

Almost-Sure Stability of a Noisy Autoparametric Vibration Absorber

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis investigates the almost-sure stability of the single mode solution of a two degree-of-freedom, noisy, nonlinear autoparametric system. While only the first mode is forced in such a system, the nonlinear coupling often transfers energy to the second mode. Equations of motion of autoparametric systems model the dynamics of a number of structural and mechanical systems, such as, a randomly excited and initially deformed shallow arch, a suspended elastic cable driven by planar excitation, or a vessel subject to longitudinal wave action. To keep things as simple as possible, we consider a very simple system, namely, a type of autoparametric vibration absorber with randomly excited base - a pendulum attached to a mass-spring oscillator.

Under the assumption of small random perturbations and small damping, the maximal Lyapunov exponent which determines the almost-sure stability of the single mode solution is calculated. Putting the maximal Lyapunov exponent to zero provides the second-order approximation of the almost-sure stability boundary in terms of the excitation intensity and the dissipation coefficients. A plot of this stability boundary reveals several trends of practical importance to engineering applications.

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Table of Contents

List of Figures	vii
List of Tables	viii
1 Introduction	1
1.1 Introduction to the Problem	1
1.2 Outline of the Thesis	4
2 Model of the Autoparametric Vibration Absorber	5
2.1 Derivation of Equations of Motion	6
2.2 Nondimensionalization of Equations of Motion	8
2.3 Periodic Excitation	12
3 Dynamics and Stability of Linear Stochastic Systems	14
3.1 Fundamental Concepts	14
3.1.1 Markov and Diffusion Processes	14
3.1.2 The Generator of a Diffusion Process	18
3.1.3 Evolution Equations for Diffusion Processes	18
3.1.4 Stochastic Differential Equations	21
3.1.5 The Feynman-Kac Formula	24
3.1.6 Stochastic Stability and Lyapunov Exponents	25

3.1.7	Power Spectral Density	29
3.2	Linear Oscillator Excited by White Noise	30
3.2.1	White Noise	31
3.2.2	System Response	32
3.2.3	Mean Response	34
3.2.4	Covariance Response	35
3.2.5	Applying White Noise Properties	37
3.2.6	Response Stationary Density	38
3.2.7	Response Power Spectral Density	39
4	Stochastic Stability of the Autoparametric Vibration Absorber	40
4.1	Asymptotic Scaling	41
4.2	Single Mode Solution	42
4.3	Linear Variational Equation	43
4.4	Coordinate Transformation	48
4.5	Furstenberg-Khas'minskii Formula	51
4.6	Asymptotic Analysis	53
4.6.1	The Leading Order Equation	54
4.6.2	Poisson Equations and the Feynman-Kac Formula	56
4.6.3	The ε -Order Equation	57
4.6.4	The ε^2 -Order Equation	60
4.7	Stability Results	66
5	Conclusion	72
	References	74
	APPENDICES	78
A	Radial and Phase Derivatives for Itô's Lemma	79

List of Figures

2.1	The autoparametric vibration absorber model	6
2.2	Variation of steady-state response amplitudes with excitation amplitude for periodic excitation	13
4.1	Almost-sure stability boundaries for $\zeta_p = 0.1$ and different values of ζ_o . . .	70
4.2	Almost-sure stability boundaries for $\zeta_o = 0.1$ and different values of ζ_p . . .	71

List of Tables

2.1	Parameter values based on [10] used for numerical simulation	13
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Chapter 1

Introduction

1.1 Introduction to the Problem

Systems in which the variables of interest, referred to as *state variables*, evolve with time according to a governing rule are called *dynamical systems*. These systems are vastly studied as they occur in a wide variety of scientific, engineering and mathematical fields, and even in other technical fields such as economics and finance. Typical examples of state variables across these fields include the concentration of a chemical in a solution during a reaction, the population of a species in a given geographical region, the motion of a physical structure (e.g., a building) under the action of an external driving force (e.g., an earthquake), or the price of a company's stocks.

Mathematically, the rules that determine how the state variables of dynamical systems evolve with time are typically modelled as *initial value problems* (IVPs) of the form

$$\dot{x}(t) = f(x(t), t), \quad t \geq 0, \quad x(0) = x_0 \in \mathbb{R}^n, \quad (1.1)$$

where $f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ and $x(t)$ is the state variable. In this thesis, we concern ourselves with a mechanical dynamical system modelled by a pair of second order differential equations of the form

$$\begin{aligned} \ddot{q}_1(t) + \zeta_1 \dot{q}_1(t) + f_1(q_1(t), q_2(t)) &= \beta(t) \\ \ddot{q}_2(t) + \zeta_2 \dot{q}_2(t) + f_2(q_1(t), q_2(t)) &= 0, \quad t \geq 0. \end{aligned} \quad (1.2)$$

Here, $q_1(t)$ and $q_2(t)$ are the *modes* of the mechanical system and are equivalently known as the *generalized coordinates*. It is worth noting that the system of two second order

differential equations (1.2) is equivalent to four first order differential equations of the form (1.1). That is, by an appropriate change of variables, (1.2) can be written in the form of (1.1) with $n = 4$. Since f_1 and f_2 are each functions of both q_1 and q_2 , q_1 acts as a parameter in the dynamics of q_2 and vice versa. This effect is known as *coupling* and systems whose dynamics are modelled by equations such as (1.2) are typically called *autoparametric systems*. This type of system often occurs in many branches of physics and engineering where the constants ζ_1 and ζ_2 represent *damping coefficients* and $\beta(t)$ is an *excitation* or *forcing* given to the system, in particular, to the first mode q_1 . We are interested in the situation where f_1 and f_2 are nonlinear functions, so there is therefore nonlinear coupling in the system which can lead to a transfer of energy from the forced mode q_1 to the unforced mode q_2 . In particular, these functions are typically of the form $f_i(q_1, q_2) = \omega_i^2 q_i + g_i(q_1, q_2)$, where ω_1 and ω_2 are parameters that represent the natural frequencies of q_1 and q_2 respectively. As will be explained in more detail shortly, it is the resonant and commensurable relationships between these natural frequencies that is the primary cause of the transfer of energy. We are then interested in analyzing how this energy transfer affects the *stability* of the system. The techniques used in performing this analysis depend greatly upon the nature of the excitation $\beta(t)$.

There have been several studies on periodically excited autoparametric systems; for example, see [5, 10, 11, 26, 36, 39]. The manner in which the excitation is applied determines the resonant frequency and hence the mode in resonance. If the mechanical system is directly excited (i.e., external excitation), then a mode whose natural frequency is close to the frequency of the excitation, is in resonance. On the other hand, if the effect of excitation is through modulating a parameter in the system (i.e., parametric excitation), then a mode is said to be in resonance if the excitation frequency is close to twice the natural frequency of the mode. In the case of parametric excitation of finite degree-of-freedom systems, a combination resonance can also occur when the excitation frequency is close to some linear combination of the natural frequencies of multiple modes. Thus, in contrast with the case of external excitation for which a small excitation produces a large response only when the excitation frequency is close to a linear natural frequency, a small parametric excitation can produce a large response when the frequency of the excitation is close to twice the natural frequency of any mode or some linear combination of the natural frequencies of the system.

The two degree-of-freedom nonlinear system (1.2) exhibits another interesting phenomenon under periodic excitation called *internal resonance*. In such a system, the internal resonance phenomenon occurs when two or more of the natural frequencies are commensurable or nearly commensurable. Depending on the order of the nonlinearity present in the system, the commensurable relationships of the linear natural frequencies can result

in strong coupling of the corresponding modes. For a system with quadratic nonlinearities, such as the one studied in this thesis, strong coupling between the modes may exist when the natural frequencies of the forced and unforced mode are in one-to-two internal resonance leading to the most fascinating situations. When the amplitude of the periodic excitation is increased, thereby increasing the energy in the forced mode, it reaches a certain value at which saturation occurs and the energy is transferred to the unforced mode. This causes the parametric excitation of the unforced mode which may or may not be desirable depending on the particular system being modelled by (1.2), and has an effect on the stability of the system.

In this thesis, we concern ourselves with the scenario where the system is given a noisy (non-deterministic) excitation. In other words, we are interested in investigating the random vibrations of a nonlinear, stochastically forced system, where the forcing $\beta(t)$ is a stationary random process. In particular, we would like to find out the value of the intensity of the noisy input at which energy saturation could potentially occur, affecting the stability of the system. Stability of a random dynamical system can be defined in different ways, one of the strongest of which is that of *almost-sure stability*. Although several papers [1, 23, 30, 31] have dealt with some aspects of these questions, none have given completely satisfactory answers. This is primarily due to the interactions between the excitation noise and the nonlinearities, and the manner in which the subsystem is parametrically excited by the original excitation noise as well as the stationary motion of the primary system. Sufficient almost-sure stability conditions were derived in [1] for the stability of the rest mode of a vibration-isolating suspension subjected to a broadband random base excitation. Onu et al. [31] applied a nonstandard method of stochastic averaging to reduce the dimension of a randomly-perturbed nonlinear vibration absorber, and numerically solved for the stationary probability distribution functions for the reduced Markov process. We would like to derive an analytical expression that states the condition on the intensity of the noisy input for the random dynamical system (1.2) to be almost-surely stable.

1.2 Outline of the Thesis

The following presents the outline for the remainder of the thesis. In Chapter 2, a model of the system of interest in this thesis is introduced and the equations of the form of (1.2) that govern the dynamics of this system are derived. A brief exposition into the results for periodic excitation is then presented. An introduction to the mathematical tools and techniques fundamental to the analysis of the dynamics and stability of linear stochastic systems is presented in Chapter 3. These concepts are essential in the study of the response and stability of dynamical systems subject to random excitations. They are then applied, in Chapter 4, to the model presented in Chapter 2 to analytically derive the condition for the almost-sure stability of the single mode solution of the stochastic system. Further, in Chapter 4, a perturbative technique based on [4, 24, 33] is utilized, which involves solving a series of partial differential equations to obtain successive terms in the expansion for the top Lyapunov exponent - a quantity that determines the almost-sure stability of stochastic systems. Chapter 5 closes the thesis by presenting relevant conclusions.

Chapter 2

Model of the Autoparametric Vibration Absorber

This chapter introduces the model being studied in this thesis and presents a derivation of the equations of motion that govern the dynamics of the system. Equations of motion of autoparametric systems model the dynamics of several mechanical and structural systems. In this thesis, the model, presented in Figure 2.1, is that of a simple autoparametric vibration absorber, just as in [10]. It consists of an *oscillator* of mass m_o vertically attached to a fixed horizontal surface by a spring with stiffness k and a damper with damping coefficient c_o . Attached to the oscillator is a *pendulum* with a massless rod of length l and a bob of mass m_p that makes an angle ϕ with the vertical and has some torsional damping c_p present in the joint where the pendulum is attached to the oscillator. The oscillator is given a random excitation in the form of a Gaussian white noise process $\Xi(t)$. The response of the oscillator, represented by $y(t)$ - its displacement from its equilibrium position, will therefore be random and so will that of the pendulum $\phi(t)$.

The oscillator can be thought of as representing a car moving on a road with a white noise profile. As the nonlinear coupling within the system results in autoparametric excitation of the pendulum, it acts as a vibration absorber that reduces the impact of the oscillations felt by the people in the car. In other words, the mass-spring oscillator represents the primary system being subjected to a random external excitation in the form of a Gaussian white noise process, and the pendulum is the parametrically excited absorber of vibrations from the primary system.

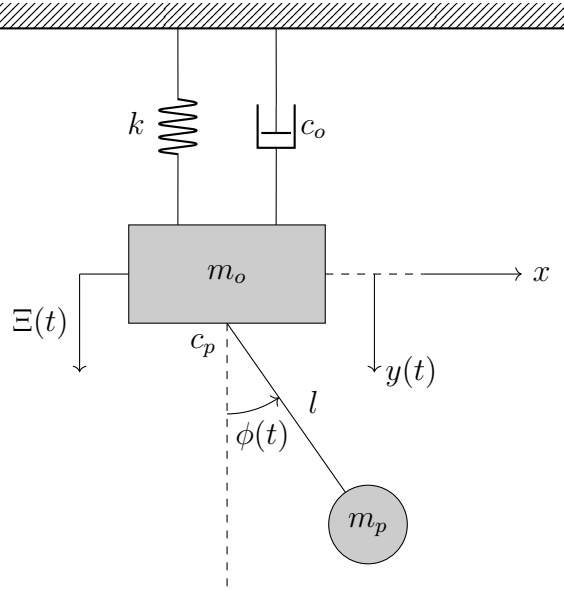


Figure 2.1: The autoparametric vibration absorber model

2.1 Derivation of Equations of Motion

While one could possibly use the Newtonian approach to derive the equations of motion for the system in Figure 2.1, the Lagrangian approach employed here is the go-to method for dealing with more complex systems, particularly those involving pendulums. This approach requires that we find a quantity creatively known as the *Lagrangian* \mathcal{L} and use the Euler-Lagrange equation to evaluate the equations of motion of the system. The Euler-Lagrange equation is given by

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} = \Lambda_j, \quad (2.1)$$

where $\mathcal{L} = T - V$

T : Total kinetic energy in the system

V : Total potential energy in the system

q_j : Generalized coordinates

Λ_j : Non-conservative forces in the generalized coordinates.

The total kinetic energy in the system is

$$T = \frac{1}{2}m_o\dot{y}^2 + \frac{1}{2}m_pv_p^2.$$

The velocity of the bob v_p , expressed in terms of its vertical and horizontal components, is

$$\begin{aligned} v_p^2 &= (\dot{y} - l\dot{\phi} \sin \phi)^2 + (l\dot{\phi} \cos \phi)^2 \\ &= \dot{y}^2 - 2l\dot{y}\dot{\phi} \sin \phi + l^2\dot{\phi}^2. \end{aligned}$$

The total kinetic energy then becomes

$$T = \frac{1}{2}(m_o + m_p)\dot{y}^2 + \frac{1}{2}m_pl^2\dot{\phi}^2 - m_pl\dot{y}\dot{\phi} \sin \phi.$$

The total potential energy in the system is

$$V = m_pgl(1 - \cos \phi) + \frac{1}{2}ky^2.$$

Note that y represents the displacement of the system from the equilibrium position after the static deflection of the spring has been accounted for. The Lagrangian then becomes

$$\begin{aligned} \mathcal{L} &= T - V \\ &= \frac{1}{2}(m_o + m_p)\dot{y}^2 + \frac{1}{2}m_pl^2\dot{\phi}^2 - m_pl\dot{y}\dot{\phi} \sin \phi - m_pgl(1 - \cos \phi) - \frac{1}{2}ky^2. \end{aligned} \quad (2.2)$$

The Euler-Lagrange equation (2.1) can now be used to obtain the system equations of motion by considering each generalized coordinate. Let $q_1 = y$. Equation (2.1) becomes

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{y}} \right) - \frac{\partial \mathcal{L}}{\partial y} = \Lambda_1.$$

Let us evaluate each term individually. Considering the non-conservative forces in the system in the generalized y -coordinate, $\Lambda_1 = \Xi(t) - c_o\dot{y}$. We have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{y}} &= (m_o + m_p)\dot{y} - m_pl\dot{\phi} \sin \phi \\ \therefore \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{y}} \right) &= (m_o + m_p)\ddot{y} - m_pl(\ddot{\phi} \sin \phi + \dot{\phi}^2 \cos \phi) \\ &\text{and} \\ \frac{\partial \mathcal{L}}{\partial y} &= -ky. \end{aligned}$$

Putting it all together, the equation of motion for the generalized coordinate y , which governs the motion of the oscillator, is obtained as follows:

$$(m_o + m_p)\ddot{y} - m_p l(\ddot{\phi} \sin \phi + \dot{\phi}^2 \cos \phi) + ky = \Xi(t) - c_o \dot{y}$$

$$\boxed{(m_o + m_p)\ddot{y} + c_o \dot{y} + ky - m_p l(\ddot{\phi} \sin \phi + \dot{\phi}^2 \cos \phi) = \Xi(t)}. \quad (2.3)$$

Similarly, letting $q_2 = \phi$ in (2.1) gives

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = \Lambda_2.$$

We again evaluate each term individually. Considering the non-conservative forces in the system in the generalized ϕ -coordinate, $\Lambda_2 = -c_p \dot{\phi}$. We have

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m_p l^2 \dot{\phi} - m_p l \dot{y} \sin \phi$$

$$\therefore \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = m_p l^2 \ddot{\phi} - m_p l(\dot{y} \dot{\phi} \cos \phi + \ddot{y} \sin \phi)$$

and

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m_p l \dot{y} \dot{\phi} \cos \phi - m_p g l \sin \phi.$$

Once again putting it all together, we obtain the equation of motion for the generalized coordinate ϕ , which governs the motion of the pendulum, as follows:

$$m_p l^2 \ddot{\phi} - m_p l(\dot{y} \dot{\phi} \cos \phi + \ddot{y} \sin \phi) + m_p l \dot{y} \dot{\phi} \cos \phi + m_p g l \sin \phi = -c_p \dot{\phi}$$

$$\boxed{m_p l^2 \ddot{\phi} + c_p \dot{\phi} + m_p l(g \sin \phi - \ddot{y} \sin \phi) = 0}. \quad (2.4)$$

2.2 Nondimensionalization of Equations of Motion

Nondimensionalization is the process of removing physical dimensions from an equation using fitting variable substitutions. Doing this is good practice as it aids in performing analyses and generalizing results irrespective of the original physical units in the system. It also facilitates the comparison of the magnitude of different parameters in the equation and their subsequent scaling, if necessary, and oftentimes simplifies the equation by reducing the number of terms in it.

The goal of this section is to nondimensionalize the equations of motion (2.3) and (2.4) of the system, derived in the previous section. We begin by recognizing that the variables in the system are y , ϕ and t , where only y and t need to be nondimensionalized since ϕ is already dimensionless. We then define dimensionless parameters

$$\hat{\eta} \stackrel{\text{def}}{=} \frac{y}{y_s} \quad \text{and} \quad \tau \stackrel{\text{def}}{=} \frac{t}{t_s}, \quad (2.5)$$

where y_s and t_s are parameters of such dimension that $\hat{\eta}$ and τ , respectively, are dimensionless. Using these definitions and the chain rule, we obtain dimensionless versions of the derivatives in the equations of motion as follows:

$$\begin{aligned} \dot{y} &= \frac{dy}{dt} = \frac{d\tau}{dt} \frac{dy}{d\tau} = \frac{1}{t_s} \frac{d(y_s \hat{\eta})}{d\tau} = \frac{y_s}{t_s} \frac{d\hat{\eta}}{d\tau} \\ \ddot{y} &= \frac{d}{dt}(\dot{y}) = \frac{d\tau}{dt} \frac{d}{d\tau} \left(\frac{y_s}{t_s} \frac{d\hat{\eta}}{d\tau} \right) = \frac{y_s}{t_s^2} \frac{d^2 \hat{\eta}}{d\tau^2} \\ \dot{\phi} &= \frac{d\phi}{dt} = \frac{d\tau}{dt} \frac{d\phi}{d\tau} = \frac{1}{t_s} \frac{d\phi}{d\tau} \\ \ddot{\phi} &= \frac{d}{dt}(\dot{\phi}) = \frac{d\tau}{dt} \frac{d}{d\tau} \left(\frac{1}{t_s} \frac{d\phi}{d\tau} \right) = \frac{1}{t_s^2} \frac{d^2 \phi}{d\tau^2}. \end{aligned} \quad (2.6)$$

Substituting these in (2.3) and normalizing the coefficient of the highest derivative to unity,

$$\begin{aligned} (m_o + m_p)\ddot{y} + c_o\dot{y} + ky - m_p l(\ddot{\phi} \sin \phi + \dot{\phi}^2 \cos \phi) &= \Xi(t) \\ (m_o + m_p) \frac{y_s}{t_s^2} \frac{d^2 \hat{\eta}}{d\tau^2} + \frac{c_o y_s}{t_s} \frac{d\hat{\eta}}{d\tau} + k y_s \hat{\eta} - m_p l \left(\frac{1}{t_s^2} \frac{d^2 \phi}{d\tau^2} \sin \phi + \left(\frac{1}{t_s} \frac{d\phi}{d\tau} \right)^2 \cos \phi \right) &= \Xi(t) \\ \hat{\eta}'' + \frac{c_o t_s}{m_o + m_p} \hat{\eta}' + \frac{k t_s^2}{m_o + m_p} \hat{\eta} - \frac{m_p l}{(m_o + m_p) y_s} (\phi'' \sin \phi + \phi'^2 \cos \phi) & \\ &= \frac{t_s^2}{(m_o + m_p) y_s} \Xi(t_s \tau). \end{aligned}$$

Now, let us introduce more definitions:

$$y_s \stackrel{\text{def}}{=} l, \quad t_s^2 \stackrel{\text{def}}{=} \frac{m_o + m_p}{k}, \quad \omega_o^2 \stackrel{\text{def}}{=} \frac{k}{m_o + m_p} \implies t_s = \frac{1}{\omega_o}. \quad (2.7)$$

Substituting these,

$$\begin{aligned}\hat{\eta}'' + \frac{c_o}{m_o + m_p} \sqrt{\frac{m_o + m_p}{k}} \hat{\eta}' + \hat{\eta} - \frac{m_p l}{(m_o + m_p)l} (\phi'' \sin \phi + \phi'^2 \cos \phi) \\ = \frac{1}{(m_o + m_p)l} \frac{m_o + m_p}{k} \Xi(\tau/\omega_o) \\ \hat{\eta}'' + \frac{c_o}{\sqrt{k(m_o + m_p)}} \hat{\eta}' + \hat{\eta} - \frac{m_p}{m_o + m_p} (\phi'' \sin \phi + \phi'^2 \cos \phi) = \frac{\Xi(\tau/\omega_o)}{kl}.\end{aligned}$$

At this stage, the equation has been nondimensionalized, however, introducing further definitions helps simplify it:

$$2\hat{\zeta}_o \stackrel{\text{def}}{=} \frac{c_o}{\sqrt{k(m_o + m_p)}}, \quad R \stackrel{\text{def}}{=} \frac{m_p}{m_o + m_p}, \quad \hat{\xi}(\tau) \stackrel{\text{def}}{=} \frac{\Xi(\tau/\omega_o)}{kl}, \quad \hat{\theta}(\tau) \stackrel{\text{def}}{=} \phi(\tau/\omega_o). \quad (2.8)$$

For convenience, t will now be used to represent the dimensionless time parameter τ and the dot notation will be employed for the derivatives. The equation of motion for the oscillator in dimensionless coordinates then becomes

$$\boxed{\ddot{\hat{\eta}} + 2\hat{\zeta}_o \dot{\hat{\eta}} + \hat{\eta} - R(\ddot{\hat{\theta}} \sin \hat{\theta} + \dot{\hat{\theta}}^2 \cos \hat{\theta}) = \hat{\xi}(t)}. \quad (2.9)$$

Following the same procedure for (2.4), we start by replacing the derivatives with their dimensionless counterparts and normalizing the coefficient of the highest derivative to unity:

$$\begin{aligned}m_p l^2 \ddot{\phi} + c_p \dot{\phi} + m_p l (g \sin \phi - \ddot{y} \sin \phi) &= 0 \\ \frac{m_p l^2}{t_s^2} \frac{d^2 \phi}{d\tau^2} + \frac{c_p}{t_s} \frac{d\phi}{d\tau} + m_p l \left(g \sin \phi - \frac{y_s}{t_s^2} \frac{d^2 \hat{\eta}}{d\tau^2} \sin \phi \right) &= 0 \\ \phi'' + \frac{c_p t_s}{m_p l^2} \phi' + \frac{t_s^2}{l} \left(g \sin \phi - \frac{y_s}{t_s^2} \hat{\eta}'' \sin \phi \right) &= 0 \\ \phi'' + \frac{c_p t_s}{m_p l^2} \phi' + \frac{g t_s^2}{l} \sin \phi - \frac{y_s}{l} \hat{\eta}'' \sin \phi &= 0 \\ \hat{\theta}'' + \frac{c_p \sqrt{m_o + m_p}}{m_p l^2 \sqrt{k}} \hat{\theta}' + \frac{g t_s^2}{l} \sin \hat{\theta} - \hat{\eta}'' \sin \hat{\theta} &= 0,\end{aligned}$$

where we have used the following previous definitions that were made:

$$t_s \stackrel{\text{def}}{=} \sqrt{\frac{m_o + m_p}{k}}, \quad y_s \stackrel{\text{def}}{=} l \quad \text{and} \quad \hat{\theta}(\tau) \stackrel{\text{def}}{=} \phi(\tau/\omega_o).$$

Introducing more definitions, and remembering that t_s has also been equivalently defined as follows: $t_s \stackrel{\text{def}}{=} \frac{1}{\omega_o}$,

$$2\hat{\zeta}_p \stackrel{\text{def}}{=} \frac{c_p \sqrt{m_o + m_p}}{m_p l^2 \sqrt{k}} \quad \omega_p^2 \stackrel{\text{def}}{=} \frac{g}{l} \quad \text{and} \quad q \stackrel{\text{def}}{=} \frac{\omega_p}{\omega_o} \quad (2.10)$$

$$\therefore \frac{gt_s^2}{l} = \frac{\omega_p^2}{\omega_o^2} = q^2.$$

Once again, t will be used to represent the dimensionless time and the dot notation will be employed for the derivatives. The equation of motion for the pendulum in dimensionless coordinates then becomes

$$\boxed{\ddot{\hat{\theta}} + 2\hat{\zeta}_p \dot{\hat{\theta}} + q^2 \sin \hat{\theta} - \hat{\eta} \sin \hat{\theta} = 0}. \quad (2.11)$$

In summary, the equations governing the dynamics of the system in dimensionless coordinates are

$$\begin{aligned} \ddot{\hat{\eta}} + 2\hat{\zeta}_o \dot{\hat{\eta}} + \hat{\eta} - R \left(\ddot{\hat{\theta}} \sin \hat{\theta} + \dot{\hat{\theta}}^2 \cos \hat{\theta} \right) &= \hat{\xi}(t) \\ \ddot{\hat{\theta}} + 2\hat{\zeta}_p \dot{\hat{\theta}} + \left(q^2 - \hat{\eta} \right) \sin \hat{\theta} &= 0, \end{aligned}$$

where

$$\begin{aligned} \omega_p^2 &\stackrel{\text{def}}{=} \frac{g}{l}, \quad \omega_o^2 \stackrel{\text{def}}{=} \frac{k}{m_o + m_p}, \quad q \stackrel{\text{def}}{=} \frac{\omega_p}{\omega_o}, \\ R &\stackrel{\text{def}}{=} \frac{m_p}{m_o + m_p}, \quad \hat{\zeta}_o \stackrel{\text{def}}{=} \frac{c_o}{2\sqrt{k}(m_o + m_p)}, \quad \hat{\zeta}_p \stackrel{\text{def}}{=} \frac{c_p \sqrt{m_o + m_p}}{2m_p l^2 \sqrt{k}}, \\ \hat{\xi}(t) &\stackrel{\text{def}}{=} \frac{\Xi(t/\omega_o)}{kl}, \quad \hat{\eta}(t) \stackrel{\text{def}}{=} \frac{y(t/\omega_o)}{l}, \quad \hat{\theta}(t) \stackrel{\text{def}}{=} \phi(t/\omega_o), \end{aligned}$$

for all $t \geq 0$.

2.3 Periodic Excitation

For autoparametric systems that are excited periodically, as mentioned in Chapter 1, things are most interesting when the natural frequencies of the oscillator and the pendulum are in 2:1 internal resonance (i.e., when $q = \frac{1}{2}$). In resonance, the pendulum is principally excited by the energy from the oscillator. When the external energy given to the oscillator by the excitation is small enough, its effect on the hanging pendulum is small compared to the stability of the same. As the external energy increases, a saturation takes place at a certain threshold, above which the pendulum noticeably moves. This phenomenon can be demonstrated by performing numerical simulations on the equations of motion (2.9) and (2.11) of the system, with $\hat{\xi}(t) \stackrel{\text{def}}{=} A \cos \mu t$. We stay close to the point of resonance by letting the ratio of the natural frequencies of the oscillator and pendulum be close to the 2:1 ratio for internal resonance, and the excitation frequency be around the natural frequency of the oscillator (i.e., $q \approx \frac{1}{2}$ and $\mu \approx 1$). Choosing reasonable values for the other system parameters (see Table 2.1), the amplitude A of the excitation is progressively increased and the corresponding amplitudes α_o and α_p of the steady-state response of each mode $\hat{\eta}(t)$ and $\hat{\theta}(t)$, respectively, are observed and recorded.

From Figure 2.2, it is clear that saturation occurs at the bifurcation point in the graph at a critical value A_c of the excitation amplitude. To the left of the bifurcation point is the *single mode solution* of the system, where only the oscillator responds to the external excitation (a locked-mass system). To the right of the same point is the *coupled mode solution* where both the oscillator and pendulum are in motion. In the regime of the single mode solution, the system is effectively a periodically excited linear mass-spring oscillator. For this type of system, it is well known [14] that the amplitude of the steady-state response and that of the periodic excitation have a linear relationship. That is, $\alpha_o = \gamma A$ where γ is a constant that depends on the other parameters in the system: $\hat{\zeta}_o$ and μ . This relationship is clearly depicted in the figure as α_o increases linearly with A . Although the numerical simulations did not capture it, in the regime of the coupled mode solution, the oscillator maintains a constant steady-state response amplitude since it is now saturated with energy. In the subsequent chapters, we seek to analytically find an analogue of A_c when the excitation $\hat{\xi}(t)$ is a Gaussian white noise process by analyzing the stochastic stability of the single mode solution.

Parameter	Value Used
$\hat{\zeta}_o$	0.02
$\hat{\zeta}_p$	0.05
R	$\frac{1}{6}$
q^2	0.26
μ	1.2

Table 2.1: Parameter values based on [10] used for numerical simulation

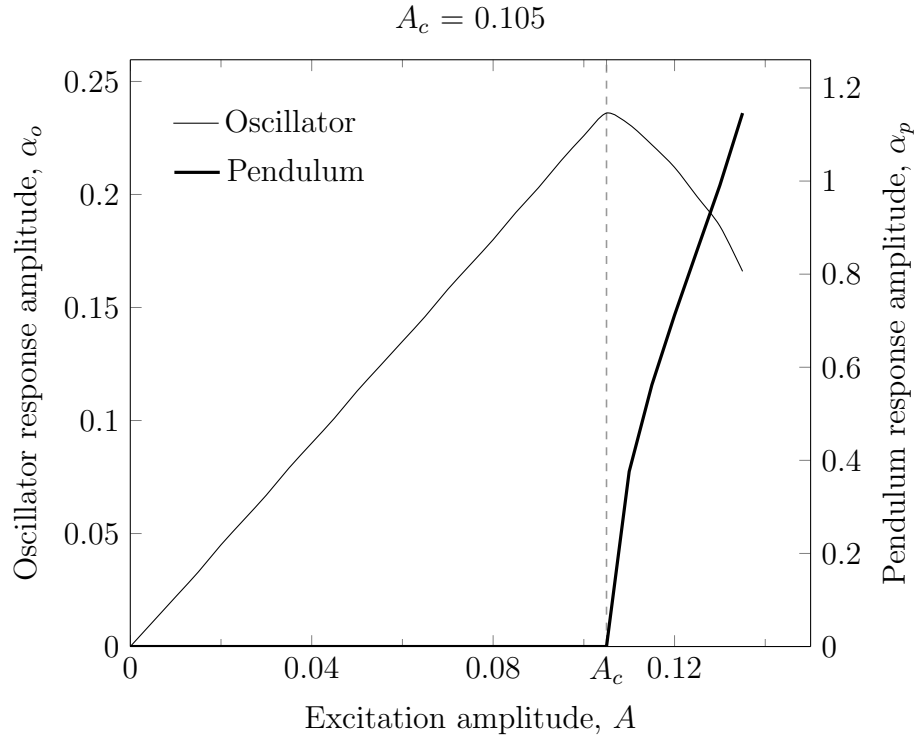


Figure 2.2: Variation of steady-state response amplitudes with excitation amplitude for periodic excitation

Chapter 3

Dynamics and Stability of Linear Stochastic Systems

This chapter is aimed at providing a description of the mathematical concepts that are fundamental to the analysis of the dynamics and stability of linear stochastic systems. These are the mathematical tools required for the analysis of the system introduced in the preceding chapter and shall therefore be greatly employed in the subsequent chapter where this system shall be investigated. It should be noted that only as much detail as is required for this investigation is covered. Readers interested in seeing more details or proofs are referred to introductory texts on the subject matter such as [2, 9, 15, 27, 29, 35].

3.1 Fundamental Concepts

3.1.1 Markov and Diffusion Processes

A stochastic process $X(t)$ is said to be a Markov process if, for time instances $t_1 < t_2 < \dots < t_n$,

$$\mathbb{P}\{X(t_n) \leq x_n \mid X(t_{n-1}) = x_{n-1}, \dots, X(t_1) = x_1\} = \mathbb{P}\{X(t_n) \leq x_n \mid X(t_{n-1}) = x_{n-1}\}. \quad (3.1)$$

This property is called the *Markov property*. It states that the probability that the process takes a value less than x_n at time t_n depends only on the value of the process at time t_{n-1} ; it is independent of the values at the times $t_{n-2}, t_{n-3}, \dots, t_1$. In other words, the

state of the process at the past times $t_{n-2}, t_{n-3}, \dots, t_1$ has no effect on its probable evolution in the future time t_n given that its state at the current time t_{n-1} is known.

In terms of probability distribution functions, the Markov property implies

$$F_{1|n-1}(x_n, t_n | x_{n-1}, t_{n-1}; x_{n-2}, t_{n-2}; \dots; x_1, t_1) = F_{1|1}(x_n, t_n | x_{n-1}, t_{n-1})$$

where the numbers in the subscript of the function denote the number of random variables being jointly considered. In terms of probability density functions,

$$p_{1|n-1}(x_n, t_n | x_{n-1}, t_{n-1}; x_{n-2}, t_{n-2}; \dots; x_1, t_1) = p_{1|1}(x_n, t_n | x_{n-1}, t_{n-1}).$$

Consider a Markov process $X(t)$ with initial condition $X(t_0) = x_0$. For the second-order joint density function $p_{2|1}$, we have

$$\begin{aligned} p_{2|1}(x_2, t_2; x_1, t_1 | x_0, t_0) &= p_{1|2}(x_2, t_2 | x_1, t_1; x_0, t_0) p_{1|1}(x_1, t_1 | x_0, t_0) \\ &= p_{1|1}(x_2, t_2 | x_1, t_1) p_{1|1}(x_1, t_1 | x_0, t_0) \end{aligned}$$

where the Markov property was invoked in the second line. In general, for $p_{n|1}$, we have

$$\begin{aligned} p_{n|1}(x_n, t_n; \dots; x_1, t_1 | x_0, t_0) &= p_{1|1}(x_n, t_n | x_{n-1}, t_{n-1}) p_{1|1}(x_{n-1}, t_{n-1} | x_{n-2}, t_{n-2}) \cdots p_{1|1}(x_1, t_1 | x_0, t_0) \\ &= \prod_{i=1}^n p_{1|1}(x_i, t_i | x_{i-1}, t_{i-1}). \end{aligned} \quad (3.2)$$

So, for a Markov process with a known initial condition, any finite dimensional density function $p_{n|1}$ can be written solely in terms of the particular conditioned density function $p_{1|1}$. In other words, the probabilistic nature of the Markov process is completely described by $p_{1|1}$ which is called the *transition density function*. Henceforth, we shall simplify the notation by dropping the subscript; $p_{1|1} \equiv p$. The mean and autocorrelation functions of the process, conditioned on the initial condition, are then given by

$$\begin{aligned} \mu_X(t) &= \mathbb{E}[X(t) | X(t_0) = x_0] \\ &= \int_{-\infty}^{\infty} x p(x, t | x_0, t_0) dx, \quad t_0 \leq t, \end{aligned} \quad (3.3a)$$

$$\begin{aligned} R_{XX}(t_1, t_2) &= \mathbb{E}[X(t_1)X(t_2) | X(t_0) = x_0] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 p(x_2, t_2 | x_1, t_1) p(x_1, t_1 | x_0, t_0) dx_1 dx_2, \quad t_0 \leq t_1 \leq t_2. \end{aligned} \quad (3.3b)$$

A Markov process is said to be homogeneous in time if its transition density $p(x, t|x_0, t_0)$ is stationary. That is, for all $\tau \in \mathbb{R}$,

$$p(x, t + \tau|x_0, t_0 + \tau) = p(x, t|x_0, t_0).$$

In other words, the transition density is time-invariant and therefore depends on time only through the difference $t - t_0$;

$$p(x, t|x_0, t_0) = p(x, t - t_0|x_0, 0), \quad t_0 \leq t. \quad (3.4)$$

Another common notation is $p(t, x_0, x)$ (or $p_t(x_0, x)$). It represents the probability of transition from x_0 to x in time t , irrespective of the actual position of the interval of length t on the time axis.

For some time-homogeneous Markov processes, it happens that

$$\lim_{(t-t_0) \rightarrow \infty} p(x, t|x_0, t_0) = p_s(x), \quad (3.5)$$

where p_s depends only on x and is called the *stationary density function*. This means that, if the stationary density function exists, the statistical properties of the process eventually become static. So, for the stationary mean, we have

$$\begin{aligned} \mu_X &= \lim_{(t-t_0) \rightarrow \infty} \mathbb{E}[X(t) | X(t_0) = x_0] \\ &= \int_{-\infty}^{\infty} x p_s(x) dx. \end{aligned} \quad (3.6a)$$

And for the stationary autocorrelation,

$$\begin{aligned} R_{XX}(t_2 - t_1) &= \lim_{(t_1-t_0) \rightarrow \infty} \mathbb{E}[X(t_1)X(t_2) | X(t_0) = x_0] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 p(x_2, t_2 - t_1|x_1, 0) p_s(x_1) dx_1 dx_2 \\ \therefore R_{XX}(\tau) &= \lim_{t_0 \rightarrow -\infty} \mathbb{E}[X(t)X(t + \tau) | X(t_0) = x_0] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 p(\tau, x_1, x_2) p_s(x) dx_1 dx_2, \end{aligned} \quad (3.6b)$$

where $\tau = |t_2 - t_1|$ and $R_{XX}(-\tau) = R_{XX}(\tau)$.

A diffusion process is a special case of a Markov process. Many physical, biological, economic, and social phenomena are either well approximated or reasonably modelled by diffusion processes. Also known as a *continuous* or *diffusive* Markov process, it is a Markov process whose sample paths are continuous functions of time. This property can be expressed as

$$\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \mathbb{P}\{|X(t + \Delta t) - X(t)| > \epsilon \mid X(t) = x\} = 0, \quad \forall \epsilon > 0, \quad (3.7)$$

which indicates that large changes in $X(t)$, of order exceeding a fixed ϵ , are very unlikely over sufficiently small time intervals. Diffusion processes are completely characterized by two parameters, known as *infinitesimal parameters* or *characterizing functions*, that describe the mean and variance of the infinitesimal displacements. Denoting the infinitesimal displacement in time Δt as $\Delta X(t) = X(t + \Delta t) - X(t)$, we define

$$\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \mathbb{E}[\Delta X(t) \mid X(t) = x] = b(x, t), \quad (3.8a)$$

$$\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \mathbb{E}[\{\Delta X(t)\}^2 \mid X(t) = x] = \sigma^2(x, t). \quad (3.8b)$$

The function $b(x, t)$ is called the *drift coefficient* or the *infinitesimal mean* and $\sigma^2(x, t)$ the *diffusion coefficient* or the *infinitesimal variance*. They are continuous functions of x and t , and $\sigma^2(x, t) \geq 0$. For a time-homogeneous diffusion process, these functions are independent of t ; so, $b(x, t) = b(x)$ and $\sigma^2(x, t) = \sigma^2(x)$.

In the multivariate case where $x \in \mathbb{R}^n$, the analog of the infinitesimal relations (3.8a) and (3.8b) are

$$\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \mathbb{E}[\Delta X_i(t) \mid X(t) = x] = b_i(x, t), \quad i = 1, 2, \dots, n, \quad (3.9a)$$

and

$$\begin{aligned} \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \mathbb{E}[\{\Delta X_i(t)\}\{\Delta X_j(t)\} \mid X(t) = x] &= a_{ij}(x, t) \\ &= \sigma_i(x, t) \sigma_j(x, t), \quad i, j = 1, 2, \dots, n, \end{aligned} \quad (3.9b)$$

where $\Delta X_i(t) = X_i(t + \Delta t) - X_i(t)$ and $x = (x_1, x_2, \dots, x_n)$. $b(x, t)$ is the drift vector and $a(x, t)$ is the diffusion matrix, which is symmetric and positive definite. Again, they are independent of t when the vector-valued diffusion process $X(t) = (X_1(t), X_2(t), \dots, X_n(t))$ is time-homogeneous.

3.1.2 The Generator of a Diffusion Process

The generator of a diffusion process is a second-order partial differential operator associated with it. Let $\{X(t), t \geq 0\}$ be a time-homogeneous diffusion process. The infinitesimal generator \mathcal{G} of $X(t)$ is defined by

$$\mathcal{G}f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}_x[f(X(t))] - f(x)}{t} \quad (3.10)$$

where $\mathbb{E}_x[\cdot]$ denotes expectation when the initial value of $X(t)$ is $X(0) = x$; that is, $\mathbb{E}_x[\bullet] = \mathbb{E}[\bullet | X(0) = x]$. The quantity $\mathcal{G}f(x)$ is interpreted as the mean infinitesimal rate of change of $f(X(0))$ in the case where $X(0) = x$, and can be formally written for every continuous twice partially differentiable function $f(x)$. It is determined by the drift and diffusion coefficients $b(x)$ and $\sigma^2(x)$ and, as such, every diffusion process is uniquely determined by its generator \mathcal{G} . The generator of a time-homogeneous diffusion process, expressed in terms of the infinitesimal parameters, is

$$\mathcal{G} = b(x) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(x) \frac{\partial^2}{\partial x^2}. \quad (3.11)$$

When $X(t)$ is vector-valued with $x \in \mathbb{R}^n$, drift vector $b(x)$ and diffusion matrix $a(x)$, the generator is

$$\mathcal{G} = \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}. \quad (3.12)$$

3.1.3 Evolution Equations for Diffusion Processes

In this section, we introduce equations that determine the time evolution of diffusion processes. We commence with an important equation for Markov processes known as the *Chapman-Kolmogorov equation*:

$$p(x_3, t_3 | x_1, t_1) = \int_{-\infty}^{\infty} p(x_3, t_3 | x_2, t_2) p(x_2, t_2 | x_1, t_1) dx_2, \quad t_1 \leq t_2 \leq t_3. \quad (3.13)$$

It is essentially a consistency condition on the transition density function p for any Markov process $X(t)$. Based on this equation, one can derive two differential equations for the so-called “forward” and “backward” time evolution of the transition density $p(x, t | x_0, t_0)$ of a

diffusion process. The first, known as the *Fokker-Planck* or *Forward Kolmogorov* equation, is given by

$$\frac{\partial}{\partial t} p(x, t|x_0, t_0) = -\frac{\partial}{\partial x} [b(x, t) p(x, t|x_0, t_0)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [\sigma^2(x, t) p(x, t|x_0, t_0)]. \quad (3.14a)$$

Given the initial condition $p(x, t = t_0|x_0, t_0) = \delta(x - x_0)$, it constitutes a t -evolution equation for $p(x, t|x_0, t_0)$, for fixed x_0 and t_0 , for the diffusion process with characterizing functions $b(x, t)$ and $\sigma^2(x, t)$. The second is known as the *Backward Kolmogorov* equation and is given by

$$-\frac{\partial}{\partial t_0} p(x, t|x_0, t_0) = b(x_0, t_0) \frac{\partial}{\partial x_0} p(x, t|x_0, t_0) + \frac{1}{2} \sigma^2(x_0, t_0) \frac{\partial^2}{\partial x_0^2} p(x, t|x_0, t_0). \quad (3.14b)$$

Given the final condition $p(x, t|x_0, t_0 = t) = \delta(x - x_0)$, it constitutes a t_0 -evolution equation for $p(x, t|x_0, t_0)$, for fixed x and t , for the same diffusion process with characterizing functions $b(x, t)$ and $\sigma^2(x, t)$.

For a time-homogeneous process, since the transition density depends on time only via the time difference $t - t_0$, it is desirable to express the time derivatives in the evolution equations with respect to this difference. Let $\tau = t - t_0$ so that

$$p(x, t|x_0, t_0) = p(x, t - t_0|x_0, 0) = p(\tau, x_0, x).$$

Then, by the chain rule,

$$\begin{aligned} \frac{\partial}{\partial t} p(\tau, x_0, x) &= \frac{\partial}{\partial \tau} p(\tau, x_0, x) \frac{\partial \tau}{\partial t} = \frac{\partial}{\partial \tau} p(\tau, x_0, x) \\ \frac{\partial}{\partial t_0} p(\tau, x_0, x) &= \frac{\partial}{\partial \tau} p(\tau, x_0, x) \frac{\partial \tau}{\partial t_0} = -\frac{\partial}{\partial \tau} p(\tau, x_0, x). \end{aligned}$$

These can now be substituted into (3.14a) and (3.14b) to obtain the evolution equations for time-homogeneous processes. The forward Kolmogorov equation is

$$\frac{\partial}{\partial \tau} p(\tau, x_0, x) = -\frac{\partial}{\partial x} [b(x) p(\tau, x_0, x)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [\sigma^2(x) p(\tau, x_0, x)] \quad (3.15a)$$

and the backward Kolmogorov equation is

$$\frac{\partial}{\partial \tau} p(\tau, x_0, x) = b(x_0) \frac{\partial}{\partial x_0} p(\tau, x_0, x) + \frac{1}{2} \sigma^2(x_0) \frac{\partial^2}{\partial x_0^2} p(\tau, x_0, x). \quad (3.15b)$$

If a stationary density exists, then, for the forward equation,

$$\lim_{\tau \rightarrow \infty} p(\tau, x_0, x) = p_s(x) \quad \text{and} \quad \lim_{\tau \rightarrow \infty} \frac{\partial}{\partial \tau} p(\tau, x_0, x) = 0, \quad (3.16a)$$

and for the backward equation,

$$\lim_{\tau \rightarrow \infty} p(\tau, x_0, x) = p_s(x_0) \quad \text{and} \quad \lim_{\tau \rightarrow \infty} \frac{\partial}{\partial \tau} p(\tau, x_0, x) = 0. \quad (3.16b)$$

The evolution equations then become

$$0 = -\frac{d}{dx} [b(x) p_s(x)] + \frac{1}{2} \frac{d^2}{dx^2} [\sigma^2(x) p_s(x)] \quad (3.17a)$$

and

$$0 = b(x_0) \frac{d}{dx_0} p_s(x_0) + \frac{1}{2} \sigma^2(x_0) \frac{d^2}{dx_0^2} p_s(x_0). \quad (3.17b)$$

In general, for a multidimensional time-homogeneous diffusion process $X(t) = (X_1(t), X_2(t), \dots, X_n(t))$ with transition density $p(t, x, y)$; $x, y \in \mathbb{R}^n$, the forward and backward Kolmogorov equations can be concisely written in terms of the generator \mathcal{G} of the process (defined in (3.12)) as

$$\frac{\partial p}{\partial t} = \mathcal{G}_y^* p \quad (3.18a)$$

and

$$\frac{\partial p}{\partial t} = \mathcal{G}_x p \quad (3.18b)$$

respectively. The subscript on the generator represents the variable it operates on, and \mathcal{G}^* is the adjoint of \mathcal{G} defined as

$$\mathcal{G}^* f(w) = -\sum_{i=1}^n \frac{\partial}{\partial w_i} [b_i(w) f(w)] + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2}{\partial w_i \partial w_j} [a_{ij}(w) f(w)], \quad w \in \mathbb{R}^n. \quad (3.19)$$

For the case when a stationary density p_s exists, we have

$$\mathcal{G}^* p_s(y) = 0 \quad (3.20a)$$

and

$$\mathcal{G} p_s(x) = 0. \quad (3.20b)$$

3.1.4 Stochastic Differential Equations

The concept of stochastic differential equations arises from the study of systems driven by noise. These systems, known as *random* or *stochastic dynamical systems*, are usually modelled by equations of the form

$$\frac{dX(t)}{dt} = f(t, X(t), \xi(t)), \quad (3.21a)$$

where $\xi(t)$ is the underlying random process driving the system. Equation (3.21a) can be studied by analyzing an equivalent integral equation

$$X(t) - X(0) = \int_0^t f(s, X(s), \xi(s)) ds. \quad (3.21b)$$

Here, we are concerned with the particular case where (3.21a) takes the form

$$\frac{dX(t)}{dt} = b(X(t), t) + \sigma(X(t), t) \xi(t), \quad (3.22)$$

where $\xi(t)$ now specifically represents the white noise process. Although white noise is just a mathematical abstraction, it serves as a reasonably good approximation of several random input processes that occur naturally in physical or biological contexts. It can *formally* be seen as the derivative of the standard Wiener process (or Brownian motion) $W(t)$ — formally, because Brownian motion paths, although almost-surely continuous, are nowhere differentiable. So, using $\xi(t) = dW(t)/dt$, (3.22) can be written as

$$dX(t) = b(X(t), t) dt + \sigma(X(t), t) dW(t). \quad (3.23a)$$

This is what is known as a *stochastic differential equation* (SDE). The differential notation used in SDEs is an informal shorthand notation for the equivalent stochastic integral equation (SIE)

$$X(t) = X(0) + \int_0^t b(X(s), s) ds + \int_0^t \sigma(X(s), s) dW(s) \quad (3.23b)$$

since, in stochastic calculus, integrals are well defined but not differentials. The stochastic differential notation makes the rules of stochastic calculus easy to use hence its wide employment. The solution of an SDE determines a diffusion process with infinitesimal mean $b(x, t)$ and infinitesimal variance $\sigma^2(x, t)$. The stochastic integral $\int \sigma(X(t), t) dW(t)$ involved in the solution has two prominent versions namely: the Itô integral (I-integral) and

the Stratonovich integral (S-integral). More precisely, (3.23a) and (3.23b) constitute an Itô SDE and SIE, and the solution is an Itô diffusion process. The Stratonovich interpretation is distinguished by the addition of a circle (\circ) before the noise term; that is,

$$\sigma(X(t), t) \circ dW(t) \implies \int \sigma(X(t), t) \circ dW(t).$$

The S-integral has the benefit that, unlike the I-integral, it satisfies most of the conventional rules of ordinary calculus with respect to transformation formulas, such as the chain rule, making its manipulation easier using familiar operations. This, however, causes it to suffer the loss of a useful property that the I-integral enjoys, namely: the I-integral is a martingale. Due to the nice properties of martingales, most theoretical work is generally done with respect to I-integrals. In fact, manipulations of the S-integral are typically reduced to calculations of related I-integrals as the S-integral only differs from the I-integral by a corrective term. We now state the precise relationship between the two integrals as follows:

The solution to the Stratonovich SDE

$$dX(t) = b(X(t), t) dt + \sigma(X(t), t) \circ dW(t) \tag{3.24}$$

is the same as the solution to the Itô SDE

$$dX(t) = \left[b(X(t), t) + \frac{1}{2} \frac{\partial \sigma}{\partial x}(X(t), t) \sigma(X(t), t) \right] dt + \sigma(X(t), t) dW(t). \tag{3.25}$$

The correction term $\frac{1}{2} \sigma_x \sigma$, discovered by Wong and Zakai [42] and consequently known as the Wong-Zakai correction, contributes to the drift coefficient. It is therefore clear that the Stratonovich SDE admits a (Stratonovich) diffusion process with the same diffusion coefficient as that of the Itô interpretation of the SDE (i.e., $\sigma^2(x, t)$), but with the drift coefficient being

$$b(x, t) + \frac{1}{2} \frac{\partial \sigma}{\partial x}(x, t) \sigma(x, t).$$

Equation (3.22) therefore has two possible solutions depending on the interpretation of the stochastic integral, and these two solutions behave quite differently. The Stratonovich solution tends to be more natural for modelling of physical systems while that of Itô is more natural for a wide range of real-world applications, particularly in financial mathematics. Random dynamical systems describing mechanical or structural engineering systems parametrically excited by white noise processes (such as the one studied in this thesis) are interpreted as Stratonovich SDEs as in (3.24). Evidently, if $\sigma(x, t)$ is independent of x so

that $\sigma_x = 0$, then the Itô and Stratonovich solutions coincide.

Remark: As SDEs define diffusion processes, the Kolmogorov equations discussed in the previous section provide a way of characterizing the solution of an SDE by solving a PDE.

Since the transformation properties of the Itô integral are not concordant with the rules of ordinary calculus, there exists a special formula, analogous to the chain rule of ordinary calculus, useful for changing variables and evaluating Itô integrals. This transformation formula, popularly known as *Itô's lemma*, is stated below.

Itô's Lemma: Consider the random process $Y(t) = g(X(t), t)$, where $X(t)$ is the solution of

$$dX(t) = b(X(t), t) dt + \sigma(X(t), t) dW(t). \quad (3.26)$$

Then $Y(t)$ is also an Itô diffusion process with its differential $dY(t)$ given by

$$dY(t) = \frac{\partial g}{\partial t}(X(t), t) dt + \frac{\partial g}{\partial x}(X(t), t) dX(t) + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(X(t), t) (dX(t))^2 \quad (3.27)$$

where $(dX(t))^2$ is computed using

$$(dt)^2 = dt dW(t) = dW(t) dt = 0, \quad (dW(t))^2 = dt. \quad (3.28)$$

Substituting (3.26) into (3.27) and using (3.28) to compute $(dX(t))^2$, we get

$$dY(t) = \left[\frac{\partial g}{\partial t}(X(t), t) + b(X(t), t) \frac{\partial g}{\partial x}(X(t), t) + \frac{1}{2} \sigma^2(X(t), t) \frac{\partial^2 g}{\partial x^2}(X(t), t) \right] dt + \sigma(X(t), t) \frac{\partial g}{\partial x}(X(t), t) dW(t). \quad (3.29)$$

In integral form, this is

$$Y(t) = Y(0) + \int_0^t \left[\frac{\partial g}{\partial s}(X(s), s) + b(X(s), s) \frac{\partial g}{\partial x}(X(s), s) + \frac{1}{2} \sigma^2(X(s), s) \frac{\partial^2 g}{\partial x^2}(X(s), s) \right] ds + \int_0^t \sigma(X(s), s) \frac{\partial g}{\partial x}(X(s), s) dW(s). \quad (3.30)$$

In the case where $X(t)$, $b(x, t)$ and $\sigma(x, t)$ are n -dimensional vectors in Euclidean space \mathbb{R}^n , (3.27) becomes

$$dY(t) = \frac{\partial g}{\partial t} dt + \sum_{i=1}^n \frac{\partial g}{\partial x_i} dX_i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 g}{\partial x_i \partial x_j} dX_i dX_j \quad (3.31)$$

where $dX_i(t) = b_i(X(t), t) dt + \sigma_i(X(t), t) dW(t)$ and (3.28) still applies. Putting them together as before, we get

$$dY(t) = \left(\frac{\partial g}{\partial t} + \sum_{i=1}^n b_i \frac{\partial g}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n \sigma_i \sigma_j \frac{\partial^2 g}{\partial x_i \partial x_j} \right) dt + \sum_{i=1}^n \sigma_i \frac{\partial g}{\partial x_i} dW(t). \quad (3.32)$$

Noting that $\sigma_i \sigma_j = a_{ij}$, this can be concisely written using the operator defined in (3.12) as

$$dY(t) = \left(\frac{\partial g}{\partial t} + \mathcal{G}g \right) dt + \langle \sigma, \nabla g \rangle dW(t) \quad (3.33)$$

where $\langle \cdot, \cdot \rangle$ represents the inner product which, in this case, is simply the dot product of the two vectors. In integral form,

$$Y(t) = Y(0) + \int_0^t \left(\frac{\partial g}{\partial s} + \mathcal{G}g \right) ds + \int_0^t \langle \sigma, \nabla g \rangle dW(s). \quad (3.34)$$

3.1.5 The Feynman-Kac Formula

The backward Kolmogorov equation may also be formulated as follows. Define

$$u(t, x) \stackrel{\text{def}}{=} \mathbb{E}[f(X(t)) \mid X(0) = x] = \mathbb{E}_x[f(X(t))], \quad (3.35)$$

where $X(t)$ is the solution of (3.23a) - an Itô diffusion process. Then, $u(t, x)$ is the unique solution of

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{G}u, & t > 0, x \in \mathbb{R}^n \\ u(0, x) &= f(x), & x \in \mathbb{R}^n. \end{aligned} \quad (3.36)$$

A generalized version of this, known as the *Feynman-Kac formula*, is stated below:

Consider Cauchy's equation with deterministic forcing and an initial condition

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{G}u(t, x) + c(t, x)u(t, x) + g(t, x), & t > 0, x \in \mathbb{R}^n \\ u(0, x) &= f(x) & x \in \mathbb{R}^n. \end{aligned} \quad (3.37)$$

The solution $u(t, x)$ can be represented in the form

$$\begin{aligned} u(t, x) &= \mathbb{E}_x \left[f(X(t)) \exp \left\{ \int_0^t c(t-s, X(s)) ds \right\} \right. \\ &\quad \left. + \int_0^t g(s, X(s)) \exp \left\{ \int_0^s c(t-\tau, X(\tau)) d\tau \right\} ds \right]. \end{aligned} \quad (3.38)$$

We shall particularly be interested in the situation where the multiplicative term and the initial condition are both identically zero; that is, $c = 0$ and $f = 0$. The Cauchy equation then reduces to

$$\begin{aligned}\frac{\partial u}{\partial t} &= \mathcal{G}u(t, x) + g(t, x), & t > 0, x \in \mathbb{R}^n \\ u(0, x) &= f(x) \equiv 0, & x \in \mathbb{R}^n,\end{aligned}\tag{3.39}$$

and the solution $u(t, x)$ is given by

$$u(t, x) = \mathbb{E}_x \left[\int_0^t g(t-s, X(s)) ds \right].\tag{3.40}$$

One can then use the integral definition of expectation

$$\mathbb{E}_x [f(X(t))] = \int_{\mathbb{R}^n} f(y) p(t, x, y) dy,$$

to write the solution in terms of the transition density $p(t, x, y)$ as

$$u(t, x) = \int_0^t \int_{\mathbb{R}^n} g(t-s, y) p(s, x, y) dy ds.\tag{3.41}$$

We will make use of the Feynman-Kac formula in the next chapter while calculating the maximal Lyapunov exponent, which is described in the next subsection.

3.1.6 Stochastic Stability and Lyapunov Exponents

Stability deals with the long-term behaviour of a dynamical system. There are several ways in which stochastic stability can be defined; for example, stability in probability, almost-sure sample stability and p -th moment stability, to name a few. A complete exposition into the available definitions of stochastic stability is provided in [18] for the interested reader. Based on the definition of stability employed, one can determine conditions on the system parameter values that dictate the stability properties of the system, and these conditions vary depending on the choice of stability definition. The stability definitions, however, are not all equal in strength. We find that the stability regions of the parameter values determined by weaker definitions are contained within those determined by stronger definitions. In other words, if stability can be established in terms of a stronger definition, then the system in question is also stable in terms of any weaker definitions.

The definition utilized in this thesis is that of almost-sure sample stability, which is one of the strongest definitions. A fixed point of a system is said to be asymptotically stable if

the state of the system approaches this equilibrium solution as time tends to infinity. For a stochastic system, a fixed point \bar{x} is said to be *almost-surely stable* if all sample functions of the response, except those belonging to a set of measure zero, converge asymptotically to \bar{x} with probability one as time progresses. This can be expressed as

$$\mathbb{P} \left\{ \lim_{t \rightarrow \infty} \|X(t) - \bar{x}\| = 0 \right\} = 1.$$

However, since the equilibrium solution is typically taken to be the trivial solution $\bar{x} = 0$, the common definition used for almost-sure stability is

$$\mathbb{P} \left\{ \lim_{t \rightarrow \infty} \|X(t)\| = 0 \right\} = 1, \quad (3.42)$$

where this relation is to be satisfied for all x such that $X(0) = x$.

It was stated earlier that conditions on the system parameters that dictate the stability properties of the system can be determined based on the stability definition employed. For almost-sure stability, this is done via a quantity known as the *maximal* or *top Lyapunov exponent*. It is defined by

$$\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \|X(t)\|, \quad (3.43)$$

from which can be deduced,

$$\|X(t)\| = \lim_{t \rightarrow \infty} e^{\lambda t}.$$

It therefore represents the exponential growth rate of the solution to the stochastic system, which determines the almost-sure asymptotic stability of the equilibrium solution $X(t) \equiv 0$. It can be seen as the stochastic analogue to the real part of the “maximal” eigenvalue of the system matrix of the corresponding deterministic system. A necessary and sufficient condition for almost-sure stability is thus that the top Lyapunov exponent be negative so that (3.42) holds. If $\lambda > 0$, then

$$\mathbb{P} \left\{ \lim_{t \rightarrow \infty} \|X(t)\| = \infty \right\} = 1,$$

which indicates almost-sure sample instability.

We now discuss how the Lyapunov exponent is computed for an n -dimensional linear stochastic system

$$\dot{x} = (A + F(t)) x, \quad x \in \mathbb{R}^n, \quad (3.44)$$

where A is a constant coefficient $n \times n$ matrix and $F(t)$ is a matrix of independent noise functions $f_{ij}(t)$, $i, j = 1, \dots, n$. It is a result of the work of Khas'minskii [16] which has

proven fundamental to the area of research in the almost-sure asymptotic stability of stochastic systems. When the noise functions $f_{ij}(t)$ are Gaussian white noise processes, (3.44) can be written as a system of n linear Itô SDEs

$$dx_i(t) = \sum_{j=1}^n b_{ij}x_j dt + \sum_{j=1}^n \sigma_{ij}x_j dW(t), \quad i = 1, \dots, n$$

which, in matrix form, gives the following n -dimensional Itô SDE

$$dX(t) = BX(t) dt + \Sigma X(t) dW(t), \quad X(0) = x \in \mathbb{R}^n \quad (3.45)$$

where $B = \|b_{ij}\|$ and $\Sigma = \|\sigma_{ij}\|$ are constant $n \times n$ matrices. The generating differential operator \mathcal{G} associated with this system is given by

$$\mathcal{G}u = \langle Bx, \nabla u \rangle + \frac{1}{2} \langle A(x) \nabla, \nabla u \rangle,$$

where $A(x) = (\Sigma x)(\Sigma x)^T$ is positive semi-definite and also assumed to be non-degenerate in the sense that there exists a constant $m > 0$ such that for any arbitrary vector $\alpha = (\alpha_1, \dots, \alpha_n)^T$,

$$\langle \alpha, A(x)\alpha \rangle \geq m\|x\|^2\|\alpha\|^2. \quad (3.46)$$

Now, we introduce logarithmic-polar coordinates in \mathbb{R}^n through the following so-called *Khas'minskii transformation*:

$$\rho \stackrel{\text{def}}{=} \ln \|x\| \in \mathbb{R}^+ \quad \text{and} \quad \phi \stackrel{\text{def}}{=} \frac{x}{\|x\|} \in S^{n-1}. \quad (3.47)$$

The angle process $\phi(t) = X(t)/\|X(t)\|$ is defined on an $(n-1)$ -dimensional unit sphere. This transformation onto a spherical projective space forms the basis for the calculation of the Lyapunov exponent. Next, we apply Itô's lemma (3.33) to $\rho(t)$ and $\phi(t)$ to obtain expressions for their differentials in Itô form:

$$\begin{aligned} d\rho(X(t)) &= \mathcal{G}\rho(X(t)) dt + \langle \Sigma X(t), \nabla \rho(X(t)) \rangle dW(t) \\ d\phi(X(t)) &= \mathcal{G}\phi(X(t)) dt + \langle \Sigma X(t), \nabla \phi(X(t)) \rangle dW(t). \end{aligned}$$

This yields

$$d\rho(t) = \left[\langle \phi(t), (B - A(\phi))\phi(t) \rangle + \frac{1}{2} \text{tr}(A(\phi)) \right] dt + \langle \phi(t), \Sigma \phi(t) \rangle dW(t), \quad (3.48a)$$

$$\begin{aligned} d\phi(t) &= \left[(B - A(\phi))\phi(t) - \langle \phi(t), (B - \frac{3}{2}A(\phi))\phi(t) \rangle \phi(t) - \frac{1}{2} \text{tr}(A(\phi))\phi(t) \right] dt \\ &\quad + [\Sigma \phi(t) - \langle \phi(t), \Sigma \phi(t) \rangle \phi(t)] dW(t). \end{aligned} \quad (3.48b)$$

Since the drift and diffusion coefficients for the angle process $\phi(t)$ depend only on $\phi(t)$, it generates a time-homogeneous diffusive Markov process on the unit sphere S^{n-1} on which it is defined. If it exists, let $\nu(\phi)$ denote the unique invariant measure of this process which satisfies the stationary Fokker-Planck equation $\mathcal{G}^{\phi*}\nu = 0$, where \mathcal{G}^{ϕ} denotes the infinitesimal generator of $\phi(t)$.

The infinitesimal coefficients of $\rho(t)$ also depend only on $\phi(t)$. Defining

$$Q(\phi(t)) \stackrel{\text{def}}{=} \langle \phi(t), (B - A(\phi))\phi(t) \rangle + \frac{1}{2}\text{tr}(A(\phi))$$

for the drift coefficient and integrating,

$$\rho(t) - \rho(0) = \int_0^t Q(\phi(\tau)) d\tau + \int_0^t \langle \phi(\tau), \Sigma\phi(\tau) \rangle dW(\tau),$$

from which can be written

$$\lim_{t \rightarrow \infty} \frac{\rho(t) - \rho(0)}{t} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t Q(\phi(\tau)) d\tau + \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \langle \phi(\tau), \Sigma\phi(\tau) \rangle dW(\tau).$$

By the martingale property of the Itô integral,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \langle \phi(\tau), \Sigma\phi(\tau) \rangle dW(\tau) = 0.$$

Therefore,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \|X(t)\| = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t Q(\phi(\tau)) d\tau.$$

Notice that the left-hand side is precisely the definition of the Lyapunov exponent as introduced in (3.43). Condition (3.46) is sufficient for the process $\phi(t)$ to be ergodic, meaning that time averaging is equivalent to ensemble averaging. We can therefore write

$$\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t Q(\phi(\tau)) d\tau = \mathbb{E}[Q(\phi(t))]. \quad (3.49)$$

So, the Lyapunov exponent can be found using the drift coefficient of $\rho(t)$ and the invariant measure of $\phi(t)$ by evaluating the integral

$$\lambda = \int_{S^{n-1}} Q(\phi) d\nu = \int_{S^{n-1}} Q(\phi)p(\phi) d\phi, \quad (3.50)$$

where $p(\phi)$ is the stationary density of the angle process $\phi(t)$ which satisfies the stationary Fokker-Planck equation $\mathcal{G}^{\phi*}p = 0$.

Now, consider, as in [3], the case where the linear stochastic system (3.44) is of the form

$$\dot{x} = A(\xi(t))x, \quad x(0) = x_0 \in \mathbb{R}^n, \quad (3.51)$$

where $\xi(t)$ is a stationary, ergodic Stratonovich diffusion process defined on a smooth Riemannian manifold M . Application of the Khas'minskii transformation to this system yields

$$\dot{\rho}(t) = Q(\phi(t), \xi(t)) \quad (3.52a)$$

$$\dot{\phi}(t) = h(\phi(t), \xi(t)) \quad (3.52b)$$

where

$$\begin{aligned} Q(\phi, \xi) &= \phi^T A(\xi) \phi \\ h(\phi, \xi) &= (A(\xi) - Q(\phi, \xi)I)\phi. \end{aligned}$$

As before, by integrating the $\dot{\rho}$ equation, we get

$$\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t Q(\phi(\tau), \xi(\tau)) d\tau. \quad (3.53)$$

If the diffusion process $(\phi(t), \xi(t))$ formed on $S^{n-1} \times M$ is ergodic, and if it has a unique invariant measure $\nu(\phi, \xi)$, then the top Lyapunov exponent is given by

$$\lambda = \int_M \int_{S^{n-1}} Q(\phi, \xi) d\nu = \int_M \int_{S^{n-1}} Q(\phi, \xi) p(\phi, \xi) d\phi d\xi, \quad (3.54)$$

where $p(\phi, \xi)$ is the stationary density of the process $(\phi(t), \xi(t))$ which satisfies the stationary Fokker-Planck equation $\mathcal{G}^{(\phi, \xi)*}p = 0$. Equations (3.50) and (3.54) constitute what is known as the *Furstenberg-Khas'minskii formula*.

3.1.7 Power Spectral Density

The power spectral density (PSD) $S_{XX}(\omega)$ of a wide-sense stationary process $X(t)$ is defined as the Fourier transform of its autocorrelation function $R_{XX}(\tau)$. So, $S_{XX}(\omega)$ and

$R_{XX}(\tau)$ constitute a Fourier transform pair, and $R_{XX}(\tau)$ is obtained as the inverse Fourier transform of $S_{XX}(\omega)$. That is,

$$S_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau, \quad (3.55a)$$

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega. \quad (3.55b)$$

These are known as the *Wiener-Khintchine relations*. For the autocorrelation function, at $\tau = 0$, we get

$$R_{XX}(0) = \mathbb{E}[X^2(t)] = \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega, \quad (3.56)$$

which shows that $S_{XX}(\omega)$ is a frequency decomposition of the expected power in the process. It provides the spectral distribution of the average energy of a stationary random process.

Since $R_{XX}(\tau)$ is an even function of τ , the imaginary part of the integrand in (3.55a), namely $-R_{XX}(\tau) \sin \omega\tau$, is an odd function of τ and therefore makes no contribution to the symmetric-limit integral. In short, by the properties of the Fourier transform, the fact that $R_{XX}(\tau)$ is a real, even function of τ means that $S_{XX}(\omega)$ is a real, even function of ω . The Wiener-Khintchine relations can then be equivalently written as

$$S_{XX}(\omega) = \frac{1}{\pi} \int_0^{\infty} R_{XX}(\tau) \cos \omega\tau d\tau, \quad (3.57a)$$

$$R_{XX}(\tau) = 2 \int_0^{\infty} S_{XX}(\omega) \cos \omega\tau d\omega. \quad (3.57b)$$

For a mean-square differentiable process $X(t)$, the PSD of the derivative process $\dot{X}(t)$ can be obtained from that of the process itself via the relation

$$S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega). \quad (3.58)$$

3.2 Linear Oscillator Excited by White Noise

In this section, we study the well-known mass-spring linear oscillator driven by additive noise. This dynamical system is modelled by the following second-order differential equation:

$$\ddot{X}(t) + 2\zeta\omega_n\dot{X}(t) + \omega_n^2 X(t) = F(t), \quad X(t_0) = X_0, \dot{X}(t_0) = \dot{X}_0, \quad (3.59)$$

where ζ is the damping ratio and ω_n is the natural frequency of the system. This equation can be cast into vector form by introducing the vector $\mathbf{X}(t) = (X(t), \dot{X}(t))^T$, so that it becomes

$$\dot{\mathbf{X}}(t) = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix} \mathbf{X}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(t). \quad (3.60)$$

We see that the system has the same form as (3.22) with the vector b being a function of $\mathbf{X}(t)$ only, and σ being a constant vector (which is why the excitation is described as being *additive*). We would like to investigate the long-term behaviour of the statistical properties of the response of the system $X(t)$ to the excitation $F(t)$, where the statistical properties of the excitation are known.

3.2.1 White Noise

As we are interested in the case where the excitation driving the linear oscillator is white noise, we commence with a heuristic description of the white noise process and its important properties. As was mentioned in the discussion about stochastic differential equations, Brownian motion paths are nowhere differentiable albeit being almost-surely continuous, and, as such, their time derivative (white noise) can only be viewed in a formal sense as a generalized stochastic process. Since Brownian motion has mean-zero independent Gaussian increments, its time derivative ends up being a mean-zero Gaussian process such that the random variables sampled at different times are independent. In other words, a white noise process $\xi(t)$ is a stationary Gaussian process with $\mu_\xi = \mathbb{E}[\xi(t)] = 0$ and autocorrelation function of the form $R_{\xi\xi}(\tau) = \mathbb{E}[\xi(t)\xi(t + \tau)] = \sigma^2\delta(\tau)$, where σ^2 is a positive constant that can be seen as a sort of “variance”.

A key, and arguably the most prominent, property of white noise is that its power spectral density is uniformly distributed over all frequencies; a characteristic of white light hence the name “white noise”. That is,

$$S_{\xi\xi}(\omega) = S_0 \quad \forall \omega \in \mathbb{R}, \quad (3.61)$$

from which it is readily observed that the process is not physically realizable since the area under the spectrum is infinite, indicating the process would have infinite energy. It however serves as a good approximation to several physical processes encountered in practice because they usually have very flat spectral densities. From (3.55b), and using the fact that the Dirac delta function can also be represented as

$$\delta(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} d\omega,$$

we obtain the autocorrelation function of the white noise process as

$$R_{\xi\xi}(\tau) = 2\pi S_0 \delta(\tau). \quad (3.62)$$

3.2.2 System Response

Let us now derive the response of the linear mass-spring system by finding the solution of (3.59), focusing on the underdamped case $\zeta < 1$. We begin by employing a change of variables to simplify the equation. Let

$$X(t) = e^{-\zeta\omega_n t} y(t).$$

Then

$$\begin{aligned} \dot{X}(t) &= e^{-\zeta\omega_n t} [-\zeta\omega_n y(t) + \dot{y}(t)], \\ \ddot{X}(t) &= e^{-\zeta\omega_n t} [\zeta^2\omega_n^2 y(t) - 2\zeta\omega_n \dot{y}(t) + \ddot{y}(t)]. \end{aligned}$$

Substituting these into (3.59) yields

$$\ddot{y}(t) + \omega_d^2 y(t) = e^{\zeta\omega_n t} F(t), \quad y(t_0) = y_0, \quad \dot{y}(t_0) = \dot{y}_0, \quad (3.63)$$

where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$. The solution to this equation can be written as the sum of a solution for the homogeneous part satisfying the initial conditions, and a particular solution for the inhomogeneous component. The linearly independent solutions for the homogeneous equation are

$$y_1(t) = \cos \omega_d t \quad \text{and} \quad y_2(t) = \sin \omega_d t,$$

for which the Wronskian is

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos \omega_d t & \sin \omega_d t \\ -\omega_d \sin \omega_d t & \omega_d \cos \omega_d t \end{vmatrix} = \omega_d.$$

Using variation of parameters, we seek a particular solution of the form

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t),$$

where

$$\begin{aligned} v_1'(t) &= \frac{-y_2(t) e^{\zeta\omega_n t} F(t)}{W[y_1, y_2](t)} = -e^{\zeta\omega_n t} F(t) \frac{\sin \omega_d t}{\omega_d}, \\ v_2'(t) &= \frac{y_1(t) e^{\zeta\omega_n t} F(t)}{W[y_1, y_2](t)} = e^{\zeta\omega_n t} F(t) \frac{\cos \omega_d t}{\omega_d}. \end{aligned}$$

Integrating these gives

$$v_1(t) = - \int_{t_0}^t e^{\zeta\omega_n\tau} F(\tau) \frac{\sin \omega_d\tau}{\omega_d} d\tau,$$

$$v_2(t) = \int_{t_0}^t e^{\zeta\omega_n\tau} F(\tau) \frac{\cos \omega_d\tau}{\omega_d} d\tau.$$

Hence the particular solution is given by

$$y_p(t) = - \cos \omega_d t \int_{t_0}^t e^{\zeta\omega_n\tau} F(\tau) \frac{\sin \omega_d\tau}{\omega_d} d\tau + \sin \omega_d t \int_{t_0}^t e^{\zeta\omega_n\tau} F(\tau) \frac{\cos \omega_d\tau}{\omega_d} d\tau.$$

The general solution is therefore

$$y(t) = A \cos \omega_d t + B \sin \omega_d t + \int_{t_0}^t \frac{e^{\zeta\omega_n\tau} F(\tau)}{\omega_d} [\sin \omega_d t \cos \omega_d \tau - \cos \omega_d t \sin \omega_d \tau] d\tau,$$

which can be simplified using a trigonometric identity to give

$$y(t) = A \cos \omega_d t + B \sin \omega_d t + \int_{t_0}^t \frac{e^{\zeta\omega_n\tau} F(\tau)}{\omega_d} \sin \omega_d(t - \tau) d\tau.$$

The constants A and B are found using the initial conditions $y(t_0) = y_0$ and $\dot{y}(t_0) = \dot{y}_0$. We have

$$y_0 = A \cos \omega_d t_0 + B \sin \omega_d t_0,$$

$$\frac{\dot{y}_0}{\omega_d} = -A \sin \omega_d t_0 + B \cos \omega_d t_0,$$

which are solved simultaneously to obtain

$$A = y_0 \cos \omega_d t_0 - \frac{\dot{y}_0}{\omega_d} \sin \omega_d t_0,$$

$$B = y_0 \sin \omega_d t_0 + \frac{\dot{y}_0}{\omega_d} \cos \omega_d t_0.$$

Substituting into the general solution and simplifying with trigonometric identities,

$$y(t) = y_0 \cos \omega_d(t - t_0) + \frac{\dot{y}_0}{\omega_d} \sin \omega_d(t - t_0) + \int_{t_0}^t \frac{e^{\zeta\omega_n\tau} F(\tau)}{\omega_d} \sin \omega_d(t - \tau) d\tau.$$

We now revert back to the original variable $X(t)$ using:

$$\begin{aligned} y(t) = e^{\zeta\omega_n t} X(t) &\implies y_0 = X_0 e^{\zeta\omega_n t_0}, \\ \dot{y}_0 &= (\zeta\omega_n X_0 + \dot{X}_0) e^{\zeta\omega_n t_0}, \end{aligned}$$

which upon substitution yields

$$X(t) = g(t - t_0)X_0 + h(t - t_0)\dot{X}_0 + \int_{t_0}^t h(t - \tau)F(\tau) d\tau \quad (3.64)$$

where

$$\begin{aligned} g(t) &= e^{-\zeta\omega_n t} \left[\cos \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d t \right], & h(t) &= \frac{e^{-\zeta\omega_n t}}{\omega_d} \sin \omega_d t, \\ \omega_d &= \sqrt{1 - \zeta^2}, & \zeta &< 1. \end{aligned}$$

It is clear that this solution is the sum of that of the homogeneous equation associated with the original equation (3.59), $X_h(t)$, and a particular solution, $X_p(t)$, of the inhomogeneous equation:

$$X(t) = \underbrace{g(t - t_0)X_0 + h(t - t_0)\dot{X}_0}_{X_h(t)} + \underbrace{\int_{t_0}^t h(t - \tau)F(\tau) d\tau}_{X_p(t)}.$$

In vibration theory, $h(t)$ is known as the *impulse response* since it represents the response of the system, initially at rest, to a unit Dirac delta function $\delta(t)$. Any single input single output linear time-invariant system is completely characterized by its impulse response.

3.2.3 Mean Response

We now proceed to find the mean of the solution (3.64) of the linear mass-spring oscillator. Let the initial conditions X_0 and \dot{X}_0 be random variables, and the mean of the excitation be $\mu_F(t)$. We have

$$\mu_X(t) = \mathbb{E}[X(t)] = g(t - t_0) \mathbb{E}[X_0] + h(t - t_0) \mathbb{E}[\dot{X}_0] + \int_{t_0}^t h(t - \tau) \mu_F(\tau) d\tau.$$

The integral can be rewritten using the substitution $u = t - \tau$ so that we now have

$$\mu_X(t) = g(t - t_0) \mathbb{E}[X_0] + h(t - t_0) \mathbb{E}[\dot{X}_0] + \int_0^{t-t_0} h(u) \mu_F(t - u) du.$$

When the excitation is stationary, its mean is constant. In that case, we have

$$\mu_X(t) = g(t - t_0) \mathbb{E}[X_0] + h(t - t_0) \mathbb{E}[\dot{X}_0] + \mu_F \int_0^{t-t_0} h(u) du.$$

The integral can now be evaluated as follows

$$\begin{aligned} \int_0^{t-t_0} h(u) du &= \frac{1}{\omega_d} \int_0^{t-t_0} e^{-\zeta\omega_n u} \sin \omega_d u du \\ &= \frac{1}{\omega_d} \operatorname{Im} \left\{ \int_0^{t-t_0} e^{(-\zeta\omega_n + i\omega_d)u} du \right\} \\ &= \frac{1}{\omega_d} \operatorname{Im} \left\{ \frac{e^{(-\zeta\omega_n + i\omega_d)(t-t_0)} - 1}{-\zeta\omega_n + i\omega_d} \right\} \\ &= \frac{1}{\omega_n^2} \left[1 - e^{-\zeta\omega_n(t-t_0)} \left(\cos \omega_d(t-t_0) + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d(t-t_0) \right) \right] \\ &= \frac{1}{\omega_n^2} [1 - g(t - t_0)]. \end{aligned}$$

So the mean response is now

$$\mu_X(t) = g(t - t_0) \mathbb{E}[X_0] + h(t - t_0) \mathbb{E}[\dot{X}_0] + \frac{\mu_F}{\omega_n^2} [1 - g(t - t_0)]. \quad (3.65)$$

Since we are interested in the steady-state behaviour, we obtain the stationary mean by taking the limit as $t_0 \rightarrow -\infty$. It is clear that $g(t - t_0)$ and $h(t - t_0)$ tend to zero in this limit. So,

$$\lim_{t_0 \rightarrow -\infty} \mu_X(t) = \frac{\mu_F}{\omega_n^2} = \mu_X, \quad (3.66)$$

which shows that the mean response of the system tends to a constant value when driven by a stationary excitation.

3.2.4 Covariance Response

Now, let us find the covariance function of the solution (3.64) of the linear mass-spring oscillator. Let the initial conditions X_0 and \dot{X}_0 be uncorrelated with the driving force $F(t)$ whose covariance function shall be denoted by $C_{FF}(t, s)$. From the definition of the covariance function,

$$\begin{aligned} C_{XX}(t, s) &= \mathbb{E}[X(t)X(s)] - \mu_X(t)\mu_X(s) \\ &= \mathbb{E}[X_h(t)X_h(s)] + \mathbb{E}[X_h(t)X_p(s)] + \mathbb{E}[X_h(s)X_p(t)] \\ &\quad + \mathbb{E}[X_p(t)X_p(s)] - \mu_X(t)\mu_X(s). \end{aligned}$$

Evaluating each term individually,

$$\begin{aligned}
\mathbb{E}[X_h(t)X_h(s)] &= g(t-t_0)g(s-t_0)\mathbb{E}[X_0^2] + h(t-t_0)h(s-t_0)\mathbb{E}[\dot{X}_0^2] \\
&\quad + \{g(t-t_0)h(s-t_0) + g(s-t_0)h(t-t_0)\}\mathbb{E}[X_0\dot{X}_0], \\
\mathbb{E}[X_h(t)X_p(s)] &= g(t-t_0)\int_{t_0}^s h(s-\tau)\mathbb{E}[X_0F(\tau)]d\tau \\
&\quad + h(t-t_0)\int_{t_0}^s h(s-\tau)\mathbb{E}[\dot{X}_0F(\tau)]d\tau, \\
\mathbb{E}[X_h(s)X_p(t)] &= g(s-t_0)\int_{t_0}^t h(t-\tau)\mathbb{E}[X_0F(\tau)]d\tau \\
&\quad + h(s-t_0)\int_{t_0}^t h(t-\tau)\mathbb{E}[\dot{X}_0F(\tau)]d\tau, \\
\mathbb{E}[X_p(t)X_p(s)] &= \int_{t_0}^t \int_{t_0}^s h(t-\tau_t)h(s-\tau_s)\mathbb{E}[F(\tau_t)F(\tau_s)]d\tau_s d\tau_t, \\
\mu_X(t)\mu_X(s) &= g(t-t_0)g(s-t_0)\mathbb{E}[X_0]^2 + h(t-t_0)h(s-t_0)\mathbb{E}[\dot{X}_0]^2 \\
&\quad + \{g(t-t_0)h(s-t_0) + g(s-t_0)h(t-t_0)\}\mathbb{E}[X_0]\mathbb{E}[\dot{X}_0] \\
&\quad + \{g(t-t_0)\mathbb{E}[X_0] + h(t-t_0)\mathbb{E}[\dot{X}_0]\}\frac{\mu_F}{\omega_n^2}[1-g(t-t_0)] \\
&\quad + \{g(s-t_0)\mathbb{E}[X_0] + h(s-t_0)\mathbb{E}[\dot{X}_0]\}\frac{\mu_F}{\omega_n^2}[1-g(s-t_0)] \\
&\quad + \frac{\mu_F^2}{\omega_n^4}[1-g(t-t_0)][1-g(s-t_0)].
\end{aligned}$$

Since the initial conditions and the driving force are uncorrelated, the cross terms (i.e., the $\mathbb{E}[X_h(\cdot)X_p(\cdot)]$ terms) vanish. Using the standard definitions of variance and covariance, the remaining terms can be simplified so that the covariance function of the response reads

$$\begin{aligned}
C_{XX}(t, s) &= g(t-t_0)g(s-t_0)\text{Var}(X_0) + h(t-t_0)h(s-t_0)\text{Var}(\dot{X}_0) \\
&\quad + \{g(t-t_0)h(s-t_0) + g(s-t_0)h(t-t_0)\}\text{Cov}(X_0, \dot{X}_0) \\
&\quad + \int_{t_0}^t \int_{t_0}^s h(t-\tau_t)h(s-\tau_s)C_{FF}(\tau_t, \tau_s)d\tau_s d\tau_t.
\end{aligned} \tag{3.67}$$

Motivated by the fact that we are interested in the steady-state behaviour and we already know $g(t-t_0)$ and $h(t-t_0)$ vanish in the long run, we omit the first three terms to simplify the algebra. The integral can be rewritten using the substitutions $u_t = t-\tau_t$ and $u_s = s-\tau_s$

so that the covariance response is

$$C_{XX}(t, s) = \int_0^{t-t_0} \int_0^{s-t_0} h(u_t)h(u_s) C_{FF}(t - u_t, s - u_s) du_s du_t. \quad (3.68)$$

When the excitation is stationary, $C_{FF}(\tau_t, \tau_s) = C_{FF}(\tau_s - \tau_t)$. So we have

$$C_{XX}(t, s) = \int_0^{t-t_0} \int_0^{s-t_0} h(u_t)h(u_s) C_{FF}(s - t + u_t - u_s) du_s du_t. \quad (3.69)$$

3.2.5 Applying White Noise Properties

Now, let the driving force be white noise. $F(t)$ is then a stationary Gaussian process with

$$\mu_F = 0, \quad C_{FF}(\tau) = R_{FF}(\tau) = 2\pi S_0 \delta(\tau).$$

Given this, it is clear from (3.66) that the stationary mean response μ_X is also zero. Substituting $C_{FF}(\tau)$ into (3.69),

$$\begin{aligned} C_{XX}(t, s) &= 2\pi S_0 \int_0^{t-t_0} \int_0^{s-t_0} h(u_t)h(u_s) \delta(s - t + u_t - u_s) du_s du_t \\ &= 2\pi S_0 \int_0^{w-t_0} h(u_t)h(|s - t| + u_t) du_t, \quad w = \min(t, s). \end{aligned}$$

Taking the limit as $t_0 \rightarrow -\infty$ to obtain the stationary covariance (or equivalently, correlation) function,

$$\begin{aligned} R_{XX}(s - t) &= \lim_{t_0 \rightarrow -\infty} C_{XX}(t, s) \\ &= 2\pi S_0 \int_0^\infty h(u)h(|s - t| + u) du \\ &= \frac{2\pi S_0}{\omega_d^2} e^{-\zeta\omega_n|s-t|} \int_0^\infty e^{-2\zeta\omega_n u} \sin \omega_d u \sin \omega_d(|s - t| + u) du. \end{aligned}$$

Letting $\tau = s - t$ and evaluating the integral, it can be shown [8] that

$$R_{XX}(\tau) = \frac{\pi S_0}{2\zeta\omega_n^3} e^{-\zeta\omega_n|\tau|} \left[\cos \omega_d|\tau| + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d|\tau| \right], \quad (3.70)$$

from which

$$\sigma_X^2 = R_{XX}(0) = \mathbb{E}[X^2(t)] = \frac{\pi S_0}{2\zeta\omega_n^3}. \quad (3.71)$$

So, as $t \rightarrow \infty$, the probabilistic behaviour of $X(t)$ becomes such that

$$\mu_X = 0 \quad \text{and} \quad \sigma_X^2 = \frac{\pi S_0}{2\zeta\omega_n^3}.$$

3.2.6 Response Stationary Density

So far, we have shown that, in the long run, the mean and variance of the response of the linear mass-spring oscillator to the white noise excitation, tend towards stationary values. Since the input excitation is Gaussian, the output response will also be Gaussian because the system is linear. Therefore, $X(t)$ has a stationary density which is given by the density function of a $\mathbb{N}(0, \pi S_0/2\zeta\omega_n^3)$ random variable. Considering the system in vector form (3.60), the vector $\mathbf{X}(t) = (X(t), \dot{X}(t))^T$ is a Gaussian vector whose stationary density function can be written explicitly since Gaussian processes are completely characterized by their mean and variance.

Constructing the stationary density of the vector requires knowledge of the stationary properties of the derivative process $\dot{X}(t)$. The stationary mean is straightforward:

$$\lim_{t_0 \rightarrow -\infty} \mathbb{E}[\dot{X}(t)] = \frac{d\mu_X}{dt} = 0. \quad (3.72)$$

The stationary covariance functions $R_{X\dot{X}}(\tau)$ and $R_{\dot{X}\dot{X}}(\tau)$ can be derived from $R_{XX}(\tau)$ using the facts

$$R_{X\dot{X}}(\tau) = \frac{d}{d\tau} R_{XX}(\tau) \quad \text{and} \quad R_{\dot{X}\dot{X}}(\tau) = -\frac{d^2}{d\tau^2} R_{XX}(\tau),$$

from which it can be shown, by setting $\tau = 0$, that

$$\lim_{t_0 \rightarrow -\infty} \mathbb{E}[X(t)\dot{X}(t)] = 0 \quad \text{and} \quad \lim_{t_0 \rightarrow -\infty} \mathbb{E}[\dot{X}^2(t)] = \sigma_{\dot{X}}^2 = \frac{\pi S_0}{2\zeta\omega_n}. \quad (3.73)$$

So, the mean vector m and covariance matrix C of $\mathbf{X}(t)$ are given by

$$m = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{\pi S_0}{2\zeta\omega_n^3} & 0 \\ 0 & \frac{\pi S_0}{2\zeta\omega_n} \end{pmatrix}.$$

The probability density function f_Z for a 2-dimensional vector $Z \sim \mathbb{N}(m, C)$ is given by

$$f_Z(z) = \frac{1}{2\pi\sqrt{\det C}} \exp \left\{ -\frac{1}{2}(z - m)^T C^{-1}(z - m) \right\}.$$

So, the stationary density function p of $\mathbf{X}(t)$ is given by

$$p(\mathbf{X}) = \frac{1}{2\pi\sqrt{\det C}} \exp \left\{ -\frac{1}{2} \mathbf{X}^T C^{-1} \mathbf{X} \right\},$$

from which we obtain

$$p(X, \dot{X}) = \frac{\zeta\omega_n^2}{\pi^2 S_0} \exp \left\{ -\frac{2\zeta\omega_n}{\pi S_0} \left(\frac{1}{2}\omega_n^2 X^2 + \frac{1}{2}\dot{X}^2 \right) \right\}. \quad (3.74)$$

3.2.7 Response Power Spectral Density

We would now like to find the spectral density of the response $X(t)$. Using the definition of the PSD (3.55a),

$$\begin{aligned} S_{XX}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi S_0}{2\zeta\omega_n^3} e^{-\zeta\omega_n|\tau|} \left[\cos \omega_d|\tau| + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d|\tau| \right] e^{-i\omega\tau} d\tau \\ &= \frac{S_0}{4\zeta\omega_n^3} \int_{-\infty}^0 e^{(\zeta\omega_n - i\omega)\tau} \left[\cos \omega_d\tau - \frac{\zeta\omega_n}{\omega_d} \sin \omega_d\tau \right] d\tau \\ &\quad + \int_0^{\infty} e^{-(\zeta\omega_n + i\omega)\tau} \left[\cos \omega_d\tau + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d\tau \right] d\tau, \end{aligned}$$

which upon simplification gives

$$S_{XX}(\omega) = \frac{S_0}{(\zeta^2\omega_n^2 - \omega^2 + \omega_d^2)^2 + 4\zeta^2\omega_n^2\omega^2}.$$

Substituting for ω_d^2 , we obtain

$$S_{XX}(\omega) = \frac{S_0}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}. \quad (3.75)$$

Chapter 4

Stochastic Stability of the Autoparametric Vibration Absorber

In this chapter, the stochastic stability of the autoparametric vibration absorber introduced in Chapter 2 is analyzed. In particular, we seek to derive an explicit result for the almost-sure stability of the single mode solution of the system by computing the maximal Lyapunov exponent. A mathematically rigorous method for the calculation of the maximal Lyapunov exponent for such a system excited by a multivariate Ornstein-Uhlenbeck process that represents a generic parametric real noise is presented in [6]. In this chapter, the focus is to approximate, via a formal expansion, the maximal Lyapunov exponent of the autoparametric vibration absorber perturbed by small intensity multiplicative real and white noise processes. The approach adopted here is different from the method developed in [6]. We follow the perturbation method developed originally by Sri Namachchivaya and Van Roessel [24] for a four dimensional system parametrically excited by real noise. This method relies on the Feynman-Kac formula introduced in Chapter 3 that establishes a link between parabolic partial differential equations and stochastic processes. The work in [6] also provides the proof of the existence and uniqueness of a certain ergodic invariant measure that is essential to the calculation of the maximal Lyapunov exponent. Furthermore, in [6], it is shown that the approximation of the maximal Lyapunov exponent is indeed asymptotic.

4.1 Asymptotic Scaling

The equations governing the dynamics of the autoparametric vibration absorber were derived in Chapter 2. They are given, in dimensionless coordinates, by

$$\ddot{\hat{\eta}} + 2\hat{\zeta}_o\dot{\hat{\eta}} + \hat{\eta} - R\left(\ddot{\hat{\theta}}\sin\hat{\theta} + \dot{\hat{\theta}}^2\cos\hat{\theta}\right) = \hat{\xi}(t) \quad (4.1a)$$

$$\ddot{\hat{\theta}} + 2\hat{\zeta}_p\dot{\hat{\theta}} + \left(q^2 - \ddot{\hat{\eta}}\right)\sin\hat{\theta} = 0, \quad (4.1b)$$

where

$$\begin{aligned} \omega_p^2 &\stackrel{\text{def}}{=} \frac{g}{l}, & \omega_o^2 &\stackrel{\text{def}}{=} \frac{k}{m_o + m_p}, & q &\stackrel{\text{def}}{=} \frac{\omega_p}{\omega_o}, \\ R &\stackrel{\text{def}}{=} \frac{m_p}{m_o + m_p}, & \hat{\zeta}_o &\stackrel{\text{def}}{=} \frac{c_o}{2\sqrt{k}(m_o + m_p)}, & \hat{\zeta}_p &\stackrel{\text{def}}{=} \frac{c_p\sqrt{m_o + m_p}}{2m_pl^2\sqrt{k}}, \\ \hat{\xi}(t) &\stackrel{\text{def}}{=} \frac{\Xi(t/\omega_o)}{kl}, & \hat{\eta}(t) &\stackrel{\text{def}}{=} \frac{y(t/\omega_o)}{l}, & \hat{\theta}(t) &\stackrel{\text{def}}{=} \phi(t/\omega_o). \end{aligned}$$

As we would like to solve the system analytically, we resort to the use of the asymptotic (or perturbation) solving technique. It is widely employed for tackling challenging problems as it decomposes the problem of interest into an infinite number of relatively easy ones which can be solved iteratively to obtain an approximate solution that is accurate up to a chosen order of accuracy. The first few steps typically reveal the important features of the solution and the remaining ones give small corrections. The first act in the process is the introduction of a small positive parameter $\varepsilon \ll 1$ to scale various important parameters in the system.

We are interested in analyzing the stability of the fixed point $(\hat{\eta}, \hat{\theta}) \equiv 0$ of the unperturbed system. The important assumption in our asymptotic analysis is that the dissipation and random perturbations are small and scaled appropriately so that the system undergoes small random vibrations about the static equilibria $\hat{\eta} = 0$ and $\hat{\theta} = 0$. We shall therefore consider these quantities on a finer resolution by letting $\hat{\eta}$ and $\hat{\theta}$ be defined by

$$\hat{\eta}(t) \stackrel{\text{def}}{=} \varepsilon\eta(t) \quad \text{and} \quad \hat{\theta}(t) \stackrel{\text{def}}{=} \varepsilon\theta(t).$$

Since we are interested in the effect of small random perturbations, we let $\hat{\xi}$ be of the form

$$\hat{\xi}(t) \stackrel{\text{def}}{=} \varepsilon\nu\xi(t),$$

where $\xi(t)$ is a white noise process of “unit variance” and ν is an empirical parameter that controls the intensity of the noise. Again, we assume dissipation to be small as the dynamics of the system is most interesting when it is not overdamped. We therefore let $\hat{\zeta}_o$ and $\hat{\zeta}_p$ be of the form

$$\hat{\zeta}_o \stackrel{\text{def}}{=} \zeta_o \quad \text{and} \quad \hat{\zeta}_p \stackrel{\text{def}}{=} \varepsilon^2 \zeta_p,$$

where ζ_o and ζ_p are positive constants less than 1. This corresponds to letting the damping of the oscillator (which can be seen, for example, as a car) be of size 1 and that of the pendulum be of size ε^2 . We are therefore letting the oscillator (car) maintain its natural damping while that of the pendulum (vibration absorber added into the car) is made to be as small as possible to facilitate our asymptotic analysis of its stability. Substituting these into (4.1), we obtain the dynamics of the system as the following perturbation problem:

$$\ddot{\eta} + 2\zeta_o \dot{\eta} + \eta - R(\ddot{\theta} \sin(\varepsilon\theta) + \varepsilon \dot{\theta}^2 \cos(\varepsilon\theta)) = \nu \xi(t) \quad (4.2a)$$

$$\ddot{\theta} + \varepsilon^2 2\zeta_p \dot{\theta} + \left(\frac{q^2}{\varepsilon} - \ddot{\eta} \right) \sin(\varepsilon\theta) = 0. \quad (4.2b)$$

The principal feature of this system is, due to the dominant deterministic component of the dynamics, the deterministic dynamics gives us a place to begin seeking structure in the midst of the randomness. Once this structure is understood, we can then examine how various system parameters affect other important quantities. In particular, we want to derive an expression for the noise intensity ν in terms of the other system parameters, which states a condition that guarantees the almost-sure stability of the vibration absorber.

4.2 Single Mode Solution

Now, assume that the pendulum initially hangs vertically at rest. Then θ will of course be identically zero (a locked-mass dynamics) and only the oscillator, excited by $\nu\xi$, moves in the vertical direction. This constitutes the single mode solution of the system as only the mode representing the displacement of the oscillator, and hence the entire rigid body, from its static equilibrium is active. Equations (4.2) then reduce to

$$\ddot{\eta}(t) + 2\zeta_o \dot{\eta}(t) + \eta(t) = \nu \xi(t), \quad (4.3)$$

where, since $\xi(t)$ is white noise, $\mu_\xi = 0$ and $S_{\xi\xi}(\omega) = S_0$, so $R_{\xi\xi}(\tau) = 2\pi S_0 \delta(\tau)$. The statistical properties of the steady-state response $\eta(t)$ can then be explicitly derived using the techniques shown in Section 3.2. Its stationary correlation function is given by

$$R_{\eta\eta}(\tau) = \frac{\nu^2 \pi S_0}{2\zeta_o} e^{-\zeta_o |\tau|} \left[\cos \omega_d |\tau| + \frac{\zeta_o}{\omega_d} \sin \omega_d |\tau| \right], \quad (4.4)$$

where $\omega_d = \sqrt{1 - \zeta_o^2}$, $\zeta_o < 1$. So, for the stationary mean and variance, we have

$$\mu_\eta = 0 \quad \text{and} \quad \sigma_\eta^2 = R_{\eta\eta}(0) = \frac{\nu^2 \pi S_0}{2\zeta_o}. \quad (4.5)$$

The stationary density can then be obtained as

$$p(\eta, \dot{\eta}) = C \exp \left\{ -\gamma \left(\frac{1}{2} \eta^2 + \frac{1}{2} \dot{\eta}^2 \right) \right\} \quad (4.6)$$

where

$$\gamma = \frac{2\zeta_o}{\nu^2 \pi S_0} \quad \text{and} \quad C = \frac{\zeta_o}{\nu^2 \pi^2 S_0}.$$

Finally, the power spectral density is given by

$$S_{\eta\eta}(\omega) = \frac{\nu^2 S_0}{(1 - \omega^2)^2 + 4\zeta_o^2 \omega^2}. \quad (4.7)$$

Since $\xi(t)$ is of unit variance as in a standard Brownian motion, we have

$$\sigma^2 = 2\pi S_0 = 1 \implies S_0 = \frac{1}{2\pi},$$

so that now,

$$\begin{aligned} \sigma_\eta^2 &= \frac{\nu^2}{4\zeta_o}, & S_{\eta\eta}(\omega) &= \frac{1}{2\pi} \frac{\nu^2}{(1 - \omega^2)^2 + 4\zeta_o^2 \omega^2} \\ \gamma &= \frac{4\zeta_o}{\nu^2}, & C &= \frac{2\zeta_o}{\nu^2 \pi}. \end{aligned} \quad (4.8)$$

4.3 Linear Variational Equation

We would like to analyze the stability of the locked mass (i.e., the pendulum) about its equilibrium state $\bar{\theta} = 0$ in the single mode solution. This is obtained by giving it a small perturbation away from the equilibrium state to get a variational equation about this state. So, remembering θ was already scaled,

$$\theta(t) = \bar{\theta} + \phi(t), \quad \dot{\theta}(t) = 0 + \dot{\phi}(t), \quad \ddot{\theta}(t) = 0 + \ddot{\phi}(t).$$

Substituting these in (4.2b) yields

$$\ddot{\phi}(t) + \varepsilon^2 2\zeta_p \dot{\phi}(t) + \left(\frac{q^2}{\varepsilon} - \ddot{\eta}(t) \right) \sin(\varepsilon \phi(t)) = 0.$$

Since we are assuming the system undergoes small oscillations, we use the first-order approximation of sine to linearize the equation. We get the variational equation for the stability of the pendulum as

$$\ddot{\phi}(t) + \varepsilon^2 2\zeta_p \dot{\phi}(t) + (q^2 - \varepsilon \ddot{\eta}(t)) \phi(t) = 0. \quad (4.9)$$

Stability results for the equation in this form already exist. An explicit expression for the maximal Lyapunov exponent is derived in [4] where parametric excitation of the random oscillator via a coloured noise process such as $\eta(t)$ is treated but will not be valid to deal with noise of the type $\ddot{\eta}(t)$. In [33], the white noise situation is treated. Instead of leaving $\ddot{\eta}(t)$ as the parametric excitation, we use the equation (4.3) governing the single mode solution to rewrite it as

$$\ddot{\eta}(t) = \nu \xi(t) - 2\zeta_o \dot{\eta}(t) - \eta(t) \implies \varepsilon \ddot{\eta}(t) = \varepsilon \nu \xi(t) - \varepsilon 2\zeta_o \dot{\eta}(t) - \varepsilon \eta(t),$$

so that the stability problem then becomes

$$\begin{aligned} \ddot{\phi}(t) + \varepsilon^2 2\zeta_p \dot{\phi}(t) + (q^2 - \varepsilon \nu \xi(t) + \varepsilon 2\zeta_o \dot{\eta}(t) + \varepsilon \eta(t)) \phi(t) &= 0, \\ \text{where } \ddot{\eta}(t) + 2\zeta_o \dot{\eta}(t) + \eta(t) &= \nu \xi(t). \end{aligned} \quad (4.10)$$

At this point, we make an interesting observation. The variational equation of the pendulum is now parametrically excited, in the form of multiplicative noise, by

- the Gaussian white noise process $\xi(t)$,
 - the coloured noise process $\eta(t)$ which is driven by the same Gaussian white noise process $\xi(t)$,
- and
- the derivative $\dot{\eta}(t)$ of the coloured noise process.

Let us write the linear variational equation (4.10) as a system of first-order equations. We define

$$\begin{aligned} u_1 = \phi &\implies \dot{u}_1 = \dot{\phi} = u_2 \\ u_2 = \dot{\phi} &\implies \dot{u}_2 = \ddot{\phi} = -q^2 u_1 - \varepsilon^2 2\zeta_p u_2 - \varepsilon(2\zeta_o \dot{\eta} + \eta) u_1 + \varepsilon \nu \xi(t) u_1, \end{aligned}$$

so that the corresponding first order system is

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -q^2 & -\varepsilon^2 2\zeta_p \end{pmatrix}}_A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 \\ -\varepsilon(2\zeta_o \dot{\eta} + \eta) & 0 \end{pmatrix}}_B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 \\ \varepsilon \nu & 0 \end{pmatrix}}_C \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \xi(t). \quad (4.11)$$

Let $u = (u_1, u_2)^T$. As stated earlier, there are stability results [4] for the system in the form $\dot{u} = Au + B(\eta(t))u$ with only a coloured noise process $\eta(t)$. In the case of only white noise, that is, the system in the form $\dot{u} = Au + Cu\xi(t)$, Khas'minskii [16] presented a general method for obtaining necessary and sufficient conditions for stability. The novelty in what is presented in this thesis is that the variational equation is parametrically excited by both white and coloured noise processes. In particular, the coloured noise is driven by the same white noise process that is driving the variational equation parametrically. To that effect, to study the stability of the pendulum, we augment the equation governing the coloured noise into (4.11). That is, we write (4.3) as a system of first-order equations by defining

$$\begin{aligned} v_1 = \eta &\implies \dot{v}_1 = \dot{\eta} = v_2 \\ v_2 = \dot{\eta} &\implies \dot{v}_2 = \ddot{\eta} = \nu\xi(t) - 2\zeta_o v_2 - v_1, \end{aligned}$$

so that the corresponding first order system is

$$\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2\zeta_o \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \nu \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xi(t). \quad (4.12)$$

Now, we can augment (4.12) into (4.11) by making use of the new state variables v_1 and v_2 to rewrite \dot{u}_2 :

$$\dot{u}_2 = -q^2 u_1 - \varepsilon^2 2\zeta_p u_2 - \varepsilon(2\zeta_o v_2 + v_1)u_1 + \varepsilon\nu\xi(t)u_1,$$

so that the full augmented system in a four dimensional state space becomes

$$\begin{pmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \dot{v}_1(t) \\ \dot{v}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -q^2 - \varepsilon(2\zeta_o v_2(t) + v_1(t)) & -\varepsilon^2 2\zeta_p & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2\zeta_o \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \\ v_1(t) \\ v_2(t) \end{pmatrix} + \nu \begin{pmatrix} 0 \\ \varepsilon u_1(t) \\ 0 \\ 1 \end{pmatrix} \xi(t), \quad (4.13)$$

which is nonlinear due to the parametric excitation of u by $v := (v_1, v_2)^T$. This system can be represented as a stochastic differential equation of the Stratonovich type:

$$\begin{pmatrix} du_1 \\ du_2 \\ dv_1 \\ dv_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ [-q^2 - \varepsilon(2\zeta_o v_2 + v_1)]u_1 - \varepsilon^2 2\zeta_p u_2 \\ v_2 \\ -v_1 - 2\zeta_o v_2 \end{pmatrix} dt + \nu \begin{pmatrix} 0 \\ \varepsilon u_1 \\ 0 \\ 1 \end{pmatrix} \circ dW(t).$$

That is, we have

$$dX(t) = \tilde{b}(X(t)) dt + \sigma(X(t)) \circ dW(t), \quad X(0) = x = \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix}, \quad (4.14)$$

where

$$\tilde{b}(x) = \begin{pmatrix} u_2 \\ [-q^2 - \varepsilon(2\zeta_o v_2 + v_1)]u_1 - \varepsilon^2 2\zeta_p u_2 \\ v_2 \\ -v_1 - 2\zeta_o v_2 \end{pmatrix} \quad \text{and} \quad \sigma(x) = \nu \begin{pmatrix} 0 \\ \varepsilon u_1 \\ 0 \\ 1 \end{pmatrix}.$$

We would like to work with an Itô stochastic differential equation. Therefore, we apply the Wong-Zakai correction to the drift vector $\tilde{b}(x)$ to obtain that of the corresponding Itô SDE. The correction is given by

$$b_i(x) = \tilde{b}_i(x) + \frac{1}{2} \sum_{j=1}^n \frac{\partial \sigma_i}{\partial x_j} \sigma_j(x); \quad 1 \leq i \leq n.$$

For $i \in \{1, 3, 4\}$, $\sigma_i(x)$ is a constant. So $\frac{\partial \sigma_i}{\partial x_j} = 0$ for all $j \in \{1, 2, 3, 4\}$ and $b_i(x) = \tilde{b}_i(x)$.

For $i = 2$, we have

$$b_2(x) = \tilde{b}_2(x) + \frac{1}{2} \underbrace{\left[\frac{\partial}{\partial u_1}(\varepsilon u_1) \cdot 0 + \frac{\partial}{\partial u_2}(\varepsilon u_1) \cdot \varepsilon u_1 + \frac{\partial}{\partial v_1}(\varepsilon u_1) \cdot 0 + \frac{\partial}{\partial v_2}(\varepsilon u_1) \cdot 1 \right]}_{=0}.$$

Therefore, the system (4.13) can be represented by an Itô stochastic differential equation

$$dX(t) = b(X(t)) dt + \sigma(X(t)) dW(t), \quad (4.15)$$

where $X(0) = x$, $b(x) = \tilde{b}(x)$ and $\sigma(x)$ are as defined in (4.14). The diffusion matrix $a(x)$ is defined as

$$a(x) = \sigma(x) \sigma^T(x) = \nu^2 \begin{pmatrix} 0 \\ \varepsilon u_1 \\ 0 \\ 1 \end{pmatrix} (0 \quad \varepsilon u_1 \quad 0 \quad 1) = \nu^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon^2 u_1^2 & 0 & \varepsilon u_1 \\ 0 & 0 & 0 & 0 \\ 0 & \varepsilon u_1 & 0 & 1 \end{pmatrix}.$$

The generator of a time-homogeneous diffusion process is given by

$$\mathcal{G} = \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}. \quad (4.16)$$

So, the generator \mathcal{L}^ε of $X(t)$ is

$$\begin{aligned} \mathcal{L}^\varepsilon = u_2 \frac{\partial}{\partial u_1} + \left([-q^2 - \varepsilon(2\zeta_o v_2 + v_1)] u_1 - \varepsilon^2 2\zeta_p u_2 \right) \frac{\partial}{\partial u_2} + v_2 \frac{\partial}{\partial v_1} - (v_1 + 2\zeta_o v_2) \frac{\partial}{\partial v_2} \\ + \frac{\nu^2}{2} \left(\varepsilon^2 u_1^2 \frac{\partial^2}{\partial u_2^2} + \varepsilon 2u_1 \frac{\partial^2}{\partial u_2 \partial v_2} + \frac{\partial^2}{\partial v_2^2} \right), \end{aligned}$$

which we rewrite as a decomposition in powers of ε :

$$\mathcal{L}^\varepsilon = \mathcal{L}_0 + \varepsilon \mathcal{L}_1 + \varepsilon^2 \mathcal{L}_2, \quad (4.17)$$

where

$$\begin{aligned} \mathcal{L}_0 &= u_2 \frac{\partial}{\partial u_1} - q^2 u_1 \frac{\partial}{\partial u_2} + v_2 \frac{\partial}{\partial v_1} - (v_1 + 2\zeta_o v_2) \frac{\partial}{\partial v_2} + \frac{\nu^2}{2} \frac{\partial^2}{\partial v_2^2} \\ \mathcal{L}_1 &= -(2\zeta_o v_2 + v_1) u_1 \frac{\partial}{\partial u_2} + \nu^2 u_1 \frac{\partial^2}{\partial u_2 \partial v_2} \\ \mathcal{L}_2 &= -2\zeta_p u_2 \frac{\partial}{\partial u_2} + \frac{\nu^2}{2} u_1^2 \frac{\partial^2}{\partial u_2^2}. \end{aligned}$$

The generator \mathcal{G} of the coloured noise process $\eta(t)$ defined by (4.3) will also be of interest to us. From the first-order system form (4.12), we have

$$b(x) = \begin{pmatrix} v_2 \\ -v_1 - 2\zeta_o v_2 \end{pmatrix} \quad \text{and} \quad \sigma(x) = \nu \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

for its Itô SDE representation since it follows from the previous calculation that the Wong-Zakai correction here will also be zero. The diffusion matrix is

$$a(x) = \sigma(x) \sigma^T(x) = \nu^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \nu^2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, from (4.16), we have

$$\mathcal{G} = v_2 \frac{\partial}{\partial v_1} - (v_1 + 2\zeta_o v_2) \frac{\partial}{\partial v_2} + \frac{\nu^2}{2} \frac{\partial^2}{\partial v_2^2}, \quad (4.18)$$

which, upon close observation, we realize constitutes part of \mathcal{L}_0 defined for the complete augmented system (4.13).

4.4 Coordinate Transformation

The basis for the calculation of the Lyapunov exponent is the transformation of the dimensions of the system of interest into logarithmic-polar coordinates. Our system $(u, v)^T$ is given by the Itô SDE (4.15). We are interested in the stability of the pendulum, which corresponds to only the u part of the system. Since the structure of the system is such that u and v are coupled only via the influence of v on u (and not vice-versa), we can effectively treat u as a system on its own that is parametrically excited by v , and apply the coordinate transformation to only u , instead of the entire system $(u, v)^T$, to obtain the maximal Lyapunov exponent dictating the almost-sure stability condition of just u . To that effect, as in [4], consider the standard transformation

$$u_1 = r \cos \varphi \quad \text{and} \quad u_2 = q r \sin \varphi. \quad (4.19a)$$

We have

$$\begin{aligned} \rho(u) := \ln |r(u)| \quad \text{where} \quad r(u) &= \sqrt{u_1^2 + \left(\frac{u_2}{q}\right)^2} \\ \text{and} \quad \varphi(u) &= \tan^{-1} \left(\frac{u_2}{q u_1} \right). \end{aligned} \quad (4.19b)$$

Since u is given by an Itô SDE, we apply Itô's lemma to $\rho(u)$ and $\varphi(u)$ to determine the respective SDEs governing them. The generic formula for a function $f(u, v)$ is

$$\begin{aligned} df(u, v) &= \mathcal{L}^\varepsilon f dt + \langle \nabla f, \sigma \rangle dW(t) \\ &= (\mathcal{L}_0 f + \varepsilon \mathcal{L}_1 f + \varepsilon^2 \mathcal{L}_2 f) dt + \left\langle \begin{pmatrix} \partial_{u_1} f \\ \partial_{u_2} f \\ \partial_{v_1} f \\ \partial_{v_2} f \end{pmatrix}, \begin{pmatrix} 0 \\ \varepsilon \nu u_1 \\ 0 \\ \nu \end{pmatrix} \right\rangle dW(t) \\ &= \left[u_2 \frac{\partial f}{\partial u_1} - q^2 u_1 \frac{\partial f}{\partial u_2} + v_2 \frac{\partial f}{\partial v_1} - (v_1 + 2\zeta_o v_2) \frac{\partial f}{\partial v_2} + \frac{\nu^2}{2} \frac{\partial^2 f}{\partial v_2^2} \right] dt \\ &\quad + \varepsilon \left[-(2\zeta_o v_2 + v_1) u_1 \frac{\partial f}{\partial u_2} + \nu^2 u_1 \frac{\partial^2 f}{\partial u_2 \partial v_2} \right] dt \\ &\quad + \varepsilon^2 \left[-2\zeta_p u_2 \frac{\partial f}{\partial u_2} + \frac{\nu^2}{2} u_1^2 \frac{\partial^2 f}{\partial u_2^2} \right] dt \\ &\quad + \left[\varepsilon \nu u_1 \frac{\partial f}{\partial u_2} + \nu \frac{\partial f}{\partial v_2} \right] dW(t). \end{aligned}$$

However, since ρ and φ are functions of only u , the Itô SDEs for ρ and φ reduce to

$$\begin{aligned} d\rho = & \left[u_2 \frac{\partial \rho}{\partial u_1} - q^2 u_1 \frac{\partial \rho}{\partial u_2} \right] dt + \varepsilon \left[-(2\zeta_o v_2 + v_1) u_1 \frac{\partial \rho}{\partial u_2} \right] dt \\ & + \varepsilon^2 \left[-2\zeta_p u_2 \frac{\partial \rho}{\partial u_2} + \frac{\nu^2}{2} u_1^2 \frac{\partial^2 \rho}{\partial u_2^2} \right] dt + \varepsilon \nu u_1 \frac{\partial \rho}{\partial u_2} dW(t) \end{aligned} \quad (4.20a)$$

$$\begin{aligned} d\varphi = & \left[u_2 \frac{\partial \varphi}{\partial u_1} - q^2 u_1 \frac{\partial \varphi}{\partial u_2} \right] dt + \varepsilon \left[-(2\zeta_o v_2 + v_1) u_1 \frac{\partial \varphi}{\partial u_2} \right] dt \\ & + \varepsilon^2 \left[-2\zeta_p u_2 \frac{\partial \varphi}{\partial u_2} + \frac{\nu^2}{2} u_1^2 \frac{\partial^2 \varphi}{\partial u_2^2} \right] dt + \varepsilon \nu u_1 \frac{\partial \varphi}{\partial u_2} dW(t). \end{aligned} \quad (4.20b)$$

Computation of the derivatives of ρ and φ requires laborious calculations, the details of which are shown in Appendix A. The derivatives we require are

$$\begin{aligned} \frac{\partial \rho}{\partial u_1} &= \frac{\cos \varphi}{r} & \frac{\partial \varphi}{\partial u_1} &= -\frac{\sin \varphi}{r} \\ \frac{\partial \rho}{\partial u_2} &= \frac{\sin \varphi}{qr} & \frac{\partial \varphi}{\partial u_2} &= \frac{\cos \varphi}{qr} \\ \frac{\partial^2 \rho}{\partial u_2^2} &= \frac{\cos^2 \varphi - \sin^2 \varphi}{q^2 r^2} & \frac{\partial^2 \varphi}{\partial u_2^2} &= -\frac{2 \sin \varphi \cos \varphi}{q^2 r^2}. \end{aligned}$$

We now substitute these into (4.20) and simplify.

The order 1 drift terms:

$$\begin{aligned} Q_0(\varphi, v) &\stackrel{\text{def}}{=} u_2 \frac{\partial \rho}{\partial u_1} - q^2 u_1 \frac{\partial \rho}{\partial u_2} \\ &= qr \sin \varphi \frac{\cos \varphi}{r} - q^2 r \cos \varphi \frac{\sin \varphi}{qr} = 0 \\ h_0(\varphi, v) &\stackrel{\text{def}}{=} u_2 \frac{\partial \varphi}{\partial u_1} - q^2 u_1 \frac{\partial \varphi}{\partial u_2} \\ &= -qr \sin \varphi \frac{\sin \varphi}{r} - q^2 r \cos \varphi \frac{\cos \varphi}{qr} = -q. \end{aligned}$$

The order ε drift terms:

$$\begin{aligned}
Q_1(\varphi, v) &\stackrel{\text{def}}{=} -(2\zeta_o v_2 + v_1) u_1 \frac{\partial \rho}{\partial u_2} \\
&= -(2\zeta_o v_2 + v_1) r \cos \varphi \frac{\sin \varphi}{qr} \\
&= -\frac{1}{q} (2\zeta_o v_2 + v_1) \sin \varphi \cos \varphi \\
h_1(\varphi, v) &\stackrel{\text{def}}{=} -(2\zeta_o v_2 + v_1) u_1 \frac{\partial \varphi}{\partial u_2} \\
&= -(2\zeta_o v_2 + v_1) r \cos \varphi \frac{\cos \varphi}{qr} \\
&= -\frac{1}{q} (2\zeta_o v_2 + v_1) \cos^2 \varphi.
\end{aligned}$$

The order ε^2 drift terms:

$$\begin{aligned}
Q_2(\varphi, v) &\stackrel{\text{def}}{=} -2\zeta_p u_2 \frac{\partial \rho}{\partial u_2} + \frac{\nu^2}{2} u_1^2 \frac{\partial^2 \rho}{\partial u_2^2} \\
&= -2\zeta_p qr \sin \varphi \frac{\sin \varphi}{qr} + \frac{\nu^2}{2} r^2 \cos^2 \varphi \frac{\cos^2 \varphi - \sin^2 \varphi}{q^2 r^2} \\
&= -2\zeta_p \sin^2 \varphi + \frac{\nu^2}{2q^2} \cos^2 \varphi (1 - 2 \sin^2 \varphi) \\
h_2(\varphi, v) &\stackrel{\text{def}}{=} -2\zeta_p u_2 \frac{\partial \varphi}{\partial u_2} + \frac{\nu^2}{2} u_1^2 \frac{\partial^2 \varphi}{\partial u_2^2} \\
&= -2\zeta_p qr \sin \varphi \frac{\cos \varphi}{qr} - \frac{\nu^2}{2} r^2 \cos^2 \varphi \frac{2 \sin \varphi \cos \varphi}{q^2 r^2} \\
&= -2\zeta_p \sin \varphi \cos \varphi - \frac{\nu^2}{q^2} \sin \varphi \cos^3 \varphi.
\end{aligned}$$

The diffusion terms:

$$\begin{aligned}
Q_D(\varphi, v) &\stackrel{\text{def}}{=} \nu u_1 \frac{\partial \rho}{\partial u_2} \\
&= \nu r \cos \varphi \frac{\sin \varphi}{qr} = \frac{\nu}{q} \sin \varphi \cos \varphi \\
h_D(\varphi, v) &\stackrel{\text{def}}{=} \nu u_1 \frac{\partial \varphi}{\partial u_2} \\
&= \nu r \cos \varphi \frac{\cos \varphi}{qr} = \frac{\nu}{q} \cos^2 \varphi.
\end{aligned}$$

So, the SDEs for $\rho(u)$ and $\varphi(u)$ are

$$\begin{aligned}
d\rho &= Q^\varepsilon(\varphi, v) dt + \varepsilon Q_D(\varphi, v) dW(t) \\
&= [Q_0(\varphi, v) + \varepsilon Q_1(\varphi, v) + \varepsilon^2 Q_2(\varphi, v)] dt + \varepsilon Q_D(\varphi, v) dW(t)
\end{aligned} \tag{4.21a}$$

$$\begin{aligned}
d\varphi &= h^\varepsilon(\varphi, v) dt + \varepsilon h_D(\varphi, v) dW(t) \\
&= [h_0(\varphi, v) + \varepsilon h_1(\varphi, v) + \varepsilon^2 h_2(\varphi, v)] dt + \varepsilon h_D(\varphi, v) dW(t)
\end{aligned} \tag{4.21b}$$

where

$$\begin{aligned}
Q_0(\varphi, v) &= 0 & h_0(\varphi, v) &= -q \\
Q_1(\varphi, v) &= -\frac{1}{q}(2\zeta_o v_2 + v_1) \sin \varphi \cos \varphi & h_1(\varphi, v) &= -\frac{1}{q}(2\zeta_o v_2 + v_1) \cos^2 \varphi \\
Q_2(\varphi, v) &= -2\zeta_p \sin^2 \varphi & h_2(\varphi, v) &= -2\zeta_p \sin \varphi \cos \varphi \\
&\quad + \frac{\nu^2}{2q^2} \cos^2 \varphi (1 - 2\sin^2 \varphi) & &\quad - \frac{\nu^2}{q^2} \sin \varphi \cos^3 \varphi \\
Q_D(\varphi, v) &= \frac{\nu}{q} \sin \varphi \cos \varphi & h_D(\varphi, v) &= \frac{\nu}{q} \cos^2 \varphi.
\end{aligned}$$

4.5 Furstenberg-Khas'minskii Formula

The process $Y(t)$, formed by $\varphi(t)$ and $v(t)$, is a diffusion process given by the Itô SDE

$$dY(t) = b(Y(t)) dt + \sigma(Y(t)) dW(t), \quad Y(0) = y = \begin{pmatrix} \varphi \\ v_1 \\ v_2 \end{pmatrix}, \tag{4.22}$$

where

$$b(y) = \begin{pmatrix} h_0(\varphi, v) + \varepsilon h_1(\varphi, v) + \varepsilon^2 h_2(\varphi, v) \\ v_2 \\ -v_1 - 2\zeta_o v_2 \end{pmatrix} \quad \text{and} \quad \sigma(y) = \begin{pmatrix} \varepsilon h_D(\varphi, v) \\ 0 \\ \nu \end{pmatrix}.$$

The diffusion matrix $a(y)$ is defined as

$$a(y) = \sigma(y) \sigma^T(y) = \begin{pmatrix} \varepsilon h_D(\varphi, v) \\ 0 \\ \nu \end{pmatrix} \begin{pmatrix} \varepsilon h_D(\varphi, v) & 0 & \nu \end{pmatrix} = \begin{pmatrix} \varepsilon^2 h_D^2(\varphi, v) & 0 & \varepsilon \nu h_D(\varphi, v) \\ 0 & 0 & 0 \\ \varepsilon \nu h_D(\varphi, v) & 0 & \nu^2 \end{pmatrix}.$$

So, using (4.16), we write the generator \mathcal{A}^ε of $Y(t)$ as a decomposition in powers of ε :

$$\mathcal{A}^\varepsilon = \mathcal{A}_0 + \varepsilon \mathcal{A}_1 + \varepsilon^2 \mathcal{A}_2, \quad (4.23)$$

where

$$\begin{aligned} \mathcal{A}_0 &= -q \frac{\partial}{\partial \varphi} + v_2 \frac{\partial}{\partial v_1} - (v_1 + 2\zeta_o v_2) \frac{\partial}{\partial v_2} + \frac{\nu^2}{2} \frac{\partial^2}{\partial v_2^2} \\ \mathcal{A}_1 &= -\frac{1}{q} (2\zeta_o v_2 + v_1) \cos^2 \varphi \frac{\partial}{\partial \varphi} + \frac{\nu^2}{q} \cos^2 \varphi \frac{\partial^2}{\partial \varphi \partial v_2} \\ \mathcal{A}_2 &= \left(-2\zeta_p \sin \varphi \cos \varphi - \frac{\nu^2}{q^2} \sin \varphi \cos^3 \varphi \right) \frac{\partial}{\partial \varphi} + \frac{\nu^2}{2q^2} \cos^4 \varphi \frac{\partial^2}{\partial \varphi^2}. \end{aligned}$$

If $Y(t)$ is ergodic, then, by the Furstenberg-Khas'minskii formula, the Lyapunov exponent for $u = (u_1, u_2)^T$ is given by

$$\begin{aligned} \lambda &= \mathbb{E}[Q^\varepsilon(\varphi, v)] \\ &= \int_v \int_\varphi Q^\varepsilon(\varphi, v) p^\varepsilon(\varphi, v) d\varphi dv, \end{aligned} \quad (4.24)$$

where $p^\varepsilon(\varphi, v)$ is the stationary density of $Y(t) = (\varphi(t), v(t))$ that satisfies the stationary Fokker-Planck equation $\mathcal{A}^{\varepsilon*} p^\varepsilon(\varphi, v) = 0$.

Computation of the Lyapunov exponent requires $Y(t)$ to be ergodic. Mathematically rigorous proof that $Y(t)$ is indeed ergodic and has a unique invariant measure $\mu(\varphi, v)$, is presented in [6] for a multivariate Ornstein-Uhlenbeck process $v(t)$ that represents a generic parametric noise. The following reasoning is also provided as a means of loosely justifying the ergodicity of $Y(t)$ since a proof is not provided here. For an ergodic process, time and ensemble averages are equivalent. Consequently, this means that, after a sufficiently long

time, a single trajectory in the system's state space will eventually pass through all possible points in the space; hence it visits all possible states of the system. Based on knowledge of a classical linear oscillator excited by white noise, presented in Section 3.2, it is reasonable to say that $v = (v_1, v_2)^T$ is, on its own, ergodic. For φ , since the leading order term in the drift coefficient is a constant that cannot be zero, it will never get trapped anywhere in the state space. By these arguments, one can hypothesize that $Y(t)$ is an ergodic process.

4.6 Asymptotic Analysis

To calculate the Lyapunov exponent in (4.24), we need to solve for $p^\varepsilon(\varphi, v)$ using the Fokker-Planck equation

$$\mathcal{A}^{\varepsilon*} p^\varepsilon(\varphi, v) = 0.$$

Instead of doing that, as in [33], we consider the adjoint problem

$$\mathcal{A}^\varepsilon u^\varepsilon(\varphi, v) = Q^\varepsilon(\varphi, v) - \lambda^\varepsilon \quad (4.25)$$

where

$$\begin{aligned} \mathcal{A}^\varepsilon &= \mathcal{A}_0 + \varepsilon \mathcal{A}_1 + \varepsilon^2 \mathcal{A}_2 \\ u^\varepsilon &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \\ Q^\varepsilon &= \varepsilon Q_1 + \varepsilon^2 Q_2 \\ \lambda^\varepsilon &= \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \dots \end{aligned}$$

In general, a necessary condition for an equation of the form $Au = f$ (where A is a linear operator) to have solution(s) is that $\langle f, v \rangle = 0$ for all v satisfying $A^*v = 0$. So, the solvability condition for (4.25) is

$$\langle Q^\varepsilon(\varphi, v) - \lambda^\varepsilon, p^\varepsilon(\varphi, v) \rangle = 0 \quad \text{where} \quad \mathcal{A}^{\varepsilon*} p^\varepsilon(\varphi, v) = 0,$$

and $\langle \cdot, \cdot \rangle$ represents the inner product

$$\langle f(\varphi, v), g(\varphi, v) \rangle = \int_v \int_\varphi f(\varphi, v) g(\varphi, v) d\varphi dv.$$

The principal idea is to solve asymptotically for $u^\varepsilon(\varphi, v)$ ¹ while obtaining the terms of λ^ε via the solvability conditions of the equations at each order of ε . Equation (4.25) can be written explicitly as

$$(\mathcal{A}_0 + \varepsilon \mathcal{A}_1 + \varepsilon^2 \mathcal{A}_2)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots) = (\varepsilon Q_1 + \varepsilon^2 Q_2) - (\lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \dots),$$

¹Clearly, this u differs from that which represents the system $u = (u_1, u_2)^T$ whose top Lyapunov exponent is being solved for.

which after expanding and grouping in orders of ε yields the following set of Poisson equations:

$$\mathcal{O}(1) \quad \mathcal{A}_0 u_0(\varphi, v) = -\lambda_0 \quad (4.26a)$$

$$\mathcal{O}(\varepsilon) \quad \mathcal{A}_0 u_1(\varphi, v) = Q_1(\varphi, v) - \mathcal{A}_1 u_0(\varphi, v) - \lambda_1 \quad (4.26b)$$

$$\mathcal{O}(\varepsilon^2) \quad \mathcal{A}_0 u_2(\varphi, v) = Q_2(\varphi, v) - \mathcal{A}_1 u_1(\varphi, v) - \mathcal{A}_2 u_0(\varphi, v) - \lambda_2. \quad (4.26c)$$

Note that we expand only up to the ε^2 -order equation as that is the order of accuracy we are looking to achieve. As shall be seen later, it is at this order we get the first non-vanishing value in the asymptotic expansion of λ^ε .

4.6.1 The Leading Order Equation

The leading order equation (4.26a) is

$$-q \frac{\partial u_0}{\partial \varphi}(\varphi, v) + \mathcal{G} u_0(\varphi, v) = -\lambda_0,$$

where \mathcal{G} is the generator associated with the coloured noise oscillator system $v = (v_1, v_2)^T$ as defined in (4.18). Its solvability condition is

$$-\langle \lambda_0, p_0(\varphi, v) \rangle = 0 \quad \text{where} \quad \mathcal{A}_0^* p_0(\varphi, v) = 0.$$

The adjoint (Fokker-Planck) equation is relatively easy to find a solution for. Since \mathcal{A}_0^* is a linear differential operator, $\mathcal{A}_0^* p_0(\varphi, v) = 0$ is a linear homogeneous partial differential equation whose solution can be written in the form

$$p_0(\varphi, v) = \Gamma(\varphi) \varrho(v). \quad (4.27a)$$

Given the composition of \mathcal{A}_0^* , we can make informed assumptions on the form of $\Gamma(\varphi)$ and $\varrho(v)$. It is reasonable to hypothesize that $\Gamma(\varphi)$ corresponds to the density of the uniform measure associated with the polar coordinate φ of the undamped, unforced system $u = (u_1, u_2)^T$ with frequency q , as evidenced by the differential equation $d\varphi = -q dt$ in (4.21b) when $\varepsilon = 0$;

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q^2 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Perhaps more evidently, $\varrho(v)$ can be reasonably assumed to be the density (4.6) of the invariant Gaussian measure associated with the coloured noise linear oscillator system

$v = (v_1, v_2)^T$ (4.12). So, we have

$$\Gamma(\varphi) = \frac{1}{2\pi} \quad \text{and} \quad \varrho(v) = C \exp \left\{ -\gamma \left(\frac{1}{2}v_1^2 + \frac{1}{2}v_2^2 \right) \right\}, \quad (4.27b)$$

where $\gamma = \frac{4\zeta_o}{\nu^2}$, $C = \frac{2\zeta_o}{\nu^2\pi}$.

We proceed to show that this does indeed satisfy the adjoint equation:

$$\begin{aligned} \mathcal{A}_0^* p_0(\varphi, v) &\implies \frac{\partial}{\partial \varphi}(q p_0) + \mathcal{G}^* p_0 \\ &= q \varrho(v) \frac{\partial \Gamma}{\partial \varphi} + \Gamma(\varphi) \mathcal{G}^* \varrho(v) \\ &= \frac{1}{2\pi} \left[-\frac{\partial}{\partial v_1}(v_2 \varrho(v)) + \frac{\partial}{\partial v_2}((v_1 + 2\zeta_o v_2) \varrho(v)) + \frac{\partial^2}{\partial v_2^2} \left(\frac{\nu^2}{2} \varrho(v) \right) \right] \\ &= \frac{1}{2\pi} \left[-v_2 \frac{\partial \varrho}{\partial v_1} + v_1 \frac{\partial \varrho}{\partial v_2} + 2\zeta_o \frac{\partial}{\partial v_2}(v_2 \varrho(v)) + \frac{\nu^2}{2} \frac{\partial^2 \varrho}{\partial v_2^2} \right], \end{aligned}$$

where

$$\begin{aligned} \frac{\partial \varrho}{\partial v_1} &= -\gamma v_1 \varrho(v) & \frac{\partial}{\partial v_2}(v_2 \varrho(v)) &= \varrho(v)(1 - \gamma v_2^2) \\ \frac{\partial \varrho}{\partial v_2} &= -\gamma v_2 \varrho(v) & \frac{\partial^2 \varrho}{\partial v_2^2} &= -\gamma \varrho(v)(1 - \gamma v_2^2). \end{aligned}$$

Substitution of these derivatives into the expression yields $\mathcal{A}_0^* p_0(\varphi, v) = 0$, confirming that the proposed solution is valid. So, from the solvability condition,

$$\begin{aligned} \langle \lambda_0, p_0(\varphi, v) \rangle &= 0 \\ \lambda_0 \int_v \int_\varphi p_0(\varphi, v) d\varphi dv &= 0 \\ \therefore \lambda_0 &= 0. \end{aligned}$$

Thus the leading order equation (4.26a) has solution $u_0(\varphi, v) = K$, where K is a constant. The ε - and ε^2 -order Poisson equations (4.26b, 4.26c) then become

$$\mathcal{O}(\varepsilon) \quad \mathcal{A}_0 u_1(\varphi, v) = Q_1(\varphi, v) - \lambda_1 \quad (4.28a)$$

$$\mathcal{O}(\varepsilon^2) \quad \mathcal{A}_0 u_2(\varphi, v) = Q_2(\varphi, v) - \mathcal{A}_1 u_1(\varphi, v) - \lambda_2. \quad (4.28b)$$

4.6.2 Poisson Equations and the Feynman-Kac Formula

Consider a generic form of the Poisson equations

$$\mathcal{A}_0 u(\varphi, v) = F(\varphi, v)$$

with solvability condition

$$\langle F(\varphi, v), p_0(\varphi, v) \rangle = 0 \quad \text{where} \quad \mathcal{A}_0^* p_0(\varphi, v) = 0,$$

so $p_0(\varphi, v)$ is as defined in (4.27a, 4.27b). We seek a solution to this generic form that can be applied to the Poisson equations as required. It is written explicitly as

$$-q \frac{\partial u}{\partial \varphi}(\varphi, v) + \mathcal{G}u(\varphi, v) = F(\varphi, v). \quad (4.29)$$

Introducing an auxiliary time t , the equation becomes [24]

$$\left(\frac{\partial}{\partial t} + q \frac{\partial}{\partial \varphi} \right) u(t, \varphi, v) = \mathcal{G}u(t, \varphi, v) - F(\varphi, v). \quad (4.30)$$

Since the inhomogeneous term $F(\varphi, v)$ is independent of t , $u(\varphi, v)$ is the stationary solution of (4.30). That is,

$$u(\varphi, v) = \lim_{t \rightarrow \infty} u(t, \varphi, v).$$

The goal is to transform the auxiliary equation (4.30) into a standard backward Kolmogorov equation and use the Feynman-Kac formula to solve it. We employ the transformation [24]

$$t = (\tau + s), \quad \varphi = q(\tau - s) \quad (4.31a)$$

and let $w(\tau, s, v) = u(t(\tau, s), \varphi(\tau, s), v)$. Since the transformation is nonsingular ($\frac{\partial(t, \varphi)}{\partial(\tau, s)} = -2q \neq 0$), it is invertible and its inverse is given by

$$\tau = \frac{1}{2} \left(t + \frac{\varphi}{q} \right), \quad s = \frac{1}{2} \left(t - \frac{\varphi}{q} \right) \quad (4.31b)$$

so that $u(t, \varphi, v) = w(\tau(t, \varphi), s(t, \varphi), v)$. Applying the chain rule, we have

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial w}{\partial \tau} \frac{\partial \tau}{\partial t} + \frac{\partial w}{\partial s} \frac{\partial s}{\partial t} = \frac{1}{2} \frac{\partial w}{\partial \tau} + \frac{1}{2} \frac{\partial w}{\partial s} \\ \frac{\partial u}{\partial \varphi} &= \frac{\partial w}{\partial \tau} \frac{\partial \tau}{\partial \varphi} + \frac{\partial w}{\partial s} \frac{\partial s}{\partial \varphi} = \frac{1}{2q} \frac{\partial w}{\partial \tau} - \frac{1}{2q} \frac{\partial w}{\partial s}. \end{aligned}$$

Substituting these into (4.30) yields

$$\frac{\partial w}{\partial \tau}(\tau, v; s) = \mathcal{G}w(\tau, v; s) + G(\tau, v; s); \quad w(0, v; s) = 0, \quad (4.32)$$

where $G(\tau, v; s) = -F(\varphi(\tau, s), v)$ and the homogeneous initial condition is used since the initial condition generally does not affect the stationary solution, which is what we are ultimately interested in. This equation constitutes an inhomogeneous backward Kolmogorov equation parametrized by s . Once it is solved for $w(\tau, v; s)$, the solution $u(t, \varphi, v)$ of the auxiliary equation (4.30) is obtained by reverting back to the original variables, then, by taking the limit as $t \rightarrow \infty$, we get the solution $u(\varphi, v)$ to the generic Poisson equation (4.29). That is,

$$u(\varphi, v) = \lim_{t \rightarrow \infty} u(t, \varphi, v) = \lim_{t \rightarrow \infty} w\left(\frac{t}{2} + \frac{\varphi}{2q}, v; \frac{t}{2} - \frac{\varphi}{2q}\right).$$

We now proceed to find the solution of (4.32). According to the Feynman-Kac formula, this solution is given by

$$w(\tau, v; s) = \mathbb{E}_v \left[\int_0^\tau G(\tau - t', V(t'); s) dt' \right], \quad (4.33)$$

where the process $V(t) = (v_1(t), v_2(t))$. This can be written in terms of the transition density function $p(\tau, v, v')$ of $V(t)$ as

$$w(\tau, v; s) = \int_0^\tau \int_{\mathbb{R}^2} G(\tau - t', v'; s) p(t', v, v') dv' dt', \quad (4.34)$$

so that

$$u(t, \varphi, v) = \int_0^{\frac{t}{2} + \frac{\varphi}{2q}} \int_{\mathbb{R}^2} G\left(\frac{t}{2} + \frac{\varphi}{2q} - t', v'; \frac{t}{2} - \frac{\varphi}{2q}\right) p(t', v, v') dv' dt'. \quad (4.35)$$

4.6.3 The ε -Order Equation

The ε -order equation (4.28a) is given by

$$\mathcal{A}_0 u_1(\varphi, v) = F_1(\varphi, v)$$

where

$$F_1(\varphi, v) = Q_1(\varphi, v) - \lambda_1 \quad \text{and} \quad Q_1(\varphi, v) = -\frac{1}{q}(2\zeta_o v_2 + v_1) \sin \varphi \cos \varphi.$$

From the solvability condition,

$$\begin{aligned}
& \langle Q_1(\varphi, v) - \lambda_1, p_0(\varphi, v) \rangle = 0 \\
\therefore \lambda_1 &= \langle Q_1(\varphi, v), p_0(\varphi, v) \rangle \\
&= \int_v \int_\varphi Q_1(\varphi, v) p_0(\varphi, v) d\varphi dv \\
&= -\frac{1}{2\pi q} \int_{\mathbb{R}^2} \int_0^{2\pi} (2\zeta_o v_2 + v_1) \sin \varphi \cos \varphi \varrho(v) d\varphi dv \\
&= -\frac{1}{4\pi q} \underbrace{\int_{\mathbb{R}^2} (2\zeta_o v_2 + v_1) \varrho(v) dv}_{\mathbb{E}[2\zeta_o v_2 + v_1]} \int_0^{2\pi} \sin 2\varphi d\varphi = 0.
\end{aligned}$$

So, by the mean zero and periodic properties of $Q_1(\varphi, v)$, we get that $\lambda_1 = 0$, meaning that $F_1(\varphi, v) = Q_1(\varphi, v)$. We have

$$\begin{aligned}
G_1(\tau, v; s) &= -F_1(\varphi(\tau, s), v) = -F_1(q(\tau - s), v) \\
&= \frac{1}{2q} (2\zeta_o v_2 + v_1) \sin 2q(\tau - s) \\
\therefore G_1(\tau - t', v'; s) &= \frac{1}{2q} (2\zeta_o v'_2 + v'_1) \sin 2q(\tau - t' - s).
\end{aligned}$$

Then,

$$\begin{aligned}
G_1\left(\frac{t}{2} + \frac{\varphi}{2q} - t', v'; \frac{t}{2} - \frac{\varphi}{2q}\right) &= \frac{1}{2q} (2\zeta_o v'_2 + v'_1) \sin 2q\left(\frac{t}{2} + \frac{\varphi}{2q} - t' - \frac{t}{2} + \frac{\varphi}{2q}\right) \\
&= \frac{1}{2q} (2\zeta_o v'_2 + v'_1) \sin(2\varphi - 2qt').
\end{aligned}$$

Therefore, from (4.35),

$$u_1(t, \varphi, v) = \frac{1}{2q} \int_0^{\frac{t}{2} + \frac{\varphi}{2q}} \int_{\mathbb{R}^2} (2\zeta_o v'_2 + v'_1) \sin(2\varphi - 2qt') p(t', v, v') dv' dt', \quad (4.36)$$

from which we obtain the steady-state solution $u_1(\varphi, v)$ that solves the ε -order Poisson equation (4.28a) by taking the limit as $t \rightarrow \infty$. So, we have

$$u_1(\varphi, v) = \frac{1}{2q} \int_0^\infty \int_{\mathbb{R}^2} (2\zeta_o v'_2 + v'_1) \sin(2\varphi - 2qt') p(t', v, v') dv' dt'. \quad (4.37)$$

Defining auxiliary variables

$$K_1(v, t') \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} v'_1 p(t', v, v') dv' \quad \text{and} \quad K_2(v, t') \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} v'_2 p(t', v, v') dv', \quad (4.38)$$

the solution $u_1(\varphi, v)$ can be concisely written as

$$u_1(\varphi, v) = \frac{1}{2q} \int_0^\infty (2\zeta_o K_2(v, t') + K_1(v, t')) \sin(2\varphi - 2qt') dt'. \quad (4.39)$$

It turns out that the φ derivative of this function is required in the next stage of the calculation process. It is given by

$$\frac{\partial u_1}{\partial \varphi}(\varphi, v) = \frac{1}{q} \int_0^\infty (2\zeta_o K_2(v, t') + K_1(v, t')) \cos(2\varphi - 2qt') dt'. \quad (4.40)$$

The auxiliary variables defined in (4.38) can be calculated explicitly by recognizing that they are, by definition, the expectation of $v' = (v'_1, v'_2)^T := v(t')$ given the initial condition $v(0) = v = (v_1, v_2)^T$. That is, $K_i(v, t') = \mathbb{E}_v[v_i(t')]$, $i \in \{1, 2\}$. Using standard techniques [22], v' can be shown to be given by

$$\begin{bmatrix} v_1(t') \\ v_2(t') \end{bmatrix} = \Phi(t') \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \int_0^{t'} \Phi(t' - s) \begin{pmatrix} 0 \\ \nu \end{pmatrix} \xi(s) ds,$$

where

$$\Phi(t') \stackrel{\text{def}}{=} e^{-\zeta_o t'} \begin{pmatrix} \cos \omega_d t' + \frac{\zeta_o}{\omega_d} \sin \omega_d t' & \frac{1}{\omega_d} \sin \omega_d t' \\ -\frac{1}{\omega_d} \sin \omega_d t' & \cos \omega_d t' - \frac{\zeta_o}{\omega_d} \sin \omega_d t' \end{pmatrix}, \quad \omega_d = \sqrt{1 - \zeta_o^2}.$$

The auxiliary variables $K_i(v, t')$ are then explicitly given by

$$\begin{aligned} \begin{bmatrix} K_1(v, t') \\ K_2(v, t') \end{bmatrix} &= \mathbb{E}_v \begin{bmatrix} v_1(t') \\ v_2(t') \end{bmatrix} \\ &= \begin{bmatrix} e^{-\zeta_o t'} \left(\cos \omega_d t' + \frac{\zeta_o}{\omega_d} \sin \omega_d t' \right) v_1 + \left(\frac{e^{-\zeta_o t'}}{\omega_d} \sin \omega_d t' \right) v_2 \\ - \left(\frac{e^{-\zeta_o t'}}{\omega_d} \sin \omega_d t' \right) v_1 + e^{-\zeta_o t'} \left(\cos \omega_d t' - \frac{\zeta_o}{\omega_d} \sin \omega_d t' \right) v_2 \end{bmatrix}. \end{aligned} \quad (4.41)$$

4.6.4 The ε^2 -Order Equation

The ε^2 -order equation (4.28b) is given by

$$\mathcal{A}_0 u_2(\varphi, v) = F_2(\varphi, v)$$

where

$$F_2(\varphi, v) = Q_2(\varphi, v) - \mathcal{A}_1 u_1(\varphi, v) - \lambda_2$$

and

$$\begin{aligned} Q_2(\varphi, v) &= -2\zeta_p \sin^2 \varphi + \frac{\nu^2}{2q^2} \cos^2 \varphi (1 - 2\sin^2 \varphi) \\ &= \zeta_p (\cos 2\varphi - 1) + \frac{\nu^2}{4q^2} \left(\frac{1}{2} + \frac{1}{2} \cos 4\varphi + \cos 2\varphi \right) \\ \mathcal{A}_1 u_1(\varphi, v) &= -\frac{1}{q} (2\zeta_o v_2 + v_1) \cos^2 \varphi \frac{\partial u_1}{\partial \varphi} + \frac{\nu^2}{q} \cos^2 \varphi \frac{\partial^2 u_1}{\partial \varphi \partial v_2} \\ &= -\frac{1}{2q^2} (2\zeta_o v_2 + v_1) (\cos 2\varphi + 1) \int_0^\infty (2\zeta_o K_2(v, t') + K_1(v, t')) \cos(2\varphi - 2qt') dt' \\ &\quad + \frac{\nu^2}{2q^2} (\cos 2\varphi + 1) \int_0^\infty \left(2\zeta_o \frac{\partial K_2}{\partial v_2}(v, t') + \frac{\partial K_1}{\partial v_2}(v, t') \right) \cos(2\varphi - 2qt') dt'. \end{aligned}$$

We use the solvability condition of this equation to find λ_2 , the first non-zero value in the asymptotic expansion of the Lyapunov exponent. The solvability condition is

$$\begin{aligned} \langle Q_2(\varphi, v) - \mathcal{A}_1 u_1(\varphi, v) - \lambda_2, p_0(\varphi, v) \rangle &= 0 \\ \therefore \lambda_2 &= \langle Q_2(\varphi, v), p_0(\varphi, v) \rangle - \langle \mathcal{A}_1 u_1(\varphi, v), p_0(\varphi, v) \rangle. \end{aligned} \quad (4.42)$$

We shall treat each of these averages separately, so, let

$$\Lambda_1 = \langle Q_2(\varphi, v), p_0(\varphi, v) \rangle \quad \text{and} \quad \Lambda_2 = \langle \mathcal{A}_1 u_1(\varphi, v), p_0(\varphi, v) \rangle.$$

For Λ_1 , we recognize that Q_2 turns out to be a function of only φ , so that

$$\begin{aligned} \Lambda_1 &= \int_v \int_\varphi Q_2(\varphi) p_0(\varphi, v) d\varphi dv \\ &= \frac{1}{2\pi} \int_0^{2\pi} Q_2(\varphi) d\varphi \int_{\mathbb{R}^2} \varrho(v) dv = \frac{1}{2\pi} \int_0^{2\pi} Q_2(\varphi) d\varphi, \end{aligned}$$

where the integral is calculated as follows:

$$\int_0^{2\pi} Q_2(\varphi) d\varphi = \begin{cases} 0 & \text{for } Q_2(\varphi) = \kappa \cos n\varphi; \\ \kappa 2\pi & \text{for } Q_2(\varphi) = \kappa \end{cases}; \quad \kappa \in \mathbb{R}, n \in \mathbb{N}.$$

Therefore,

$$\Lambda_1 = -\zeta_p + \frac{\nu^2}{8q^2}. \quad (4.43)$$

The evaluation of Λ_2 is a lot more involved, and we would like to obtain the results in terms of power spectral density. We have

$$\begin{aligned} \Lambda_2 &= \int_v \int_{\varphi} \mathcal{A}_1 u_1(\varphi, v) p_0(\varphi, v) d\varphi dv \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_0^{2\pi} \mathcal{A}_1 u_1(\varphi, v) \varrho(v) d\varphi dv \\ &= -\frac{1}{2\pi} \int_{\mathbb{R}^2} \int_0^{2\pi} \left[\frac{1}{2q^2} (2\zeta_o v_2 + v_1) (\cos 2\varphi + 1) \right. \\ &\quad \left. \int_0^{\infty} (2\zeta_o K_2(v, t') + K_1(v, t')) \cos(2\varphi - 2qt') dt' \right] \varrho(v) d\varphi dv \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_0^{2\pi} \left[\frac{\nu^2}{2q^2} (\cos 2\varphi + 1) \right. \\ &\quad \left. \int_0^{\infty} \left(2\zeta_o \frac{\partial K_2}{\partial v_2}(v, t') + \frac{\partial K_1}{\partial v_2}(v, t') \right) \cos(2\varphi - 2qt') dt' \right] \varrho(v) d\varphi dv. \end{aligned}$$

We interchange the integrals to compute the φ integrals first:

$$\begin{aligned} \Lambda_2 &= -\frac{1}{2\pi} \int_{\mathbb{R}^2} \left[\frac{1}{2q^2} (2\zeta_o v_2 + v_1) \int_0^{\infty} (2\zeta_o K_2(v, t') + K_1(v, t')) \right. \\ &\quad \left. \int_0^{2\pi} (\cos 2\varphi + 1) \cos(2\varphi - 2qt') d\varphi dt' \right] \varrho(v) dv \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{R}^2} \left[\frac{\nu^2}{2q^2} \int_0^{\infty} \left(2\zeta_o \frac{\partial K_2}{\partial v_2}(v, t') + \frac{\partial K_1}{\partial v_2}(v, t') \right) \right. \\ &\quad \left. \int_0^{2\pi} (\cos 2\varphi + 1) \cos(2\varphi - 2qt') d\varphi dt' \right] \varrho(v) dv. \end{aligned}$$

These phase average terms yield

$$\begin{aligned}
& \int_0^{2\pi} (\cos 2\varphi + 1) \cos(2\varphi - 2qt') d\varphi \\
&= \int_0^{2\pi} \cos 2\varphi \cos(2\varphi - 2qt') d\varphi + \underbrace{\int_0^{2\pi} \cos(2\varphi - 2qt') d\varphi}_{=0} \\
&= \frac{1}{2} \cos 2qt' \underbrace{\int_0^{2\pi} d\varphi}_{=2\pi} + \frac{1}{2} \underbrace{\int_0^{2\pi} \cos(4\varphi - 2qt') d\varphi}_{=0} \\
&= \pi \cos 2qt'.
\end{aligned}$$

Substituting this into the expression for Λ_2 , we get

$$\begin{aligned}
\Lambda_2 &= -\frac{1}{4q^2} \int_{\mathbb{R}^2} \left[(2\zeta_o v_2 + v_1) \int_0^\infty (2\zeta_o K_2(v, t') + K_1(v, t')) \cos 2qt' dt' \right] \varrho(v) dv \\
&\quad + \frac{\nu^2}{4q^2} \int_{\mathbb{R}^2} \left[\int_0^\infty \left(2\zeta_o \frac{\partial K_2}{\partial v_2}(v, t') + \frac{\partial K_1}{\partial v_2}(v, t') \right) \cos 2qt' dt' \right] \varrho(v) dv.
\end{aligned}$$

Again, we interchange integrals; this time, so that we can compute the v integrals first. We have

$$\begin{aligned}
\Lambda_2 &= -\frac{1}{4q^2} \int_0^\infty \left[\int_{\mathbb{R}^2} (2\zeta_o v_2 + v_1) (2\zeta_o K_2(v, t') + K_1(v, t')) \varrho(v) dv \right] \cos 2qt' dt' \\
&\quad + \frac{\nu^2}{4q^2} \int_0^\infty \left[\int_{\mathbb{R}^2} \left(2\zeta_o \frac{\partial K_2}{\partial v_2}(v, t') + \frac{\partial K_1}{\partial v_2}(v, t') \right) \varrho(v) dv \right] \cos 2qt' dt'.
\end{aligned}$$

Now, let us make the following definitions:

$$\begin{aligned}
R_1(t') &\stackrel{\text{def}}{=} \int_{\mathbb{R}^2} (2\zeta_o v_2 + v_1) (2\zeta_o K_2(v, t') + K_1(v, t')) \varrho(v) dv \\
R_2(t') &\stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \left(2\zeta_o \frac{\partial K_2}{\partial v_2}(v, t') + \frac{\partial K_1}{\partial v_2}(v, t') \right) \varrho(v) dv,
\end{aligned} \tag{4.44}$$

then, Λ_2 can be written concisely as

$$\Lambda_2 = -\frac{1}{4q^2} \int_0^\infty R_1(t') \cos 2qt' dt' + \frac{\nu^2}{4q^2} \int_0^\infty R_2(t') \cos 2qt' dt'. \tag{4.45}$$

The stationary autocorrelation function of a process $f(V(t))$ is

$$\begin{aligned} R_{ff}(t') &= \mathbb{E}[f(V(t))f(V(t+t'))] \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(v) f(v') p(t', v, v') \varrho(v) dv dv'. \end{aligned} \quad (4.46)$$

With this in mind, and from the definitions of $K_1(v, t')$ and $K_2(v, t')$ given in (4.38), for $R_1(t')$, we have

$$\begin{aligned} R_1(t') &= 4\zeta_o^2 \int_{\mathbb{R}^2} v_2 K_2(v, t') \varrho(v) dv + 2\zeta_o \int_{\mathbb{R}^2} v_2 K_1(v, t') \varrho(v) dv \\ &\quad + 2\zeta_o \int_{\mathbb{R}^2} v_1 K_2(v, t') \varrho(v) dv + \int_{\mathbb{R}^2} v_1 K_1(v, t') \varrho(v) dv \\ &= 4\zeta_o^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} v_2 v'_2 p(t', v, v') \varrho(v) dv dv' + 2\zeta_o \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} v_2 v'_1 p(t', v, v') \varrho(v) dv dv' \\ &\quad + 2\zeta_o \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} v_1 v'_2 p(t', v, v') \varrho(v) dv dv' + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} v_1 v'_1 p(t', v, v') \varrho(v) dv dv' \\ &= 4\zeta_o^2 R_{v_2 v_2}(t') + 2\zeta_o (R_{v_2 v_1}(t') + R_{v_1 v_2}(t')) + R_{v_1 v_1}(t'). \end{aligned}$$

The stationary autocorrelation functions for $v(t) = (v_1(t), v_2(t))^T$ are given by [22]

$$\begin{aligned} \begin{pmatrix} R_{v_1 v_1}(t') & R_{v_1 v_2}(t') \\ R_{v_2 v_1}(t') & R_{v_2 v_2}(t') \end{pmatrix} &= \mathbb{E} \begin{bmatrix} v_1(t)v_1(t+t') & v_1(t)v_2(t+t') \\ v_2(t)v_1(t+t') & v_2(t)v_2(t+t') \end{bmatrix} \\ &= \frac{\nu^2 e^{-\zeta_o |t'|}}{4\zeta_o} \begin{pmatrix} \cos \omega_d |t'| + \frac{\zeta_o}{\omega_d} \sin \omega_d |t'| & \frac{1}{\omega_d} \sin \omega_d |t'| \\ -\frac{1}{\omega_d} \sin \omega_d |t'| & \cos \omega_d |t'| - \frac{\zeta_o}{\omega_d} \sin \omega_d |t'| \end{pmatrix}. \end{aligned} \quad (4.47)$$

Thus, we have $R_{v_2 v_1}(t') + R_{v_1 v_2}(t') = 0$, which reduces $R_1(t')$ to

$$R_1(t') = 4\zeta_o^2 R_{v_2 v_2}(t') + R_{v_1 v_1}(t'). \quad (4.48)$$

For $R_2(t')$, we have

$$R_2(t') = 2\zeta_o \int_{\mathbb{R}^2} \frac{\partial K_2}{\partial v_2}(v, t') \varrho(v) dv + \int_{\mathbb{R}^2} \frac{\partial K_1}{\partial v_2}(v, t') \varrho(v) dv.$$

Using the expressions for $K_1(v, t')$ and $K_2(v, t')$ obtained in (4.41), we calculate

$$\frac{\partial K_1}{\partial v_2}(v, t') = \frac{e^{-\zeta_o t'}}{\omega_d} \sin \omega_d t' \quad \text{and} \quad \frac{\partial K_2}{\partial v_2}(v, t') = e^{-\zeta_o t'} \left(\cos \omega_d t' - \frac{\zeta_o}{\omega_d} \sin \omega_d t' \right). \quad (4.49)$$

Therefore,

$$\begin{aligned}
R_2(t') &= 2\zeta_o e^{-\zeta_o t'} \left(\cos \omega_d t' - \frac{\zeta_o}{\omega_d} \sin \omega_d t' \right) \int_{\mathbb{R}^2} \varrho(v) dv + \frac{e^{-\zeta_o t'}}{\omega_d} \sin \omega_d t' \int_{\mathbb{R}^2} \varrho(v) dv \\
&= e^{-\zeta_o t'} \left(2\zeta_o \cos \omega_d t' + \frac{1 - 2\zeta_o^2}{\omega_d} \sin \omega_d t' \right). \tag{4.50}
\end{aligned}$$

Substituting (4.48) and (4.50) into (4.45) yields

$$\begin{aligned}
\Lambda_2 &= -\frac{1}{4q^2} \int_0^\infty [4\zeta_o^2 R_{v_2 v_2}(t') + R_{v_1 v_1}(t')] \cos 2qt' dt' \\
&\quad + \frac{\nu^2}{4q^2} \int_0^\infty e^{-\zeta_o t'} \left(2\zeta_o \cos \omega_d t' + \frac{1 - 2\zeta_o^2}{\omega_d} \sin \omega_d t' \right) \cos 2qt' dt' \\
&= -\frac{\zeta_o^2}{q^2} \int_0^\infty R_{v_2 v_2}(t') \cos 2qt' dt' - \frac{1}{4q^2} \int_0^\infty R_{v_1 v_1}(t') \cos 2qt' dt' \\
&\quad + \frac{\nu^2 \zeta_o}{2q^2} \int_0^\infty e^{-\zeta_o t'} \cos \omega_d t' \cos 2qt' dt' + \frac{\nu^2 (1 - 2\zeta_o^2)}{4q^2 \omega_d} \int_0^\infty e^{-\zeta_o t'} \sin \omega_d t' \cos 2qt' dt' \\
&= -\frac{1}{4q^2} \int_0^\infty R_{v_1 v_1}(t') \cos 2qt' dt' - \frac{\zeta_o^2}{q^2} \int_0^\infty R_{v_2 v_2}(t') \cos 2qt' dt' \\
&\quad + \underbrace{\frac{\nu^2 \zeta_o}{4q^2} \int_0^\infty e^{-\zeta_o t'} [\cos(\omega_d + 2q)t' + \cos(\omega_d - 2q)t'] dt'}_{I_1} \\
&\quad + \underbrace{\frac{\nu^2 (1 - 2\zeta_o^2)}{8q^2 \omega_d} \int_0^\infty e^{-\zeta_o t'} [\sin(\omega_d + 2q)t' + \sin(\omega_d - 2q)t'] dt'}_{I_2}.
\end{aligned}$$

Using standard integral identities, the integrals I_1 and I_2 can be shown to simplify to

$$I_1 = \frac{\nu^2}{4q^2} \left[\frac{2\zeta_o^2(1 + 4q^2)}{(1 + 4q^2)^2 - (4q\omega_d)^2} \right] \quad \text{and} \quad I_2 = \frac{\nu^2}{4q^2} \left[\frac{(1 - 2\zeta_o^2)(1 - 4q^2)}{(1 + 4q^2)^2 - (4q\omega_d)^2} \right],$$

so that Λ_2 becomes

$$\begin{aligned}
\Lambda_2 &= -\frac{1}{4q^2} \int_0^\infty R_{v_1 v_1}(t') \cos 2qt' dt' - \frac{\zeta_o^2}{q^2} \int_0^\infty R_{v_2 v_2}(t') \cos 2qt' dt' \\
&\quad + \frac{\nu^2}{4q^2} \left[\frac{2\zeta_o^2(1 + 4q^2) + (1 - 2\zeta_o^2)(1 - 4q^2)}{(1 + 4q^2)^2 - (4q\omega_d)^2} \right].
\end{aligned}$$

The power spectral density defined in terms of the cosine function is

$$S_{v_i v_i}(\omega) = \frac{1}{\pi} \int_0^\infty R_{v_i v_i}(t') \cos \omega t' dt'. \quad (4.51)$$

Hence Λ_2 can be written in terms of power spectral density functions as

$$\Lambda_2 = -\frac{\pi}{4q^2} S_{v_1 v_1}(2q) - \frac{\pi \zeta_o^2}{q^2} S_{v_2 v_2}(2q) + \frac{\nu^2}{4q^2} \left[\frac{2\zeta_o^2(1+4q^2) + (1-2\zeta_o^2)(1-4q^2)}{(1+4q^2)^2 - (4q\omega_d)^2} \right].$$

The power spectral density functions of v_1 and v_2 are related as follows:

$$S_{\dot{m}\dot{m}}(\omega) = \omega^2 S_{\eta\eta}(\omega) \implies S_{v_2 v_2}(\omega) = \omega^2 S_{v_1 v_1}(\omega), \quad (4.52)$$

so that Λ_2 may be written in terms of only $S_{v_1 v_1}$ as

$$\begin{aligned} \Lambda_2 &= -\frac{\pi}{4q^2} [S_{v_1 v_1}(2q) + 4\zeta_o^2(2q)^2 S_{v_1 v_1}(2q)] + \frac{\nu^2}{4q^2} \left[\frac{2\zeta_o^2(1+4q^2) + (1-2\zeta_o^2)(1-4q^2)}{(1+4q^2)^2 - (4q\omega_d)^2} \right] \\ &= -\frac{\pi}{4q^2} [S_{v_1 v_1}(2q)(1+16\zeta_o^2 q^2)] + \frac{\nu^2}{4q^2} \left[\frac{2\zeta_o^2(1+4q^2) + (1-2\zeta_o^2)(1-4q^2)}{(1+4q^2)^2 - (4q\omega_d)^2} \right]. \end{aligned} \quad (4.53)$$

Now that we have finally obtained an expression for Λ_2 , we substitute the expressions in (4.43) and (4.53), for Λ_1 and Λ_2 respectively, into equation (4.42) to obtain an expression for λ_2 , the second order approximation of the maximal Lyapunov exponent. We have

$$\begin{aligned} \lambda_2 &= \Lambda_1 - \Lambda_2 \\ &= -\zeta_p + \frac{\nu^2}{8q^2} + \frac{\pi}{4q^2} S_{v_1 v_1}(2q)(1+16\zeta_o^2 q^2) \\ &\quad - \frac{\nu^2}{4q^2} \left[\frac{2\zeta_o^2(1+4q^2) + (1-2\zeta_o^2)(1-4q^2)}{(1+4q^2)^2 - (4q\omega_d)^2} \right], \end{aligned} \quad (4.54)$$

where the spectral density function is given explicitly by

$$S_{v_1 v_1}(\omega) = \frac{1}{2\pi} \frac{\nu^2}{(1-\omega^2)^2 + 4\zeta_o^2 \omega^2}.$$

Since the ultimate goal is to obtain an expression for the noise intensity ν in terms of the other system parameters, we rewrite (4.54) as

$$\begin{aligned} \lambda_2 &= -\zeta_p + \frac{\nu^2}{8q^2} + \frac{\pi \nu^2}{4q^2} S_{v_1 v_1}(2q)(1+16\zeta_o^2 q^2) \\ &\quad - \frac{\nu^2}{4q^2} \left[\frac{2\zeta_o^2(1+4q^2) + (1-2\zeta_o^2)(1-4q^2)}{(1+4q^2)^2 - (4q\omega_d)^2} \right], \end{aligned} \quad (4.55)$$

where

$$S_{v_1 v_1}(2q) = \frac{1}{2\pi} \frac{1}{(1 - 4q^2)^2 + 16\zeta_o^2 q^2} \quad \text{and} \quad \omega_d = \sqrt{1 - \zeta_o^2}.$$

Substituting for $S_{v_1 v_1}(2q)$ and ω_d , (4.55) eventually simplifies to

$$\lambda_2 = -\zeta_p + \frac{2q^2\nu^2}{(1 - 4q^2)^2 + 16\zeta_o^2 q^2}. \quad (4.56)$$

4.7 Stability Results

We have found that the maximal Lyapunov exponent that governs the almost-sure stability of the pendulum in the single mode solution of the system being studied in this thesis is given by

$$\lambda = \varepsilon^2 \left[-\zeta_p + \frac{2q^2\nu^2}{(1 - 4q^2)^2 + 16\zeta_o^2 q^2} \right] + \mathcal{O}(\varepsilon^4), \quad \varepsilon \rightarrow 0. \quad (4.57)$$

This result agrees perfectly with that of Baxendale and Sri Namachchivaya [6] where mathematically rigorous asymptotic results for a multivariate Ornstein-Uhlenbeck process that represents a generic parametric real noise is presented for the same system. There, further analysis is performed to show that the expression (4.57) for the maximal Lyapunov exponent is indeed asymptotic, and that the higher order terms in it are $\mathcal{O}(\varepsilon^4)$ as $\varepsilon \rightarrow 0$.

The result also validates those of Onu et al. [31]. There, the variational equation was considered in the form of (4.9), that is,

$$\ddot{\phi}(t) + \varepsilon^2 2\zeta_p \dot{\phi}(t) + (q^2 - \varepsilon \ddot{\eta}(t))\phi(t) = 0.$$

Then, based on the results of [4, 33], an explicit expression for the maximal Lyapunov exponent of the single mode solution was approximated, for the parametric real noise excitation $\ddot{\eta}(t)$, to be

$$\lambda \approx \varepsilon^2 \left(-\zeta_p + \frac{\pi}{4q^2} S_{\ddot{\eta}\ddot{\eta}}(2q) \right), \quad (4.58)$$

where, although its existence was not rigorously shown, the power spectral density function of $\ddot{\eta}$ was taken to be

$$S_{\ddot{\eta}\ddot{\eta}}(\omega) = \frac{1}{2\pi} \frac{\omega^4 \nu^2}{(1 - \omega^2)^2 + 4\zeta_o^2 \omega^2}.$$

The logic behind this approximation of $S_{\ddot{\eta}\ddot{\eta}}(\omega)$ is as follows:

Since, for a differentiable real noise process $\eta(t)$, the power spectral density functions of η

and $\dot{\eta}$ are related by $S_{\dot{\eta}\dot{\eta}}(\omega) = \omega^2 S_{\eta\eta}(\omega)$, this implies

$$S_{\dot{\eta}\dot{\eta}}(\omega) = \omega^2 S_{\eta\eta}(\omega) = \omega^4 S_{\eta\eta}(\omega).$$

Therefore, combining these ideas, (4.58) is then

$$\lambda \approx \varepsilon^2 \left(-\zeta_p + \frac{\pi}{4q^2} \cdot \frac{1}{2\pi} \frac{(2q)^4 \nu^2}{(1 - 4q^2)^2 + 4\zeta_o^2 (2q)^2} \right),$$

which simplifies to

$$\lambda \approx \varepsilon^2 \left(-\zeta_p + \frac{2q^2 \nu^2}{(1 - 4q^2)^2 + 16\zeta_o^2 q^2} \right). \quad (4.59)$$

This is precisely the second-order approximation of λ as seen in (4.57) obtained via the calculations presented in this chapter. It is remarkable that the expression that was derived in this thesis with more rigor matches the approximation presented in Onu et al. [31].

We now use the second-order approximation of the maximal Lyapunov exponent (4.59) to derive a condition on the white noise excitation intensity ν that guarantees the almost-sure stability of the pendulum. To do this, let $\lambda = 0$ so that

$$-\zeta_p + \frac{2q^2 \nu_c^2}{(1 - 4q^2)^2 + 16\zeta_o^2 q^2} = 0,$$

where ν_c represents the critical noise intensity at the stability boundary. Rearranging for ν_c yields

$$\nu_c = \sqrt{\frac{\zeta_p [(1 - 4q^2)^2 + 16\zeta_o^2 q^2]}{2q^2}}. \quad (4.60)$$

The necessary condition for almost-sure stability is $\lambda < 0$. Using this, we get that the necessary condition on the excitation noise intensity for the pendulum to be almost-surely stable is

$$\boxed{\nu < \sqrt{\frac{\zeta_p [(1 - 4q^2)^2 + 16\zeta_o^2 q^2]}{2q^2}}}. \quad (4.61)$$

So, when the excitation noise intensity ν is small, the size of the motion of the oscillator η is similarly small, and the pendulum undergoes small random motion near the stable point $\theta \equiv 0$; the system is stable. However, as the excitation noise intensity is further increased, the top Lyapunov exponent λ becomes positive when $\nu > \nu_c$ and the system becomes unstable.

To analyze the effects the system's parameters have on its stability, we consider plots of (4.60) to observe the behaviour of the almost-sure stability boundary of the single mode solution. In particular, we are able to see the destabilizing effect of internal resonance by plotting against the frequency q for different values of the dissipation coefficients of the primary system (the oscillator) ζ_o , and of the absorber system (the pendulum) ζ_p . These plots are presented in Figures 4.1 and 4.2. In each plot, the region of stability represented by (4.61) corresponds to the area below the graph and above the horizontal axis.

From the figures, it is evident that the stability boundary curves are centred upon a key value of the frequency; they all trough at $q = 0.5$. That is, the minimum point of each curve occurs at this value of q , meaning that the threshold level of the excitation intensity parameter ν required to cause instability of the single mode solution is obtained when $q = 0.5$. This shows that unstable random motions of the coupled system are most likely to occur when the natural frequency q of the absorber system is tuned to be in the vicinity of half of that of the primary system, which, in this case, was normalized to unity. Therefore, although no particular attention was given to the 1:2 resonance in the analysis of the linearized system, the stability boundary curves immediately show the significance of internal resonance in determining the behaviour of the system.

The graphs of the stability boundary also show the effect of the dissipation coefficients ζ_o and ζ_p on the stability region. Increasing the damping in either of the primary or absorber systems causes the region of stability to increase. In particular, the minimum point at $q = 0.5$ where the boundary troughs moves upwards, implying that the threshold level of the excitation intensity ν that results in instability of the single mode solution is increased. This effect is clearly more prevalent in Figure 4.1 where the damping of the absorber system ζ_p is kept constant while that of the primary system ζ_o is varied. The behaviour displayed in this figure mimics that of the instability tongues and transition curves in the stability chart of Mathieu's equation with linear viscous damping [17]. This equation is a linear second-order ordinary differential equation with cosine-type periodic forcing of the stiffness coefficient as follows:

$$\ddot{x}(t) + c\dot{x}(t) + (\delta + a \cos t)x(t) = 0.$$

The variational equation of the absorber system has a similar form except that the forcing of the stiffness coefficient is random. In particular, this forcing consists of the white noise process that excites the primary system and the real noise processes that represent the random motion of the primary system generated in response to the white noise excitation. This variational equation is given, prior to the introduction of asymptotic scaling, by

$$\ddot{\phi}(t) + 2\zeta_p\dot{\phi}(t) + (q^2 - \nu\xi(t) + 2\zeta_o\dot{\eta}(t) + \eta(t))\phi(t) = 0.$$

Let us first consider the scenario where only the white noise $\xi(t)$ is present in the parametric forcing. In this case, the second-order approximation of the top Lyapunov exponent is given by [33]

$$\lambda^\xi \approx \varepsilon^2 \left(-\zeta_p + \frac{\nu^2}{8q^2} \right),$$

and the almost-sure stability boundary is thus given by

$$\nu_c^\xi = \sqrt{8\zeta_p q^2}.$$

So, in this scenario, the stability boundary is simply a straight line from the origin whose slope is determined by ζ_p . Now, notice that (4.60) can equivalently be expressed as

$$\nu_c = \sqrt{8\zeta_p q^2 \frac{(1 - 4q^2)^2 + 16\zeta_o^2 q^2}{16q^4}}.$$

Therefore, the expression for the stability boundary is a product of the contributions of the white and real noise processes. The white noise has a sort of amplification effect, introducing ζ_p as a vertical scaling parameter, the effects of which are evident in Figure 4.2. The fact that the stability boundary is a curve that always troughs at $q = 0.5$ is the effect of the real noise. It brings in ζ_o as a parameter that controls how far up from the horizontal axis this minimum point occurs, leading to the Mathieu-like behaviour observed in Figure 4.1.

In summary, because of the nature of the parametric excitation of the variational equation of the absorber system containing correlated white and real noise processes, the dissipation coefficients of the primary and absorber systems have a similar effect on the almost-sure stability of the single mode solution of the overall system, but the manner in which this effect manifests itself depends upon the ratio between the natural frequencies of the primary and absorber systems, represented by q . As mentioned earlier, increasing either of these damping parameters results in an increase in stability, which is the similar effect they both unsurprisingly have. From Figures 4.1 and 4.2, it can be observed that for a given frequency reasonably far away from $q = 0.5$, an increase in ζ_p results in a significantly larger increase in the maximum intensity required to maintain stability than a corresponding increase in ζ_o . In other words, when the frequency q is outside the vicinity of the 1:2 internal resonance, ζ_p plays a far more crucial role in stabilizing the vibration absorber system. This makes sense from a physical point of view as one would expect the damping in the vibration absorber to be its primary source of stability. However, the story changes when the frequency $q \approx \frac{1}{2}$. In this resonance regime, ζ_o becomes the key stabilizing parameter because, as mentioned before, it controls how far up the minimum point of

the stability boundary occurs, which corresponds to the threshold level of the excitation intensity required to cause instability of the single mode solution in 1:2 internal resonance.

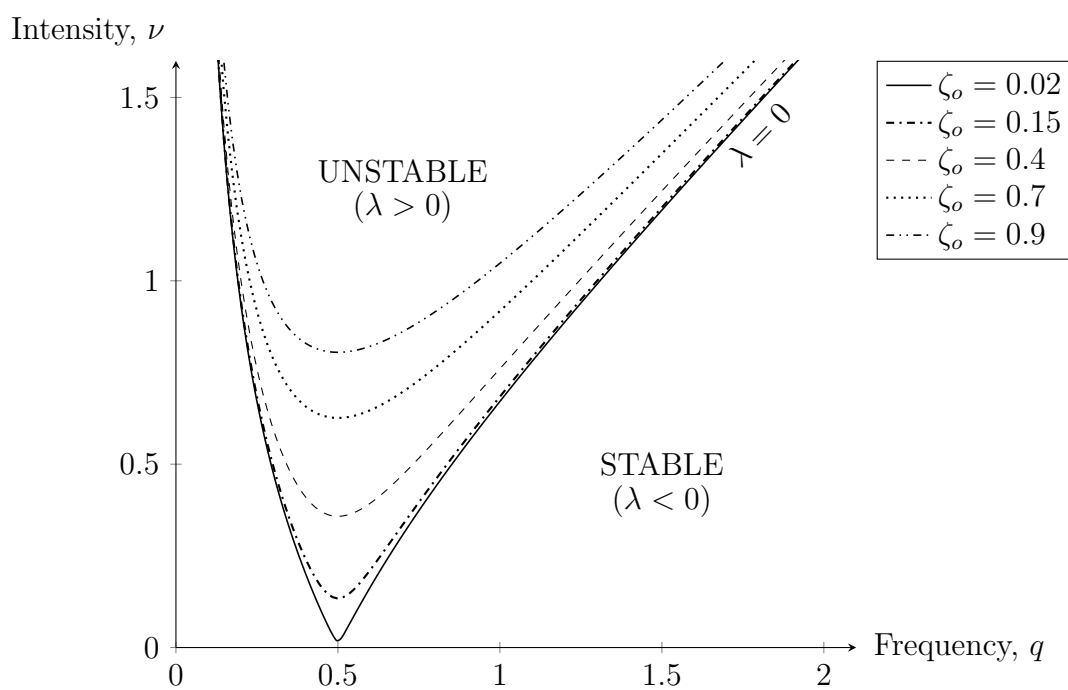


Figure 4.1: Almost-sure stability boundaries for $\zeta_p = 0.1$ and different values of ζ_o

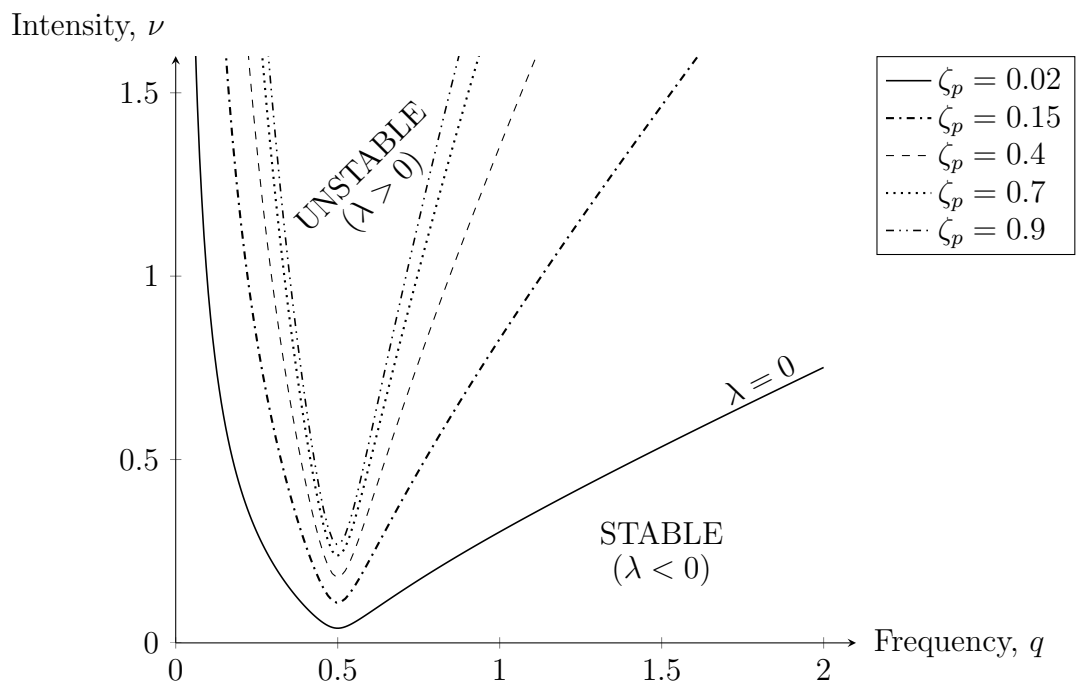


Figure 4.2: Almost-sure stability boundaries for $\zeta_o = 0.1$ and different values of ζ_p

Chapter 5

Conclusion

In this thesis, the almost-sure stability of the single mode solution of a two degree-of-freedom autoparametric vibration absorber is studied. Modelled by a pendulum, which represents the vibration absorber, attached to a mass-spring oscillator excited by a small-intensity additive white noise, the maximal Lyapunov exponent for the pendulum is computed to obtain the bifurcation points of its equilibrium solution $\bar{\theta} = 0$ in the parameter space of the system. The novelty in this work is that the variational equation governing the stability of the pendulum is parametrically excited by the white noise exciting the oscillator, as well as the real noise (and its derivative) generated in response to the white noise excitation. Calculation of the Lyapunov exponent involves taking advantage of the special structure of the system of equations governing the stability of the single mode solution, and employing perturbative techniques. The asymptotic analysis leads a sequence of linear Poisson equations to be solved at each order of ε . The solvability condition of these equations at each order yields the terms in the asymptotic expansion of the Lyapunov exponent. The asymptotic nature of this expansion is rigorously proven in [6].

Using the second-order approximation of the maximal Lyapunov exponent, the almost-sure stability boundary with respect to the system parameters is obtained, and several trends of practical importance to engineering applications are observed. In particular, it is observed that the damping parameter ζ_p plays a pivotal role in maintaining the almost-sure stability of the single mode solution of the vibration absorber, which is logical since it controls the energy dissipation in the vibration absorber and should therefore have a direct impact on its stability. The other damping parameter ζ_o , which controls the energy dissipation in the primary system represented by the mass-spring oscillator directly excited by the Gaussian white noise, is the key stabilizing parameter for frequencies close to the one-to-two internal resonance frequency $q = \frac{1}{2}$ at which the almost-sure stability of the

single mode solution of the vibration absorber is least. In general, increasing either of these damping parameters increases the stability of the system.

The work presented in this thesis can be complemented by explicit calculations of other related stability quantities to provide a more complete description of the asymptotic behaviour of the randomly excited autoparametric vibration absorber considered. Closely related to the Lyapunov exponent λ is the rotation number α . Roughly speaking, whereas λ determines the growth or decay properties of the solutions $\mathbf{X}(t) = (X(t), \dot{X}(t))$ of a stochastic system, α determines the asymptotic rate of rotation of the unit vector $\mathbf{X}(t)/\|\mathbf{X}(t)\|$. It is analogous to the imaginary part of the “maximal” eigenvalue of the system matrix of the corresponding deterministic system. Another important quantity is the moment Lyapunov exponent $g(p)$ which describes the exponential growth rate of $\mathbb{E}[\|\mathbf{X}(t)\|^p]$. It is therefore a quantity used to characterize the stability of the p -th moment of a random dynamical system, and it provides insight into the rate of convergence of the system response towards its steady-state value [21].

It is important to recognize the critical limitation of this study is that the model is not very robust. This stems from the fact that the random external excitation given to the mass-spring oscillator in the model is assumed to take the form of a Gaussian white noise process which, as has been discussed in this thesis, is merely a mathematical abstraction that is not physically realizable. It serves the purpose of simplifying the analysis while still providing reasonable results because it is a good approximation to physical processes encountered in practice that have very flat power spectral density functions. Therefore, a natural progression of this work is to consider a more realistic external noise excitation with a non-flat spectral density function. Although the analysis will be more difficult, the results are likely to be more realistically representative of mechanical and structural engineering systems subject to random external forcing.

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APPENDICES

Appendix A

Derivatives of $\rho(u)$ and $\varphi(u)$ in Section 4.4

From the coordinate transformation defined in (4.19), we have

$$u_1 = r \cos \varphi, \quad u_2 = q r \sin \varphi \quad (\text{A.1a})$$

and

$$\begin{aligned} \rho(u) &:= \ln |r(u)| \quad \text{where} \quad r(u) = \sqrt{u_1^2 + \left(\frac{u_2}{q}\right)^2}, \\ \varphi(u) &= \tan^{-1} \left(\frac{u_2}{qu_1} \right). \end{aligned} \quad (\text{A.1b})$$

Differentiation of ρ clearly requires the chain rule. We have

$$\begin{aligned} \frac{\partial \rho}{\partial u_1} &= \frac{\partial \rho}{\partial r} \frac{\partial r}{\partial u_1} = \frac{1}{r} \frac{\partial r}{\partial u_1} \\ \frac{\partial \rho}{\partial u_2} &= \frac{\partial \rho}{\partial r} \frac{\partial r}{\partial u_2} = \frac{1}{r} \frac{\partial r}{\partial u_2} \\ \frac{\partial^2 \rho}{\partial u_1^2} &= \frac{\partial}{\partial u_1} \left(\frac{1}{r} \frac{\partial r}{\partial u_1} \right) = \frac{1}{r} \frac{\partial^2 r}{\partial u_1^2} - \frac{1}{r^2} \left(\frac{\partial r}{\partial u_1} \right)^2 \\ \frac{\partial^2 \rho}{\partial u_2^2} &= \frac{\partial}{\partial u_2} \left(\frac{1}{r} \frac{\partial r}{\partial u_2} \right) = \frac{1}{r} \frac{\partial^2 r}{\partial u_2^2} - \frac{1}{r^2} \left(\frac{\partial r}{\partial u_2} \right)^2 \\ \frac{\partial^2 \rho}{\partial u_1 \partial u_2} &= \frac{\partial}{\partial u_2} \left(\frac{1}{r} \frac{\partial r}{\partial u_1} \right) = \frac{1}{r} \frac{\partial^2 r}{\partial u_1 \partial u_2} - \frac{1}{r^2} \left(\frac{\partial r}{\partial u_1} \right) \left(\frac{\partial r}{\partial u_2} \right). \end{aligned} \quad (\text{A.2})$$

We now compute the first-order derivatives of r and φ :

$$\frac{\partial r}{\partial u_1} = \frac{1}{2} \left(u_1^2 + \left(\frac{u_2}{q} \right)^2 \right)^{-\frac{1}{2}} \cdot 2u_1 = \frac{u_1}{r} = \cos \varphi \quad (\text{A.3})$$

$$\frac{\partial r}{\partial u_2} = \frac{1}{2} \left(u_1^2 + \left(\frac{u_2}{q} \right)^2 \right)^{-\frac{1}{2}} \cdot \frac{2u_2}{q^2} = \frac{u_2}{q^2 r} = \frac{\sin \varphi}{q}$$

$$\frac{\partial \varphi}{\partial u_1} = \frac{-\frac{u_2}{qu_1^2}}{1 + \left(\frac{u_2}{qu_1} \right)^2} = \frac{-\frac{u_2}{q}}{u_1^2 + \left(\frac{u_2}{q} \right)^2} = -\frac{u_2}{qr^2} = -\frac{\sin \varphi}{r} \quad (\text{A.4})$$

$$\frac{\partial \varphi}{\partial u_2} = \frac{\frac{1}{qu_1}}{1 + \left(\frac{u_2}{qu_1} \right)^2} = \frac{\frac{u_1}{q}}{u_1^2 + \left(\frac{u_2}{q} \right)^2} = \frac{u_1}{qr^2} = \frac{\cos \varphi}{qr}.$$

The first-order derivatives of ρ are then

$$\begin{aligned} \frac{\partial \rho}{\partial u_1} &= \frac{1}{r} \frac{\partial r}{\partial u_1} = \frac{\cos \varphi}{r} \\ \frac{\partial \rho}{\partial u_2} &= \frac{1}{r} \frac{\partial r}{\partial u_2} = \frac{\sin \varphi}{qr}. \end{aligned} \quad (\text{A.5})$$

We now compute the second-order derivatives of r :

$$\begin{aligned} \frac{\partial^2 r}{\partial u_1^2} &= \frac{\partial}{\partial u_1} (\cos \varphi) = \frac{\partial}{\partial \varphi} (\cos \varphi) \frac{\partial \varphi}{\partial u_1} = \frac{\sin^2 \varphi}{r} \\ \frac{\partial^2 r}{\partial u_2^2} &= \frac{1}{q} \frac{\partial}{\partial u_2} (\sin \varphi) = \frac{1}{q} \frac{\partial}{\partial \varphi} (\sin \varphi) \frac{\partial \varphi}{\partial u_2} = \frac{\cos^2 \varphi}{q^2 r} \\ \frac{\partial^2 r}{\partial u_1 \partial u_2} &= \frac{\partial}{\partial u_2} (\cos \varphi) = \frac{\partial}{\partial \varphi} (\cos \varphi) \frac{\partial \varphi}{\partial u_2} = -\frac{\sin \varphi \cos \varphi}{qr}. \end{aligned} \quad (\text{A.6})$$

The second-order derivatives of ρ are then

$$\begin{aligned} \frac{\partial^2 \rho}{\partial u_1^2} &= \frac{1}{r} \frac{\partial^2 r}{\partial u_1^2} - \frac{1}{r^2} \left(\frac{\partial r}{\partial u_1} \right)^2 = \frac{\sin^2 \varphi - \cos^2 \varphi}{r^2} \\ \frac{\partial^2 \rho}{\partial u_2^2} &= \frac{1}{r} \frac{\partial^2 r}{\partial u_2^2} - \frac{1}{r^2} \left(\frac{\partial r}{\partial u_2} \right)^2 = \frac{\cos^2 \varphi - \sin^2 \varphi}{q^2 r^2} \\ \frac{\partial^2 \rho}{\partial u_1 \partial u_2} &= \frac{1}{r} \frac{\partial^2 r}{\partial u_1 \partial u_2} - \frac{1}{r^2} \left(\frac{\partial r}{\partial u_1} \right) \left(\frac{\partial r}{\partial u_2} \right) = -\frac{2 \sin \varphi \cos \varphi}{qr^2}. \end{aligned} \quad (\text{A.7})$$

We now compute the second-order derivatives of φ :

$$\begin{aligned}
\frac{\partial^2 \varphi}{\partial u_1^2} &= -\frac{\partial}{\partial u_1} \left(\frac{\sin \varphi}{r} \right) = -\frac{r \cos \varphi \frac{\partial \varphi}{\partial u_1} - \sin \varphi \frac{\partial r}{\partial u_1}}{r^2} = \frac{2 \sin \varphi \cos \varphi}{r^2} \\
\frac{\partial^2 \varphi}{\partial u_2^2} &= \frac{1}{q} \frac{\partial}{\partial u_2} \left(\frac{\cos \varphi}{r} \right) = \frac{1}{q} \frac{-r \sin \varphi \frac{\partial \varphi}{\partial u_2} - \cos \varphi \frac{\partial r}{\partial u_2}}{r^2} = -\frac{2 \sin \varphi \cos \varphi}{q^2 r^2} \\
\frac{\partial^2 \varphi}{\partial u_1 \partial u_2} &= -\frac{\partial}{\partial u_2} \left(\frac{\sin \varphi}{r} \right) = -\frac{r \cos \varphi \frac{\partial \varphi}{\partial u_2} - \sin \varphi \frac{\partial r}{\partial u_2}}{r^2} = \frac{\sin^2 \varphi - \cos^2 \varphi}{qr^2}
\end{aligned} \tag{A.8}$$

In summary, the derivatives of $\rho(u)$ and $\varphi(u)$, up to the second order, are

$$\begin{aligned}
\frac{\partial \rho}{\partial u_1} &= \frac{\cos \varphi}{r} & \frac{\partial \varphi}{\partial u_1} &= -\frac{\sin \varphi}{r} \\
\frac{\partial \rho}{\partial u_2} &= \frac{\sin \varphi}{qr} & \frac{\partial \varphi}{\partial u_2} &= \frac{\cos \varphi}{qr} \\
\frac{\partial^2 \rho}{\partial u_1^2} &= \frac{\sin^2 \varphi - \cos^2 \varphi}{r^2} & \frac{\partial^2 \varphi}{\partial u_1^2} &= \frac{2 \sin \varphi \cos \varphi}{r^2} \\
\frac{\partial^2 \rho}{\partial u_2^2} &= \frac{\cos^2 \varphi - \sin^2 \varphi}{q^2 r^2} & \frac{\partial^2 \varphi}{\partial u_2^2} &= -\frac{2 \sin \varphi \cos \varphi}{q^2 r^2} \\
\frac{\partial^2 \rho}{\partial u_1 \partial u_2} &= -\frac{2 \sin \varphi \cos \varphi}{qr^2} & \frac{\partial^2 \varphi}{\partial u_1 \partial u_2} &= \frac{\sin^2 \varphi - \cos^2 \varphi}{qr^2}.
\end{aligned} \tag{A.9}$$