Symmetries in Black Hole Spacetimes

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see the Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis is divided into three parts based on the following publications:

1. *Conformally Coupled Scalar in Rotating Black Hole Spacetimes*, **F. Gray**, I. Holst, D. Kubizňak, G. Odak, D. Pirvu, T. R. Perche, Phys. Rev. **D**, 101, 8, 2020.

2. *Symmetry operators for the conformal wave equation in rotating black hole spacetimes*, **F. Gray**, T. Houri, D. Kubizňák, Y. Yasui, Phys. Rev. **D**, 104, 8, 2021

3. *Principal tensor strikes again: Separability of vector equations with torsion*, R. Cayuso, **F. Gray**, D. Kubizňák, A. Margalit, R. G. Souza, L. Thiele, Phys. Lett. **B**, 795, 2019

4. *Massive vector fields in Kerr-Newman and Kerr-Sen black hole spacetimes*, R. Cayuso, Ó. J. C. Dias, **F. Gray**, D. Kubizňák, A. Margalit, J. E. Santos, R. Gomes Souza, L. Thiele, JHEP, 04,2020

5. *Slowly rotating black holes with exact Killing tensor symmetries,* **F. Gray**, D. Kubizňák, Phys. Rev. **D**, 105, 6, 2022

6. *Generalized Lense–Thirring metrics: higher-curvature corrections and solutions with matter*, **F. Gray**, R.A. Hennigar, D. Kubizňak, R.B. Mann, M. Srivastava, JHEP, 4, 70,2022

Part I is derived from papers 1-2 in which I did (and checked) all the calculations. I co-authored both the manuscripts. For the second paper, I contributed the original idea, and all the calculations. I wrote most of the document along with editing help from D.K., my advisor. T.H. and Y.Y. contributed several useful ideas and checked some of the calculations. For paper 1, I co-supervised (with D.K.) the students I.H., G.O., D.P., and T.R.P. during a PSI winterschool. They contributed to calculations and manuscript writing.

Part II is derived from papers 3-4 wherein I did (and checked) all the analytic calculations. The numerical section of the work is due to Ó.J.C.D. and J.E.S our collaborators. I co-authored both the manuscripts. For both papers 3-4 I co-supervised (with D.K.) the students R.C., A.M., R.G.S, and L.T. during another PSI winterschool. Again they contributed to calculations and manuscript writing.

Part III is derived from papers 5-6 in which I did (and checked) all the analytic calculations except for the contributions due to R.A.H. with regards to the general results relating to higher curvature gravities. I co-authored both the manuscripts with D.K., R.A.H, R.B.M., and, M.S., who also contributed to some calculations.

Work during my PhD not included in this thesis:

1. *Thermodynamics of charged, rotating, and accelerating black holes*, A. Anabalón, **F. Gray**, R. Gregory, D. Kubizňák, R. B. Mann, JHEP, 04, 96,2019

2. *Thermodynamics and Phase Transitions of NUTty Dyons*, A.B. Bordo, **F. Gray**, D. Kubizňák, JHEP, 07, 119, 2019

3. *Misner Gravitational Charges and Variable String Strengths*, A.B. Bordo, **F. Gray**, R.A. Hennigar, D. Kubizňák, Class. Quant. Grav., 36, 19, 2019

4. *Thermodynamics of Rotating NUTty Dyons*, A.B. Bordo, **F. Gray**, D. Kubizňák, JHEP, 05, 84, 2020

5. *The First Law for Rotating NUTs*, A.B. Bordo, **F. Gray**, R.A. Hennigar, D. Kubizňák, Phys. Lett. **B**, 798, 2019

6. *Quantum imprints of gravitational shockwaves*, **F. Gray**, D. Kubizňák, T. May, S. Timmerman, E. Tjoa, JHEP, 11, 54, 2021

7. *Holographic reconstruction of asymptotically flat spacetimes*, E. Tjoa and **F. Gray**, **Honorable Mention**: Gravity Research Foundation Essay, Int. J. Mod. Phys **D**, 31, 14, 2022

8. *Modest holography and bulk reconstruction in asymptotically flat spacetimes*, E. Tjoa and **F. Gray**, Phys. Rev. **D**, 106, 2, 2022

9. *Carrollian Motion in Magnetized Black Hole Horizons*, **F. Gray**, D. Kubizňák, T. R. Perche, and J. Redondo–Yuste, Preprint: arXiv: 2211.13695, 2022

Abstract

This thesis covers the role of explicit and hidden symmetries in some selected topics on the properties and excitations of black hole spacetimes. To this end, the symmetries of classical physics in Lorentzian manifolds are reviewed. In particular, explicit and hidden symmetries are presented from the Hamiltonian phase space perspective, and then, their role in the separability and integrability of geodesics and field equations is covered.

Next, I present some applications of hidden symmetries and separability of fields on black hole backgrounds. Specifically this is divided into three parts. First, the intrinsic separability of the conformal wave equation is characterized for the entire conformal class of Kerr–NUT–(Anti)-de Sitter spacetimes in all dimensions. Second, the separability of the Maxwell and Proca equations is demonstrated two examples of spacetimes, beyond general relativity, which posses these hidden symmetries. The results are applied in the four dimensions, to compare the unstable quasi-normal modes of the Proca field in the Kerr–Sen example to that of ordinary Kerr–Newman black holes of general relativity. Third, I present a new class of slowly rotating black holes which can be applied to many theories beyond general relativity and are the first physically motivated example of spacetimes which posses more hidden symmetries than explicit.

Finally to conclude, I very briefly mention some possible future directions for separability of physical equations and fields in rotating black hole spacetimes.

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Perimeter Institute and the University of Waterloo are situated on the Haldimand Tract, land that was promised to, and stolen from, the Haudenosaunee of the Six Nations of the Grand River, and is within the territory of the Neutral, Anishinaabeg, and Haudenosaunee peoples.

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Chapter 1

Introduction

Physics, at its heart, is the study of symmetries, be it conservation of momentum in billiards, or gauge "symmetry" in classical and quantum Yang–Mills theories. In particular Noether's theorem [1, 2] linking a symmetry of a given system to a conserved quantity is one of the deepest insights and guiding principles of physics. Historically symmetry played an important role in reducing the dynamics of physical systems to tractable equations of motion, and this is preserved in e.g. undergraduate mechanics when making a physical assumption of conservation of energy (ignoring friction) or translational invariance (ignoring full Newtonian gravity)[1]. Or complementarily, by imposing certain (in a sense arbitrary) symmetry properties, new solutions to equations of motion may be obtained—the physical relevance of which is to be determined later.

This latter approach has played an extremely important role in General Relativity where the full nonlinear Einstein equations[2]

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \tag{1.1}$$

are extremely hard to solve even numerically and analytic solutions are few and far between. Requiring spherical symmetry reduces the ten coupled nonlinear equations in the independent variables of the metric to two functions. This, and the assumption of time independence, lead directly to the Schwarzschild black hole solution [5], found just the

[1]See also spherical cows [3].

[2]Here as usual the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ is given by the Ricci tensor and scalar curvatures, $R_{ab} = R^d{}_{adb}$ and $R = R^a{}_a$ respectively, built from the Riemann curvature $R^a{}_{bcd}$ of the metric g_{ab}. Also, Λ is the cosmological constant, and T_{ab} is the stress energy tensor of the matter fields—I will follow Wald's conventions [4] and as usual set the physical constants $G = c = \hbar = 1$ unless otherwise stated.

following year after Einstein published his field equations. What's more, the Birkoff theorem [6, 7] shows that this is the unique solution to the (vacuum) field equations with just the assumption of spherical symmetry alone—time independence then falls out as a consequence of the field equations.

The case of exact solutions is not always so easy. In fact, even just relaxing spherical symmetry in the metric to axial symmetry (i.e. one rotational symmetry axis), while keeping the assumption of time independence (stationarity), complicated the situation so much it took almost another fifty years for the Kerr solution [8], representing a rotating black hole, to be found. This was achieved by looking for an algebraically special (Petrov type type D [9, 10]) solution to the vacuum Einstein equations (see [11] for the historical build up). Soon after, the Kerr solution was generalized to include electromagnetic charge [12,13]. However, it is still (perhaps) the most important vacuum spacetime we know, being both the unique solution with these (and a couple of other) assumptions [14–19] and astrophysically relevant[3]. For example, the Kerr spacetime provides one starting point for modeling gravitational wave emission in black hole merger events detected by LIGO [20, 21].

Moreover this spacetime turns out to have an extra "hidden" symmetry encoded in Carter's constant [22, 23] for particle motion. This was shown to arise as the contraction of a Killing tensor with two momentum vectors along the particle's trajectory [24,25]. In fact, for the Kerr spacetime the Killing tensor is the square of a Killing–Yano [26] two form [27]. Carter also generalized this construction to the Kerr–Newman–Unti-Tamburino–(Anti)-de Sitter (Kerr–NUT–(A)dS) type D spacetime [28] which includes an arbitrary cosmological constant and the NUT parameter [29, 30]. This parameter adds a "twist" to the spacetime due to the presence rotating "Misner strings" along the north-south pole axis leading to conical singularities.

Subsequently the complete class of type D spacetimes was found to be characterized by the Plebański–Demiański [31] group of metrics which also include acceleration (see [32] for an examination of these metrics in coordinates where the nature of the type D solutions and their subclasses is clear). In fact, all type-D spacetimes without acceleration (encoded in a conformal factor) admit Killing–Yano tensors [33]. However, due to this conformal factor the generic Plebański–Demiański class only possesses conformal Killing and Killing–Yano tensors.

Ultimately, much later it was shown that the fundamental object underlying the hidden symmetry properties of the Kerr–NUT–(A)dS spacetime is the principal tensor [34–36]—the Hodge dual of the Killing–Yano tensor. These hidden symmetries lead to a much richer structure in higher dimensions and their applications will form the body of this thesis. It

[3]At least the uncharged version.

is important to stress that these extra symmetries are not just mathematical niceties—as a starting point for studying physical properties of black holes it is extremely useful to take advantage of them (see for instance the review article [37] and the references therein). Essentially the corresponding conserved quantities allow one to completely integrate the geodesic equations for the motion of particles. This enables the study of e.g. the shadow of black holes [38] which have recently been imaged for the first time [39].

Following the discovery of Carter's constant the classical age of separatists came. That is, the separation of variables to find the mode functions was achieved for many different field equations. Teukolsky [40, 41] separated the equations for the electromagnetic and gravitational perturbations and obtained a master equation for spin 0, 1, 2 perturbations. Additionally, the decoupling of the spin $1/2$ case of massless neutrinos was achieved by Unruh [42] and Teukolsky [41], and the massive Dirac field by Chandrasekhar [43] and Page [44]. Further applications are found in the study of emission rates of black holes via Hawking radiation [45–47]. The Teukolsky equation continues to be exploited in gravitational wave analysis, the study of quasinormal modes [48], superradiance phenomena [49–53], and also with applications to the recent proof of the stability of black holes [54–56].

While we appear to live in 4 spacetime dimensions there are good motivations for studying higher dimensions. First, there are many higher dimension black hole solutions which come from Supergravity and String Theory inspired models [57–63]. These present new features and more general horizon topologies [63]. By understanding the consequences of these features one can search for deviations from General Relativity. Second, understanding physics in higher dimensions can allow one to make progress in four dimensions by taking advantage of, for example, large D limits [64–66]. Third, the symmetry structure becomes much more apparent and can lead to new insights in four dimensions. In particular, one example is the higher dimensional Schwarzschild black hole, discovered by Tangherlini [67]. It is contemporaneous with the Kerr spacetime and is the most general solution with an $SO(D-1)$ symmetry group.

The Kerr spacetime itself was found in arbitrary dimensions by Myers and Perry [68, 69]—again it was achieved by using symmetry, notably the group structure of the multiple axes of rotation. It was then generalized to include cosmological constant [70] and NUT parameters by Chen, Lü, and Pope [71]. Unfortunately as yet there is no higher dimensional exact solution for the Plebański–Demiański class and more surprisingly there is not even a generalization of the Kerr–Newman black hole for the Einstein–Maxwell equations to all dimensions. Here one can, for example, construct weakly charged solutions to the Maxwell equations on the Kerr–NUT–(A)dS background [72] but one cannot "backreact" these by modifying the metric functions as was the case in 4 dimensions [28, 73].

The form of the metric in [71] is actually a generalization of Carter's canonical form [28] and the hidden symmetry structure is manifest [36, 74, 75]. It is the unique spacetime to possess the above mentioned principal tensor [76–78]. This principal tensor generates the tower of hidden and explicit symmetries, i.e. Killing tensors, and Killing vectors. It is this object that lead to the separatist renaissance and underlies all of the separability results that have been achieved[4], e.g. geodesics [79–82], spin 0 [80, 83–85], spin 1/2 [86–88], spin 1 [89–93] amongst other results [94–96]. The principal tensor has also been generalized to a version with torsion [97–102], which finds application in the separability of equations on the Kerr–Sen [57] and Chong–Cvetič–Lü–Pope [60] black holes.

At its heart, understanding the symmetries of spacetimes in general D dimensions allows one to characterize when it is possible to separate these physical field equations and is still an ongoing mathematics program. Generally, one needs to have D conserved quantities for geodesic motion [103, 104] and D "symmetry operators" for the particular field equation in question (see e.g. [105–109]). The separability of geodesic motion [110–112] and the scalar (Klein–Gordon) field [83, 84, 113, 114] is completely characterized for generic spacetimes. We shall call these Benenti spacetimes of which the D dimensional Kerr–NUT–(A)dS metric is a member. On the other hand, for field equations of higher spins only certain cases are understood [115–120] and the problem of the separability of gravitational (spin 2) perturbations is only partially understood for rotating black holes [121–124].In higher dimensions, particularly, the question is open [125, 126] although progress can be made when there is no rotation [127] or when the rotation parameters are restricted [128–131].

To complete this understanding it is important to have a good language for describing explicit and hidden symmetries. There are many such languages to study classical mechanics, symmetries and Noether's theorem depending on one's perspective and desired level of mathematical sophistication. On the higher end, I refer the reader to the excellent textbooks of J.-M. Souriau [132] or (for a relatively modern treatment of symplectic geometry treatment of classical mechanics) J.E. Mardsen and T.S. Ratiu [133], and also, to the set of lecture notes by M.J. Gotay et al. [134, 135] for a multi-symplectic jet bundle approach to covariant field theory.

However, this thesis will on focus three main applications of separability and hidden symmetries: separating the conformally coupled Klein–Gordon equation in the Kerr–NUT–(A)dS class, separating the massless (Maxwell) and massive (Proca) vector field equations in the Kerr–Sen [57] and Chong–Cvetič–Lü–Pope [60] black holes, and finally studying hidden symmetries for generic slowly rotating black holes. So, for our purposes here we

[4]For a detailed review of the history here including the steps towards higher dimensional integrability and separability see [37].

will not need all the details in [132–135]. In the next section I will try to introduce, in a self contained manner, the necessary tools for the following chapters where I present the applications to black hole physics. In particular it will be necessary to introduce the symplectic picture of Hamiltonian mechanics on phase space to talk about hidden symmetries, integrability, and separability. I will assume a working knowledge of manifolds, Lie groups and General Relativity (and refer readers to [4,136] for the essential definitions).

1.1 Symmetries and Motion in Spacetimes

As mentioned above, although there are many different (equivalent) descriptions of motion, the most useful one for our purposes will be the Hamiltonian phase space. So in this section I present the symplectic structure and discuss symmetries, Noether's theorem, integrability and separability, and the properties of higher dimensional rotating black holes.

1.1.1 Phase Space and Hamiltonian Mechanics

Let us begin by recalling the standard undergraduate Hamiltonian description of the dynamics of a point particle in $\mathbb{R}^3$ subject to a potential $V(\mathbf{q})$. One begins by writing the total energy of the system, i.e. the Hamiltonian $\mathcal{H}$, in terms of the position and momentum

$$\mathcal{H}(\mathbf{q},\mathbf{p}) = \frac{1}{2m}p^2 + V(\mathbf{x})\,, \tag{1.2}$$

which is related to the Lagrangian $\mathcal{L}$ by the Legendre transformation

$$\mathcal{L} = \mathbf{p}\cdot\frac{d}{dt}\mathbf{q} - \mathcal{H}(\mathbf{q},\mathbf{p})\,. \tag{1.3}$$

The equations of motion follow from the variation of the action $S = \int_{t_i}^{t_f} dt\mathcal{L}$ with respect to the position and momentum[5] yielding

$$\frac{dq^i}{dt} = \frac{\partial\mathcal{H}}{\partial p_i}\,, \quad \frac{dp_i}{dt} = -\frac{\partial\mathcal{H}}{\partial q^i}\,. \tag{1.4}$$

[5] Writing the Lagrangian in the form of (1.3) explicitly in terms of the positions and momenta is known as the first order formalism. Boundary conditions are only specified for the initial and final positions $q(t_i) = q_i$, $q(t_f) = q_f$, and *not* on the momenta.

Thus the second order system is reduced to a first order system. This is one advantage of the Hamiltonian formulation. Another we will encounter later on is that it allows us to deal with more general kinds of symmetries.

How does Hamiltonian mechanics generalize to manifolds? Consider a spacetime $(\mathcal{M}, g)$, i.e. a D dimensional manifold $\mathcal{M}$ equipped with a (mostly minus) Lorentzian signature metric[6] g and coordinates x^a. For a relativistic point particle $\mathcal{M}$ plays the role of *configuration* space, that is the space of all possible trajectories. While the co-tangent bundle $\mathcal{P} \equiv T^*\mathcal{M}$, with coordinates (x^a, p_b), defines the phase space—this is of course $2D$ dimensional and has a canonical projection $\pi : \mathcal{P} \to \mathcal{M}$, $(x^a, p_b) \mapsto x^a$.

In this thesis lower case roman letters $a, b, \ldots$ will stand for indices on $\mathcal{M}$ while capital letters $A, B, \ldots$ will be indices on $\mathcal{P}$. We will not distinguish abstract indices from coordinate indices as this is mostly clear from context—see [4, 138] for a discussion of these. Smooth functions will be denoted by $C^\infty(\mathcal{N})$, sections of tangent vectors $\mathfrak{X}(\mathcal{N})$, and $\Lambda^n(\mathcal{N})$ the antisymmetric n-th tensor product, of the manifold $\mathcal{N}$.

Here, as suggested by the label, the coordinates p_b play the role of momentum, so that given a trajectory (curve) $\gamma : \mathbb{R} \to \mathcal{M}$, $t \mapsto x^a(t)$, $p_a(t)$ defines the momentum along γ. Note that t is now a parameter along the curve describing the internal time of the system. Now, since the Lorentzian spacetime $\mathcal{M}$ already has a time direction, we are over-counting the degrees of freedom—for a point particle its motion must be time-like (or null[7]) and likewise for the momentum. Thus $\mathcal{P}$ is in fact the extended phase space and we should properly be speaking of the constrained dynamics (see e.g. [133] for a discussion of this). The constraints are related to time reparametrization invariance and that the Hamiltonian is conserved, i.e. constant $\mathcal{H}(x^a, p_b) = -m^2/2$, on phase space. For our purposes it is enough to fix this at the end by working with proper time and normalizing the momentum.

The phase space forms a symplectic manifold, i.e., has a closed nondegenerate[8] two-form Ω, so that for $X, Y \in T\mathcal{P}$

$$d\Omega = 0\,, \quad \Omega(X, Y) = 0 \implies X = 0\,. \tag{1.5}$$

which governs the dynamics of the system. A symplectic manifold need not arise as the cotangent bundle to some base space. It simply needs to be a $2D$ manifold with such a two

[6] I will not make assumptions about the topology/causal structure of the manifold in this section and be explicit when it is necessary to do so. For a discussion of these properties I refer the reader to the classic work by Hawking and Ellis [137].

[7] I.e. its norm vanishes, $g_{ab}\dot{x}^a\dot{x}^b = 0$. All light-rays are null vectors. The causal structure of a spacetime is determined by whether events (points) can be connected by null geodesics, see ref. [137] for details.

[8] Again we are ignoring constraints and the degenerate directions they introduce.

form. Moreover, in canonical (Darboux) coordinates the symplectic form takes the simple expression

$$\Omega = dp_a \wedge dx^a \,. \tag{1.6}$$

In these coordinates Ω is exact being the (phase space) exterior derivative of the canonical one form $\Omega = d\theta$,

$$\theta = p_a dx^a \,, \tag{1.7}$$

sometimes known as the Cartan one-form or symplectic potential. Being a 2-form Ω on $\mathcal{P}$ provides a volume on the phase space by taking D wedge products of itself. In Darboux coordinates the volume reads

$$\Omega^{\wedge^D} \equiv \frac{1}{D!} \underbrace{\Omega \wedge \Omega \cdots \wedge \Omega}_{D \text{ times}} = \prod_{i=1}^{D} dp_i \wedge dx^i \,. \tag{1.8}$$

This has applications in statistical mechanics, where one is concerned with ensembles of configurations, since it provides a measure on the phase space [132].

The observables of the theory are scalar functions of the phase space, i.e. $F : \mathcal{M} \to \mathbb{R}$, $(x,p) \mapsto F(x,p)$, from which Hamiltonian vector fields can be constructed by using the inverse symplectic form $\Omega^{-1} \in \Lambda^2(T\mathcal{P})$[9]. That is,

$$X_{\mathcal{F}} = \Omega^{-1} \cdot d\mathcal{F} \iff \Omega(X_F, \cdot) = -dF \,. \tag{1.9}$$

Explicitly this is

$$X_F = \frac{\partial F}{\partial p_a}\frac{\partial}{\partial x^a} - \frac{\partial F}{\partial x^a}\frac{\partial}{\partial p_a} \,. \tag{1.10}$$

The integral curves of these generate flows in phase space that whichpreserve the symplectic form. That is to say, (at the infinitesimal level) the Lie derivative along X_F of Ω vanishes,

$$L_{X_F}\Omega = i_{X_F}(d\Omega) + d(i_{X_F}\Omega) = i_{X_F}(0) + -d^2 F = 0 \,. \tag{1.11}$$

Here, we used Cartan's magic formula for the Lie derivative of forms $L_X(\cdot) = i_X(d\,\cdot) + d(i_X\,\cdot)$, the fact that Ω is closed, and (1.9). Often (1.11) is used to define a Hamiltonian vector.

In particular the time evolution in phase space is generated by some specified Hamiltonian $\mathcal{H}$ which does not depend on t. The integral curves, $\gamma(t)$, associated with the Hamiltonian vector $X_{\mathcal{H}}$ of $\mathcal{H}$ specify the dynamical trajectories. That is,

$$\dot{\gamma}(t) = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{p}_a \frac{\partial}{\partial p_a} \equiv X_{\mathcal{H}} \,. \tag{1.12}$$

[9]Since Ω is nondegenerate its inverse exists and is unique. This means that $\Omega \cdot \Omega^{-1} = \mathbb{1} \in T^*\mathcal{P} \bigotimes T\mathcal{P}$ and the symplectic structure provides an natural isomorphism between TP and $T^*\mathcal{P}$

Thus Hamilton's equations of motion

$$\dot{x}^a = \frac{\partial \mathcal{H}}{\partial p_a}\,, \quad \dot{p}_a = -\frac{\partial \mathcal{H}}{\partial x^a}\,, \tag{1.13}$$

are recovered by comparing this to (1.10). In fact for any function $F(x^a, q_a)$ on the phase space this construction of the evolution of F is generated by the flow of the Hamiltonian vector field,

$$\dot{F} \equiv X_{\mathcal{H}}(F) = \Omega(X_{\mathcal{H}}, X_F) = \frac{\partial \mathcal{H}}{\partial p_a}\frac{\partial F}{\partial x^a} - \frac{\partial \mathcal{H}}{\partial x^a}\frac{\partial F}{\partial p_a}\,. \tag{1.14}$$

Notice (1.14) defines also the Poisson bracket[10] of F and $\mathcal{H}$, or more generally of any two functions F, G on $\mathcal{P}$,

$$\{F, G\} = \Omega^{-1}(dF, dG) = -\Omega(X_F, X_G) = \frac{\partial F}{\partial x^a}\frac{\partial G}{\partial p_a} - \frac{\partial F}{\partial p_a}\frac{\partial G}{\partial x^a}\,. \tag{1.15}$$

Note that in Darboux coordinates we recover the canonical Poisson brackets for the coordinate functions themselves

$$\{x^a, p_b\} = \delta^a{}_b\,, \quad \{x^a, x^b\} = 0 = \{p_a, p_b\}\,. \tag{1.16}$$

The Poisson bracket has some important algebraic properties. Along with the bilinearity and the anti-symmetry in F and G, the Jacobi identity is satisfied for the Poisson bracket,

$$\{\{F, G\}, H\} + \{\{F, G\}, H\} + \{\{F, G\}, H\} = 0\,, \tag{1.17}$$

which follows from the closed-ness of the symplectic form by evaluating

$$i_{X_F} i_{X_G} i_{X_H} d\Omega = 0\,. \tag{1.18}$$

It also obeys a Leibniz identity, as any derivation does,

$$\{FG, H\} = F\{G, H\} + \{F, H\}G\,. \tag{1.19}$$

Together these four properties actually can be used to define a Poisson structure and hence Poisson manifold, i.e. a manifold $\mathcal{P}$ with a bracket, $\{\cdot, \cdot\} : C^\infty(\mathcal{P}) \times C^\infty(\mathcal{P}) \to C^\infty(\mathcal{P})$,

[10] A note on conventions. Most people seem to take the sign convention as in the rightmost equality of (1.15) as the definition for the Poisson bracket. However, the symplectic form often takes the opposite sign $\Omega = -d\theta = dx^a \wedge dp_a$. To compensate for this the Poisson bracket is defined by the bi-vector Π such that $\Pi \cdot \Omega = -\mathbb{1}$. This also changes the sign in of the second equation in (1.9).

such that the four properties are satisfied. Every symplectic manifold is Poisson, but, the converse is not necessarily true (because the Poisson Bracket is not required to be nondegenerate). Furthermore, the Lie bracket between the Hamiltonian vector fields of arbitrary functions on phase space takes the nice form

$$\big[X_F, X_G\big] = -X_{\{F,G\}}\,. \tag{1.20}$$

That is, the Lie bracket is a representation of the Poisson algebra[11]. Poisson manifolds are very useful when discussing gauge theories where the degenerate directions of the (now pre-)symplectic form correspond to gauge transformations [139, 140]. Poisson manifolds are foliated by leaves (orbits) which are symplectic. Then "symplectic reduction" [133–135, 139, 140] provides a mathematical way to get to physical configurations.

Finally, in Hamiltonian mechanics one often talks about canonical transformations $\Phi : \mathcal{P} \to \mathcal{P}$, $(x^a, p_b) \mapsto \Phi(x^a, p_b) = (X^a(x^a, p_b), P^a(x^a, p_b))$, which preserve the Poisson brackets of the phase space variables, i.e. Hamilton's equations. In the symplectic language these transformations have a natural geometric formulation—they are symplectomorphisms. That is, they preserve the form of the symplectic structure

$$\Phi^*\Omega = \Omega\,, \iff \Omega = dp_a \wedge dx^a = dP_a \wedge dX^a\,. \tag{1.21}$$

If Φ is generated by some X_Φ then the infinitesimal version of (1.21) is

$$L_{X_\Phi}\Omega = 0\,. \tag{1.22}$$

Consequently, canonical transformations preserve the phase volume (1.8) and every Hamiltonian vector field is a generator of one (see (1.11)).

These canonical transformations can be induced by a generating function that is a function of the old and new coordinates $F = F(x^a, p_b, X^c, P_d; t)$. For example, if one takes $F = F(x^a, X^b; t)$ then one can show in order to preserve Hamilton's equations

$$[\dot{x}^a\,, \dot{p}_a] = \left[\frac{\partial \mathcal{H}}{\partial p_a}\,, -\frac{\partial \mathcal{H}}{\partial x^a}\right] \to \left[\dot{X}^a\,, \dot{P}_a\right] = \left[\frac{\partial \mathcal{H}'}{\partial P_a}\,, -\frac{\partial \mathcal{H}'}{\partial X^a}\right] \tag{1.23}$$

F must satisfy

$$p_a = \frac{\partial F}{\partial x^a}\,, \quad P_a = -\frac{\partial F}{\partial X^a}\,, \quad \mathcal{H}' = \mathcal{H} + \frac{\partial F}{\partial t}\,. \tag{1.24}$$

[11] Actually, they are anti-isomorphic due to the minus sign [133].

If we can find a particular generating function S such that the new Hamiltonian is identically zero $\mathcal{H}' = 0$, then the solution to the equations of motion (1.23) is immediate—it is just the level sets of the new coordinates, i.e.,

$$X^a(t) = A^a = \text{const.}\,, \quad P_a(t) = -B_a = \text{const.} \tag{1.25}$$

The middle equation in (1.24) becomes a consistency relation for $S(x^a, A^a, t)$ which can in principle be inverted:

$$P_a = -\frac{\partial S}{\partial X^a}\,, \quad \Longleftrightarrow \quad B_a = \frac{\partial S(x^b, A^c, t)}{\partial A^a}\,, \quad \Longleftrightarrow \quad x^a = x^a(A^a, B_b, t)\,. \tag{1.26}$$

Then we are left with two equations: if we substitute the first, $p_a = \frac{\partial S}{\partial x_a}$ into the second $\mathcal{H} + \frac{\partial S}{\partial t} = 0$ we obtain the *Hamilton–Jacobi equation*:

$$\mathcal{H}(x^a, \frac{\partial S}{\partial x^a}, t) + \frac{\partial S}{\partial t} = 0\,. \tag{1.27}$$

This is one partial differential equation (PDE) for Sm which is called Hamilton's principal function and is related to the action as a function of the (final) coordinates $S = S(x^a, t) = \int_{t_i}^{t} dt'[\dot{x}^a p_a - \mathcal{H}(x^a, p_b)]$. This and the Hamilton–Jacobi equation play a key role in the notion of integrability and separation of variables as we will see later. In the case of a time independent Hamiltonian we can write $S(x^a, t) = -\frac{J_0}{2}t + S(x^a)$ and (1.27) reduces to

$$\mathcal{H}(x^a, \frac{\partial S}{\partial x^a}, t) = \frac{J_0}{2}\,. \tag{1.28}$$

Covariant formulation

These statements above are coordinate dependent and given that we are working on a Lorentzian Manifold with metric g (and metric connection ∇[12] such that $\nabla_a g_{bc} = 0$) we have a natural way to promote these to covariant statements. This presentation will largely follow refs [81, 85] where it was introduced.

[12] Recalling that in components for a mixed tensor $X^{a_1\dots a_p}{}_{b_1\dots b_q}$, $\nabla_c X^{a_1\dots a_p}{}_{b_1\dots b_q} = \partial_c X^{a_1\dots a_p}{}_{b_1\dots b_q} + \Gamma^{a_1}{}_{dc} X^{d\dots a_p}{}_{b_1\dots b_q} + \dots - \Gamma^{d}{}_{b_1 c} X^{a_1\dots a_p}{}_{d\dots b_q}$, where $\Gamma^a{}_{bc} = \frac{1}{2} g^{am}(\partial_b g_{mc} + \partial_c g_{bm} - \partial_m g_{bc})$ is the Christoffel symbol. Also, recall that the Riemann tensor is defined by the failure of the covariant derivative to commute

$$[\nabla_a, \nabla_b] X^c = R^c{}_{dab} X^d\,.$$

Things are relatively simple for derivatives in the momentum direction since these are taken in a fixed co-tangent space $T^*_x\mathcal{M}$ of a point x. Thus, the space is linear in the momentum and we have for the change $p_a \to p_a + \epsilon f_a$, the derivative in the direction f_a is

$$f_a \frac{\partial F}{\partial p_a} = \frac{d}{d\epsilon} F(x^a, p_b + \epsilon f_b)\Big|_{\epsilon=0}, \tag{1.29}$$

which generalizes to any tensor on $\mathcal{P}$. To construct the derivative in the position direction we make use of the extra structure of the metric connection to parallel transport objects from one co-tangent space $T^*_x\mathcal{M}$ to another $T^*_{x_\epsilon}\mathcal{M}$, where $x^a_\epsilon = x^a + \epsilon u^a$. That is, defining $\bar{p}_\epsilon$ to be the parallel transported co-vector[13] in $T^*_{x_\epsilon}\mathcal{M}$ of p_a, the covariant derivative in the u^a direction is

$$u^a \frac{\nabla_a F}{\partial x} = \frac{d}{d\epsilon} F(x_\epsilon, \bar{p}_\epsilon)\Big|_{\epsilon=0}. \tag{1.30}$$

In particular, this is generating a splitting of a given direction X in the tangent to the phase space $X \in T\mathcal{P}$ into a direction u^a tangent to the configuration space $u \in T\mathcal{M}$, and a direction tangent to the momentum direction $f_a \in T^*\mathcal{M}$, see Figure 1.1. Note that u^a is canonically defined by the pushforward $\pi_* : T\mathcal{P} \to T\mathcal{M}$ of the projection $\pi : \mathcal{P} \to \mathcal{M}$, i.e. $u = \pi_* X$. Given a curve, $z(t) = [x(t), p(t)]$ in $\mathcal{P}$, and $u(t)$ the part tangent to the configuration direction, then the momentum direction is the covariant derivative of $p(t)$

$$f_a = \frac{\nabla}{dt} p_a = \dot{p}_a - u^c \Gamma^b{}_{ca}. \tag{1.31}$$

As an example consider any phase space variable of the form $F(x,p) = \phi^{a_1 a_2 \cdots}(x) p_{a_1} p_{a_2} \ldots$. Then the covariant derivative in phase space is simply

$$\frac{\nabla_b}{\partial x} F = \nabla_b(\phi^{a_1 a_2 \cdots}) p_{a_1} p_{a_2} \ldots, \quad \frac{\partial}{\partial p_b} F = \phi^{b a_2 \cdots} p_{a_2} \cdots + \phi^{a_1 b \cdots} p_{a_1} \ldots. \tag{1.32}$$

More generally, for an arbitrary tensor on $A^{a\cdots}{}_{b\ldots}$ the derivative in the direction $X \in T\mathcal{P}$ tangent to the curve $[x(t), p(t)]$ is,

$$\frac{\nabla}{dt} \nabla_X A^{a\ldots}{}_{b\ldots} = u^c \frac{\nabla_c}{\partial x} A^{a\ldots}{}_{b\ldots} + f_a \frac{\partial}{\partial p_a} A^{a\ldots}{}_{b\ldots}. \tag{1.33}$$

In other words, for each of these cases, (1.29), (1.30), we can identify the derivatives themselves (since the objects are linear and ultra-local in f and u) as the mixed tensors

$$\frac{\nabla^A_a}{\partial x} \in T\mathcal{P} \otimes T^*\mathcal{M}, \quad \frac{\partial^A}{\partial p_a} \in T\mathcal{P} \otimes T\mathcal{M}, \tag{1.34}$$

[13] This simply means $u \cdot \nabla p = 0$ along the curve x_ϵ.

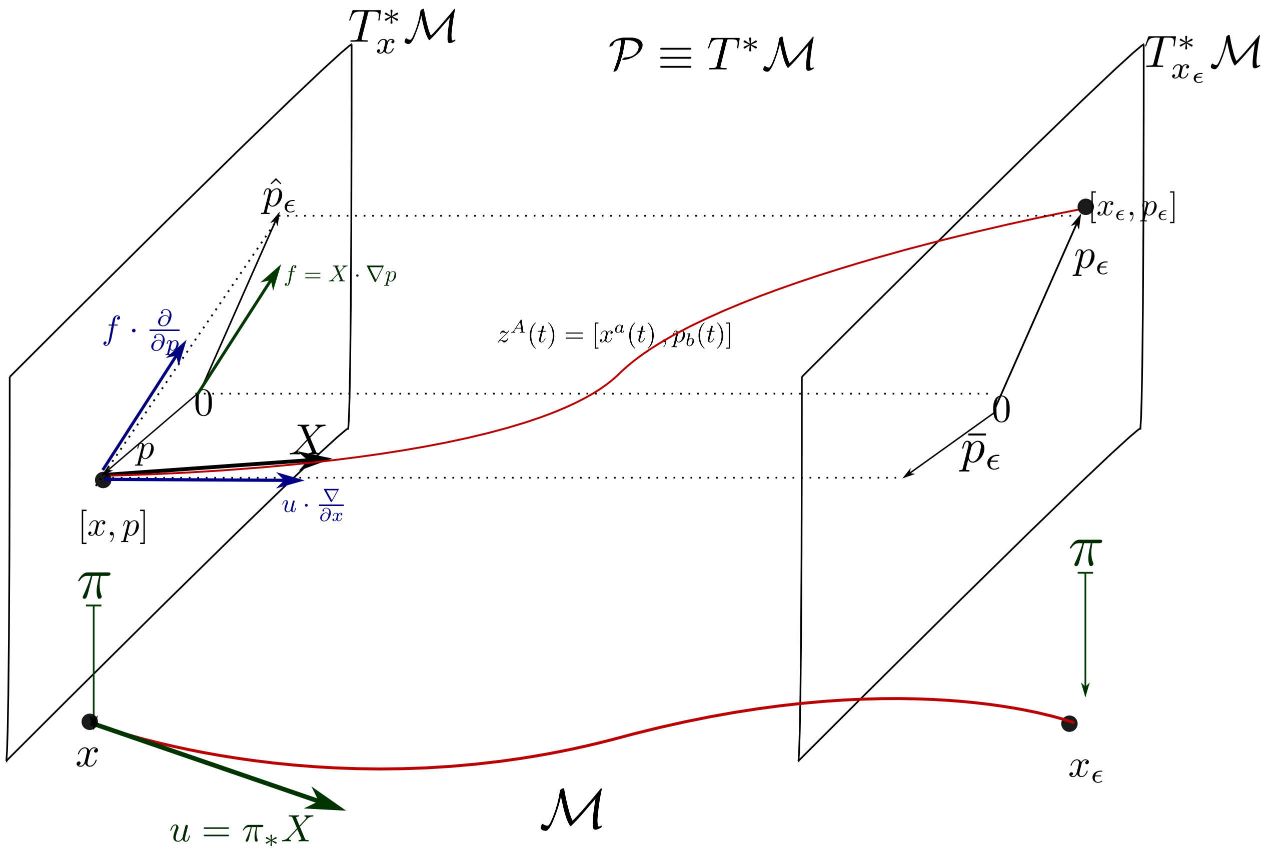

Figure 1.1: **Covariant cotangent space splitting**: Here the vector X in phase space is decomposed into its position and momentum directions (shown in blue). Let $z^A(t) = [x^a(t), p_b(t)]$ be the integral curve of X (shown in red) connecting $[x, p]$ to $[x_\epsilon, p_\epsilon]$. The dashed lines represent the parallel transporting of various objects using the connection ∇ on $\mathcal{M}$, e.g., $\bar{p}_\epsilon$ is the parallel transport of p from $[x, p]$ to $[x_\epsilon, p_\epsilon]$ and $\hat{p}_\epsilon$ is the parallel transport of p_ϵ in the opposite direction from $[x_\epsilon, p_\epsilon]$ to $[x, p]$. The direction f then points from p to $\hat{p}_\epsilon$. The covariant projections of X into x and p directions are shown in dark green.

which are a generalization of the phase space coordinate vectors $(\frac{\partial^A}{\partial x^a}, \frac{\partial^A}{\partial p_a})$. Of course we can introduce the dual quantities $D^a_A x$ and $\nabla_A p_a$ (corresponding to the coordinate forms $d_A x^a$ and $d_A p_a$) such that

$$\frac{\nabla^A_a}{\partial x} D^b_A x = \delta^a{}_b = \frac{\nabla^A}{\partial p_a} \nabla_B p_b \,, \quad \frac{\nabla^A_a}{\partial x} \nabla_B p_b = 0 = \frac{\partial^A}{\partial p_a} D^b_A x \,, \tag{1.35}$$

with the completeness relation

$$D^a_A x \frac{\nabla^B_a}{\partial x} + \nabla_A p_a \frac{\partial^B}{\partial p_a} = \delta_A{}^B \,. \tag{1.36}$$

The differential $D^a_A x$ is the push-forward of the canonical projection $\pi_* : T\mathcal{P} \to T\mathcal{M}$—c.f. above. That is, if $X^A \in T\mathcal{P}$ is tangent to the phase space then $D^a_A x$ projects out the component in the tangent fibre $D^a_A x : T_{[x,p]}\mathcal{P} \to T_x\mathcal{M}$, $u^a = X^A D^a_A x$.

This construction can also used to define the canonical lift $\sharp : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{P})$ of a vector field on the configuration space $X_\mathcal{M} \in \mathfrak{X}(\mathcal{M})$ to a vector field on $X^\sharp_\mathcal{M} \in \mathfrak{X}(\mathcal{P})$. For brevity we just present the covariant coordinate form (see [133] for the index free language formulation). Explicitly this is

$$X^\sharp_\mathcal{M} = X^a_\mathcal{M} \frac{\nabla_a}{\partial x} + (\nabla_a X^b_\mathcal{M}) p_b \frac{\partial}{\partial p_a} \,. \tag{1.37}$$

The new derivatives and differentials being related to the coordinate forms means that the symplectic and Hamiltonian structures introduced in the previous section can predictably be written in this language. In particular, the Hamiltonian vector (1.10), Hamiltonian equations (1.13), and Poisson bracket (1.15) become[14]

$$X^A_\mathcal{H} = \frac{\partial \mathcal{H}}{\partial p_a} \frac{\nabla^A_a}{\partial x} - \frac{\nabla^A_a \mathcal{H}}{\partial x} \frac{\partial^A}{\partial p_a} \,, \tag{1.38}$$

$$\left(\dot{x}^a, \frac{\nabla}{dt} p_a \right) = \left(\frac{\partial \mathcal{H}}{\partial p_a}, -\frac{\nabla_a \mathcal{H}}{\partial x} \right) , \tag{1.39}$$

$$\{F, G\} = \frac{\nabla_a F}{\partial x} \frac{\partial G}{\partial p_a} - \frac{\partial F}{\partial p_a} \frac{\nabla_a G}{\partial x} \,. \tag{1.40}$$

Finally, the *Schouten–Nijenhuis (SN)* $[\cdot, \cdot]_{\text{SN}}$ [141, 142] bracket arises naturally in this setting by considering the Poisson brackets between observables which are monomials in the momenta. That is given $A = \alpha^{a_1 \dots a_p} p_{a_1} \dots p_{a_p}$ and $B = \beta^{b_1 \dots b_q} p_{b_1} \dots p_{b_q}$ then their Poisson bracket defines the SN bracket in the following way

$$\{A, B\} \equiv -[\alpha, \beta]_{\text{SN}}^{c_1 \dots c_{p+q-1}} p_{c_1} \dots p_{c_{p+q-1}} \,, \tag{1.41}$$

where one can check from (1.40) that the explicit expression,

$$[\alpha, \beta]_{\text{SN}}^{a_1 \dots a_{p-1} c b_1 \dots b_{q-1}} \equiv p \alpha^{(d a_1 \dots a_{p-1}} \nabla_d \, \beta^{c b_1 \dots b_{q-1})} - q \beta^{d (b_1 \dots b_{q-1}} \nabla_d \, \alpha^{c a_1 \dots a_{p-1})} \,, \tag{1.42}$$

[14] There are extra terms which appear if one considers a connection on $\mathcal{M}$ which has torsion, see [85] for details.

follows.

Note, the following two results: first, if α and β are vectors then the SN bracket is just their Lie bracket ((1.20)). Second, if α is a vector and β an arbitrary contravariant tensor then,

$$[\alpha, \beta]_{\text{SN}} = L_\alpha \beta \,, \tag{1.43}$$

i.e. the SN bracket is the Lie derivative of β along α. Based on these statements one can think of the SN bracket as a generalization of the Lie derivative to tensors and it will be very useful later when discussing symmetries, constants of motion, and integrability and separability.

1.1.2 Noether's Theorem and Explicit and Hidden Symmetries

The nice thing about working on the phase space with a symplectic manifold is that the role of symmetries and conserved quantities in Noether's theorem can be reversed. That is to say, one uses the symplectic structure to convert a conserved quantity of a given Hamiltonian $\mathcal{H}$ into a symmetry flow on phase space.

Specifically, suppose $\mathcal{J} = \mathcal{J}(x^a, p_b)$ is conserved along the flow generated by $\mathcal{H}$,

$$\dot{\mathcal{J}} = \{\mathcal{J}, \mathcal{H}\} = 0 \,. \tag{1.44}$$

In other words $\mathcal{J}$ is an integral of motion. Then, using the symplectic structure, we have a symmetry flow on phase space generated by the Hamiltonian vector associated with $\mathcal{J}$,

$$X_{\mathcal{J}} = \Omega^{-1}(\cdot, d\mathcal{J}) \,. \tag{1.45}$$

By (1.20) this implies

$$[X_{\mathcal{J}}, X_{\mathcal{H}}] = 0 \,. \tag{1.46}$$

This commutation means that a given dynamical trajectory (a solution of the equations of motion of the Hamiltonian) can be Lie dragged by the flow generated by $\mathcal{J}$ and remain a dynamical trajectory. Thus, these conserved quantities generate symmetries of the equations of motion. Notice that the symmetry flow is now in phase space which allows for more general symmetries than just those of the manifold/configuration space $\mathcal{M}$.

If $\mathcal{C}$ commutes with all functions on the phase space, $\{\mathcal{C}, F\} = 0$ for all $F : \mathcal{P} \to \mathbb{R}$, then $\mathcal{C}$ is called a Casimir, and the space of $\mathcal{C}$ is the centre of the Poisson algebra. For a symplectic manifold this implies (by the nondegeneracy of Ω) $d\mathcal{C} = 0$, whereas, if $\mathcal{P}$ is a Poisson manifold then the centre can be nontrivial even when the cohomology of $\mathcal{P}$ is trivial.

Suppose $\mathcal{J}_1$ and $\mathcal{J}_2$ are conserved along the flow of $\mathcal{H}$ then by the Jacobi identity we can generate another conserved quantity $\mathcal{J}_3 = \{\mathcal{J}_1, \mathcal{J}_2\}$. However $\mathcal{J}_3$ need not be new and independent from $\mathcal{J}_1, \mathcal{J}_2$—one must separately check that it is functionally independent. Again, using (1.20), we can make similar statements about the flows $X_{\mathcal{J}_3} = X_{\{\mathcal{J}_1,\mathcal{J}_2\}}$.

Formally what we have done here is essentially introduce the concept of a momentum map which underlies the previously mentioned symplectic reduction. Again this is important when considering symmetries resulting from the action of a Lie group. For example, in gauge theories and constrained systems, see e.g. [133–135, 139, 140] for details and applications.

To be more precise, consider the left action, $\Phi : G\times\mathcal{P} \to \mathcal{P}$, $z \mapsto (g, z) \mapsto \Phi_g(z) = g\cdot z$, of a Lie group G on $(\mathcal{P}, \Omega)$. At the Lie algebra level, for $\xi \in \mathrm{Lie}\, G$, the infinitesimal generator of the action in the tangent space of $\mathcal{P}$, $\xi_{\mathcal{P}} \in \mathfrak{X}(\mathcal{P})$, is given by[15]

$$\xi_{\mathcal{P}}(z) = \frac{d}{dt}\left[\exp(t\xi)\cdot z\right]\Big|_{t=0}, \tag{1.47}$$

and the flow along $\xi_{\mathcal{P}}$ is $\varphi_t = \Phi_{\exp t\xi}$.

Suppose that the action is canonical on $\mathcal{P}$, i.e. for any $g \in G$ the symplectic structure is preserved

$$\Phi^*_{\exp t\xi}\Omega = \Omega\,, \quad \Longleftrightarrow \quad L_{\xi_{\mathcal{P}}}\Omega = 0\,. \tag{1.48}$$

Suppose further that $\xi_{\mathcal{P}}$ globally Hamiltonian, i.e. when we can write

$$X_{\mathcal{J}(\xi)} = \xi_{\mathcal{P}}\,, \tag{1.49}$$

for some function $\mathcal{J}(\xi)$ on $\mathcal{P}$ (note that on a connected symplectic manifold this defines $\mathcal{J}(\xi)$ uniquely up to a constant). On a Poisson manifold $\mathcal{J}$ is uniquely defined up to the Casimirs. This map is linear in ξ on the left hand side and the right hand side can be also made linear—this is called the co-momentum map. Since $\mathcal{J} : \mathrm{Lie}\, G \to \mathcal{F}(\mathcal{P})$ is linear we can construct its dual. This defines the momentum map $\mathcal{J}^* : P \to \mathrm{Lie}\, G^*$[16]

$$\langle \mathcal{J}^*(z), \xi\rangle = \mathcal{J}(\xi)(z)\,. \tag{1.50}$$

[15] This map is a Lie algebra anti-homomorphism between the Lie bracket on $T\mathcal{P}$ and $\mathrm{Lie}\, G$. That is, $[\eta_{\mathcal{P}}, \xi_{\mathcal{P}}] = -[\eta, \xi]_{\mathcal{P}}$ for $\eta, \xi \in \mathrm{Lie}\, G$.

[16] Here $\mathrm{Lie}\, G^*$ is the dual Lie algebra defined by the inner product on $\mathrm{Lie}\, G$, $\langle\cdot,\cdot\rangle$.

One can check that we have the following algebraic relation between the charges at the level of the Hamiltonian vectors

$$X_{\{\mathcal{J}(\eta),\mathcal{J}(\xi)\}} = X_{\mathcal{J}([\eta,\xi])} \,. \tag{1.51}$$

If this holds at the level of the Poisson brackets then the Poisson brackets (i.e. "charge algebra") of $\mathcal{J}(\xi)$, $\xi \in \mathrm{Lie}\, G$, form a linear representation of the algebra $\mathrm{Lie}\, G$, i.e.

$$\{\mathcal{J}(\eta), \mathcal{J}(\xi)\} = \mathcal{J}([\eta,\xi]) \,, \tag{1.52}$$

and the momentum map is called equivariant. This holds for symplectic manifolds. But for Poisson manifolds the left and right side can differ by a Casimir, see [133] for details. This has important consequences when one tries to quantize these charge algebras (see [139,140] for a classical discussion with references to quantization) but will not concern us here.

Finally, we can now formally state

Theorem 1.1.1 (Noether [1, 2, 133]) *If the Lie algebra* $\mathrm{Lie}\, G$ *acts canonically as above,* (1.48)*, on the symplectic manifold* $\mathcal{P}$ *with Hamiltonian* $\mathcal{H}$*, and admits a momentum map* $\mathcal{J}^* : \mathcal{P} \to \mathrm{Lie}\, G^*$*, such that* $\mathcal{H}$ *is* $\mathrm{Lie}\, G$ *invariant (i.e.* $L_{\xi_\mathcal{P}}\mathcal{H} = \xi_\mathcal{P}[\mathcal{H}] = 0$*) for all* $\xi \in \mathrm{Lie}\, G$*, then* $\mathcal{J}^*$ *is a constant of motion for* $\mathcal{H}$*. That is, if* φ_t *is the flow generated by* $X_\mathcal{H}$ *then*

$$\mathcal{J}^* \circ \varphi_t = \mathcal{J}^* \iff \{\mathcal{J}(\xi), \mathcal{H}\} = 0 \,. \tag{1.53}$$

Explicit and Hidden Symmetries

Given a vector field on phase space $X \in \mathfrak{X}(\mathcal{P})$ one can ask if it is projectable back to the configuration space. That is, is $\pi_* X \in \mathfrak{X}(\mathcal{M})$? Roughly speaking working with the covariant splitting introduced above where $X^A(x,p)\partial_A = u^a(x,p)\frac{\nabla_a^A}{\partial x} + f^a(x,p)\frac{\partial^A}{\partial p_a}$ the pushforward of the projection acts (like before) as

$$(\pi_* X)^a = u^a(x,p) \,. \tag{1.54}$$

We can see that in order for u^a to be well defined on the whole bundle $T\mathcal{M}$, and *not* just point-wise for a given fibre, it must be independent of p.

Suppose that, as in Noether's theorem 1.1.1, we have a left acting Lie algebra $\mathrm{Lie}\, G$ on $\mathcal{P}$, with co-momentum map $\mathcal{J}(\xi)$ and associated Hamiltonian vector $X_{\mathcal{J}(\xi)} = \xi_\mathcal{P}(z) \in \mathfrak{X}(\mathcal{P})$, such that $\mathcal{H}$ is $\mathrm{Lie}\, G$ invariant. An *explicit symmetry* is one which can be projected down to the configuration space i.e.

$$\pi_* X_{\mathcal{J}(\xi)} = \pi_* \xi_\mathcal{P} = \xi_\mathcal{M} \in \mathfrak{X}(\mathcal{M}) \,. \tag{1.55}$$

This also means that the Lie algebra G on acting on $\mathcal{P}$ can be projected down to a subalgebra acting on $\mathcal{M}$.

Similarly, one can start from a Lie algebra acting on $\mathcal{M}$, use the canonical lift (1.37) to raise the vectors to $\mathfrak{X}(\mathcal{P})$, and then ask if they are symmetries of the Hamiltonian. These are of course then explicit symmetries. For example, one could have isometries of the metric on the configuration space. That is, directions/rotations which preserve the metric[17] and are generated by Killing vectors k,

$$(L_k g)_{ab} = 0 \iff \nabla_{(a} k_{b)} = 0 \,. \tag{1.56}$$

These lead to symmetries of any Hamiltonian built out of invariants of the metric. In fact in such cases the co-momentum map is canonically defined,

$$\mathcal{J}(k) \equiv i_k \theta = k^a p_a \,, \tag{1.57}$$

and is equivariant (in the sense of (1.52) when k is viewed as a representation of the algebra of isometries): see [133] for details.

On the other hand *Hidden symmetries* are those which cannot be projected to the configuration space. Hidden symmetries are often called dynamical symmetries because they depend on the dynamics (momentum) of the particular trajectory on which they are evaluated. An early example of this is the Laplace–Runge–Lenz vector which underlies the in integrability of orbital motion and Kepler's laws [143].

To be concrete, let us now focus on a particular Hamiltonian of interest, that for point particles undergoing geodesic motion. That is,

$$\mathcal{H} = -\frac{1}{2} g^{ab} p_a p_b \,. \tag{1.58}$$

Applying the covariant form of Hamilton's equations (1.39) one immediately finds the geodesic equation for the velocity $\dot{x}^a$,

$$\frac{\nabla}{dt} p_a = \dot{x}^b \nabla_b p_a = 0 \,, \quad \dot{x}^a = -g^{ab} p_b \,. \tag{1.59}$$

Notice at the price of all the notation in the covariant formulation writing the equations of motion is extremely easy.

The simplest conserved quantity one could imagine is one that is linear in momentum $\mathcal{J}(k) = k^a(x) p_a$. As above, demanding its conservation leads to the Killing vector equation

$$\{\mathcal{J}, \mathcal{H}\} = \nabla_a k_b p^a p^b = 0 \iff \nabla_{(a} k_{b)} = 0 \,, \tag{1.60}$$

[17] E.g. the Poincaré group for Minkowski spacetime.

where we use the covariant Poisson brackets (1.40). The Hamiltonian vector field of $\mathcal{J}(k)$ follows from (1.38)

$$X_{\mathcal{J}(k)} = k^a(x)\frac{\nabla_a}{\partial x} - (\nabla_a k^b)p_b\frac{\partial}{\partial p_a}\,, \tag{1.61}$$

which is clearly projectable (being the canonical lift (1.37) of k) as it should be for a Killing vector/isometry.

One should then consider arbitrary monomials in the momenta $\mathcal{J}(K) = K^{a_1\ldots a_q}(x)p_{a_1\ldots a_q}$ where $K^{a_1\ldots a_q}$ is an arbitrary symmetric tensor on $\mathcal{M}$. Again we demand conservation of $\mathcal{J}$ but now this leads to the Killing tensor equation

$$\{\mathcal{J}(K), \mathcal{H}\} = 0 \iff \nabla_{(a}K_{b_1\ldots b_q)} = 0\,, \tag{1.62}$$

where again we use (1.40). This time the Hamiltonian vector field is not projectable

$$X_{\mathcal{J}(K)} = qK^{ab_1\ldots b_{q-1}}p_{b_1}\ldots p_{b_{q-1}}\frac{\nabla_a}{\partial x} - (\nabla_a K^{b_1\ldots b_q})p_{b_1}\ldots p_{b_q}\frac{\partial}{\partial p_a}\,, \tag{1.63}$$

since the first term depends on the momentum. Thus Killing tensors correspond to Hidden symmetries! Notice also that the Killing vector and tensor equations are captured by the SN bracket (1.42), that is

$$\nabla^{(a}K^{b_1\ldots b_q)} = \left[K, g^{-1}\right]^{ab_1\ldots b_q}\,. \tag{1.64}$$

By the Leibniz property of the Poisson bracket (1.19) the product of two conserved charges, $\mathcal{J}_3 = \mathcal{J}_1\mathcal{J}_2$ is also a conserved charge, and in the case of those built from Killing tensors, this implies that the symmetrized product of two Killing tensors $K_3^{a_1\ldots a_p b_1\ldots b_q} = {K_1}^{(a_1\ldots a_p}{K_2}^{b_1\ldots b_q)}$ is also a Killing tensor. Such Killing tensors are of course not independent and are called reducible. Since the Poisson bracket of two conserved quantities generates a new one (as mentioned above) so the SN bracket of two Killing tensors is a new one $K_3^{a_1\ldots a_{p-1}b_1\ldots b_{q-1}} = [K_1, K_2]_{SN}^{a_1\ldots a_{p-1}b_1\ldots b_{q-1}}$. This time it is not clear if K_3 is reducible or in fact an independent quantity.

One can construct a general polynomial of these monomial conserved quantities of different degrees. Then one calls an inhomogeneous Killing tensor to be the formal sum of Killing tensors of these different ranks corresponding to the degrees of the monomials. Such objects then form a Lie algebra under the Schouten–Nijenhuis bracket.

There are two final extensions of these hidden symmetries to introduce: the Killing–Yano tensor and its conformal generalization mentioned in the opening. Suppose one has an antisymmetric q-form $f_{a_1\ldots a_q}$, and we demand conservation of the $(q-1)$-form

$F_{a_1\ldots a_{q-1}} = f_{a_1\ldots a_{q-1}b}p^b$ along the geodesic trajectory then f must satisfy the Killing–Yano equation

$$\dot{F} = 0 \implies \nabla_{(a}f_{a_1)a_2\ldots a_q} = 0\,. \tag{1.65}$$

Any two Killing–Yano forms f_1, f_2 "square" to a Killing tensor. That is,

$$K_f^{ab} = f_1{}^{(a}{}_{c_2\ldots c_q} f_2{}^{b)c_2\ldots c_q} \tag{1.66}$$

is a Killing tensor. Killing–Yano (KY) tensors find use in the parallel transport of frames [144–147].

To consider the generalization one can decompose the covariant derivative of an arbitrary q-form $\nabla_a\omega_{a_1\ldots a_q}$ [37, 148]. This generically splits into an anti-symmetric (exact) part, a divergence part, and a "twistor" part (here we will, for brevity, consider only those forms for which this vanishes; see [37] for details). These forms satisfy the conformal Killing–Yano (CKY) equation

$$\nabla_a\omega_{a_1\ldots a_q} = \nabla_{[a}\omega_{a_1\ldots a_q]} + \frac{p}{D-p+1} g_{a[a_1}\nabla^b\omega_{|b|a_2\ldots a_q]}\,. \tag{1.67}$$

We have already dealt with the CKY forms which are divergence free—these are just KY forms. The other important case is those forms h which are closed Conformal Killing–Yano (CCKY) forms, i.e. $dh = 0$, so that

$$\nabla_a h_{a_1\ldots a_q} = \frac{p}{D-p+1} g_{a[a_1}\nabla^b h_{|b|a_2\ldots a_q]}\,. \tag{1.68}$$

When h is a two-form h_{ab} this equation defines the principal tensor of the Kerr–NUT–(A)dS class of spacetimes.

All of these equations are very strong and impose certain "integrability" requirements relating their second derivatives to the curvature of the spacetime—e.g. in general a metric will not even possess a Killing vector. One way to see this is to consider the second derivative of, for example, the Killing vector equation and use the Bianchi identity to find,

$$\nabla_a\nabla_b k_c = -R^d_{abc}k_d\,. \tag{1.69}$$

See ref. [149] for more information on the Killing tensor equation and [37] for a discussion of these in the context of all the Killing objects and in particular the principal tensor.

Killing vectors and Killing tensors can also be generalized to conformal versions. I summarize all the hidden symmetry objects and their abbreviations in appendix A (see also the review [37] for an extensive exposition of their properties). For now note that the Hodge dual of a CCKY form $f = \star h$ is a KY form and vice versa; the wedge product of any two CCKY forms $h_1 \wedge h_2$ is also a CCKY form; and a certain h will play a fundamental role for the Kerr–NUT–(A)dS class of spacetimes.

1.1.3 Integrability of Hamiltonian Systems, Geodesic Motion, and Separability of Fields

We have now introduced all the necessary concepts to talk about complete integrability and separability for Hamiltonian systems and will discuss how this is very similar to the separability of fields. Recall, the time-independent Hamilton–Jacobi equation (1.28)

$$\mathcal{H}\left(x^a, \frac{\partial S}{\partial x^a}, t\right) = \frac{J_0}{2}\,,$$

whose corresponding trajectories are given by

$$\frac{dx^a}{dt} = (\pi_* X_{\mathcal{H}})^a = \frac{\partial \mathcal{H}(x^b, p_c)}{\partial p_a}\Big|_{p_a = \frac{\partial S}{\partial x^a}}\,. \tag{1.70}$$

The *integrability* of the trajectories is characterized by the following:

Theorem 1.1.2 (Liouville–Arnold [103, 104]) *A Hamiltonian system with D degrees of freedom (living on a $2D$ phase space) and with D independent integrals of motion $\mathcal{J}_i$ in involution, i.e.,*

$$\{\mathcal{J}_i, \mathcal{J}_j\} = 0\,, \quad i, j = 0, \ldots D-1\,, \tag{1.71}$$

is completely integrable—a solution to the equations of motion can be obtained by quadratures, i.e. a finite number of integrations and algebraic manipulations

Integrable systems represent a great simplification since the equations of motion can be formally solved. See [104] but for extended discussions of this in the context of classical and celestial mechanics.

Related to this is the stronger concept of *separability.* That is, a system is *separable* when there are coordinates in which a solution to the time independent Hamiltonian–Jacobi equation can be solved by the additive separation of variables, i.e.,

$$S = \sum_{a=1}^{D} S_a(x^a, \mathbf{J})\,, \tag{1.72}$$

where $\mathbf{J} = (J_0, \ldots J_{D-1})$. Moreover, for each of the constants of motion we get another equation for S, i.e.

$$\mathcal{J}_i\left(x^a, \frac{\partial S}{\partial x^b}\right) = J_i\,. \tag{1.73}$$

In the time independent case $\mathcal{J}_0$ corresponds to the Hamiltonian and, for the geodesic particle (1.58), $J_0 = -m^2$ is the energy. Finally (1.73) is actually a set of ordinary differential equations (ODEs) using (1.72). That is we have taken the PDE (1.27) and turned it into D ODEs in D variables

$$\mathcal{J}_i\left(x^a, \frac{dS_b}{dx^b}\right) = J_i\,, \quad \text{(no sum over } b\text{)}\,. \tag{1.74}$$

This is the explicit sense of separability. Any separable systems is completely integrable but the converse is not always true.

Of course such a separable system of ODEs, (1.74), can be easily solved (in principle) formally, or numerically, or more rarely analytically in closed book form. In any case (1.74) represents a very tractable system which greatly aids in studying motion in different spacetimes.

Separability Structures for the Hamilton–Jacobi equation

One can ask when geodesic motion is separable, i.e. what kind of spacetimes $(\mathcal{M}, g)$ allow for a separation of variables solution to the Hamilton–Jacobi equation. The characterization of this problem goes back a long time, notably to 1904 when Levi-Civita [150] provided the following condition on the Hamiltonian (see also [151, 152]). The time independent Hamilton–Jacobi equation (1.28) admits a separable solution if and only if

$$\partial^a\partial^b\mathcal{H}\,\partial_a\mathcal{H}\,\partial_b\mathcal{H} + \partial_a\partial_b\mathcal{H}\,\partial_a\mathcal{H}\,\partial_b\mathcal{H} - \partial^a\partial_b\mathcal{H}\,\partial_a\mathcal{H}\,\partial^b\mathcal{H} - \partial_a\partial^b\mathcal{H}\,\partial^a\mathcal{H}\,\partial_b\mathcal{H} = 0\,, \qquad (a \neq b,\ \text{no sum}) \tag{1.75}$$

where $\partial_a = \partial/\partial x^a$, $\partial^a = \partial/\partial p_a$. Notice that this is a condition on the spacetime geometry whenever $\mathcal{H}$ depends on the metric, e.g. as in the geodesic Hamiltonian (1.58).

This horrible equation was solved explicitly for the geodesic Hamiltonian (1.58) and geometrically characterized by S. Benenti [110–112] and M. Francaviglia [114] in terms of a *separability structure*. The statement is as follows.

Theorem 1.1.3 (Benenti) *A D-dimensional manifold $(\mathcal{M}, g)$ admits a separation of variables solution to the Hamilton–Jacobi equation if and only if it has a separability structure. That is, the following conditions hold:*

1. *There exist r independent commuting Killing vectors $X_{(j)}$*

$$[X_{(j)}, X_{(k)}] = 0\,, \quad j = 0, \ldots, r-1. \tag{1.76}$$

2. *There exist $D-r$ independent Killing tensors $K_{(\mu)}$, $\mu = 1, \dots D-r$, which satisfy the relations*

$$[K_{(\mu)}, K_{(\nu)}]_{SN} = 0 \ , \quad [X_{(j)}, K_{(\mu)}]_{SN} = 0 \ , \tag{1.77}$$

3. *The Killing tensors $K_{(\mu)}$ share $D-r$ eigenvectors $m_{(\mu)}$ such that*

$$[m_{(\mu)}, m_{(\nu)}] = 0 \ , \quad [m_{(\mu)}, X_{(j)}] = 0 \ , \quad g(m_{(\mu)}, X_{(j)}) = 0 \ . \tag{1.78}$$

Here independence means that the r linear charges $\mathcal{J}(X_{(j)})$ associated with the Killing vectors and the $D-r$ quadratic charges $\mathcal{J}(K_{(\mu)})$ associated with the Killing tensors are functionally independent and commute with respect to Poisson brackets.

This separability structure implies the existence of a *normal separable coordinate system* $x^a = (x_\mu, \psi_j)$[18]. The ψ_j coordinates are Killing coordinates corresponding to the directions preserved by the Killing vectors, i.e.

$$\partial_j \, g^{ab} \equiv \frac{\partial}{\partial \psi_j} \, g^{ab} = 0 \ . \tag{1.79}$$

The x_μ correspond to non-ignorable coordinates. The Greek indices $\mu, \nu, \cdots$ will label these directions while the Latin indices $i, j, \dots$ label the Killing (ignorable) directions.

In these normal separable coordinates the metric can generically be written as[19]

$$\left(\frac{\partial}{\partial s}\right)^2 = \sum_{\mu=1}^{D-r} g^{\mu\mu} \left(\frac{\partial}{\partial x_\mu}\right)^2 + \sum_{j,k=1}^{r} g^{jk} \frac{\partial}{\partial \psi_j} \frac{\partial}{\partial \psi_k} \ . \tag{1.80}$$

where by inserting (1.80) into (1.75) one gets differential equations for $g^{\mu\mu}$ and g^{jk}. The general solution to these equations is:

$$g^{\mu\mu} = (\mathcal{A}^{-1})^{\mu}{}_{(1)} \ , \quad g^{\mu a} = 0 \quad (a \neq \mu) \ , \quad g^{jk} = \sum_{\mu=0}^{m} \mathcal{B}^{jk}_{(\mu)} (\mathcal{A}^{-1})^{\mu}{}_{(1)} \ , \tag{1.81}$$

where $m = D-r$, $(\mathcal{A}^{-1})^{\mu}{}_{(1)}$ $(\mu = 1, \cdots, m)$ is the 1-st row of the inverse of an $m \times m$ Stäckel matrix A, i.e., $(\mathcal{A}^{-1})^{\mu}{}_{(\rho)} \mathcal{A}^{(\rho)}{}_{\nu} = \delta_{\mu\nu}$, and $\mathcal{B}^{jk}_{(\mu)}$ is the (j,k)-element of an $r \times r$

[18]Note the index placement here is just a label of the particular coordinate.

[19]Here we use a slightly non-standard notation for the inverse metric $\left(\frac{\partial}{\partial s}\right)^2 \equiv g^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^b}$, c.f. the notation $ds^2 = g_{ab} dx^a dx^b$ for the metric.

symmetric matrix $\mathcal{B}_{(\mu)}$ which is a function of the single variable x_μ. A Stäckel matrix is an $m \times m$ matrix such that each element $\mathcal{A}^{(\mu)}{}_\nu$ depends only on x_ν.

Moreover, in normal separable coordinates, the same M and N matrices determine all the Killing tensors of the separability structures. Explicitly these are given by

$$K^{\mu\mu}_{(\nu)} = (\mathcal{A}^{-1})^{\mu}{}_{(\nu)} \ , \quad K^{\mu a}_{(\nu)} = 0 \quad (a \neq \mu) \ , \quad K^{jk}_{(\nu)} = \sum_{\mu=1}^{m} \mathcal{B}^{jk}_{(\mu)} (\mathcal{A}^{-1})^{\mu}{}_{(\nu)} \ , \tag{1.82}$$

where the metric is the trivial first Killing tensor, i.e., $K_{(1)} = g$.

Thus, the separability of geodesics is fully characterized mathematically. Later we will show how this applies to the physically motivated case of rotating black holes. Notice the integrability is only explicit in this particular set of coordinates.

Separability of Field Equations

Another physically relevant question is that of the separability[20] of test fields on background spacetimes. Typically one is interested in fields of various spin (scalar, spinor, vector, and tensor) as these represent (linear) perturbations to these spacetimes and can be used to study the stability of, for example, black holes to disturbances. One is also often interested in quantizing these fields, and in either case, to find analytic or numerical solutions to the field equations is a challenging task. However, when separation of variables applies the task becomes much simpler.

Typically the problem consists of a linear differential operator $\mathcal{D}$ on the D-dimensional background spacetime $(\mathcal{M}, g)$ acting on a field Ψ (where we suppress the indices[21]),

$$\mathcal{D}\Psi = 0 \, . \tag{1.83}$$

[20] That is when can make a separation of variables ansatz for the field. This time it is a multiplicative separation of variables, i.e., a product of functions of one variable.

[21] In practical terms what one typically does is to either construct some scalar invariants of the particular field à la the Newman-Penrose formalism [153] in, e.g., the Teukolsky equation [40], or to make separation of variables ansatz like below in (1.84) and have the tensor structure carried by a prefactor depending on the separation constants. E.g., for the electromagnetic field in Minkowski space one writes $A_a(x^b) = A_a(k)e^{ik_b x^b}$. Sometimes the prefactor is geometric in origin and also depends on the coordinates e.g. as in [43, 87, 90]. In this case often such an ansatz is made to convert the tensor equation into an equation for a scalar potential.

In this context the analogue of integrability/separability is to make a separation of variables ansatz, i.e.,

$$\Psi(x^a) = \prod_{i=1}^{D} \Psi_i(x^i)\,, \tag{1.84}$$

where $\Psi_i(x^i)$ is a function of only the i-th coordinate x^i.

This kind of ansatz is possible when one has a set of D mutually commuting operators (see e.g. [105–109]). That is $\mathcal{K}_{(i)}$, such that $\mathcal{K}_{(0)} = \mathcal{D}$, and for all i, j

$$\left[\mathcal{K}_{(i)}, \mathcal{K}_{(j)}\right] = \mathcal{K}_{(i)} \circ \mathcal{K}_{(j)} - \mathcal{K}_{(j)} \circ \mathcal{K}_{(i)} = 0\,. \tag{1.85}$$

Any operator that commutes as above with a field equation is known as a *symmetry operator*. This commutation relation implies that one can find a basis which diagonalizes all the $\mathcal{K}_{(i)}$ simultaneously

$$\mathcal{K}_{(i)}\Psi = C_i\Psi\,, \tag{1.86}$$

for some constants C_i known as the separation constants. It is this basis in which the separation of variables (1.84) is possible. Thence one ends up with a system of D ODEs depending on the D functions Ψ_i and D separation constants C_i. The question of when this is possible for generic operators is not fully characterized and is a difficult problem. See refs. [121–123] for some general discussions on and characterizations of symmetry operators for different field equations in various contexts. Even when this kind of separation of variables is possible it may only be obvious in certain coordinate systems.

In the case of the Klein–Gordon equation for a minimally coupled scalar field Φ

$$\mathcal{K}_{(0)}\Phi \equiv \left(-\nabla_a g^{ab} \nabla_b\right)\Phi = -m^2\Phi\,, \tag{1.87}$$

the problem is well understood. In fact, in analogy with the Hamiltonian for a free particle, making the heuristic classical to quantum substitution of the momentum $p_a \to -i\hbar\nabla_a$ one obtains the Klein–Gordon equation (typically one sets $\hbar = 1$)[22]. Thus it is natural to consider first order operators of the form $\mathcal{X}_{(i)} = -iX^a_{(i)}\nabla_a$ and second order operators of the form $\mathcal{K}_{(\mu)} = -\nabla_a K^{ab}_{(\mu)}\nabla_b$ for some vectors $X_{(i)}$ and symmetric tensors $K_{(\mu)}$. It is straightforward to show that the $X_{(i)}$ must be independent Killing vectors in order to commute with (1.87).

[22] Alternatively, one can recover the Hamilton–Jacobi equation (1.27) in a semi-classical approximation to the Klein–Gordon equation (1.87). Specifically one makes an ansatz of the form $\Phi = \exp\left(\frac{i}{\hbar}S\right)$ and expands to lowest order in the limit $\hbar \to 0$.

Moreover, Carter [154], (see also [155, 156]) showed that in order for any operator, of the form $\mathcal{K}_{(\mu)} = -\nabla_a K^{ab}_{(\mu)} \nabla_b$, to commute with (1.87), not only must $K^{ab}_{(\mu)}$ be a Killing tensor, but also satisfy the geometric property

$$\frac{1}{3}\nabla_b (K_a{}^c R_c{}^b - R_a{}^c K_c{}^b) = 0\,. \tag{1.88}$$

Carter and McLenaghan [157] showed that this condition is automatically satisfied when K is the square of a Killing–Yano tensor. See also [155–158] for a characterization of the commutation relation (1.85) between the second order operators built out of two nontrivial Killing tensors. Generically this mutual commutation between all operators is a much stronger geometric requirement [158].

However, for spacetimes with a separability structure and normal separable coordinates (1.88) is automatically satisfied as one can show that the Ricci tensor has the same eigenvectors as the Killing tensors [155, 156] and hence they commute as matrices. This mutual eigenvector basis also guarantees the commutation relations (1.85) are satisfied [155, 156] and thus these spacetimes admit a separation of variables ansatz for the Klein–Gordon equation. Based on the classical/quantum relationship between the Klein–Gordon and Hamilton–Jacobi equation it is perhaps not surprising that their integrability properties are intimately connected.

In this thesis, I will present some applications of this separability: in chapters 2 and 3 I will characterize the separability of the conformally coupled Klein–Gordon equation in rotating black hole spacetimes. Moreover, in chapters 4 and 5 the separability of the Proca equation is demonstrated for two example spacetimes beyond General Relativity and in the four dimensional case used to study the unstable quasinormal modes of the Proca field. Finally, in chapters 6 and 7 I present a new class of physically motivated spacetimes which possesses more hidden symmetries than explicit and allow exact separability and integrability of the Hamiltonian–Jacobi and Klein–Gordon equations. In the final chapter I mention some future directions and the landscape (as I see it) for separability of physical fields in rotating black hole spacetimes.

At this point it is time to see some physical examples of spacetimes with separability structures.

1.1.4 Rotating Black Holes and the Killing Tower

Kerr Black Holes

As mentioned in the introduction one of the most interesting solutions to the Einstein equations is the Kerr metric discovered in 1963 [8]. It represents a black hole of mass M and angular momentum $J = Ma$[23]. In asymptotically flat spacetimes its line element reads

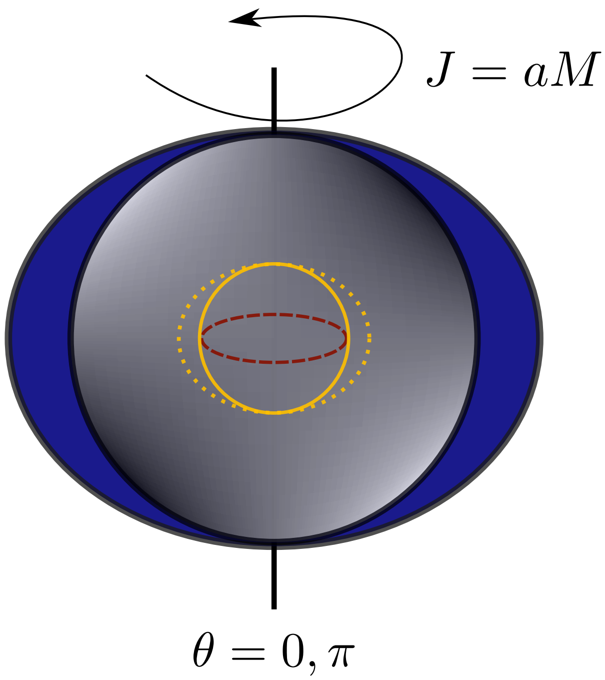

Figure 1.2: **Kerr black hole**: Schematically shown above is the Kerr black hole rotating about the symmetry axis ($\theta = 0, \pi$) corresponding to the line between the north and south poles. Special regions are indicated in different colours. The ergo-surface where an observer can no longer remain stationary and must rotate is depicted in blue. The outer horizon is shaded grey. Then comes the inner ergo-surface and the inner horizon indicated by the dashed and solid yellow lines respectively. Finally the ring singularity at $r = 0$ is shown by the red dashed line.

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mra\sin^2\theta}{\rho^2}\, dt\, d\phi + \frac{\Sigma \sin^2\theta}{\rho^2}\, d\phi^2 + \frac{\rho^2}{\Delta_r}\, dr^2 + \rho^2\, d\theta^2 \,, \quad (1.89)$$

[23] These charges can be calculated in many ways, e.g. the Arnowitt–Deser–Misner formalism [159–161].

where the metric functions are

$$\rho^2 = r^2 + a^2 \cos^2\theta\,, \quad \Delta_r = r^2 - 2Mr + a^2\,, \quad \Sigma = (r^2+a^2)^2 - \Delta_r\, a^2 \sin^2\theta\,. \tag{1.90}$$

There are many interesting properties of the spacetime and I will only mention a few here, see e.g. the text book expositions [4, 161, 162] for (many) more details. Pertinent to this work and as mentioned above it has two explicit symmetries; the Killing vectors $t^a = (\partial_t)^a$ and $\phi^a = (\partial_\phi)^a$, plus the hidden Killing tensor K^{ab} (see a little later for its explicit form).

The spacetime rotates for a zero angular momentum asymptotic observer (ZAMO) with angular velocity given by[24]

$$\Omega = \frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}}\,. \tag{1.91}$$

Of particular note are the ergoregions, horizons, and the singularity. The ergoregions are defined as the place where the time–like vector $t^a = (\partial_t)^a$ becomes null, i.e.

$$g_{ab}t^a t^b = 0 \implies r_E^{\pm} = M \pm \sqrt{M^2 + a^2\cos^2\theta}\,. \tag{1.92}$$

Since massive particles must travel along time–like trajectories inside the ergoregions they start to rotate in this region, i.e. pick up an angular component in the ϕ^a direction.

However, there are still outward pointing time-like vectors that can escape the ergoregion (i.e. those with angular velocity). The horizon occurs when any time-like vector becomes null, and represents the point at which nothing can escape the black hole classically. For the Kerr black hole this occurs when $\Delta_r = 0$, i.e.,

$$r_\pm = M \pm \sqrt{M^2 - a^2}\,. \tag{1.93}$$

Notice (see Figure 1.2) there are two horizons that collapse to one, for an extremal black hole, when $a = M$. Note also that if $a > M$ there is no horizon which means the ring-shaped curvature singularity at $\rho = 0$, signaled by the divergence of curvature scalars is visible. That is the quantity known as the the Kretschmann scalar[25]

$$I \equiv R_{abcd}R^{abcd}\,, \tag{1.94}$$

$$\tag{1.95}$$

[24]Zero angular momentum means $L = \dot{x}^a g_{ab}\phi^b = \dot{t}g_{t\phi} + \dot{\phi}g_{\phi\phi} = 0$

[25]Note that the Kretschmann scalar is regular at the horizon indicating that the apparent singularity indicated by $\Delta_r = 0$ is only a coordinate artifact. In fact one can construct infalling/Painlevé coordinates which are manifestly regular across the horizon.

diverges

$$I \sim \frac{48m^2(r^2 - a^2\cos^2\theta)(\rho^4 - 16a^2r^2\cos^2\theta)}{\rho^{12}}\,, \text{ as } \rho \to 0\,. \tag{1.96}$$

The Cosmic Censorship Conjecture [163] assumes that this singularity must be hidden behind a horizon (e.g. there are no extremal black holes) and is the subject of ongoing debate (see e.g. [164]).

Moreover, the outer horizon has some very important properties. It rotates with angular velocity

$$\Omega_H = \Omega|_{r=r_+} = \frac{a}{r_+^2 + a^2}\,, \tag{1.97}$$

and is generated by the Killing vector $\xi^a = t^a + \Omega_H \phi^a$. Its surface area is

$$A_H = 4\pi(r_+^2 + a^2) \tag{1.98}$$

and has surface gravity defined by $\xi^a \nabla_a \xi^b = \kappa \xi^b$ representing the acceleration of a particle, as measured asymptotically, required to remain at rest on surface of the horizon without falling in. This is explicitly given by

$$\kappa = \frac{\Delta_r'(r_+)}{2(r^2 + a^2)}\,. \tag{1.99}$$

Notice that the surface gravity is constant across the Horizon— this is the zeroth law of black hole mechanics [165].

The mass, M, angular momentum, J, and horizon area A satisfy the Smarr relation [166]

$$M = 2\left(\Omega_H J + \frac{\kappa}{2\pi}\frac{A}{4}\right)\,, \tag{1.100}$$

and the first law of black hole mechanics for a variation of the parameters M, J

$$\delta M = \frac{\kappa}{2\pi}\delta\left(\frac{A}{4}\right) + \Omega_H \delta J\,. \tag{1.101}$$

Thus we see that in analogy with the first law of thermodynamics that the area plays the role of entropy [167] and the surface gravity the role of temperature. Both this (1.101) and the Smarr relation (1.100) arise more generically as conservation laws [168]—in particular, the entropy is a Noether charge like the mass and angular momentum, and the (1.100) is the integrated version of the first law (1.101).

Moreover one can show a second law: that the area of the horizon only grows $\delta A \geq 0$ as matter falls in [169]. Finally the third law is that the surface gravity can never be reduced to zero within a finite time [170]. These four laws of black hole mechanics become a concrete a thermodynamic relation through Hawking radiation [171,172] where the black hole actually has a temperature proportional to the surface gravity. This is a fruitful and on going line of research providing insight into the microscopic degrees of freedom and macroscopic consequences of a possible quantum theory of gravity.

The ergoregion is responsible for the Penrose process [161, 173] where the additional angular velocity the particle picks up allows energy to be extracted from black holes. Ultimately it is responsible for the *superradiance* phenomena of fields [49]. Heuristically superradiance is when a wave with frequency $\omega = \omega_R + i\omega_I$ is scattered off a potential barrier and picks up energy. As usual the wave is partially transmitted through the barrier but unusually the reflection coefficient is greater than 1—meaning that the reflected wave has a larger amplitude than the incoming due to a kick from the potential (see e.g. [162] for detailed but introductory discussion). For integer spin fields with azimuthal angular momentum number m_ϕ this superradiance occurs when $0 < \omega_R m_\phi < \Omega_H$ and again the field gains energy from the rotation of the black hole [174]. If the bosonic particle has mass it can form bound states around the black hole and can be characterized by a complex frequency. These states which will grow if $\omega_I > 0$ triggering an instability [50–52]. This provides an intriguing avenue for the detection of (massive bosonic) dark matter [175].

There are many different coordinates that highlight different features of the Kerr geometry (see e.g. [4, 161, 176]) and, as mentioned above, the separability of the physical equations is a coordinate dependent statement. Carter's form [28] turns out to be the natural setting for separability and has become known as the canonical form [37].

It is obtained by the following coordinate transformation: set $y = a\cos\theta\,, \psi = \phi/a\,, \tau = t - a\phi$, and $\Delta_y = a^2 - y^2$:

$$ds^2 = \Sigma\left[\frac{dr^2}{\Delta_r} + \frac{dy^2}{\Delta_y}\right] + \frac{1}{\Sigma}\left[-\Delta_r(d\tau + y^2 d\psi)^2 + \Delta_y(d\tau - r^2 d\psi)^2\right] \tag{1.102}$$

The metric functions can be generalized to include cosmological constant Λ and the NUT parameter N,

$$\Delta_r = (r^2 + a^2)(1 - \Lambda r^2/3) - 2Mr\,, \tag{1.103}$$

$$\Delta_y = (a^2 - y^2)(1 + \Lambda y^2/3) - 2Ny\,. \tag{1.104}$$

Notice the symmetric form of the metric and metric functions. In fact this is in Benenti [110–112] form, where, the Stäkel matrix and coordinate $\mathcal{B}_r = \mathcal{B}_r(r)$ and $\mathcal{B}_y = \mathcal{B}_y(y)$

matrices are [37, 114]

$$\mathcal{A} = \begin{pmatrix} \frac{r^2}{\Delta_r} & \frac{y^2}{\Delta_y} \\ -\frac{1}{\Delta_r} & \frac{1}{\Delta_y} \end{pmatrix}, \quad \mathcal{B}_r = -\frac{1}{\Delta_r^2}\begin{pmatrix} r^4 & r^2 \\ r^2 & 1 \end{pmatrix} \quad \mathcal{B}_y = \frac{1}{\Delta_y^2}\begin{pmatrix} y^4 & -y^2 \\ -y^2 & 1 \end{pmatrix}. \tag{1.105}$$

Moreover, due to generality of the Benenti form of the metric, the Killing vectors and tensors hold for arbitrary “off-shell” $\Delta_r(r)$ and $\Delta_y(y)$. Thus the separability of both the Hamilton–Jacobi and Klein–Gordon equations is guaranteed. Of course, as stated earlier, the higher spin fields separate as well but it is *not* a straightforward consequence of the separability structure and Benenti's theorem 1.1.3 [110–112].

In fact, as mentioned in the opening, actually the separability properties descend from the principal tensor h—in Carter's form it is generated by the potential $b = -\frac{1}{2}(r^2 - y^2)d\tau + r^2y^2d\psi$,

$$h = db = ydy \wedge (d\tau - r^2d\psi) - rdr \wedge (d\tau + y^2d\psi) \tag{1.106}$$

This satisfies the defining CCKY equation

$$\nabla_a h_{bc} = g_{ab}\,\xi_c - g_{ac}\,\xi_b\,, \quad \xi = \partial_\tau\,, \tag{1.107}$$

and again its Hodge dual $f = \star h$ is a Killing–Yano tensor,

$$f = rdy \wedge (d\tau - r^2d\psi) + ydr \wedge (d\tau + y^2d\psi)\,. \tag{1.108}$$

The Killing tensor is then $k_{ab} = f_a{}^c f_{bc}$

$$k_{ab}dx^a dx^b = \frac{1}{\Sigma}\left[y^2\Delta_r(d\tau + y^2d\psi)^2 + r^2\Delta_y(d\tau - r^2d\psi)^2\right] + \Sigma\left[\frac{r^2dy^2}{\Delta_y} - \frac{y^2d\tau^2}{\Delta_r}\right], \tag{1.109}$$

and the last symmetry $l = -\partial_\psi$ is generated from k^{ab} and ξ

$$l^a = k^a{}_b\xi^b\,. \tag{1.110}$$

Clearly in 4 dimensions the principal tensor does not aid much. However it is everything in higher dimensions.

Finally, it is this form of the Kerr–NUT–(A)dS metric (1.102) which best generalizes to arbitrary dimensions. Typically this is done in a Wick rotated Euclidean form $r \to ir$ and $M \to iM$. Thus in the next section we will work with a Riemannian metric.

Canonical Metric for Higher Dimensional Rotating Black Holes

We now turn to the higher dimensional Kerr geometry. In canonical coordinates the *off-shell Kerr–NUT–(A)dS geometry* in $D = 2n + \varepsilon$ dimensions (with $\varepsilon = 0$ in even and $\varepsilon = 1$ in odd dimensions) takes the form [71]

$$ds^2 = \sum_{\mu=1}^{n} \left[\frac{U_\mu}{X_\mu}\, dx_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{j=0}^{n-1} A_\mu^{(j)} d\psi_j \Big)^2 \right] + \frac{\varepsilon c}{A^{(n)}} \Big(\sum_{k=0}^{n} A^{(k)} d\psi_k \Big)^2 . \tag{1.111}$$

As before the coordinates $y^a = \{x_\mu, \psi_k\}$ naturally split into two sets: Killing coordinates ψ_k ($k = 0, \ldots, n{-}1{+}\varepsilon$) associated with the explicit symmetries $l_{(k)} = \partial_{\psi_k}$, and (Wick rotated) *radial and longitudinal coordinates* x_μ ($\mu = 1, \ldots, n$). For a discussion of the connection to the original Myers–Perry [68] coordinates and the ranges of these coordinates see the review article [37]. The functions $A^{(k)}$, $A_\mu^{(j)}$, and U_μ are "symmetric polynomials" of the coordinates x_μ, and are defined by:

$$\begin{gathered} A^{(k)} = \sum_{\substack{\nu_1,\ldots,\nu_k=1 \\ \nu_1<\cdots<\nu_k}}^{n} x_{\nu_1}^2 \ldots x_{\nu_k}^2 \, , \quad A_\mu^{(j)} = \sum_{\substack{\nu_1,\ldots,\nu_j=1 \\ \nu_1<\cdots<\nu_j \\ \nu_i\neq\mu}}^{n} x_{\nu_1}^2 \ldots x_{\nu_j}^2 \, , \\ U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\nu^2 - x_\mu^2) \, , \quad U = \prod_{\substack{\mu,\nu=1 \\ \mu<\nu}}^{n} (x_\mu^2 - x_\nu^2) = \det\big(A_\mu^{(j)}\big) \, , \end{gathered} \tag{1.112}$$

where we have fixed $A^{(0)} = 1 = A_\mu^{(0)}$. Being off-shell each metric function X_μ is an unspecified function of a single coordinate x_μ:

$$X_\mu = X_\mu(x_\mu) \, . \tag{1.113}$$

And finally c is a free constant parameter appearing only in odd dimensions. The square root of the determinant of the metric reads

$$\sqrt{|g|} = \big(cA^{(n)}\big)^{\frac{\varepsilon}{2}}\, U \, . \tag{1.114}$$

In addition the inverse metric takes the form:

$$\left(\frac{\partial}{\partial s}\right)^2 = \sum_{\mu=1}^{n} \left[\frac{X_\mu}{U_\mu} \partial_{x_\mu}^2 + \frac{U_\mu}{X_\mu} \left(\sum_{k=0}^{n-1+\varepsilon} \frac{(-x_\mu^2)^{n-1-k}}{U_\mu} \partial_{\psi_k} \right)^2 \right] + \varepsilon\, \frac{1}{cA^{(n)}}\, \partial_{\psi_n}^2 \, .$$

From which one can show that the Stäkel matrix $\mathcal{A}$ and coordinate matrices $\mathcal{B}_{(\mu)}$ are given by [177]

$$\mathcal{A}^{(j)}{}_{\mu} = \frac{(-1)^j x_\mu^{2(n-j-1)}}{X_\mu} \;, \quad (\mathcal{A}^{-1})^{\mu}{}_{(j)} = \frac{A_\mu^{(j)} X_\mu}{U_\mu} \;, \tag{1.115}$$

$$\mathcal{B}^{k\ell}_{(\mu)} = \frac{(-x_\mu^2)^{2(n-1)-(k+\ell)}}{X_\mu^2} + \epsilon \frac{\delta_{nk}\delta_{n\ell}}{c x_\mu^2 X_\mu} \;. \tag{1.116}$$

Thus (1.115) and the Kerr–NUT–(A)dS spacetimes are again exactly in Benenti form. Hence, it follows from (1.80), (1.82) and a certain identity (which we will make heavy use of)[26] of the symmetric polynomials

$$\sum_\mu \frac{A_\mu^{(j)}}{x_\mu^2 U_\mu} = \frac{A^{(j)}}{A^{(n)}} \,, \tag{1.117}$$

that the Killing tensors are expressed as

$$k_{(j)} = \sum_{\mu=1}^{n} A_\mu^{(j)} \left[\frac{X_\mu}{U_\mu} \partial_{x_\mu}^2 + \frac{U_\mu}{X_\mu} \left(\sum_{k=0}^{n-1+\varepsilon} \frac{(-x_\mu^2)^{n-1-k}}{U_\mu} \partial_{\psi_k} \right)^2 \right] + \varepsilon \frac{A^{(j)}}{c A^{(n)}} \partial_{\psi_n}^2 \,, \tag{1.118}$$

for $(j = 0, \ldots, n-1)$. Fitting within the class of metrics with a separability structure they all mutually Schouten–Nijenhuis commute (see [37] for details):

$$\left[l_{(i)}, k_{(j)} \right]_{\text{SN}} = 0 \;, \quad \left[l_{(i)}, l_{(j)} \right]_{\text{SN}} = 0 \;, \quad \left[k^{(i)}, k^{(j)} \right]_{\text{SN}} = 0 \;. \tag{1.119}$$

The Riemann and Ricci curvatures of the complicated metric are explicitly calculated in [178]. This is best done in an orthonormal basis for the metric $(e^\mu, \hat{e}^\mu, \hat{e}^0)$, $\mu = 1, \ldots n$, i.e.

$$g = \sum_{\mu=1}^{n} (e^\mu e^\mu + \hat{e}^\mu \hat{e}^\mu) + \epsilon \hat{e}^0 \hat{e}^0 \,, \tag{1.120}$$

where we define $Q_\mu = X_\mu / U_\mu$,

$$e^\mu = \frac{dx_\mu}{\sqrt{Q_\mu}} \,, \quad \hat{e}^\mu = \sqrt{Q_\mu} \sum_k A_\mu^{(k)} d\psi_k \,, \quad e^0 = \sqrt{\frac{c}{A^{(n)}}} \sum_k A^{(k)} d\psi_k \,, \tag{1.121}$$

[26] See appendix D of the review [37] for this and many more identities of the symmetric polynomials.

and the inverse basis is

$$e_\mu = \sqrt{Q_\mu}\partial_{x_\mu}\,, \quad \hat{e}_\mu = \frac{1}{\sqrt{Q_\mu}}\sum_{k=0}^{n-1+\varepsilon}\frac{(-x_\mu^2)^{n-1-k}}{U_\mu}\partial_{\psi_k}\,, \quad e^0 = \sqrt{\frac{1}{cA^{(n)}}}\partial_{\psi_n}\,. \tag{1.122}$$

Also in this basis the Killing tensors are diagonal too (as they should be based on the Benenti Theorem 1.1.3),

$$k_{(j)} = \sum_{\mu=1}^{n} A_\mu^{(j)}\left[e_\mu e_\mu + \hat{e}_\mu \hat{e}_\mu\right] + \varepsilon A^{(j)} e_0 e_0\,. \tag{1.123}$$

This means that they commute as matrices; for any $i, j = 0, \ldots, n-1$

$$k^a_{(i)\,b} k^b_{(j)\,c} - k^a_{(j)\,b} k^b_{(i)\,c} = 0\,. \tag{1.124}$$

In fact, then the Ricci tensor is also diagonal [37, 178] (again as it should be due to Benenti)

$$\mathrm{Ric} = -\sum_\mu \hat{r}_\mu \left(e^\mu e^\mu + \hat{e}^\mu \hat{e}^\mu\right) - \varepsilon \hat{r}_0 e^0 e^0\,, \tag{1.125}$$

where we have introduced

$$\hat{r}_\mu = \frac{\hat{X}_\mu'' + \frac{\epsilon \hat{X}_\mu'}{x_\mu}}{2U_\mu} + \sum_{\nu\neq\mu}\frac{x_\nu \hat{X}_\nu' - x_\mu \hat{X}_\mu' - (1-\epsilon)(\hat{X}_\nu - \hat{X}_\mu)}{\left(x_\nu^2 - x_\mu^2\right)U_\nu}\,,$$
$$\hat{r}^0 = \sum_\nu \frac{\hat{X}_\nu'}{x_\nu U_\nu}\,, \quad \hat{X}_\mu = X_\mu + \varepsilon c/x_\mu^2\,. \tag{1.126}$$

One can then calculate the remarkably simple Ricci scalar

$$R = \sum_{\mu=1}^{n}\frac{r_\mu}{U_\mu}\,, \quad r_\mu = -X_\mu'' - \frac{2\varepsilon X_\mu'}{x_\mu} - \frac{2\varepsilon c}{x_\mu^4}\,. \tag{1.127}$$

Importantly, each r_μ only depends on a single variable x_μ.

Imposing the vacuum Einstein equations, $G_{ab} + \Lambda g_{ab} = 0$, the metric functions X_μ become rational functions of x_μ:

$$X_\mu(x_\mu) = \sum_{k=\varepsilon}^{n} c_k x_\mu^{2k} - 2b_\mu x_\mu^{1-\varepsilon} - \frac{\varepsilon c}{x_\mu^2}\,. \tag{1.128}$$

Here c_n represents the cosmological constant $\Lambda = \frac{1}{2}(-1)^n(d-1)(d-2)c_n$, while the other parameters c_k, b_μ, c are related to rotations, mass, and NUT charges. Fixing these one obtains on-shell Kerr–NUT–(A)dS metrics constructed by Chen, Lü, and Pope [71]. However, as the separability properties do not require the Einstein equations we stick with the general off-shell Kerr–NUT–(A)dS metrics unless otherwise specified.

Killing Tower

The generalization of the Kerr metric to all dimensions is best understood as a symmetry requirement that is stronger than the requirement to be in Benenti form. That is, demanding the existence of the aforementioned principal tensor [35, 36] (a CCKY form) uniquely determines the metric to be the off-shell Kerr–NUT–(A)dS spacetime [76–78][27].

The principal tensor [35, 36] is a nondegenerate closed conformal Killing–Yano 2-form h, i.e. it obeys c.f. (1.68), (1.107)

$$\nabla_a h_{bc} = g_{ab}\,\xi_c - g_{ac}\,\xi_b\,, \quad \xi^a = \frac{1}{D-1}\nabla_b h^{ba}\,. \tag{1.129}$$

The nondegeneracy implies that h has the maximum number[28] n of eigenvalues x_μ which are moreover *required by assumption* to be functionally independent and so these form local coordinates. The remaining coordinates ψ_k are the Killing coordinates. Thus the canonical coordinates $\{x_\mu, \psi_k\}$ are completely determined by the principal Killing–Yano tensor.

Now, the principal tensor takes the form (i.e. is simultaneously diagonalized with the metric (1.120), c.f. Darboux coordinates.)

$$h = \sum_{\mu=1}^{n} x_\mu e^\mu \wedge \hat{e}^\mu\,. \tag{1.130}$$

It generates towers of explicit and hidden symmetries, see [37]. That is by taking wedge products we obtain the following tower of closed conformal Killing–Yano tensors:

$$h^{(j)} = \frac{1}{j!}\underbrace{h\wedge\cdots\wedge h}_{j\ \text{times}}\,. \tag{1.131}$$

[27]This uniqueness is proved in the Wick rotated Riemannian case. When going back to Lorentzian signature there are some interesting limits in the null directions which can generalize this statement [179]

[28]Being antisymmetric the eigenvalues of h come in pairs corresponding to each block diagonal part—thus in odd dimensions there is in fact an extra final 1×1 block with zero eigenvalue. So more precisely h is as nondegenerate as possible.

Their Hodge duals are Killing–Yano tensors $f^{(j)} = *h^{(j)}$, whose square gives rise to a tower of rank-2 Killing tensors:

$$k^{ab}_{(j)} = \frac{1}{(d-2j-1)!} f^{(j)a}{}_{c_1 \ldots d_{d-2j-1}} f^{(j)bc_1 \ldots c_{d-2j-1}} \,. \tag{1.132}$$

which finally generate the Killing vectors:

$$l_{(j)} = k_{(j)} \cdot \xi = \partial_{\psi_j} \,. \tag{1.133}$$

There is an additional Killing vector in odd dimensions, $l_{(n)} = \partial_{\psi_n}$. Note that the $j = 0$ Killing tensor is just the inverse metric (1.115), and the zeroth Killing vector is the primary Killing vector, $l_{(0)} = \xi = \partial_{\psi_0}$.

Since all of the symmetries are generated by this single object h, it ultimately underlies the separability properties of the field equations. In particularly when dealing with field equations beyond scalars it is especially useful for constructing the symmetry operators [88, 180]. However, it is an open question as to the role of the principal tensor in separability of gravitational perturbations.

We now have all the material ready to turn to the body of this thesis. In the next part we discuss the separability of the conformally coupled Klein–Gordon equation, then we look at the Proca equations in the Kerr–Sen [57] and Chong–Cvetič–Lü–Pope [60] black holes, and finally present new solutions for slowly rotating black holes which have more hidden than explicit symmetries.

Part I

Separability of Conformally Coupled Scalar Fields

This part is derived (or lifted from) the two works [181, 182] in which we demonstrate separability of conformally coupled scalar field equation in general (off-shell) Kerr–NUT–AdS spacetimes arbitrary D dimensions.

In the first chapter of this part we perform calculations in the canonical coordinates and demonstrate that the separability is characterized by the existence of a complete set of mutually commuting operators that can be constructed from the principal Killing–Yano tensor. The separability also works for any Weyl rescaled (off-shell) metrics and especially interesting in four dimensions where it guarantees separability of a conformally coupled scalar field in the general Plebański–Demiański spacetime. Moreover, by employing the WKB approximation we derive the associated Hamilton–Jacobi equation with a scalar curvature potential term and show its separability in the Kerr–NUT–AdS spacetimes.

In the second chapter we show that the symmetry operators have a covariant expression constructed from the principal Killing–Yano tensor, its "symmetry descendants", and the curvature tensor. We next discuss the general theory of "conformal quantization", how these operators give fit into this theory, and rise to a full set of conformally invariant mutually commuting operators. For the conformally rescaled spacetimes this underlies the R-separability of the conformal wave equation.

Chapter 2

Conformally Coupled Scalar in Rotating Black Hole Spacetimes

As mentioned in the introduction the entire higher-dimensional off–shell Kerr–NUT–(A)dS family of vacuum black holes [68, 70, 71, 76, 77] admits a hidden symmetry of the *principal Killing–Yano tensor*, a non-degenerate closed conformal Killing–Yano 2-form [36], which in its turn generates towers of explicit and hidden symmetries and implies separability of a number of test field equations in these backgrounds [37].

The purpose of the current chapter is to extend the result on separability of the massive scalar equation demonstrated in [80, 83] and show that also the equation for a conformally coupled scalar,

$$\Box_R \Phi \equiv \big(\Box - \eta R\big)\Phi = 0\,, \quad \eta = \frac{1}{4}\frac{D-2}{D-1}\,, \tag{2.1}$$

separates in the general off-shell Kerr–NUT–(A)dS spacetime. Here, D stands for the number of spacetime dimensions, R is the Ricci scalar of the background metric g, and prefactor η is chosen so that the equation enjoys conformal symmetry, see e.g. Appendix D in [4]. The equation (2.1) is of fundamental importance and has a number of applications, see e.g. recent study of the asymptotic structure of Kerr spacetime via conformal compactification [183]. Exploiting the conformal symmetry, the separability remains valid for any Weyl rescaled metrics and in particular implies the separability of the conformal scalar equation in the general class of four-dimensional Plebański–Demiański spacetimes.

Moreover, applying the WKB approximation we derive a Hamilton–Jacobi from (2.1) with an extra scalar curvature potential term,

$$g^{ab}\partial_a S\,\partial_b S + \eta R = 0\,, \tag{2.2}$$

and demonstrate its separability in the Kerr–NUT–(A)dS spacetimes. The equation (2.2) has a long history, going back at least to a paper by DeWitt [184] which considers quantum Hamiltonians arising from classical systems. Therein, couplings to the geometrical objects can naturally arise. In a similar vein, the extra term we find in the Hamiltonian can arise due to ambiguities in operator ordering when quantizing non-linear systems [185]. It has also found use when considering the quantum mechanics of the motion of a free particle constrained to a Riemannian surface [186,187]. Here we understand it as a purely classical equation that describes certain modification of the free particle motion in a curved space.

2.1 Separability of Conformal Wave Equation

2.1.1 Conformal Operators

In order to separate the conformally coupled scalar field equation (2.1), let us first consider the following operators:

$$\widehat{\mathcal{K}}_{(j)} = \nabla_a k^{ab}_{(j)} \nabla_b \,, \tag{2.3}$$

whose explicit action on a scalar Φ reads

$$\tilde{\mathcal{K}}_{(j)}\Phi = \nabla_a k^{ab}_{(j)} \nabla_b \Phi = \frac{1}{\sqrt{|g|}} \partial_a \Big(\sqrt{|g|} k^{ab}_{(j)} \partial_b \Phi \Big) \,. \tag{2.4}$$

To find the coordinate form of these operators, we use (1.118) to obtain

$$\widehat{\mathcal{K}}_{(j)}\Phi = \sum_{\mu=1}^{n} \frac{1}{\sqrt{|g|}} \partial_\mu \Big(\sqrt{|g|} \frac{A^{(j)}_\mu X_\mu}{U_\mu} \partial_\mu \Phi \Big) + \sum_{\mu=1}^{n} \frac{A^{(j)}_\mu}{U_\mu X_\mu} \Big(\sum_{k=0}^{n-1+\varepsilon} (-x_\mu^2)^{n-1-k} \partial_k \Big)^2 \Phi + \varepsilon \frac{A^{(j)}}{A^{(n)}} \partial_n^2 \Phi \,, \tag{2.5}$$

where we have abbreviated $\partial_\mu = \partial_{x_\mu}$, $\partial_k = \partial_{\psi_k}$, and $\partial_n = \partial_{\psi_n}$. Employing the expression for the metric determinant (1.114) and the fact that neither $A^{(j)}_\mu$ nor

$$U/U_\mu = \frac{\prod_{\substack{\nu,\rho=1 \\ \rho<\nu}}^{n} (x_\rho^2 - x_\nu^2)}{\prod_{\substack{\nu=1 \\ \sigma\neq\mu}}^{n} (x_\sigma^2 - x_\mu^2)} = (-1)^\mu \prod_{\substack{\rho,\nu\neq\mu \\ \rho<\nu}}^{n} (x_\rho^2 - x_\nu^2) \tag{2.6}$$

depend on coordinate x_μ, we have

$$\frac{1}{\sqrt{|g|}}\partial_\mu\Big(\sqrt{|g|}\frac{A_\mu^{(j)}X_\mu}{U_\mu}\partial_\mu\Phi\Big) = \frac{A_\mu^{(j)}}{U_\mu}\frac{\partial_\mu\Big(\big(cA^{(n)}\big)^{\frac{\varepsilon}{2}}X_\mu\partial_\mu\Phi\Big)}{\big(cA^{(n)}\big)^{\frac{\varepsilon}{2}}} = \frac{A_\mu^{(j)}}{U_\mu}\Big[\partial_\mu\big(X_\mu\partial_\mu\Phi\big) + \varepsilon\frac{X_\mu}{x_\mu}\partial_\mu\Phi\Big]\,. \tag{2.7}$$

Finally using the now familiar identity (1.117) we arrive at the following explicit form of these operators:

$$\widehat{\mathcal{K}}_{(j)}\Phi = \sum_{\mu=1}^{n}\frac{A_\mu^{(j)}}{U_\mu}\widehat{\mathcal{K}}_{(\mu)}\Phi\,, \tag{2.8}$$

where each $\widehat{\mathcal{K}}_{(\mu)}$ involves only one coordinate x_μ and reads

$$\widehat{\mathcal{K}}_{(\mu)} = \partial_\mu\big(X_\mu\partial_\mu\big) + \frac{1}{X_\mu}\Big(\sum_{k=0}^{n-1+\varepsilon}(-x_\mu^2)^{n-1-k}\partial_k\Big)^2 + \frac{\varepsilon}{cx_\mu^2}\partial_n^2 + \varepsilon\frac{X_\mu}{x_\mu}\partial_\mu\,, \tag{2.9}$$

which is the form derived in [83].

Now introducing the following scalar functions

$$R_{(j)} = \sum_{\mu=1}^{n}\frac{A_\mu^{(j)}}{U_\mu}r_\mu\,, \tag{2.10}$$

which are reminiscent of the Ricci scalar and where r_μ is the same as in (1.127). One can now upgrade the operators $\widehat{\mathcal{K}}_{(j)}$ above to the following "*conformal operators*":

$$\mathcal{K}_{(j)} = \widehat{\mathcal{K}}_{(j)} - \eta R_{(j)}\,. \tag{2.11}$$

We immediately find

$$\mathcal{K}_{(j)} = \sum_{\mu=1}^{n}\frac{A_\mu^{(j)}}{U_\mu}\mathcal{K}_{(\mu)}\,, \tag{2.12}$$

where

$$\mathcal{K}_{(\mu)} = \partial_\mu\big(X_\mu\partial_\mu\big) + \frac{1}{X_\mu}\Big(\sum_{k=0}^{n-1+\varepsilon}(-x_\mu^2)^{n-1-k}\partial_k\Big)^2 - \eta r_\mu + \frac{\varepsilon}{cx_\mu^2}\partial_n^2 + \varepsilon\frac{X_\mu}{x_\mu}\partial_\mu\,. \tag{2.13}$$

2.1.2 Separability

Since $\mathcal{K}_{(0)} = \Box_R$, the conformally coupled scalar field equation (2.1) can be written as

$$\mathcal{K}_{(0)}\Phi = 0\,. \tag{2.14}$$

Although it breaks the conformal invariance, for generality, we can include a mass term and consider the equation

$$\big(\mathcal{K}_{(0)} - m^2\big)\Phi = 0\,. \tag{2.15}$$

To separate this equation we seek the solution in the multiplicative separated form,

$$\Phi = \prod_{\mu=1}^{n} Z_\mu(x_\mu) \prod_{k=0}^{n-1+\varepsilon} e^{i\Psi_k \psi_k}, \tag{2.16}$$

where Ψ_k are (Killing vector) separation constants and each of the Z_μ is a function of the single corresponding variable x_μ only. With this ansatz we have

$$\partial_k\Phi = i\Psi_k\Phi\,, \quad \partial_\mu\Phi = \frac{Z'_\mu}{Z_\mu}\Phi\,, \quad \partial^2_\mu\Phi = \frac{Z''_\mu}{Z_\mu}\Phi\,, \tag{2.17}$$

which allows us to rewrite (2.14) in the following form:

$$\Phi(x)\sum_{\mu=1}^{n}\frac{G_\mu}{U_\mu} = 0\,, \tag{2.18}$$

where $G_\mu = G_\mu(x_\mu)$ are functions of one variable only

$$G_\mu = X_\mu\frac{Z''_\mu}{Z_\mu} + X'_\mu\frac{Z'_\mu}{Z_\mu} - \frac{1}{X_\mu}\Big(\sum_{k=0}^{n-1+\varepsilon}(-x_\mu^2)^{n-1-k}\Psi_k\Big)^2\eta r_\mu - \frac{\varepsilon}{c x_\mu^2}\Psi_n^2 + \varepsilon\frac{X_\mu}{x_\mu}\frac{Z'_\mu}{Z_\mu} - m^2(-x_\mu^2)^{n-1}\,. \tag{2.19}$$

Here we have used the another identity (see [80])

$$1 = \sum_{\mu=1}^{n}\frac{(-x_\mu^2)^{n-1}}{U_\mu}\,. \tag{2.20}$$

Now let us use the following result due to Frolov, Krtouš, and Kubizňák [80, 188]

Lemma 2.1.1 (FKK separability) *The most general solution of*

$$\sum_{\mu=1}^{n} \frac{f_\mu(x_\mu)}{U_\mu} = 0\,, \tag{2.21}$$

where U_μ is defined in (1.112), is given by

$$f_\mu = \sum_{k=1}^{n-1} C_k (-x_\mu^2)^{n-1-k}\,, \tag{2.22}$$

where C_j are arbitrary (separation) constants.

Thus, we see that the most general solution of (2.18) is

$$G_\mu = \sum_{k=1}^{n-1} C_k (-x_\mu^2)^{n-1-k}\,. \tag{2.23}$$

That is, the equation (2.15) is satisfied for our ansatz (2.16) provided the functions $Z_\mu = Z_\mu(x_\mu)$ satisfy the following ordinary differential equations (ODEs):

$$Z''_\mu + Z'_\mu\Big(\frac{X'_\mu}{X_\mu} + \frac{\varepsilon}{x_\mu}\Big) - \frac{Z_\mu}{X^2_\mu}\Big(\sum_{k=0}^{n-1+\varepsilon} (-x^2_\mu)^{n-1-k}\Psi_k\Big)^2 \\ - \frac{Z_\mu}{X_\mu}\Big(\eta r_\mu + \frac{\varepsilon}{c x^2_\mu}\Psi^2_n + \sum_{k=0}^{n-1} C_k (-x^2_\mu)^{n-1-k}\Big) = 0\,, \tag{2.24}$$

where we have set $C_0 = m^2$. When the coefficient η is set to zero, we recover the result from [80] on separability of the massive Klein–Gordon equation in the off-shell Kerr–NUT–(A)dS spacetime in canonical coordinates. On the other hand, setting $m = 0$ we have successfully separated the conformal equation (2.1) in these spacetimes.

2.1.3 Commuting Operators

In a similar manner to [83] it is possible to show that the above separability descends from the existence of a complete set of mutually commuting operators—this is exactly the situation described in the introduction. To construct such a set, one can take the conformal operators $\mathcal{K}_{(j)}$ and the Killing vector operators $\mathcal{L}_{(j)}$,

$$\begin{aligned} \mathcal{K}_{(j)} &= \nabla_a k^{ab}_{(j)} \nabla_b - \eta R_{(j)} \,, && (2.25)\\ \mathcal{L}_{(j)} &= -i\, l^a_{(j)} \nabla_a \,. && (2.26) \end{aligned}$$

It is easiest to demonstrate their mutual commutation by considering their explicit coordinate form

$$\begin{aligned} \mathcal{L}_{(j)} &= -i\, \frac{\partial}{\partial \psi_j} \,, && (2.27)\\ \mathcal{K}_{(j)} &= \sum_{\mu=1}^{n} \frac{A^{(j)}_\mu}{U_\mu} \mathcal{K}_{(\mu)} \,, && (2.28) \end{aligned}$$

where $\mathcal{K}_{(\mu)}$ were derived above and are given by equation (2.13). Obviously, we have

$$\big[\mathcal{K}_{(k)}, \mathcal{L}_{(l)}\big] = 0 \,, \quad \big[\mathcal{L}_{(k)}, \mathcal{L}_{(l)}\big] = 0 \,. \tag{2.29}$$

since the operators $\mathcal{K}_{(j)}$ and $\mathcal{L}_{(l)}$ do not depend on the Killing coordinates ψ_k. Furthermore, to show that the Killing tensor operators commute,

$$\big[\mathcal{K}_{(k)}, \mathcal{K}_{(l)}\big] = 0 \,, \tag{2.30}$$

we can re-apply the methodology presented in [83]. First, note that for $\mu \neq \nu$ we have $[\mathcal{K}_{(\mu)}, \mathcal{K}_{(\nu)}] = 0$ because these operators depend on different $x^\mu \neq x^\nu$ and so any derivatives terms will commute. Next we can employ the first of the following expressions [80], which are nothing more than the fact than Stäkel matrix and its inverse (1.115) multiplied together to give the identity matrix:

$$\sum_{k=0}^{n-1} A^{(k)}_\nu \frac{(-x_\mu^2)^{n-1-k}}{U_\mu} = \delta^\nu_\mu \,, \quad \sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-1-k}}{U_\mu} A^{(j)}_\mu = \delta^j_k \,. \tag{2.31}$$

This relationship "inverts" the expression in (2.28) so one can write

$$\mathcal{K}_{(\mu)} = \sum_{k=0}^{n-1} (-x_\mu^2)^{n-1-k} \mathcal{K}_{(k)} \,. \tag{2.32}$$

Hence, using the fact that $\mathcal{K}_{(\mu)}$ only involves functions and derivatives of the single coordinate x_μ, it is clear that $[\mathcal{K}_{(\mu)}, (-x_\nu^2)^{n-1-l}] = 0$ when $\mu \neq \nu$. One can finally express the commutation of the $\mathcal{K}_{(\mu)}$'s as

$$0=[\mathcal{K}_{(\mu)},\mathcal{K}_{(\nu)}]=\sum_{k,l=0}^{n-1}(-x_\mu^2)^{n-1-k}(-x_\nu^2)^{n-1-l}[\mathcal{K}_{(l)},\mathcal{K}_{(k)}]\ . \tag{2.33}$$

In particular as the $(-x_\mu^2)^{n-1-k}$ are non-vanishing in general this shows that $[\mathcal{K}_{(k)},\mathcal{K}_{(l)}]=0$, as required.

Of course, following the logic of the introduction, the separated solution (2.16) is nothing else than the common eigenfunction of these operators and the separation constants $\{\Psi_k, C_j\}$ are the corresponding eigenvalues. That is, for our solution (2.16) obeying (2.24) we have

$$\mathcal{K}_{(j)}\Phi = C_j\Phi\,,\quad \mathcal{L}_{(j)}\Phi=\Psi_j\Phi\,. \tag{2.34}$$

To see the first equality (the second following immediately) we write

$$\begin{aligned}\frac{1}{\Phi(x)}\mathcal{K}_{(j)}\Phi(x) =&\frac{1}{\Phi(x)}\sum_{\mu=1}^{n}\frac{A_\mu^{(j)}}{U_\mu}\mathcal{K}_{(\mu)}\Phi(x)=\sum_{\mu=1}^{n}\frac{A_\mu^{(j)}}{U_\mu}\Big(G_\mu+m^2(-x_\mu^2)^{n-1}\Big)\\ =&\sum_{\mu=1}^{n}\frac{A_\mu^{(j)}}{U_\mu}\sum_{k=0}^{n-1}C_k(-x_\mu^2)^{n-1-k}=\sum_{k=0}^{n-1}C_k\sum_{\mu=1}^{n}\frac{A_\mu^{(j)}}{U_\mu}(-x_\mu^2)^{n-1-k}\\ =&C_j\,.\end{aligned} \tag{2.35}$$

Here, the definition (2.28) has been applied, then one uses the separability Lemma and (2.23), and finally applies the second identity (2.31).

2.2 Separability of Associated Hamilton–Jacobi Equation

Now we can turn to study the natural extension (2.2) of the Hamiltonian–Jacobi equation that arises from the the geometric optics (WKB) approximation of the conformal wave equation. Consider the following α-modified conformal wave equation:

$$\big(\alpha^2\Box-\eta R\big)\Phi=0\,. \tag{2.36}$$

Then, upon employing the geometric optics ansatz

$$\Phi=\Phi_0\exp\Big(\frac{i}{\alpha}S\Big)\,, \tag{2.37}$$

while taking the WKB limit $\alpha \to 0$, we arrive at the corresponding Hamilton–Jacobi equation:

$$g^{ab}\partial_a S \partial_b S + \eta R = 0\,. \tag{2.38}$$

This equation is obviously not conformally invariant. However, it is consistent with the particle Hamiltonian,

$$H = g^{ab} p_a p_b + \eta R\,, \tag{2.39}$$

where this kind of coupling to the Ricci scalar can arise from quantum corrections to the particle motion [184, 185]. The equations of motion, (1.39), for this Hamiltonian yield the following modified geodesic equation

$$\frac{dp_a}{d\tau} = -\eta \partial_a R\,. \tag{2.40}$$

Let us stress that the procedure of deriving (2.38) is similar to how one arrives at the massive Hamilton–Jacobi equation starting from the massive (α-modified) Klein–Gordon one, e.g. as in the introduction with the heuristic correspondence $-i\hbar\nabla \leftrightarrow p$ [83]. There is, however, a fundamental difference. Namely, the α-modified equation (2.36) is not conformally invariant, unless $\alpha = 1$. This is the reason why the WKB limit $\alpha \to 0$ does not produce a conformally invariant Hamilton–Jacobi equation. If instead, one started with the conformal wave equation, setting $\alpha = 1$ in (2.36), the WKB approximation would then yield the massless Hamilton–Jacobi equation, which of course is conformally invariant.

In what follows we consider the Hamilton–Jacobi equation (2.38) of potential physical interest and show its separability in the off-shell Kerr–NUT–(A)dS spacetimes. Let us make an additive separation ansatz like in the introduction, (1.72):

$$S = \sum_{\mu=1}^{n} S_\mu(x_\mu) + \sum_k \Psi_k \psi_k\,. \tag{2.41}$$

Using the form of the inverse metric given by (1.118) for $j = 0$, the Hamilton–Jacobi equation (2.38) takes the following explicit form:

$$\sum_{\mu=1}^{n}\left[\frac{X_\mu}{U_\mu} S_\mu'^2 + \frac{1}{U_\mu X_\mu}\left(\sum_{k=0}^{n-1+\varepsilon}(-x_\mu^2)^{n-1-k}\Psi_k\right)^2\right] + \varepsilon\,\frac{1}{cA^{(n)}}\,\Psi_n^2 + \eta\sum_{\mu=1}^{n}\frac{r_\mu}{U_\mu} = 0\;, \tag{2.42}$$

Again (using (1.117)) we can rewrite the previous equation as

$$\sum_\mu \frac{G_\mu}{U_\mu} = 0\,, \tag{2.43}$$

where

$$G_\mu = X_\mu S'^2_\mu + \frac{1}{X_\mu}\left(\sum_{k=0}^{n-1+\varepsilon}(-x_\mu^2)^{n-1-k}\,\Psi_k\right)^2 + \varepsilon\,\frac{\Psi_n^2}{cx_\mu^2} + \eta r_\mu\,. \tag{2.44}$$

To proceed, once more we use the FKK separation lemma 2.1.1 and so we find n first order ODEs in the n functions S_μ

$$X_\mu S'^2_\mu + \frac{1}{X_\mu}\left(\sum_{k=0}^{n-1+\varepsilon}(-x_\mu^2)^{n-1-k}\Psi_k\right)^2 + \varepsilon\,\frac{\Psi_n^2}{cx_\mu^2} + \eta r_\mu = \sum_{k=1}^{n-1} C_k(-x_\mu^2)^{n-1-k}\,. \tag{2.45}$$

Inverting this expression and identifying the canonical momenta $p = dS$ the corresponding constants of motion of the modified geodesic equation (2.40) are given by

$$C_j = k^{(j)}_{ab}p^a p^b + \eta R_{(j)}\,, \tag{2.46}$$

where $R_{(j)}$ are given by (3.9). It would be interesting to understand what these constants of motion represent physically, e.g. in the quantum systems [184, 185], as this would give a natural interpretation for the functions $R_{(j)}$.

Now with the separability of the conformal wave and corresponding modified Hamilton–Jacobi equation guaranteed we can turn to applications involving metrics conformally related to the general metric (1.111).

2.3 Separability in Weyl rescaled metrics

The equation (2.1) enjoys a conformal symmetry. This means that under a Weyl scaling of the metric,

$$g \to \; \tilde{g} = \Omega^2 g\,, \tag{2.47}$$

we have [4]

$$\big(\tilde{\Box} - \eta\tilde{R}\big)\big[\Omega^{1-D/2}\Phi\big] = \Omega^{-1-D/2}\big(\Box - \eta R\big)\Phi\,. \tag{2.48}$$

In other words, provided Φ is a solution to the equation (2.1) in the spacetime with metric g,

$$\tilde{\Phi} = \Omega^{1-D/2}\,\Phi\,, \tag{2.49}$$

is a solution of (2.1) in the spacetime with metric $\tilde{g}$.

In particular, this implies that in any spacetime $\tilde{g}$ related to the off-shell Kerr–NUT–(A)dS metric by the Weyl transformation, we can find a solution of the corresponding conformal equation (2.1) in the form (2.49), where Φ is the separated solution (2.16) and functions Z_μ obey (2.24). Strictly speaking, due to the pre-factor $\Omega^{1-D/2}$ the corresponding solution (3.11) is no longer formally written in a multiplicative separation form and the corresponding separability is called *R-separability*.

Let us also note that this result is non-trivial as the principal tensor no longer exists in the Weyl scaled metrics and consequently only towers of conformal hidden symmetries (as opposed to full hidden symmetries) exist in the Weyl rescaled spacetimes. Specifically, if ω is a conformal Killing–Yano p-form in spacetime with g, then $\tilde{\omega} = \Omega^{p+1}\omega$ is a conformal Killing–Yano p-form in spacetime with $\tilde{g}$. In particular

$$\tilde{h} = \Omega^3 h \tag{2.50}$$

is a new principal conformal Killing–Yano tensor, which however need no longer be closed and is a much weaker structure. This implies, in turn, that each Killing tensor, generated from j copies of h with $j+1$ contractions with the inverse metric, c.f. (1.132), becomes a conformal Killing tensor:

$$\tilde{K}^{ab}_{(j)} = K^{ab}_{(j)}\,, \tag{2.51}$$

and the former explicit symmetries become conformal Killing vectors, $\tilde{l}^a_{(j)} = l^a_{(j)}$. In the next chapter we will derive the underlying geometrical properties which guarantee the separability of conformal wave equations in these spacetimes.

2.4 Four-Dimensional Examples

2.4.1 Carter's Spacetime

To apply the above machinery, let us now specify to $d = 4$ dimensions. Recall from the introduction that, upon the Wick rotation of one of the x_μ coordinates,

$$\psi_0 = \tau\,, \quad \psi_1 = \psi\,, \quad x_1 = y\,, \quad x_2 = ir\,, \tag{2.52}$$

and setting

$$X_1 = -\Delta_y\,, \quad X_2 = -\Delta_r\,, \quad U_2 = \Sigma = r^2 + y^2 = -U_1\,, \tag{2.53}$$

with arbitrary

$$\Delta_r = \Delta_r(r)\,, \quad \Delta_y = \Delta_y(y)\,, \tag{2.54}$$

the off-shell Kerr–NUT–(A)dS spacetime yields the off-shell Lorentzian Carter's metric [23], (1.102)

$$ds^2 = -\frac{\Delta_r}{\Sigma}\left(d\tau + y^2 d\psi\right)^2 + \frac{\Delta_y}{\Sigma}\left(d\tau - r^2 d\psi\right)^2 + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{\Delta_y}dy^2\,,$$

and the following Ricci scalar:

$$R = -\frac{\Delta_r'' + \Delta_y''}{\Sigma}\,. \tag{2.55}$$

The conformal scalar field equation (2.1) reduces to

$$(\Box - \frac{R}{6})\Phi = 0\,. \tag{2.56}$$

Its solution can be found in a separable form,

$$\Phi = Z(r)\, Y(y)\, e^{i\omega\tau} e^{i\Psi\psi}\,, \tag{2.57}$$

where functions Z and Y satisfy the following ordinary differential equations:

$$(\Delta_r Z')' \quad + \quad Z\Big(\frac{1}{\Delta_r}(\Psi + r^2\omega)^2 + \frac{\Delta_r''}{6} - C\Big) = 0\,, \tag{2.58}$$

$$(\Delta_y Y')' \quad + \quad Y\Big(-\frac{1}{\Delta_y}(\Psi - y^2\omega)^2 + \frac{\Delta_y''}{6} + C\Big) = 0\,. \tag{2.59}$$

Of course, this result remains valid for the on-shell Carter spacetime [23], a solution to the Einstein–Maxwell–Λ theory, for which

$$\Delta_r \;=\; \left(r^2 + a^2\right)\left(1 - \Lambda r^2/3\right) - 2mr + e^2 + g^2\,, \tag{2.60}$$

$$\Delta_y \;=\; \left(a^2 - y^2\right)\left(1 + \Lambda y^2/3\right) + 2Ny\,. \tag{2.61}$$

Here, e and g are electric and magnetic charges, and M, a, N are related to mass[1], rotation, and NUT charge parameters, while the metric is accompanied by the $U(1)$ gauge potential

$$A = -\frac{er}{\Sigma}\left(d\tau + y^2\, d\psi\right) - \frac{gy}{\Sigma}\left(d\tau - r^2\, d\psi\right)\,. \tag{2.62}$$

[1] I use lowercase m here as with the cosmological constant Λ it is no longer exactly equal to the asymptotic mass M.

2.4.2 Plebański–Demiański Class

Another, more general, class of 4-dimensional black hole spacetimes is encoded in the Plebański–Demiański spacetime [31]. The off-shell metric is given by

$$\tilde{g} = \Omega^2 g \,, \tag{2.63}$$

where g is given in (1.102) and the conformal prefactor takes the following form:

$$\Omega = \frac{1}{1 - yr} \,. \tag{2.64}$$

By the above theory, this spacetime admits a solution of the conformal equation (2.56), which can be found in the R-separated form

$$\Phi = \frac{1}{\Omega} Z(r) Y(y) e^{i\omega\tau} e^{i\Psi\psi} \,, \tag{2.65}$$

where functions Z and Y obey the ordinary differential equations (2.58).

One particular example of a spacetime in this class is the original on-shell Plebański–Demiański metric [31], for which the metric functions Δ_r and Δ_y take the following specific form:

$$\Delta_r = k + e^2 + g^2 - 2mr + \epsilon r^2 - 2Nr^3 - (k + \Lambda/3) r^4, \tag{2.66}$$

$$\Delta_y = k + 2Ny - \epsilon y^2 + 2my^3 - \big(k + e^2 + g^2 + \Lambda/3\big)\, y^4 \,, \tag{2.67}$$

where e, g, N, k, m, and ϵ are free parameters that are related to the electric and magnetic charges, NUT parameter, rotation, mass, and acceleration. Due to the conformal invariance of Maxwell equations in $4d$, the gauge potential remains given by (2.62). In this special case, the separability of the conformal scalar equation follows from the results presented in [189], see also [115] for its intrinsic characterization.

Another example of a spacetime which belongs to the off-shell Plebański–Demiański class is the hairy black hole solution constructed in [190, 191]. See also [192] for a more general spacetime that can be written in the form (2.63) with a more general conformal pre-factor.

2.5 Summary

In this chapter we have separated the conformal wave equation in general off-shell Kerr–NUT–(A)dS spacetimes in all dimensions, generalizing the work [80] on separability of the massive Klein–Gordon equation in these spacetimes. Let us emphasize that although the two results formally coincide in vacuum with cosmological constant—for the on-shell Kerr–NUT–(A)dS spacetime [71]—they are very different for a more general matter content.

We have also introduced a modified Hamilton–Jacobi equation for a single particle with a Ricci scalar potential term. This equation naturally arises from the WKB limit of the "α-modified" conformal wave equation. This limit breaks the conformal invariance and the resulting equation no longer enjoys conformal symmetry. We have shown that this equation also separates in the Kerr–NUT-AdS spacetimes – the corresponding non-trivial constants of motion are given by the Killing tensors and the scalar functions $R_{(j)}$, giving a natural setting for the interpretation of the latter. For the future one possible avenue is study to the physical context and implications of the newly derived (non-minimal coupling) Hamilton–Jacobi equation.

We have further shown that the demonstrated separability can be characterized by a complete set of mutually commuting operators. To the leading order in derivatives, these operators are constructed from Killing tensors and Killing vectors generated from the hidden symmetry of the off-shell Kerr–NUT–(A)dS spacetime encoded in the principal Killing–Yano tensor. The second order operators also pick up an "anomalous" absolute term, see (2.10) and (2.11), which in the case of the original conformal wave operator is simply given by the Ricci scalar of the spacetime and guarantees the conformal invariance of the corresponding equation. In the next chapter we will study the intrinsic geometric nature of these operators and how they fit within the theory of symmetry operators.

We have also discussed the Weyl rescaled metrics and shown how our results imply separability of the conformal wave equations in those spacetimes. As a concrete application we have considered the most general type D spacetime described by the Plebański–Demiański family and constructed the associated R-separated test field solution of the conformal wave equation. We expect that this construction applies to a wide class of solutions with various matter content, similar to what happens in four dimensions [31, 190–192].

The obtained separated solution (2.16) is general – it depends on $D-1$ separation constants $\{\Psi_k, C_j\}$ – any solution to the conformal scalar equation can be written as a superposition of these separated modes. Note, however, that this chapter we have used the canonical "symmetric gauge" (1.111), where the (Wick rotated) radial and longitudinal coordinates x_μ are treated on the same footing and there is no clear distinction between the time and angle Killing coordinates both being encoded in ψ_k. Consequently, the resultant ordinary differential equations (2.24) all "look the same". In order to apply our result to study the behavior of the scalar field in the black hole vicinity, one needs to transform to the "physical space". See [37] where this is explicitly done.

Upon this one of the separated equations (2.24) becomes a (distinguished) radial equation while the other equations are the angular ones. In order to solve this system (which is only coupled through parameters of the solution and separation constants), one needs to impose the regularity conditions on the axes, as well as proper boundary conditions for the radial modes. This then distinguishes various physical modes one wants to study. For example, the quasinormal modes are characterized by ingoing boundary conditions on the horizon together with the appropriate asymptotic conditions, this will become relevant in chapter 5.

This in turn restricts the admissible values of the separation and integration constants, and poses the "non-linear eigenvalue problem", see e.g. [91, 193, 194] for how this is done in similar settings. In particular, a similar approach to [91, 195] (where comparable ODEs were obtained by exploiting the hidden symmetries for the case of massive vector fields) can be used to numerically analyze the quasinormal modes arising from the coupled ODEs (2.58) in the physically interesting Plebański–Demiański family of spacetimes. See the next part of the the thesis for an application to the massive vector field equations in the Kerr–Sen spacetime. We now turn the the geometric interpretation of the symmetry operators for the conformal wave equations

Chapter 3

Geometric Characterization of Conformal Symmetry Operators

The purpose of the present chapter is to further our understanding of the conformal wave equation (2.1) in the Kerr–NUT–(A)dS spacetime —filling some important gaps in the previous analysis. In particular, we want to intrinsically and geometrically characterize the obtained separability by finding an explicit covariant form of the corresponding symmetry operators. It turns out that these operators can be written in terms of the principal Killing–Yano tensor, its symmetry descendants, and the curvature tensor. Moreover, following [118], such operators can be lifted to conformal operators and guarantee R-separability of the conformal wave equation in any conformally related spacetime (2.47).

In the previous chapter (derived from [181]) the separability of the conformal wave equation arose from the symmetry operators of (2.1) which were found to be

$$\mathcal{K}_{(j)} = \nabla_a k^{ab}_{(j)} \nabla_b - \eta R_{(j)}\,, \quad \mathcal{L}_{(j)} = -i\, l^a_{(j)} \nabla_a\,. \tag{3.1}$$

where the functions $R_{(j)}$ take the following coordinate form

$$R_{(j)} = \sum_{\mu=1}^{n} \frac{A^{(j)}_\mu}{U_\mu} r_\mu\,, \tag{3.2}$$

Here r_μ are the Ricci scalar functions appearing in (1.127). The commutation of these operators with the conformal wave equation (i.e. $\mathcal{K}_{(0)}$ and (2.1)) is a special case of the result presented in [154, 155]. Therein, it is shown that the commutation of any operators,

$$\left[\Box + g, \nabla_a K^{ab} \nabla_b + f\right] = 0\,, \tag{3.3}$$

where $f, g \in \mathcal{C}^\infty(\mathcal{M})$ and K^{ab} is a Killing tensor is guaranteed provided

$$\nabla_a f = K_a{}^b \nabla_a g - \frac{1}{3}\nabla_b (K_a{}^c R_c{}^b - R_a{}^c K_c{}^b)\,. \tag{3.4}$$

This is of course a generalization of (1.88).

In the case of the off-shell Kerr–NUT–(A)dS metrics the final term on the right hand side vanishes as the Killing and Ricci tensors are diagonal in the same basis [37, 178] (see (1.125) and (1.123)). Thus, this equation reduces to a relationship between $R_{(j)}$ and the Ricci scalar. Hence these functions $R_{(j)}$ can be understood geometrically as follows. Let us define the following 1-forms $\kappa^{(j)}$:

$$\kappa_a^{(j)} = k_{(j)\,a}{}^b \nabla_b R\,. \tag{3.5}$$

Then, (3.4) with $g = R$ and $f = R_{(j)}$ implies that, $R_{(j)}$ must be potentials for the 1-forms:

$$\kappa^{(j)} = dR_{(j)}\,. \tag{3.6}$$

To verify this explicitly one starts with (3.2) and (1.127) to find the derivative of $R_{(j)}$;

$$\nabla_a R_{(j)} \overset{a=\nu}{=} \sum_{\mu=1}^{n} \frac{\partial_\nu r_\mu\, A_\mu^{(j)}}{U_\mu} + 2x_\nu A_\nu^{(j)} \sum_{\mu\neq\nu} \frac{\frac{r_\nu}{U_\nu} + \frac{r_\mu}{U_\mu}}{x_\mu^2 - x_\nu^2} = \frac{r'_\nu\, A_\nu^{(j)}}{U_\nu} + 2x_\nu A_\nu^{(j)} \sum_{\mu\neq\nu} \frac{\frac{r_\nu}{U_\nu} + \frac{r_\mu}{U_\mu}}{x_\mu^2 - x_\nu^2}\,. \tag{3.7}$$

Then calculating the left hand side of (3.5) (using (1.118)) one recovers (3.5)

$$\begin{aligned} k_{(j)\,a}{}^b \nabla_b R &\overset{a=\nu}{=} \sum_\mu A_\nu^{(j)}\, \partial_\nu \frac{r_\nu}{U_\nu} \\ &= \frac{r'_\nu\, A_\nu^{(j)}}{U_\nu} + 2x_\nu A_\nu^{(j)} \sum_{\mu\neq\nu} \frac{\frac{r_\nu}{U_\nu} + \frac{r_\mu}{U_\mu}}{x_\mu^2 - x_\nu^2} = \nabla_a R_{(j)}\,, \end{aligned} \tag{3.8}$$

exactly as required.

However if we want to know how the operators (3.1) transform under a Weyl re-scaling $g \to \Omega^2 g$ we need an expression for them built entirely from geometric quantities not just their derivatives. Here we amend this situation. That is to say, we show in the appendix B.1 that $R_{(j)}$ are given in terms of the principal Killing–Yano tensor, its symmetry descendants, and the curvature tensor by the following covariant formula:

$$\begin{aligned} R_{(j)} &= k_{(j)}^{ab} R_{ab} + \frac{D-4}{2(D-2)} \Box\, \mathrm{Tr}\big(k_{(j)}\big) + \alpha_j k_{(j-1)}^{ac} h_c{}^b\, (d\xi)_{ab} - \beta_j l_{(j-1)}^a\, \xi_a \\ &= k_{(j)}^{ab} R_{ab} + \frac{D-4}{2(D-2)} \Box\, \mathrm{Tr}\big(k_{(j)}\big) - k_{(j-1)}^{ab} \Big(\alpha_j h_a{}^c\, (d\xi)_{cb} + \beta_j \xi_a\, \xi_b\Big)\,, \end{aligned} \tag{3.9}$$

where $\xi = l_{(0)}$ is the primary Killing vector, for $j = 0$ we defined $k_{(-1)} \equiv 0 \equiv l_{(-1)}$, and the constants α_j and β_j are given by

$$\alpha_j = \frac{(n - j + \frac{\epsilon}{2})}{(n - 1 + \frac{\epsilon}{2})} \quad \beta_j = 2\frac{(n - j + \frac{\epsilon}{2})}{(n - 1 + \frac{\epsilon}{2})}(2j - 3)\,. \tag{3.10}$$

3.1 Symmetry Operators in Conformally Related Spacetimes

As mentioned in section 2.3, the wave equation (2.1) is conformally invariant. That is, provided we have a solution Φ in the spacetime g, then

$$\widetilde{\Phi} = \Omega^w \Phi\,, \quad w = 1 - D/2 \tag{3.11}$$

is a solution of the same equation in the conformally rescaled spacetime

$$\widetilde{g} = \Omega^2 g\,. \tag{3.12}$$

Notice in this way we can think of the conformal wave equation as scaling as an operator (under the transformation (3.12)) in the following sense (c.f. (2.48))

$$\widetilde{\Box}_R = \Omega^{w-2} \circ \Box_R \circ \Omega^{-w}\,. \tag{3.13}$$

It is interesting to ask if such R-separability can also be intrinsically characterized by some complete set of mutually commuting operators. In what follows we explicitly construct such operators and discuss their properties. First, starting from the special conformal frame with $\Omega = 1$, we scale the operators $\{\mathcal{K}_{(j)}, \mathcal{L}_{(j)}\}$ to construct a complete set of mutually commuting operators for the metric $\widetilde{g}$, (3.12). Second, following [118], we show that such operators can in fact be lifted to conformally invariant operators. Thence, providing a complete set of, conformally invariant, mutually commuting operators for the conformal wave equation (2.1), in any spacetime related to the Kerr–NUT–(A)dS metric by a conformal transformation.

3.1.1 Mutually Commuting Operators

Starting from the mutually commuting operators $\{\mathcal{K}_{(j)}, \mathcal{L}_{(j)}\}$ in the special frame with $\Omega = 1$, let us define new operators $\{\widetilde{\mathcal{O}}_{(j)}, \widetilde{\mathcal{P}}_{(j)}\}$ for general Ω by requiring the following conformal rescaling properties.

$$\widetilde{\mathcal{O}}_{(j)} \equiv \Omega^w \circ \mathcal{K}_{(j)} \circ \Omega^{-w}\,, \quad \widetilde{\mathcal{P}}_{(j)} \equiv \Omega^w \circ \mathcal{L}_{(j)} \circ \Omega^{-w}\,. \tag{3.14}$$

By construction such operators mutually commute, as we have

$$\left[\widetilde{\mathcal{O}}_{(i)}, \widetilde{\mathcal{O}}_{(j)}\right] = \Omega^w \circ \left[\mathcal{K}_{(i)}, \mathcal{K}_{(j)}\right] \circ \Omega^{-w} = 0\,, \tag{3.15}$$

$$\left[\widetilde{\mathcal{O}}_{(i)}, \widetilde{\mathcal{P}}_{(j)}\right] = \Omega^w \circ \left[\mathcal{K}_{(i)}, \mathcal{L}_{(j)}\right] \circ \Omega^{-w} = 0\,, \tag{3.16}$$

$$\left[\widetilde{\mathcal{P}}_{(i)}, \widetilde{\mathcal{P}}_{(j)}\right] = \Omega^w \circ \left[\mathcal{L}_{(i)}, \mathcal{L}_{(j)}\right] \circ \Omega^{-w} = 0\,. \tag{3.17}$$

Moreover, it follows that when Φ satisfies the eigenvalue problem (2.34) in the spacetime g, $\widetilde{\Phi} = \Omega^w \Phi$ given by (2.49) obeys the associated eigenvalue problem:

$$\widetilde{\mathcal{O}}_{(j)} \widetilde{\Phi} = C_j \widetilde{\Phi}\,, \quad \widetilde{\mathcal{P}}_{(j)} \widetilde{\Phi} = \Psi_j \widetilde{\Phi}\,, \tag{3.18}$$

in the conformal spacetime $\widetilde{g}$. In other words, the operators $\{\widetilde{\mathcal{O}}_{(j)}, \widetilde{\mathcal{P}}_{(j)}\}$, (3.14), intrinsically characterize the separability of the conformal wave equation in the conformal spacetime (3.12).

The only difficulty with (3.14) is that the new operators $\{\widetilde{\mathcal{O}}_{(j)}, \widetilde{\mathcal{P}}_{(j)}\}$ remain expressed in terms of the old connection ∇_a, the old Ricci tensor R_{ab}, and other objects associated with the metric g rather than the conformally rescaled metric $\widetilde{g}$. Ideally we would like to write this in a frame independent way. However, using the well known transformation properties of the connection and curvature tensor, one can straightforwardly amend this situation.

For example, let us define the following tilded objects:[1]

$$\widetilde{k}^{ab}_{(j)} = \Omega^{-2} k^{ab}_{(j)}\,, \quad \widetilde{h}_{ab} = \Omega^2 h_{ab}\,, \quad \widetilde{l}^a_{(j)} = \Omega^{-2} l^a_{(j)}\,, \tag{3.19}$$

and raise or lower their indices with the metric $\widetilde{g}$ and its inverse. We further denote by $\widetilde{\nabla}_a$ the covariant derivative in the spacetime $\widetilde{g}$ and by $\widetilde{R}_{ab}$ its Ricci tensor.

[1]We stress that these objects are not the conformal symmetries of the spacetime $\widetilde{g}$, although it is possible to define such symmetries. Namely, (as mentioned in the previous chapter) the following objects:

$$k^{ab}_{(j>0)}\,, \quad \Omega^3 h_{ab}\,, \quad l^a_{(j\geq 0)}\,,$$

are the conformal Killing tensors, conformal Killing–Yano 2-form, and conformal Killing vectors of the spacetime $\widetilde{g}$. Notice that in doing this, necessarily $k_{(0)} = g$ transforms differently to the rest of the Killing tensors. One could, of course, use these objects to define the transformed operators, leading to different (seemingly more complex) expressions. We will adopt this strategy for the Killing tensors at least in the next section (3.1.2).

Using these definitions, and adding (the identically zero in the $\Omega = 1$ frame[2]) quantities

$$\widetilde{\nabla}_b \left[\widetilde{k}^{ab}_{(j)} + \frac{1}{2} \widetilde{k}^{c}_{(j)\,c} \widetilde{g}^{ab} \right] \quad \text{and} \quad \widetilde{\nabla}_a \widetilde{l}^{a}_{(j)} \tag{3.20}$$

to the operators in (3.1), we find that operators (3.14) can be expressed as follows (see appendix B.2 for details):

$$\widetilde{\mathcal{O}}_{(j)} := \Omega^2 \left(\widetilde{\mathcal{K}}_{(j)} + \eta \left[\left(\widetilde{\nabla}_a \widetilde{\nabla}_b \big(\widetilde{k}^{ab}_{(j)} + \frac{1}{2} \widetilde{k}^{c}_{(j)\,c} \widetilde{g}^{ab} \big) \right) \right] \right) , \tag{3.21}$$

$$\widetilde{\mathcal{P}}_{(j)} := \Omega^2 \left(\widetilde{\mathcal{L}}_{(j)} - \frac{w}{D-2} \widetilde{\nabla}_a \widetilde{l}^{a}_{(j)} \right) , \tag{3.22}$$

where $\widetilde{\mathcal{K}}_{(j)}$ and $\widetilde{\mathcal{L}}_{(j)}$ are given by the expressions (3.1) and (3.9), with all the objects replaced by the tilded ones.

Moreover, $\widetilde{\mathcal{O}}_{(0)}$ is just a conformally rescaled $\widetilde{\mathcal{K}}_{(0)}$,

$$\widetilde{\mathcal{K}}_{(0)} = \Omega^{-2} \circ \widetilde{\mathcal{O}}_{(0)} = \Omega^{w-2} \circ \mathcal{K}_{(0)} \circ \Omega^{-w} \,, \tag{3.23}$$

highlighting the conformal invariance of this operator. The other operators, however, take a more complicated form, as is to be expected from the privileged role of the conformal frame with $\Omega = 1$[3]. Also one needs to know which conformal frame one is in to be able to write (3.21) and (3.22). Ideally one can characterize things in a totally frame independent manner. This is the subject of the next subsection.

3.1.2 Conformal Symmetry Operators

Conformal symmetry operators for the conformal wave equation in general settings have been studied for many years, see e.g. [115, 121, 154, 156, 197–202]. This work culminated in ref. [118] where a complete and constructive theory was finally formulated. Our goal for the remainder of this chapter is to review this theory in a more physics community oriented language, and briefly discuss how it applies to the problem at hand.

Essentially the work [118] deals with the process of a “quantization map” which takes any symmetric tensor and promotes it to an operator. In fact it deals with conformally

[2]These extra quantities vanish when the Killing vector and tensor equations (1.60), (1.62) hold, which is only when $\Omega = 1$.

[3]This is the only frame where the spacetime admits full (not only conformal) Killing tensors [196] and the Ricci tensor is diagonal in the natural orthonormal frame.

invariant quantization maps which do this in a conformally invariant way. Let us now include a few technical details before returning to the problem at hand. To be precise we will follow the notation/definitions of [118].

The best, well most "invariant", way to deal with the conformal rescaling we have been talking about is to promote scalar functions to scalar densities of weight λ and talk about operators mapping between different weights. That is to say, if $\mathcal{F}_\lambda(\mathcal{M})$ denotes the space of scalar densities then given, $\phi \in C^\infty(\mathcal{M})$ consider the isomorphism $f : C^\infty(\mathcal{M}) \to \mathcal{F}_\lambda(\mathcal{M})$ defined by,

$$f : \phi \mapsto \phi_\lambda \equiv |g|^{\lambda/2}\phi\,. \tag{3.24}$$

Then consider conformally invariant linear differential operators between scalar densities of different weights λ, μ,

$$\mathcal{D}_{\lambda,\mu} : \mathcal{F}_\lambda \to \mathcal{F}_\mu\,. \tag{3.25}$$

Notice that by using the square root of the metric determinant a scalar density of weight λ transforms, under a Weyl rescaling $g_{ab} \to \tilde{g}_{ab} = \Omega^2 g_{ab}$ as

$$\tilde{\phi}_\lambda = \Omega^{\lambda D/2}\phi_\lambda\,. \tag{3.26}$$

Thus by working with the one object (the scalar densities) one can deal with the entire conformal class at once. In particular, in this language the conformal Laplacian $\square_R : \mathcal{F}_{\lambda_0} \to \mathcal{F}_{\mu_0}$ maps scalar densities $\phi_{\lambda_0} \to \phi_{\mu_0}$ of weights $\lambda_0 = -w/D\,, \quad \mu_0 = (2-w)/D\,.$

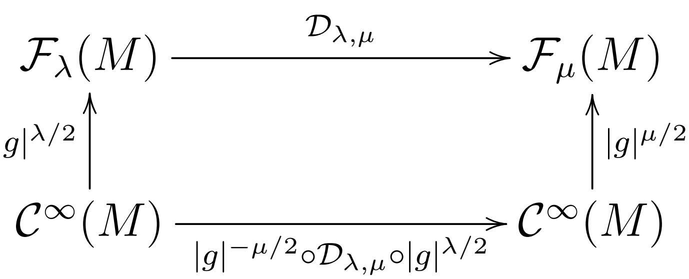

Figure 3.1: Commuting diagram representing how to construct conformally invariant operators from maps between scalar densities of different weights.

If D is an operator in the space of differential operators of order k, $\mathcal{D}^k_{\lambda,\mu}(\mathcal{M})$, then one can consider the principal symbol of D. This is a map $\sigma_k : \mathcal{D}^k_{\lambda,\mu}(\mathcal{M}) \to S_\delta(M)$, where $S_\delta(M)$ is the space of symmetric tensor densities of weight $\delta = \mu - \lambda$, such that if the highest order term of D is, in local coordinates, $D^{a_1\dots a_k}\partial_{a_1}\dots\partial_{a_k}$

$$\sigma_k(D) = D^{a_1\dots a_k}\,. \tag{3.27}$$

Then a quantization map $\mathcal{Q}_{\lambda,\mu}$, of order k, is the linear bijection between the space of differential operators $\mathcal{D}^k_{\lambda,\mu}$ such that

$$\sigma_k\left(\mathcal{Q}_{\lambda,\mu}\left(S\right)\right) = S\,. \tag{3.28}$$

Finally, it has been proved that there exists a unique such quantization map [199] that is natural (ie. respects the pushforward of any diffeomorphisms on $\mathcal{M}$) and conformally invariant. See [118] for the explicit construction of first and second order quantization maps and the proofs of the following statements exploiting the conformal geometry.

In any case, this whole process of constructing densities (3.26) defines a commutative diagram (see Figure 3.1) and so one can naturally use the $\mathcal{D}_{\lambda,\mu}$ to define a differential map between functions. That is to say, set $\mathfrak{D}_{s_\mu,s_\lambda} \equiv |g|^{-\mu/2}\circ\mathcal{D}_{\lambda,\mu}\circ|g|^{\lambda/2}$, which thence transforms as:

$$\widetilde{\mathfrak{D}}_{s_\lambda,s_\mu} \equiv \Omega^{s_\mu}\circ\mathfrak{D}_{s_\lambda,s_\mu}\circ\Omega^{s_\lambda} \tag{3.29}$$

when $g_{ab}\to\widetilde{g}_{ab}=\Omega^2 g_{ab}$ and where $s_\mu=-D\lambda$, $s_\mu=-D\mu$. Thus we can use the quantization map at the level of operators between functions (even if this may not be the natural mathematical setting).

In what follows, we are going to concentrate on conformal operators of equal weights, $s_1=s_2=s$, acting on scalars. In particular, as shown in [118][4] the previously mentioned conformal quantization map, now with weight s, that is built out of a symmetric tensor K^{ab} is given by[5]

$$\begin{aligned} Q_s(K) =& \nabla_a K^{ab}\nabla_b + \Big(\gamma_1[\nabla_a K^{ab}] + \gamma_2[\nabla^b\,\mathrm{Tr}\,K]\Big)\nabla_b + \gamma_3(\nabla_a\nabla_b K^{ab}) + \gamma_4(\Box\,\mathrm{Tr}\,K) \\ &+ \gamma_5\,R_{ab}K^{ab} + \gamma_6\,R\,\,\mathrm{Tr}\,K + f\;. \end{aligned} \tag{3.30}$$

Here $f\in C^\infty(\mathcal{M})$ so does not scale under conformal transformation, and we assume also that $\widetilde{K}^{ab}=K^{ab}$, and the coefficients are

$$\begin{aligned} &\gamma_1 = 2\gamma_2 = -\frac{(2s+D)}{D+2}\,,\ \gamma_3 = \frac{(s-1)s}{(D+1)(D+2)}\,,\quad \gamma_4 = \frac{s(D+2s-1)}{2(D+1)(D+2)}\,,\\ &\gamma_5 = \frac{s(D+s)}{(D-2)(D+1)}\,,\quad \gamma_6 = -\frac{2s(D+s)}{(D-2)(D-1)(D+1)(D+2)}\,. \end{aligned} \tag{3.31}$$

Similarly, given a vector L^a, the corresponding conformal operator is given by

$$Q_s(L) = L^a\nabla_a - \frac{s}{D}(\nabla_a\,L^a)\;. \tag{3.32}$$

[4]In [118] they deal with the general case, of unequal weights, which we do not need.

[5]Since we deal with differential operators on functions we use the symbol Q here rather than $\mathcal{Q}$.

In particular, we consider conformal operators of weight $w = 1 - D/2$, c.f. (3.11),

$$\widetilde{Q}_w = \Omega^w \circ Q_w \circ \Omega^{-w}\,, \tag{3.33}$$

that are *symmetry* operators of the conformal wave operator $\mathcal{K}_{(0)}$, that is, they satisfy the following relation:

$$\mathcal{K}_{(0)} \circ Q_w = \mathcal{D} \circ \mathcal{K}_{(0)}\,, \tag{3.34}$$

for some operator $\mathcal{D}$; in fact, it is easy to see that the conformal invariance implies $\mathcal{D} \equiv \mathcal{D}_{-2+w}$. Note that the equation (3.34) obviously preserves the kernel of $\mathcal{K}_{(0)}$.

To find such symmetry operators we can use the following theorem [118]:

Theorem 3.1.1 (Michel, Radoux, Şilhan) *Let K^{ab} be a (special) Killing tensor of the metric g, so that the following conformally invariant geometric obstruction built from the Weyl tensor C_{abcd}:*

$$Obs(K)_a = \frac{2(D-2)}{3(D+1)}\Big(\nabla_b K^{cd} C^b{}_{cda} - \frac{3}{D-3} K^{cd} \nabla_b C^b{}_{cda}\Big) \tag{3.35}$$

is exact, that is,

$$\mathbf{Obs}(K) = -2df\,. \tag{3.36}$$

Then (3.30) with f given by (3.36) (up to a constant) is a symmetry operator for the conformal wave operator and in fact satisfies

$$\mathcal{K}_{(0)} \circ Q_w(K) = Q_{-2+w}(K) \circ \mathcal{K}_{(0)}\,. \tag{3.37}$$

When K^{ab} is a Killing tensor we can simplify the operator given by (3.30) via the Killing equation,

$$\nabla_{(a} K_{bc)} = 0\,, \tag{3.38}$$

however this will only hold for a particular metric of the conformal class. For this particular metric, we then have

$$\begin{aligned} Q_w(K) &= Q_{w-2}(K) \\ &= \nabla_a K^{ab} \nabla_b - \frac{(D-2)}{8(D+1)}\left[\Box \operatorname{Tr} K\right] - \frac{(D+2)}{4(D+1)} R_{ab} K^{ab} + \frac{R \operatorname{Tr} K}{2(D+1)(D-1)} + f\,. \end{aligned} \tag{3.39}$$

In this case, therefore the corresponding symmetry operator (3.34) actually commutes with the conformal wave equation

$$\left[Q_w, \mathcal{K}_{(0)}\right] = 0\,, \tag{3.40}$$

and more generally, we have the conformal commutation

$$\left[\widetilde{Q}_w,\, \Omega^2\, \widetilde{\mathcal{K}}_{(0)}\right] = 0\,, \tag{3.41}$$

valid in any conformal frame.

In particular, taking the Killing tensors $k_{(j)}$ ($j>0$) in the Kerr–NUT–(A)dS metric g, we find that they satisfy the obstruction condition (3.36) with $f_{(j)}$ given by

$$\begin{aligned} f_{(j)} =& \frac{1}{4(1-D^2)}\Big[2D\, k^{ab}_{(j)} R_{ab} + 3\Box \operatorname{Tr} k_{(j)} + (D+1)(D-2)k^{ab}_{(j-1)}\Big(\alpha_j h_a{}^c\,(d\xi)_{cb} + \beta_j \xi_a\, \xi_b\Big) \\ & - 2R\, \operatorname{Tr} k_{(j)}\Big]\,. \end{aligned} \tag{3.42}$$

It can then easily be checked that[6] the corresponding operators, from (3.30), given by

$$\mathcal{K}^{(j)}_w \equiv Q_w(k_{(j)})\,, \tag{3.43}$$

coincide with the operators $\mathcal{K}_{(j)}$, (2.25),

$$\mathcal{K}^{(j)}_w = \mathcal{K}_{(j)}\,. \tag{3.44}$$

Since all of these operators commute with one another for $\Omega=1$, their conformal versions $\widetilde{\mathcal{K}}^{(j)}_w$, (3.33) also mutually commute in the spacetime $\widetilde{g}$. Of course, these are nothing else than the operators $\widetilde{\mathcal{O}}_{(j)}$, (3.14), this time, however, written in a conformally invariant way (3.30)[7]. The remaining commutation relations are then guaranteed by (3.41), since we define for $j=0$

$$\widetilde{\mathcal{K}}^{(0)}_w \equiv \widetilde{\mathcal{O}}_{(0)} = \Omega^2 \widetilde{\mathcal{K}}_{(0)}\,, \tag{3.45}$$

reflecting the fact that the metric transforms differently than the other Killing tensors under the conformal transformation.

[6] Of course, the expression (3.35) is only defined in this coordinate invariant way in the $\Omega=1$ frame although its coordinate expression will be unchanged no matter the frame.

[7] Although the formulae (3.21) and (3.30) look rather different, they represent the same operators, and in particular, the coordinate expressions for the operators $\widetilde{\mathcal{O}}_{(j)}$ and $\widetilde{\mathcal{K}}^{(j)}_w$ will coincide in any conformal frame. The apparent differences arise from how we choose to scale the Killing tensors.

Similarly one can define a quantization map for first order operators built out of vectors [118]. This will take $\mathcal{L}_{(j)}$, (2.26), to the conformal ones (as in (3.32) and c.f. (3.22) where the Killing vectors transform differently)

$$\mathcal{L}_w^{(j)} = -i\, l^a_{(j)} \nabla_a + i\frac{w}{D}(\nabla_a\, l^a_{(j)})\,, \tag{3.46}$$

where the second term identically vanishes in the frame $\Omega = 1$ when $l^a_{(j)}$ are now (full *not* conformal) Killing vectors. Of course, these will coincide with $\widetilde{\mathcal{P}}_{(j)}$, (3.22), in any coordinate system.

To summarize, we have found a conformally invariant generalization $\{\mathcal{K}_w^{(j)}, \mathcal{L}_w^{(j)}\}$ of the symmetry operators (2.25) and (2.26), with the two being equal in the Kerr-NUT–AdS conformal frame g. Writing $\widetilde{\Phi} = \Omega^w \Phi$ in any conformal frame $\widetilde{g}$, these operators obey the following eigenvalue problem:

$$\widetilde{\mathcal{K}}_w^{(j)} \widetilde{\Phi} = C_j \widetilde{\Phi}\,, \quad \widetilde{\mathcal{L}}_w^{(j)} \widetilde{\Phi} = \Psi_j \widetilde{\Phi}\,, \tag{3.47}$$

guaranteeing R-separability of $\widetilde{\Phi}$ in any of these frames.

3.2 Summary

In this chapter we have found covariant forms of the symmetry operators (2.25) and (2.26) of the conformal wave equation in the Kerr–NUT–(A)dS background (1.111). These operators are built out of the principal Killing–Yano tensor, its symmetry descendants, and the curvature tensor. Moreover their commutativity descends naturally from the commutation properties of the Killing tensors and the special character of the Ricci scalar functions $R_{(j)}$, (3.9). We then showed how to lift these to a full set of conformally invariant mutually commuting symmetry operators $\{\mathcal{K}_w^{(j)}, \mathcal{L}_w^{(j)}\}$ that guarantee R-separability of the conformal wave equation in any conformally related spacetime $\widetilde{g}$, providing thus a highly non-trivial example to the beautiful theory developed in [118].

The conformal wave equation (2.1) is characterized by a specific value of η. In principle one can consider more general wave equations, where η takes any value. It is easy to see that all such equations still separate in the Kerr–NUT–(A)dS backgrounds; the operators (2.25) and (2.26) commute for any value of η. However, for general η the corresponding wave equations are not conformally invariant and will not separate in a generic conformally related spacetime. In this case, one could use the conformal properties outlined in appendix B.2 to construct an equation which separates in the conformal spacetime. However there is no clear physical interpretation for such an equation. One other possible future direction is to use these results (and e.g. [117]) to understand separability of conformal fields with higher spin.

Part II

Separability of Vector Fields in Rotating Black Holes beyond General Relativity

This material in this part is derived (and taken) from the papers [203, 204] in which we separate the massive vector field equations (corresponding to ultralight vector bosons) in two spacetimes with a generalization of the principal tensor. Namely, we consider the Chong–Cvetič–Lü–Pope black hole spacetime of $D = 5$ minimal gauged supergravity (SUGRA) and also the low-energy heterotic string theory inspired $D = 4$ Kerr–Sen black hole. In each case the spacetime possesses a torsion generalization of the principal Killing–Yano tensor. Although this is a weaker object than the full principal tensor it still underlies the separability properties of these spacetimes. In particular, we are able to apply the Lunin–Frolov–Kubizňák–Krtouš ansatz employing the principal tensor with torsion.

I also present results comparing the superradiant instability modes of these ultralight massive vector bosons for weakly charged rotating black holes in Einstein–Maxwell gravity (the Kerr–Newman solution) to those of the Kerr–Sen black hole. The ordinary differential equations from the separation of variables are solved numerically, thanks to Ó.J.C Dias and J.E. Santos, to find the most unstable modes of the Proca field in the two backgrounds and compared to the vacuum (Kerr black hole) case.

Chapter 4

Separability of Vector Equations with Torsion

As mentioned in the introduction, the classical age of separatists followed the discovery of Carter's constant in the late '60s and '70s. This saw a flurry of separable solutions for the Kerr black hole. Similarly in the mid '00s, after the generalization of Kerr geometry to higher dimensions by Myers and Perry [68], the renaissance came, and there was a string of integrability and separability results for geodesic motion [79], and test scalar [34] and spinor [86] fields to arbitrary dimensions—all thanks to the principal tensor. However, the appropriate separation scheme for vector and tensor in dimensions $D > 4$ remained elusive, due to the fact that the methods of Teukolsky [40, 41] do not generalize to higher dimensions [205, 206]. Moreover, for massive vectors fields they do not even work in four dimensions [207, 208].

The motivation for studying massive vector fields comes from their potential physical relevance as ultralight bosons. Ultralight bosons feature in many different extensions of the standard model, such as string theory [209], and provide compelling candidates for explaining dark matter [210]. One particular model for these ultralight bosons is a massive spin-1 particle known as the *Proca field*. Considered first by Proca [211] as a way to understand short-range nuclear forces in flat spacetime (see also [212, 213]), the Proca equation is presently an integral part of the Standard Model where it is used for describing the massive spin-1 Z and W bosons, as well can be generalized to curved spacetime, e.g. [214].

A breakthrough for massless vector fields came in 2017 when Lunin [89] demonstrated the separability of Maxwell's equations in Myers–Perry-(A)dS geometry [70]. Lunin's approach was novel in that it provided a separable ansatz for the vector potential rather than the field strength, a method that had previously seen success in $D = 4$ dimensions [40, 41]. In 2018, Frolov, Krtouš, and Kubizňák showed that Lunin's ansatz can be written in a covariant form, in terms of the principal tensor [90–92], allowing them to extend Lunin's result to general (possibly off-shell) Kerr–NUT–(A)dS spacetimes [71]. The separation of massive vector (*Proca*) field perturbations in these spacetimes (an achievement previously absent even for the four-dimensional Kerr geometry) followed shortly after that [91], see also [93, 215, 216].

The separability of the vector field hinges on the existence of the principal tensor. Such a tensor: i) determines the canonical (preferred) coordinates in which the separation occurs; ii) generates the towers of explicit and hidden symmetries linked to the symmetry operators of the separated vector equation; and iii) explicitly enters the separation ansatz for the vector potential P. Namely, this ansatz can be written in the following covariant form:

$$P^a = B^{ab}\nabla_b Z\,, \quad B^{ab}(g_{bc} + i\mu h_{bc}) = \delta^a_c\,, \tag{4.1}$$

where, as shown, the tensor B is determined by the principal tensor h and the metric g, with μ a separation constant.

The solution for the scalar functions Z is then sought in a standard multiplicative separable form, e.g. (1.84). We will denote the ansatz with the B tensor (4.1) as the *Lunin–Frolov–Krtouš–Kubizňák (LFKK) ansatz*. Remarkably, the LFKK ansatz works equally well for both massless and massive vector perturbations. It is also valid in *any* dimension, even or odd. Finally, the corresponding symmetry operators for the scalar potential Z were also recently found [180] thus completing this story[1].

The goal of this section is to apply this ansatz to massive vector fields in spacetimes that possess only the weaker structure of the principal tensor with torsion [100, 102, 218]. In this first chapter we will consider the mathematical separability for massive vector fields while in the next chapter we will motivate this physically with an example application. We will discus this in more detail below but let us now briefly review how the LFFK ansatz works in the Kerr–NUT–(A)dS spacetimes.

[1] See also [96] for the extension to p-forms and [217] for an alternative way of separating the Maxwell equations in the Wahlquist metrics.

4.1 Separability of Proca Equation in Kerr–NUT–(A)dS Spacetimes

Although the original work only presented the results in even dimensions we will display the formulae in arbitrary even or odd dimensions so that we may then compare the five-dimensional Kerr–NUT–(A)dS result to the five-dimensional SUGRA black hole. As usual we work in $D = 2n + \epsilon$ dimensions and now are interested in finding a general solution of the Proca equation [211]

$$\nabla_a F^{ab} - m^2 P^b = 0\,, \tag{4.2}$$

in the background (1.111), where $F = dP$. This equation represents a massive vector field. Due to the mass term the equation no longer has the usual $U(1)$ gauge symmetry $P \to P + d\lambda$. However an immediate consequence of (4.2) is the "Lorenz condition",

$$\nabla_a P^a = 0\,, \tag{4.3}$$

following from the antisymmetry of F.

Employing the LFKK ansatz (4.1) we seek the solution in the separated form

$$Z = \prod_{\nu=1}^{n} R_\nu\left(x_\nu\right) \exp\Bigl(i \sum_{j=0}^{n-1+\epsilon} L_j \psi_j\Bigr)\,. \tag{4.4}$$

Following [90,91], one can show the Lorenz condition (4.3) provides an explicit linear ODE for the functions $R_\nu(x_\nu)$ appearing in the separation ansatz (4.4). Namely,

$$0 = \nabla_a P^a = \frac{Z}{A} \sum_{\nu=1}^{n} \frac{A_\nu}{U_\nu} \frac{1}{R_\nu} \mathcal{D}_\nu R_\nu\,, \tag{4.5}$$

where[2]

$$A = \prod_{\mu=1}^{n} q_\mu\,, \quad A_\mu = \prod_{\nu\neq\mu}^{n} q_\nu\,, \text{ and, } q_\nu = 1 - \mu^2 x_\nu^2\,. \tag{4.6}$$

[2] Notice A and A_μ are generating functions for the symmetric polynomials (1.112) that enter the metric (1.111). That is,

$$A^{(j)} = (-1)^j \frac{d^j}{d(\mu^2)^j} A\,, \text{ and, } A^{(j)}_\mu = (-1)^j \frac{d^j}{d(\mu^2)^j} A_\mu\,.$$

So that

$$A = \sum_{j=0} (-\mu^2)^j A^{(j)} \quad A_\mu = \sum_{j=0} (-\mu^2)^j A^{(j)}_\mu\,.$$

The differential operator $\mathcal{D}_\nu$ is given by

$$\mathcal{D}_\nu = \frac{q_\nu}{x_\nu^\epsilon}\partial_\nu\Big[\frac{X_\nu x_\nu^\epsilon}{q_\nu}\partial_\nu\Big] - \frac{1}{X_\nu}\Big[\sum_{j=0}^{n-1+\epsilon}(-x_\nu^2)^{N-1-j}L_j\Big]^2 + \mu\big(\frac{2}{q_\nu}+\epsilon-1\big)\sum_{j=0}^{n-1+\epsilon}(-\mu^2)^{j+1-n}L_j + \epsilon\frac{L_n^2 q_\nu}{c^2 x_\nu^2}\,. \quad (4.7)$$

The Lorenz condition (4.3) can be solved if the differential operators satisfy

$$\mathcal{D}_\nu R_\nu = f_\nu R_\nu\,, \quad (4.8)$$

where the polynomials f_ν are given by

$$f_\nu = \sum_{k=0}^{n-1} C_k(-x_\mu^2)^{n-2-k}\,, \quad (4.9)$$

and are characterized by $(n-1)$ separation constants $C_j, j = 0,\dots,n-2$. In this case, using (2.31), $\nabla_a P^a = 0$ implies

$$\sum_{\nu=1}^{n}\frac{A_\nu f_\mu}{U_\nu} = \sum_{j=0}^{n}(-\mu^2)^j\sum_{k=0}^{n-1}C_k\sum_{\nu=1}^{n}\frac{A_\nu^{(j)}(-x_\nu^2)^{n-2-k}}{U_\nu} = \sum_{j=0}^{n-1}C_j(-\mu^2)^j = 0\,. \quad (4.10)$$

That is, the separation constants are subject to the constraint

$$\sum_{j=0}^{n-1}C_j(-\mu^2)^j = 0\,. \quad (4.11)$$

One can further show (in a coordinate independent manner) that the Proca equation takes the form [90,93]

$$\nabla_b F^{ba} - m^2 P^a = B^{ab}\nabla_b J\,, \quad (4.12)$$

where the scalar J is given by

$$J = \Box Z - 2i\mu\xi_a B^{ab}\partial_b Z - m^2 Z\,. \quad (4.13)$$

Note here the $\Box$ operator is the ordinary scalar $\nabla_a g^{ab}\nabla_a$ wave operator that we saw in the previous section and has the explicit expression (2.7). Plugging everything in yields the following explicit coordinate dependent formula for J:

$$J = Z\left[-m^2 + \sum_{\nu=1}^{n}\frac{1}{U_\nu R_\nu}\mathcal{D}_\nu R_\nu\right]. \quad (4.14)$$

Then substituting (4.8) and (4.9) into (4.14) implies

$$m^2 - \sum_{k=0}^{n-1} C_k \sum_{\nu=1}^{n} \frac{(-x_\nu^2)^{n-1-k}}{U_\nu} = 0\,. \tag{4.15}$$

Finally, using the identities (2.31) this is zero if and only if we have

$$C_0 = m^2\,. \tag{4.16}$$

To summarize, the solution of the Proca equation (4.2) in the general Kerr–NUT–(A)dS spacetimes in all dimensions can be found in the LFKK form (4.1), (4.4), for the mode functions R_ν satisfying the ordinary differential equations (4.8). Such a solution is general in that it is characterized by $(D-1)$ independent separation constants $\{\mu, C_j, L_k\}$; the constant C_0 is not independent and is fixed by the mass of the vector according to equation (4.16). Moreover, in four dimensions all three massive polarizations can be extracted from this solution [215]. It remains to be seen whether the same remains true in a general dimension D, that is, whether all $D-1$ polarizations of the massive and $(D-2)$ polarizations of the massless vector field are encoded in this solution.

However useful the above results are, they have one drawback: the existence of the principal tensor is limited to the off shell Kerr–NUT–(A)dS metrics [77]. For this reason various generalizations of the notion of hidden symmetries, that would allow for more general spacetimes while preserving integrability features of the symmetry, have been sought. One such generalization, that of the Killing–Yano tensor with torsion [97–99], turns out to be quite fruitful. Such a symmetry exists in a number of supergravity and string theory spacetimes, where the torsion can be naturally identified with a defining 3-form of the theory. Although less restrictive, the principal tensor with torsion still implies the essential integrability features of its torsion-less cousin and underlies separability of Hamilton–Jacobi, Klein–Gordon, and torsion-modified Dirac equations on the black hole background.

With this in mind a natural question arises: is the vector field separability described above limited to the vacuum spacetimes? It is the purpose of the present section to show that it is not so. To this end, we first concentrate on a prototype non-vacuum black hole spacetime known to admit the principal tensor with torsion [99], the Chong–Cvetič–Lü–Pope black hole [60] of $D = 5$ minimal gauged supergravity. In the next chapter we consider the Kerr–Sen black hole [57]. We will show that the LFKK ansatz can be used for separability of the (properly torsion modified) vector equation in these background.

4.2 Killing–Yano Tensors with Torsion

Let us start by briefly recapitulating a "torsion generalization" [99–102] of Killing–Yano tensors which has applications for a variety of supergravity black hole solutions (See also [219] for a discussion Killing tensors in supergravity spacetimes).

In what follows we assume that the torsion is completely antisymmetric and described by a 3-form T, with the standard torsion tensor given by $T^d{}_{ab} = T_{abc}g^{cd}$, where g is the metric. The torsion connection ∇^T acting on a vector field X is, in this context, defined as

$$\nabla^T_a X^b = \nabla_a X^b + \frac{1}{2} T^b{}_{ac} X^c \,, \tag{4.17}$$

where ∇ is the Levi-Civita (torsion-free) connection. The connection ∇^T satisfies the metricity condition, $\nabla^T g = 0$, and has the same geodesics as ∇. It induces a connection acting on forms. Namely, let ω be a p-form, then

$$\nabla^T_X \omega = \nabla_X \omega - \frac{1}{2}\big(X \cdot T\big) \underset{1}{\wedge} \omega \,, \tag{4.18}$$

where we have used a notation of contracted wedge product introduced in [101], defined for a p-form α and q-form β as

$$(\alpha \underset{m}{\wedge} \beta)_{a_1 \ldots a_{p-m} b_1 \ldots b_{q-m}} = \frac{(p+q-2m)!}{(p{-}m)!(q{-}m)!}\, \alpha_{c_1 \ldots c_m [a_1 \ldots a_{p-m}} \beta^{c_1 \ldots c_m}{}_{b_1 \ldots b_{q-m}]} \,. \tag{4.19}$$

Equipped with this, one can define the following two operations:

$$d^T \omega \equiv \nabla^T \wedge \omega = d\omega - T \underset{1}{\wedge} \omega \,, \tag{4.20}$$

$$\delta^T \omega \equiv -\nabla^T \cdot \omega = \delta\omega - \frac{1}{2} T \underset{2}{\wedge} \omega \,. \tag{4.21}$$

Note that in particular, for $\omega = T$, we have $\delta^T T = \delta T$.

A *conformal Killing–Yano tensor with torsion* k is a p-form which for any vector field X satisfies the following equation [97–99]:

$$\nabla^T_X k - \frac{1}{p+1} X \cdot d^T k + \frac{1}{D-p+1} X \wedge \delta^T k = 0 \,, \tag{4.22}$$

where D stands for the total number of spacetime dimensions. In analogy with the Killing–Yano tensors defined with respect to the Levi-Civita connection, a conformal Killing–Yano tensor with torsion f obeying $\delta^T f = 0$ is called a *Killing–Yano tensor with torsion*, and a conformal Killing–Yano tensor with torsion h obeying $d^T h = 0$ is a *closed conformal Killing–Yano tensor with torsion.*

These conformal Killing–Yano tensors with torsion possess many remarkable properties of which the following three are especially important for generating other symmetries and separability of test field equations (see [99, 100] for the proof and other properties):

1. The Hodge star $\star$ maps conformal Killing–Yano with torsion p-forms to conformal Killing–Yano with torsion $(D-p)$-forms. In particular, the Hodge star of a closed conformal Killing–Yano with torsion p-form is a Killing–Yano with torsion $(D-p)$-form and vice versa.

2. Closed conformal Killing–Yano tensors with torsion form a (graded) algebra with respect to a wedge product. Namely, let h_1 and h_2 be a closed conformal Killing–Yano tensor with torsion p-form and q-form, respectively, then $h_3 = h_1 \wedge h_2$ is a closed conformal Killing–Yano with torsion $(p+q)$-form.

3. Let h and k be two (conformal) Killing–Yano tensors with torsion of rank p. Then

$$K_{ab} = h_{(a|c_1 \dots c_{p-1}|} k_{b)}{}^{c_1 \dots c_{p-1}} \tag{4.23}$$

   is a (conformal) Killing tensor of rank 2.

In what follows, we shall concentrate on a *principal tensor* with torsion, h, which is a non-degenerate closed conformal Killing–Yano with torsion 2-form. It obeys (c.f. with the principal CCKY tensor (1.129))

$$\nabla^T_X h = X \wedge \xi \,, \quad \xi = -\frac{1}{D-p+1} \delta^T h \,. \tag{4.24}$$

One key difference between this and the previously defined principal tensor is that, as we have seen, the existence of the principal tensor uniquely determines the Kerr–NUT–(A)dS class of black hole spacetimes [76–78] (see also [179]), but no full classification is available for spacetimes with torsion. Nor is it clear if such spacetimes have to admit any isometries (i.e. Killing vectors; in particular ξ need not be a Killing vector) [102]. On the other hand, several explicit example solutions with a principal tensor where the torsion is naturally identified with a 3-form of the particular theory (from which they are derived) are known. Among them, the D-dimensional Kerr–Sen spacetimes [100] and black holes of $D = 5$ minimal gauged supergravity [99] are what we focus on here.

Starting from one such tensor, one can generate (via the three properties above) the towers of Killing tensors, and (closed conformal) Killing–Yano tensors with torsion. In their turn, as for the torsion free case such symmetries can typically be associated with symmetry operators for a given field operator. For example, the principal tensor with torsion in the Chong–Cvetič–Lü–Pope and Kerr–Sen black hole guarantees the integrability of the geodesic motion [220–222], as well as separability of scalar [100, 220, 223] and modified Dirac [220], [100, 224, 225] equations. Our aim is to show that it also guarantees the separability of properly torsion modified (massive) vector field equations.

4.3 Black holes in Minimal Gauged Supergravity

The bosonic sector of $D = 5$ minimal gauged supergravity is governed by the Lagrangian [60]

$$\pounds = \star(R + \Lambda) - \frac{1}{2} F \wedge \star F + \frac{1}{3\sqrt{3}} F \wedge F \wedge A \,, \tag{4.25}$$

where Λ is the cosmological constant, A is the $U(1)$ gauge field and F its field strength. This yields the following set of Maxwell and Einstein equations:

$$dF = 0 \,, \quad d \star F - \frac{1}{\sqrt{3}} F \wedge F \;=\; 0 \,, \tag{4.26}$$

$$R_{ab} - \frac{1}{2}\Big(F_{ac}F_b{}^c - \frac{1}{6}\, g_{ab}F^2\Big) + \frac{1}{3}\Lambda g_{ab} \;=\; 0 \,. \tag{4.27}$$

In this case the torsion can be identified with the Maxwell field strength according to [99]

$$T = -\frac{1}{\sqrt{3}} \star F \,. \tag{4.28}$$

Having done so, the Maxwell equations can be written as

$$\delta^T T = 0 \,, \quad d^T T = 0 \,. \tag{4.29}$$

In other words, the torsion T is "T-harmonic". Here, the first equality follows from the fact that $\delta^T T = \delta T$, while the second is related to the special property in five dimensions (with Lorentzian signature),

$$d^T \omega = d\omega + (\star T) \wedge (\star \omega) \,, \tag{4.30}$$

which valid for any 3-form ω. The principal tensor equation (4.24) can now explicitly be written as

$$
\begin{aligned}
\nabla_c h_{ab} &= 2g_{c[a}\xi_{b]} + \frac{1}{\sqrt{3}}(\star F)_{cd[a}h^d{}_{b]}\,,\\
\xi^a &= \frac{1}{4}\nabla_b h^{ba} - \frac{1}{2\sqrt{3}}(\star F)^{abc}h_{bc}\,.
\end{aligned} \tag{4.31}
$$

A general doubly spinning charged black hole solution in this theory has been constructed by Chong, Cvetič, Lü, and Pope [60]. It can be written in a symmetric Wick-rotated form [99]:

$$
g = \sum_{\mu=1,2}\left(e^\mu e^\mu + \hat{e}^\mu\hat{e}^\mu\right) + \hat{e}^0\hat{e}^0\,, \tag{4.32}
$$

$$
A = \sqrt{3}c(A_1 + A_2)\,, \tag{4.33}
$$

where the orthonormal basis and metric functions are given by

$$
\begin{aligned}
e^1 &= \sqrt{\frac{U_1}{X_1}}\,dx_1\,, \quad \hat{e}^1 = \sqrt{\frac{X_1}{U_1}}(d\psi_0 + x_2^2 d\psi_1)\,,\\
e^2 &= \sqrt{\frac{U_2}{X_2}}\,dx_2\,, \quad \hat{e}^2 = \sqrt{\frac{X_2}{U_2}}(d\psi_0 + x_1^2 d\psi_1)\,,\\
\hat{e}^0 &= \frac{ic}{x_1x_2}\Big[d\psi_0 + (x_1^2+x_2^2)d\psi_1 + x_1^2x_2^2 d\psi_2 - x_2^2A_1 - x_1^2A_2\Big]\,,\\
A_1 &= -\frac{e_1}{U_1}(d\psi_0 + x_2^2 d\psi_2)\,, \quad A_2 = -\frac{e_2}{U_2}(d\psi_0 + x_1^2 d\psi_1)\,,\\
U_1 &= x_2^2 - x_1^2 = -U_2\,.
\end{aligned} \tag{4.34}
$$

The solution is stationary and axisymmetric, corresponding to three Killing vectors $\partial_{\psi_0}, \partial_{\psi_1}, \partial_{\psi_2}$, and possesses two non-trivial coordinates x_1 and x_2. Here we choose $x_2 > x_1 > 0$ and note that the metric written in this form has $\det g < 0$. We have also used a "symmetric gauge" for the $U(1)$ potential; the electric charge of the Maxwell field $F = dA$ depends on a difference $(e_1 - e_2)$.

In order to satisfy the Einstein–Maxwell equations, the metric functions take the following form:

$$
\begin{aligned}
X_1 &= A + Cx_1^2 - \frac{\Lambda}{12}x_1^4 + \frac{c^2(1+e_1)^2}{x_1^2}\,,\\
X_2 &= B + Cx_2^2 - \frac{\Lambda}{12}x_2^4 + \frac{c^2(1+e_2)^2}{x_2^2}\,,
\end{aligned} \tag{4.35}
$$

where of the four free parameters A, B, C, c only three are physical (one can be scaled away) and are related to the mass and two rotations. As usual, the separability property shown below does not need the special form (4.35) and occurs off-shell, for arbitrary functions

$$X_1 = X_1(x_1)\,, \quad X_2 = X_2(x_2)\,. \tag{4.36}$$

As shown in [99], the spacetime admits a principal tensor with torsion, which takes the form

$$h = \sum_{\mu=1,2} x_\mu e^\mu \wedge \hat{e}^\mu\,. \tag{4.37}$$

Interestingly, the torsion (4.28) in Chong–Cvetič–Lü–Pope spacetimes is very special as it satisfies the following conditions:

$$(\star F)_{d[ab}h^d{}_{c]} = 0\,, \quad (\star F)_{abc}h^{bc} = 0\,. \tag{4.38}$$

This implies that the tensor is not only d^T-closed (as it must be), but it is also d-closed and obeys:

$$d^T h = dh = 0\,, \quad \xi = -\frac{1}{4}\delta^T h = -\frac{1}{4}\delta h = \partial_{\psi_0}\,. \tag{4.39}$$

Therefore it can be locally written in terms of a potential

$$h = db\,, \quad b = -\frac{1}{2}\Big[(x_1^2 + x_2^2)d\psi_0 + x_1^2 x_2^2 d\psi_1\Big]\,. \tag{4.40}$$

Using the properties of closed conformal Killing–Yano tensors with torsion, the principal tensor generates a Killing–Yano with torsion 3-form $\star h$, and a rank-2 Killing tensor

$$K_{ab} = (\star h)_{acd}(\star h)_b{}^{cd} = h_{ac}h_b{}^c - \frac{1}{2}g_{ab}h^2\,. \tag{4.41}$$

Such symmetries are responsible for separability of the Hamilton–Jacobi, Klein–Gordon, and torsion-modified Dirac equations in these spacetimes.

4.4 Separability of Vector Perturbations

4.4.1 Troca Equation

Let us now proceed and consider a test massive vector field P on the above background. It is reasonable to expect that, similar to the Dirac case [99, 101], the corresponding Proca equation will pick up the a torsion generalization. In what follows we shall argue that the natural sourceless massive vector equation to consider is

$$\nabla \cdot \mathcal{F} - m^2 P = 0\,, \tag{4.42}$$

where m is the mass of the field, and the field strength $\mathcal{F}$ is defined via the torsion exterior derivative,

$$\mathcal{F} \equiv d^T P = dP - P \cdot T\,. \tag{4.43}$$

Being a torsion generalization of the Proca equation (4.2), we shall refer to the equation (4.42) as a *Troca equation*[3]. It implies the "Lorenz condition"

$$\nabla \cdot P = 0\,. \tag{4.44}$$

To motivate the above form of the Troca equation, we demand that it is linear in P, reduces to the Proca equation in the absence of torsion, and would obey the current conservation in the presence of sources. We have three natural candidates for generalizing the Maxwell operator $\nabla \cdot dP$, namely:

$$O_1 = \nabla \cdot d^T P\,, \quad O_2 = \nabla^T \cdot dP\,, \quad O_3 = \nabla^T \cdot d^T P\,. \tag{4.45}$$

However, the last two do not obey the current conservation equation. Indeed, due to $\nabla^T \cdot (\nabla^T \cdot \omega) \neq 0$ (for any form ω), we have $\nabla \cdot O_2 = \nabla^T \cdot O_2 \neq 0$, and similarly for O_3. So we are left with O_1 which, when extended to the massive case, yields the Troca equation (4.42).

Let us also note that the choice of operator O_1 is "consistent" with the Maxwell equation for the background Maxwell field. Namely, due to the identity

$$X \cdot \star\omega = \star(\omega \wedge X)\,, \tag{4.46}$$

valid for any vector X and any p-form ω, the field equations (4.26) can be written as

$$\begin{aligned} 0 &= d \star F - \frac{1}{\sqrt{3}} F \wedge F = d \star dA - d\Big(\frac{1}{\sqrt{3}} F \wedge A\Big) \\ &= d \star dA + d \star \Big(A \cdot \frac{1}{\sqrt{3}} \star F\Big) \\ &= d \star d^T A\,. \end{aligned} \tag{4.47}$$

That is, identifying A with the Proca field in the test field approximation gives

$$\nabla \cdot d^T P = 0\,, \tag{4.48}$$

which is the massless Troca equation (4.42) (upon treating the torsion as an independent field).

[3]Thanks to L.T. for the naming.

4.4.2 Separability

Having identified the Troca equation (4.42), let us now find its general solution in the supergravity background (4.32). To this purpose we employ the LFKK ansatz (4.1) and seek the solution in a separated form

$$Z = \prod_{\nu=1,2} R_\nu\,(x_\nu)\exp\Big[i\sum_{j=0}^{2} L_j\psi_j\Big]\,, \tag{4.49}$$

where $\{\mu, L_0, L_1, L_2\}$ are four separation constants.

As in the case without torsion, it is useful to start with the Lorenz condition (4.44). We find

$$\nabla_a P^a = Z\sum_{\nu=1,2}\frac{1}{U_\nu}\frac{\mathcal{D}_\nu R_\nu}{q_\nu R_\nu}\,, \tag{4.50}$$

where the differential operator $\mathcal{D}_\nu$ is now given by

$$\mathcal{D}_\nu = \frac{q_\nu}{x_\nu}\partial_\nu\left(\frac{X_\nu x_\nu}{q_\nu}\partial_\nu\right) - \frac{1}{X_\nu}\Big(\sum_{j=0}^{2}(-x_\nu^2)^{1-j}L_{j\nu}\Big)^2 + \frac{2\mu}{q_\nu}\sum_{j=0}^{2}(-\mu^2)^{j-1}L_{j\nu} + \frac{L_2^2 q_\nu}{c^2 x_\nu^2}\,, \tag{4.51}$$

and we have defined

$$q_\nu = 1-\mu^2 x_\nu^2\,,\quad L_{j\nu} = L_j(1+\delta_{j2}e_\nu)\,. \tag{4.52}$$

The latter definition is essentially the only difference when compared to the five-dimensional torsion-less case, c.f. (4.7).

In order to impose the Lorenz condition, this time we will use the LFKK separability lemma 2.1.1 from chapter 2. Thus, demanding $\nabla_a P^a = 0$, using the expression (4.50), and the above lemma for $n=2$ and $f_\nu = \mathcal{D}_\nu R_\nu/(q_\nu R_\nu)$, yields the separated equations

$$\mathcal{D}_\nu R_\nu = q_\nu f_\nu R_\nu\,, \tag{4.53}$$

where f_ν is given by (2.22) with $n=2$, that is, $f_\nu = C_0$.

With this at hand, let us now turn to the Troca equation (4.42). Using the ansatz (4.1) and the Lorenz condition (4.44), the L.H.S. of the Troca equation takes the following form:

$$\nabla_b\mathcal{F}^{ba} - m^2 P^a = B^{ab}\nabla_b J\,, \tag{4.54}$$

where the scalar J is, as in the Kerr–NUT–(A)dS case [90,91], c.f. (4.13), given by

$$J = \Box Z - 2i\mu\xi_a B^{ab}\partial_b Z - m^2 Z\,, \tag{4.55}$$

or more explicitly,

$$J = Z \sum_{\nu=1,2} \frac{1}{U_\nu R_\nu} \Big[\mathcal{D}_\nu - m^2 \left(-x_\nu^2 \right) \Big] R_\nu \,. \tag{4.56}$$

One can then repeat the argument presented above for the Maxwell case with $n = 2$ to find[4]

$$C_0 = m^2 \,, \quad C_1 = \frac{m^2}{\mu^2} \,. \tag{4.57}$$

Again C_0 is not an independent separation constant and, in the case of massless vectors, we can set $m = 0 = C_0$.

To summarize, we have shown that one can apply a separation of variables for the Troca equation (4.42) in the Chong–Cvetič–Lü–Pope black hole spacetime. The solution can be found in the form of the LFKK ansatz (4.1), where the scalar function Z is written in the multiplicative separated form (4.49), and the modes R_ν satisfy the ordinary differential equations (4.53) with $C_1 = m^2/\mu^2$. The obtained solution is general in that it depends on four independent separation constants $\{\mu, L_0, L_1, L_2\}$. It remains to be seen if, similar to the Kerr–NUT–(A)dS case [215], all polarizations (four in the case of massive field and three for $m = 0$) are captured by our solution.

4.5 Summary

The principal tensor is a very powerful object and proves key to separating the vector equations. In particular, one needs to concentrate *not* on the field strength (as previously thought) but rather employ a new LFKK separability ansatz (4.1) for the vector potential itself. The leads to new insights for massive vector fields in four dimensions where the previously much used Teukolsky methods using the Newman–Penrose formalism fails.

In this chapter we have shown that the applicability of the LFKK ansatz goes far beyond the realm previously expected. Namely, we have demonstrated the separability of the vector field equation in the background of the Chong–Cvetič–Lü–Pope black hole of minimal gauged supergravity. Such a black hole no longer possesses a principal tensor. However, upon identifying the Maxwell 3-form of the theory with torsion, a weaker structure, the principal tensor with torsion, *is* present. Remarkably, such a structure enters the LFKK ansatz in precisely the same way as the standard (vacuum) principal tensor and allows one to separate the naturally torsion modified vector (Troca) field equations: "*principal*

[4]There is actually a misapplication of the separability lemma 2.1.1 in the paper [203] stating $C_0 = m^2/\mu^2$ which I fix here.

tensor strikes again". This result opens future horizons for applicability of both the LFKK ansatz and the torsion modified principal tensor. In the next chapter we consider a physical application of this separability to the Kerr–Sen black hole.

Chapter 5

Massive Vector Fields in Kerr–Sen and Kerr–Newman Spacetimes

Although direct detection of dark matter proves to be very difficult, recently a new line of investigation has opened up with a flurry of papers considering the interplay of these ultralight bosons and *superradiance* from black holes [53,175,226–229]. In particular, it has been shown that the instabilities from these superradiant modes can, in principle, be used to detect beyond Standard Model particles [175] and put bounds on the potential masses of dark matter candidates, e.g. [227]. For example, in the LIGO/LISA era ultralight bosons and superradiance can leave signatures in the signals of detected gravitational waves [226, 230–232]; see [53] for recent fully relativistic calculation of the resulting gravitational wave signals. The first step towards studying these instabilities is to consider test fields on a background spacetime. Naturally, almost all the previous studies have focused on the Kerr rotating black hole of Einstein's general relativity likely to be the most astrophysically relevant.

However, taking seriously the low energy limits of string theory leads to new kinds of black holes, and, if one is extending the Standard Model to include ultralight bosons it is a natural question to ask how the superradiant instabilities generated by these particles are modified by extensions to general relativity. Such extensions lead to black holes that typically carry extra fields and charges. In the astrophysical $D = 4$ dimensions the Kerr–Sen geometry [57] arises from the low energy limit of heterotic string theory. It represents a black hole with mass M and $U(1)$ charge Q and contains two extra background fields; the scalar dilaton Φ and the 3-form H. In the limit where these two fields vanish the spacetime reduces to the Kerr black hole.

On the other hand, this spacetime should be compared to the Kerr–Newman solution [13] which is the unique stationary black hole solution to the Einstein equations with $U(1)$ charge (it can also be understood as a solution of $N = 2$, $D = 4$ supergravity). Both Kerr–Newman and Kerr–Sen black holes are stationary and axisymmetric spacetimes, possessing two Killing vectors that aid in understanding the behaviour of test fields in these backgrounds. In the Kerr–Newman case there exists an additional hidden symmetry of the *principal* Killing–Yano tensor, which gives rise to Carter's constant for charged geodesics [37]. For the Kerr–Sen spacetime only a *generalized principal tensor* with torsion exists, which is a weaker but still rather useful structure [100]. However as we have seen in the previous chapter this still proves very useful for the integrability of geodesics and separability of fields.

The aim of this chapter is to present results on the *superradiant instabilities* of the Kerr–Sen and Kerr–Newman black holes, as triggered by the ultralight massive bosons. These are well understood in the case of massive scalar fields, see [233] and [234], but the corresponding study for massive vectors is currently missing. The reason is simple, since as mentioned previously, the methods of Teukolsky did not work [205, 206]. Even for vacuum (Kerr) black holes, the corresponding Proca equations have only recently been separated [89] and the corresponding instabilities studied [91]. As a consequence the problem was investigated either using approximations [207, 208, 227] or employing serious numerical analysis [50, 51, 228].

However, as we have seen in the previous chapter there is a new window into the separability for vector fields due to the LFKK ansatz (4.1) by Lunin [89], and simplified and written in covariant form by Frolov–Krtouš–Kubizňák [90, 91]. We also saw that this can be applied to study the Proca equations in the background of the Chong–Cvetič–Lü–Pope black hole of $D = 5$ minimal gauged supergravity [60] where, as for Kerr–Sen, the principal tensor must be generalized to the case with torsion and is a weaker construction [99, 149]. Despite these significant differences, the LFKK ansatz still applies and the (torsion) modified Proca equations decouple and separate [203].

In this chapter, we will apply the LFKK ansatz to separate the Proca equations in the Kerr–Sen black hole background. Similar to the black hole of minimal gauged supergravity, the corresponding Proca equations have to be modified in the presence of torsion, now naturally identified with the 3-form field H. Again we will call this the Troca equation. To account for the dilaton field Φ, we work in the string frame. The corresponding unstable superradiant modes were studied numerically by Ó.J.C Dias and J.E. Santos in ref. [204] from which this chapter is derived. This was done in an astrophysically viable situation where the black holes are fast spinning (close to extremal) and weakly charged [235, 236]. Here I reproduce the most relevant figure and physical discussion, but not the numerical details, as I was not involved in this work. The interested reader is referred to the original paper [204] for the details.

5.1 Kerr–Sen and Kerr–Newman Black Holes

In this section we present the Kerr–Sen and the Kerr–Newman metrics, the background spacetimes in which we will study the instability modes of the Proca equation. The two metrics we consider both describe rotating and charged black hole spacetimes. However, there are some key differences in the Kerr–Sen case due to modifications of general relativity coming from the low energy heterotic string theory effective action.

5.1.1 Kerr–Newman Geometry

The Kerr–Newman solution [13] is the most general solution of the Einstein–Maxwell equations for an asymptotically flat, stationary and axisymmetric black hole. Its line element and vector potential read:

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\sin^2\theta}{\rho^2}\left[a\,dt - (r^2+a^2)d\phi\right]^2 + \rho^2 d\theta^2\,, \\ A &= -\frac{Qr}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)\,, \end{aligned} \tag{5.1}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta\,, \qquad \Delta = r^2 - 2Mr + a^2 + Q^2\,. \tag{5.2}$$

The solution describes a black hole with mass M, charge Q, angular momentum $J = Ma$, and a magnetic dipole moment $\mu_g = Qa$. Notice that this is only a minor modification of the Kerr metric (1.89).

The metric possesses a curvature singularity at $\rho^2 = 0$, which is protected by an event horizon at $r = r_+ \equiv M + \sqrt{M^2 - a^2 - Q^2}$ provided that $a^2 + Q^2 \leq M^2$. As for the Kerr black hole, discussed in the introduction, when the equality holds we have an extremal black hole. Likewise, the rotation of the Kerr–Newman black hole also causes inertial frame dragging whose extreme manifestation is the existence of the ergosphere, for $r_+ < r < r_E \equiv M + \sqrt{M^2 - Q^2 - a^2\cos^2\theta}$. In this region the time-like vector ∂_t becomes null and therefore any massive particle must rotate. As before, it is this region that leads to superradiant emission and is responsible for the instability modes for perturbations on this spacetime.

The black hole horizon rotates with angular velocity

$$\Omega_H = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{a}{r_+^2 + a^2} \,, \tag{5.3}$$

and can be assigned the following Hawking temperature, entropy, and electrostatic potential:

$$T_H = \frac{\Delta'(r_+)}{r_+^2 + a^2} = \frac{r_+^2 - a^2 - Q^2}{4\pi r_+(r_+^2 + a^2)} \,, \quad S = \pi(r_+^2 + a^2) \,, \quad \phi_H = \frac{Q r_+}{r_+^2 + a^2} \,. \tag{5.4}$$

These quantities satisfy the first law of black hole thermodynamics

$$\delta M = T_H \delta S + \Omega_H \delta J + \phi_H \delta Q \,, \tag{5.5}$$

as well as the associated Smarr relation, $M = 2(T_H S + \Omega_H J) + \phi_H Q$. Notice this is exactly the same as the first law (1.101) but with an additional work term related to the electromagnetic charge and potential. In fact this is the most general first law one can find in asymptotically flat spacetimes (see e.g. [168]). Relaxing the asymptotic flatness condition leads to a much richer thermodynamic structure (see e.g. [237–244])

The Kerr–Newman metric continues to admit a hidden symmetry of the *principal tensor* (PT), which is a non-degenerate closed conformal Killing–Yano 2-form h, obeying the usual CCKY equation (1.107). In these Boyer–Lindquist coordinates it explicitly reads

$$h = r(\mathrm{d}t - a\sin^2\theta\, \mathrm{d}\phi) \wedge \mathrm{d}r - a\, \cos\theta \big[a\, \mathrm{d}t - (r^2 + a^2)\mathrm{d}\phi\big] \wedge \mathrm{d}\cos\theta \,, \tag{5.6}$$

and gives rise to the associated Killing tensor

$$K_{ab} = h_{ac} h^c{}_b - \frac{1}{2} g_{ab} h^2 \,. \tag{5.7}$$

This now generates the generalized Carter's constant for charged geodesics. It also yields the two independent isometries of the spacetime: ξ^a in (1.107) and $\eta^a = K^{ab}\xi_b$ [37].

Let us now compare this to the Kerr–Sen spacetime.

5.1.2 Kerr–Sen Geometry

The Kerr–Sen black hole [57] is an exact classical solution of the low-energy effective theory describing heterotic string theory given by the following action:

$$S = \frac{1}{16\pi} \int \mathrm{d}^4x \sqrt{-g}\, e^{\Phi} \left(R + g^{ab}\partial_a \Phi \partial_b \Phi - F_{ab}F^{ab} - \frac{1}{12} H_{abc} H^{abc} \right) \,, \tag{5.8}$$

where g_{ab} represents the metric in the string frame, Φ is the dilaton field, $F = \mathrm{d}A$ is the Maxwell field strength, and $H = \mathrm{d}\mathfrak{B} - 2A \wedge F$ is a 3-form defined in terms of the vector potential A and a 2-form potential $\mathfrak{B}$[1]. The action is invariant under a $U(1)$ transformation $A \to A + \mathrm{d}\lambda$ provided we also send $\mathfrak{B} \to \mathfrak{B} + 2\lambda F$ and the corresponding equations of motion for the background fields A and H are,

$$\nabla^a \Big[e^{\Phi} (F_{ab} - H_{abc} A^c) \Big] = 0 \,, \quad \nabla^a (e^{\Phi} H_{abc}) = 0 \,. \tag{5.9}$$

These will be important in section 5.2 where we motivate a generalization of the Proca equation to this background. The full set of equations of motion is supplemented by the Einstein and dilaton equations. Since these will not play any role in the further discussion we do not write them here explicitly and refer the interested reader to for example [100].

In any case, the Kerr-Sen metric in the standard Boyer–Lindquist-type coordinates and the string frame reads [57, 100, 223]:

$$\begin{aligned} \mathrm{d}s^2 &= e^{-\Phi} \Big\{ -\frac{\Delta_b}{\rho_b^2} \big(\mathrm{d}t - a \sin^2\theta \, \mathrm{d}\phi\big)^2 + \frac{\rho_b^2}{\Delta_b} \mathrm{d}r^2 + \frac{\sin^2\theta}{\rho_b^2} \Big[a \, \mathrm{d}t - (r^2 + 2\, b\, r + a^2) \mathrm{d}\phi \Big]^2 + \rho_b^2 \mathrm{d}\theta^2 \Big\} \,, \\ \mathfrak{B} &= \frac{2\, b\, r}{\rho_b^2}\, a \, \sin^2\theta \mathrm{d}t \wedge \mathrm{d}\varphi \,, \quad A = -\frac{Q\, r}{\rho^2} e^{-\Phi} \big(\mathrm{d}t - a \, \sin^2\theta \mathrm{d}\phi\big) \,, \quad e^{-\Phi} = \frac{\rho^2}{\rho_b^2} \,, \end{aligned} \tag{5.10}$$

where the metric functions are given by

$$\rho^2 = r^2 + a^2 \cos^2\theta \,, \quad \rho_b^2 = \rho^2 + 2br \,, \quad \Delta_b = r^2 - 2(M - b) r + a^2 \,. \tag{5.11}$$

The 3-form H reads

$$H = -\frac{2\, b\, a}{\rho_b^4} \, \mathrm{d}t \wedge \mathrm{d}\phi \wedge \Big[\big(r^2 - a^2 \cos^2\theta\big) \sin^2\theta \, \mathrm{d}r - r \Delta_b \sin 2\theta \mathrm{d}\theta \Big] \,. \tag{5.12}$$

Note that, the transformation $g_{ab} \to e^{\Phi} g_{ab}$ can be implemented to go from the string frame to the Einstein frame. Our choice for the string frame is guided by the fact that, in the context of separability, the string frame seems to be more fundamental than the Einstein one, as are the hidden symmetries present in the Kerr–Sen spacetime, see [100][2].

[1]Note that we have rescaled the vector potential $A \to 2\sqrt{2} A$ so that the Maxwell Lagrangian has the canonical prefactor [57].

[2]However, perhaps a generalization of the conformal techniques presented in the previous part of the thesis would allow the separability (of certain) fields to continue to hold in the Einstein fame.

As mentioned in the introduction, the solution describes a black hole with mass M, $U(1)$ charge [57][3]

$$Q = \frac{1}{4\pi} \int_{S^2_\infty} e^{\Phi} \star (F - A \cdot H)\,, \tag{5.13}$$

angular momentum $J = Ma$, and magnetic dipole moment $\mu_g = Qa$. When the "twist parameter"[4]

$$b = \frac{Q^2}{2M} \tag{5.14}$$

is set to zero, the solution reduces to the Kerr geometry. The horizon of the Kerr–Sen black hole is located at $r = r_+ \equiv M - b + \sqrt{(M-b)^2 - a^2}$ when the inequality $M - b \geq |a|$ holds. As in the Kerr–Newmann case the ergosphere is present and responsible for the instability modes but it is now located at $r_+ < r < r_e \equiv M - b + \sqrt{(M-b)^2 - a^2\cos^2\theta}$. Moreover, the Kerr-Sen black hole also obeys the first law, (5.5), where now the (Einstein frame) thermodynamic quantities are given by

$$\begin{aligned}
\Omega_H &= \frac{a}{r_+^2 + 2br_+ + a^2}\,, \quad \phi_H = \frac{Qr_+}{r_+^2 + 2br_+ + a^2}\,, \\
T_H &= \frac{r_+^2 - a^2}{4\pi r_+(r_+^2 + 2br_+ + a^2)}\,, \quad S = \pi(r_+^2 + 2br_+ + a^2)\,.
\end{aligned} \tag{5.15}$$

The spacetime no longer possesses the hidden symmetry of the principal tensor. However, as shown in [180] a weaker structure of the *principal tensor with torsion* exists [99]. This obeys the generalized principal tensor equation of the previous chapter (4.24). More explicitly we have

$$\nabla^T_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_a\,, \quad \xi^a = \frac{1}{3}\nabla^T_c h^{ca}\,. \tag{5.16}$$

Here the torsion is simply identified [180] with the 3-form H, (5.12),

$$T_{abc} = H_{abc}\,. \tag{5.17}$$

In these coordinates the principal tensor with torsion reads,

$$h = e^{-\Phi}\Big[r(\mathrm{d}t - a\sin^2\theta \mathrm{d}\phi)\wedge \mathrm{d}r - a\cos\theta\big[a\,\mathrm{d}t - (r^2 + 2br + a^2)\mathrm{d}\phi\big]\wedge \mathrm{d}\cos\theta\Big]\,. \tag{5.18}$$

[3]That this is the conserved charge of the system follows from the equation of motion (5.9) for F. See for instance [245] for an explicit calculation of the charges in the Kerr–Sen spacetime.

[4]This is distinct from, and not to be confused with, the NUT parameter(s). Kerr–Sen black holes can have these see [100] and appendix C.

Despite being a weaker structure, the principal tensor with torsion still gives rise to standard Killing tensor, via (5.7). This can be generalized to all dimensions and has the corresponding Killing tower. However, the isometries (Killing vectors) of the spacetime are no longer straightforwardly generated from h [180].

5.2 Separability of Proca Equations

In this section, in a similar manner, to the previous chapter we motivate the new form of the Proca equations for the Kerr–Sen spacetime. We then outline our ansatz and resulting separated equations using the (generalized) hidden symmetries of these two spacetimes. As the methods are, more or less, the same as the previous chapter the full details are relegated to appendix C.

5.2.1 Proca in Kerr–Newman Spacetime

The separability of the Proca equation in the Kerr–Newmann background was demonstrated in [91]—the Kerr–Newman metric being a special case of the $D = 4$ canonical metric for which the separability was shown there. In Boyer–Lindquist type coordinates the scalar potential Z entering the LFKK anastaz (4.1) assumes the standard multiplicative separation form,

$$Z = R(r)\, S(\theta)\, e^{im_\phi \phi}\, e^{-i\omega t}\,, \tag{5.19}$$

where m_ϕ and ω are the eigenvalues of $i\partial_t$ and $-i\partial_\phi$. Note that ϕ has period 2π, and regularity of the spherical harmonics $S(\theta)\, e^{im_\phi \phi}$ requires that $m_\phi \in \mathbb{Z}$.

Since the details of the separation follow the same form as the previous chapter we just present the result here and refer the reader to the original material [90, 91]. With this ansatz, the Proca equation (4.2) reduces to two differential equations in r and θ, respectively, which only couple to each other via their dependence on the Killing parameters $\{\omega, m_\phi\}$, the separation constant μ, the Proca mass parameter m, and the black hole parameters $\{M, Q, a\}$. These equations take the explicit form in Boyer-Lindquist type coordinates,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\Delta}{q_r}\frac{\mathrm{d}R}{\mathrm{d}r}\right] + \left[\frac{K_r^2}{\Delta\, q_r} + \frac{2-q_r}{q_r^2}\frac{\sigma}{\mu} - \frac{m^2}{\mu^2}\right] R = 0\,, \tag{5.20a}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{\sin\theta}{q_\theta}\frac{\mathrm{d}S}{\mathrm{d}\theta}\right] - \left[\frac{K_\theta^2}{q_\theta \sin^2\theta} + \frac{2-q_\theta}{q_\theta^2}\frac{\sigma}{\mu} - \frac{m^2}{\mu^2}\right] S = 0\,, \tag{5.20b}$$

where

$$K_r = a\,m_\phi - (a^2{+}r^2)\omega\,, \quad K_\theta = m_\phi - a\,\omega\,\sin^2\theta\,, \\ q_r = 1 + \mu^2 r^2\,, \quad q_\theta = 1 - \mu^2 a^2 \cos^2\theta\,, \\ \sigma = a\mu^2\,(m_\phi - a\omega) + \omega\,, \tag{5.21}$$

and Δ is as in (5.2).

This separation depends crucially on the existence of the principal tensor in a, by now hopefully familiar, number of ways. First, the separation occurs in geometrically preferred coordinates determined by the principal tensor – coordinates r and $\cos\theta$ are related to the eigenvalues of the principal tensor, e.g. [37]. Second, the principal tensor explicitly enters the separation ansatz (4.1) via the polarization tensor B. Third, the principal tensor gives rise to a complete set of mutually commuting operators that guarantee this separability [90, 180, 217]. Namely, apart from the (trivial) ones connected with Killing vectors, the following two (2nd order) operators directly link to the separation ansatz:

$$\hat{g} = \nabla_a(g^{ab}\nabla_b) - 2i\mu V_a g^{ab}\nabla_b\,, \quad \hat{K} = \nabla_a(K^{ab}\nabla_b) - 2i\mu V_a K^{ab}\nabla_b\,, \tag{5.22}$$

where K^{ab} is the Killing tensor (4.41) and $V^a = \xi_b B^{ba}$. See appendix C for more details.

5.2.2 Generalized Proca in Kerr–Sen Spacetime

Test fields in the Kerr–Sen background naturally pick up modifications due to the presence of background fields ϕ, A, and H. See for example [99–101] for a modification of the Dirac equation. To motivate the generalized Proca equation, we assume, as in the previous chapter (and ref. [203]), that the massive vector field P couples to the background fields Φ and H in analogy to the massless Maxwell field already present in the Kerr–Sen action (5.8). Moreover, we also require that the modified Proca equation be linear in P, reduce to the Proca equation in the absence of the background fields, and obey current conservation in the presence of sources.

It follows that there are two key modifications to the Proca equation in the Kerr–Sen background. First, in the string frame the dilaton enters the field equation, now called the Troca equation

$$\nabla^a\left(e^\Phi \mathcal{F}^T_{ab}\right) - m^2 e^\Phi P_b = 0\,. \tag{5.23}$$

Second, the 3-form $H_{abc} = T_{abc}$ contributes to the field strength in a torsion-like fashion

$$\mathcal{F}^T_{ab} = (d^T P)_{ab} = \nabla_a P_b - \nabla_b P_a - P^c H_{cab}\,, \tag{5.24}$$

where d^T is the torsion generalization of the exterior derivative, $d^T = \nabla^T\wedge$[5]. With this definition and using the equation of motion for H, (5.9), the Troca field equation (5.23) takes the same form as the equation of motion of the Maxwell field with the addition of the standard mass term. (5.23) also implies a modified "Lorenz condition"

$$\nabla_a(e^\Phi P^a) = 0\,. \tag{5.26}$$

To separate the Troca equation (5.23) in the Kerr–Sen background we exploit the same machinery as for the Kerr–Newman case, with the only difference that the principal tensor (5.6) is now replaced with the principal tensor with torsion (5.18). Upon this, the LFKK ansatz (4.1) continues to work (see appendix C) and we recover the following separated equations:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\Delta_b}{q_r}\frac{\mathrm{d}R}{\mathrm{d}r}\right] + \left[\frac{K_r^2}{\Delta_b\, q_r} + \frac{2-q_r}{q_r^2}\frac{\sigma}{\mu} - \frac{m^2}{\mu^2} - \frac{4br\omega\mu}{q_r^2}\right] R = 0\,, \tag{5.27a}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{\sin\theta}{q_\theta}\frac{\mathrm{d}S}{\mathrm{d}\theta}\right] - \left[\frac{K_\theta^2}{q_\theta\sin^2\theta} + \frac{2-q_\theta}{q_\theta^2}\frac{\sigma}{\mu} - \frac{m^2}{\mu^2}\right] S = 0\,, \tag{5.27b}$$

where

$$\begin{gathered} K_r = a\, m_\phi - (a^2{+}r^2 + 2rb)\omega\,, \quad K_\theta = m_\phi - a\,\omega\,\sin^2\theta\,, \\ q_r = 1 + \mu^2 r^2\,, \quad q_\theta = 1 - \mu^2 a^2\cos^2\theta\,, \\ \sigma = a\mu^2\,(m_\phi - a\omega) + \omega\,, \end{gathered} \tag{5.28}$$

which are to be compared to the Proca equations in the Kerr–Newman spacetime (5.20), and upon setting $b = 0$ reduce to the those in the Kerr spacetime. Note that the angular equation in all three cases is exactly the same, while the radial one picks up some small modifications.

Similar to the Kerr–Newman case, the separability is underlain by a complete set of mutually commuting symmetry operators, one of which is constructed from the generalized principal tensor,

$$\hat{g} = e^{-\Phi}\nabla_a(e^\Phi g^{ab}\nabla_b) - 2i\mu V_a g^{ab}\nabla_b\,, \quad \hat{K} = e^{-\Phi}\nabla_a(e^\Phi K^{ab}\nabla_b) - 2i\mu V_a K^{ab}\nabla_b\,. \tag{5.29}$$

See appendix C for more details.

[5] Note that, again, of the 3 possible generalizations of the Maxwell operator $\nabla\cdot dP$:

$$O_1 = \nabla\cdot(d^T P)\,, \quad O_2 = \nabla^T\cdot dP\,, \quad O_3 = \nabla^T\cdot(d^T P)\,, \tag{5.25}$$

it is only the first one which obeys the current conservation equation and (upon including the dilatonic modification) consequently appears in (5.23).

Now that we have shown the separated the Proca and Troca equations for the Kerr–Newman and Kerr–Sen black holes we can present some of the results of the numerical investigations in ref. [204]. Again we will not discuss the numerical details and refer the original work.

5.3 Finding Unstable Modes

5.3.1 Setting up the Problem—Boundary Conditions

To begin the numerical analysis it is important to choose the appropriate boundary conditions. Then one can employ the methods outlined in [204] and the references therein[6] to numerically solve the resulting ODEs (5.20) and (5.27).

In any case, it is useful to parameterize things in terms of the horizon radius and work with compact coordinates. For example, on can use $\Delta(r_+) = 0$ (for Kerr–Newman) or $\Delta_b(r_+) = 0$ (for Kerr–Sen) to solve for M as a function of the outer horizon radius parameter r_+. Next we can define $y \in [0, 1]$ and a nicer polar coordinate by

$$r = \frac{r_+}{1 - y^2}\,, \qquad \cos\theta = 2x - 1\,. \tag{5.30}$$

Then the horizon is located at $y = 0$ and asymptotic infinity is at $y = 1$[7].

Let us now consider the appropriate boundary conditions for the radial and angular functions $R(y)$ and $S(x)$[8]. We are seeking modes that can trigger superradiance instabilities as these will determine the signatures of the Proca fields in gravitational wave signals (e.g. [53]). In order to be bound states, these modes must have frequencies whose real part is smaller than the potential barrier height set by the Proca field mass, $\mathrm{Re}(\omega) < m$ (see the discussion in the introduction). A Frobenius analysis at asymptotic infinity $y = 1$ then indicates that unstable modes must decay as

$$R\Big|_{y\to 1} \sim e^{-\frac{\sqrt{\widetilde{m}^2 - \widetilde{\omega}^2}}{1-y^2}} (1 - y^2)^{\Sigma}\,, \quad \text{where} \quad \Sigma \equiv i\,(1 + \widetilde{a}^2 + \widetilde{Q}^2)\,\frac{(\widetilde{m}^2 - 2\widetilde{\omega}^2)}{2\sqrt{\widetilde{m}^2 - \widetilde{\omega}^2}}\,. \tag{5.31}$$

[6] Or ask collaborators kindly!

[7] For intermediate numerical calculations one also works with dimensionless quantities

$$\{\widetilde{a} = a/r_+\,,\ \widetilde{Q} = Q/r_+\,,\ \widetilde{m} = m\,r_+\,,\ \widetilde{\omega} = \omega\,r_+\,,\ \widetilde{\mu} = \mu\,r_+\}\,.$$

[8] Many more details of this can be found in [246, 247] and pedagogically described in [194].

Here, we have already imposed a boundary condition that eliminates a solution that grows unbounded at infinity as $e^{\sqrt{\widetilde{m}^2-\widetilde{\omega}^2}/(1-y^2)}$.

At the horizon, regularity of the perturbation in ingoing Eddington–Finkelstein coordinates requires that we impose the boundary condition,

$$R\Big|_{y\to 0} \sim y^{-i\,\frac{\omega - m_\phi \Omega_H}{2\pi T_H}}\,, \tag{5.32}$$

where Ω_H and T_H are the horizon angular velocity and temperature, given in see (5.3), (5.4) and (5.15), respectively. This condition excludes outgoing modes, $y^{i\,(\omega-m_\phi\Omega_H)/(2\pi T_H)}$, at the horizon.

We need to consider north and south poles of the S^2 for the angular function $S(x)$. Here, regularity of the perturbations requires that m_ϕ be an integer. We are interested in unstable modes which must co-rotate with the black hole because these will extract energy from the black hole (see also the discussion in the introduction). Thus we must have $m_\phi > 0$. Under these conditions, regularity requires that the perturbations behave as

$$S\Big|_{x\to 0} \sim x^{\frac{1}{2}|m_\phi|}\,, \qquad S\Big|_{x\to 1} \sim (1-x)^{\frac{1}{2}|m_\phi|}\,, \tag{5.33}$$

which eliminates irregular modes that would diverge as $x^{-\frac{1}{2}|m_\phi|}(1-x)^{-\frac{1}{2}|m_\phi|}$.

The boundary conditions (5.32) and (5.33) are straightforwardly imposed if we define the new functions q_i, $i = 1, 2$, as

$$R(y) = e^{-\frac{\sqrt{\widetilde{m}^2-\widetilde{\omega}^2}}{1-y^2}}(1-y^2)^{\Sigma} y^{-i\,\frac{\omega-m_\phi\Omega_H}{2\pi T_H}}\, q_1(y)\,, \quad S(x) = x^{\frac{1}{2}|m_\phi|}(1-x)^{\frac{1}{2}|m_\phi|}\, q_2(x)\,, \tag{5.34}$$

and search numerically for regular functions $q_1(y)$ and $q_2(x)$ [204]. Our pair of Proca ODEs are coupled only via the eigenvalues ω and μ, but this is a non-linear eigenvalue problem for ω and μ. Finally we mention that all the results that we present in the next section are accurate at least up to the 11th decimal digit [204].

5.3.2 Parameter Space for the Proca System

The Proca-black hole system has various scales; e.g. for the black hole: angular momentum, mass, and charge; and for the Proca field: frequency, mass, and angular eigenvalue. So we present the dimensionless physical quantities measured in black hole mass units, namely[9]

$$\{J/M^2, Q/M, mM, \omega M, \mu M\}\,. \tag{5.35}$$

[9]Confusingly, in the natural units we are using, the black hole mass is measured in units length as can be seen in (5.11). Hence the black hole mass multiplied by the Proca mass is dimensionless $[Mm] = 0$.

Note that $J = Ma$ so the dimensionless rotation a/M also gives the dimensionless angular momentum of the black hole: $J/M^2 = a/M$.

We are considering non-extremal black holes which reduce to the (non-extremal) Kerr black hole when the charge is switched off. Thus we have a 3-parameter phase space parameterized by the Proca mass the black hole angular momentum, and the electric charge, which respectively satisfy

$$mM \geq 0\,, \quad 0 \leq J/M^2 \leq 1\,, \quad 0 \leq Q/M \leq Q/M\big|_{\text{ext}}\,. \tag{5.36}$$

For Kerr–Newman and the Kerr–Sen black hole extremality (*i.e.* zero temperature) for a given a/M is attained at

$$Q/M\big|_{\text{ext}} = \sqrt{1 - J^2/M^4}\,, \quad Q/M\big|_{\text{ext}} = \sqrt{2}\sqrt{1 - J/M^2}\,, \tag{5.37}$$

respectively.

In this thesis, I will present just the following case. We consider a fixed angular momentum $J/M^2 = a/M$ of the black hole and the Proca mass mM also fixed. Then I reproduce the plot in [204] which shows the change of the frequency ωM and angular eigenvalue μM as the asymptotic $U(1)$ charge of the Kerr-Newman and Kerr–Sen black holes are varied $0 \leq Q/M \leq Q/M\big|_{\text{ext}}$. The qualitative features for other values of J/M^2 and mM are similar. In four dimensions Proca fields have three polarization degrees of freedom, see the discussion in [90,91] and particularly [215] for how to extract these from the LFKK ansatz. Here, we will only discuss the most unstable polarization. For the same reason, we also only consider its lowest radial eigenvalue and lowest azimuthal number $m_\phi = 1$.

5.3.3 Instabilities of Proca Fields in Kerr–Newman and Kerr–Sen Black Holes

Generically we find that the Kerr system, $Q/M = 0$, is unstable due to the rotation. However we want to distinguish the effect of the charge in the Kerr–Sen and Kerr–Newman black holes. Thus it is appropriate to consider Proca masses such that the instability effect is maximized. This happens when the Compton wavelength is comparable to the size of the black hole [215], i.e. restoring the physical units

$$\frac{GM}{c^2} \Big/ \left(\frac{\hbar}{mc}\right) \sim \mathcal{O}(1)\,. \tag{5.38}$$

This fixes the Proca mass to be $m \lesssim 10^{-11} \times M_\odot/M\text{eV}$ which fits within the recent model with a mass of 10^{-22}eV [248]. Thus we choose the mass such that $mM = 0.51$. We find this to give a significant instability in the Kerr spacetime. To see the full discussion and numerical work to find appropriate parameter ranges I refer the reader to our work [204]. I will now display the results for the unstable modes of the Kerr–Newman and Kerr–Sen black holes in Figure 5.1.

The brown diamond here in Figure 5.1 represents the Kerr black hole which has $J/M^2 = a/M = 0.998$ and $mM = 0.51$, for which one finds that:

$$\begin{aligned} \omega M &\simeq 0.42212572826 + 0.00041466190852\, i\,, \\ \mu M &\simeq -0.83369985496 - 0.00080656941824\, i\,. \end{aligned} \tag{5.39}$$

Then Figure 5.1 plots the (real and imaginary parts of) frequency and angular eigenvalue for this particular Proca–Kerr solution with $Q/M = 0, J/M^2 = 0.998, mM = 0.51$ as the charge increases from the Kerr limit $Q/M = 0$ all the way up to the extremal limit $Q/M = Q/M\big|_{\text{ext}}$. The upper frames show the imaginary (left) and real part (right) of the dimensionless frequency ωM, while the lower frames give the imaginary (left) and real (right) parts of the dimensionless angular eigenvalue μM. The red disks describe the solution in the Kerr–Newman background while the black squares describe the unstable Troca modes in the Kerr–Sen black hole.

We indicate the extremal limit by the vertical black dashed line which occurs at $Q/M = 0.063213922517$ for Kerr–Newman and at $Q/M = 0.0632455553203$ for the Kerr–Sen black hole (these are very close so the two extremal locations cannot be distinguished in the plots).

The most important plot is the left-upper plot where we display the imaginary part of the frequency. As discussed above, the system is already unstable ($\text{Im}(\omega M) > 0$) in the $Q/M = 0$ Kerr limit (brown diamond). We then see that as the electric charge Q/M is turned on, $\text{Im}(\omega M)$ decreases monotonically, in the Kerr–Newman *and* Kerr–Sen black holes, until it reaches a (positive) minimum at extremality. Thus the electric charge decreases the strength of the superradiant instability. The upper left plot of Figure 5.1 moreover shows that, in the parameter space range where both co-exist, *Kerr–Sen black holes are more unstable than Kerr–Newman black holes*[10]. The other panels of Fig. 5.1 show similar behaviour in that they have start at the expected Proca–Kerr solution and show a monotonic decrease as the charge increases.

[10] One needs to be a little cautious in interpreting these results regarding the use of the string vs. the Einstein frame (see e.g the discussion in [249]). Namely, our calculation for the Kerr–Sen black hole has been carried out in the string frame, where the Troca equations decouple and separate. It is an open question whether this can be directly compared to the Kerr–Newman case where the dilaton field identically vanishes.

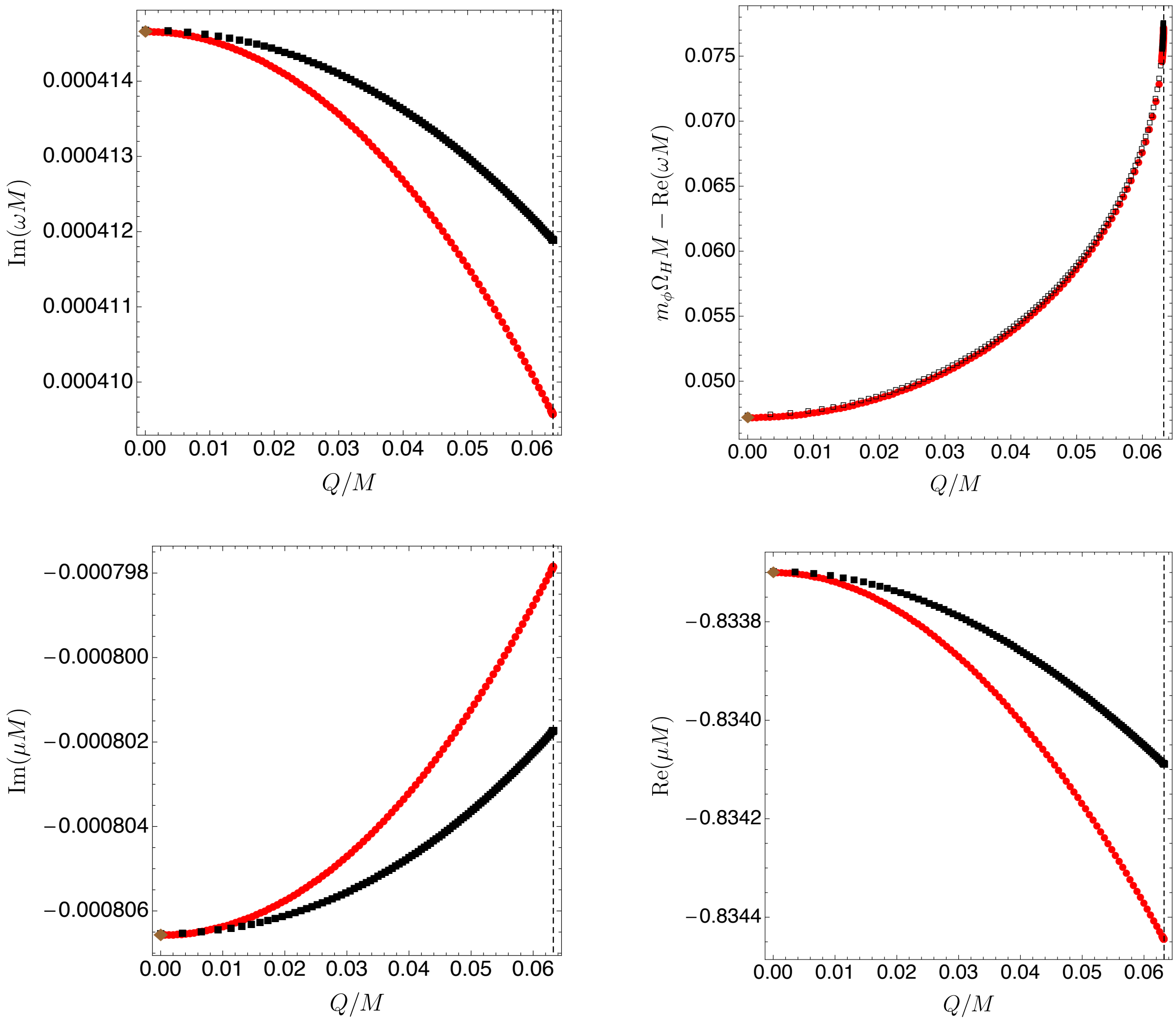

Figure 5.1: **Kerr–Sen vs. Kerr–Newman black holes: effect of charge.** [204] The unstable Proca modes are plotted with, mass $mM = 0.51$ and $m_\phi = 1$ and angular momentum $J/M^2 = a/M = 0.998$, for Kerr–Newman (red disks) and Kerr–Sen (black squares) black holes as a function of the dimensionless hole charge Q/M. The upper frames describe the imaginary and real part of the dimensionless frequency ωM, while the lower panels give the imaginary and real parts of the dimensionless angular eigenvalue μM. Larger imaginary parts of ωM signal higher instabilities. The brown diamond at $Q/M = 0$ describes the Kerr solution. The vertical black dashed line signals extremality, which occurs at $Q/M = 0.063213922517$ for Kerr–Newman and at $Q/M = 0.063245553203$ for the Kerr–Sen black hole (so the two extremal vertical lines cannot be distinguished in the plots).

These plots illustrate in a clear way the main properties of unstable Proca fields in Kerr–Newman and Kerr–Sen black holes and other combinations of the parameters J/M^2 and mM (for which the instability is already present in the Kerr limit) would show similar qualitative features as those illustrated in Fig. 5.1.

5.4 Summary

In this chapter we have shown that the LFKK ansatz can be used to separate the Troca equations in the Kerr–Sen black hole background of the low energy heterotic string theory. This happens for a (well motivated) modification of these equations and in the string frame. As for the previous chapter, this is an extension beyond the usual bounds of the LFKK ansatz and makes use of the much weaker principal tensor with torsion. Since the principal tensor with torsion applies to the D dimensional Kerr–Sen black hole, and it generates the tower of hidden symmetries. These results suggest that it should be possible to separate both test Maxwell and the Troca fields in the general dimension case. One possible difficulty with this is that the explicit symmetries are no longer generated.

We have then used the resulting separated ordinary differential equations to study the corresponding instability modes of the Troca field in the Kerr–Sen background and compared them to the instability modes in the Kerr and Kerr–Newman backgrounds. This is the first study of the Proca instability modes around rotating black holes that considered the possibility of weakly charged solutions. Moreover we have considered an astrophysically viable setting where the black holes are highly spinning (close to extremal) and weakly charged. Our results allow one to compare the prediction of the two theories: the Einstein–Maxwell theory (represented by the Kerr–Newman solution) and the low energy heterotic string theory (with the corresponding Kerr–Sen black hole). Our findings indicate that, at equal asymptotic charges i.e. M and Q, Kerr-Newman black holes are more stable than Kerr–Sen ones.

Part III

The Hidden Symmetries of Slowly Rotating Black Holes

This part is derived (or lifted from) the two works [250, 251] and constructs a new class of solutions with exact Killing tensors. Namely, we pick up the threads on a recent extension [252, 253] to the well known *Lense–Thirring spacetime*, which describes a field of a slowly rotating body. These spacetimes and solve the field equations to linear order in the rotation parameter and admit an exact Killing tensor. We show that this result can be extended to a very general metric representing slowly rotating black holes.

In the first chapter we focus on the the mathematical structure of the new metric: in particular, its rich (exact) hidden symmetry structure, and its manifest regularity at the horizon, being able to be put in *Painlevé–Gullstrand* form. We also show these symmetries are inherited from the principal Killing–Yano tensor of the exact rotating black hole geometry in the slow rotation limit. This provides a missing link as to how the exact hidden symmetries emerge as rotation is switched on. Remarkably, in higher dimensions the novel generalized Lense–Thirring spacetimes feature a rapidly growing number of exact irreducible rank-2, as well as higher-rank, Killing tensors—giving a first example of a physical spacetime with more hidden than explicit symmetries.

In the second chapter, we present a few applications of this result showing how it encapsulates the slow rotation limits of black holes we have already seen in this thesis: the Einstein–Maxell Kerr–Newman, supergravity Chong–Cvetič–Lü–Pope, and the Kerr–Sen black holes. Each of these cases demonstrates certain features of the new Lense–Thirring metrics. Moreover we present a theorem derived in [251] (largely due to calculations of R.A. Hennigar) which demonstrates that the standard form of the (vacuum) Lense–Thirring metric is unique to Einstein gravity. That is, in Einstein gravity, the Lense–Thirring spacetime is fully characterized by a single metric function of the corresponding static (Schwarzschild) solution. However, this is the only non-trivial theory amongst all up to quartic curvature gravities that admits a Lense–Thirring solution characterized by a single metric function.

Chapter 6

Slowly Rotating Black Holes with Exact Killing Tensor Symmetries

As we have discussed many times in the thesis rotating black holes are very special and have many interesting properties. Of particular importance (at least for our purposes here) are the integrability and separability properties linked to their hidden symmetry structure. When one considers static black holes these properties arise directly from the isometries (time translation and rotational invariance) of the spacetime and there are no nontrivial hidden symmetries. It is then an interesting question as to how these arise as rotation is switched on. This is especially true in the context of the Newman-Janis trick [12] wherein a complex coordinate transformation can take a static spacetime to a rotating one. To investigate this we will consider here some slowly rotating black holes.

First let us recall the Kerr metric, and its hidden symmetries, in Boyer–Lindquist coordinates for context. Its line element reads

$$ds^2 = -\left(1-\frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mra\sin^2\theta}{\rho^2} dt\, d\phi + \frac{\Sigma\, \sin^2\theta}{\rho^2} d\phi^2 + \frac{\rho^2}{\Delta_r} dr^2 + \rho^2\, d\theta^2\,, \qquad (6.1)$$

and is algebraically special (i.e. type D). It has fundamental hidden symmetry, encoded in the principal Killing–Yano tensor h, which obeys (1.107), and is explicitly given by $h = db$, where

$$2b = r^2\big(dt - a\sin^2\theta\frac{d\phi}{\Xi}\big) - a^2\cos^2\theta\big(dt - \frac{ad\phi}{\Xi}\big)\,. \qquad (6.2)$$

The corresponding irreducible Killing tensor, constructed from h according to $K_{ab} =$

$(\star h)_{ac}(\star h)_b{}^c$, reads:

$$K = \frac{a^2\cos^2\theta}{\Delta\Sigma}\Big((r^2+a^2)\partial_t + a\Xi\partial_\phi\Big)^2 - \frac{a^2\cos^2\theta\Delta}{\Sigma}(\partial_r)^2 + \frac{r^2}{\Sigma S\sin^2\theta}(a\sin^2\theta\partial_t + \Xi\partial_\phi)^2 + \frac{Sr^2}{\Sigma}(\partial_\theta)^2\,. \quad (6.3)$$

Together with the two explicit symmetries, ∂_t and ∂_ϕ, it guarantees the complete integrability of geodesic motion in these spacetimes [23, 28].

Now let us consider slowly rotating objects, i.e. ones for which the dimensionless rotation parameter $a \ll 1$. The prototypical solution for slowly rotating black holes in Einstein gravity in four dimensions is the well-known Lense–Thirring metric [254], and was discovered very soon after the Schwarzschild black hole and nearly fifty years before the Kerr solution. Its metric reads

$$\begin{aligned} ds^2 &= -fdt^2 + \frac{dr^2}{f} + 2a(f-1)\sin^2\theta dt d\phi + r^2(\sin^2\theta d\phi^2 + d\theta^2)\,, \\ f &= 1 - \frac{2M}{r}\,. \end{aligned} \quad (6.4)$$

It solves the vacuum Einstein equations to leading order in the rotation parameter a, and describes spacetime outside of a rotating body. The metric also arises as the slow rotation ($a \to 0$) limit of the Kerr metric [8].

A simple observation about the Lense–Thirring metric (6.4), which will be relevant in the second chapter of this section, is that it is completely characterized by the static Schwarzschild solution: the metric component $g_{t\phi}$ is written in terms of the static metric function f. In the context of four-dimensional Einstein gravity, this point underlies to an extent the Newman–Janis trick [12]. The fact that the full Kerr solution (1.89) can be generated from the "seed" static solution of course implies that the Lense–Thirring metric is obtained when the trick is truncated at $\mathcal{O}(a)$.

Of course, the spacetime inherits the hidden symmetries of the full solution which solve their respective equations to the linear order in a. These are given by

$$2b = r^2dt - ar^2\sin^2\theta d\phi + O(a^2)\,, \quad (6.5)$$

$$K = 2a\partial_t\partial_\phi + \frac{1}{\sin^2\theta}(\partial_\phi)^2 + (\partial_\theta)^2 + O(a^2)\,. \quad (6.6)$$

Note that since the metric is stationary and axisymmetric, the first term in (6.6) is trivial being just a product of Killing vectors and can be excluded.

A tempting possibility is to truncate the $O(a^2)$ terms in (6.4), and treat the resultant fields as "exact" (not necessarily a solution of the field equations). However, the spacetime has several "drawbacks". Namely, as exact metric, it is singular on what would be the black hole horizon $f = 0$, noting for example that the Kretschmann scalar (1.94) diverges there at $O(a^2)$. If one wants to interpret this spacetime as that of a slowly rotating black hole then it is crucial to have a regular horizon. Although these divergences come in at second order nonetheless one cannot say a divergence is negligible. Second, both (truncated to $O(a)$) hidden symmetries (6.5) and (6.6) remain only approximate. Finally, the metric cannot be cast in the PG form [252].

6.1 Generalized Lense–Thirring metric

To fix the above "drawbacks", let us instead consider the following modification due to Baines et al. [252, 253] of the above slowly rotating solution:

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 \sin^2\theta \Big(d\phi + \frac{a(f-1)}{r^2} dt\Big)^2 + r^2 d\theta^2 \,,. \tag{6.7}$$

with metric function f given in (6.4). In what follows, we shall call it the *generalized Lense–Thirring solution*. Formally, it can be obtained by "completing the square" in the truncated solution i.e. adding extra $O(a^2)$ terms. As such, it still solves the Einstein equations to $O(a)$, as well as admitting the approximate hidden symmetries (6.5) and (6.6).

However, when understood as an exact (filled with matter) spacetime[1], It is a much better approximation for a slowly rotating black hole than the above truncated solution since it is regular on the horizon – the curvature scalars, such as I (1.94), no longer diverge at $f = 0$ and the metric can be cast (at least in the vicinity of the horizon) in the manifestly regular PG form [252]. It also has an ergosphere and will feature superradiant phenomena, e.g. [49]. Most remarkably, the generalized Lense–Thirring spacetime (6.7) falls into a class of the Benenti metrics [110–112]. This means that not only does the metric posses an

[1]One should be a bit cautious about the physicality of the extra matter supporting the (modified) Lense–Thirring spacetimes. In particular, when $Q = 0 = \Lambda$, the tetrad component of the Einstein tensor, $G^{\hat{0}\hat{0}} = -\frac{1}{2}R = -(3aM\sin\theta/r^4)^2$, yielding a negative energy density [252]. This is partly amended in the presence of the additional ordinary matter—for example in our case the EM field dominates near infinity and guarantees there positive energy density (see e.g. (7.4) in the next chapter). However, we expect that the generic Lense–Thirring spacetimes will be (at least partly) supported by exotic matter violating the standard energy conditions.

exact Killing tensor, the separability of the scalar wave equation and integrability of the geodesics are guaranteed[2].

The corresponding exact Killing tensor is given by

$$K = \frac{1}{\sin^2\theta}(\partial_\phi)^2 + (\partial_\theta)^2 \,, \tag{6.8}$$

and can be understood as a slow rotation (truncated) version of the approximate Killing tensor (6.6). Interestingly, this Killing tensor can be written in the following suggestive form:

$$K = L_x^2 + L_y^2 + L_z^2 \,, \tag{6.9}$$

where $L_z = \partial_\phi$ is a Killing vector of (6.7) and vectors L_x and L_y are given by

$$\begin{aligned} L_x &= + \cot\theta \cos\phi \partial_\phi + \sin\phi \partial_\theta \,, \\ L_y &= - \cot\theta \sin\phi \partial_\phi + \cos\phi \partial_\theta \,, \end{aligned} \tag{6.10}$$

which upon recovering the spherical symmetry ($a \to 0$) would be the remaining two $SO(3)$ Killing vectors. Since L_z and ∂_t are the only two Killing vectors present in the spacetime (6.7), it can be checked that the above Killing tensor is irreducible.

6.2 Higher-Dimensional Lense–Thirring Spacetimes

Interestingly the Lense–Thirring metrics can be generalized to higher dimensions, including rotations in multiple planes. To illustrate this, let us start from the full Kerr-AdS metric in d spacetime dimensions. Throughout this section we will use $d = 2m + 1 + \varepsilon$, where $\varepsilon = 1, 0$[3]. The reason is due to the constraint on the angular variables in Myers–Perry

[2]In particular, one can check that Carter's criterion [154] $\nabla_a(k_\gamma{}^{[\alpha} R^{\beta]\gamma})=0$ is satisfied. Interestingly, one can also check that the criteria required for separability of the conformally coupled scalar equation are not satisfied, even though this equation does separate for Kerr–NUT–(A)dS spacetimes [182].

[3]Compare this to the previous parts where $D = 2n + \epsilon$ and $\epsilon = 0, 1$ in even and odd dimensions respectively.

coordinates. To be precise, in Boyer–Lindquist type coordinates the metric reads [255]:

$$
\begin{aligned}
ds^2 &= -W(1+r^2/\ell^2)dt^2 + \frac{2M}{U}\Big(W dt + \sum_{i=1}^{m}\frac{a_i\mu_i^2 d\phi_i}{\Xi_i}\Big)^2 \\
&\quad + \sum_{i=1}^{m}\frac{r^2+a_i^2}{\Xi_i}\big(\mu_i^2 d\phi_i^2 + d\mu_i^2\big) + \frac{U dr^2}{V-2M} + \epsilon r^2 d\nu^2 \\
&\quad + \frac{1}{W(l^2-r^2)}\Big(\sum_{i=1}^{m}\frac{r^2+a_i^2}{\Xi_i}\mu_i d\mu_i + \epsilon r^2 \nu d\nu\Big)^2 , \qquad (6.11)
\end{aligned}
$$

where

$$
\begin{aligned}
W &= \sum_{i=1}^{m}\frac{\mu_i^2}{\Xi_i} + \epsilon\nu^2 , \quad V = r^{\epsilon-2}(1+r^2/\ell^2)\prod_{i=1}^{m}(r^2+a_i^2) , \\
U &= \frac{V}{1+r^2/\ell^2}\Big(1 - \sum_{i=1}^{m}\frac{a_i^2\mu_i^2}{r^2+a_i^2}\Big) , \quad \Xi_i = 1 - \frac{a_i^2}{\ell^2} . \qquad (6.12)
\end{aligned}
$$

Here, $\epsilon = 1, 0$ for even, odd dimensions, $m = \left[\frac{d-1}{2}\right]$ (where $[A]$ denotes the whole part of A), and the coordinates μ_i and ν obey a constraint

$$
\sum_{i=1}^{m}\mu_i^2 + \epsilon\nu^2 = 1 . \qquad (6.13)
$$

The metric admits [35, 37] a principal Killing–Yano tensor, $h = db$,

$$
2b = \Big(r^2 + \sum_{\mu=1}^{m} a_i^2\mu_i^2(1 + \frac{r^2+a_i^2}{\ell^2\Xi_i}\Big)dt + \sum_{i=1}^{m} a_i\mu_i^2\frac{r^2+a_i^2}{\Xi_i}d\phi_i , \qquad (6.14)
$$

which generates the towers of explicit and hidden symmetries, see [37].

Expanding to linear order in a_i we following slowly rotating generalized Lense–Thirring solution (written now in non-rotating at infinity coordinates): Lense–Thirring metric reads [256].

$$
ds^2 = -f dt^2 + \frac{dr^2}{f} + \sum_{i=1}^{m}\mu_i^2 a_i(f-1)dt d\phi_i + r^2\big(\sum_{i=1}^{m} d\mu_i^2 + \mu_i^2 d\phi_i^2\big) + \varepsilon r^2 d\nu^2 , \qquad (6.15)
$$

for $m = \lfloor \frac{d-1}{2} \rfloor$ independent rotation parameters a_i, where $\lfloor A \rfloor$ denotes the whole part of A, and

$$f = 1 - \frac{16\pi M}{(d-2)\Omega_{d-2} r^{d-3}} \tag{6.16}$$

with M the mass of the rotating body, and Ω_d the volume of the d-dimensional sphere. In the above, the coordinates μ_i and ν obey the following constraint:

$$\sum_{\mu=1}^{m} \mu_i^2 + \varepsilon \nu^2 = 1\,, \tag{6.17}$$

where $\varepsilon = 1, 0$ in even, odd dimensions.

The metric (6.15) solves the d-dimensional Einstein equations to $\mathcal{O}(a)$ in the rotation parameter, and is still characterized completely by the static metric function. It has $m+1$ Killing vectors ∂_t and ∂_{ϕ_i} and the approximate Killing tensors inherited from the full Myers–Perry black hole (which are of course best seen in the canonical coordinates used previously—see the review [37] for how to translate between the two systems).

The same remains true even in the presence of a cosmological constant Λ, where f above is replaced with[4] [255]:

$$f = 1 - \frac{16\pi M}{(d-2)\Omega_{d-2} r^{d-3}} + \frac{r^2}{\ell^2}\,, \quad \Lambda = -\frac{(d-1)(d-2)}{2\ell^2}\,, \tag{6.18}$$

and ℓ is known as the AdS radius[5].

6.3 Improved Lense–Thirring spacetimes

The higher dimensional version (6.15) has the same drawbacks as the four dimensional case, i.e. the Kretschmann scalar diverges at the horizon, there are no regular infalling coordinates, and the hidden symmetries are approximate. Given this, it is natural to ask

[4]This higher-dimensional case is less obviously equivalent to the Newman–Janis trick, as, to the best of our knowledge, it is not yet known whether such a trick exists for arbitrary rotations in arbitrary dimensions—see [257] for the singly spinning Kerr–Newman metric in all dimensions, and [258] for the general five dimensional Myers–Perry case, and a discussion of the problems of further generalization. Nonetheless, it is clearly in the same spirit.

[5]This works equally for positive cosmological constant, sending $\ell \to i\ell$, i.e. for asymptotically de Sitter black holes.

if we can find an improved version à la refs. [252, 253]. Moreover, thus far, variants of the Lense–Thirring spacetime considered in the literature have focused on vacuum Einstein gravity. And as mentioned in the opening of this chapter, the Einstein form has a number of very special properties, for which there is no good reason to expect those properties to hold in general.

Allowing for general functions entering the mixed time and azimuthal directions, $g_{t\phi_i}$, and then completing the square of (6.15) we take the following as a generalized Lense–Thirring ansatz (6.15) for multiply-spinning black holes:

$$ds^2 = -Nfdt^2 + \frac{dr^2}{f} + r^2 \sum_{i=1}^{m} \mu_i^2 \Big(d\phi_i + \frac{\sum_{j=1}^{m} p_{ij} a_j}{r^2} dt\Big)^2 + r^2 (\sum_{i=1}^{m} d\mu_i^2) + \varepsilon r^2 d\nu^2 \,. \quad (6.19)$$

Obviously, the metric possesses the same "freedom" as the metric (6.15). The functions f, N, p_{ij} are functions of the radial coordinate r, and the coordinates μ_i are ν obey the constraint (6.17). It reduces to this Einstein form when

$$N = 1\,, \quad p_{ij} = (f-1)\delta_{ij}\,, \quad (6.20)$$

As we shall see below, this apparently small modification has far reaching consequences for the properties of the corresponding spacetime.

First, the metric admits a Killing horizon generated by the following Killing vector:

$$\xi = \partial_t + \sum_{i=1}^{m} \Omega_i \partial_{\phi_i}\,, \quad \Omega_i = -\sum_{j=1}^{m} \frac{p_{ij} a_j}{r^2}\Big|_{r=r_+}\,, \quad (6.21)$$

where r_+ is the location of the horizon—the largest root of $f(r_+) = 0$. The horizon is surrounded by the ergoregion, inside of which the Killing vector ∂_t has negative norm. Due to this ergoregion, the metric will exhibit superradiant phenomena [49]. It also possesses the m angular Killing vectors ∂_{ϕ_i}.

Second, the metric is regular on the horizon, and near its vicinity admits the Painlevé–Gullstrand form. Under the following coordinate transformation:

$$\begin{aligned} dt &= dT - \sqrt{\frac{1-f}{N}} \frac{dr}{f}\,, \\ d\phi_i &= d\Phi_i + \frac{\sum_j p_{ij} a_j}{r^2} \sqrt{\frac{1-f}{N}} \frac{dr}{f}\,, \end{aligned} \quad (6.22)$$

we recover

$$ds^2 = -NdT^2 + \Big(dr + \sqrt{N(1-f)}dT\Big)^2 + r^2\sum_{i=1}^{m}\mu_i^2\Big(d\Phi_i + \frac{\sum_j p_{ij}a_j}{r^2}dT\Big)^2$$
$$+r^2(\sum_{i=1}^{m}d\mu_i^2) + \varepsilon r^2 d\nu^2\,, \tag{6.23}$$

which is manifestly regular on the horizon, and the $T = const.$ slices are manifestly flat.[6]

6.4 New Hidden Symmetries

Most importantly, the metric (6.23) also admits a rapidly growing tower of Killing tensors. These can be generated as follows. Define the set $S = \{1, .., m\}$ and let $I \in P(S)$ where $P(S)$ is the power set of S. Then we may define the following objects:

$$2b^{(I)} \equiv r^2(dt + \sum_{i\in I} a_i\mu_i^2 d\phi_i)\,, \quad h^{(I)} \equiv db^{(I)}\,, \tag{6.24}$$

$$f^{(I)} \equiv \frac{\sqrt{N}}{(|I|+1)!} * \big(\underbrace{h^{(I)}\wedge\cdots\wedge h^{(I)}}_{|I|+1 \text{ times}}\big)\,, \tag{6.25}$$

where $|I|$ denotes the size of the set I. These potentials correspond to various limits on the rotation parameters in the principal tensor (6.14).

In turn these $f^{(I)}$ generate the following exact rank-2 Killing tensors:

$$K^{(I)}_{ab} = \big(\prod_{i\in I.} a_i\big)^{-2}(f^{(I)}\cdot f^{(I)})_{ab}\,, \quad K^{(I)}_{(ab;c)} = 0\,, \tag{6.26}$$

where we have defined

$$(\omega_1\cdot\omega_2)_{ab} = \frac{1}{p!}\omega_{ac_1\ldots c_p}\omega_b{}^{c_1\ldots c_p} \tag{6.27}$$

for any $(p+1)$-forms ω_1, ω_2.

[6]If the $\mathcal{O}(a^2)$ corrections are considered to have physical effects, then it should be noted that the stress tensor associated with them fails to satisfy the classical energy conditions. Of course, this does not affect the results below concerning the hidden symmetry structure of the metrics, and moreover the falloff of the $\mathcal{O}(a^2)$ terms is sufficiently fast that inclusion of classical matter, e.g. an electromagnetic field, restores the energy conditions.

Explicitly, these Killing tensors can be written as:

$$K^{(I)} = \sum_{i\notin I}^{m-1+\varepsilon} \left[\left(1 - \mu_i^2 - \sum_{j\in I} \mu_j^2\right)(\partial_{\mu_i})^2 - 2 \sum_{j\notin I\cup\{i\}} \mu_i \mu_j \, \partial_{\mu_i} \partial_{\mu_j} \right] + \sum_{i\notin I}^{m} \left[\frac{1 - \sum_{j\in I} \mu_j^2}{\mu_i^2} (\partial_{\phi_i})^2 \right] . \tag{6.28}$$

See appendix D.2 for details of the Killing tensors in the orthonormal frame.

Of course, in a given dimension, not all of these are non-trivial. In fact, it is only $K^{(\emptyset)}$ which exists in all dimensions $d \geq 4$, and is given by the formula (6.8). It turns out, however, that $\sum_{i=0}^{m-3} \binom{m}{i}$ of these are reducible, leaving in total

$$k = \sum_{i=0}^{m-2+\varepsilon} \binom{m}{i} - \sum_{i=0}^{m-3} \binom{m}{i} = \frac{1}{2} m(m - 1 + 2\varepsilon) \tag{6.29}$$

irreducible rank-2 Killing tensors in d dimensions.

Note that this tower increases quadratically with the number of dimensions, contrary to the tower of rank-2 Killing tensors in exact Kerr–NUT–(A)dS spacetimes [37], which only grows linearly with d. For example, already in $d = 8$ we have (for distinct rotation parameters) 6 irreducible rank-2 Killing tensors and only $m + 1 = 4$ independent Killing vectors—that is the number of hidden symmetries exceeds the number of the explicit ones (more so once we also count higher rank Killing tensors obtained by various combinations of SN brackets—see below). However, this is still much smaller than the maximum possible number of rank-2 Killing tensors in a given dimension d, which for rank-p Killing tensor ($p \geq 1$) reads, e.g. [149]:

$$k_{\max} = \frac{1}{d} \binom{d+p}{p+1} \binom{d+p-1}{p} , \tag{6.30}$$

and for $d = 8$ and $p = 2$ gives $k_{\max} = 540$.

Moreover, this construction "coincides" with the one for the full Kerr-AdS geometry [37], replacing the principal Killing–Yano tensor h with its appropriate limits $h^{(I)}$ at the relevant order of the small rotation parameters expansion. However these potentials in (6.24) and (6.25) are not even approximate CCKY and KY forms respectively.

Let us also stress that all the above hidden symmetries exist regardless of the form of the functions $p_{ij} = p_{ij}(r)$ and $N = N(r)$. In particular, when all p_{ij} vanish,

$$p_{ij} = 0 \, , \tag{6.31}$$

or alternatively, when all a_i are zero, we recover the static spherically symmetric metric and all of the above Killing tensors become reducible—given as sums of products of subsets of rotational Killing vectors.

In addition to the above rank-2 Killing tensors, one can also generate (potentially irreducible) higher-rank Killing tensors via the SN brackets (1.42). Our construction thus provides a physically well motivated example in the long-standing search for spacetimes with higher-rank Killing tensors [259–261]. In particular, we have verified up to $d = 13$ that the SN bracket of any two Killing tensors vanishes if the intersection of the two labels equals the first. That is,

$$[K^{(I_1)}, K^{(I_2)}]_{\text{SN}} = 0\,, \tag{6.32}$$

if $I_1 \cap I_2 = I_1$. Otherwise, a new Killing tensor is generated. We expect this to remain true also in higher dimensions. In particular, in d dimensions this implies (taking into account explicit symmetries and the metric as well) the existence of d mutually commuting symmetry objects—a necessary requirement for complete (and again exact) integrability of geodesic motion in the spacetime (6.19). In fact in the orthonormal frame one can show exactly which set of d Killing tensors are diagonal in the same frame, guaranteeing separability of the Hamilton–Jacobi equations for geodesics—see appendix D.2. Moreover, Carter's condition (1.88) is satisfied so Klein-Gordon equation for scalar fields separates. Interestingly however, one can check that the geometric obstruction to separating the conformal wave equation (see chapter 3) in the theorem of ref. [118] is not exact. Therefore the conformal wave equation is not guaranteed to separate exactly.

Finally we close by illustrating the above construction in $d = 6$ dimensions. In this case $k = 3$ and we have the following irreducible rank-2 Killing tensors:

$$\begin{aligned}
K^{(\emptyset)} &= \frac{1}{\mu_1^2}(\partial_{\phi_1})^2 + \frac{1}{\mu_2^2}(\partial_{\phi_2})^2 + (1-\mu_1^2)(\partial_{\mu_1})^2 \\
&\quad -2\mu_1\mu_2(\partial_{\mu_1})(\partial_{\mu_2}) + (1-\mu_2^2)(\partial_{\mu_2})^2\,, \\
K^{(1)} &= \frac{1-\mu_1^2}{\mu_2^2}(\partial_{\phi_2})^2 - (1-\mu_1^2-\mu_2^2)(\partial_{\mu_2})^2\,, \\
K^{(2)} &= \frac{1-\mu_2^2}{\mu_1^2}(\partial_{\phi_1})^2 - (1-\mu_1^2-\mu_2^2)(\partial_{\mu_1})^2\,.
\end{aligned} \tag{6.33}$$

Their SN brackets are

$$[K^{(\emptyset)}, K^{(1)}]_{\text{SN}} = 0 = [K^{(\emptyset)}, K^{(2)}]_{\text{SN}}\,,\ \ M = [K^{(1)}, K^{(2)}]_{\text{SN}}\,, \tag{6.34}$$

where M is the new rank-3 Killing tensor with the following components:

$$
\begin{aligned}
M^{\mu_1\mu_2\mu_2} &= -\frac{4\mu_1(\mu_1^2+\mu_2^2-1)}{3} = \mu_2^2 M^{\phi_2\phi_2\mu_1}\,, \\
M^{\mu_1\mu_1\mu_2} &= \frac{4\mu_2(\mu_1^2+\mu_2^2-1)}{3} = \mu_1^2 M^{\phi_1\phi_1\mu_2}\,.
\end{aligned}
\tag{6.35}
$$

Provided no additional irreducible rank-2 Killing tensors exist in this spacetime, M is also irreducible. This tensor further generates rank-4 Killing tensors via SN brackets with $K^{(1)}$ and $K^{(2)}$, and so on.

6.5 Summary

Starting in four dimensions, we have seen how a "small modification" of the linear in a expansion of the exact Kerr–(A)dS black hole solution gives rise to an, in many ways, preferred slowly rotating geometry. This *generalized Lense–Thirring spacetime*, is (when taken as an exact metric) manifestly regular on the black hole horizon and admits an exact Killing tensor.

This observation fills an important gap in understanding as to how the exact hidden symmetries of the full Kerr geometry emerge as the rotation is switched on. While the non-rotating (spherical) solution admits an exact principal Killing–Yano and Killing tensor, these are trivial, the latter being reducible—given by a product of Killing vectors derived from the rotational symmetry (and possibly time independence). Adding a small rotation to $O(a)$ breaks the full rotational symmetry and the approximate hidden symmetries become non-trivial. Remarkably a simple modification of the metric at $O(a^2)$ yields a spacetime which in 4 dimensions is of the Benenti class of spacetimes [110–112] in which separability of the Klein–Gordon and Hamilton–Jacobi equations is guaranteed. The exact irreducible Killing tensor can be understood as a (truncated) version of the approximate Killing tensor generated from the approximate principal Killing–Yano tensor.

Naturally, a similar construction also works in higher dimensions, which we have explicitly demonstrated for Kerr–AdS spacetimes in all dimensions, however the structure is much richer. The corresponding generalized Lense–Thirring spacetimes admit a rapidly growing tower of exact rank-2 and higher-rank Killing tensors, that is a "slow rotation seed" of the associated (much smaller) tower of rank-2 Killing tensors for the full Kerr-AdS geometry. Although the higher-dimensional Lense–Thiring spacetime (6.19) is not explicitly in the Benenti form, there are enough mutually commuting ones to guarantee the exact integrability and separability of the Hamilton–Jacobi and Klein–Gordon equations. It is an open question regarding higher spin fields and moreover whether these improved Lense–Thirring spacetimes can be put into canonical form. Intriguingly the tower of Killing tensors grows faster than the number of Killing vectors—-providing the first example of a physically interesting spacetime with larger number of hidden symmetries than the explicit ones.

It remains an interesting open direction whether similar construction would also work for higher order expansions in rotation parameters, providing thus even a more complete link between the generalized Lense–Thirring spacetimes and the exact black hole solutions and perhaps signaling how to do the Newman–Janis trick in new settings.

Chapter 7

Generalized Lense–Thirring Metrics: Applications

Although a number of exact solutions with rotation are known in Einstein gravity [8, 12, 57, 60, 255, 256], for general theories of gravitation (or in the presence of various matter fields) they are quite difficult to obtain. On the other hand slowly rotating solutions are somewhat easier to come by, as we saw in the previous chapter, in the context of Einstein gravity.

Perhaps the nicest example is that of Lovelock gravity, where the slowly rotating solution for a (single) rotation parameter has been known for some time [262, 263], and the equations of motion can be solved exactly[1]. There are other interesting higher-curvature theories, beyond Lovelock gravity, which have been the focus of some study—examples appear in Einstein Gauss–Bonnet gravity in $d = 4$ and $d = 5$ [268]. Moreover, the slowly rotating solutions of Einsteinian Cubic Gravity were studied in [269], while the case of five-dimensional cubic and quartic quasi-topological gravities were studied in [270], allowing for two independent rotation parameters in five dimensions. These cases are interesting examples where theories that generally have fourth-order equations of motion reduce to second-order equations of motion for a particular case of interest.

However, in each case the slowly rotating solutions must be obtained numerically, and are more complicated than the corresponding Einstein gravity solutions. More general solutions have been obtained in the context of four-dimensional effective field theory, see

[1] Attempts to analytically push beyond the slowly rotating regime have generally been met with failure, except in certain special cases [264–266], though numerical studies suggest the full rotating solutions exist [267].

e.g. [271], [272–274], and dynamical Chern–Simons gravity [275,276]. Even within the realm of Einstein-Maxwell gravity, exact charged solutions are not known in higher dimensions, or even in four dimensions when non-linear generalizations of Maxwell's theory are considered. Here one of our aims is to study slowly rotating solutions in more general theories of gravity, and also with matter.

The purpose of this chapter is to take the general slowly rotating template presented in (6.19) and see how it fits within these examples of non-vacuum or non Einstein spacetimes. We will also state the theorem proved in ref. [251] from which this chapter (and parts of the previous chapter are derived). This theorem, due to explicit calculations of R. A. Hennigar, demonstrates how special Einstein gravity is and what one needs to relax if one seeks to potentially construct rotating solutions beyond the bounds of ordinary (vacuum) general relativity. We will then present three examples demonstrating that (6.19) fits all of these known examples and shows why its form needs to be generic.

7.1 Einstein Slowly Rotating Black Holes are Special

For completeness we now summarize one of the main results in the original work [251] from which this chapter is based. The Einstein-like form (6.20) of any slowly rotating black hole is very special, and will not be generic enough to remain a solution to the field equations when higher-curvature terms and/or matter are added to the action.

Theorem 7.1.1 *In $d = 4$ dimensions, Einstein gravity is the only nontrivial gravitational theory up to powers quartic in the Riemann curvature tensor, whose (vacuum) generalized Lense–Thirring solutions* (6.19) *are of the form* (6.20)*.*

The proof of this essentially follows from direct calculations. One considers all the possible independent curvature scalars up to quartic power in the curvature [277] (at this order there are 26 such terms). One then substitutes the ansatz (6.19) into the equations of motion and imposes the Einstein like form. Then one constructs an asymptotic solution and finds, by going to high enough powers in a $1/r$ expansion, that the requirement for $p_i = (1 - f)$ identically fixes the relevant extra coupling constants[2] in the extended action to zero. Thus the only theories that are left are the trivial ones.

But what does trivial mean in this context? Here nontrivial means theories that lead to genuine corrections to the metric, rather than simply admitting the Einstein gravity

[2]Strictly speaking it is the independent combinations that are not related by curvature identities.

solution as one possible solution. There are a few different kinds of theories that we lump into the term "trivial". For example:

1. There are dimension-dependent identities for the Riemann tensor that force certain terms to vanish identically for all metrics below a critical dimension. For example, the cubic Lovelock density vanishes identically in five and lower dimensions.

2. There are certain terms that are topological invariants and so make no dynamical contributions to the field equations in certain dimensions. For example, this is the case for the k^{th} Lovelock Lagrangian in $d = 2k$.

3. There are certain combinations of curvature tensors that, while not identically zero for all metrics, may make no contribution to a certain class of metrics, but are not topological invariants.

4. Theories that are constructed solely from the Ricci curvature. Such theories, in the asymptotically flat case, will admit the Einstein gravity solution as one particular solution to the equations of motion.

In higher dimensions there are other nontrivial theories that admit solutions of the form (6.20). Examples include Lovelock gravity in all dimensions, and a subset of quartic generalized quasi topological gravities [278] in five dimensions. When matter is included, one easily finds examples with more general N or p_{ij}. As we shall see in Sec. 7.2, an especially interesting example is that of a slowly rotating black hole in minimal gauged supergravity where the "Chern–Simons" term mixes rotation parameters in various rotation 2-planes, resulting in distinct p_{ij} in each of the planes.

This theorem implies that the regular Newman–Janis trick, which generates slowly rotating solutions (valid to linear order in the rotation parameter) in four dimensions, will only work for Einstein gravity (or these trivial extensions). This is because the trick preserves the Einstein form (6.20). However, in higher dimensions or with matter, it may be possible to generate such slowly rotating solutions via the trick. Of course, this does not immediately extend to the construction of fully rotating solutions in these theories, but one may wonder if the construction of full rotating solutions is easier in these theories than in the general case.

7.2 Generalized Lense–Thirring Solutions with Matter

To demonstrate the required versatility of the improved Lense-Thirring metric (6.19) let us apply it to a few examples. Namely, we write down the (Maxwell) charged Kerr-AdS solutions (example 1), the slowly rotating Kerr–Sen solution (example 2), and the solution of the $d = 5$ minimal gauged supergravity (example 3). These three examples are illustrative of the following features.

- Example 1 shows that even with matter fields, one may still find solutions in the form (6.20).
- Example 2 provides an example of the generalized Lense–Thirring spacetime with non-trivial N.
- Example 3 shows the possibility of having distinct metric functions p_{ij} (distinct also from $f - 1$) and illustrates a very interesting effect of mixing of rotation parameters, induced by the Chern–Simons coupling.

The latter two examples are constructed by taking the slow rotation limit of the full exact black hole solutions.

7.2.1 Example 1: Charged Kerr-AdS in All Dimensions

Importantly we can actually find slowly rotating but (possibly strongly) charged solutions of Einstein–Maxwell theory:

$$\mathcal{L}_M = \frac{1}{16\pi}\Big(R - F_{ab}F^{ab} + \frac{(d-1)(d-2)}{\ell^2}\Big)\,. \tag{7.1}$$

Let us begin in 4 dimensions and perform the linear in a expansion to the exact Kerr–Newman–(A)dS metric (5.1). This yields, upon completing the square, the following approximate to $O(a)$ solution of the Einstein–Maxwell-Λ equations:

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 \sin^2\theta \Big(d\phi + \frac{a(f-1)}{r^2} dt\Big)^2 + r^2 d\theta^2\,, \\ A &= -\frac{q}{r}\big(dt - a\sin^2\theta d\phi\big) + O(a^2)\,, \end{aligned} \tag{7.2}$$

where

$$f = 1 - \frac{2M}{r} + \frac{q^2}{r^2} + \frac{r^2}{\ell^2}\,. \tag{7.3}$$

Thus we see the $4D$ Einstein Maxwell systems has the special property (6.20). Let us finally mention that we have the following asymptotic charges:

$$M = m\,, \quad J = ma\,, \quad Q = q\Big(1 + \frac{2a^2}{3\ell^2}\Big)\,, \tag{7.4}$$

and (7.2) is surrounded by (charged) matter. To linear order in a, the corresponding first law of black hole thermodynamics coincides with that of the spherical charged AdS black hole, e.g. [237].

Moreover, we may define the following 2-form:

$$h^{(0)} = db^{(0)} \quad 2b^{(0)} = r^2 dt\,, \tag{7.5}$$

obtained by the $a \to 0$ limit of the 2-form (6.5). While this is not a principal tensor even to the linear order in a, it yields the above exact Killing tensor (6.8). We also note that

$$\xi^{(0)} = -\frac{1}{3}\nabla \cdot h^{(0)} = \partial_t + a\Big(\frac{q^2}{3r^4} - \frac{1}{\ell^2}\Big)\partial_\phi\,, \tag{7.6}$$

which is an *exact* Killing vector when $q = 0$.

Interestingly, the property (6.20) holds in all dimensions, and the solution takes the form (6.19), with $N = 1$ and

$$p_i = f - 1 = -\frac{m}{r^{d-3}} + \frac{q^2}{r^{2(d-3)}} + \frac{r^2}{\ell^2}\,, \tag{7.7}$$

where m and q are parameters related to mass and charge, respectively. The metric is accompanied by the Maxwell field, $F = dA$, where the vector potential A takes the following form:

$$A = -\sqrt{\frac{d-2}{2(d-3)}}\frac{q}{r^{d-3}}\Big[dt - \sum_{i=1}^{m}\Big(a_i\mu_i^2 d\phi_i - \frac{a_i^2\mu_i^2 p_i}{r^2 f}dr\Big)\Big]\,, \tag{7.8}$$

where the last term was introduced in order that the field invariant $F_{ab}F^{ab}$ be finite on the horizon, $f = 0$. Note, that there is no known higher dimensional version of the Kerr–Newman spacetime but perhaps novel insights can be found from these new slowly rotating solutions.

7.2.2 Example 2: Slowly Rotating Kerr–Sen Spacetime

Let us next consider the 4-dimensional Kerr–Sen [57] low-energy effective heterotic string theory rotating black hole solution, of chapter 5, which in the standard Boyer–Lindquist-type coordinates and the string frame reads [57, 100] (see also (5.10)):

$$ds^2 = e^{-\Phi}\Big(-\frac{\Delta_b}{\rho_b^2}\big(dt - a\sin^2\theta d\phi\big)^2 + \frac{\rho_b^2}{\Delta_b}dr^2 + \frac{\sin^2\theta}{\rho_b^2}\Big[adt - (r^2 + 2\,b\,r + a^2)d\phi\Big]^2 + \rho_b^2 d\theta^2\Big)\,,$$
$$\mathfrak{B} = \frac{2abr}{\rho_b^2}\,\sin^2\theta dt\wedge d\phi\,, \quad A = -\frac{Q\,r}{\rho_b^2}\big(dt - a\,\sin^2\theta d\phi\big)\,, \quad e^{-\Phi} = \frac{\rho^2}{\rho_b^2}\,, \tag{7.9}$$

where the metric functions are given by (5.11) (see also (5.12))in chapter 5. Again M is the mass of the black hole, Q its charge, a its rotation parameter, and $b = Q^2/(2M)$ is the twist parameter.

Taking the $\mathcal{O}(a)$ expansion, and completing the square we recover the following generalized Lense–Thirring solution with non-trivial N and p:

$$\begin{aligned} ds^2 &= -Nf dt^2 + \frac{dr^2}{f} + r^2\sin^2\theta\Big(d\phi + \frac{ap}{r^2}dt\Big)^2 + r^2 d\theta^2\,, \\ A &= -\frac{Q}{r+2b}\big(dt - a\sin^2\theta d\phi\big)\,, \\ B &= \frac{2ab}{r+2b}\sin^2\theta dt\wedge d\phi\,, \quad e^{\Phi} = 1 + \frac{2b}{r}\,, \end{aligned} \tag{7.10}$$

where

$$f = 1 - \frac{2(M-b)}{r}\,, \quad N = \Big(1 + \frac{2b}{r}\Big)^{-2}\,, \quad p = N\Big(f - 1 - \frac{2b}{r}\Big)\,. \tag{7.11}$$

Being in the similar Benenti form as the 4D Einstein and Einstein Maxwell case the corresponding exact Killing tensor is the same as above in (6.8).

7.2.3 Example 3: Chong–Cvetič–Lü–Pope Solution

Finally, let us consider the $d = 5$ minimal gauged supergravity Chong–Cvetič–Lü–Pope solution [60] of chapter 4:

$$\mathcal{L} = \frac{1}{16\pi}\Big(*(R - 2\Lambda) - \frac{1}{2}F\wedge *F + \frac{1}{3\sqrt{3}}F\wedge F\wedge A\Big)\,, \tag{7.12}$$

where $F = dA$. Contrary to the original paper and chapter 4, we write it in a coordinate system that rotates at infinity as follows:

$$
\begin{aligned}
ds^2 &= d\gamma^2 - \frac{2q\nu\omega}{\Sigma} + \frac{\sigma\omega^2}{\Sigma^2} + \frac{\Sigma dr^2}{\Delta} + \frac{\Sigma d\theta^2}{S}\,, \\
A &= \frac{\sqrt{3}q\omega}{\Sigma}\,,
\end{aligned}
\tag{7.13}
$$

where we have defined

$$
\begin{aligned}
\nu &= \frac{ab}{\ell^2}dt - b\sin^2\theta d\phi - a\cos^2\theta d\psi\,, \quad \omega = dt + \frac{a\sin^2\theta d\phi}{\Xi_a} + \frac{b\cos^2\theta d\psi}{\Xi_b}\,, \\
d\gamma^2 &= \frac{\sin^2\theta}{\Xi_a}\Big[(r^2+a^2)d\phi^2 - \frac{2a}{\ell^2}(r^2+a^2)dtd\phi - \frac{dt^2}{\ell^2}\big(\rho^2 - (r^2+a^2)\frac{a^2}{\ell^2}\big)\Big] \\
&\quad + \frac{\cos^2\theta}{\Xi_b}\Big[(r^2+b^2)d\psi^2 - \frac{2b}{\ell^2}(r^2+b^2)dtd\psi - \frac{dt^2}{\ell^2}\big(\rho^2 - (r^2+b^2)\frac{b^2}{\ell^2}\big)\Big]\,,
\end{aligned}
\tag{7.14}
$$

and

$$
\begin{aligned}
S &= \Xi_a\cos^2\theta + \Xi_b\sin^2\theta\,, \quad \Delta = \frac{(r^2+a^2)(r^2+b^2)\rho^2/\ell^2 + q^2 + 2abq}{r^2} - 2m\,, \\
\Sigma &= r^2 + a^2\cos^2\theta + b^2\sin^2\theta\,, \quad \rho^2 = r^2 + \ell^2\,, \\
\Xi_a &= 1 - \frac{a^2}{\ell^2}, \quad \Xi_b = 1 - \frac{b^2}{\ell^2}, \quad \sigma = 2m\Sigma - q^2 + \frac{2abq}{\ell^2}\Sigma\,.
\end{aligned}
\tag{7.15}
$$

The black hole rotates in two different directions, corresponding to the rotation parameters a and b, while the parameter q is related to the black hole charge[3], and m to its mass. As we have seen in detail, the spacetime admits a principal Killing–Yano tensor with torison [99], which generates an exact Killing tensor.

Taking the linear $\mathcal{O}(a)$ and $\mathcal{O}(b)$ limit, and completing the square we obtain the following generalized Lense–Thirring solution:

$$
\begin{aligned}
ds^2 &= -fdt^2 + \frac{dr^2}{f} + r^2d\theta^2 + r^2\sin^2\theta\Big(d\phi + \frac{ap_{aa}+bp_{ab}}{r^2}dt\Big)^2 + r^2\cos^2\theta\Big(d\psi + \frac{bp_{bb}+ap_{ba}}{r^2}dt\Big)^2\,, \\
A &= \frac{\sqrt{3}q}{r^2}\Big(dt - a\sin^2\theta d\phi - b\cos^2\theta d\psi + \frac{a[ap_{aa}+bp_{ab}]\sin^2\theta + b[bp_{bb}+ap_{ba}]\cos^2\theta}{r^2 f}dr\Big)\,,
\end{aligned}
\tag{7.16}
$$

[3]Compare the form of the metric (4.32), and to the symmetric gauge used previously for the potential (4.33).

where

$$f = 1 - \frac{2m}{r^2} + \frac{q^2}{r^4} + \frac{r^2}{\ell^2}\,, \quad p_{aa} = f - 1 = p_{bb}\,, \quad p_{ab} = \frac{q}{r^2} = p_{ba}\,. \tag{7.17}$$

This solution provides a unique example where the presence of Chern–Simons charge mixes the rotation parameters in the metric functions p_{ab}, which are distinct from each other and distinct from $(f-1)$. The spacetime admits the following Killing tensor

$$K = \frac{1}{\sin^2\theta}(\partial_\phi)^2 + \frac{1}{\cos^2\theta}(\partial_\psi)^2 + (\partial_\theta)^2\,, \tag{7.18}$$

inherited from the approximate Killing–Yano tensor with torsion.

7.3 Summary

In this chapter we have explored the generalized Lense–Thirring metric of (6.19) in various different settings to demonstrate that, it is not just a nice mathematical spacetime, but also forms a good ansatz for slowly rotating black holes. In particular we have stated the theorem of [251] which shows how special the Einstein form (6.20) is. Moreover, we have considered slowly rotating approximations to a number of known exact solution rotating black holes that include matter. In doing so, we have provided examples where each additional term in the generalized ansatz is non-trivial.

Finally, for future work, it would be interesting to expand on the calculation of R.A. Hennigar to go beyond the asymptotic $(1/r^n)$ solutions in [251] and construct the full generalized Lense–Thirring solutions in higher curvature gravities. At the same time, it would be interesting to extend our considerations to other effective theories with matter fields, for example, involving a metric and a scalar. In this context, it is known that the Horndeski theory corresponding to the four-dimensional limit of Gauss-Bonnet gravity [279–281] allows for a Lense–Thirring metric characterized by the static metric function [282]. One may then wonder if this is the unique Horndeski theory with that property.

Chapter 8

Summary and Outlook

The work presented here is the culmination of six different projects, on three related themes, exploiting and exploring hidden symmetries in black holes spacetimes. Thus, this thesis is naturally divided into three parts: part I on the separability of the conformal wave equation [181, 182], part II on the separability of massive vector fields in spacetimes beyond Einstein gravity [203, 204], and part III on the hidden symmetries of rotating black holes [250, 251].

The first two themes presented here arose as the results of two Perimeter Scholars International Winterschool (PSI) projects which I co-supervised along with my advisor D. Kubizňák and naturally involved collaboration. This is similar to the second chapter of the third theme. This collaboration, and chance for supervision, has been one of the most beneficial and fruitful aspects of my PhD and I am grateful for the multiple ongoing opportunities I have had. Having said that all the content presented here, except where explicitly stated, is my own original work or arising as a result of these collaborations and having been checked by me.

I now summarize the main results before turning to some possible future directions in the web of hidden symmetries, separability, and integrability.

8.1 Recapitulation of Main Results

The first part of the thesis was devoted to demonstrating separability of the conformally coupled scalar field equation in general (off-shell) Kerr–NUT–(A)dS spacetimes in arbitrary D dimensions. At first we presented calculations in the canonical coordinates, in which the separability properties for these spacetimes are manifest, and demonstrated that the separability is characterized by the existence of a complete set of mutually commuting operators. Importantly these can be constructed from the principal Killing–Yano tensor. The associated Hamilton–Jacobi equation with a scalar curvature potential was also separated.

Then we saw that the symmetry operators have a covariant expression constructed from the principal Killing–Yano tensor, its "symmetry descendants", and the curvature tensor. Also, using this covariant construction we demonstrated these operators fit into the general theory of symmetry operators, and give rise to the full set of conformally invariant mutually commuting operators. This underlies the R-separability of the conformal wave equation in the entire conformal class of the Kerr–NUT–(A)dS spacetimes.

The second part was devoted to applying the LFKK ansatz 4.1 to separate the massive vector field (Proca) equations (corresponding to ultralight vector bosons) in two spacetimes beyond general relativity. Importantly, these spacetimes only possess a weaker version of the principal tensor from which the usual separability properties are derived. Specifically, we considered the Chong–Cvetič–Lü–Pope black hole spacetime of $D = 5$ minimal gauged supergravity (SUGRA) and the low-energy heterotic string theory inspired $D = 4$ Kerr–Sen black hole posses a torsion generalization of the principal Killing–Yano tensor. Here the torsion is identified with an inherent 3-form in the spacetime. In each case, this torsion modifies the Proca equations and it is these equations that separate using the LFKK ansatz. Then, as a physical application of the formal frame work we saw a comparison of the superradiant instability modes of the Proca fields for the Kerr–Newman to the Kerr–Sen black hole. This showed the Kerr–Sen black holes are more unstable than Kerr–Newman ones.

The final part of the thesis constructs of a new class of solutions with exact Killing tensors. That is, we considered a very general extension of the long standing Lense–Thirring spacetime, which describes a field of a slowly rotating body. These spacetimes, solve the field equations to linear order in the rotation parameters and admit an exact Killing tensor. The improved spacetimes feature many benefits compared to the old Lense–Thirring templates. In the first chapter we focused on these benefits, in particular, its rich (exact) hidden symmetry structure, and ability to be put in Painlevé–Gullstrand form (which also implies manifest regularity at the horizon). It is this feature which allows them to be interpreted as slowly rotating *black holes* not just massive bodies. This is the first example of a physically motivated spacetime with more hidden than explicit symmetries. Next with the theorem derived in [251] we demonstrated that the standard form of the (vacuum) Lense–Thirring metric is unique to Einstein gravity. Finally, we discussed a few applications of this result to show how it reflects the slow rotation limits of known solutions beyond the vacuum Einstein equations. Each of these cases required a particular facet of the new Lense–Thirring metrics.

8.2 Outlook and Future Directions

As remarkable and powerful as the principal tensor is, so is it restrictive. In fact as mentioned previously, it uniquely determines the Kerr–NUT–(A)dS class of spacetimes. Moreover, its best known generalization (the case with torsion we encountered in part II) only fits a couple of known examples and does not have a general characterization. Thus one may wonder how much further can we push these results, and, how seriously should we take such particular spacetimes?

The answer to the second part, I hope, is apparent at this stage given the number of astrophysically relevant and mathematically interesting examples we have seen in the introduction and discussed in detail in the thesis. So I shall try to address the second question with what I believe to be a fruitful road map as a separatist.

First and foremost (and most concretely) we should aim to close the gaps that yet remain for physical equations in the Kerr–NUT–(A)dS class of black holes. In particular, while it is known for this class of black holes how the principal tensor allows for the complete integrability of supersymmetric spinning particles in all dimensions [94], it is not clear how this applies to the case of classical spinning particles [283–285]. This area is currently undergoing a renaissance as it has become relevant for the extreme mass ratio formalism for inspiral processes in gravitational wave generation (e.g. [286]). The integrability of spinning particles beyond linear order (in the spin vector) remains an open question particularly, in higher dimensions. In a similar vein, the separability of geodesic deviation has be solved by the principal tensor [95], but one may wonder if this leaves any effects in the asymptotic regime. That is, is there some extra memory type effect around Kerr black holes due to Carter's constant (or more directly the principal tensor)? Finally, and most pressing in this direction, is the question of tensor perturbations. Can we use the principal tensor (in some kind of generalization of the LFKK ansatz (4.1)) to understand the separability of gravitational waves on the Kerr spacetime? In four dimensions this seems to be a straightforward next step but the general dimension case is (of course) much more involved.

Going beyond the Kerr–NUT–(A)dS class of spacetimes must also be at the forefront of new directions. If I may speculate there seem to be three natural directions to follow. First, near horizons and null surfaces, in general, there are many extra symmetries that emerge for generic spacetimes. So do hidden symmetries and the principal tensor manifest themselves in some way for spacetimes that are approximately Kerr like? Second, can we find a way to fully classify à la ref. [102] the torsion extension of the principal tensor? Third, the Benenti class of spacetimes which admit a separability structure is much larger than the Kerr–NUT–(A)dS one, so does this class admit more physically interesting spacetimes.

Ultimately the separatist must be asking: can we mathematically characterize the kind of spacetimes which allow for the separation of variables for the most interesting kinds of physical fields, i.e. spinor, vector, and tensor perturbations?

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APPENDICES

Appendix A

Killing objects

Here as a reference for completeness I list the plethora of Killing objects and their acronyms which appear in the text. Recalling the motivation comes from the particle motion governed by the Hamiltonian $H = -\frac{1}{2}g^{\mu\nu}p_\mu p_\nu$ and geodesics equations of motion $p^\mu \nabla_\mu p_\nu = 0$.

- Linear charge $\iff$ Killing vector (KV) equation: Projectable Hamiltonian vector (Explicit)

$$\mathcal{J}(k) = k^a p_a \iff \nabla_{(a}k_{b)} = 0\,, \quad \pi_*(X_{\mathcal{J}(k)}) = k^a \frac{\nabla_a}{\partial x}\,. \tag{A.1}$$

- Higher order $\iff$ Killing tensor (KT) equation

$$\mathcal{J}(K) = K^{a_1\ldots a_s}p_{a_1}\ldots p_{a_s} \iff \nabla_{(a}K_{a_1\ldots a_s)} = 0 \tag{A.2}$$

  Not projectable $\iff$ Dynamical/Hidden

$$\pi_*(X_{\mathcal{J}(K)}) = s\,K^{aa_1\ldots a_{s-1}}p_{a_1}\ldots p_{a_{s-1}}\frac{\nabla_a}{\partial x} \tag{A.3}$$

- Killing–Yano (KY) forms $f_{a_1\ldots a_p}$:

$$\nabla_a f_{a_1\ldots a_p} = \nabla_{[a}f_{a_1\ldots a_p]} \tag{A.4}$$

  which square to Killing tensors

$$K^f_{ab} = f_{ac_1\ldots c_{s-1}}f_b{}^{c_1\ldots c_{s-1}} \tag{A.5}$$

We can also generalize these symmetries to the conformal class of spacetimes $g \to \Omega^2 g$:

- Conformal Killing vectors (CKVs)

$$\nabla_{(a}k_{a)} = \alpha g_{ab} \tag{A.6}$$

- Conformal Killing tensors (CKTs)

$$\nabla_{(a}K_{a_1 \ldots a_s)} = g_{(aa_1}\alpha_{a_2 \ldots a_s)} \tag{A.7}$$

- Conformal Killing-Yano tensor (CKYTs)

$$\nabla_a f_{a_1 \ldots a_p} = \nabla_{[a} f_{a_1 \ldots a_p]} + \frac{p}{D-p+1} g_{a[a_1} \nabla^b f_{|b|a_2 \ldots a_{p-1}]}$$

  Closed-Conformal-Killing-Yano (CCKY) forms also satisfy $df = 0$. Any two Conformal Killing–Yano forms f_1, f_2 "square" to a conformal Killing tensor. That is,

$$K_f^{ab} = f_1{}^{(a}{}_{c_2 \ldots c_q} f_2{}^{b)c_2 \ldots c_q} \tag{A.8}$$

  is a CKT.

- The Principal tensor h that uniquely determines the Kerr–NUT–(A)dS class of spacetimes is a non degenerate closed conformal Killing–Yano (CCKY) 2 form

$$\nabla_a h_{bc} = g_{ab}\xi_c - g_{ac}\xi_b \tag{A.9}$$

  (A.9) $\implies$ ξ is a Killing vector.

Appendix B

Conformal Symmetry Operator Calculations

In this appendix we find the covariant form of functions $R_{(j)}$ appearing in the symmetry operators that guarantee the separability of the conformal wave equation in Kerr–NUT–(A)dS spacetimes. We also calculate their conformal behaviour.

B.1 Covariant Form of $R_{(j)}$

To start recall the Ricci tensor (1.125) and Killing tensors (1.123) are diagonal in the same orthonormal basis (1.121). A natural guess is that these functions depend on the Ricci tensor so we can calculate the difference $R_{(j)} - k_{(j)}^{ab} R_{ab}$ making use of another identity for the symmetric polynomials [37];

$$\sum_{\nu \neq \mu} \frac{A_\mu^{(j)} - A_\nu^{(j)}}{x_\nu^2 - x_\mu^2} = (n - j) A_\mu^{(j-1)} \, . \tag{B.1}$$

Then we have

$$R_{(j)} - k_{(j)}^{ab} R_{ab} = \sum_\mu \left[\varepsilon \frac{A_\mu^{(j-1)} x_\mu \hat{X}'_\mu}{U_\mu} + 2 \sum_{\nu\neq\mu} \frac{A_\mu^{(j)}}{x_\nu^2 - x_\mu^2} \left(\frac{x_\mu \hat{X}'_\mu - (1-\varepsilon)\hat{X}_\mu}{U_\mu} + \frac{x_\nu \hat{X}'_\nu - (1-\varepsilon)\hat{X}_\nu}{U_\nu} \right) \right]$$
$$= \sum_\mu \left[\varepsilon \frac{A_\mu^{(j-1)} x_\mu \hat{X}'_\mu}{U_\mu} + 2 \frac{x_\mu \hat{X}'_\mu - (1-\varepsilon)\hat{X}_\mu}{U_\mu} \sum_{\nu\neq\mu} \frac{A_\mu^{(j)} - A_\nu^{(j)}}{x_\nu^2 - x_\mu^2} \right]$$
$$= 2 \sum_\mu \frac{A_\mu^{(j-1)}}{U_\mu} \Big([n - j + \varepsilon/2] x_\mu \hat{X}'_\mu - (n-j)(1-\varepsilon)\hat{X}_\mu \Big) \ . \tag{B.2}$$

Moreover it is also natural to expect that $R_{(j)}$ may depend on the Killing vectors themselves. Since they satisfy $\nabla_{(a} l^{(j)}{}_{b)} = 0$ the only information in their derivatives is contained in their exterior derivative. In particular, since in the orthonormal basis (1.121)

$$l^{(j)} = \sum_\mu A_\mu^{(j)} \sqrt{Q_\mu} \hat{e}^\mu + \epsilon A^{(j)} \sqrt{\frac{c}{A^{(n)}}} e^0 \,, \tag{B.3}$$

we have that

$$dl^{(j)} = \sum_\mu \left[\left(A_\mu^{(j)} Q_\mu \frac{X'_\mu}{X_\mu} - \varepsilon \frac{2}{x_\mu} \frac{cA^{(j)}}{A^{(n)}} + 2x_\mu \sum_{\nu\neq\mu} \frac{Q_\mu A_\mu^{(j)} + Q_\nu A_\nu^{(j)}}{x_\nu^2 - x_\mu^2} \right] e^\mu \wedge \hat{e}^\mu \right.$$
$$\left. + \varepsilon 2 x_\mu \sqrt{Q_\mu} \sqrt{\frac{c}{A^{(n)}}} A_\mu^{(j-1)} e^\mu \wedge e^0 + \sum_{\nu\neq\mu} 2x_\nu \sqrt{Q_\mu Q_\nu} \frac{A_\mu^{(j)} - A_\nu^{(j)}}{x_\nu^2 - x_\mu^2} e^\nu \wedge \hat{e}^\mu \right) \ . \tag{B.4}$$

Furthermore, introducing the Killing co-potential

$$(D - 2j - 1)\omega_{ab}^{(j)} = k_{(j)}{}^n{}_a h_{nb} = \sum_\mu A_\mu^{(j)} x_\mu e^\mu \wedge \hat{e}^\mu \ . \tag{B.5}$$

which generates the Killing tensors [37]

$$l_{(j)}^a = \nabla_b \, \omega_{(j)}^{ba} \,, \tag{B.6}$$

we can calculate

$$k_{(j)n}^{\ \ a} h^{nb} \, dl_{ab}^{(k)} = 2 \sum_\mu \frac{1}{U_\mu} \left(A_\mu^{(j)} A_\mu^{(k)} x_\mu \hat{X}'_\mu + \sum_{\nu\neq\mu} \frac{2\hat{X}_\mu (A_\mu^{(j)} A_\mu^{(k)} x_\mu^2 - A_\nu^{(j)} A_\nu^{(k)} x_\nu^2) - \varepsilon c A_\mu^{(j)} \left(\frac{A_\nu^{(k)}}{U_\nu} - \frac{A_\mu^{(k)}}{U_\mu} \right)}{x_\nu^2 - x_\mu^2} \right) \ . \tag{B.7}$$

Notice the last term proportional to is ε and vanishes when $k = 0$. Finally let us calculate $\Box\,\mathrm{Tr}\big(k_{(j)}\big)$. First, we have

$$\mathrm{Tr}\big(k_{(j)}\big) = \varepsilon A^{(j)} + \sum_{\mu} 2A^{(j)}_{\mu} = (2(n-j)+\varepsilon)A^{(j)}\ . \tag{B.8}$$

Since this expression only depends on x_μ we can use the form of the wave operator (2.7) to write

$$\begin{aligned}\nabla_a(k^{ab}_{(j)}\nabla_b\,\mathrm{Tr}\big[k_{(j)}\big]) &= \sum_{\mu}\frac{A^{(j)}_{\mu}}{U_\mu}\left[X_\mu\partial^2_\mu\,\mathrm{Tr}\big(k_{(j)}\big) + \partial_\mu\,\mathrm{Tr}\big(k_{(j)}\big)\left(X'_\mu + \frac{\varepsilon}{x_\mu}X_\mu\right)\right] \\ &= 4\sum_{\mu}\frac{A^{(j)}_{\mu}A^{(j-1)}_{\mu}}{U_\mu}[n-j+\frac{\varepsilon}{2}]\left(x_\mu X'_\mu + (1+\varepsilon)X_\mu\right)\ . \end{aligned}\tag{B.9}$$

Putting this together we have

$$\begin{aligned}&\alpha_j k^{\ \ a}_{(j-1)n}h^{nb}\,dl^{(0)}_{ab} - \beta_j l^a_{(j-1)}\,l^{(0)}_a + \frac{D-4}{2(D-2)}\Box\,\mathrm{Tr}\big(k_{(j)}\big) \\ &= 2\sum_{\mu}\frac{1}{U_\mu}\left(A^{(j-1)}_{\mu}\left(\left[\alpha_j + \frac{(D-4)(n-j+\frac{\varepsilon}{2})}{D-2}\right]x_\mu\hat{X}'_\mu - \left[\frac{\beta_j}{2} - \frac{(D-4)(n-j+\frac{\varepsilon}{2})(1+\varepsilon)}{D-2}\right]\hat{X}_\mu\right)\right. \\ &\qquad\qquad \left. -2\alpha_j\hat{X}_\mu\sum_{\nu\neq\mu}\frac{A^{(j)}_{\mu}-A^{(j)}_{\nu}}{x^2_\nu - x^2_\mu}\right) \\ &= 2\sum_{\mu}\frac{A^{(j-1)}_{\mu}}{U_\mu}\left(\left[\alpha_j + \frac{(D-4)(n-j+\frac{\varepsilon}{2})}{D-2}\right]x_\mu\hat{X}'_\mu\right. \\ &\qquad\qquad \left. -\left[\frac{\beta_j}{2} + 2(n-j)\alpha_j - \frac{(D-4)(n-j+\frac{\varepsilon}{2})(1+\varepsilon)}{D-2}\right]\hat{X}_\mu\right)\ . \end{aligned}\tag{B.10}$$

Thus, using the fact that $\varepsilon = \{0, 1\}$ we can choose the coefficients to be

$$\alpha_j = \frac{2(n-j+\frac{\varepsilon}{2})}{D-2}\ , \tag{B.11}$$

$$\beta_j = \frac{4(n-j+\frac{\varepsilon}{2})}{D-2}(D-3-2(n-j+\frac{\varepsilon}{2}))\,. \tag{B.12}$$

Thence we obtain our covariant expression for $R_{(j)}$

$$R_{(j)} = k^{ab}_{(j)}R_{ab} + \frac{D-4}{2(D-2)}\Box\,\mathrm{Tr}\big(k_{(j)}\big) + \alpha_j k^{\ \ a}_{(j-1)n}h^{nb}\,dl^{(0)}_{ab} - \beta_j l^a_{(j-1)}\,l^{(0)}_a\ , \tag{B.13}$$

which matches the form in chapter 3, (3.9), upon noting $l_0 = \xi$ and $D = 2n + \varepsilon$.

B.2 Conformal Transformations

Given the spacetime $(\mathcal{M}, g)$ we now consider a conformal transformation of the metric, Killing tensors, and scalar field ($k_{(j)} \to \Omega^{-2} k_{(j)}$, $\Phi \to \Omega^w \Phi$ for $w = 1 - D/2$) to the conformal spacetime $(\mathcal{M}, g, \Omega)$. The goal of this section is to find a conformally covariant form of our wave operators

$$\big(\hat{\mathcal{K}}_{(j)} - \eta R_{(j)}\big)\Phi\,, \quad \hat{\mathcal{K}}_{(j)} = \nabla_a k^{ab}_{(j)} \nabla_b\,, \quad \eta = \frac{1}{4}\frac{D-2}{D-1}\,. \tag{B.14}$$

Using the conformal properties of the Ricci tensor and covariant derivatives, we find the following transformations

$$\begin{aligned}\Omega^2\, \hat{\mathcal{K}}_{(j)}\Phi \to &\\ &\Omega^w \Big(\hat{\mathcal{K}}_{(j)} + w\nabla_a (k^{ab}_{(j)} \nabla_b \log\Omega) \\ &+ w(w-2+D)\nabla_a \log\Omega\, k^{ab}_{(j)} \nabla_b \log\Omega\Big)\,\Phi\end{aligned} \tag{B.15}$$

and

$$\begin{aligned}\Omega^2\, k^{ab}_{(j)} R_{ab} \to &\\ &k^{ab}_{(j)} R_{ab} - \left[(D-2)k^{ab}_{(j)} + k_{(j)\,c}^{c}\, g^{ab}\right] \nabla_a \nabla_b \log\Omega \\ &+ (D-2)\left[k^{ab}_{(j)} - k_{(j)\,c}^{c}\, g^{ab}\right] \nabla_a \log\Omega \nabla_b \log\Omega\,.\end{aligned} \tag{B.16}$$

Thence we have

$$\begin{aligned}&\Omega^2 \left(\hat{\mathcal{K}}_{(j)}\Phi - \eta k^{ab}_{(j)} R_{ab}\Phi\right)/\Phi \to \\ &(\hat{\mathcal{K}}_{(j)}\Phi - \eta k^{ab}_{(j)} R_{ab}\Phi)/\Phi + w(\nabla_a k^{ab}_{(j)})\nabla_b \log\Omega + ((w + \eta(D-2))k^{ab}_{(j)} + \eta k^{c}_{(j)c} g^{ab})\,[\nabla_a \nabla_b \log\Omega] \\ &+ (w(w-2+D) - (D-2)\eta)k^{ab}_{(j)} + (D-2)\eta k^{c}_{(j)c} g^{ab})\,[\nabla_a \log\Omega \nabla_b \log\Omega] \\ &= \left(\hat{\mathcal{K}}_{(j)}\Phi - \eta k^{ab}_{(j)} R_{ab}\Phi\right)/\Phi + w(\nabla_a k^{ab}_{(j)})\nabla_b \log\Omega - \eta D\, \hat{k}^{ab}_{(j)}\,[(D-2)\nabla_a \log\Omega \nabla_b \log\Omega + \nabla_a \nabla_b \log\Omega]\,.\end{aligned} \tag{B.17}$$

Here we have introduced the traceless Killing tensor $\hat{k}^{ab}_{(j)} = k^{ab}_{(j)} - k_{(j)\,c}^{c}\, g^{ab}/D$. Clearly this vanishes when $j = 0$ so the first operator is conformally invariant. Notice that the last term contains two derivatives of the conformal factor, so consider the identically zero term (following from the Killing tensor equation (4.41))

$$\nabla_a \nabla_b \left(k^{ab}_{(j)} + \frac{1}{2} k_{(j)\,c}^{c}\, g^{ab}\right) \equiv 0\,. \tag{B.18}$$

Under the transformation $k_{(j)} \to \Omega^2 k_{(j)}$ this becomes

$$\Omega^2\, \nabla_a \nabla_b \left(k^{ab}_{(j)} + \frac{1}{2} k^{c}_{(j)\,c} g^{ab} \right) \to \nabla_a \nabla_b \left(k^{ab}_{(j)} + \frac{1}{2} k^{c}_{(j)\,c} g^{ab} \right) + (D+2)(\nabla_a k^{ab}_{(j)}) \nabla_b \log \Omega \\ + D\, \hat{k}^{ab}_{(j)} \left[(D-2) \nabla_a \log \Omega \nabla_b \log \Omega + \nabla_a \nabla_b \log \Omega \right] . \qquad \text{(B.19)}$$

So we have

$$\Omega^2 \left(\hat{\mathcal{K}}_{(j)} \Phi - \eta \left[k^{ab}_{(j)} R_{ab} - \left\{ \nabla_a \nabla_b \left(k^{ab}_{(j)} + \frac{1}{2} k^{c}_{(j)\,c} g^{ab} \right) \right\} \right] \Phi \right) / \Phi \to \\ \left(\hat{\mathcal{K}}_{(j)} \Phi - \eta \left[k^{ab}_{(j)} R_{ab} - \left\{ \nabla_a \nabla_b \left(k^{ab}_{(j)} + \frac{1}{2} k^{c}_{(j)\,c} g^{ab} \right) \right\} + (D-4)(\nabla_a k^{ab}_{(j)}) \nabla_b \log \Omega \right] \Phi \right) / \Phi . \qquad \text{(B.20)}$$

Note that, as the covariant derivatives and Killing tensors in the second line are in the $\Omega = 1$ frame, we have
$(D-4)(\nabla_a k^{ab}_{(j)}) \nabla_b \log \Omega = -(D-4)/2\, (\nabla_a k^{c}_{(j)\,c}) \nabla_b \log \Omega$. Thus this term will be cancelled by the transformation of $\Box\, \mathrm{Tr}(k_{(j)})$. That is,

$$\frac{D-4}{2(D-2)} \Box\, \mathrm{Tr}(k_{(j)}) \to \Omega^{-2} \left(\frac{D-4}{2(D-2)} \Box\, \mathrm{Tr}(k_{(j)}) + \frac{D-4}{2} \nabla_a \left[\mathrm{Tr}(k_{(j)}) \right] \nabla^a \log \Omega \right) . \qquad \text{(B.21)}$$

We now consider the conformal transformation of the final piece;

$$\mathcal{R}_{(j)} := \alpha_j k_{(j-1)n}^{\;\;a} h^{nb}\, dl^{(0)}_{ab} - \beta_j l^{a}_{(j-1)}\, l^{(0)}_{a} . \qquad \text{(B.22)}$$

Now, if $k_{(j)} \to \Omega^{-2} k_{(j)}$ consistency demands that $h \to \Omega^2 h$ and that $l^a_{(j)} \to \Omega^{-2} l^a_{(j)}$. That is, one can show on a p form $\star \to \Omega^{d-2p} \star$. Assuming $h \to \Omega^r h$; $h^j \to \Omega^{jr} h^j$ then $f^{(j)} = \star h^j \to \Omega^{d-4j+jr} f^{(j)}$. So

$$k^{(j)}_{ab} \propto f_{ac_1 \dots c_{D-2j-1}} f_b^{\;c_1 \dots c_{D-2j-1}} \to \Omega^{2(d-4j+jr)+2(D-2j-1)} k^{(j)}_{ab} = \Omega^{2+2j(-2+r)} k^{(j)}_{ab} . \qquad \text{(B.23)}$$

Hence demanding for all j that $k^{(j)}_{ab} \to \Omega^2 k^{(j)}_{ab}$ fixes $r = 2$. Then, we are left with $\mathcal{R}_{(j)}$ as a scalar density of weight -2:

$$\mathcal{R}_{(j)} \to \Omega^{-2}\, \mathcal{R}_{(j)} . \qquad \text{(B.24)}$$

Thus putting this all together we have

$$\Omega^2 \left[\left(\hat{\mathcal{K}}_{(j)} - \eta \left[R_{(j)} - \left\{ \nabla_a \nabla_b \left(k^{ab}_{(j)} + \frac{1}{2} k^{c}_{(j)\,c} g^{ab} \right) \right\} \right] \right) \Phi \right] / \Phi \to \\ \left[\left(\hat{\mathcal{K}}_{(j)} - \eta \left[R_{(j)} - \left\{ \nabla_a \nabla_b \left(k^{ab}_{(j)} + \frac{1}{2} k^{c}_{(j)\,c} g^{ab} \right) \right\} \right] \right) \Phi \right] / \Phi , \qquad \text{(B.25)}$$

which gives us the form we use in the chapter 3.

Appendix C

Separation of Proca equations in Kerr–Sen background

C.1 Carter Form of the Metric

The separation of the modified Proca equation (5.23) in the Kerr–Sen background is easiest when the metric is expressed in the pseudo-Euclidean Carter-like coordinates $(\psi_0, \psi_1, x_1, x_2)$ [37]. In fact, in these coordinates a more general solution to the heterotic string theory action (5.8), which includes a NUT parameter, can easily be written and reads [100]:

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{U_1}{X_1}\mathrm{d}x_1^2 + \frac{U_2}{X_2}\mathrm{d}x_2^2 + \frac{X_1}{U_1}A_1^2 + \frac{X_2}{U_2}A_2^2\,, \quad e^{\Phi} = \frac{U_m}{U_1}\,,\\
A &= \frac{2\,c\,s}{U_m}\Big(m_1 x_1(\mathrm{d}\psi_0 + x_2^2\,\mathrm{d}\psi_1) - m_2 x_2(\mathrm{d}\psi_0 + x_1^2\mathrm{d}\psi_1)\Big)\,,\\
\mathfrak{B} &= \frac{s}{c}\big(\mathrm{d}\psi_0 - c_0\mathrm{d}\psi_1\big)\wedge A\,,
\end{aligned} \tag{C.1}$$

with the field strengths $F = \mathrm{d}A$ and $H = \mathrm{d}\mathfrak{B} - A\wedge F$.[1] Here

$$\begin{aligned}
U_1 &= x_2^2 - x_1^2 = -U_2\,, \quad U_m = x_2^2 - x_1^2 - 2m_1 s^2 x_1 + 2m_2 s^2 x_2\,,\\
A_1 &= \frac{U_1}{U_m}\Big[\mathrm{d}\psi_0 + \mathrm{d}\psi_1(x_2^2 + 2m_2 x_2 s^2)\Big]\,, \quad A_2 = \frac{U_1}{U_m}\Big[\mathrm{d}\psi_0 + \mathrm{d}\psi_1(x_1^2 + 2m_1 x_1 s^2)\Big]\,,
\end{aligned} \tag{C.2}$$

[1] To recover the notations in the main text, one has to set $A \to A/\sqrt{2}$. The other fields Φ, $\mathcal{B}$, and H are unchanged but the coupling between H and A picks up a factor of 2 to compensate the redefinition of A, that is, $H \to \mathrm{d}\mathcal{B} - 2A\wedge F = H$.

and the metric functions take the following form:

$$X_1 = c_0 - 2m_1 x_1 + x_1^2\,, \quad X_2 = c_0 - 2m_2 x_2 + x_2^2\,. \tag{C.3}$$

Here, $s = \sinh\delta$, $c = \cosh\delta$, and c_0, m_1, m_2, δ are arbitrary constants, related to the rotation parameter, mass and NUT charges, and the twist parameter.

The Kerr–Sen solution in the main text is recovered upon the following change of coordinates and parameters:

$$(\psi_0, \psi_1, x_1, x_2) = (t - a\,\phi, \phi/a, i\,r, a\cos\theta)\,. \tag{C.4}$$

Here we also send $m_1 s^2 \to ib$, $im_1 \to (b - M)$, turn off the NUT parameter by setting $m_2 = 0$, and send $c_0 \to -a^2$. Thence the metric functions become

$$\begin{aligned} X_1 &= 2r(M-b) - r^2 - a^2 \equiv -\Delta_b\,, \quad X_2 = -a^2\sin^2\theta\,, & \text{(C.5)}\\ U_1 &= r^2 + a^2\cos^2\theta = \rho^2\,, \quad U_m = \rho_b^2 = \rho^2 + 2br\,. & \text{(C.6)} \end{aligned}$$

In what follows we shall work with the more general metric (C.1)–(C.3). In fact, as already observed for the Kerr–NUT-(A)dS metrics [90,91], the separability actually works for a more general class of *off-shell metrics* where

$$X_\mu = X_\mu(x_\mu) \tag{C.7}$$

are arbitrary functions of one variable. Thence in what follows we shall leave $X_\mu(x_\mu)$ arbitrary.

The off-shell metric admits a generalized principal tensor with torsion, which reads [180]

$$h = x_1\,\mathrm{d}x_1 \wedge A_1 + x_2\,\mathrm{d}x_2 \wedge A_2\,, \tag{C.8}$$

and obeys the defining equation (5.16) with

$$\xi = e^{\Phi}\partial_{\psi_0}\,, \tag{C.9}$$

and the torsion identified with the 3-form H,

$$T = H = \frac{s}{c}F \wedge \xi = -\left(\frac{\partial\Phi}{\partial x_1}\,\mathrm{d}x_1 \wedge A_1 + \frac{\partial\Phi}{\partial x_2}\,\mathrm{d}x_2 \wedge A_2\right) \wedge \xi\,. \tag{C.10}$$

The associated irreducible Killing tensor is given by

$$K_{ab} = h_{ac}h^c_{\ b} - \frac{1}{2}g_{ab}h^2\,. \tag{C.11}$$

C.2 Separability of Proca Equations

Let us now apply the LFKK ansatz to separate the Proca equations in the generalized beckground (C.1). As argued in Sec. 5.2 the Proca equation in this background reads

$$\nabla_n \left(e^{\Phi}\mathcal{F}^{na}\right) - m^2 e^{\Phi} P^a = 0\,, \tag{C.12}$$

and implies the corresponding Lorenz condition

$$\nabla_a \left(e^{\Phi} P^a\right) = 0\,. \tag{C.13}$$

In order to separate these equations, we employ the LFKK ansatz [89–92],

$$P^a = B^{ab}\nabla_b Z\,, \quad B^{ab}(g_{bc} + i\mu h_{bc}) = \delta^a_c\,, \tag{C.14}$$

where as before μ is a complex parameter, h_{bc} in the generalized principal tensor (C.8), and the potential function Z is written in the multiplicative separated form

$$Z = R_1(x_1)R_2(x_2)e^{iL_0\psi_0}e^{iL_1\psi_1}\,. \tag{C.15}$$

Similar to refs. [90, 91] we first concentrate on the Lorenz condition (C.13), for which the ansatz (C.14) yields:

$$\nabla_a \left(e^{\Phi} P^a\right) = e^{\Phi}\frac{Z}{q_1 q_2}\left(\frac{q_2}{U_1}\frac{1}{R_1(x_1)}\mathcal{D}_1 R_1(x_1) + \frac{q_1}{U_2}\frac{1}{R_2(x_2)}\mathcal{D}_2 R_2(x_2)\right)\,, \tag{C.16}$$

where the differential operators are given by

$$\begin{aligned}\mathcal{D}_\mu = q_\mu \frac{\partial}{\partial x_\mu}\left[\frac{X_\mu}{q_\mu}\frac{\partial}{\partial x_\mu}\right] - \frac{1}{X_\mu}\Big[(-x_\mu^2 - 2m_\mu s^2 x_\mu)L_0 + L_1\Big]^2 \\ - \frac{2-q_\mu}{\mu q_\mu}\Big[L_0 + (-\mu^2)L_1\Big] - \frac{4\mu L_0\, m_\mu s^2 x_\mu}{q_\mu}\,,\end{aligned} \tag{C.17}$$

and

$$q_\nu = 1 - \mu^2 x_\nu^2\,. \tag{C.18}$$

The Lorenz condition (C.13) will be satisfied provided the mode functions R_ν obey the separated equations

$$\mathcal{D}_\nu R_\nu = (C_1 - x_\nu^2 C_0)R_\nu\,. \tag{C.19}$$

Here C_0 and C_1 are two new separation constants. Then expression in (C.16) reduces to

$$\nabla_a(e^{\Phi}P^a) = e^{\Phi}\frac{Z}{q_1 q_2}\left[C_0 + C_1(-\mu^2)\right] , \tag{C.20}$$

and we see that the Lorenz condition holds provided we fix

$$C_1 = \frac{C_0}{\mu^2} . \tag{C.21}$$

At this stage we are left with one new separation constant C_0 but this will be fixed by solving the full Proca equations.

The results of [90] can be also used to find the representation of the Proca equation (C.12) for the ansatz (C.14). Employing the Lorenz condition (C.13) one finds

$$\nabla_n\left(e^{\Phi}\mathcal{F}^{na}\right) - m^2 e^{\Phi}P^a = e^{\Phi}B^{am}\nabla_m J . \tag{C.22}$$

Here we have introduced the object (c.f. (4.13))

$$J = e^{-\Phi}\nabla_a(e^{\Phi}g^{ab}\nabla_b Z) - 2i\mu\xi_a B^{ab}\nabla_b Z - m^2 Z . \tag{C.23}$$

At this stage, by employing the LFFK ansatz and enforcing the Lorenz condition, the Proca equation has been reduced to solving a scalar "wave equation".

In particular this "wave equation" may be written as an eigenvalue problem

$$\hat{g}Z = m^2 Z , \tag{C.24}$$

where

$$\hat{g} = e^{-\Phi}\nabla_a(e^{\Phi}g^{ab}\nabla_b) - 2i\mu V_a g^{ab}\nabla_b , \quad V^a = \xi_b B^{ba} . \tag{C.25}$$

In this suggestive form, where we consider the metric tensor as the trivial Killing tensor $K^{(0)}_{ab} = g_{ab}$, one can guess from the Kerr–Newman case that this operator can be generalized to the two commuting operators in 4 dimensions, which are enough to guarantee the separability of this equation. Thus we define

$$\hat{K} = e^{-\Phi}\nabla_a(e^{\Phi}K^{ab}\nabla_b) - 2i\mu V_a K^{ab}\nabla_b , \tag{C.26}$$

where K_{ab} is the Killing tensor generated from the principal tensor (C.11). Then one can explicitly check that these two operators commute, i.e.

$$\left[\hat{g}, \hat{K}\right] = 0 , \tag{C.27}$$

and that the solution Z is also an eigenvector of $\hat{K}$

$$\hat{K}Z = \left(\frac{m^2}{\mu^2} + \frac{L_0}{\mu} + \mu L_1\right) Z \,. \tag{C.28}$$

These operators are just a torsion generalization of those presented in [90,180,217] and we expect that this construction can be generalized to all dimensions.

Thus the separability of J (C.23) is guaranteed and in fact J separates in the form

$$J = Z \sum_{\nu=1}^{n} \frac{1}{U_\nu} \frac{1}{R_\nu} \big[\mathcal{D}_\nu - m^2(-x_\nu^2)\big] R_\nu \,, \tag{C.29}$$

where D_ν is same the operator defined in (C.17). In the above expression we have used the identity,

$$\sum_{\nu=1}^{n} \frac{1}{U_\nu} \, (-x_\nu^2)^{1-j} = \delta_0^j \tag{C.30}$$

for $j = 0$, to rewrite the mass term. This identity further ensures $J = 0$, provided the modes $R_\nu(x_\nu)$ obey separated equations (C.19) and additionally the extra free separation constant C_0 is given by the Proca mass,

$$C_0 = m^2 \,. \tag{C.31}$$

Summarizing, the Proca equation (C.12) for the vector field P in the off-shell Kerr–Sen–NUT background (C.1) can be solved by using the LFKK ansatz (C.14), (C.15), where the mode functions R_ν satisfy ODEs (C.19). Moreover the separation constant C_0 is given by (C.31) and μ satisfies (C.21). To translate our separation (C.19) back into the Boyer–Lindquist form presented in the main text, (5.27), we perform the map outlined above in (C.4), (C.5). Furthermore, we need to modify the eigenvalues L_0 and L_1, to the eigenvalues of $i\partial_t$ and $-i\partial_\phi$, ω and m_ϕ. This is simply done via the linear map

$$L_0 = -\omega \,, \qquad L_1 = a(m_\phi - a\omega) \,. \tag{C.32}$$

Appendix D

Some Details of the Slowly Rotating Black Holes

D.1 Regular Slow Rotation Expansion of Kerr

In this appendix we attempt to physically motivate the improved Lense–Thirring form of the metric (6.19). To do this, we construct a slowly rotating variant of the Lense–Thirring solution starting from the Kerr metric, writing it in Kerr ingoing coordinates, expanding to linear order in the rotation parameter a, and returning back to the Boyer–Lindquist coordinates. This yields a metric that is a vacuum solution of Einstein equations to linear order in the rotation parameter a, and that is manifestly regular on the horizon when taken 'as is'. Since the transformation between Kerr and Boyer–Lindquist coordinates involves the rotation parameter, this metric carries certain $\mathcal{O}(a^2)$ corrections, slightly distinct, however, from the improved Lense–Thirring solution (6.19).

Let us start from the Kerr metric written in the standard Boyer–Lindquist coordinates:

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\Big[adt - (r^2+a^2)d\phi\Big]^2\,, \\ \Sigma &= r^2 + a^2\cos^2\theta\,, \quad \Delta = r^2 + a^2 - 2mr\,, \end{aligned} \tag{D.1}$$

and perform the following coordinate transformation to Kerr coordinates $\{v, \chi, r, \theta\}$:

$$dv = dt + \frac{r^2+a^2}{\Delta}dr\,, \quad d\chi = d\phi + \frac{a}{\Delta}dr\,. \tag{D.2}$$

By expanding the resultant metric to linear order in a we obtain:

$$\begin{aligned} ds^2 &= -f dv^2 + 2dvdr - \frac{4Ma}{r}\sin^2\theta dv d\chi - 2a\sin^2\theta d\chi dr + r^2\sin^2\theta d\chi^2 + r^2 d\theta^2\,, \\ f &= 1 - \frac{2M}{r}\,. \end{aligned} \tag{D.3}$$

Using now the inverse transform to linear order in a:

$$dv = dt + \frac{dr}{f}\,, \quad d\chi = d\phi + \frac{a}{r^2 f}dr\,, \tag{D.4}$$

we thus recover the following metric:

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\theta^2 + r^2\sin^2\theta\Big(d\phi + \frac{a(f-1)}{r^2}dt\Big)^2 \\ &\quad -\frac{a^2\sin^2\theta}{r^2 f^2}\Big(f(1-f)dt + dr\Big)^2\,. \end{aligned} \tag{D.5}$$

Note that the first line is the improved Lense–Thirring metric (6.19), while dropping all $\mathcal{O}(a^2)$ terms gives the ordinary Lense-Thirring metric.

D.2 Improved Lense–Thirring Spacetimes: Further Considerations

D.2.1 Orthonormal Frame

Consider the following generalized Lense–Thirring metric (6.19) with a slight change of notation: rather than denoting by ν, the constrained coordinate in even dimensions, we use $\mu_{m+\varepsilon}$, then the metric and constraint simplify to

$$ds^2 = -Nf dt^2 + \frac{dr^2}{f} + r^2\sum_{i=1}^{m}\mu_i^2\Big(d\phi_i + \sum_{j=1}^{m}\frac{a_j p_{ij}}{r^2}dt\Big)^2 + r^2\sum_{i=1}^{m+\varepsilon}d\mu_i^2\,, \tag{D.6}$$

and

$$\sum_{i=1}^{m+\varepsilon}\mu_i^2 = 1\,. \tag{D.7}$$

Then one can calculate the orthonormal vielbeins $e^A = e^A_a\, dx^a$ which satisfy

$$g_{ab} = \eta_{AB} e^A_a e^B_b \,. \tag{D.8}$$

We find

$$\begin{aligned} e^{\hat{t}} &= \sqrt{Nf}dt\,, \quad e^{\hat{r}} = \frac{dr}{\sqrt{f}}\,, \quad e^{\hat{\phi}_i} = r\mu_i \left(d\phi_i + \sum_{j=1}^{m} \frac{a_j p_{ij}}{r^2} dt \right)\,, \\ e^{\hat{\mu}_i} &= \frac{r}{\sqrt{\mu^2_{m+\varepsilon} + \sum\limits_{j=1}^{i-1} \mu_j^2}} \left(\sqrt{\mu^2_{m+\varepsilon} + \sum_{k=1}^{i} \mu_k^2}\; d\mu_i + \sum_{l>i}^{m-(1-\varepsilon)} \frac{\mu_i \mu_l d\mu_l}{\sqrt{\mu^2_{m+\varepsilon} + \sum_{k=1}^{i} \mu_k^2}} \right)\,, \end{aligned} \tag{D.9}$$

with the following inverse $e_A = e^a_A \partial_a$

$$\begin{aligned} e_{\hat{t}} &= \frac{1}{\sqrt{Nf}} \left(\partial_t - \sum_{j=1}^{m} \frac{a_j p_{ij}}{r^2} \partial_{\phi_j} \right)\,, \quad e_{\hat{r}} = \sqrt{f}\partial_r\,, \quad e_{\hat{\phi}_i} = \frac{1}{r\mu_i}\,\partial_{\phi_i}\,, \\ e_{\hat{\mu}_i} &= \frac{\sqrt{\mu^2_{m+\varepsilon} + \sum\limits_{j=1}^{i-1} \mu_j^2}}{r} \left(\frac{\partial_{\mu_i}}{\sqrt{\mu^2_{m+\varepsilon} + \sum_{k=1}^{i} \mu_k^2}} - \left(\frac{\mu_i}{\mu^2_{m+\varepsilon} + \sum_{k=1}^{i-1} \mu_k^2} \right) \sum_{l=1}^{i-1} \frac{\mu_l \partial_{\mu_l}}{\sqrt{\mu^2_{m+\varepsilon} + \sum_{k=1}^{i} \mu_k^2}} \right)\,. \end{aligned} \tag{D.10}$$

D.2.2 Killing Tensors

The separability and integrability of the spacetime requires m Killing tensors[1] in addition to the $1 + (m + \varepsilon)$ Killing vectors ∂_t and ∂_{ϕ_j}. Now, we have seen the the metric (6.19) has a fast growing (with the number of dimensions) tower of exact Killing tensors. Explicitly, as a reminder let us recall the following: given the set $S = \{1, .., m\}$, let $I \in P(S)$ where $P(S)$ is the power set of S, then we have the exact rank-2 Killing tensors c.f (6.28):

$$K^{(I)} = \sum_{i \notin I}^{m-1+\varepsilon} \left[\Big(1 - \mu_i^2 - \sum_{j\in I} \mu_j^2\Big)(\partial_{\mu_i})^2 - 2 \sum_{j \notin I \cup \{i\}} \mu_i \mu_j\, \partial_{\mu_i} \partial_{\mu_j} \right] + \sum_{i \notin I}^{m} \left[\frac{1 - \sum_{j\in I} \mu_j^2}{\mu_i^2} (\partial_{\phi_i})^2 \right]. \tag{D.11}$$

[1]One of these is, of course, the trivial Killing tensor, i.e. the metric.

Moreover, we have verified up to $d = 13$ that the SN bracket (1.42) of any two Killing tensors vanishes if the intersection of the two set labels equals one of the two. That is, if $I_1 \cap I_2 = I_1$ or if $I_1 \cap I_2 = I_2$,

$$[K^{(I_1)}, K^{(I_2)}]_{\text{SN}} = 0\,. \tag{D.12}$$

In particular, we find there is a subset of these Killing tensors that are diagonal in the orthonormal basis. Let us denote $\tilde{m} = m - (1-\varepsilon)$, and then define $Q \subset P(S)$ by[2]

$$Q = \Big\{\emptyset, \{\tilde{m}\}, \{\tilde{m}, \tilde{m}-1\}, \ldots, \{\tilde{m}, \tilde{m}-1, \tilde{m}-2, \ldots, 2\}\Big\}\,. \tag{D.13}$$

Then for all $J \in Q$

$$K^{(J)} = r^2\left(1 - \sum_{j\in J}\mu_j^2\right)\left(\sum_{i\notin J}^{\tilde{m}} e_{\hat{\mu}_i}e_{\hat{\mu}_i} + \sum_{i\notin J}^{m} e_{\hat{\phi}_i}e_{\hat{\phi}_i}\right)\,. \tag{D.14}$$

Since these elements of Q satisfy the following nesting property:

$$\emptyset \subset \{\tilde{m}\} \subset \{\tilde{m}, \tilde{m}-1\} \subset \cdots \subset \{\tilde{m}, \tilde{m}-1, \tilde{m}-2, \ldots, 2\}\,, \tag{D.15}$$

these Killing tensors all mutually Schouten–Nijenhuis (SN) commute by (D.12), guaranteeing the separability of the Hamilton–Jacobi equation [37].

Moreover, one can check they satisfy Carter's criterion [154] $\nabla_a(k_\gamma{}^{[\alpha}R^{\beta]\gamma}) = 0$. Hence they define commuting operators

$$\mathcal{K}^{(J)} \equiv \nabla^a K_{ab}^{(J)}\nabla^b \tag{D.16}$$

with the Klein–Gordon operator $\nabla^a g_{ab}\nabla^b$ for scalars. That is,

$$\left[\mathcal{K}^{(J)}, \nabla^a g_{ab}\nabla^b\right] = 0\,. \tag{D.17}$$

Thus, we have now d commuting operators for the Klein–Gordon equation (including $\mathcal{L}_j = \xi_a^{(j)}\nabla^a$ for the Killing vectors $\xi^{(j)} = \partial_t, \partial_{\phi_j}$). Therefore we know the Hamilton–Jacobi and Klein–Gordon equations separate in these spacetimes [37].

[2] Note that there are exactly $m-1$ elements in Q – one from each 'level' of subsets in $P(S)\backslash S$.