Mathematical Actualism

An Alternative Realist Philosophy of Mathematics

by

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Author's Declaration:

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners. I understand that my thesis may be made electronically available to the public.

Abstract:

The present thesis argues for a philosophy of mathematics, herein dubbed Actualism, which is contrasted with several existing views on the philosophy of mathematics. It begins with a brief introduction to the classical (Platonist) view on the philosophy of mathematics and examines some of the major problems with the account. Thereafter, two alternative philosophies of mathematics (mathematical Constructivism and mathematical Finitism) are examined. Constructivism is detailed in the first chapter through the work of Brouwer and Dummett and, in the second chapter, a description of Finitism is provided through the work of Dantzig and Mayberry. In the final chapter of the thesis, the underpinnings of mathematical Actualism are articulated. The central motivation behind Actualism as an alternative philosophy of mathematics arises from the desire to restore a realist thesis to mathematics that is consistent with the semantics of our modern scientific discourse, or else, with a naturalistic worldview.

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MATHEMATICAL ACTUALISM: AN ALTERATIVE REALIST PHILOSPHY OF MATHEMATICS

INTRODUCTION:

In this thesis I argue for a particular view on the nature of mathematics that I have dubbed Actualism. The details of this preliminary account itself will be spelled out in Chapter 3. Actualism offers an account of the nature of mathematical truth and the ontological basis of mathematical claims for which I will claim certain philosophical advantages. While I do not want the Hegelian architecture of my argument to be taken too seriously, I think it is convenient and efficient to take mathematical Platonism as a starting "thesis," present some well-established but more recent anti-realist views as "antithesis," then offer Actualism as a kind of synthesis which preserves the important advantages of both views.

While of course there have always been philosophical alternatives to it (Aristotelian accounts, nominalism, Kant, J.S. Mill, to name just some well-known examples), a Platonist conception of mathematics has served as the standard philosophy of mathematics throughout much of the history of thought. Around the turn of the 20th Century, however, the variety of alternative philosophies of mathematics expanded considerably into developments internal to mathematics itself and in response to the serious issues which have always been present within mathematical Platonism. For present purposes, it will be useful to focus on two alternatives to mathematical Platonism, namely Constructivism and Finitism. Both differ dramatically from their Platonic predecessor in that they are inherently anti-realist positions.

The present thesis begins with the assumption that the rudiments of Platonism are well-known and so, instead of a detailed presentation, offers, by way of an introduction, a brief detailing of the flaws which are inherent in mathematical Platonism. Thereafter, the alternative philosophies of mathematics (Constructivism and Finitism) will each be considered and broadly detailed in their own respective chapters. With respect to Constructivism, the present thesis examines Brouwer's Intuitionism and Dummett's Constructivism as two examples of what Constructivist mathematics look like. Concerning the second alternative to mathematical Platonism, Finitism, the present thesis provides an illustration of the central tenets, features, and elements of mathematical Finitism by examining the work of Dantzig and Mayberry.

Both of these alternatives to mathematical Platonism, as will be seen, offer an anti-realist antithesis to mathematical Platonism's realist thesis. However, and notwithstanding the dilapidated state and philosophical disrepute of mathematical Platonism, the realist thesis provided by mathematical Platonism contains a number of very important and desirable elements. Consequently, there is more than a little intuitive appeal to mathematical Platonism's realist thesis today since, even in spite of the difficulties encountered within the account, it satisfies certain desiderata that are not fulfilled by the Constructivist anti-realist antithesis. The aim of the present thesis is to offer an alternative philosophy of mathematics to all of these, occurring in the final chapter, which forfeits neither the claim to a realist thesis nor a defensible epistemology. The alternative philosophy of mathematics that is articulated in this thesis, mathematical Actualism, differs from the other alternatives to mathematical Platonism in that it offers an alternative *realist* philosophy of mathematics. In this way, Actualism offers a synthesis of the Platonist thesis and Constructivist anti-thesis.

It is useful to frame the advantages and disadvantages of Platonism and its rivals by employing the framing offered in Benacerraf's paper "Mathematical Truth". Within this paper, Benacerraf describes two desiderata for any account of mathematical truth. The first of these he takes to be “the concern for having a homogeneous semantical theory in which semantics for the propositions of mathematics parallel the semantics for the rest of the language” (Benacerraf, 2, 1973). The second of these is “the concern that the account of mathematical truth mesh with a reasonable epistemology” (Benacerraf, 2, 1973). Benacerraf offers the observation that nearly all conceptions of mathematical truth appear to serve one of these motivations to the detriment of the other (Benacerraf, 2, 1973). Mathematical Platonism is seen to be an example of an account that sacrifices the second of these motivations since, as will be seen shortly, it entails a number of untenable epistemic and metaphysical consequences. On the other hand, Constructivist philosophies of mathematics, resting upon an anti-realist thesis, while avoiding the epistemic problems of mathematical Platonism, do not, Benacerraf suggests, satisfy the first desideratum, as these anti-realist views are based on *constructivist semantics* for mathematical claims, which differs from the Tarskian, referential semantics Benacerraf takes to be correct for “the rest of language”. In contrast to these, Actualism as a philosophy of mathematics aims to synthesize both of these motivations insofar as it aims to offer an account that synthesizes a theory of mathematical truth that allows for a referential semantics and a reasonable account of knowledge that is homogenous with the rest of our language. The great advantage which Actualism will offer is that, because of its commitment to a kind of realism, it is compatible with a referential semantics that directly parallels that of our empirical scientific discourse, and which further is able to offer an epistemic explanation of how we both come to mathematical knowledge and how this knowledge causally connects to our cognitive faculties.

Before moving into the specific chapters and details of Constructivism and Finitism as alternative philosophies of mathematics, it is first necessary to broadly examine the nature of mathematical Platonism. Initially, it is important to disentangle mathematical Platonism from

Plato's general philosophy – as the two are not synonymous. In this respect, the nature of mathematical Platonism has been identified with the following three central theses: a) that there exist mathematical objects, b) that these mathematical objects are abstract, and c) that these mathematical objects are independent of our conception of them (Linnebo, 1, 2009). Mathematical Platonism, in its contemporary articulations, is said to entail no more and no less than is contained in these three theses and, in this respect, it is said to be attached to Plato's metaphysics but not his epistemology[1]. In this sense, mathematical Platonism is understood to be a metaphysical but not an epistemic position. Two important consequences of mathematical Platonism should immediately be recognized: a) that it entails the notion that reality, being more than the physical, also includes the abstract, and b) the notion that there are objects which are not part of the physical causal chain. There is another methodological view of mathematical Platonism, dubbed working realism, which maintains that we ought to practice mathematics as though Platonism were true (Linnebo, 1.5, 2009). This methodological view is pragmatically beneficial (insofar as it allows mathematicians to go about their work without worrying about the underlying philosophical issue), however, were it to be taken up by the philosopher it would amount to a neglecting of the underlying metaphysical question if not to a kind of Platonic 'faith'. Consequently, and insofar as one is in the manner of a philosopher concerned with the deeper philosophical question and foundations of mathematics, as opposed to solely being concerned for the practice of mathematics, it is not sufficient to take mathematical Platonism in the sense of working realism without examining the underlying truth of the matter. Moreover, the unacceptability of neglecting the question of whether or not Platonism is true (which is

[1] This is to say that it is attached to the notion that mathematical objects exist in an independent and abstract sense (like the forms) but not to the more clearly absurd epistemology of 'recollection' whereby an immortal soul remembers its knowledge about the pure forms which it previously encountered, such as occurs in Plato's *Meno*.

tantamount to taking mathematical Platonism on faith) is made readily apparent by the presence of very serious philosophical problems within the account.

Just one of these serious problems concerns the epistemology, or lack of it, which is entailed by mathematical Platonism. For, although contemporary mathematical Platonism has distanced itself from Plato's epistemology (from the notion of 'recollection'), it has been left with the non-benign problem that it renders the abstract objects of mathematics epistemically inaccessible to us. This is because, through its combination of theses, it maintains that mathematical objects exist as abstract objects which are independent of our conception of them. Because of this, it entails the notion that mathematical objects are things beyond the material/physical world that exist in an abstract sense akin to Plato's forms. However, because we – so far as concerns our cognitive faculties – are at least in part most definitely physical, the question emerges: how it is that we come to knowledge about these abstract mathematical objects? But this was just the problem identified by Benacerraf in "Mathematical Truth" and is why he maintained that mathematical Platonism failed because it was unable to offer an account of mathematical truth that was able to mesh well with a reasonable epistemology (Benacerraf, 1973). Insofar as mathematical Platonism maintains that mathematical objects are abstract, a notion which is essential to the account, then mathematical objects cannot reasonably be said to be causally connected to the physical world. It then becomes a dubious posit if not a Quixotian fantasy to maintain that our clearly physical[2] minds/brains interact with and come to knowledge about these non-physical form-like mathematical objects. Consequently, mathematical Platonism runs into both the problem of interaction and the serious epistemic question of how it is our

[2] This is not to presume the presence of a clear and definitive answer to the theory of mind as a mind-brain identity theory, only, that it is incontrovertible that at least some (and almost certainly most) of out cognitive capacities are in fact physical if not directly related to our physical faculties.

physical brains/organisms can ever be said to access or come to knowledge about abstract mathematical objects.

While the epistemic problems associated with the account are in themselves sufficient to render mathematical Platonism dubious, there are other metaphysical difficulties that further problematize the account. We began by noting that there are philosophical benefits to Platonism that explain why people have accepted it for centuries. There are, of course, also the challenges that motivated its critics for centuries, including the challenges that have motivated nominalists. One such difficulty concerns the number of problematic metaphysical consequences that result from mathematical Platonism. Chief among these is that, owing to its extension of existence to abstracta, mathematical Platonism is an inherently dualist metaphysical position. Because of this, mathematical Platonism requires a defense for a number of highly dubious metaphysical notions including but not limited to: a) the problem of interaction between abstractly existing things and physically existing things, b) the question of the origin of these abstract mathematical objects in the first place, and c) where it is that these abstract objects exist. Although these problems are related to the epistemic issue raised above, they are prior to the epistemic question - how we can come to know about mathematical objects which are rendered epistemically inaccessible – since they instead question the initial metaphysical intelligibility of positing that such abstract objects ‘exist’ in the first place.

For reasons such as these and many more (for there is no shortage of arguments against mathematical Platonism), alternative philosophies of mathematics have proliferated, each attempting to circumvent the epistemic and metaphysical difficulties of the Platonic conception. The alternative philosophies of mathematics we turn to next (Intuitionism, Constructivism in the broader sense, and Finitism) generally have the shared feature of offering an anti-realist thesis.

Because of this anti-realism, these alternative philosophies of mathematics are able to sidestep the philosophical issues facing mathematical Platonism. However, while these alternatives are successful in this aim, rejecting the robust, Platonic commitment to the real ontological status of mathematical objects, this comes at the cost of a realist referential semantics. It is for this reason that the present thesis proposes an alternative philosophy of mathematics, mathematical Actualism, which aims to offer the kind of account that Benacerraf suggested has hitherto not been achieved. In the following two chapters, a general description of mathematical Constructivism and Finitism will be given and, in the final chapter, a preliminary articulation of mathematical Actualism as an alternative realist philosophy of mathematics will be given.

CHAPTER ONE: CONSTRUCTIVISM

An Introduction to Constructivism:

The first variety of anti-Platonist (or non-classical) philosophy of mathematics which will here be considered is Constructivism. It is immediately critical to recognize that, although what follows is a broad survey of Constructivist positions, there are a great number of different kinds of Constructivism in the philosophy of mathematics. In fact, this variance and disparity (not uncommon across general philosophical 'isms') is so pronounced in the case of Constructivism that it has been asserted that "Constructivism in mathematics is generally a business of practice rather than principle: there are no significant mathematical axioms or attitudes characteristic of Constructivism and statable succinctly that absolutely all constructivists, across the spectrum, endorse" (McCarty, 105, 2009). Consequently, it should be recognized that the account of Constructivism as it occurs in this chapter, while indicative of Constructivism as a philosophy of mathematics generally, is not exhaustive of the philosophical position as a whole.

Notwithstanding this variance, there is one general principle to be declared which differentiates Constructivist philosophies of mathematics from their classical counterparts. This principle is the translation of the phrase 'there exists some X' into the form 'we can construct some X'. In contrast to classical mathematics' inherently realist orientation, Constructivism instead is an inherently anti-realist position. Where Platonic Realism sees its mathematical objects as unchanging, mind-independent entities (alike the forms), Constructivism instead is concerned with mind-dependent constructions. This move has critical philosophical consequences and implications since, insofar as the constructivists offer a reinterpretation of what it means to say that X exists. According to Constructivism, mathematical objects exist in the same way as obviously mind-dependent entities like humour or dollars, and commitment to their existence does not involve the same sort of ontological commitments that Platonic realism does. In other words, the Constructivist position does not have the ontological commitment of maintaining that mathematical objects possess a mind-independent existence – that they are ‘real’ objects out in the world.

With this defining principle in mind, this chapter will provide an overview of Constructivism as a philosophy of mathematics by examining both Brouwer’s Intuitionism and Dummett’s Constructivist position. This examination will be structured around the following questions: a) what are Constructivist proofs, practice, and methods comprised of, i.e., what do they take mathematical proofs to be, b) in what do Constructivists take mathematical truth to consist (how does their conception of mathematical truth relate to mathematical proof), c) what do different varieties of Constructivism take mathematical objects to be, and d) where do various Constructivists lay the foundations of mathematics? Through an exegesis of Brouwer’s and Dummett’s answers to these questions, this chapter will provide a general illustration of Constructivism as a philosophy of mathematics and will outline the epistemic and ontological commitments of the philosophy by clarifying how it differs from the classical Platonic realist position.

Constructivist Proofs, Practices, and Methods:

The philosophical underpinnings of both Brouwer’s Intuitionism and Dummett’s Constructivism are intricately bound to their technical foundations within intuitionistic mathematics and intuitionistic logic respectively. Nonetheless, it will be the goal of this subsection to provide a general description of the philosophy behind the more technical proofs and methods of Brouwer’s Intuitionism and Dummett’s Constructivism.

The foundations of Brouwer's philosophical Intuitionism are rooted within his conception of intuitionistic mathematics. Brouwer's intuitionistic mathematics mark one of the first alternatives to classical mathematical Platonism (at least in the modern period, i.e., setting aside details of Aristotelian or Kantian mathematics, for instance) and serve as an origin of contemporary Constructivist mathematics broadly construed. Moreover, Brouwer's intuitionistic mathematics differs so dramatically from classical mathematics that not all the propositions which are held to be true within classical mathematics remain so within Brouwer's intuitionism[3]. Atten gives the following statement as a general characterization of the intuitionistic approach to mathematics: "Intuitionistic mathematics consists primarily in the act of effecting mental constructions of a certain kind; its objects, relations, and proofs exist only in so far as they have been constructed in these acts" (Atten, 37, 2018).

Further, Brouwer's intuitionistic mathematics features a division between 'separable mathematics' and his theory of the continuum (Posy, 4, 2007). These separable mathematics concern finite mathematics in the sense that all the mathematical objects involved are finite. Specifically, they concern the natural number sequence, basic arithmetical operations, integers, and rational numbers (Posy, 5, 2007). Of course, it is tempting to notice that there are *infinitely many* natural numbers, integers, or rational numbers; however, the point of referring to mathematics as *finite* is that no operations are carried out on these collections (because such complete collections cannot be constructed), and each individual natural/integer/rational number is, considered as an individual, finite. Brouwer's operation for deriving (or constructing) these aspects of mathematics occurs through the sequential formulation of ordered pairs and the

[3] One example of this concerns the 'trichotomy' – the technical treatment of which occurs in Posy's (2007) "Intuitionism and Philosophy".

repetition of this process (Posy, 5, 2007). In contrast, Brouwer's method for generating infinite mathematical objects and the continuum comes from the reiterative application of a law to some finite sequence - otherwise referred to as a 'choice sequence' (Posy, 5, 2007). The clear difference here for Brouwer is that, while one may generate/construct an infinite object or sequence through the process of an infinitely reflexive repetition of a law or operator, they will not arrive at any stop point where the sequence can be said to be 'completed'. Consequently, Brouwer's infinite sequences never reach an 'end point' or final ordinal at which the series ends – thus we do not arrive at a completed infinity.

Another important way in which these choice sequences of Brouwer differ from pre-intuitionistic mathematics is that they are not fully deterministic, which is to say, they at times allow for multiple possible values to be 'chosen' for a given place along a sequence (Posy, 5, 2007). Although the notion that we at some point along a sequence 'chose' what value belongs there may appear arbitrary if not fantastical at first glance, if we are charitable to Brouwer's intuitionistic project this very idea will be seen to emerge as both a necessary and desired consequence of his position. The reason for this is that the disinclination to follow Brouwer to this point results from the Platonic temptation to think that as of yet undetermined sequences, even before we have conceived of them, are endowed with some 'matter of fact' or 'already determined' status. In other words, the thought that all sequences run off into some already determined point, such that all their values are already necessitated, relates to the Platonic temptation to think of these things as already 'existing'. However since, for Brouwer, these sequences can not be said to exist until they have been constructed by us, their values are not 'determinate' or 'written' (which is to say they allow for multiple values at a given place) until they have been constructed by us. It should further be recognized that Brouwer's intuitionistic mathematics features other technical idiosyncrasies, particularly concerning sets, or 'species' as he refers to them, the most prominent of which are spreads, fans, and refinement[4]; however, details about these are not essential to the purposes of the present paper.

Perhaps the most distinctive, and almost certainly the most controversial, aspect of Brouwer's intuitionism is his conception of the 'creating subject', which is understood to be an idealized mathematician. The defining characteristic of this creating subject is that it is able to perform (successfully) whatever mathematics can be done in principle (Atten, 2018). For the creating subject to be able to do so it is envisioned as having several traits such as: a) an indefinitely long-life span, b) a perfect memory, and c) being infallible in the sense of not making mistakes (Atten, 39, 2018). However, idealized as this creating subject may be, Brouwer does not conceive of it as being able to do anything that the classical mathematician regards as being "in principle possible," but rather, just as being able to do substantially more than a human can do – owing to its properties just described. For example, while it is in principle possible to do things such as: a) forming a set containing all the subsets of the set of natural numbers (the so-called Powerset of N), or b) collecting a member of each non-empty subset of P(N), even Brouwer's creating subject would not be able to do so. The reason for this is that, despite its indefinitely long lifespan, it is not able to work infinitely quickly so as to create a completed infinite collection. This highlights an area of great disagreement between Brouwer and classical mathematicians as, while the later think that certain sorts of mathematics (such as either of the two procedures above described) make sense, because even the creating subject cannot perform them Brouwer instead thinks it is nonsense to talk about the powerset of N or of performing a

[4] For the technical description of these aspects of Brouwer's intuitionistic mathematics see "Intuitionism and Philosophy" (2007) by Posy.

choice function on N. A nuance in this way emerges between what the classical mathematician regards as being “in principle possible” and what mathematics can be in principle done. Here it will be seen that Brouwer’s creating subject relates closely to his conception of choice sequences and plays an essential role in constructing the infinite or non-deterministic sequences previously mentioned — having no limits on its lifespan, it can construct finite sequences or arbitrary length, but it cannot complete an infinite collection. Further, it is important to recognize the essential role which the creating subject plays within Brouwer’s intuitionistic mathematics. This is because, as opposed to serving as a hypothetical thought experiment as is often the case in philosophy, Brouwer’s creating subject plays a critical role in his very method - so much so in fact that Brouwer’s usage of it has since been referred to as ‘the method of the creating subject’ (Niekus, 1987; Atten, 2018; Posy, 2007).

With his conception of the creating subject, and his use of it as a method for establishing proofs[5], Brouwer’s Intuitionism has rightly been understood as accepting provability in principle as a legitimate means of mathematical argumentation. This is because, through demonstrating that the creating subject, which does not suffer some of the limitations of organisms like us, is able to arrive at a proof of *x* we are also able to conclude that *x* is provable. Consequently, even if we are not ourselves able to perform some operation, we are able to show that a creating subject that has fewer physical/practical impediments could. Moreover, because Brouwer equates truth to provability, the claim that *x* is provable is tantamount to its being true. We can take a toy example to illustrate this. While no human is able to count to 100,000,000 in their lifetime, the claim that 100,000,001 is the next number to occur in the natural number sequence can be said to

[5] For several examples of Brouwer’s usage of the creating subject within formal arguments see “The Creating Subject, the Brouwer-Kripke Schema, and infinite proofs” (2018) by Mark Atten.

be true since the creating subject, endowed with an indefinitely long lifespan, could count this high. Therefore, while we may not personally have the direct experience or 'proof' of the claim that 100,000,001 is the successor of 100,000,000, seeing as we cannot personally count to this magnitude, the claim is provable in principle and can be said to be true. Brouwer's legitimatization and acceptance of provability in principle is also carried into other forms of Constructivism to such a degree that, as will later be seen, whether or not a mathematician accepts provability in principle as opposed to in practice serves as a diagnostic criterion for differentiating Finitist and Constructivist philosophies of mathematics.

This acceptance of proof in principle is also present in Dummett's Constructivism, particularly, in his consideration of disjunctions and existential quantification (Dummett, 1975). With respect to a logical disjunction, this similarly is seen through Dummett's recognition that a construction of the proposition $A \vee B$ can be asserted even without an actual proof of which part of the disjunction obtains (Dummett, 31, 1975). Dummett explains this point with a division between canonical (actual) proofs and demonstrations. One way to illustrate this division is to consider the question, "Is the number ν (here defined as a number larger than anything which has hitherto been discovered by any human or advanced computer), a prime number (p)?" While we know all numbers must either be a prime number or not be a prime number, and while we can assert that: $\forall n\ (Pn \vee \neg Pn)$ is true, no actual proof can be provided to say which side of the disjunction in this case obtains (at least that is until ν has been constructed/discovered by an advanced computer). In other words, while it is not currently feasible to arrive at a canonical proof of whether ν is a prime number or not, we can demonstrate that such a proof is in principle possible – and even necessary in this case, insofar as ν will either be prime or not be prime. For Dummett, a *canonical* proof of a disjunction $A \vee B$ is a proof of A or a proof of B, followed by an application of disjunction introduction. Since, by hypothesis, ν is a number larger than any hitherto discovered, neither "ν is prime" nor "ν is not prime" has been proved, so we do not have a canonical proof of "ν is prime or ν is not prime." Nevertheless, Dummett allows for another sense in which we do know that to be true. This is because we can still offer the following *demonstration* of the claim that "for any n, n is prime or n is not prime" - $\forall n\ (Pn \vee \neg\ Pn)$. Let us now reconsider our number ν . Initially, regardless of the value of ν it will be easy to test if ν is even. Next, we are able to check whether "ν is divisible by 3 – and we may further continue in this way until either the answer is 'yes', after which, ν will be seen not to be prime, or until ν/2 - 0.5 is tested. If our answer is always no, then ν is prime. This is a finite procedure that might be undertaken for any number, and, from it, we are able to infer the overall truth of the claim that $\forall n\ (Pn \vee \neg\ Pn)$. For Dummett, this allows us to prove *in principle* that for any number it will either be prime or not be prime insofar as we have a *demonstration* of this claim. However, Dummett maintains a more strenuous criterion for what suffices as a *canonical proof* wherein a *canonical proof*, in the case of ν, would further need to contain a proof about which side of the disjunction obtains, a proof of whether ν is prime or is not prime. It is important to not here misconstrue Dummett's usage of the words 'canonical' or 'actual' proofs as delegitimizing their contrary (demonstrations), which instead as we have seen provide proofs in principle as he states "A demonstration is just as cogent a ground for the assertion of its conclusion as is a canonical proof, and is related to it in this way: that a demonstration of a proposition provides an effective means for finding a canonical proof" (Dummett, 32, 1975).

Notwithstanding their shared acceptance of propositions that are in principle provable, and notwithstanding that the classes of statements and patterns of inference Brouwer and Dummett are willing to accept as correct are in the final analysis quite similar, the philosophical differences between the two are nonetheless profound. Brouwer's intuitionistic mathematics implies that certain principles of classical logic are not valid and, in this sense, the logical changes are seen to follow from mathematical principles. On the other hand, at least in some presentations of his view, Dummett starts with the changes in logic and from there the differences from classical mathematics follow. Moreover, where Brouwer fundamentally maintains that mathematics is characteristically non-linguistic and exhibits a thoroughly anti-linguistic tendency in his Intuitionism, language, meaning, and use play a key explanatory role within Dummett's Constructivism which is itself grounded in ideas taken from the philosophy of language. This dramatic linguistic turn in Dummett's Constructivism is characteristically Wittgensteinian and is epitomized by his statement that "Any justification for adopting one logic rather than another as the logic for mathematics must turn on questions of *meaning*" (Dummett, 5, 1975).

The manifest difference between Brouwer's practice of intuitionistic mathematics and Dummett's Constructivism can be best exhibited by juxtaposing what they take mathematical proofs to be. While, on the one hand, Brouwer conceives of mathematical proofs as being both private and non-linguistic, on the other Dummett conceives of mathematical proofs both as being essentially linguistic and publicly constructed. As is to be expected, such differing conceptions of mathematical proof as are had by Brouwer and Dummett culminate in their proofs, regardless of how similar some of their conclusions are, looking rather different from one another. This difference can be further illustrated by looking at the means by which Brouwer and Dummett object to the classical methods. For one thing, Brouwer's arguments against classical mathematics are inherently formal in nature – they centre around the construction of formal symbolic proofs (Posy, 2007; Atten, 2018). In contrast, Dummett's arguments are inherently linguistic and feature ideas from the philosophy of language – especially the notion that publicly constructed meaning and use play a key role in our understanding of language.

For example, Dummett maintains, following Wittgenstein, that because the meaning of a statement is exhaustively determined by its use, and because this use is observable through a person's behavior, “a grasp of the meaning of a mathematical statement must, in general, consist of a capacity to use that statement in a certain way, or to respond in a certain way to its use by others” (Dummett, 7, 1975). Dummett takes this as being problematic for classical mathematical reasoning since it typically rests upon a kind of ‘verbalizable knowledge’ which is here understood as “knowledge which consists in the ability to state the rules in accordance with which the expression or symbol is used or the way in which it may be replaced by an equivalent expression or sequence of symbols” (Dummett, 7, 1975). Dummett argues that this results in an infinite regress since if we maintain that knowledge of the meaning of something is found in the ability to verbalize or state the meaning of it, then it would be impossible to learn a language unless someone already possessed enough of an understanding of the language so as to be able to verbally formulate its meaning. Here we have arrived at the notion that mathematical knowledge and meaning are not acquired but rather are ‘implicit’ or ‘innate’ (a notion that is not unfamiliar to Platonic philosophy). However Dummett, operating within a Wittgensteinian framework, takes this to be problematic in a way that is greatly reminiscent of Wittgenstein’s private language argument. He maintains that in order to meaningfully ascribe implicit knowledge to a person, there *must* be an observable mark in the behavior of a person possessing this knowledge that is not seen in a person who is lacking it – the implicature being that without an observable mark the ascription of implicit knowledge would be meaningless and unfounded (Dummett, 7, 1975). Consequently, Dummett argues that classical logics relate to a Platonic notion of truth and the ascription of implicit knowledge that is inaccessible to us (involving something hidden or ineffable which cannot be observed) and which is also in conflict with the notion that use exhaustively determines meaning (Dummett, 10, 1975).

Despite the differences between Brouwer and Dummett, differences which generally revolve around the role that language plays in mathematics, both remain characteristically Constructivist – though they differ in their views about the nature of constructing proofs in mathematics. Brouwer's Intuitionism utilizes the method of the creating subject to construct proofs and views proofs as private mental phenomena while, in contrast, Dummett's Constructivism views proofs as publicly constructed, determined, or else constituted by the human practitioner's use of language. Notwithstanding these differences, both Brouwer's Intuitionism and Dummett's Constructivism share the key characteristic of identifying the truth of a mathematical proposition with its provability.

Mathematical Truth in Constructivism:

Concerning the question of what constitutes as mathematical truth among the Constructivists, the answer to this, at least on Brouwer's part, is well evinced from his reason for rejecting the law of the excluded middle and from his corollary rejection of Hilbert's formalist mathematical project. Nevertheless, and before detailing this, it is necessary to establish a few general remarks regarding Brouwer's conception of mathematical proofs and truth. Both the most important and the most descriptive of which is the idea that *proof is the criterion of truth* for Brouwer (Posy, 2007; Hansen, 2016)[6]. In other words, in order for us to know that something

[6] Brouwer's philosophy is on this point particularly difficult to interpret and may not lend to a single interpretation or conclusion. Nonetheless, the common sense is found in both Posy 2007 and Hansen 2016 that proof is certainly constitutive, at least to some degree, of truth for Brouwer. For a greater elaboration on both the technical and conceptual intricacies of this point see Hansen 2016.

is true, we must have a constructed proof (or demonstration) of it. Moreover, it would further be accurate to extend this claim in the form: in order for something *to be* true it must be provable – and it is in this extension that Constructivism is differentiated from other philosophies of mathematics. When taken in this stronger sense, it is seen that Brouwer maintains that truth presupposes provability (Posy, 2007). Here, however, the question comes – what kind of provability is here presupposed? As we have seen in the last section, the answer is that at the least proof in principle is presupposed by Brouwer as a necessary criterion of mathematical truth (and that Brouwer seems to accept proof in principle as legitimate).

Brouwer's reason for maintaining this proof criterion of truth falls analytically out of his general Intuitionist project. Because of his staunch rejection of the Platonic mathematical assumption (the assumption that mathematical objects exist in a sense that is akin to the Platonic forms), mathematical truths as much as mathematical proofs, for Brouwer, are only said to 'exist' insofar as they have been mentally constructed by us[7]. Without recourse to a Platonic sense of a mathematical object as existing independently of the mind, a mathematical proof that purports to appeal to something that is not mentally constructed is plainly nonsense for Brouwer. The reason for this is that, once one has supplanted existence with mental constructability, it is no longer sensible to appeal to anything beyond that which we *have* or, at the very least, which we *could* construct[8].

It is with this sense that Brouwer's rejection of the law of the excluded middle is made sensible. According to the law, it is *true, for every proposition A,* that $A \vee \sim A$. For a particular instance of the law, a canonical proof would be a proof of one of the disjuncts, but, even for

[7] And it is this which further serves as the general escutcheon of the Constructivist camp.
[8] Here again the parallel occurs that that which we *have* constructed corresponds to the idea of provability in practice and that which we *could* construct corresponds to the idea of somethings being provable in principle.

Brouwer, a proof of the law needn't involve a proof of one of the disjuncts for every instance: the truth of the law would be established if we knew each instance was true, so could therefore be established by showing (i.e., by describing a method) that for any given instance we *could* prove one of the disjuncts. But this is not something which we have at present. Although we can certainly imagine cases where this is not problematic, cases where we can construct a proof of which side of the disjunction obtains (consider, for instance, the example given when discussing Dummett above, but in fact for any proposition B that we can prove we have a known instance of B ∨ ∼B by disjunction introduction), the law of the excluded middle becomes greatly problematic in cases where which side of the operator obtains is undecidable. To take the ready-at-hand example, we can imagine the question of whether or not γ, defined as the proposition that some specific sequence of digits appears in the decimal expansion of π, is provable or refutable. Moreover, for our present purpose, let us further define γ as involving a sequence of digits that is not as of yet determined to be found in the decimal expansion of π. In order to better understand Brouwer's position, we can then ask the question: what are we to make of the claim γ is true or not, i.e. that γ ∨ ∼γ?

While it certainly appears *prima facie* that it is true that γ ∨ ∼γ, from within Brouwer's Intuitionism, the truth of γ is indeterminate insofar as the decimal expansion of π has only been constructed up to a given point. In order to appreciate Brouwer's point, it is essential to recognize that the decimal expansion of π only 'exists' insofar as it has been mentally constructed by us. In order to positively conclude that ∼γ is true one would need to have a proof that γ does not occur in the decimal expansion of π. From Brouwer's perspective, we cannot refute γ on the basis of such a small fragment of the total evidence (the sequence of π we have currently constructed, a sequence which since we know (i.e., have proven) the decimal expansion to be potentially infinite and non-repeating is an infinitesimal part of the whole of the decimal expansion) and have not been able to derive a contradiction from the assumption that ∼γ, and nor can it be said that we have evidence that γ is true. Consequently, we can neither refute nor assert the truth of γ and so we lack sufficient knowledge to say that the disjunction obtains one way or the other. Even worse yet, "Assuming that it [in this case γ] is true or false in the absence of such knowledge amounts to assuming that the decimal expansion of π has extra-mental existence" (Hansen, 1, 2016). Therefore, in the case of γ as it has been afore defined (as asserting the occurrence of a sequence the decisional expansion of π that has not as of yet been definitively found/constructed) the 'truth' of not γ does in fact appear in π is undecided, and for all we know at present a proof cannot even be in principle constructed one way or the other. On the other hand, for all we know at present ∼γ might be proved or refuted tomorrow. Thus, it is critical here to recognize that this reasoning is not unique to the specific example (γ). There are *many* examples of instances of LEM that are at present neither refuted nor proven, and presumably always will be as long as mathematics is a discipline worth pursuing. In other words, Brouwer's Intuitionism maintains in a general sense that we can neither reject nor affirm the law of the excluded middle as there are a number of given cases for which we do not know whether either side of the disjunction is provable. Despite the possibility of our discovering a proof of any specific given instance of the law of the excluded middle in the future, there will still be a number of cases that such that we would not know whether either side of the disjunction is provable. And it is for this reason that we must, at least in cases where this fine point might be relevant, recognize that while Brouwer *rejects* LEM, he does not *deny* it in the sense of being willing to assert its negation: we cannot *assert* LEM because we do not know it to be true; but we cannot deny it, because we have not refuted it by, for instance, presenting a counter-example that we know could never be decided.

In contrast with Brouwer, much of Dummett's conception of mathematical truth takes on a characteristically linguistic aspect and may be understood in relation to his 'case' against realism. One author has described this case as occurring in four stages, where in the first stage Dummett identifies disputes over realism with disputes over the principle of bivalence (Rosen, 600, 1995). Moreover, it is said that: "Dummett holds that to be a realist is to accept bivalence, to reject it to be some sort of anti-realist (p. 230)" (Rosen, 600, 1995). According to this principle, all statements (otherwise also referred to as declarative sentences or judgments) are maintained to be either true or false. Rosen identifies the second stage of Dummett's case as maintaining that the principle of bivalence only holds in those cases where the meanings of our sentences allow us to, at least in principle, ascertain their truth value (Rosen, 600, 1995).

However here, with the notion of meaning, what Rosen identifies as the 'destructive part' of Dummett's case is introduced. In this destructive third part of the case, Dummett's characteristically Wittgensteinian aspect appears as he, maintaining that meaning is use, is said to hold that "whatever it is to understand a sentence as possessing certain truth-conditions, this state must be "exhaustively manifestable": it must be the sort of thing that can (and frequently does) *show up* in one's overt behavior, otherwise languages would be unlearnable and communication impossible" (Rosen, 601, 1995). Although this criterion of understanding as being observably manifestable is benign in clear-cut cases where the given truth conditions are easily evaluable through one's behavior, it becomes greatly problematic in those cases where the truth conditions cannot be made manifest in someone's observable behavior. By Rosen's characterization, this stage concludes in the notion that "since there is no way for a speaker to manifest his assignment of this or that verification transcendent truth-condition to a sentence, no sentence can possess such a truth-condition; hence we are never entitled in the interesting cases to accept the principle of bivalence, and realism must therefore be abandoned" (Rosen, 601, 1995). It should be noted that this conclusion is an especial concern for mathematics since, owing to its highly abstract subject matter and the complicated nature of mathematical proofs, the truth conditions, which must be exhaustively manifestable if the proposition is properly understood, are often not manifestable at all.

In the fourth stage of the case, by Rosen's recapitulation, Dummett offers an alternative theory (based upon an intuitionistic semantics) to redress this problem concerning our understanding of the truth conditions of mathematical language. According to this picture, whether or not someone understands a mathematical sentence can be manifested by their acceptance or rejection of proposed proofs (Rosen, 601, 1995). On this point, it is said that "The statements of such a language do have truth-conditions; but the notions of truth and falsity play no central role in an explanation of their meanings, this function having been shifted to the notions of proof and refutation" (Rosen, 601, 1995). Here Dummett displays a characteristically Constructivist orientation as he is more concerned with whether or not one can manifest an understanding of the meaning of constructed mathematical statements (through the acceptance or refutation of them) than with whether or not the statements themselves obtain as true. Dummett's position can be better illustrated by reconsidering his treatment of the law of the excluded middle, a treatment that parallels Brouwer's rather closely, which was touched upon earlier in relation to canonical proofs. According to this, one may make clear that they understand the meaning of the law of the excluded middle by the proposition $A \vee \neg A$ (through affirming that the overall proposition is true), even without a canonical proof of which side of the disjunction obtains. And with carefully selected examples like “n is prime” there will be such a non-canonical demonstration available. But for γ as discussed above, we possess no such demonstration, so we cannot accept $\gamma \vee \neg\gamma$. Consequently, for Dummett, as much as for Brouwer, the value of the law of the excluded middle is in this way to be rejected (though not denied).

Constructivism and Mathematical Objects:

The definitive feature of Constructivism – the replacement of the idea that $\exists xPx$ with the idea that ‘we can construct an x *that has the property P*’– in itself already contains a great number of ontological consequences. Chief among these is that the only mathematical objects which enter into our ontology are those which we have either already constructed or which are constructible. Brouwer’s Intuitionism serves as the quintessential exemplar of this notion as he explicitly maintained that the only legitimate mathematical objects are constructed, and consequently, that only those objects which have been constructed exist. This carries the rather strong consequence that, with respect to mathematical objects, a great number of ‘objects’ that are taken to be legitimate under classical mathematics lose their positive ontological status within Brouwer’s Intuitionism. The foremost example of these is completed infinities which, as was seen earlier in this chapter, Brouwer’s Intuitionism does not admit. Much of the reason for this ontological variation between the objects which are classically and intuitionistically accepted is that, being devoid of the Platonic posit of an independent form-like existence, Intuitionism only accepts mathematical objects in a mind-dependent sense and only qualifies them as ‘existing’ insofar as they are constructible. In discussing the construction of mathematical objects, Brouwer speaks of these constructions as occurring through ‘generative acts’ after which an object may be said to exist (Posy, 2007).

What he takes a construction to be, and so what distinguishes him from the classical mathematician, can be better evidenced by considering what does *not* satisfy Brouwer as a legitimate mathematical proof of existence. In this respect, we can revisit Brouwer's repudiation of the law of the excluded middle with a greater focus on why he rejects proofs of existence that are founded upon this law. An earlier means of rejecting the law of the excluded middle took issue outright with the fact that it does not 'prove' either $A \vee \neg A$ since, instead of indicating which side of the disjunction obtains, it instead just indicates that one side must obtain, and for some examples of A we do not know that one or the other disjunct is provable. However here, concerning proofs for the existence of mathematical objects, Brouwer's repudiation of the law of the excluded middle takes on a more distinctly ontological accent. Proofs of this sort take on the form: if $A \vee \neg A$; then$\neg \neg A \rightarrow A$; and $A \rightarrow \neg \neg A$. These proofs generally function by showing that, so long as we can show that one side of the disjunction does not obtain, it would be absurd not to suppose that the opposite side does obtain. In other words, one can prove the truth of $\neg A$ by disproving A, or vice versa, since according to the law one side of the general disjunction must obtain. Consequently, this law allows for the proof of $A \vee \neg A$ through the negation of the opposite side of the disjunction. For example, one could theoretically offer a proof for the existence of a completed infinity, within a classical framework, by proving that it is absurd to maintain that there is not a completed infinity (η) by proving $\neg (\neg \eta)$ through a double negation elimination.

Brouwer objects to existence proofs of this sort outright since they do not 'construct' a mathematical object. He maintains that such proofs cannot culminate in a proof of the existence of a mathematical object - since such an existence is only endowed by a generative act of construction. In the above case, Brouwer would respond that a proof demonstrating $\neg (\neg \eta)$ does not entail that η exists because such a proof does not contain a construction of η – and moreover, Brouwer would further object to the idea that a construction of η is possible since not even the creating subject can collect/construct a completed infinity. A similar sentiment to this has also been identified in how Brouwer handles Cantor's diagonal argument (Posy, 2007). Here it is said that Brouwer "happily accepts Cantor's diagonal argument as a proof that the real unit interval is uncountable" (Posy, 13, 2007); however, he does not accept the conclusion that an uncountable cardinal number exists since no construction of such a mathematical object is present therein. This carries a consequence for Brouwer's Intuitionism, relating to the notion of a 'generative act', which is that the construction of an object is inherently an *explicit* act for Brouwer. In other words, not only do indirect existence proofs not satisfy Brouwer, but further, the construction of a mathematical object is seen as an explicit process whereby a mathematician – through a generative act – constructs an object which can then be said to 'exist'. It is for this reason that Brouwer refuses to accept not just LEM, but also Double-negation elimination (or, as it is sometimes called, indirect proof.)

Before moving on to Dummett, one final qualification of Brouwer's conception of mathematical objects should be made. This qualification is that Brouwer maintains that an object may be considered to have been legitimately constructed even if all of its possible properties are not completely determined (Posy, 13, 2007)[9]. The benefits afforded by this qualification of Brouwer are borne out by the following example. We can consider (or 'generate') a right triangle as an example of a mathematical object which may be constructed in the form: 'a three-sided polygon whose interior angle sum equals 180 degrees where one angle is equal to 90 degrees'.

[9] Another interesting consideration found in this article is that Brouwer does not require that legitimate mathematical objects are distinguishable from each other (Posy, 13, 2007).

By Brouwer's account, we can recognize this as being a legitimate construction of a right triangle despite the fact that this construction does not include the Pythagorean theorem. Moreover, if we were to maintain the opposite position (that a construction is only legitimate when it completely determines all of the potential properties of an object), then we would have to similarly maintain that no one before Pythagoras had constructed a 'right triangle' and that, before him, there was no such object (a notion which Pythagoras himself would surely find heretical)[10].

In contrast with Brouwer, Dummett's treatment and discussion of mathematical objects displays more of a philosophical, as opposed to a formal, orientation. Dummett's treatment of mathematical objects as it is here recapitulated is that which he articulates in the article "What is Mathematics About?" - though it should be noted that he had also discussed the subject earlier in other writings[11]. Dummett's perspective, as it is here contained, is both charitable and partial to the logicist project of Frege, Russell, and Whitehead. This partiality is revealed when Dummett satisfies himself with the logicists' answer to the question: 'what is mathematics about' in the statement that "the logicist answer, if not the exact truth of the matter, is closer to the truth than any other than has been put forward" (Dummett, 21, 1993). Moreover, Dummett reaffirms this partiality in his conclusion to this article with the further statement that "I have argued that it is useless to cast around for new answers to the question what mathematics is about: the logicists already had essentially the correct answer" (Dummett, 29, 1993).

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[10] Another example to the same effect of this occurs in Posy 2007 and concerns the indiscernibility of the real numbers.

[11] The earlier conception of Dummett occurred in his 1973 book *Frege: Philosophy of Language* and has been critiqued by Noonan in his article "Dummett on Abstract Objects". It is in part due to this critique, but is more so due to the development of Dummett's thought on the subject which occurs in the more recent book, that I am here concerned with Dummett's position as it occurs in the 1993 "What is Mathematics About?".

Dummett identifies the logicist's answer to this question as maintaining that mathematics is concerned with generalities - as opposed to anything particular (Dummett, 21, 1993). This notion, that logicism is the view that mathematics is a part of logic dealing with general claims, was well articulated by Frege’s persuasive argument that logical principles can be differentiated from those of the particular empirical sciences in the sense that they apply across all domains without restriction. It is in this sense that Dummett suggests that the logicists essentially had the right answer in identifying mathematics with generalities – since a key aspect of mathematical truths and principles is that they apply in a general sense regardless of what specific objects one is concerned with. This position is greatly appealing as it services an explanation for why it is mathematical truths apply generally across all domains (why 2 + 2 of any four given things equals 4). Moreover, this explanation also serves to distinguish claims which are true in some but not all domains, such as empirical claims like ‘all things are affected by some gravitational force’, from claims like 2 + 2 = 4 which remain true across all domains – such as within the hypothetical domain of abstract and non-physical objects. The logicists' answer, as Dummett maintains, is therefore the ‘right’ way of thinking about what mathematics is about because their conception accounts for the generality of mathematical claims whose formal truth/structure both underlies and is independent of the particular objects concerned. However, while Dummett gives logicism credit for having provided the best available answer to the question of what mathematics is about, he also holds that the logicist project falters when trying to address the problem of whether or not mathematical objects exist (Dummett, 21-22, 1993).

The general concern regarding the existence of mathematical objects (that which Dummett finds the logicists’ faltering upon), is how we are to make sense of references to abstract mathematical objects. In this regard, Dummett suggests that one must either justify or explain away how it is we refer to abstract objects (Dummett, 23, 1993). For his own part, Dummett seems to be amendable to Frege's defence of abstract objects and vehemently opposed to a strict nominalism. He makes this rather explicit in the statement, "Once we have abandoned the superstitious nominalist horror of abstract object in general, there would be nothing problematic about the existence of real numbers in the context of some empirical theory involving quantities of one or another kind" (Dummett, 24, 1993). Dummett's move here, the idea that we should understand the 'existence' of real numbers within the context of an empirical theory, incites the question: does this then entail that the real numbers possess an object status in the context of the empirical theory and, if so, does this not just reintroduce the Platonic problem of how/where it is the real numbers exist as abstract objects?

Dummett, however, anticipates this concern and answers it by suggesting that we can rid ourselves of 'the superstitious nominalist horror' which refuses to admit any kind of existence whatsoever to abstract objects by considering contingent abstract objects which serve a role in an empirical theory. To support this point, Dummett makes recourse to Frege's example that the equator is an abstract object whose existence is contingent upon other concrete objects or 'the relations between them' (Dummett, 24, 1993). Moreover, he repudiates the orthodox distinction that is made between abstract and concrete objects (where mathematical objects are taken to be nearly synonymous with abstract objects) and argues that we should instead be drawing a distinction between mathematical objects and all other objects (Dummett, 21, 1993). Dummett uses Frege's example of the equator to make sensible the notion that the existence of certain abstract objects depends upon concrete objects and their relations within an empirical theory; however, he nonetheless recognizes that "By contrast, the existence of mathematical objects is assumed to be independent of what concrete objects the world contains" (Dummett, 24, 1993).

The difference for Dummett then, in this case, between mathematical objects and other abstract objects like the equator is that mathematical objects are independent of what concrete objects there are, whereas the equators existence is dependent upon concrete objects within a given domain. Nevertheless, the question remains – what is the object status and similarly, the claim to an ontological status which is granted to mathematical objects within Dummett's Constructivism? A difficulty is encountered here since (as we have seen) if we view mathematical objects as abstract objects which, like the equator, are contingent on an empirical theory, then they would cease to be logical objects that are generally applicable across every domain. While there is no issue with making something like the earth's equator contingent upon an empirical domain, there is an issue with making mathematical objects (which are supposed to be logical objects that generalize across any and all domains) contingent upon an empirical theory.

Although some difficulty can be said to remain around this point, Dummett is able to relieve some of the tension by offering a distinction between mathematical objects and other objects on the basis of what 'domain' is relevant. For concrete objects and abstract objects which have an existence contingent upon the concrete objects and their relations (like the equator), the external reality is the relevant domain (Dummett, 25, 1993). In contrast, Dummett treats mathematical objects differently since the question of "what mathematical objects there are within a fundamental domain of quantification is supposed to be independent of how things happen in the world, and so, if it is to be determinate, *we* must determine it" (Dummett, 25, 1993). This differing treatment of mathematical and of other objects is both similar to Brouwer as well as generally representative of the Constructivist position since, at least as far as the former is concerned, Dummett takes the existence of mathematical objects in the anti-realist sense of being said to exist insofar as they have been constructed or else determined by us.

In summary, while Brouwer and Dummett differ in what exactly they take mathematical objects to be, with the former viewing them as private mental constructions and the latter viewing them as public constructions, they arrive at the same place of qualifying mathematical objects as existing insofar as they either have been, or can be, constructed. Consequently, for both Brouwer and Dummett the mathematical objects which are said to exist are those which can be constructed or proven. This reflects the general Constructivist tendency to avoid all metaphysical speculation and complication by replacing the idea that '*x* exists' with the idea that 'we can construct *x*'.

The Constructivist Foundations of Mathematics:

While earlier, in the first section of this chapter, we examined how Brouwer's intuitionistic mathematics related to his conception of mathematical proof, here I will aim to describe the philosophical foundations of Brouwer's Intuitionism. In order to do so, it will be essential to understand what Brouwer takes intuitionistic mathematics (and consequently mathematics as a whole) to be founded upon. It is also, however, equally important to understand what Brouwer explicitly takes the foundations of mathematics *not* to be, namely, either logic or language. Brouwer's avowed anti-linguism is evident as early as his doctoral thesis, and he explicitly maintains that mathematics is prior to both language and logic – both of which he maintains are dependent upon mathematics. In addition to this negative characterization, Brouwer follows Kant in positively grounding mathematics in *a priori* mathematical intuition insofar as he takes this mathematical intuition to be both temporal and abstract (Posy, 12, 2007).

Consequently, it is beneficial to discuss some of Kant's philosophy in order to better situate the philosophical foundations (as well as the *intuitive* appeal) of Brouwer's Intuitionism.

Brouwer's understanding of mathematical intuition, as being inherently temporal and abstract, relates to Kant's conception of the Pure Intuition of Time (conceived there as the *inner sense*), which Kant argues is intrinsically related to our understanding of simultaneity and succession (Kant, 67, 1781)[12]. Moreover, Kant strongly maintains that it is only through this inner sense of Time that alteration is made possible (Kant, 68, 1781). How Kant's conception relates to mathematics, inasmuch as why it was appealing to Brouwer as a philosophical foundation for his Intuitionism, is made immediately apparent by considering the *series* of natural numbers. The reason for this is that the very conception of the natural numbers as a sequence or ordered series already implies the conception of succession – the series begins at one and successively increases at a fixed rate. Moreover, Kant provided a philosophical foundation, and legitimacy, to Brouwer's inclusion of temporal dimensions to mathematics in the form of past, present, and future times which he controversially takes to play a positive explanatory role in proofs involving the 'creating subject' (Atten, 2018).

Notwithstanding the (perhaps) questionable status of Brouwer's extrapolation of Kant's conception of the Pure Intuition of Time, his utilization of Kant's notion of *a priority* appears to be less contentious. It is said that Brouwer took mathematics to be *a priori* in the same sense that Kant did, namely, that it was independent and prior to our sensory experience (Posy, 13, 2007). Brouwer's reason for maintaining that mathematics is independent of sensory experience derives

[12] Although this relation is explicit in Brouwer, it is important to recognize that Brouwer understands the Intuition very differently from the way that Kant does. The staunch difference between the two is made apparent insofar as Brouwer understands both arithmetic and geometry as coming from the Pure Intuition of Time, whereas, Kant notoriously related geometry to the Pure Intuition of Space.

from the means by which we go about justifying mathematical judgements. Unlike *a posteriori* judgements (which are justified in light of empirical experience), mathematical judgements do not require empirical experience to justify them, but rather, their proofs are taken, titularly, to be knowable prior to sensory experience. Moreover, Brouwer is further attributed with maintaining that mathematics underlies, and is thus primordial to, the empirical sciences (Posy, 13, 2007). In this sense, Brouwer remains faithful to Kant's conception - as Kant similarly maintained that the Pure Intuitions of Time and Space were presupposed, and thus primordial, to the very possibility of having empirical experience.

In contrast to Brouwer, Dummett views mathematics as a fundamentally linguistic enterprise whose proofs and objects are, as opposed to being the private mental constructions of a mathematician, public linguistic objects. Consequently, for Dummett, the truth or correctness of a proof is subject to rules within a public language game, and more, a mathematician's understanding is also manifested in a public sense as has been seen. Although this linguistic focus of Dummett would be anathema to Brouwer, who vitriolically maintained that mathematics was not dependent on either logic or language, their positions nonetheless remain greatly similar across a number of varying respects. The most important of these similarities is that both Brouwer's Intuitionism and Dummett's Constructivism are inherently anti-realist theses that aim to correct the Platonic proliferation of abstract objects which clutter up one's ontology. Moreover, while they might differ in what they take the constructive process to entail, they both share the characteristic Constructivist tendency of supplanting the notion that *x* exists with the idea that *x* can be constructed and see the truth of a claim in a way that relates to its provability.

CHAPTER TWO: FINITISM

An Introduction to Finitism:

Within this chapter, I will provide a short exegesis of the general principles which underlay the philosophy behind Mathematical Finitism. In doing so, I will explore two examples or 'kinds' of Mathematical Finitism as they occur in the literature. The first of these concerns Dantzig's article "Is 10^10^10 a Finite Number?" which, while by no means offering a systematic Finitist account of mathematics, illustrates a number of important Finitist principles. In contrast, the second example of Finitism explored in this chapter is Mayberry's Euclidean Finitism which offers a *bona fide* Finitist system of mathematics. This chapter's analysis will take the form of assessing both the benefits which Mathematical Finitism offers as a foundation of mathematics, as well as the philosophical cost of maintaining the position. Furthermore, it should be noted that, although the dialogue concerning Mathematical Finitism is well developed within the literature, two initial problems present themselves for anyone attempting to treat of the subject. The first, and greatest, of these problems, is the mosaic of varying Finitist positions (some examples of which include Classical Finitism, Strict Finitism, and Ultrafinitism); and the second, recognized in Mawby 2005, is the relative rarity of Finitism's proponents. Consequently, it is important to recognize that the subset of Finitist positions which are considered within this chapter – while representative of Mathematical Finitism generally – are not intended to suffice as an exhaustive treatment of all Finitist positions.

Notwithstanding this, some sufficient criteria for identifying those philosophies of mathematics that are examples of Mathematical Finitism may be identified. One such criterion takes the form of a 'family resemblance' by which Mathematical Finitist positions are all related in their shared desire/principle of only accepting finite and 'concrete' mathematical objects as legitimate, or rather, their shared principle of not accepting infinite mathematical objects as legitimate (however much their reasons for this differ). Furthermore, the extent to which a Mathematical Finitist philosophy remains inflexible around the illegitimacy of infinite mathematical objects also serves as a useful means of broadly differentiating the various kinds of Finitism. In this respect, Classical Finitist positions can be classified as those softer forms of Finitism which, despite opposing the existence of actual mathematical infinities, are willing to admit of potential mathematical infinities to some extent[13]. Contrarily, Strict or Ultrafinitist positions generally repudiate the legitimacy of infinite mathematical objects outright and thereby reject any predication of even a potential (or formal) construction of infinity from the natural numbers[14]. This distinction has been supported by others in the literature who make similar classifications between the different forms of Mathematical Finitism (Tiles, 1989; Mawby, 2005).

Before beginning this chapter's analysis of specific formulations of Mathematical Finitism, it is essential to first situate Finitism generally in relation to both Platonism and Constructivism. Mathematical Finitism, much like Constructivism, offers an alternative to Mathematical Platonism's ontology regarding the mind-independent existence of the numbers. In this capacity, Finitistic philosophies of mathematics are generally aligned with an anti-realist ontology though, as will be seen, some Finitist positions such as Mayberry's are softer on this point. However, this alignment is even further pronounced in the stronger example of Finitism, such as in Strict Finitism, which is recognized to be "fundamentally committed to an anti-realist

[13] Hilbert's position of trying to 'found' Cantor's work within "finitary" arithmetic and his classical declaration that 'no one shall remove us from the paradise which Cantor created' has led many to place him here as a kind of Finitist. See "Varieties of Finitism" by Manuel Bremer for one such example.

[14] These positions are also sometimes referred to as Actualism or Ultra-intuitionism and are generally associated with Alexander Yessenin-Volpin.

position with respect to mathematics. That is to say that the numbers, statements, and proofs of mathematics are mind-dependent" (Mawby, 9, 2005). In this way, Finitism is similar to Constructivism, and more, this similarity is further demonstrated by certain Finitists utilizing Constructivist methods (as occurs in Mayberry 2000) and is even made declaratively explicit in others stating "Indeed, it [Strict Finitism] is a Constructivist theory - it stems from the idea that mathematics is constructed by the mathematician, and hence numbers (for example) are only 'real' if they are constructible. This is a key motivation behind strict finitism" (Mawby, 9). However, other Finitists – most notably Mayberry – appear as a pendulum shift back towards mathematical realism insofar as they defend a sense in which (finite) numbers are said to exist in even a mind-independent sense.

Dantzig's Finitism:

Dantzig's Limitations on Mathematical Considerations:

For the purposes of this paper, I will consider Dantzig's Finitist position as it occurs within his essay "Is 10^10^10 a Finite Number?". Dantzig's position within this essay differs from that of Mayberry's which will be next considered in that, instead of offering a systematic framework for doing mathematics, Dantzig instead centres his inquiry around the question of whether considerably large numbers can be said to be finite or natural numbers. It is very important to first recognize the framework from which Dantzig is here operating, namely, that he does not want to admit of "fictitious superior minds like Laplace's intelligence, Maxwell's demon or Brouwer's creating subject [ideal mathematician]" (Dantzig, 1, 1955). What these 'fictitious superior minds' all share in common is their unbounded or else omniscient intelligence. Consequently, Dantzig argues that it is necessary to consider mathematics with respect to the limitation of the human mind and those machinations which assist it (Dantzig, 1, 1955). This notion I will refer to as Dantzig's *cognitivist* thesis and it may be summarized in the form: 'we ought only to concern ourselves with mathematics as they are able to occur within our limited (*finite*) cognitive capacities'.

Furthermore, Dantzig lays out another foundational limitation on the notion of a natural number. He argues that, regardless of the definition of number with which one is operating, it is a requirement of any 'sequence of printed signs' that it be uniquely identifiable – i.e., that for any number it is necessary that said number is distinguished from the other numbers (Dantzig, 1, 1955). This requirement is not unique to Dantzig, as it is a general requirement of 'identity' to maintain that, for any given thing to be a given thing, it needs be distinguishable from other things. Another example of this requirement within mathematics also comes from Mayberry who spends a great deal of time treating of how to establish the unique identity of an *arithmos*[15].

In addition to Dantzig's *cognitivist* thesis, and his identity requirement of numbers, he articulates a third limitation on mathematical considerations – the requirement of actual constructability (Dantzig, 1, 1955). In this regard, Dantzig argues against the possibility of constructing certain prodigiously large numbers (such as 10^10^10) which outstrip the physical limitations of the universe. Ultimately, he argues against mathematicians who imagine the construction of 'arbitrarily large natural numbers' because "this would imply the rejection of at least one of the fundamental statements of modern physics (quantum theory, finiteness of the universe, necessity of at least one quantum jump for every mental act). Modern physics implies an upper limit, by far surpassed by 10^10^10 for numbers which actually can be constructed in this way" (Dantzig, 1, 1955). This third limitation shares a similar spirit to Dantzig's *cognitivist*

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[15] For one example of this within Mayberry's work see section 3.1 "Objects and Identity" of *The Foundations of Mathematics in the Theory of Sets*, though this point will be greatly detailed in the next subsection of this chapter.

thesis insofar as both centre around the idea that our mathematical considerations must consider the real or practical limitations which are afforded by circumstance, i.e., our own cognitive limitations or the limitations imposed by the physical universe.

Dantzig on Discerning Between Finite and Infinite Numbers:

Dantzig's general argument within this paper centres around the notion that arbitrarily large numbers like 10^10^10 only appear to be natural numbers when one has "unconsciously changed the meaning of the term "natural number" (Dantzig, 2, 1955). In order to demonstrate this point, Dantzig provides a number of constructions in which he compares numbers/sets that have been constructed up to a definite point (dubbed n_1 and s_1) to numbers/sets that are constructed in a second purely formal sense (dubbed n_2 and s_2 respectively) (Dantzig, 2, 1955). Dantzig refers to numbers/sets of this second sense as 'fictitious' and considers proofs regarding their properties as 'postulates' (Dantzig, 2, 1955). Further, he recognizes that this first set will not contain all of the fictitious natural numbers of the second, but rather, that it will only contain those "for which sufficiently simple abbreviations have been introduced" (Dantzig, 2, 1955). In other words, the first set will only contain those natural numbers which have been definitely constructed – those which are surveyable.

Concerning the application of mathematical operations to fictitious numbers, Dantzig argues that "the statement that 10^10^10 + 10^20^20 = 10^20^20 + 10^10^10 can not be said to have been proved, but is only a formal rule for handling formally the symbols" (Dantzig, 3, 1955). With this, Dantzig objects to the legitimacy of formally extending arithmetical operations beyond the bounds of numbers that have been definitely constructed - n_1 or s_1 (Dantzig, 3, 1955). Consequently, he rejects Poincare's notion that complete induction is the creative principle of mathematics. His reason for this rejection is that complete induction beyond the definitely constructed numbers n_1 and s_1 does not afford us any proof, but rather, results only in postulates which could be true but have not yet been proven.

Furthermore, Dantzig argues that our differentiation between the finite and transfinite numbers may not be operationally defined. In order to demonstrate this point, he imagines two mathematicians, A and B, where A is understood to be a Transfinitist and B is a Finitist. Dantzig notes that it is possible that whenever mathematician A refers to a transfinite number (ω), mathematician B instead interprets the number as a finite number (Ω). Moreover, he states that when mathematician B interprets mathematician A's transfinite ω as the finite Ω, he may do so without coming to an inconsistency (Dantzig, 3, 1955). However, the matter is more complicated in light of Dantzig's recognition that mathematician B, in interpreting mathematician A's transfinite number as a finite number, will not always interpret it as the same one (Dantzig, 3, 1955). This possibility poses a serious problem for the transfinitist, since this example seems to illustrate that transfinite numbers are not uniquely identifiable, allowing for a hidden indeterminacy to slip into mathematics. This hidden indeterminacy takes the form of an equivocation possibly entering into our mathematical discourse. While it may certainly appear that, in both mathematician A and B's usages of Ω and ω, no outward inconsistency results in how the mathematicians operate upon the symbols, a semantic indeterminacy can be said to result insofar as when B operates upon ω (reinterpreting it as the finite number Ω) they may, and at times will, be referring to an entirely distinct number. The critical point in this case is that while mathematician B may engage in discourse about the number Ω/ω with mathematician A, and while mathematician B may possess an operationally sufficient translation of ω as Ω such that no direct inconsistency is apparent, there is a hidden indeterminacy in their dialogue insofar as the two are unable to determine if they are referring to entirely separate numbers. However, because transfinite numbers are so large, or, as Dantzig is getting at, because numbers so prodigious as 10^10^10 are so large, neither Ω nor ω are uniquely identifiable as numbers in the same way that clearly finite numbers are. This inconsistency is tantamount to an imperfect translation, whereby something of the original sense and meaning is lost (in this case the sense of the number being ‘transfinite’) but in such a way that this loss is not noticed because the translation is operationally functional. These considerations help to illustrate the sense behind Dantzig’s notion that finite and transfinite numbers may not be operationally defined since, at a certain point (beyond the limits of numbers that are surveyable to us), it is possible for mathematicians A and B to suffer from an equivocation in referring to fundamentally different numbers without any inconsistency being apparent to them.

The possibility that gives rise to this issue results from the fact that, in the case of transfinite numbers, we are not dealing with definite or surveyable constructions that allow for us to clearly identify or define a precise number/object to which we are referring. Dantzig makes another suggestion that if mathematician B were to use transfinite symbols such as ω or $\aleph^0$ he would do so only with the sense of referring to “numbers surpassing everything I can ever obtain but not as anything essentially different from those he *can* obtain” (Dantzig, 4, 1955). Consequently, Dantzig’s ultimate argument is that “The difference between finite and infinite numbers is not an essential, but a gradual one” (Dantzig, 4, 1955). This conclusion is based on the recognition that, as we approach increasingly prodigious numbers, our ability to understand them wanes and, at a certain point, we begin to deal with numbers that are no longer uniquely identifiable.

Mayberry’s Euclidean Finitism:

An Introduction to Mayberry’s Position:

The Finitist system of mathematics which will be here considered, dubbed Euclidean Finitism, is derived from Mayberry’s *The Foundations of Mathematics in the Theory of Sets*. Whereas in the previous chapter, Constructivism was seen to be a through and through anti-realist alternative to mathematical Platonism which did not provide a positive answer to the ontological question, Mayberry’s system at times begins to again reintroduce the ontological questions. Notwithstanding this, it would be inaccurate to characterize Mayberry as a realist since, while he does raise and address the ontological question in a limited extent, he both: a) does not himself adopt a realist title, and b) explicitly distances himself from the ontological question, and most importantly c) even the ‘realist’ elements of his account (the *arithmoi*) are themselves fundamentally derived from Euclid’s *constructed* definition of them as ‘finite pluralities’. Moreover, the primary contribution and merits of Mayberry’s system is syntactical in nature and concerns his reinterpretation of Cantor’s set theory along Finitist grounds.

Mayberry’s conception of Euclidean Finitism is peculiar among Finitist accounts in that it simultaneously rejects conceptions of the transfinite (conceptions which are most ubiquitously associated with the work of Cantor) while simultaneously offering an animated defense of the theory of sets which Cantor founded. While Mayberry comes to the defense of set theory, and even at times speaks from a Cantorian viewpoint, he argues against Cantor’s ‘non-Euclideanism’ and maintains that the central principles of set theory “are really *finiteness principles*” (Mayberry, xv, 2000). In this respect, Mayberry goes so far as to reinterpret Cantor and the axiom of infinity along finite grounds. Nonetheless, and as the title of the book suggests, while Mayberry sees set theory as playing a critical role in our mathematical foundations, he further contends that the notion of a set itself is founded within the classical Greek conception of the *arithmoi* as finite totalities (a notion which is in dramatic contrast with the prevalent tendency of founding set theory on axiomatic grounds). In summary, Mayberry's position offers Euclidean set theory as an alternative to Cantorian set theory – a position which is ultimately founded in the notion of the *arithmoi* as being the primary presupposition on which mathematics rests.

Mayberry on the Foundations of Mathematics:

The first chapter of *The Foundations of Mathematics in the Theory of Sets* concerns Mayberry's argument that all attempts to establish the foundations of mathematics must meet certain criteria. Moreover, Mayberry argues that the criteria from which the foundations of mathematics are built centre around a distinction between the nature of the finite and infinite. The most explicit articulation of Mayberry's position occurs in his identification of the 'two central tasks which must be fulfilled in the foundations of mathematics which he argues are:

1. To determine what it is to be *finite*, that is to say, to discover what basic principles apply to finite pluralities by *virtue of their being finite*.
2. To determine what logical principles should govern our reasoning about *infinite* and *indefinite* pluralities, pluralities that are *not* finite in size. (Mayberry, xix, 2000).

On this point, Mayberry makes the strong claim that "all disputes about the proper foundations for mathematics arise out of differing solutions to these two central problems" (Mayberry, xix, 2000).

While these central problems and criteria serve to frame Mayberry's discussion of the foundations of mathematics, it is also essential to answer the question: what would constitute as a foundation for mathematics by his account? A potential answer to this question may be found in the following suggestion of Mayberry, “If we were to carry out such a complete analysis on all mathematical proofs, the totality of ultimate presuppositions we should then arrive at would obviously constitute the foundations upon which mathematics rests” (Mayberry, 5, 2000). While Mayberry does not himself take up this task (and while it may be either *in practice* or *in principle* impossible to fully implement) its suggestion entails a certain philosophical orientation concerning the nature of mathematics’ foundations. Despite the fact that what would constitute as a ‘complete analysis’ is unclear, the statement displays a philosophical sentiment that at the bottom mathematics will be founded upon the ‘totality of its ultimate presuppositions’, or, in other words, mathematics ultimately rests upon the totality of its presuppositions. From this, the inquiry into the foundations of mathematics, as Mayberry conceives of them, will take the form of an inquiry into what things are presupposed by mathematics. Furthermore, while an explicit definition of what counts as an ultimate presupposition is not given by Mayberry, he does suggest that these mathematical presuppositions are ‘unproven assertions or else undefined concepts’ (Mayberry, 5, 2000). Much of Mayberry’s Finitist system can be understood through precisely this notion (that mathematics is founded upon certain presuppositions), insofar as his general project, as will be seen, is to articulate the concept of the *arithmoi* – for it is these *arithmoi* which he takes to be the central presupposition of mathematics.

In this respect Mayberry’s position has something in common with ‘orthodox’ set theorists, however, the two positions differ dramatically in what they take these ultimate presuppositions to be. The principal difference between the two is that where Mayberry ultimately lays the foundations of mathematics in the *arithmoi* (a concept to be further unpacked in the next subsection), the ‘orthodox’ set theorists rest the foundations of mathematics upon specified axioms. This difference runs deeper than a mere preference of some methods over others, as Mayberry further makes the strong claim that "*we cannot use the modern axiomatic method to establish the theory of sets*" (Mayberry, 7, 2000). Mayberry's argument centres around the notion that the current axiomatizations of set theory are circular insofar as they presuppose that set theory is already in place in order to establish their account[16]. This statement may be made more comprehensible in light of Mayberry's identification that the modern axiomatic set theory is ultimately 'a matter of logic' – taken in the sense of its consisting in a defined system of set principles (axioms). Mayberry here means to say that the axiomatizations themselves presuppose the theory of sets to first be in place[17] and, therefore, an axiomatic method cannot establish set theory. Why Mayberry takes the axiomatic theory to already presuppose a theory of sets can be made apparent by considering the question: what are the axioms of set theory taken to apply to and, if they were taken entirely on their own would they have a subject matter? The essential point in this case is that, taken on their own, the axioms of set theory merely lay out rules *for* what can be done by, or said about, sets but, critically, Mayberry is here pointing out that these axiomatic 'rules' for sets presuppose these sets in the first place. Consequently, in much the same way that the rules for how a chess piece may move *first* presupposes that there is a chess piece, Mayberry argues that the modern axiomatic method first requires sets (the theory of sets) to be in place.

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[16] Mayberry's phrasing of this point includes the statement that the axioms of set theory "are fundamental truths expressed in a language whose fundamental vocabulary must be understood *prior* to the laying down of the axioms" (pg, 8). In some sense then, this critique of circularity seems to entail the (seemingly Kantian) notion that a thing which presupposes something else cannot serve as the ultimate bedrock or foundation. Furthermore, it is interesting to consider this point apropos Wittgenstein's notion that we need already 'have in place' an understanding of a language game or concept before we 'make moves' in, or with, it; i.e., that we must already understand something before we can learn how the king chess piece moves, or what it is called.
[17] Here Mayberry's implicature is that a theoretical understanding (a 'theory' taken in a less formal sense) is presupposed as already being in place by the axioms of set theory.

Having seen what Mayberry does *not* take the foundations of mathematics to rest in (namely the modern axioms of set theory), we are now well situated to better understand what he *does* take them to consist in. In this regard, Mayberry states, “the foundations of mathematics comprise those ideas, principles, and techniques that make rigorous proof and rigorous definition possible” (Mayberry, 8, 2000). Moreover, Mayberry argues that a systematic foundation of mathematics must provide an account of the following three things: a) the elements of mathematics, b) the principles of mathematics, and c) the methods of mathematics (Mayberry, 8, 2000). He then proceeds to define these three things. Mayberry takes the ‘elements’ of mathematics to be “the fundamental *concepts* of mathematics, the *objects* that fall under those concepts, and the fundamental *relations* and *operations* that apply to them” (Mayberry, 8, 2000). Furthermore, Mayberry maintains that propositions about these objects have an objective truth value - that the truth or falsity of basic mathematical propositions will be a question of objective fact (Mayberry, 8, 2000). In contrast, Mayberry takes the ‘principles’ of mathematics to be fundamental propositions or ‘axioms’ which, while true, do not ‘require or admit of proof’ and instead act as the primary assumptions on which mathematics rests (Mayberry, 8, 2000). Finally, Mayberry states that the ‘methods’ of mathematics are “given by laying down the canons of definition and of argument that govern the introduction of new concepts and the construction of proofs” (Mayberry, 9, 2000). In other words, the ‘methods’ of mathematics can be understood like the ‘rules’ of the mathematical language game.

Now that we have detailed Mayberry’s positive conception of what the foundations of mathematics must include, it is essential to the project of this paper to also analyze Mayberry’s negative classification of what the foundations of mathematics need *not* include (for he devotes an entire subsection to this point). What Mayberry generally wants to exclude from the foundations of mathematics is philosophy. He states, "*we should make every effort to avoid incorporating purely speculative philosophical ideas into mathematical foundations, properly so called*" (Mayberry, 11, 2000). Nonetheless, it is here important not to misconstrue Mayberry's position as dogmatic, for he does recognize the inevitability of philosophical questions occurring; however, he favours a position of resting mathematics on what 'minimal philosophical presuppositions' are required (Mayberry, 11, 2000). While Mayberry's view does not appear as dogmatic, it is doubtlessly restrictive as he makes clear, "I take the view that the foundations of mathematics do not require, and therefore should not include, a general theory of the meaning of mathematical propositions, or a general theory of mathematical truth, or a general theory about how mathematical knowledge is acquired" (Mayberry, 11, 2000).

While there is some veracity in Mayberry's point here, insofar as a foundation of mathematics should not be expected to include a full-fledged account of the meaning, truth, or epistemology of mathematical propositions, there is good reason to suppose that a proper foundation of mathematics should nonetheless be able to provide a general framework and account of these things. Moreover, and with respect to the ontological question, a philosophically defensible general ontology should certainly be taken as a requirement for a proper foundation of mathematics – for it was just here, with its highly problematic metaphysics and epistemology, that the critical flaws of mathematical Platonism emerged. Although the failings of mathematical Platonism were far from unbeknownst to Mayberry, he maintains that his Euclidean Finitism is able to avoid the issues which arose within the Platonic account. He does so through attempting to 'dodge' the ontological question as to what the nature of mathematical 'objects' are by instead redefining these objects as 'structures' which he argues "do not give rise to the [same] ontological and epistemic difficulties" (Mayberry, 12, 2000).

Regardless of whether or not this semantic redefinition is satisfactory (and despite the fact that Mayberry himself seems to indicate in a footnote that he knows it is not), it should be recognized that avoiding the 'traditional' ontological and epistemic problems concerning mathematical objects would not render these mathematical 'structures' impervious to new, and perhaps even similar, philosophical problems. While the ontological and epistemic problems facing Mayberry's account will be discussed fully in a later subsection, it is sufficient to state in the conclusion of this one that an inquiry into what constitutes a "genuine" or "*real*" foundation for mathematics is as intrinsically bound to ontological considerations as a concern for "objectively determined truth values" is to epistemological considerations.

The Arithmoi:

With Mayberry's account of what would constitute as a proper foundation of mathematics now in place, let us examine where Mayberry lays the roots of mathematics – the concept of the *arithmos*. The *arithmoi* are utterly essential to Mayberry's position as he makes explicit in the statement, "the point of view embodied in this book [is that] all of mathematics is rooted in arithmetic, for the central concept in mathematics is the concept of a plurality limited, or bounded, or determinate, or definite—in short, finite—in size, the ancient concept of number (*arithmos*)" (Mayberry, xix, 2000). Mayberry's conception of the *arithmos* is based upon Klein's scholarship surrounding the Greek conceptions of *arithmoi* as finite pluralities and Euclid's seventh book in which it is asserted "A number (*arithmos*) is a multitude composed of units" (Mayberry, 18, 2000). He maintains that this Greek conception of number (*arithmos*) is what we today refer to as a set and that the 'units' composing an *arithmos* correspond to what we refer to as a set's 'members' or 'individuals'[18].

Despite Mayberry's extended analogy between the *arithmoi* and sets, he draws some key differences between the two – namely, that not everything which is a set is an *arithmos* (Mayberry, 70, 2000). The reason for this follows directly from the aforementioned conception of an *arithmos* as a 'finite plurality' since, according to this definition, neither the null set nor singleton sets will be *counted* among the *arithmoi*. Mayberry notes that this position is reflected in the fact that the Greeks did not view either one or zero as 'numbers' (Mayberry, 70, 2000). Furthermore, he argues that because the *arithmoi* do not include the empty set, they avoid certain ontological difficulties associated with empty or singular sets (pluralities). In a (perhaps) provocative passage, Mayberry charges modern set theory with building up the universe of sets from a kind of creation *ex nihilo* since it is 'balanced' upon the null set (Mayberry, 71, 2000). The basis of this charge follows from the fact that, notwithstanding infinite totalities, the empty set axiom is the only axiom within the finitary part of modern set theory that makes an unconditional existence claim (while all the others are conditional form: If X is a set, then f(X) is also a set). It is in this sense that modern set theory can be said to build its 'universe' upon a *creatio ex nihlo* since it follows from but one single existence claim which asserts the existence of the empty set/object that, definitionally, has nothing in it (no members). Consequently, Mayberry instead argues that we take the notion of the *arithmos* as fundamental and instead define sets in accordance with these *arithmoi*. Mayberry contends that once this is done, "the fundamental assumption upon which set theory, and with it all of mathematics, rests comes to

[18] This thesis is dispersed all throughout Mayberry's work but for one specific articulation of it one may see page 70 at the start of subsection 3.2.

this: *whatever objects there are, there are also finite pluralities composed of those objects, namely, arithmoi; moreover, these arithmoi are themselves objects in their turn and, as such, can serve as units in further arithmoi*" (Mayberry, 71-71, 2000).

The position expressed in this quote is analogous to set theory insofar as sets (here understood as *arithmoi*) may serve as members (or units) in other sets. Additionally, another more implicit consequence (but a consequence which is nonetheless as important as it is contentious) slips into Mayberry's position here, namely, that no 'infinite' sets are here permitted as legitimate *arithmoi*. The reason for this follows from the redefinition of sets into the concept of *arithmoi*, but critically, because an *arithmos* is understood to be a '*finite* plurality' infinite sets go the way of the null and singleton sets within this account. This is because Mayberry's redefinition takes on an even stronger sense of a *reidentification*[19] of what sets are with the *arithmoi* (which are understood to be inherently finite objects). Consequently, Mayberry's position is most clearly Finitist in nature, and more, it is best understood as a form of Strict Finitism insofar as it appears to strip all legitimacy from even the idea of an 'infinite set' (seeing as these are excluded from counting as *arithmoi* by their very definition).

In tandem with this reidentification of sets with the *arithmoi*, Mayberry also (rather contentiously) reinterprets Cantor's infinities through a finite lens – a sizable part of his project is to offer a Finitist interpretation of Cantor. He maintains that the critical aspect of Cantor's work was that he extended the domain of the finite in a way that includes totalities that were otherwise thought to be infinite as a part of it (Mayberry, 47, 2000). Mayberry calls these infinite totalities 'non-finite pluralities' (or infinite species) which he interprets in an idealized or else

[19] It should be noted that the very specific wording that Mayberry is *re*-defining and *re*-identifying sets and the theory of sets with the *arithmoi* highlights a critical aspect of his project – since he maintains that the *arithmoi* is both the conceptually presupposed and historically precedented 'original face' of what sets are.

potential sense (Mayberry, 265, 2000). Moreover, while Mayberry does not go so far as to reject Cantorian set theory, or the axiom of infinity in particular, he does suggest that there is a 'serpent in Cantor's paradise'. This serpent he takes to be the fact that infinite collections are barred from nature/reality as they are "phantasms corresponding to nothing in reality" (Mayberry, 269, 2000). Consequently, the view that emerges from Mayberry's account is that 'non-finite pluralities' (infinities), while being possible in a potential or ideal sense, are both: a) not accepted if they are taken in the sense of being actual infinities, and b) barred from being included among the *arithmoi*.

Mayberry's treatment of infinite totalities seems to, generally, amount to a dialogue between Mayberry's Euclidean Finitism and classical mathematics. However, Constructivists could push back on Mayberry with the notion that, even if we were to accept Mayberry's point that infinite collections do not qualify as legitimate *arithmoi*, this does not mean that they cannot be understood as collections in a different sense. Moreover, set theorists may argue that the notion that infinite collections do not meet the criteria of the *arithmoi* does not entail that they cannot be understood or described by the same logical principles and reasoning which apply to the *arithmoi*. In other words, even if we grant that Mayberry has, definitionally, barred infinite collections from set theory (insofar as sets have been redefined as 'finite pluralities'), this does not render infinite collections as conceptually illegitimate in themselves, but rather, this just signifies that they are something else other than *arithmoi*. Something of this idea was pre-emptively addressed by Mayberry's acceptance of non-finite pluralities as potential phantasms, however, insofar as our logical reasonings and principles which describe the *arithmoi* are taken to be *general* in nature (in the sense of being applicable across any and all domains as was discussed in the previous chapter), then any legitimate reasonings about them should be *in* *principle* equally applicable to infinite collections, regardless of whether or not those infinite collections qualify as *arithmoi* by Mayberry's account. Consequently, while Mayberry's treatment of non-finite collections may have something to say against classical mathematicians, it does not necessarily speak against Constructivists who were not concerned with asserting the existence of infinite collections, beyond their potentially being mind-dependent constructions, in the first place.

Moreover, insofar as one wants to object to the conceptual legitimacy of 'completed infinities', it is useful to contrast the way that Mayberry's system handles infinities to Constructivists like Brouwer. Where Mayberry's definitional barring of infinite collections from the *arithmoi* does not directly concern the formulation of a completed infinite collection, Brouwer's repudiation of completed infinities, as we have seen, instead directly attacks the very possibility of legitimately forming a completed infinite collection. This discrepancy should be borne in mind as it suggests that, if one takes issue with the notion of completed infinite collections, a better means of rejecting them will likely come from taking issue with the very legitimacy of forming them in the first place. On the other hand, because it has been seen that Mayberry's account allows for the same principles of reasoning and classical logic to be applied to 'phantasms' that are applied to the *arithmoi*, it may be seen as a virtue of Mayberry's account that it is also consistent with the practices of most mathematicians (relatively few of whom are Constructivists).

The Ontology of the Arithmoi:

We may further unpack the nature of the *arithmoi*, as Mayberry conceives of them, through considering the following question: notwithstanding the formal definition of the *arithmoi*, what does Mayberry take them to 'be'? Within his book, Mayberry provides an extensive answer to this question through his ontological treatment of how the *arithmoi* are said to exist. The beginning of Mayberry's answer to this question takes the form of an extended analogy concerning a herd of twenty-five horses from which we can pick out a 'number' of triples (2300) from this herd[20]. He gives an example of one such triple composed of the particular horses 'Trigger, Champion, and Red Rum' and argues that by the mere fact of their being the particular horses that they are they compose the number of horses that they are (Mayberry, 22, 2000). Within this example, Mayberry makes the point that the horses, inasmuch as the various triples from within the herd, already are there or 'composed' before anyone conceives of them. This leads Mayberry to the assertion that "A number of horses is no more a creature of the mind than are the individual horses that compose it. Since we can count such numbers, it is natural that we "count" them as things (Mayberry, 22, 2000). However, critically, Mayberry likewise maintains the same position towards the *arithmos* of the 2300 horse triples which are also contained in the herd (Mayberry, 22, 2000). In this essay's later considerations, it will be seen that this position is ontologically fraught.

As a consequence of Mayberry's conception of the *arithmoi*, he argues that we should understand 'number words' not as naming abstract 'objects', but rather, as standing for what he calls 'species' of number (Mayberry, 24, 2000). In this respect, Mayberry takes number words not as referring to any *specific* abstract objects so much as he takes them as referring, in the more general sense, to all sets/*arithmoi* of their given numbers. According to this, and following Frege, Mayberry argues that by the 'original' conception of number we should express number statements in the form "There is a five of horses in the field" or, "The number of horses in the field is a five" (Mayberry, 24, 2000). By this conception, the 'number words' which enter into

[20] This example begins on page 21 of Mayberry's *The Foundations of Mathematics in the Theory of Sets*.

our propositions refer to numerical species – and it is this procedure that contributes to the referential semantics of Mayberry's account. However, concerning the nature of these 'numerical species' which are picked out by the ascriptions of our number words, Mayberry explicitly leaves the door open to Platonism and attempts to avoid the ontological question when he states: "you may still be tempted to take each such species itself to be an abstract object, just as you may be tempted to take species words in the category of substance (e.g. "horse" or "man") as standing for particular abstract objects – "universals", or "Platonic Ideas" … But such ontological extravagance (if it be extravagance) is not forced upon you" (Mayberry, 29, 2000).

Part of Mayberry's impetus for wanting to avoid ontological questions regarding the *arithmoi* is illuminated in light of his later identification of the following two problems. The first of these problems concerns the *semantics* of mathematical discourse, and Mayberry frames it in the form of a question: "what conditions must we place upon the things referred to in mathematical discourse, what features must we suppose them to possess, in order for mathematical definition and proofs to work; and how can we formulate those conditions in a mathematically usable way?" (Mayberry, 68, 2000). This semantic problem of mathematics, therefore, is seen to concern the question of what mathematical discourse is about and, in particular, it concerns the question of what conditions are required of the things referred to in mathematical discourse for them to work and function within proofs. For Mayberry's part, he takes definiteness and identity to be the minimum necessary conditions required by the things referred to in mathematical discourse (Mayberry, 68, 2000). In addition to this semantic problem, Mayberry identifies a second *ontological* problem which he also frames in the form of a question: "what kinds of things, if any, satisfy those [the semantic] conditions?" (Mayberry, 68, 2000). Whereas the former semantic question concerned the formal and operational functionality of the things referred to in mathematics, this second ontological question concerns whether or not any 'real' things can be said to satisfy the question. This signifies a shift in Mayberry's work towards realism. For the Constructivist, their positive inquiry stops at the semantic question, however, much of Mayberry's account (notwithstanding his declared desire to avoid the ontological question) is interwoven with an ontological thesis through his prioritization of the *arithmoi*. Consequently, while Mayberry states that it is 'vital' to keep these problems separate for the reason that, while the semantic problem can be solved, the ontological problem remains 'open-ended', he, unlike the Constructivist, does entertain a positive answer to the ontological question (Mayberry, 69, 2000). Concerning Mayberry's positive account, he takes the *arithmoi* as the kind of things that satisfy those earlier semantic conditions and makes the further assertion that the *arithmoi* exist in even a mind-independent sense[21].

In light of these considerations, and insofar as he offers a direct treatment of the ontological question, it is important to evaluate the ontology contained within Mayberry's account. Much of Mayberry's ontology ultimately rests upon Aristotle's statement that "Things are said to be in many ways" (Aristotle, Metaphysics, Z) – a notion which appears, at times, as a kind of Catechism or assumed dogma within Mayberry's ontology. Notwithstanding this, Mayberry does not take his position to be an absolute answer, but rather, he asserts that he is personally contented with a 'relative' answer to the ontological question (Mayberry, 31, 2000). His positive conception offers a distinction between the two 'essences' of a number where the first is its 'material aspect' and the second is its 'formal' or 'arithmetical' aspect (Mayberry, 32, 2000). The material aspect of a number he relates to the 'units' which compose the number, and

[21] Some central places where these ideas occur in Mayberry are: 2.4 "Number words and ascriptions of number"; 2.5 "The existence of numbers"; as well as parts 3.1; 3.2; 3.3; 3.4; 3.5; 3.6; and 3.7 composing the section "Semantics, Ontology, and Logic" within Mayberry's *The Foundations of Mathematics in the Theory of Sets*.

the formal or arithmetical aspect he relates to ‘how many’ of said units there are (Mayberry, 32, 2000). Mayberry offers a summary of his ontological position in the claim that “The view that emerges from these considerations is that a number composed of things of a certain kind has the same kind of claim to existence as have individual things of that kind – its units, for example” (Mayberry, 34, 2000). Moreover, a further ontological commitment of Mayberry’s account is here made visible, that being that numbers are said to possess the *same kind* of existence as the individual units that compose them do. That this commitment may be fairly charged to Mayberry’s account is demonstrated through his subsequent statement that “Whatever kinds of things there are, and in whatever sense things of those kinds are said to exist, there are numbers whose units are things of those kinds, and those numbers may be said to exist in a way analogous to the way in which the things that are their units exist” (Mayberry, 35, 2000). There is a nuance here within Mayberry’s account which should be noted, namely, that the numbers exist in the same kind of way to their units. Consequently, if the units are themselves abstract – or if the units are said to possess a weak or contentious claim to existence – then their ‘number’ exists in a similarly abstract way. It is this nuance that Mayberry has in mind when he recognizes that a number of colors is abstract in a way that a herd of horses is not (Mayberry, 35, 2000). However, if we return to the aforementioned example of a herd of horses, Mayberry is committed to maintaining that the number species ‘25’ is existent in the *same kind* of way as the individual twenty-five horses that compose it are. It is this commitment that I will later argue is highly problematic.

A final aspect of Mayberry’s ontology should be considered before moving on, namely, what does Mayberry take to be an ‘object’? Here he adopts Frege’s account. He states “What I am proposing to call “objects” is that *propositions asserting the identity of objects must always* *have deterministic truth values*" (Mayberry, 69, 2000). According to this Fregean sense of objects, *x* counts as an object if and only if for any *y*, *x*=*y* is determinately true or false. This conception of what constitutes an object, while stated in formal terms, has an intuitive basis, as it amounts to the insistence that definite things/objects are objectively distinct from all other things/objects. Consequently, Mayberry understands objects, in the Fregean sense, to be definite things whose identity may be objectively and *uniquely* determined in the sense that they are distinguished from other objects. For Mayberry, this treatment satisfies the problem of articulating a clear referential semantics of objects, however, we have seen that Mayberry understands there to still be the further ontological problem of whether or not any things can be said to fulfill the semantic criteria.

Notwithstanding Mayberry's bifurcation between the semantic and ontological problems and subsequent suggestion that the ontological problem may be insoluble, he does offer a positive ontological account of the *arithmoi* in which he explicitly asserts that they exist (Mayberry, 72, 2000). Moreover, he makes the further claim that "they [the *arithmoi*] have the same kind of claim to real existence as definite, independent entities as have the objects that make them up. But this then becomes the sum total of the mathematician's "ontological commitment". The only "mathematical objects" he need acknowledge are *arithmoi*" (Mayberry, 72, 2000)[22]. It should here be recognized that Mayberry's account entails a commitment to the idea that the *arithmoi* have a *real* existence as objects - critically, they are accounted as

[22] The reader should recall that Mayberry is consistent on this point, as this claim is identical to that he makes concerning the 'species of numbers' in section 2.5 on "The existence of numbers". However, and as will be taken up in later sections of this paper, Mayberry is incorrect to state that the only ontological commitment of the mathematician is to the existence of the *arithmoi*. Another such commitment has already been pointed out, which is, that Mayberry is further committed to the *arithmoi* possessing "the same kind" of existence as their units. Moreover, in the same way that being committed to the existence of Plato's forms entails a great number of other ontological commitments, so too does Mayberry's commitment to the *arithmoi* entail further ontological consequences.

possessing a *bona fide* ontological status within Mayberry's account. This quote reveals a stark difference between Mayberry and Constructivists, since the former makes the further assertion that the *arithmoi* exist (so long as their members exist) independent to our consideration/construction of them. Consequently, Mayberry's essential position is not captured by the Constructivist statement 'we can construct some *arithmoi*', as rather he contends that 'there independently exists some *arithmoi*' and, in this way, Mayberry's Finitism differs dramatically from Constructivism. And although it is true that the only ontological commitment in Mayberry is to the *arithmoi*, any ontological commitment whatsoever entails a host of ontological consequences which Mayberry then is similarly committed to addressing.

Before detailing one of these consequences, one final quality of the *arithmoi* within Mayberry's account should be clarified. Mayberry maintains that the 'principle of unity' for the *arithmoi* is found in their 'finitude' (Mayberry, 73, 2000). With this point, Mayberry pre-emptively addresses the question of how an *arithmos* can be said to be an 'object' in the singular sense when it is by its very definition a plurality. The answer he gives is that following from the principle of unity in finitude of an *arithmos*, "simply by being objects, and therefore, by hypothesis, having, severally, claim to real, independent existence, *and* by being, conjointly, finite in multitude, the units of an *arithmos* together and collectively constitute a single well-defined thing, viz. an object" (Mayberry, 73, 2000). This notion relates to Mayberry's implementation of the Fregean sense of object as he is here concerned with defending the legitimacy of treating an *arithmos* as a singular and well-defined thing despite the fact that it is, definitionally, a manifold. Given his commitment to the Fregean account of what constitutes an object, we can see why: the arithmos X, including Trigger, Champion and Red Rum, will have determinate answers for any c, whether c = X (assuming that Trigger, Champion, and Red Rum are objects). If c is not an arithmoi, the answer is no. If c is an arithmoi, then the answer is yes exactly when c has three members and they are the three horses in question, and the question “are they the same three horses?” will have a definite answer. For example, one might ask the question: ‘in what sense can the number/*arithmoi* of Mayberry’s herd of 25 horses be held to be a singular and existent ‘object’ when it is essentially a plurality/multitude’? According to Mayberry’s conception, it is perfectly legitimate to take the units (horses) of the herd together in a singular sense such that they constitute a single thing (in this case a number of horses). It would be uncharitable not to grant Mayberry this point, as he is not maintaining any more contentious a claim than we do when we commonly view composite things as well-defined unities – such as how we view treating a horse’s tail as singular and perfectly legitimate despite its being composed of a multitude of hairs.

There are two central problems that emerge from Mayberry’s ontology – the first of which relates to how it is he extends claims of kinds of existence to the *arithmoi*. It is an explicit aspect of Mayberry's account that the *arithmoi* have a real claim to existence for he states that “we must accept that a set [*arithmos*] has just as legitimate a claim to existence as have the objects that compose it” (Mayberry, 70, 2000). However, the way that Mayberry goes about extending an independent existence to the *arithmoi*, at times, sounds dubiously similar to a fallacy of composition which infers that something (in this case the kind of existence) is true of the whole (*arithmos*/set) on the basis that it is true of its parts (units/members). This problem becomes apparent in light of Mayberry’s further statement that: “Whatever kinds of things there are, and in whatever sense things of those kinds are said to exist, there are numbers whose units are things of those kinds, and those numbers may be said to exist in a way analogous to the way in which the things that are their units exist” (Mayberry, 35, 2000). Mayberry infers what *kind* of existence a given *arithmoi* has from the *kind* of existence that its members have. If we recall his earlier treatment of abstract colours, he only qualifies 'numbers/*arithmoi*' of colours as existing in an abstract way because the units composing them only exist in an abstract way, whereas, in the case of horses, he extends a stronger claim to existence. But here Mayberry appears to be inferring a property, in this case a kind of existence, is had by the whole (for a number/*arithmoi*) because it is had by its parts (members). In other words, Mayberry infers the kind of existence that an *arithmoi* from the kind of existence that its members have – but this is just a fallacy of composition. The problem here results from the fact that, even within Mayberry's account, a set/*arithmos* only has a claim to existence *through* its units/members (that is unless one returns to Platonically asserting it exists as a form or thing in itself).

Mayberry could here offer the counter objection, which admittedly is somewhat intuitively appealing, that the property of existence/being is unique in that the kind of existence had by all of the members/parts of a whole does imply the same kind of existence is had by the whole. However, we can imagine a whole (collection) like 'all of the kings of France who were at one time the present king of France' and, despite the fact that all of the members/units of this collection have existed in a physical kind of way, this does not entail that this set/*arithmoi* itself physically exists. Importantly, the issue of this present case can not be alleviated in the same way that Mayberry handled the abstract existence of numbers of colours. This is because, unlike with the case of colours, the members of the collection 'all of the kings of France who were at one time the present king of France' are physical, as opposed to abstract, members.

The second central problem of Mayberry's ontology also relates to his statement that "The view that emerges from these considerations is that a number composed of things of a certain kind has the same kind of claim to existence as have individual things of that kind – its units, for example" (Mayberry, 34, 2000), The primary reason why this claim is problematic is that, in many cases, such as is apparent even within Mayberry's own example of a herd of horses, the units/members exist in an actual and physical way – their claim to existence is a physical one. A problem therefore results from the fact that nothing actual/physical is either lost or gained by the horses 'togetherness' when in a herd as opposed to their being dispersed in the way that Mayberry describes (Mayberry, 35, 2000). Moreover, the idea that something actual/physical is lost or gained by the horses 'togetherness' appears *prima facie* to violate the law of the conservation of matter – for it would be an odd universe where something actual/physical came into existence when the horses got together and that this something was lost when they dispersed. Although these considerations do not cause an issue for the formal definition of objects that Mayberry borrows from Frege, for these may be constructed in an anti-realist sense, Mayberry's ontology is itself highly problematic because he makes the further claim that the number/*arithmos* exists in the same way as the individual members which compose it.

Mayberry's Anti-Operationalism:

In addition to Mayberry's positive account, centred around the notion of the *arithmos*, he also offers a negative critique of operationalism in mathematics. Mayberry's critique of operationalism may be characterized by his rejection of the idea that "the foundations of mathematics are to be discovered in the activities (actual or idealized) of mathematicians when they count, calculate, write down proofs, invent symbols, draw diagrams and so on" (Mayberry, 15, 2000). The central notion of Mayberry's anti-operationalism is the idea that, as opposed to rooting the foundations of mathematics in the operations or activities associated with mathematics, we should seek to root mathematics on the fundamental presuppositions which are themselves presupposed by mathematical operations. This notion relates to Mayberry's criticism of axiomatic set theory which, as was seen earlier, was based on the fact that the axioms of set theory presuppose that there are sets in the first place. Similarly, Mayberry does not see the mathematical operations as primary for the same reason – as these too can be said to presuppose the 'things' (the *arithmoi* as Mayberry maintains) which are operated upon.

Basser has succinctly categorized Mayberry's anti-operationalism into the following three theses:

Thesis 1 "If . . . we see the notion of natural number as a secondary growth on the more fundamental notion of *arithmos* . . . then the principles of proof by induction and definition by recursion are no longer just 'given' as part of the raw data, so to speak, but must be established from more fundamental, set-theoretical principles." (pp. xvi–xvii)

Thesis 2 "Nor are the operations of counting out or calculating to be taken as primary data: they too must be analysed in terms of more fundamental notions." (p. xvii)

Thesis 3 "[The] operationalist conception of natural number is the central fallacy that underlies all our thinking in the foundations of mathematics." (p. xvii). (Basser, 10, 2005).

Taken together, these three theses help to contextualize why Mayberry takes the *arithmos* to be so central and fundamental to mathematics. The reason for this, as Mayberry maintains, is that the operations of mathematics, and the natural number sequence as a consequence, only emerge after the *arithmoi* are already in place. Consequently, it is a fallacy by Mayberry's account to lay the foundation of mathematics on operations that are not themselves fundamental, but which instead are founded upon the more fundamental notions of the *arithmoi* which are presupposed by the operations.

Notwithstanding Mayberry's thoroughgoing critique of the operationalist fallacy with respect to the foundations of mathematics, it is important to recognize that Mayberry is (obviously) not opposed to the use of operators in mathematical practice. While operations clearly play a central role when it comes to our doing mathematics, Mayberry's anti-operationalist point is just that these operations are not the primary data of mathematics – they are not the ultimate presuppositions that constitute the foundation on which mathematics is built. The 'operationalist fallacy' which is so opposed by Mayberry consists in the tendency to view the mathematical operators as the foundation on which mathematics is built. This point may be better drawn out through considering the specific example of 'counting'. Even before the possibility of counting is made available, it is necessary to first have 'objects' that one could count, since counting presupposes things to be counted. These 'objects', as we have seen, Mayberry contends are the *arithmoi*.

Having now provided a description of mathematical Finitism through the work of Dantzig and Mayberry, a clear narrative will be seen to emerge in the progression from mathematical Platonism to Constructivism to Finitism. Whereas standard Constructivist accounts, such as those of Brouwer and Dummett, are thoroughgoingly anti-realist, it has been seen that certain realist tendencies begin to be reincorporated at the fringes of Finitism such as occurs in Mayberry's Euclidean Finitism. Notwithstanding the fact that the central issues within Mayberry's account came precisely from his pseudo-realist ontology, this pendulum swing back towards a pseudo-realist thesis can be seen as an important step in recapturing the *desiderata* which mathematical Platonism captured (albeit in a problematic way).

CHAPTER THREE: ACTUALISM

An Introduction to Actualism:

In the previous chapters, through an examination of several key authors, I have broadly detailed the Constructivist and Finitist alternatives to mathematical Platonism. The aim of this final chapter will be to provide an alternative philosophy of mathematics (Actualist Mathematics) to all of these. Where both Constructivism and Finitism (generally) offer anti-realist alternatives to Platonism, Actualist Mathematics differs in that it explicitly endorses a realist alternative to mathematical Platonism which is also founded upon a referential semantics. In this respect, the greatest benefit of the present account is that Actualism is able to conjoin the important and desirable elements from both the realist and anti-realist camps – that it is able to synthesize the two and meet Benacerraf's challenge of simultaneously offering a referential semantics which parallels the rest of language while allowing for a satisfactory epistemology. Actualism does this by avoiding Platonism's metaphysical and epistemological issues while simultaneously avoiding Constructivism's need for an alternative non-referential semantic theory.

Narratively speaking, Finitism – by moving away from the unfettered idealistic constructions of Brouwer's creating subject – appears already as a step towards a synthesis between the Realist and Constructivist theses. The foremost indication of this was seen to be the pseudo-realism of Mayberry's Euclidean Finitism, wherein the realism occurs through Mayberry's implicitly if not explicitly endowing the *arithmoi* with a mind-independent ontological status. Notwithstanding this, Finitism is generally still understood as being as strictly an anti-realist philosophy of mathematics as Constructivism is although, as we have seen, some Finitists like Mayberry wax more towards realism than others and begin to reincorporate some realist elements (Mawby, 2005). Still, even in the case of Mayberry, his Euclidean Finitism can be seen to carry the same anti-realist tendencies of Constructivism insofar as the central notion within his account – the *arithmoi* – is based upon Euclid's constructed definition of them as 'finite pluralities'. Consequently, a lacuna remains in the shape of a missing realist alternative to mathematical Platonism.

The impetus for providing such an alternative arises from the joint recognition that Platonism is an ontologically untenable position, which nonetheless fulfills a number of desiderata to mathematics, and that the anti-realist alternatives are themselves fraught. In other words, mathematical Platonism offers a desirable semantics, which promises a reality behind the reference, but whose reference is founded upon Plato's ontologically untenable idealization of the forms. In contrast, Constructivism and Finitism alike instead ignore the issue of providing a semantics that attaches well to a realist ontology all together and, at the cost of offering a rather ontologically bankrupt account, avoid the ontological errors of mathematical Platonism by adopting an anti-realist thesis. While a realist ontology is not a necessary requirement for mathematics (seen singularly from the fact that Constructivism does not cease to be mathematics without it), having a semantics which is attached, or which can attach, to a *sensible* realist ontology has the benefit of fitting well within the semantics of our empirical scientific discourse. This reflects Benacerraf's recognition that realist theses do offer something important – even if those which are based in Platonism are clearly flawed. The motivation to come to an alternative semantics, one which is amenable to scientific discourse, is itself an attempt to meet the following imperative: let us not throw the material out with the aether, the realist out with the Platonic.

The central argument of the present chapter is that we do not need to abandon the mathematical realist thesis outright because of the fraught Platonic account – and that neither should we grant Platonism the monopoly on realist mathematical theses; instead, we can restore a mathematical realism, restore a reality behind the reference, by rooting our mathematical reference in the actual physical/material universe as is treated by science. By way of arguing for this alternative, I will begin this chapter by detailing the beginning foundations and primary principles of Actualist Mathematics. I will then address several initial challenges which come to the position and, thereafter, I will argue for the viability and benefits of Actualist Mathematics (and recognize some limitations on the claims for its advantages with respect to Benacerraf's challenge) as an alternative realist thesis to mathematical Platonism. By the end of the chapter, it will be seen that the central aspect of Actualism's account is its realist referential semantics; however, it will also be seen that – most unlike mathematical Platonism – Actualism's realist semantics do not extend to the entirety of mathematics. Instead of this, Actualism maintains that only the basic core aspects of mathematics are accounted for in realist terms, whereas it views the more complex aspects of mathematics as anti-real constructions which are built upon the more basic realist foundations.

Towards an Actualist Mathematics:

Actualism and Alternative Foundations of Mathematics:

The defining characteristic of Actualist Mathematics is the prioritization of the actual (here understood as the physical) and the subsequent notion that claims of being real or existent (a positive ontological status) can therefore only be extended to mathematical referents which are actual/physical. In this way, Actualism as a philosophy of mathematics, unlike Constructivism, is far more concerned with *actual* ontology insofar as it aims to provide both a realist referential semantics as well as a positive answer to the ontological question. On the other hand, like Constructivism, the Actualistic thesis is also entirely opposed to Platonism's posit of immaterially existent and mind-independent forms; though it rejects them without having to hold that mathematical objects exist only in the attenuated sense appropriate to mentally constructed 'objects'. Contrary to these, an Actualistic philosophy of mathematics insists on a stricter sense of 'existence' and only extends the positive ontological status of 'existing' to that which is actual/physical. Notwithstanding the ontological question, the Actualistic thesis is sympathetic to Constructivism regarding the more complex aspects of mathematics, such as set theory, which are seen to be constructed extensions of that which is physically/actually given. In other words, Actualism as an ontological thesis does not take issue with the syntax, methods, or proofs of either set theory or Constructivism (*it does not attack them from within*), but rather, it is ancillary to these and instead aims to offer an ontologically realist foundation of mathematics which is based upon the actual/physical world, which the more complex mathematics are built upon. We will return to the question of what to say about the ontological status of apparently referring terms in more complex mathematical discourse below, when a bit of development of the Actualist account will provide better tools to make the answer clear.

One immediate question which confronts such an aim is: What is meant here by the 'foundation of mathematics' and what does Actualism take this foundation to be? Corollary to Actualism's general emphasis on ontology, its treatment of the foundations of mathematics takes on a similarly ontological aspect. I admit to using this terminology in a non-standard sense. Generally, "mathematical foundations" are taken to refer to the logical or philosophical basis that underlies mathematics, and many courses in "foundations of mathematics" quickly devolve into courses in set theory demonstrating the extent to which other branches of mathematics can be reconstructed as part of set theory — arguing, in effect, that the logical and philosophical basis of all practical mathematics is set theory, i.e., its axiomatic principles and, though this second part usually receives less attention, whatever ontology is presumed by set theory. I am therefore focusing on one part of the more standard version of the question 'what are the foundations of mathematics' by focusing on the question 'upon what (things) is mathematics founded/based'; it is part of the Actualist answer to reformulate this question as 'what physical/actual things give rise to mathematics'? As noted, there is a sense in which the Actualist answer applies directly to the basic areas of mathematics. But, as an answer about the foundations of mathematics in the sense just described, Actualism serves as the general *foundation* of mathematics — where the more complex areas of mathematics are themselves seen to be constructed upon these more basic areas, and so share the same foundations. Aphoristically, the Actualistic thesis is contained in the statement that the ultimate foundation of mathematics – inasmuch as anything else – needs to be laid upon something which can itself *bear weight*, which is to say, upon something actual/physical.

Working within this conception of the foundation of mathematics, a number of different 'foundations of mathematics' appear depending upon which conceptual system one is considering. For Constructivists and Finitists, the foundations of mathematics are generally taken to be mind-dependent; consequently, anti-realists about mathematics lay the foundations of mathematics upon that which we have ourselves constructed. However, Actualism as a philosophy of mathematics operates with the prior ontological concern of asking the question: upon what, if anything, are these anti-real foundations *themselves* founded[23]? Unless one here

[23] Here I do not mean to dismiss the Constructivist project/Syntax, rather, my intention is more essentially to provide the ontology/semantics that it is lacking.

resorts to a Berkeleyan idealism, two distinct answers present themselves to this question. In the anti-realist sense (which is to say the mind-dependent sense), the foundations of mathematics, insofar as we have constructed/defined them (as is the case in set theory for example), are themselves founded upon the evolved cognitive capacities of our organism/brain. Furthermore, in the realist sense, a mind-independent foundation of mathematics may be *derived/substantiated* insofar as we are able to relate the content of our mathematical propositions to the actual/physical world. In other words, we may endow some of mathematics with the unique claim of being about mind-independent objects (of possessing a reality behind their reference), insofar as they can be externally substantiated as describing the actual/physical world. It will be the aim of that which proceeds in this chapter to outline and defend some principles for substantiating the claim that at least a certain subset of mathematics rests upon mind-independent and realist foundations.

Intellectual Instincts as Inherited a Priori Judgements:

There is one common notion that unifies the realist sense behind the idea that both the mind-dependent and the mind-independent aspects of mathematics can be founded upon the actual/physical. This common notion is found in rejecting the 'primary dogma of rationalism' - that being the distinction between *a priori* and *a posteriori* judgement[24]. By which I mean to say that the distinction between *a priori* and *a posteriori* judgement is an illegitimate one, seeing as all *a priori* judgements are at the bottom ultimately just *a posteriori* judgements. In other words, ontologically speaking there is no such 'thing' as a judgement that is made/known prior to experience. This is because, although certain judgements may appear as *a priori* to us, unless one

[24] It is imperative to recognize that the distinction between *a priori* and *a posteriori* judgements which is discussed in the present thesis is limited to those stringent characterizations which maintain a very strong sense in which *a priori* judgements must be independent of any and all experience.

is to ascribe to the scientifically disparaged notion of Wallace that evolution ‘ends at the head’, what are taken to be *a priori* judgements (insofar as one wants to entertain of such things) can instead be seen as the culmination of thousands of years of inherited evolutionary experience. Consequently, it is empirically false to say that any judgment is made *a priori* to all experience, as rather, *a priori* claims are made at best *a priori* to one's own experience. In other words, *a priori* judgements (insofar as we posit them at all) are really just *a posteriori* judgements inherited from our evolutionary ancestors.

When *a priori* judgements are conceived of in this way they appear in the form of what we should instead call ‘intellectual instincts’ - a phrasing which itself relates to the Intuitionism of Brouwer and the Pure Intuitions of Kant. However, there is a great difference between the Actualistic posit of inherited ‘intellectual instincts’ and Kant’s Pure Intuitions; this being that the according to Actualism these judgements are based upon the empirical world (upon the actual/physical) and not upon idealistic abstractions. From this perspective, there is a certain truth within Kant’s conception of Pure Intuitions and the notion that certain ‘intuitions’ must first be in place before we can either have sensations or form the judgments of reason. However, the Actualistic thesis maintains that Kant was wrong to posit this *de abstracta* when they are (at least today) able to be seen as the result of an inherited evolutionary history and are thus better understood as biologically inherited intellectual instincts. Contrary to Kant’s conception, Actualism instead views these intellectual instincts as physical parts of our cognitive structure which, owing to the selective pressures of evolution, over time have ensured that our intellectual instincts offer a (generally) veridical representation of the world in which we live.

This conception is also different from a related view of the Logical Positivists who maintained, in a more strictly pluralistic sense, that there are many possible conceptual frameworks within which we might do science and that we adopt one as a matter of convention. Actualism varies from this in that it instead maintains that we possess one physical conceptual framework, our cognitive structure, which is not conventional but evolutionarily endowed as a part of our physical organism (in the form of what have here been called intellectual instincts). However, Actualism shares the sentiment of the Logical Positivists that there are conceptual features to the conceptual frameworks within which we do science and so remains pluralistic, albeit in a softer sense — it holds that the conventional portion of our conceptual frameworks are additional formal constructions that are built on top of our basic, physically endowed conceptual framework. That is, Actualism is pluralistic regarding our conceptual frameworks but features a bifurcation between what may be called our 'first-order' conceptual framework, which is the innate physical endowment of our conceptual structure, and what may be called 'second-order' conceptual frameworks which are constructed upon it

When the present reconceptualization of Kant's Pure Intuitions in the form of inherited intellectual instincts is applied to the nature of our mathematical judgments, it has the consequence of entailing that the ultimate foundation of mathematics lies in both our inherited evolutionary experience and in the external world which shaped/selected them (through the application of adaptive pressures). In a heuristic sense, this Actualistic thesis has a conceptual similarity to Chomsky's notion of a universal grammar; however, as opposed to positing an inborn syntax/grammar, it instead posits an inborn/inherited mathematical/logical ability in the form of what has here been called the 'intellectual instincts'. Although the idea that mathematical/logical judgments such as that 2 + 2 = 4 or that geometric judgements, such as Kant treats of in relation to the Pure Intuition of Space, are biologically heritable judgements might *prima facie* appear fantastical, not only does the evolutionary literature provide a multitude of examples of inherited instinctual behaviors or ‘judgements’ which appear equally if not even more fantastical than these, but more, it even presents us with clear empirical indications that the rudiments of mathematical judgements, such as numerosity and even a basic geometric understanding of shapes, are found in a surprising number of other species both within and beyond the great apes[25]. Consequently, not only is the assertion that the foundations of our mathematical/logical *a priori* judgments arise from the inheritance of ‘intellectual instincts’ theoretically plausible, it is already an empirically supported/substantiated hypothesis within the psychological/ecological literature.

Notwithstanding the scientifically secure claim that the ultimate foundations of our intellectual capacities are biologically inherited[26], this claim does not include the fallacious overstatement that all of our intellectual judgements are inherited. In other words, one can recognize that we are endowed with certain *a priori* judgements[27], in the form of intellectual instincts, without taking this idea to the dogmatic point of asserting that all of our judgements are inherited. On the contrary, and with respect to mathematics, it would be speculative if not absurd to state that all mathematical judgements, especially those of a more complex nature, are evolutionarily endowed. Consequently, it is neither the intention nor the conclusion of the present thesis to assert that *all* of mathematics reduces to the actual/physical, but rather, it is the aim of the present thesis to provide an ontologically realist foundation from which the more

[25] What exactly these judgements look like, as well as the empirical support for hypothesizing them, will be discussed in greater detail later in this subsection.

[26] In addition to the empirical support as to why we should suppose that this, a more directly philosophical case comes from considering what it would mean to maintain otherwise. If we did not maintain that the ultimate foundations of our intellectual capacities were based in our physically inherited cognitive structure, where else would we imagine them to come from? Here it appears that someone would need to recourse to some Platonic idea about a non-physical intellect or *nous*.

[27] The only sense in which these judgements are here said to be *a priori* is that they occur prior to our own individual experience but, critically, they are still acquired from our inherited evolutionary experience.

complicated mathematical constructions arise. It is in this sense that the Actualistic thesis is sympathetic to Constructivism (and is even maintainable alongside it), insofar as it offers an ontological account which the latter is lacking. And although the Constructivist can define 'objects' and consistently manipulate symbols within an axiomatized syntax in an a-ontological language game (*for one may surely speak in this way if they wish*), it is explanatorily beneficial to found this practice within a consistent ontology that itself connects to the actual/physical world. The perspective that emerges from these considerations is that the more complex aspects of mathematics (things like the Axiom of infinity, Powersets, and other Pure Mathematical constructions like Lie group E8) have their foundations in the more basic and immediate judgements which are a part of our cognitive structure. These more complex aspects of mathematics, according to Actualism, are seen to occur within 'second-order' formal conceptual frameworks which are themselves grounded and built upon the more basic physical conceptual framework which was evolutionarily inherited. To take an example, Actualism maintains that our judgements about the more complex algebraic constructions studied in group theory, like Lie group E8, that deal with abstract dimensionalities are built off of our more basic and immediate judgements about simple physical spaces. Moreover, because we know that these more basic and immediate judgements are the result of our evolutionarily inherited cognitive structure, it is in this sense that Actualism includes an account of the foundation of our more complex 'second-order' conceptual frameworks.

One very important question here arises in the form: what does the present account take these more basic evolutionarily endowed '*a priori*' judgements (intellectual instincts) to be and what might one look like? These intellectual instincts are here taken in the sense of being basic judgements that are inborn parts of our core cognitive system which have been acquired through evolution. One such basic system (which under the present system is to be classified as a 'first-order' or physical conceptual framework), is numerosity, which concerns judgements about the size of groups[28] of objects. The judgements which result from this cognitive system, for example, are those which differentiate between the number of objects in groups. Critically, the fact that these most basic mathematical judgements (such as the ability to discriminate between the numbers 2 and 4, or the ability to ordinally sequence the numerals 1-9) have been found in a plethora of other animal species, albeit in the great apes most consistently, is what provides the indication that the most basic elements of mathematics are an evolutionary endowment[29]. In addition to this first physical conceptual framework, which shares a conceptual similarity to Kant's conception of the Pure Intuition of Time, there is a further indication of a second physical conceptual framework - that of a core geometric sense. This additional 'first-order' conceptual system, which similarly shares a conceptual resemblance to Kant's conception of a Pure Intuition of Space, has also been found in a myriad of other animal species[30]. Specifically, the judgements which are endowed by this core cognitive system have been related to "the three fundamental

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[28] Here I am using 'group' as another way of referring to collections of individuals so that I do not call to mind unwanted associations (e.g. with Cantor, or even with Mayberry's *arithmoi*). I do not intend to suggest a group in the modern mathematical sense, i.e. I do not presume the presence of an associative, symmetric relation nor an inverse on the members of the collection.

[29] Some good papers which demonstrate this in the literature include: a) Inoue, Sana, and Tetsuro Matsuzawa. "Working memory of numerals in chimpanzees." *Current Biology* 17.23 (2007): R1004-R1005. Piantadosi, Steven T., and Jessica F. Cantlon. "True numerical cognition in the wild." *Psychological science* 28.4 (2017): 462-469; b) Tomonaga, Masaki. "Relative numerosity discrimination by chimpanzees (Pan troglodytes): evidence for approximate numerical representations." *Animal cognition* 11.1 (2008): 43-57; c) Nieder, Andreas. "Neuroethology of number sense across the animal kingdom." *Journal of Experimental Biology* 224.6 (2021): jeb218289; and d) Lorenzi, Elena, Matilde Perrino, and Giorgio Vallortigara. "Numerosities and other magnitudes in the brains: a comparative view." *Frontiers in psychology* (2021): 1104.

[30] Two good indications of this come from: a) Spelke, Elizabeth S., and Sang Ah Lee. "Core systems of geometry in animal minds." *Philosophical Transactions of the Royal Society B: Biological Sciences* 367.1603 (2012): 2784-2793; and b) Spelke, Elizabeth, Sang Ah Lee, and Véronique Izard. "Beyond core knowledge: Natural geometry." *Cognitive science* 34.5 (2010): 863-884.

Euclidean relationships of *distance* (or *length*), *angle*, and *direction* (or *sense*)" (Spelke, Lee, and Izard, 2010). Moreover, there is a clear empirical indication that these 'first order' or physical conceptual frameworks are, even in the case of other animals, not the result of an organism's *own* experience and learning, but rather, that they are innate evolutionarily endowments (Chiandetti and Vallortigara, 2008; Vallortigara, Sovrano, and Chiandetti, 2009). The view which emerges from these considerations is that our most basic and fundamental understanding of mathematics, which is to say our most basic judgements and core conceptual mathematical systems, are evolutionarily acquired at the most basic level as 'first-order' or physical conceptual frameworks.

Notwithstanding this, it is very important to recognize that, in the case of our more complex and higher order mathematics, not all of our judgements are directly reducible to these most basic and core cognitive systems. In the case of more complex mathematical constructions, such as branches of Pure Mathematics that deal with judgements concerning things as high as 11 dimensions, it would be clearly absurd to reduce our judgements and understanding of higher dimensionalities to the evolutionary endowment of a basic geometric sense. Actualism does not maintain this over-reductionist assertion, but rather, maintains that these more complicated constructions, inasmuch as our more complex judgements and understanding about them, relate to 'second-order' or formal conceptual frameworks which are essentially constructed or 'chosen'. However, with respect to the foundation of mathematics, Actualism maintains that these 'second-order' formal conceptual frameworks are themselves ultimately built upon our more basic and core 'first-order' physical conceptual frameworks, and so share their foundations[31]. In other words, the most basic building blocks of mathematics (of our mathematical understanding) is founded upon our evolved cognitive structure, and our more complex mathematical systems, while distinct and not reducible to this core structure, are founded upon it[32]. This is to say, our judgements and understandings about higher dimensionalities so abstract as an 11th dimension are founded upon our evolved cognitive structure, upon our more basic judgements and understandings about the 3 physical dimensions which are evolutionarily acquired. It should here be noted that the justification for viewing the more basic claims of math as inherited judgements has not yet been provided, as the justification for this assertion occurs below.

The Semantic Question - Retaining a Realism of Reference:

The theory of reference which I mean to here employ is generally derived from Frege's "Sense and Reference" and Russell's "On Denoting". In much that same way that, as was seen earlier, Dummett remarked that the logicist's answer to the question 'what is mathematics about' generally had the right idea, I mean here to similarily affirm that the theory of reference employed by the logicists "if not the exact truth of the matter, is closer to the truth than any other than has been put forward" (Dummett, 21, 1993). This theory entailed that the criterion for being an object is having a determinate truth value (stated briefly in the form: x is an object if and only if $a = x$ for all values of a). While I accept this formal construction of objecthood, it allows as objects both *actual* things and other abstract 'things' that an Actualist will not qualify as actual (such as some of the clearly non-physical constructions of higher mathematics). In other words,

[31] It needs to be recognized, both here and in what follows throughout the rest of the thesis, that the account at present remains metaphorical. What exactly constitutes this relation (the relation between the first-order and second-order frameworks), as well as the specific metaphysical status of how it is that the second-order frameworks are said to be built/founded upon the first-order frameworks, is a subject which requires further elaboration.

[32] It is here, in this passage, that the metaphorical nature of this relation as it is presented in the current thesis is the most pronounced. The explication of what this relation precisely consists in is a complex matter for a future paper.

while this account of objecthood is formally acceptable, it is not sensitive enough to distinguish between *actual* objects and fictitious objects. Consequently, the Actualist makes a further distinction between *actual referents* and *referents of convenience*. According to this distinction, actual referents are physical objects which are independent of the mind and extended in the world, whereas referents of convenience are abstract mind-dependent concepts/constructions which are not physically substantiated. For example, the referent Venus, the morning/evening star, is here taken to be an actual referent whereas Venus, the Roman goddess of love and beauty, is instead taken to be a referent of convenience (that is, if one is to grant that the goddess is a genuinely constructed object at all). Consequently, the proposition 'Venus is the second planet from the Sun when Earth is taken as the frame of reference' describes a fact of the Universe whereas the proposition 'Venus is a more beautiful love goddess than Aphrodite' refers to an abstract idea/sentiment.

With respect to mathematics, the Actualistic thesis maintains that a form of realism may be maintained for those mathematical claims which similarly refer to actual referents. Moreover, a realist theory of reference is entirely essential to Actualism insofar that it serves as a 'loadstone' in the sense that, without a referential realism, Actualism would possess neither an ontological nor an explanatory claim of connecting or describing the actual/physical world[33]. However, in the case of mathematical propositions, the case is not as clear insofar as mathematical 'objects/numbers' are not immediately seen to be actual in the way that physical objects such as the planet Venus are. Because of this, it is necessary to first defend the claim that it is sensible to treat certain mathematical 'objects/numbers' as actual referents as opposed to mind-dependent referents of convenience. The difficulty, in this case, results from the fact that

[33] Just what this claim entails (Actualism's account of truth), will be detailed in the next subsection.

mathematical ‘objects/numbers’ such as ‘Two’ are not encountered as objects in the world in the way that planets, tables, or readers are. Consequently, in what sense can it be said that mathematical propositions ever refer to actual referents when they appear to reference abstract ‘objects/numbers’ as opposed to any individually existent physical thing?

Although the point must be granted that mathematical ‘objects/numbers’ are in important ways different from physical things such as planets or desks, a sense of realism can be restored by maintaining that mathematical propositions can be said to refer to physical things insofar as both: a) their referents are reducible to physical objects, and b) they truly describe a given state of affairs of actual referents[34]. In other words, insofar as the content of a mathematical proposition describes an actual state of affairs that is itself substantiated in the world, then that proposition may be said to possess a claim of referring to the actual/physical world. For example, if we take the claim ‘two trees plus two other trees equals four trees’, Actualism maintains that it can be taken to describe a physically substantiated state of affairs since the referents, in this case, can be taken to refer to any four actual objects (four trees). Concerning the more abstract and general claim that 2 + 2 = 4, Actualism also maintains that this can be substantiated in a realist sense insofar as such abstract truths can be seen as *generalizations* of the truths recognized in the mixed mathematical/empirical claims about individual objects. In this way, Actualism’s metaphysical view on the generalizability of number has a great affinity to Aristotle’s theory of abstractions (Aristotle, *De Anima* 429; *Metaphysics*, 1084b-1087a). Here Aristotle articulates the idea that it is a mistake to separate the universal from the individual and, most directly concerning mathematics, he reasons that numbers cannot exist apart from things (Aristotle,

[34] A full explanation of how Actualism’s referential semantics relate to the truth of mathematical propositions will occur in the next section.

Metaphysics, 1084b-1087a). With respect to the present thesis, a great affinity may be observed between this account of Aristotle and Actualism's notion that the more abstract (or universal) generalizations about numbers are preceded by empirical claims about individual sensible things.

One way to account for the generalizability of mathematical claims according to this picture may be given as follows: one begins with the more specific conception of 'a set of two trees' and observes the property of 'twoness'. We further observe that the union of two disjoint sets of two trees amounts to a set of four trees and the property of 'fourness' – and a similar story may be told for the other numbers. Thereafter we can be said to arrive at the generalization that for all sorts of X of countable individuals, two X plus two X equals 4 X. However, speaking of generalizations in this way should not be taken to signify that this process is deliberative or conscious within the Actualistic account, for, these generalizations are understood as having been formed throughout the course of the evolutionary process. Within this account, the 'reason' as to why we formed these specific generalizations is said to result from evolutionary pressures. Actualism maintains that reliable beliefs and generalizations about at least the basic mathematical truths conferred a selective advantage insofar as these general beliefs offered a reliable reflection of the actual world. It is from this notion, the notion that the basic core of our mathematical judgements and intuitions are evolutionarily inherited, that Actualism's realist thesis emerges since here mathematics is said to 'connect' to veridical claims about the external world.

From this may be derived the central principle which underlies Actualism's referential semantics, that being, the method of aiming to establish a *reductio ad materiam*. It is through this that certain mathematical propositions such as 2 + 2 = 4 may be said to possess a realism of reference insofar as its contents can be reduced and related to a veridical state of physical affairs. The application of a *reductio ad materiam* takes the form of attempting to reduce/relate the referents of a mathematical proposition to actual referents which are veridically instantiated in a state of affairs within the external world. The role that the *reductio ad materiam* plays within Actualism is to provide a means of identifying what mathematical propositions can be said to take actual referents and, consequently, which propositions can be said to possess a realist sense/meaning. When a *reductio ad materiam* is able to be established in this way (i.e., when a mathematical proposition can be said to physically obtain) it can be said to be true in a mind-independent sense as it truly describes a physical state of affairs in the world. It is in this sense of reference that Actualistic Mathematics offers an alternative realist thesis to mathematical Platonism since, as opposed to the ontologically dubious posit of the existence of things like the actual or completed infinities, the realist sense in Actualism is instead derived from the actual/physical world. Although this sense of realism does not endow mathematical 'objects/numbers' with a mind-independent existence in the same way that a planet or a rock exists, it does offer a sense in which a certain subset of mathematical claims (those which can be said to be physical veridical and instantiated through a *reductio ad materiam*), can be said to be empirical, a posteriori, and mind-independently true.

It is useful, for the purposes of explicating this distinction, to juxtapose cases of mathematical propositions which take referents of convenience and those which, through a *reductio ad materiam,* can be said to take actual/physical referents. A case of the former are mathematical propositions that refer to completed infinities as, while completed infinities may be axiomatically defined and we can derive true claims about them within mathematical theories, an actual or completed infinity cannot be said to physically exist in a finite universe. Consequently, although for the sake of the smooth functioning of the semantics of our language, and in order to facilitate the application of higher mathematics in our descriptions of the world, we may grant that one can refer to a completed infinity, it has no real reference, only a "referent of convenience.", Had we never invented language games like Cantorian set theory, the referent in question would not have existed in any sense at all. . On the other hand, in the case of mathematical propositions whose referents can be reduced to claims true in the physical world (including generalizations about that world) the states of affairs described are independent of whether we had ever conceived of them[35].

Moreover, our ability to extend a claim of independent reality to those areas of mathematics that are reducible, through a *reductio ad materiam*, to actual/physical states of affairs is not impeded by the idea that the syntax/concepts themselves are mind-dependent. In other words, the fact the natural numbers and the basic arithmetical operators can be or have been constructed does not diminish the fact that these constructions can be empirically observed as offering veridical descriptions of the world. We can defend the sense that a certain subset of mathematics is actual/physical (we can defend a realism about a certain subset of mathematics) even if we grant that the idea of 'numbers/functions/operators' are constructed within our mind-dependent system and syntax provided that those constructed concepts refer to (or can be reduced/related to) actual/physical referents which are instantiated in the world. To deny this would, of course, be tantamount to denying that realism about anything is viable because the claims about which one hopes to be a realist are articulated in a human language.

[35] This statement is of course disputable if one wants to maintain, alongside Berkley, the notion that *esse est percepi* and the subsequent denial of the physical/external world.

One initial question that emerges is: according to Actualism, how much of mathematics should we be realists about, and what do we say about the rest of mathematics? Concerning the first of these questions, the answer is at this provisional point indeterminate but, as will be seen later, there is reason to suppose that a not insignificant portion of mathematics may be interpretable in realist terms. However, a large portion of mathematics likely will not be reducible to accurate descriptions of the physical world through their correspondence to a physically instantiated state of affairs, but this does not mean that they must be given up. Rather, Actualism maintains that these instead take mentally constructed concepts as referents of convenience within their propositions. Moreover, this only becomes problematic without the earlier distinction between actual referents and referents of convenience. In the case of the Constructivist language game, where a reference to actual objects is not a concern, recognizing the entirely mind-dependent nature of much of mathematics, particularly its more complicated branches such as occurs in set theory, appears as a desired consequence. The referents of the terms in these realms have the status that the logical positivists attributed to all mathematical terms: and here it appears as a kind of philosophical confusion to ask after their referents at all. It is merely an efficient and useful way to talk using names like "Aleph Null", to talk as if that name has a referent. However, what the Actualistic orientation and the method of a *reductio ad materiam* offers (that being the claim that a certain subset of mathematics are true in a realist and mind-independent sense insofar as they offer veridical descriptions of actual referents in the world) fits together well with a referential semantics which permits an ontologically realist foundation for mathematics which is both scientifically and philosophically defensible.

Truth Apropos Actualism:

As was indicated earlier in this chapter, the truth of basic mathematical claims can be reduced to true descriptions, often in the form of generalizations, which correspond to facts about the actual world. For Actualism, however, there is another residual problem that is: in what sense are the claims of more complex mathematical realms, those which are not directly reducible to the actual world, true? In order to account for the truths of our ‘second-order’ abstract conceptual schemas, Actualism asserts a composite theory of truth whereby it maintains that there are two different ‘kinds’ of mathematical truth. The first of these, *actual truth*, is said to occur in cases where a proposition corresponds to a physically instantiated state of affairs by describing it - when a proposition gives us some information or knowledge about the external world in the way earlier described. The second of these, *formal truth*, is said to occur through the correct/consistent application of the rules, axioms, or syntax of a formal symbolic system to referents of convenience. Whereas *formal truths* do not require a realist account of correspondence (since they do not purport to refer to the external physical world), *actual truths* will be seen to require an articulation and defence of just how it is that the sense of a mathematical proposition can be said to correspond to some actual/physical state of affairs.

Within Actualism, correspondence is understood to consist solely in the fact that certain mathematical propositions offer veridical descriptions of some state of affairs[36] in the external world which is physically instantiated and empirically verifiable. The sense of correspondence here meant is therefore taken to consist entirely in a proposition conveying some true information about the external world - in their offering a veridical description of something

[36] A state of affairs is here taken in the somewhat nonstandard sense of including generalizations about the actual world.

actual/physical. With respect to the earlier example: ‘two trees plus two other trees equal four trees’, this statement is said to correspond to the extended world insofar as they convey true information about it[37]. However, this is not to say that they latch onto or connect to an actual/physical referent or state of affairs in any necessary, definite, or essential way; but rather, that they offer some accurate approximation of an actual referent or physical state of affairs. In other words, Actualism does not maintain that the correspondence between a proposition and reality is itself actual/physical, but rather, that there is some correspondence *to* the actual when a proposition offers an accurate description of some external state of affairs that is physically embodied (and it is in this sense that a mathematical proposition is said to be ‘realist’ when it conveys an *actual truth*). In contrast, propositions whose referents do not concern the actual, propositions which are essentially abstract/ideal, their truth is context dependent on the system in which they occur and is understood to consist in their correctly applying the rules/axioms of that system. This is to be starkly contrasted from those propositions which, through a *reductio ad materiam*, may be said to refer to actual things and whose truth consists in their accurately describing (corresponding to) a physically instantiated state of affairs. It is the fact that certain mathematical claims can be reduced to *actual truths* about the physical world that stands behind Actualism’s claim that we are correct to be realists with respect to these claims.

The residual problem (of how it is that Actualism accounts for the sense in which claims of more complex mathematical realms, those which are not directly reducible to the actual world, are true) nonetheless remains. According to Actualism, such claims are formally true in an anti-realist sense – which is to say, they are said to be *formal truths* about referents of convenience.

[37] While closely related, the sense in which a proposition may as a whole be related to the actual (in the form of offering a veridical description of a physically instantiated state of affairs) should be consciously differentiated from the sense in which its referents may be taken to refer to actual/physical things.

As was earlier seen, Actualism accepted Frege's conception of objecthood and identifiability. Consequently, the more complex claims of mathematics are just seen to pertain to well constructed objects which take referents of convenience. Concerning these more complex mathematical realms, Actualism likewise outright accepts the Constructivist project for, once we have left the world of referents that are reducible to actual objects, the Actualist's project has ended.

Actual Admissibility:

In order to further clarify the Actualist position, it is useful to have recourse to the notion of 'actual admissibility' which is where Actualism's conception of physical possibility arises. By the term actual admissibility, I mean something which could be physically instantiated – something which could exist be a spatio-temporal (actual) object. Within the present account, actual admissibility is a notion predicated of referents that, regardless of whether or not they presently exist, could possibly exist as actual objects. In this way, what it means for something to be actually admissible is just that it is physically possible – that it could be physically instantiated as an occurrent state of affairs or as an object within a physically instantiated state of affairs. On the other hand, what it means to say that some thing is not actually admissible is that it could not physically exist in the material universe. For example, a square circle is not actually admissible insofar as no such object could be physically instantiated (such an object could not exist). However, neither is a square circle abstractly admissible – for it is also a conceptual contradiction. However, actual admissibility can be seen to be a more restrictive criterion than abstract or conceptual admissibility when we consider examples like completed infinities which, despite being abstractly admissible (which we know because they are referents of convenience in consistent systems), are not actually admissible. It is here, when we consider things that are abstractly admissible while not simultaneously being actually admissible, that a great similarity between Actualism and Finitism emerges – for the primary examples of such things are actual or completed infinities or of unnaturally large finite numbers like 10^10^10[38].

While Actualism reupdates these numbers alongside Finitism, the Actualistic reason for rejecting things like the actually infinite, completed infinities, or of numbers like 10^10^10 dramatically differs from the Finitist's. As has been seen, Finitists reject such numbers because of the impossibility of creatures sufficiently like us being able to construct them, whereas an Actualist rejects such constructions for ontological reasons. Where Finitists object for reasons such as: a) the lack of a unique identifiability of such constructions, b) the lack of suveyabiltiy or provability of such constructions, or c) the lack of a definite/concrete construction in these cases, Actualism instead repudiates these constructions on the ontological basis that they are not actually admissible (that they could not be physically instantiated).

How dramatically the two positions differ, despite the likeness of their conclusion, can be evinced if we consider a case where a construction that is not actually admissible *is* imagined to have been definitely/concretely constructed in such a way that it is surveyable. From the Finitist perspective, it is legitimate to imagine that tomorrow a mathematician might be born who works exponentially faster than any who have come before and, while not having the prodigious powers of Brouwer's idealized mathematician, is capable of arriving at a construction of 10^10^10 over a few weeks. Since the limitations for the Finitist have to do with our limitations, it seems that there are imaginable circumstances in which he would have to grant the existence of this number,

[38] While it is important to recognize that future science may always discover something about the universe which could render these specific examples false, in which case new examples could be derived, at least in the case of completed/actual infinities the point is well substantiated. This is because if, as our current sciences indicate, the universe is finite then collections/objects cannot be infinite in any real sense – no actual infinite collections or objects are physically admissible within a finite space (though potential infinities are abstractly admissible).

what with it having been successfully constructed. On the other hand, from the Actualistic perspective, even if we could definitely construct a number so large as 10^10^10, these constructions would still fail to meet the criterion of actual admissibility – they would not be physically instantiated insofar as we exist in a finite, physical universe. Consequently, even if we should grant the possibility of a definite construction of an actually unadmissible 'thing', it would be in principle excluded from entering positively into our ontology insofar as no *reductio ad materiam* can be made for a construction that is not actually admissible.

The notion of actual admissibility further offers a heuristic means of determining if the reference of a mathematical proposition is to an actual referent as opposed to a referent of convenience. The reason for this is that, if the referent of a mathematical proposition is to something which is not actually admissible, then it is necessarily made to a referent of convenience. For example, if we assume that we exist within a finite universe, then all mathematical propositions which make reference to the infinite (actual, potential, completed, or otherwise) take as their referent an abstract referent of convenience. Consequently, within an Actualistic language game, such propositions cannot be said to describe anything with a positive ontological status since they are not reducible to any actual/physical referent or state of affairs. And although one may choose to define/extend existence to that which has been constructed in only an anti-realist sense, the case that it is either sensible or meaningful to predicate the existence of things that are not actual/physical (and the case that this use of existence does not culminate in a kind of ontological dualism), is outstanding.

Some Initial Challenges and Objections to the Actualistic Thesis:

What Criterion Marks the Physically Actual:

One initial challenge which confronts the Actualistic account is whether or not a sufficient criterion can be identified by which something can be qualified as being physically or materially ‘real’. In other words, Actualism requires a defensible case as to why it is justified in identifying certain referents as being actual/physical and others as ideal/abstract constructions of convenience. Moreover, it is also important that this criterion serves a genuine explanatory value – otherwise it should appear as nothing more than an empty division. Concerning the nature of these referents, it is also important to distinguish between the ontological status of the referents themselves and our epistemic ability to discern or understand them. For the purposes of the present thesis, the criterion desired is of the first kind and seeks to identify an ontological, as opposed to an epistemological, criterion for justifying the bifurcation of actual referents and referents of convenience. Consequently, the central question at present is whether or not there is a sufficient justification for asserting that some referents possess the positive ontological status of being actually/physically instantiated and not the secondary question of whether or not we have a complete epistemological understanding or awareness of the referents themselves.

Notwithstanding this, a certain degree of epistemic understanding should be expected of the present account insofar as we have already critiqued Platonism for positing the existence of referents to which we have no access. Although Actualism is similar to Platonism in the sense that both offer a realist thesis, the reality behind the Actualistic reference is made to actual/physical objects in the external world as opposed to Platonic forms with which we have no explicable sort of interaction given our cognitive apparatus. Therefore, so long as one does not, with Berkeley, object to the existence of material things, and so long as one does not outwardly disavow all of the empirical observations made by the physical sciences, then the claim that we are able to access the actual/physical referents in at least some capacity is a peremptory claim. Because of this, the Actualistic thesis avoids some very serious flaws found within the Platonic account – particularly the posit of forms to which we have no access – since the only entities which are here asserted to be mind-independent are the objects of the external world itself. It is in this way that Actualism's assertion of a mind-independent reality and reference grounded in the external world avoids the problems associated with recollection and remembrance of forms (or thoughts, in Frege's case) which trouble mathematical Platonism.

The question nonetheless remains, what is the criterion that qualifies something as actual/physical in the present account? A number of possible answers could be given at this point, such as: a) that it is a substance, b) that it has causal physical properties, c) that it is materially extended, d) that it exerts a gravitational force, or even e) that it has mass and is able to hold weight. However, what is here being taken as a criterion of physicality should be taken as a first approximation as it is implausible to claim that a perfect or exhaustive criterion can be identified and be expected to hold for all time, and also, because this criterion could itself be further developed as the central topic of another inquiry. Notwithstanding these limitations, a present general criterion can be offered which maintains that something is said to be actual/physical when the empirical observations of our best physical sciences substantiate the claim that the thing is, at least in some veridical sense, embodied in the external and extended world.

Moreover, such a criterion may be made more intelligible by contrasting it to Quine's notion that "To be is to be the value of a variable," i.e. that the things that exist are the things that our best scientific theories quantify over. Whereas Quine's is primarily theory-driven, and thus, whereas his criterion concerns ontology primarily for the relation that objects have to theories, the Actualistic criterion is more directly concerned with ontology. In contrast to the Quinean slogan, the Actualistic reinterpretation of the slogan would instead run – to be the value of a variable, in a veridical scientific theory, is a good indication that something really is. However, Actualism does not endow 'being' to something because it is quantified over by a scientific theory, but rather, it views this as an epistemic reason and justification for supposing that something really is. In other words, Actualism takes a thing featuring within a good scientific theory or model as providing good evidence that the thing we are positing really exists in the way we suppose, but the actual existence and ontology of the thing occurs irrespective of our theories and models for the Actualist. And although the Actualistic criterion does seem, at least to some extent, to introduce a temporal dimension into Actualism's account (insofar as the postulates of our best sciences will change over time) it does so in a much more benign sense than Brouwer does through his method of the 'creating subject'. It is worthwhile to note that a criterion such as this, one which is able to justify such a heavy reliance on the postulates of its present science, is a contemporary privilege afforded only by the developments that science has attained up to the present point. However, in light of the scientific body of knowledge upon which we stand – built veritably as it were on the shoulders of those such as Ymir – we are contemporarily justified in staking our criterion of what qualifies something as being an actual/physical referent in the empirical observations of our present science.

It is with this sense that the *reductio ad materiam* method is applied, whereby the reference of a proposition *can* be said to reduce, relate, or else to correspond to some actual/physical referent when it offers a veridical description of a physical state of affairs in the external world. In this sense, propositions like 2 + 2 = 4 can be said to have an actual/physical correspondence, or reference, insofar as the referents of the proposition are reducible to objects in a state of observable affairs in the external world. Moreover, the claim is said to be 'true' according to Actualism insofar as it offers a description of a veridical state of affairs in the world that may be empirically tested and verified. Therefore, the claim that 2 + 2 = 4 can be said to be true in a mind-independent sense insofar as any two actual/physical things added to another two actual/physical things will result in four things. Should one question what the words two and four denote within the above statement, we can instead qualify, to whatever degree of detail that is desired, that they denote the sizes of groups of actual/physical objects to which we have arbitrarily assigned these words as descriptions of[39]. In contrast, constructions such as a 'square circle', or else of 'a number larger than anything or any set of things in the physical universe', are not physically admissible and can neither be said to, through a *reductio ad materiam*, denote any actual/physical object or state of affairs.

What Does Actualism 'Do' with Constructions not Meeting This Criterion:

Concerning the Actualistic prioritization of actual/physical referents the following questions arise, how does Actualism explain references which are made to referents of convenience, and also, how does it account for the proofs resulting from the anti-realist branches of mathematics? The first of these questions relates to the problem that the logicists faced regarding cases of negative quantification, and references to non-existent entities like a Pegasus, some of which take the form 'there is not some X', or otherwise, 'some X does not exist'. However, the treatment employed by Russell in "On Denoting" which centred around an analysis of definite descriptions will fail to satisfy the present question for two reasons: a) we are not at

[39] The case that it is sensible to speak about denotation in this way (that being in the sense of eventually arriving at a definite description that denote some object which may be expressed in the form of the existential quantifier), is made sufficiently in Russell's paper "On Denoting" and is therefore assumed for the present purpose.

present solely concerned with singularly definite descriptions (seeing as the claims of mathematics are general in nature), and b) we have already granted the syntactical legitimacy of referents of convenience (it would therefore be contradictory to now be as dismissive of them as Russell is of Meinongian entities). The potential problem thus comes to a head at the question: what is the ontological status of legitimately constructed referents of convenience which, despite not satisfying a *reductio ad materiam*, feature meaningfully in mathematical proofs?

Unlike existential claims which assert the existence or non-existence of physical impossibilities, such as square circles, there are a number of references to mathematical referents of convenience, such as the empty set, which can feature meaningfully in our proofs and propositions. It is in these cases that the sentiment of one such as Meinong may seem more appealing – seeing as we do certainly seem to be referring to something (or at the least some idea) when we discuss the empty set or an infinite set. Moreover, although Actualism is able to account for referents that are blatantly contradictory, such as a square circle (in the sense that they are said to be 'nothings' owing to the fact that they are neither actually nor conceptually admissible), it is a different matter to explain the meaningfulness of proofs that make reference to abstract constructions of convenience which are nonetheless not reducible to actual objects. While Platonism may shamelessly presuppose the form-like existence of such things, and while Constructivism, resting upon an anti-realist thesis, can with all good intellectual conscience disregard the question, Actualism – insofar as it is interested in offering a realist thesis – does require an answer to this question.

One way of answering this question is that the Actualistic notion that we are able to reduce a certain subset of mathematical claims, through a *reductio ad materiam*, as offering veridical descriptions of physical states of affairs, does not preclude us from also maintaining that the rest of the mathematical claims, as well as the formal syntax itself, are merely a human construction. However, insofar as these constructions do not feature actual referents, it does not make sense to endow them with a positive ontological status by way of maintaining that they exist. Moreover, it is possible to maintain the claim that the mathematical syntax and notation (at least insofar as it is formalized contemporarily) is itself anti-real (and thus not interpreted in realist terms) while also maintaining that it is able to play a role in veridical descriptions of the extended world. A good example of this comes from the pragmatic utility that infinitesimals afford to calculus which itself is part of a veridical description of how continuous change operates in physically extended states of affairs. In this way, although calculus may incorporate non-actual referents and concepts in order to function (infinitesimals as a referent of convenience, or completed infinite sequences or equivalence classes of Dedekind cuts, depending on your tastes), it still serves as a tool that when paired with some additional content allows the formulation of physical principles that culminate in a truthful description of the world. Moreover, it is likely that nearly all of the actual aspects of mathematics also rely in some sense on referents of convenience or ‘second-order’ conceptual frameworks which possess an anti-real aspect. Although it is important to recognize that the case will likely become more complicated with respect to mathematical propositions owing to the earlier bifurcation between those which take actual referents as opposed to those which take referents of convenience. For mathematical propositions which take actual referents, we have seen that they may be said to be true insofar as they describe a state of affairs that corresponds to an instantiated state of affairs in the external world. In contrast, for those mathematical claims that only take referents of convenience, it has been seen that Actualism maintains that their reference is to non-actual, empty, or else ‘ideal’ constructions which do not exist.

While it may initially confound to maintain that constructions that act as meaningful referents within mathematical proofs do not exist, this conclusion is seen to be required when we instead consider what it would mean were this not the case. If we instead maintained that, despite not being actual/physical, these constructions existed in any real sense (such as is the case with mathematical Platonism), then we would be forced into the ontologically dubious position of substance dualism. This position requires the posit of a non-actual kind of existence, such as subsistence as it was formulated by Meinong, whereby things are said to exist in a non-actual but possible sense. Instead of this, Actualism maintains that our mind-dependent linguistic constructions which are not reducible to any actual/physical referents do not exist in any real sense, and it is a mistake to endow them with an ontological status at all. If pressed, the Actualist will say that their apparent referents do not exist.

Although it may sound unintuitive to describe any referent as a ‘nothing’, this description is defensible insofar that it is preferable to calling referents that do not exist as ‘somethings’ (the latter phrasing being that which lends jointly to Meinong and Plato’s ontological confusions). Moreover, the intelligibility of this description can be made more readily available by considering obviously non-existent entities such as the Pegasus, Zeus, or Fenrir. Despite the fact that one might ‘construct’ such a referent as ‘the immortal god son of the titan Cronus who lives atop mount Ida and who is the cause of lightning’, the construction can not be defensibly said to refer to any existent thing/object. While one *could* here say that the construction instead refers to a concept or a non-existent thing, it is precisely this way of speaking which lead to the errors of Meinong or, in the case of less clearly absurd constructions, the Platonic posit of form-like otherworldly entities. Actualism, in contrast, is unwilling to extend a positive ontological status of existence or ‘thinghood’ to any non-actual referents. Consequently, references which refer to an anti-real construction, as opposed to an actual/physical thing, are here said to refer to 'nothings'.

While traditionally this problem has often been handled by suggesting that terms which do not refer to anything simply do not have a referent, for many there will likely remain a strong intuitive appeal to the notion that statements which feature the names of fictitious entities such as Zeus are really referring to something. Part of the impetus behind this intuitive appeal derives from the fact that, for those at all familiar with the relevant mythology, statements like "Zeus the dread son of Cronus is king of the Olympians" or "Zeus the dread son of Cronus defeated his father" both make sense to us and worse they (putatively) appear to be *true*. Consequently, the Russellian account, in rendering all substantive claims that seem to refer to non-existent entities false, is greatly at odds with the intuitive sense that such propositions appear to be true. Although there are many virtues to the traditional manner of handling the problem (and although Russell's treatment in "On Denoting" is cogent), Actualism, in lieu of engaging in the more technical question of what constitutes as a *bona fide* referent, instead grants the reference while noting that the statements referents are 'nothings'. It is here that the Actualist, if they are pressed, will recognize the oddity of referring to 'nothings' and say that the referents do not exist.

With respect to the second question, how does Actualism account for the proofs resulting from the anti-realist branches of mathematics, it does so through its demarcation of a secondary (anti-realist) sense in which mathematical propositions that refer to referents of convenience can be said to be true. This secondary sense of truth is derived from the formal consistency of a syntactical or logical system, that is, in the correct application of a constructed system's rules for symbolic manipulation. In this way, Actualism is seen to be highly sympathetic to Constructivism – it takes nothing away from it insofar as Constructivism did not aspire to mind-independent truths in the first place. However, it has still more in common with Positivism, since it does not take Constructivism to be the only possible formally consistent system extending the core of mathematics that comes as standard equipment for humans, and indeed it recognizes that for most purposes we have chosen other options. With Carnap, we see the truths of higher mathematics as (in the suitably hedged sense Carnap and others spent considerable effort trying to clarify) *conventional*. But Actualism differs from Carnap and the Construtivists by re-endowing a certain subset of mathematics with realism, showing how they are true in a mind-independent and external sense.

How Much of Modern Mathematics Qualifies as Actualistic:

The concern that, through its prioritization of the actual and physical, Actualism leaves out a substantial portion of modern mathematics, such as the highly axiomatized systems of set theory, may appear as initially troubling. This is a concern that can be relieved. This concern would be particularly problematic for a more dogmatic form of Actualism which, as opposed to ontologically prioritizing the actual and physical, instead outwardly rejected any sense in purely mind-dependent constructions. However, insofar as the present Actualistic thesis only highlights an ontological priority (in the form of entering positively into our ontology) for those things which are actual/physical, it does not 'take away' or object to the internal consistency of any standing formalized mathematical systems or constructions. Consequently, Actualism can be said to leave all of modern mathematics standing. It does not object to the internally consistent structure of 'Cantor's paradise'; only, it offers the ontological stipulation that such abstract constructions are no more real actual/physical things than Eden or the Elysian fields are.

Notwithstanding this, or perhaps as a consequence of this, one may ask the question: are 'enough' of mathematics reducible to the actual/physical to either be of explanatory value or to be interesting enough in their own right? This question is likely the most concerning for the Actualistic account since, if a rather insubstantial portion of mathematics meets the standards of Actualism, then Actualistic mathematics may appear as trite if not entirely unimportant. Although this concern, should it be borne out, would be entirely devastating to Actualism; the present preponderance of applied mathematics within the hard sciences is sufficient in and of itself to address this concern. The applied branches of mathematics, most particularly in the case of their use in physics, can defensibly be said to relate to actual/physical referents or states of affairs in the external world, and thus, being no small part of mathematics, they are sufficient to defend the notion that a substantial portion of mathematics meets the Actualistic criterion. Moreover, even those more ideal aspects and abstract constructions of Pure Mathematics are incorporated into Actualism's account insofar as these more complex 'second-order' conceptual systems are seen to be built upon the more basic 'first-order' core mathematical systems which are inherent to our cognitive structure.

Another way of expressing this concern may take the form of asking: how are the physically-based mathematical truths and the "ideal" mathematical truths related, or, in what way is it accurate to call both of these mathematics? When expressed in this form Actualism may answer this concern by first making the recognition that the more complicated and abstract branches of Pure mathematics take the more basic 'first-order' or physical mathematics as a common core. Consequently, a unified sense of mathematics may be provided as being that language game that is both founded upon and which operates within that core cognitive conceptual framework (the intellectual instincts) that deal with judgements of numerical or geometrical things.

It is further possible to defend a sense in which even a certain subset of pure and abstract mathematics can be said to meet the Actualistic standards. Significantly, one can through the *reductio ad materiam* method defend the notion that the natural number sequence (at least to some indeterminate point), as well as the basic arithmetic operations, are 'real' and true in a mind-independent sense. Even more interestingly, one may defend a sense in which propositions referring to the empty said to express a mind-independent truth via a *reductio ad materiam*. One reasoning for supposing this may arise from the recognition of how the concept of a void, zero, or *nihil* may offer a veridical description of a given material state of affairs. For example, this notion can be taken as offering a true description of an actual world state in the case of temperatures reaching absolute zero (zero kelvin or −273.15 °C). Another indication of this also arises from certain empirical studies in numerosity which indicate that other species have at least a rudimentary conception of an empty set (Kirschhock, Ditz, and Nieder, 2021). Within the present account, this finding indicates that the empty set is an evolutionarily endowed intellectual instinct that is, therefore, to be understood as a part of our 'first order' and core cognitive conceptual system. Although these things may not be said to 'exist' as Platonic objects in their own right, propositions about them may be said to be true insofar as their senses express true information/facts about the way that things or groups of things will be. Consequently, we are justified in asserting that 'nature gave us the natural numbers all else is the work of man', and in maintaining that a rather substantial portion of mathematics expresses mind-independent truths about the external world. Furthermore, Actualistic mathematics is in this way able to avoid all of the complications which arise from attempting to formalize/axiomatize a consistent foundational system of mathematics - since their foundations are manifestly true and substantiated by their already obtaining within the extended world. Most importantly, Actualism's foundation of mathematics has the unique quality of not being compromised by Gödel's incompleteness theorems – owing to the fact that its foundations obtain manifestly in the external world itself and have no need for a complete or consistent formal axiomatization.

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