

## COMPLEXITY DICHOTOMY FOR LIST-5-COLORING WITH A FORBIDDEN INDUCED SUBGRAPH

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ABSTRACT. For a positive integer  $r$  and graphs  $G$  and  $H$ , we denote by  $G + H$  the disjoint union of  $G$  and  $H$  and by  $rH$  the union of  $r$  mutually disjoint copies of  $H$ . Also, we say  $G$  is  $H$ -free if  $H$  is not isomorphic to an induced subgraph of  $G$ . We use  $P_t$  to denote the path on  $t$  vertices. For a fixed positive integer  $k$ , the LIST- $k$ -COLORING PROBLEM is to decide, given a graph  $G$  and a list  $L(v) \subseteq \{1, \dots, k\}$  of colors assigned to each vertex  $v$  of  $G$ , whether  $G$  admits a proper coloring  $\phi$  with  $\phi(v) \in L(v)$  for every vertex  $v$  of  $G$ , and the  $k$ -COLORING PROBLEM is the LIST- $k$ -COLORING PROBLEM restricted to instances with  $L(v) = \{1, \dots, k\}$  for every vertex  $v$  of  $G$ . We prove that for every positive integer  $r$ , the LIST-5-COLORING PROBLEM restricted to  $rP_3$ -free graphs can be solved in polynomial time. Together with known results, this gives a complete dichotomy for the complexity of the LIST-5-COLORING PROBLEM restricted to  $H$ -free graphs: For every graph  $H$ , assuming  $P \neq NP$ , the LIST-5-COLORING PROBLEM restricted to  $H$ -free graphs can be solved in polynomial time if and only if  $H$  is an induced subgraph of either  $rP_3$  or  $P_5 + rP_1$  for some positive integer  $r$ . As a hardness counterpart, we also show that the  $k$ -COLORING PROBLEM restricted to  $rP_4$ -free graphs is NP-complete for all  $k \geq 5$  and  $r \geq 2$ .

### 1. INTRODUCTION

Throughout this paper, all graphs are finite and simple. We denote the set of positive integers by  $\mathbb{N}$ , and for every  $k \in \mathbb{N}$ , we define  $[k] = \{1, \dots, k\}$ . Let  $G$  be a graph. We denote by  $V(G)$  and  $E(G)$  the vertex set and the edge set of  $G$ , respectively. By a *clique* in  $G$  we mean a set of pairwise adjacent vertices, and a *stable set* in  $G$  is a set of pairwise nonadjacent vertices. For every  $d \in \mathbb{N}$  and every vertex  $v \in V(G)$ , we denote by  $N_G^d(v)$  the set of all vertices in  $G$  at distance  $d$  from  $v$ , and by  $N_G^d[v]$  the set of all vertices in  $G$  at distance at most  $d$  from  $v$ . In particular, we write  $N_G(v)$  for  $N_G^1(v)$ , which is the set of neighbors of  $v$  in  $G$ , and  $N_G[v]$  for  $N_G^1[v] = N_G(v) \cup \{v\}$ . Also, for every  $X \subseteq V(G)$ , we define  $N_G^d[X] = \bigcup_{x \in X} N_G^d[x]$  and  $N_G^d(X) = N_G^d[X] \setminus X$ . Again, we write  $N_G(X)$  for  $N_G^1(X)$  and  $N_G[X]$  for  $N_G^1[X]$ . For every  $Z$  which is either a vertex or a subset of vertices of  $G$ , we write  $G - Z$  for the graph obtained from  $G$  by removing  $Z$ . A graph  $H$  is an *induced subgraph* of a graph  $G$  if  $H$  is isomorphic to  $G - X$  for some  $X \subseteq V(G)$ , and otherwise  $G$  is  $H$ -free. Also, for every graph  $G$  and every  $X \subseteq V(G)$ , we denote the *subgraph of  $G$  induced on  $X$* , that is,  $G - (V(G) \setminus X)$ , by  $G|X$ . For  $r \in \mathbb{N}$  and graphs  $G$  and  $H$ , we denote by  $G + H$  the disjoint union of  $G$  and  $H$  and by  $rH$  the union of  $r$  pairwise disjoint copies of  $H$ . For all  $t \in \mathbb{N}$ , we use  $P_t$  to denote the path on  $t$  vertices.

Let  $G$  be a graph and  $k \in \mathbb{N}$ . By a  $k$ -coloring of  $G$ , we mean a function  $\phi : V(G) \rightarrow [k]$ . A coloring  $\phi$  of  $G$  is said to be *proper* if  $\phi(u) \neq \phi(v)$  for every edge  $uv \in E(G)$ . In other words,  $\phi$  is proper if and only if for every  $i \in [k]$ ,  $\phi^{-1}(i)$  is a stable set in  $G$ . We say  $G$  is  $k$ -colorable

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if  $G$  has a proper  $k$ -coloring. For fixed  $k \in \mathbb{N}$ , the  $k$ -COLORING PROBLEM asks, given graph  $G$ , whether  $G$  is  $k$ -colorable.

A  $k$ -list-assignment of  $G$  is a map  $L : V(G) \rightarrow 2^{[k]}$ . For  $v \in V(G)$ , we refer to  $L(v)$  as the *list of  $v$* . Also, for every  $i \in [k]$ , we define  $L^{(i)} = \{v \in V(G) : i \in L(v)\}$ . An  $L$ -coloring of  $G$  is a proper  $k$ -coloring  $\phi$  of  $G$  with  $\phi(v) \in L(v)$  for all  $v \in V(G)$ . For example, if  $L(v) = \emptyset$  for some  $v \in V(G)$ , then  $G$  admits no  $L$ -coloring. Also, if  $V(G) = \emptyset$ , then  $G$  vacuously admits an  $L$ -coloring for every  $k$ -list-assignment  $L$ . For fixed  $k \in \mathbb{N}$ , the LIST- $k$ -COLORING PROBLEM is to decide, given an instance  $(G, L)$  consisting of a graph  $G$  and a  $k$ -list-assignment  $L$  for  $G$ , whether  $G$  admits an  $L$ -coloring. Note that the  $k$ -coloring problem is in fact the LIST- $k$ -COLORING PROBLEM restricted to instances  $(G, L)$  where  $L(v) = [k]$  for every  $v \in V(G)$ .

The  $k$ -COLORING PROBLEM, and so the LIST- $k$ -COLORING PROBLEM, are well-known to be NP-complete for all  $k \geq 3$  [18]. This motivates studying the complexity of these problems restricted to graphs with a fixed forbidden induced subgraph, that is,  $H$ -free graphs for some fixed graph  $H$ . As a narrowing start, the following two theorems show that there is virtually no hope for a polynomial-time algorithm unless  $H$  is a disjoint union of paths.

**Theorem 1** (Kamiński and Lozin [17]). *For all  $k \geq 3$ , the  $k$ -COLORING problem restricted to  $H$ -free graphs is NP-complete if  $H$  contains a cycle.*

**Theorem 2** (Holyer [15]). *For all  $k \geq 3$ , the  $k$ -COLORING PROBLEM restricted to  $H$ -free graphs is NP-complete if  $H$  contains a ‘claw’ (a vertex with three pairwise nonadjacent neighbors).*

Accordingly, an extensive body of work has been devoted to show that excluding certain paths (or their disjoint unions) makes the  $k$ -COLORING and the LIST- $k$ -COLORING problem easier. Here is a list of known results in this direction.

**Theorem 3.** *The  $k$ -COLORING PROBLEM restricted to  $H$ -free graphs can be solved in polynomial time if:*

- $H = P_6$  for  $k = 4$  (Chudnovsky, Spirkl and Zhong [4]);
- $H = rP_2$  for all fixed  $k, r \in \mathbb{N}$  (Golovach, Johnson, Paulusma and Song [12]);
- $H = rP_3$  for  $k = 3$  and all fixed  $r \in \mathbb{N}$  (Broersma, Golovach, Paulusma and Song [2]);

*and the LIST- $k$ -COLORING PROBLEM restricted to  $H$ -free graphs can be solved in polynomial time if:*

- $H = P_5$  for all fixed  $k \in \mathbb{N}$  (Hoàng, Kamiński, Lozin, Sawada and Shu [14]);
- $H = P_7$  for  $k = 3$  (Bonomo, Chudnovsky, Maceli, Schaudt, Stein and Zhong [1]);
- $H = P_6 + rP_3$  for  $k = 3$  and all  $r \in \mathbb{N}$  (Chudnovsky, Huang, Spirkl and Zhong [3]);
- $H = P_5 + rP_1$  for all  $r, k \in \mathbb{N}$  (Couturier, Golovach, Kratsch and Paulusma [6]).

On the other hand, the following hardness results are known.

**Theorem 4.** *The  $k$ -COLORING PROBLEM restricted to  $H$ -free graphs is NP-Complete if:*

- $H = P_6$  for  $k = 5$ , or  $H = P_7$  for  $k = 4$  (Huang [16]);
- $H = P_5 + P_2$  for  $k = 5$  (Chudnovsky, Huang, Spirkl and Zhong [3]);

*and the LIST- $k$ -COLORING PROBLEM restricted to  $H$ -free graphs is NP-Complete if:*

- $H = P_6$  for  $k = 4$  (Golovach, Paulusma and Song [11]);
- $H = P_4 + P_2$  for  $k = 5$  (Couturier, Golovach, Kratsch and Paulusma [6]).

Our main result is the following.

**Theorem 5.** *For every  $r \in \mathbb{N}$ , the LIST-5-COLORING PROBLEM restricted to  $rP_3$ -free graphs can be solved in polynomial time.*

Note that in addition to extending the third bullet of Theorem 3, this completely classifies the complexity of the LIST-5-COLORING PROBLEM restricted to  $H$ -free instances. Let us prove this formally:

**Theorem 6.** *Let  $H$  be a graph. Assuming  $P \neq NP$ , the LIST-5-COLORING PROBLEM restricted to  $H$ -free graphs can be solved in polynomial time if and only if  $H$  is an induced subgraph of  $rP_3$  or  $P_5 + rP_1$  for some  $r \in \mathbb{N}$ .*

*Proof of Theorem 6 assuming Theorem 5.* If  $H$  is an induced subgraph of  $rP_3$ , then the result follows from Theorem 5, and if  $H$  is an induced subgraph of  $P_5 + rP_1$ , then the result follows from Theorem 3. So we may assume that neither is the case.

If  $H$  is not a disjoint union of paths, then either  $H$  is not a forest, in which case the result follows from Theorem 1, or  $H$  is a forest with a vertex of degree at least three, and so the result follows from Theorem 2. Therefore, we may assume that  $H$  is a disjoint union of paths. Since  $H$  is not an induced subgraph of  $rP_3$ , there is a connected component  $C$  of  $H$  which is isomorphic to  $P_t$  for some  $t \geq 4$ . If  $t \geq 6$ , then the result follows from the first bullet of Theorem 4. So we may assume that  $t \in \{4, 5\}$ . Now, since  $H$  is not an induced subgraph of  $P_5 + rP_1$ , it follows that  $H - V(C)$  contains an edge. But now  $H$  contains  $P_4 + P_2$  as an induced subgraph, and thus the result follows from the fourth bullet of Theorem 4. This completes the proof. ■

As a hardness counterpart to Theorem 5, using a reduction similar to the one in [6], we also show that:

**Theorem 7.** *The  $k$ -COLORING PROBLEM restricted to  $2P_4$ -free graphs (and hence  $rP_4$ -free graphs for every fixed  $r \geq 2$ ) is NP-complete for all  $k \geq 5$ .*

In view of Theorem 4, this leaves open the complexity of the 5-COLORING PROBLEM restricted to  $H$ -free graphs only if exactly one connected component  $C$  of  $H$  isomorphic to  $P_4$ , and  $H - V(C)$  is an induced subgraph of  $rP_3$  containing at least one edge for some  $r \geq 1$ .

The remainder of this paper is organized as follows. In Sections 2-4, we prepare the tools required for the proof of Theorem 5. In Section 5, we prove Theorem 5, and finally in Section 6, we prove Theorem 7. It is worth noting that the main results of Sections 2 and 3, namely Theorems 13 and 17, respectively, are in fact proved for the LIST- $k$ -COLORING PROBLEM restricted to  $rP_3$ -free graphs with arbitrary  $k$ . However, our results from Section 4 fail to extend to this general setting for  $k \geq 6$ . On the other hand, we were not able to decide whether there exists  $k \in \mathbb{N}$  for which the LIST- $k$ -COLORING PROBLEM restricted to  $rP_3$ -free graphs is NP-hard.

## 2. REFINEMENTS, PROFILES AND FRUGALITY

We begin with introducing the notions of a *refinement* and a *profile* as a unified terminology we employ pervasively in this paper. Let  $k \in \mathbb{N}$  and  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM. By a  $(G, L)$ -refinement we mean an instance  $(G', L')$  of the LIST- $k$ -COLORING PROBLEM where  $G'$  is an induced subgraph of  $G$  and  $L'(v) \subseteq L(v)$  for all  $v \in V(G')$ . The  $(G, L)$ -refinement  $(G', L')$  is *spanning* if  $G' = G$ . Also, a  $(G, L)$ -profile is a set of  $(G, L)$ -refinements. A  $(G, L)$ -profile is *spanning* if all its elements are spanning. A large portion of this work deals with how the feasibility of an instance of the LIST- $k$ -COLORING PROBLEM is tied to the feasibility of certain refinements. For example, the following is easily observed.

**Lemma 8.** *Let  $k \in \mathbb{N}$  be fixed and  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM and  $(G, L')$  be a spanning  $(G, L)$ -refinement. If  $G$  admits an  $L'$ -coloring, then  $G$  admits an  $L$ -coloring.*

Let  $k \in \mathbb{N}$  and  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM. An  $L$ -coloring  $\phi$  of  $G$  is said to be *frugal* if for all  $v \in V(G)$  and every  $i \in L(v)$ ,  $v$  has at most one neighbor in  $\phi^{-1}(i)$ . This could be viewed as a list-variant of the so-called *frugal coloring* introduced by Hind et al [13]. Also, it is crucially different from another list-variant of frugal coloring studied in [5], where the restriction applies to all colors, not just those in the list of  $v$ . The following lemma is straightforward to verify.

**Lemma 9.** *Let  $k \in \mathbb{N}$  be fixed and  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM,  $(G', L')$  be a  $(G, L)$ -refinement, and  $\phi$  be a frugal  $L$ -coloring of  $G$ . If  $\phi(v) \in L'(v)$  for every  $v \in V(G')$  (that is,  $\phi|_{V(G')}$  is an  $L'$ -coloring of  $G'$ ), then it is a frugal  $L'$ -coloring of  $G'$ .*

Note that for an instance  $(G, L)$  of the LIST- $k$ -COLORING PROBLEM, if  $|L(v)| = 1$  for some  $v \in V(G)$ , then we may remove  $v$  from  $G$  and also remove the single color in  $L(v)$  from the lists of all neighbors of  $v$  in  $G$ , obtaining an instance with the same state of feasibility. To remain precise, let us state this simple observation formally, as follows.

**Theorem 10.** *Let  $k \in \mathbb{N}$  be fixed and  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM. Then there exists a  $(G, L)$ -refinement  $(\hat{G}, \hat{L})$  with the following specifications.*

- $(\hat{G}, \hat{L})$  can be computed from  $(G, L)$  in time  $\mathcal{O}(|V(G)|^2)$ .
- $|\hat{L}(v)| \neq 1$  for all  $v \in V(\hat{G})$ .
- If  $G$  admits a frugal  $L$ -coloring, then  $\hat{G}$  admits a frugal  $\hat{L}$ -coloring.
- If  $\hat{G}$  admits an  $\hat{L}$ -coloring, then  $G$  admits an  $L$ -coloring.

*Proof.* The proof is easy, so we only give a sketch and leave it to the reader to check the details. One can find in time  $\mathcal{O}(|V(G)|)$  a vertex  $v \in V(G)$  with  $|L(v)| = 1$ , or confirm that there is none. In the former case, we replace  $G$  by  $G - v$  and  $L(w)$  by  $L(w) \setminus L(v)$  for every vertex  $w \in N_G(v)$ . In the latter case, we output the current instance and stop. Applying the same procedure iteratively, it is straightforward to check that we obtain a  $(G, L)$ -refinement  $(\hat{G}, \hat{L})$  satisfying Theorem 10. This completes the proof.  $\blacksquare$

The main goal of this section, though, is to establish a reduction from list-coloring to frugal list-coloring restricted to  $rP_3$ -free graphs. To achieve this, we need the main result of [7], the statement of which calls for a few definitions. A *hypergraph*  $H$  is an ordered pair  $(V(H), E(H))$  where  $V(H)$  is a finite set of vertices and  $E(H)$  is a collection of nonempty subsets of  $V(H)$ , usually referred to as *hyperedges*. A *matching* in  $H$  is a set of pairwise disjoint hyperedges, and a *vertex-cover* in  $H$  is a set of vertices meeting every hyperedge. We denote by  $\nu(H)$  the maximum size of a matching in  $H$ , and by  $\tau(H)$  the minimum size of a vertex-cover in  $H$ . Also, we denote by  $\Lambda(H)$  the maximum  $k \geq 2$  for which there exists hyperedges  $e_1, \dots, e_k \in E(H)$  with the following property. For all distinct  $i, j \in [k]$ , there exists a vertex  $v_{i,j} \in e_i \cap e_j$  which belongs to no other hyperedge among  $e_1, \dots, e_k$ . If there is no such  $k$  (that is, if the elements of  $E(H)$  are mutually disjoint), then we set  $\Lambda(H) = 2$ .

**Theorem 11** (Ding, Seymour and Winkler [7]). *For every hypergraph  $H$ , we have*

$$\tau(H) \leq 11\Lambda(H)^2(\Lambda(H) + \nu(H) + 3) \binom{\Lambda(H) + \nu(H)}{\nu(H)}^2.$$

We apply Theorem 11 to prove the following.

**Lemma 12.** *Let  $r \in \mathbb{N}$  and  $\eta(r) = 11(r+1)^2(2r+3) \binom{2r}{r-1}$ . Let  $G$  be an  $rP_3$ -free graph and  $A, B$  be two disjoint stable sets in  $G$ , such that every vertex in  $B$  has at least two neighbors in  $A$ . Then there exists  $S \subseteq A$  with  $|S| \leq \eta(r)$  such that every vertex in  $B$  has a neighbor in  $S$ .*

*Proof.* For every vertex in  $b \in B$ , let  $A(b) = N_G(b) \cap A$ . Let  $H$  be the hypergraph with  $V(H) = A$  and  $E(H) = \{A(b) : b \in B\}$ .

(1) *We have  $\nu(H) \leq r - 1$ .*

Suppose not. Let  $b_1, \dots, b_r \in B$  be distinct such that hyperedges  $A(b_1), \dots, A(b_r)$  of  $H$  are pairwise disjoint. By the assumption, for each  $i \in [r]$ , there exist two distinct vertices

$a_i, c_i \in A(b_i)$ . But then  $G|\{a_i, b_i, c_i : i \in [r]\}$  is isomorphic to  $rP_3$ , a contradiction. This proves (1).

(2) We have  $\Lambda(H) \leq r + 1$ .

For otherwise  $\Lambda(H) \geq r + 2 \geq 3$ , and so there exist distinct vertices  $b_1, \dots, b_{r+2} \in B$  with the following property. For all distinct  $i, j \in [r + 2]$ , there exists a vertex  $c_{i,j} \in A(b_i) \cap A(b_j)$  which belongs to no other set among  $A(b_1), \dots, A(b_{r+2})$ . But then  $G|\{b_i, c_{i,r+1}, c_{i,r+2} : i \in [r]\}$  is isomorphic to  $rP_3$ , a contradiction. This proves (2).

From (1) and (2) combined with Theorem 11, we obtain  $\tau(H) \leq 11(r+1)^2(2r+3)\binom{2r}{r-1} = \eta(r)$ , and so  $H$  has a vertex-cover of size at most  $\eta(r)$ . In other words, there exists  $S \subseteq A$  with  $|S| \leq \eta(r)$  such that for every vertex  $b \in B$ ,  $S \cap A(b) \neq \emptyset$ ; that is, every vertex in  $B$  has a neighbor in  $S$ . This completes the proof of Lemma 12. ■

Now we can prove the main theorem of this section.

**Theorem 13.** *For all fixed  $k, r \in \mathbb{N}$ , there exists  $\pi(k, r) \in \mathbb{N}$  with the following property. Let  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM where  $G$  is  $rP_3$ -free. Then there exists a spanning  $(G, L)$ -profile  $\Pi(G, L)$  with the following specifications.*

- $|\Pi(G, L)| \leq \mathcal{O}(|V(G)|^{\pi(k,r)})$  and  $\Pi(G, L)$  can be computed from  $(G, L)$  in time  $\mathcal{O}(|V(G)|^{\pi(k,r)})$ .
- If  $G$  admits an  $L$ -coloring, then for some  $(G, L') \in \Pi(G, L)$ ,  $G$  admits a frugal  $L'$ -coloring.

*Proof.* Let  $\eta(r)$  be as in Lemma 12. Let  $\mathcal{S}$  be the set of all  $k$ -tuples  $(S_1, \dots, S_k)$  of subsets of  $V(G)$  where

- (S1)  $S_i \subseteq L^{(i)}$  and  $|S_i| \leq (k-1)\eta(r)$  for all  $i \in [k]$ ;
- (S2)  $S_i$  is a stable set; and
- (S3)  $S_i \cap S_j = \emptyset$  for all distinct  $i, j \in [k]$ .

For each  $S = (S_1, \dots, S_k) \in \mathcal{S}$ , we define a  $k$ -list-assignment  $L_S$  of  $G$  as follows. Let  $v \in V(G)$ .

- (L1) If  $v \in S_i$  for some  $i \in [k]$ , then let  $L_S(v) = \{i\}$ .
- (L2) Otherwise, if  $v \in V(G) \setminus (\bigcup_{i=1}^k S_i)$ , then let  $L_S(v) = L(v) \setminus \{i \in [k] : N_G(v) \cap S_i \neq \emptyset\}$ .

This definition immediately yields the following.

(3) For all  $S = (S_1, \dots, S_k) \in \mathcal{S}$ ,  $i \in [k]$  and  $v \in V(G) \setminus S_i$  with a neighbor in  $S_i$ , we have  $i \notin L_S(v)$ .

Note that for every  $S \in \mathcal{S}$ ,  $(G, L_S)$  is a spanning  $(G, L)$ -refinement. Consider the spanning  $(G, L)$ -profile  $\Pi(G, L) = \{(G, L_S) : S \in \mathcal{S}\}$ .

(4)  $|\Pi(G, L)| \leq \mathcal{O}(|V(G)|^{k(k-1)\eta(r)})$  and  $\Pi(G, L)$  can be computed from  $(G, L)$  in time  $\mathcal{O}(|V(G)|^{k(k-1)\eta(r)+2})$ .

Let  $\mathcal{T}$  be the set of all  $k$ -tuples  $(S_1, \dots, S_k)$  of subsets of  $V(G)$  satisfying (S1). Then for each  $i \in [k]$ , there are at most  $((k-1)\eta(r) + 1)|V(G)|^{(k-1)\eta(r)}$  possibilities for  $S_i$ . As a result, we have  $|\mathcal{T}| \leq ((k-1)\eta(r) + 1)^k |V(G)|^{k(k-1)\eta(r)}$ , which along with  $|\Pi(G, L)| = |\mathcal{S}|$  and  $\mathcal{S} \subseteq \mathcal{T}$  proves the first assertion. For the second, it is straightforward to observe that the elements of  $\mathcal{T}$  can be enumerated in time  $\mathcal{O}(|\mathcal{T}|)$ . Then, for each  $S = (S_1, \dots, S_k) \in \mathcal{T}$ , one can check in constant time whether  $S$  satisfies (S2) and (S3). Thus,  $\mathcal{S}$  can be computed in time  $\mathcal{O}(|\mathcal{T}|)$ . Also, for each  $S \in \mathcal{S}$  and every  $v \in V(G)$ , it is readily seen from (L1) and (L2) that  $L_S(v)$

can be computed in time  $\mathcal{O}(|V(G)|)$ . Therefore,  $\Pi(G, L)$  can be computed from  $(G, L)$  in time  $\mathcal{O}(|\mathcal{T}||V(G)|^2) = \mathcal{O}(|V(G)|^{k(k-1)\eta(r)+2})$ . This proves (4).

(5) Let  $\phi$  be an  $L$ -coloring of  $G$  and  $i, j \in [k]$  be distinct. Let  $B_{i,j}$  be the set of all vertices in  $\phi^{-1}(j)$  with at least two neighbors in  $\phi^{-1}(i)$ . Then there exists  $S_{i,j} \subseteq \phi^{-1}(i)$  with  $|S_{i,j}| \leq \eta(r)$  such that every vertex in  $B_{i,j}$  has a neighbor in  $S_{i,j}$ .

A direct application of Lemma 12 to  $G$  and stable sets  $A = \phi^{-1}(i)$  and  $B = B_{i,j}$  proves (5).

(6) If  $G$  admits an  $L$ -coloring, then for some  $S \in \mathcal{S}$ ,  $G$  admits a frugal  $L_S$ -coloring.

Let  $\phi$  be an  $L$ -coloring of  $G$ . For distinct  $i, j \in [k]$ , let  $B_{i,j}$  be the set of all vertices in  $\phi^{-1}(j)$  with at least two neighbors in  $\phi^{-1}(i)$ . By (5), there exists  $S_{i,j} \subseteq \phi^{-1}(i)$  with  $|S_{i,j}| \leq \eta(r)$  such that every vertex in  $B_{i,j}$  has a neighbor in  $S_{i,j}$ . For each  $i \in [k]$ , let  $S_i = \bigcup_{j \in [k], j \neq i} S_{i,j}$ . Then from (5), we have  $S_i \subseteq \phi^{-1}(i) \subseteq L^{(i)}$ ,  $|S_i| \leq (k-1)\eta(r)$  and  $S_i \cap S_j \subseteq \phi^{-1}(i) \cap \phi^{-1}(j) = \emptyset$  for every  $j \in [k]$ . It follows that  $S = (S_1, \dots, S_k)$  satisfies both (S1), (S2) and (S3), and so  $S \in \mathcal{S}$ . Let  $L_S$  be the corresponding list-assignment defined in (L1) and (L2). We claim that  $\phi$  is a frugal  $L_S$ -coloring of  $G$ . To see this, note that being an  $L$ -coloring,  $\phi$  is proper. In addition, for every  $v \in V(G)$ , if  $v \in S_i \subseteq \phi^{-1}(i)$  for some  $i \in [k]$ , then by (L1), we have  $L_S(v) = \{i\} = \{\phi(v)\}$ . Otherwise, if  $v \in V(G) \setminus (\bigcup_{i=1}^k S_i)$ , then since  $\phi^{-1}(\phi(v))$  is a stable set of  $G$  containing  $S_{\phi(v)} \cup \{v\}$ ,  $v$  has no neighbor in  $S_{\phi(v)}$ , and so by (L2), we have  $\phi(v) \in L_S$ . As a result, we have  $\phi(v) \in L_S$  for every  $v \in V(G)$ , and  $\phi$  is in fact an  $L_S$ -coloring. It remains to argue the frugality of  $\phi$ . For each  $i \in [k]$ , let  $B_i = \bigcup_{j \in [k], j \neq i} B_{i,j}$ ; that is,  $B_i$  is the set of all vertices in  $G$  with at least two neighbors in  $\phi^{-1}(i)$ . Note that  $B_i \subseteq V(G) \setminus S_i$ . By (5), every vertex in  $B_i$  has a neighbor in  $S_i$ . Therefore, by (3), we have  $i \notin L_S(v)$  for every  $v \in B_i$ . In other words, for all  $v \in V(G)$  and every  $i \in L_S(v)$ ,  $v$  has at most one neighbor in  $\phi^{-1}(i)$ , and so  $\phi$  is frugal. This proves (6).

Finally, by setting  $\pi(k, r) = k(k-1)\eta(r) + 2$ , from (4) and (6), we conclude that  $\Pi(G, L)$  satisfies Theorem 13. This completes the proof.  $\blacksquare$

### 3. GOOD $P_3$ 's

Let  $G$  be a graph and  $\{x_1, x_2, x_3\} \subseteq V(G)$  with  $E(G|\{x_1, x_2, x_3\}) = \{x_1x_2, x_2x_3\}$ . Then  $G|\{x_1, x_2, x_3\}$  is isomorphic to  $P_3$ , and we refer to it as an *induced  $P_3$*  in  $G$ , denoting it by  $x_1 - x_2 - x_3$ . Also, for all  $Z, W \subseteq V(G)$ , we say  $Z$  is *complete* (*anticomplete*) to  $W$  if  $Z \cap W = \emptyset$  and every vertex in  $Z$  is adjacent (nonadjacent) to every vertex in  $W$ . If  $Z = \{z\}$  and  $Z$  is complete (anticomplete) to  $W$ , then we say  $z$  is *complete* (*anticomplete*) to  $W$ . For two induced  $P_3$ 's  $P$  and  $Q$  in  $G$ , we say  $P$  is *anticomplete* to  $Q$  (or  *$P$  and  $Q$  are anticomplete*), if their vertex sets are anticomplete in  $G$ .

Let  $k \in \mathbb{N}$  be an integer and  $\gamma = (I_1, I_2, I_3)$  be a triple of subsets of  $[k]$ . We say  $\gamma$  is *good* if  $|I_1|, |I_2|, |I_3| \geq 2$  and  $I_1 \cap I_2, I_1 \cap I_3$  and  $I_2 \cap I_3$  are all nonempty. Let  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM and  $x_1 - x_2 - x_3$  be an induced  $P_3$  in  $G$ . We refer to  $(L(x_1), L(x_2), L(x_3))$  as the  *$L$ -type* of  $x_1 - x_2 - x_3$  and to  $|L(x_1)| + |L(x_2)| + |L(x_3)|$  as the  *$L$ -weight* of  $x_1 - x_2 - x_3$ . Also, an  *$L$ -good  $P_3$*  in  $G$  is an induced  $P_3$  with good  $L$ -type. The following lemma, easy to check, asserts that excluding good  $P_3$ 's is inherited by refinements.

**Lemma 14.** *Let  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM and  $(G', L')$  be a  $(G, L)$ -refinement. Suppose that  $G$  has no  $L$ -good  $P_3$ . Then  $G'$  has no  $L'$ -good  $P_3$ .*

In this section, we show how to reduce an  $rP_3$ -free instance of the LIST- $k$ -COLORING PROBLEM to polynomially many instances with no good  $P_3$  in polynomial time. This goal is attained in Theorem 17. First, we need two lemmas.

**Lemma 15.** *Let  $k \in \mathbb{N}$  be fixed and  $\gamma = (I_1, I_2, I_3)$  be a good triple of subsets of  $[k]$ . Let  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM and  $x_1 - x_2 - x_3$  be an induced  $P_3$  in  $G$  of  $L$ -type  $(I_1, I_2, I_3)$ . Then there exists a spanning  $(G, L)$ -profile  $\Upsilon_1(G, L)$  with the following specifications.*

- $|\Upsilon_1(G, L)| \leq \mathcal{O}(|V(G)|^{3k-3})$  and  $\Upsilon_1(G, L)$  can be computed from  $(G, L)$  in time  $\mathcal{O}(|V(G)|^{3k-2})$ .
- For every  $(G, L_1) \in \Upsilon_1(G, L)$ , every induced  $P_3$  in  $G$  of  $L_1$ -type  $(I_1, I_2, I_3)$  is anticomplete to (and thus disjoint from)  $x_1 - x_2 - x_3$ .
- If  $G$  admits a frugal  $L$ -coloring, then for some  $(G, L_1) \in \Upsilon_1(G, L)$ ,  $G$  admits a frugal  $L_1$ -coloring.

*Proof.* Let  $\mathcal{S}$  be the set of all pairs  $(S, \psi)$  where

- (T1)  $S$  is a subset of  $N_G[\{x_1, x_2, x_3\}]$  containing  $\{x_1, x_2, x_3\}$  with  $|S| \leq 3k$ , and
- (T2)  $\psi$  is an  $L|_S$ -coloring of  $G|_S$ , where for every  $i \in \{1, 2, 3\}$  and every  $j \in L(x_i)$ ,  $x_i$  has at most one neighbor  $v$  in  $S$  with  $\psi(v) = j$ .

We deduce:

- (7)  $|\mathcal{S}| \leq \mathcal{O}(|V(G)|^{3k-3})$  and  $\mathcal{S}$  can be computed from  $(G, L)$  in time  $\mathcal{O}(|V(G)|^{3k-3})$ .

Let  $\mathcal{T}$  be the set of all pairs  $(S, \psi)$ , consisting of a set  $S$  satisfying (T1) and a coloring  $\psi : S \rightarrow [k]$  of  $G|_S$ . Note that for each  $(S, \psi) \in \mathcal{T}$ , there are at most  $(3k-2)|V(G)|^{3k-3}$  choices for  $S$ , and for each such choice, there are  $k^{|S|} \leq k^{3k}$  possibilities for  $\psi$ . So we have  $|\mathcal{T}| \leq (3k-2)k^{3k}|V(G)|^{3k-3}$ , which along with  $\mathcal{S} \subseteq \mathcal{T}$  proves the first assertion. To see the second, note that the elements of  $\mathcal{T}$  can be enumerated in time  $\mathcal{O}(|\mathcal{T}|) = \mathcal{O}(|V(G)|^{3k-3})$ . Also, for every  $(S, \psi) \in \mathcal{T}$ , since  $|S| \leq 3k$ , it can be checked in constant time whether  $\psi$  satisfies (T2), or equivalently  $(S, \psi) \in \mathcal{S}$ . Hence,  $\mathcal{S}$  can be computed from  $(G, L)$  in time  $\mathcal{O}(|\mathcal{T}|) = \mathcal{O}(|V(G)|^{3k-3})$ . This proves (7).

For every  $\sigma = (S, \psi) \in \mathcal{S}$ , consider the  $k$ -list-assignment  $L_\sigma$  of  $G$ , defined as follows. Let  $v \in V(G)$ .

- (M1) If  $v \in S$ , then  $L_\sigma(v) = \{\psi(v)\}$ .
- (M2) If  $v \in N_G[\{x_1, x_2, x_3\}] \setminus S$ , then  $L_\sigma(v) = L(v) \setminus (\bigcup_{j \in \{1, 2, 3\}, v \in N_G(x_j)} I_j)$ .
- (M3) If  $v \notin N_G[\{x_1, x_2, x_3\}]$ , then  $L_\sigma(v) = L(v)$ .

Note that for every  $\sigma \in \mathcal{S}$ ,  $(G, L_\sigma)$  is a spanning  $(G, L)$ -refinement. Consider the spanning  $(G, L)$ -profile  $\Upsilon_1(G, L) = \{(G, L_\sigma) : \sigma \in \mathcal{S}\}$ .

- (8)  $|\Upsilon_1(G, L)| \leq \mathcal{O}(|V(G)|^{3k-3})$  and  $\Upsilon_1(G, L)$  can be computed from  $(G, L)$  in time  $\mathcal{O}(|V(G)|^{3k-2})$ .

The first assertion follows from (7) and the fact that  $|\Upsilon_1(G, L)| = |\mathcal{S}|$ . For the second, we need to compute  $\mathcal{S}$ , which by (7) is attainable in time  $\mathcal{O}(|V(G)|^{3k-3})$ . Then, for every  $\sigma \in \mathcal{S}$ , it is easily seen from (M1), (M2) and (M3) that  $L_\sigma$  can be computed in time  $\mathcal{O}(|V(G)|)$ . Therefore,  $\Upsilon_1(G, L)$  can be computed in time  $\mathcal{O}(|V(G)|^{3k-3} + |V(G)||\mathcal{S}|) = \mathcal{O}(|V(G)|^{3k-2})$ , where the last equality follows from (7). This proves (8).

- (9) For every  $\sigma = (S, \psi) \in \Upsilon_1(G, L)$ , every induced  $P_3$  in  $G$  of  $L_\sigma$ -type  $\gamma$  is anticomplete to  $x_1 - x_2 - x_3$ .

Let  $Q = y_1 - y_2 - y_3$  be an induced  $P_3$  in  $G$  and  $(L_\sigma(y_1), L_\sigma(y_2), L_\sigma(y_3)) = \gamma$ . Since  $\gamma$  is good, we have  $|L_\sigma(y_i)| = |I_i| \geq 2$  for all  $i \in \{1, 2, 3\}$ , while  $|L_\sigma(v)| = 1$  for all  $v \in S$ . So  $S \cap \{y_1, y_2, y_3\} = \emptyset$ . Also, if  $y_i \in N_G(x_j) \setminus S$  for some  $i, j \in \{1, 2, 3\}$ , then by (M2),

we have  $I_i = L_\sigma(y_i) \subseteq L(y_i) \setminus I_j$ , and so  $I_i \cap I_j = \emptyset$ , which violates  $\gamma$  being good. Hence,  $N_G[\{x_1, x_2, x_3\}] \cap \{y_1, y_2, y_3\} = \emptyset$ , and so  $Q$  is anticomplete to  $x_1 - x_2 - x_3$ . This proves (9).

(10) *If  $G$  admits a frugal  $L$ -coloring, then for some  $\sigma \in \mathcal{S}$ ,  $G$  admits a frugal  $L_\sigma$ -coloring.*

Let  $\phi$  be a frugal  $L$ -coloring of  $G$ . For every  $j \in \{1, 2, 3\}$ , we define  $A_j = \{v \in N_G(x_j) : \phi(v) \in I_j\}$ . Let  $S = \{x_1, x_2, x_3\} \cup A_1 \cup A_2 \cup A_3$ . Then we have  $\{x_1, x_2, x_3\} \subseteq S \subseteq N_G[\{x_1, x_2, x_3\}]$ , and since  $\phi$  is a frugal  $L$ -coloring of  $G$ , we have  $|A_j| \leq |I_j| - 1 \leq k - 1$  for every  $i \in \{1, 2, 3\}$ , which in turn implies that  $|S| \leq 3k$ . Thus,  $S$  satisfies (T1). We define  $\psi = \phi|_S$ . Then, again by the frugality of  $\phi$ , for every  $i \in \{1, 2, 3\}$  and every  $j \in L(x_i)$ ,  $x_i$  has at most one neighbor  $v$  in  $S$  with  $\psi(v) = j$ . In other words,  $\psi$  satisfies (S2), and so  $\sigma = (S, \psi) \in \mathcal{S}$ . Let  $L_\sigma$  be the corresponding  $k$ -list-assignment defined in (M1), (M2) and (M3). We claim that  $\phi$  is a frugal  $L_\sigma$ -coloring of  $G$ . Let  $v \in V(G)$ . If  $v \in S$ , then by (M1), we have  $\phi(v) = \psi(v) \in \{\psi(v)\} = L_\sigma(v)$ . Also, if  $v \in N_G(\{x_1, x_2, x_3\}) \setminus S = N_G(\{x_1, x_2, x_3\}) \setminus (A_1 \cup A_2 \cup A_3)$ , then for every  $j \in \{1, 2, 3\}$  with  $v \in N_G(x_j)$ , we have  $\phi(v) \notin I_j$ , and so by (M2), we have  $\phi(v) \in L(v) \setminus (\bigcup_{j \in \{1, 2, 3\}, v \in N_G(x_j)} I_j) = L_\sigma(v)$ . Finally, if  $v \notin N_G(\{x_1, x_2, x_3\})$ , then by (M3), we have  $\phi(v) \in L(v) = L_\sigma(v)$ . In summary, we have  $\phi(v) \in L_\sigma(v)$  for every  $v \in V(G)$ . Therefore, by Lemma 9,  $\phi$  is a frugal  $L_\sigma$ -coloring of  $G$ . This proves (10).

Finally, from (8), (9) and (10), we conclude that  $\Upsilon_1(G, L)$  satisfies Lemma 15. This completes the proof.  $\blacksquare$

Let  $k \in \mathbb{N}$  and  $\gamma = (I_1, I_2, I_3)$  be a triple of subsets of  $[k]$ . Also, let  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM. We denote by  $f_{L, \gamma}(G)$  be the maximum number of mutually anticomplete induced  $P_3$ 's in  $G$  of  $L$ -type  $\gamma$ .

**Lemma 16.** *Let  $k \in \mathbb{N}$  be fixed and  $\gamma = (I_1, I_2, I_3)$  be a good triple of subsets of  $[k]$ . Let  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM. Also, suppose that no  $L$ -good  $P_3$  in  $G$  is of  $L$ -weight strictly larger than  $|I_1| + |I_2| + |I_3|$ . Then there exists a spanning  $(G, L)$ -profile  $\Upsilon_2(G, L)$  with the following specifications.*

- $|\Upsilon_2(G, L)| \leq \mathcal{O}(|V(G)|^{(3k-3)f_{L, \gamma}(G)})$  and  $\Upsilon_2(G, L)$  can be computed from  $(G, L)$  in time  $\mathcal{O}(|V(G)|^{(3k-2)f_{L, \gamma}(G)})$ .
- For every  $(G, L_2) \in \Upsilon_2(G, L)$ ,  $G$  has no induced  $P_3$  of  $L_2$ -type  $\gamma$  (and, of course, none of  $L_2$ -weight strictly larger than  $|I_1| + |I_2| + |I_3|$ ).
- If  $G$  admits a frugal  $L$ -coloring, then for some  $(G, L_2) \in \Upsilon_2(G, L)$ ,  $G$  admits a frugal  $L_2$ -coloring.

*Proof.* For fixed  $\gamma$ , we proceed by induction on  $f_{L, \gamma}(G)$ . If  $f_{L, \gamma}(G) = 0$ , then  $\Upsilon_2(G, L) = \{(G, L)\}$  satisfies Lemma 16. So we may assume that  $k \geq 2$ , and we may choose  $x_1 - x_2 - x_3$  as an induced  $P_3$  in  $G$  with  $(L(x_1), L(x_2), L(x_3)) = \gamma$ . Consequently, we may apply Lemma 15 to  $(G, L)$ ,  $\gamma$  and  $x_1 - x_2 - x_3$ , obtaining a spanning  $(G, L)$ -profile  $\Upsilon_1(G, L)$  satisfying Lemma 15.

(11) *For every  $(G, L_1) \in \Upsilon_1(G, L)$ , we have  $f_{L_1, \gamma}(G) < f_{L, \gamma}(G)$ .*

Suppose for a contradiction that  $f_{L_1, \gamma}(G) \geq f_{L, \gamma}(G) = t \geq 1$ . We may consider a collection  $\{q_1^i - q_2^i - q_3^i : i \in [t]\}$  of  $t$  mutually anticomplete induced  $P_3$ 's in  $G$ , such that for every  $i \in [t]$ ,  $(L_1(q_1^i), L_1(q_2^i), L_1(q_3^i)) = \gamma$ . Now, for each  $i \in [t]$ , by the second bullet of Lemma 15,  $q_1^i - q_2^i - q_3^i$  is anticomplete to  $x_1 - x_2 - x_3$ . Also, since  $(G, L_1)$  is a  $(G, L)$ -refinement, for every  $j \in \{1, 2, 3\}$ , we have  $I_j = L_1(q_j^i) \subseteq L(q_j^i)$ . Therefore,  $\gamma$  being good,  $q_1^i - q_2^i - q_3^i$  is an  $L$ -good  $P_3$  in  $G$  of  $L$ -weight at least  $|I_1| + |I_2| + |I_3|$ . This, along with the assumption of Lemma 16 that no  $L$ -good  $P_3$  in  $G$  is of  $L$ -weight strictly larger than  $|I_1| + |I_2| + |I_3|$ , implies that  $L(q_j^i) = I_j$  for every  $j \in \{1, 2, 3\}$ . In other words,  $q_1^i - q_2^i - q_3^i$  is an induced  $P_3$  in  $G$  of  $L$ -type  $\gamma$  which is anticomplete



to  $x_1 - x_2 - x_3$ . Hence,  $\{q_1^i - q_2^i - q_3^i : i \in [t]\} \cup \{x_1 - x_2 - x_3\}$  comprises  $t + 1$  mutually anticomplete  $P_3$ 's in  $G$  of  $L$ -type  $\gamma$ , which is impossible. This proves (11).

(12) *Let  $(G, L_1) \in \Upsilon_1(G, L)$ . Then there exists a spanning  $(G, L_1)$ -profile  $\Upsilon_2(G, L_1)$  with the following specifications.*

- $|\Upsilon_2(G, L_1)| \leq \mathcal{O}(|V(G)|^{(3k-3)(f_{L,\gamma}(G)-1)})$  and  $\Upsilon_2(G, L_1)$  can be computed from  $(G, L_1)$  in time  $\mathcal{O}(|V(G)|^{(3k-2)(f_{L,\gamma}(G)-1)})$ .
- For every  $(G, L_2) \in \Upsilon_2(G, L_1)$ ,  $G$  has no induced  $P_3$  of  $L_2$ -type  $\gamma$  (and, of course, none of  $L_2$ -weight strictly larger than  $|I_1| + |I_2| + |I_3|$ ).
- If  $G$  admits a frugal  $L_1$ -coloring, then for some  $(G, L_2) \in \Upsilon_2(G, L)$ ,  $G$  admits a frugal  $L_2$ -coloring.

By the assumption of Lemma 16,  $G$  has no  $L$ -good  $P_3$  of  $L_1$ -weight strictly larger than  $|I_1| + |I_2| + |I_3|$ . So since  $(G, L_1)$  is a  $(G, L)$ -refinement,  $G$  has no  $L_1$ -good  $P_3$  of  $L_1$ -weight strictly larger than  $|I_1| + |I_2| + |I_3|$ . This, along with (11) and the induction hypothesis, proves (12).

Finally, we define  $\Upsilon_2(G, L) = \bigcup_{(G, L_1) \in \Upsilon_1(G, L)} \Upsilon_2(G, L_1)$ , where for every  $(G, L_1) \in \Upsilon_1(G, L)$ ,  $\Upsilon_2(G, L_1)$  is as promised in (12). By the first bullet of (12) and the first bullet of Lemma 15, we have  $|\Upsilon_2(G, L)| \leq \mathcal{O}(|V(G)|^{(3k-3)f_{L,\gamma}(G)})$  and  $\Upsilon_2(G, L)$  can be computed from  $(G, L)$  in time  $\mathcal{O}(|V(G)|^{(3k-2)}) + \mathcal{O}(|V(G)|^{(3k-3)}) \mathcal{O}(|V(G)|^{(3k-2)(f_{L,\gamma}(G)-1)}) = \mathcal{O}(|V(G)|^{(3k-2)f_{L,\gamma}(G)})$ . So  $\Upsilon_2(G, L)$  satisfies the first bullet of Lemma 16. Also, the second bullet of Lemma 16 for  $\Upsilon_2(G, L)$  follows from the second bullet of (12), and the third bullet of Lemma 16 for  $\Upsilon_2(G, L)$  follows from the third bullet of (12) together with the third bullet of Lemma 15. Hence,  $\Upsilon_2(G, L)$  satisfies Lemma 16. This completes the proof.  $\blacksquare$

Here is the main theorem of this section.

**Theorem 17.** *For all  $k, r \in \mathbb{N}$ , there exists  $v(k, r) \in \mathbb{N}$  with the following property. Let  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM where  $G$  is  $rP_3$ -free. Then there exists a spanning  $(G, L)$ -profile  $\Upsilon(G, L)$  with the following specifications.*

- $|\Upsilon(G, L)| \leq \mathcal{O}(|V(G)|^{v(k,r)})$ , and  $\Upsilon(G, L)$  can be computed from  $(G, L)$  in time  $\mathcal{O}(|V(G)|^{v(k,r)})$ .
- For every  $(G, L') \in \Upsilon(G, L)$ ,  $G$  has no  $L'$ -good  $P_3$ .
- If  $G$  admits a frugal  $L$ -coloring, then for some  $(G, L') \in \Upsilon(G, L)$ ,  $G$  admits a frugal  $L'$ -coloring.

*Proof.* If  $k = 1$ , then by setting  $v(1, r) = 1$  and  $\Upsilon(G, L) = \{(G, L)\}$ , we are done. So we may assume that  $k \geq 2$ . Let  $(\gamma_i = (I_1^i, I_2^i, I_3^i) : i \in [m])$  be an enumeration of all good triples of

subsets of  $[k]$ , such that for  $i, j \in [m]$ ,  $i > j$  implies  $|I_1^i| + |I_2^i| + |I_3^i| \leq |I_1^j| + |I_2^j| + |I_3^j|$ . Note that  $m \leq 2^{3k}$ , and so this enumeration can be computed in constant time.

(13) For every list assignment  $K$  of  $G$  and every  $i \in [m]$ , we have  $f_{K, \gamma_i}(G) \leq r - 1$ .

For otherwise  $G$  contains  $r$  mutually anticomplete induced  $P_3$ 's, which violates the assumption of Theorem 17 that  $G$  is  $rP_3$ -free. This proves (13).

(14) There exists a sequence  $(\Lambda_0, \dots, \Lambda_m)$  of spanning  $(G, L)$ -profiles, where  $\Lambda_0 = \{(G, L)\}$ , and for each  $i \in [m]$ , the following hold.

- $|\Lambda_i| \leq \mathcal{O}(|V(G)|^{i(r-1)(3k-3)})$  and  $\Lambda_i$  can be computed from  $\Lambda_{i-1}$  in time  $\mathcal{O}(|V(G)|^{i(r-1)(3k-2)})$ .
- For every  $(G, L') \in \Lambda_i$ ,  $G$  has no induced  $P_3$  of  $L'$ -type in  $\{\gamma_j : j \in [i]\}$ .
- If  $G$  admits a frugal  $L''$ -coloring for some  $(G, L'') \in \Lambda_{i-1}$ , then for some  $(G, L') \in \Lambda_i$ ,  $G$  admits a frugal  $L'$ -coloring.

We generate this sequence recursively. To initiate, note that  $G$  has no  $L$ -good  $P_3$  of  $L$ -weight larger than  $|I_1^1| + |I_2^1| + |I_3^1|$ . Thus, we may apply Lemma 16 to  $(G, L)$  and  $\gamma_1$ , obtaining a spanning  $(G, L)$ -profile  $\Upsilon_2(G, L)$  which satisfies Lemma 16. As result, defining  $\Lambda_1 = \Upsilon_2(G, L)$ , then by (13),  $\Lambda_1$  satisfies the bullet conditions of (14) for  $i = 1$ . Next, assume that for some  $i \in \{2, \dots, m\}$ , the  $(G, L)$ -profile  $\Lambda_{i-1}$ , satisfying the bullet conditions of (14), is computed. In particular, for every  $(G, L'') \in \Lambda_{i-1}$ ,  $G$  has no induced  $P_3$  of  $L''$ -type in  $\{\gamma_j : j \in [i-1]\}$ . As a result,  $G$  has no  $L''$ -good  $P_3$  of  $L''$ -weight larger than  $|I_1^i| + |I_2^i| + |I_3^i|$ . Thus, we may apply Lemma 16 to  $(G, L'')$  and  $\gamma_i$ , obtaining a spanning  $(G, L)$ -profile  $\Upsilon_2(G, L'')$  which satisfies Lemma 16. Let  $\Lambda_i = \bigcup_{(G, L'') \in \Lambda_{i-1}} \Upsilon_2(G, L'')$ . We claim that  $\Lambda_i$  satisfies the bullet conditions of (14). To see this, from (13) and the first bullet of Lemma 16, we deduce that for every  $(G, L'') \in \Lambda_{i-1}$ ,  $|\Upsilon_2(G, L'')| \leq \mathcal{O}(|V(G)|^{f_{L'', \gamma_i}(G)(3k-3)}) = \mathcal{O}(|V(G)|^{(r-1)(3k-3)})$  and  $\Upsilon_2(G, L'')$  can be computed from  $(G, L'')$  in time  $\mathcal{O}(|V(G)|^{f_{L'', \gamma_i}(G)(3k-2)}) = \mathcal{O}(|V(G)|^{(r-1)(3k-2)})$ . This, along with the fact that  $\Lambda_{i-1}$  satisfies the first bullet of (14), implies that  $|\Lambda_i| \leq \mathcal{O}(|V(G)|^{(i-1)(r-1)(3k-3)})\mathcal{O}(|V(G)|^{(r-1)(3k-3)}) = \mathcal{O}(|V(G)|^{i(r-1)(3k-3)})$ , and  $\Lambda_i$  can be computed from  $\Lambda_{i-1}$  in time  $\mathcal{O}(|V(G)|^{(i-1)(r-1)(3k-3)})\mathcal{O}(|V(G)|^{(r-1)(3k-2)}) = \mathcal{O}(|V(G)|^{i(r-1)(3k-2)})$ . Therefore,  $\Lambda_i$  satisfies the first bullet of (14). Moreover, for every  $(G, L') \in \Lambda_i$ , say  $(G, L') \in \Upsilon_2(G, L'')$  for some  $(G, L'') \in \Lambda_{i-1}$ , by second bullet of Lemma 16,  $G$  has no induced  $P_3$  of  $L'$ -type  $\gamma_i$ . Also, since  $(G, L')$  is a  $(G, L'')$ -refinement, by the second bullet (14) for  $\Lambda_{i-1}$ ,  $G$  has no induced  $P_3$  of  $L'$ -type in  $\{(I_1^j, I_2^j, I_3^j) : j \in [i-1]\}$ . It follows that  $G$  has no induced  $P_3$  of  $L'$ -type in  $\{(I_1^j, I_2^j, I_3^j) : j \in [i]\}$ . So  $\Lambda_i$  satisfies the second bullet of (14). Finally, the third bullet of Lemma 16 implies that, if  $G$  admits a frugal  $L''$ -coloring for some  $(G, L'') \in \Lambda_{i-1}$ , then for some  $(G, L') \in \Lambda_i$ ,  $G$  admits a frugal  $L'$ -coloring. So  $\Lambda_i$  satisfies the third bullet of (14). This proves (14).

Now, let  $(\Lambda_1, \dots, \Lambda_m)$  be as in (14). Let  $\Upsilon(G, L) = \Lambda_m$ . Then, since  $m \leq 2^{3k}$ , by the first bullet of (14) for  $i = m$ , we have  $|\Upsilon(G, L)| \leq \mathcal{O}(|V(G)|^{(r-1)(3k-3)2^{3k}})$ , and by the first bullet of (14) for  $i = 0, 1, \dots, m$ ,  $\Upsilon(G, L)$  can be computed in time  $\mathcal{O}(|V(G)|^{(r-1)(3k-2)2^{3k}})$ . So by setting  $v(k, r) = (r-1)(3k-2)2^{3k}$ ,  $\Upsilon(G, L)$  satisfies the first bullet of Theorem 17. Also, by the second bullet of (14) for  $i = m$ , for every  $(G, L') \in \Upsilon(G, L)$ ,  $G$  has no induced  $P_3$  of  $L'$ -type in  $\{\gamma_i : i \in [m]\}$ , and so  $G$  has no  $L'$ -good  $P_3$ . Therefore,  $\Upsilon(G, L)$  satisfies the second bullet of Theorem 17. Finally, applying the third bullet of (14) to  $i = 0, 1, \dots, m$  consecutively, it follows that if  $G$  admits a frugal  $L$ -coloring, then for some  $(G, L') \in \Upsilon(G, L)$ ,  $G$  admits a frugal  $L'$ -coloring. Hence,  $\Upsilon(G, L)$  satisfies the third bullet of Theorem 17. This completes the proof.  $\blacksquare$

## 4. FIVE COLORS AND VERTICES WITH LARGE LISTS

In this section, we take the last major step towards the proof of Theorem 5: we show that essentially every instance of the LIST-5-COLORING PROBLEM which has at least one vertex of list-size three or more and no good  $P_3$ 's can be reduced in polynomial time to a “smaller” instance. We prove this formally in Theorem 20, whose proof relies crucially on two lemmas, and in order to state them, we need another definition. Let  $k \in \mathbb{N}$  and  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM. We denote by  $G^L$  the graph with  $V(G^L) = V(G)$  and  $E(G^L) = \{uv \in E(G) : L(u) \cap L(v) \neq \emptyset\}$ . Note that  $(G, L)$  and  $(G^L, L)$  have the same state of feasibility. But  $G^L$  is not necessarily an induced subgraph of  $G$ , and so for our purposes, it seems dangerous to consider  $(G^L, L)$  as a ‘simplified’ instance to investigate. However, it turns out that we may still take advantage of certain properties of  $G^L$ . For example, the following lemma proposes a useful interaction between frugality and good  $P_3$ 's in terms of vertex degrees in  $G^L$ .

**Lemma 18.** *Let  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM such that  $|L(v)| \neq 1$  for every  $v \in V(G)$ , and  $G$  has no  $L$ -good  $P_3$ . If  $G$  admits a frugal  $L$ -coloring  $\phi$ , then for every vertex  $v \in V(G)$ ,  $\phi$  assigns mutually distinct colors to all vertices in  $N_{G^L}[v]$ , and in particular, we have  $|N_{G^L}(v)| < k$ .*

*Proof.* Suppose not. Then since  $\phi$  is proper, there exist two vertices  $u, w \in N_{G^L}(v)$  such that  $u$  and  $w$  are nonadjacent in  $G$  and  $\phi(u) = \phi(w) \in L(u) \cap L(w)$ . Also, since  $u, w \in N_{G^L}(v)$ , both  $L(u) \cap L(v)$  and  $L(v) \cap L(w)$  are nonempty. But then  $u - v - w$  is an  $L$ -good  $P_3$  in  $G$ , a contradiction. This completes the proof.  $\blacksquare$

The following technical lemma also unravels the structural properties of the second neighborhood of certain vertices in  $G^L$ . We use this lemma extensively while proving Theorem 20.

**Lemma 19.** *Let  $(G, L)$  be an instance of the LIST-5-COLORING PROBLEM such that  $|L(u)| \in \{0, 2, 3\}$  for all  $u \in V(G)$  and  $G$  has no  $L$ -good  $P_3$ . Moreover, suppose that there exists a vertex  $u_0 \in V(G)$  with  $L(u_0) = \{1, 2, 3\}$ . In addition, let  $A = N_{G^L}(u_0) \cap L^{(4)}$ ,  $B = N_{G^L}(u_0) \cap L^{(5)}$ ,  $A' = \{w \in N_{G^L}^2(u_0) : N_{G^L}(w) \cap A \neq \emptyset\}$ ,  $B' = \{w \in N_{G^L}^2(u_0) : N_{G^L}(w) \cap B \neq \emptyset\}$ . Then the following hold.*

- $|L(u)| \in \{2, 3\}$  for every  $u \in N_{G^L}^2[u_0]$ .
- $L(w) = \{4, 5\}$  for every  $w \in N_{G^L}^2(u_0)$ .
- $N_{G^L}^2(u_0) = A' \cup B'$ .
- Both  $A$  and  $B$  are cliques of  $G^L$ .
- For every vertex  $w \in N_{G^L}^2(u_0)$ ,  $w$  is either complete or anticomplete to  $A$  in  $G^L$ , and either complete or anticomplete to  $B$  in  $G^L$ . In particular,  $A'$  is complete to  $A$  in  $G^L$ , and  $B'$  is complete to  $B$  in  $G^L$ .
- Both  $A'$  and  $B'$  are cliques of  $G^L$ .
- If in addition,  $|N_{G^L}^2(u_0)| \geq 2$  and  $|A'|, |B'| \leq 1$ , then
  - $|A'| = |B'| = 1$  and  $A' \cap B' = \emptyset$ ;
  - $A, B \neq \emptyset$  and  $A \cap B = \emptyset$ ;
  - $A'$  is anticomplete to  $B$  in  $G$  and  $B'$  is anticomplete to  $A$  in  $G$ , and;
  - for every  $a \in A$  and every  $b \in B$ , we have  $L(a) \cap L(b) = \emptyset$ .

*Proof.* The first bullet follows directly from  $L(u_0) = \{1, 2, 3\} \neq \emptyset$ ,  $u_0 \in N_{G^L}^2[u_0]$  and  $G^L|N_{G^L}^2[u_0]$  being connected.

To see the second bullet, let  $w \in N_{G^L}^2(u_0)$  and  $v \in N_{G^L}(u_0) \cap N_{G^L}(w)$ . If  $L(u_0) \cap L(w) \neq \emptyset$ , then  $u_0w \notin E(G)$ , and so from  $u_0v, vw \in E(G^L)$  and first bullet of Lemma 19, it follows that  $u_0 - v - w$  is an  $L$ -good  $P_3$  in  $G$ , which is impossible. Consequently, we have  $L(w) \subseteq [5] \setminus L(u_0) =$

$\{4, 5\}$ , and so by the first bullet of Lemma 19, we have  $L(w) = \{4, 5\}$ . This proves the second bullet of Lemma 19.

To verify the third bullet, note that the inclusion  $A' \cup B' \subseteq N_{G^L}^2(u_0)$  is clear. Now, let  $w \in N_{G^L}^2(u_0)$  and  $v \in N_{G^L}(u_0) \cap N_{G^L}(w)$ . Then by the second bullet of Lemma 19, we have  $L(v) \cap \{4, 5\} = L(v) \cap L(w) \neq \emptyset$ , and so  $v \in A \cup B$ , which in turn implies that  $w \in A' \cup B'$ . So  $N_{G^L}^2(u_0) \subseteq A' \cup B'$ , and the third bullet of Lemma 19 follows.

To see the fourth bullet, suppose for a contradiction that there exist  $a_1, a_2 \in A$  with  $a_1 a_2 \notin E(G^L)$ . Therefore, since  $4 \in L(a_1) \cap L(a_2)$ , we have  $a_1 a_2 \notin E(G)$ . This, together with  $a_1, a_2 \in N_{G^L}(u_0) \subseteq N_G(u_0)$  and the first bullet of Lemma 19, implies that  $a_1 - u_0 - a_2$  is an  $L$ -good  $P_3$  in  $G$ , a contradiction. So  $A$  is a clique of  $G^L$ . Similarly, one can show that  $B$  is also a clique of  $G^L$ , and so the fourth bullet of Lemma 19 follows.

Now we argue the fifth bullet. Suppose for a contradiction that there exists  $w \in N_{G^L}^2(u_0)$ , such that in  $G^L$ ,  $w$  has both a neighbor  $x$  and a non-neighbor  $y$  in either  $A$  or  $B$ , say the former. By the fourth bullet of Lemma 19, we have  $xy \in E(G^L) \subseteq E(G)$ . Also, by the second bullet of Lemma 19, we have  $4 \in L(w) \cap L(x) \cap L(y)$ , which in turn implies that  $wy \notin E(G)$ ; that is,  $w - x - y$  is an induced  $P_3$  in  $G$ . From this and the second bullet of Lemma 19, it follows that  $w - x - y$  is an  $L$ -good  $P_3$  in  $G$ , a contradiction. The case  $x, y \in B$  can be handled similarly. This proves the fifth bullet of Lemma 19.

For the sixth bullet, suppose for a contradiction that  $a'_1, a'_2 \in A'$  are not adjacent in  $G^L$ . Then by the second bullet of Lemma 19, we have  $L(a'_1) = L(a'_2) = \{4, 5\}$ , and so  $a'_1$  and  $a'_2$  are not adjacent in  $G$ . Note that since  $A' \neq \emptyset$ , by the definition of  $A'$ , we have  $A \neq \emptyset$ , and so we may pick a vertex  $a_0 \in A$ . Therefore, by the sixth bullet of Lemma 19,  $a'_1 - a_0 - a'_2$  is an induced  $P_3$  in  $G$  with  $4 \in L(a'_1) \cap L(a_0) \cap L(a'_2)$ , which along with the first bullet of Lemma 19, implies that  $a'_1 - a_0 - a'_2$  is an  $L$ -good  $P_3$  in  $G$ , a contradiction. So the sixth bullet of Lemma 19 follows.

The rest of the proof aims to verify the seventh bullet of Lemma 19. For the first dash, from  $|N_{G^L}^2(u_0)| \geq 2$  and the third bullet of Lemma 19, we have  $|A' \cup B'| \geq 2$ , which along with  $|A'|, |B'| \leq 1$ , implies that  $|A'| = |B'| = 1$  and  $A' \cap B' = \emptyset$ , as desired. Henceforth, we assume  $A' = \{a'\}$  and  $B' = \{b'\}$  for distinct  $a', b'$ . Note that by the second bullet of Lemma 19, we have  $L(a') = L(b') = \{4, 5\}$ .

For the second dash, note that the fact that  $|A'| = |B'| = 1$  along with the definition of  $A'$  and  $B'$ , implies that  $A, B \neq \emptyset$ . Next we show that  $A \cap B = \emptyset$ . Suppose not. Let  $z \in A \cap B$ . By the fifth bullet of Lemma 19,  $a'$  and  $b'$  are adjacent to  $z$  in  $G^L$ . Thus, from  $z \in A \cap B$  and again the fifth bullet of Lemma 19, we deduce that  $A' \cup B' = \{a', b'\}$  is complete to  $A \cup B$  in  $G^L$ . But then from the definition of  $A'$  and  $B'$ , it follows that  $A = B$ , which in turn implies that  $A' = B'$ , a contradiction with the first dash. So  $A \cap B = \emptyset$ , as desired.

To see the third dash, suppose for a contradiction that  $a'$  has a neighbor  $b \in B$  in  $G$ . Then, since  $5 \in L(a') \cap L(b)$ , we have  $a'b \in E(G^L)$ . But then from the definition of  $B'$ , we have  $a' \in B'$ , and so  $a' \in A' \cap B'$ , which violates the first dash. Note that if  $b'$  has a neighbor in  $A$ , then a contradiction can be derived similarly.

Finally, we prove the fourth dash. Suppose not. Let  $L(a) \cap L(b) \neq \emptyset$  for distinct  $a \in A$  and  $b \in B$ . If  $ab \notin E(G)$ , then from  $a, b \in N_{G^L}(u_0)$  and the first bullet of Lemma 19, we deduce that  $a - u_0 - b$  is an  $L$ -good  $P_3$ , which is impossible. As a result, we have  $ab \in E(G)$ , and so  $ab \in E(G^L)$ . By the fifth bullet of Lemma 19,  $a'$  is adjacent to  $a$  in  $G^L$ , and by the third dash,  $a'$  is not adjacent to  $b$  in  $G$ . So  $a' - a - b$  is an induced  $P_3$  in  $G$ . This, along with  $4 \in L(a') \cap L(a)$ ,  $5 \in L(a') \cap L(b)$  and  $L(a) \cap L(b) \neq \emptyset$  implies that  $a' - a - b$  is an  $L$ -good  $P_3$  in  $G$ , a contradiction. This proves the fourth dash of the seventh bullet of Lemma 19, and so concludes the proof. ■

Let  $k \in \mathbb{N}$  and  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM. We define  $p(G, L) = |V(G)| + \sum_{v \in V(G)} |L(v)|$ . It is immediate from the definition that  $p(G, L) \leq (k+1)|V(G)|$ . For a  $(G, L)$ -refinement  $(G', L')$ , we say that  $(G', L')$  represents  $(G, L)$  if the following hold.

- (R1)  $p(G', L') < p(G, L)$ .
- (R2) If  $G$  admits a frugal  $L$ -coloring, then  $G'$  admits a frugal  $L'$ -coloring.
- (R3) If  $G'$  admits an  $L'$ -coloring, then  $G$  admits an  $L$ -coloring.

**Theorem 20.** *Let  $(G, L)$  be an instance of the LIST-5-COLORING PROBLEM such that  $|L(v)| \neq 1$  for all  $v \in V(G)$  and  $G$  has no  $L$ -good  $P_3$ . Moreover, suppose that there exists a vertex  $u_0 \in V(G)$  with  $|L(u_0)| \geq 3$ . Then there exists a  $(G, L)$ -refinement  $(\tilde{G}, \tilde{L})$  with the following specifications.*

- $(\tilde{G}, \tilde{L})$  can be computed from  $(G, L)$  in time  $\mathcal{O}(|V(G)|^2)$ .
- $|\tilde{L}(v)| \neq 1$  for all  $v \in V(\tilde{G})$ .
- $(\tilde{G}, \tilde{L})$  represents  $(G, L)$ .

*Proof.* Without loss of generality, we may assume that  $\{1, 2, 3\} \subseteq L(u_0)$ . We define the four sets  $A = N_{GL}(u_0) \cap L^{(4)}$ ,  $B = N_{GL}(u_0) \cap L^{(5)}$ ,  $A' = \{w \in N_{GL}^2(u_0) : N_{GL}(w) \cap A \neq \emptyset\}$  and  $B' = \{w \in N_{GL}^2(u_0) : N_{GL}(w) \cap B \neq \emptyset\}$  as in Lemma 19. For every vertex  $u \in V(G)$ , let  $\Phi_u$  be the set of all frugal  $L|_{N_{GL}^2[u]}$ -colorings of  $G|_{N_{GL}^2[u]}$ . Consider the following algorithm, called **algorithm A**, which, given  $G, L$  and  $u_0$ , computes a  $(G, L)$ -refinement  $(G^*, L^*)$ .

- Step 1:* Using BFS, compute  $N_{GL}(v)$  and  $N_{GL}^2(v)$  for every  $v \in V(G)$ . Go to step 2, and from each step, proceed to the one below unless instructed otherwise.
- Step 2:* Compute  $A, B, A'$  and  $B'$ .
- Step 3:* If  $|N_{GL}(u)| \geq 5$  for some  $u \in V(G)$ , then compute  $G^* = G$ ,  $L^*(v) = \emptyset$  for every  $v \in V(G^*)$ . Return  $(G^*, L^*)$ .
- Step 4:* If  $|N_{GL}(u)| < |L(u)|$  for some  $u \in V(G)$ , then compute  $G^* = G - u$ ,  $L^*(v) = L(v)$  for every  $v \in V(G^*)$ . Return  $(G^*, L^*)$ .
- Step 5:* If  $|N_{GL}^2(u)| \leq 1$  for some  $u \in V(G)$ , then
  - (a) Compute  $\Phi_u$  by brute-forcing.
  - (b) If  $\Phi_u = \emptyset$ , then compute  $G^* = G$  and  $L^*(v) = \emptyset$  for every  $v \in V(G^*)$ . Return  $(G^*, L^*)$ .
  - (c) Otherwise, Compute  $G^* = G - N_{GL}[u]$ ,  $L^*(v) = \{i \in L(v) : \phi(v) = i \text{ for some } \phi \in \Phi_u\}$  for every  $v \in N_{GL}^2(u)$  and  $L^*(v) = L(v)$  for every  $v \in V(G^*) \setminus N_{GL}^2(u)$ . Return  $(G^*, L^*)$ .
- Step 6:* If  $|A'| \geq 2$ , then compute  $G^* = G$ ,  $L^*(a) = L(a) \setminus \{4, 5\}$  for every  $a \in A$  and  $L^*(v) = L(v)$  for every  $v \in V(G^*) \setminus A$ . Return  $(G^*, L^*)$ .
- Step 7:* If  $|B'| \geq 2$ , then compute  $G^* = G$ ,  $L^*(b) = L(b) \setminus \{4, 5\}$  for every  $b \in B$  and  $L^*(v) = L(v)$  for every  $v \in V(G^*) \setminus B$ . Return  $(G^*, L^*)$ .
- Step 8:* If there exist distinct vertices  $a_1, a_2 \in A$  with  $L(a_1) = L(a_2)$  and  $|L(a_1)| = |L(a_2)| = 2$ , then compute  $G^* = G$ ,  $L^*(a') = L(a') \setminus \{4\}$  for every  $a' \in A'$  and  $L^*(v) = L(v)$  for every  $v \in V(G^*) \setminus A'$ . Return  $(G^*, L^*)$ .
- Step 9:* If there exist distinct vertices  $b_1, b_2 \in B$  with  $L(b_1) = L(b_2)$  and  $|L(b_1)| = |L(b_2)| = 2$ , then compute  $G^* = G$ ,  $L^*(b') = L(b') \setminus \{5\}$  for every  $b' \in B'$  and  $L^*(v) = L(v)$  for every  $v \in V(G^*) \setminus B'$ . Return  $(G^*, L^*)$ .
- Step 10:* If  $|N_{GL}(u_0)| \geq 4$ , then compute  $G^* = G$ ,  $L^*(a') = L(a') \setminus \{4\}$  for every  $a' \in A'$ ,  $L^*(b') = L(b') \setminus \{5\}$  for every  $b' \in B'$  and  $L^*(v) = L(v)$  for every  $v \in V(G^*) \setminus (A' \cup B')$ . Return  $(G^*, L^*)$ .
- Step 11:* Compute a minimal subset  $M$  of  $L(u_0)$  such that  $M \cap L(a) \neq \emptyset$  for some  $a \in A$  and  $M \cap L(b) \neq \emptyset$  for some  $b \in B$ . Choose  $i \in M \cap L(a)$  and  $j \in M \cap L(b)$ . Compute  $G^* = G \setminus (\{u_0, a, b\} \cup (V(G) \setminus N_{GL}[u_0]))$ ,  $L^*(u_0) = M$ ,  $L(a) = \{i, 4\}$ ,  $L(b) = \{j, 5\}$  and  $L^*(v) = L(v)$  for every  $v \in V(G^*) \setminus \{u_0, a, b\}$ . Return  $(G^*, L^*)$ .

As a general property of algorithm **A**, note that for each step other than steps 1, 2 and 5a, if the corresponding ‘if condition’ is satisfied, then the algorithm terminates at that step. In particular,

(15) *Suppose that  $|L(u)| \geq 4$  for some  $u \in V(G)$ . Then algorithm **A** terminates at or before step 5.*

Let  $i \in [5]$  with  $[5] \setminus L(u) \subseteq \{i\}$ . We claim that  $N_{G^L}^2(u) = \emptyset$ . Suppose not. Let  $w \in N_{G^L}^2(u)$ . Then since  $G$  has no good  $P_3$ , we have  $L(u) \cap L(w) = \emptyset$ , and so  $L(w) \subseteq \{i\}$ . Also, there exists  $v \in N_{G^L}(u)$  which is adjacent to  $w$  in  $G^L$ , and so  $L(v) \cap L(w) \neq \emptyset$ . Thus,  $L(w) = \{i\}$ , and so  $|L(w)| = 1$ , a contradiction. This proves the claim. But then the ‘if condition’ in step 5 is satisfied, and so algorithm **A** terminates at or before step 5. This proves (15).

We deduce:

(16) *If algorithm **A** does not stop at steps 3-7, then all seven bullets of Lemma 19 hold.*

Note that algorithm **A** does not terminate at step 5. This has two consequences. First, by (15) and the assumption of Theorem 20, we have  $|L(u)| \in \{0, 2, 3\}$  for every  $u \in V(G)$ . In particular, we have  $L(u_0) = \{1, 2, 3\}$ . Second, the ‘if condition’ of step 5 is not satisfied, and in particular  $|N_{G^L}^2(u_0)| \geq 2$ . In addition, since algorithm **A** does not stop at steps 6 and 7, the ‘if condition’ in these two steps is not satisfied, and so  $|A'|, |B'| \leq 1$ . Therefore, all seven bullets of Lemma 19 hold. This proves (16).

(17) *Algorithm **A** terminates in finite time. Indeed, it runs in time  $\mathcal{O}(|V(G)|^2)$ .*

Note that if the algorithm does not terminate in steps 3-10, then it arrives at step 11, and by (16), all seven bullets of Lemma 19 hold. In particular, by the second dash of the seventh bullet of Lemma 19, we have  $A, B \neq \emptyset$ . As a result, the set  $M$  mentioned in step 11 is well-defined. So algorithm **A** executes step 11, and stops in finite time. We leave the reader to check the straightforward fact that the overall running time of algorithm **A** is  $\mathcal{O}(|V(G)|^2)$ . This proves (17).

We need to show that the output  $(G^*, L^*)$  of algorithm **A** represents  $(G, L)$ . The proof is broken into several statements, below.

(18) *If algorithm **A** terminates at step 3, then  $(G^*, L^*)$  represents  $(G, L)$ .*

Note that  $|L^*(u_0)| = 0 < 3 \leq |L(u_0)|$ , and so  $(G^*, L^*)$  satisfies (R1). Also, the ‘if condition’ of step 3 is satisfied, and there exists a vertex  $u \in V(G)$  with  $|N_{G^L}(u)| \geq 5$ . Thus,  $L$  being a 5-list assignment, by Lemma 18,  $G$  admits no frugal  $L$ -coloring, and so  $(G^*, L^*)$  vacuously satisfies (R2). Moreover, since  $L^*(v) = \emptyset$  for every  $v \in V(G^*)$ ,  $G^*$  admits no  $L^*$ -coloring, and so  $(G^*, L^*)$  vacuously satisfies (R3), as well. This proves (18).

(19) *If algorithm **A** terminates at step 4, then  $(G^*, L^*)$  represents  $(G, L)$ .*

The ‘if condition’ of step 4 is satisfied, and so we have  $|N_{G^L}(u)| < |L(u)|$  for some  $u \in V(G)$ ,  $G^* = G - u$  and  $L^*(v) = L(v)$  for every  $v \in V(G^*)$ . As a result,  $|V(G^*)| < |V(G)|$ , and so  $(G^*, L^*)$  satisfies (R1). Moreover, if  $G$  admits a frugal  $L$ -coloring  $\phi$ , then by Lemma 9,  $\phi|_{V(G^*)}$  is a frugal  $L^*$ -coloring of  $G^*$ , and so  $(G^*, L^*)$  satisfies (R2). Now, suppose that  $G^*$  admits an  $L^*$ -coloring  $\psi$ . Since  $|\{\psi(v) : v \in N_{G^L}(u)\}| \leq |N_{G^L}(u)| < |L(u)|$ , there exists a color  $j \in L(u) \setminus \{\psi(v) : v \in N_{G^L}(u)\}$ . Hence, extending  $\phi$  to  $G$  by defining  $\psi(u) = j$ , we obtain an  $L$ -coloring of  $G$ , and so  $(G^*, L^*)$  satisfies (R3). This proves (19).

(20) *If algorithm **A** terminates at step 5, then  $(G^*, L^*)$  represents  $(G, L)$ .*

The ‘if condition’ of step 5 is satisfied, and so we have  $|N_{G^L}^2(u)| \leq 1$  for some  $u \in V(G)$ . Now, suppose that  $\Phi_u = \emptyset$ . Then algorithm **A** terminates at step 5b,  $G^* = G$  and  $L^*(v) = \emptyset$  for every  $v \in V(G)$ . Note that  $|L^*(u_0)| = 0 < 3 \leq |L(u_0)|$ , and so  $(G^*, L^*)$  satisfies (R1). Also, from  $\Phi_u = \emptyset$ , it follows that  $G|N_{G^L}^2[u]$  admits no frugal  $L|N_{G^L}^2[u]$ -coloring, and so  $G$  admits no frugal  $L$ -coloring. Thus,  $(G^*, L^*)$  vacuously satisfies (R2). Moreover, since  $L^*(v) = \emptyset$  for every  $v \in V(G^*)$ ,  $G^*$  admits no  $L^*$ -coloring, and  $(G^*, L^*)$  vacuously satisfies (R3), as well.

Therefore, we may assume that  $\Phi_u \neq \emptyset$ . Then algorithm **A** terminates at step 5c,  $G^* = G - N_{G^L}[u]$ ,  $L^*(v) = \{i \in L(v) : \phi(v) = i \text{ for some } \phi \in \Phi_u\}$  for every  $v \in N_{G^L}^2(u)$  and  $L^*(v) = L(v)$  for every  $v \in V(G^*) \setminus N_{G^L}^2(u)$ . It follows that  $|V(G^*)| < |V(G)|$ , and so  $(G^*, L^*)$  satisfies (R1). For (R2), suppose that  $G$  admits a frugal  $L$ -coloring  $\psi$ . Then  $\phi = \psi|_{N_{G^L}^2[u]}$  is readily seen to be a frugal  $L|N_{G^L}^2[u]$ -coloring of  $G|N_{G^L}^2[u]$ ; that is,  $\phi \in \Phi_u$ . So for every  $v \in N_{G^L}^2(u)$ ,  $\psi(v) = \phi(v) \in L^*(v)$ . Also  $\psi(v) \in L(v) = L^*(v)$  for all  $v \in V(G^*) \setminus N_{G^L}^2(u)$ . Thus,  $\psi$  being a frugal  $L$ -coloring of  $G$ , it follows from Lemma 9 that  $\psi|_{V(G^*)}$  is a frugal  $L^*$ -coloring of  $G^*$ . This shows that  $(G^*, L^*)$  satisfies (R2). Now we need to argue that  $(G^*, L^*)$  satisfies (R3). Suppose that  $G^*$  admits an  $L^*$ -coloring  $\psi'$ . Since  $|N_{G^L}^2(u)| \leq 1$  and  $\psi'(c) \in L^*(c)$  for every  $c \in N_{G^L}^2(u)$ , there exists  $\phi' \in \Phi_u$  such that  $\phi'(c) = \psi'(c)$  for all  $c \in N_{G^L}^2(u)$ . We define a coloring  $\theta : V(G) \rightarrow [5]$  as follows. For every  $v \in N_{G^L}[u_0]$ , let  $\theta(v) = \phi'(v)$ . For all  $c \in N_{G^L}^2(u)$ , let  $\theta(c) = \phi'(c) = \psi'(c)$ . For every  $v \in V(G) \setminus N_{G^L}^2[u_0] = V(G^*) \setminus N_{G^L}^2(u_0)$ , let  $\theta(v) = \psi'(v)$ . We claim that  $\theta$  is an  $L$ -coloring of  $G$ . To see this, note that since  $\phi' \in \Phi_u$ , we have  $\theta(v) = \phi'(v) \in L(v)$  for every  $v \in N_{G^L}^2[u_0]$ . Also,  $\theta(v) = \psi'(v) \in L^*(v) = L(v)$  for every  $v \in V(G) \setminus N_{G^L}^2[u_0]$ . So  $\theta(v) \in L(v)$  for all  $v \in V(G)$ . It remains to show that  $\theta$  is proper. Let  $xy \in E(G)$ . If  $x, y \in N_{G^L}^2[u_0]$ , then since  $\phi'$  is proper, we have  $\theta(x) = \phi'(x) \neq \phi'(y) = \theta(y)$ . If  $x, y \in V(G) \setminus N_{G^L}^2[u_0] = V(G^*)$ , then since  $\psi'$  is proper,  $\theta(x) = \psi'(x) \neq \psi'(y) = \theta(y)$ . Finally, let  $x \in N_{G^L}[u_0]$  and  $y \in V(G) \setminus N_{G^L}^2[u_0]$ . Then  $xy \notin E(G^L)$ , and so  $L(x) \cap L(y) = \emptyset$ . Hence  $\theta(x) \neq \theta(y)$ , and  $\theta$  is proper. This proves (20).

(21) *If algorithm **A** terminates at step 6, then  $(G^*, L^*)$  represents  $(G, L)$ .*

The ‘if condition’ of step 6 is satisfied, and so we have  $|A'| \geq 2$ ,  $G^* = G$ ,  $L^*(a) = L(a) \setminus \{4, 5\}$  for every  $a \in A$  and  $L^*(v) = L(v)$  for every  $v \in V(G^*) \setminus A$ . Note that from  $|A'| \geq 2$  and the definition of  $A'$ , it follows that  $A \neq \emptyset$ , and so for every  $a \in A$ , we have  $4 \in L(a) \setminus L^*(a)$ , which in turn implies that  $|L^*(a)| < |L(a)|$ . This shows that  $(G^*, L^*)$  satisfies (R1).

For (R2), suppose that  $G$  admits a frugal  $L$ -coloring  $\phi$ . Let  $a'_1, a'_2 \in A'$  be distinct. Since the algorithm does not stop at step 5, by (15) and the assumption of Theorem 20, we have  $|L(u)| \in \{0, 2, 3\}$  for every  $u \in V(G)$ . In particular, we have  $L(u_0) = \{1, 2, 3\}$ . So by the second bullet of Lemma 19, we have  $L(a'_1) = L(a'_2) = \{4, 5\}$ , by the fifth bullet of Lemma 19,  $a'_1$  and  $a'_2$  are complete to  $A$  in  $G^L$  (and so in  $G$ ), and by the sixth bullet of Lemma 19,  $a'_1$  and  $a'_2$  are adjacent in  $G^L$  (and so in  $G$ ). As a result, we have  $\{\phi(a'_1), \phi(a'_2)\} = \{4, 5\}$ , and for every  $a \in A$ , we have  $\phi(a) \in L(v) \setminus \{\phi(a'_1), \phi(a'_2)\} = L(v) \setminus \{4, 5\} = L^*(a)$ . In addition, we have  $\phi(v) \in L(v) = L^*(v)$  for every  $v \in V(G^*) \setminus A$ . Thus,  $\phi$  being a frugal  $L$ -coloring  $\phi$ , it follows from Lemma 9 that  $\phi|_{V(G^*)}$  is an  $L^*$ -coloring of  $G^*$ , and so  $(G^*, L^*)$  satisfies (R2). Finally, note that  $(G^*, L^*)$  is a spanning  $(G, L)$ -refinement, and by Lemma 8,  $(G^*, L^*)$  satisfies (R3). This proves (21).

The reader may have noticed that steps 6 and 7 of algorithm **A** are symmetrical with respect to  $A$  and  $B$ . As a result, the proof of (22), the following, is identical to that of (21), and so we omit it.

(22) *If algorithm **A** terminates at step 7, then  $(G^*, L^*)$  represents  $(G, L)$ .*

Then we continue with the following.

(23) *If algorithm **A** terminates at step 8, then  $(G^*, L^*)$  represents  $(G, L)$ .*

The ‘if condition’ of step 8 is satisfied, and so there exist distinct vertices  $a_1, a_2 \in A$  with  $L(a_1) = L(a_2)$  and  $|L(a_1)| = |L(a_2)| = 2$ . Also, we have  $G^* = G$ ,  $L^*(a') = L(a') \setminus \{4\}$  for every  $a' \in A'$  and  $L^*(v) = L(v)$  for every  $v \in V(G) \setminus A'$ .

By (16), all seven bullets of Lemma 19 hold. In particular, by the first dash of the seventh bullet of Lemma 19, we have  $|A'| = 1$ , say  $A' = \{a'_0\}$ , and by the second bullet of Lemma 19, we have  $L(a'_0) = \{4, 5\}$ . As a result, we have  $4 \in L(a'_0) \setminus L^*(a'_0)$ , which in turn implies that  $|L^*(a'_0)| < |L(a'_0)|$ . So  $(G^*, L^*)$  satisfies (R1).

To argue the validity of (R2), suppose that  $G$  admits a frugal  $L$ -coloring  $\phi$ . Note that by the fourth bullet of Lemma 19,  $a_1$  and  $a_2$  are adjacent in  $G$ . So from  $L(a_1) = L(a_2)$ ,  $|L(a_1)| = |L(a_2)| = 2$  and  $4 \in L(a_1) \cap L(a_2)$ , we have  $4 \in \{\phi(a_1), \phi(a_2)\}$ . Also, by the fifth bullet of Lemma 19,  $a'_0$  is adjacent to both  $a_1$  and  $a_2$ . Therefore, we have  $\phi(a'_0) \in L(a'_0) \setminus \{\phi(a_1), \phi(a_2)\} \subseteq L(a'_0) \setminus \{4\} = L^*(a'_0)$ . In other words, for every  $v \in A' = \{a'_0\}$ , we have  $\phi(v) \in L^*(v)$ . Moreover, for every  $v \in V(G^*) \setminus A'$ , we have  $\phi(v) \in L(v) = L^*(v)$ . Hence,  $\phi$  being a frugal  $L$ -coloring of  $G$ , by Lemma 9,  $\phi|_{V(G^*)}$  is a frugal  $L^*$ -coloring of  $G^*$ , and so  $(G^*, L^*)$  satisfies (R2). Finally, note that  $(G^*, L^*)$  is spanning  $(G, L)$ -refinement, and so by Lemma 8,  $(G^*, L^*)$  satisfies (R3). This proves (23).

Again, we observe that steps 8 and 9 of algorithm **A** are symmetrical with respect to  $A$  and  $B$ . For this reason, the proof of (24) below is identical to that of (23), and so we omit it.

(24) *If algorithm **A** terminates at step 9, then  $(G^*, L^*)$  represents  $(G, L)$ .*

(25) *If algorithm **A** terminates at step 10, then  $(G^*, L^*)$  represents  $(G, L)$ .*

The ‘if condition’ of step 10 is satisfied, that is  $|N_{G^L}(u_0)| \geq 4$ . Also, since the algorithm **A** does not terminate at step 3, the ‘if condition’ of step 3 does not hold. In particular, we have  $|N_{G^L}(u_0)| \leq 4$ , and so  $|N_{G^L}(u_0)| = 4$ . In addition, we have  $G^* = G$ ,  $L^*(a') = L(a') \setminus \{4\}$  for every  $a \in A'$ ,  $L^*(b') = L(b') \setminus \{5\}$  for every  $b' \in B'$  and  $L^*(v) = L(v)$  for every  $v \in V(G) \setminus (A' \cup B')$ .

By (16), all seven bullets of Lemma 19 hold. In particular, by the first dash of the seventh bullet of Lemma 19, we have  $A' \cap B' = \emptyset$  and  $|A'| = |B'| = 1$ , say  $A' = \{a'_0\}$  and  $B' = \{b'_0\}$  for distinct  $a'_0, b'_0$ , and by the second bullet of Lemma 19, we have  $L(a'_0) = L(b'_0) = \{4, 5\}$ . As a result, we have  $4 \in L^*(a'_0) \setminus L(a'_0)$ , which in turn implies that  $|L^*(a'_0)| < |L(a'_0)|$ . Thus,  $(G^*, L^*)$  satisfies (R1).

To see (R2), suppose that  $G$  admits a frugal  $L$ -coloring  $\phi$ . Since  $|N_{G^L}(u_0)| = 4$ , by Lemma 18, we have  $\phi(N_{G^L}[u_0]) = [5]$ . Thus, from  $L(u_0) = \{1, 2, 3\}$  and the definition of  $A$  and  $B$ , we deduce that there exists  $a_0 \in A$  with  $\phi(a_0) = 4$  and  $b_0 \in B$  with  $\phi(b_0) = 5$ . On the other hand, by the fifth bullet of Lemma 19,  $a'_0$  is adjacent to  $a_0$  and  $b'_0$  is adjacent to  $b_0$  in  $G^L$  (and so in  $G$ ). Therefore, we have  $\phi(a'_0) \in L(a'_0) \setminus \{\phi(a_0)\} = L(a'_0) \setminus \{4\} = L^*(a'_0)$  and  $\phi(b'_0) \in L(b'_0) \setminus \{\phi(b_0)\} = L(b'_0) \setminus \{5\} = L^*(b'_0)$ . In other words, for every  $v \in \{a'_0, b'_0\} = A' \cup B'$ , we have  $\phi(v) \in L^*(v)$ . Moreover, for every  $v \in V(G^*) \setminus (A' \cup B')$ , we have  $\phi(v) \in L(v) = L^*(v)$ . Hence,  $\phi$  being a frugal  $L$ -coloring of  $G$ , by Lemma 9,  $\phi|_{V(G^*)}$  is a frugal  $L^*$ -coloring of  $G^*$ , and so  $(G^*, L^*)$  satisfies (R2). Finally, note that  $(G^*, L^*)$  is spanning  $(G, L)$ -refinement, and so by Lemma 8,  $(G^*, L^*)$  satisfies (R3). This proves (25).

From (18)-(25), we deduce:

(26) *The output  $(G^*, L^*)$  of algorithm **A** represents  $(G, L)$ .*

If algorithm **A** terminates at one of the steps 3-10, then by (18)-(25), we are done. Therefore, we



may assume that algorithm **A** stops at step 11. Since algorithm **A** does not terminate at steps 3 and 10, the ‘if condition’ in these two steps is not satisfied, and so  $3 \leq |L(u_0)| \leq |N_{G^L}(u_0)| \leq 3$ ; that is,  $|N_{G^L}(u_0)| = |L(u_0)| = 3$  and  $L(u_0) = \{1, 2, 3\}$ . By (16), all seven bullets of Lemma 19 hold. In particular, by the first and the second dash of the seventh bullet of Lemma 19, we have  $A' \cap B' = \emptyset$  and  $|A'| = |B'| = 1$ , say  $A' = \{a'_0\}$  and  $B' = \{b'_0\}$  for distinct  $a'_0, b'_0$ ,  $A, B \neq \emptyset$  and  $A \cap B = \emptyset$ . Also, by the fifth bullet and the third dash of the second bullet of Lemma 19,  $a'_0$  is complete to  $A$  in  $G^L$  (and so in  $G$ ) and anticomplete to  $B$  in  $G$  (and so in  $G^L$ ), and  $b'_0$  is complete to  $B$  in  $G^L$  (and so in  $G$ ) and anticomplete to  $A$  in  $G$  (and so in  $G^L$ ). Moreover, by the second bullet of Lemma 19, we have  $L(a'_0) = L(b'_0) = \{4, 5\}$ .

Let  $M$ ,  $i$  and  $j$  be as in step 11 of algorithm **A**. Then we have  $G^* = G \setminus (\{u_0, a, b\} \cup (V(G) \setminus N_{G^L}[u_0]))$ ,  $L^*(u_0) = M$ ,  $L^*(a) = \{i, 4\}$ ,  $L^*(b) = \{j, 5\}$  and  $L^*(v) = L(v)$  for every  $v \in V(G^*) \setminus \{u_0, a, b\}$ . Also, by the minimality of  $M$ , we have  $M = \{i, j\} \subseteq \{1, 2, 3\}$ .

To verify validity of (R1), note that from  $|N_{G^L}(u_0)| = 3$ , one may deduce  $|V(G^*)| = 3 + |V(G) \setminus N_{G^L}[u_0]| < 4 + |V(G) \setminus N_{G^L}[u_0]| \leq |V(G)|$ . So  $(G^*, L^*)$  satisfies (R1).

For (R2), suppose that  $G$  admits a frugal  $L$ -coloring  $\phi$ . From  $L(u_0) = \{1, 2, 3\}$ ,  $|N_{G^L}(u_0)| = 3$  and Lemma 18, we observe that either there exists  $a_0 \in A$  with  $\phi(a_0) = 4$  or there exists  $b_0 \in B$  with  $\phi(b_0) = 5$ . On the other hand, by the fifth bullet of Lemma 19,  $a'_0$  is complete to  $A$  in  $G^L$  (and so in  $G$ ) and  $b'_0$  is complete to  $B$  in  $G^L$  (and so in  $G$ ). Consequently, since  $\phi$  is proper, either  $\phi(a'_0) = 5$  or  $\phi(b'_0) = 4$ . In the former case, let  $\psi(a) = 4$ ,  $\psi(b) = j$ ,  $\psi(u_0) = i$  and  $\psi(v) = \phi(v)$  for every  $v \in V(G^*) \setminus \{u_0, a, b\} = V(G) \setminus N_{G^L}[u_0]$ . In the latter case, let  $\psi(a) = i$ ,  $\psi(b) = 5$ ,  $\psi(u_0) = j$  and again  $\psi(v) = \phi(v)$  for every  $v \in V(G^*) \setminus \{u_0, a, b\} = V(G) \setminus N_{G^L}[u_0]$ . We leave the reader to check that, from  $\phi$  being a frugal  $L$ -coloring of  $G$ , it follows that  $\psi$  is a frugal  $L^*$ -coloring of  $G^*$ . This verifies (R2) for  $(G^*, L^*)$ .

It remains to argue the truth of (R3) for  $(G^*, L^*)$ . Suppose  $\psi$  is an  $L^*$ -coloring of  $G^*$ . Due to  $|N_{G^L}(u_0)| = 3$ , let  $N_{G^L}(u_0) \setminus \{a, b\} = \{c\}$ . By the first bullet of Lemma 19, we have  $|L(a)|, |L(b)|, |L(c)| \geq 2$ . Note that either  $\psi(a'_0) = 5$  or  $\psi(b'_0) = 4$ , for otherwise from  $\psi(a'_0) = 4$  and  $\psi(b'_0) = 5$ , it follows that  $\psi(a) = i$ ,  $\psi(b) = j$ , and so  $\psi(u_0) \in M = \{i, j\} = \{\psi(a), \psi(b)\}$ , which contradicts  $\psi$  being proper. We deduce (R3) for cases  $\psi(a'_0) = \psi(b'_0)$  and  $\psi(a'_0) \neq \psi(b'_0)$  separately, below.

First, suppose that  $\psi(a'_0) = \psi(b'_0)$ , and by symmetry, let  $\psi(a'_0) = \psi(b'_0) = 4$ . We define a coloring  $\phi$  of  $G$  as follows. Let  $\phi(b) = 5$  and  $\phi(v) = \psi(v)$  for every  $v \in V(G) \setminus N_{G^L}[u_0]$ . In order to determine  $\phi(a)$  and  $\phi(c)$ , we need to consider two cases. If  $c \in A$ , then from  $A \cap B = \emptyset$ , we have  $5 \notin L(a) \cup L(c)$ , and so we may choose two distinct colors  $k$  and  $l$  with  $k \in L(a) \setminus \{4, 5\} = L(a) \setminus \{\phi(a'_0), \phi(b'_0), \phi(b)\}$  and  $l \in L(c) \setminus \{4\} = L(c) \setminus \{\phi(a'_0), \phi(b'_0), \phi(b)\}$ , since otherwise  $L(a) = L(c)$  and  $|L(a)| = |L(c)| = 2$ , and so algorithm **A** should have terminated at step 8, a contradiction. Otherwise, if  $c \notin A$ , then we have  $4 \notin L(c)$ . So since  $|L(a)| \geq 2$ , there exists  $k \in L(a) \setminus \{4, 5\} = L(a) \setminus \{\phi(a'_0), \phi(b'_0), \phi(b)\}$ , and since  $|L(c)| \geq 2$ , there exists  $l \in L(c) \setminus \{k, 5\} = L(c) \setminus \{k, 4, 5\} = L(c) \setminus \{k, \phi(a'_0), \phi(b'_0), \phi(b)\}$ , as otherwise  $c \in B$  and  $L(a) \cap L(c) \neq \emptyset$ , which violates the fourth dash of the seventh bullet of Lemma 19. We define  $\phi(a) = k$  and  $\phi(c) = l$ . Eventually, we choose  $\phi(u_0) \in L(u_0) \setminus \{k, l, 4\} = \{1, 2, 3\} \setminus \{k, l\}$ . We leave it to the reader to check that since  $\psi$  is an  $L^*$ -coloring of  $G^*$ ,  $\phi$  is an  $L$ -coloring of  $G$ , and so the third bullet of (26) follows. Note that the argument for the case  $\psi(a'_0) = \psi(b'_0) = 5$  is analogous, with an additional caveat that this time we rely on the fact that algorithm **A** does not terminate at step 9, instead.

Next, suppose that  $\psi(a'_0) \neq \psi(b'_0)$ ; that is,  $\psi(a'_0) = 5$  and  $\psi(b'_0) = 4$ . We define a coloring  $\phi$  of  $G$  as follows. Let  $\phi(a) = 4$ ,  $\phi(b) = 5$ ,  $\phi(v) = \psi(v)$  for every  $v \in V(G) \setminus N_{G^L}[u_0]$ . Also since  $A \cap B = \emptyset$ , either  $4 \notin L(c)$  or  $5 \notin L(c)$ , and from  $|L(c)| \geq 2$ , there exists  $k \in L(c) \setminus \{4, 5\} = L(c) \setminus \{\phi(a), \phi(b), \psi(a'_0), \psi(b'_0)\}$ , and we set  $\phi(c) = k$ . Finally, we choose  $\phi(u_0) \in$

$L(u_0) \setminus \{k, 4, 5\} = \{1, 2, 3\} \setminus \{k\}$ . Then it is easy to check that since  $\psi$  is an  $L^*$ -coloring of  $G^*$ ,  $\phi$  is an  $L$ -coloring of  $G$ , and so  $(G^*, L^*)$  satisfies (R3). This proves (26).

(27) *There exists a  $(G^*, L^*)$ -refinement  $(\tilde{G}, \tilde{L})$  with the following specifications.*

- $(\tilde{G}, \tilde{L})$  can be computed from  $(G^*, L^*)$  in time  $\mathcal{O}(|V(G^*)|^2)$ .
- We have  $|\tilde{L}(v)| \neq 1$  for all  $v \in V(\tilde{G})$ .
- If  $G^*$  admits a frugal  $L^*$ -coloring, then  $\tilde{G}$  admits a frugal  $\tilde{L}$ -coloring.
- If  $\tilde{G}$  admits an  $\tilde{L}$ -coloring, then  $G^*$  admits an  $L^*$ -coloring.

We may apply Theorem 10 to  $(G^*, L^*)$ , obtaining a  $(G^*, L^*)$ -refinement  $(\hat{G}^*, \hat{L}^*)$  satisfying the bullet conditions of Theorem 10. Then, defining  $\tilde{G} = \hat{G}^*$  and  $\tilde{L} = \hat{L}^*$ , it follows that  $(\tilde{G}, \tilde{L})$  satisfies the bullet conditions of (27). This proves (27).

To conclude the proof, let  $(\tilde{G}, \tilde{L})$  be as in (27). We show that  $(\tilde{G}, \tilde{L})$  satisfies Theorem 20.

By (17), algorithm **A** computes  $(G^*, L^*)$  from  $(G, L)$  in time  $\mathcal{O}(|V(G)|^2)$ . Also, by the first bullet of (27),  $(\tilde{G}, \tilde{L})$  can be computed from  $(G^*, L^*)$  in time  $\mathcal{O}(|V(G^*)|^2) = \mathcal{O}(|V(G)|^2)$ . So  $(\tilde{G}, \tilde{L})$  can be computed from  $(G, L)$  in time  $\mathcal{O}(|V(G)|^2)$ ; that is,  $(\tilde{G}, \tilde{L})$  satisfies the first bullet of Theorem 20.

For the second bullet of Theorem 20, we argue the validity of (R1), (R2) and (R3) for  $(\tilde{G}, \tilde{L})$  separately. By (26),  $(G^*, L^*)$  satisfies (R1), and so being a  $(G^*, L^*)$ -refinement, it follows that  $(\tilde{G}, \tilde{L})$  satisfies (R1), as well.

For (R2) suppose that  $G$  admits a frugal  $L$ -coloring. Then by (26),  $(G^*, L^*)$  satisfies (R2), and so  $G^*$  admits a frugal  $L^*$ -coloring. Therefore, by the third bullet of (27),  $\tilde{G}$  admits a frugal  $\tilde{L}$ -coloring, and so  $(\tilde{G}, \tilde{L})$  satisfies (R2).

Finally, for (R3), suppose that  $\tilde{G}$  admits an  $\tilde{L}$ -coloring. Then by the fourth bullet of (27),  $G^*$  admits an  $L^*$ -coloring. Also, by (26),  $(G^*, L^*)$  satisfies (R3), and so  $G$  admits an  $L$ -coloring. Hence,  $(\tilde{G}, \tilde{L})$  satisfies (R3). This completes the proof.  $\blacksquare$

## 5. PROOF OF THEOREM 5

In this section, we combine Theorems 13, 17 and 20 to deduce Theorem 5. First, Theorems 13 and 17 are applied to deduce the following.

**Theorem 21.** *For all fixed  $k, r \in \mathbb{N}$ , there exists  $\eta(k, r) \in \mathbb{N}$  with the following property. Let  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM where  $G$  is  $rP_3$ -free graph. Then there exists a  $(G, L)$ -profile  $\Xi(G, L)$  with the following specifications.*

- $|\Xi(G, L)| \leq \mathcal{O}(|V(G)|^{\eta(k, r)})$  and  $\Xi(G, L)$  can be computed from  $(G, L)$  in time  $\mathcal{O}(|V(G)|^{\eta(k, r)})$ .
- For every  $(G', L') \in \Xi(G, L)$  and every  $v \in V(G')$ , we have  $|L'(v)| \neq 1$ .
- For every  $(G', L') \in \Xi(G, L)$ ,  $G'$  has no  $L'$ -good  $P_3$ .
- If  $G$  admits an  $L$ -coloring, then for some  $(G', L') \in \Xi(G, L)$ ,  $G'$  admits a frugal  $L'$ -coloring.
- If  $G'$  admits an  $L'$ -coloring for some  $(G', L') \in \Xi(G, L)$ , then  $G$  admits an  $L$ -coloring.

*Proof.* Applying Theorem 13 to  $(G, L)$ , we obtain a spanning  $(G, L)$ -profile  $\Pi(G, L)$  satisfying the bullet conditions of Theorem 13. Also, for every  $(G, K) \in \Pi(G, L)$ , applying Theorem 17 to  $(G, K)$ , we obtain a spanning  $(G, K)$ -profile  $\Upsilon(G, L)$  satisfying the bullet conditions of Theorem 17. Let  $\Theta(G, L) = \bigcup_{(G, K) \in \Pi(G, L)} \Upsilon(G, K)$ . Then, for every  $(G, J) \in \Theta(G, L)$ , we may apply Theorem 10 to  $(G, J)$ , obtaining a  $(G, J)$ -refinement  $(\hat{G}, \hat{J})$  satisfying bullet conditions of Theorem 10.

Let  $\Xi(G, L) = \{(\hat{G}, \hat{J}) : (G, J) \in \Theta(G, L)\}$ . We claim that  $\Xi(G, L)$  satisfies Theorem 21. Clearly,  $\Xi(G, L)$  is a  $(G, L)$ -profile. Also, let  $\pi(k, r)$  be as in Theorem 13 and  $v(k, r)$  be as in Theorem 17. Now, assuming  $\eta(k, r) = \pi(k, r) + v(k, r) + 2$ , by the first bullet of Theorems 13 and 17 and 10,  $\Xi(G, L)$  satisfies the first bullet of Theorem 21. Also, by the second bullet of Theorem 10,  $\Xi(G, L)$  satisfies the second bullet of Theorem 21. Moreover, the second bullet of Theorem 17 along with Lemma 14 implies that  $\Xi(G, L)$  satisfies the third bullet of Theorem 21. The fourth bullet of Theorem 21 for  $\Xi(G, L)$  follows from the second bullet of Theorem 13 and the third bullets of Theorems 17 and 10. Finally, by Lemma 8 and the fourth bullet of Theorem 10,  $\Xi(G, L)$  satisfies the fifth bullet of Theorem 21. This completes the proof.  $\blacksquare$

Next, we prove the following as an application of Theorem 20. Recall the definition  $p(G, L) = |V(G)| + \sum_{v \in V(G)} |L(v)|$  for every instance  $(G, L)$  of the LIST- $k$ -COLORING PROBLEM,  $k \in \mathbb{N}$ .

**Theorem 22.** *Let  $(G, L)$  be an instance of the LIST-5-COLORING PROBLEM such that  $|L(v)| \neq 1$  for all  $v \in V(G)$  and  $G$  has no  $L$ -good  $P_3$ . Then there exists a  $(G, L)$ -refinement  $(G^b, L^b)$  with the following specifications.*

- $(G^b, L^b)$  can be computed from  $(G, L)$  in time  $\mathcal{O}(p(G, L)|V(G)|^2) = \mathcal{O}(|V(G)|^3)$ .
- $|L^b(v)| \in \{0, 2\}$  for all  $v \in V(G^b)$ .
- If  $G$  admits a frugal  $L$ -coloring, then  $G^b$  admits a frugal  $L^b$ -coloring.
- If  $G^b$  admits an  $L^b$ -coloring, then  $G$  admits an  $L$ -coloring.

*Proof.* Let  $(G, L)$  be a counterexample with  $p(G, L)$  as small as possible. If  $|L(u)| \in \{0, 2\}$  for every  $u \in V(G)$ , then we define  $G^b = G$ ,  $L^b = L$ , and it is immediately seen that  $(G^b, L^b)$  satisfies the bullet conditions of Theorem 22, a contradiction. So we may assume that there exists a vertex  $u_0 \in V(G)$  with  $|L(u_0)| \geq 3$ . Applying Theorem 20 to  $(G, L)$  and  $u_0$ , we obtain a  $(G, L)$ -refinement  $(\tilde{G}, \tilde{L})$ , satisfying the bullet conditions of Theorem 20. In particular, by the second bullet of Theorem 20, we have  $|\tilde{L}(v)| \neq 1$  for all  $v \in V(\tilde{G})$ . Also, since  $G$  has no  $L$ -good  $P_3$ , by Lemma 14,  $\tilde{G}$  has no  $\tilde{L}$ -good  $P_3$ . Moreover, by the third bullet of Theorem 20,  $(\tilde{G}, \tilde{L})$  satisfies (R1); that is,  $p(\tilde{G}, \tilde{L}) < p(G, L)$ . This, together with the minimality of  $p(G, L)$ , implies that there exists a  $(\tilde{G}, \tilde{L})$ -refinement  $(\tilde{G}^b, \tilde{L}^b)$ , satisfying the bullet conditions of Theorem 22. Now, let  $G^b = \tilde{G}^b$  and  $L^b = \tilde{L}^b$ . Since  $(\tilde{G}^b, \tilde{L}^b)$  is a  $(\tilde{G}, \tilde{L})$ -refinement and  $(\tilde{G}, \tilde{L})$  is a  $(G, L)$ -refinement, it follows that  $(G^b, L^b) = (\tilde{G}^b, \tilde{L}^b)$  is a  $(G, L)$ -refinement. Moreover, since  $(G^b, L^b) = (\tilde{G}^b, \tilde{L}^b)$ , it is easy to see that

- the first bullet of Theorem 22 for  $(G, L)$  and  $(G^b, L^b)$  follows from the first bullet of Theorem 20 for  $(G, L)$  and  $(\tilde{G}, \tilde{L})$  and the first bullet of Theorem 22 for  $(\tilde{G}, \tilde{L})$  and  $(\tilde{G}^b, \tilde{L}^b)$ ;
- the second bullet of Theorem 22 for  $(G^b, L^b)$  follows from the second bullet of Theorem 22 for  $(\tilde{G}^b, \tilde{L}^b)$ ;
- the third bullet of Theorem 22 for  $(G, L)$  and  $(G^b, L^b)$  follows from the third bullet of Theorem 20 (in particular, (R2)) for  $(G, L)$  and  $(\tilde{G}, \tilde{L})$  and the third bullet of Theorem 22 for  $(\tilde{G}, \tilde{L})$  and  $(\tilde{G}^b, \tilde{L}^b)$ ; and
- the fourth bullet of Theorem 22 for  $(G, L)$  and  $(G^b, L^b)$  follows from the third bullet of Theorem 20 (in particular, (R3)) for  $(G, L)$  and  $(\tilde{G}, \tilde{L})$  and the fourth bullet of Theorem 22 for  $(\tilde{G}, \tilde{L})$  and  $(\tilde{G}^b, \tilde{L}^b)$ .

But this violates  $(G, L)$  being a counterexample to Theorem 22, and so completes the proof.  $\blacksquare$

As the last ingredient, we need the following, which is proved via a reduction to 2SAT, and has been discovered independently by many authors [8, 9, 19].

**Theorem 23** (Edwards [8]). *Let  $k \in \mathbb{N}$  be fixed and  $(G, L)$  be an instance of the LIST- $k$ -COLORING PROBLEM with  $|L(v)| \leq 2$  for every  $v \in V(G)$ . Then it can be decided in time  $\mathcal{O}(|V(G)|^2)$  whether  $G$  admits an  $L$ -coloring.*

Now we are in a position to prove Theorem 5, which we restate.

**Theorem 24.** *Let  $r \in \mathbb{N}$  be fixed. Then there exists a polynomial-time algorithm which solves the LIST-5-COLORING PROBLEM restricted to  $rP_3$ -free instances.*

*Proof.* Given an  $rP_3$ -free instance  $(G, L)$  of the LIST-5-COLORING PROBLEM, let  $\Xi(G, L)$  be as in Theorem 21. Then, for every  $(G', L') \in \Xi(G, L)$ , by the second bullet of Theorem 21,  $|L'(v)| \neq 1$  for all  $v \in V(G)$ , and by the third bullet of Theorem 21,  $G'$  has no  $L$ -good  $P_3$ . Therefore, we may apply Theorem 22 to  $(G', L')$  obtaining a  $(G, L)$ -refinement  $(G^{fb}, L^{fb})$  satisfying the bullet conditions of Theorem 22. Now, consider the  $(G, L)$ -profile  $\Gamma(G, L) = \{(G^{fb}, L^{fb}) : (G', L') \in \Xi(G, L)\}$ . For all  $k, r \in \mathbb{N}$ , let  $\eta(k, r)$  be as in Theorem 21. Then, statement (28) below follows immediately from the first bullet of Theorem 21 for  $\Xi(G, L)$ , and the first bullet of Theorem 22 for every  $(G^{fb}, L^{fb})$ , where  $(G', L') \in \Xi(G, L)$ .

$$(28) \quad |\Gamma(G, L)| \leq \mathcal{O}(|V(G)|^{\eta(k, r)}), \text{ and } \Gamma(G, L) \text{ can be computed from } (G, L) \text{ in time } \mathcal{O}(|V(G)|^{\eta(k, r)+3}).$$

Also, we deduce:

$$(29) \quad G \text{ admits an } L\text{-coloring if and only if there exists } (G^{fb}, L^{fb}) \in \Gamma(G, L) \text{ such that } G^{fb} \text{ admits an } L^{fb}\text{-coloring for some } (G', L') \in \Xi(G, L).$$

Suppose that  $G$  admits an  $L$ -coloring. By the fourth bullet of Theorem 21, for some  $(G', L') \in \Xi(G, L)$ ,  $G'$  admits a frugal  $L'$ -coloring. As a result, by the third bullet of Theorem 22,  $(G^{fb}, L^{fb}) \in \Gamma(G, L)$  admits a frugal  $L^{fb}$ -coloring, and so an  $L^{fb}$ -coloring.

Conversely, suppose that for some  $(G', L') \in \Xi(G, L)$ ,  $G^{fb}$  admits an  $L^{fb}$ -coloring. Then by the fourth bullet of Theorem 22,  $G'$  admits an  $L'$ -coloring. Therefore, by the fifth bullet of Theorem 21,  $G$  admits an  $L$ -coloring. This proves (29).

Now, the algorithm is as follows. First, we compute  $\Gamma(G, L)$ . By (28), this is doable in time  $\mathcal{O}(|V(G)|^{\max\{\eta(k, r), 3\}})$ . Then, by the second bullet of Theorem 22, for each  $(G^{fb}, L^{fb}) \in \Gamma(G, L)$ , we have  $|L^{fb}(v)| \in \{0, 2\}$ . Therefore, since  $|\Gamma(G, L)| \leq \mathcal{O}(|V(G)|^{\eta(k, r)})$  by (28), applying the algorithm from Theorem 23, we decide in polynomial time whether there exists  $(G^{fb}, L^{fb}) \in \Gamma(G, L)$  such that  $G^{fb}$  admits an  $L^{fb}$ -coloring. If the answer is yes, then by (29),  $G$  admits an  $L$ -coloring. If the answer is no, again by (29),  $G$  admits no  $L$ -coloring. This completes the proof.  $\blacksquare$

## 6. PROOF OF THEOREM 7

In this section, we prove Theorem 7 via a reduction from *monotone* NAE3SAT, defined as follows. The NOT-ALL-EQUAL-3-SATISFIABILITY PROBLEM (NAE3SAT) is to decide, given an instance  $I$  consisting of  $n$  Boolean variables  $x_1, \dots, x_n$  and  $m$  clauses  $C_1, \dots, C_m$ , each containing three literals, whether there exists a true/false assignment for each variable such that each clause contains at least one true literal and one false literal. We say  $I$  is *satisfiable* if it admits such an assignment. By *monotone* NAE3SAT, we mean NAE3SAT restricted to *monotone* instances; that is, instances with no negated literals.

**Theorem 25** (Garey and Johnson [10]). *Monotone NAE3SAT is NP-complete.*

Now we can prove Theorem 7, which we restate.

**Theorem 26.** *The  $k$ -COLORING PROBLEM restricted to  $2P_4$ -free graphs (and hence  $rP_4$ -free graphs for every fixed  $r \geq 2$ ) is NP-complete for all  $k \geq 5$  and  $r \geq 2$ .*

*Proof.* Clearly, the  $k$ -COLORING PROBLEM restricted to  $2P_4$ -free graphs belongs to NP. For the hardness, given a monotone NAE3SAT instance  $I$  with variables  $x_1, x_2, \dots, x_n$  and clauses  $C_1, C_2, \dots, C_m$ , we construct a graph  $G$  as an instance of the 5-COLORING PROBLEM, as follows. Let  $C = \{c_i : i \in [5]\}$ ,  $X = \{x_i : i \in [n]\}$ ,  $Y = \{y_j, z_j : j \in [m]\}$  and  $U = \{u_j^k, w_j^k : j \in [m], k \in [3]\}$ . Then we let  $V(G) = C \cup X \cup Y \cup U$ , and the adjacency in  $G$  is as follows.

- $C$  is a clique of  $G$ .
- $\{c_3, c_4, c_5\}$  is complete to  $X$ .
- $\{c_1, c_2\}$  is complete to  $Y$ .
- For each  $j \in [m]$  and  $k \in [3]$ , we have  $c_1 u_j^k, c_2 w_j^k \in E(G)$ .
- For each  $j \in [m]$ , we have  $c_i u_j^k, c_i w_j^k \in E(G)$  for all pairs  $(i, k)$  with  $i \in \{3, 4, 5\}$ ,  $k \in \{1, 2, 3\}$ , and  $i \neq k + 2$ .
- $X$  is complete to  $Y$ .
- For each  $j \in [m]$  and all  $k \in [3]$ , we have  $y_j u_j^k, z_j w_j^k \in E(G)$ .
- For each  $j \in [m]$ , if  $C_j$  contains  $x_{i_1}, x_{i_2}$ , and  $x_{i_3}$ , then we have  $x_{i_k} u_j^k, x_{i_k} w_j^k \in E(G)$  for all  $k \in [3]$ .

There are no edges in  $E(G)$  other than those described above. It is easily seen that the construction is of polynomial size and can be computed in polynomial time.

(30)  *$I$  is satisfiable if and only if  $G$  is 5-colorable.*

First, let  $\phi : V(G) \rightarrow [5]$  be a 5-coloring of  $G$ . Since  $C$  is a clique of  $G$ , we may assume without loss of generality that  $\phi(c_i) = i$  for every  $i \in [5]$ . Thus,  $\phi(x_i) \in \{1, 2\}$  for every  $i \in [n]$ . Now, let  $j \in [m]$  and let  $x_{i_1}, x_{i_2}$ , and  $x_{i_3}$  be the literals in  $C_j$ . If  $\phi(x_{i_1}) = \phi(x_{i_2}) = \phi(x_{i_3}) = 2$ , then we have  $\phi(u_j^1) = 3$ ,  $\phi(u_j^2) = 4$  and  $\phi(u_j^3) = 5$ . But then  $y_j$  has a neighbour of each color in  $[5]$ , which is a contradiction. As a result, at least one of  $\phi(x_{i_1}), \phi(x_{i_2})$  and  $\phi(x_{i_3})$  is equal to 1. Similarly, by considering the vertex  $w_j^k$  for  $k \in \{1, 2, 3\}$ , we deduce that at least one of  $\phi(x_{i_1}), \phi(x_{i_2})$  and  $\phi(x_{i_3})$  is 2. Thus, by setting  $x_i$  to be True if  $\phi(x_i) = 1$  and False if  $\phi(x_i) = 2$ , we conclude that  $I$  is satisfiable.

Next, suppose that  $I$  is satisfiable. We define a coloring  $\phi : V(G) \rightarrow [5]$  of  $G$  as follows. Let  $\phi(c_i) = i$  for every  $i \in [5]$ . Let  $\phi(x_i) = 1$  if  $x_i$  is assigned True and  $\phi(x_i) = 2$  otherwise. For each  $j \in [m]$  and  $C_j$  with literals  $x_{i_1}, x_{i_2}$ , and  $x_{i_3}$ , and each  $k \in [3]$ , if  $\phi(x_{i_k}) = 1$ , then we set  $\phi(u_j^k) = 2$  and  $\phi(w_j^k) = k + 2$ , and if  $\phi(x_{i_k}) = 2$ , we set  $\phi(u_j^k) = k + 2$  and  $\phi(w_j^k) = 1$ . Since at least one of  $x_{i_1}, x_{i_2}$  and  $x_{i_3}$  is assigned True and at least one of them is assigned False, there exists  $k_1, k_2 \in [3]$  with  $k_1 \neq k_2$  such that  $\phi(u_j^{k_1}) = 2$  and  $\phi(w_j^{k_2}) = 1$ . We set  $\phi(y_j) = k_1 + 2$  and  $\phi(z_j) = k_2 + 2$ . We leave it to the reader to check that  $\phi$  is 5-coloring of  $G$ . This proves (30).

(31) *The vertex set of every induced  $P_4$  in  $G$  intersects either  $C$  or both  $X$  and  $Y$ .*

Let  $P$  be an induced  $P_4$  in  $G$  with  $V(P) = \{v_1, v_2, v_3, v_4\}$  and  $E(P) = \{v_1 v_2, v_2 v_3, v_3 v_4\}$  such that  $V(P) \cap C = \emptyset$ . If  $v_2 \in U$ , then without loss of generality, we may assume that  $v_1 \in X$  and  $v_3 \in Y$ , as desired. So  $v_2 \notin U$ , and similarly,  $v_3 \notin U$ . It follows that  $v_2, v_3 \in X \cup Y$ . Therefore, since  $X$  and  $Y$  are stable sets of  $G$  and  $v_2 v_3 \in E(G)$ , one of  $v_2$  and  $v_3$  belongs to  $X$  and the other one belongs to  $Y$ . This proves (31).

(32)  *$G$  is  $2P_4$ -free.*

Suppose not. Let  $P$  and  $Q$  be two induced  $P_4$ 's in  $G$  with  $V(P)$  anticomplete to  $V(Q)$ . Since  $C$  is a clique of  $G$ , we may assume without loss of generality that  $V(P) \cap C \neq \emptyset$ . By (31), we may choose vertices  $x \in V(P) \cap X$  and  $y \in V(P) \cap Y$ . If there exists  $c_i \in V(Q) \cap C$ , then depending

on whether  $i \in \{1, 2\}$  or not, either  $c_i y \in E(G)$  or  $c_i x \in E(G)$ , which is impossible. Therefore, by (31), we may choose a vertex  $x' \in V(Q) \cap X$ . But then  $x'y \in E(G)$ , a contradiction. This proves (32).

From (30), (32) and Theorem 25, it follows that the 5-COLORING PROBLEM restricted to  $2P_4$ -free graphs is NP-hard. This completes the proof. ■

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