{T}_2\}),$$

or

$$\phi_p(a_i, \mathrm{S}_k) = \phi_p(a_i, \mathrm{S}_{k-t}) + \sum_{\mathbf{T}_1 \subseteq \mathbf{S}_{k-t}} \sum_{\emptyset \neq \mathbf{T}_2 \subseteq \mathbf{S}_t} \Delta_p(\{a_i\} \cup \mathbf{T}_1 \cup \mathbf{T}_2). \square$$

(Note that the summation includes a $\mathbf{T}_1 = \emptyset$ term.)

A.6 Proof of Theorem 5.4

According to (4.2) when $\mathbf{A}^0 = \emptyset$

$$\phi_p(\mathrm{S}_1 \cup \mathrm{S}_2) = c_p(\mathrm{S}_1 \cup \mathrm{S}_2) - c_p(\mathrm{S}_1) - c_p(\mathrm{S}_2). \tag{A.22}$$

Similar to Appendix A.5, we have

$$c_p(\mathrm{S}_1 \cup \mathrm{S}_2) = \sum_{i \in S_1} c_p(i) + \sum_{j \in S_2} c_p(j) + \sum_{\substack{T_1 \subseteq S_1 \\ |T_1| \geq 2}} \Delta_p(\mathbf{T}_1) + \sum_{\substack{T_2 \subseteq S_2 \\ |T_2| \geq 2}} \Delta_p(\mathbf{T}_2)$$
$$+ \sum_{\emptyset \neq \mathbf{T}_1 \subseteq \mathbf{S}_1} \sum_{\emptyset \neq \mathbf{T}_2 \subseteq \mathbf{S}_2} \Delta_p(\mathbf{T}_1 \cup \mathbf{T}_2). \tag{A.23}$$

Substituting (A.23), $c_p(\mathbf{S}_1)$ and $c_p(\mathbf{S}_2)$ into (A.22) gives

$$\phi_p(\mathbf{S}_1, \mathbf{S}_2) = \sum_{\emptyset \neq \mathbf{T}_1 \subseteq \mathbf{S}_1} \sum_{\emptyset \neq \mathbf{T}_2 \subseteq \mathbf{S}_2} \Delta_p(\mathbf{T}_1 \cup \mathbf{T}_2). \square \quad (A.24)$$

Bibliography

[1] Aaker, D.A. and Tyebjee, T.T. [1978] "A Model for the Selection of Interdependent R&D Projects", *IEEE Transactions on Engineering Management*, Vol. EM-25, pp. 30-36.

[2] Abu-Taleb,M.F. and Mareschal, B. [1995] "Water resources Planning in the Middle East: Application of the PROMETHEE V Multi-criterion Method", *European Journal of Operational Research*, Vol. 81, pp. 500-511.

[3] Arrow, K. and Raynaud, H. [1986] *Social Choice and Multi-criteria Decision-making*, MIT Press, Cambridge, Mass.

[4] Associated Engineering [1994] *Regional Municipality of Waterloo, Long Term Water Strategy, Phase 1 Report.*

[5] Athanassopoulos, A.D. and Podinovski, V.V. [1997] "Dominance and Potential Optimality in Multiple Criteria Decision Analysis with Imprecise Information", *Journal of the Operational Research Society*, Vol. 48, pp. 142-150.

[6] Bana e Costa, C. [1990a] "An Additive Value Function Technique with a Fuzzy Outranking Relation for Dealing with Poor Inter-criteria Preference

Information", In Reading in Multiple Criteria Decision Aid, Bana e. Costa (ed.), Springer-verlag, Berlin, pp. 383,412.

[7] Bana e Costa, C.(ed.)[1990b] *Reading in Multiple Criteria Decision Aid*, Springer-verlag, Berlin.

[8] Benayoun, R., Demontgolfier, J., Tergny, J., and Lartitchev, O. [1971] " Linear programming with multi-objective function: Step method (STEM)", *Mathematical Programming*, 1, pp. 366-375.

[9] Bitran, Gabriel R. [1977] "Linear Multiple Objective Programs With Zero-One Variables", *Mathematical Programming*, 13, pp. 121-139.

[10] Bitran, Gabriel R. [1979] "Theory and Algorithms for Linear Multiple Objective Programs With Zero-One Variables", *Mathematical Programming* 17, pp. 362-390.

[11] Bitran, Gabriel R. and Lawrence, Kenneth D. [1980] "Locating Service Offices: a Multicriteria Approach", *OMEGA*, Vol. 8, No. 2, pp. 201 - 206.

[12] Bitran, Gabriel R. and Rivera, Jorge M. [1982] "A Combined Approach to Solve Binary Multicriteria Problems", *Naval Research Logistics Quarterly*, Vol. 29, No. 2, pp. 181-201.

[13] Bouyssou, D. [1990] "Building Criteria: A Prerequisite for MCDA," In *Reading in Multiple Criteria Decision Aid*, Bana e Costa(ed.), Springer-Verlag, Berlin, pp. 58-80.

[14] Bouyssou, D., Perny, P., Pirlot, M., Tsoukias,A. and Vincke, P. [1993] "A Manifesto for the New MCDA Era", *Journal of Multi-Criteria Decision Analysis*, vol. 2, pp. 125-127.

[15] Brans, J.P. and Mareshall, B. [1992] "PROMETHEE V: MCDM Problems with Segmentation Constraints", *INFOR*, 30:2, 85-96.

[16] Brualdi, R. A.[1992], *Introductory Combinatorics*, 2nd Edition, North-Holland, New York.

[17] Burkard, R., Krarup, J., and Pruzan, P. [1982], "Efficiency and Optimality in Minisum, Minimax 0-1 Programmming Problems", *Journal of Operational research Society*, Vol. 33. pp. 137-151.

[18] Chankong, V.. Haimes, Y.Y.. Thadathil, J. and Zionts, S. [1984] "Multiple Criteria Optimization: A State of the Art Review". In *Decision making with multiple objectives*, Haimes, Y. Y. and Chankong, V. (eds.), pp. 36-90.

[19] Charnes. A. and Cooper, W. W. [1961] *Management Models and Industrial Applications of Linear Programming*, Wiley. New York.

[20] Cochrane, James L. and Zeleny, Milan (eds.) [1973] *Multiple Criteria Decision Making*, University of South Carolina Press, Columbia.

[21] Cohon, J.L., and Marks, D.H. [1973] "Multi-objective Screening Models and Water Resources Investment", *Water Resources Research*, 9, pp. 826-836.

[22] Cohon, J.L., Marks, D.H. [1975] "A Review and Evaluation of Multi-objective Programming Techniques" *Water Resources research*, 11, pp. 208-220.

[23] Cohn, [1980] *Measure Theory*, Birkhauser, Boston.

[24] Deckro, R.F. and Winkofsky, E.P. [1983] "Solving Zero-One Multiple Objective Programs Through Implicit Enumeration", *European Journal of Operational Research*, 12 (4), pp. 362-374.

[25] Dyer, J.S., Fishburn, P.C., Steuer, R. S., Wallenius J. and Zionts S. [1992] "Multiple Criteria Decision Making, Multi Attribute Utility Theory: The Next Ten Years". *Management Sciences*, Vol.38, No.5, pp. 645-654.

[26] Eilon, S. [1971] "Goals and Constraints in Decision Making", *Operational Research Quarterly*, Vol. 23, No. 1, pp. 3-16.

[27] Enthoven, A. C. [1975] "Ten Practical Principles for policy and Program Analysis", In *Benefit Cost and Policy Analysis*, 1974, S. Zeckhauser *et al.*, (eds.), pp. 456-465.

[28] Evan, G.[1984] "An Overview of Techniques for Solving Multi Objective Mathematical Programs", *Management Sciences* 30(11), pp. 1268-1282.

[29] Evans, G.W. and Fairbairn, R. [1989] "Selection and Scheduling of Advanced Missions for NASA Using 0-1 Integer Linear Programming", *Journal of the Operational Research Society*, Vol.40, No. 11, pp. 971-981.

[30] Fishburn, P. C. [1970] *Utility Theory For Decision Making*, Wiley, New York.

[31] Fishburn, P.C., and LaValle I.H. [1996] "Binary Interactions and Subset Choice", *European Journal of Operational Research*, 92, 182-192.

[32] French, Simon [1986] *Decision Theory; An Introduction to the Mathematics of Rationality*, Ellis Horwood Series in Mathematics and Its Application, Chichester.

[33] Friend, J., and A. Hickling [1987] *Planning Under Pressure: The Strategic Choice Approach*, Pergamon Press, Oxford, England.

[34] Gabani, D. and Magazine, M. [1986] "An Interactive Heuristic Approach for Multi-Objective Integer-Programming Problems", *Journal of the Operational Research Society*, Vol. 37, No. 3, pp. 285-291.

[35] Geoffrion, A. [1968] "Proper Efficiency and Theory of Vector Maximization", *Journal of Mathematical Analysis and Applications*, No. 22. pp. 618 - 630.

[36] Geoffrion, A. M., Dyer, J.S., and Feinberg, A. [1972] "An Interactive Approach for Multi-criteria Optimization, with an Application to Operation of an Academic Department". *Management Science*. Vol. 19. No. 4. pp. 357-368.

[37] Gershon, Mark and Duckstein, Lucien [1982] "An Algorithm For Choosing A Multi-objective Technique". *In Essay and Survey on MCDM*, Hansen, P. (ed.), Springer-Verlag, Berlin.

[38] Glover, F.and Woolsey, E. [1974] "Converting the 0-1 Polynomial Programming Problem to a 0-1 Linear Program". *Operations Research*. 22/1. pp 180-182.

[39] Goicoechea, A., D. R. Hansen and L. Ducksten [1982] *Multi-objective Decision Analysis with Engineering and Business Applications*, John Wiley and Sons, New York.

[40] Goicoechea, A., E. Z. Stakhiv and F. Li [1992] "Experimental Evaluation of Multiple Criteria Decision Models for Application to Water Resources Planning", *Water Resources Bulletin*, Volume 28, No. 1, K.W. Hipel (ed.), Bethesda, Maryland.

[41] Goicoechea, A. [1989] *On the Use of Multiple Attributes and Imperfect Information with ARIADNE to Rank Alternative Systems*, Working Paper, Information Systems and Systems Engineering Department, George Mason University, Fairfax.

[42] Gomes, Luiz F.A. [1992] "Modelling Interdependencies Among Urban Transportation System Alternatives within A Multi-criteria Ranking Framework", *Journal of Advanced Transportation*. Vol. 24, No. 1, pp. 77-85.

[43] Haimes, Y.Y.. Hall. W. A.. and Freedman. H. T. [1975] *Multi-objective Optimization in Water Resources Systems: The Surrogate Worth Trade-Off Method*, Elsevier. Amsterdam.

[44] Halpern [1978] "Finding Minimal Center-Median Convex Combination of a Graph", *Management Science*. 24, pp. 535-544.

[45] Halpern [1980] "Duality in the Cent-Median of a Graph, *Operations Research*. 28, pp. 722-735.

[46] Hannan. E. L. [1980] "Non-dominance in Goal Programming", *INFOR. Canadian Journal of Operational Research and Information Processing*, Vol. 18, No. 4. pp. 300-309.

[47] Hazen, Gordon B. [1986] "Partial information, Dominance, and Potential Optimality in Multi-attribute Utility Theory", *Operations Research*, Vol. 34, No. 2. pp. 296-310.

[48] Herwijnen, M.. Rietveld, P., Thevenet, K., and Tol R. [1995], "Sensitivity Analysis with Interdependent Criteria for Multi-criteria Decision Making: the Case of Soil Pollution Treatment", *Journal of Multi-Criteria Decision Analysis*. V. 4. No. 1. pp. 57-70.

[49] Hipel, Keith W. (ed.) [1992] *Multiple Objective Decision Making in Water Resources*, AWRA (American Water Resources Association), Monograph Series No.18, Herndon, Virginia.

[50] Hipel, K., W., Radford, J. and Fang, L. [1993] "Multiple Participant - Multiple Criteria Decision Making", *IEEE Transaction on Systems, Man, and Cybernetics*, SMC-23, No.4.

[51] Hipel, K., W., Kilgour, D.M., and Rajabi, S. [1997] "Operations Research and Refinement of Courses of Actions", in *Handbook of Systems Engineering* edited by Sage A., Wiley, To appear.

[52] [1979] "Analytical Multiobjective Decision Methods for Power Plant Siting: A Review of Theory and Applications", National Center for Analysis of Energy Systems, New York.

[53] Hobbs, B. F. [1986] "What Can We Learn From Experiments in Multiobjective Decision Analysis?", *IEEE Transactions on Systems, Man, and Cybernetics*, Vol. SMC-16, No. 3, pp. 384-394.

[54] Hobbs, B. F. [1996] "Energy vs. the Environment", *OR/MS Today*, Vol.23, No. 6, pp. 30-33.

[55] Hobbs, B. F., Chankong, V., and Hamadeh, W. [1990]"Screening Water Resources Plans under Risk and Multiple Objectives: A Comparision of Methods" *The U.S. Army Corps of Engineers.*

[56] Hwang, Ching-Lai; Yoon, Kwangsun [1981] *Multiple Attribute Decision Making*, Springer-Verlag, Berlin.

[57] Ignizio, J.P. [1976] *Goal Programming and Extensions*, Lexington Books, Massachusetts.

[58] Ignizio, J.P.[1981] "The Determination of A Subset of Efficient Solutions Via Goal Programming", *Computers and Operations Research*, Vol. 8, pp. 9-16.

[59] Ignizio, J.P. and Cavalier T.M. [1994] *Linear Programming*, Prentice Hall,New Jersey.

[60] Insua, David Rios [1990] *Sensitivity Analysis in Multiple Objective Decision Making*, Springer-Verlag, Berlin.

[61] Karaivanova, J., Narula, S., and Vassilev, V. [1993] "An Interactive Procedure for Multiple Objective Integer Linear Programming Problems", *European Journal of Operational Research*, 68, pp. 344-351.

[62] Karpak, B. and Zionts, S. (eds.)[1989] *Multiple Criteria Decision Making and Risk Analysis Using Microcomputers*, NATO ASI Series, Series F: Vol. 56.

[63] Keeney, R. L. [1992] *Value Focused Thinking, A Path to Creative Decision Making*, Harvard University Press, Cambridge, Massachusetts.

[64] Keeney, R. L. [1994] "Using Values in Operations Research", *Operations Research*, Vol. 42, No. 5, pp. 793.

[65] Keeney, R. L. and Raiffa, H. [1976] *Decision with Multiple Objectives: Preferences and Value Tradeoffs*, Wiley, New York.

[66] Keeney, R. L.. McDaniels, T. L., and Ridge-Cooney, V. L. [1997] "Using Values in Planning Wastewater Facilities for Metropolitan Seattle", Forthcoming in *Water Resources Bulletin.*

[67] Keeney, R. L. [1996] "Value-Focused Thinking: Identifying Decision Opportunities and Creating Alternatives", *European Journal of Operational Research*, Vol. 92. No.3, pp. 537-549.

[68] Kirkwood, C.W. & Sarin, R.K. [1985] "Ranking with Partial Information: A Method and an Application", *Operations Research*, vol. 33. No.1, pp. 38-48.

[69] Koksalan, M. M. [1989] "Identifying and Ranking a Most Preferred Subset of Alternatives in the Presence of Multiple Criteria", *Naval Research Logistics*, Vol. 36, pp. 359-372.

[70] Koksalan, M. M., Karwan, M. H., and Zionts, S. [1984] "An Improved Method for Solving Multiple Criteria Problem Involving Discrete Alternatives", *IEEE Transactions on systems, Man, and Cybernetics*, SMC-14, pp. 24-34.

[71] Koksalan, M. M., Karwan, M. H., and Zionts, S. [1986] "Approaches for Discrete Alternative Multiple Criteria Problems for Different Types of Criteria", *Journal of Institute of Industrial Engineering Transactions*, 16, pp. 262-270.

[72] Korhonen, Pekka [1992] "Multi Criteria Decision Support: The State of Research and Future Direction", *Computers and Operations Research*, Vol. 19, No. 7, pp. 549-551.

[73] Korhonen, P., and Soismaa, M. [1981] "An Interactive Multiple Criteria Approach to Ranking alternatives", *Journal of the Operational Research Society*, 32, pp. 577-585.

[74] Korhonen, P., Wallenius, J. and Zionts, S. [1984] "Solving the Discrete Multiple Criteria Problem Using Convex Cones", *Management Sciences*, 30(11), pp. 1336-1345.

[75] Krantz, D., Duncan Luce, R., Suppes, P. and Tversky, A. [1971] *Foundation of Measurement; Volume I: Additive and Polynomial Representations*, Academic Press, New York.

[76] Lee, S. M., and Luebbe, R.L. [1987] " A Zero-One Goal Programming Algorithm Using Partitioning and Constraint Aggregation", *Journal of Operational Research Society*, Vol. 38, No. 7, pp. 633-641.

[77] Lewandowski, A., Kreglewski, T., Rogowski, T., and Wierzbicki, A.P. [1989] "Decision Support Systems of DIDAS Family (Dynamic Interactive Decision Analysis & Support)", In *Aspiration Based Decision Support Systems*, pp. 21-37, Springer-Verlag, Berlin.

[78] Lowe, T.J, Thisse, J. F., Ward, J. E. and Wendell, R. E. [1984] "On Efficient Solutions to Multiple Objective Mathematical Programming", *Management Science*, Vol. 30, No. 11, pp. 1346-1349.

[79] Loucks, D. P., Stedinger, J. R., and Haith, D. A. [1981] *Water Resource Systems Planning and Analysis*, Prentice-Hall, New Jersey.

[80] MacCrimmon, Kenneth R. [1973] "An Overview Of Multiple Objective Decision Making", *In Multiple Criteria Decision Making*, J.L.Cochrance and M. Zeleny(eds.), University of South Carolina Press, Columbia, pp. 18-44.

[81] Merkhofer, Miley W. and Keeney, Ralph L. [1987] "A Multi-attribute Utility Analysis of Alternative Sites for the Disposal of Nuclear Waste", *Risk Analysis*, Vol. 7, No. 2, pp. 173-193.

[82] Miser, H. J. and Quade, E. S. (eds.) [1988] *Handbook of System Analysis*, North Holland, New York.

[83] Moskowitz, H., Preckel, p., and Yang, A. [1992] "Multiple Criteria Robust Interactive Decision Analysis (MCRID), for Optimizing Public Policies", *European Journal of Operational Research*, 56, pp. 219-236.

[84] Nakayama, H. [1995] "Aspiration Level Approach to Interactive Multi-Objective Programming and Its Applications", in *Advances in Multi-criteria Analysis*, edited by Pardalos, P., Siskos, Y., and Zopounidis C., Kluwer, Dordrecht, The Netherland.

[85] Netto, O.C., Parent, E., and Duckstein, L. [1997] "Multi-criterion Design of Long-Term Water Supply in Southern France", Forthcoming in *Journal of Water Resources Planning and Management.*

[86] Ogryczak, W. [1994] "A Goal Programming Model of the Reference Point Method", *Annals of Operations Research*, 51, pp. 33-44.

[87] Olson, D. L. [1996] *Decision Aids for Selection Problems*, Springer, New York.

[88] Osyczka, Andrzej [1984] *Multi-criterion Optimization in Engineering*, Ellis Horwood Limited, Series in Engineering Science, Chichester, United Kingdom.

[89] Ozernoy, Vladimir M. [1988] "MCDM in the USSR: A Survey", *Naval Research Logistics*, Vol. 35, pp.543-556.

[90] Pasternak, H. and Passy, U. [1973] "Bicriterion Mathematical Programs with Boolean Variables", In *Multiple Criteria Decision Making*, J.L. Cochrane and M. Zeleny (eds.), University of South Carolina, Columbia, pp. 327-348.

[91] Rajabi, S., Hipel, K.W., Kilgour, D.M. [1995a] "Multiple Criteria Decision Making under Interdependence of Actions", *Proc. of IEEE Int. Conf. on Systems, Man, and Cybernetics, Vancouver, Canada, 1995*, pp. 2365-2370.

[92] Rajabi, S., Kilgour, D.M., and Hipel K.W. [1996] " Zero-One Goal Programming under Interdependence of Actions", *Proceedings of Twelfth International Conference on MCDM, Hagen, Germany, 1995*, Springer-Verlag, Berlin, pp. 383-392.

[93] Rajabi, S., Hipel, K.W., Kilgour, D.M. [1996] "Multiple Objective Water Supply Planning", *Proc. of International Conference on Water Resources & Environmental Research: Towards the 21st Century*, Vol. 2, 167-174, Water Resources Research Center, Disaster Prevention Research Institute, Kyoto University, Kyoto, Japan.

[94] Rajabi, S., Kilgour, D.M., Hipel K. W. [1997] "Modelling Action Interdependence in Multiple Criteria Decision Making", Submitted for publication.

[95] Rajabi,S., Hipel, K.W., and Kilgour,D.M. [1997] "Multiple Criteria Water Supply Planning", Accepted for Publication in *Proc. of IEEE Int. Conf. on Systems, Man, and Cybernetics, Orlando, Florida, 1997.*

[96] Rajabi, S., Kilgour, D.M., Hipel K. W. [1997] "Screening Actions in Multiple Criteria Subset Selection", Submitted for publication.

[97] Ramesh, R. [1985] *Multi-criteria Integer Programming*, Ph.D. Dissertation, State University of New York at Buffalo.

[98] Ramesh,R., Karwan, M. H., and Zionts, S. [1989] *Interactive Multi-criteria Linear Programming: An Extension of the Method of Zionts and Wallenius*, Naval Research Logistics, Vol. 36, pp. 321-335.

[99] Rasmussen, L.M. [1986] "Zero-One Programming with Multiple Criteria", *European Journal of Operational Research*, 26, pp. 83-95.

[100] Romero, Carlos [1986] "A Survey of Generalized Goal Programming (1970 - 1982)", *European Journal of Operational Research*, 25, pp. 183-191.

[101] Romero, Carlos [1990] *Handbook of Critical Issues in Goal Programming*, Pergamon Press, New York.

[102] Ringuest, Jeffery L. [1992] *Multi-objective Optimization: Behavioral and Computational Considerations*, Kluwer, Boston.

[103] Ringuest, Jeffery L. [1997] "L_p-Metric Sensitivity Analysis for Single and Multi-Attribute Decision Analysis", *European Journal of Operational Research*, Vol. 98, pp. 563-570.

[104] Roy, Bernard [1990a] "Outranking Approach and the Foundation of ELECTRE Methods". In *Reading in Multi Criteria Decision Aid*, Bana e Costa (ed.), Springer-Verlag, Berlin.

[105] Roy, Bernard [1990b] "Decision-Aid and Decision Making", In *Readings in Multiple Criteria Decision Aid*, Bana e Costa (ed.), Springer-Verlag, Berlin, pp. 17-35.

[106] Roy, B., Slowinski, R., and Treichel, W. [1992] " Multi-criteria Programming of Water Supply Systems for Rural Areas", *Water Resources Bulletin,* Vol. 28, No.1, pp. 13-31.

[107] Ruhe, G., [1988] "Complexity Result of Multicriteria and Parametric Network Flows Using a Pathological Network of Zadeh", *Operational Research*, No. 32, pp. 9-27.

[108] Saaty, Thomas L. [1980] *Analytic Hierarchy Process*, McGraw Hill , New York.

[109] Sage, A. P. and White, C. C. [1984] "ARIADNE: A Knowledge-Based Interactive System for Planning and Decision Support", *IEEE Transaction on Man, System, and Cybernetic* . SMC-14, pp. 35-47.

[110] Sawaragi, Y., Nakayama, H. and Tanino, T. [1985] *Theory of Multi-objective Optimization*, Academic Press, New York.

[111] Salo, Ahti A. and Hamalainen, Raimo P. [1992] "Preference Assessment by Imprecise Ratio Statements", *Operations Research*, Vol, 40, No. 6, pp. 1053-1061.

[112] Santhanam, R. and Kyparisis, G. [1996] "A Decision Model for Interdependent Information System Project Selection", *European Journal of Operational Research*, Vol. 89, pp. 380-399.

[113] Schniederians, M. J., [1995] *Goal Programming, Methodology and Applications*, Kluwer, Boston.

[114] Simon, H.A. [1976] *Administrative Behavior*, 3rd edition, Macmillan, New York.

[115] Stadler, W. (ed.) [1988] *Multi-criteria Optimization in Engineering and in the Sciences*, Plenum Press, New York.

[116] Stadler, W. [1979] "A Survey of Multi-criteria Optimization or the Vector Maximum Problem. Part 1: 1776-1960", *Journal of Optimization Theory and Application*, Vol. 29, No. 1, pp. 1-52.

[117] Stanoulov, N. [1995] "A Parsimonious Outranking Method for Individual and Group Decision Making and Its Computerized Support", *IEEE Transactions on Systems, Man, and Cybernetics*, Vol. 25, No, 2.

[118] Steuer, R. E. [1986] *Multiple Criteria Optimization: Theory, Computation and Application*, Wiley, New York.

[119] Steuer, R. E., and Choo, E.U. [1983] " An Interactive Weighted Tchebycheff Procedure for Multiple Objective Programming", *Mathematical Programming*, Vol. 26, No. 1, pp. 2835-2843.

[120] Stewart, T. J. [1992] "A Critical Survey on the Status of Multiple Criteria Decision Making Theory and Practice", *OMEGA*, Vol. 20, No. 5/6, pp. 569-586.

[121] Stewart, T.J. and Scott, L. [1995] " A Scenario-Based Framework for Multi-criteria Decision Analysis in Water Resources Planning", *Water Resources Research*, Vol. 31, No. 11, pp. 2835-2843.

[122] Szidarovszky, F., M.E. Gershon, and L. Duckstein [1986], *Techniques for Multi-objective Decision Making in Systems Management*, Elsevier, Amsterdam.

[123] Tamiz, M. and Jones, D.F. [1997] "Interactive Framework for Investigation of GP Models: Theory and Practice", *Journal of Multi-Criteria Decision Analysis*, V. 6, No. 1, pp. 52-60.

[124] Tecle, Aregai [1992] "Selecting A Multi-criterion Decision Making Technique For Watershed Resources management", *Water Resources Bulletin*, Volume 28, No. 1, K.W. Hipel (ed.), Monograph Series, No. 18, Bethesda, Maryland.

[125] Teghem, J., and Kunsch, P.[1986] "Survey of Technniques to Determine the Efficient solutions for Multi-Objective Integer Linear Programming", *Asia-Pacific Journal of Operations Research*, 3, pp. 95-108.

[126] Tzeng, G.H. and Teng, J.Y. [1993] "Transportation Investment Project Selection with Fuzzy Multiobjectives", *Transportation Planning and Technology*, Vol. 17, pp. 91-112.

[127] Ulungu, E.L. and Teghem, J. [1994] "Multi-Objective Combinatorial Optimization Problems: A Survey", *Journal of Multi-criteria Decision Analysis*, pp. 83-104.

[128] Villareal, B., and Karwan, M.H. [1981] "Multicriteria Integer Programming: A (hybrid) Dynamic Programming Recursive Approach", *Mathematical Programming*, 21, pp. 204-223.

[129] Vincke, Philippe [1992] *Multicriteria Decision-Aid*, John Wiley & Sons, New York.

[130] Warburton, A. [1987] "Approximation of Pareto Optima in Multiple-Objective, Shortest-Path Problems", *Operations Research*, Vol. 35, No. 1, pp. 70-79.

[131] White, C.C., Sage, A.P.[1980] "A Multiple Objective Optimization-Based Approach to Choice Making", *IEEE Transactions on Systems, Man, and Cybernetics*, SMC-10, pp. 315-326.

[132] White, C.C., Sage, A.P. and Dozono, S. [1984] "A Model of Multiattribute Decision Making and Trade Off Weight Determination Under Uncertainity", *IEEE Transactions on Systems, Man, and Cybernetics*, SMC-14, pp. 223-229.

[133] White, D. J. [1980] *Optimality and Efficiency*, Wiley, Chichester.

[134] White, D. J. [1990] "A Bibliography on the Application of Mathematical Programming Multiple-Objective Methods", *Journal of Operational Research Society*, Vol. 41, No. 8, pp. 669-691.

[135] Wierzbicki, A. [1982] "A Mathematical Bases for Satisfying Decision making", *Mathematical Programming*, Vol. 3, pp. 391-405.

[136] Wierzbicki, A. [1986] "On the Completeness and Constructiveness of Parametric Characterizations to Vector Optimization Problems", *OR-Spektrum*, Vol. 8, pp. 73-87.

[137] Yu,P. L. [1985] *Multiple Criteria Decision Making*, Plenum, New York.

[138] Zeleny, M. [1981] "The Pros and Cons of Goal Programming", *Computers and Operations Research*, No. 8, pp. 357-359.

[139] Zeleny, M.[1982] *Multiple Criteria Decision Making*, McGraw-Hill, New York.

[140] Zeleny, M.[1992] "An Essay into A Philosophy of MCDM: A Way of Thinking or Another Algorithm?", *Computers and Operations Research*, Vol. 19, No.7, pp. 563- 566.

[141] Zeleny, M. and Cochrane, J. L. [1973] "A Priori and Posteriori Goals in Macroeeconomic Policy Making". In *Multiple Criteria Decision Making*, Cochrane. J. L. and Zeleny, M. (Eds.), Univeersity of South Carolina Prss, Columbia, pp. 373-391.

[142] Zionts, S. and Wallenius, J. [1976] "An Interactive Programming Method for Solving the Multiple Criteria Problem". *Management Sciences*, Vol.22, No. 6, pp. 652-663.

[143] Zionts, S. and Wallenius, J. [1983] "An Interactive Multiple Objective Linear Programming Method for a Class of Underlying Nonlinear Utility Functions", *Management Sciences*. 29(5), pp. 519-529.

[144] Zionts, S. [1992] "Some Thoughts on Research in MCDM", *Computers and Operations Research*, Vol. 19, No. 7, pp. 567-570.