On coloring digraphs with forbidden induced subgraphs

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis includes work from 1 upcoming coauthored paper in which I played a major role in producing. Many of the words in this thesis will appear verbatim in this paper, which I wrote. The paper is listed below.

- On heroes in digraphs with forbidden induced forests, joint work with Hidde Koerts, Benjamin Moore, and Sophie Spirkl.

Abstract

This thesis mainly focuses on the structural properties of digraphs with high dichromatic number. The dichromatic number of a digraph D, denoted by $\vec{\chi}(D)$, is designed to be the directed analog of the chromatic number of a graph G, denoted by $\chi(G)$. The field of χ-boundedness studies the induced subgraphs that need to be present in a graph with high chromatic number. In this thesis, we study the equivalent of χ-boundedness but with dichromatic number instead. In particular, we study the induced subgraphs of digraphs with high dichromatic number from two different perspectives which we describe below.

First, we present results in the area of heroes. A digraph H is a hero of a class of digraphs $\mathcal{C}$ if there exists a constant c such that every H-free digraph $D \in \mathcal{C}$ has $\vec{\chi}(D) \leq c$. It is already known that when $\mathcal{C}$ is the family of F-free digraphs for some digraph F, the existence of heroes that are not transitive tournaments TT_k implies that F is the disjoint union of oriented stars. In this thesis, we narrow down the characterization of the digraphs F which have heroes that are not transitive tournaments to the disjoint union of oriented stars of degree at most 4. Furthermore, we provide a big step towards the characterization of heroes in $\{rK_1 + K_2\}$-free digraphs, where $r \geq 1$. We achieve the latter by developing mathematical tools for proving that a hero in F-free digraphs is also a hero in $\{K_1 + F\}$-free digraphs.

Second, we present results in the area of $\vec{\chi}$-boundedness. In this area, we try to determine the classes of digraphs for which transitive tournaments are heroes. In particular, we ask whether, for a given class of digraphs $\mathcal{C}$, there exists a function f such that, for every $k \geq 1$, $\vec{\chi}(D) \leq f(k)$ whenever $D \in \mathcal{C}$ and D is TT_k-free. We provide a comprehensive literature review of the subject and outline the χ-boundedness results that have an equivalent result in $\vec{\chi}$-boundedness. We conclude by generalizing a key lemma in the literature and using it to prove $\{\mathcal{B}, \mathcal{B}'\}$-free digraphs are $\vec{\chi}$-bounded, where $\mathcal{B}$ and $\mathcal{B}'$ are small brooms whose orientations are related and have an additional particular property.

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Chapter 1

Introduction

One of the most studied graph invariants in graph theory is the chromatic number. The chromatic number of a graph G, denoted by $\chi(G)$, which we will formally define soon, can be thought of as a measurement of complexity of G. Graphs with few edges and a high chromatic number have become a subject of study in graph theory. They have inspired new research fields, such as χ-boundedness, with conjectures that have resisted decades of work trying to settle them. The field of χ-boundedness, roughly speaking, deals with the question: which structures are we guaranteed to find in a graph G if G has small chromatic number locally but $\chi(G)$ is really large?

In this thesis, we present results related to the directed analog of the question: what structures are we guaranteed to find in a digraph D when D has large dichromatic number, denoted by $\vec{\chi}(D)$? As we will see, the dichromatic number of digraphs, which is a directed analog of the chromatic number of graphs, has multiple interesting properties. For instance, our question is interesting even when D is dense. We proceed to formalize these concepts in the following section.

Important note: Throughout this thesis, graphs and digraphs are simple and finite. In particular, for two vertices u, v in a digraph, not both of the arcs uv and vu are present.

1.1 Background definitions

A *graph* $G = (V, E)$ is an ordered pair composed of a *vertex set* $V(G) = V$, and an *edge set* $E(G) = E \subseteq \binom{V(G)}{2}$. A *digraph* $D = (V, A)$ is an ordered pair composed of a *vertex set* $V(D) = V$ and an *arc set* $A(D) = A \subseteq \{(u, v) : u, v \in V(D), u \neq v\}$. For simplicity, we use the notation uv to denote the edge $\{u, v\}$ in a graph or the arc (u, v) in a digraph. Furthermore, when considering a digraph $D = (V, A)$ where $uv \in A(D)$, we say *u sees v*, and *v is seen by u*. An *orientation* of a graph G is a digraph D such that if $\{u, v\} \in E(G)$, then either $(u, v) \in A(D)$ or $(v, u) \in A(D)$, but not both. The *underlying undirected graph* of a digraph D is the graph G with $V(G) = V(D)$ and where uv is an edge in G if either $uv \in A(D)$ or $vu \in A(D)$.

A path P on m vertices is the graph with m vertices $v_1, \ldots, v_m$ whose edges are of the form $v_i v_{i+1}$ for every $i \in \{1, \ldots, m-1\}$. Regarding orientations of a path, we use arrows $\rightarrow$ and $\leftarrow$ to denote the direction of the arcs. For instance, the orientation $P : v_1 \rightarrow v_2 \leftarrow v_3 \rightarrow v_4 \leftarrow \ldots$ is an orientation where the directions of the arcs are alternating. When the names of the vertices are not important, we omit them. For instance, $\rightarrow\rightarrow\leftarrow\leftarrow$ denotes a digraph isomorphic to $(\{v_1, \ldots, v_5\}, \{v_1v_2, v_2v_3, v_4v_3, v_5v_4\})$. Finally, the *directed path* on m vertices refers to a path on m vertices with orientation $\rightarrow\rightarrow \cdots \rightarrow$.

A (di)graph H is an *induced subgraph* of a (di)graph G if by deleting vertices of G we can get a (di)graph isomorphic to H. Equivalently, we say H is an induced subgraph of G if there exists a set $S \subseteq V(G)$ such that $G[S]$ is isomorphic to H. We call S a *copy* of H in G. If G has no copy of H, then we say G is *H-free*. Furthermore, if $\mathcal{C}$ is a set of (di)graphs, then G is *$\mathcal{C}$-free* if for every $H \in \mathcal{C}$, G is H-free. Note, for instance, that complete graphs are $2K_1$-free graphs. In directed graphs, $2K_1$-free digraphs, or equivalently oriented complete graphs, are called *tournaments*. *Transitive tournaments* are tournaments oriented such that there is no directed cycle.

Let D_1, D_2, and D_3 be induced subgraphs of a digraph D whose vertex sets are pairwise disjoint. We use $D_1 \Rightarrow D_2$ to indicate that every vertex in D_1 sees every vertex in D_2. We will also use this notation constructively. That is, for two digraphs F_1 and F_2 with $V(F_1) \cap V(F_2) = \emptyset$, the notation $F_1 \Rightarrow F_2$ denotes the digraph F with induced subgraphs F_1 and F_2 with non-overlapping vertex sets where $V(F) = V(F_1) \cup V(F_2)$, and every vertex in the copy of F_1 sees every vertex in the copy of F_2. Furthermore, we say $D = \Delta(D_1, D_2, D_3)$ if $V(D) = V(D_1) \cup V(D_2) \cup V(D_3)$, and $D_1 \Rightarrow D_2$, $D_2 \Rightarrow D_3$, and $D_3 \Rightarrow D_1$. For convenience, if D_1 is a transitive tournament on m vertices, then we write $D = \Delta(m, D_2, D_3)$ for $\Delta(D_1, D_2, D_3)$. We use $D_1 + D_2$ to denote the disjoint union of D_1 and D_2, and we use rD_1 for an integer $r \geq 0$ to denote the disjoint union of r copies of D_1.

Let G be a graph. For an integer m, let $[m] := \{1, \ldots, m\}$. A set $S \subseteq V(G)$ is a *stable set* if for every $u, v \in S$, we have that $uv \notin E(G)$. An *m-coloring* of G is a function $c : V(G) \rightarrow [m]$ such that for every color $i \in [m]$, the set $c^{-1}(i)$ is a stable set. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number m such that there exists an m-coloring of G. Figure 1.1 shows an example of a graph with chromatic number

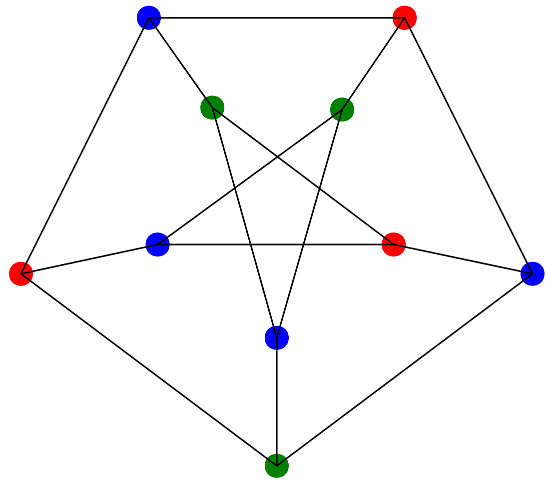


Figure 1.1: The Petersen graph with a 3-coloring

3.

Assume D is a digraph. Neumann-Lara [23] introduced the following analogs of m-colorings and χ for digraphs. An *m-dicoloring* of D is a function $c : V(D) \to [m]$ such that for every color $i \in [m]$, the set $c^{-1}(i)$ is an *acyclic set* (that is, a set that contains no directed cycle). The *dichromatic number* of D, denoted by $\vec{\chi}(D)$, is the minimum number m such that there exists an m-dicoloring of D. For convenience, if D is a digraph and $S \subseteq V(D)$, then we use $\vec{\chi}(S)$ to denote $\vec{\chi}(D[S])$. Figure 1.2 shows an example of a digraph with dichromatic number 2.

For a graph G, we say a set $S \subseteq V(G)$ is a *clique* if for every $u, v \in S$, where $u \neq v$, we have $uv \in E(G)$. Furthermore, the clique number of G, denoted by $\omega(G)$, is the maximum cardinality of a set $S \subseteq V(G)$ such that S is a clique. On the other hand, if D is a digraph, then $\omega(D)$ denotes $\omega(G)$ where G is the underlying undirected graph of D. Moreover, the *girth* of a graph G, denoted by $g(G)$, is the size of the smallest cycle in G, or ∞ if G is a forest. If D is a digraph, then the *undirected girth* of D, denoted by $g(D)$, refers to the girth of the underlying undirected graph of D. It is worth pointing out that different definitions of the girth of a digraph exists. For instance, some define it as the size of the shortest directed cycle. This is a different concept which we will not use. Notice that if $g(G) > 2$, then $\omega(G) \leq 2$.

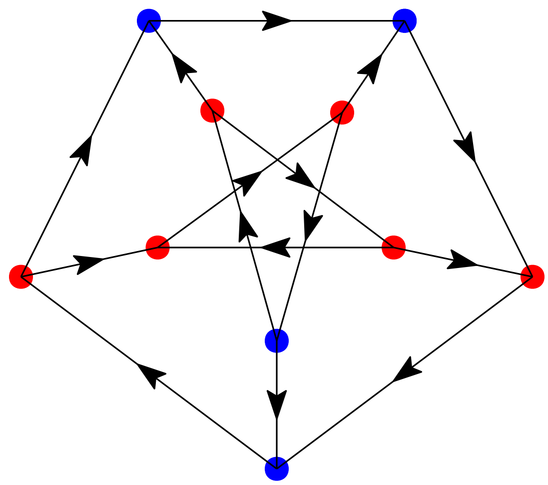


Figure 1.2: An orientation of the Petersen graph with a 2-dicoloring

1.2 Heroes

One of the easiest observations regarding the chromatic number is that, for every graph G, we have $\omega(G) \leq \chi(G)$. The equivalent in digraphs does not hold. An acyclic tournament TT_k, for instance, has $k = \omega(TT_k) > \vec{\chi}(TT_k) = 1$. In fact, it is not immediately obvious, as it is for χ, that there exists a digraph D with $\vec{\chi}(D) > 2$. One of the easiest ways to get arbitrarily high dichromatic number is the following construction due to Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour, and Thomassé [5]. Let $T_1 = K_1$, and for every $n \geq 1$, let $T_{n+1} = \Delta(1, T_n, T_n)$. A quick proof by induction yields $\vec{\chi}(T_n) = n$. The simplicity of the construction motivates the following question. What makes a tournament have high dichromatic number?

Following the notation of Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour, and Thomassé [5], we say that a digraph H is a *hero* of a family of digraphs $\mathcal{C}$ if there exists a constant c such that every H-free digraphs $D \in \mathcal{C}$ have $\vec{\chi}(D) \leq c$. Given that tournaments are $2K_1$-free digraphs, we can ask our question again as follows. What are the heroes of tournaments?

Many properties of the dichromatic number of digraphs can be seen as analogous to properties of undirected graphs. For instance, Erdős [13] proved the counter-intuitive result that a graph G need not have many edges to have high chromatic number.

Theorem 1.2.1 (Erdős [13]). *For every pair of integers g, k, there exists a graph G such that $g(G) \geq g$ and $\chi(G) \geq k$.*

Harutyunyan and Mohar [19] proved that the dichromatic number shares this property as well.

Theorem 1.2.2 (Harutyunyan and Mohar [19])**.** *For every pair of integers g, k, there exists a digraph D such that $g(D) \geq g$ and $\vec{\chi}(D) \geq k$.*

This very surprising property indicates that the dichromatic number, like the chromatic number, is hard to bound. In this section and the following, we focus on questions related to bounding the dichromatic number by forbidding induced subgraphs.

In [4], Aboulker, Charbit, and Naserasr organize related results and open questions into the following unified theory. They observe that, due to Theorem 1.2.2, for $\{F, H\}$-free digraphs to have bounded dichromatic number, one of F and H needs to be a directed forest. Furthermore, given that $2K_1$ is an induced subgraph of every directed forest except the path on two vertices $\rightarrow$ and K_1, it follows that if H is a hero in F-free digraphs where $F \notin \{\rightarrow, K_1\}$ is an oriented forest, then H is a hero in $2K_1$-free digraphs. That is, Aboulker, Charbit, and Naserasr [4] proved the following.

Theorem 1.2.3 (Aboulker, Charbit, and Naserasr [4])**.** *Given digraphs H and F, if $\{H, F\}$-free digraphs have bounded dichromatic number, then one of them, say F, is an oriented forest. Furthermore, if F has at least 3 vertices, then H is a hero in $2K_1$-free digraphs.*

Thus, it is important to know the structure of heroes of $2K_1$-free digraphs. In the landmark paper [5], Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour, and Thomassé characterized them completely.

Theorem 1.2.4 (Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour, and Thomassé [5])**.** *H is a hero in tournaments if and only if one of the following holds:*

- *$H = K_1$,*
- *$H = H_1 \Rightarrow H_2$ where H_1 and H_2 are heroes in tournaments, or*
- *$H = \Delta(1, H_1, m)$ or $H = \Delta(1, m, H_1)$ where $m \geq 1$ and H_1 is a hero in tournaments.*

As noted in [4], the question about bounding the dichromatic number of $\{F, H\}$-free digraphs is the directed version of the famous Gyárfás-Sumner conjecture.

Conjecture 1.2.5 (Gyárfás [17] and Sumner[31])**.** *For every forest F and every clique K_k on k vertices, the $\{F, K_k\}$-free graphs have bounded chromatic number.*

Aboulker, Charbit, and Naserasr [4] proved that if the underlying undirected graph of F has a path on four vertices and H is a hero in F-free digraphs, then H is a transitive tournament. Using this, the authors of [4] made the following two conjectures. An oriented star is an orientation of the star $K_{1,t}$ for some integer t.

Conjecture 1.2.6 (Aboulker, Charbit, and Naserasr [4]). *If F is a directed forest and H is a transitive tournament, then H is a hero in F-free digraphs.*

Conjecture 1.2.7 (Aboulker, Charbit, and Naserasr [4]). *If F is the disjoint union of oriented stars, and H is a hero in tournaments, then $\{F, H\}$-free digraphs have bounded dichromatic number.*

Thus, Conjecture 1.2.6 can be seen as the directed version of Conjecture 1.2.5, the Gyárfás-Sumner conjecture. Although some progress has been made (for instance, Cook, Masařík, Pilipczuk, Reinald, and Souza [12] proved the conjecture for every orientation of the path on four vertices), the conjecture remains wide open. We will discuss Conjecture 1.2.5 and Conjecture 1.2.6 in more detail in the next section.

On the other hand, Aboulker, Aubian, and Charbit [1] disproved Conjecture 1.2.7.

Theorem 1.2.8 (Aboulker, Aubian, and Charbit [1]). *If C is the cyclic triangle and F contains a copy of $K_1 + K_2$, then $\Delta(1, 2, C), \Delta(1, C, 2), \Delta(1, 2, 3)$, and $\Delta(1, 3, 2)$ are not heroes in F-free digraphs.*

Despite the conjecture being false, there are positive results in the literature. The following theorems summarize some of them.

Theorem 1.2.9 (Harutyunyan, Le, Newman, and Thomassé [18]). *Let $r \geq 2$ be an integer. H is a hero in rK_1-free digraphs if and only if:*

- *$H = K_1$;*
- *$H = H_1 \Rightarrow H_2$ where H_1 and H_2 are heroes in rK_1-free digraphs; or*
- *$H = \Delta(1, m, H_1)$ or $H = \Delta(1, H_1, m)$ where $m \geq 1$, and H_1 is a hero in rK_1-free digraphs.*

In other words, the heroes of rK_1-free digraphs are precisely the heroes of tournaments.

Theorem 1.2.10 (Aboulker, Aubian, and Charbit [1]). *A digraph H is a hero in $\{K_1 + K_2\}$-free digraphs if:*

- $H = K_1$;
- $H = H_1 \Rightarrow H_2$ *where* H_1 *and* H_2 *are heroes in* $\{K_1 + K_2\}$*-free digraphs; or*
- $H = \Delta(1, 1, H_1)$ *where* H_1 *is a hero in* $\{K_1 + K_2\}$*-free digraphs.*

Along with Theorem 1.2.8, this characterizes heroes in $\{K_1 + K_2\}$-free digraphs almost completely: only the status of $\Delta(1, 2, 2)$ remains to be decided.

Theorem 1.2.11 (Chudnovsky, Scott, and Seymour [9])**.** *If* F *is an oriented star and* H *is a transitive tournament, then* H *is a hero in* F*-free digraphs.*

Theorem 1.2.12 (Steiner [30])**.** *If* $F = \leftarrow\!\!\rightarrow$ *and* $H = \vec{C_3} \Rightarrow TT_k$ *for some integer* $k \geq 1$*, then* H *is a hero in* F*-free digraphs.*

As we will see, Theorems 1.2.9 to 1.2.11 are very important to our paper. Regarding the second bullet point of Theorem 1.2.10, Aboulker, Aubian, and Charbit proved the following much stronger result.

Theorem 1.2.13 (Aboulker, Aubian, and Charbit [1])**.** *Let* H_1*,* H_2 *and* F *be digraphs such that* $H_1 \Rightarrow H_2$ *is a hero in* F*-free digraphs, and* H_1 *and* H_2 *are heroes in* $\{K_1 + F\}$*-free digraphs. Then* $H_1 \Rightarrow H_2$ *is a hero in* $\{K_1 + F\}$*-free digraphs.*

In this thesis, we include the proof of the following three results which were proven in collaboration with Hidde Koerts, Benjamin Moore, and Sophie Spirkl. The first result is a full characterization of heroes in F-free digraphs when the underlying undirected graph of F is a star with degree at least 5.

Theorem 1.2.14. *If* F *is an orientation of a star of degree at least 5, then* H *is a hero in* F*-free digraphs if and only if* H *is a transitive tournament.*

Note that, previously, the smallest case known to be false for Conjecture 1.2.7 was $K_1 + K_2$. Thus, it was still possible that Conjecture 1.2.7 holds for directed stars, rather than the stronger statement of the disjoint union of directed stars. Theorem 1.2.14 proves that this weaker statement is false as well.

Regarding oriented stars of degree 4, we prove the following.

Theorem 1.2.15. *If* F *is an oriented star of degree 4 and* H *is a hero in* F*-free digraphs, then either* H *is a transitive tournament or* $H = \Delta(1, m, m')$ *where* $m, m' \geq 1$*.*

Like Theorem 1.2.14, Theorem 1.2.15 contradicts Conjecture 1.2.7 for cases that were previously still open. Furthermore, we prove the following strengthening of Theorem 1.2.10.

Theorem 1.2.16. *For every $r \geq 0$, a digraph H is a hero in $\{rK_1 + K_2\}$-free digraphs if:*

- *$H = K_1$,*
- *$H = H_1 \Rightarrow H_2$ where H_1 and H_2 are heroes in $\{rK_1 + K_2\}$-free digraphs.*
- *$H = \Delta(1, 1, H_1)$ where H_1 is a hero in $\{rK_1 + K_2\}$-free digraphs.*

We prove this theorem by developing conditions that imply that heroes in F-free digraphs are also heroes in $K_1 + F$-free digraphs.

By Theorem 1.2.8 and Theorem 1.2.10, to characterize heroes in $\{K_1 + K_2\}$-free digraphs, it is enough to determine whether $H = \Delta(1, 2, 2)$ is a hero in $\{K_1 + K_2\}$-free digraphs. By the same logic, Theorem 1.2.16 characterizes heroes in $\{rK_1 + K_2\}$-free digraphs up to $\Delta(1, 2, 2)$.

1.3 $\vec{\chi}$-boundedness

We now discuss the directed analog of the field of χ-boundedness.

A set of graphs $\mathcal{C}$ is *hereditary* if for every $G \in \mathcal{C}$, the induced subgraphs of G are also in $\mathcal{C}$. A hereditary set of graphs $\mathcal{C}$ is *χ-bounded* if there exists a function f such that $\chi(G) \leq f(\omega(G))$ when $G \in \mathcal{C}$. Furthermore, we say $\mathcal{C}$ is *χ-bounded by f*. Gyárfás [17] was the first person to study χ-boundedness systematically. We elaborate on his contributions later.

The concept of χ-boundedness is a generalization of the concept of perfect graphs, where a graph G is *perfect* if every induced subgraph G' of G satisfies that $\chi(G') = \omega(G')$. Recall that $\omega(G) \leq \chi(G)$. Thus, perfect graphs are those graphs where equality is achieved for every induced subgraph. The most important result regarding perfect graphs is the Strong Perfect Graph Theorem by Chudnovsky, Robertson, Seymour, and Thomas [8] which fully characterizes them. The *complement* of a graph G, denoted by $\bar{G}$, is the graph such that $V(G) = V(\bar{G})$ and $E(\bar{G}) = \{uv : uv \notin E(G), u \neq v\}$.

Theorem 1.3.1 (Chudnovsky, Robertson, Seymour, and Thomas [8])**.** *A graph G is perfect if and only if G is $\{C_5, \bar{C}_5, C_7, \bar{C}_7, \dots\}$-free.*

Notice that the family of perfect graphs is χ-bounded by $f(x) = x$, the identity function. The next most natural question, then, is: which other families of graphs $\mathcal{C}$ have the property that $\mathcal{C}$-free graphs are χ-bounded as well? The study of χ-bounded families is an active area of study in graph theory. We refer the interested reader to the most recent survey on the subject by Scott and Seymour [28]. One of the most important conjectures in the field is the Gyárfás-Sumner conjecture, which was already mentioned in an equivalent form in the previous section (Conjecture 1.2.5).

Conjecture 1.3.2 (Gyárfás [17] and Sumner[31]). *F-free graphs are χ-bounded if and only if F is a forest.*

The following argument proves the only if part of the conjecture. If F is not a forest, then, for every k, we can use Theorem 1.2.1 to find a graph G with $g(G) \geq g(F) + 1$ and $\chi(G) \geq k$. That is, G is F-free, has $\omega(G) = 2$, yet it has an arbitrarily high chromatic number, so there cannot exist a function f such that F-free graphs are χ-bounded by f.

Much work has been put into proving the other direction of the Gyárfás-Sumner conjecture. In this thesis, we deal with the directed analog of the conjecture. In the directed case, we consider $\vec{\chi}$-boundedness. A hereditary family $\mathcal{C}$ of digraphs is *$\vec{\chi}$-bounded* if there exists a function f such that $\vec{\chi}(D) \leq f(\omega(D))$ for every $D \in \mathcal{C}$. Notice that Conjecture 1.2.6 is equivalent to the following:

Conjecture 1.3.3 (Aboulker, Charbit, and Naserasr [4]). *F-free digraphs are $\vec{\chi}$-bounded if and only if F is a directed forest.*

Unlike Conjecture 1.2.7, this conjecture is yet to be decided. We proceed to compare some of the results from the field of χ-boundedness to those in $\vec{\chi}$-boundedness. First, the only if part of Conjecture 1.3.3 follows by using the same argument but instead of Theorem 1.2.1, we use Theorem 1.2.2. Second, Conjecture 1.3.3 can also be reduced to oriented trees. As pointed out by Scott and Seymour [28] in their survey on the subject, if F_1-free graphs and F_2-free graphs are χ-bounded, then $\{F_1+F_2\}$-free graphs are χ-bounded. $\vec{\chi}$-boundedness has the equivalent result.

Theorem 1.3.4 (Steiner [30]). *If F_1-free digraphs and F_2-free digraphs are $\vec{\chi}$-bounded, then $\{F_1 + F_2\}$-free digraphs are $\vec{\chi}$-bounded.*

Note that the equivalent is not true for heroes: if H is a hero in F_1-free digraphs and F_2-free digraphs, then it does not necessarily follow that H is a hero in $\{F_1 + F_2\}$-free digraphs. Take, for instance, the case where $H = \Delta(1, 2, 3)$, $F_1 = K_1$, and $F_2 = K_2$.

Gyárfás [17] started the systematic study of χ-boundedness by proving the result for every path. Let P_n be the path on n vertices. Furthermore, two non-overlaping sets of vertices X and Y in a graph G are complete (resp. anticomplete) to each other if $xy \in E(G)$ (resp. $xy \notin E(G)$) for every $x \in X$ and $y \in Y$. Lastly, let $N[v] = N(v) \cup \{v\}$.

Theorem 1.3.5 (Gyarfas [17]). *For every $n \geq 1$, P_n-free graphs are χ-bounded.*

Description of the proof: We proceed by induction on the clique number. Evidently, if $\omega(G) = 1$, then $\chi(G) = 1$. For the inductive step, let G be a P_n-free graph, and assume that the statement is true for every graph with clique number less than $\omega(G)$. Let γ be the constant such that if G' is a P_n-free graph and $\omega(G') < \omega(G)$, then $\chi(G') \leq \gamma$. We want to bound $\chi(G)$ by using γ and $\omega(G)$ only. For this purpose, we may assume that G is connected as otherwise we can color every component independently.

The first key observation is noticing that, for every vertex $v \in V(G)$, the neighborhood of v has clique number strictly less than $\omega(G)$, so $\chi(N(v)) \leq \gamma$. The way in which we use this fact is as follows. Let $G_1 = G$, and $v_1 = v$. Consider $G_2 = G_1 \setminus N[v_1]$ (that is, the graph that results from deleting every vertex in $N[v_1]$ from G_1). What do we know about $\chi(G_2)$? If it is small, that is, if it can be bounded by using γ and $\omega(G)$, then we win: $\chi(G_1) \leq \chi(G_2) + \gamma$. So assume $\chi(G_2)$ is large. Furthermore, for simplicity we assume G_2 is connected as otherwise the rest of the argument can be applied to the component of G_2 with largest chromatic number.

What is different between G_1 and G_2? There is one important difference: there exists a vertex, v_1, anticomplete to G_2. Other than that, they have the same properties: both have large chromatic number, both have the same clique number (as otherwise $\chi(G_2) \leq \gamma$), and both are P_n-free. We take advantage of this by repeating the same process: pick a vertex, v_2, in $N(v_1)$ with a neighbor in G_2, so $v_2 \notin V(G_2)$, and now do the same. We set $G_3 = G_2 \setminus N[v_2]$, assume G_3 is connected as before, and we ask again: what do we know about $\chi(G_3)$? Again, if it is small, then we win: $\chi(G_2) \leq \chi(G_3) + \gamma$, which contradicts that G_2 has large chromatic number. So assume G_3 has large chromatic number.

What have we gained? Now there exists a P_2, the one with vertex set $\{v_1, v_2\}$, that is anticomplete to G_3. And as before, there is nothing stopping us from repeating the same argument. Thus, if $\chi(G)$ is indeed large (that is, it cannot be bounded by using γ and $\omega(G)$), then we can repeat this process forever. But once we get to step n, there is a problem: the set $\{v_1, \ldots, v_n\}$ induces a copy of P_n, contradicting that G is P_n-free. ∎

This argument is famously known as the Gyárfás path argument. It works in many contexts, with the appropiate necessary modifications, when trying to prove statements about P_n-free graphs, and it gives a lesson: forbidding a tree, at least one as simple as a path, restricts our ability to infinitely "explore" the graph, and this leads to a bounded chromatic number.

The original Gyárfás path argument, although elegant and a cornerstone of the field, is not terribly complicated. For instance, one does not need more than half a page to formalize it. So it is reasonable to expect oriented paths to have an analogous argument, at the very least the directed path $\rightarrow \cdots \rightarrow$. Such argument, however, is yet to be found and it is not likely to exist because of a very important yet subtle difference between graphs and digraphs. Shortest paths in graphs are induced paths; this is not the case in digraphs. This small difference makes the Gyárfás path argument fail very early: once we pick v_2, how do we reach G_2? How does a shortest path between v_2 and G_2 help us if it is not induced? By making strong additional assumptions, it is possible to make the Gyárfás path argument work, as we explain later.

Arguments with a similar objective as the Gyárfás path argument have been found for other cases. One known in the folklore with no known first author is the proof that $K_{1,t}$-free graphs are χ-bounded. Stars allow for a very short argument. First, we observe that in $K_{1,t}$-free graphs, stable sets in the neighborhood of a vertex have size at most t. Second, since in an inductive proof on $\chi(G)$ we bound the clique number, this leads to a bound on the degree of every vertex, a bound provided by Ramsey's theorem [24]. Third, we use the bounded degree to bound $\chi(G)$. The equivalent for $\vec{\chi}$-boundedness is true:

Theorem 1.3.6 (Chudnovsky, Scott, and Seymour [9])**.** *If F is an oriented star, then F-free digraphs are $\vec{\chi}$-bounded.*

Note that this is equivalent to Theorem 1.2.11. Chudnovsky, Scott and Seymour [9] in fact proved the stronger result that F-free digraphs are χ-bounded. This notion is stronger since $\vec{\chi}(D) \leq \chi(D)$ where $\chi(D)$ refers to the chromatic number of the underlying undirected graph of D. Proving this result is much harder than proving that $K_{1,t}$-free graphs are χ-bounded.

In their breakthrough paper, Kierstead and Penrice [22] developed the notion of templates, which in short are maximal complete multipartite parts, to prove that if T is a tree of radius two, then T-free graphs are χ-bounded. This idea of using templates was improved in [21], and then again to obtain the following:

Theorem 1.3.7 (Scott and Seymour [27])**.** *If T is a tree obtained from a radius two tree by subdividing some of the edges incident to the center vertex, then T-free graphs are χ-bounded.*

This is the strongest known result regarding trees of bounded radius. Indeed, templates are a powerful idea. A closed tournament in a digraph is the smallest, by vertex size, strongly connected subgraph which contains a tournament of maximal size. These structures can be seen as the directed analog of templates. Closed tournaments are used to prove the current strongest result regarding oriented paths. Furthermore, we use them to prove a related result. We elaborate on this subject later.

The last technique which we will discuss is levelings. This technique fixes a vertex v, and then partitions the graph by distance to v. Leveling has been used to prove results such as:

Theorem 1.3.8 (Chudnovsky, Scott, and Seymour [10]). *If T is a tree obtained from a star and the subdivision of a star by adding a path joining their centers, then T-free graphs are χ-bounded.*

Theorem 1.3.9 (Chudnovsky, Scott, Seymour, and Spirkl [29]). *If T is a tree obtained from two paths by joining them with an edge, then T-free graphs are χ-bounded.*

These results are among the most significant advances towards the Gyárfás-Sumner conjecture. More relevant to our subject, leveling can be used to prove $\mathcal{C}$-free graphs are χ-bounded for the case where $\mathcal{C}$ is an infinite set of graphs. For instance, the technique can be used to prove the following theorem.

Theorem 1.3.10 (Chudnovsky, Scott, Seymour, and Spirkl [11]). *For every $l \geq 2$, the $\{C_{2l+1}, C_{2l+3}, \dots\}$-free graphs are χ-bounded.*

In [7], with Hompe, Moore, and Spirkl, we proved that Theorem 1.3.10 does not have an analog in $\vec{\chi}$-boundedness:

Theorem 1.3.11 (Carbonero, Hompe, Moore, and Spirkl [7]). *Digraphs with no induced directed cycle of odd length at least 5 are not $\vec{\chi}$-bounded.*

Indeed, $\vec{\chi}$-bounded families are rare. To illustrate why, define a *t-chordal digraph* as a digraph D whose induced directed cycles all have size t. The same authors as [7] proved the following (where the case $t = 3$ was first proven by Aboulker, Bousquet, and de Verclos [3]).

Theorem 1.3.12 (Carbonero, Hompe, Moore, and Spirkl [6]). *For every $t \geq 3$, the t-chordal digraphs are not $\vec{\chi}$-bounded.*

Notice that 3-chordal digraphs are the directed analog of chordal graphs, that is, graphs in which every induced cycle has size 3 . Chordal graphs are perfect, so they are χ-bounded by the identity function. Yet, when we consider the equivalent of chordal graphs in digraphs, or the more general concept of t-chordal graphs, we do not even get $\vec{\chi}$-boundedness.

On the upside, and coming back to paths, Theorem 1.3.12 is, in some sense, tight. In the same paper, the authors proved the following.

Theorem 1.3.13 (Carbonero, Hompe, Moore, and Spirkl [6]). *If $t \geq 1$ and P is the directed path on t vertices, then P-free t-chordal digraphs are $\vec{\chi}$-bounded.*

Theorem 1.3.13 is proven by using the Gyárfás path argument. Indeed, one needs the strong assumption that D is t-chordal for the Gyárfás path argument to work in P-free graphs. Despite paths being probably the next most complicated case after oriented stars, which, again, is not a simple case, they are still hard. This is true despite the rich theory and library of techniques that has been developed for χ-boundedness. In regards to paths, the following two are the best results.

Theorem 1.3.14 (Aboulker, Aubian, Charbit, and Thomassé [2]). *Let P be the directed path on 6 vertices. If D is P-free and $\omega(D) \leq 2$, then $\vec{\chi}(D) \leq 382$.*

Theorem 1.3.15 (Chudnovsky, Scott, and Seymour [9] and Cook, Masařík, Pilipczuk, Reinald, and Souza [12]). *If P is an orientation of the path on 4 vertices, then P-free digraphs are $\vec{\chi}$-bounded.*

For Theorem 1.3.15, Chudnovsky, Scott, and Seymour [9] proved the following: if $P = \rightarrow\leftarrow\leftarrow$ or $P = \leftarrow\rightarrow\rightarrow$, then P-free digraphs are χ-bounded. Interestingly, the other orientations of P do not have this property. Cook, Masařík, Pilipczuk, Reinald, and Souza [12] designed a proof technique to prove that, for any given orientation P of P_4, the P-free digraphs are $\vec{\chi}$-bounded. They achieve this by using closed tournaments, which, as discussed before, can be seen as the directed analog of templates.

In this thesis, we include the proof of the following result which was proven in collaboration with Hidde Koerts, Benjamin Moore, and Sophie Spirkl. This is a strengthening of an unpublished result by Linda Cook and Seokbeom Kim (private communication). We present a way to use the proof technique from [12] to prove the following.

For an integer $r \geq 1$, let the *r-broom*, denoted by B_r, be the graph defined as follows:

$$B_r := (V = \{v_1, v_2, v_3, w_1, \ldots, w_r\}, E = \{v_1v_2, v_2v_3, v_3w_1, \ldots, v_3w_r\}).$$

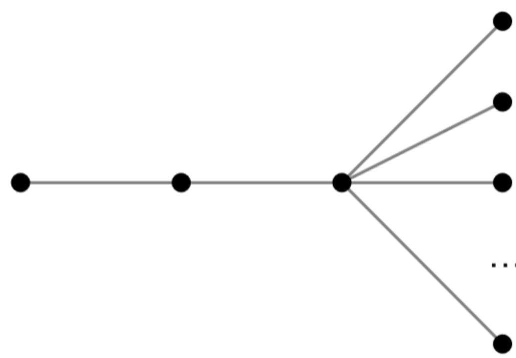


Figure 1.3: An illustration of B_r.

See Figure 1.3. If $\mathcal{B}$ is an orientation of B_r and $\mathcal{B}'$ is an orientation of B_s, then we say $\mathcal{B}$ and $\mathcal{B}'$ have *opposing orientations* if $v_2v_3 \in A(\mathcal{B})$ and $v_3v_2 \in A(\mathcal{B}')$. Furthermore, a *valid orientation* $\mathcal{B}$ of B_r is an orientation such that either $\{v_3w_1, \ldots, v_3w_r\} \subseteq A(\mathcal{B})$ or $\{w_1v_3, \ldots, w_rv_3\} \subseteq A(\mathcal{B})$.

Theorem 1.3.16. *Let r and s be positive integers. If $\mathcal{B}$ and $\mathcal{B}'$ have valid opposing orientations of B_r and B_s respectively, then $\{\mathcal{B}, \mathcal{B}'\}$-free digraphs are $\vec{\chi}$-bounded.*

The analogous result for χ-boundedness is true and stronger. A broom is a tree where an end-vertex of a path is identified with the center of a star. Notice that the underlying undirected graph of an r-broom is a broom where the path has 3 vertices. Gyárfás [16] proved that if B is a broom, then B-free graphs are χ-bounded.

To prove Theorem 1.3.16, we expand the proof technique in [12] to this context by generalizing a key lemma from the literature. This lemma was first proven in [4], but independently discovered in [12]. A set $S \neq \emptyset$ of vertices of D is *nice* if there exists a partition S_1, S_2 of S such that every vertex in S_1 (resp. S_2) only has in-neighbors (resp. out-neighbors) in $V(D) \setminus S$.

Lemma 1.3.17 (Aboulker, Charbit, and Naserasr [4])**.** *Let $\mathcal{C}$ be a hereditary class of digraphs. If there exists integers c, k such that every $D \in \mathcal{C}$ has a nice set S with $\vec{\chi}(S) \leq c$, then $\vec{\chi}(D) \leq 2c$ for every $D \in \mathcal{C}$.*

By generalizing this lemma to the following result, we are able to use the proof technique in [12] used to prove Theorem 1.3.16. A set S of vertices of D is *k-nice* if there exists a partition S_1, S_2 of $S \neq \emptyset$ such that every vertex in S_1 (resp. S_2) has at most k in-neighbors (resp. k out-neighbors) in $V(D) \setminus S$.

Lemma 1.3.18. *Let $k \geq 0$, and let $\mathcal{C}$ be a hereditary class of digraphs. If there exists an integer c such that every $D \in \mathcal{C}$ has a k-nice set S with $\vec{\chi}(S) \leq c$, then $\vec{\chi}(D) \leq 2c(k+1)$ for every $D \in \mathcal{C}$.*

Chapter 2

Heroes

In this chapter, we present the proofs related to the results highlighted in Section 1.2. We prove Theorem 1.2.14 in Section 2.1, Theorem 1.2.15 in Section 2.2, and Theorem 1.2.16 in Sections 2.3 and 2.4. These results and their proofs are part of an upcoming publication co-authored with Hidde Koerts, Benjamin Moore, and Sophie Spirkl.

Throughout the section, we use the following definitions. For a digraph D, when we say that $X_1 \subseteq V(D)$ is complete (resp. anticomplete) to $X_2 \subseteq V(D)$, we mean that this is the case for the underlying undirected graph of D. Additionally, X_1 is *in-complete* (resp. *out-complete*) to X_2 if every vertex in X_1 is seen by (resp. sees) every vertex in X_2.

2.1 Proof of Theorem 1.2.14

In this section, we prove Theorem 1.2.14, which we restate for the reader's convenience:

Theorem 1.2.14. *If F is an orientation of a star of degree at least 5, then H is a hero in F-free digraphs if and only if H is a transitive tournament.*

To prove this theorem, as well as Theorem 1.2.15, we need the following family of graphs. Let n and k be integers such that $n > 2k > 2$. The *k-tuple shift-graph* with indices in $\{1, ..., n\}$ is the graph whose vertices are of the form $(x_1, \ldots, x_k)$, where $x_i \in \{1, \ldots, n\}$ for every $i \in \{1, \ldots, n\}$ and $x_i < x_{i+1}$ for every $i \in \{1, \ldots, n\}$. Furthermore, two vertices $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ are adjacent if $a_{i+1} = b_i$ for every $i \in \{1, \ldots, k-1\}$ or vice versa. In [14], Erdős proved the following.

Theorem 2.1.1 (Erdős [14])**.** *For every fixed k, if G_n is the k-tuple shift-graph with indices in $\{1, \dots, n\}$, then $\chi(G_n) \to \infty$ as $n \to \infty$.*

We will also use the Gallai-Roy Theorem ([15, 25]):

Theorem 2.1.2 (Gallai [15] and Roy [25])**.** *If D has no directed path of length t, and G is the underlying undirected graph of D, then $\chi(G) \leq t$.*

The following theorem gives a construction which will allow us to prove one direction of Theorem 1.2.14. The construction is inspired by the construction in [1] used to prove Theorem 1.2.8. They add edges to shift graphs to create a complete multipartite digraph with certain properties. We use shift graphs as well but we add edges in a different way. We use the same strategy to prove Theorem 1.2.15.

Theorem 2.1.3. *There exists digraphs $F_1, F_2, \dots$ such that:*

- *$\vec{\chi}(F_n) \to \infty$ as $n \to \infty$;*
- *for every $n \geq 1$ and $v \in V(F_n)$, the neighborhood of v can be partitioned into four tournaments; and*
- *for every $n \geq 1$, the digraph F_n has no cyclic triangle $\Delta(1, 1, 1)$.*

Proof. Let G_n be the 7-tuple shift-graph with indices in $\{1, ..., n\}$, and let D_n be the orientation of G_n where $(a_1, \dots, a_7)(b_1, \dots, b_7) \in A(D)$ if $b_i = a_{i+1}$ for every $i \in \{1, \dots, 6\}$. For every $v = (a_1, \dots, a_7) \in V(G_n)$, define $m(v) = a_4$. Let $X := A(D_n)$. That is, X is the set of edges of the form $(\bullet, b, c, d, e, f, g) \to (b, c, d, e, f, g, \bullet)$. Moreover, let D'_n be the digraph with $V(D'_n) = V(D_n)$ and $A(D'_n) = X \cup Y$ where Y is the set of arcs of the form $(a, b, c, d, \bullet, \bullet, \bullet) \to (\bullet, \bullet, \bullet, a, b, c, d)$. Note that as m is strictly increasing along arcs in X, it follows that X is acyclic. Likewise, m is strictly decreasing in Y, so Y is acyclic.

(2.1) *For every $n \geq 1$, $\chi(G_n)/3 \leq \vec{\chi}(D'_n)$.*

Proof. We will prove the claim by proving that a set of vertices that induces an acyclic set in D'_n also induces a subgraph with chromatic number at most 3 in G_n. Let Λ be a set of vertices that induces an acyclic set in D'_n. Notice that $D_n[\Lambda]$ does not have a directed path of length 3 because if such a path $v_1 \to v_2 \to v_3 \to v_4$ exists, then $v_4v_1 \in A(D'_n)$ contradicting that Λ is an acyclic set in D'_n. Thus, by Theorem 2.1.2, we have $\chi(G_n[\Lambda]) \leq 3$ as desired. ■

Finally, let F_n be the digraph with $V(F_n) = V(D'_n)$ and $A(F_n) = X \cup Y \cup Z_1 \cup Z_2$ where we define Z_1 and Z_2 as follows. Let $<$ be a total ordering of $V(F_n)$. Define Z_1 (resp. Z_2) as the set of edges such that $uv \in Z_1$ (resp. $uv \in Z_2$) if $u < v$ and there exists numbers a, b, c, d (resp. d, e, f, g) such that both u and v are of the form $(a, b, c, d, \bullet, \bullet, \bullet)$ (resp. $(\bullet, \bullet, \bullet, d, e, f, g)$).

(2.2) *If $v \in V(F_n)$, then $N_{F_n}(v)$ can be partitioned into four tournaments.*

Proof. Fix $v = (a, b, c, d, e, f, g) \in V(F_n)$. The neighbors u of v such that $uv \in X$ or $vu \in X$ are of the form $(\bullet, a, b, c, d, e, f)$ and $(b, c, d, e, f, g, \bullet)$ respectively. By the definition of edges in $Z_1 \cup Z_2$, vertices of these forms each induce a tournament.

Denote by A and B the neighbors of v of the forms $(\bullet, \bullet, \bullet, a, b, c, d)$ and $(d, e, f, g, \bullet, \bullet, \bullet)$, respectively. Notice that these sets partition the neighbors of v connected to v via edges in Y. Denote by M and N the neighbors of v of the form $(a, b, c, d, \bullet, \bullet, \bullet)$ and $(\bullet, \bullet, \bullet, d, e, f, g)$, respectively. Notice that these sets partition the neighbors of v connected to v via edges in $Z_1 \cup Z_2$. By the definition of edges in $Z_1 \cup Z_2$, each of the sets A, B, M and N induces a tournament. Furthermore, M is complete to A, and B is complete to N via edges in Y. Since $A(F_n) = X \cup Y \cup (Z_1 \cup Z_2)$, these are all the neighbors of v, thus finishing the proof. ■

(2.3) *F_n has no cyclic triangle.*

Proof. Assume for a contradiction that there exist vertices u, v, w such that u sees v, v sees w, w sees u, and where $u = (a, b, c, d, e, f, g)$.

We claim that no edge in the cyclic triangle is in $Z_1 \cup Z_2$. For a contradiction, assume without loss of generality that $uv \in Z_1 \cup Z_2$, so $m(v) = d$. Assume first that $wu \in Z_1 \cup Z_2$. Consequently, $m(w) = d$ as well, so $vw \in Z_1 \cup Z_2$, which contradicts that the edges in $Z_1 \cup Z_2$ form an acyclic orientation. Therefore, $wu \notin Z_1 \cup Z_2$. Assume next that $wu \in X$. Consequently, $m(w) = c$, so $vw \in X$. Thus, $v = (\bullet, \bullet, a, b, c, d, e)$, which contradicts that $m(v) = d$. Therefore, $wu \notin X$. Thus, $wu \in Y$, so $w = (d, e, f, g, \bullet, \bullet, \bullet)$. But then $vw \notin X \cup (Z_1 \cup Z_2)$, so $vw \in Y$. Thus, the first index of v is g. This contradicts that $m(v) = d$ since $d < g$. We conclude that no edge in the cyclic triangle is in $Z_1 \cup Z_2$.

We claim that no edge in the cyclic triangle is in X. For a contradiction, assume without loss of generality that $uv \in X$, so $v = (b, c, d, e, f, g, \bullet)$. Assume $vw \in X$. Consequently, $w = (c, d, e, f, g, \bullet, \bullet)$, so by definition $wu \notin Y$. But $wu \notin X$ since $m(u) \neq g$, which contradicts that wu is an arc and $wu \notin Z_1 \cup Z_2$. Thus, $vw \in Y$, so $w = (\bullet, \bullet, \bullet, b, c, d, e)$.

But then $wu \notin X$ and $wu \notin Y$. This contradicts that wu is an arc and $wu \notin Z_1 \cup Z_2$. We conclude that no edge in the cyclic triangle is in X. But then every edge in the cyclic triangle is in Y, which contradicts that the edges in Y induce an acyclic digraph. This finishes the proof. ■

The second and third bullet points are proven in (2.2) and (2.3) respectively. Since F_n contains D'_n as a subgraph, it follows that $\chi(G_n)/3 \leq \vec{\chi}(F'_n)$ as well. As mentioned, the sequence $\chi(G_n) \to \infty$ as $n \to \infty$, so $\vec{\chi}(F_n) \to \infty$ as $n \to \infty$ as well. Thus, the first bullet point holds. ▮

Proof of Theorem 1.2.14: Assume F is a directed star of degree at least 5. By Theorem 1.2.11, every transitive tournament is a hero in F-free digraphs. For the other direction, assume that H is a hero in F-free digraphs. If H is not transitive, then H contains a cyclic triangle. Thus, the cyclic triangle is a hero in F-free digraphs. This, however, contradicts Theorem 2.1.3 which provides a family of digraphs of arbitrarily high dichromatic number with no cyclic triangles and which is F-free (the construction is F-free because a copy of F contains a vertex whose neighborhood has a stable set with at least 5 vertices, contradicting that the neighborhood of every vertex can be partitioned into four tournaments). Thus, we conclude that H is transitive, which finishes the proof. ▮

2.2 Proof of Theorem 1.2.15

In this section, we prove Theorem 1.2.15, which we restate for the reader's convenience:

Theorem 1.2.15. *If F is an oriented star of degree 4 and H is a hero in F-free digraphs, then either H is a transitive tournament or $H = \Delta(1, m, m')$ where $m, m' \geq 1$.*

We do so by using a proof technique very similar to the one we used to prove Theorem 1.2.14. We start by first restricting some of the heroes in F-free digraphs when F is an oriented star of degree 4. Let the *in-triangle*, denoted by IT, be the digraph on 4 vertices a, b, c, d where d is in-complete from a, b, c and where $\{a, b, c\}$ induces the cyclic triangle. The first step towards proving Theorem 1.2.15 is proving the following.

Theorem 2.2.1. *If ST is a directed star of degree at least 4, then no hero in ST-free digraphs contains the in-triangle as a subgraph.*

This theorem is an immediate consequence to the following theorem.

Theorem 2.2.2. *There exists digraphs $F_1, F_2, \ldots$ such that:*

- $\vec{\chi}(F_n) \to \infty$ *as* $n \to \infty$*;*
- *for every* $n \geq 1$ *and* $v \in V(F_n)$*, the neighborhood of* v *can be partitioned into three tournaments; and*
- *for every* $n \geq 1$*, the digraph* IT *is not a subgraph of* F_n*.*

Proof. Let G_n be the 5-tuple shift-graph with indices in $\{1, \ldots, n\}$, and let D_n be the orientation of G_n where $(a_1, \ldots, a_5)(b_1, \ldots, b_5) \in A(D)$ if $b_i = a_{i+1}$ for every $i \in \{1, \ldots, 4\}$. For every $v = (a_1, \ldots, a_5) \in V(G_n)$, define $m(v) = a_3$. Let $X = A(D_n)$. That is, X is the set of edges of the form $(\bullet, b, c, d, e) \to (b, c, d, e, \bullet)$. Moreover, let D'_n be the digraph with $V(D'_n) = V(D_n)$ and $A(D'_n) = X \cup Y$ where Y is the set of arcs of the form $(a, b, c, \bullet, \bullet) \to (\bullet, \bullet, a, b, c)$. Note that as m is strictly increasing along arcs in X, it follows that X is acyclic. Likewise, m is strictly decreasing in Y, so Y is acyclic.

(2.1) *For every* $n \geq 1$*, we have* $\chi(G_n)/2 \leq \vec{\chi}(D'_n)$*.*

Proof. We will prove the claim by proving that a set of vertices that induces an acyclic set in D'_n also induces a bipartite subgraph in G_n. Let Λ be a set of vertices that induces an acyclic set in D'_n. For a contradiction, assume that Λ does not induce a bipartite subgraph in G_n. Without loss of generality, we may assume that Λ induces an odd cycle in G_n. Since $G_n[\Lambda]$ is an odd cycle and G_n is the undirected underlying graph of D_n, it follows that $D_n[\Lambda]$ contains an induced directed path on 3 vertices as a subgraph. Let $v_1 \to v_2 \to v_3$ be such a directed path in $D_n[\Lambda]$. By the way in which the edges of D_n are oriented, and by the definition of Y, it follows that $v_3 v_1 \in Y$. This, however, contradicts that Λ induces an acyclic set in D'_n, thus proving the claim. ∎

Finally, let F_n be the digraph with $V(F_n) = V(D'_n)$ and $A(F_n) = X \cup Y \cup Z_1 \cup Z_2$ where we define Z_1 and Z_2 as follows. Let $<$ be a complete ordering of $V(F_n)$. Define Z_1 (resp. Z_2) as the set of edges such that $uv \in Z_1$ (resp. $uv \in Z_2$) if $u < v$ and there exists numbers a, b, c (resp. c, d, e) such that both u and v are of the form $(a, b, c, \bullet, \bullet)$ (resp. $(\bullet, \bullet, c, d, e)$.

(2.2) *If* $v \in V(F_n)$*, then* $N_{F_n}(v)$ *can be partitioned into three tournaments.*

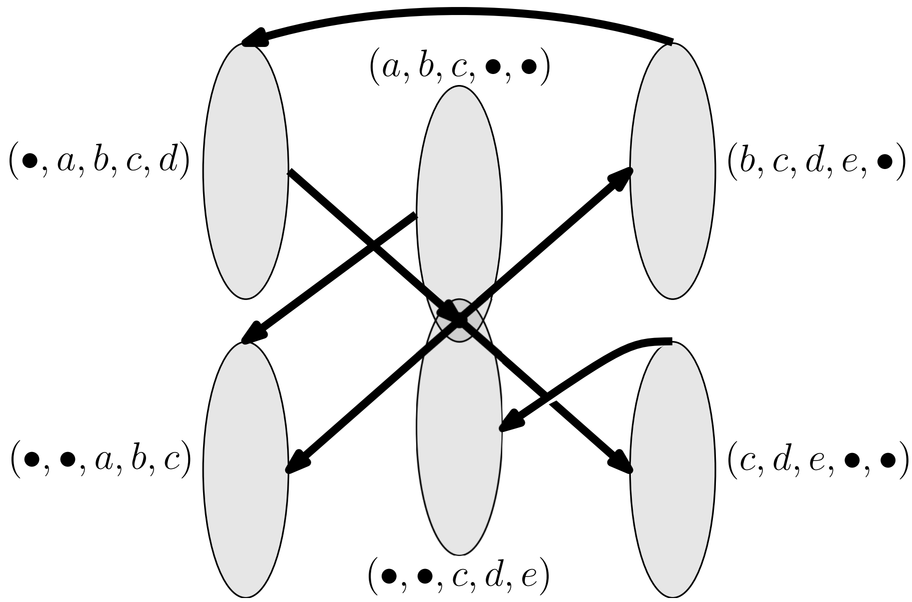


Figure 2.1: Illustration of the neighborhood of the vertex (a, b, c, d, e) in F_n.

Proof. Fix $v = (a, b, c, d, e) \in V(F_n)$. The neighbors u of v such that $uv \in X$ or $vu \in X$ are of the form $(\bullet, a, b, c, d)$ and $(b, c, d, e, \bullet)$. Vertices of the former type are complete to the vertices of the latter type by edges in Y. Thus, vertices adjacent to v via an edge in X form a clique.

Denote by A and B the neighbors of v of the forms $(\bullet, \bullet, a, b, c)$ and $(c, d, e, \bullet, \bullet)$, respectively. Notice that these sets partition the neighbors of v connected to v via edges in Y. Denote by M and N the neighbors of v of the form $(a, b, c, \bullet, \bullet)$ and $(\bullet, \bullet, c, d, e)$, respectively. Notice that these sets partition the neighbors of v connected to v via edges in $Z_1 \cup Z_2$. By the definition of edges in $Z_1 \cup Z_2$, each of the sets A, B, M and N induce a tournament. Furthermore, M is complete to A, and B is complete to N via edges in Y. Since $A(F_n) = X \cup Y \cup (Z_1 \cup Z_2)$, these are all the neighbors of v, thus finishing the proof. Figure 2.1 illustrates the neighborhood of a vertex.. ■

(2.3) *Every cyclic triangle in F_n has two edges in X and one edge in Y.*

Proof. Let u, v, and w be vertices such that u sees v, v sees w, and w sees u. For a contradiction, assume that $uv \in Z_1 \cup Z_2$, and set $v = (a, b, c, d, e)$. Since $uv \in Z_1 \cup Z_2$, we have $m(u) = c$. If $vw \in X$, then $m(w) = d$. Since $m(u) < m(w)$, it follows that $wu \in Y$, so $m(u) = b$, a contradiction. If $vw \in Z_1 \cup Z_2$, then $m(w) = c$, contradicting the fact that edges in $Z_1 \cup Z_2$ induce an acyclic graph. Thus, $vw \in Y$, so $m(w) = a$. Since $m(w) < m(u)$, it must be that $wu \in X$, so $m(u) = b$, a contradiction. We conclude that every edge in the directed triangle is not in $Z_1 \cup Z_2$. Since each of X and Y span acyclic graphs, we may assume $uv \in X$ and $vw \in Y$. Consequently, $m(u) = b$ and $m(w) = a$, so $m(w) < m(u)$. This implies that $wu \in X$, proving that directed triangles have two edges in X and one edge in Y. ■

The second bullet point is true by (2.1). Since F_n contains D'_n as a subgraph, by (2.2), we have $\chi(G_n)/2 \leq \vec{\chi}(F'_n)$ as well. As mentioned before, we have $\chi(G_n) \to \infty$ as $n \to \infty$. Thus, $\vec{\chi}(F_n) \to \infty$. This proves the first bullet point. We prove the third bullet point by contradiction. Assume IT is a subgraph of F_n. Let $u \to v \to w \to u$ be the directed cycle in F_n and x be the vertex in-complete from $\{u, v, w\}$. Without loss of generality, by (2.1), we may assume that $uv, vw \in X$. Set $v = (a, b, c, d, e)$. If $ux \in X$, then $m(x) = c$, and since $m(w) = d > m(x)$, it follows that $wx \in Y$. This implies that $m(x) = b$, a contradiction to $m(x) = c$. If $ux \in Y$, then $m(x) < a$, so $m(x) < m(v)$. It follows that $vx \in Y$, implying $m(x) = a$, a contradiction. Thus, $ux \in Z_1 \cup Z_2$, so $m(x) = b$. Since $m(x) < m(v)$, it follows that $vx \in Y$, so $m(x) = a$, a contradiction. This shows that F_n is IT-free, which finishes the proof. ■

Proof of Theorem 1.2.15: Let the out-triangle, denoted by OT, be IT with arcs reversed.

(2.1) *OT is not a hero in ST-free digraphs.*

Proof. Let ST' be ST with arcs reversed. By Theorem 2.2.1, the $\{ST', IT\}$-free digraphs do not have bounded dichromatic number. By reversing arcs, we get that $\{ST, OT\}$-free digraphs do not have bounded dichromatic number. ■

Assume for a contradiction that there exists a hero H such that H is not acyclic and $H \neq \Delta(1, m, m')$ for integers $m, m' \geq 1$. If H is not strongly connected, then there exists non-empty tournaments $H_1 \subseteq H$ and $H_2 \subseteq H$ such that $V(H_1) \cap V(H_2) = \emptyset$, H_1 is not transitive, and either H_1 is out-complete to H_2, or H_2 is out-complete to H_1. Since H_1 is not transitive, it contains a directed triangle T. If T is out-complete to H_1, then H contains a copy of IT, a contradiction. Thus, H_2 is out-complete to T, but then H contains a copy of OT, a contradiction. We conclude H is strongly connected.

Since H is strongly connected, by Theorem 1.2.4, it follows that $H = \Delta(1, m, H')$ or $H = \Delta(1, H', m)$ where H' is a hero in ST-free digraphs. It is then enough to prove that

H' is acyclic. Suppose not. That is, assume that H' contains a directed triangle T. In either case, by the structure of strongly connected heroes, H contains a copy of IT, a contradiction. ∎

2.3 Localized and colocalized digraphs

In this section, we introduce the concept of localized and colocalized digraphs and how these conditions relate to Theorem 1.2.16. Broadly speaking, we want to build upon Theorem 1.2.13 by using the proof strategies devised by Harutyunyan, Le, Newman, and Thomassé [18] to prove Theorem 1.2.9. To elaborate, we need some definitions.

In a digraph D, we say the *out-neighborhood* (resp. *in-neighborhood*) of a set of vertices $S \subseteq V(D)$, denoted by $N^+(S)$ (resp. $N^-(S)$), is the set of vertices not in S that vertices $v \in S$ see (resp. $v \in S$ is seen by). Furthermore, the neighborhood of S is $N(S) := N^+(S) \cup N^-(S)$. When $S = \{v\}$, we use $N(v), N^+(v)$, and $N^-(v)$ to denote $N(S), N^+(S)$, and $N^-(S)$ respectively.

A digraph D is *k-local* if, for every $v \in V(D)$, we have $\vec{\chi}(N^+(v)) \leq k$. Furthermore, it is *$k$-colocal* if, for every $v \in V(D)$, we have $\vec{\chi}(N^-(v)) \leq k$. The concept of k-local digraphs was introduced by Harutyunyan, Le, Newman, and Thomassé [18]. A digraph F *cooperates* if H is a hero in F-free digraphs when one of the following three conditions hold:

- $H = K_1$;
- $H = H_1 \Rightarrow H_2$ where H_1 and H_2 are heroes in F-free digraphs; or
- $H = \Delta(1, 1, H_1)$ where H_1 is a hero in F-free digraphs.

In other words, a digraph F cooperates when their heroes can be used to construct bigger heroes by using the operations described above. Notice that Theorem 1.2.10 is equivalent to proving $K_1 + K_2$ cooperates, and Theorem 1.2.16 is equivalent to proving that $rK_1 + K_2$ cooperates for every $r \geq 1$.

The definitions of localized and colocalized digraphs are more technical. We say a digraph F is *localized* (resp. *colocalized*) if for every $r \geq 1$, the following two:

- $\Delta(1, 1, H)$ is a hero in $\{(r-1)K_1 + F\}$-free digraphs; and
- H is a hero in $\{rK_1 + F\}$-free digraphs

imply that, for every fixed $k \geq 1$, $\Delta(1,1,H)$ is a hero in k-local (resp. colocal) $\{rK_1 + F\}$-free digraphs. These definitions are meant to describe the properties a digraph F needs to have for the proof strategy of Theorem 1.2.9 to apply to F, where in Theorem 1.2.9, $F = 2K_1$.

The first step to proving Theorem 1.2.16 is proving the following.

Theorem 2.3.1. *Let F be a localized and colocalized digraph. If F cooperates, then rK_1+F cooperates for every $r \geq 0$.*

Aboulker, Aubian, and Charbit [1], in Question 5.4, asks the following. If H is a hero in $\{K_1 + F\}$-free digraphs and $\Delta(1,1,H)$ is a hero in F-free digraphs, does it follow that $\Delta(1,1,H)$ is a hero in $\{K_1 + F\}$-free digraphs? Theorem 2.3.1 proves that the question has an affirmative answer if F is localized and colocalized.

The following lemma simplifies using Theorem 2.3.1 for certain cases.

Lemma 2.3.2. *Let F be a digraph isomorphic to F with every arc reversed. If F is localized, then F is colocalized.*

Proof. Assume that $\Delta(1,1,H)$ is a hero in $\{(r-1)K_1 + F\}$-free digraphs, and assume that H is a hero in $\{rK_1 + F\}$-free digraphs. Fix $k \geq 1$. We want to prove that $\Delta(1,1,H)$ is a hero in k-colocal $\{rK_1+F\}$-free digraphs. Let D be a k-colocal $\{rK_1+F, \Delta(1,1,H)\}$-free digraph. Let D^* be D with every arc reversed, and notice that D^* is k-local and $\{rK_1+F, \Delta(1,1,H)\}$-free. Since F is localized, $\Delta(1,1,H)$ is a hero in k-local $\{rK_1+F\}$-free digraphs. Thus, there exists an integer c such that $\vec{\chi}(D^*) \leq c$. This implies that $\vec{\chi}(D) \leq c$. Equivalently, $\Delta(1,1,H)$ is a hero in $\{rK_1 + F\}$-free digraphs. This finishes the proof. ∎

In this section, we prove Theorem 2.3.1. The following lemma simplifies the proof of Theorem 2.3.1 to the case where $r = 1$.

Lemma 2.3.3. *If F is localized, then $K_1 + F$ is localized. Similarly, if F is colocalized, then $K_1 + F$ is colocalized.*

Proof. Assume that F is localized. To prove that $K_1 + F$ is localized, fix $r \geq 0$, and assume that (1) $\Delta(1,1,H)$ is a hero in $\{rK_1 + F\}$-free digraphs, and that (2) H is a hero in $\{(r+1)K_1 + F\}$-free digraphs. We want to prove that, for every $k \geq 1$, the digraph $\Delta(1,1,H)$ is a hero in k-local $\{(r+1)K_1 + F\}$-free digraphs. But since F is localized, this follows by definition of localized graphs. The statement for colocalized has an analogous proof with arcs reversed. ∎

To prove Theorem 2.3.1, for $r = 1$, we rely heavily on two key results from the literature. First, we use Theorem 1.2.13 to prove that $H_1 \Rightarrow H_2$ is a hero in $\{K_1 + F\}$-free digraphs whenever H_1 and H_2 are heroes in F-free digraphs. Second, we use the proof strategy devised by Harutyunyan, Le, Newman, and Thomassé [18] to prove that if H is a hero in tournaments, then $\Delta(1, m, H)$, where $m \geq 1$, is a hero in rK_1-free digraphs, for $r \geq 2$.

This proof strategy uses the concept of bags. A (c, β)-*bag* is a subset B of $V(D)$ such that $\vec{\chi}(B) = \beta$, and a (c, β)-*bag-chain* is a sequence of (c, β)-bags $B_1, \dots, B_t$ such that for every $1 \leq i \leq t$ and $v \in B_i$, we have:

- $\vec{\chi}(N^+(v) \cap B_{i-1}) \leq c$, and
- $\vec{\chi}(N^-(v) \cap B_{i+1}) \leq c$.

The *length* of the (c, β)-bag-chain is t. The proof strategy is as follows. For a $\{\Delta(1, 1, H), rK_1 + F\}$-free digraph D, we want to prove the following three objectives.

1. First, we prove that for a large enough c, and for every $\beta \in \mathbb{N}$, the absence of a (c, β)-bag-chain of length 8 implies that the digraph has bounded dichromatic number.
2. Second, we prove that, for large enough c and β', (c, β')-bag-chains have a bounded dichromatic number.
3. Lastly, we prove that for large enough c and β', if there is a (c, β')-bag-chain, then vertices not in a maximal (c, β')-bag-chain have bounded dichromatic number as well.

To accomplish the second objective, we use the following lemma.

Lemma 2.3.4. *Assume that there exists an integer m such that:*

- *$\{\Delta(1, 1, H), F\}$-free digraphs D have $\vec{\chi}(D) \leq m$; and*
- *$\{H, K_1 + F\}$-free digraphs D have $\vec{\chi}(D) \leq m$.*

If D is a $\{\Delta(1, 1, H), K_1 + F\}$-free digraph with a partition $(X_1, \dots, X_n)$ of $V(D)$, and m' an integer such that:

- *for every $1 \leq i \leq n$, we have $\vec{\chi}(X_i) \leq m'$;*
- *for every $1 \leq i \leq n$ and for every $v \in X_i$, we have $\vec{\chi}(N^+(v) \cap (X_1 \cup \dots \cup X_{i-1})) \leq m'$; and*

- *for every $1 \leq i \leq n$ and for every $v \in X_i$, we have $\vec{\chi}(N^-(v) \cap (X_{i+1} \cup \cdots \cup X_n)) \leq m'$;*

then $\vec{\chi}(D) \leq 6(m + m') + 2$.

This is a generalization of Lemma 3.8 by Aboulker, Aubian, and Charbit [1], and our proof differs from theirs only slightly. The proof also needs the following result.

Lemma 2.3.5 (Aboulker, Aubian, and Charbit [1])**.** *Let D be a digraph and let $(X_1, \ldots, X_n)$ be a partition of $V(D)$. Suppose that k is an integer such that:*

- *for every $1 \leq i \leq n$, we have $\vec{\chi}(X_i) \leq k$, and*
- *for every $1 \leq i < j \leq n$, if there is an arc uv with $u \in X_j$ and $v \in X_i$, then $\vec{\chi}(X_{i+1} \cup \cdots X_j) \leq k$.*

Then $\vec{\chi}(D) \leq 2k$.

For a vertex v in a digraph D, let $N^0(v)$ be the set of non-neighbors of v.

Proof of Lemma 2.3.4. We start with the following claim.

(2.1) $\quad$ *$\vec{\chi}(N^0(v)) \leq m$ for every $v \in D$.*

Proof. Since D is $K_1 + F$-free, it follows that $N^0(v)$ is F-free. Furthermore, since D is $\Delta(1, 1, H)$-free, we have by the definition of m that $\vec{\chi}(N^0(v)) \leq m$. ■

Set $k' = 2(m+m')+m+1$. It suffices to show that the partition $(X_1, \ldots, X_n)$ satisfies the hypothesis of Lemma 2.3.5 with $k = k' + m'$. Let uv be an edge such that $u \in X_j$, $v \in X_i$, and $i < j$, and set $X = X_{i+1} \cup \cdots \cup X_{j-1}$. For a contradiction, assume that $\vec{\chi}(X) > k'$. Let $A = (N^-(v) \cup N^0(v)) \cap X$. By the hypothesis and by (2.1), the dichromatic number of A is at most $m + m'$. Similarly, the set $B = (N^+(u) \cup N^0(u)) \cap X$ has dichromatic number at most $m + m'$. Thus, the set $X' = X \setminus (A \cup B)$ has $\vec{\chi}(X') > k' - 2(m + m') > m$. Consequently, there exists a copy X'' of H in X'. But then, by the definitions of A and B, it follows that $\{u, v\} \cup X''$ induces a copy of $\Delta(1, 1, H)$, a contradiction. Thus, $\vec{\chi}(X) \leq k'$, so $\vec{\chi}(X \cup X_j) \leq k' + m'$, as desired. ▮

We dedicate the rest of the section to proving Theorem 2.3.1.

Proof of Theorem 2.3.1. By Lemma 2.3.3, it is enough to prove the result for $r = 1$. That is, we want to prove that $K_1 + F$ cooperates. Evidently, $H = K_1$ is a hero in every class of graphs. Assume then that H_1 and H_2 are heroes in $\{K_1 + F\}$-free digraphs. Consequently, they are heroes in F-free digraphs, and since F cooperates, it follows that $H_1 \Rightarrow H_2$ is a hero in F-free digraphs. Thus, by Theorem 1.2.13, it follows that $H_1 \Rightarrow H_2$ is a hero in $K_1 + F$-free digraphs.

It remains to show that $\Delta(1, 1, H)$ is a hero in $\{K_1 + F\}$-free digraphs whenever H is a hero in $\{K_1 + F\}$-free digraphs. The proof of this fact uses the strategy used to prove Theorem 1.5 in [18]. In particular, we use the concept of bag chains and apply them in the same way, just in a more general context and with many small modifications, such as the use of Lemma 2.3.4. Some of our proofs, thus, are very similar to those in [18]. However, since we exclude $\Delta(1, 1, H)$ instead of $\Delta(1, k, H)$, in some places we are able to simiplify the proofs.

Assume H is a hero in $\{K_1+F\}$-free digraphs. Let c be an integer such that $\{K_1+F, H\}$-free digraphs D have $\vec{\chi}(D) \leq c$. Since H is a hero in $K_1 + F$-free digraphs, H is a hero in F-free digraphs as well, and since F cooperates, it follows that $\Delta(1, 1, H)$ is a hero in F-free digraphs. Let b' be such that $\{F, \Delta(1, 1, H)\}$-free digraphs D have $\vec{\chi}(D) \leq b'$. Since F is localized, set $f_1(r, k, H)$ as the function such that $\{\Delta(1, 1, H), rK_1 + F\}$ k-local digraphs D have $\vec{\chi}(D) \leq f_1(r, k, H)$ whenever $\Delta(1, 1, H)$ is a hero in $\{(r-1)K_1 + F\}$-free digraphs, and H is a hero in $\{rK_1 + F\}$-free digraphs. Let $f_2(r, k, H)$ be the equivalent but from the fact that F is colocalized. Now let $f(r, k, H) = \max\{f_1(r, k, H), f_2(r, k, H)\}$. Set

$$\hat{f}(\beta) := 2f\left(1, 2f\left(1, 2f\left(1, \beta, H\right) + 1, H\right) + 1, H\right).$$

Finally, set $\beta' = 2|V(H)|(c + b') + b' + 1$. We will show that $\{\Delta(1, 1, H), K_1 + F\}$-free digraphs D have $\vec{\chi}(D) \leq b$ where

$$b = 6(\max\{b', c\} + \beta') + 3\hat{f}(\beta') + 2.$$

Assume that D is a $\{\Delta(1, 1, H), K_1 + F\}$-free digraph. Henceforth, we will use the terms *β-bags* and *β-bag-chains* to refer to (c, β)-bags and (c, β)-bag-chains. To achieve the first objective, we start by proving that the absence of a β-bag-chain of length 2 bounds the dichromatic number. Call a vertex v *β-red* if $N^+(v) \leq \beta$, and *β-blue* if $N^-(v) \leq \beta$. The following two claims are the equivalent of Lemma 4.11 in [18], although our proof is significantly simpler as we deal with $\Delta(1, 1, H)$ instead of $\Delta(1, m, H)$ for some m.

(2.1) *For every $\beta \in \mathbb{N}$, if D does not have a β-bag-chain of length 2, then $\vec{\chi}(D) \leq 2f(1, \beta, H)$.*

Proof. Set R, B, and U as the sets of β-red, β-blue and uncolored vertices respectively. We start by proving that U is empty. For the sake of a contradiction, assume that $u \in U$. Set $B_1 = N^-(u)$ and $B_2 = N^+(u)$. We claim that B_1, B_2 is a β-bag-chain. Let $v \in B_1$. If $\vec{\chi}(N^-(v) \cap B_2) > c$, then there exists a copy X of H in B_2. But then $\{u, v\} \cup X$ induces a copy of $\Delta(1, 1, H)$ in D, a contradiction. A symmetric argument proves that if $v \in B_2$, then $\vec{\chi}(N^+(v) \cap B_1) \leq c$. That is, B_1, B_2 is β-bag-chain of length 2, a contradiction. Thus, U is empty. Notice that $D[R]$ is d-local, so $\vec{\chi}(R) \leq f(1, d, H)$. Similarly, $D[B]$ with arcs reversed is β-colocal as well, so $\vec{\chi}(B) \leq f(1, \beta, H)$, and hence $\vec{\chi}(D) \leq 2f(1, \beta, H)$, as claimed. ■

(2.2) *For every $\beta \in \mathbb{N}$, if D does not have a β-bag-chain of length 8, then $\vec{\chi}(D) \leq \hat{f}(\beta)$.*

Proof. We proceed by contrapositive. Assume that $\vec{\chi}(D) > \hat{f}$. By (2.1), there exists a $(2f(1, 2f(1, \beta, H) + 1, H) + 1)$-bag-chain of length 2, say A_1, A_2. By definition of a bag and by (2.1), it follows that A_1 contains a $(2f(1, \beta, H) + 1)$-bag-chain of length 2 consisting of bags A_1^1, A_1^2. Similarly, A_2 contains the $(2f(1, \beta, H) + 1)$-bag-chain A_2^1, A_2^2. Finally, using the same reasoning, we can split each of these bags into the β-bag-chain $B_1, \ldots, B_8$ where B_1, B_2 is the β-bag-chain of A_1^1, where B_3, B_4 is the β-bag-chains of A_1^2, and so on. But then $B_1, \ldots, B_8$ is a β-bag-chain of length 8, finishing the proof. ■

With the first objective achieved, we now prove the second objective. From now on, we assume $B_1, \ldots, B_t$ is a β'-bag-chain in D with t maximum, where $\beta' = 2|V(H)|(c+b')+b'+1$. For convenience, define $B_{i,j}$, where $i < j$, as the union of the bags $B_i, \ldots, B_j$.

(2.3) *$\vec{\chi}(N^0(v)) \leq b'$ for every $v \in V(D)$.*

Proof. This is a consequence of the fact that the set of non-neighbors of v is $\{F, \Delta(1, 1, H)\}$-free. Thus, the result holds by the definition of b'. ■

The following is the equivalent of Claim 4.3 in [18], although we are able to prove a stronger statement.

(2.4) *For every $i \geq 1$, $v \in B_i$, and $s > 1$,*

- $N^+(v) \cap B_{i-s} = \emptyset$, and
- $N^-(v) \cap B_{i+s} = \emptyset$.

Proof. For a contradiction, let $s > i$ be the smallest integer such that there exists vertices u and v such that $u \in N^+(v) \cap B_{i-s}$ or $u \in N^-(v) \cap B_{i+s}$. We deal first with the former.

Suppose first that $s = 2$. Let $A = (N^-(u) \cup N^+(v)) \cap B_{i-1}$, and $B = (N^0(u) \cup N^0(v)) \cap B_{i-1}$. By the definition of a β-bag-chain and (2.3), $\vec{\chi}(A) \leq 2c$ and $\vec{\chi}(B) \leq 2b'$. Thus, $\vec{\chi}(B_{i-1} \setminus (A \cup B)) \geq \beta' - 2c - 2b' > c$. By the definition of c, there exists a copy X of H in $B_{i-1} \setminus (A \cup B)$. But by the definition of A and B, this implies that $\{u, v\} \cup X$ induces a copy of $\Delta(1, 1, H)$, a contradiction.

Suppose then that $s > 2$. The proof for this case is very similar. Let $A = (N^-(u) \cup N^+(v)) \cap B_{i-1})$ and $B = (N^0(u) \cup N^0(v)) \cap B_{i-1}$. By the minimality of s, and since $s > 1$, we have $A = N^+(v) \cap B_{i-1}$. By (2.3), it follows that $\vec{\chi}(B) \leq 2b'$. Thus, $\vec{\chi}(B_{i-1} \setminus (A \cup B)) \geq \beta' - 2b' - c > c$. By the definition of c, there exists a copy X of H in $B_{i-1} \setminus (A \cup B)$. But then, by the definition of A and B, this implies that $\{u, v\} \cup X$ induces a copy of $\Delta(1, 1, H)$, a contradiction.

The proof for the case where $u \in N^-(v) \cap B_{i+s}$ is analogous with arcs reversed. ■

The following is the equivalent of Claim 4.4 and Claim 4.5 in [18].

(2.5) *For every i and $v \in B_i$,*

- $\vec{\chi}(N^+(v) \cap B_{1,i-1}) \leq c$, *and*
- $\vec{\chi}(N^-(v) \cap B_{i+1,t}) \leq c$.

Proof. The result is immediate from (2.4) and the definition of β'-bag-chains. ■

We can now prove our second objective, which is the equivalent of Claim 4.6 in [18]. Our proof however is significantly simpler as we deal with $\Delta(1, 1, H)$ instead of $\Delta(1, m, H)$ for some m.

(2.6) $\vec{\chi}(B_{1,t}) \leq 6(\max\{b', c\} + \beta') + 2$.

Proof. Applying Lemma 2.3.4 with $m = \max\{b', c\}$, and $m' = \beta'$, where the hypothesis holds by (2.5), it follows that $\vec{\chi}(B_{1,t}) \leq 6(\max\{b', c\} + \beta') + 2$. ■

For our final objective, we will partition the vertices of $V(D) \setminus B_{1,t}$ in such a way that we can apply Lemma 2.3.4 again. Partition $V(D) \setminus B_{1,t}$ into sets Z_i we call *zones* such that $v \in Z_i$ if i is the largest index such that $\vec{\chi}(N^-(v) \cap B_i) > c$, and $v \in Z_0$ if no such i exists.

Furthermore, for convenience, set $Z_{i,j} := Z_i \cup \cdots \cup Z_j$ where $i < j$. We proceed to prove claims that will allow us to bound $\vec{\chi}(Z_{0,t})$ by using Lemma 2.3.4. (2.7-2.9) might seem unrelated to this objective at first, but they will allow us to bound $\vec{\chi}(Z_i)$. The following is the equivalent of Claim 4.7 in [18].

(2.7) *For every i and every $v \in Z_i$,*

- $\vec{\chi}(N^-(v) \cap B_{i+r}) \leq c$ for $r \geq 1$, and
- $N^+(v) \cap B_{i-r} = \emptyset$ for $r \geq 2$.

Proof. The first bullet point is true by the definition of Z_i. We prove the second. For a contradiction, assume that there exists a vertex u such that $u \in N^+(v) \cap B_{i-r}$. We claim that $\vec{\chi}(N^-(v) \cap B_{i-1}) \leq b' + 2c$. For a contradiction, assume this is not the case. Set

$$A := (N^0(u) \cup N^-(u)) \cap (N^-(v) \cap B_{i-1}).$$

By (2.3) and (2.5), $\vec{\chi}(A) \leq b' + c$, so $\vec{\chi}((N^-(v) \cap B_{i-1}) \setminus A) > c$. Thus, there exists a copy X of H in $(N^-(v) \cap B_{i-1}) \setminus A$. But then $\{u, v\} \cup X$ induces a copy of $\Delta(1, 1, H)$, a contradiction.

Thus, $\vec{\chi}(N^-(v) \cap B_{i-1}) \leq b' + 2c$. Since $\vec{\chi}(N^0(v) \cap B_{i-1}) \leq b'$ by (2.3), and since B_{i-1} is a β'-bag, it follows that $\vec{\chi}(N^+(v) \cap B_{i-1}) \geq |V(H)|(b' + c) + 1$. By the definition of a zone, there exists a copy X' of H in $N^-(v) \cap B_i$. Set

$$A' := \bigcup_{x \in X'} (N^0(x) \cup N^+(x)) \cap (N^+(v) \cap B_{i-1}).$$

By (2.3) and (2.5), it follows that $\vec{\chi}(A') \leq |V(H)|(b' + c)$. Thus, $\vec{\chi}((N^+(v) \cap B_{i-1}) \setminus A') > 0$, so there exists a vertex u' in $(N^+(v) \cap B_{i-1}) \setminus A'$. This, however, implies that $\{u', v, X'\}$ induces a copy of $\Delta(1, 1, H)$, a contradiction. ■

The following two claims are the equivalent of Claim 4.8 in [18], although our statement is stronger given that we deal with $\Delta(1, 1, H)$ instead of $\Delta(1, m, H)$ for some m.

(2.8) *For every $i \geq 0$, $v \in B_i$, and $r \geq 2$, we have $N^+(v) \cap Z_{i-r} = \emptyset$.*

Proof. For a contradiction, assume that there exists a vertex u such that $u \in N^+(v) \cap Z_{i-r}$. Now let

$$A = (N^0(u) \cup N^-(u)) \cap B_{i-1},$$

and let

$$B = (N^+(v) \cup N^0(v)) \cap B_{i-1}.$$

By the definition of zones and by (2.3), $\vec{\chi}(A) \leq b' + c$, and by the definition of a β-bag-chain and (2.3), $\vec{\chi}(B) \leq b' + c$. Thus, $\vec{\chi}(B_{i-1} \setminus (A \cup B)) \geq \beta' - (b' + c) - (b' + c) > c$. Consequently, there exists a copy X of H in $B_{i-1} \setminus (A \cup B)$. But by the definition of A and B, $\{u, v\} \cup X$ induces a copy of $\Delta(1, 1, H)$, a contradiction. ■

(2.9) *For every i, $v \in B_i$, and $r \geq 3$, we have $N^-(v) \cap Z_{i+r} = \emptyset$.*

Proof. For a contradiction, assume that there exists a vertex u such that $u \in N^-(v) \cap Z_{i+r}$. Now let

$$A := (N^0(u) \cup N^+(u)) \cap B_{i+1},$$

and let

$$B := (N^0(v) \cup N^-(v)) \cap B_{i+1}.$$

By (2.3) and (2.8), $\vec{\chi}(A) \leq b'$. Furthermore, by (2.3) and the definition of bags, $\vec{\chi}(B) \leq b' + c$. Thus, $\vec{\chi}(B_{i+1} \setminus (A \cup B)) \geq \beta' - b' - (b' + c) > c$. Consequently, there exists a copy X of H in $B_{i+1} \setminus (A \cup B)$. But by the definition of A and B, it follows that $\{u, v\} \cup X$ induces a copy of $\Delta(1, 1, H)$, a contradiction. ■

Finally, we are ready to bound $\vec{\chi}(Z_i)$. The following is the equivalent of Claim 4.10 in [18].

(2.10) *For every i, $\vec{\chi}(Z_i) \leq \hat{f}(\beta')$.*

Proof. By (2.2), it is enough to prove that zones do not have a β'-bag-chain of length 8. We will do this by using the maximality of t. Assume for a contradiction that $Y_1, \dots, Y_8$ is a β'-bag-chain of length 8 in Z_i. By (2.7), (2.8) and (2.9), $B_1, \dots, B_{i-3}, Y_1, \dots, Y_8, B_{i+3}, \dots, B_t$ is a longer β'-bag-chain than $B_1, \dots, B_t$ which contradicts the maximality of t. ■

To finish the proof, it remains to show we can partition $Z_{0,t}$ such that we are able to color each part. The following is the equivalent of Claim 4.9 in [18].

(2.11) *For every i and $v \in Z_i$,*

- $N^+(v) \cap Z_{0,i-3} = \emptyset$, and
- $N^-(v) \cap Z_{i+3,t} = \emptyset$.

Proof. Let us prove the first bullet point. Suppose for a contradiction that there exists a vertex u such that $u \in N^+(v) \cap Z_{0,i-3}$. Now let

$$A := (N^0(u) \cup N^-(u)) \cap B_{i-2},$$

and

$$B := (N^0(v) \cup N^+(v)) \cap B_{i-2}.$$

By (2.3) and the definition of zones, $\vec{\chi}(A) \le b' + c$. Similarly, $\vec{\chi}(B) \le b'$ by (2.3) and (2.7). Since B_{i-2} is a β'-bag, we have $\vec{\chi}(B_{i-2} \setminus (A \cup B)) > \beta' - (b' + c) - b' > c$. By the definition of c, there exists a copy X of H in $B_{i-2} \setminus (A \cup B)$. But then, by the definitions of A and B, it follows that $\{u, v\} \cup X$ induces a copy of $\Delta(1, 1, H)$, a contradiction. A similar argument, using the established claims, gives the second bullet point. ■

We are ready to prove that $\vec{\chi}(Z_{0,t})$ is bounded.

(2.12) $\quad \vec{\chi}(Z_{0,t}) \le 3\hat{f}(\beta')$.

Proof. Let $\mathcal{Z}_i = \bigcup_{j \cong i \mod 3} Z_j$. By (2.11), every strongly connected component in $\mathcal{Z}_i$ is contained in a zone Z_j. Thus, by (2.10), $\vec{\chi}(\mathcal{Z}_i) \le \hat{f}(\beta')$. Since $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$ partitions $Z_{0,t}$, it follows that $\vec{\chi}(Z_{0,t}) \le 3\hat{f}(\beta')$ as claimed. ■

We are ready to finish the proof. Since $V(D) = B_{1,t} \cup Z_{0,t}$, and by (2.6) and (2.12), we have:

$$\vec{\chi}(D) \le \vec{\chi}(B_{1,t}) + \vec{\chi}(Z_{0,t}) \le 6(\max\{b', c\} + \beta') + 2 + 3\hat{f}('\beta)$$

as claimed. ∎

2.4 Proof of Theorem 1.2.16

In this section, we prove Theorem 1.2.16, which we restate for the reader's convenience.

Theorem 1.2.16. *For every $r \ge 0$, a digraph H is a hero in $\{rK_1 + K_2\}$-free digraphs if:*

- $H = K_1$,
- $H = H_1 \Rightarrow H_2$ *where H_1 and H_2 are heroes in $\{rK_1 + K_2\}$-free digraphs.*

- $H = \Delta(1, 1, H_1)$ *where* H_1 *is a hero in* $\{rK_1 + K_2\}$*-free digraphs.*

Equivalently, we will prove that for every $r \geq 1$, the digraph $rK_1 + K_2$ cooperates. We will use Theorem 2.3.1 to do this. Thus, we need to prove that K_2 cooperates, and that K_2 is localized and colocalized. Notice that by Lemma 2.3.2, we only need to show that K_2 is localized. Notice that K_2-free digraphs are stable sets, which have bounded dichromatic number. Thus, every digraph is a hero in K_2-free digraphs, so K_2 cooperates.

To prove that K_2 is localized, we use domination. We say a set of vertices S_1 *dominates* a set of vertices S_2, or equivalently S_1 is a *dominating set* for S_2, if every vertex in $S_2 \setminus S_1$ is seen by a vertex in S_1. A digraph F *dominates* if, for every $r \geq 1$, the following two:

- $\Delta(1, 1, H)$ is a hero in $\{(r-1)K_1 + F\}$-free digraphs;
- H is a hero in $\{rK_1 + F\}$-free digraphs;

imply that there exists a function $g(r, k, H)$ such that for every $\{\Delta(1, 1, H), rK_1 + F\}$-free k-local digraph D, either $\vec{\chi}(D) \leq g(r, k, H)$, or F-free acyclic induced subsets S of $V(D)$ have a dominating set in D of size at most $g(r, k, H)$.

We want to prove that if F dominates, then F is localized. The concept that a digraph F dominates, as well as how this implies that F is localized, is meant to generalize the proof strategy devised by Harutyunyan, Le, Newman, and Thomassé [18] to prove that k-local rK_1-free digraphs, where $r \geq 2$, have bounded dichromatic number.

To prove that digraphs that dominate are localized, we use a concept introduced in [18]. A family of digraphs $\mathcal{C}$ is *tamed* if, for every m, there exists integers M and l such that if $D \in \mathcal{C}$ has $\vec{\chi}(D) \geq M$, then there exists a subset $X \subseteq V(D)$ such that $|X| \leq l$ and $\vec{\chi}(X) \geq m$. The following proof is a slight generalization of the proof of Claim 2.4 in [18].

Lemma 2.4.1. *If F dominates and the following two hold:*

- $\Delta(1, 1, H)$ *is a hero in* $\{(r-1)K_1 + F\}$*-free digraphs, and*
- H *is a hero in* $\{rK_1 + F\}$*-free digraphs,*

then, for every $k \geq 1$, the family of $\{\Delta(1, 1, H), rK_1 + F\}$-free k-local digraphs is tamed.

Proof. We proceed by induction on m. The case when $m = 1$ is immediate. Assume the statement holds for m. Let M and l be the corresponding integers. Let c be an integer such that $\{rK_1 + F, H\}$-free digraphs D have $\vec{\chi}(D) \leq c$, and let b be an integer such that $\{(r-1)K_1 + F, \Delta(1,1,H)\}$-free digraphs D have $\vec{\chi}(D) \leq b$. Since F dominates, let $g(r,k,H)$ be the associated function. Furthermore, let $p = M + bl + kl + 1$, and let $d = m((g(r,k,H) + r)p + 1) + 1$. Note that, by the pigeonhole principle, d is the smallest number such that if a set S of size d is m-colored, then there exists a monochromatic susbset of size at least $(g(r,k,H) + r)p + 2$. We claim that the statement holds for $m + 1$ when $M' = \max\{g(r,k,H) + 1, kd, M + d(b + k + 1)\}$ and $l' = d + l + l\binom{d}{(g(r,k,H)+r)p+2}$.

Assume that D is a $\{\Delta(1,1,H), rK_1 + F\}$-free k-local digraph, and assume $\vec{\chi}(D) \geq M'$. We start with the following claim.

(2.1) $\quad$ *$\vec{\chi}(N^0(v)) \leq b$ for every $v \in V(D)$.*

Proof. Since D is $\{rK_1 + F, \Delta(1,1,H)\}$-free, it follows that $N^0(v)$ is $\{(r-1)K_1 + F, \Delta(1,1, H)\}$-free, so the claim follows by definition of b. ■

Since $\vec{\chi}(D) \geq M'$, we have $\vec{\chi}(D) > g(r,k,H)$. Let B be a minimum dominating set for D. Since D is k-local, it follows that $\vec{\chi}(D) \leq |B|k$, so $|B| \geq M'/k \geq d$. Pick $W \subseteq B$ such that $|W| = d$. By the choice of M' and the size of B, we know this subset exists. Notice that $\vec{\chi}(\bigcup_{w \in W} N^0(w)) \leq bd$ by (2.1), and $\vec{\chi}(\bigcup_{w \in W} N^+(w)) \leq kd$ since D is k-local. Since $\vec{\chi}(D \setminus W) \geq M' - d$, it follows that the set $\mathcal{A}$ of vertices out-complete to W has dichromatic number at least $M' - d - bd - kd \geq M$. By the definition of M, there exists a set A out-complete to W of size at most l and dichromatic number at least m.

We will define a set A_S for every subset S of W of size $(g(r,k,H) + r)p + 2$ as follows. Let S be such a set, and let $Y = \bigcup_{s \in S} N^+(s)$. For a contradiction, assume that $\vec{\chi}(Y) \leq p$. Let $Y_1, \ldots, Y_p$ be a partition of Y into p acyclic sets. For each set Y_i, pick a vertex y_i^1 with no in-neighbors. Having picked vertex y_i^j for some $1 \leq j \leq r-1$, pick another vertex y_i^{j+1} in $Y_i \setminus \bigcup_{k \leq j} N^+[y_i^k]$ (unless this set is empty) with no in-neighbors in $Y_i \setminus \bigcup_{1 \leq k \leq j} N^+[y_i^k]$. Then, for every i, the vertices $y_i^1, \ldots, y_i^r$ form a stable set, and so the set $Y_i' = Y \setminus \bigcup_{1 \leq k \leq r} N^+[y_i^k]$ is acyclic and F-free. Since $\vec{\chi}(D) > g(r,k,H)$, there exists a dominating set Z_i for Y_i' of size at most $g(r,k,H)$, so the set $Z_i' = Z_i \cup \{y_i^1, \ldots, y_i^r\}$ is a dominating set for Y_i of size at most $g(r,k,H) + r$.

Thus, the set $Z = Z_1' \cup \cdots \cup Z_p'$ is a dominating set for Y of size at most $(g(r,k,H) + r)p$. Adding a vertex z from A, we get a dominating set for $N^+[S]$ of size at most $(g(r,k,H) + r)p + 1$. Then $(B \setminus S) \cup Z \cup \{z\}$ is a dominating set for D of size at most $|B| - 1$, contradicting that B is a smallest dominating set. Thus, $\vec{\chi}(Y) > p$.

Because $|A| \leq l$, by (2.1), and by the fact that D is k-local, we have

$$\vec{\chi}(N^0(A) \cap Y) \leq bl,$$

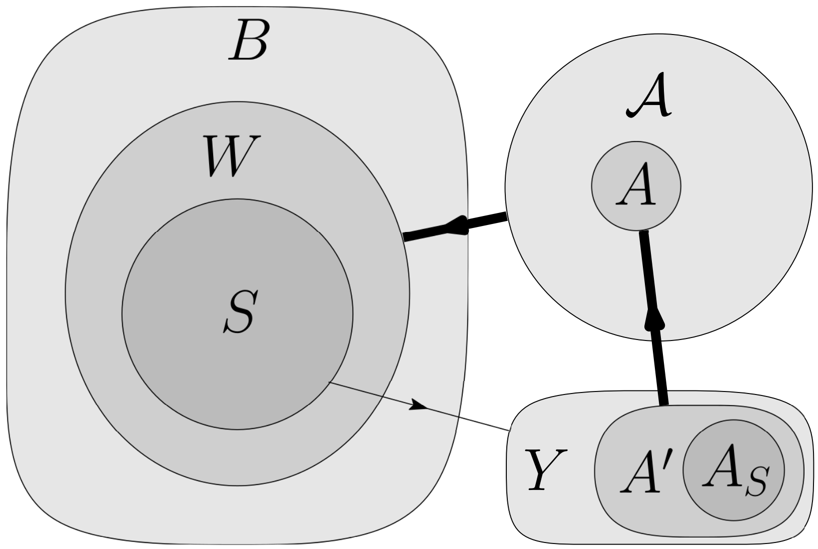


Figure 2.2: Illustration of the proof of Lemma 2.4.1.

and

$$\vec{\chi}(N^+(A) \cap Y) \leq kl.$$

Thus, the set A' of vertices of Y out-complete to A has dichromatic number at least $p - bl - kl > M$, which implies by the inductive hypothesis that A' contains a set A_S with $\vec{\chi}(A_S) \geq m$ and $|A_S| \leq l$. This is how we define A_S for every subset S of W where $|S| = (g(r, k, H) + r)p + 2$. Figure 2.2 illustrates this process.

Finally, take

$$V := W \cup A \cup \bigcup A_S.$$

where the union happens over all subsets S of W of size exactly $(g(r, k, H) + r)p + 2$. This set has size at most $d + l + l\binom{d}{g(r,k,H)p+2} = l'$. By the definition of d, every m-coloring f of V contains a monochromatic set $S \subseteq W$ of size $g(r, k, H)p + 2$. Let $f(S) = \{\gamma\}$. Since $\vec{\chi}(A), \vec{\chi}(A_S) \geq m$, it follows that there exists $a \in A$ and $a' \in A_S$ with $f(a) = f(a') = \gamma$. Now let $s \in S$ be an in-neighbor of a' (which exists since $A_S \subseteq N^+(S)$). It follows that $\{a, a', s\}$ is a cyclic triangle monochromatic under f. Since f was an arbitrary m-coloring, this argument applies to every m-coloring of V. We conclude that $\vec{\chi}(V) \geq m + 1$, and so V is the desired set for $m + 1$, finishing the inductive argument. ∎

The following is analogous to the proof of Theorem 2.3 in [18].

Lemma 2.4.2. *If F dominates, then F is localized.*

Proof. Assume that

- $\Delta(1,1,H)$ is a hero in $\{(r-1)K_1+F\}$-free digraphs, and
- H is a hero in $\{rK_1+F\}$-free digraphs.

Let c be an integer such that $\{rK_1+F,H\}$-free digraphs D have $\vec{\chi}(D)\leq c$. Furthermore, let b be an integer such that $\{(r-1)K_1,\Delta(1,1,H)\}$-free digraphs D have $\vec{\chi}(D)\leq b$. Fix an integer $k\geq 1$. By Lemma 2.4.1, $\{rK_1+F,\Delta(1,1,H)\}$-free k-local digraphs are tamed. Let M and l be the corresponding integers following the definition of tameness when $m=k+b+1$.

Let D be a $\{rK_1+F,\Delta(1,1,H)\}$-free k-local digraph. To prove that F is localized, it is enough to show that $\vec{\chi}(D)\leq\max\{M,lk\}$. Assume that $\vec{\chi}(D)>M$. By definition, there exists a set $X\subseteq D$ such that $|X|\leq l$ and $\vec{\chi}(X)\geq m$. We claim that X is a dominating set of D. Assume for a contradiction that there exists a vertex v not in $\bigcup_{x\in X}N^+(x)$. Consequently, $X\subseteq N^0(v)\cup N^+(v)$. By the definition of k and b, it follows that $\vec{\chi}(X)\leq k+b$, a contradiction. Thus, X dominates D. But D is k-local, so $\vec{\chi}(D)\leq kl$ thus finishing the proof. ∎

Now that we have proven that digraphs F that dominate are localized, it only remains to show that K_2 dominates.

Lemma 2.4.3. *K_2 dominates.*

Proof. Fix an integer $r\geq 1$. Assume that

- b is an integer such that $\{\Delta(1,1,H),(r-1)K_1+K_2\}$-free digraphs D have $\vec{\chi}(D)\leq b$; and
- c is an integer such that $\{H,rK_1+K_2\}$-free digraphs D have $\vec{\chi}(D)\leq c$.

Fix k, and assume that D is a $\{rK_1+K_2,\Delta(1,1,H)\}$-free k-local digraph. Set $g(r,k,H)=c+k+|V(H)|(kr+k+b)+2$.

Notice that K_2-free sets are stable sets, so let S be a stable set. We may assume by possibly adding vertices that S is a maximal stable set in D. Set $T=V(D)\setminus S$. Since S is maximal, every vertex in T has a neighbor in S.

(2.1) $\quad$ *$\vec{\chi}(N^0(v))\leq b$ for every $v\in V(D)$.*

Proof. The proof is a consequence of the fact that $N^0(v)$ is $\{(r-1)K_1+K_2, \Delta(1,1,H)\}$-free, so the result follows by the definition of b. ■

(2.2) *Every vertex $v \in T$ has fewer than r non-neighbors in S.*

Proof. For a contradiction, assume that a vertex v has r non-neighbors in S. Let Λ be the set containing these non-neighbors. Let $u \in S$ be a neighbor of v. It follows that $\Lambda \cup \{u, v\}$ induces a copy of $r'K_1 + K_2$ where $r' \geq r$, a contradiction. ■

(2.3) *If there exists a vertex in T with only out-neighbors in S, then there exists r vertices that dominate S.*

Proof. Suppose that there exists a vertex $v \in T$ whose neighbors in S are only out-neighbors. Let Λ be the set of non-neighbors of v in S. By (2.2), we have that $|\Lambda| \leq r-1$, making $\Lambda \cup \{v\}$ a dominating set for S of size at most r as desired. ■

Since $r \leq g(r,k,H)$, we may assume that every vertex in T has an in-neighbor in S. Consequently, $\bigcup_{s\in S} N^+(s) \setminus S = T$. If $\vec{\chi}(T) \leq g(r,k,H) - 1$, then $\vec{\chi}(D) \leq g(r,k,H)$, so we may assume that $\vec{\chi}(T) > g(r,k,H)$. Let $d := |V(H)|(kr+k+b)+1$. By removing one vertex at a time, we create a subset S' of S such that $d \leq \vec{\chi}(\bigcup_{s\in S'} N^+(s) \setminus S) \leq d+k$. This is possible because D is k-local. Set $X = T \setminus (\bigcup_{s\in S'} N^+(s))$. It follows that $\vec{\chi}(X) \geq g(r,k,H) - d - k \geq c$. By the definition of c, there exists a copy X' of H in X.

Notice that, by the dichromatic number of $\bigcup_{s\in S'} N^+(s)$ and since D is k-local, we have that $|S'| \geq r|V(H)|+1$. Let S'' be the set of vertices $s \in S'$ such that s is a neighbor of every vertex in X'. By (2.2), every vertex in T has at most r non-neighbors in S. Thus, S'' contains at most $r|V(H)|$ fewer vertices than S', and so S'' is non-empty. It follows that X' is complete to S''. Let $Y = \bigcup_{s\in S''} N^+(s) \setminus S$. Since $\vec{\chi}(\bigcup_{s\in S'} N^+(s) \setminus S) \geq d$ and D is k-local, it follows that $\vec{\chi}(Y) \geq d - kr|V(H)|$.

Let $A = \bigcup_{x\in X'} N^0(x) \cap Y$ and $B = \bigcup_{x\in X} N^+(x) \cap Y$. Then, $\vec{\chi}(A) \leq |V(H)|b$ and $\vec{\chi}(B) \leq |V(H)|k$ by (2.1) and the fact that D is k-local. Consequently, $\vec{\chi}(Y \setminus (A \cup B)) \geq d - kr|V(H)| - |V(H)|b - |V(H)|k > 0$ by the definition of d. Thus, $Y' = Y \setminus (A \cup B)$ is non-empty. Figure 2.3 illustrates the situation.

Let $y \in Y'$, and let $s \in S''$ be an in-neighbor of y in S'', which exists by the definition of Y. By definition, s is in-complete from X', and X' is in-complete from y. Thus, the set $\{s, y\} \cup X'$ induces $\Delta(1,1,H)$, a contradiction. ■

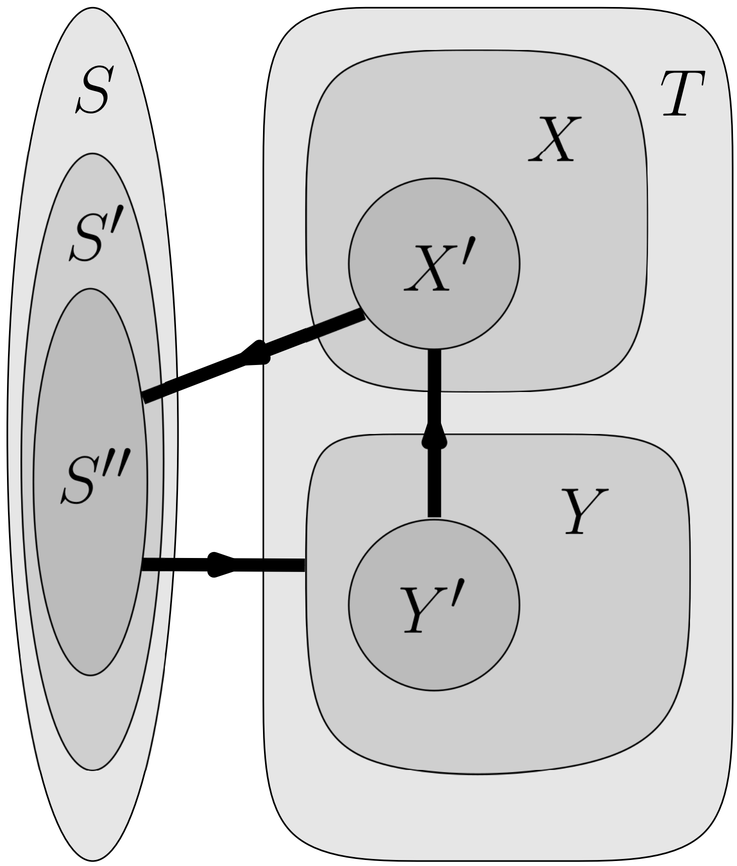


Figure 2.3: Illustration of the proof of Lemma 2.4.3.

Proof of Theorem 1.2.16. By Theorem 2.3.1, it suffices to show that K_2 cooperates, is localized, and is colocalized. Since K_2-free digraphs are stable sets, K_2 cooperates. By Lemma 2.4.3, K_2 dominates, so by Lemma 2.4.2 K_2 is localized. By Lemma 2.3.2, K_2 is colocalized as well, thus finishing the proof.

Chapter 3

$\vec{\chi}$-boundedness

In this chapter, we prove Theorem 1.3.16, which we restate for the reader's convenience.

Theorem 1.3.16. *Let r and s be positive integers. If $\mathcal{B}$ and $\mathcal{B}'$ have valid opposing orientations of B_r and B_s respectively, then $\{\mathcal{B}, \mathcal{B}'\}$-free digraphs are $\vec{\chi}$-bounded.*

We do so by imitating the technique designed by Cook, Masařík, Pilipczuk, Reinald, and Souza [12] to prove that if P is an orientation of P_4, then P-free digraphs are $\vec{\chi}$-bounded. We achieve this by generalizing Lemma 1.3.17. In particular, we prove Lemma 1.3.18, which we restate for the reader's convenience.

Lemma 1.3.18. *Let $k \geq 0$, and let $\mathcal{C}$ be a hereditary class of digraphs. If there exists an integer c such that every $D \in \mathcal{C}$ has a k-nice set S with $\vec{\chi}(S) \leq c$, then $\vec{\chi}(D) \leq 2c(k+1)$ for every $D \in \mathcal{C}$.*

We remind the reader that a set $S \neq \emptyset$ is k-nice if there exists a partition S_1, S_2 of S such that every vertex in S_1 (resp. S_2) has at most k in-neighbors (resp. k out-neighbors) in $V(D) \setminus S$.

3.1 Proof of Lemma 1.3.18

Proof of Lemma 1.3.18. Fix $\mathcal{C}$. We proceed by induction on $|V(D)|$. The statement holds if $|V(D)| = 1$. Assume the statement holds for digraphs with fewer than $|V(D)|$ vertices. We proceed to prove the statement for $|V(D)|$. By assumption, D has a k-nice set S with $\vec{\chi}(S) \leq c$. Let S_1 and S_2 be the sets as in the definition of a k-nice set.

Using the induction hypothesis, let $f_0 : (V(D) \setminus S) \to \{1, \ldots, k+1\} \times \{1, \ldots, 2c\}$ be a $(2c(k+1))$-dicoloring of the digraph induced by the vertex set $V(D) \setminus S$. Furthermore, let f_1 be a c-dicoloring of S_1 using colors in $\{1, \ldots, c\}$, and let f_2 be a c-dicoloring of S_2 using colors in $\{c+1, \ldots, 2c\}$.

We define a function $m : S \to \{1, \ldots, k+1\}$ as follows. Let $u \in S$. If $u \in S_1$, then u has at most k in-neighbors in $V(D) \setminus S$. Thus, $|f_0(N^-(u) \cap (V(D) \setminus S))| \leq k$. Consequently, there exists a number $m(u)$ such that no color in $f_0(N^-(u) \cap (V(D) \setminus S))$ has $m(u)$ as its first coordinate. We define $m(v)$ when $v \in S_2$ similarly, where we use its out-neighborhood in $V(D) \setminus S$ instead. Using these, we can define the following coloring.

$$f(v) = \begin{cases} f_0(v) & \text{if } v \notin S; \\ (m(v), f_1(v)) & \text{if } v \in S_1; \\ (m(v), f_2(v)) & \text{if } v \in S_2. \end{cases}$$

We claim that f is a $(2c(k+1))$-dicoloring of D. The first index of the coordinate has $k+1$ values, and the second index at most $2c$. Thus, this indeed uses at most $2c(k+1)$ colors. For a contradiction, assume that C is a directed monochromatic cycle in D. Since f_0, f_1 and f_2 are dicolorings, C is not contained in neither of the sets S_1, S_2 and $V(D) \setminus S$. Since f_1 and f_2 use colors that do not overlap, it follows that C does not overlap with S_1 and S_2, so C is not contained in S. By the same reason, if C overlaps with S and $V(D) \setminus S$, then C overlaps with only one of S_1 and S_2.

Thus, either C overlaps with $V(D) \setminus S$ and S_1, or C overlaps with $V(D) \setminus S$ and S_2. We proceed to prove that both lead to a contradiction. Assume C overlaps with S_1. Thus, there is an edge $e = uv$ in C such that $u \in V(D) \setminus S$ and $v \in S_1$. But then, by the definition of $m(v)$, the first coordinate of $f(u)$ is not equal to the first coordinate of $f(v)$, contradicting that C is monochromatic. A similar argument shows a contradiction when C overlaps with S_2 and $V(D) \setminus S$. This finishes the proof. ∎

3.2 Proof of Theorem 1.3.16

We restate Theorem 2.3.1 one last time.

Theorem 1.3.16. *Let r and s be positive integers. If $\mathcal{B}$ and $\mathcal{B}'$ have valid opposing orientations of B_r and B_s respectively, then $\{\mathcal{B}, \mathcal{B}'\}$-free digraphs are $\vec{\chi}$-bounded.*

With Lemma 1.3.18, we can now prove Theorem 1.3.16 by using the same argument line that Cook, Masařík, Pilipczuk, Reinald, and Souza [12] used to prove that for every orientation P of P_4, the P-free digraphs are $\vec{\chi}$-bounded (Theorem 1.3.15). To do so, we need to define the concepts developed in [12] to prove the result.

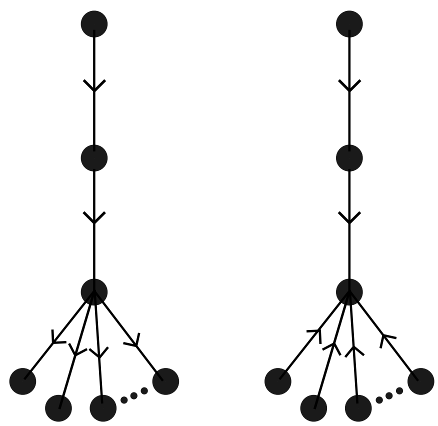
For a not strongly connected tournament K, let $K_1, \dots, K_k$ be the partition of $V(K)$ into its strongly connected components. Let K^* be the tournament that results from contracting each of these parts into a single vertex each. K^* has vertices u^* and v^* such that $N^-_{K^*}(u) \cap K^* = \emptyset$ and $N^+_{K^*}(v) \cap K^* = \emptyset$. If u is in the component that got contracted to the vertex u^*, then we call u a *source vertex*. If v is in the component that got contracted to the vertex v^*, then we call v a *sink vertex*.

We say C is a *path-minimizing closed tournament* (PMCT) if either $V(C) = K$, where K is a strongly connected tournament with $\omega(D) = |K|$, or $V(C) = K \cup V(P)$ where K is a tournament that is not strongly connected, $\omega(D) = |K|$, and P is a directed path from a sink vertex to a source vertex of K. Furthermore, K is picked such that $|V(C)| = |V(K) \cup V(P)|$ is minimized. Notice that if D has a strongly connected tournament on $\omega(D)$ vertices, then every PMCT is a tournament. Otherwise, if C is a PMCT, then C is not a tournament, and K is picked such that $|V(P)|$ is as small as possible.

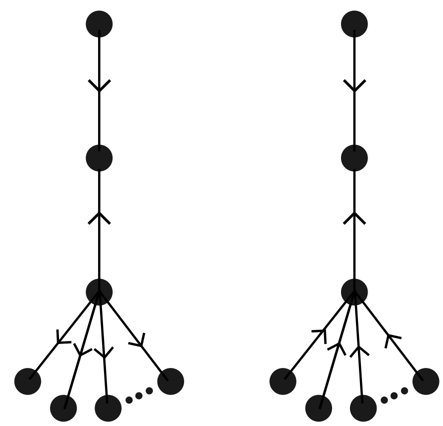
The proof strategy from Cook, Masařík, Pilipczuk, Reinald, and Souza [12] that proves Theorem 1.3.15 is as follows. They to prove that $f(\omega(D)) \leq \vec{\chi}(D)$ by induction on $\omega(D)$. Assuming that the case has been solved for $\omega < \omega(D)$, it follows that, for every $v \in V(D)$, $\vec{\chi}(N(v)) \leq f(\omega(D) - 1)$. By using PMCTs, they find a nice set. Finally, they finish the proof by proving that this nice set has bounded dichromatic number, which by Lemma 1.3.18 finishes their inductive step. In our case, we follow the same line of logic with some minor differences. In particular, instead of finding a nice set, we find a k-nice set, which allows us to find brooms instead of paths on four vertices. The rest of the proof is similar to theirs but with extra consideration for the fact that we now deal with a k-nice set, as opposed to simply nice sets.

Eventually, we need to go into four different cases. For that, we will illustrate the different cases that we will have. There are 8 types of orientations to consider that we separate into four types. These are illustrated on Figure 3.1a, Figure 3.1b, Figure 3.2a, and Figure 3.2b. Since $\mathcal{B}$ and $\mathcal{B}'$ are of opposing orientation, we may assume that $\mathcal{B}$ is of type 1 or type 3, and that $\mathcal{B}$ is of type 2 or type 4, giving four cases.

Proof of Theorem 1.3.16: Let $\mathcal{C}$ be the set of $\{\mathcal{B}, \mathcal{B}'\}$-free digraphs. To prove that $\mathcal{C}$ is $\vec{\chi}$-bounded, we proceed by induction on $\omega(D)$. The result is immediate if $\omega(D) = 1$. For a digraph D, assume that the statement holds for every $\omega < \omega(D)$. That is, assume that there exists a number γ such that if $\omega(D') < \omega(D)$ and D' is $\{\mathcal{B}, \mathcal{B}\}$-free, then $\vec{\chi}(D') \leq \gamma$.

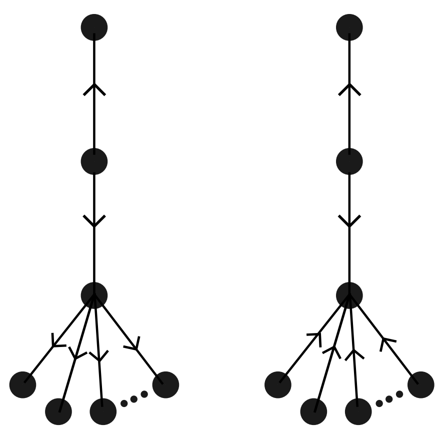


(a) Type 1 brooms.

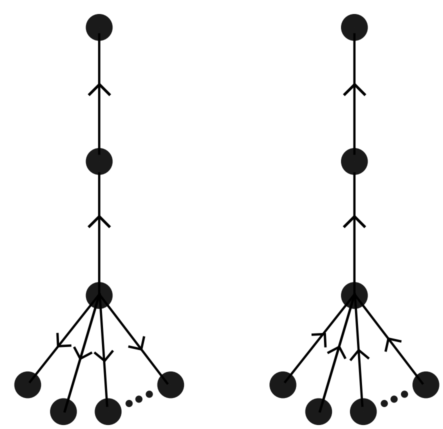


(b) Type 2 brooms.

Figure 3.1: Type 1 and type 2 brooms.



(a) Type 3 brooms.



(b) Type 4 brooms.

Figure 3.2: Type 3 and type 4 brooms.

Finally, let $k = \max\{R(r, \omega(D)), R(s, \omega(D))\}$ where R is the graph Ramsey number. We want to prove that

$$\vec{\chi}(D) \leq 2(\omega(D)(\gamma + 1) + \gamma(6k + 25) + 2)(k + 1).$$

We may assume that D is strongly connected as the strongly connected components of a digraph can be colored independently. Let C be a PMCT, which exists since D is strongly connected. Let X be the set of vertices $v \notin C$ such that v has an in-neighbor and an out-neighbor in C, $Z = N(V(C)) \setminus X$, and $Y = N(X) \setminus N[V(C)]$.

(3.1) *If S is a set of vertices in D such that $|S| \geq k$, then S contains a stable set of size at least* $\max\{r, s\}$.

Proof. The proof is immediate from the definition of the graph Ramsey number. ■

The following is the analog of the proof of Lemma 3.1 from [12].

(3.2) $N[C \cup X]$ *is a k-nice set.*

Proof. We want to prove that if $v \in N[C \cup X]$, then either v has at most k in-neighbors in $V(D) \setminus N[C \cup X]$, or v has at most k out-neighbors in $V(D) \setminus N[C \cup X]$. For this purpose, notice that if $v \in C \cup X$, then the result follows immediately.

For a contradiction, assume that there exists a vertex $v \in N(C \cup X)$ such that v has at least k in-neighbors and out-neighbors not in $N[C \cup X]$. Let $S^- := N^-(v) \setminus N[C \cup X]$ and $S^+ := N^+(v) \setminus N[C \cup X]$. Either $v \in Y$ or $v \in Z$. If $v \in Y$, then by the definition of Y, there exists $x \in X$ such that x is a neighbor of v. Since $x \in X$, there exists vertices $c_1, c_2 \in C$ such that $c_1x, xc_2 \in A(D)$. Note that as $v \in Y$, v is non-adjacent to c_1, c_2. Furthermore, notice that $\{x, c_1, c_2\}$ is anticomplete to $S^- \cup S^+$. Since $\mathcal{B}$ and $\mathcal{B}'$ have opposing orientations, both cases $xv \in A(D)$ and $vx \in A(D)$ each imply that there exists a copy of $\mathcal{B}$ or $\mathcal{B}'$ in $\{c_1, c_2, x, v\} \cup S^- \cup S^+$. Since D is $\{\mathcal{B}, \mathcal{B}'\}$-free, we conclude $v \notin Y$.

It follows that $v \in Z$. Since $v \notin X$, v has either only in-neighbors or only out-neighbors in C. Furthermore, since C contains a clique of maximal size, $N^0(v) \cap C$ is nonempty. Thus, since C is strongly connected, there is an arc from $N^0(v) \cap C$ to $N(v) \cap C$, and an arc from $N(v) \cap C$ to $N^0(v) \cap C$. Let these arcs be x_1y_1 and y_2x_2, respectively. Note that $\{x_1, x_2, y_1, y_2\}$ are anti-complete to $S^+ \cup S^-$. As before, both cases where v has only in-neighbors in C or out-neighbors in C imply that the set $\{x_1, x_2, y_1, y_2, v\} \cup S^- \cup S^+$ contain a copy of $\mathcal{B}$ or $\mathcal{B}'$. Since D is $\{\mathcal{B}, \mathcal{B}'\}$-free, both lead to contradictions. We conclude $N[C \cup X]$ is a k-nice set. ■

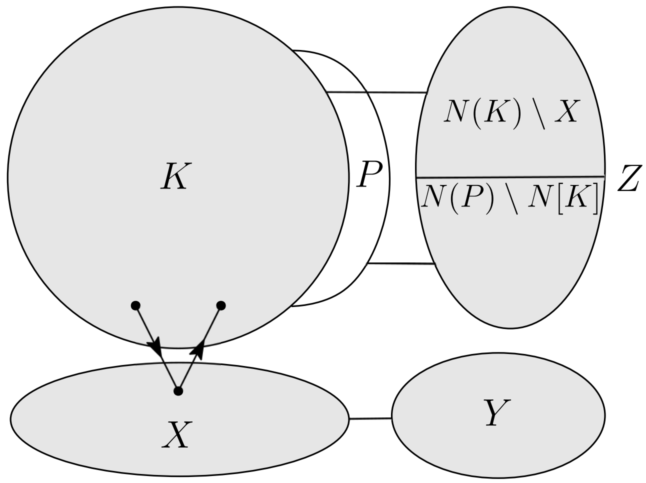


Figure 3.3: An illustration of $N[C \cup X]$.

By using Lemma 1.3.18, it is enough to bound $\vec{\chi}(N[C \cup X])$. If C is a strongly connected tournament, then we consider P to be the empty path. As noted by Cook, Masařík, Pilipczuk, Reinald, and Souza [12],

$$\vec{\chi}(N[C \cup X]) \leq \vec{\chi}(N[K]) + \vec{\chi}(P) + \vec{\chi}(N(P) \setminus N[K]) + \vec{\chi}(Y).$$

For an illustration of $N[C \cup X]$, see Figure 3.3. Thus, we want to bound each of these. By the minimality of $|V(P)|$ and by Observation 4.1 in [12], we have $\vec{\chi}(P) \leq 2$. Furthermore, since $\vec{\chi}(N(v)) \leq \gamma$ for every $v \in V(D)$ by the definition of γ, we have $\vec{\chi}(N[K]) \leq \omega(D) + \omega(D)\gamma = \omega(D)(\gamma + 1)$. We proceed to bound $\vec{\chi}(Y)$. The following is the analog of Lemma 4.3 and Corollary 4.4 in [12]. However, we use k-nice sets to get brooms rather than paths.

(3.3) $\qquad \vec{\chi}(Y) \leq 2\gamma(k + 1)$.

Proof. We proceed by proving that every non-empty induced subgraph Y' of Y has a k-nice set S such that $\vec{\chi}(S) \leq \gamma$, which finishes the proof by Lemma 1.3.18. The statement is true for $Y' = \emptyset$, so we may assume Y' is not empty.

By the definition of Y, there exists a vertex $x \in X$ such that $N(x) \cap Y' \neq \emptyset$. By the definition of X, there exists vertices $c_1, c_2 \in C$ such that $c_1x, xc_2 \in A(D)$. Set $S = N(x) \cap Y'$. By the definition of γ, we get $\vec{\chi}(S) \leq \gamma$. It suffices to prove that S is a k-nice set. For a contradiction, assume that there exists a vertex $s \in S$ which has at least k in-neighbors and out-neighbors in $Y' \setminus N(x)$. Let $S^- := N^-(s) \cap (Y' \setminus N(x))$ and $S^+ := N^+(s) \cap (Y' \setminus N(X))$. If $xs \in A(D)$, then $\{c_1, x, s\} \cup S^+$ induces a subgraph that contains a copy of $\mathcal{B}'$ by Claim (3.1). A similar argument works if $sx \in A(D)$. This proves that S is a k-nice set, thus finishing the proof. ∎

It remains to bound $\vec{\chi}(N(P) \setminus N[K])$. How we do it mimics the technique used in Section 5 of [12]. If C is a strongly connected tournament, then P is the empty path, so $\vec{\chi}(N(P)) = 0$. Assume then that K is not strongly connected. If P has at most four vertices, then $\vec{\chi}(N(P)) \setminus N[K] \leq 4\gamma$. Assume then that P has more than four vertices. Let P' be the path P with the first, the second first, the last, and the second-to-last vertices deleted. Let Q be the set of these four vertices. We want to bound the dichromatic number of $N(P') \setminus (N[K] \cup N(Q))$.

Let $v_1, \ldots, v_n$ denote the vertices of P' labeled such that $v_i v_{i+1} \in A(D)$ for every $i \in \{1, \ldots, m-1\}$. When talking about vertices v in $N(P')$, we use *first (in/out-)neighbor* of v to refer to the vertex v_i that is a (in/out-)neighbor of v such that i is minimized. Similarly, the *last (in/out-)neighbor* of v refers to the vertex v_i that is a (in/out-)neighbor of v such that i is maximized.

The rest of the proof is by cases. Notice that since $\mathcal{B}$ and $\mathcal{B}'$ have opposing consistent orientations, we may assume without loss of generality that $\mathcal{B}$ is of type 1 or type 3, and $\mathcal{B}'$ is of type 2 or type 4.

Let A^- (resp A^+) be the set of vertices in $N(P') \setminus (N[K] \cup N(Q))$ such that their first neighbor is an in-neighbor (resp out-neighbor). Furthermore, let B^+ (resp. B^-) be the set of vertices in $N(P') \setminus (N[K] \cup N(Q))$ such that their last neighbor is an out-neighbor (resp. in-neighbor). The following claim will make the proofs of each case less repetitive.

(3.4) *The following are true.*

- *If $\mathcal{B}$ is of type 1, then $\vec{\chi}(A^-) \leq 2\gamma(k+1)$.*
- *If $\mathcal{B}'$ is of type 2, then $\vec{\chi}(A^+) \leq 2\gamma(k+1)$.*
- *If $\mathcal{B}$ is of type 3, then $\vec{\chi}(B^-) \leq 2\gamma(k+1)$.*
- *If $\mathcal{B}'$ is of type 4, then $\vec{\chi}(B^+) \leq 2\gamma(k+1)$.*

Proof. Let us prove the first bullet point. We will use Lemma 1.3.18 to bound $\vec{\chi}(A^-)$. Let i be the smallest integer such that $N^+(v_i) \cap A^- \neq \emptyset$. By the definition of γ, we have $\vec{\chi}(N^+(v_i) \cap A^-) \leq \gamma$. We claim $N^+(v_i) \cap A^-$ is a k-nice set in $D[A^-]$. Let $v \in N^+(v_i) \cap A^-$

be such that v has at least k in-neighbors and at least k out-neighbors not in $A^- \setminus N^+(v_i)$. Let $S^+ := N^+(v) \cap (A^- \setminus N^+(v_i))$ be the set of out-neighbors of v in A^-, and let $S^- := N^-(v) \cap (A^- \setminus N^+(v_i))$ be the set of out-neighbors of v in $A^- \setminus N^+(v_i)$.

Since vertices in A^- have v_i as their first neighbor, v_{i-1}, where we pick v_{i-1} as the second vertex of P if $i = 1$, is anticomplete to $S^+ \cup S^-$. Furthermore, since vertices in $S^+ \cup S^-$ are not in $N^+(v_i)$, and $S^+ \cup S^- \subseteq A^-$, we have that v_i is anticomplete to $S^- \cup S^+$.

This, however, implies that $\{v_{i-1}, v_i, v\} \cup S^+ \cup S^-$ contains a copy of $\mathcal{B}$. We conclude that every vertex in $N^+(v_i) \cap A^-$ contains at most k out-neighbors in $A^- \setminus N^+(v_i)$. This finishes the proof that $N^+(v_i) \cap A^-$ is a k-nice set in $D[A^-]$, and so the proof that $\vec{\chi}(A^-) \leq 2\gamma(k+1)$.

Similar arguments prove the remaining bullet points. ■

In the following sections, we will prove that, for every case, $\vec{\chi}(N(P') \setminus (N[K] \cup N(Q))) \leq \gamma(4k+19)$. This will finish the proof since then,

$$\begin{aligned}\vec{\chi}(N[K \cup X]) &\leq \vec{\chi}(N[K]) + \vec{\chi}(P) + \vec{\chi}(N(P) \setminus N[k]) + \vec{\chi}(Y) \\ &\leq \omega(D)(\gamma+1) + 2 + 4\gamma + \gamma(4k+19) + \gamma(2k+2) \\ &= \omega(D)(\gamma+1) + \gamma(6k+25) + 2,\end{aligned}$$

where the 4γ in the second line came from $N(Q)$, the neighborhood of the vertices in P that are not in P'. By Lemma 1.3.18,

$$\vec{\chi}(D) \leq 2(\omega(D)(\gamma+1) + \gamma(6k+25) + 2)(k+1),$$

as claimed.

3.2.1 Brooms of type 1 and type 2

Assume that $\mathcal{B}$ is a broom of type 1, and that $\mathcal{B}'$ is a broom of type 2. By (3.4), $\vec{\chi}(A^- \cup A^+) \leq 4\gamma(k+1)$. However, A^-, A^+ partitions $N(P') \setminus (N[K] \cup N(Q))$, so $\vec{\chi}(N(P') \setminus (N[K] \setminus N(Q))) \leq \gamma(4k+4) \leq \gamma(4k+19)$, as claimed.

3.2.2 Brooms of type 3 and type 4

Assume that $\mathcal{B}$ is a broom of type 3, and that $\mathcal{B}'$ is a broom of type 4. By (3.4), $\vec{\chi}(B^- \cup B^+) \leq 4\gamma(k+1)$. However, B^-, B^+ partitions $N(P') \setminus (N[K] \setminus N(Q))$, so $\vec{\chi}(N(P') \setminus (N[K] \setminus N(Q))) \leq \gamma(4k+4) \leq \gamma(4k+19)$, as claimed.

3.2.3 Brooms of type 2 and type 3

Assume that $\mathcal{B}$ is a broom of type 3, and that $\mathcal{B}'$ is a broom of type 2. By (3.4), $\vec{\chi}(A^+ \cup B^-) \leq 4\gamma(k+1)$. Let $C' = N(P') \setminus (N[K] \cup N(Q) \cup A^+ \cup B^-)$, and for every $i \in \{1, \dots, m\}$, let L_i be the set of vertices v in C' such that $v_i v \in A(D)$ and i is minimized.

(3.5) *For every $i \geq 1$ and $j \geq i+3$, if $v \in L_i$, then $N^+(v) \cap L_j = \emptyset$.*

Proof. Assume for a contradiction that there exists a vertex $u \in N^+(v) \cap L_j$. Since $u \notin A^+$, v_j is the first neighbor of v in P. Since $u \notin B^-$, u has an out-neighbor v_l in P with $l > j$. But then we can shorten the path P by replacing vertices $v_{i+1}, \dots, v_{l-1}$ with v and u. This contradicts C is a PMCT. ■

Set

$$C_i = \{c \in L_j : i \text{ is congruent to } j \text{ modulo } 3\}.$$

That is, for example, C_1 is the union of L_1, L_4, L_7, and so on. Furthermore, by (3.5), every strongly connected subdigraph D' of C_i is contained in a set L_i. Since L_i is contained in the neighborhood of v_i, it follows that $\vec{\chi}(L_i) \leq \gamma$. Thus, $\vec{\chi}(C_i) \leq \gamma$.

We can now finish the proof. Since C_1, C_2, C_3 is a partition of C', it follows that $\vec{\chi}(C') \leq 3\gamma$. Thus,

$$\begin{aligned}
\vec{\chi}(N(P') \setminus N[K]) &\leq \vec{\chi}(A^- \cup B^-) + \vec{\chi}(C') \\
&\leq 4\gamma(k+1) + 3\gamma \\
&\leq \gamma(4k+7) \\
&\leq \gamma(4k+19).
\end{aligned}$$

as claimed.

3.2.4 Brooms of type 1 and type 4

Assume that $\mathcal{B}$ is a broom of type 1, and that $\mathcal{B}'$ is a broom of type 4. By (3.4), $\vec{\chi}(A^- \cup B^+) \leq 4\gamma(k+1)$. Let $C' = N(P') \setminus (N[K] \cup N(Q) \cup A^- \cup B^+)$. C' does not contain a strongly connected tournament on $\omega(D)$ vertices by the minimality of $|V(P)|$. For every $i \in \{1, \dots, m\}$, let L_i be the set of vertices v in C' such that $vv_i \in A(D)$ and i is minimized. Notice that since $v \notin A^- \cup B^+$, it follows that v has both an in-neighbor and out-neighbor in P', and so $L_1, \dots, L_m$ partitions C'. Finally, let $C_1, \dots, C_5$ be such that:

$$C_i = \{c \in L_j : i \text{ is congruent to } j \text{ modulo } 5\}.$$

That is, for example, C_1 is the union of L_1, L_6, L_{11}, and so on. We will bound $\vec{\chi}(C_i)$ by partitioning each of $C_1, \ldots, C_5$ into three sets each with a clique number strictly smaller than $\omega(D)$. This will imply that $\vec{\chi}(C') \leq 15\gamma$. The following claim will allow us to make such a partition.

(3.6) *Let $1 \leq i \leq 5$, and let $v \in C_i$. If K_1 and K_2 are tournaments in C_i each of size $\omega(D)$, then v is not both a sink vertex of K_1 and a source vertex of K_2.*

Proof. For a contradiction, assume the claim does not hold. That is, suppose there exists a vertex v and tournaments K_1 and K_2 each of size $\omega(D)$ such that v is a source vertex of K_1 and a sink vertex of K_2. Let u be a sink vertex in K_1 and w be a source vertex in K_2. Note that this implies that $wv, vu \in A(D)$. Let v_i and v_j be the first out-neighbor and in-neighbor of v respectively. Furthermore, let v_x be the first in-neighbor of w, and let v_y be the last out-neighbor of u. Since $v \notin A^- \cup B^+$, we have $i < j$. Furthermore, if $i \leq x$, then K_1 and $v_i \to \cdots \to v_x$ contradict the minimality of C as a PMCT. Thus, $x < i$. Using similar logic, we also have $j < y$.

By the definition of C_i, we have $x \cong j \mod 5$, and since $x < j$, we have that $|x - j| \geq 5$. Consequently, the path P'' which is P' with vertices $v_x, \ldots, v_y$ replaced by v_x, w, v, u, v_y, is strictly smaller. This contradicts the minimality of C, thus finishing the proof. ■

(3.7) *For every $1 \leq i \leq 5$, we have $\vec{\chi}(C_i) \leq 3\gamma$.*

Proof. Let X_i (resp. Y_i) be the set of vertices $v \in N(P') \setminus (N[K] \cup N(Q))$ such that there exists a tournament K' with $|K'| = \omega(D)$ in C_i where v is a sink vertex (resp. source vertex) of K', and let $Z_i = C_i \setminus (X_i \cup Y_i)$. If $\omega(X_i) = \omega(D)$, then there exists a tournament K' in X_i with a source vertex v. But since $v \in X_i$, then v is a sink vertex of another tournament, this contradicts (3.6). Thus, $\omega(X_i) < \omega(D)$. By similar logic, $\omega(Y_i) < \omega(D)$. As for Z_i, each $\omega(D)$-vertex tournament in Z_i is strongly connected by the choice of X_i and Y_i. But since $P \neq \emptyset$, this contradicts that C is a PMCT. Thus, $\omega(Z_i) < \omega(D)$. We conclude $\vec{\chi}(C_i) \leq 3\gamma$. ■

We can now finish the proof. Since $C_1, \ldots, C_5$ is a partition of C', it follows that $\vec{\chi}(C') \leq 15\gamma$. Thus,

$$\begin{aligned}\vec{\chi}(N(P') \setminus (N[K] \cup N(Q))) &\leq \vec{\chi}(A^- \cup B^+) + \vec{\chi}(C') \\ &\leq 4\gamma(k+1) + 15\gamma \\ &\leq \gamma(4k + 19).\end{aligned}$$

as claimed, which finishes the proof of this case, and so the proof of Theorem 1.3.16. ■

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