Consensus Problems in Hybrid Multi-Agent Systems

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

A multi-agent system (MAS) is a dynamic system that consists of a group of interacting agents distributed over a network. In the past decades, the study of distributed coordination of multi-agent systems has been widely attracted by many groups of researchers such as mathematicians, engineers, physicists, and others. This is partly due to various applications in many areas, including spacecraft formation flying, multiple robot coordination, flocking, consensus or synchronization, cooperative control of vehicle formations, etc. As one of the most important problems in distributed coordination, consensus means that a group of agents achieves an agreement on a common value by designing the control law which is based on the information received by interacting with neighbors. There are many consensus methods that have been studied in recent years. Some problems focused on seeking the consensus of continuous-time (CT) multi-agent systems or discrete-time (DT) multi-agent systems, the others considered consensus problems on hybrid systems which are dynamical systems involving the interaction of continuous and discrete dynamics. Most consensus algorithms have been proposed for the multi-agent systems, but most results of consensus analysis are on the situation that all agents are continuous-time or discrete-time dynamic behavior. There are, however, some practical problems that the discrete-time and continuous-time dynamic agents coexist and interact with each other at the same time. Thus, it is reasonable to study consensus problems in such hybrid multi-agent systems (HMASs).

Generally, the consensus protocols are designed to ensure that the states of all agents converge to a common value. However, up to date, in many practical problems, the states of agents may converge to prescribed ratios rather than a common value, such as compartmental mass-action systems, water distribution systems, and multiscale coordination control between spacecrafts and their simulating vehicles on ground. To deal with this problem, the scaled consensus problem has been introduced, where all agents will converge to the assigned proportions. Different from the standard consensus, where a group of agents seek to agree on a common quantity depending on the states of agents, scaled consensus implies that the state of each agent will approach prescribed ratios in the asymptote.

So this work aims to study the (scaled) consensus problems in hybrid multi-agent systems under fixed and switching topologies including linear and nonlinear dynamics. Furthermore, we study consensus problems with communication delays, external perturbations, finite-time (scaled) consensus problems and also apply to the random networks.

Acknowledgements

First and foremost, I would like to thank my supervisor, Prof. Xinzhi Liu. Thank you for providing the guidance and support in my research and life. I would also like to thank my examining committee members, Prof. Jun Liu, Prof. Henry Shum, Prof. Sherman Shen and Prof. Albert C.J. Luo, for their invaluable feedback to enhance this thesis.

I would like to extend my thanks to all the members of my research group for the help and conversations during my program at Waterloo. I would also like to express my sincere gratitude to all my friends and family members for their encouragement.

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Dedication

This thesis is dedicated to my parents.

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Chapter 1

Introduction

1.1 Motivations

An agent is a computational mechanism like a computer program or a robot, which can interact with its neighbors or environment, and also adapt when its environment changes. During to the ability of adaptation and interaction, it is considered as an autonomous agent. An agent has appeared in several areas depending on the purpose of applications such as in medicine, military, agriculture, transportation and other areas (see e.g.,in Figure 1.1).

(a) (b) (c)

Figure 1.1: An autonomous agent

A multi-agent system is a dynamical system consisting of a group of agents which can interact with each other or their environment (see e.g.,Figure 1.2).

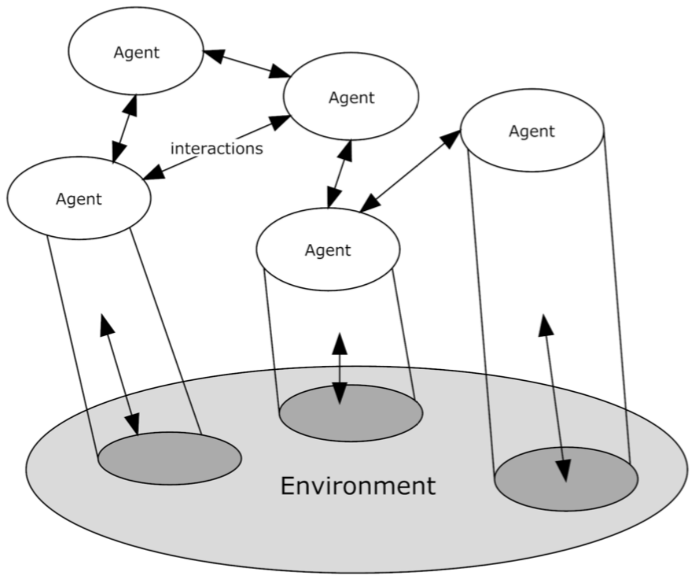

Figure 1.2: Muti-agent systems

Over the past decades, multi-agent systems have been broadly studied due to its wide applications, such as in 1980s and 1990s, many scientists, especially in computer science, have actively studied in multi-agent systems, typically referred to software agents. Moreover, multi-agent systems can be applied to artificial intelligence. Since most problems can be simplified by dividing into subproblems, so it is important to study how all agents work together as a team to reach the goal. In particular, Weiss (1999, [110]) replaced the single agents by the multi-agent systems as the computing paradigm in artificial intelligence.

Furthermore, in robotic society, multi-agent systems are referred to multi-robot systems (as the agents can be robots), and have been extensively studied since the early 1990s. For example, Sugihara and Suzuki (1990, [92]) studied the distributed motion coordination of multiple robots (see e.g.,Figure 1.3).

Figure 1.3: Multi-robots

In the recent years, many research topics in multi-agent systems have been actively studied, such as consensus or agreement problems (2007, [73]), flocking (2006, [72]), formation control (2009, [117]), coverage control (2004, [16]), containment control (2011, [10]), distributed estimation (2008, [126]), and others (see e.g., Figure 1.4).

Figure 1.4: Formation control

In mathematics, a hybrid multi-agent system can be represented as a communication network or directed graph [78], where all agents are regarded as the nodes and the interaction between two agents has been represented by the edge in a graph. This implies that $(v_i, v_j) \in \mathcal{E}$ corresponds to an available information link from agent i to agent j. Besides, each agent updates its current state based on the information received from its neighbours (see e.g., Figure 1.5).

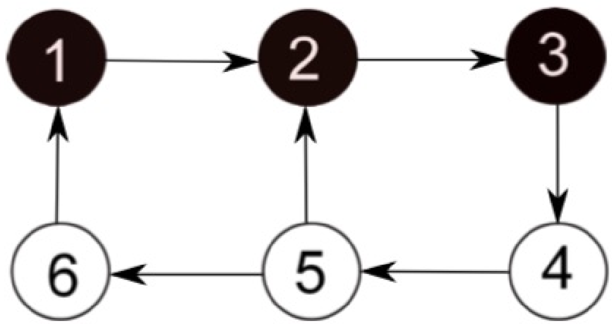

Figure 1.5: A communication network of a multi-agent system

As one of the most fundamental and important research topics in multi-agent coordination, consensus algorithms have received significant attention. The objective of consensus algorithm is to design an appropriate control input based on the local information that enables all agents to reach an agreement on some common features, which can be velocities, positions, attitudes, and many other quantities. The original work of consensus problems was proposed by Degroot (1974, [18]). In the recent years, many consensus algorithms were proposed based on the dynamic model of agents. In 1995, for example, Vicsek *et al* [98] presented the discrete time model of agents moving with the same speed and proved by simulation that all agents can move to one direction. In 2003, Jadbabaie *et al* [40] used the nearest neighbor rules for proving the model of Vicsek in [98]. Moreover, Wang *et al* (2007, [116]) proposed the new method for solving consensus problem in discrete time multi-agent systems with time-delays, more results about consensus seeking in discrete time multi-agent systems can be seen in ([55], [6]) and references therein.

For continuous time dynamic agents, many consensus algorithms have been proposed, such as in 2004, Olfati-Saber and Murray [74] showed the consensus results of continuous time dynamic agents with switching topology and time-delays. In addition, Ren and Beard (2005, [80]) also studied the consensus in continuous time multi-agent systems and used some concepts from graph theory and matrix theory to extend the results in [74], which gave more relaxation conditions than the previous works. More results about consensus seeking in continuous time multi-agent systems can be seen in ([122], [79]) and references therein. Inspired by the result of Halloy *et al* ([32], 2007) who studied a group decision-

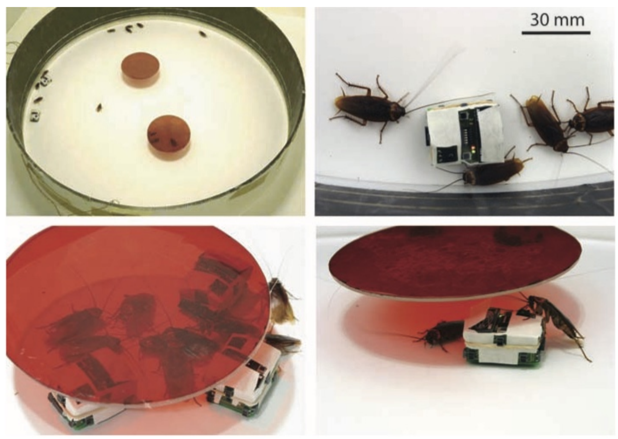

Figure 1.6: Group decision-making between robots and cockroaches [32]

making between animals and autonomous robots by proving that the group of autonomous robots mixed with cockroaches can share shelter together under some conditions(see e.g., Figure 1.6). Hence, it is reasonable to study consensus problems in the dynamical systems involving the interaction of continuous-time and discrete-time dynamics, which is typically called as the hybrid systems (Antsaklis(2000), [4]). In particular, the study of consensus problems under switching topologies have been actively attracted by many researchers, such as in 2008, Sun *et al* [95] showed the average consensus results of dynamic agents with switching topologies and time-varying delays. Moreover, Zheng and Wang (2015, [133]) studied the consensus of switched multi-agent systems. See more results for consensus seeking of switched multi-agent systems in ([137], [124]) and extensive references.

According to the above discussion, most consensus problems have been studied only when all the agents are discrete-time or continuous-time dynamic behaviours. As a result, Zheng *et al* (2017, [132]) introduced the consensus problems of hybrid multi-agent systems (HMASs), where the continuous-time and discrete-time dynamic agents coexist and interact with each other. This system has actively been studied my many groups of researchers, such as, Zheng *et al* [134] who extended the consensus results of [132] into the second order case in 2019. Furthermore, the consensus problems of hybrid multi-agent systems with heterogeneous dynamics have been studied by Zhao *et al* [131] in 2020. See more results about consensus problems in hybrid multi-agent systems in [67, 105, 87, 37] and references therein.

However, in some real multi-agent systems, a state of agents may encounter some abrupt changes (or agents exchange information to their neighbors instantaneously) at a certain time moments and cannot be considered continuously, which the continuous-time consensus algorithms cannot be applied. To deal with this problem, impulsive consensus protocols have been introduced. The primary idea of the impulsive consensus method is based on the strategy of impulsive control which is to change the state instantaneously at certain instants, so it can reduce control cost and the amount of transmitted information dramatically. Furthermore, impulsive control can deal with systems that cannot be controlled by the continuous control methods or impossible to provide continuous control inputs. For example, consider a multi-agent system where each agent represents a deep-space spacecraft. A consensus problem is to design a control law that allows all spacecrafts reach an agreement upon certain quantity of interests, such as velocity, position, and direction. If each spacecraft has only limited fuel supply, it cannot leave its engine on and exchange information with the others continuously. Hence, it is more practical for the spacecrafts to communicate with their neighbors once in a while at discrete moments which can be modeled by impulsive consensus protocols. Consequently, impulsive consensus protocols have been rapidly developed lately (see, e.g., [42, 26, 130, 127, 129, 65, 66, 68, 24, 107, 114, 115]).

The aforementioned works primarily focus on complete consensus problems, that is, the consensus protocols are designed to ensure that the states of all agents converge to a common value. However, up to date, in many practical problems, the states of agents may converge to prescribed ratios rather than a common value, such as compartmental mass-action systems [30], water distribution systems, and multiscale coordination control between spacecrafts and their simulating vehicles on ground [28, 58]. In order to deal with this problem Roy (2015, [81]) introduced a new type of consensus called "scaled consensus". Different from the standard consensus that a group of agents seek to agree on a common quantity depending on the states of agents, scaled consensus implies that the state of each agent will converge to prescribed ratios in the asymptote.

In the recent years, scaled consensus problems have been attracted by many researchers, for instance, scaled consensus problem of multi-agent systems under fixed and strongly connected topology have been studied by Roy[81] in 2015. In 2017, Ebrahimkhani *et al* [20] applied the idea of [81] to the descriptor multi-agent systems. Moreover, since time-delay cannot be avoided in some real applications, some people extended scaled consensus to the delayed multi-agent systems, such as Aghoblagh *et al* (2015, [1]), Aghoblagh *et al*

(2016, [2]), Shang (2017, [84]), Ebrahimkhani and Hashemzadeh (2018, [21]), Xing and Deng (2018, [123]).

On the other hand, scaled consensus problems have been broadly investigated in switching networks. For example, Meng and Jia (2016, [69]), Shang (2019, [86]), Chen *et al* (2020, [11]), and Li *et al* (2020, [48]). Furthermore, scaled consensus problems have been extended to heterogeneous systems by Liu *et al* (2016, [61]), and to the multiple non-identical linear autonomous agents by Cheng-Liu and Liu in 2017[62] and other problems (see, e.g., [83, 85, 128, 34, 35, 44, 53, 52]).

In the actual network, it can be seen that each agent receiving information from its neighbors may has time delays induced by the distance among them, and sometimes has self-delay obtained by processing its information. These delays will lead in general to a reduction of the performance or instability of the system. Therefore, investigating the time-delay problem of the multi-agent system comes to be important (see e.g.,[108, 121, 66, 63, 65]).

In some practical applications, it is important to drive the multi-agent systems (MASs) to a desirable state as soon as possible. Therefore, the convergence rate is an important indicator in the design of protocols. It is often required that the eventual consensus is reached in a finite time, whether or not this depends on the initial configuration of the system. Hence, the finite-time consensus problems have been actively studied in hybrid dynamic systems, specifically, the switched system that consists of continuous-time and discrete-time subsystems. For example, Jiang and Wang (2009, [41]) proved that the multi-agent system under nonlinear interaction can reach the consensus state in finite time. Lin, Yu and Chen (2016, [64]) proposed switching protocols to solve the finite-time consensus of multi-agent systems. Besides, in 2017, Lin and Zheng [56] proposed the finite-time consensus protocol to guarantee reaching consensus of switched multi-agent systems in finite time and also studied fixed-time consensus problems of switched multi-agent systems. See more works about finite-time consensus problems in MASs in [118, 119, 13, 94].

It is motivated by above discussion, this thesis aims to investigate some problems in HMASs under such as (scaled) consensus problems and finite-time (scaled) consensus problems, see the contributions of the thesis in the next section.

1.2 Contribution of the thesis

The main contributions of this thesis are summarized as follows:

1. The considered hybrid multi-agent system is quite general as the system consists of a group of continuous-time discrete-time dynamics agents that can interact with each other which can be utilized to model the practical networking agent systems. Furthermore, it can be seen that if there is no continuous (or discrete) time dynamic agents, the system is a discrete (or continuous) system. Moreover, we also investigate the (scaled) consensus problems of nonlinear (hybrid) multi-agent systems.

2. According to above discussion, the consensus results are guaranteed only on the situation that the interactions among agents occur at the sampling times t_k [132], but in real applications the communications among continuous-time dynamic agents can happen in real time. To deal with this problem, we introduce the impulsive consensus protocols and show that the system reaches consensus under appropriate conditions.

3. Since time-delay and external perturbations cannot be avoidable in some practical problems and the communication delays have not been studied in hybrid multi-agent system yet, so the consensus problems of hybrid multi-agent system with communication delays and impulsive delays have been studied. In addition, scaled consensus problems of multi-agent systems with external perturbation are also investigated.

4. Compared with the usual consensus problems focus on reaching an agreement on a common quantity, the scaled consensus problem means that the state of each agent will converge to a prescribed ratio in the asymptote, which implies the generalization of consensus. In addition, by selecting appropriate scalar scales, the scaled consensus problem can solve the group consensus problems, bipartite consensus problems, etc.

5. As mentioned above, an impulsive control method is a powerful control method, in particular, when an agent exchanges information instantaneously and cannot be considered continuously. To the best of our knowledge and from the previous works [81, 20, 1, 2, 84, 21, 123, 69, 86, 11, 48, 61, 62, 83, 85, 128, 34, 35, 44], the impulsive control has not been studied on scaled consensus problems yet. This work aims to investigate scaled consensus problems by using the impulsive consensus protocols.

6. Inspired by the above discussion and aforementioned works [118, 119, 13, 94, 64, 56], this work also studies the finite-time scaled consensus problems of hybrid multi-agent systems via impulsive control and multi-agent systems with impulsive perturbations, respectively.

7. The communication among agents may change over the time because of link or node failure, package drops etc, which can happen randomly and cannot avoid in some practical problems. Hence, in this work, we extend the scaled consensus ideas to the hybrid multi-agent systems over random networks.

1.3 Thesis Organization

The organization of this thesis is summarized as follows.

In Chapter 1, the introduction of this thesis is given. Chapter 2 introduces the notations, the mathematical background information of algebraic graph theory, matrix theory and the Kronecker product, hybrid multi-agent systems, and consensus problems. Moreover, the stabilization, Razumikhin techniques, impulsive mechanism, some useful definitions, lemmas, and properties are also provided in this chapter.

In Chapter 3, consensus problems of hybrid multi-agent systems including with the nonlinear and linear dynamics have been investigated. Moreover, the consensus of hybrid multi-agent systems with and without communication delays have also been studied.

In Chapter 4, we study scaled consensus problems of hybrid multi-agent systems with and without communication delays, scaled consensus problems of multi-agent systems under fixed and switched topologies via impulsive protocols and also for the systems with external perturbations.

In Chapter 5, the finite-time consensus and finite-time scaled consensus problems of hybrid multi-agent systems have been studied, respectively. Moreover, the finite-time scaled consensus of multi-agent systems with impulsive perturbations have also been studied in this chapter.

In Chapter 6, we apply the ideas of scaled consensus problems into the hybrid multi-agent systems over random networks.

In Chapter 7, we summarize the results and also discuss some future research directions along the line of this thesis.

Chapter 2

Background

This chapter establishes the basic knowledge and notations to understand the thesis. Consensus is usually studied through graph theoretical methods and matrix theory. Hence, a short summary on the subjects are presented as follows.

2.1 Notations

Throughout the thesis, we denote by $\mathbb{R}$ the real number set, $\mathbb{N}$ the positive integer set, $\mathbb{R}^n$ the $n-$ dimensional real vector space. For a given vector $\mathbf{y} = [y_1, y_2, \ldots, y_n] \in \mathbb{R}^n$, $\mathbf{y}^T$ denotes its transpose. A vector is non-negative if all its elements are non-negative. The Euclidean norm (also called the vector magnitude, Euclidean length, or 2-norm) of a vector $\mathbf{y}$ with n elements is defined by $\|\mathbf{y}\| = \sqrt{\sum_{i=1}^{n} y_i^2}$. The $\mathbf{1}_n$ and $\mathbf{0}_n$ are the column vector with all entries are equal to one and zero, respectively. $\mathbf{I}_n$ is an $n-$dimensional identity matrix and the diagonal matrix with diagonal elements being $a_1, a_2, ..., a_n$ is denoted by diag $\{a_1, a_2, ..., a_n\}$.

Furthermore, $\mathbb{R}^{n\times n}$ stands for the set of $n \times n$ matrix, for the matrix $\mathbf{B} = [b_{ij}]_{n\times n} \in \mathbb{R}^{n\times n}$, $\mathbf{B}^T$ and $\mathbf{B}^{-1}$ denote its transpose and inverse of the matrix $\mathbf{B}$, respectively. The spectral norm of $\mathbf{B}$ is defined by $\|\mathbf{B}\| = \sqrt{\lambda_{max}(\mathbf{B}\mathbf{B}^T)}$. A matrix $\mathbf{B} = [b_{ij}]_{n\times n}$ is said to be non-negative, denoted by $\mathbf{B} \geq 0$, if all its entries are non-negative. If $\mathbf{B} > 0$, then $\mathbf{B}$ is symmetric positive definite, i.e., $\mathbf{x}^T\mathbf{B}\mathbf{x} > 0$, for all $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0$. The notation $\otimes$ denotes the Kronecker product.

2.2 Algebraic Graph Theory

Generally, the consensus problem is strongly related to graph theory and use intensely graph theoretical methods for description and analysis of networks (e.g., [74, 80, 73]). Hence, some basic notions of the subject are presented based on the quoted works and specialized books as [15, 39] in this section. The following definitions are modified for the purpose of this thesis from the standard notions of graph theory.

Definition 2.2.1. A directed graph (or digraph) with n nodes is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where

- $\mathcal{V} = \{v_1, v_2, ..., v_n\}$ is a set of vertices,
- $\mathcal{E} = \{e_{ij} = (v_i, v_j)\} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of edges, and
- $\mathcal{A} = [a_{ij}]_{n \times n}$ is a weighted adjacency matrix.

Note that in a directed graph, all edges have unique direction and each edge is described as an ordered pair of vertices (v_i, v_j) representing and edge that starts at vertex i and terminates at vertex j, and there is one-way adjacency between the ordered pairs (see Figure 2.1 for the example of a directed graph).

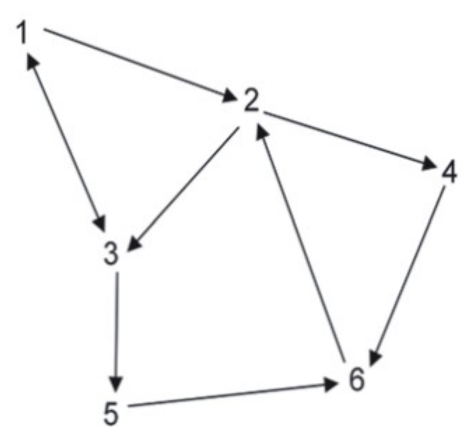

Figure 2.1: A directed graph $\mathcal{G}$ with 6 vertices

On the other hands, an undirected graph is described to be a graph with collection of vertices connected in a way where all the edges are bidirectional, and there is no orientation. In other words, if we consider the ordered pair (v_i, v_j) is the same as (v_j, v_i). Figure 2.2 shows the example of the undirected graph.

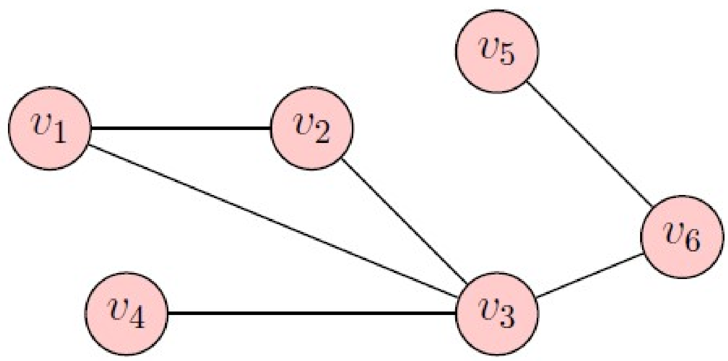

Figure 2.2: An undirected graph with 6 vertices

In the context of this thesis, the nodes correspond to agents, and the information flow from agent i to j is represented by the edge of a communication network denoted by e_{ij} or (v_i, v_j). Moreover, in this thesis we do not consider information flow from an agent to itself and assume that underlying information graph is simple.

Definition 2.2.2. The in-degree of node v_i denoted by $deg_{in}(v_i)$ is defined as the number of edges which are coming into v_i and the out-degree of v_i denoted by $deg_{out}(v_i)$ is the number of its outgoing edges.

Definition 2.2.3. A graph is said to be balanced if the out-degree and in-degree of each node are equal i.e., $deg_{in}(v_i) = deg_{out}(v_i)$, for all $v_i \in \mathcal{V}$.

Definition 2.2.4. The set of all neighbors of a node $v_i \in \mathcal{V}$ in a directed graph $\mathcal{G}$ is denoted by $N_i = \{j : (v_i, v_j) \in \mathcal{E}\}$.

Definition 2.2.5. A directed path of $\mathcal{G}$ is a sequence of edges $(v_1, v_2), (v_2, v_3), (v_3, v_4), \ldots$ in a digraph $\mathcal{G}$.

Definition 2.2.6. A digraph $\mathcal{G}$ is said to be strongly connected if there is a directed path connecting any two arbitrary nodes in $\mathcal{G}$.

Definition 2.2.7. A directed tree is a digraph such that there is only one root (that is, no edge points to this vertex) in it, and every vertex except the root has exactly one parent. Moreover, if a directed tree connects all the vertices of a graph $\mathcal{G}$, it is called as a directed spanning tree. Obviously, a graph may have more than one spanning tree (Figure 2.3 shows an example of a spanning tree of a digraph $\mathcal{G}$ in Figure 2.1).

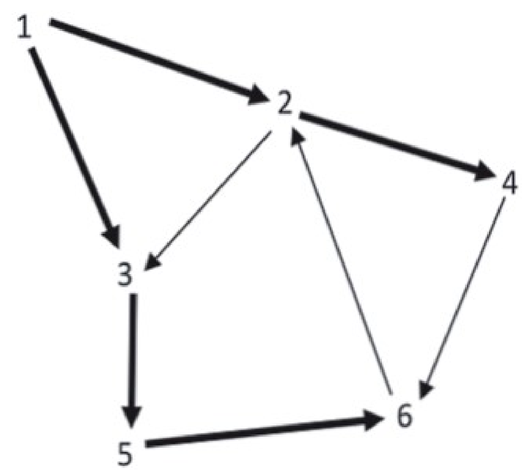

Figure 2.3: A spanning tree of $\mathcal{G}$ in Figure 2.1

Furthermore, an undirected tree is the undirected graph where all the nodes can be connected by the way of a single undirected path. A directed tree is defined as spanning when it connects all the nodes in the graph and a graph is said to have or contain a directed spanning tree if a subset of the edges forms a directed spanning tree. This is equivalent to saying that the graph has at least one node with directed paths to all other nodes. For undirected graphs, the existence of a directed spanning tree is equivalent to being connected. However, in directed graphs, the existence of a directed spanning tree is a weaker condition than being strongly connected. A strongly connected graph contains at least one directed spanning tree.

Associated with the communication graph is its **adjacency matrix** denoted by $\mathcal{A} = [a_{ij}]_{n\times n}$, where the element a_{ij} denotes the connection between the agent i and agent j. As mentioned earlier, in our work graph is simple (there is no self-loops) and therefore $a_{ii} = 0$. Thus, for $i \neq j$,

$$a_{ij} = \begin{cases} 1, & if \quad (v_i, v_j) \in \mathcal{E} \\ 0, & otherwise. \end{cases}$$

It can be seen that the structure of a weighted graph can be represented by adjacency matrix. Hence, one can study, using algebraic graph theory, all the properties of a graph by only looking at its associated adjacency matrix. Two of these properties are weighted in-degree and out-degree of node v_i that are defined respectively as follows

$$deg_{in}(v_i) = \sum_{i=1}^{n} a_{ji}$$

and

$$deg_{out}(v_i) = \sum_{i=1}^{n} a_{ij}.$$

Additionally, if the adjacency matrix of a graph is symmetric, then the graph is undirected. Another matrix that we may assign to a weighted graph is **Laplacian matrix** and it is one of the most important matrices in studying of multi-agent systems. It is denoted by $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D}$ denotes diagonal weighted out-degree matrix defined by $\mathcal{D} = [d_{ij}]_{n\times n}$, where

$$d_{ij} = \begin{cases} deg_{out}(i), & if \quad i = j \\ 0, & otherwise. \end{cases}$$

For example, consider the undirected graph $\mathcal{G}$ with 6 vertices as shown in Figure 2.4. Then,the degree matrix of $\mathcal{G}$ can be demonstrated as follows:

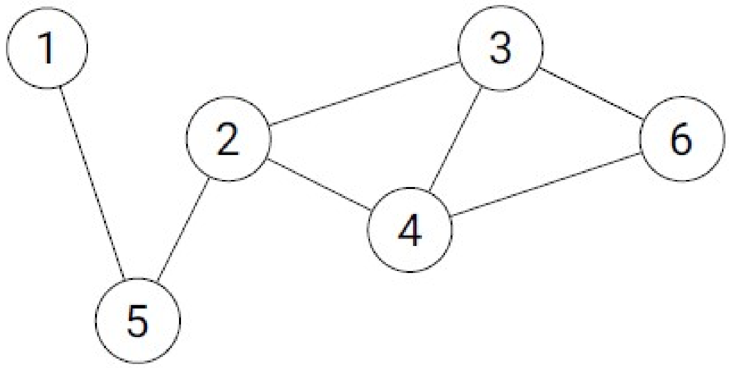

Figure 2.4: An undirected graph $\mathcal{G}$ with 6 vertices

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Hence, the Laplacian matrix of $\mathcal{G}$ can be calculate as follows:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 3 & -1 & -1 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & -1 & -1 & 3 & 0 & -1 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{bmatrix}$$

The following properties are some characteristics of the Laplacian matrix for undirected connected graph $\mathcal{G}$ that are applicable to the multi-agent systems obtained from [113]:

- $\mathcal{L}$ is a symmetric matrix.
- $\mathcal{L}$ consists of n non-negative, real-valued eigenvalues ($0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$), where the minimum eigenvalue is zero.
- Row sum of the Laplacian matrix for each row is equal to zero i.e., $\mathcal{L}\mathbf{1}_n = 0$ with a vector $\mathbf{1}_n = (1, \ldots, 1)^T$. In addition, since $\mathcal{G}$ is undirected and connected, the column sum of the Laplacian matrix for each column is equal to zero i.e., $\mathbf{1}_n^T\mathcal{L} = 0$, where $\mathbf{1}_n = (1, \ldots, 1)^T$.

2.3 Matrix Theory

In this section, some mathematical notations and basic definitions from matrix theory that will be used in the remainder of this thesis are provided. The main references in this section are [39, 60].

2.3.1 Stochastic Matrix

For the set of nonnegative matrices, we define an order as follows:

- The matrix $\mathbf{A} = [a_{ij}]$ is said to be a non-negative matrix if all the elements of $\mathbf{A}$ are equal to or greater than zero i.e., $a_{ij} \geq 0$ for all i, j.

- If $\mathbf{A}$ and $\mathbf{B}$ are non-negative matrices, then $\mathbf{A} \geq \mathbf{B}$ implies $\mathbf{A} - \mathbf{B}$ is a nonnegative matrix.
- $\mathbf{A}$ is a stochastic matrix if $\mathbf{A}$ is non-negative and all its row sums are equal to one.
- A stochastic matrix $\mathbf{P}$ is called indecomposable and aperiodic (SIA) if there exists a column vector $\mathbf{y}$ such that $\lim_{k\to\infty} \mathbf{P}^k = \mathbf{1}_n\mathbf{y}^T$, where $\mathbf{1}_n = (1, 1, ..., 1)^T$ is a $n \times 1$ vector.

The following results are interesting lemmas corresponding to the Stochastic matrix and the SIA matrix that will be used in this thesis.

Lemma 2.3.1. [80] Let $\mathcal{A} = [a_{ij}]_{n\times n}$ be a stochastic matrix. If $\mathcal{A}$ has an eigenvalue $\lambda = 1$ with algebraic multiplicity equal to one, and all the other eigenvalues satisfy $|\lambda| < 1$, then $\mathcal{A}$ is SIA, that is,

$$\lim_{k\to\infty} \mathcal{A}^k = \mathbf{1}_n\mathbf{y}^T,$$

where $\mathbf{y}$ satisfies $\mathcal{A}^T\mathbf{y} = \mathbf{y}$ and $\mathbf{1}_n^T\mathbf{y} = 1$. Furthermore, each element of $\mathbf{y}$ is non-negative.

Lemma 2.3.2. [132] Let $\mathbf{H} = diag\{h_1, h_2, ..., h_n\}$ and $0 < h_i < \frac{1}{d_{ii}}, \quad i \in \mathcal{I}_n$. Then,

$$\mathbf{I}_n - \mathbf{H}\mathcal{L} \text{ is SIA, i.e., } \lim_{k\to\infty} [\mathbf{I}_n - \mathbf{H}\mathcal{L}]^k = \mathbf{1}_n\mathbf{y}^T$$

if and only if graph $\mathcal{G}$ has a spanning tree. Furthermore, $[\mathbf{I}_n - \mathbf{H}\mathcal{L}]^T\mathbf{y} = \mathbf{y}$, $\mathbf{1}_n^T\mathbf{y} = 1$ where each element of $\mathbf{y}$ is non-negative.

2.3.2 Kronecker product

The Kronecker product is an operation performed on two matrices or vectors with a random size in which results in a block matrix. This operation is denoted with symbol $\otimes$. For example, let $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $\mathbf{B} = [b_{ij}] \in \mathbb{R}^{p\times q}$. Then, the Kronecker product of $\mathbf{A}$ and $\mathbf{B}$ denoted by $\mathbf{A} \otimes \mathbf{B}$ is defined as follows [91]:

$$\begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix} \tag{2.1}$$

Let $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{m\times n}$, $\mathbf{B} = [b_{ij}] \in \mathbb{R}^{p\times q}$, $\mathbf{C} = [c_{ij}] \in \mathbb{R}^{n\times k}$ and $\mathbf{D} = [d_{ij}] \in \mathbb{R}^{q\times r}$, some important properties of the Kronecker product used in this thesis are as follows [39]

- $\mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) = \mathbf{A} \otimes \mathbf{B} + \mathbf{A} \otimes \mathbf{C}$
- $(\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes \mathbf{C} + \mathbf{B} \otimes \mathbf{C}$
- $(\beta\mathbf{A}) \otimes \mathbf{B} = \mathbf{A} \otimes (\beta\mathbf{B}) = \beta\mathbf{A} \otimes \mathbf{B}$
- $(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C})$
- $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$
- $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$
- $(\mathbf{A} \otimes \mathbf{B}) \otimes (\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$
- If $\mathbf{A}$ and $\mathbf{B}$ are (semi-)positive definite, then $\mathbf{A} \otimes \mathbf{B}$ is also (semi-)positive definite.

2.4 Multi-agent systems

In this section the correlation between algebraic graph theory and cooperative control of multi-agent is presented. To introduce hybrid multi-agent systems we first need to define agents.

Definition 2.4.1. An agent is a dynamical system with a state vector which evolves through time based on its past value and a control input vector. Here, the state of the agents is not dependent on any other agent, but control input is a function of the agent and some other agents state vectors.

Definition 2.4.2. A multi-agent system is a set of agents that exchange information and collaborate to each other based on a common control strategy to achieve a goal as a single entity which cannot be done by each agent alone.

As mentioned in the previous chapter, multi-agent systems have been attractively studied in the recent years because of various applications in many areas (see [92, 16, 72, 73, 126, 10]). If the communication among agents allows continuous information sharing, the system could be modelled as a continuous-time multi-agent system and the dynamics of each agent can be described as follows[80]:

$$\dot{x}_i(t) = u_i(t), \tag{2.2}$$

on the other hand, if the communication among agents allows discrete-time information sharing, the system could be modelled as a discrete-time multi-agent system and the dynamics of each agent can be described as follows[80]:

$$x_i(k+1) = x_i(k) + u_i(k), k \in \mathbb{N}, \tag{2.3}$$

where $x_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ are the state and control input of agent i at time t, respectively.

Next, we introduce the formal definition of hybrid multi-agent systems based on the continuous-time and discrete-time dynamics agents as follows.

Definition 2.4.3. A hybrid multi-agent system is a set of agents consisting of continuous-time dynamics agents and discrete-time dynamics agents that can exchange information and collaborate to each other based on a control strategy to achieve a goal.

2.5 Consensus problem

According to the previous section, one of the most attractive problems in multi-agent systems is consensus that aims to design distributed control protocols to drive a group of agents to achieve agreement on states, such as position and velocity[80]. Next, some important definitions from [80] that are the key to understand consensus algorithm are introduced.

Definition 2.5.1. (Distributed Control Protocols [80]):
The control given by $u_i = g_i(x_{i_1}, x_{i_2}, \ldots, x_{i_{m_i}})$ for some function $g_i(\cdot)$ is said to be distributed if $m_i < N$ for all i, that is, the control input of each node depends on some proper subset of all the nodes. It is said to be a protocol with topology $\mathcal{G}$ if $u_i = g_i(x_i, x_j | j \in N_i)$, that is, each node can obtain information about the state only of itself and its neighbors in N_i

Definition 2.5.2. (Consensus Problem [80]): Find a distributed control protocol that drives all states to the common value, that is, $x_i = x_j, \forall i, j$: this value is called consensus value.

Definition 2.5.3. The protocol u_i is said to solve the consensus problem if for any initial conditions,

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \quad \text{for all } \ i, j.$$

In the recent years, many consensus algorithms were proposed to solve the consensus problems based on the dynamic model of agents [40, 116, 55, 6].

2.6 Useful Definitions Lemmas and Properties

In this section, we introduce some useful definitions, Lemmas and properties that are used in this thesis.

Definition 2.6.1. [26] (Mirror Graphs) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted digraph and $\tilde{\mathcal{E}}$ be the set of reverse edges of $\mathcal{G}$ obtained by reversing the order of nodes of all the pair in $\mathcal{E}$. The mirror of $\mathcal{G}$ is denoted by $\hat{\mathcal{G}} = (\mathcal{V}, \hat{\mathcal{E}}, \hat{\mathcal{A}})$ with the same set of nodes as $\mathcal{G}$, the set of edges $\hat{\mathcal{E}} = \mathcal{E} \bigcup \tilde{\mathcal{E}}$, and the symmetric adjacency matrix $\hat{\mathcal{A}} = [\hat{a}_{ij}]_{n\times n}$ with elements

$$\hat{a}_{ij} = \hat{a}_{ji} = \frac{a_{ij} + a_{ji}}{2} \geq 0.$$

Definition 2.6.2. [15] Let $\mathcal{G}$ be an undirected graph with the Laplacian matrix $\mathcal{L}$, the algebraic connectivity is defined as

$$\lambda_2(\mathcal{L}) = \min_{x\neq 0,\ 1^T x=0} \frac{x^T \mathcal{L} x}{x^T x},$$

where $\lambda_2(\mathcal{L})$ is the second smallest eigenvalue of $\mathcal{L}$.

Definition 2.6.3. [23] Let $\mathbf{A}$ be a Hermitian positive definite matrix of size n. For n nonzero vectors $p_1, \ldots, p_m \in \mathbb{R}^n$, if

$$< \mathbf{A}p_i, p_j >= 0, \quad i \neq j, \quad i, j = 1, 2, \ldots, m,$$

then $p_1, \ldots, p_m$ is called $\mathbf{A}$-conjugate, where $< \cdot, \cdot >$ denote the vector inner product.

Proposition 1. [82] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a digraph with an adjacency matrix $\mathcal{A} = [a_{ij}]$ satisfying $a_{ii} = 0, \ \ \forall i$. Then, all the following statements are equivalent:
$i)$ $\mathcal{G}$ is balanced,
$ii)$ $1^T \mathcal{L} = 0$, and
$iii)$ $\sum_{i=1}^n u_i = 0, \ \ \forall x \in \mathbb{R}^n$ with $u = -\mathcal{L}x$.

Lemma 2.6.1. [74] Let $\mathcal{L}$ be the Laplacian matrix of a directed graph $\mathcal{G}$ and $\hat{\mathcal{G}}$ be the mirror graph of $\mathcal{G}$. Then

$$\hat{\mathcal{L}} = Sym(\mathcal{L}) = \frac{1}{2}(\mathcal{L} + \mathcal{L}^T)$$

is a valid Laplacian matrix for $\hat{\mathcal{G}}$ if and only if $\mathcal{G}$ is balanced.

Next, lemmas regarding the properties of the Laplacian matrix and having a spanning tree of a network play an important role in the analysis of consensus of multi-agent systems:

Lemma 2.6.2. [79] Given a directed graph $\mathcal{G}$ with Laplacian Matrix $\mathcal{L}$, $\mathcal{L}$ has at least one zero eigenvalue with an associated eigenvector $\mathbf{1}_n$, and all the nonzero eigenvalues are in the open right half plane. Furthermore, $\mathcal{L}$ has exactly one zero eigenvalue if and only if the $\mathcal{G}$ contains a directed spanning tree.

Lemma 2.6.3. [111] Let $P_1, P_2, \ldots, P_n$ be a finite set of SIA matrices with the property that, for each sequence $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ of positive length, the matrix product $P_{i_1} \cdot P_{i_2} \cdots P_{i_k}$ is SIA. Then, for each infinite sequence $P_{i_1}, P_{i_2}, \ldots, P_{i_k}, \ldots$, there exists a column vector y such that $\lim_{k\to\infty} P_{i_k} \cdot P_{i_{k-1}} \cdots P_{i_1} = 1_n y^T$.

Lemma 2.6.4. [40] Let $\mathcal{G}$ be a digraph and $\mathcal{G}_{i_1}, \mathcal{G}_{i_2}, \ldots, \mathcal{G}_{i_k}$ be directed graphs. If $\bigcup_{j=1}^{k} \mathcal{G}_{i_j} \subset \mathcal{G}$ has a spanning tree, then the matrix product $P_{i_k} \cdot P_{i_{k-1}} \cdots P_{i_1}$ is SIA, where P_{i_j} is a stochastic matrix corresponding to each directed graph $\mathcal{G}_{i_j}$.

Lemma 2.6.5. [40] Let $\bar{\mathcal{G}}$ be a set of all possible interaction graphs for the multi-agent networks based on the impulsive system. If $\bigcup_{j=1}^{k} \mathcal{G}_{i_j} \subset \bar{\mathcal{G}}$ has a spanning tree and P_{t_i} is a stochastic matrix corresponding to each directed graph $\mathcal{G}_{t_i}$, then the matrix product $e^{P_{t_k}\Delta t_k} \cdot e^{P_{t_{k-1}}\Delta t_{k-1}} \cdots e^{P_{t_1}\Delta t_1}$ is SIA, where $\Delta t_i > 0$ are bounded.

Lemma 2.6.6. [80] A stochastic matrix has algebraic multiplicity equal to one for eigenvalue $\lambda = 1$ if and only if the graph associated with matrix has a spanning tree. Furthermore, a stochastic matrix with positive diagonal elements has the property that $|\lambda| < 1$ for every eigenvalue not equal to one.

Lemma 2.6.7. [80] Let $\mathbf{A} = [a_{ij}]_{n\times n}$ be a stochastic matrix. If $\mathbf{A}$ has an eigenvalue $\lambda = 1$ with algebraic multiplicity equal to one, and all the other eigenvalues satisfy $|\lambda| < 1$, then $\mathbf{A}$ is SIA, that is, $\lim_{k\to\infty} \mathbf{A}^k = \mathbf{1}_n \mathbf{y}^T$,where $\mathbf{y}$ is nonnegative and satisfies $\mathbf{A}^T\mathbf{y} = \mathbf{y}$, $\mathbf{1}_n^T \mathbf{y} = 1$.

Lemma 2.6.8. [41] Suppose function $\phi : \mathbb{R}^2 \to \mathbb{R}$ satisfies $\phi(x_i, x_j) = -\phi(x_j, x_i)$, $i, j \in \mathcal{I}_N, i \neq j$. Then, for any undirected graph $\mathcal{G}$ and a set of numbers $y_1, y_2, \ldots, y_N$,

$$\sum_{i=1}^{N} \sum_{j\in N_i} a_{ij} y_i \phi(x_j, x_i) = -\frac{1}{2} \sum_{(v_i,v_j)\in\mathcal{E}} a_{ij}(y_j - y_i)\phi(x_j, x_i).$$

Lemma 2.6.9. [49] For $x_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $\quad 0 < p \leq 1$, then

$$\Big(\sum_{i=1}^{n} |x_i|\Big)^p \leq \sum_{i=1}^{n} |x_i|^p \leq n^{1-p} \Big(\sum_{i=1}^{n} |x_i|\Big)^p.$$

Lemma 2.6.10. [72] For a connected undirected graph $\mathcal{G}$, the Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ has the following properties.

1. For any $\mathbf{x} = [x_1, \dots, x_n]^T \in \mathbb{R}^n$,

$$\mathbf{x}^T \mathcal{L} \mathbf{x} = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x_j - x_i)^2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \in N_i} a_{ij}(x_j - x_i)^2;$$

2. 0 is a simple eigenvalue of $\mathcal{L}$ and $\mathbf{1}_n$ is the associated eigenvector;

3. If the eigenvalues of $\mathcal{L}$ are denoted by $0 \leq \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$, then the second smallest eigenvalue $\lambda_2 > 0$. Furthermore, if $\mathbf{1}_n^T \mathbf{x} = 0$, then $\mathbf{x}^T \mathcal{L} \mathbf{x} \geq \lambda_2 \mathbf{x}^T \mathbf{x}$.

Lemma 2.6.11. [135] Let $\mathcal{L}$ be the Laplacian matrix of a connected undirected graph $\mathcal{G}$ with N vertices and $0 \leq \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$ be the eigenvalues of $\mathcal{L}$. Then,
(1) the eigenvalues of $\mathcal{L}^2$ are $0 \leq \lambda_1^2 < \lambda_2^2 \leq \cdots \leq \lambda_N^2$;
(2) if $\eta \geq 1$, then $\mathcal{L}^2 \leq \eta \lambda_N \mathcal{L}$.

Lemma 2.6.12. [73] Consider a network of agents $x_i(k+1) = x_i(k) + u_i(k)$ with topology $\mathcal{G}$ applying the distributed consensus algorithm

$$x_i(k+1) = x_i(k) + h \sum_{j=1}^{n} a_{ij}[x_j(k) - x_i(k)],$$

where $0 < h < 1/\Delta$ and Δ is the maximum degree of the network.
Let $\mathcal{G}$ be a strongly connected digraph. Then,
(i) a consensus is asymptotically reached for all initial states;
(ii) the group decision value is $\bar{x} = \sum_i w_i x_i(0)$, where $\sum_i w_i = 1$;
(iii) if the digraph is balanced, an average-consensus is asymptotically reached and $\bar{x} = \dfrac{1}{n} \sum_i x_i(0)$.

Lemma 2.6.13. [5] Let $\mathbf{A}$ be a Hermitian positive definite matrix of size n. Then the conjugate gradient algorithm finds the solution of $\mathbf{A}\mathbf{x} = b$ within n iterations in the absence of roundoff errors.

Lemma 2.6.14. [15, 74] Let $\mathcal{L}$ be the Laplacian of an undirected graph $\mathcal{G}$ with N vertices, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ be the eigenvalues of $\mathcal{L}$. Then
(1) 0 is an eigenvalue of $\mathcal{L}$ and $\mathbf{1}_N$ is the associate eigenvector, that is, $\mathcal{L}\mathbf{1}_N = 0$;
(2) If $\mathcal{G}$ is connected, then $\lambda_1 = 0$ is the algebraically simple eigenvalue of $\mathcal{L}$ and
(3) If 0 is the simple eigenvalue of $\mathcal{L}$, then it is an $n-$multiplicity eigenvalue of $\mathcal{L} \otimes I_n$ and the corresponding eigenvalues are $\mathbf{1}_N \otimes e_i, i = 1, 2, \dots, n$.

2.7 Stabilization

As one of the most crucial issues in dynamical system analysis, stability of solutions of the dynamical systems has been widely studied in the dynamic systems and control community(see, e.g., [31, 9, 71, 100, 102, 29, 101, 76]).

Generally, solving consensus problem in multi-agent systems, we can always transform the network models into the equivalent consensus error system, then study the asymptotic stability of the trivial state of the error system. It has been shown that the Lyapunov's stability method can be extended to analyze the stability of dynamical systems by employing a positive definite function $V(t)$. Thus, the consensus problem of hybrid multi-agent systems becomes the stability problem of the error system. However, due to the complexities of node dynamics and topological structures of networks, all the nodes cannot achieve the goal themselves. Therefore, appropriate consensus protocols are introduced to guarantee reaching consensus of the systems.

2.7.1 Stability via Razumikhin techniques

In this section, we introduce the stabilization of the systems via Razumikhin techniques being used to prove one of our main results.

Let $t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ with $lim_{k\to\infty} t_k = \infty$. Consider the impulsive functional differential equation

$$\begin{cases} x'(t) = f(t, x_t), & t \geq t_0, \\ x(t_k) = J_k(x(t_k^-)), & k \in \mathbb{N}, \end{cases} \tag{2.4}$$

where $\mathbb{N}$ is the set of all positive integers, $f : [t_0, \infty) \times PC \to \mathbb{R}^n$ and $J_k(x) : S(\rho) \to \mathbb{R}^n$ for each $k \in \mathbb{N}$, and $PC = PC([-\tau, 0], \mathbb{R}^n) = \{\phi : [-\tau, 0] \to \mathbb{R}^n, \phi(t)$ is continuous everywhere except at a finite nuber of points $\bar{t}$ at which $\phi(\bar{t}^+)$ and $\phi(\bar{t}^-)$ exist and $\phi(\bar{t}^+) = \phi(\bar{t})\}$, $S(\rho) = \{x \in \mathbb{R}^n : |x| < \rho\}$, and $x'(t)$ denotes the right-hand derivative of $x(t)$.
For each $t \geq t_0$, $x_t \in PC$ is defined by $x_t(r) = x(t + r)$, $-\tau \leq r \leq 0$. For $\phi \in PC$, the norm of ϕ is defined by $\|\phi\| = \sup_{-\tau \leq r \leq 0} |\phi(r)|$, where $|\cdot|$ denotes the norm of the vector in $\mathbb{R}^n$.

Throughout this work, we assume that there exists a $\rho_1 > 0$, $\rho_1 \leq \rho$ such that $x \in S(\rho_1)$ implies $J_k(x) \in S(\rho)$ for all $k \in N$. Let $K, K^*$and Ω be defined by

$K = \{w \in C(R^+, R^+),$ strictly increasing and $w(0) = 0\}$,

$K^* = \{\psi \in K, \psi(s) < s$ for $s > 0\}$,

$\Omega = \{H \in C(R^+, R^+), H(0) = 0, H(s) > 0$ for $s > 0\}$.

Lemma 2.7.1. [125]Assume that there exist functions $V \in V_0,\ w_1, w_2 \in K, \psi \in K^*$ and $H \in \Omega$ such that

(i) $w_1(|x|) \leq V(t,x) \leq w_2(|x|)$, for $(t,x) \in [t_0,\infty) \times S(\rho)$

(ii) for all $x \in S(\rho_1)$ and $k \in N$,

$$V(t_k, J_k(x)) \leq \psi(V(t_k^-, x)).$$

(iii) For any solution $x(t)$ of Eq.(2.4), $V(t+s, x(t+s)) \leq \psi^{-1}(V(t,x(t)))$, $-\tau \leq s \leq 0$, implies that

$$D^+V(t,x(t)) \leq g(t)H(V(t,x(t))),$$

where $g : [t_0,\infty) \to R^+$ locally integrable, ψ^{-1} is the inverse function of ψ.

(iv) H is nondecreasing and the exist constants $\alpha_2 \geq \alpha_1 > 0$ and $\eta > 0$ such that for all $k \in N$ and $\mu > 0$,

$$\alpha_1 \leq t_k - t_{k-1} \leq \alpha_2 \text{ and } \int_{\psi(\mu)}^{\mu} \frac{du}{H(u)} - \int_{t_{k-1}}^{t_k} g(s)ds \geq \eta.$$

Then, the zero solution of Eq.(2.4) is uniformly asymptotically stable.

2.7.2 Impulsive mechanism

Discrete-time linear impulsive system

Consider the following impulsive system

$$\begin{cases} x(t_{k+1}) = Ax(t_k), & t_k \neq t_l, \\ \Delta x(t_l) = x(t_l^+) - x(t_l^-) = B_k x(t_l), & t_k = t_l, \quad k,l \in \mathbb{N}, \end{cases} \tag{2.5}$$

where $x(t_k) \in \mathbb{R}^n,\ A, B_k \in \mathbb{R}^{n\times n}$. The discrete time instant t_k satisfy $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \ldots$ and $\lim_{k\to\infty} t_k = \infty$. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim\limits_{t\to t_k^+} x(t_k)$ and $x(t_k^-) = \lim\limits_{t\to t_k^-} x(t_k)$. Without loss of generality, we assume that $\lim\limits_{t\to t_k^+} x(t_k) = x(t_k)$, which implies that the solution $x(t,t_0,x_0)$ is right continuous at time t_k. Then,

$$x(t,t_0,x_0) = A^{(t-t_k)} \prod_{i=1}^{k} (I_n + B_i) A^{(t_i - t_{i-1})} x(0), \tag{2.6}$$

where $t_k \leq t < t_{k+1}, k \in \mathbb{N}_+$.

Lemma 2.7.2. [46] All solutions of system (2.5) are asymptotically stable if the following conditions are satisfied:
(H1) $0 < \alpha_1 \leq t_k - t_{k-1} \leq \alpha_2 < \infty$,
(H2) $||(I_n + B_k)A^{(t_k - t_{k-1})}|| \leq b < 1, k \in \mathbb{N}_+$.

Continuous-time linear impulsive system

Consider the following impulsive system

$$\begin{cases} \dot{x}(t) = Ax(t), & t \neq t_k, \\ \Delta x(t_k) = B_k x(t_k), & k \in \mathbb{N}, \end{cases} \tag{2.7}$$

where $x(t) \in \mathbb{R}^n$, $A, B_k \in \mathbb{R}^{n \times n}$. The discrete time instant t_k satisfy $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \ldots$ and $\lim_{k\to\infty} t_k = \infty$. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim\limits_{t \to t_k^+} x(t_k)$ and $x(t_k^-) = \lim\limits_{t \to t_k^-} x(t_k)$. Without loss of generality, we assume that $\lim\limits_{t \to t_k^+} x(t_k) = x(t_k)$, which implies that the solution $x(t, t_0, x_0)$ is right continuous at time t_k. Then,

$$x(t, t_0, x_0) = e^{A(t-t_k)} \prod_{i=1}^{k} (I_n - B_i) e^{A(t_i - t_{i-1})} x(0), \tag{2.8}$$

where $t_k \leq t < t_{k+1}, k \in \mathbb{N}_+$.

Lemma 2.7.3. [46] All solutions of system (2.7) are asymptotically stable if the following conditions are satisfied:
(H1) $0 < \alpha_1 \leq t_k - t_{k-1} \leq \alpha_2 < \infty$,
(H2) $||(I_n + B_k)e^{A(t_k - t_{k-1})}|| \leq b < 1, k \in \mathbb{N}_+$.

Lemma 2.7.4. [70] Consider the nonlinear impulsive dynamical system given by

$$\begin{cases} \dot{\mathbf{x}}(t) = f_c(\mathbf{x}(t)), \ \mathbf{x}(0) = x_0, \ (t, \mathbf{x}(t)) \notin \mathcal{S} \ \text{ for } \ t \in \mathcal{I}_{x_0}, \\ \Delta \mathbf{x}(t) = f_d(\mathbf{x}(t)), \ \ (t, \mathbf{x}(t)) \in \mathcal{S}, \end{cases} \tag{2.9}$$

where $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f_c : \mathcal{D} \to \mathbb{R}^n$ is continuous, $f_d : \mathcal{S} \to \mathbb{R}^n$ is continuous, and $\mathcal{S} \subset [0, \infty) \times \mathcal{D}$ is the resetting set.
Assume there is a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}_+$, satisfying

$V(0) = 0,\ V(\mathbf{x}) > 0,\ x \in \mathcal{D},\ x \neq 0$, and

$$\begin{cases} V'(\mathbf{x}) f_c(\mathbf{x}) \leq -\gamma (V(\mathbf{x}))^{\alpha},\ \mathbf{x} \notin \mathcal{Z}, \\ V(\mathbf{x} + f_d(\mathbf{x}) \leq V(\mathbf{x}),\ \ \mathbf{x} \in \mathcal{Z}, \end{cases} \tag{2.10}$$

where $\mathcal{Z} \subset \mathcal{D}$, $\gamma > 0,\ \alpha \in (0, 1)$. Then the zero solution $\mathbf{x}(t) \equiv 0$ to (2.9) is finite-time stable. In addition, if $\mathcal{D} = \mathbb{R}^n,\ V(\cdot)$ is radially unbounded, then the zero solution $\mathbf{x}(t) \equiv 0$ to (2.9) is globally finite-time stable.

Chapter 3

Consensus of Hybrid multi-agent systems

This chapter studies consensus problems of hybrid multi-agent systems (HMASs) under fixed topologies. By introducing the impulsive consensus protocols to the continuous-time dynamics agents with and without communication delays, the consensus problems of HMASs are, respectively, solved in Section 3.2 and Section 3.3. In Section 3.2, the consensus protocol for hybrid multi-agent systems with no communication delays is proposed. In Section 3.3, the consensus protocol for hybrid multi-agent systems with communication delays is studied. In particular, we assume that the communication between continuous-time dynamic agents have delays, but the communication among discrete-time dynamic agents has no delay and the communication between the discrete-time dynamic agents and the continuous-time dynamic agents has also no delay. In addition, we also study consensus problems when the dynamics of agents are linear (called linear hybrid multi-agent systems (LHMASs)) and nonlinear (called nonlinear hybrid multi-agent systems (NHMASs)) in Section 3.4 and Section 3.5, respectively. In Section 3.6, numerical examples are provided to illustrate the effectiveness of the theoretical results.

3.1 Problem formulation

In this section, we assume that the hybrid multi-agent system consists of n agents which are continuous-time and discrete-time dynamic agents, labelled 1 through n, where the number of continuous-time dynamic agents is c, $c < n$. Without loss of generality, we assume that agent 1 through c are continuous-time dynamic agents. Moreover, $\mathcal{I}_c = \{1, 2, 3, ..., c\}$, $\mathcal{I}_n \backslash \mathcal{I}_c = \{c+1, c+2, c+3, ..., n\}$. Then, each agent has the dynamics as follows:

$$\begin{cases} \dot{x}_i(t) = u_i(t), & \text{for} \quad i \in \mathcal{I}_c, \\ x_l(t_{k+1}) = x_l(t_k) + u_l(t_k), \quad t_k = kh, \quad k \in \mathbb{N} & \text{for} \quad l \in \mathcal{I}_n \backslash \mathcal{I}_c, \end{cases} \tag{3.1}$$

where h is the sampling period, $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the state and control input of agent i, respectively. The initial conditions are $x_i(0) = x_{i0}$, and $x(0) = [x_{10}, x_{20}, ..., x_{n0}]^T$.

Moreover, the hybrid multi-agent system (3.1) is modelled as a connected directed graph, where all agents are regarded as the nodes and the interaction between two agents has been represented by the edge in a graph. This implies that $(v_i, v_j) \in \mathcal{E}$ corresponds to an available information link from agent i to agent j. Besides, each agent updates its current state based on the information received from its neighbours. Furthermore, we suppose that there exists communication behaviour as in hybrid multi-agent system (3.1), that is, there are agent i and agent j which make $a_{ij} > 0$.

Definition 3.1.1. The hybrid multi-agent system (3.1) is said to reach consensus if for any initial conditions,

$$\lim_{t_k \to \infty} \|x_i(t_k) - x_j(t_k)\| = 0, \quad \text{for} \quad i, j \in \mathcal{I}_n, \tag{3.2}$$

and

$$\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0, \quad \text{for} \quad i, j \in \mathcal{I}_c. \tag{3.3}$$

According to the results of Zheng et al.[132], two classes of consensus protocols were proposed for solving consensus in hybrid multi-agent systems if all agents communicate with their neighbours and update their control inputs in the sampling time t_k. In this section, the consensus problems of hybrid multi-agent systems with and without communication delays have been studied when the continuous-time dynamic agents can interact with their neighbors in real time. By using graph theory, matrix theory and Lyapunov method, the consensus results of hybrid multi-agent systems can be guaranteed under some necessary and sufficient conditions.

3.2 Consensus of HMASs with no communication delays

In this section, we assume that all continuous-time dynamic agents communicate with their neighbours and update their control inputs in real time, while all discrete-time dynamic agents communicate with their neighbours and update their control inputs at a sampling time t_k. In addition, the interactions between the discrete-time dynamic agents and the continuous-time dynamic agents happen only at $t = t_k$.

In this work, we assume that the hybrid multi-agent system (3.1) has been modelled as a connected digraph $\mathcal{G} = \mathcal{G}_c \cup \mathcal{G}_d \cup \mathcal{G}'$, where $\mathcal{G}_c,\ \mathcal{G}_d,\ \mathcal{G}'$ are the communication networks of continuous-time dynamic agents, discrete-time dynamic agents, and the interactions between each other, respectively. Then the consensus protocol for the hybrid multi-agent system (3.1) is defined as follows: for $t \in (t_{k-1}, t_k]$,

$$\begin{cases} u_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}[x_j(t) - x_i(t)] + \sum_{k=1}^{\infty} \sum_{s \in \mathcal{N}_i'} a_{is}'[x_s(t) - x_i(t)]\delta(t - t_k), & \text{for} \quad i \in \mathcal{I}_c \\ u_l(t_k) = h \sum_{j \in \mathcal{N}_l} b_{lj}[x_j(t_k) - x_l(t_k)], & \text{for} \quad l \in \mathcal{I}_n/\mathcal{I}_c \end{cases} \tag{3.4}$$

where $\mathcal{A} = [a_{ij}]$ and $\mathcal{B} = [b_{lj}]$ are the weighted adjacency matrices associated with the graph $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$, respectively. Moreover, $h = t_k - t_{k-1}$ is the sampling period, $\mathcal{N}_i$ and $\mathcal{N}_i'$ are the neighbor sets of i in $\mathcal{G}_c \cup \mathcal{G}'$ at time $t \neq t_k$ and $t = t_k$, respectively. $\mathcal{N}_l$ is a neighbor set of agent l in $\mathcal{G}_d \cup \mathcal{G}'$ at time t_k and $\delta(\cdot)$ is the Dirac delta function; i.e.,

$$\delta(t - t_k) = \begin{cases} 1, & t = t_k \\ 0, & t \neq t_k. \end{cases}$$

To establish our main results, some assumptions are provided as follows:

(A1) $0 < h < \dfrac{1}{max_{i \in \mathcal{I}_n}\{d_{ii}\}}$;

(A2) there exists a constant $0 < \alpha \leq 1$ such that

$$(1 - \alpha)\mathbf{I} - \mathcal{L}' - \mathcal{L}'^T + \mathcal{L}'^T \mathcal{L}' \leq 0,$$

where $\mathcal{L}'$ is the Laplacian matrix of $\mathcal{G}_c \cup \mathcal{G}'$ at $t = t_k$.

Now, we are in the position to introduce our main result.

Theorem 3.2.1. Let $\mathcal{G}$ be a directed connected communication network of the hybrid multi-agent system (3.1). Assume that the assumptions **(A1)** and **(A2)** hold. Then, the hybrid multi-agent system (3.1) with the protocol (3.4) reaches consensus if and only if $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$ are both balanced and contain a spanning tree.

Proof. **(Sufficiency)** First of all, consider for each $i \in \mathcal{I}_c$. Without loss of generality, we assume that all discrete-time dynamic agents have interacted with some continuous-time dynamic agents. Hence, the system (3.1) with the protocol (3.4) can be described as an impulsive system on the communication network $\mathcal{G}_c \cup \mathcal{G}'$ with n nodes, where $n = |\mathcal{G}_c \cup \mathcal{G}'|$.

For simplicity of presentation, agents which maintain communication with agent i for a period of time are called as regular neighbors of agent i, while the agents which maintain information exchange at impulsive time are called impulsive neighbors of agent i. The sets of regular neighbors and impulsive neighbors of agent i are denoted by $\mathcal{N}_i$ and $\mathcal{N}_i'$, respectively. In addition, let $\mathcal{D}$ be diagonal matrix with the out-degree of each vertex along the diagonal, where the out-degree of node i is denoted by $\sum_{j\in\mathcal{N}_i} a_{ij}$. Then, the Laplacian matrix of $\mathcal{G}_c \cup \mathcal{G}'$ at $t \neq t_k$ is denoted by $\mathcal{L} = [l_{ij}]_{n\times n}$, defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where

$$l_{ij} = \begin{cases} \displaystyle\sum_{j\in\mathcal{N}_i} a_{ij}, & i = j \\ -a_{ij}, & i \neq j. \end{cases}$$

On the other hand, for $t = t_k$, the out-degree of node i is denoted by $\sum_{j\in\mathcal{N}_i'} a_{ij}'$. Then, the Laplacian matrix of $\mathcal{G}_c \cup \mathcal{G}'$ at $t = t_k$ is denoted by $\mathcal{L}' = [l_{ij}']_{n\times n}$, where

$$l_{ij}' = \begin{cases} \displaystyle\sum_{j\in\mathcal{N}_i'} a_{ij}', & i = j \\ -a_{ij}', & i \neq j. \end{cases}$$

Hence, for $i \in \mathcal{I}_c$, the system (3.1) with the protocol (3.4) can be written as an impulsive system on the communication network $\mathcal{G}_c \cup \mathcal{G}'$ with n nodes as follows:

$$\begin{cases} \dot{x}_i(t) = \displaystyle\sum_{j\in\mathcal{N}_i} a_{ij}[x_j(t) - x_i(t)], \quad t \neq t_k, \\ \Delta x_i(t_k) = \displaystyle\sum_{j\in\mathcal{N}_i'} a_{ij}'[x_j(t_k) - x_i(t_k)], \end{cases} \tag{3.5}$$

where $t \in \mathbb{R}^+$, $x_i(t) \in \mathbb{R}$ is the state of agent i at time t, $i = 1, 2, ...,$ $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$: $x_i(t_k^+) = \lim\limits_{\epsilon\to 0^+} x_i(t_k + \epsilon)$ and $x_i(t_k^-) = \lim\limits_{\epsilon\to 0^+} x_i(t_k - \epsilon)$.

This implies that an agent i can intermittently update its state on the basis of the state information of itself and its neighbours at time t_k. Without loss of generality, we assume that $\lim\limits_{\epsilon \to 0^+} x_i(t_k - \epsilon) = x_i(t_k)$, that is, $x_i(t_k)$ is left-continuous. The sequence $\{t_k\}$ satisfies $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$.

Letting $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, the system (3.5) can be written as the form:

$$\begin{cases} \dot{x}(t) = -\mathcal{L}x(t), & t \neq t_k, \\ \Delta x(t) = -\mathcal{L}'x(t), & t = t_k, \qquad k \in \mathbb{N} \end{cases} \tag{3.6}$$

where $\mathcal{L}$ and $\mathcal{L}'$ are the Laplacian matrix of $\mathcal{G}_c \cup \mathcal{G}'$ when $t \neq t_k$ and $t = t_k$, respectively. Since $\mathcal{G}_c \cup \mathcal{G}'$ is balanced, from the consequence of Proposition 4 [18], then $\bar{x} = Ave(x) = \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} x_j$ is invariant quantity i.e.,

$$\bar{x}(t) = \bar{x}(0) = \frac{1}{n}\sum_{j=1}^{n} x_j(0),$$

which is not true for an arbitrary digraph. The invariant of $\bar{x}$ allows decomposition of x_i for $i = 1, 2, ..., n$ as in the following equation:

$$\eta_i(t) = x_i(t) - \bar{x}, \quad t \in (t_{k-1}, t_k],$$

$\eta_i(t_k^+) = x_i(t_k^+) - \bar{x}$ and $\eta_i(t_k^-) = \eta_i(t_k)$, $\; i = 1, 2, 3, ..., n$, with initial conditions $x(t_0) = x(0) = [x_{10}, x_{20}, ..., x_{n0}]^T$, where $\eta = (\eta_1, \ldots, \eta_n)^T$ is the error vector or disagreement vector. Thus,

$$\begin{cases} \dot{\eta}(t) = -\mathcal{L}\eta(t), & t \neq t_k \\ \eta(t_k^+) = [\mathbf{I} - \mathcal{L}']\eta(t_k), & t = t_k, \quad k \in \mathbb{N}. \end{cases} \tag{3.7}$$

Consider the Lyapunov function candidate as follows:

$$V(\eta) = \eta^T \eta.$$

Let $V(\eta) =: V(\eta(t))$. Since $\mathcal{G}_c \cup \mathcal{G}'$ is balanced, by Lemma 2.6.1 [74] $\hat{\mathcal{L}} = Sym(\mathcal{L}) = (\mathcal{L} + \mathcal{L}^T)/2$, the total derivation of $V(\eta)$ with respect to (3.7) is

$$\begin{aligned} \dot{V}(t) &= \dot{\eta}^T(t)\eta(t) + \eta^T(t)\dot{\eta}(t) \\ &= -\eta^T(t)(\mathcal{L}^T + \mathcal{L})\eta(t) \\ &= -2\eta^T(t)(\hat{\mathcal{L}})\eta(t) \quad t \in (t_{k-1}, t_k). \end{aligned}$$

Since the mirror graph $\hat{\mathcal{G}}$ of $\mathcal{G}_c \cup \mathcal{G}'$ is a connected undirected graph, the definition 2.6.2 [15] gives us $\min\limits_{x\neq 0,\ 1^T x=0} \dfrac{x^T \hat{\mathcal{L}} x}{x^T x} = \lambda_2(\hat{\mathcal{L}})$. Hence,

$$\dot{V}(t) \leq -2\lambda_2(\hat{\mathcal{L}})V(t) \quad \text{for} \quad t \in (t_{k-1}, t_k),$$

which implies that, for $t \in (t_{k-1}, t_k]$,

$$V(t) \leq e^{-2\lambda_2(\hat{\mathcal{L}})(t-t_{k-1})} V(t_{k-1}^+).$$

On the other hand, when $t = t_{k-1}$, by using $(1-\alpha)\mathbf{I} - \mathcal{L}' - \mathcal{L}'^T + \mathcal{L}'^T\mathcal{L}' \leq 0$, for $0 < \alpha \leq 1$, one obtains

$$\begin{aligned}
V(t_{k-1}^+) &= \eta^T(t_{k-1})(\mathbf{I} - \mathcal{L}')^T(\mathbf{I} - \mathcal{L}')\eta(t_{k-1}) \\
&= \eta^T(t_{k-1})[\mathbf{I} - \mathcal{L}'^T - \mathcal{L}' + \mathcal{L}'^T\mathcal{L}' - \alpha\mathbf{I} + \alpha\mathbf{I}]\eta(t_{k-1}) \\
&= \eta^T(t_{k-1})[(1-\alpha)\mathbf{I} - \mathcal{L}'^T - \mathcal{L}' + \mathcal{L}'^T\mathcal{L}']\eta(t_{k-1}) + \alpha\eta^T(t_{k-1})\eta(t_{k-1}) \\
&\leq \alpha\eta^T(t_{k-1})\eta(t_{k-1}) \\
&= \alpha V(t_{k-1}).
\end{aligned}$$

In general, for $t \in (t_{k-1}, t_k]$,

$$V(t) \leq \alpha^{k-1} e^{-2\lambda_2(\hat{\mathcal{L}})(t-t_0^+)} V(t_0^+).$$

Hence, for $t \in (t_{k-1}, t_k]$,

$$|\eta(t)| \leq \alpha^{(k-1)/2} e^{-\lambda_2(\hat{\mathcal{L}})(t-t_0)} |\eta(t_0^+)|.$$

Thus,

$$\|x_i(t) - \bar{x}\| \to 0 \quad \text{as} \quad t \to \infty \quad \text{or} \quad \lim_{t\to\infty} x_i(t) = \bar{x}, \quad \forall i \in \mathcal{I}_c.$$

This implies that, for $t \in (t_{k-1}, t_k]$,

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0 \qquad \text{for} \quad i, j \in \mathcal{I}_c. \tag{3.8}$$

Now, we will show that

$$\lim_{t_k\to\infty} \|x_i(t_k) - x_j(t_k)\| = 0 \qquad \text{for} \quad i, j \in \mathcal{I}_n.$$

Consider, for $i, j \in \mathcal{I}_n$,

$$\|x_i(t_k) - x_j(t_k)\| \leq \|x_i(t_k) - x_i(t)\| + \|x_i(t) - x_j(t)\| + \|x_j(t) - x_j(t_k)\|. \tag{3.9}$$

The proof can be separated into three cases as follows:
Case 1. If $i, j \in \mathcal{I}_c$, the above discussion gives

$$\lim_{t_k \to \infty} x_i(t_k) = \lim_{t \to \infty} x_i(t) = \bar{x}, \quad \forall i \in \mathcal{I}_c.$$

This implies that

$$\lim_{t_k \to \infty} \|x_i(t_k) - x_j(t_k)\| = 0 \qquad \text{for} \;\; i, j \in \mathcal{I}_c \subset \mathcal{I}_n. \tag{3.10}$$

Case 2. If $i, j \in \mathcal{I}_n \backslash \mathcal{I}_c = \{c+1, c+2, ..., n\}$, the problem can be simplified by considering the communication network of $\mathcal{G}_d \cup \mathcal{G}'$. Since the discrete-time dynamic agents interact with their neighbours at time $t = t_k$, one obtains

$$x_i(t_{k+1}) = x_i(t_k) + h \sum_{(i,j) \in \mathcal{E}'} b_{ij}[x_j(t_k) - x_i(t_k)], \tag{3.11}$$

where $h = t_k - t_{k-1}$ is a sampling period, $\mathcal{E}'$ is the set of edges and $B = [b_{ij}]_{r \times r}$ is the adjacency matrix of $\mathcal{G}_d \cup \mathcal{G}'$, where $|\mathcal{G}_d \cup \mathcal{G}'| = r \leq n$ is the number of the discrete-time dynamic agents and continuous-time dynamic agents that interact with them.
Letting $x(t_k) = [x_1(t_k), x_2(t_k), ..., x_r(t_k)]^T$, the equation (3.11) can be written as

$$x(t_{k+1}) = [\mathbf{I}_r - h\mathcal{L}_d]x(t_k),$$

where I_r is an identity matrix and $\mathcal{L}_d$ is the Laplacian matrix of $\mathcal{G}_d \cup \mathcal{G}'$.
According to Lemma 2.3.2, since $\mathcal{G}_d \cup \mathcal{G}'$ has a directed spanning tree and $h < \dfrac{1}{max_{i \in \mathcal{I}_n}\{d_{ii}\}}$, there exists a column vector y such that

$$\lim_{k \to \infty} [\mathbf{I}_r - h\mathcal{L}_d]^k = \mathbf{1}\mathbf{y}^T \;\; \text{where} \;\; [\mathbf{I}_r - h\mathcal{L}_d]^T \mathbf{y} = \mathbf{y}.$$

Thus,

$$\lim_{t_k \to \infty} x(t_k) = \lim_{k \to \infty} [\mathbf{I}_r - h\mathcal{L}_d]^k x(0) = \mathbf{1}\mathbf{y}^T x(0) \;\; \text{and} \;\; \mathcal{L}_d^T \mathbf{y} = 0.$$

This implies that

$$\lim_{t_k \to \infty} \|x_i(t_k) - x_j(t_k)\| = 0 \qquad \text{for} \;\; i, j \in \mathcal{I}_n \backslash \mathcal{I}_c. \tag{3.12}$$

Moreover, there exists a column vector y such that

$$\lim_{t_k\to\infty} x_i(t_k) = \mathbf{y}^T x(0) \quad \text{for all} \quad i \in \mathcal{I}_n \backslash \mathcal{I}_c.$$

Case 3. If $j \in \mathcal{I}_n \backslash \mathcal{I}_c$ and $i \in \mathcal{I}_c$ (or $i \in \mathcal{I}_n \backslash \mathcal{I}_c$ and $j \in \mathcal{I}_c$), we consider, for $i, l \in \mathcal{I}_c$ and $j \in \mathcal{I}_n \backslash \mathcal{I}_c$,

$$\begin{aligned}\|x_i(t) - x_l(t)\| \le &\|x_i(t) - x_i(t_k)\| + \|x_i(t_k) - x_j(t_k)\| + \|x_j(t_k) - x_l(t_k)\| \\ &+ \|x_l(t_k) - x_l(t)\|.\end{aligned}$$

Since, for $i, l \in \mathcal{I}_c$,

$$\lim_{t\to\infty} \|x_i(t) - x_l(t)\| = 0, \quad \text{and} \quad \lim_{t\to\infty} x_i(t) = \bar{x}, \quad \forall i \in \mathcal{I}_c.$$

When $t \to \infty$, we have $t_k \to \infty$. Thus,

$$\lim_{t_k\to\infty} \|x_i(t) - x_i(t_k)\| = 0 \quad \text{and} \quad \lim_{t_k\to\infty} \|x_l(t_k) - x_l(t)\| = 0.$$

This implies that $\lim\limits_{t_k\to\infty} \|x_i(t_k) - x_j(t_k)\| = 0$ and $\lim\limits_{t_k\to\infty} \|x_j(t_k) - x_l(t_k)\| = 0$.
Hence,

$$\lim_{t_k\to\infty} \|x_i(t_k) - x_j(t_k)\| = 0, \quad \text{for} \quad j \in \mathcal{I}_n \backslash \mathcal{I}_c \quad \text{and} \quad i \in \mathcal{I}_c.$$

From **Case 1,2** and **3**, we can conclude that

$$\lim_{t_k\to\infty} \|x_i(t_k) - x_j(t_k)\| = 0 \qquad \text{for} \quad i, j \in \mathcal{I}_n. \tag{3.13}$$

Therefore, from (3.8) and (3.13), the hybrid multi-agent system (3.1) with protocol (3.4) reaches consensus.
(Necessity) Suppose that $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$ are not balanced and do not contain a spanning tree. Then, by Lemma 2.3.2, we have $\lim\limits_{k\to\infty} [\mathbf{I} - h\mathcal{L}_d]^k \neq \mathbf{1}\mathbf{y}^T$. Hence,

$$\lim_{t_k\to\infty} \|x_i(t_k) - x_j(t_k)\| \neq 0 \qquad \text{for} \quad i, j \in \mathcal{I}_n.$$

This implies that the hybrid multi-agent system (3.1) cannot achieve consensus. □

Remark. It can be seen that if $c = n$, then the hybrid multi-agent systems can reduce as a continuous-time dynamic system. On the other hand if $c = 0$, the hybrid multi-agent systems is a discrete-time dynamic system.

Remark. It is easy to see that the results from Theorem 3.2.1 are more general than the results of Zheng *et al* [132], the interactions among agents are assumed to occur only at the sampling time t_k.

3.3 Consensus of HMASs with communication delays

In this section, the consensus problems of hybrid multi-agent systems with communication delays have been studied, where the information (from v_i to v_j for all $i, j \in \mathcal{I}_c$) passes through edge (v_i, v_j) with the coupling time delays $\tau(t)$. Here, we assume that the communications among discrete-time dynamic agents have no delays and the communications between the discrete-time dynamic agents and the continuous-time dynamic agents have also no delays. Then, the consensus protocol for hybrid multi-agent system (3.1) is defined as follows: for $t \in (t_{k-1}, t_k]$,

$$\begin{cases} u_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}[x_j(t - \tau(t)) - x_i(t - \tau(t))] \\ \qquad\quad + \sum_{k=1}^{\infty} \sum_{s \in \mathcal{N}'_i} a'_{is}[x_s(t) - x_i(t)]\delta(t - t_k), & \text{for} \quad i \in \mathcal{I}_c \\ u_l(t_k) = h \sum_{j \in \mathcal{N}_l} b_{lj}[x_j(t_k) - x_l(t_k)], & \text{for} \quad l \in \mathcal{I}_n \backslash \mathcal{I}_c, \end{cases} \tag{3.14}$$

where $\mathcal{A} = [a_{ij}]$ and $\mathcal{B} = [b_{lj}]$ are the weighted adjacency matrices associated with the graph $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$, respectively. Moreover, $h = t_k - t_{k-1}$ is the sampling period, $\mathcal{N}_i$ and $\mathcal{N}'_i$ are the neighbor sets of i in $\mathcal{G}_c \cup \mathcal{G}'$ at time $t \neq t_k$ and $t = t_k$, respectively. $\mathcal{N}_l$ is a neighbor set of agent l in $\mathcal{G}_d \cup \mathcal{G}'$ at time t_k and $\delta(\cdot)$ is the Dirac delta function.

To establish our main results, some assumptions are provided as follows:

(A1) $0 < h < \dfrac{1}{max_{i \in \mathcal{I}_n}\{d_{ii}\}}$;

(A2) there exists positive constants α, β such that for all $k \in \mathbb{N}$ the following conditions are satisfied:

(i) $[1 + 2\lambda_2(\hat{\mathcal{L}}) + \lambda_2(\mathcal{L}'\mathcal{L}'^T)] \cdot \|\mathcal{L}\| \leq \alpha$;

(ii) $ln[1 + 2\lambda_2(\hat{\mathcal{L}}) + \lambda_2(\mathcal{L}'\mathcal{L}'^T)] - \alpha(t_k - t_{k-1}) \geq \beta > 0$,

where $\hat{\mathcal{L}}$ is a symmetric matrix of $\mathcal{L}$, which has zero row sums ; $\mathcal{L}$ and $\mathcal{L}'$ are the Laplacian matrices of $\mathcal{G}_c \cup \mathcal{G}'$ when $t \neq t_k$ and $t = t_k$, respectively.
Now, we are in the position to introduce our main result.

Theorem 3.3.1. Let $\mathcal{G}$ be a communication network of the hybrid multi-agent system (3.1), which is undirected. Assume that the assumptions **(A1)** and **(A2)** are satisfied. Then, the hybrid multi-agent system (3.1) with the protocol (3.14) reaches consensus if and only if $\mathcal{G}$ is connected.

Proof. **(Sufficiency)** Consider a communication network $\mathcal{G} = \mathcal{G}_c \cup \mathcal{G}_d \cup \mathcal{G}'$ defined as a previous section. Since there are interactions among discrete-time dynamic agents and continuous-time dynamic agents. Hence, for $i \in \mathcal{I}_c$, the system (3.1) with the protocol (3.14) can be described as an impulsive system on the communication network $\mathcal{G}_c \cup \mathcal{G}'$ with r nodes, where $|\mathcal{G}_c \cup \mathcal{G}'| = r \leq n$ as follows:

$$\begin{cases} \dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}[x_j(t-\tau(t)) - x_i(t-\tau(t))], & t \in (t_{k-1}, t_k), \\ \Delta x_i(t_k) = \sum_{j \in \mathcal{N}_i'} a'_{ij}[x_j(t_k) - x_i(t_k)], \end{cases} \tag{3.15}$$

where $t \in \mathbb{R}^+$, $x_i(t) \in \mathbb{R}$ is the state of agent i at time t, $i = 1, 2, ..., r$. $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$: $x_i(t_k^+) = \lim_{h \to 0^+} x_i(t_k + h)$ and $x_i(t_k^-) = \lim_{h \to 0^+} x_i(t_k - h)$.

Letting $x = (x_1, x_2, ..., x_r)^T \in \mathbb{R}^r$, the system (3.15) can be written as the form:

$$\begin{cases} \dot{x}(t) = -\mathcal{L}x(t-\tau(t)), & t \neq t_k, \\ \Delta x(t) = -\mathcal{L}'x(t), & t = t_k, \qquad k \in \mathbb{N} \end{cases} \tag{3.16}$$

where $\mathcal{L}$ and $\mathcal{L}'$ are the Laplacian matrix of $\mathcal{G}_c \cup \mathcal{G}'$ when $t \neq t_k$ and $t = t_k$, respectively. Since $\mathcal{G}_c \cup \mathcal{G}'$ is connected, from the consequence of Proposition 4 [18], then $\bar{x} = Ave(x) = \frac{1}{r}\sum_{j=1}^{r} x_j$ is invariant quantity i.e.,

$$\bar{x}(t) = \bar{x}(0) = \frac{1}{r}\sum_{j=1}^{r} x_j(0),$$

which is not true for an arbitrary digraph. The invariant of $\bar{x}$ allows decomposition of x_i for $i = 1, 2, ..., r$ as in the following equation:

$$\xi_i(t) = x_i(t) - \bar{x}, \quad t \in (t_{k-1}, t_k],$$

$\xi_i(t_k^-) = x_i(t_k^-) - \bar{x}$ and $\xi_i(t_k^+) = \xi_i(t_k)$, $i = 1, 2, 3, ..., r$, with initial conditions $x(t_0) = x(0) = [x_{10}, x_{20}, ..., x_{r0}]^T$, where $\xi = (\xi_1, ..., \xi_r)^T$ is the error vector or disagreement vector. Thus,

$$\begin{cases} \dot{\xi}(t) = -\mathcal{L}\xi(t-\tau(t)), & t \neq t_k \\ [I + \mathcal{L}']\xi(t_k) = \xi(t_k^-), & t = t_k, \quad k \in \mathbb{N}. \end{cases} \tag{3.17}$$

Since the graph $\mathcal{G}_c \cup \mathcal{G}'$ is connected, it follows from Lemma 3.3 in [80] that the Laplacian $\mathcal{L}'$ has exactly one zero eigenvalue and the rest $n-1$ eigenvalues all have positive real-parts.

Furthermore, $\hat{\mathcal{L}} = \dfrac{1}{2}(\mathcal{L}' + \mathcal{L}'^T)$ is a symmetric matrix and has zero row sums. Thus, the eigenvalues of matrices $\hat{\mathcal{L}}$ and $\mathcal{L}'\mathcal{L}'^T$ can be ordered as

$$0 = \lambda_1(\hat{\mathcal{L}}) < \lambda_2(\hat{\mathcal{L}}) \le \cdots \le \lambda_r(\hat{\mathcal{L}}),$$

and

$$0 = \lambda_1(\mathcal{L}'\mathcal{L}'^T) < \lambda_2(\mathcal{L}'\mathcal{L}'^T) \le \cdots \le \lambda_r(\mathcal{L}'\mathcal{L}'^T).$$

On the other hand, since $\hat{\mathcal{L}}$ and $\mathcal{L}'\mathcal{L}'^T$ are symmetric, by definition 2.6.2, we know

$$\lambda_2(\hat{\mathcal{L}}) = \min_{\xi \neq 0,\ 1^T\xi = 0} \frac{\xi^T \hat{\mathcal{L}} \xi}{\xi^T \xi}. \tag{3.18}$$

$$\lambda_2(\mathcal{L}'\mathcal{L}'^T) = \min_{\xi \neq 0,\ 1^T\xi = 0} \frac{\xi^T (\mathcal{L}'\mathcal{L}'^T) \xi}{\xi^T \xi}. \tag{3.19}$$

Consider the Lyapunov function candidate as follows:

$$V(t, \xi(t)) = \frac{1}{2}\xi^T(t)\xi(t).$$

For $t = t_k$, for all $\xi(t) \in S(\rho_1),\ 0 < \rho_1 \le \rho$, we have

$$\begin{aligned}
\xi^T(t_k^-)\xi(t_k^-) &= \xi^T(t_k)(I + \mathcal{L}'^T)(I + \mathcal{L}')\xi(t_k) \\
&= \xi^T(t_k)[I + \mathcal{L}' + \mathcal{L}'^T + \mathcal{L}'^T\mathcal{L}']\xi(t_k) \\
&\ge \left[1 + 2\lambda_2(\hat{\mathcal{L}}) + \lambda_2(\mathcal{L}'\mathcal{L}'^T)\right]\xi^T(t_k)\xi(t_k).
\end{aligned} \tag{3.20}$$

That is

$$V(t_k, \xi(t_k)) \le \frac{1}{\left[1 + 2\lambda_2(\hat{\mathcal{L}}) + \lambda_2(\mathcal{L}'\mathcal{L}'^T)\right]} V(t_k^-, \xi(t_k^-)).$$

Let

$$\psi(t) = \frac{t}{\left[1 + 2\lambda_2(\hat{\mathcal{L}}) + \lambda_2(\mathcal{L}'\mathcal{L}'^T)\right]},$$

then $\psi(t)$ is strictly increasing and $\psi(0) = 0$ with $\psi(t) < t$ for all $t > 0$. Hence, the condition (ii) of Lemma 2.7.1 is satisfied. Also, by letting $w_1(|x|) = w_2(|x|) = \dfrac{|x|^2}{2}$, the condition (i) of Lemma 2.7.1 is satisfied. For any solution of (3.17), if

$$V(t - \tau(t), \xi(t - \tau(t))) \le \psi^{-1}(V(t, \xi(t))),$$

by calculating the upper Dini derivative of $V(t)$ along the solutions of Eqs. (3.17) and using the inequality $x^T y + y^T x \le \epsilon x^T x + \epsilon^{-1} y^T y$, one obtains

$$\begin{aligned} D^+V(t) &= -\xi^T(t)\mathcal{L}\xi(t-\tau(t)) \\ &\le \|\mathcal{L}\| \cdot \big[V(t,\xi(t)) + \sup_{t-\tau\le s\le t} V(s,\xi(s))\big] \\ &\le \big[1 + 2\lambda_2(\hat{\mathcal{L}}) + \lambda_2(\mathcal{L}'\mathcal{L}'^T)\big] \cdot \|\mathcal{L}\| V(t,\delta(t)) \\ &\le \alpha V(t,\xi(t)). \end{aligned} \tag{3.21}$$

Letting $g(t) \equiv 1$ and $H(t) = \alpha t$. Thus, the condition (iii) of Lemma 2.7.1 is satisfied. Moreover, the condition (**A2**)(ii) implies that

$$\begin{aligned} \int_{\psi(\mu)}^{\mu} \frac{du}{H(u)} - \int_{t_{k-1}}^{t_k} g(s)ds &= \frac{1}{\alpha}\{ln(\mu) - ln[\frac{\mu}{1 + 2\lambda_2(\hat{\mathcal{L}}) + \lambda_2(\mathcal{L}'\mathcal{L}'^T)}]\} - (t_k - t_{k-1}) \\ &= \frac{ln\big[1 + 2\lambda_2(\hat{\mathcal{L}}) + \lambda_2(\mathcal{L}'\mathcal{L}'^T)\big]}{\alpha} - (t_k - t_{k-1}) \\ &\ge \frac{\beta}{\alpha} > 0. \end{aligned} \tag{3.22}$$

The condition (iv) of Lemma 2.7.1 is satisfied. Therefore, all the conditions of Lemma 2.7.1 are satisfied. This implies that, for $t \in (t_{k-1}, t_k]$,

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0 \qquad \text{for } \ i,j \in \mathcal{I}_c. \tag{3.23}$$

On the other hand, since the communication of discrete-time dynamic agents has no delay and the communication between the discrete-time dynamic agents and the continuous-time dynamic agents has also no delay, then showing that

$$\lim_{t_k\to\infty} \|x_i(t_k) - x_j(t_k)\| = 0 \qquad \text{for } \ i,j \in \mathcal{I}_n.$$

can be done by using the similar way of proving in Theorem 3.2.1. Hence, the remain part will be omitted. Therefore, the hybrid multi-agent system (3.1) with protocol (3.14) reaches consensus.

(Necessity) Suppose that $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$ are not connected, which implies that there is no any spanning tree. Then, using similar idea in the proof of Necessity part of Theorem 3.2.1, the hybrid multi-agent system (3.1) cannot achieve consensus. □

3.4 Consensus of LHMASs via impulsive protocols

Consider a linear hybrid multi-agent system (LHMAS) consisting of N agents which are continuous-time and discrete-time dynamic agents, labelled 1 through N, where the number of continuous-time dynamic agents is c, $c < N$. Without loss of generality, we assume that agent 1 through c are continuous-time dynamic agents. Moreover, $\mathcal{I}_c = \{1, 2, 3, ..., c\}$, $\mathcal{I}_N \backslash \mathcal{I}_c = \{c+1, c+2, c+3, ..., N\}$. Then, each agent has the dynamics as follows:

$$\begin{cases} \dot{x}^i(t) = \mathbf{A}x^i(t) + u^i(t), & \text{for } \ i \in \mathcal{I}_c, \\ x^i(t_{k+1}) = \mathbf{C}x^i(t_k) + u^i(t_k), \quad t_k = kh, & \text{for } \ i \in \mathcal{I}_N \backslash \mathcal{I}_c, \end{cases} \tag{3.24}$$

where h is the sampling period, $x^i(t) = [x^i_1(t), x^i_2(t), ..., x^i_n(t)]^T \in \mathbb{R}^n$ and $u^i \in \mathbb{R}^n$ are the state and control input of agent i at time t, respectively. In this work, we assume that there exists communication behaviour as in hybrid multi-agent system (3.24), that is, there are agent i and agent j which make $a_{ij} > 0$ and the following assumption is provided to obtain the main results:

(A1): $\|\mathbf{A}\| \neq 0$ and $\|\mathbf{C}\| \neq 0$, where $\mathbf{A}, \mathbf{C} \in \mathbb{R}^{n \times n}$.

Definition 3.4.1. The hybrid multi-agent system (3.24) is said to reach consensus if for any initial conditions,

$$\lim_{t_k \to \infty} \|x^i(t_k) - x^j(t_k)\| = 0, \quad \text{for } \ i, j \in \mathcal{I}_N, \tag{3.25}$$

and

$$\lim_{t \to \infty} \|x^i(t) - x^j(t)\| = 0, \quad \text{for } \ i, j \in \mathcal{I}_c. \tag{3.26}$$

Assume that all agents communicate with their neighbors only at the sampling time t_k, then the dynamics of each agent is designed as

$$\begin{cases} u^i(t) = \sum_{k=1}^{\infty} \delta(t - t_k)\mathbf{B}_k \sum_{j \in \mathcal{N}_i} a_{ij}(x^j(t) - x^i(t)), & \text{for } \ i \in \mathcal{I}_c, \\ u^i(t_l) = \sum_{k=1}^{\infty} \delta(t_l - t_k)\mathbf{C}_k \sum_{j \in \mathcal{N}_i} a_{ij}(x^j(t_l) - x^i(t_l)), & \text{for } \ i \in \mathcal{I}_N \backslash \mathcal{I}_c, \end{cases} \tag{3.27}$$

where $\mathbf{B}_k \in \mathbb{R}^{n \times n}$ and $\mathbf{C}_k \in \mathbb{R}^{n \times n}$ are impulsive matrices to be determined later. The discrete time instant t_k satisfy $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \dots$ and $\lim_{k \to \infty} t_k = \infty$, $\delta(t)$ is the Dirac delta function.

To establish our main results, the following assumptions are provided:

(A2) There exist positive constants α_1, α_2 such that $0 < \alpha_1 \le t_k - t_{k-1} \le \alpha_2 < \infty$,

(A3) $\|(\mathbf{I} - \lambda_i^c \mathbf{B}_k) e^{A(t_k - t_{k-1})}\| \le \alpha_3 < 1, k \in \mathbb{N}_+$, for some $\alpha_3 > 0$,

(A4) $\|(\mathbf{I} + \lambda_i^d \mathbf{C}_k) \mathbf{A}^{(t_k - t_{k-1})}\| \le \alpha_4 < 1, k \in \mathbb{N}_+$, for some $\alpha_4 > 0$,

where λ_i^c and λ_i^d are the eigenvalues of the Laplacian matrix $\mathcal{L}_1 = \mathcal{L}(\mathcal{G}_c \cup \mathcal{G}')$ and $\mathcal{L}_2 = \mathcal{L}(\mathcal{G}_d \cup \mathcal{G}')$, respectively.

Theorem 3.4.1. Let $\mathcal{G}$ be a directed connected communication network of (3.24). Assume that the assumptions **(A1)-(A4)** hold. Then, the hybrid multi-agent system (3.24) with the protocol (3.27) reaches consensus if $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$ are both balanced and contain a spanning tree.

Proof. Assume that a digraph $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$ both balanced and contain spanning tree. For $i \in \mathcal{I}_c$, it can be seen that the system (3.24) with the protocol (3.27) can be described as an impulsive system on the communication network $\mathcal{G}_c \cup \mathcal{G}'$ with r nodes, where $|\mathcal{G}_c \cup \mathcal{G}'| = r \le N$. WLOG, we assume that all discrete-time dynamics agents have a communication with some continuous-time dynamics agents. Thus, $r = N$ and the dynamics of each agent can be described as follows:

$$\begin{cases} \dot{x}^i(t) = \mathbf{A} x^i(t), & t \ne t_k, \\ \Delta x^i(t_k) = \mathbf{B}_k \sum_{j \in \mathcal{N}_i} a_{ij}(x^j(t_k) - x^i(t_k)), & k \in \mathbb{N}. \end{cases} \tag{3.28}$$

Let $x(t) = [x^1(t), x^2(t), \ldots, x^N(t)]^T$, then the system (3.28) can be written as

$$\begin{cases} \dot{x}(t) = (\mathbf{I}_N \otimes \mathbf{A}) x(t), & t \ne t_k, \\ \Delta x(t_k) = (\mathbf{I}_N \otimes \mathbf{B}_k)(-\mathcal{L}_1 \otimes \mathbf{I}_n) x(t_k), & k \in \mathbb{N}. \end{cases} \tag{3.29}$$

Since $\mathcal{G}_c \cup \mathcal{G}'$ is strongly connected and balanced, then $\mathcal{L}_1 = \mathcal{L}(\mathcal{G}_c \cup \mathcal{G}')$ is symmetric. Then, there exists an orthogonal matrix $\mathbf{U} \in \mathbb{R}^{N \times N}$ such that

$$\mathbf{U} \mathcal{L}_1 \mathbf{U}^{-1} = \mathbf{U} \mathcal{L}_1 \mathbf{U}^T = \mathbf{D} = diag\{\lambda_1^c, \lambda_2^c, \ldots, \lambda_N^c\},$$

where $\{\lambda_i^c,\} = \sigma(\mathcal{L}_1)$ is the spectrum of $\mathcal{L}_1$. Inspired by Wang *et al* (2008)[99], let

$$\bar{x}(t) = (\mathbf{U} \otimes \mathbf{I}_n) x(t).$$

Thus

$$x(t) = (\mathbf{U} \otimes \mathbf{I}_n)^{-1}\bar{x}(t) = (\mathbf{U}^{-1} \otimes \mathbf{I}_n)\bar{x}(t).$$

Using the Kronecker product properties, we have when $t \in [t_k, t_{k+1}), k \in \mathbb{N}_+$,

$$\begin{aligned}
\frac{d\bar{x}(t)}{dt} &= (\mathbf{U} \otimes \mathbf{I}_n)\dot{x}(t) \\
&= (\mathbf{U} \otimes \mathbf{I}_n)(\mathbf{I}_N \otimes \mathbf{A})x(t) \\
&= (\mathbf{U} \otimes \mathbf{I}_n)(\mathbf{I}_N \otimes \mathbf{A})(\mathbf{U}^{-1} \otimes \mathbf{I}_n)\bar{x}(t) \\
&= (\mathbf{U}\mathbf{I}_N\mathbf{U}^{-1}) \otimes (\mathbf{I}_n\mathbf{A}\mathbf{I}_n)\bar{x}(t) \\
&= (\mathbf{I}_N \otimes \mathbf{A})\bar{x}(t)
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}
\Delta\bar{x}(t_k) &= (\mathbf{U} \otimes \mathbf{I}_n)\Delta x(t_k) \\
&= (\mathbf{U} \otimes \mathbf{I}_n)(\mathbf{I}_N \otimes \mathbf{B}_k)(-\mathcal{L}_1 \otimes \mathbf{I}_n)\bar{x}(t_k^-) \\
&= (\mathbf{U} \otimes \mathbf{I}_n)(\mathbf{I}_N \otimes \mathbf{B}_k)(-\mathcal{L}_1 \otimes \mathbf{I}_n)(\mathbf{U}^{-1} \otimes \mathbf{I}_n)\bar{x}(t_k^-) \\
&= (-\mathbf{U}\mathbf{I}_N\mathcal{L}_1\mathbf{U}^{-1}) \otimes (\mathbf{I}_n\mathbf{B}_k\mathbf{I}_n\mathbf{I}_n)\bar{x}(t_k^-) \\
&= (-\mathbf{D} \otimes \mathbf{B}_k)\bar{x}(t_k^-), \qquad k \in \mathbb{N}_+.
\end{aligned} \tag{3.31}$$

From (3.30) and (3.31), the system (3.29) becomes

$$\begin{cases} \dfrac{d\bar{x}(t)}{dt} = (\mathbf{I}_N \otimes \mathbf{A})\bar{x}(t), & t \neq t_k, \\ \Delta\bar{x}(t_k) = (-\mathbf{D} \otimes \mathbf{B}_k)\bar{x}(t_k^-), & k \in \mathbb{N}_+. \end{cases} \tag{3.32}$$

Therefore

$$\begin{cases} \dfrac{d\bar{x}^i(t)}{dt} = \mathbf{A}\bar{x}^i(t), & & t \neq t_k, \\ \Delta\bar{x}^i(t_k) = (-\lambda_i^c\mathbf{B}_k)\bar{x}^i(t_k^-), & i = 1, 2, \ldots, N, & k \in \mathbb{N}_+. \end{cases} \tag{3.33}$$

Since $\mathcal{G}_c \cup \mathcal{G}'$ contains a spanning tree, then $\lambda_1^c = 0$ is the algebraically simple eigenvalue of $\mathcal{L}_1$ and λ_i^c are positive for $i > 1$. Thus, we have

$$0 = \lambda_1^c < \lambda_2^c \leq \cdots \leq \lambda_N^c.$$

Assume that the $\lim_{t \to t_k^+} \bar{x}(t) = \bar{x}(t_k)$ i.e., the solution $\bar{x}(t)$ is right continuous at time t_k. Then

$$\bar{x}(t, t_0, x_0) = e^{\mathbf{A}(t-t_k)} \prod_{i=1}^{k} (\mathbf{I} - \lambda_i^c\mathbf{B}_k)e^{\mathbf{A}(t_i - t_{i-1})}x(0), \tag{3.34}$$

where $t_k \leq t < t_{k+1}, \quad k \in \mathbb{N}_+$.
Hence, by the assumptions **(A1)**, **(A2)** and Lemma 2.7.3, the system (3.33) is asymptotically stable i.e., $\bar{x}^i(t) \to 0$ as $t \to \infty, \quad i = 2, \dots, N$. It can be verified that

$$\begin{aligned}(\mathcal{L}_1 \otimes \mathbf{I}_n)x(t) &= (\mathbf{U} \otimes \mathbf{I}_n)^{-1}(\mathbf{U} \otimes \mathbf{I}_n)(\mathcal{L}_1 \otimes \mathbf{I}_n)(\mathbf{U}^{-1} \otimes \mathbf{I}_n)\bar{x}(t) \\ &= (\mathbf{U} \otimes \mathbf{I}_n)^{-1}(\mathbf{D} \otimes \mathbf{I}_n)\bar{x}(t) \\ &= (\mathbf{U} \otimes \mathbf{I}_n)^{-1}[0 \ \ \lambda_2^c \bar{x}^2(t) \cdots \lambda_N^c \bar{x}^N(t)]^T.\end{aligned}$$

Because $\bar{x}^i(t) \to 0$ as $t \to \infty, \quad i = 2, \dots, N$. So, $(\mathcal{L}_1 \otimes \mathbf{I}_n)x(t) \to 0$ as $t \to \infty$. Since $\mathcal{G}_c \cup \mathcal{G}'$ contains a spanning tree, by Lemma 2.6.14, 0 is the eigenvalue of $\mathcal{L}_1 \otimes \mathbf{I}_n$ with multiplicity n. The n linearly independent eigenvalues associated with the eigenvalue 0 of $\mathcal{L}_1 \otimes \mathbf{I}_n$ are $\mathbf{1}_N \otimes e_i, \quad i = 1, 2, \dots, n$. Therefore

$$x(t) \to \mathbf{1}_N \otimes s, \text{ as } t \to \infty; \ s = \sum_{i=1}^{n} \alpha_i e_i, \quad \alpha_i \in \mathbb{R}, \quad i = 1, 2, \dots, n.$$

This implies that

$$\lim_{t \to \infty} \|x^i(t) - x^j(t)\| = 0, \qquad \forall i, j \in \mathcal{I}_c.$$

Next, we will show that

$$\lim_{t \to \infty} \|x^i(t_k) - x^j(t_k)\| = 0, \qquad \forall i, j \in \mathcal{I}_N.$$

The proof can be separated into three cases as follows:

Case 1. If $i, j \in \mathcal{I}_c$, the above discussion gives

$$\lim_{t_k \to \infty} \|x^i(t_k) - x^j(t_k)\| = 0, \quad \forall i, j \in \mathcal{I}_c.$$

Case 2. If $i, j \in \mathcal{I}_N \backslash \mathcal{I}_c = \{c+1, c+2, ..., N\}$, the problem can be simplified by considering the communication network of $\mathcal{G}_d \cup \mathcal{G}'$. For each $i \in \mathcal{I}_N \backslash \mathcal{I}_c$, the dynamics of agent i can be described as the discrete-time linear multi-agent system (DLMAS) on the communication network $\mathcal{G}_d \cup \mathcal{G}'$ with d nodes where $|\mathcal{G}_d \cup \mathcal{G}'| = d \leq N$. For simplicity, we assume that all continuous-time dynamic agents have communications with some discrete-time dynamic agents. Thus, $|\mathcal{G}_d \cup \mathcal{G}'| = d = N$. From 3.24 and protocol 3.27, for each $i \in \mathcal{I}_N \backslash \mathcal{I}_c$ we have

$$x^i(t_{l+1}) = \mathbf{C}x^i(t_l) + \sum_{k=1}^{\infty} \delta(t_l - t_k)\mathbf{C}_k \sum_{j \in \mathcal{N}_i} a_{ij}(x^j(t_l) - x^i(t_l)), \ \text{ for } \ i \in \mathcal{I}_N \tag{3.35}$$

Let $x(t) = [x^1(t), x^2(t), \ldots, x^N(t)]^T$, then the system (3.24) can be written as

$$\begin{cases} x^i(t_{l+1}) = (\mathbf{I}_N \otimes \mathbf{C})x(t_l), & t_l \neq t_k, \\ \Delta x(t_l) = (\mathbf{I}_N \otimes \mathbf{C}_k)(-\mathcal{L}_2 \otimes \mathbf{I}_n)x(t_l), & k, l \in \mathbb{N}. \end{cases} \tag{3.36}$$

Since $\mathcal{G}_d \cup \mathcal{G}'$ is strongly connected and balanced, then $\mathcal{L}_2 = \mathcal{L}(\mathcal{G}_d \cup \mathcal{G}')$ is symmetric. Then, there exists an orthogonal matrix $\mathbf{W} \in \mathbb{R}^{N \times N}$ such that

$$\mathbf{W}\mathcal{L}_2\mathbf{W}^{-1} = \mathbf{W}\mathcal{L}_2\mathbf{W}^T = \mathbf{P} = diag\{\lambda_1^d, \lambda_2^d, \ldots, \lambda_N^d\},$$

where $\{\lambda_i^d\} = \sigma(\mathcal{L}_2)$ is the spectrum of $\mathcal{L}_2$. Motivated by Wang et al.(2008), let

$$\bar{x}(t_l) = (\mathbf{W} \otimes \mathbf{I}_n)x(t_l).$$

Thus

$$x(t_l) = (\mathbf{W} \otimes \mathbf{I}_n)^{-1}\bar{x}(t_l) = (\mathbf{W}^{-1} \otimes \mathbf{I}_n)\bar{x}(t_l).$$

Using the Kronecker product properties, we have when $t_l \neq t_k$ for $k, l \in \mathbb{N}_+$

$$\begin{aligned} \bar{x}(t_{l+1}) &= (\mathbf{W} \otimes \mathbf{I}_n)x(t_{l+1}) \\ &= (\mathbf{W} \otimes \mathbf{I}_n)(\mathbf{I}_N \otimes \mathbf{C})(\mathbf{W}^{-1} \otimes \mathbf{I}_n)\bar{x}(t_l) \\ &= (\mathbf{I}_N \otimes \mathbf{C})\bar{x}(t_l) \end{aligned} \tag{3.37}$$

and when $t_l = t_k$ for $k, l \in \mathbb{N}_+$, we have

$$\begin{aligned} \Delta\bar{x}(t_l) &= (\mathbf{W} \otimes \mathbf{I}_n)\Delta x(t_l) \\ &= (\mathbf{W} \otimes \mathbf{I}_n)(\mathbf{I}_N \otimes \mathbf{C}_k)(-\mathcal{L}_2 \otimes \mathbf{I}_n)(\mathbf{W}^{-1} \otimes \mathbf{I}_n)\bar{x}(t_l) \\ &= (-\mathbf{P} \otimes \mathbf{C}_k)\bar{x}(t_l), \qquad l \in \mathbb{N}_+. \end{aligned} \tag{3.38}$$

From (3.37) and (3.38), the system (3.36) becomes

$$\begin{cases} \bar{x}(t_{l+1}) = (\mathbf{I}_N \otimes \mathbf{C})\bar{x}(t_l), & t_l \neq t_k, \\ \Delta\bar{x}(t_l) = (-\mathbf{P} \otimes \mathbf{C}_k)\bar{x}(t_l), & l \in \mathbb{N}_+. \end{cases} \tag{3.39}$$

Therefore

$$\begin{cases} \bar{x}^i(t_{l+1}) = \mathbf{C}\bar{x}^i(t_l), & & t_l \neq t_k, \\ \Delta\bar{x}^i(t_l) = (-\lambda_i^d\mathbf{C}_k)\bar{x}^i(t_l), & i = 1, 2, \ldots, N, & l \in \mathbb{N}_+. \end{cases} \tag{3.40}$$

It can be seen that $\lambda_1^d = 0$ is the algebraically simple eigenvalue of $\mathcal{L}_2$ and the others are positive since $\mathcal{G}_d \cup \mathcal{G}'$ contains a spanning tree. Thus, we have

$$0 = \lambda_1^d < \lambda_2^d \leq \cdots \leq \lambda_N^d.$$

Assume that the $\lim\limits_{t \to t_k^+} \bar{x}(t) = \bar{x}(t_k)$ i.e., the solution $\bar{x}(t)$ is right continuous at time t_k. Then

$$\bar{x}(t, t_0, x_0) = \mathbf{C}^{(t-t_k)} \prod_{i=1}^{k} (\mathbf{I} - \lambda_i^d \mathbf{C}_k) \mathbf{C}^{(t_i - t_{i-1})} x(0), \tag{3.41}$$

where $t_l \leq t < t_{l+1}, \quad l \in \mathbb{N}_+$.

Hence, by the assumptions **(A1)**, **(A2)** and Lemma 2.7.2, the system (3.40) is asymptotically stable i.e., $\bar{x}^i(t_l) \to 0$ as $t_l \to \infty, \quad i = 2, \ldots, N$. It can be verified that

$$\begin{aligned}
(\mathcal{L}_2 \otimes \mathbf{I}_n) x(t_l) &= (\mathbf{W} \otimes \mathbf{I}_n)^{-1} (\mathbf{U} \otimes \mathbf{I}_n)(\mathcal{L}_2 \otimes \mathbf{I}_n)(\mathbf{W}^{-1} \otimes \mathbf{I}_n) \bar{x}(t_l) \\
&= (\mathbf{W} \otimes \mathbf{I}_n)^{-1} (\mathbf{P} \otimes \mathbf{I}_n) \bar{x}(t_l) \\
&= (\mathbf{W} \otimes \mathbf{I}_n)^{-1} [0 \quad \lambda_2^d \bar{x}^2(t) \cdots \lambda_N^d \bar{x}^N(t_l)]^T.
\end{aligned}$$

Because $\bar{x}^i(t_l) \to 0$ as $t_l \to \infty, \quad i = 2, \ldots, N$. So, $(\mathcal{L}_2 \otimes \mathbf{I}_n) x(t) \to 0$ as $t \to \infty$. Since $\mathcal{G}_d \cup \mathcal{G}'$ contains a spanning tree, by Lemma 2.6.14, 0 is the eigenvalue of $\mathcal{L}_2 \otimes \mathbf{I}_n$ with multiplicity n. The n linearly independent eigenvalues associated with the eigenvalue 0 of $\mathcal{L}_2 \otimes \mathbf{I}_n$ are $\mathbf{1}_N \otimes e_i, \quad i = 1, 2, \ldots, n$. Therefore

$$x(t_l) \to \mathbf{1}_N \otimes s, \text{ as } t_l \to \infty; \; s = \sum_{i=1}^{n} \alpha_i e_i, \quad \alpha_i \in \mathbb{R}, \quad i = 1, 2, \ldots, n.$$

This implies that

$$\lim_{t_k \to \infty} \|x^i(t_k) - x^j(t_k)\| = 0, \qquad \forall i, j \in \mathcal{I}_N \backslash \mathcal{I}_c.$$

Case 3. If $j \in \mathcal{I}_N \backslash \mathcal{I}_c$ and $i \in \mathcal{I}_c$ (or $i \in \mathcal{I}_N \backslash \mathcal{I}_c$ and $j \in \mathcal{I}_c$),
we consider, for $i, l \in \mathcal{I}_c$ and $j \in \mathcal{I}_N \backslash \mathcal{I}_c$,

$$\begin{aligned}
\|x^i(t) - x^l(t)\| \leq \; &\|x^i(t) - x^i(t_k)\| + \|x^i(t_k) - x^j(t_k)\| + \|x^j(t_k) - x^l(t_k)\| \\
&+ \|x^l(t_k) - x^l(t)\|.
\end{aligned}$$

Since, for $i, l \in \mathcal{I}_c$,

$$\lim_{t \to \infty} \|x^i(t) - x^l(t)\| = 0, \quad \text{i.e.,} \quad \lim_{t \to \infty} x_i(t) = q, \quad \forall i \in \mathcal{I}_c,$$

where q is a constant. When $t \to \infty$, we have $t_k \to \infty$. Thus,

$$\lim_{t_k \to \infty} \|x^i(t) - x^i(t_k)\| = 0 \quad \text{and} \quad \lim_{t_k \to \infty} \|x^l(t_k) - x^l(t)\| = 0.$$

This implies that $\lim_{t_k \to \infty} \|x^i(t_k) - x^j(t_k)\| = 0$ and $\lim_{t_k \to \infty} \|x^j(t_k) - x^l(t_k)\| = 0$. Hence,

$$\lim_{t_k \to \infty} \|x^i(t_k) - x^j(t_k)\| = 0, \quad \text{for } \ j \in \mathcal{I}_N \backslash \mathcal{I}_c \ \text{ and } \ i \in \mathcal{I}_c.$$

From **Case 1, Case 2** and **Case 3**, we can conclude that

$$\lim_{t_k \to \infty} \|x^i(t_k) - x^j(t_k)\| = 0 \qquad \text{for } \ i, j \in \mathcal{I}_N. \tag{3.43}$$

This completes the proof.

□

3.5 Consensus of NHMASs via impulsive protocols

Consider a nonlinear hybrid multi-agent system (NHMAS) consisting of N agents which are continuous-time and discrete-time dynamic agents, labelled 1 through N, where the number of continuous-time dynamic agents is c, $c < N$. Without loss of generality, we assume that agent 1 through c are continuous-time dynamic agents. Moreover, $\mathcal{I}_c = \{1, 2, 3, ..., c\}$, $\mathcal{I}_N \backslash \mathcal{I}_c = \{c+1, c+2, c+3, ..., N\}$. Then, each agent has the dynamics as follows:

$$\begin{cases} \dot{x}_i(t) = f(t, x_i(t)) + u_i(t), & \text{for } \ i \in \mathcal{I}_c, \\ x_i(t_{l+1}) = x_i(t_l) + hf(t_l, x_i(t_l)) + u_i(t_l), & \text{for } \ i \in \mathcal{I}_N \backslash \mathcal{I}_c, \end{cases} \tag{3.44}$$

where h is the sampling period, $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the state and control input of agent i at time t, respectively. $f(\cdot, \cdot)$ is a nonlinear function. The initial conditions are $x_i(0) = x_{i0}$, and $x(0) = [x_{10}, x_{20}, ..., x_{N0}]^T$.

In this work, we assume that all agents can update their states and interact with their neighbours only at the sampling time $t = t_k$. Thus, the impulsive consensus protocols can be described as follows:

$$\begin{cases} u_i(t) = \sum_{k=1}^{\infty} \delta(t - t_k) \mathbf{B}_k \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)), & \text{for } \ i \in \mathcal{I}_c, \\ u_i(t_l) = \sum_{k=1}^{\infty} \delta(t_l - t_k) \mathbf{C}_k \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(t_l) - x_i(t_l)), & \text{for } \ i \in \mathcal{I}_N / \mathcal{I}_c, \end{cases} \tag{3.45}$$

where $\mathbf{B}_k \in \mathbb{R}^{N\times N}$ and $\mathbf{C}_k \in \mathbb{R}^{N\times N}$ are impulsive matrices to be determined later. The discrete time instant t_k satisfy $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \ldots$ and $\lim_{k\to\infty} t_k = \infty$, $\delta(t)$ is the Dirac delta function.

To establish our main results, the following assumptions are provided:

(A1) For any $x(t), y(t) \in \Omega \subset \mathbb{R}^n$, there exists a constant β, such that

$$(x(t) - y(t))^T(f(t, x(t)) - f(t, y(t))) \leq \beta(x(t) - y(t))^T(x(t) - y(t)),$$

where Ω is a bounded set.

(A2) There exist two constants τ_1 and τ_2 such that $0 < \tau_1 \leq t_k - t_{k-1} \leq \tau_2 < \infty, k \in \mathbb{N}_+$.

(A3) There exist some constants $0 < \alpha_k < 1$ and $0 < \gamma < 1$ such that

$$(1 - \alpha_k)\mathcal{L} - 2b_k\mathcal{L}\mathcal{L} + (b_k)^2\mathcal{L}\mathcal{L}\mathcal{L} \leq 0$$

and

$$\alpha_k e^{\beta(t_k - t_{k-1})} \leq \gamma < 1, k \in \mathbb{N},$$

where $\mathcal{L}$ is the Laplacian matrix of $\mathcal{G}_c \cup \mathcal{G}'$.

(A4) There exist some constants $\alpha > 0$ such that

$$(f(k, x(k)) - f(k, y(k))) \leq \alpha(x(k) - y(k)), k \in \mathbb{N},$$

where $f(\cdot)$ is a nonlinear function.

Theorem 3.5.1. Let $\mathcal{G}$ be a directed connected communication network of the hybrid multi-agent system (3.44). Assume that the assumptions **(A1)-(A4)** hold. Then, the multi-agent system (3.44) with the protocol (3.45) reaches consensus if $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$ are both balanced and contain a spanning tree.

Proof. Assume that a digraph $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$ both balanced and contain spanning tree. For $i \in I_c$, it can be seen that the system (3.44) with the protocol (3.45) can be described as an impulsive system on the communication network $\mathcal{G}_c \cup \mathcal{G}'$ with r nodes, where $|\mathcal{G}_c \cup \mathcal{G}'| = r \leq N$. WLOG, we assume that all DT agents have a communication with some CT agents. Thus, $r = N$. For simplicity, in the following we choose $\mathbf{B}_k = b_k\mathbf{I}, k \in \mathbb{N}$ and the dynamics of agents can be described as follows:

$$\begin{cases} \dot{x}_i(t) = f(t, x_i(t)), & t \neq t_k, \\ \Delta x_i(t_k) = b_k \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t_k^-) - x_i(t_k^-)), & k \in \mathbb{N}. \end{cases} \tag{3.46}$$

Let $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T$, then the system (3.46) can be written as

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x}(t)), & t \neq t_k, \\ \Delta \mathbf{x}(t_k) = (-b_k \mathcal{L})\mathbf{x}(t_k^-), & k \in \mathbb{N}, \end{cases} \tag{3.47}$$

where $\mathbf{F}(t, x(t)) = (f(t, x_1(t)), f(t, x_2(t)), \ldots, f(t, x_N(t)))^T$. Then, we have

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x}(t)), & t \neq t_k, \\ \mathbf{x}(t_k^+) = (\mathbf{I} - b_k \mathcal{L})\mathbf{x}(t_k^-), & k \in \mathbb{N}, \end{cases} \tag{3.48}$$

where $\mathcal{L}$ is the Laplacian matrix of $\mathcal{G}_c \cup \mathcal{G}'$.

Let $V_i(x(t)) = \sum\limits_{j \in \mathcal{N}_i} a_{ij}(x_j(t) - x_i(t))^T (x_j(t) - x_i(t))$. Consider the Lyapunov function candidate

$$\begin{aligned} V(\mathbf{x}(t)) &= \sum_{i=1}^{N} V_i(\mathbf{x}(t)) \\ &= \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(t) - x_i(t))^T (x_j(t) - x_i(t)) \\ &= \mathbf{x}^T(t) \mathcal{L} \mathbf{x}(t). \end{aligned}$$

Taking the Dini derivative of $V(\mathbf{x}(t))$ for $t \in [t_{k-1}, t_k), k \in \mathbb{N}_+$, by the assumption **(A1)**, we obtain

$$\begin{aligned} \mathbf{D}^+ V(\mathbf{x}(t)) &= \sum_{i=1}^{N} \mathbf{D}^+ V_i(\mathbf{x}(t)) \\ &= 2 \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(t) - x_i(t))^T (\dot{x}_j(t) - \dot{x}_i(t)) \\ &= 2 \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(t) - x_i(t))^T (f(t, x_j(t)) - f(t, x_i(t))) \\ &\leq 2\beta \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(t) - x_i(t))^T (x_j(t) - x_i(t)) \\ &= 2\beta V(\mathbf{x}(t)). \end{aligned}$$

Then

$$V(\mathbf{x}(t)) \leq e^{2\beta(t-t_{k-1})}V(\mathbf{x}(t_{k-1}^{+})), t \in [t_{k-1}, t_k), k \in \mathbb{N}_+. \tag{3.49}$$

On the other hand, when $k \in \mathbb{N}_+$, by the assumption **(A3)**, we have

$$\begin{aligned}
V(\mathbf{x}(t_k^+)) &= \mathbf{x}^T(t_k^+)\mathcal{L}\mathbf{x}(t_k^+) \\
&= \mathbf{x}^T(t_k^-)[(1-\alpha_k)\mathcal{L} - 2b_k\mathcal{L}\mathcal{L} + (b_k)^2\mathcal{L}\mathcal{L}\mathcal{L}]\mathbf{x}^T(t_k^-) \\
&= \mathbf{x}^T(t_k^-)[(1-\alpha_k)\mathcal{L} - 2b_k\mathcal{L}\mathcal{L} + (b_k)^2\mathcal{L}\mathcal{L}\mathcal{L} - \alpha_k\mathcal{L}]\mathbf{x}^T(t_k^-) + \alpha_k\mathbf{x}^T(t_k^-)\mathcal{L}\mathbf{x}^T(t_k^-) \\
&\leq \alpha_k V(\mathbf{x}(t_k^-)).
\end{aligned}$$

By mathematical induction, one obtains that, for $t \in [t_{k-1}, t_k), k \in \mathbb{N}_+, k \geq 2$,

$$V(\mathbf{x}(t)) \leq e^{2\beta(t-t_{k-1})}\prod_{i=1}^{k-1}\alpha_j e^{2\beta(t_j-t_{j-1})}V(\mathbf{x}(t_0^+)). \tag{3.50}$$

It follows from **(A2)** and **(A3)** that

$$V(\mathbf{x}(t)) \leq e^{2|\beta|\tau_2}\gamma^k V(\mathbf{x}(t_0^+)), \quad t \in [t_{k-1}, t_k), k \in \mathbb{N}_+, k \geq 2.$$

Hence, $V(\mathbf{x}(t)) \to 0$ as $t \to \infty$. Since $\mathcal{G}$ is connected, one obtains that

$$\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = 0, \quad \forall i, j \in \mathcal{I}_c.$$

Next, we will show that

$$\lim_{t_k\to\infty}||x_i(t_k) - x_j(t_k)|| = 0, \quad \forall i, j \in \mathcal{I}_N.$$

The proof can be separated into three cases as follows:
Case 1. If $i, j \in \mathcal{I}_c$, the above discussion gives

$$\lim_{t_k\to\infty}\|x_i(t_k) - x_j(t_k)\| = 0, \quad \forall i, j \in \mathcal{I}_c.$$

Case 2. If $i, j \in \mathcal{I}_N \backslash \mathcal{I}_c = \{c+1, c+2, ..., N\}$, the problem can be simplified by considering the communication network of $\mathcal{G}_d \cup \mathcal{G}'$, where $|\mathcal{G}_d \cup \mathcal{G}'| = d \leq N$. For simplicity, we assume that all CT agents have communications with some DT agents. Thus, $|\mathcal{G}_d \cup \mathcal{G}'| = d = N$. From (3.44) and protocol (3.45), for each $i \in \mathcal{I}_N \backslash \mathcal{I}_c$, we have

$$x_i(t_{l+1}) = x_i(t_l) + hf(t_l, x_i(t_l)) + \sum_{k=1}^{\infty}\delta(t_l - t_k)\mathbf{C}_k\sum_{j\in\mathcal{N}_i}a_{ij}(x_j(t_l) - x_i(t_l)). \tag{3.51}$$

For simplicity, we choose $\mathbf{C}_k = c_k I, k \in \mathbb{N}$. Thus, by the definition of the Dirac delta function, we get

$$\begin{cases} x_i(t_{l+1}) & = x_i(t_l) + hf(t_l, x_i(t_l)), \quad t_l \neq t_k, \\ \Delta x_i(t_k) & = hf(t_k, x_i(t_k)) + u_i(t_k), \quad k \in \mathbb{N}. \end{cases} \tag{3.52}$$

Let $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T$, then the system (3.52) can be written as

$$\begin{cases} \mathbf{x}(t_{l+1}) & = \mathbf{x}(t_l) + h\mathbf{F}(t_l, \mathbf{x}), \quad t_l \neq t_k, \\ \mathbf{x}(t_{k+1}) & = (I - c_k\mathcal{L}')\mathbf{x}(t_k) + h\mathbf{F}(t_k, \mathbf{x}), \quad k \in \mathbb{N}. \end{cases} \tag{3.53}$$

where $\mathbf{F}(t_l, \mathbf{x}) = (f(t_l, x_1), f(t_l, x_2), \ldots, f(t_l, x_N))^T$.

It follows from the results of Han [33] that the system (3.53) can achieve exponential consensus, that is, there exist two constants $M_0 > 0$ and $\sigma > 0$ such that

$$\|x_i(t_k) - x_j(t_k)\| \leq M_0 e^{-\sigma(t_k - t_0)},$$

which leads to

$$\lim_{t_k \to \infty} ||x_i(t_k) - x_j(t_k)|| = 0, \quad \forall i, j \in \mathcal{I}_N \backslash \mathcal{I}_c.$$

Case 3. If $j \in \mathcal{I}_N \backslash \mathcal{I}_c$ and $i \in \mathcal{I}_c$ (or $i \in \mathcal{I}_N \backslash \mathcal{I}_c$ and $j \in \mathcal{I}_c$), we consider, for $i, l \in \mathcal{I}_c$ and $j \in \mathcal{I}_N \backslash \mathcal{I}_c$,

$$\begin{aligned} \|x_i(t) - x_l(t)\| \leq & \|x_i(t) - x_i(t_k)\| + \|x_i(t_k) - x_j(t_k)\| + \|x_j(t_k) - x_l(t_k)\| \\ & + \|x_l(t_k) - x_l(t)\|. \end{aligned}$$

It follows from **Case 1** that for $i, l \in \mathcal{I}_c$,

$$\lim_{t \to \infty} \|x_i(t) - x_l(t)\| = 0.$$

As $t \to \infty$, we have $t_k \to \infty$. Then, we get

$$\lim_{t_k \to \infty} \|x_i(t) - x_i(t_k)\| = 0 \text{ and } \lim_{t_k \to \infty} \|x_l(t_k) - x_l(t)\| = 0,$$

which gives us $\lim_{t_k \to \infty} \|x_i(t_k) - x_j(t_k)\| = 0$ and $\lim_{t_k \to \infty} \|x_j(t_k) - x_l(t_k)\| = 0$. Therefore,

$$\lim_{t_k \to \infty} \|x_i(t_k) - x_j(t_k)\| = 0, \quad \text{for } j \in \mathcal{I}_N \backslash \mathcal{I}_c \text{ and } i \in \mathcal{I}_c.$$

It follows from **Case 1,2** and **3** that

$$\lim_{t_k \to \infty} \|x_i(t_k) - x_j(t_k)\| = 0 \qquad \text{for} \;\; i, j \in \mathcal{I}_N. \tag{3.54}$$

This completes the proof.

□

3.6 Simulations and Discussion

In this section, two examples have been provided to demonstrate the effectiveness of theoretical results in this work. With out loss of generality, we assume that all discrete-time dynamic agents have interactions to some continuous-time dynamic agents and for simplicity, we consider the equidistant impulsive interval $t_k - t_{k-1} \equiv h$.

Example 1. Assume that there are 8 agents consisting of six continuous-time dynamic agents and two discrete-time dynamic agents, denoted by $1-6$ and $7-8$, respectively. In the following, all networks with $0-1$ weights will be needed. Let $x(0) = [-6 \quad 4 \quad -2 \quad 1 \quad -1 \quad 2 \quad -4 \quad 6]^T$ and $h = 0.3$. The communication network $\mathcal{G}$ is shown in Figure 3.1, where the dashed lines mean that each agent exchanges information at time $t = t_k$.

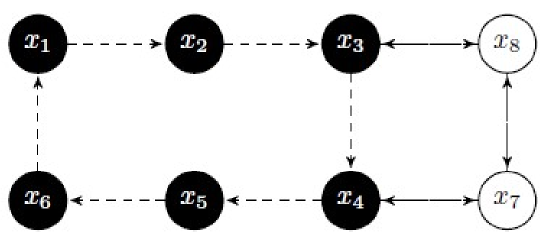

Figure 3.1: A connected directed network $\mathcal{G}$.

Consider a communication network $\mathcal{G}$ in Figure 3.1, it can be seen that $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$ are balanced and contain a directed spanning tree with $d_{max} = 2$. The Laplacian matrix of a network $\mathcal{G}_c \cup \mathcal{G}'$ is described as following:

$$\mathcal{L}' = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Choosing the sampling period $h = 0.3 < 0.5 = (d_{max})^{-1}$ and by using MATLAB, it easy to calculate that

$$(1-\alpha)I - \mathcal{L}' - \mathcal{L}'^T + \mathcal{L}'^T\mathcal{L}' \leq 0,$$

for some appropriate α. Thus, the assumption **(A1)** and **(A2)** are satisfied. By using the consensus protocol (3.4), the state trajectories of all agents are shown in Figure 3.2, which is consistent with the sufficiency of Theorem 3.2.1.

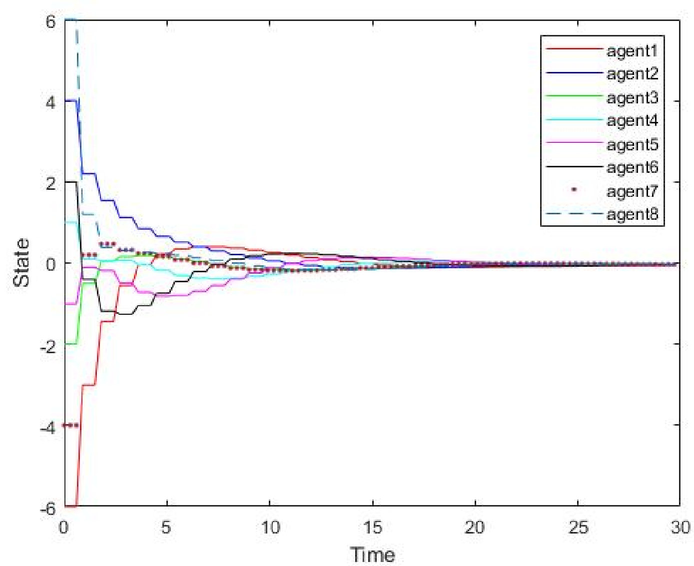

Figure 3.2: The state trajectories of all agents using consensus protocol (3.4) for $h = 0.3$.

In addition, it can be seen that if $h = 0.675 > 0.5$, which implies that the condition **(A1)** is not satisfied, and hence the consensus protocol (3.4) cannot guarantee achieving consensus as shown in Figure 3.3.

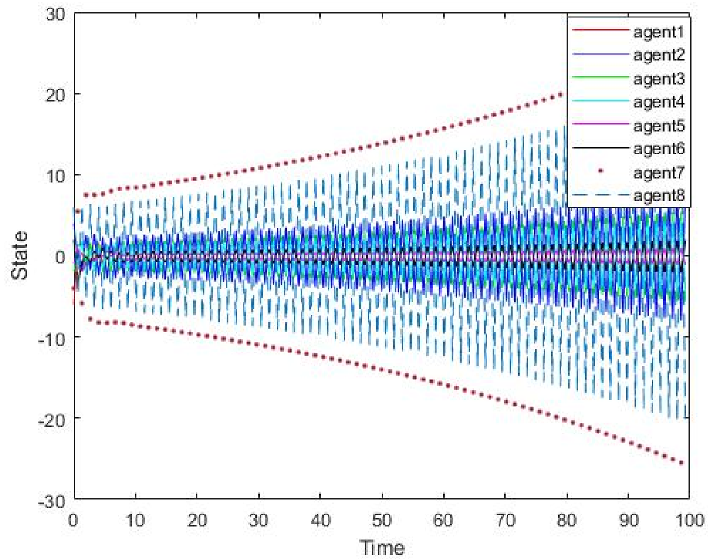

Figure 3.3: The state trajectories of all agents using consensus protocol (3.4) for $h = 0.675$

Example 2. Assume that there are 8 agents consisting of six continuous-time dynamic agents and two discrete-time dynamic agents, denoted by $1-6$ and $7-8$, respectively. In the following, all networks with $0-1$ weights will be needed and let $x(0) = [-6 \quad 4 \quad -2 \quad 1 \quad -1 \quad 2 \quad -4 \quad 6]^T$. The communication network $\mathcal{G}$ is shown in Figure 3.02, where the dashed lines mean that each agent exchanges information at time $t = t_k$.

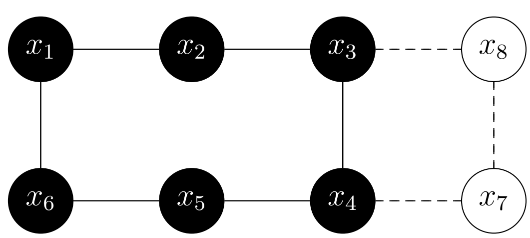

Figure 3.02: A connected undirected network $\mathcal{G}$.

It is easy to see that $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$ are both connected with $d_{max} = 3$. By choosing the sampling period $h = 0.15 < 0.33 \leq \frac{1}{max_{i=1}^{8} d_{ii}}$ and for some appropriate α (using MATLAB), we have

$$(1-\alpha)I - 2\mathcal{L}' + \mathcal{L}'^T \mathcal{L}' \leq 0.$$

Thus, the assumption **(A1)** and **(A2)** hold. According to Theorem 3.2.1, the consensus problems can be solved and the state trajectories of all agents are shown in Figure 3.4.

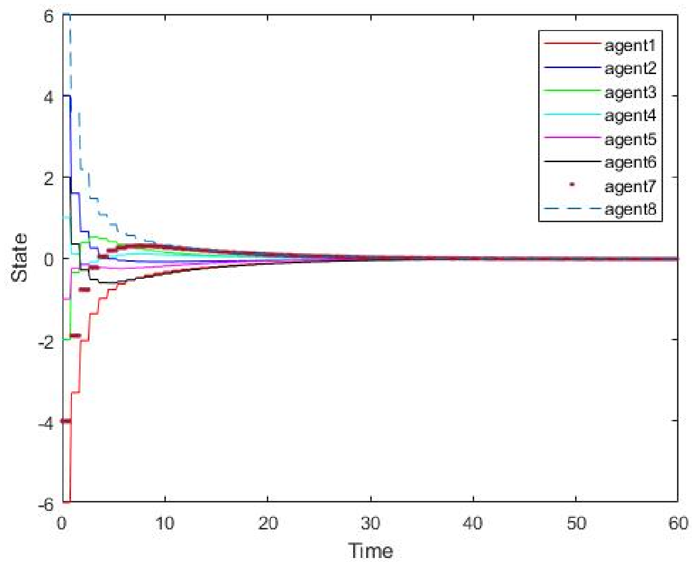

Figure 3.4: The state trajectories of all agents using the consensus protocol (3.4) under the communication network $\mathcal{G}$ with $h = 0.15$.

Furthermore, if some conditions are not satisfied, the system (3.1) cannot reach consensus under protocol (3.4), for example, pick $h = 0.57 > 0.33$, then the state of agents cannot converge (see Figure 3.5).

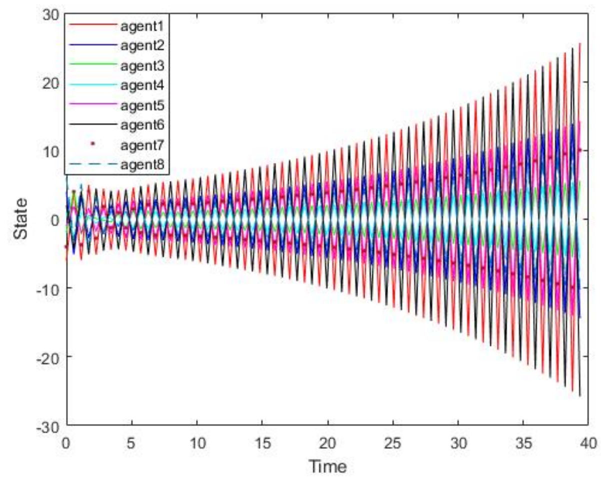

Figure 3.5: The state trajectories of all agents using the consensus protocol (3.4) under the communication network $\mathcal{G}$ with $h = 0.57$.

Next, we will show the effectiveness of Theorem 3.3.1. Consider the networked topology $\mathcal{G}$ which has the dynamics as in Figure 3.02 with $0 - 0.345$ weights and assume that the communication delays process only in the impulsive interval (t_{k-1}, t_k), which implies that there is no delays at the sampling time t_k. It can be seen by using MATLAB that there exists positive constants α, β such that for all $k \in \mathbb{N}$,

$$[1 + 2\lambda_2(\hat{\mathcal{L}}) + \lambda_2(\mathcal{L}'\mathcal{L}'^T)] \cdot 1.38 \leq \alpha$$

and

$$ln[1 + 2\lambda_2(\hat{\mathcal{L}}) + \lambda_2(\mathcal{L}'\mathcal{L}'^T)] - \alpha(t_k - t_{k-1}) \geq \beta > 0.$$

If we choose the sampling period $h = 0.2$ and $\tau = 3.4$, then all conditions are satisfied. Hence, by Theorem 3.3.1 under protocol (3.14), the consensus problem is solved and the state of all agents are as in Figure 3.6.

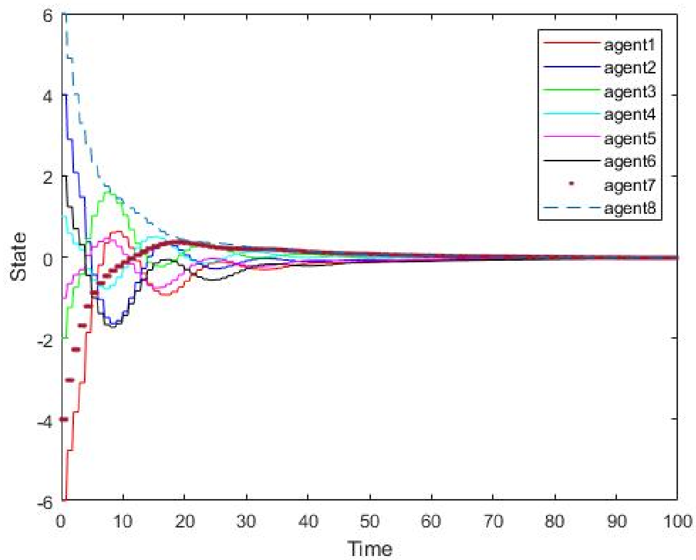

Figure 3.6: The state trajectories of all agents using the consensus protocol (3.14) for $h = 0.2$ and $\tau = 3.4$.

However, the consensus protocol (3.14) cannot guarantee solving consensus problem if all conditions are not satisfied. For example, choosing $h = 0.5$ and $\tau = 3.4$, obviously the assumption **(A1)** is not satisfied, and hence the state of agents are shown as in Figure 3.7.

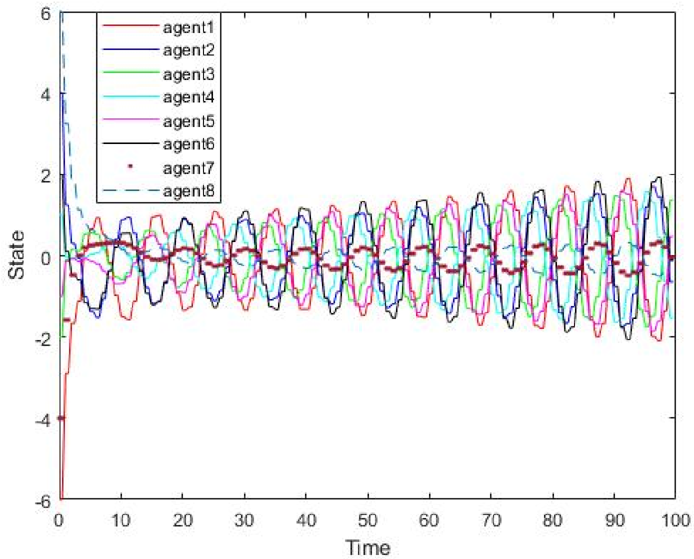

Figure 3.7: The state trajectories of all agents using the consensus protocol (3.14) for $h = 0.5$ and $\tau = 3.4$.

Discussion

In this chapter, consensus problems of hybrid multi-agent systems under fixed topology with and without communication delays have been studied. We assume that all continuous-time dynamic agents communicate with their neighbors and update their own states in real time, while the discrete-time dynamic agents communicate with their neighbors and update their own states at time t_k. Firstly, we assume that the hybrid multi-agent system (3.1) described as a graph $\mathcal{G} = \mathcal{G}_c \cup \mathcal{G}_d \cup \mathcal{G}'$, where $\mathcal{G}_c,\ \mathcal{G}_d,\ \mathcal{G}'$ is the communication network of continuous-time dynamic agents, discrete-time dynamic agents, and the interactions between each other, respectively. If the sampling period $0 < h < (d_{max})^{-1}$, where $h = t_k - t_{k-1}, k \in \mathbb{N}$ and the assumption **(A2)** holds, Theorem 3.2.1 and protocol (3.4) shows that the hybrid multi-agent system (3.1) reaches consensus if $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$ are balanced and contain a spanning tree. Obviously, the protocol (3.4) is a generalization of [132]., where the interactions among agents occur only in the sampling time t_k (see *Case 1* and *Case 2*).

Secondly, the impulsive consensus protocol has been introduced to solve consensus problems of hybrid multi-agent systems with communication delays based on the continuous-time dynamics. Theorem 3.3.1 under the protocol (3.14) shows that the consensus can be guaranteed if the sampling period $0 < h < (d_{max})^{-1}$ and the condition **(A2)** holds. However, if one of conditions in our theorems is not satisfied as showed in the examples, our protocols cannot guarantee consensus.

Finally, the linear and nonlinear hybrid multi-agent systems are introduced in this chapter, and the impulsive consensus protocols have been proposed to solve the consensus problems.

Chapter 4

Scaled consensus of Hybrid multi-agent systems

As one of the fundamental problems in multi-agent coordination, consensus means that all agents reach an agreement on some common features, which can be velocities, positions, attitudes, and many other quantities. In this chapter, we aim to extend the results of [132] to more general problems called scaled consensus problems, where all agents are not necessary to achieve on the same value. In section 4.1, we employ the same methodology as Roy[81] to extend the results of [132] by introducing the scaled consensus protocols for the HMASs with no communication delays. Section 4.2.2 studies scaled consensus problems of MASs under fixed and switched topologies via impulsive protocols. In Section 4.2.3, the scaled consensus problems of HMASs with external perturbations via impulsive mechanism have been investigated. Finally, in Section 4.2.4, we investigate the scaled consensus of MASs with impulsive time delays under fixed and switching topologies.

4.1 Scaled consensus problems in Hybrid multi-agent systems

4.1.1 Problem formulation

Consider the hybrid multi-agent system consists of n agents which are continuous-time and discrete-time dynamic agents, labelled 1 through n, where the number of continuous-time dynamic agents is c, $c \leq n$. Without loss of generality, we assume that agent 1 through c

are continuous-time dynamic agents and the scalar scaled of agent i is denoted by $\beta_i, \beta_i \neq 0$. Moreover, $\mathcal{I}_c = \{1, 2, 3, ..., c\}$, $\mathcal{I}_n \backslash \mathcal{I}_c = \{c+1, c+2, c+3, ..., n\}$. Then, each agent has the dynamics as follows:

$$\begin{cases} \beta_i \dot{x}_i(t) = u_i(t), & \text{for} \quad i \in \mathcal{I}_c, \\ \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + u_i(t_k), \quad t_k = kh, & \text{for} \quad i \in \mathcal{I}_n \backslash \mathcal{I}_c, \end{cases} \tag{4.1}$$

where h is the sampling period, $x_i \in \mathbb{R}$, $u_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}/\{0\}$ are the state, control input and the scalar scale of agent i, respectively. The initial conditions are $x_i(0) = x_{i0}$, and $x(0) = [x_{10}, x_{20}, ..., x_{n0}]^T$.

Moreover, the hybrid multi-agent system (4.1) is modelled as a connected directed graph, where all agents are regraded as the nodes and the interaction between two agents has been represented by the edge in a graph. This implies that $(v_i, v_j) \in \mathcal{E}$ corresponds to an available information link from agent i to agent j. Besides, each agent updates its current state based on the information received from its neighbours. Furthermore, we suppose that there exists communication behaviour as in hybrid multi-agent system (4.1), that is, there are agent i and agent j which make $a_{ij} > 0$.

Definition 4.1.1. Given any scalar scale $\beta_i \neq 0$ for the agent i, the hybrid multi-agent system (4.1) is said to reach scaled consensus to $(\beta_1, \ldots, \beta_n)$ if for any initial conditions, we have

$$\lim_{t_k \to \infty} \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| = 0, \quad \text{for} \quad i, j \in \mathcal{I}_n, \tag{4.2}$$

and

$$\lim_{t \to \infty} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0, \quad \text{for} \quad i, j \in \mathcal{I}_c. \tag{4.3}$$

Remark. If a scalar scale $\beta_i = 1$ for all i, the scaled consensus can reduce to the standard consensus.

The following Lemma is one of the most important Lemmas that will be used to prove our main results in this thesis.

Lemma 4.1.1. Given the scalar scale $\mathcal{B} = (\beta_1, \beta_2, \ldots, \beta_n)$, $\beta_i \neq 0$. Define $\beta_{max} = max_{1 \leq i \leq n} |\beta_i|$, $H = diag\{h_1, h_2, ..., h_n\}$ such that $0 < h_i < \dfrac{1}{d_{max}\beta_{max}}$, $i \in \mathcal{I}_n$, and $|\mathcal{B}| = diag(|\beta_1|, |\beta_2|, ..., |\beta_n|)$. If the communication network $\mathcal{G}$ contains a spanning tree, then $\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}$ is SIA, i.e., there exists a column vector $\mathbf{y}$ such that

$$\lim_{k \to \infty} [\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^k = \mathbf{1}_n \mathbf{y}^T.$$

Proof. Since $h \in (0, \dfrac{1}{d_{max}\beta_{max}})$, one obtains

$$\mathbf{I}_n - H|\mathcal{B}|\mathcal{L} = (\mathbf{I}_n - H|\mathcal{B}|\mathcal{D}) + H|\mathcal{B}|\mathcal{A}$$

is a stochastic matrix with positive diagonal entries, where $\mathcal{D} = diag(d_1, \dots, d_n)$ and $\mathcal{A}$ are the degree matrix and adjacency matrix of $\mathcal{G}$, respectively. Obviously, for all $i, j \in \mathcal{I}_n$; $i \neq j$, the (i, j)th entry of $\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}$ is positive if and only if $a_{ij} > 0$. Then, $\mathcal{G}$ is the graph associated with $\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}$. Since $\mathcal{G}$ contains a spanning tree, it follows from Lemma 2.6.6 and Lemma 2.6.7 that

$$\lim_{k\to\infty} [\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^k = \mathbf{1}_n \mathbf{y}^T$$

for some a column vector $\mathbf{y}$.

□

4.1.2 Consensus results

In this section, the scaled consensus problems of hybrid multi-agent system (4.1) have been studied under two kinds of control inputs (consensus protocols), respectively.

Case I: We assume that all agents communicate with their neighbours and update their control inputs in a sampling time t_k. Then, the consensus protocol for hybrid multi-agent system (4.1) is defined as follows:

$$\begin{cases} u_i(t) = |\beta_i| \sum\limits_{j\in\mathcal{N}_i^c} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } \ t \in (t_k, t_{k+1}], \quad i \in \mathcal{I}_c \\ u_i(t_k) = h \cdot |\beta_i| \sum\limits_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } \ i \in \mathcal{I}_n \backslash \mathcal{I}_c, \end{cases} \tag{4.4}$$

where $\mathcal{A} = [a_{ij}]$ is the weighted adjacency matrices associated with the graph $\mathcal{G}$, $h = h_i = t_{k+1} - t_k$ for all i is the sampling period.

Theorem 4.1.2. Let $\mathcal{G}$ be a directed connected communication network of the hybrid multi-agent system (4.1) and $\beta_i \neq 0$ be any scalar scale of agent i. Assume that $0 < h < \dfrac{1}{d_{max}\beta_{max}}$ and $\mathcal{G}$ contains a spanning tree. Then, the hybrid multi-agent system (4.1) with the protocol (4.4) reaches scaled consensus to $(\beta_1, \ldots, \beta_n)$.

Proof. Let $\beta_i \neq 0$ be any scalar scale of agent i, we first show that equation (4.2) holds. From (4.4) we have, for $t \in (t_k, t_{k+1}]$,

$$\begin{cases} \beta_i x_i(t) = \beta_i x_i(t_k) + (t - t_k)|\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } \ i \in \mathcal{I}_c \\ \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } \ i \in \mathcal{I}_n \backslash \mathcal{I}_c. \end{cases} \tag{4.5}$$

Therefore, it follows that

$$\beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], \quad \text{for } \ i \in \mathcal{I}_n \tag{4.6}$$

Let $x(t_k) = (x_1(t_k), x_2(t_k), ..., x_n(t_k))^T \in \mathbb{R}^n$, $\mathcal{B} = diag(\beta_1, \beta_2, ..., \beta_n) \in \mathbb{R}^{n \times n}$, $|\mathcal{B}| = diag(|\beta_1|, |\beta_2|, ..., |\beta_n|) \in \mathbb{R}^{n \times n}$ and $H = diag(h_1, h_2, \ldots, h_n)$. Then, equation (4.6) can be written as

$$\mathcal{B}x(t_{k+1}) = [\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]\mathcal{B}x(t_k). \tag{4.7}$$

Since $\mathcal{G}$ has a directed spanning tree and $0 < h < \dfrac{1}{d_{max}\beta_{max}}$, by Lemma 4.1.1, we have $\lim_{k \to \infty}[\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^k = \mathbf{1}_n \mathbf{y}^T$, where $\mathbf{y}$ is a column vector. Thus

$$\lim_{k \to \infty} \mathcal{B}x(t_k) = \lim_{k \to \infty} [\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^k \mathcal{B}x(0) = \mathbf{1}_n \mathbf{y}^T \mathcal{B}x(0).$$

As a consequence, equation (4.2) holds. Furthermore,

$$\lim_{t_k \to \infty} \beta_i x_i(t_k) = \mathbf{y}^T \mathcal{B}x(0) \ \text{ for } \ i \in \mathcal{I}_n. \tag{4.8}$$

Now, we will show that

$$\lim_{t \to \infty} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0 \qquad \text{for } \ i, j \in \mathcal{I}_c.$$

Consider, for $i, j \in \mathcal{I}_c$ and any $\beta_i \neq 0$,

$$\begin{aligned}\|\beta_i x_i(t) - \beta_j x_j(t)\| \leq \|\beta_i x_i(t) - \beta_i x_i(t_k)\| + \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| \\ + \|\beta_j x_j(t_k) - \beta_j x_j(t)\|.\end{aligned} \tag{4.9}$$

From equation (4.5), one obtains, for $t \in (t_k, t_{k+1}]$,

$$\|\beta_i x_i(t) - \beta_i x_i(t_k)\| \leq h|\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij} \|\beta_j x_j(t_k) - \beta_i x_i(t_k)\|.$$

As $t \to \infty$, we have $t_k \to \infty$. Thus,

$$\lim_{t \to \infty} \|\beta_i x_i(t) - \beta_i x_i(t_k)\| = 0 \qquad \text{for} \quad i, j \in \mathcal{I}_c.$$

Taking the limit as $t \to \infty$ on both sides of equation (4.9), one obtains

$$\lim_{t \to \infty} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0 \qquad \text{for} \quad i, j \in \mathcal{I}_c.$$

Furthermore,

$$\lim_{t \to \infty} \beta_i x_i(t) = \lim_{t_k \to \infty} \beta_i x_i(t_k) = \mathbf{y}^T \mathcal{B} x(0) \quad \text{for} \quad i \in \mathcal{I}_c,$$

which implies that equation (4.3) holds. Therefore, the hybrid multi-agent system (4.1) with protocol (4.4) reaches scaled consensus.

□

Case II: All agents communicate with their neighbours and update their control inputs in a sampling time t_k. However, different from **Case I**, we assume that each continuous-time dynamic agent can observe its own state in real time. Then, the consensus protocol for hybrid multi-agent system (4.1) is defined by:

$$\begin{cases} u_i(t) = |\beta_i| \sum_{j \in \mathcal{N}} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t)], & \text{for} \quad t \in (t_k, t_{k+1}], \quad i \in \mathcal{I}_c \\ u_i(t_k) = h \cdot |\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for} \quad i \in \mathcal{I}_n \backslash \mathcal{I}_c, \end{cases} \tag{4.10}$$

where $\mathcal{A} = [a_{ij}]$ is the weighted adjacency matrices associated with the graph $\mathcal{G}$, $h = h_i = t_{k+1} - t_k$ for all i is the sampling period.

Theorem 4.1.3. Let $\mathcal{G}$ be a directed connected communication network of the hybrid multi-agent system (4.1) and $\beta_i \neq 0$ be any scalar scale of agent i. Assume that $0 < h < \dfrac{1}{d_{max}\beta_{max}}$ and $\mathcal{G}$ contains a spanning tree. Then, the hybrid multi-agent system (4.1) with the protocol (4.10) achieves scaled consensus to $(\beta_1, \ldots, \beta_n)$, where

$$H = diag\left\{\frac{1-e^{-\sum_{j=1}^{n} a_{1j}|\beta_1|h}}{\sum_{j=1}^{n} a_{1j}|\beta_1|} \ldots, \frac{1-e^{-\sum_{j=1}^{n} a_{cj}|\beta_c|h}}{\sum_{j=1}^{n} a_{cj}|\beta_c|}, h, \ldots, h\right\}.$$

Proof. We first show that equation (4.2) holds. From (4.10) we know that for $t \in (t_k, t_{k+1}]$,

$$\begin{cases} \beta_i x_i(t) = \beta_i x_i(t_k) \\ \qquad\qquad +|\beta_i|\left(\dfrac{1-e^{-\sum_{j=1}^{n} a_{ij}|\beta_i|(t-t_k)}}{\sum_{j=1}^{n} a_{ij}|\beta_i|}\right)\displaystyle\sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], \quad \text{for} \quad i \in \mathcal{I}_c \\ \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i| \displaystyle\sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], \quad \text{for} \quad i \in \mathcal{I}_n \backslash \mathcal{I}_c. \end{cases} \tag{4.11}$$

Accordingly, at time t_{k+1}, the states of agents are

$$\begin{cases} \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) \\ \qquad\qquad +|\beta_i|\left(\dfrac{1-e^{-\sum_{j=1}^{n} a_{ij}|\beta_i|h}}{\sum_{j=1}^{n} a_{ij}|\beta_i|}\right)\displaystyle\sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], \quad \text{for} \quad i \in \mathcal{I}_c \\ \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i| \displaystyle\sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], \quad \text{for} \quad i \in \mathcal{I}_n \backslash \mathcal{I}_c. \end{cases} \tag{4.12}$$

Letting $x(t_k) = (x_1(t_k), x_2(t_k), ..., x_n(t_k))^T \in \mathbb{R}^n$, $\mathcal{B} = diag(\beta_1, \beta_2, ..., \beta_n) \in \mathbb{R}^{n\times n}$, $|\mathcal{B}| = diag(|\beta_1|, |\beta_2|, ..., |\beta_n|) \in \mathbb{R}^{n\times n}$, equation (4.12) can be written as

$$\mathcal{B}x(t_{k+1}) = [\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]\mathcal{B}x(t_k), \tag{4.13}$$

where $H = diag\left\{\dfrac{1-e^{-\sum_{j=1}^{n} a_{1j}|\beta_1|h}}{\sum_{j=1}^{n} a_{1j}|\beta_1|} \ldots, \dfrac{1-e^{-\sum_{j=1}^{n} a_{cj}|\beta_c|h}}{\sum_{j=1}^{n} a_{cj}|\beta_c|}, h, \ldots, h\right\}$.

Since $\dfrac{1-e^{-\sum_{j=1}^{n} a_{ij}|\beta_i|h}}{\sum_{j=1}^{n} a_{ij}|\beta_i|} < \dfrac{1}{d_{ii}|\beta_i|}$ for $i \in \mathcal{I}_c$, and $h < \dfrac{1}{d_{max}\beta_{max}}$, one obtains $0 < h_i < \dfrac{1}{d_{max}\beta_{max}}$ for H. Since $\mathcal{G}$ has a spanning tree, by Lemma 4.1.1, $\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}$ is an SIA, i.e., there exists a column vector $\mathbf{y}$ such that

$$\lim_{k\to\infty}[\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^k = \mathbf{1}_n\mathbf{y}^T.$$

Thus

$$\lim_{k\to\infty} \mathcal{B}x(t_k) = \lim_{k\to\infty} [\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^k \mathcal{B}x(0) = \mathbf{1}_n \mathbf{y}^T \mathcal{B}x(0).$$

As a consequence, equation (4.2) holds. Moreover,

$$\lim_{t_k\to\infty} \beta_i x_i(t_k) = \mathbf{y}^T \mathcal{B}x(0) \quad \text{for} \quad i \in \mathcal{I}_n. \tag{4.14}$$

Now, we will show that

$$\lim_{t\to\infty} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0 \qquad \text{for} \quad i, j \in \mathcal{I}_c.$$

From equation (4.11), one obtains, for $t \in (t_k, t_{k+1}]$,

$$\|\beta_i x_i(t) - \beta_i x_i(t_k)\| \leq \frac{|\beta_i|}{\sum_{j=1}^n a_{ij}|\beta_i|} \sum_{j\in\mathcal{N}_i} a_{ij}\|\beta_j x_j(t_k) - \beta_i x_i(t_k)\|. \tag{4.15}$$

As $t \to \infty$, we have $t_k \to \infty$. Thus,

$$\lim_{t\to\infty} \|\beta_i x_i(t) - \beta_i x_i(t_k)\| = 0 \qquad \text{for} \quad i, j \in \mathcal{I}_c. \tag{4.16}$$

Consider, for $i, j \in \mathcal{I}_c$ and any $\beta_i \neq 0$,

$$\|\beta_i x_i(t) - \beta_j x_j(t)\| \leq \|\beta_i x_i(t) - \beta_i x_i(t_k)\| + \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| + \|\beta_j x_j(t_k) - \beta_j x_j(t)\|. \tag{4.17}$$

Thus, by (4.16), we get

$$\lim_{t\to\infty} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0 \qquad \text{for} \quad i, j \in \mathcal{I}_c. \tag{4.18}$$

Furthermore,

$$\lim_{t\to\infty} \beta_i x_i(t) = \lim_{t_k\to\infty} \beta_i x_i(t_k) = \mathbf{y}^T \mathcal{B}x(0) \quad \text{for} \quad i \in \mathcal{I}_c,$$

which implies that equation (4.3) holds. Therefore, the hybrid multi-agent system (4.1) with protocol (4.10) reaches scaled consensus.

□

4.1.3 Simulations and Discussion

In this section, two examples have been provided to demonstrate the effectiveness of theoretical results in this work.

Example 1. Assume that there are 8 agents consisting of six continuous-time dynamic agents and two discrete-time dynamic agents, denoted by $1-6$ and $7-8$, respectively. Let $x(0) = [-6 \quad 4 \quad -2 \quad 1 \quad -1 \quad 2 \quad -4 \quad 6]^T$. The communication network $\mathcal{G}$ with $0-1$ weights is shown in Figure 4.1, where the dashed lines mean that each agent exchanges information at time $t = t_k$.

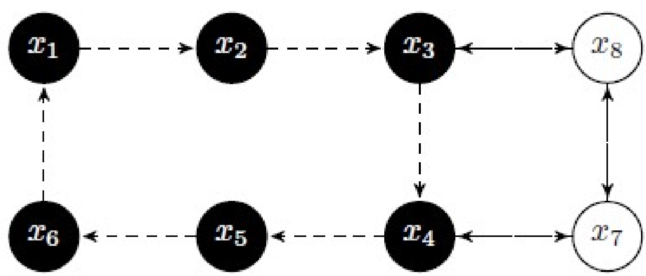

Figure 4.1: A connected directed network $\mathcal{G}$.

Consider a communication network $\mathcal{G}$ in Figure 4.1, it can be seen that $\mathcal{G}$ is balanced and contains a directed spanning tree with $d_{max} = 2$ and the Laplacian matrix of a network $\mathcal{G}$ as

$$\mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Let the scalar scales be $(2, -2, 1, -1, 3, 1.5, -2, -1)$, once obtains that $\beta_{max} = 3$. Clearly, $h = 0.02 < (2 \cdot 3)^{-1} = (d_{max}\beta_{max})^{-1}$. By using the consensus protocol (4.4), the state trajectories of all agents are shown in Figure 4.2, which is consistent with the sufficiency of Theorem 4.1.2.

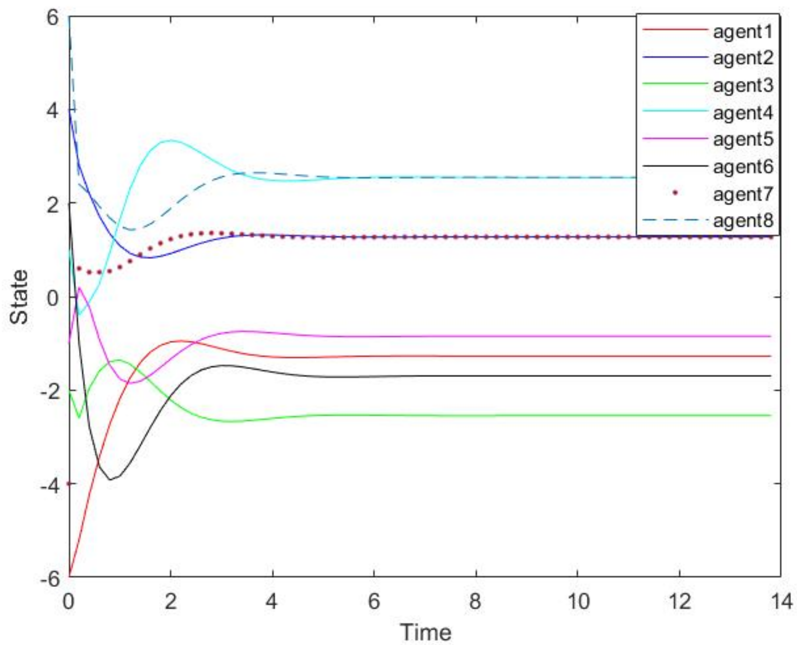

Figure 4.2: The state trajectories of all agents using the consensus protocol (4.4) with $h = 0.02$.

Furthermore, the Figure 4.3 shows the state trajectories of all agents with scalar scales $(2, -2, 1, -1, 3, 1.5, -2, -1)$ using the consensus protocol (4.4) and communication network $\mathcal{G}$ with $h = 0.02$.

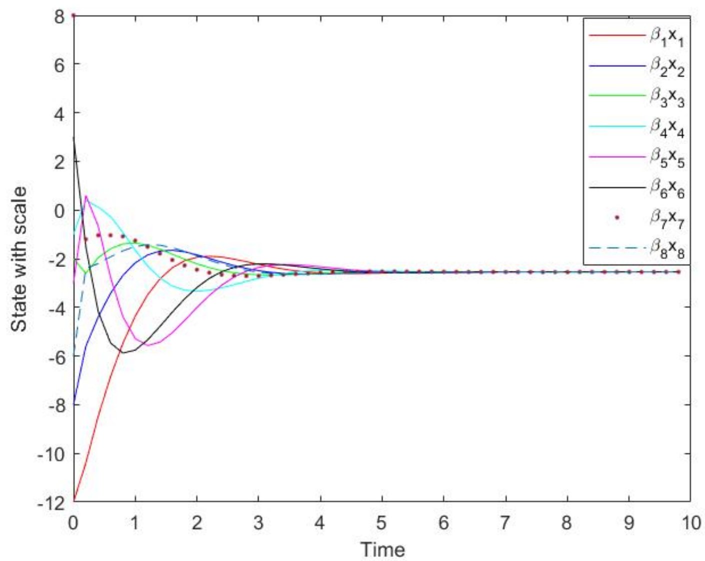

Figure 4.3: The state trajectories of all agents with scalar scales $(2, -2, 1, -1, 3, 1.5, -2, -1)$ and $h = 0.02$.

Moreover, If the scalar scale $\beta_i = 1$ for all i, the state trajectories of all agents under the consensus protocol (4.4) can be shown as in Figure 4.4.

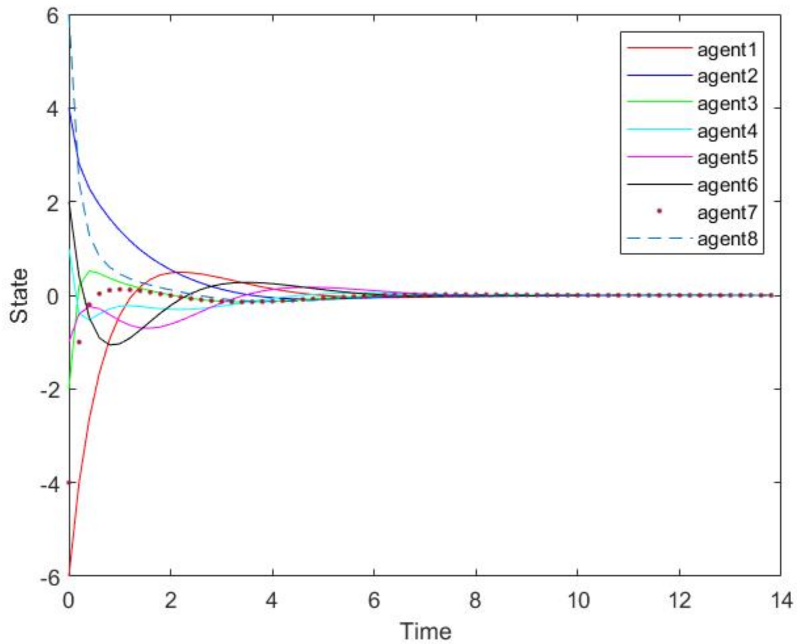

Figure 4.4: The state trajectories of all agents with scalar scales $(1, 1, 1, 1, 1, 1, 1, 1)$ and $h = 0.02$.

In addition, if the sampling period $h = 0.4 > 0.33 = (d_{max}\beta_{max})^{-1}$ the state trajectories of all agent under the consensus protocol (4.4) are divergent as in Figure 4.5.

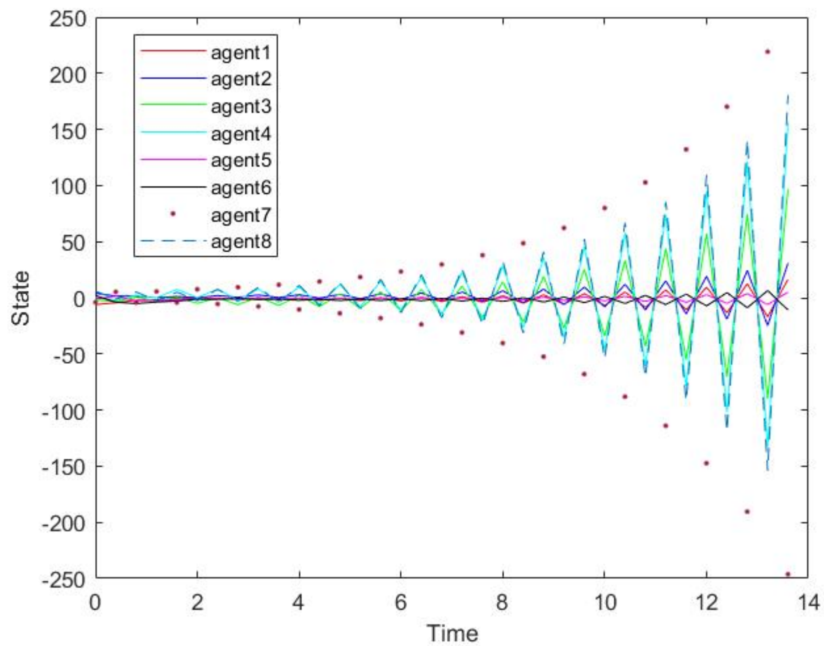

Figure 4.5: The state trajectories of all agents with scalar scales $(2, -2, 1, -1, 3, 1.5, -2, -1)$ and $h = 0.4$.

Example 2. Assume that there are 8 agents consisting of six continuous-time dynamic agents and two discrete-time dynamic agents, denoted by $1-6$ and $7-8$, respectively. The communication network $\mathcal{G}$ with $0-1$ weights is shown in Figure 4.6, where the continuous-time dynamic agents can observe their own state in real time, while the interactions among agents happen in the sampling time t_k. It can be seen in Figure 4.6 that a network $\mathcal{G}$ is

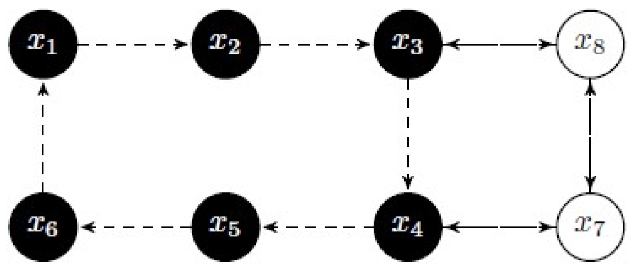

Figure 4.6: A connected directed network $\mathcal{G}$.

balanced and contains a spanning tree with $d_{max} = 2$. Moreover, the Laplacian matrix of $\mathcal{G}$ can be described as following:

$$\mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Let the initial states of all agents be $x(0) = [-6 \quad 4 \quad -2 \quad 1 \quad -1 \quad 2 \quad -4 \quad 6]^T$ and the scalar scales be $(2, -2, 1, -1, 3, 1.5, -2, -1)$. Thus, $\beta_{max} = 3$ and by selecting the sampling period $h = 0.02 < (2 \cdot 3)^{-1} = (d_{max}\beta_{max})^{-1}$, all conditions of Theorem 4.1.3 are satisfied. Hence, the consensus protocol (4.10) can guarantee reaching scaled consensus of the system and the state trajectories of all agents are shown in Figure 4.7, which is consistent with the sufficiency of Theorem 4.1.3.

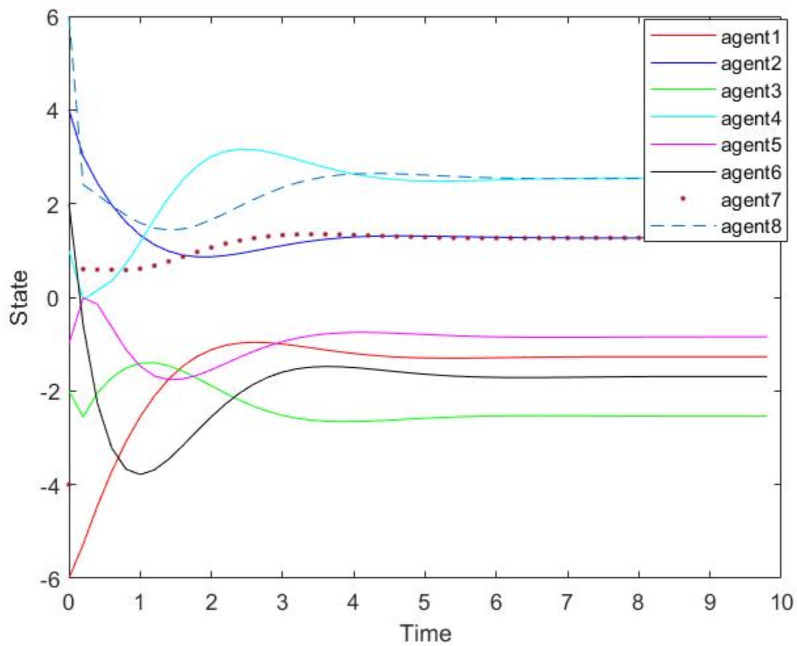

Figure 4.7: The state trajectories of all agents using the consensus protocol (4.10) and communication network $\mathcal{G}$ with $h = 0.02$.

In addition, the state trajectories of all agents with scalar scales $(2, -2, 1, -1, 3, 1.5, -2, -1)$ using the consensus protocol (4.10) and communication network $\mathcal{G}$ with $h = 0.02$ are described as in Figure 4.8.

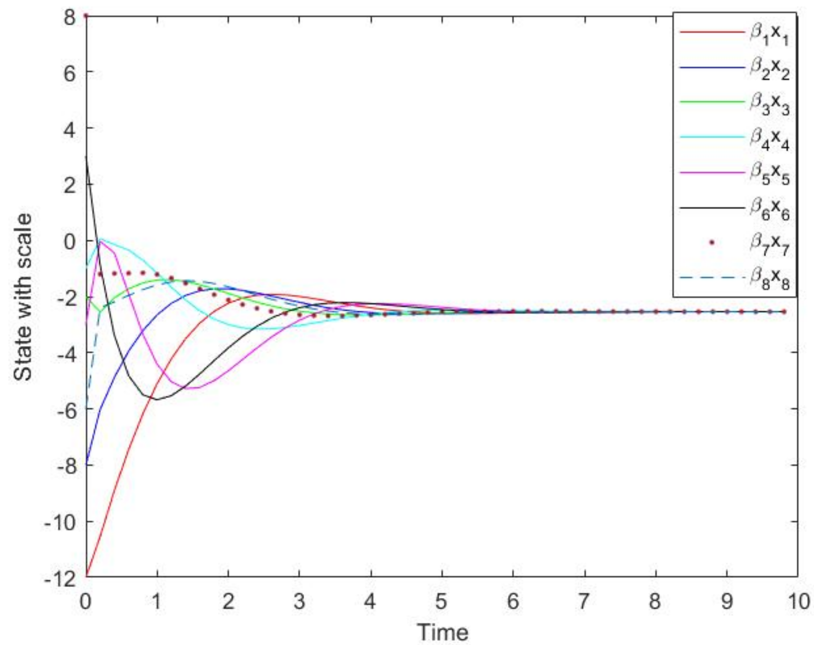

Figure 4.8: The state trajectories of all agents with scalar scales $(2, -2, 1, -1, 3, 1.5, -2, -1)$ under protocol (4.10) and $h = 0.02$.

Furthermore, if the scalar scales $\beta_i = 1$ for all i, the state trajectories of all agents under the consensus protocol (4.10) with $h = 0.02$ can be described as in Figure 4.9.

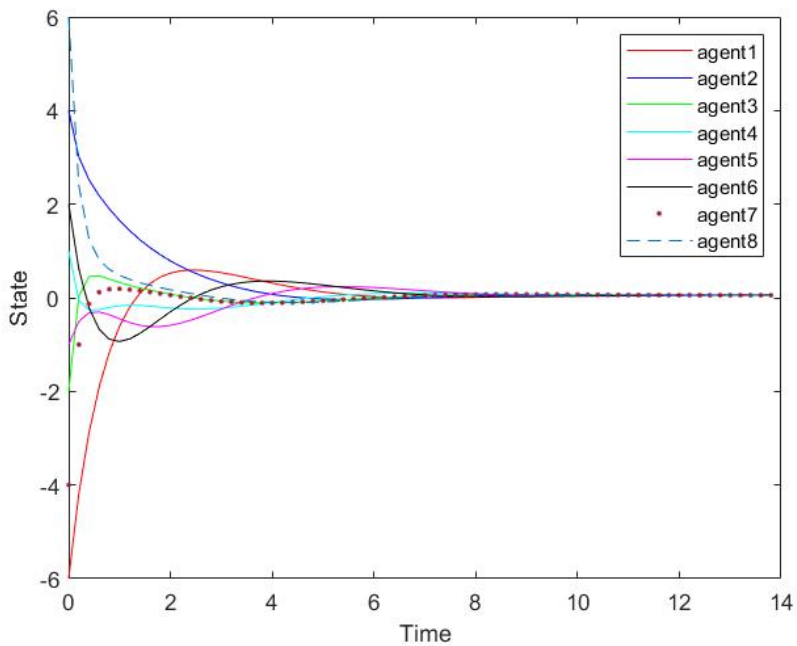

Figure 4.9: The state trajectories of all agents with scalar scales $(1, 1, 1, 1, 1, 1, 1, 1)$ and $h = 0.02$.

Moreover, if the sampling period $h = 0.42 > 0.33 = (d_{max}\beta_{max})^{-1}$ the state trajectories of all agent under the consensus protocol (4.10) are divergent as in Figure 4.10.

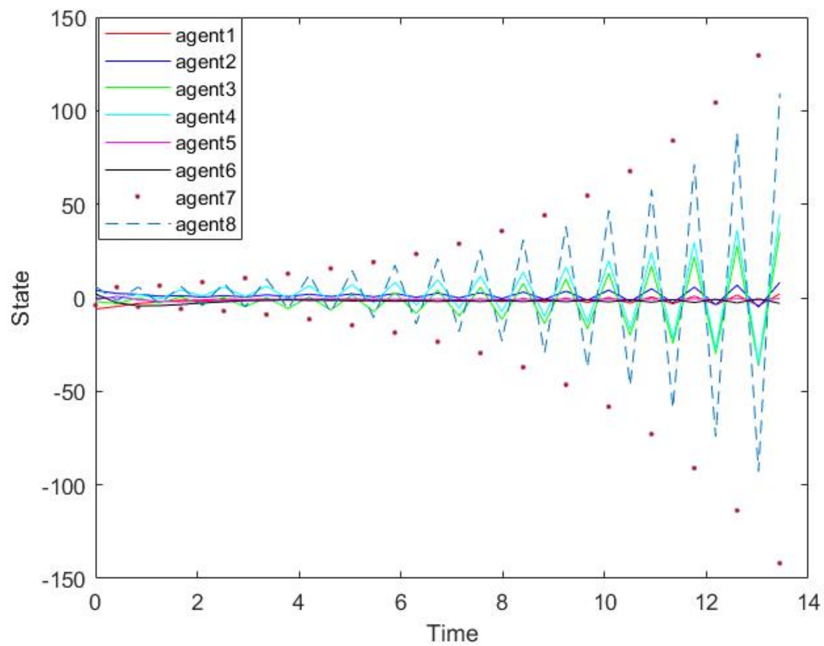

Figure 4.10: The state trajectories of all agents with scalar scales $(2, -2, 1, -1, 3, 1.5, -2, -1)$ under protocol (4.10) with $h = 0.42$.

Discussion

In this section, scaled consensus problems for the hybrid multi-agent system (4.1) consisting of directed communication networks have been studied. Two consensus protocols are proposed based on the interactions among agents. Firstly, we assume that the directed communication networks $\mathcal{G}$ contains a spanning tree with $0 < h < (d_{max}\beta_{max})^{-1}$ and interactions among agents occur in the sampling time t_k. Hence, by Theorem 4.1.2 and protocol (4.4), the hybrid multi-agent system (4.1) achieves scaled consensus to $(\beta_1, \ldots, \beta_n)$. Secondly, assume that the directed communication networks $\mathcal{G}$ contains a spanning tree with $0 < h < (d_{max}\beta_{max})^{-1}$ and interactions among agents occur in the sampling time t_k but the continuous-time dynamic agents can observe their own states in real time. By Theorem 4.1.3 and protocol (4.10), we show that the hybrid multi-agent system (4.1) achieves scaled consensus to $(\beta_1, \ldots, \beta_n)$.

Moreover, under the consensus protocols (4.4) and (4.10), we see that if $\beta_i = 1$ for all i, the state trajectories of all agents are as in Figure 4.4 and Figure 4.9. This shows that our scaled consensus results are more general than the consensus results of Zheng [132].

In addition, if the sampling period $0 < h < (d_{max}\beta_{max})^{-1}$, Theorem 4.1.2 and Theorem 4.1.3 can guarantee reaching scaled consensus to $(\beta_1, \ldots, \beta_n)$ as shown in Figure 4.2 and Figure 4.7. However, if $h > (d_{max}\beta_{max})^{-1}$, the hybrid multi-agent system (4.1) cannot achieves scaled consensus to $(\beta_1, \ldots, \beta_n)$ under protocols (4.4) and (4.10) as shown in Figure 4.5 and Figure 4.10.

4.2 Scaled consensus problems in Multi-agent systems

In this section, scaled consensus problems of multi-agent systems have been investigated by using impulsive consensus protocols

4.2.1 Scaled consensus of Nonlinear Multi-agent systems

In this section, we study scaled consensus problems of nonlinear multi-agent systems by using impulsive consensus protocols

Consider a nonlinear multi-agent system (NMAS) consisting of N agents labelled 1 through N. Then, each agent has the dynamics as follows:

$$\beta_i \dot{x}_i(t) = f(t, x_i(t)) + u_i(t), \quad \text{for} \;\; i \in \mathcal{I}_N \tag{4.19}$$

where $\beta_i \neq 0$ is a scalar scaled of agent i, $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the state and control input of agent i at time t, respectively. $f(\cdot)$ is a nonlinear function. The initial conditions are $x_i(0) = x_{i0}$, and $x(0) = [x_{10}, x_{20}, ..., x_{N0}]^T$.

In this work, we assume that all agents can update their states and interact with their neighbours only at the sampling time $t = t_k$. Thus, the impulsive consensus protocols can be described as follows:

$$u_i(t) = \sum_{k=1}^{\infty} \delta(t - t_k) \mathbf{B}_k |\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij}(\beta_j x_j(t) - \beta_i x_i(t)), \;\; \text{for} \;\; i \in \mathcal{I}_N, \tag{4.20}$$

where $\mathbf{B}_k \in \mathbb{R}^{N \times N}$ is an impulsive matrix to be determined later. The discrete time instant t_k satisfy $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \dots$ and $\lim_{k \to \infty} t_k = \infty$, $\delta(t)$ is the Dirac delta function.

Theorem 4.2.1. Let $\mathcal{G}$ be a directed connected communication network of the multi-agent system (4.19). The nonlinear multi-agent system (4.19) under protocol (4.20) reaches consensus if the following conditions are satisfied:

(i) $\mathcal{G}$ contains a spanning tree;

(ii) For any $x_i(t), x_j(t) \in \Omega \subset \mathbb{R}^n; i \neq j$ there exists a constant $\gamma > 0$, such that

$$(\beta_i x_i(t) - \beta_j x_j(t))^T (f(t, \beta_i x_i(t)) - f(t, \beta_j x_j(t))) \leq \gamma(\beta_i x_i(t) - \beta_j x_j(t))^T (\beta_i x_i(t) - \beta_j x_j(t));$$

(iii) There exist two constants τ_1 and τ_2 such that $0 < \tau_1 \leq t_k - t_{k-1} \leq \tau_2 < \infty, k \in \mathbb{N}_+$;

(iv) There exist some constants b_k, $0 < \alpha_k < 1$ and $0 < \eta < 1$ such that

$$(1 - \alpha_k)|\mathcal{B}|\mathcal{L} - 2b_k(|\mathcal{B}|\mathcal{L})^2 + (b_k)^2(|\mathcal{B}|\mathcal{L})^3 \leq 0$$

and

$$\alpha_k e^{\gamma(t_k - t_{k-1})} \leq \eta < 1, k \in \mathbb{N},$$

where $\mathcal{L}$ is the Laplacian matrix of $\mathcal{G}$ and Ω is a bounded set.

Proof. Let $\mathbf{B}_k = b_k \mathbf{I}_N$, where b_k is a constant. Then, the system (4.19) with protocol (4.20) can be written as

$$\begin{cases} \beta_i \dot{x}_i(t) = f(t, x_i(t)), & t \neq t_k, \\ \Delta \beta_i x_i(t_k) = b_k |\beta_i| \sum\limits_{j \in \mathcal{N}_i} a_{ij}(\beta_j x_j(t_k^-) - \beta_i x_i(t_k^-)), & k \in \mathbb{N}. \end{cases} \tag{4.21}$$

Let $\mathcal{B}\mathbf{x}(t) = [\beta_1 x_1(t), \beta_2 x_2(t), ..., \beta_N x_N(t)]^T$, $\mathcal{B} = diag(\beta_1, \beta_2, ..., \beta_N) \in \mathbb{R}^{N \times N}$, $H = diag(h_1, h_2, \ldots, h_N)$, and $|\mathcal{B}| = diag(|\beta_1|, |\beta_2|, ..., |\beta_N|) \in \mathbb{R}^{N \times N}$ then the system (4.21) can be written as

$$\begin{cases} \mathcal{B}\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), t), & t \neq t_k, \\ \Delta \mathcal{B}\mathbf{x}(t_k) = (-b_k |\mathcal{B}|\mathcal{L})\mathcal{B}\mathbf{x}(t_k^-), & k \in \mathbb{N}, \end{cases} \tag{4.22}$$

where $\mathbf{F}(t, x(t)) = (f(t, x_1(t)), f(t, x_2(t)), \ldots, f(t, x_N(t)))^T$. Then, we have

$$\begin{cases} \mathcal{B}\dot{\mathbf{x}}(t) = \mathbf{F}(t, \mathbf{x}(t)), & t \neq t_k, \\ \mathcal{B}\mathbf{x}(t_k^+) = (\mathbf{I}_N - b_k |\mathcal{B}|\mathcal{L})\mathcal{B}\mathbf{x}(t_k^-), & k \in \mathbb{N}. \end{cases} \tag{4.23}$$

Let $V_i(t) := V_i(\mathcal{B}\mathbf{x}(t)) = |\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij}(\beta_j x_j(t) - \beta_i x_i(t))^T(\beta_j x_j(t) - \beta_i x_i(t))$. Consider the Lyapunov function candidate

$$\begin{aligned}
V(t) &= \sum_{i=1}^{N} V_i(t) \\
&= |\beta_i| \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a_{ij}(\beta_j x_j(t) - \beta_i x_i(t))^T(\beta_j x_j(t) - \beta_i x_i(t)) \\
&= (\mathcal{B}\mathbf{x}(t))^T(|\mathcal{B}|\mathcal{L})(\mathcal{B}\mathbf{x}(t)).
\end{aligned}$$

Taking the Dini derivative of $V(t)$ for $t \in [t_{k-1}, t_k), k \in \mathbb{N}_+$, by condition (ii), one obtains

$$\begin{aligned}
\mathbf{D}^+V(t) &= \sum_{i=1}^{N} \mathbf{D}^+V_i(t) \\
&= 2|\beta_i| \sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i} a_{ij}(\beta_j x_j(t) - \beta_i x_i(t))^T(\beta_j \dot{x}_j(t) - \beta_i \dot{x}_i(t)) \\
&\le 2\gamma V(t).
\end{aligned}$$

Then

$$V(\mathbf{x}(t)) \le e^{2\gamma(t-t_{k-1})}V(t_{k-1}^+), \qquad t \in [t_{k-1}, t_k), k \in \mathbb{N}_+. \tag{4.24}$$

On the other hand, when $k \in \mathbb{N}_+$, by the condition (iv), we have

$$\begin{aligned}
V(t_k^+) &= (\mathcal{B}\mathbf{x}(t_k^+))^T|\mathcal{B}|\mathcal{L}(\mathcal{B}\mathbf{x}(t_k^+)) \\
&= (\mathcal{B}\mathbf{x}(t_k^-))^T[(1-\alpha_k)|\mathcal{B}|\mathcal{L} - 2b_k(|\mathcal{B}|\mathcal{L})^2 + (b_k)^2(|\mathcal{B}|\mathcal{L})^3](\mathcal{B}\mathbf{x}(t_k^-)) \\
&= (\mathcal{B}\mathbf{x}(t_k^-))^T[(1-\alpha_k)|\mathcal{B}|\mathcal{L} - 2b_k(|\mathcal{B}|\mathcal{L})^2 + (b_k)^2(|\mathcal{B}|\mathcal{L})^3 - \alpha_k|\mathcal{B}|\mathcal{L}](\mathcal{B}\mathbf{x}(t_k^-)) \\
&+ \alpha_k(\mathcal{B}\mathbf{x}(t_k^-))^T|\mathcal{B}|\mathcal{L}(\mathcal{B}\mathbf{x}(t_k^-)) \\
&\le \alpha_k V(t_k^-).
\end{aligned}$$

By mathematical induction, one obtains that, for $t \in [t_{k-1}, t_k), k \in \mathbb{N}_+, k \ge 2$,

$$V(t) \le e^{2\gamma(t-t_{k-1})}\prod_{i=1}^{k-1}\alpha_j e^{2\gamma(t_j - t_{j-1})}V(\mathbf{x}(t_0^+)). \tag{4.25}$$

It follows from the conditions (iii) and (iv) that

$$V(t) \leq e^{2|\beta|\tau_2}\eta^k V(\mathbf{x}(t_0^+)), \quad t \in [t_{k-1}, t_k), k \in \mathbb{N}_+, k \geq 2.$$

Hence, $V(\mathbf{x}(t)) \to 0$ as $t \to \infty$. Since $\mathcal{G}$ is connected, one obtains that

$$\lim_{t\to\infty} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0, \quad \forall i, j \in \mathcal{I}_N.$$

□

4.2.2 Scaled consensus of Multi-agent systems

In this section, scaled consensus problems of directed multi-agent systems with fixed and switching topologies have been studied.

Fixed topology

Consider a multi-agent system consists of n agents which are continuous-time dynamic agents, labelled 1 through n, and let $\mathcal{I}_n = \{1, 2, 3, ..., n\}$. Then, the dynamics of each agent with a scalar scale can be described as follows:

$$\beta_i \dot{x}_i(t) = u_i, \quad \text{for} \;\; i \in \mathcal{I}_n, \tag{4.26}$$

where $\beta_i \neq 0$, $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the scalar scale, state and control input of agent i, respectively. The initial conditions are $x_i(0) = x_{i0}$, and $x(0) = [x_{10}, x_{20}, ..., x_{n0}]^T$.

Definition 4.2.1. Given any scalar scale $\beta_i \neq 0$ for the agent i, the multi-agent system (4.26) is said to reach scaled consensus to $(\beta_1, \ldots, \beta_n)$ if for any initial conditions,

$$\lim_{t\to\infty} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0, \quad \text{for} \;\; i, j \in \mathcal{I}_n, \tag{4.27}$$

Remark. It can be seen that the scaled consensus problem is more general than the usual consensus problems, that is, if $\beta_i = 1$ for all i, then the scaled consensus problem can reduce to the standard consensus problems that have been studied in [18, 97, 98, 40, 104, 8, 6, 74, 80, 27, 79].

Assuming that the multi-agent system (4.26) has been modelled as a connected digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}) = \mathcal{G}^c \cup \mathcal{G}^d$, where $\mathcal{G}^c = (\mathcal{V}^c, \mathcal{E}^c, \mathcal{A})$ and $\mathcal{G}^d = (\mathcal{V}^d, \mathcal{E}^d, \mathcal{A}')$; $\mathcal{V}^c = \mathcal{V}^d = \mathcal{V}$ are the communication networks of system (4.26) at time $t \neq t_k$ called 'continuous graph' and at time $t = t_k$ called 'discrete graph', respectively. Given any scalar scales $\beta_i \neq 0$ for $i = 1, 2, \dots, n$, the scaled consensus protocol of multi-agent system (4.26) based on the continuous graph and discrete graph is defined as follows: for $t \in (t_{k-1}, t_k]$,

$$\begin{aligned} u_i(t) = |\beta_i| &\sum_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t) - \beta_i x_i(t)] \\ &+ h \cdot |\beta_i| \sum_{k=1}^{\infty} \sum_{l \in \mathcal{N}_i'} a_{il}'[\beta_l x_l(t) - \beta_i x_i(t)]\delta(t - t_k), \quad \text{for} \quad i \in \mathcal{I}_n, \end{aligned} \tag{4.28}$$

where β_i is the scalar scale of agent i; $h = t_k - t_{k-1}$ is a sampling period; $\mathcal{A} = [a_{ij}]$ (or $\mathcal{A}' = [a_{il}']$) is the weighted adjacency matrices associated with the graph $\mathcal{G}^c$ (or $\mathcal{G}^d$) and $\delta(\cdot)$ is the Dirac delta function. Thus, for each scalar scale $\beta_i \neq 0$, the multi-agent system (4.26) with protocol (4.28) can be written as:

$$\begin{aligned} \beta_i \dot{x}_i(t) = |\beta_i| &\sum_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t) - \beta_i x_i(t)] \\ &+ h|\beta_i| \sum_{k=1}^{\infty} \sum_{l \in \mathcal{N}_i'} a_{il}'[\beta_l x_l(t) - \beta_i x_i(t)]\delta(t - t_k), \quad \text{for} \quad i \in \mathcal{I}_n. \end{aligned} \tag{4.29}$$

By the definition of the Dirac delta function, the system (4.29) can be described as the impulsive system:

$$\begin{cases} \beta_i \dot{x}_i(t) = |\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t) - \beta_i x_i(t)], \quad t \in (t_{k-1}, t_k), \\ \Delta \beta_i x_i(t_k) = h|\beta_i| \sum_{l \in \mathcal{N}_i'} a_{il}'[\beta_l x_l(t_k) - \beta_i x_i(t_k)], \end{cases} \tag{4.30}$$

where $\Delta \beta_i x_i(t_k) = \beta_i x_i(t_k^+) - \beta_i x_i(t_k^-) : x_i(t_k^+) = \lim_{h \to 0^+} x_i(t_k + h)$ and $x_i(t_k^-) = \lim_{h \to 0^+} x_i(t_k - h)$.

With out loss of generality, we assume that the solution of system (4.30) is left continuous, that is, $\beta_i x_i(t_k^-) = \beta_i x_i(t_k)$ and let $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$, $\mathcal{B} = diag(\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{R}^{n \times n}$, $H = diag(h_1, h_2, \dots, h_n)$, and $|\mathcal{B}| = diag(|\beta_1|, |\beta_2|, \dots, |\beta_n|) \in \mathbb{R}^{n \times n}$. Then, the system (4.30) can be written as the form:

$$\begin{cases} \mathcal{B}\dot{x}(t) = -|\mathcal{B}|\mathcal{L}\mathcal{B}x(t), \quad & t \neq t_k, \\ \mathcal{B}x(t_k^+) = \big(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'\big)\mathcal{B}x(t_k), \quad & k \in \mathbb{N}. \end{cases} \tag{4.31}$$

Theorem 4.2.2. Let $\mathcal{G}$ be a directed connected communication network of the multi-agent system (4.26) and $\beta_i \neq 0$ be a scalar scale of agent i. The multi-agent system (4.26) with protocol (4.28) reaches scaled consensus to $(\beta_1, \ldots, \beta_n)$ if the following conditions are satisfied:

(i) $\mathcal{G}^c \cup \mathcal{G}^d$ are both balanced and contain a spanning tree;

(ii) $0 < h < \dfrac{1}{d_{max}\beta_{max}}$;

(iii) there exists a constant $0 < \alpha \leq 1$ such that

$$(1-\alpha)\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}' - (H|\mathcal{B}|\mathcal{L}')^T + \mathcal{L}'^T(H|\mathcal{B}|)^2\mathcal{L}' \leq 0,$$

where $d_{max} = \max_i\{d_{ii}\}$ and $\beta_{max} = \max_i\{|\beta_i|\}$, for $i \in \mathcal{I}_n$, $\mathcal{L}'$ is the Laplacian matrix of $\mathcal{G}^c \cup \mathcal{G}^d$ at $t = t_k$.

Proof. Since $\mathcal{G}^c \cup \mathcal{G}^d$ is balanced, from the consequence of [18], then $\bar{x} = Ave(x) = \dfrac{1}{n}\sum_{j=1}^{n} x_j$ is invariant quantity i.e., $\bar{x}(t) = \bar{x}(0) = \dfrac{1}{n}\sum_{j=1}^{n} x_j(0)$, which is not true for an arbitrary digraph. The invariant of $\bar{x}$ allows decomposition of x_i for $i = 1, 2, ..., n$ as in the following equation:

$$\beta_i\delta_i(t) = \beta_i x_i(t) - \beta_i\bar{x}, \quad t \in (t_{k-1}, t_k],$$

$\beta_i\delta_i(t_k^+) = \beta_i x_i(t_k^+) - \beta_i\bar{x}$ and $\beta_i\delta_i(t_k^-) = \beta_i\delta_i(t_k)$, $i = 1, 2, 3, ..., n$, with initial conditions $x(t_0) = x(0) = [x_{10}, x_{20}, ..., x_{n0}]^T$, where $\delta = (\delta_1, ..., \delta_n)^T$ is the error vector or disagreement vector. Thus,

$$\begin{cases} \mathcal{B}\dot{\delta}(t) = -|\mathcal{B}|\mathcal{L}\mathcal{B}\delta(t), & t \neq t_k \\ \mathcal{B}\delta(t_k^+) = [\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}']\mathcal{B}\delta(t_k), & t = t_k, \quad k \in \mathbb{N}. \end{cases} \tag{4.32}$$

Consider the Lyapunov function candidate as follows:

$$V(\delta) = (\mathcal{B}\delta)^T(\mathcal{B}\delta).$$

Let $V(\delta) =: V(\delta(t))$. Since $\mathcal{G}^c \cup \mathcal{G}^d$ is balanced, by Lemma 2.6.1 [74], we have

$$\hat{\mathcal{L}} = Sym(\mathcal{L}) = \frac{\mathcal{L} + \mathcal{L}^T}{2}$$

and by definition 2.6.2 [15], the total derivation of $V(\delta)$ with respect to (4.32) is

$$\begin{aligned}\dot{V}(t) &= (\mathcal{B}\dot{\delta})^T(\mathcal{B}\delta) + (\mathcal{B}\delta)^T(\mathcal{B}\dot{\delta}) \\ &= -(\mathcal{B}\delta)^T((|\mathcal{B}|\mathcal{L})^T + |\mathcal{B}|\mathcal{L})(\mathcal{B}\delta) \\ &= -2(\mathcal{B}\delta)^T(|\mathcal{B}|\hat{\mathcal{L}})(\mathcal{B}\delta) \\ &\leq -2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}})V(t).\end{aligned}$$

This implies that, for $t \in (t_{k-1}, t_k]$,

$$V(t) \leq e^{-2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}})(t-t_{k-1})}V(t_{k-1}^+).$$

On the other hand, when $t = t_{k-1}$, using the fact from the condition (iii) that there exists $0 < \alpha \leq 1$ such that

$$(1-\alpha)\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}' - (H|\mathcal{B}|\mathcal{L}')^T + \mathcal{L}'^T(H|\mathcal{B}|)^2\mathcal{L}' \leq 0,$$

one obtains that

$$\begin{aligned}V(t_{k-1}^+) &= (\mathcal{B}\delta)^T(t_{k-1})(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}')^T(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}')(\mathcal{B}\delta)(t_{k-1}) \\ &= (\mathcal{B}\delta)^T(t_{k-1})[\mathbf{I}_n - (H|\mathcal{B}|\mathcal{L}')^T - H|\mathcal{B}|\mathcal{L}' + \mathcal{L}'^T(H|\mathcal{B}|)^2\mathcal{L}' - \alpha\mathbf{I}_n + \alpha\mathbf{I}_n](\mathcal{B}\delta)(t_{k-1}) \\ &= (\mathcal{B}\delta)^T(t_{k-1})[(1-\alpha)\mathbf{I}_n - (H|\mathcal{B}|\mathcal{L}')^T - H|\mathcal{B}|\mathcal{L}' + \mathcal{L}'^T(H|\mathcal{B}|)^2\mathcal{L}'](\mathcal{B}\delta)(t_{k-1}) \\ &+ \alpha(\mathcal{B}\delta)^T(t_{k-1})(\mathcal{B}\delta)(t_{k-1}) \\ &\leq \alpha(\mathcal{B}\delta)^T(t_{k-1})(\mathcal{B}\delta)(t_{k-1}) \\ &= \alpha V(t_{k-1}).\end{aligned}$$

For $t \in (t_0, t_1]$,

$$V(t) \leq e^{-2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}})(t-t_0)}V(t_0^+),$$

which leads to

$$V(t_1^+) \leq \alpha e^{-2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}})(t_1-t_0)}V(t_0^+).$$

Similarly, for $t \in (t_1, t_2]$,

$$V(t) \leq e^{-2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}})(t-t_1)} V(t_1^+)$$
$$\leq \alpha e^{-2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}})(t-t_0)} V(t_0^+).$$

In general, for $t \in (t_{k-1}, t_k]$, we have

$$V(t) \leq \alpha^{k-1} e^{-2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}})(t-t_0^+)} V(t_0^+).$$

Hence,

$$|\delta(t)| \leq \alpha^{(k-1)/2} e^{-\lambda_2(|\mathcal{B}|\hat{\mathcal{L}})(t-t_0)} |\delta(t_0^+)|, t \in (t_{k-1}, t_k].$$

Therefore,

$$\|\beta_i x_i(t) - \beta_i \bar{x}\| \to 0 \quad \text{as} \quad t \to \infty \quad \text{or} \quad \lim_{t\to\infty} \beta_i x_i(t) = \beta_i \bar{x}, \quad \forall i \in \mathcal{I}_n.$$

This implies that, for $t \in (t_{k-1}, t_k]$,

$$\lim_{t\to\infty} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0 \qquad \text{for} \quad i, j \in \mathcal{I}_n.$$

TThis completes the proof. $\square$

Switching topologies

In this section, the consensus problems of multi-agent systems under switching topologies described by the impulsive systems have been studied. Using similar notations, the scaled consensus protocol for multi-agent system based on the scalar scale $\beta_i \neq 0$ under switching topology is defined as follows: for $t \in (t_{k-1}, t_k]$,

$$\begin{aligned} u_i(t) = {} & |\beta_i| \sum_{j\in\mathcal{N}_i(\sigma(t))} a_{ij}(\sigma(t))[\beta_j x_j(t) - \beta_i x_i(t)] \\ & + h \cdot |\beta_i| \sum_{k=1}^{\infty} \sum_{l\in\mathcal{N}_i'(s(k))} a'_{il}(s(k))[\beta_l x_l(t) - \beta_i x_i(t)]\delta(t - t_k), \quad \text{for} \quad i \in \mathcal{I}_n, \end{aligned} \tag{4.33}$$

where $h = t_k - t_{k-1}$ is a sampling period; $\mathcal{A} = [a_{ij}]$ (or $\mathcal{A}' = [a'_{ij}]$) is the weighted adjacency matrices associated with the graph $\mathcal{G}^c$ (or $\mathcal{G}^d$) and $\delta(\cdot)$ is the dirac delta function.

Furthermore, for some $r, m \in \mathbb{N}$, $\sigma : [0, \infty) \to \{1, 2, 3, \dots, r\}$ is a piecewise constant function and $s : \mathbb{N} \to \{1, 2, 3, \dots, m\}$ is a constant function called the continuous-time switching signal and discrete-time switching signal, respectively.

In this work, we assume that there is no switching on each impulsive interval, that is, $\sigma(t) = \sigma(k)$ for $t \in (t_{k-1}, t_k]$. Consequently, for any $\beta_i \neq 0$, the multi-agent system (4.26) with protocol (4.33) can be written as

$$\begin{cases} \dot{x}_i(t) = |\beta_i| \displaystyle\sum_{j \in \mathcal{N}_i(\sigma(k))} a_{ij}(\sigma(k))[\beta_j x_j(t) - \beta_i x_i(t)], \quad t \in (t_{k-1}, t_k), \\ \Delta x_i(t_k) = h \cdot |\beta_i| \displaystyle\sum_{l \in \mathcal{N}'_i(s(k))} a'_{il}(s(k))[\beta_l x_l(t_k) - \beta_i x_i(t_k)]. \end{cases} \tag{4.34}$$

Let $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$, $H = diag(h_1, h_2, \dots, h_n)$, and $|\mathcal{B}| = diag(|\beta_1|, |\beta_2|, ..., |\beta_n|) \in \mathbb{R}^{n \times n}$, the system (4.34) can be written as the form:

$$\begin{cases} \mathcal{B}\dot{x}(t) = -|\mathcal{B}|\mathcal{L}_{\sigma(k)}\mathcal{B}x(t), \quad t \neq t_k, \\ \Delta \mathcal{B}x(t) = -H|\mathcal{B}|\mathcal{L}'_{s(k)}\mathcal{B}x(t), \quad t = t_k, \qquad k \in \mathbb{N}, \end{cases} \tag{4.35}$$

where $\mathcal{L}_{\sigma(k)}$ and $\mathcal{L}'_{s(k)}$ are the Laplacian matrix of $\mathcal{G}^c \cup \mathcal{G}^d$. The time sequence t_k satisfies $0 < t_1 < t_2 < t_3 < \cdots < t_k < \cdots$, $\lim\limits_{t \to \infty} t_k = \infty$. Without loss of generality, we assume that $\lim\limits_{h \to 0^+} x_i(t_k - h) = x_i(t_k)$, that is, $x_i(t_k)$ is left-continuous.
Then, the system (4.35) can be written as

$$\begin{cases} \mathcal{B}\dot{x}(t) = -|\mathcal{B}|\mathcal{L}_{\sigma(k)}\mathcal{B}x(t), \quad t \neq t_k, \\ \mathcal{B}x(t_k^+) = \big(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(k)}\big)\mathcal{B}x(t_k), \qquad k \in \mathbb{N}. \end{cases} \tag{4.36}$$

Theorem 4.2.3. Let $\mathcal{G}$ be a directed connected communication network of the multi-agent system (4.26). The multi-agent system (4.26) under protocol (4.33) reaches scaled consensus to $(\beta_1, \dots, \beta_n)$ if the following conditions are satisfied:

(i) $\mathcal{G}^c \cup \mathcal{G}^d$ are both balanced and contain a spanning tree;

(ii) $0 < h < \dfrac{1}{d_{max}\beta_{max}}$;

(iii) there exists constants $0 < \alpha_i \leq 1$ such that

$$(1 - \alpha_i)\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(k-1)} - (H|\mathcal{B}|\mathcal{L}'_{s(k-1)})^T + \mathcal{L}'^T_{s(k-1)}(H|\mathcal{B}|)^2\mathcal{L}'_{s(k-1)} \leq 0,$$

where $d_{max} = \max_i\{d_i\}$ and $\beta_{max} = \max_i\{|\beta_i|\}$, for $i \in \mathcal{I}_n$; $\mathcal{L}'_{s(k-1)}$ is the Laplacian matrix of $\mathcal{G}^c \cup \mathcal{G}^d$ at $t = t_k$.

Proof. Let $\beta_i\delta_i(t) = \beta_i x_i(t) - \beta_i\bar{x}$, for $t \in (t_{k-1}, t_k]$, where $\bar{x} = \dfrac{1}{n}\sum_{j=1}^{n} x_j(0)$; $\beta_i\delta_i(t_k^+) = \beta_i x_i(t_k^+) - \beta_i\bar{x}$ and $\beta_i\delta_i(t_k^-) = \beta_i\delta_i(t_k)$, $i = 1, 2, 3, ..., n$, with initial conditions $x(t_0) = x(0) = [x_{10}, x_{20}, ..., x_{n0}]^T$, where $\delta = (\delta_1, ..., \delta_n)^T$ is the error vector or disagreement vector. Thus,

$$\begin{cases} \mathcal{B}\dot{\delta}(t) = -|\mathcal{B}|\mathcal{L}_{\sigma(k)}\mathcal{B}\delta(t), & t \neq t_k \\ \mathcal{B}\delta(t_k^+) = \big(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(k)}\big)\mathcal{B}\delta(t_k), & t = t_k, \quad k \in \mathbb{N}. \end{cases} \tag{4.37}$$

Consider the Lyapunov function candidate as follows:

$$V(\delta) = (\mathcal{B}\delta)^T(\mathcal{B}\delta).$$

Let $V(\delta) =: V(\delta(t))$. Since $\mathcal{G}^c \cup \mathcal{G}^d$ is balanced, by Lemma 2.6.1 [74] we have

$$\hat{\mathcal{L}} = Sym(\mathcal{L}) = \frac{1}{2}(\mathcal{L} + \mathcal{L}^T).$$

Hence, the total derivation of $V(\delta)$ with respect to (4.37) is

$$\begin{aligned} \dot{V}(t) &= (\mathcal{B}\dot{\delta})^T(\mathcal{B}\delta) + (\mathcal{B}\delta)^T(\mathcal{B}\dot{\delta}) \\ &= -(\mathcal{B}\delta)^T((|\mathcal{B}|\mathcal{L}_{\sigma(k)})^T + |\mathcal{B}|\mathcal{L}_{\sigma(k)})(\mathcal{B}\delta) \\ &= -2(\mathcal{B}\delta)^T(|\mathcal{B}|\hat{\mathcal{L}}_{\sigma(k)})(\mathcal{B}\delta) \\ &\leq -2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}}_{\sigma(k)})V(t), \qquad t \in (t_{k-1}, t_k). \end{aligned}$$

Then, for $t \in (t_{k-1}, t_k]$,

$$V(t) \leq e^{-2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}}_{\sigma(k)})(t - t_{k-1})}V(t_{k-1}^+).$$

On the other hand, when $t = t_{k-1}$, from the condition (iii), there exists $0 < \alpha_i \leq 1$ such that

$$(1 - \alpha_i)\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(k-1)} - (H|\mathcal{B}|\mathcal{L}'_{s(k-1)})^T + \mathcal{L}'^T_{s(k-1)}(H|\mathcal{B}|)^2\mathcal{L}'_{s(k-1)} \leq 0.$$

One obtains that

$$\begin{aligned} V(t_{k-1}^+) &= (\mathcal{B}\delta)^T(t_{k-1})(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(k-1)})^T(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(k-1)})(\mathcal{B}\delta)(t_{k-1}) \\ &\leq \alpha_{s(k-1)}(\mathcal{B}\delta)^T(t_{k-1})(\mathcal{B}\delta)(t_{k-1}) \\ &= \alpha_{s(k-1)}V(t_{k-1}). \end{aligned}$$

Then, for $t \in (t_{k-1}, t_k]$,

$$V(t) \leq \alpha_{s(1)}\alpha_{s(2)} \cdots \alpha_{s(k-1)} e^{-2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}}_{\sigma(k)})(t-t_{k-1})} \cdots e^{-2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}}_{\sigma(2)})(t_1-t_0)} V(t_0^+).$$

Let $\alpha = \max_k\{\alpha_{s(k)}\}$ and $\lambda_2(|\mathcal{B}|\hat{\mathcal{L}}) = \min_k\{\lambda_2(|\mathcal{B}|\hat{\mathcal{L}}_{\sigma(k)})\}$, then we have

$$V(t) \leq \alpha^{k-1} e^{-2\lambda_2(|\mathcal{B}|\hat{\mathcal{L}})(t-t_0)} V(t_0^+).$$

This implies that, for $t \in (t_{k-1}, t_k]$,

$$|\delta(t)| \leq \alpha^{(k-1)/2} e^{-\lambda_2(|\mathcal{B}|\hat{\mathcal{L}})(t-t_0)} |\delta(t_0^+)|.$$

Therefore,

$$\|\beta_i x_i(t) - \beta_i \bar{x}\| \to 0 \quad \text{as} \quad t \to \infty \quad \text{or} \quad \lim_{t\to\infty} \beta_i x_i(t) = \beta_i \bar{x}, \quad \forall i \in \mathcal{I}_n.$$

This implies that, for $t \in (t_{k-1}, t_k]$,

$$\lim_{t\to\infty} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0 \qquad \text{for} \quad i, j \in \mathcal{I}_n.$$

□

Theorem 4.2.4. Let $\mathcal{G}$ be a directed connected communication network of the multi-agent system (4.26) with the sampling period $0 < h < (d_{max}\beta_{max})^{-1}$. Assume that for $k = 1, 2, \ldots$ the interval $(t_{k-1}, t_k]$ are uniformly bounded. Then, the multi-agent system (4.26) under the protocol (4.33) achieves scaled consensus to $(\beta_1, \ldots, \beta_n)$ if $\mathcal{G}^c \cup \mathcal{G}^d$ contains a spanning tree.

Proof. For any $t > 0$, there exists a positive integer k such that $t \in (t_{k-1}, t_k]$. Then the solution of (4.36) with initial conditions $x(t_0) = x(0)$ can be obtained by Mathematical induction: for $t \in (t_{k-1}, t_k]$,

$$\begin{aligned}\mathcal{B}x(t) = {} & e^{-|\mathcal{B}|\mathcal{L}_{\sigma(k)}(t-t_{k-1})}(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(k-1)})\times \\ & \times e^{-|\mathcal{B}|\mathcal{L}_{\sigma(k-1)}(t_{k-1}-t_{k-2})} \cdots (\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(1)}) e^{-|\mathcal{B}|\mathcal{L}_{\sigma(1)}(t_1-t_0)} \mathcal{B}x(0),\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}x(t_k^+) = {} & (\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(k)}) e^{-|\mathcal{B}|\mathcal{L}_{\sigma(k)}(t_k-t_{k-1})}(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(k-1)})\times \\ & \times e^{-|\mathcal{B}|\mathcal{L}_{\sigma(k-1)}(t_{k-1}-t_{k-2})} \cdots (\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(1)}) e^{-|\mathcal{B}|\mathcal{L}_{\sigma(1)}(t_1-t_0)} \mathcal{B}x(0).\end{aligned}$$

Since $(t_{k-1}, t_k]$ are uniformly bounded for $k = 1, 2, \ldots,$ then by Lemma 2.6.5 we have $e^{-|\mathcal{B}|\mathcal{L}_{\sigma(k)}(t_k - t_{k-1})}$ is a stochastic indecomposable and aperiodic (SIA) matrix. Furthermore, Lemma 4.1.1 implies that $(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(k)})$ is also SIA because $0 < h < (d_{max}\beta_{max})^{-1}$ giving the other eigenvalues of $H|\mathcal{B}|\mathcal{L}'$ are less than 1 except one simple eigenvalue 0. Using the Lemma 2.6.3 and 2.6.4, there exists a column vector y such that

$$\lim_{t\to\infty} e^{-|\mathcal{B}|\mathcal{L}_{\sigma(k)}(t-t_{k-1})} \cdots (\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(1)}) e^{-|\mathcal{B}|\mathcal{L}_{\sigma(1)}(t_1 - t_0)} \mathcal{B}x(0) = 1y^T$$

and

$$\lim_{t_k\to\infty} (\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(k)}) e^{-|\mathcal{B}|\mathcal{L}_{\sigma(k)}(t_k-t_{k-1})} \cdots (\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(1)}) e^{-|\mathcal{B}|\mathcal{L}_{\sigma(1)}(t_1 - t_0)} \mathcal{B}x(0) = 1y^T.$$

Therefore, the multi-agent network described by system (4.37) reaches scaled consensus to $(\beta_1, \ldots, \beta_n)$.

□

4.2.3 Scaled consensus of MASs with external perturbations

In some practical applications, the external perturbations or noises often exist in communications between agents. In this section, we study scaled consensus problem in MASs with external perturbations. The dynamic of agent i with a nonzero scalar scale β_i is defined as follows:

$$\beta_i \dot{x}_i(t) = u_i(t) + w_i(t), \quad \text{for } \ i \in \mathcal{I}_n \tag{4.38}$$

where $w_i(t)$ is an external disturbance for agent i and

$$\begin{aligned} u_i(t) = |\beta_i| &\sum_{j\in\mathcal{N}_i(\sigma(t))} a_{ij}(\sigma(t))[\beta_j x_j(t) - \beta_i x_i(t)] \\ &+ h \cdot |\beta_i| \sum_{k=1}^{\infty} \sum_{l\in\mathcal{N}'_i(s(k))} a'_{il}(s(k))[\beta_l x_l(t) - \beta_i x_i(t)]\delta(t - t_k), \quad \text{for } \ i \in \mathcal{I}_n, \end{aligned}$$

where $t \in (t_{k-1}, t_k]$ and β_i is a nonzero scalar scale of agent i is the control input.

By the definition of Dirac delta function, (4.38) can be described as

$$\begin{cases} \mathcal{B}\dot{x}(t) = -|\mathcal{B}|\mathcal{L}_{\sigma(k)}\mathcal{B}x(t) + w(t), & t \neq t_k, \\ \mathcal{B}x(t_k^+) = \big(\mathbf{I}_n - \mathbf{H}|\mathcal{B}|\mathcal{L}'_{s(k)}\big)\mathcal{B}x(t_k), & k \in \mathbb{N} \end{cases} \tag{4.39}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t), ..., x_n(t))^T$ and $\mathbf{w}(t) = (w_1(t), w_2(t), ..., w_n(t))^T \in \mathbb{R}^n$, $\mathbf{H} = diag(h_1, h_2, \ldots, h_n)$, and $|\mathcal{B}| = diag(|\beta_1|, |\beta_2|, ..., |\beta_n|) \in \mathbb{R}^{n\times n}$.

To suppress disturbances in the network and make all agents reach consensus, we define an output function $z_i(t) = \beta_i \eta_i(t) = \beta_i x_i - \beta_i \bar{x},\ i = 1, 2, \ldots, n$, where $\bar{x} = \dfrac{1}{n}\sum_{j=1}^{n} x_j(0)$. Then, system (4.39) can be described as

$$\begin{cases} \mathcal{B}\dot{\eta}(t) = -|\mathcal{B}|\mathcal{L}_{\sigma(k)}\mathcal{B}\eta(t) + \mathbf{w}(t), & t \neq t_k \\ \mathcal{B}\eta(t_k^+) = \big(\mathbf{I}_n - \mathbf{H}|\mathcal{B}|\mathcal{L}'_{s(k)}\big)\mathcal{B}\eta(t_k), & t = t_k, \quad k \in \mathbb{N} \\ \mathbf{z}(t) = \mathcal{B}\eta(t), \\ \mathcal{B}\eta(t_0^+) = \mathcal{B}\eta(0) \end{cases} \tag{4.40}$$

where $\mathbf{z}(t)$ is the controlled output, $t_0 \geq 0$ is the initial time and $\eta = (\eta_1, \eta_2, \ldots, \eta_n)^T$ is an error vector.

For the disturbance signal $\mathbf{w}(\cdot) \in \mathbb{R}^n$, define

$$\|\mathbf{w}\|_T = \Big[\int_0^T \|\mathbf{w}(t)\|^2 dt\Big]^{1/2} = \Big[\int_0^T \mathbf{w}(t)^T\mathbf{w}(t)dt\Big]^{1/2},$$

where $T > 0$ is an arbitrary constant. Then, $\mathbf{w}(t)$ is said to belong to $L_2[0, T]$, if $\|\mathbf{w}\|_T < \infty$. Throughout this work, it is assumed that the disturbance input $\mathbf{w}(t) \in L_2[0, T]$.

Then, the robust $\mathbf{H}_\infty$ problem to be addressed can be formulated to achieve the following objectives:

i) System (4.40) is exponentially stable when $\mathbf{w}(t) = 0$.

ii) Under the zero-initial condition, the controlled output $z(t)$ satisfies

$$\|\mathbf{z}(t)\|_T \leq \gamma\|\mathbf{w}(t)\|_T,$$

for any nonzero $\mathbf{w}(t) \in L_2[0, T]$, where $\gamma > 0$ is a prescribed scalar. Moreover, the above conditions are often called robust $\mathbf{H}_\infty$ criteria for system (4.40).

Theorem 4.2.5. Let $\mathcal{G}$ be a directed connected communication network of the multi-agent system (4.38) with the sampling period $0 < h < (d_{max}\beta_{max})^{-1}$. If there exist positive scalars $\lambda, \gamma, 0 < \alpha_r < 1$ and positive-definite matrices $\mathbf{P}_i$ such that

$$\frac{1}{\gamma^2}\mathbf{P}_i\mathbf{P}_i^T - \mathbf{P}_i|\mathcal{B}|\mathcal{L}_i - (|\mathcal{B}|\mathcal{L}_i)^T\mathbf{P}_i + 2\lambda\mathbf{P}_i + \mathbf{I}_n < 0 \tag{4.41}$$

and

$$\left(\mathbf{I}_n - \mathbf{H}|\mathcal{B}|\mathcal{L}'\right)^T \mathbf{P}_i \left(\mathbf{I}_n - \mathbf{H}|\mathcal{B}|\mathcal{L}'\right) - \alpha_r \mathbf{P}_r \leq 0, \tag{4.42}$$

where $i, r = 1, 2, \ldots, l$ and $j = 1, 2, \ldots, m$, then, the impulsive system (4.40) satisfies the robust $\mathbf{H}_\infty$ criteria i) and ii).

Proof. For the system (4.40), the Lyapunov functions candidate can be defined as

$$\mathbf{V}_{\sigma(k)}(\delta) = (\mathcal{B}\eta)^T \mathbf{P}_{\sigma(k)} (\mathcal{B}\eta),$$

where $\mathcal{B} = diag(\beta_1, \beta_2, ..., \beta_n) \in \mathbb{R}^{n\times n}$ and $\sigma(k) \in \{1, 2, \ldots, l\}$.
Let $\mathbf{V}_{\sigma(k)} =: \mathbf{V}_{\sigma(k)}(\eta(t))$, $\eta =: \eta(t)$ and define

$$\psi_{\sigma(\mathbf{k})}(t) = \dot{\mathbf{V}}_{\sigma(k)} + 2\lambda \mathbf{V}_{\sigma(k)}(t) + \|\mathbf{z}\|^2 - \gamma^2 \|\mathbf{w}\|^2.$$

It follows from (4.40) and (4.41) that

$$\begin{aligned}
&\psi_{\sigma(\mathbf{k})}(t) \\
&= (\mathcal{B}\eta)^T \Big(- \mathbf{P}_{\sigma(k)} |\mathcal{B}| \mathcal{L}_{\sigma(k)} - (|\mathcal{B}|\mathcal{L}_{\sigma(k)})^T P_{\sigma(k)} + 2\lambda \mathbf{P}_{\sigma(k)} + \mathbf{I}_n \Big)(\mathcal{B}\eta) + (\mathcal{B}\eta)^T \mathbf{P}_{\sigma(k)} \mathbf{w}(t) \\
&+ \mathbf{w}(t)^T \mathbf{P}_{\sigma(k)} (\mathcal{B}\eta) - \gamma^2 \mathbf{w}(t)^T \mathbf{w}(t) \\
&= -\Big(\gamma \mathbf{w}(t) - \frac{1}{\gamma} P^T_{\sigma(k)} (\mathcal{B}\eta)\Big)^T \Big(\gamma \mathbf{w}(t) - \frac{1}{\gamma} \mathbf{P}^T_{\sigma(k)} (\mathcal{B}\eta)\Big) \\
&+ (\mathcal{B}\eta)^T \Big(\frac{1}{\gamma^2} \mathbf{P}_{\sigma(k)} \mathbf{P}^T_{\sigma(k)} - \mathbf{P}_{\sigma(k)} |\mathcal{B}| \mathcal{L}_{\sigma(k)} - (|\mathcal{B}|\mathcal{L}_{\sigma(k)})^T \mathbf{P}_{\sigma(k)} + 2\lambda \mathbf{P}_{\sigma(k)} + \mathbf{I}_n \Big)(\mathcal{B}\eta) \\
&\leq (\mathcal{B}\eta)^T \Big(\frac{1}{\gamma^2} \mathbf{P}_{\sigma(k)} \mathbf{P}^T_{\sigma(k)} - \mathbf{P}_{\sigma(k)} |\mathcal{B}| \mathcal{L}_{\sigma(k)} - (|\mathcal{B}|\mathcal{L}_{\sigma(k)})^T \mathbf{P}_{\sigma(k)} + 2\lambda \mathbf{P}_{\sigma(k)} + \mathbf{I}_n \Big)(\mathcal{B}\eta) \\
&< 0, \quad t \in (t_{k-1}, t_k].
\end{aligned}$$

Namely, for $t \in (t_{k-1}, t_k]$,

$$\dot{\mathbf{V}}_{\sigma(k)} + 2\lambda \mathbf{V}_{\sigma(k)}(t) + \|\mathbf{z}\|^2 - \gamma^2 \|\mathbf{w}\|^2 < 0. \tag{4.43}$$

It follows from (4.43) that for $t \in (t_{k-1}, t_k]$,

$$\int_{t_{k-1}}^{t_k} \Big(\dot{\mathbf{V}}_{\sigma(k)} + 2\lambda \mathbf{V}_{\sigma(k)}(t) + \|\mathbf{z}\|^2 - \gamma^2 \|\mathbf{w}\|^2\Big) dt < 0. \tag{4.44}$$

For any $T \in (t_{k-1}, t_k]$ we denote that

$$\Lambda_1 = \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \dot{\mathbf{V}}_{\sigma(i)}(t)dt + \int_{t_{k-1}}^{T} \dot{\mathbf{V}}_{\sigma(k)}(t)dt$$
$$\Lambda_2 = \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} 2\lambda \mathbf{V}_{\sigma(i)}(t)dt + \int_{t_{k-1}}^{T} 2\lambda \mathbf{V}_{\sigma(k)}(t)(t)dt$$
$$\Lambda_3 = \int_{t_0}^{T} \Big(\|\mathbf{z}\|^2 - \gamma^2\|\mathbf{w}\|^2\Big)dt.$$

Applying (4.44) successively on each subinterval from t_0 to T with $\eta(t_0) = 0$ gives

$$\Lambda_1 + \Lambda_2 + \Lambda_3 < 0. \tag{4.45}$$

It follows (4.42) from

$$\begin{aligned}
&\mathbf{V}_{\sigma(k)}(t_{k-1}^{+}) - \mathbf{V}_{\sigma(k)}(t_{k-1}) \\
&= (\mathcal{B}\eta(t_{k-1}))^T\Big(\mathbf{I}_n - \mathbf{H}|\mathcal{B}|\mathcal{L}'_{s(k-1)}\Big)^T\mathbf{P}_{\sigma(k)}\Big(\mathbf{I}_n - \mathbf{H}|\mathcal{B}|\mathcal{L}'_{s(k-1)}\Big)(\mathcal{B}\eta(t_{k-1})) \\
&- (\mathcal{B}\eta(t_{k-1}))^T\mathbf{P}_{\sigma(k-1)}(\mathcal{B}\eta(t_{k-1})) \\
&= (\mathcal{B}\eta(t_{k-1}))^T\Big[\big(\mathbf{I}_n - \mathbf{H}|\mathcal{B}|\mathcal{L}'_{s(k-1)}\big)^T\mathbf{P}_{\sigma(k)}\big(\mathbf{I}_n - \mathbf{H}|\mathcal{B}|\mathcal{L}'_{s(k-1)}\big) \\
&- \mathbf{P}_{\sigma(k-1)}\Big](\mathcal{B}\eta(t_{k-1})) \\
&= (\mathcal{B}\eta(t_{k-1}))^T\Big[\big(\mathbf{I}_n - \mathbf{H}|\mathcal{B}|\mathcal{L}'_{s(k-1)}\big)^T\mathbf{P}_{\sigma(k)}\big(\mathbf{I}_n - \mathbf{H}|\mathcal{B}|\mathcal{L}'_{s(k-1)}\big) \\
&- \alpha_{\sigma(k)}\mathbf{P}_{\sigma(k-1)}\Big](\mathcal{B}\eta(t_{k-1})) + (\mathcal{B}\eta(t_{k-1}))^T\big(\alpha_{\sigma(k)}\mathbf{P}_{\sigma(k-1)} - \mathbf{P}_{\sigma(k-1)}\big)(\mathcal{B}\eta(t_{k-1})) \\
&\leq -(1-\alpha_{\sigma(k)})\mathbf{V}_{\sigma(k-1)}(t_{k-1}), \quad k = 1, 2, \ldots
\end{aligned}$$

that

$$\begin{aligned}
&\int_{t_0}^{t_1} \dot{\mathbf{V}}_{\sigma(1)}(t)dt + \cdots + \int_{t_{k-2}}^{t_{k-1}} \dot{\mathbf{V}}_{\sigma(k-1)}(t)dt + \int_{t_{k-1}}^{T} \dot{\mathbf{V}}_{\sigma(k)}(t)dt \\
&= \mathbf{V}_{\sigma(k)}(T) - \sum_{i=1}^{k-1}\Big(\mathbf{V}_{\sigma(i+1)}(t_i^{+}) - \mathbf{V}_{\sigma(i)}(t_i)\Big) \\
&> 0.
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\dot{\mathbf{V}}_{\sigma(i)}(t)dt+\int_{t_{k-1}}^{T}\dot{\mathbf{V}}_{\sigma(k)}(t)dt+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}2\lambda\mathbf{V}_{\sigma(i)}(t)dt+\int_{t_{k-1}}^{T}2\lambda\mathbf{V}_{\sigma(k)}(t)(t)dt>0. \tag{4.46}$$

It follows from (4.45) and (4.46) that

$$\|\mathbf{z}\|_T^2-\gamma^2\|\mathbf{w}\|_T^2<0, \tag{4.47}$$

which immediately yields that $\|\mathbf{z}\|_T<\gamma\|\mathbf{w}\|_T$.

If $\mathbf{w}=0$, it follows from (4.43) that

$$\dot{\mathbf{V}}_{\sigma(k)}+2\lambda\mathbf{V}_{\sigma(k)}(t)<0,\quad t\in(t_{k-1},t_k]$$

which implies that for $t\in(t_{k-1},t_k]$,

$$\mathbf{V}_{\sigma(k)}(t)\leq\mathbf{V}_{\sigma(k)}(t_{k-1}^+)e^{-2\lambda(t-t_{k-1})}. \tag{4.48}$$

At time instant $t=t_k^+$,

$$\begin{aligned}\mathbf{V}_{\sigma(k+1)}(t_k^+)&=(\mathcal{B}\eta(t_k))^T\Big(\mathbf{I}_n-\mathbf{H}|\mathcal{B}|\mathcal{L}'_{s(k)}\Big)^T\mathbf{P}_{\sigma(k+1)}\Big(\mathbf{I}_n-\mathbf{H}|\mathcal{B}|\mathcal{L}'_{s(k)}\Big)(\mathcal{B}\eta(t_k))\\&\leq\alpha_{\sigma(k)}\mathbf{V}_{\sigma(k)}(t_k).\end{aligned} \tag{4.49}$$

The following results come from (4.48) and (4.49). For $t\in(t_0,t_1]$,

$$\mathbf{V}_{\sigma(1)}(t)\leq\mathbf{V}_{\sigma(1)}(t_0^+)e^{-2\lambda(t-t_0)},$$

which leads to

$$\mathbf{V}_{\sigma(1)}(t_1)\leq\mathbf{V}_{\sigma(1)}(t_0^+)e^{-2\lambda(t_1-t_0)},$$

and

$$\mathbf{V}_{\sigma(2)}(t_1^+)\leq\alpha_{\sigma(1)}\mathbf{V}_{\sigma(1)}(t_1)\leq\mathbf{V}_{\sigma(1)}(t_0^+)\alpha_{\sigma(1)}e^{-2\lambda(t_1-t_0)}.$$

In general, for $t\in(t_{k-1},t_k]$,

$$\mathbf{V}_{\sigma(k)}(t)\leq\mathbf{V}_{\sigma(1)}(t_0^+)\alpha_{\sigma(1)}\cdots\alpha_{\sigma(k-1)}e^{-2\lambda(t-t_0)}.$$

That is, system (4.39) is exponentially stable. This completes the proof. □

Simulations and Discussion

In this section, two examples have been provided to demonstrate the effectiveness of theoretical results in this work.

Example 1. Assume that there are 4 agents denoted by $1-4$ and the initial conditions are denoted by $x(0) = [1 \quad -1 \quad 2 \quad -2]^T$. Consider the fixed communication network $\mathcal{G}$ shown in Figure 4.11, where the dashed lines mean that each agent exchanges information at time $t = t_k$.

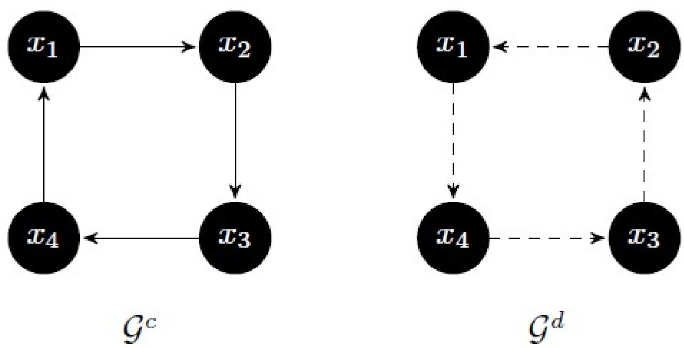

Figure 4.11: A connected directed network $\mathcal{G}$ at time $t \neq t_k$ and $t = t_k$, respectively.

It is obviously that $\mathcal{G}^c \cup \mathcal{G}^d$ is balanced and contain a directed spanning tree with $d_{max} = 1$. Let scalar scales $\beta = (1 \quad 0.4 \quad 1.5 \quad -2)^T$ and choose the sampling period $h = 0.2$, then $h = 0.2 < 0.5 = (\beta_{max} d_{max})^{-1}$ and by using MATLAB, one obtains

$$(1-\alpha)\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}' - (H|\mathcal{B}|\mathcal{L}')^T + (H|\mathcal{B}|)^2 \mathcal{L}'^T \mathcal{L}' \leq 0,$$

for some appropriate α. Thus, the assumption (ii) and (iii) are satisfied.

Thus, using Theorem 4.2.2 and consensus protocol (4.28), the scaled consensus problem is solved and the state trajectories of all agents are shown in Figure 4.12.

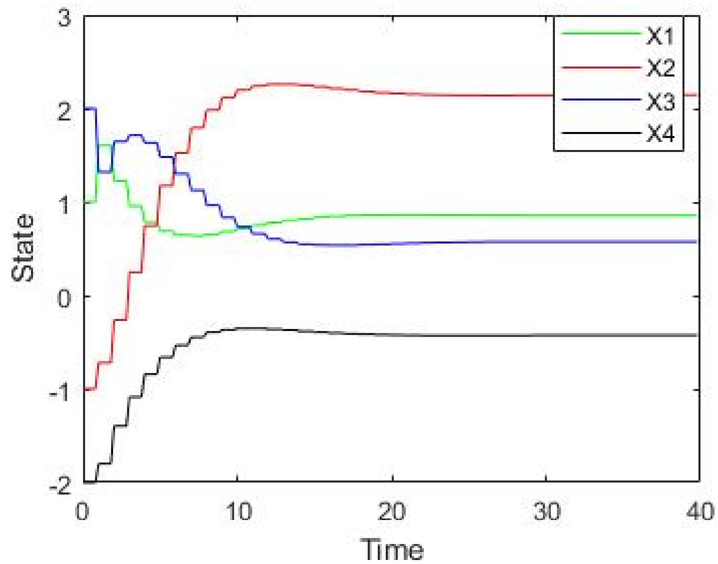

Figure 4.12: The state trajectories of all agents using scaled consensus protocol (4.28) with scalar scales $\beta = (1 \quad 0.4 \quad 1.5 \quad -2)^T$ for $h = 0.2$.

In addition, it can be seen that if $h = 0.84 > 0.5$, which implies that the condition (ii) is not satisfied, and hence the consensus protocol (4.28) cannot guarantee achieving scaled consensus to $\beta = (1 \;\; 0.4 \quad 1.5 \quad -2)^T$ as shown in Figure 4.13.

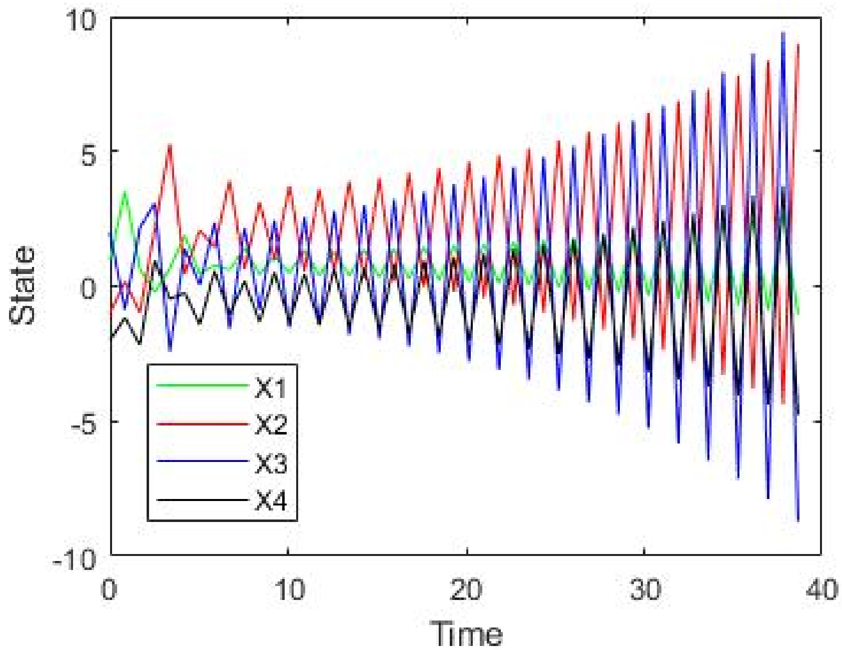

Figure 4.13: The state trajectories of all agents using scaled consensus protocol (4.28) with scalar scales $\beta = (1 \;\; 0.4 \quad 1.5 \quad -2)^T$ for $h = 0.84$.

Moreover, if the scalar scales of all agent are equal to one i.e., $\beta_i = 1$ for all i, the state trajectories of agents under protocol (4.28) with $h = 0.2$ can be described as in Figure 4.14.

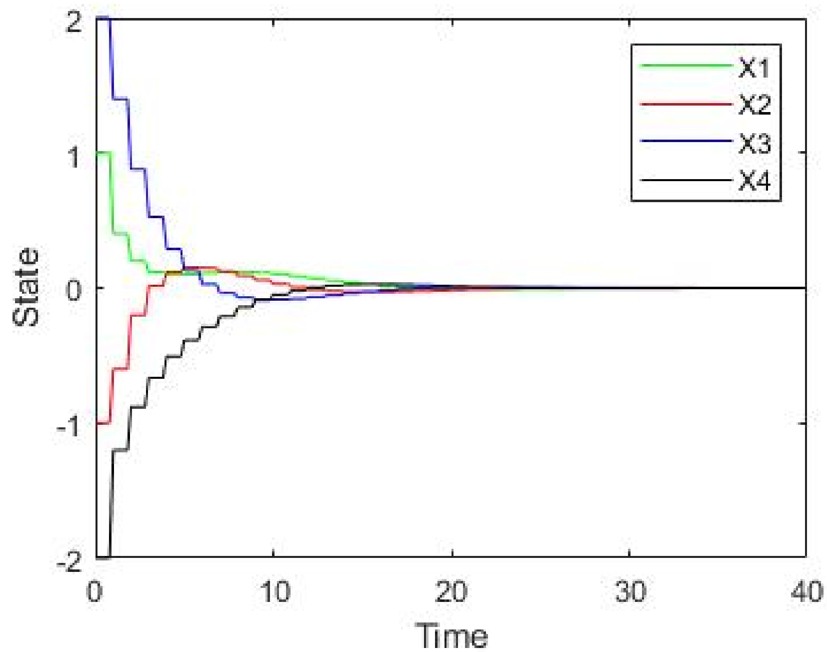

Figure 4.14: The state trajectories of all agents using scaled consensus protocol (4.28) with scalar scales $\beta = (1 \;\; 1 \;\; 1 \;\; 1)^T$ and $h = 0.2$.

Example 2. Consider the communication networks of the multi-agent system (4.26) with switching topologies consisting of 4 agents denoted by $1-4$ as shown in Figure 4.15. Assume that the impulse and switching occur simultaneously at the sampling time t_k and the switching happens in order of $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ and $\{\mathcal{G}'_1, \mathcal{G}'_2\}$, respectively.

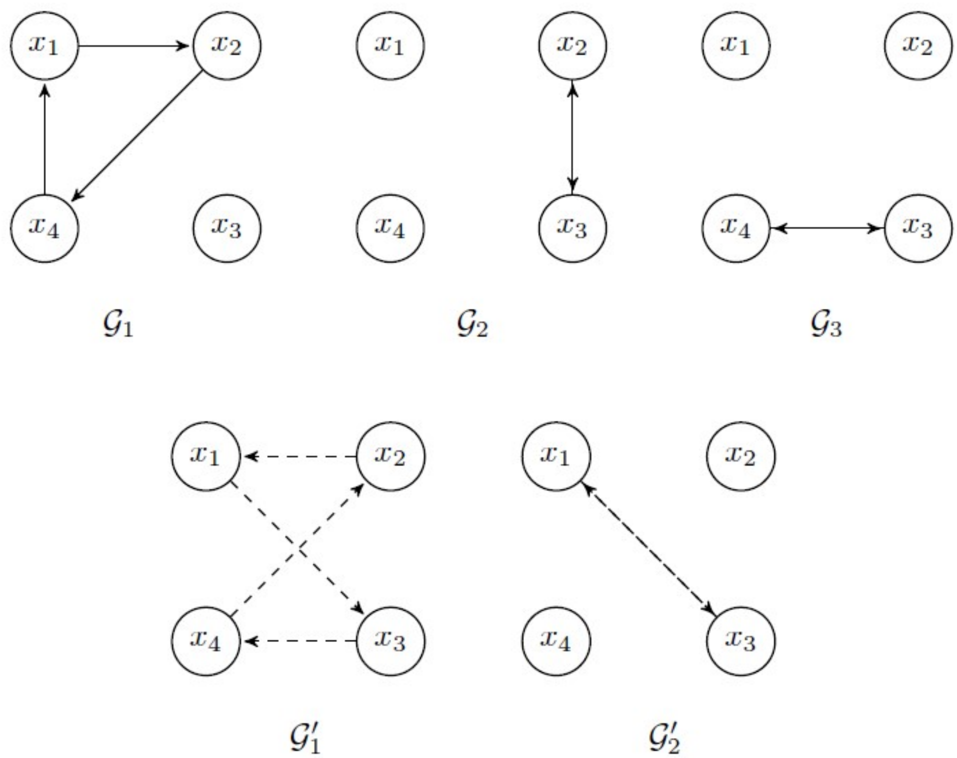

Figure 4.15: Switching topologies.

Clearly, each digraph of $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ and $\{\mathcal{G}'_1, \mathcal{G}'_2\}$ is balanced and contains a spanning tree. Given scalar scales $\beta = (1 \quad 0.4 \quad 1.5 \quad -2)^T$ and the initial conditions $x(0) = [1 \quad -1 \quad 2 \quad -2]^T$, then $\beta_{max} = 2$ and $d_{max} = 1$.

Choosing the sampling period $h = 0.34 < 0.5 = (\beta_{max} d_{max})^{-1}$ and for some suitable α_i (using MATLAB), we have

$$(1 - \alpha_i)\mathbf{I}_n - \mathbf{H}|\mathcal{B}|\mathcal{L}'_{s(k-1)} - (\mathbf{H}|\mathcal{B}|\mathcal{L}'_{s(k-1)})^T + \mathcal{L}'^T_{s(k-1)}(\mathbf{H}|\mathcal{B}|)^2\mathcal{L}'_{s(k-1)} \leq 0.$$

Thus, the assumption (ii) and (iii) hold. Hence, the scaled consensus problem is solved by using protocol (4.33) and the state of all agents are as in Figure 4.16.

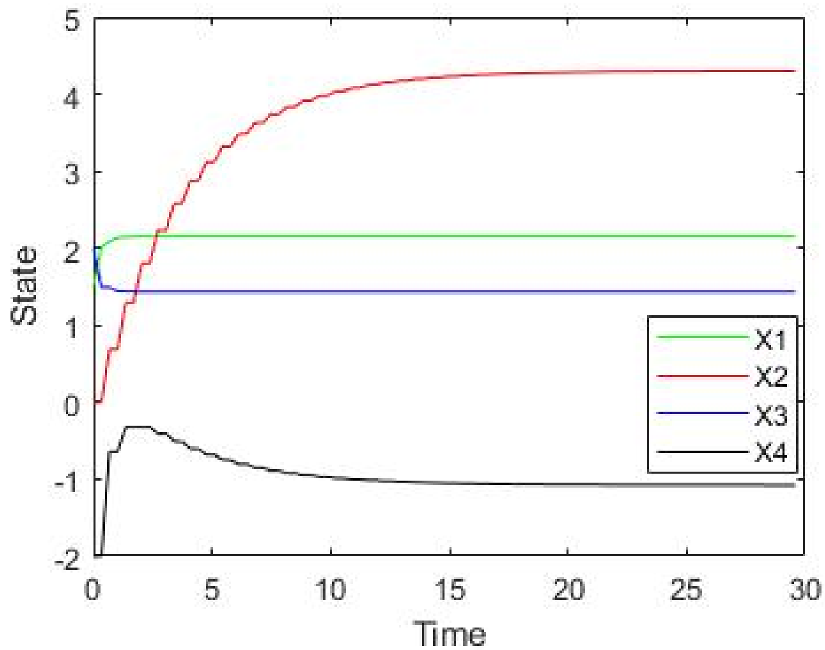

Figure 4.16: The state trajectories of all agents using the consensus protocol (4.33) with scalar scales $\beta = (1 \;\; 0.4 \;\;\; 1.5 \;\;\; -2)^T$ and $h = 0.34$.

However, if we choose $h = 0.2$ and $\beta = (1 \;\; 0.5 \quad 9.45 \quad -2)^T$, it can be seen that the assumption (ii) is not satisfied. This implies that the consensus protocol (4.33) cannot guarantee solving scaled consensus problem if all conditions are not satisfied (see Figure 4.17 for state trajectories).

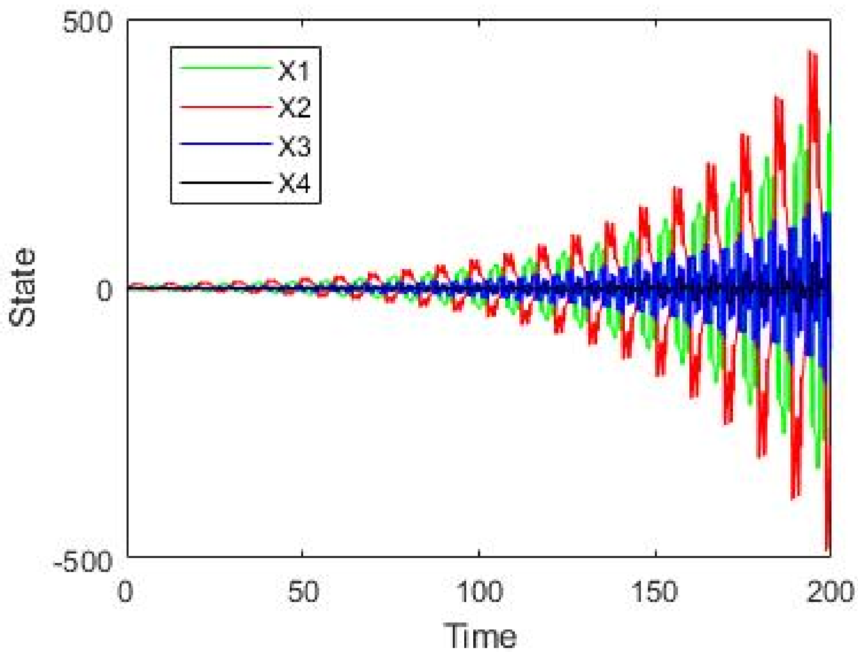

Figure 4.17: The state trajectories of all agents using the consensus protocol (4.33) with scalar scales $\beta = (1 \;\; 0.5 \quad 9.45 \quad -2)^T$ and $h = 0.2$.

4.2.4 Scaled consensus problems of MASs with impulsive time delays

Fixed topology

In this section, the scaled consensus problems of multi-agent systems under fixed topology have been studied.

Assuming that the multi-agent system (4.26) has been modelled as a connected digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A}) = \mathcal{G}^c \cup \mathcal{G}^d$, where $\mathcal{G}^c = (\mathcal{V}^c, \mathcal{E}^c, \mathcal{A})$ and $\mathcal{G}^d = (\mathcal{V}^d, \mathcal{E}^d, \mathcal{A}'); \mathcal{V}^c = \mathcal{V}^d = \mathcal{V}$ are the communication networks of system (4.26) at time $t \neq t_k$ called 'continuous graph' and at time $t = t_k$ called 'discrete graph', respectively. Given any scalar scales $\beta_i \neq 0$ for $i = 1, 2, \ldots, n$ and $\mathcal{I}_n = \{1, 2, \ldots, n\}$, the scaled consensus protocol of multi-agent system (4.26) based on the continuous graph and discrete graph is defined as follows: for $t \in (t_{k-1}, t_k]$,

$$\begin{aligned} u_i(t) = |\beta_i| &\sum_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t) - \beta_i x_i(t)] \\ &+ h \cdot |\beta_i| \sum_{k=1}^{\infty} \sum_{l \in \mathcal{N}'_i} a'_{il}[\beta_l x_l(t-\tau) - \beta_i x_i(t-\tau)]\delta(t-t_k), \qquad \text{for} \quad i \in \mathcal{I}_n, \end{aligned} \tag{4.50}$$

where β_i is the scalar scale of agent i; $h = t_k - t_{k-1}$ is a sampling period; $\mathcal{A} = [a_{ij}]$ (or $\mathcal{A}' = [a'_{il}]$) is the weighted adjacency matrices associated with the graph $\mathcal{G}^c$ (or $\mathcal{G}^d$); τ is the time-delay as processing the impulsive information according to graph $\mathcal{G}^d$ and $\delta(\cdot)$ is the dirac delta function.

Thus, for each scalar scale $\beta_i \neq 0$, the multi-agent system (4.26) with protocol (4.50) can be written as:

$$\begin{aligned} \beta_i \dot{x}_i(t) = |\beta_i| &\sum_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t) - \beta_i x_i(t)] \\ &+ h|\beta_i| \sum_{k=1}^{\infty} \sum_{l \in \mathcal{N}'_i} a'_{il}[\beta_l x_l(t-\tau) - \beta_i x_i(t-\tau)]\delta(t-t_k), \qquad \text{for} \quad i \in \mathcal{I}_n. \end{aligned} \tag{4.51}$$

By the definition of the delta function, the system (4.51) can be described as the impulsive system:

$$\begin{cases} \beta_i \dot{x}_i(t) = |\beta_i| \sum\limits_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t) - \beta_i x_i(t)], \quad t \in (t_{k-1}, t_k), \\ \Delta\beta_i x_i(t_k) = h|\beta_i| \sum\limits_{l\in\mathcal{N}_i'} a'_{il}[\beta_l x_l(t_k - \tau) - \beta_i x_i(t_k - \tau)], \end{cases} \tag{4.52}$$

where $\Delta\beta_i x_i(t_k) = \beta_i x_i(t_k^+) - \beta_i x_i(t_k^-)$: $x_i(t_k^+) = \lim\limits_{h\to 0^+} x_i(t_k + h)$ and $x_i(t_k^-) = \lim\limits_{h\to 0^+} x_i(t_k - h)$.

With out loss of generality, we assume that the solution of system (4.52) is left continuous, that is, $\beta_i x_i(t_k^-) = \beta_i x_i(t_k)$ and let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$, $\mathcal{B} = diag(\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{R}^{n\times n}$, $H = diag(h_1, h_2, \ldots, h_n)$, and $|\mathcal{B}| = diag(|\beta_1|, |\beta_2|, \ldots, |\beta_n|) \in \mathbb{R}^{n\times n}$. Then, the system (4.52) can be written as the form:

$$\begin{cases} \mathcal{B}\dot{x}(t) = -|\mathcal{B}|\mathcal{L}\mathcal{B}x(t), \qquad t \neq t_k, \\ \mathcal{B}x(t_k^+) = \big(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'\big)\mathcal{B}x(t_k - \tau), \qquad k \in \mathbb{N}. \end{cases} \tag{4.53}$$

To establish our main results, the following assumptions are provided:

(A1) $e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}'$ is a stochastic matrix with positive diagonal entries;

(A2) $\mathcal{G}^c \cup \mathcal{G}^d$ contains a spanning tree,

where $\mathcal{L}'$ is the Laplacian matrix of $\mathcal{G}^c \cup \mathcal{G}^d$ at $t = t_k$.

Theorem 4.2.6. Let $\mathcal{G}$ be a directed connected communication network of the multi-agent system (4.26) with the sampling period $0 < h < (d_{max}\beta_{max})^{-1}$. Assume that the assumption **(A1)** and **(A2)** hold and for $k = 1, 2, \ldots$ the interval $(t_{k-1}, t_k]$ are uniformly bounded, that is, there exist positive constants t_{min} and t_{max} such that $t_{min} \le t_k - t_{k-1} \le t_{max}$. Then, the multi-agent system (4.26) under the protocol (4.50) reaches scaled consensus to $(\beta_1, \ldots, \beta_n)$.

Proof. For any initial conditions $x(t_0) = x(0)$ and for $t > 0$, the solution of system (4.53) is as the form (by Mathematical induction):

$$\begin{aligned} \mathcal{B}x(t) = {} & e^{-|\mathcal{B}|\mathcal{L}(t-t_{k-1})}(e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}') \\ & \times e^{-|\mathcal{B}|\mathcal{L}(\Delta t_{k-1}-\tau)} \cdots (e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}')e^{-|\mathcal{B}|\mathcal{L}(\Delta t_1-\tau)}\mathcal{B}x(0), \end{aligned} \tag{4.54}$$

where $t \in [t_{k-1}, t_k)$ and $\Delta t_i = t_i - t_{i-1}$.

It can be seen that the protocol (4.50) solves scaled consensus problems if and only if $\mathcal{B}x(t) \to \mathbf{1}\xi$ as $t \to \infty$, for some $\xi \in \mathbb{R}$.

Next, we will show that $\mathcal{B}x(t) \to \mathbf{1}\xi$ as $t \to \infty$ is equivalent to $\mathcal{G}^c \cup \mathcal{G}^d$ containing a spanning tree.

Since $e^{-|\mathcal{B}|\mathcal{L}t}$ is a stochastic matrix with positive diagonal entries (SPD) for any $t > 0$. Furthermore, for any $t > 0$,

$$(e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}')e^{-|\mathcal{B}|\mathcal{L}t} \geq c\big[(e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}') + e^{-|\mathcal{B}|\mathcal{L}t}\big],$$

where c is a positive constant. In addition, from **(A1)**, $(e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}')$ is SPD, and hence

$$(e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}')e^{-|\mathcal{B}|\mathcal{L}t}$$

is also SPD when $t > 0$.

Next, we claim that $(e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}')e^{-|\mathcal{B}|\mathcal{L}t}$ has a spanning tree. Let $d_{max} = \max_i\{d_i\}$ and $M = d_{max}\mathbf{I}_n - |\mathcal{B}|\mathcal{L}$, then for each $i \neq j$, the $(i,j)th$ entry of M is a_{ij} which implies the graph $\mathcal{G}^c$ and M have the same edge set. For any $t > 0$, there exists $\rho > 0$ such that $e^{-|\mathcal{B}|\mathcal{L}t} = e^{-d_{max}t}e^{Mt} \geq \rho M$, and hence the edge set of $\mathcal{G}^c$ is a subset of the edge set of a graph associated with $e^{-|\mathcal{B}|\mathcal{L}t}$.

On the other hand, the graph of $\mathcal{G}^d$ and the graph of $H|\mathcal{B}|\mathcal{L}'$ share the same edge set. Hence, the union of $\mathcal{G}^c$ and $\mathcal{G}^d$ has a spanning tree implies that the union of $e^{|\mathcal{B}|\mathcal{L}\tau}$ and $H|\mathcal{B}|\mathcal{L}'$ has a spanning tree. This implies that $e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}'$ has a spanning tree. Therefore, $(e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}')e^{-|\mathcal{B}|\mathcal{L}t}$ contains a spanning tree.

Based on the above discussion, we have shown that $(e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}')e^{-|\mathcal{B}|\mathcal{L}t}$, for $t > 0$ is SPD and the graph of it contains a spanning tree. Thus, by Lemma 2.6.1 and Lemma 2.6.3, $(e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}')e^{-|\mathcal{B}|\mathcal{L}t}$ is SIA.

Let $\Xi = \{(e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}')e^{-|\mathcal{B}|\mathcal{L}t} : t \in [t_{min}, t_{max}]\}$. Since the time interval $[t_{k-1}, t_k)$ are uniformly bounded, for $k \in \mathbb{N}$, then Ξ is compact and all of its elements are SIA matrices. Thus, there exists a column vector y such that

$$\begin{aligned}\lim_{k\to\infty} &(e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}')e^{-|\mathcal{B}|\mathcal{L}(\Delta t_{k-1}-\tau)} \cdots (e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}')e^{-|\mathcal{B}|\mathcal{L}(\Delta t_1-\tau)} \\ &= \mathbf{1}_n y^T.\end{aligned}$$

Since $e^{-|\mathcal{B}|\mathcal{L}(t-t_{k-1})}$ is a stochastic matrix, we have $e^{-|\mathcal{B}|\mathcal{L}(t-t_{k-1})}\mathbf{1}_n = \mathbf{1}_n$. Thus, for any $t - t_{k-1} \in [t_{min}, t_{max}]$,

$$\begin{aligned}\lim_{t\to\infty} &e^{-|\mathcal{B}|\mathcal{L}(t-t_{k-1})}(e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}')e^{-|\mathcal{B}|\mathcal{L}(\Delta t_{k-1}-\tau)} \cdots (e^{-|\mathcal{B}|\mathcal{L}\tau} - H|\mathcal{B}|\mathcal{L}')e^{-|\mathcal{B}|\mathcal{L}(\Delta t_1-\tau)}\mathcal{B}x(0) \\ &= \mathbf{1}_n y^T \mathcal{B}x(0).\end{aligned}$$

This implies that

$$\lim_{t\to\infty} \mathcal{B}x(t) = \mathbf{1}_n y^T \mathcal{B}x(0).$$

$\square$

Switching topologies

In this subsection, the consensus problems of multi-agent systems under switching topologies described by the impulsive system have been studied. Using similar notations, the scaled consensus protocol for multi-agent system based on the scalar scale $\beta_i \neq 0$ under switching topology is defined as follows: for $t \in (t_{k-1}, t_k]$,

$$\begin{aligned} u_i(t) = |\beta_i| &\sum_{j\in\mathcal{N}_i(\sigma(t))} a_{ij}(\sigma(t))[\beta_j x_j(t) - \beta_i x_i(t)] \\ &+ h \cdot |\beta_i| \sum_{k=1}^{\infty} \sum_{l\in\mathcal{N}_i'(s(k))} a'_{il}(s(k))[\beta_l x_l(t-\tau) - \beta_i x_i(t-\tau)]\delta(t-t_k), \quad \text{for} \quad i \in \mathcal{I}_n, \end{aligned} \tag{4.55}$$

where $h = t_k - t_{k-1}$ is a sampling period; $\mathcal{A} = [a_{ij}]$ (or $\mathcal{A}' = [a'_{ij}]$) is the weighted adjacency matrices associated with the graph $\mathcal{G}^c$ (or $\mathcal{G}^d$); $sgn(\cdot)$ is the signum function defined as above and $\delta(\cdot)$ is the dirac delta function. Furthermore, for some $r, m \in \mathbb{N}$, $\sigma : [0, \infty) \to \{1, 2, 3, \dots, r\}$ is a piecewise constant function and $s : \mathbb{N} \to \{1, 2, 3, \dots, m\}$ is a constant function called the continuous-time switching signal and discrete-time switching signal, respectively.

In this work, we assume that there is no switching on each impulsive interval, that is, $\sigma(t) = \sigma(k)$ for $t \in (t_{k-1}, t_k]$. Consequently, for any $\beta_i \neq 0$, the multi-agent system (4.26) with protocol (4.55) can be written as

$$\begin{cases} \dot{x}_i(t) = |\beta_i| \displaystyle\sum_{j\in\mathcal{N}_i(\sigma(k))} a_{ij}(\sigma(k))[\beta_j x_j(t) - \beta_i x_i(t)], \quad t \in (t_{k-1}, t_k), \\ \Delta x_i(t_k) = h \cdot |\beta_i| \displaystyle\sum_{l\in\mathcal{N}_i'(s(k))} a'_{il}(s(k))[\beta_l x_l(t_k-\tau) - \beta_i x_i(t_k-\tau)]. \end{cases} \tag{4.56}$$

Multiplying both sides of (4.56) by β_i and letting $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$, $\mathcal{B} = diag(\beta_1, \beta_2, ..., \beta_n) \in \mathbb{R}^{n\times n}$, $H = diag(h_1, h_2, \dots, h_n)$, and $|\mathcal{B}| = diag(|\beta_1|, |\beta_2|, ..., |\beta_n|) \in \mathbb{R}^{n\times n}$, the system (4.56) can be written as the form:

$$\begin{cases} \mathcal{B}\dot{x}(t) = -|\mathcal{B}|\mathcal{L}_{\sigma(k)}\mathcal{B}x(t), \qquad t \neq t_k, \\ \Delta\mathcal{B}x(t) = -H|\mathcal{B}|\mathcal{L}'_{s(k)}\mathcal{B}x(t-\tau), \qquad t = t_k, \qquad k \in \mathbb{N}, \end{cases} \tag{4.57}$$

where $\mathcal{L}_{\sigma(k)}$ and $\mathcal{L}'_{s(k)}$ are the Laplacian matrix of $\mathcal{G}^c \cup \mathcal{G}^d$ at the impulse instant t_k. The time sequence t_k satisfies $0 < t_1 < t_2 < t_3 < \cdots < t_k < \cdots$, $\lim_{t\to\infty} t_k = \infty$. Without loss of generality, we assume that $\lim_{h\to 0^+} x_i(t_k - h) = x_i(t_k)$, that is, $x_i(t_k)$ is left-continuous. Then, the system (4.57) can be written as

$$\begin{cases} \mathcal{B}\dot{x}(t) = -|\mathcal{B}|\mathcal{L}_{\sigma(k)}\mathcal{B}x(t), \quad & t \neq t_k, \\ \mathcal{B}x(t_k^+) = \big(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'_{s(k)}\big)\mathcal{B}x(t_k - \tau), \quad & k \in \mathbb{N}. \end{cases} \tag{4.58}$$

Now, we are in the position to introduce our main result.

Theorem 4.2.7. Let $\mathcal{G}$ be a directed connected communication network of the multi-agent system (4.26) with $0 < h < (d_{max}\beta_{max})^{-1}$. The switched multi-agent system (4.58) under the protocol (4.55) is said to reach scaled consensus to $(\beta_1, \ldots, \beta_n)$ if the following conditions are satisfied:

(i) $\mathcal{G}^c \cup \mathcal{G}^d$ is balanced and contains a spanning tree;

(ii) $\mathcal{D}_{ij} := e^{-|\mathcal{B}|\mathcal{L}_i\tau_j} - H|\mathcal{B}|\mathcal{L}'_j$ is a stochastic matrix with positive diagonal entries, where $\mathcal{L}'$ is the Laplacian matrix of $\mathcal{G}^c \cup \mathcal{G}^d$ at $t = t_k$;

(iii) there exist positive constants t_{min} and t_{max} such that $t_{min} \le t_k - t_{k-1} \le t_{max}$ and $\tau_{\sigma(k)} < t_k - t_{k-1}$ for $k = 1, 2, \ldots$;

(iv) there exists a subsequence $t_{k_j} \subseteq t_k$ such that $(t_{k-1}, t_k]$ for $j \in \mathbb{N}$ are uniformly bounded from above.

Proof. Condition (ii) implies that initial conditions $x(0) = x_0$ is well-defined for system (4.58) and it can be obtained from condition (iii) that there exists a positive constant T such that $t_{k_j} - t_{k_{j-1}} \le T$ for all $j \in \mathbb{N}$. Then, for any interval $(t_{k_{j-1}}, t_{k_j}]$, the following matrix

$$\prod_{i=k_{j-1}+1}^{k_j} \mathcal{D}_{\sigma(t_i),s(i)} e^{-|\mathcal{B}|\mathcal{L}_{\sigma(t_i)}(\Delta(t_i) - \tau_{s(i)})} \tag{4.59}$$

is a product of finite number of matrices.

Following the similar argument in Theorem 4.2.6, we have the matrix (4.59) is SIA since the union of graphs across the interval $(t_{k_{j-1}}, t_{k_j}]$ has a spanning tree. Next, define the following set

$$\Omega = \Big\{ \prod_{i=1}^{l} \mathcal{D}_{p_i,q_i} e^{-|\mathcal{B}|\mathcal{L}_{p_i}(\Delta_i - \tau_{q_i})} | l \in \mathbb{Z}, 1 \le l \le T/t_{min}; \Delta_i \in [t_{min}, t_{max}] \text{ for } i = 1, 2, \ldots, l;$$
$$\text{the union of graphs } \mathcal{G}_{p_1}, \mathcal{G}_{p_2}, \ldots, \mathcal{G}_{p_l} \text{ and } \mathcal{G}'_{q_1}, \mathcal{G}'_{q_2}, \ldots, \mathcal{G}'_{q_l} \text{ has a spanning tree} \Big\}.$$

As discussed for matrix (4.59), we can see that Ω is an SIA matrix set. Furthermore, since all Δ_i's belong to a closed interval and $\sum_{i=1}^{l} \Delta_i$ is bounded, the set Ω is compact. For any $t > 0$, there exist non-negative integers k and $\hat{j}$ such that $t \in (t_k, t_{k+1}] \subseteq (t_{k_{\hat{j}}}, t_{k_{\hat{j}+1}}]$, and then for $t \in (t_k, t_{k+1}]$, the solution $x(t)$ can be obtained by mathematical induction, and then combine the matrices product according to each interval $(t_{k_j}, t_{k_{j+1}}]$ to get the form:

$$\begin{aligned}\mathcal{B}x(t) = &\big(e^{-|\mathcal{B}|\mathcal{L}_{\sigma(t_{k+1})}(t-t_k)} \prod_{i=k_{j-1}+1}^{k_j} \mathcal{D}_{\sigma(t_i),s(i)} e^{-|\mathcal{B}|\mathcal{L}_{\sigma(t_i)}(\Delta(i)-\tau_{s(i)})}\big) \\ &\times \big(\sum_{j=0}^{\hat{j}-1} \prod_{i=k_j+1}^{k_{j+1}} \mathcal{D}_{\sigma(t_i),s(i)} e^{-|\mathcal{B}|\mathcal{L}_{\sigma(t_i)}(\Delta(t_i)-\tau_{s(i)})}\big)\mathcal{B}x(0). \qquad (4.60)\end{aligned}$$

Since $\prod_{i=k_j+1}^{k_{+1}} \mathcal{D}_{\sigma(t_i),s(i)} e^{-|\mathcal{B}|\mathcal{L}_{\sigma(t_i)}(\Delta(t_i)-\tau_{s(i)})} \in \Omega$, for $j \geq 0$, then, by Lemma 2.6.3, there exists a column vector y such that

$$\lim_{\hat{j}\to\infty} \sum_{j=0}^{\hat{j}-1} \prod_{i=k_j+1}^{k_{j+1}} \mathcal{D}_{\sigma(t_i),s(i)} e^{-|\mathcal{B}|\mathcal{L}_{\sigma(t_i)}(\Delta(t_i)-\tau_{s(i)})} = \mathbf{1}_n y^T. \qquad (4.61)$$

In addition, $t_{k_{\hat{j}+1}} - t_{k_{\hat{j}}} \leq T$ implies that

$$e^{-|\mathcal{B}|\mathcal{L}_{\sigma(t_{k+1})}(t-t_k)} \prod_{i=k_{j-1}+1}^{k_j} \mathcal{D}_{\sigma(t_i),s(i)} e^{-|\mathcal{B}|\mathcal{L}_{\sigma(t_i)}(\Delta(i)-\tau_{s(i)})} \qquad (4.62)$$

is bounded. Moreover, the matrix (4.62) is a stochastic matrix since it is a product of stochastic matrices. This implies that

$$e^{-|\mathcal{B}|\mathcal{L}_{\sigma(t_{k+1})}(t-t_k)} \prod_{i=k_{j-1}+1}^{k_j} \mathcal{D}_{\sigma(t_i),s(i)} e^{-|\mathcal{B}|\mathcal{L}_{\sigma(t_i)}(\Delta(i)-\tau_{s(i)})}\mathbf{1}_n = \mathbf{1}_n. \qquad (4.63)$$

From (4.61) and (4.63), it follows that $\lim_{t\to\infty} \mathcal{B}x(t) = \mathbf{1}_n y^T \mathcal{B}x(0)$, i.e., the protocol (4.55) solves scaled consensus problems.

$\square$

Simulations and Discussion

In this section, two examples are provided to show the effectiveness of theoretical results in this work.

Example 3. Assume that there are 4 agents denoted by x_1, x_2, x_3, x_4 and the initial conditions are denoted by $x(0) = [1 \quad -1 \quad 2 \quad -2]^T$. Consider the fixed communication network $\mathcal{G}$ shown in Figure 4.18, where the dashed lines mean that each agent exchanges information at time $t = t_k$.

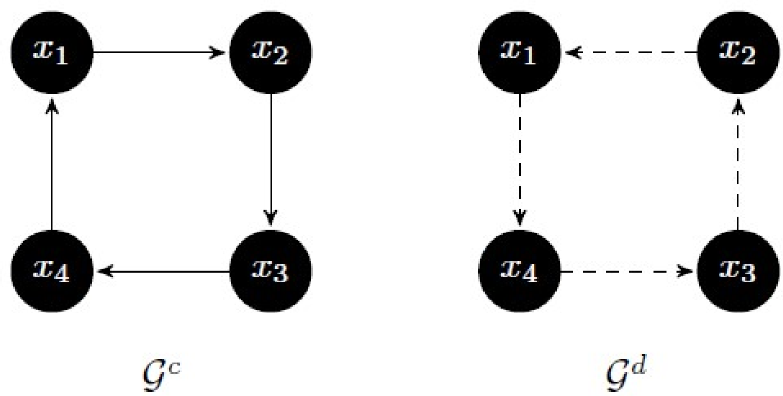

Figure 4.18: A connected directed network $\mathcal{G}$ at time $t \neq t_k$ and $t = t_k$, respectively.

It can be seen that $\mathcal{G}^c \cup \mathcal{G}^d$ is balanced and contains a directed spanning tree with $d_{max} = 1$. Let scalar scales $\beta = (1 \quad 0.4 \quad 1.5 \quad -2)^T$ and choose the sampling period $h = 0.22$, then $h = 0.22 < 0.5 = (\beta_{max} d_{max})^{-1}$.

Moreover, choosing $\tau = 6$, it can be seen by MATLAB that the condition (ii) are satisfied. Therefore, the scaled consensus problem is solved by Theorem 4.2.6 under protocol (4.50) and the state trajectories of all agents are shown in Figure 4.19.

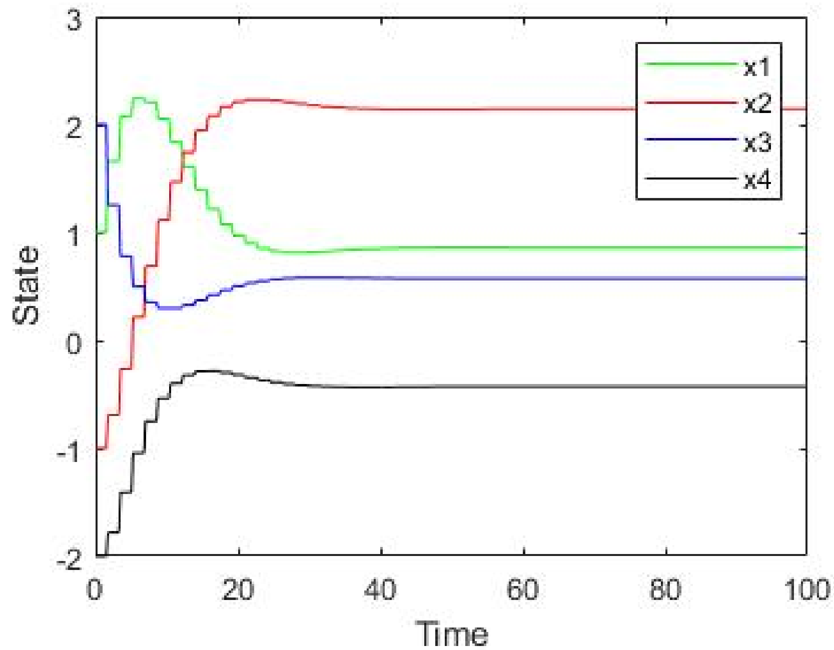

Figure 4.19: The state trajectories of all agents using scaled consensus protocol (4.50) with scalar scales $\beta = (1 \;\; 0.4 \quad 1.5 \quad -2)^T$ for $h = 0.2$.

In addition, if $h = 0.83 > 0.5 = (\beta_{max} d_{max})^{-1}$, it can be seen that the consensus protocol (4.50) cannot guarantee achieving scaled consensus to $\beta = (1 \;\; 0.4 \quad 1.5 \quad -2)^T$ as shown in Figure 4.20.

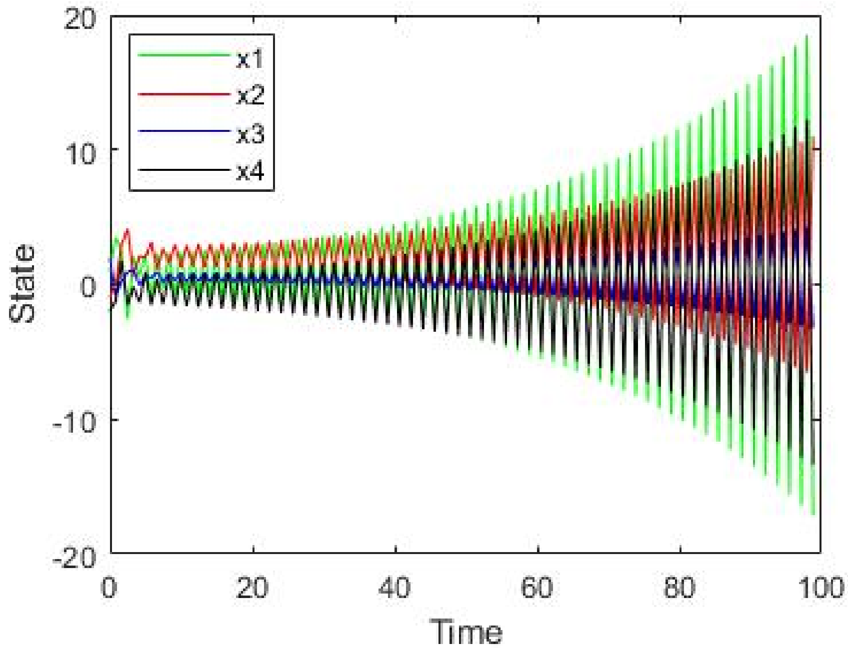

Figure 4.20: The state trajectories of all agents using scaled consensus protocol (4.50) with scalar scales $\beta = (1 \;\; 0.4 \quad 1.5 \quad -2)^T$ for $h = 0.83$.

Moreover, if the scalar scales of all agent are equal to one i.e., $\beta_i = 1$ for all i, the state trajectories of agents under protocol (4.50) with $h = 0.2$ can be described as in Figure 4.21

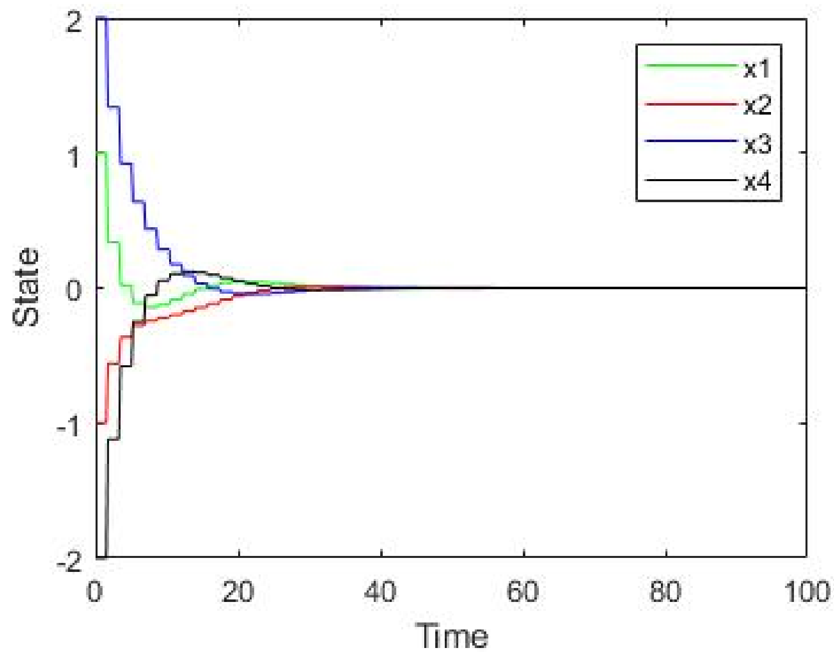

Figure 4.21: The state trajectories of all agents using scaled consensus protocol (4.50) with scalar scales $\beta = (1 \ \ 1 \ \ 1 \ \ 1)^T$ and $h = 0.2$.

Example 4. Consider the communication networks of the multi-agent system (4.26) with switching topologies consisting of 4 agents denoted by x_1, x_2, x_3, x_4 as described in Figure 4.22.

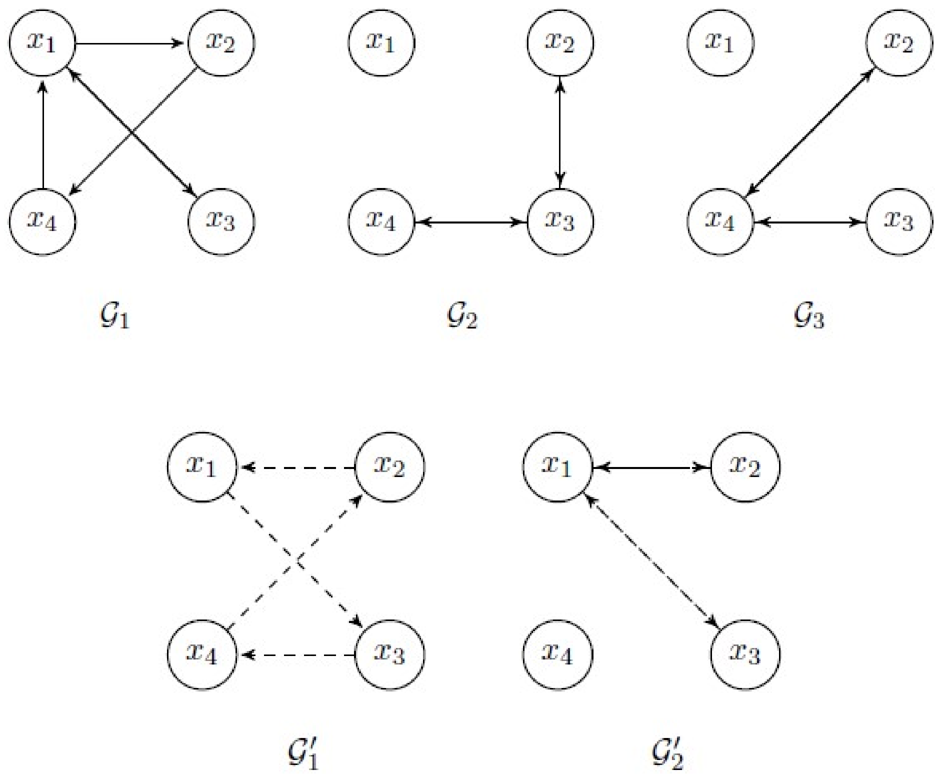

Figure 4.22: Switching topologies.

Assume that the impulse and switching occur simultaneously at the sampling time t_k and the switching happens in order of $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ and $\{\mathcal{G}_1', \mathcal{G}_2'\}$, respectively. Clearly, each digraph of $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ and $\{\mathcal{G}_1', \mathcal{G}_2'\}$ is balanced and contains a spanning tree.

Given scalar scales $\beta = (1 \quad 2.1 \quad 1.5 \quad -2)^T$ and the initial conditions $x(0) = [1 \quad -1 \quad 2 \quad -2]^T$. It can be seen that $d_{max} = 2$ and $\beta_{max} = 2.1$, by choosing the sampling period $h = 0.2$, one obtains that $h = 0.2 < 0.23 = (\beta_{max} d_{max})^{-1}$. Selecting $\tau = 6$, we can see by MATLAB that the assumption (ii) is satisfied. Hence, the scaled consensus problem is solved by using protocol (4.55) and the states of all agents are as in Figure 4.23.

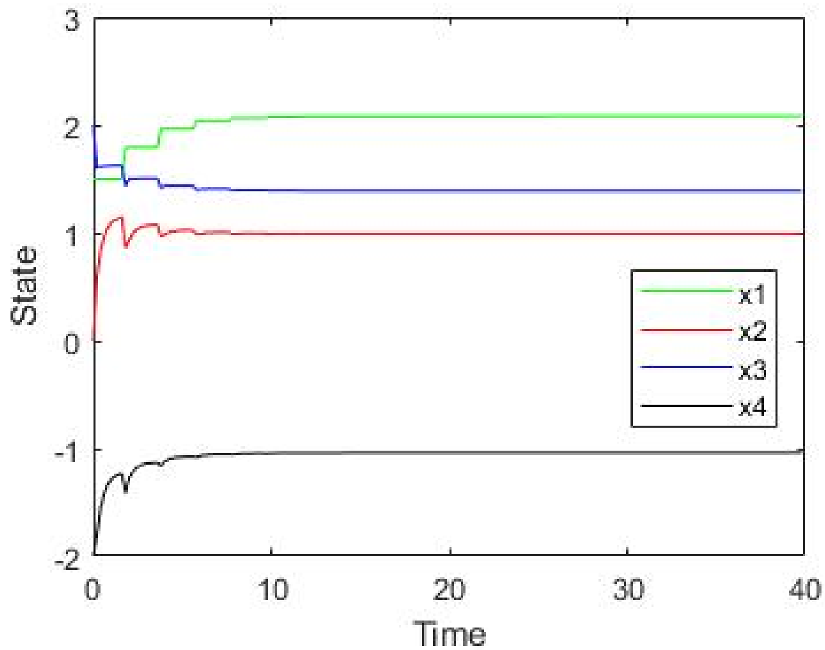

Figure 4.23: The state trajectories of all agents using the consensus protocol (4.55) with scalar scales $\beta = (1 \quad 2.1 \quad 1.5 \quad -2)^T$ and $h = 0.2$.

However, if we choose $h = 0.2$ and $\beta = (5 \;\; 1.5 \quad -2 \;\; 1)^T$, it follows that $h = 0.2 > 0.1 = (\beta_{max} d_{max})^{-1}$, which implies the consensus protocol (4.55) cannot guarantee solving scaled consensus problem (see Figure 4.24 for state trajectories).

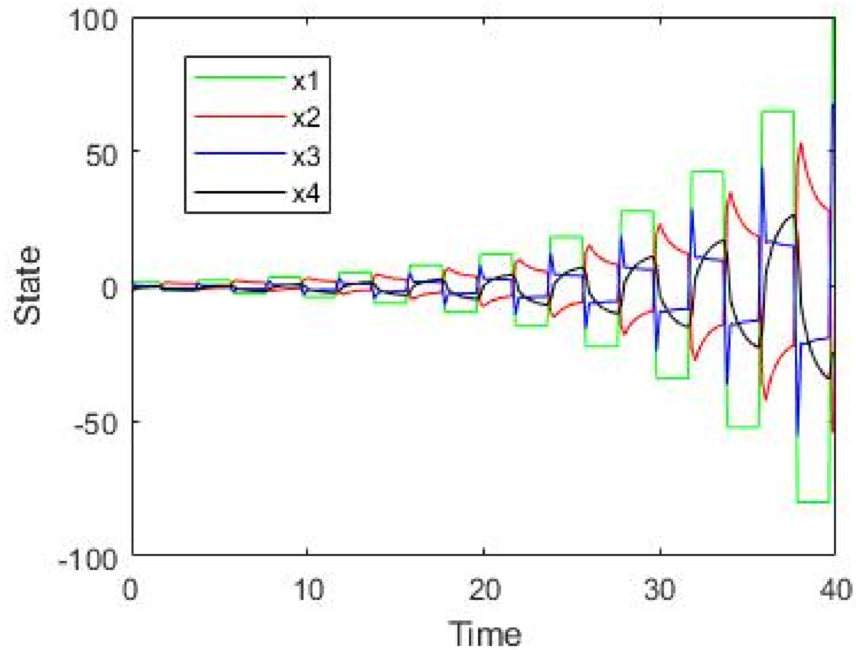

Figure 4.24: The state trajectories of all agents using the consensus protocol (4.55) with scalar scales $\beta = (5 \;\; 1.5 \quad -2 \;\; 1)^T$ and $h = 0.2$.

In addition, if the scalar scales of all agents are equal to 1, the state trajectories of all agents under protocol (4.55) with $h = 0.2$ can be described as in Figure 4.25.

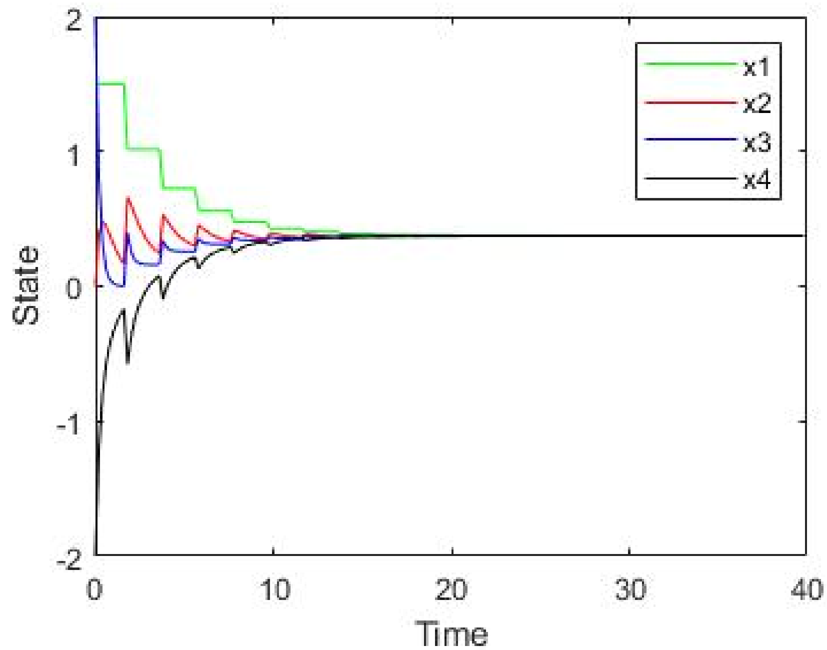

Figure 4.25: The state trajectories of all agents using the consensus protocol (4.55) with scalar scales $\beta = (1 \;\; 1 \;\; 1 \;\; 1)^T$ and $h = 0.2$.

Chapter 5

Finite-time consensus problems in Hybrid multi-agent systems

In this chapter, we study the finite-time (scaled) consensus problems in HMASs by using the impulsive consensus protocols. In Section 5.1, the impulsive consensus protocols have been proposed to solve finite-time consensus problems in HMASs. In Section 5.2.1, finite-time scaled consensus problems of HMASs have been studied by using the impulsive consensus protocols. Furthermore, the finite-time scaled consensus of MASs with impulsive perturbations have been studied in Section 5.2.2. Finally, the numerical examples are illustrated in the last section to show the effectiveness of our main results in Section 5.3.

5.1 Finite-time consensus of Hybrid multi-agent systems

In this section, finite-time scaled consensus problems of hybrid multi-agent systems via impulsive control have been studied.

Problem formulation

In this section, we assume that the hybrid multi-agent system consists of n agents which are continuous-time and discrete-time dynamic agents, labelled 1 through n, where the number of continuous-time dynamic agents is c, $c < n$. Without loss of generality, we assume that agent 1 through c are continuous-time dynamic agents. Moreover, $\mathcal{I}_c = \{1, 2, 3, ..., c\}$, $\mathcal{I}_n \backslash \mathcal{I}_c = \{c+1, c+2, c+3, ..., n\}$. Then, each agent has the dynamics as follows:

$$\begin{cases} \dot{x}_i(t) = u_i, & \text{for } \ i \in \mathcal{I}_c, \\ x_j(t_{k+1}) = x_j(t_k) + u_j(t_k), \quad t_k = kh, & \text{for } \ j \in \mathcal{I}_n \backslash \mathcal{I}_c, \end{cases} \tag{5.1}$$

where h is the sampling period, $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the state and control input of agent i, respectively. The initial conditions are $x_i(0) = x_{i0}$, and $x(0) = [x_{10}, x_{20}, ..., x_{n0}]^T$.

Definition 5.1.1. The hybrid multi-agent system (5.1) is said to reach finite-time consensus if for any initial conditions, there is a setting time T such that

$$\lim_{t \to T} \|x_i(t_k) - x_j(t_k)\| = 0, \quad \text{and} \quad x_i(t_k) = x_j(t_k), \quad \forall t_k \geq T \qquad \text{for } \ i, j \in \mathcal{I}_n, \tag{5.2}$$

and

$$\lim_{t \to T} \|x_i(t) - x_j(t)\| = 0, \quad \text{and} \quad x_i(t) = x_j(t), \quad \forall t \geq T \qquad \text{for } \ i, j \in \mathcal{I}_c. \tag{5.3}$$

Now, we are in a position to present our protocol that solves the consensus problem in finite time:

$$\begin{cases} u_i(t) & = c_1 \sum\limits_{l \in \mathcal{N}_i} a_{il} \varphi(x_l(t) - x_i(t)) + c_2 \sum\limits_{l \in \mathcal{N}_i} a_{il} (x_l(t) - x_i(t))^{\alpha} \\ & \quad + \sum\limits_{k=1}^{\infty} \delta(t - t_k) g(t) \sum\limits_{l \in \mathcal{N}'_i} a'_{il} (x_l(t) - x_i(t)), \quad \text{for } \ i \in \mathcal{I}_c \\ u_j(t_k) & = h \sum\limits_{l \in \mathcal{N}_i} b_{jl} [x_l(t_k) - x_j(t_k)], \quad \text{for } \ j \in \mathcal{I}_n \backslash \mathcal{I}_c, \end{cases} \tag{5.4}$$

where $\mathcal{A} = [a_{ij}]$ and $\mathcal{B} = [b_{ij}]$ are the weighted adjacency matrices associated with the graph $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$, respectively. Moreover, $h = t_k - t_{k-1}$ is the sampling period and N_i is the set of neighbours of i. Moreover, $c_1, c_2 > 0$, $\underline{c} = min\{c_1, c_2\}$; the odd function $\varphi(\cdot)$ satisfying that $y \cdot \varphi(y) \geq y^{\beta+1} > 0$, $\forall y \in \mathbb{R} \setminus \{0\}$, $\ \varphi(0) = 0$; $\ \alpha, \beta$ are ratios of odd integers with $\alpha + \beta \in (-2, 2)$; the discrete instant t_k, $\ k \in \mathbb{N}_+$ satisfy $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k \to \infty} t_k = \infty$ and $\delta(\cdot)$ is the Dirac delta function.

Assume that there exist two constants r_1 and r_2 such that $0 < r_1 \leq t_{k+1} - t_k \leq r_2 < \infty$; $g(t)$ is a function to be designed later. Define the left and right value of the state at time

t_k as follows:

$$x_i(t_k^+) = \lim_{h\to 0^+} x_i(t_k+h) \ \text{ and } \ x_i(t_k^-) = \lim_{h\to 0^+} x_i(t_k-h), \quad h>0.$$

Equivalently, the system (5.1) with the protocol (5.4) can be written as follows:

$$\begin{cases} \begin{cases} \dot{x}_i(t) & = c_1 \sum_{l\in\mathcal{N}_i} a_{il}\varphi(x_l(t)-x_i(t)) + c_2 \sum_{l\in\mathcal{N}_i} a_{il}(x_l(t)-x_i(t))^{\alpha}, \quad t\neq t_k, \\ \Delta x_i(t_k) & = \sum_{l\in\mathcal{N}_i'} a_{il}'[x_l(t_k)-x_i(t_k)], \qquad \text{for } \ i\in\mathcal{I}_c \end{cases} \\ x_j(t_{k+1}) = x_j(t_k) + h\sum_{l\in\mathcal{N}_i} b_{jl}[x_l(t_k)-x_j(t_k)], \qquad \text{for } \ j\in\mathcal{I}_n\backslash\mathcal{I}_c, \end{cases} \tag{5.5}$$

where $x_i(t)\in\mathbb{R}$ is the state of agent i at time t, $i=1,2,...,n$. $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$.

To obtain our main results, some assumptions are provided as follows:

(A1) Let $p_1,\ldots,p_n\in\mathbb{R}^n$ be C-conjugate and for any initial value $x(0)\in\mathbb{R}^n$, the discrete-time dynamic protocol algorithm is designed as: for $k=1,2,\ldots,n$

$$x(k) = x(k-1) + \frac{< b - Cx^{k-1}, p_k >}{< Cp_k, p_k >}\cdot p_k, \ C = \rho I + \mathcal{L}_2, \quad b = \rho x^*, \quad 0<\rho<1;$$

(A2) $0 < h < \dfrac{1}{\max_{i\in\mathcal{I}_n}\{d_i\}}$;

(A3) $0\le g(t)\le \dfrac{2}{\gamma\lambda_N}, \quad \gamma\ge 1,$

where $\mathcal{L}_2$ is the Laplacian matrix of $\mathcal{G}_d\cup\mathcal{G}'$ and $x^* = \dfrac{1}{m}\sum_{j=1}^{m} x_j(0), \ \ m=|\mathcal{G}_d\cup\mathcal{G}'|$ is the convergence value of average consensus.

Theorem 5.1.1. Let $\mathcal{G}$ be an undirected communication network of the hybrid multi-agent system (5.1). Assume that **(A1)-(A3)** are satisfied, then the hybrid multi-agent system (5.1) with the protocol (5.4) reaches consensus in finite time if and only if $\mathcal{G}_c\cup\mathcal{G}'$ and $\mathcal{G}_d\cup\mathcal{G}'$ are connected.

Proof. **(Sufficiency)** Let $\mathcal{G} = \mathcal{G}_c\cup\mathcal{G}_d\cup\mathcal{G}'$ be an undirected communication network of the hybrid multi-agent system (5.1), where $\mathcal{G}_c, \mathcal{G}_d, \mathcal{G}'$ are defined as previous discussion. Firstly, consider for each $i\in\mathcal{I}_c$. Without loss of generality, we assume that all discrete-time dynamic agents interact with some continuous-time dynamic agents.

Thus, the system (5.1) with the protocol (5.4) can be described as an impulsive system on the communication network $\mathcal{G}_c \cup \mathcal{G}'$ with r nodes, where $|\mathcal{G}_c \cup \mathcal{G}'| = r \leq n$. Hence, for each $i \in \mathcal{I}_c$, the system (5.1) with the protocol (5.4) can be written as an impulsive system on the communication network $\mathcal{G}_c \cup \mathcal{G}'$ as follows:

$$\begin{cases} \dot{x}_i(t) = c_1 \sum_{l\in\mathcal{N}_i} a_{il}\varphi(x_l(t) - x_i(t)) + c_2 \sum_{l\in\mathcal{N}_i} a_{il}(x_l(t) - x_i(t))^\alpha, \quad t \neq t_k, \\ \Delta x_i(t_k) = \sum_{l\in\mathcal{N}'} a'_{il}[x_l(t_k) - x_i(t_k)], \end{cases} \tag{5.6}$$

where $x_i(t) \in \mathbb{R}$ is the state of agent i at time t, $i = 1, 2, ..., r$. $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$: $x_i(t_k^+) = \lim_{h\to 0^+} x_i(t_k + h)$ and $x_i(t_k^-) = \lim_{h\to 0^+} x_i(t_k - h)$.

Letting $x(t) = (x_1(t), x_2(t), ..., x_r(t))^T \in \mathbb{R}^r$, the system (5.6) can be written as:

$$\begin{cases} \dot{x}(t) = F(t, x(t)), \qquad t \neq t_k, \\ \Delta x(t_k) = -g(t_k)\mathcal{L}' x(t_k), \qquad k \in \mathbb{N}, \end{cases} \tag{5.7}$$

where $F(t, x(t)) = (f_1(t, x(t)), \ldots, f_r(t, x(t)))^T$ and

$$f_i(t, x(t)) = c_1 \sum_{l\in\mathcal{N}_i} a_{il}\varphi(x_l(t) - x_i(t)) + c_2 \sum_{l\in\mathcal{N}_i} a_{il}(x_l(t) - x_i(t))^\alpha.$$

Here, we assume that $x_i(t_k^-) = x_i(t_k)$, which implies that the solution is left continuous at time t_k. Then, one obtains

$$\begin{cases} \dot{x}(t) = F(t, x(t)), \qquad t \neq t_k, \\ x(t_k^+) = (\mathbf{I}_r - g(t_k)\mathcal{L}')x(t_k), \qquad k \in \mathbb{N}, \end{cases} \tag{5.8}$$

where $\mathcal{L}$ and $\mathcal{L}'$ are the Laplacian matrix of $\mathcal{G}_c \cup \mathcal{G}'$ when $t \neq t_k$ and $t = t_k$, respectively.

Let $\bar{x}(t) = \frac{1}{r}\sum_{j=1}^{r} x_i(t)$, which is time-invariant, i.e. $\bar{x} = \bar{x}(t) = \bar{x}(0) = Ave(x(0)) = \frac{1}{r}(\sum_{i=1}^{r} x_i(0))$. As a results of the invariant of $\bar{x}(t)$, it allows the decomposition of x_i as follows

$$\eta_i(t) = x_i(t) - \bar{x}, \quad \text{for} \quad i = 1, 2, \ldots, r.$$

Consider the Lyapunov function:

$$V(\eta) = \frac{1}{2}\eta^T(t)\eta(t) = \frac{1}{2}\sum_{i=1}^{r}\eta_i^2(t),$$

where $\eta = (\eta_1, \eta_2, \dots, \eta_r)^T$ is the error vector.

For $t \neq t_k$, from Lemmas 2.6.8-2.6.10, one obtains that

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{r}\eta_i(t)\dot{\eta}_i(t) \\
&= \sum_{i=1}^{r}\eta_i\Big(c_1\sum_{l\in\mathcal{N}_i}a_{il}\varphi(\eta_l(t)-\eta_i(t)) + c_2\sum_{l\in\mathcal{N}_i}a_{il}(\eta_l(t)-\eta_i(t))^{\alpha}\Big) \\
&= -\frac{1}{2}c_1\sum_{i=1}^{r}\sum_{l\in\mathcal{N}_i}a_{il}(\eta_i(t)-\eta_l(t))\varphi(\eta_i(t)-\eta_l(t)) - \frac{1}{2}c_2\sum_{i=1}^{r}\sum_{l\in\mathcal{N}_i}a_{il}(\eta_i(t)-\eta_l(t))^{\alpha+1} \\
&\leq -\frac{1}{2}c_1\sum_{i=1}^{r}\sum_{l\in\mathcal{N}_i}a_{il}(\eta_i(t)-\eta_l(t))^{\beta+1} - \frac{1}{2}c_2\sum_{i=1}^{r}\sum_{l\in\mathcal{N}_i}a_{il}(\eta_i(t)-\eta_l(t))^{\alpha+1} \\
&\leq -\frac{1}{2}\underline{c}\sum_{i=1}^{r}\sum_{l\in\mathcal{N}_i}a_{il}\Big((\eta_i(t)-\eta_l(t))^{\beta+1} + (\eta_i(t)-\eta_l(t))^{\alpha+1}\Big) \\
&\leq -\underline{c}\sum_{i=1}^{r}\sum_{l\in\mathcal{N}_i}a_{il}(\eta_i(t)-\eta_l(t))^{\frac{\alpha+\beta+2}{2}} \leq -\underline{c}\left(\sum_{i=1}^{r}\sum_{l\in\mathcal{N}_i}a_{il}^{\frac{4}{\alpha+\beta+2}}(\eta_i(t)-\eta_l(t))^2\right)^{\frac{\alpha+\beta+2}{4}} \\
&= -\underline{c}\Big(2\eta^T(t)\mathcal{L}_B\eta(t)\Big)^{\frac{\alpha+\beta+2}{4}} \leq -\underline{c}\Big(4\lambda_2(\mathcal{L}_B)V(t)\Big)^{\frac{\alpha+\beta+2}{4}} \\
&= -\underline{c}\cdot 2^{\frac{\alpha+\beta+2}{2}}\cdot\Big(\lambda_2(\mathcal{L}_B)\Big)^{\frac{\alpha+\beta+2}{4}}\cdot\Big(V(t)\Big)^{\frac{\alpha+\beta+2}{4}},
\end{aligned}$$

where $\lambda_2(\mathcal{L}_B)$ is the second smallest eigenvalue of $\mathcal{L}_B$; $\mathcal{L}_B$ is the Laplacian matrix of graph $\mathcal{G}(B)$, and $B = [a_{il}^{\frac{4}{\alpha+\beta+2}}] \in \mathbb{R}^{r\times r}$.
On the other hand, by conditions of the theorem, one has

$$\Big(2g_i(t_k) - \gamma\lambda_N g_i^2(t_k)\Big)\mathcal{L} \geq 0.$$

It follows from Lemma 2.6.11 that

$$2g_i(t_k)\mathcal{L} - g_i^2(t_k)\mathcal{L}^2 \geq 0,$$

or

$$\left(\mathbf{I}_r - g_i(t_k)\mathcal{L}\right)^2 \leq \mathbf{I}_r.$$

As $t = t_k$, one obtains

$$\begin{aligned}
V(t_k^+) &= \frac{1}{2}\sum_{i=1}^{r}\eta_i^2(t_k^+) = \frac{1}{2}\eta^T(t_k^+)\eta(t_k^+) \\
&= \frac{1}{2}\eta^T(t_k)(I - g_i(t_k)\mathcal{L}')^2\eta(t_k) \\
&\leq \frac{1}{2}\eta^T(t_k)\eta(t_k) \\
&= \frac{1}{2}\sum_{i=1}^{r}\eta_i^2(t_k) = V(t_k).
\end{aligned}$$

Hence, from Lemma 2.7.4, we have $\eta \equiv 0$ is finite-time stable, i.e., there is a time T such that $x_i(t) = \bar{x}, \ \ \forall t \geq T, \ \ \forall i \in \mathcal{I}_c$. Thus, the consensus can be achieved in finite time with the protocol (5.4). This implies that, for $t \in (t_{k-1}, t_k]$,

$$\lim_{t\to T}\|x_i(t) - x_j(t)\| = 0 \ \text{ and } \ x_i(t) = x_j(t), \ \ \forall t \geq T \qquad \text{for } \ i, j \in \mathcal{I}_c. \tag{5.9}$$

Now, we will show that

$$\lim_{t_k\to T}\|x_i(t_k) - x_j(t_k)\| = 0, \ \text{ and } \ x_i(t_k) = x_j(t_k), \ \ \forall t_k \geq T \qquad \text{for } \ i, j \in \mathcal{I}_n.$$

Consider, for $i, j \in \mathcal{I}_n$,

$$\|x_i(t_k) - x_j(t_k)\| \leq \|x_i(t_k) - x_i(t)\| + \|x_i(t) - x_j(t)\| + \|x_j(t) - x_j(t_k)\|. \tag{5.10}$$

The proof can be separated into three cases as follows:

Case 1. If $i, j \in \mathcal{I}_c$, the proof can be done by the above discussion.

Case 2. If $i, j \in \mathcal{I}_n \backslash \mathcal{I}_c = \{c+1, c+2, ..., n\}$, the problem can be simplified by considering the communication network of $\mathcal{G}_d \cup \mathcal{G}'$. Since the discrete-time dynamic agents interact with their neighbours at time $t = t_k$, one obtains

$$x_i(t_{k+1}) = x_i(t_k) + h\sum_{j\in\mathcal{N}_i} b_{ij}[x_j(t_k) - x_i(t_k)], \tag{5.11}$$

where $h = t_k - t_{k-1}$ is a sampling period and $B = [b_{ij}]_{m\times m}$ is the adjacency matrix of $\mathcal{G}_d \cup \mathcal{G}'$, where $|\mathcal{G}_d \cup \mathcal{G}'| = m \leq n$ is the number of the discrete-time dynamic agents and continuous-time dynamic agents that interact with them.

According to Lemma 2.6.12, applying the distributed consensus algorithm (5.11) and **(A2)**, then a consensus is asymptotically reached for all initial states. Let the convergence value be equal to x^*. Since $\mathcal{G}$ is balanced, the average consensus is asymptotically reached and $x^* = \dfrac{1}{m}\sum_{i=1}^{m} x_i(0), \quad i = 1, 2, \ldots, m$. Let $t_k \to \infty$, then equation (5.11) can be rewritten as

$$x^* = x^* - h\mathcal{L}_2 x^*, \tag{5.12}$$

where $\mathcal{L}_2$ is the Laplacian matrix of $\mathcal{G}_d \cup \mathcal{G}'$. Furthermore, (5.12) can be simplified for

$$\mathcal{L}_2 x^* = 0. \tag{5.13}$$

By **(A1)**, we have an arbitrarily small positive constant ρ, which satisfies $0 < \rho < 1$ such that $C = \rho I + \mathcal{L}_2$, and $b = \rho x^*$. Moreover, equation (5.13) is equivalent to the following equation

$$Cx^* = (\rho I + \mathcal{L}_2)x^* = \rho x^* = b.$$

According to the above discussion, we can easily get that C is a symmetrical positive definite matrix. By Lemma 2.6.13, the consensus of the multi-agent system can be reached in a maximum of n step. Let an iterative step be a time unit, then the consensus of the discrete multi-agent system can be reached in finite-time $n = T'$ i.e., there is a time T' such that

$$\lim_{t \to T'} \|x_i(t_k) - x_j(t_k)\| = 0 \;\text{ and }\; x_i(t_k) = x^*, \quad \forall t_k \geq T', \quad \forall i, j \in \mathcal{I}_n \backslash \mathcal{I}_c.$$

Case 3. If $j \in \mathcal{I}_n \backslash \mathcal{I}_c$ and $i \in \mathcal{I}_c$, we consider $i, l \in \mathcal{I}_c$ and $j \in \mathcal{I}_n \backslash \mathcal{I}_c$,

$$\|x_i(t) - x_l(t)\| \leq \|x_i(t) - x_i(t_k)\| + \|x_i(t_k) - x_j(t_k)\| + \|x_j(t_k) - x_l(t_k)\| + \|x_l(t_k) - x_l(t)\|.$$

Since, for $i, l \in \mathcal{I}_c$,

$$\lim_{t \to T} \|x_i(t) - x_l(t)\| = 0 \;\text{ and }\; x_i(t) = \bar{x}, \quad \forall t \geq T, \quad \forall i \in \mathcal{I}_c.$$

When $t \to \infty$, we have $t_k \to \infty$. Thus, for any $i, l \in \mathcal{I}_c$,

$$\lim_{t \to T} \|x_i(t) - x_i(t_k)\| = 0, \quad \lim_{t \to T} \|x_l(t_k) - x_l(t)\| = 0,$$

and

$$x_i(t) = x_l(t) = \bar{x}, \quad \forall t \geq T.$$

By choosing $T^* = max\{T, T'\}$, then there exists a real number $\underline{x}$ such that

$$\lim_{t \to T^*} \|x_i(t_k) - x_j(t_k)\| = 0, \lim_{t \to T^*} \|x_j(t_k) - x_l(t_k)\| = 0,$$

and

$$x_i(t) = x_j(t) = x_l(t) = \underline{x}, \quad \forall t \geq T^*.$$

Hence, for each $j \in \mathcal{I}_n \backslash \mathcal{I}_c$ and $i \in \mathcal{I}_c$,

$$\lim_{t_k \to T^*} \|x_i(t_k) - x_j(t_k)\| = 0, \quad \text{and} \quad x_i(t_k) = x_j(t_k) = \underline{x}, \quad \forall t_k \geq T^*. \tag{5.14}$$

Claim that $\underline{x} = x^* = \bar{x}$. Consider, for each $j \in \mathcal{I}_n \backslash \mathcal{I}_c$ and $i \in \mathcal{I}_c$,

$$\|x_i(t_k) - x_j(t_k)\| \leq \|x_i(t_k) - \bar{x}\| + \|\bar{x} - x^*\| + \|x^* - x_j(t_k)\|. \tag{5.15}$$

Taking the limit on both sides of equations (5.15) as $t_k \to T^*$, one obtains

$$\lim_{t_k \to T^*} \|\bar{x} - x^*\| = 0, \quad \text{i.e.,} \quad \bar{x} = x^*.$$

It follows form **Case 1,2** and **3** that there is an T^* such that

$$\lim_{t_k \to T^*} \|x_i(t_k) - x_j(t_k)\| = 0, \quad \text{and} \quad x_i(t_k) = x_j(t_k) = \bar{x}, \quad \forall t_k \geq T^*, \quad \text{for} \quad i, j \in \mathcal{I}_n. \tag{5.16}$$

Therefore, the hybrid multi-agent system (5.1) with protocol (5.4) reaches consensus in finite time.
(Necessity) Suppose that $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$ are not connected. Then, by Lemma 2.6.12, we have $\lim_{t_k \to \infty} x_i(t_k) \neq x^*$ for some $i \in \mathcal{I}_n \backslash \mathcal{I}_c$. Hence, there exist $i, j \in \mathcal{I}_n \backslash \mathcal{I}_c$ such that

$$\lim_{t_k \to \infty} \|x_i(t_k) - x_j(t_k)\| \neq 0.$$

This implies that the hybrid multi-agent system (5.1) cannot achieve consensus. □

Remark. The results in this paper establish a unified viewpoint for the finite-time consensus of continuous-time and discrete-time multi-agent systems. In other words, if $c = n$, the hybrid multi-agent system (5.1) becomes a continuous-time multi-agent system. And if $c = 0$, the hybrid multi-agent system (5.1) becomes a discrete-time multi-agent system.

5.2 Finite-time scaled consensus problems in Multi-agent systems

5.2.1 Finite-time scaled consensus of MASs via impulsive protocols

In this section, we study finite-time scaled consensus problems of multi-agent systems by using the impulsive consensus protocols.

Consider a multi-agent system consists of n agents, labeled 1 through n. Let $\mathcal{I}_n = \{1, 2, 3, ..., n\}$ and $\beta_i \neq 0$ be a constant, then the dynamics of each agent with scalar scale is as follows:

$$\beta_i \dot{x}_i(t) = u_i, \quad \text{for} \;\; i \in \mathcal{I}_n, \tag{5.17}$$

where $\beta_i \in \mathbb{R}/\{0\}$, $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the scalar scale, state and control input of agent i, respectively. The initial conditions are $x_i(0) = x_{i0}$, and $x(0) = [x_{10}, x_{20}, ..., x_{n0}]^T$.

Definition 5.2.1. The multi-agent system (5.17) is said to reach finite-time scaled consensus to $(\beta_1, \ldots, \beta_n)$ if for any initial conditions, there is a setting time T such that

$$\lim_{t\to T} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0, \;\; \text{and} \;\; \beta_i x_i(t) = \beta_j x_j(t), \;\; \forall t \geq T, \;\; \text{for} \;\; i, j \in \mathcal{I}_n. \tag{5.18}$$

Now, we are in a position to present our protocol that solves the scaled consensus problem in finite time:

$$\begin{aligned} u_i(t) = {} & c_1|\beta_i| \sum_{l\in\mathcal{N}_i} a_{il}\varphi(\beta_l x_l(t) - \beta_i x_i(t)) + c_2|\beta_i| \sum_{l\in\mathcal{N}_i} a_{il}(\beta_l x_l(t) - \beta_i x_i(t))^{\alpha} \\ & + h \cdot |\beta_i| \sum_{k=1}^{\infty} \delta(t - t_k) \sum_{l\in\mathcal{N}'_i} a'_{il}(\beta_l x_l(t) - \beta_i x_i(t)), \qquad \text{for} \;\; i \in \mathcal{I}_n. \end{aligned} \tag{5.19}$$

where β_i is a nonzero scalar scale of agent i, $\mathcal{A} = [a_{ij}]$ and $\mathcal{A}' = [a'_{ij}]$ are the weighted adjacency matrices associated with the graph $\mathcal{G}$ and $\mathcal{G}'$, respectively. Moreover, $h = t_k - t_{k-1}$ is the sampling period and N_i is the set of neighbors of i.

Furthermore, $c_1, c_2 > 0$, $\underline{c} = min\{c_1, c_2\}$; the odd function $\varphi(\cdot)$ satisfying that $y \cdot \varphi(y) \geq y^{\gamma+1} > 0$, $\forall y \in \mathbb{R} \setminus \{0\}$, $\varphi(0) = 0$; α, γ are ratios of odd integers with $\alpha + \gamma \in (-2, 2)$; the discrete instant t_k, $k \in \mathbb{N}_+$ satisfy $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$ and $\delta(\cdot)$ is the Dirac delta function.

Assume that there exist two constants r_1 and r_2 such that $0 < r_1 \le t_{k+1} - t_k \le r_2 < \infty$. Equivalently, the system (5.17) with the protocol (5.19) can be written as follows:

$$\begin{cases} \beta_i \dot{x}_i(t) &= c_1|\beta_i| \sum\limits_{l\in\mathcal{N}_i} a_{il}\varphi(\beta_l x_l(t) - \beta_i x_i(t)) + c_2|\beta_i| \sum\limits_{l\in\mathcal{N}_i} a_{il}(\beta_l x_l(t) - \beta_i x_i(t))^{\alpha}, \quad t \neq t_k, \\ \Delta\beta_i x_i(t_k) &= h\cdot|\beta_i| \sum\limits_{l\in\mathcal{N}'_i} a'_{il}(\beta_l x_l(t_k) - \beta_i x_i(t_k)), \qquad \text{for} \quad k \in \mathbb{N} \end{cases} \tag{5.20}$$

where $x_i(t) \in \mathbb{R}$ is the state of agent i at time t, $i = 1, 2, ..., n$. $\Delta\beta_i x_i(t_k) = \beta_i x_i(t_k^+) - \beta_i x_i(t_k^-)$; $\beta_i x_i(t_k^+) = \lim\limits_{h\to 0^+} \beta_i x_i(t_k + h)$ and $\beta_i x_i(t_k^-) = \lim\limits_{h\to 0^+} \beta_i x_i(t_k - h)$, $h > 0$.

Theorem 5.2.1. Let $\mathcal{G}$ be a communication network of the multi-agent system (5.17) and $\beta_i \neq 0$ be a scalar scale of agent i. The multi-agent system (5.17) with the protocol (5.19) is said to reach scaled consensus to $(\beta_1, \dots, \beta_n)$ in finite time if the following conditions are satisfied :

(i) $0 < h < \dfrac{1}{d_{max}\beta_{max}}$;

(ii) there exists a constant $0 < \epsilon \le 1$ such that

$$(1-\epsilon)\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}' - (H|\mathcal{B}|\mathcal{L}')^T + \mathcal{L}'^T(H|\mathcal{B}|)^2\mathcal{L}' \le 0;$$

(iii) $\mathcal{G} \cup \mathcal{G}'$ is connected,

where $d_{max} = \max_i\{d_{ii}\}$ and $\beta_{max} = \max_i\{|\beta_i|\}$, for $i \in \mathcal{I}_n$; $\mathcal{L}'$ is the Laplacian matrix of $\mathcal{G} \cup \mathcal{G}'$ at $t = t_k$.

Proof. **(Sufficiency)** For simplicity, we denote that $\hat{x}_i(t) =: \beta_i x_i(t)$ for all i.
Then, the system (5.20) can be written as

$$\begin{cases} \dfrac{d\hat{x}_i(t)}{dt} &= c_1|\beta_i| \sum\limits_{l\in\mathcal{N}_i} a_{il}\varphi(\hat{x}_l(t) - \hat{x}_i(t)) + c_2|\beta_i| \sum\limits_{l\in\mathcal{N}_i} a_{il}(\hat{x}_l(t) - \hat{x}_i(t))^{\alpha}, \quad t \neq t_k, \\ \Delta\hat{x}_i(t_k) &= h\cdot|\beta_i| \sum\limits_{l\in\mathcal{N}'_i} a'_{il}(\hat{x}_l(t_k) - \hat{x}_i(t_k)), \qquad \text{for} \quad k \in \mathbb{N}. \end{cases} \tag{5.21}$$

Assume that $x_i(t_k^-) = x_i(t_k)$, which implies that the solution is left continuous at time t_k and denote $\hat{x}(t) = (\hat{x}_1(t), \hat{x}_2(t), ..., \hat{x}_n(t))^T \in \mathbb{R}^n$, $H = diag(h, h, \dots, h)$,
$\mathcal{B} = diag(\beta_1, \beta_2, ..., \beta_n) \in \mathbb{R}^{n\times n}$ and $|\mathcal{B}| = diag(|\beta_1|, |\beta_2|, ..., |\beta_n|) \in \mathbb{R}^{n\times n}$. Then, the system (5.21) can be described as:

$$\begin{cases} \dfrac{d\hat{x}(t)}{dt} = F(t, \hat{x}(t)), \qquad t \neq t_k, \\ \hat{x}(t_k^+) = (\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}')\hat{x}(t_k), \qquad k \in \mathbb{N} \end{cases} \tag{5.22}$$

where $F(t, \hat{x}(t)) = (f_1^T(t, \hat{x}(t)), f_2^T(t, \hat{x}(t)), ..., f_n^T(t, \hat{x}(t)))^T$;

$$f_i(t, \hat{x}(t)) = c_1|\beta_i| \sum_{l \in \mathcal{N}_i} a_{il}\varphi(\hat{x}_l(t) - \hat{x}_i(t)) + c_2|\beta_i| \sum_{l \in \mathcal{N}_i} a_{il}(\hat{x}_l(t) - \hat{x}_i(t))^\alpha;$$

$\mathcal{L}$ and $\mathcal{L}'$ are the Laplacian matrices of $\mathcal{G} \cup \mathcal{G}'$ when $t \neq t_k$ and $t = t_k$, respectively.

Let $\bar{x}(t) = \dfrac{1}{n} \displaystyle\sum_{i=1}^{n} \beta_i x_i(t)$ and denote that

$$\eta_i(t) = \hat{x}_i(t) - \bar{x}, \quad t \in (t_{k-1}, t_k],$$

$\eta_i(t_k^+) = \hat{x}_i(t_k^+) - \bar{x}$ and $\eta_i(t_k^-) = \eta_i(t_k)$, $\quad i = 1, 2, 3, ..., n$, with initial conditions $\hat{x}(t_0) = \hat{x}(0) = [\hat{x}_{10}, \hat{x}_{20}, ..., \hat{x}_{n0}]^T$. Thus,

$$\begin{cases} \dot{\eta}(t) = F(t, \eta(t)), & t \neq t_k \\ \eta(t_k^+) = [\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}']\eta(t_k), & t = t_k, \quad k \in \mathbb{N}. \end{cases} \tag{5.23}$$

Consider the Lyapunov function:

$$V(\eta) = \frac{1}{2}\eta^T(t)\eta(t) = \frac{1}{2}\sum_{i=1}^{n} \eta_i^2(t),$$

where $\eta = (\eta_1, ..., \eta_n)^T$ is the error vector or disagreement vector.
For $t \neq t_k$, using the fact from Lemmas 2.6.8-2.6.10, one obtains

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{n} \eta_i(t)\dot{\eta}_i(t) \\ &= \sum_{i=1}^{n} \eta_i \Big(c_1|\beta_i| \sum_{l \in \mathcal{N}_i} a_{il}\varphi(\eta_l(t) - \eta_i(t)) + c_2|\beta_i| \sum_{l \in \mathcal{N}_i} a_{il}(\eta_l(t) - \eta_i(t))^\alpha \Big) \\ &= -\frac{1}{2}c_1|\beta_i| \sum_{i=1}^{n} \sum_{l \in \mathcal{N}_i} a_{il}(\eta_i(t) - \eta_l(t))\varphi(\eta_i(t) - \eta_l(t)) \\ &\quad - \frac{1}{2}c_2|\beta_i| \sum_{i=1}^{n} \sum_{l \in \mathcal{N}_i} a_{il}(\eta_i(t) - \eta_l(t))^{\alpha+1} \end{aligned}$$

$$
\begin{aligned}
&\leq -\frac{1}{2}c_1|\beta_i|\sum_{i=1}^{n}\sum_{l\in\mathcal{N}_i}a_{il}(\eta_i(t)-\eta_l(t))^{\gamma+1}-\frac{1}{2}c_2|\beta_i|\sum_{i=1}^{n}\sum_{l\in\mathcal{N}_i}a_{il}(\eta_i(t)-\eta_l(t))^{\alpha+1}\\
&\leq -\frac{1}{2}\underline{c}|\beta_i|\sum_{i=1}^{n}\sum_{l\in\mathcal{N}_i}a_{il}\Big((\eta_i(t)-\eta_l(t))^{\gamma+1}+(\eta_i(t)-\eta_l(t))^{\alpha+1}\Big)\\
&\leq -\underline{c}|\beta_i|\sum_{i=1}^{n}\sum_{l\in\mathcal{N}_i}a_{il}(\eta_i(t)-\eta_l(t))^{\frac{\alpha+\gamma+2}{2}}\\
&\leq -\underline{c}|\beta_i|\left(\sum_{i=1}^{n}\sum_{l\in\mathcal{N}_i}a_{il}^{\frac{4}{\alpha+\gamma+2}}(\eta_i(t)-\eta_l(t))^{2}\right)^{\frac{\alpha+\gamma+2}{4}}\\
&= -\underline{c}|\beta_i|\Big(2\eta^T(t)\mathcal{L}_B\eta(t)\Big)^{\frac{\alpha+\gamma+2}{4}} \leq -\underline{c}|\beta_i|\Big(4\lambda_2(\mathcal{L}_B)V(t)\Big)^{\frac{\alpha+\gamma+2}{4}}\\
&= -\underline{c}|\beta_i|\cdot 2^{\frac{\alpha+\gamma+2}{2}}\cdot\Big(\lambda_2(\mathcal{L}_B)\Big)^{\frac{\alpha+\gamma+2}{4}}\cdot\Big(V(t)\Big)^{\frac{\alpha+\gamma+2}{4}},
\end{aligned}
$$

where $\lambda_2(\mathcal{L}_B)$ is the second smallest eigenvalue of $\mathcal{L}_B$, where $\mathcal{L}_B$ is the Laplacian matrix of graph $\mathcal{G}(B)$, and $B = [a_{il}^{\frac{4}{\alpha+\gamma+2}}] \in \mathbb{R}^{n\times n}$. On the other hand, when $t = t_{k-1}$, there exists $0 < \epsilon \leq 1$ such that

$$
(1-\epsilon)\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}' - (H|\mathcal{B}|\mathcal{L}')^T + \mathcal{L}'^T(H|\mathcal{B}|)^2\mathcal{L}' \leq 0.
$$

Hence,

$$
\begin{aligned}
V(t_{k-1}^{+}) &= \frac{1}{2}\eta^T(t_{k-1})(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}')^T(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}')\eta(t_{k-1})\\
&= \frac{1}{2}\eta^T(t_{k-1})[\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'^T - H|\mathcal{B}|\mathcal{L}' + (H|\mathcal{B}|)^2\mathcal{L}'^T\mathcal{L}' - \epsilon\mathbf{I}_n + \epsilon\mathbf{I}_n]\eta(t_{k-1})\\
&= \frac{1}{2}\eta^T(t_{k-1})[(1-\epsilon)\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}'^T - H|\mathcal{B}|\mathcal{L}' + (H|\mathcal{B}|)^2\mathcal{L}'^T\mathcal{L}']\eta(t_{k-1})\\
&\quad + \frac{1}{2}\epsilon\eta^T(t_{k-1})\eta(t_{k-1})\\
&\leq \epsilon\frac{1}{2}\eta^T(t_{k-1})\eta(t_{k-1})\\
&= \epsilon V(t_{k-1}).
\end{aligned}
$$

Thus, by Lemma 2.7.4, one obtains that $\eta(t) \equiv 0$ is finite-time stable i.e., there exists a time T such that $\hat{x}_i(t) = \beta_i x_i(t) = \bar{x}$, $\forall t \geq T$, $\forall i \in \mathcal{I}_n$. Therefore, the scaled consensus can be achieved in finite time. □

Remark. It is obvious that if $\beta_i = 1$ for all i, then the scaled consensus protocol (5.19) can be written as

$$\begin{aligned} u_i(t) = c_1 \sum_{l \in \mathcal{N}_i} a_{il} \varphi(x_l(t) - x_i(t)) + c_2 \sum_{l \in \mathcal{N}_i} a_{il} (x_l(t) - x_i(t))^{\alpha} \\ + h \cdot \sum_{k=1}^{\infty} \delta(t - t_k) \sum_{l \in \mathcal{N}_i'} a_{il}' (x_l(t) - x_i(t)), \qquad \text{for} \quad i \in \mathcal{I}_n. \end{aligned}$$

Moreover, if $\beta_i = 1$ for all i and there is no intermittent communications among agents (at time t_k), the protocol (5.19) can be reduced as

$$u_i(t) = c_1 \sum_{l \in \mathcal{N}_i} a_{il} \varphi(x_l(t) - x_i(t)) + c_2 \sum_{l \in \mathcal{N}_i} a_{il} (x_l(t) - x_i(t))^{\alpha},$$

which was studied in [41].

5.2.2 Finite-time scaled consensus of MASs with impulsive perturbations

Consider a group of n identical agents described as in (5.17), in this section, the finite-time scaled consensus problems of multi-agent system with impulsive perturbations have been studied. To solve the finite-time scaled consensus problems, the scaled consensus protocol is designed as follows:

$$\begin{aligned} u_i(t) = |\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij} sgn(\beta_j x_j(t) - \beta_i x_i(t)) |\beta_j x_j(t) - \beta_i x_i(t)|^{\gamma} \\ + \sum_{k=1}^{\infty} p_k \beta_i x_i(t) \delta(t - t_k), \qquad \text{for} \quad i \in \mathcal{I}_n. \end{aligned} \tag{5.24}$$

where β_i is a nonzero scalar scale of agent i and $0 < \gamma < 1$; a_{ij} is the element in the adjacency matrix of digraph $\mathcal{G}$; $sgn(\cdot)$ is a signum function defined as above; $p_k > 0, \quad k \in \mathbb{Z}_+$ denotes the impulsive perturbation coefficient; $\delta(\cdot)$ is the Dirac delta function and t_k is an impulsive instant for $k \in \mathbb{Z}_+$ satisfying $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, which $lim_{t \to \infty} t_k = \infty$.

By the definition of the Dirac delta function, the system (5.17) with protocol (5.24) can be written as the impulsive differential equation as follows:

$$\begin{cases} \beta_i \dot{x}_i(t) = |\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij} sgn(\beta_j x_j(t) - \beta_i x_i(t)) |\beta_j x_j(t) - \beta_i x_i(t)|^{\gamma}, & t \neq t_k, \\ \Delta \beta_i x_i(t_k) = p_k \beta_i x_i(t_k), \qquad \text{for} \quad k \in \mathbb{N}, \end{cases} \tag{5.25}$$

where $\Delta\beta_i x_i(t_k) = \beta_i x_i(t_k^+) - \beta_i x_i(t_k^-)$; $\beta_i x_i(t_k^+) = \lim_{h\to 0^+} \beta_i x_i(t_k + h)$ and $\beta_i x_i(t_k^-) = \lim_{h\to 0^+} \beta_i x_i(t_k - h)$, $h > 0$.

Without loss of generality, we assume that the solution $x(t)$ is left-continuous i.e., $x_i(t_k^-) = x_i(t_k)$. Thus, system (5.25) can be written as:

$$\begin{cases} \beta_i \dot{x}_i(t) & = |\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij} sgn(\beta_j x_j(t) - \beta_i x_i(t))|\beta_j x_j(t) - \beta_i x_i(t)|^\gamma, \quad t \neq t_k, \\ \beta_i x_i(t_k^+) & = (1 + p_k)\beta_i x_i(t_k), \qquad \text{for} \quad k \in \mathbb{N}. \end{cases} \tag{5.26}$$

Theorem 5.2.2. Let β_i be a nonzero scalar scale of agent i. The multi-agent system (5.17) under protocol (5.24) is said to reach scaled consensus to $(\beta_1, \dots, \beta_n)$ in finite time if the following assumptions are satisfied:

(A1) The communication network $\mathcal{G}$ is balanced and contains a spanning tree;

(A2) There exist constant $\gamma \in (0, 1)$, $p_k > 0$, and $\alpha \in [1, \infty)$ such that

$$\frac{(1 + p_k)^2}{2} \leq \alpha^{\frac{2}{1-\gamma}}.$$

Proof. Let $\bar{x}(t) = \frac{1}{n}\sum_{i=1}^{n} \beta_i x_i(t)$. It follows from **(A2)** that $\sum_{j\in\mathcal{N}_i} a_{ij} = \sum_{i\in\mathcal{N}_j} a_{ji}$.

Thus,

$$\begin{aligned} D^+\bar{x}(t) &= \frac{1}{n}\sum_{i=1}^{n} D^+\beta_i x_i(t) \\ &= \frac{1}{n}|\beta_i| \sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} a_{ij} sgn(\beta_j x_j(t) - \beta_i x_i(t))|\beta_j x_j(t) - \beta_i x_i(t)|^\gamma \\ &= \frac{1}{n}|\beta_i| \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} sgn(\beta_j x_j(t) - \beta_i x_i(t))|\beta_j x_j(t) - \beta_i x_i(t)|^\gamma \\ &= \frac{1}{2n}|\beta_i| \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} sgn(\beta_j x_j(t) - \beta_i x_i(t))|\beta_j x_j(t) - \beta_i x_i(t)|^\gamma \\ &+ \frac{1}{2n}|\beta_i| \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} sgn(\beta_j x_j(t) - \beta_i x_i(t))|\beta_j x_j(t) - \beta_i x_i(t)|^\gamma \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2n}|\beta_i| \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} a_{ij} sgn(\beta_j x_j(t) - \beta_i x_i(t)) |\beta_j x_j(t) - \beta_i x_i(t)|^{\gamma} \\
&+ \sum_{j=1}^{n} a_{ij} sgn(\beta_i x_i(t) - \beta_j x_j(t)) |\beta_i x_i(t) - \beta_j x_j(t)|^{\gamma} \Big) \\
&= \frac{1}{2n}|\beta_i| \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} a_{ij} sgn(\beta_j x_j(t) - \beta_i x_i(t)) |\beta_j x_j(t) - \beta_i x_i(t)|^{\gamma} \\
&- \sum_{j=1}^{n} a_{ij} sgn(\beta_j x_j(t) - \beta_i x_i(t)) |\beta_i x_i(t) - \beta_j x_j(t)|^{\gamma} \Big) \\
&= 0, \quad t \neq t_k, \quad k \in \mathbb{Z}_+.
\end{aligned}
$$

Similarly, it holds that $\Delta \bar{x}(t) = 0$. This implies that $\bar{x}$ is time-invariant, i.e.

$$\bar{x} = \bar{x}(t) = \bar{x}(0) = Ave(x(0)) = \frac{1}{n}(\sum_{i=1}^{n} \beta_i x_i(0)).$$

The invariance of $\bar{x}$ allows decomposition of $\beta_i x_i$ for $i = 1, 2, ..., n$ as in the following equation:

$$\eta_i(t) = \beta_i x_i(t) - \bar{x}, \quad t \in (t_{k-1}, t_k],$$

$\eta_i(t_k^+) = \beta_i x_i(t_k^+) - \bar{x}$ and $\eta_i(t_k^-) = \eta_i(t_k)$, $\;i = 1, 2, 3, ..., n$, where $\eta = (\eta_1, ..., \eta_n)^T$ is the error vector or disagreement vector. Hence, the error system between $\beta_i x_i(t)$ and $\bar{x}$ can be written as

$$
\begin{cases}
\dot{\eta}_i(t) = |\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij} sgn(\eta_j - \eta_i)|\eta_j - \eta_i|^{\gamma}, & t \neq t_k \\
\eta_i(t_k^+) = (1 + p_k)\eta_i(t_k), & t = t_k, \quad k \in \mathbb{N}.
\end{cases}
\tag{5.27}
$$

Consider the Lyapunov function:

$$V(t) = \frac{1}{2} \sum_{i=1}^{n} \eta_i^2(t). \tag{5.28}$$

It follows from **(A2)** that when $t = t_{k-1}$,

$$
\begin{aligned}
V(t_{k-1}^{+}) &= \frac{1}{2}\sum_{i=1}^{n}\eta_i^2(t_{k-1}^{+}) \\
&= \frac{(1+p_k)^2}{2}\sum_{i=1}^{n}\eta_i^2(t_{k-1}) \\
&\leq \alpha^{\frac{2}{1-\gamma}}V(t_{k-1}). \qquad (5.29)
\end{aligned}
$$

On the other hand, by calculating the upper right-hand Dini derivative of $V(t)$ along the state trajectory of system (5.27) as $t \neq t_k, \quad k \in \mathbb{N}$, we get

$$
\begin{aligned}
D^+V(t) &= \sum_{i=1}^{n}\eta_i(t)\dot{\eta}_i(t) \\
&= \sum_{i=1}^{n}\eta_i(t)\Big(|\beta_i|\sum_{j\in\mathcal{N}_i}a_{ij}sgn(\eta_j(t)-\eta_i(t))|\eta_j(t)-\eta_i(t)|^{\gamma}\Big) \\
&= \frac{1}{2}|\beta_i|\sum_{i=1}^{n}\sum_{j=1}^{n}\eta_i(t)a_{ij}sgn(\eta_j(t)-\eta_i(t))|\eta_j(t)-\eta_i(t)|^{\gamma} \\
&+ \frac{1}{2}|\beta_i|\sum_{i=1}^{n}\sum_{j=1}^{n}\eta_i(t)a_{ij}sgn(\eta_j(t)-\eta_i(t))|\eta_j(t)-\eta_i(t)|^{\gamma} \\
&= \frac{1}{2}|\beta_i|\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\eta_i(t)a_{ij}sgn(\eta_j(t)-\eta_i(t))|\eta_j(t)-\eta_i(t)|^{\gamma} \\
&+ \sum_{j=1}^{n}\eta_j(t)a_{ji}sgn(\eta_i(t)-\eta_j(t))|\eta_i(t)-\eta_j(t)|^{\gamma}\Big) \\
&= \frac{1}{2}|\beta_i|\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\eta_i(t)a_{ij}sgn(\eta_j(t)-\eta_i(t))|\eta_j(t)-\eta_i(t)|^{\gamma} \\
&- \sum_{j=1}^{n}\eta_j(t)a_{ji}sgn(\eta_j(t)-\eta_i(t))|\eta_i(t)-\eta_j(t)|^{\gamma}\Big) \\
&= -\frac{1}{2}|\beta_i|\sum_{i=1}^{n}\sum_{j=1}^{n}(\eta_j(t)-\eta_i(t))a_{ij}sgn(\eta_j(t)-\eta_i(t))|\eta_j(t)-\eta_i(t)|^{\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}|\beta_i| \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} |\eta_j(t) - \eta_i(t)|^{\gamma+1} \\
&\leq -\frac{1}{2}|\beta_i| \left(\sum_{i=1}^{n} \sum_{l \in \mathcal{N}_i} a_{ij}{}^{\frac{2}{1+\gamma}} (\eta_j(t) - \eta_i(t))^2 \right)^{\frac{1+\gamma}{2}} \\
&= -\frac{1}{2}|\beta_i| \Big(2\eta^T(t) \mathcal{L}_\gamma \eta(t) \Big)^{\frac{1+\gamma}{2}} \\
&\leq -\frac{1}{2}|\beta_i| \Big(4\lambda_2(\mathcal{L}_\gamma) \Big)^{\frac{1+\gamma}{2}} \cdot \Big(V(t) \Big)^{\frac{1+\gamma}{2}}, \qquad (5.30)
\end{aligned}
$$

where $\lambda_2(\mathcal{L}_\gamma)$ is the second smallest eigenvalue of $\mathcal{L}_\gamma$, $\mathcal{L}_\gamma$ is the Laplacian matrix of graph $\mathcal{G}(\gamma)$, and $\gamma = [a_{ij}^{\frac{2}{\gamma+1}}] \in \mathbb{R}^{n \times n}$.

Therefore, by Lemma 2.7.4, the multi-agent system (5.17) with the protocol (5.24) reaches scaled consensus to $(\beta_1, \ldots, \beta_n)$ in finite time.

$\square$

5.3 Simulations and Discussion

In this section, two examples have been provided to demonstrate the effectiveness of theoretical results in this work.

Example 1. Consider the multi-agent system (5.17) under protocol (5.19) with 4 agents described as in Figure 5.1.

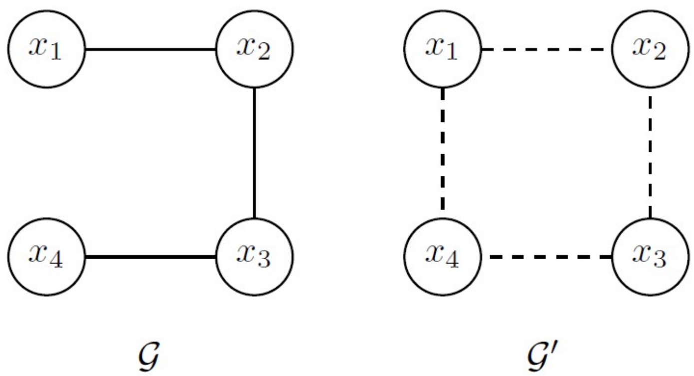

Figure 5.1: A communication network $\mathcal{G}$ at time $t \neq t_k$ and $t = t_k$, respectively.

It is easy to see that the Laplacian matrices of $\mathcal{G}$ and $\mathcal{G}'$ are as follows:

$$\mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad \mathcal{L}' = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

According to the Laplacian matrix $\mathcal{L}$, the eigenvalues are $\lambda_1 = 0, \lambda_2 = 0.5858, \lambda_3 = 2, \lambda_4 = 3.4142$. Let the initial conditions $x(0) = [2 \quad 1 \quad 0 \quad -1]^T$ and scalar scales $\beta_1 = -2, \quad \beta_2 = 1, \quad \beta_3 = 2, \quad \beta_4 = -1.5$. By choosing $\alpha = 3/5, \beta = 1/3, \varphi(y) = y^{\beta}, c_1 = c_2 = 2$, and $h = 0.1$, one obtains that $0 \leq h \leq (\beta_{max} d_{max})^{-1} = 0.25$.

Hence, all conditions of Theorem 5.2.1 are satisfied. Thus, the multi-agent system (5.17) reaches scaled consensus under protocol (5.19) and the states of all agents can be seen in Figure 5.2.

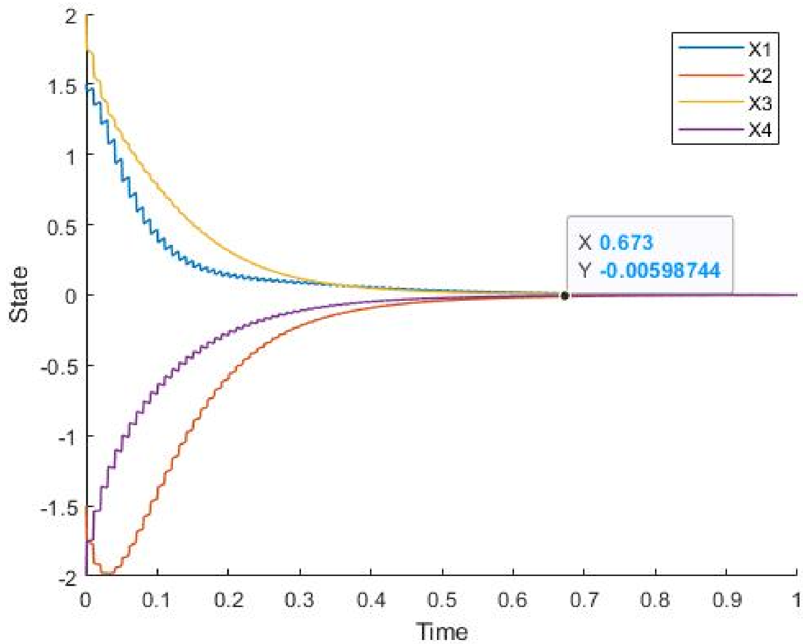

Figure 5.2: The state trajectories of all agents under protocol (5.19) with scalar scales $\beta = (-2,\ 1,\ 2,\ -1.5)$.

Moreover, if the scalar scales $\beta_i = 1$ for all i, the state trajectories of all agents under protocol can be described as in Figure 5.3.

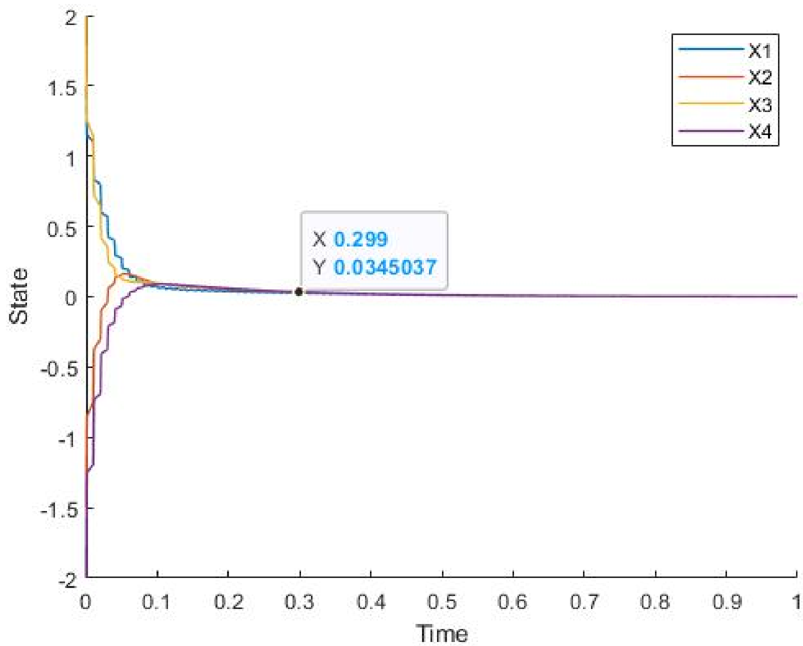

Figure 5.3: The state trajectories of all agents under protocol (5.19) with scalar scales $\beta = (1,\ 1,\ 1,\ 1)$.

On the other hand, if $h = 0.45 > \dfrac{1}{d_{max}\beta_{max}}$, which implies that the assumption **(A1)** is not satisfied. Hence, the multi-agent system (5.17) cannot reach scaled consensus under protocol (5.19) and the states of all agents are described in Figure 5.4.

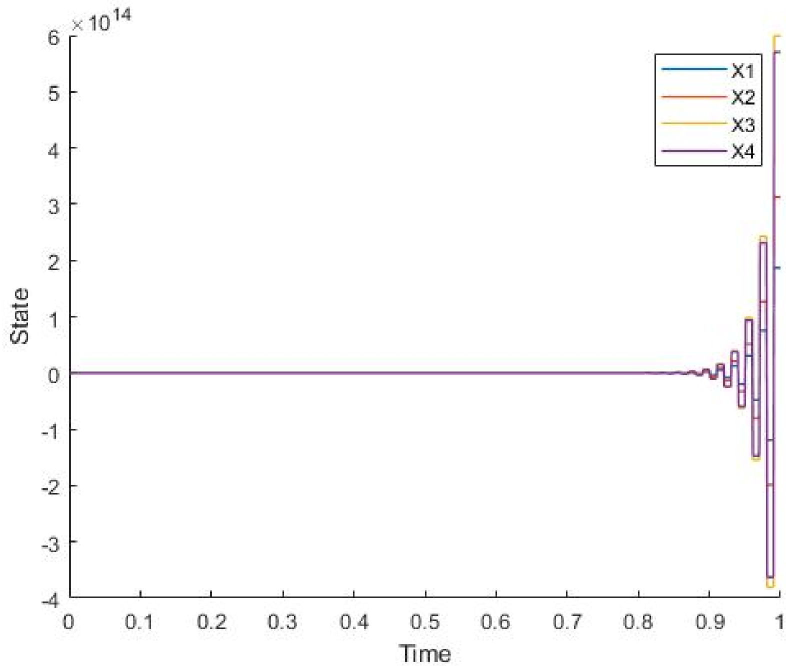

Figure 5.4: The state trajectories of all agents using scaled consensus protocol (5.19) with $h = 0.40$.

In addition, if $\alpha = \gamma = 1$, the finite-time scaled consensus protocol (5.19)

$$u_i(t) = c_1|\beta_i| \sum_{l \in \mathcal{N}_i} a_{il}\varphi(\beta_l x_l - \beta_i x_i) + c_2|\beta_i| \sum_{l \in \mathcal{N}_i} a_{il}(\beta_l x_l - \beta_i x_i)$$
$$+ h \cdot |\beta_i| \sum_{k=1}^{\infty} \delta(t - t_k) \sum_{l \in \mathcal{N}_i'} a'_{il}(\beta_l x_l - \beta_i x_i), \qquad \text{for} \;\; i \in \mathcal{I}_n. \tag{5.31}$$

Thus, by choosing $h = 0.1$ together with $\beta = (-2,\ 1,\ 2,\ -1.5)$, the state trajectories of all agents are described in Figure 5.5

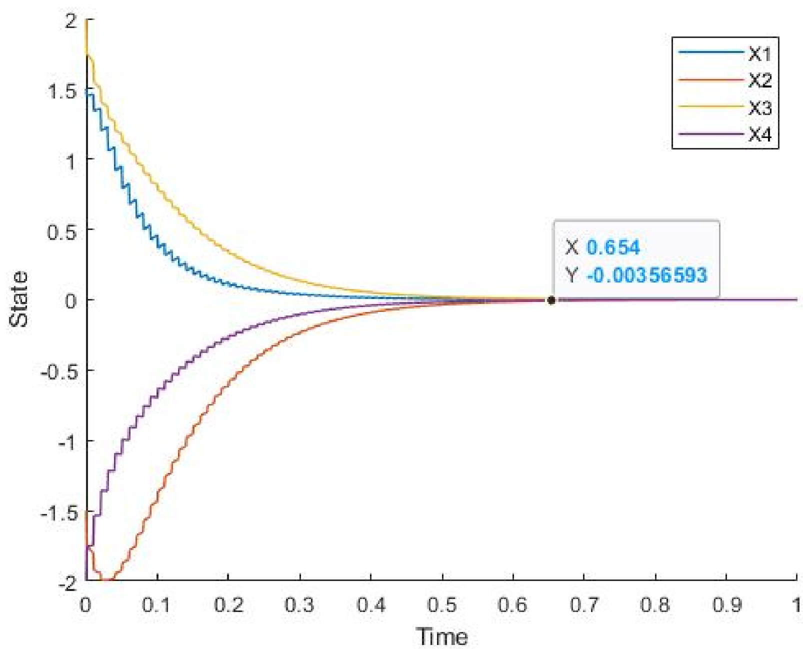

Figure 5.5: The state trajectories of all agents under protocol (5.31) with scalar scales $\beta = (-2,\ 1,\ 2,\ -1.5)$.

If scalar scales $\beta_i = 1$ for all i, the state trajectories of all agents under protocol (5.31) can be depicted as in Figure 5.6.

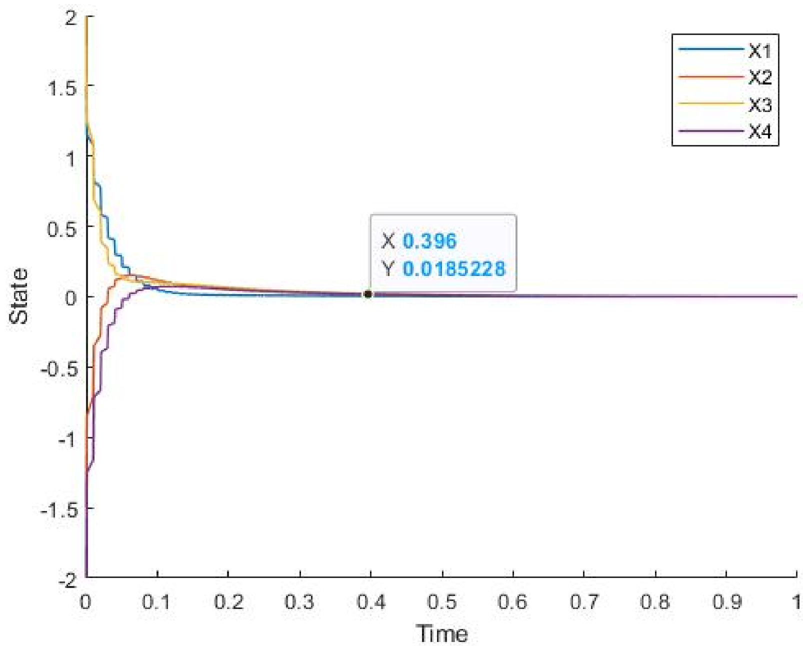

Figure 5.6: The state trajectories of all agents under protocol (5.31) with scalar scales $\beta = (1,\ 1,\ 1,\ 1)$.

If there is no instantaneous communications among agents, the consensus protocol (5.19) can be considered as follows:

$$u_i(t) = c_1|\beta_i| \sum_{l \in \mathcal{N}_i} a_{il}\varphi(\beta_l x_l - \beta_i x_i) + c_2|\beta_i| \sum_{l \in \mathcal{N}_i} a_{il}(\beta_l x_l - \beta_i x_i)^{\alpha}. \tag{5.32}$$

By selecting $h = 0.1$ and $\beta = (-2,\ 1,\ 2,\ -1.5)$, the state trajectories of all agents under protocol (5.32) can be seen as in Figure 5.7.

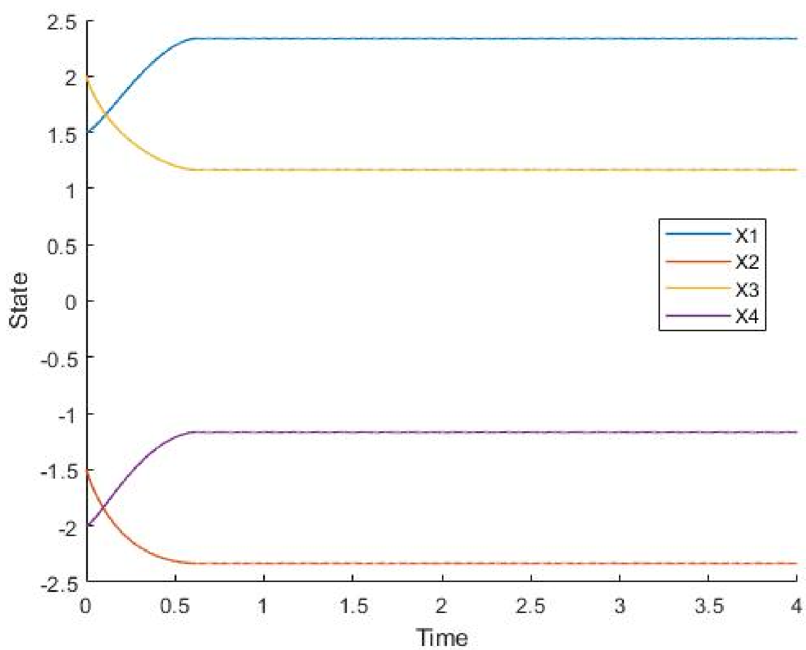

Figure 5.7: The state trajectories of all agents under protocol (5.32) with scalar scales $\beta = (-2,\ 1,\ 2,\ -1.5)$.

Moreover, if scalar scales $\beta = (1,\ 1,\ 1,\ 1)$ and $h = 0.1$, the state trajectories of all agents under protocol (5.32) can be depicted as in Figure 5.8.

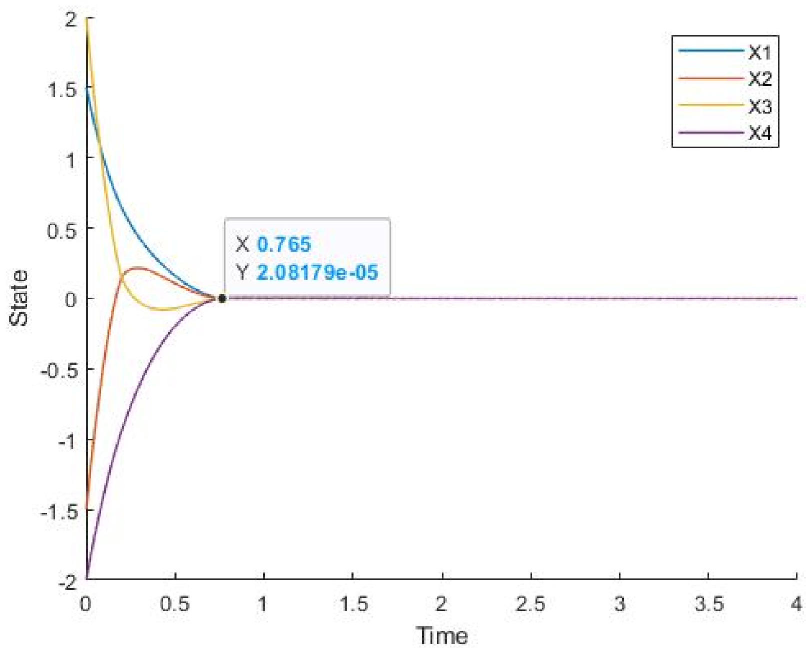

Figure 5.8: The state trajectories of all agents under protocol (5.32) with scalar scales $\beta = (1,\ 1,\ 1,\ 1)$.

If there is no instantaneous communications among agents and $\alpha = \beta = 1$, the consensus of system (5.17) can be achieved asymptotically under the linear scaled consensus protocol (5.17) as follows:

$$u_i(t) = c \cdot sgn(\beta_i) \sum_{l \in \mathcal{N}_i} a_{il}(\beta_l x_l - \beta_i x_i). \tag{5.33}$$

If the sampling period $h = 0.1$ and scalar scales are $\beta = (-2,\ 1,\ 2,\ -1.5)$, the state trajectories of all agents under protocol (5.33) can be described in Figure 5.9.

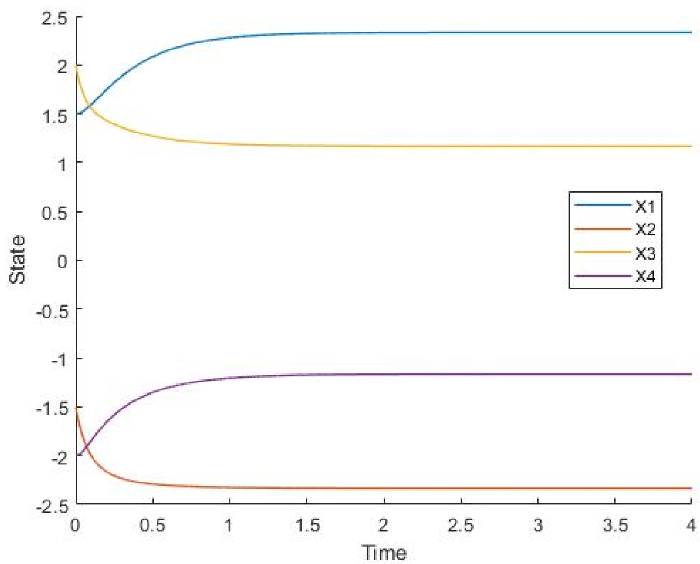

Figure 5.9: The state trajectories of all agents under protocol (5.33) with scalar scales $\beta = (-2,\ 1,\ 2,\ -1.5)$.

If the sampling period $h = 0.1$ and the scalar scales $\beta_i = 1$ for all i, The state trajectories of all agents under protocol (5.33) can be described as in Figure 5.10.

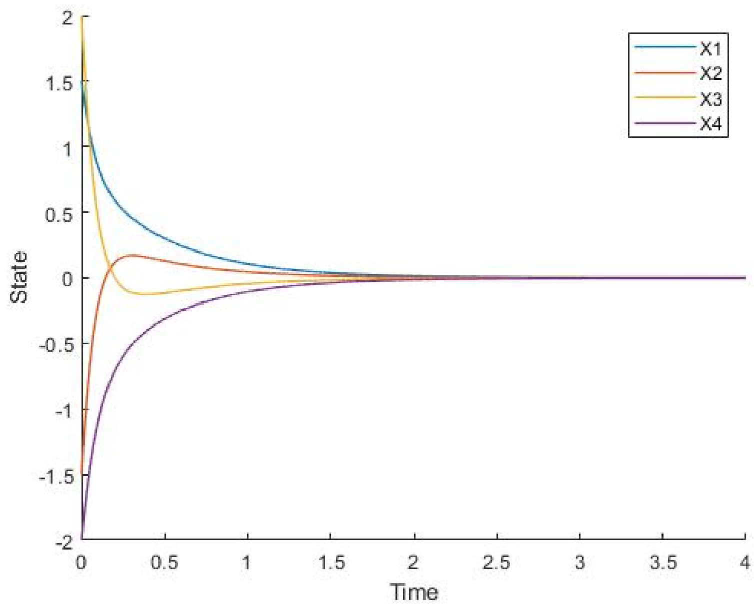

Figure 5.10: The state trajectories of all agents under protocol (5.33) with scalar scales $\beta = (1,\ 1,\ 1,\ 1)$.

Example 2. Consider the communication networks of the multi-agent system (5.17) with impulsive perturbations under protocol (5.24) consisting of 4 agents denoted by x_1, x_2, x_3, x_4 as described Figure 5.11. Assume that the perturbation $p_k = 5 \times 10^{-6}$ for all k and the

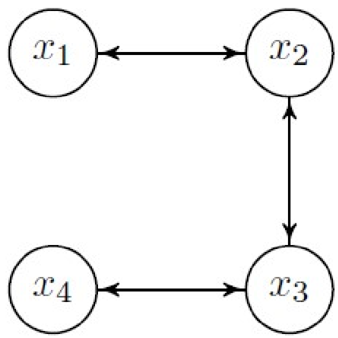

Figure 5.11: A communication network $\mathcal{G}$ of system (5.17).

initial states are $x(0) = [3, 0, 2, -1]^T$, then there exists some constants α and γ which satisfy the assumption (**A2**). Moreover, it is obvious that the digraph $\mathcal{G}$ is balanced and contains a spanning tree, thus the assumption (**A1**) also satisfies. Hence, Theorem 5.2.2 under protocol (5.24) can guarantee achieving scaled consensus.

By choosing $\gamma = 0.2$, the states of all agents with scalar scales $\beta = (-2,\ 1,\ 2,\ -1.5)$ are described in Figure 5.12.

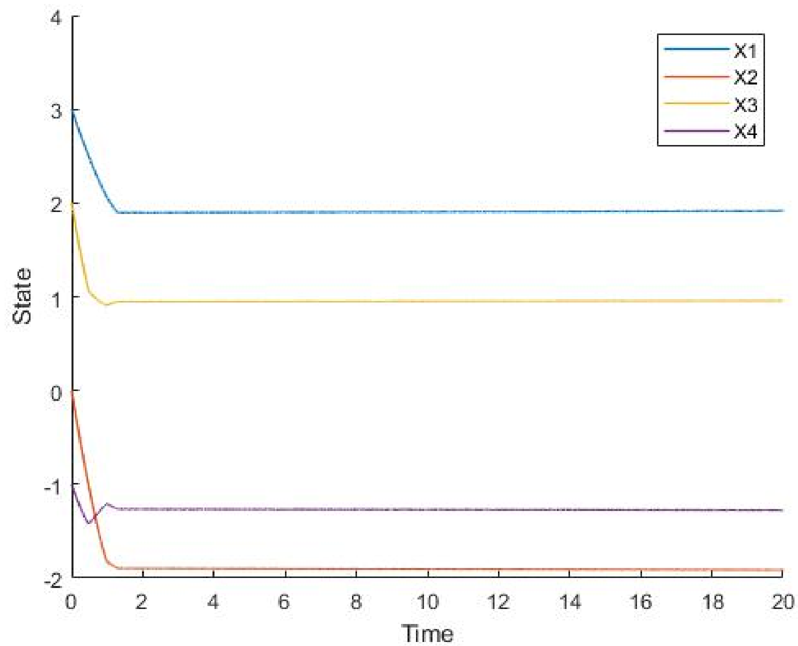

Figure 5.12: The state trajectories of all agents under protocol (5.24) with scalar scales $\beta = (-2,\ 1,\ 2,\ -1.5)$.

Moreover, if the sampling period $h = 0.1$ and the scalar scales $\beta_i = 1$ for all i, The state trajectories of all agents under protocol (5.24) can be described as in Figure 5.13.

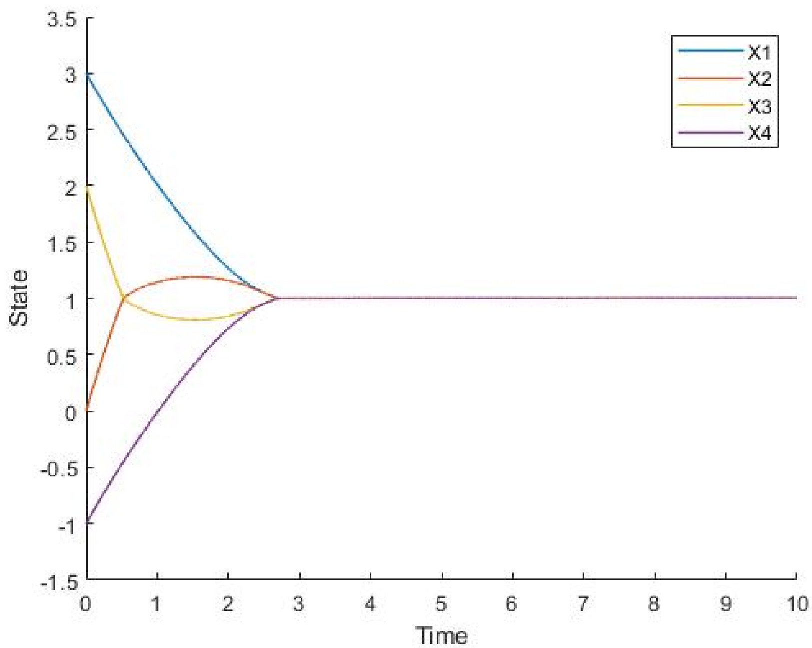

Figure 5.13: The state trajectories of all agents under protocol (5.24) with scalar scales $\beta = (1,\ 1,\ 1,\ 1)$.

However, if the perturbation is not small enough, the Theorem 5.2.1 under protocol (5.24) cannot guarantee reaching scaled consensus. For example, choosing $p_k = 5 \times 10^{-3}$ and scalar scales $\beta = (-2,\ 1,\ 2,\ -1.5)$, the states of all agents are divergent (see Figure 5.14).

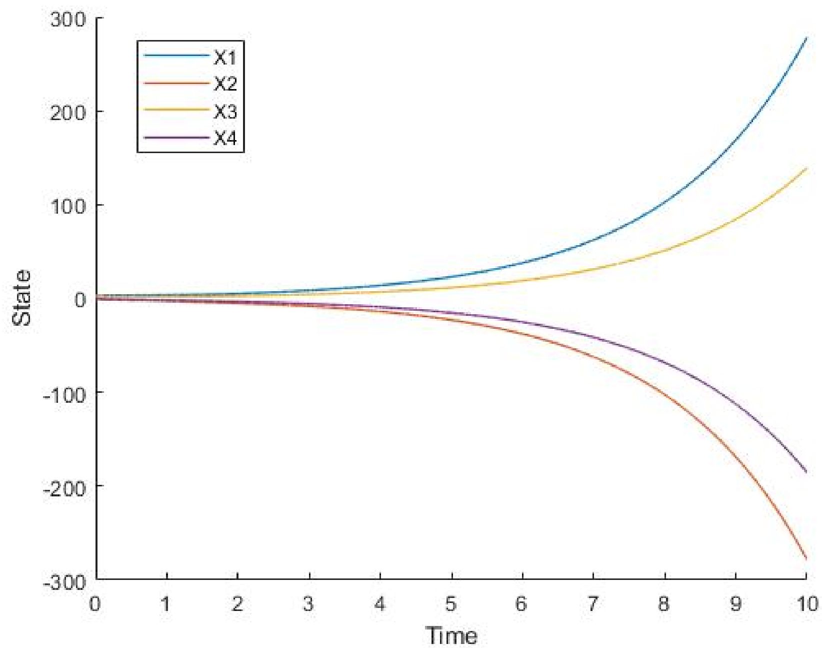

Figure 5.14: The state trajectories of all agents using scaled consensus protocol (5.24) with $p_k = 5 \times 10^{-3}$.

Chapter 6

Applications to Hybrid Multi-agent systems over random networks

In some practical problems, the communication among agents may change over the time due to link failures, packet drops, node failure, etc. Such variations in the network can happen randomly, which attracts researchers' great attention concerning random networks. For instant, in 2005, Hatano and Mesbahi [36] studied consensus problem of multi-agent systems under random networks. In 2007, Porfiri and Stilwell [77] extended the results of Hatano and Mesbahi (2005) by introducing the relaxable conditions to solve the consensus of MASs under random networks. Moreover, in 2008, Tahbaz-Salehi and Jadbabaie [96] gave some necessary and sufficient conditions to solve consensus problem for stochastic discrete-time linear dynamical systems. Other research topics of consensus problems under random networks have also been addressed in [43, 59] and extensive references.

This chapter studies the scaled consensus problem of hybrid multi-agent systems (HMASs) over random networks. Two scaled consensus protocols are introduced depending on the communication among agents. Firstly, we assume that all agents communicate with their neighbours and update their control inputs in a sampling time t_k. Secondly, we study scaled consensus problem when all agents communicate with their neighbours and update their control inputs in a sampling time t_k, where each continuous-time dynamic agent can observe its own state in real time.

6.1 Scaled Consensus of Hybrid Multi-agent systems with random networks

Network Topology

In this subsection, we present some basic concepts of algebraic graph theory and random networks that will be used in this thesis.

The information flow among the nodes of an undirected random network can be described by a sequence of undirected random graphs $\mathcal{G}_i$. At each time i, the random graph is $\mathcal{G}_i = (\mathcal{V}, \mathcal{E}_i)$, where $\mathcal{V} = \{v_i, i = 1, \ldots, n\}$ is the determinate vertex set and $\mathcal{E}_i = \{e_{ij}\} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges where e_{ij} denotes that agent i and j communicate with each other. In the random graph on n vertices, we assume that the existence of $e_{ij} \in \mathcal{E}_i$ is determined randomly and independently of other edges with probability $p_{ij} \in [0, 1]$, for $i, j = 1, 2, \ldots, n$ and $i \neq j$ (not consider self-loops and multiple edges). We define the edge probability matrix $P = P^T \in \mathbb{R}^{n\times n}$, $0 \leq p_{ij} \leq 1$ and $p_{ii} = 0$. The adjacency matrix of $\mathcal{G}_i$ is denoted by $\mathcal{A}_i = [a_{ij}] \in \mathbb{R}^{n\times n}$, where

$$a_{ij} = \begin{cases} 1, & \text{with probability} \;\; p_{ij}, \\ 0, & \text{with probability} \;\; 1 - p_{ij}, \end{cases}$$

for $i \neq j$ and $a_{ii} = 0, \forall i$. The degree matrix denoted by $\mathcal{D}_i \in \mathbb{R}^{n\times n}$ is a diagonal matrix with $d_{ii} = \sum_{j\in N_i} a_{ij}$ and $max_{i\in\mathcal{I}_n}\{d_{ii}\}$ is maximum degree of agent under any random network. The Laplacian matrix of $\mathcal{G}_i$ is $\mathcal{L}_i = [l_{ij}]_{n\times n} = \mathcal{D}_i - \mathcal{A}_i$. Due to the nature of $\mathcal{A}_i$, the Laplacian matrix $\mathcal{L}_i$ is also random. Moreover, it can be seen that $\mathcal{L}_i\mathbf{1} = 0$ and $\mathbf{1}^T\mathcal{L}_i = 0$.

Let $\mathcal{G}, \mathcal{A}, \mathcal{L}$ denote the sample space of all random space of all random graphs, all adjacency matrices and all Laplacian matrices, respectively. The following properties are used in this thesis [57]:

- The expected value of the adjacency matrix of $\mathcal{G}_i$ is denoted by $\bar{\mathcal{A}} = \mathbb{E}[\mathcal{A}_i]$.
- The expected value of the Laplacian matrix is denoted by $\bar{\mathcal{L}} = \mathbb{E}[\mathcal{L}_i] = [\bar{l}_{ij}]_{n\times n}$, where

$$\bar{l}_{ij} = \begin{cases} \sum_{j=1}^{n} p_{ij}, & \text{if} \;\; i = j, \\ -p_{ij}, & \text{otherwise.} \end{cases}$$

- Matrix $\bar{\mathcal{L}}$ corresponds to graph $\bar{\mathcal{G}}$ which does not necessary to belong to $\mathcal{G}$.
- $\bar{\mathcal{G}}$ is the expected graph i.e., the average graph over time.
- $\bar{\mathcal{G}}$ is connected if and only if $\lambda_2(\bar{\mathcal{L}}) > 0$.
- $0 = \lambda_1(\mathcal{L}_i) < \lambda_2(\mathcal{L}_i) \leq \cdots \leq \lambda_n(\mathcal{L}_i)$.
- $\mathcal{L}_i$ is positive semi-definite and has a simple zero eigenvalue when $\mathcal{G}_i$ is a connected undirected graph.
- If $\mathcal{A} \in \mathbb{R}^{n\times n}, \bar{\mathcal{A}} = \mathbb{E}[\mathcal{A}] \in \mathbb{R}^{n\times n}; \bar{a}_{ij} = \mathbb{E}[\mathcal{A}_{ij}]$.
- $\mathbb{E}[A+B] = \mathbb{E}(A) + \mathbb{E}(B); A, B \in \mathbb{R}^{n\times n}$.
- $\mathbb{E}[aA] = a\mathbb{E}[A], a \in \mathbb{R}$.

Problem Formulation

In this work, we assume that the hybrid multi-agent system consists of n agents which are continuous-time and discrete-time dynamic agents, labelled 1 through n, where the number of continuous-time dynamic agents is c, $c < n$. Without loss of generality, we assume that agent 1 through c are continuous-time dynamic agents. Moreover, $\mathcal{I}_c = \{1, 2, 3, ..., c\}$, $\mathcal{I}_n \backslash \mathcal{I}_c = \{c+1, c+2, c+3, ..., n\}$. Then, the dynamics of agent i with non-zero scalar scale β_i are as follows:

$$\begin{cases} \beta_i \dot{x}_i(t) = u_i, & \text{for } \ i \in \mathcal{I}_c, \\ \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + u_i(t_k), \quad t_k = kh, k \in \mathbb{N} & \text{for } \ i \in \mathcal{I}_n \backslash \mathcal{I}_c, \end{cases} \tag{6.1}$$

where h is the sampling period, $x_i \in \mathbb{R}$ and $u_i \in \mathbb{R}$ are the state and control input of agent i, respectively. The initial conditions are $x_i(0) = x_{i0}$, and $x(0) = [x_{10}, x_{20}, ..., x_{n0}]^T$.

Definition 6.1.1. The hybrid multi-agent system (6.1) is said to reach scaled consensus to $(\beta_1, \ldots, \beta_n)$ in **mean square** if for any initial conditions, we have

$$\lim_{t_k \to \infty} \mathbb{E}(\|\beta_i x_i(t_k) - \beta_j x_j(t_k)\|^2) = 0, \quad \text{for } \ i, j \in \mathcal{I}_n, \tag{6.2}$$

and

$$\lim_{t \to \infty} \mathbb{E}(\|\beta_i x_i(t) - \beta_j x_j(t)\|^2) = 0, \quad \text{for } \ i, j \in \mathcal{I}_c. \tag{6.3}$$

Definition 6.1.2. The hybrid multi-agent system (6.1) converges scaled consensus to $(\beta_1, \ldots, \beta_n)$ in **almost surely** if for any initial conditions, it holds that

$$\mathbb{P}\{\lim_{t_k\to\infty} \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| = 0\} = 1, \quad \text{for} \;\; i, j \in \mathcal{I}_n, \tag{6.4}$$

and

$$\mathbb{P}\{\lim_{t\to\infty} \|\beta_i x_i(t) - \beta_j x_j(t)\| = 0\} = 1, \quad \text{for} \;\; i, j \in \mathcal{I}_c. \tag{6.5}$$

Remark. It is obvious that reaching scaled consensus in mean square (almost surely) can guarantee reaching standard consensus in mean square (almost surely) when scalar scales $\beta_i = 1$ for all i.

6.1.1 Consensus results

In this section, the scaled consensus problems of hybrid multi-agent system (6.1) have been studied under two kinds of control inputs (consensus protocols), respectively.

Case I

In this section, we assume that all agents communicate with their neighbours and update their control inputs in a sampling time t_k. Then, the consensus protocol for hybrid multi-agent system (6.1) is defined as follows:

$$\begin{cases} u_i(t) = |\beta_i| \sum\limits_{j\in\mathcal{N}_i^c} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for} \;\; t \in (t_k, t_{k+1}], \;\; i \in \mathcal{I}_c \\ u_i(t_k) = h \cdot |\beta_i| \sum\limits_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for} \;\; i \in \mathcal{I}_n \backslash \mathcal{I}_c, \end{cases} \tag{6.6}$$

where $\mathcal{A} = [a_{ij}]$ is the weighted adjacency matrices associated with the graph $\mathcal{G}$, $h = h_i = t_{k+1} - t_k$ for all i is the sampling period.

Theorem 6.1.1. Let $\beta_i \neq 0$ be any scalar scale of agent i and assume that $0 < h < (d_{max}\beta_{max})^{-1}$. Then, the hybrid multi-agent system (6.1) with the protocol (6.6) reaches scaled consensus to $(\beta_1, \ldots, \beta_n)$ in **mean square** if and only if $\bar{\mathcal{G}}$ is connected.

Proof. **(Sufficiency)** Let $\beta_i \neq 0$ be any scalar scale of agent i, we first show that equation (6.2) holds. From (6.1) and (6.6), one has, for $t \in (t_k, t_{k+1}]$,

$$\begin{cases} \beta_i x_i(t) = \beta_i x_i(t_k) + (t - t_k)|\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for} \quad i \in \mathcal{I}_c \\ \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for} \quad i \in \mathcal{I}_n \backslash \mathcal{I}_c. \end{cases} \tag{6.7}$$

Therefore, it follows that

$$\beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], \quad \text{for} \quad i \in \mathcal{I}_n \tag{6.8}$$

Let $x(t_k) = (x_1(t_k), x_2(t_k), ..., x_n(t_k))^T \in \mathbb{R}^n$, $\mathcal{B} = diag(\beta_1, \beta_2, ..., \beta_n) \in \mathbb{R}^{n \times n}$, $|\mathcal{B}| = diag(|\beta_1|, |\beta_2|, ..., |\beta_n|) \in \mathbb{R}^{n \times n}$ and $H = diag(h_1, h_2, \ldots, h_n)$. Then, equation (6.8) can be written as

$$\mathcal{B}x(t_{k+1}) = [\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]\mathcal{B}x(t_k), \tag{6.9}$$

which is equivalent to

$$\mathcal{B}x(t_k) = [\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^k \mathcal{B}x(0). \tag{6.10}$$

Since $\bar{\mathcal{G}}$ is connected and $h < \dfrac{1}{2d_{max}\beta_{max}}$, by Lemma 4.1.1, we know that $[\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]$ is SIA. Then, by Lemma 2.6.3, one obtains

$$W_{t_k} = [\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^k$$

is also SIA. Moreover, it follows from [57] that the eigenvalues of $W_{t_k} - \dfrac{\mathbf{1}_n\mathbf{1}_n^T}{n}$ are as the form

$$\lambda_i = \big(1 - H\lambda_i(|\mathcal{B}|\mathcal{L}_{t_k})\big)^{d_k}, \quad i = 2, \ldots, n.$$

Since $\mathcal{L}_{t_k}$ is a symmetric positive semi-definite matrix, so all its eigenvalues are non-negative and

$$0 = \lambda_1(|\mathcal{B}|\mathcal{L}_{t_k}) \leq \lambda_2(|\mathcal{B}|\mathcal{L}_{t_k}) \leq \cdots \leq \lambda_n(|\mathcal{B}|\mathcal{L}_{t_k}). \tag{6.11}$$

It follows that $\lambda_1(W_{t_k}) = 1$ with the corresponding eigenvector $u_1 = \dfrac{1}{\sqrt{n}}\mathbf{1}_n$. Also,

$$\rho\big(W_{t_k} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big) = \max\big(|\lambda_2(W_{t_k})|, \ldots, |\lambda_n(W_{t_k})|\big). \tag{6.12}$$

By Lemma 2.4 [57], we have

$$\|\mathcal{B}x(t_{k+1}) - \bar{x}\|^2 \le \rho^2\big(W_{t_k} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\|\mathcal{B}x(t_k) - \bar{x}\|^2 \tag{6.13}$$

which implies that

$$\|\mathcal{B}x(t_k) - \bar{x}\|^2 \le \rho^2\big(W_{t_0} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\cdots\rho^2\big(W_{t_{k-1}} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\|\mathcal{B}x(0) - \bar{x}\|^2. \tag{6.14}$$

Taking the expectation on both sides of (6.14) and using the independent property of random matrix W_{t_k}, we have

$$\begin{aligned}\mathbb{E}\big(\|\mathcal{B}x(t_k) - \bar{x}\|^2\big) &\le \mathbb{E}\big(\rho^2\big(W_{t_0} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\big)\cdots\mathbb{E}\big(\rho^2\big(W_{t_{k-1}} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\big)\|\mathcal{B}x(0) - \bar{x}\|^2.\\ &\le \max_{i=0,1,\ldots k-1}\mathbb{E}\big(\rho^2\big(W_{t_i} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\big)^k\|\mathcal{B}x(0) - \bar{x}\|^2. \end{aligned}\tag{6.15}$$

In order to prove that the state vector sequence of system (6.15) converges in mean square, we only need to prove that $\mathbb{E}\big(\rho^2\big(W_{t_i} - \dfrac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\big) < 1$ for $i = 0, 1, \ldots$. By Lemma 2.2 [57], it can be seen that the eigenvalues of $W_{t_i} - \dfrac{\mathbf{1}_n\mathbf{1}_n^T}{n}$ are λ_i, where

$$\lambda_i = (1 - h\lambda_{t_i}(|\mathcal{B}|\mathcal{L}_{t_i}))^{d_i}, \quad i = 2, 3, \ldots, n.$$

Based on Gersgorin Disc Theorem [19], we have the sampling period

$$0 < h < \frac{1}{d_{max}\beta_{max}} \le \frac{1}{\bar{\lambda}_n}$$

with non-zero probability where $\bar{\lambda}_n = \max\limits_{i=0,1,\ldots k-1}\{\lambda_n(\mathcal{L}_{t_i})\}$. This implies that

$$(1 - h\lambda_{t_i}(|\mathcal{B}|\mathcal{L}_{t_i}))^{d_i} \ge 0 \;\text{ and }\; \rho^2\big(W_{t_i} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big) = (1 - h\lambda_2(|\mathcal{B}|\mathcal{L}_{t_i}))^{2d_i} \le 1.$$

Since the expected graph $\bar{\mathcal{G}}$ is connected, there is at least one graph $\mathcal{G}$ with non-zero probability for $\lambda_2(\mathcal{L}) > 0$. Thus,

$$(1 - h\lambda_i(|\mathcal{B}|\mathcal{L}))^{2d} < 1 \implies \mathbb{E}\big(\rho^2\big(W_{t_i} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\big) < 1.$$

Hence, $\lim_{t_k\to\infty} \mathbb{E}\big[\|\mathcal{B}x(t_k) - \bar{x}\|^2\big] = 0$. i.e.,

$$\lim_{t_k\to\infty} \mathbb{E}\big[\|\beta_i x_i(t_k) - \beta_j x_j(t_k)\|^2\big] = 0, \quad \text{for} \quad i, j \in \mathcal{I}_n. \tag{6.16}$$

Now, we will show that

$$\lim_{t\to\infty} \mathbb{E}(\|\beta_i x_i(t) - \beta_j x_j(t)\|^2) = 0 \qquad \text{for} \quad i, j \in \mathcal{I}_c.$$

Consider, for $i, j \in \mathcal{I}_c$ and any $\beta_i \neq 0$,

$$\begin{aligned} \|\beta_i x_i(t) - \beta_j x_j(t)\| \leq {} & \|\beta_i x_i(t) - \beta_i x_i(t_k)\| + \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| \\ & + \|\beta_j x_j(t_k) - \beta_j x_j(t)\|. \end{aligned} \tag{6.17}$$

It is obvious that

$$\begin{aligned} \|\beta_i x_i(t) - \beta_j x_j(t)\|^2 \leq {} & \|\beta_i x_i(t) - \beta_i x_i(t_k)\|^2 + \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\|^2 \\ & + \|\beta_j x_j(t_k) - \beta_j x_j(t)\|^2, \end{aligned} \tag{6.18}$$

and hence

$$\begin{aligned} \mathbb{E}\big[\|\beta_i x_i(t) - \beta_j x_j(t)\|^2\big] \leq {} & \mathbb{E}\big[\|\beta_i x_i(t) - \beta_i x_i(t_k)\|^2\big] + \mathbb{E}\big[\|\beta_i x_i(t_k) - \beta_j x_j(t_k)\|^2\big] \\ & + \mathbb{E}\big[\|\beta_j x_j(t_k) - \beta_j x_j(t)\|^2\big]. \end{aligned} \tag{6.19}$$

From equation (6.7), one obtains, for $t \in (t_k, t_{k+1}]$,

$$\|\beta_i x_i(t) - \beta_i x_i(t_k)\|^2 \leq h|\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij} \|\beta_j x_j(t_k) - \beta_i x_i(t_k)\|^2. \tag{6.20}$$

Hence,

$$\mathbb{E}\big[\|\beta_i x_i(t) - \beta_i x_i(t_k)\|^2\big] \leq h|\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij} \mathbb{E}\big[\|\beta_j x_j(t_k) - \beta_i x_i(t_k)\|^2\big]. \tag{6.21}$$

As $t \to \infty$, we have $t_k \to \infty$. Thus,

$$\lim_{t\to\infty} \mathbb{E}\big[\|\beta_i x_i(t) - \beta_i x_i(t_k)\|^2\big] = 0 \qquad \text{for} \;\; i, j \in \mathcal{I}_c.$$

Taking the limit as $t \to \infty$ on both sides of equation (6.19), one obtains

$$\lim_{t\to\infty} \mathbb{E}\big[\|\beta_i x_i(t) - \beta_i x_i(t)\|^2\big] = 0 \qquad \text{for} \;\; i, j \in \mathcal{I}_c.$$

Therefore, the hybrid multi-agent system (6.1) with protocol (6.6) reaches scaled consensus.

(Necessity) If the expected graph is $\bar{\mathcal{G}}$ which is not connected, there exist at least two components with zero probability of communication between each other. This implies that there is no path between two components. Hence, the information of these two components cannot reach consensus for any initial condition. □

Theorem 6.1.2. Assume that the sampling period $0 < h < \dfrac{1}{d_{max}\beta_{max}}$. Then, the HMAS (6.1) reaches scaled consensus to $(\beta_1, \ldots, \beta_n)$ almost surely if and only if the expected graph $\bar{\mathcal{G}}$ is connected.

Proof. **(Sufficiency)** Assume that $\bar{\mathcal{G}}$ is connected. Let $x(t) = [x_1(t), \ldots, x_n(t)]^T$, as a result of Markovapo's inequality, for any $a > 0$, we have

$$\mathbb{P}\{\|\beta x(t_k) - \bar{x}\|^2 \geq a^2\} \leq \frac{\mathbb{E}\big[\|\beta x(t_k) - \bar{x}\|^2\big]}{a^2}. \tag{6.22}$$

Because $\|\beta x(t_k) - \bar{x}\|^2 \geq a^2$ is equivalent to $\|\beta x(t_k) - \bar{x}\| \geq a$, inequality (6.22) can be written as

$$\mathbb{P}\{\|\beta x(t_k) - \bar{x}\| \geq a\} \leq \frac{\mathbb{E}\big[\|\beta x(t_k) - \bar{x}\|^2\big]}{a^2}. \tag{6.23}$$

It follows from Theorem 6.1.1 that

$$\|\beta x(t_{k+1}) - \bar{x}\|^2 \leq \rho^2\big(W_{t_k} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\|\beta x(t_k) - \bar{x}\|^2.$$

Therefore,

$$\sum_{k=0}^{\infty} \mathbb{P}\{\|\beta x(t_k) - \bar{x}\| \geq a\} \leq \frac{\mathbb{E}\big[\|\beta x(t_k) - \bar{x}\|^2\big]}{a^2(1-\alpha)}, \tag{6.24}$$

where $\alpha = \max_{i=0,1,\ldots k-1}\{\mathbb{E}\big[\rho^2(W_{t_k} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n})\big]\}$. Using the Borel-Cantelli Lemma leads to $\mathbb{P}\{\|\beta x(t_k) - \bar{x}\| \geq a \quad i.o.\} = 0$. Thus, we conclude that agents reach scaled consensus almost surely. i.e.,

$$\mathbb{P}\{\lim_{t_k\to\infty} \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| = 0\} = 1, \quad \text{for} \;\; i,j \in \mathcal{I}_n$$

or $\beta_j x_j(t_k) \to \beta_i x_i(t_k)$ almost surely for all $i, j \in \mathcal{I}_n$.

Consider, for $i, j \in \mathcal{I}_c$ and any $\beta_i \neq 0$,

$$\begin{aligned} \|\beta_i x_i(t) - \beta_j x_j(t)\| \leq \|\beta_i x_i(t) - \beta_i x_i(t_k)\| + \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| \\ + \|\beta_j x_j(t_k) - \beta_j x_j(t)\|. \end{aligned} \tag{6.25}$$

From equation (6.7), one obtains, for $t \in (t_k, t_{k+1}]$,

$$\|\beta_i x_i(t) - \beta_i x_i(t_k)\| \leq h|\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij}\|\beta_j x_j(t_k) - \beta_i x_i(t_k)\|. \tag{6.26}$$

As $t \to \infty$, $t_k \to \infty$ and from the previous discussion, we know that $\beta_j x_j(t_k) \to \beta_i x_i(t_k)$ almost surely for all $i, j \in \mathcal{I}_n$. It can be seen from (6.26) that $\beta_j x_j(t) \to \beta_i x_i(t_k)$ almost surely for all $i, j \in \mathcal{I}_c$ as $t \to \infty$. And hence, from (6.25) , one obtains $\beta_j x_j(t) \to \beta_i x_i(t)$ almost surely for all $i, j \in \mathcal{I}_c$ as $t \to \infty$.

(Necessity) The proof is similar to the argument used in Theorem 6.1.1. □

Corollary 6.1.2.1. Assume that the sampling period $0 < h < \dfrac{1}{d_{max}\beta_{max}}$ and the expected graph $\bar{\mathcal{G}}$ is connected. Then, the following statements are equivalent

(a) the HMAS(6.1) reaches scaled consensus in mean square;

(b) the HMAS(6.1) reaches scaled consensus almost surely.

Proof. Based on Theorem 6.1.1 and Theorem 6.1.2, it is obvious that $(a) \iff (b)$. □

Case II

In this section, we assume that all agents communicate with their neighbours and update their control inputs in a sampling time t_k, where each continuous-time dynamic agent can observe its own state in real time. Then, the scaled consensus protocol for hybrid multi-agent system (6.1) is defined by:

$$\begin{cases} u_i(t) = |\beta_i| \sum_{j\in\mathcal{N}} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t)], & \text{for } \ t \in (t_k, t_{k+1}], \quad i \in \mathcal{I}_c \\ u_i(t_k) = h \cdot |\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], & \text{for } \ i \in \mathcal{I}_n \backslash \mathcal{I}_c, \end{cases} \tag{6.27}$$

where $\mathcal{A} = [a_{ij}]$ is the weighted adjacency matrices associated with the graph $\mathcal{G}$, $h = h_i = t_{k+1} - t_k$ for all i is the sampling period and $\beta_i \neq 0$ is a scalar scale of agent i.

Theorem 6.1.3. Let $\beta_i \neq 0$ be any scalar scale of agent i and

$$H = diag\Big\{\frac{1 - e^{-\sum_{j=1}^n a_{1j}|\beta_1|h}}{\sum_{j=1}^n a_{1j}|\beta_1|}, \ldots, \frac{1 - e^{-\sum_{j=1}^n a_{cj}|\beta_c|h}}{\sum_{j=1}^n a_{cj}|\beta_c|}, h, \ldots, h\Big\}.$$

Assume that $0 < h < \dfrac{1}{d_{max}\beta_{max}}$. Then, the hybrid multi-agent system (6.1) with the protocol (6.27) achieves scaled consensus to $(\beta_1, \ldots, \beta_n)$ in mean square if and only if the expected graph $\bar{\mathcal{G}}$ is connected.

Proof. **(Sufficiency)** We first show that equation (6.2) holds. From (6.27) we know that for $t \in (t_k, t_{k+1}]$,

$$\begin{cases} \beta_i x_i(t) = \beta_i x_i(t_k) \\ \qquad\qquad + |\beta_i| \Big(\dfrac{1 - e^{-\sum_{j=1}^n a_{ij}|\beta_i|(t-t_k)}}{\sum_{j=1}^n a_{ij}|\beta_i|}\Big) \sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], \quad \text{for } \ i \in \mathcal{I}_c \\ \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], \quad \text{for } \ i \in \mathcal{I}_n \backslash \mathcal{I}_c. \end{cases} \tag{6.28}$$

Accordingly, at time t_{k+1}, the states of agents are

$$\begin{cases} \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) \\ \qquad\qquad + |\beta_i| \Big(\dfrac{1 - e^{-\sum_{j=1}^n a_{ij}|\beta_i|h}}{\sum_{j=1}^n a_{ij}|\beta_i|}\Big) \sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], \quad \text{for } \ i \in \mathcal{I}_c \\ \beta_i x_i(t_{k+1}) = \beta_i x_i(t_k) + h|\beta_i| \sum_{j\in\mathcal{N}_i} a_{ij}[\beta_j x_j(t_k) - \beta_i x_i(t_k)], \quad \text{for } \ i \in \mathcal{I}_n \backslash \mathcal{I}_c. \end{cases} \tag{6.29}$$

Letting $x(t_k) = (x_1(t_k), x_2(t_k), ..., x_n(t_k))^T \in \mathbb{R}^n$, $\mathcal{B} = diag(\beta_1, \beta_2, ..., \beta_n) \in \mathbb{R}^{n\times n}$, $|\mathcal{B}| = diag(|\beta_1|, |\beta_2|, ..., |\beta_n|) \in \mathbb{R}^{n\times n}$, equation (6.29) can be written as

$$\mathcal{B}x(t_{k+1}) = [\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]\mathcal{B}x(t_k) \tag{6.30}$$

which is equivalent to

$$\mathcal{B}x(t_k) = [\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}]^k\mathcal{B}x(0), \tag{6.31}$$

where $H = diag\bigg\{\dfrac{1-e^{-\sum_{j=1}^n a_{1j}|\beta_1|h}}{\sum_{j=1}^n a_{1j}|\beta_1|}, \dots, \dfrac{1-e^{-\sum_{j=1}^n a_{cj}|\beta_c|h}}{\sum_{j=1}^n a_{cj}|\beta_c|}, h, \dots, h\bigg\}$.
It follows from

$$\frac{1-e^{-\sum_{j=1}^n a_{ij}|\beta_i|h}}{\sum_{j=1}^n a_{ij}|\beta_i|} < \frac{1}{d_{ii}|\beta_i|} \quad \text{for} \quad i \in \mathcal{I}_c$$

and $h < \dfrac{1}{d_{max}\beta_{max}}$, that $0 < h_i < \dfrac{1}{d_{max}\beta_{max}}$ for H.
Because the expected graph $\bar{\mathcal{G}}$ is connected, by Lemma 4.1.1 and Lemma 2.6.3, one obtains

$$U_{t_k} = \big(\mathbf{I}_n - H|\mathcal{B}|\mathcal{L}\big)^k$$

is SIA. In addition, we know from Lemma 2.2[57] that the eigenvalues of $U_{t_k} - \dfrac{\mathbf{1}\mathbf{1}^T}{n}$ are

$$\lambda_i = \big(1 - H\lambda_i(|\mathcal{B}|\mathcal{L})\big)^k, \quad i = 2, \dots, n.$$

Since $\mathcal{L}$ is a symmetric positive semi-definite matrix, all its eigenvalues are non-negative and

$$0 = \lambda_1(|\mathcal{B}|\mathcal{L}) \le \lambda_2(|\mathcal{B}|\mathcal{L}) \le \dots \le \lambda_n(|\mathcal{B}|\mathcal{L}). \tag{6.32}$$

It follows that $\lambda_1(U_{t_k}) = 1$ with the corresponding eigenvector $u_1 = \dfrac{1}{\sqrt{n}}\mathbf{1}$. Also,

$$\rho\big(U_{t_k} - \frac{\mathbf{1}\mathbf{1}^T}{n}\big) = \max\big(|\lambda_2(U_{t_k})|, \dots, |\lambda_n(U_{t_k})|\big). \tag{6.33}$$

By Lemma 2.4 [57], we know that

$$\|\beta x(t_{k+1}) - \bar{x}\|^2 \le \rho^2\big(U_{t_k} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\|\beta x(t_k) - \bar{x}\|^2 \tag{6.34}$$

which implies that

$$\|\beta x(t_k)-\bar{x}\|^2 \le \rho^2\big(U_{t_0}-\frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\cdots\rho^2\big(U_{t_{k-1}}-\frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\|\beta x(0)-\bar{x}\|^2. \tag{6.35}$$

Taking the expectation on both sides of (6.35) and using the independent property of random matrix U_{t_k}, we have

$$\begin{aligned}\mathbb{E}\big(\|\beta x(t_k)-\bar{x}\|^2\big) &\le \mathbb{E}\big(\rho^2\big(U_{t_0}-\frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\big)\cdots\mathbb{E}\big(\rho^2\big(U_{t_{k-1}}-\frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\big)\|\beta x(0)-\bar{x}\|^2. \\ &\le \max_{i=0,1,\ldots k-1}\mathbb{E}\big(\rho^2\big(U_{t_i}-\frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\big)^k\|\beta x(0)-\bar{x}\|^2 \end{aligned} \tag{6.36}$$

From [57], we know that the eigenvalues

$$\lambda_i=(1-h\lambda_{t_i}(|\beta|\mathcal{L}_{t_i}))^{d_i}, \quad i=2,3,\ldots,n.$$

Based on Gersgorin Disc Theorem [19], we have the sampling period

$$0<h<\frac{1}{d_{max}\beta_{max}}\le\frac{1}{\bar{\lambda}_n}$$

with non-zero probability where $\bar{\lambda}_n=\max\limits_{i=0,1,\ldots k-1}\{\lambda_n(\mathcal{L}_{t_i})\}$. This implies that

$$(1-h\lambda_{t_i}(|\beta|\mathcal{L}_{t_i}))^{d_i}\ge 0 \ \text{ and } \ \rho^2\big(U_{t_i}-\frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)=(1-h\lambda_2(|\beta|\mathcal{L}_{t_i}))^{2d_i}\le 1.$$

Indeed, the expected graph $\bar{\mathcal{G}}$ which is connected shows that there is at least one graph $\mathcal{G}$ with non-zero probability for $\lambda_2(\mathcal{L})>0$. It is obvious that

$$(1-h\lambda_i(|\beta|\mathcal{L}))^{2d}<1 \implies \mathbb{E}\big(\rho^2\big(U_{t_i}-\frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\big)<1.$$

Thus, $\lim\limits_{t_k\to\infty}\mathbb{E}\big[\|\beta x(t_k)-\bar{x}\|^2\big]=0$. This implies that

$$\lim_{t_k\to\infty}\mathbb{E}\big[\|\beta_i x_i(t_k)-\beta_j x_j(t_k)\|^2\big]=0, \quad \text{for } \ i,j\in\mathcal{I}_n. \tag{6.37}$$

Now, we will show that

$$\lim_{t\to\infty}\mathbb{E}(\|\beta_i x_i(t)-\beta_j x_j(t)\|^2)=0 \qquad \text{for } \ i,j\in\mathcal{I}_c.$$

Consider, for $i, j \in \mathcal{I}_c$ and any $\beta_i \neq 0$,

$$\begin{aligned}\|\beta_i x_i(t) - \beta_j x_j(t)\| \leq \|\beta_i x_i(t) - \beta_i x_i(t_k)\| + \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| \\ + \|\beta_j x_j(t_k) - \beta_j x_j(t)\|.\end{aligned} \tag{6.38}$$

It is obvious that

$$\begin{aligned}\|\beta_i x_i(t) - \beta_j x_j(t)\|^2 \leq \|\beta_i x_i(t) - \beta_i x_i(t_k)\|^2 + \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\|^2 \\ + \|\beta_j x_j(t_k) - \beta_j x_j(t)\|^2,\end{aligned} \tag{6.39}$$

and hence

$$\begin{aligned}\mathbb{E}\big[\|\beta_i x_i(t) - \beta_j x_j(t)\|^2\big] \leq \mathbb{E}\big[\|\beta_i x_i(t) - \beta_i x_i(t_k)\|^2\big] + \mathbb{E}\big[\|\beta_i x_i(t_k) - \beta_j x_j(t_k)\|^2\big] \\ + \mathbb{E}\big[\|\beta_j x_j(t_k) - \beta_j x_j(t)\|^2\big].\end{aligned} \tag{6.40}$$

From equation (6.28), one obtains, for $t \in (t_k, t_{k+1}]$,

$$\|\beta_i x_i(t) - \beta_i x_i(t_k)\|^2 \leq h|\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij} \|\beta_j x_j(t_k) - \beta_i x_i(t_k)\|^2. \tag{6.41}$$

Hence,

$$\mathbb{E}\big[\|\beta_i x_i(t) - \beta_i x_i(t_k)\|^2\big] \leq h|\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij} \mathbb{E}\big[\|\beta_j x_j(t_k) - \beta_i x_i(t_k)\|^2\big]. \tag{6.42}$$

As $t \to \infty$, we have $t_k \to \infty$. Thus,

$$\lim_{t \to \infty} \mathbb{E}\big[\|\beta_i x_i(t) - \beta_i x_i(t_k)\|^2\big] = 0 \qquad \text{for} \;\; i, j \in \mathcal{I}_c.$$

Taking the limit as $t \to \infty$ on both sides of equation (6.40), one obtains

$$\lim_{t \to \infty} \mathbb{E}\big[\|\beta_i x_i(t) - \beta_j x_j(t)\|^2\big] = 0 \qquad \text{for} \;\; i, j \in \mathcal{I}_c.$$

Therefore, the hybrid multi-agent system (6.1) with protocol (6.27) reaches scaled consensus.

(Necessity) The proof is similar to the necessity part of the previous theorem, so it is omitted. □

Theorem 6.1.4. Assume that the sampling period $0 < h < \dfrac{1}{d_{max}\beta_{max}}$. Then, the HMAS (6.1) with protocol (6.27) reaches scaled consensus to $(\beta_1, \ldots, \beta_n)$ almost surely if and only if the expected graph $\bar{\mathcal{G}}$ is connected.

Proof. **(Sufficiency)** Assume that $\bar{\mathcal{G}}$ is connected. Let $x(t) = [x_1(t), \ldots, x_n(t)]^T$, as a result of Markovapo's inequality, for any $a > 0$, we have

$$\mathbb{P}\{\|\beta x(t_k) - \bar{x}\|^2 \geq a^2\} \leq \frac{\mathbb{E}\big[\|\beta x(t_k) - \bar{x}\|^2\big]}{a^2}. \tag{6.43}$$

Because $\|\beta x(t_k) - \bar{x}\|^2 \geq a^2$ is equivalent to $\|\beta x(t_k) - \bar{x}\| \geq a$, inequality (6.43) can be written as

$$\mathbb{P}\{\|\beta x(t_k) - \bar{x}\| \geq a\} \leq \frac{\mathbb{E}\big[\|\beta x(t_k) - \bar{x}\|^2\big]}{a^2}. \tag{6.44}$$

By Theorem 6.1.3, it is easy to know that

$$\|\beta x(t_{k+1}) - \bar{x}\|^2 \leq \rho^2\big(W_{t_k} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\big)\|\beta x(t_k) - \bar{x}\|^2.$$

Therefore,

$$\sum_{k=0}^{\infty}\mathbb{P}\{\|\beta x(t_k) - \bar{x}\| \geq a\} \leq \frac{\mathbb{E}\big[\|\beta x(t_k) - \bar{x}\|^2\big]}{a^2(1-\alpha)}, \tag{6.45}$$

where $\alpha = \max_{i=0,1,\ldots k-1}\{\mathbb{E}\big[\rho^2(W_{t_k} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n})\big]\}$. Using the Borel-Cantelli Lemma leads to

$$\mathbb{P}\{\|\beta x(t_k) - \bar{x}\| \geq a \;\; i.o.\} = 0.$$

Thus, we conclude that agents reach scaled consensus almost surely. i.e.,

$$\mathbb{P}\{\lim_{t_k\to\infty}\|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| = 0\} = 1, \quad \text{for} \;\; i, j \in \mathcal{I}_n$$

or $\beta_j x_j(t_k) \to \beta_i x_i(t_k)$ almost surely for all $i, j \in \mathcal{I}_n$. Consider, for $i, j \in \mathcal{I}_c$ and any $\beta_i \neq 0$,

$$\begin{aligned}\|\beta_i x_i(t) - \beta_j x_j(t)\| \leq{}& \|\beta_i x_i(t) - \beta_i x_i(t_k)\| + \|\beta_i x_i(t_k) - \beta_j x_j(t_k)\| \\ &+ \|\beta_j x_j(t_k) - \beta_j x_j(t)\|.\end{aligned} \tag{6.46}$$

From equation (6.28), one obtains, for $t \in (t_k, t_{k+1}]$,

$$\|\beta_i x_i(t) - \beta_i x_i(t_k)\| \leq h|\beta_i| \sum_{j \in \mathcal{N}_i} a_{ij} \|\beta_j x_j(t_k) - \beta_i x_i(t_k)\|. \tag{6.47}$$

As $t \to \infty$, $t_k \to \infty$ and from the previous discussion, we know that $\beta_j x_j(t_k) \to \beta_i x_i(t_k)$ almost surely for all $i, j \in \mathcal{I}_n$. It can be seen from (6.47) that $\beta_j x_j(t) \to \beta_i x_i(t_k)$ almost surely for all $i, j \in \mathcal{I}_c$ as $t \to \infty$. And hence, from (6.46), one obtains $\beta_j x_j(t) \to \beta_i x_i(t)$ almost surely for all $i, j \in \mathcal{I}_c$ as $t \to \infty$.

(Necessity) The proof is similar to the argument used in Theorem 6.1.3. □

Corollary 6.1.4.1. Assume that the sampling period $0 < h < \dfrac{1}{d_{max}\beta_{max}}$ and the expected graph $\bar{\mathcal{G}}$ is connected. Then, the following statements are equivalent

(a) the HMAS(6.1) with protocol (6.27) reaches scaled consensus in mean square;

(b) the HMAS(6.1) with protocol (6.27) reaches scaled consensus almost surely.

Proof. Based on Theorem 6.1.3 and Theorem 6.1.4, it is obvious that (a) $\iff$ (b). □

Chapter 7

Conclusions and Future Research

In this chapter, we summarize the results of this thesis and suggest possible future work related to the topics that we have studied in the thesis. In the present thesis, some consensus problems of hybrid multi-agent systems have been studied as follows.

In Chapter 3, consensus problems of directed hybrid multi-agent systems under fixed topology with (and without) communication delays have been studied. We assume that all continuous-time dynamic agent communicate with their neighbors and update their own states in real time, while the discrete-time dynamic agents communicate with their neighbors and update their own states at time t_k. Our results show that the hybrid multi-agent system system (3.1) reaches consensus if $\mathcal{G}_c \cup \mathcal{G}'$ and $\mathcal{G}_d \cup \mathcal{G}'$ are balanced and contain a spanning tree under the sampling period $0 < h < (d_{max})^{-1}$ and some conditions.

In Chapter 4, scaled consensus problems for the hybrid multi-agent system have been investigated by using impulsive consensus protocols if the directed communication networks $\mathcal{G}$ contains a spanning tree with $0 < h < (d_{max}\beta_{max})^{-1}$. Moreover, the scaled consensus problems of multi-agent systems under fixed and switching topologies have been studied by introducing impulsive protocols together with some appropriate conditions. In addition, scaled consensus problems have been studied in multi-agent systems with external disturbances, some conditions are introduced to guarantee reaching scaled consensus and satisfy robust H_∞ performance. Furthermore, our results show that the scaled consensus can be guaranteed if the sampling period h is bounded by some values depending on the scalar scales and degree of networks.

In Chapter 5, finite-time scaled consensus problems of multi-agent system have been studied. By using Lyapunov finite-time consensus theorem, algebraic graph theory and matrix theory, some conditions are provided to guarantee achieving scaled consensus in finite time. Firstly, a scaled consensus protocol have been proposed for solving finite-time consensus problems of hybrid multi-agent systems via impulsive control. Secondly, the finite-time scaled consensus of multi-agent systems with (impulsive) perturbations have been investigated and the impulsive consensus protocols are introduced which can guarantee reaching scaled consensus if the perturbations are small enough.

In Chapter 6, the scaled consensus problems of hybrid multi-agent systems have been studied in random networks. Two scaled consensus protocols are introduced depending on the communication among agents. Firstly, we assume that all agents communicate with their neighbours and update their control inputs in a sampling time t_k. Secondly, we study scaled consensus problem when all agents communicate with their neighbours and update their control inputs in a sampling time t_k, but continuous-time dynamic agent can observe its own state in real time.

In the future, it might be possible to study the scaled consensus problems of hybrid multi-agent systems under nonlinear protocols, partial scaled consensus problems, complex consensus problems and examine how to apply the idea of scaled consensus into edge dynamics problems.

References

[1] Hassan Dehghani Aghbolagh, Elham Ebrahimkhani, and A. Ghiasi. Scaled consensus of multi-agent systems under time-varying delay. *2015 2nd International Conference on Knowledge-Based Engineering and Innovation (KBEI)*, pages 729–732, 2015.

[2] Hassan Dehghani Aghbolagh, Elham Ebrahimkhani, and F. Hashemzadeh. Scaled consensus tracking under constant time delay. *IFAC-PapersOnLine*, 49:240–243, 2016.

[3] C. Altafini. Consensus problems on networks with antagonistic interactions. *IEEE Transactions on Automatic Control*, 58(4):935–946, 2013.

[4] Panos J. Antsaklis. A brief introduction to the theory and applications of hybrid systems, 2000.

[5] Owe Axelsson and Vincent A. Barker. *Finite Element Solution of Boundary Value Problems: Theory and Computation*. Society for Industrial and Applied Mathematics, USA, 2001.

[6] Tuncer Can Aysal, Mehmet Ercan Yildiz, Anand D. Sarwate, and Anna Scaglione. Broadcast gossip algorithms for consensus. *IEEE Transactions on Signal Processing*, 57(7):2748–2761, 2009.

[7] Dario Bauso, Laura Giarré, and Raffaele Pesenti. Non-linear protocols for optimal distributed consensus in networks of dynamic agents. *Syst. Control. Lett.*, 55:918–928, 2006.

[8] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah. Randomized gossip algorithms. *IEEE Transactions on Information Theory*, 52(6):2508–2530, June 2006.

[9] R. Brayton and C. Tong. Stability of dynamical systems: A constructive approach. *IEEE Transactions on Circuits and Systems*, 26(4):224–234, 1979.

[10] Yongcan Cao, Daniel Stuart, Wei Ren, and Ziyang Meng. Distributed containment control for multiple autonomous vehicles with double-integrator dynamics: Algorithms and experiments. *IEEE Transactions on Control Systems Technology*, 19(4):929–938, 2011.

[11] L. Chen, Yuan Gao, Libing Bai, and Y. Cheng. Scaled consensus control of heterogeneous multi-agent systems with switching topologies. *Neurocomputing*, 408:13–20, 2020.

[12] S. Chen and Z. Zhu. Practical stability of impulsive functional differential systems via analysis techniques. *Miskolc Mathematical Notes*, 15:373, 2014.

[13] Tao Chen, Shiguo Peng, and Zhenhua Zhang. Finite-time consensus of leader-following non-linear multi-agent systems via event-triggered impulsive control. *IET Control Theory & Applications*, 15(7):926–936, 2021.

[14] L. Cheng, Y. Wang, Z. Hou, and M. Tan. Stochastic consensus of linear multi-agent systems: Communication noises and markovian switching topologies. In *The 26th Chinese Control and Decision Conference (2014 CCDC)*, pages 274–279, 2014.

[15] Godsil Chris and Royle Gordon. *Algebraic Graph Theory*, volume 207. 01 2001.

[16] J. Cortes, S. Martinez, T. Karatas, and F. Bullo. Coverage control for mobile sensing networks. *IEEE Transactions on Robotics and Automation*, 20(2):243–255, 2004.

[17] Jorge Cortés. Finite-time convergent gradient flows with applications to network consensus. *Automatica*, 42(11):1993 – 2000, 2006.

[18] Morris H. Degroot. Reaching a consensus. *Journal of the American Statistical Association*, 69(345):118–121, 1974.

[19] Rick Durrett. *Probability: Theory and Examples*. Thomson, 2005.

[20] Elham Ebrahimkhani, A. Ghiasi, F. Hashemzadeh, and M. Baradarannia. Scaled consensus of descriptor multi-agent systems. *2017 10th International Conference on Electrical and Electronics Engineering (ELECO)*, pages 938–943, 2017.

[21] Elham Ebrahimkhani and F. Hashemzadeh. Scaled consensus of multi-agent systems under time delay. *2018 Australian & New Zealand Control Conference (ANZCC)*, pages 216–221, 2018.

[22] J. A. Fax and R. Murray. Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, 49:1465–1476, 2004.

[23] Xin-Lei Feng and Ting-Zhu Huang. A finite-time consensus protocol of the multi-agent systems. *International Journal of Mathematical and Computational Sciences*, 5(3):357 – 359, 2011.

[24] Zhiwen Fu, Shiguo Peng, T. Chen, and Zizhao Zheng. Consensus of hybrid switched multi-agent systems under different topologies via impulsive control. *IEEE Access*, 8:135458–135467, 2020.

[25] Veysel Gazi and Kevin M. Passino. *Swarm Coordination and Control Problems*, pages 15–25. Springer Berlin Heidelberg, Berlin, Heidelberg, 2011.

[26] Z. Guan, Y. Wu, and G. Feng. Consensus analysis based on impulsive systems in multiagent networks. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 59(1):170–178, Jan 2012.

[27] Guangming Xie and Long Wang. Consensus control for a class of networks of dynamic agents: Fixed topology. In *Proceedings of the 44th IEEE Conference on Decision and Control*, pages 96–101, Dec 2005.

[28] Giorgio Guglieri, Franco Maroglio, Pasquale Pellegrino, and Liliana Torre. Design and development of guidance navigation and control algorithms for spacecraft rendezvous and docking experimentation. *Acta Astronautica*, 94(1):395–408, 2014.

[29] Wassim M. Haddad and VijaySekhar Chellaboina. Stability and dissipativity theory for nonnegative dynamical systems: a unified analysis framework for biological and physiological systems. *Nonlinear Analysis: Real World Applications*, 6(1):35–65, 2005.

[30] Wassim M Haddad, VijaySekhar Chellaboina, and Qing Hui. *Nonnegative and compartmental dynamical systems*. Princeton University Press, 2010.

[31] Jack K Hale. Dynamical systems and stability. *Journal of Mathematical Analysis and Applications*, 26(1):39–59, 1969.

[32] J. Halloy, G. Sempo, G. Caprari, C. Rivault, M. Asadpour, F. Tâche, I. Saïd, V. Durier, S. Canonge, J. M. Amé, C. Detrain, N. Correll, A. Martinoli, F. Mondada, R. Siegwart, and J. L. Deneubourg. Social integration of robots into groups of cockroaches to control self-organized choices. *Science*, 318(5853):1155–1158, 2007.

[33] Yiyan Han, Chuandong Li, Zhigang Zeng, and Hongfei Li. Exponential consensus of discrete-time non-linear multi-agent systems via relative state-dependent impulsive protocols. *Neural Networks*, 108:192–201, 2018.

[34] Kenta Hanada, T. Wada, I. Masubuchi, T. Asai, and Y. Fujisaki. On a new class of structurally balanced graphs for scaled group consensus. *2019 58th Annual Conference of the Society of Instrument and Control Engineers of Japan (SICE)*, pages 1671–1676, 2019.

[35] Kenta Hanada, T. Wada, I. Masubuchi, T. Asai, and Y. Fujisaki. Scaled group consensus over weakly connected structurally balanced graphs. *IFAC-PapersOnLine*, 53:3037–3042, 2020.

[36] Y. Hatano and M. Mesbahi. Agreement over random networks. *IEEE Transactions on Automatic Control*, 50(11):1867–1872, 2005.

[37] Sulan He, Wenwu Yu, Yuezu Lv, Zhenling Wang, and Dong Zhang. Consensus for hybrid multi-agent systems with pulse-modulated protocols. *Nonlinear Analysis: Hybrid Systems*, 36:100867, 2020.

[38] Yiguang Hong, Jiangping Hu, and Linxin Gao. Tracking control for multi-agent consensus with an active leader and variable topology. *Automatica*, 42(7):1177 – 1182, 2006.

[39] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, New York, NY, USA, 2nd edition, 2012.

[40] A.and Jie Lin Jadbabaie and A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.

[41] Fangcui Jiang and Long Wang. Finite-time information consensus for multi-agent systems with fixed and switching topologies. *Physica D: Nonlinear Phenomena*, 238(16):1550 – 1560, 2009.

[42] Haibo Jiang, Jianjiang Yu, and C. Zhou. Consensus of multi-agent linear dynamic systems via impulsive control protocols. *International Journal of Systems Science*, 42:967 – 976, 2011.

[43] Soummya Kar and JosÉ M. F. Moura. Sensor networks with random links: Topology design for distributed consensus. *IEEE Transactions on Signal Processing*, 56(7):3315–3326, 2008.

[44] Bilal J. Karaki and M. Mahmoud. Scaled consensus design for multiagent systems under dos attacks and communication-delays. *J. Frankl. Inst.*, 358:3901–3918, 2021.

[45] D. B. Kingston and R. W. Beard. Discrete-time average-consensus under switching network topologies. In *2006 American Control Conference*, pages 6 pp.–, 2006.

[46] V Lakshmikantham, D D Bainov, and P S Simeonov. *Theory of Impulsive Differential Equations*. WORLD SCIENTIFIC, 1989.

[47] M. Li, L. Su, and G. Chesi. Consensus of heterogeneous multi-agent systems with diffusive couplings via passivity indices. *IEEE Control Systems Letters*, 3(2):434–439, April 2019.

[48] Mengling Li, F. Deng, and Hong wei Ren. Scaled consensus of multi-agent systems with switching topologies and communication noises. *Nonlinear Analysis: Hybrid Systems*, 36:100839, 2020.

[49] Shihua Li, Haibo Du, and Xiangze Lin. Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics. *Automatica*, 47(8):1706 – 1712, 2011.

[50] X. Li, M. Bohner, and C. Wang. Impulsive differential equations: Periodic solutions and applications. *Autom.*, 52:173–178, 2015.

[51] X. Li, Q. Zhu, and D. O'Regan. pth moment exponential stability of impulsive stochastic functional differential equations and application to control problems of nns. *J. Frankl. Inst.*, 351:4435–4456, 2014.

[52] Xiaodi Li and Peng Li. Input-to-state stability of nonlinear systems: Event-triggered impulsive control. *IEEE Transactions on Automatic Control*, 67(3):1460–1465, 2022.

[53] Xiaodi Li, Haitao Zhu, and Shiji Song. Input-to-state stability of nonlinear systems using observer-based event-triggered impulsive control. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 51(11):6892–6900, 2021.

[54] Yanjiang Li, Chong Tan, Liyong Luo, Fengyang Tan, and Mingzhe Hou. Consensus of multi-agent systems with communication delays. *2015 34th Chinese Control Conference (CCC)*, pages 6571–6575, 2015.

[55] Peng Lin and Yingmin Jia. Consensus of second-order discrete-time multi-agent systems with nonuniform time-delays and dynamically changing topologies. *Automatica*, 45(9):2154–2158, 2009.

[56] X. Lin and Y. Zheng. Finite-time consensus of switched multiagent systems. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 47(7):1535–1545, 2017.

[57] Xue Lin, Yuanshi Zheng, and Long Wang. Consensus of switched multi-agent systems with random networks. *International Journal of Control*, 90(5):1113–1122, 2017.

[58] ZHAO Lin et al. Similitude design method for motion reconstruction of space cooperative vehicles. *Journal of astronautics*, 35(7):802, 2014.

[59] Zhiyun Lin, Jian Hou, Gangfeng Yan, and Changbin (Brad) Yu. Reach almost sure consensus with only group information. *Automatica*, 52:283–289, 2015.

[60] Bolian Liu. *Matrices in combinatorics and graph theory / Bolian Liu, Hong-Jian Lai.* Network theory and applications ; v. 3. Kluwer Academic Publishers, 2000.

[61] C. Liu, Hong-Peng Wang, and S. Liu. Scaled consensus problem of heterogeneous first-order multi-agent systems. *2016 IEEE International Conference on Information and Automation (ICIA)*, pages 1284–1289, 2016.

[62] Cheng-Lin Liu. Scaled consensus seeking in multiple non-identical linear autonomous agents. *ISA transactions*, 71 Pt 1:68–75, 2017.

[63] Kaien Liu and Zhijian Ji. Consensus of multi-agent systems with time delay based on periodic sample and event hybrid control. *Neurocomputing*, 270:11–17, 2017. Distributed Control and Optimization with Resource-Constrained Networked Systems.

[64] X. Liu, J. Lam, W. Yu, and G. Chen. Finite-time consensus of multiagent systems with a switching protocol. *IEEE Transactions on Neural Networks and Learning Systems*, 27(4):853–862, 2016.

[65] X. Liu, Kexue Zhang, and W. Xie. Consensus seeking in multi-agent systems via hybrid protocols with impulse delays. *Nonlinear Analysis: Hybrid Systems*, 25:90–98, 2017.

[66] X. Liu, Kexue Zhang, and W. Xie. Consensus of multi-agent systems via hybrid impulsive protocols with time-delay. *Nonlinear Analysis: Hybrid Systems*, 30:134–146, 2018.

[67] Jingying Ma, Maojiao Ye, Yuanshi Zheng, and Yunru Zhu. Consensus analysis of hybrid multiagent systems: A game-theoretic approach. *International Journal of Robust and Nonlinear Control*, 29:1840–1853, 2019.

[68] Tiedong Ma, Zhengle Zhang, and Bing Cui. Variable impulsive consensus of nonlinear multi-agent systems. *Nonlinear Analysis: Hybrid Systems*, 31:1–18, 2019.

[69] Deyuan Meng and Y. Jia. Scaled consensus problems on switching networks. *IEEE Transactions on Automatic Control*, 61:1664–1669, 2016.

[70] Sergey G. Nersesov and Wassim M. Haddad. Finite-time stabilization of nonlinear impulsive dynamical systems. *Nonlinear Analysis: Hybrid Systems*, 2(3):832 – 845, 2008. Special Issue Section: Analysis and Design of Hybrid Systems.

[71] Y. Ohta, T. Wada, and D.D. Siljak. Stability analysis of discontinuous nonlinear systems via piecewise linear lyapunov functions. In *Proceedings of the 2001 American Control Conference. (Cat. No.01CH37148)*, volume 6, pages 4852–4857 vol.6, 2001.

[72] R. Olfati-Saber. Flocking for multi-agent dynamic systems: algorithms and theory. *IEEE Transactions on Automatic Control*, 51(3):401–420, 2006.

[73] R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, 2007.

[74] R. Olfati-Saber and R.M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004.

[75] Zhaoxia Peng, Shichun Yang, Guoguang Wen, Ahmed Rahmani, and Yongguang Yu. Adaptive distributed formation control for multiple nonholonomic wheeled mobile robots. *Neurocomputing*, 173:1485 – 1494, 2016.

[76] Mohammad Pirani, Thilan Costa, and Shreyas Sundaram. Stability of dynamical systems on a graph. In *53rd IEEE Conference on Decision and Control*, pages 613–618, 2014.

[77] Maurizio Porfiri and Daniel J. Stilwell. Consensus seeking over random weighted directed graphs. *IEEE Transactions on Automatic Control*, 52(9):1767–1773, 2007.

[78] Amirreza Rahmani, Meng Ji, Mehran Mesbahi, and Magnus Egerstedt. Controllability of multi-agent systems from a graph-theoretic perspective. *SIAM Journal on Control and Optimization*, 48(1):162–186, 2009.

[79] W. Ren. On consensus algorithms for double-integrator dynamics. *IEEE Transactions on Automatic Control*, 53(6):1503–1509, July 2008.

[80] Wei Ren and R.W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, 2005.

[81] Sandip Roy. Scaled consensus. *Automatica*, 51:259–262, 2015.

[82] R. O. Saber and R. M. Murray. Agreement problems in networks with directed graphs and switching topology. In *42nd IEEE International Conference on Decision and Control (IEEE Cat. No.03CH37475)*, volume 4, pages 4126–4132 vol.4, Dec 2003.

[83] Y. Shang. Finite-time scaled consensus through parametric linear iterations. *International Journal of Systems Science*, 48:2033 – 2040, 2017.

[84] Y. Shang. On the delayed scaled consensus problems. *Applied Sciences*, 7:713, 2017.

[85] Y. Shang. Finite-time scaled consensus in discrete-time networks of agents. *Asian Journal of Control*, 20:2351–2356, 2018.

[86] Y. Shang. Scaled consensus of switched multi-agent systems. *IMA J. Math. Control. Inf.*, 36:639–657, 2019.

[87] Y. Shang. Consensus of hybrid multi-agent systems with malicious nodes. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 67:685–689, 2020.

[88] Yilun Shang. Finite-time consensus for multi-agent systems with fixed topologies. *International Journal of Systems Science*, 43(3):499–506, 2012.

[89] Yilun Shang. A combinatorial necessary and sufficient condition for cluster consensus. *Neurocomputing*, 216:611–616, 2016.

[90] J. Shen and J. Yan. Razumikhin type stability theorems for impulsive functional differential equations 1 1 research was. *Nonlinear Analysis-theory Methods & Applications*, 1998.

[91] Willi-Hans Steeb and Yorick Hardy. *Matrix Calculus and Kronecker Product*. WORLD SCIENTIFIC, 2nd edition, 2011.

[92] K. Sugihara and I. Suzuki. Distributed motion coordination of multiple mobile robots. In *Proceedings. 5th IEEE International Symposium on Intelligent Control 1990*, pages 138–143 vol.1, 1990.

[93] Y. G. Sun and L. Wang. Consensus of multi-agent systems in directed networks with nonuniform time-varying delays. *IEEE Transactions on Automatic Control*, 54:1607–1613, 2009.

[94] Yongzheng Sun, Wang Li, Hongjun Shi, Donghua Zhao, and Sandro Azaele. Finite-time and fixed-time consensus of multiagent networks with pinning control and noise perturbation. *SIAM Journal on Applied Mathematics*, 79(1):111–130, 2019.

[95] Yuan Gong Sun, Long Wang, and Guangming Xie. Average consensus in networks of dynamic agents with switching topologies and multiple time-varying delays. *Systems & Control Letters*, 57(2):175–183, 2008.

[96] Alireza Tahbaz-Salehi and Ali Jadbabaie. A necessary and sufficient condition for consensus over random networks. *IEEE Transactions on Automatic Control*, 53(3):791–795, 2008.

[97] John Nikolas Tsitsiklis. Problems in decentralized decision making and computation. Technical report, Massachusetts Inst of Tech Cambridge Lab for Information and Decision Systems, 1984.

[98] Tamás Vicsek, András Czirók, Eshel Ben-Jacob, Inon Cohen, and Ofer Shochet. Novel type of phase transition in a system of self-driven particles. *Phys. Rev. Lett.*, 75:1226–1229, Aug 1995.

[99] Jinhuan Wang, Daizhan Cheng, and Xiaoming Hu. Consensus of multi-agent linear dynamic systems. *Asian Journal of Control*, 10(2):144–155, 2008.

[100] K. Wang and A.N. Michel. On the stability of a family of nonlinear time-varying systems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 43(7):517–531, 1996.

[101] Kaining Wang and A.N. Michel. Lyapunov stability of dynamical systems determinined by differential inequalities with applications to robust stability1. *IFAC Proceedings Volumes*, 26(2, Part 1):43–46, 1993. 12th Triennal Wold Congress of the International Federation of Automatic control. Volume 1 Theory, Sydney, Australia, 18-23 July.

[102] Kaining Wang and A.N. Michel. Stability of dynamical systems determined by differential inequalities with applications to nonlinear circuits. In *Proceedings of 36th Midwest Symposium on Circuits and Systems*, pages 637–640 vol.1, 1993.

[103] L. Wang and F. Xiao. Finite-time consensus problems for networks of dynamic agents. *IEEE Transactions on Automatic Control*, 55(4):950–955, 2010.

[104] Long Wang and Feng Xiao. A new approach to consensus problems in discrete-time multiagent systems with time-delays. *Science in China Series F: Information Sciences*, 50:625–635, 08 2007.

[105] X. Wang and Housheng Su. Consensus of hybrid multi-agent systems by event-triggered/self-triggered strategy. *Appl. Math. Comput.*, 359:490–501, 2019.

[106] Xiaoli Wang and Yiguang Hong. Finite-time consensus for multi-agent networks with second-order agent dynamics. *IFAC Proceedings Volumes*, 41(2):15185 – 15190, 2008. 17th IFAC World Congress.

[107] Yaqi Wang and J. Lu. Some recent results of analysis and control for impulsive systems. *Commun. Nonlinear Sci. Numer. Simul.*, 80:104862, 2020.

[108] Jia Wei, Huajing Fang, and Hua O. Wang. Consensus of multi-agent system with communication delays by self-delay pd control. In *2010 8th World Congress on Intelligent Control and Automation*, pages 2295–2300, 2010.

[109] Wei Ren and R. W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, 2005.

[110] Gerhard Weiss, editor. *Multiagent Systems: A Modern Approach to Distributed Artificial Intelligence*. MIT Press, Cambridge, MA, USA, 1999.

[111] J. Wolfowitz. Products of indecomposable, aperiodic, stochastic matrices. *Proceedings of the American Mathematical Society*, 14(5):733–737, 1963.

[112] Michael Wooldridge. *An Introduction to MultiAgent Systems*. Wiley Publishing, 2nd edition, 2009.

[113] Chai Wah Wu. Synchronization in networks of nonlinear dynamical systems coupled via a directed graph. *Nonlinearity*, 18(3):1057–1064, feb 2005.

[114] Q. Wu, Li Xu, H. Zhang, and J. Zhou. Impulsive consensus in networks of multi-agent systems with any communication delays. In *ICSI*, 2011.

[115] Q. Wu, J. Zhou, and L. Xiang. Impulsive consensus seeking in directed networks of multi-agent systems with communication time delays. *International Journal of Systems Science*, 43:1479 – 1491, 2012.

[116] Feng Xiao and Long Wang. Consensus protocols for discrete-time multi-agent systems with time-varying delays. *Autom.*, 44:2577–2582, 2008.

[117] Feng Xiao, Long Wang, Jie Chen, and Yanping Gao. Finite-time formation control for multi-agent systems. *Automatica*, 45(11):2605–2611, 2009.

[118] Feng Xiao, Long Wang, and Tongwen Chen. Finite-time consensus of multi-agent systems with directed and intermittent links. In *Proceedings of the 30th Chinese Control Conference*, pages 6539–6543, 2011.

[119] Feng Xiao, Long Wang, and Tongwen Chen. Finite-time consensus in networks of integrator-like dynamic agents with directional link failure. *IEEE Transactions on Automatic Control*, 59(3):756–762, 2014.

[120] Lin Xiao and Stephen Boyd. Fast linear iterations for distributed averaging. *Systems & Control Letters*, 53(1):65 – 78, 2004.

[121] Xia Xiao and Xiaowu Mu. Consensus of linear multi-agent systems with communication delays by using the information of second-order neighbours under intermittent communication topology. *International Journal of Systems Science*, 48(1):200–208, 2017.

[122] Guangming Xie and Long Wang. Consensus control for a class of networks of dynamic agents. *International Journal of Robust and Nonlinear Control*, 17(10-11):941–959, 2007.

[123] Mali Xing and F. Deng. Scaled consensus for multi-agent systems with communication time delays. *Transactions of the Institute of Measurement and Control*, 40:2651 – 2659, 2018.

[124] Y. Zhu Y. Zheng and L. Wang. Consensus of heterogeneous multi-agent systems. *IET Control Theory & Applications*, 5:1881–1888(7), November 2011.

[125] Jurang Yan and Jianhua Shen. Impulsive stabilization of functional differential equations by lyapunov–razumikhin functions. *Nonlinear Analysis: Theory, Methods & Applications*, 37(2):245 – 255, 1999.

[126] Peng Yang, Randy A. Freeman, and Kevin M. Lynch. Multi-agent coordination by decentralized estimation and control. *IEEE Transactions on Automatic Control*, 53(11):2480–2496, 2008.

[127] Zhiyong Ye, Y. Chen, and H. Zhang. Leader-following consensus of multiagent systems with time-varying delays via impulsive control. *Mathematical Problems in Engineering*, 2014:1–10, 2014.

[128] Junyan Yu and Li He. Scaled group consensus in agent networks with finite subnetworks under continuous/discrete-time settings. *J. Frankl. Inst.*, 355:8780–8801, 2018.

[129] H. Zhang, Huihui Ji, Zhiyong Ye, Tian Sen-ping, and Chenlong Li. Impulsive consensus of multi-agent systems with stochastically switching topologies. *Nonlinear Analysis: Hybrid Systems*, 26:212–224, 2017.

[130] H. Zhang and Jin Zhou. Distributed impulsive consensus for second-order multi-agent systems with input delays. *Iet Control Theory and Applications*, 7:1978–1983, 2013.

[131] Q. Zhao, Yuanshi Zheng, and Yunru Zhu. Consensus of hybrid multi-agent systems with heterogeneous dynamics. *International Journal of Control*, 93:2848 – 2858, 2020.

[132] Yuanshi Zheng, Jingying Ma, and Long Wang. Consensus of hybrid multi-agent systems. *IEEE Transactions on Neural Networks and Learning Systems*, 29(4):1359–1365, 2018.

[133] Yuanshi Zheng and Long Wang. Consensus of switched multiagent systems. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 63(3):314–318, 2016.

[134] Yuanshi Zheng, Qi Zhao, J. Ma, and L. Wang. Second-order consensus of hybrid multi-agent systems. *Syst. Control. Lett.*, 125:51–58, 2019.

[135] Y. Zhu, C. Chen, and X. Guan. Finite-time consensus for multi-agent systems via impulsive control. In *2016 35th Chinese Control Conference (CCC)*, pages 7690–7694, July 2016.

[136] Ya-Kun Zhu, Xin-Ping Guan, and Xiao-Yuan Luo. Finite-time consensus of heterogeneous multi-agent systems. *Chinese Physics B*, 22(3):038901, mar 2013.

[137] Yunru Zhu, Yuanshi Zheng, and Long Wang. Containment control of switched multi-agent systems. *International Journal of Control*, 88(12):2570–2577, 2015.