Turnpike Property for Generalized Linear-Quadratic Optimal Control Problem

by

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Author’s Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The turnpike phenomenon describes the long time behavior of optimally controlled systems whose optimal trajectories over a sufficiently large time horizon stay for most of the time close to a prescribed trajectory of the system. This thesis is devoted to the characterization of the turnpike property for generalized LQ optimal control problem.

Through our research, we derive both sufficient and necessary conditions for the turnpike property in infinite dimensional setting. It is shown that the turnpike property is closely related to certain structural properties of the control system. In particular, we deduce an equivalent condition of the turnpike property in terms of the exponential stabilizability and detectability of the system for finite dimensional case and point spectrum case. We also show in our thesis that the turnpike property for generalized LQ optimal control problem is equivalent to the turnpike property for LQ optimal control problem plus an algebraic condition. Next, we investigate the applications of our results to the generalized LQ optimal control problem subject to the parabolic equations, wave equations, delay equations and in relation with model predictive control schemes.

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Dedication

To University of Waterloo.

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List of Notation

$\mathcal{H}$	The state space
$\mathcal{U}$	The input space
$\mathcal{Y}$	The output space
$L^p(J, W)$	The space of W-valued p-integrable (in Bochner sense) functions defined on interval J
$L^p_{loc}(J, \mathcal{S})$	The space of W-valued locally p-integrable (in Bochner sense) functions defined on interval J
$\mathcal{L}(W, G)$	Bounded linear operators from W to G
$\mathcal{L}(W)$	Bounded linear operators from W to W
$\Sigma(W)$	Self-adjoint operators on W
$\Sigma^+(W)$	Non-negative operators on W
$C_s(J, \Sigma^+(W))$	The space of all $f : J \to \Sigma^+(W)$ such that for any $w \in W$, $f(\cdot)w$ is continuous
$C^1_s(J, \Sigma^+(W))$	The space of all $f : J \to \Sigma^+(W)$ such that for any $w \in W$, $f(\cdot)w$ is continuously differentiable
∂D	The boundary of D
$\overline{D}$	The closure of D
int D	The interior of D

$H^d(J,W)$	The set of functions f in $L^2(J,W)$ such that f and its weak derivatives up to order d have a finite L^2-norm
$H^d_{loc}(J,W)$	The set of functions $f \in L^2_{loc}(J,W)$ such that f and its weak derivatives up to order d locally have a finite L^2-norm
$H^1_0(J,W)$	The set of functions f in $H^1(J,W)$ such that the Dirichlet trace on ∂J equals to 0.
W'	The space of bounded linear functionals on W
$\mathbb{P}_V$	Projection operator to V
I	Identity operator
$\rho(S)$	The resolvent set of S
$\sigma(S)$	The spectrum of S
$\mathbb{R}^+$	The set of positive real numbers
$\mathbb{C}^+$	The set of complex numbers with positive real part
$\overline{\mathbb{R}}$	$\mathbb{R} \cup \{+\infty, -\infty\}$
S^*	The adjoint operator of S
b^*	The conjugate transpose of matrix b or the complex conjugate of vector b
$S\|_V$	The restriction of S on V
$\rightharpoonup$	The weak convergence
$\ker S$	The kernel space of S
$\operatorname{ran} S$	The range of S

Chapter 1

Introduction

1.1 Optimal control theory for infinite dimensional systems

Infinite dimensional systems are used to describe many phenomena in the real world, such as heat conduction, control of elastic structures, chemical processes, fluid dynamics, fusion reactors, metal casting processes, etc., all lie within this area. In this thesis, we focus on infinite dimensional problems such that the state evolution is described by an abstract evolution equation. This is a very standard framework with reasonably wide range of applications. A large amount of Partial Differential Equations (PDEs) and Functional Differential Equations (FDEs), including Delay differential equations, can all be formulated into this framework.

It is widely recognized that the Pontryagin's maximum principle, the Bellman's dynamic programming method, and the optimal linear regulator theory are three big milestones of modern optimal control theory for finite dimensional systems. Optimal control theory for infinite-dimensional systems, on the other hand, dates back to the 1960s, with the main goal of establishing the infinite-dimensional version of the three fundamental theories. Over the past decades, many mathematicians and control theorists have made significant contributions in this research area. Here we refer to the book [29] by Li and Yong, and the book [15] by Fattorini for an excellent overview of the results in this direction.

In particular, linear-quadratic (LQ) optimal control problem is one of the most (if not the most) important problem in optimal control theory and has motivated many popular control designs. Since 1960s, the works of Bellman-Glicksberg-Gross [5] and R. Kalman [27] have placed the LQ optimal control problem at the forefront of control theory. The latter first discovered the optimal linear state feedback control of LQ optimal control problem, and his work was soon generalized to the infinite dimensional context. In the mid 1960s, Lions first considered the LQ optimal control problem for partial differential equations in his book [30], which is a widely renowned text in this field whose significance continues to be felt even today. The LQ optimal control problem for general evolution equations with bounded controls was also studied by Lukes-Russell [31] in 1969 and by Curtain-Pritchard [10] in 1976. In the 1970s, the LQ optimal control problem with unbounded controls was investigated by Lions [30], Curtain-Pritchard [12] and Balakrishnan [4]. Since the mid-1980s, Lasiecka and Triggiani systematically investigated the LQ optimal control problem for parabolic, hyperbolic, and other equations with boundary controls and point controls. Their book [28] provides a nice survey on the theory of LQ optimal control problem for PDEs with boundary or point controls in a unified framework.

As the name suggests, generalized LQ optimal control problem is a generalized version of LQ optimal control problem. The term 'generalized' here refers to the fact that both quadratic and linear terms with respect to state and control are contained in the cost functional. Compared to LQ optimal control problem, generalized LQ optimal control problem is less often considered in existing literature, but as a special case of generalized LQ optimal control problem, tracking problem with point reference has received pretty much attention. A simple treatment of such a problem is discussed in Bensoussan [6, Section 5.1, Chapter 1, Part V] in connection with the nonhomogeneous state equation. Our thesis is fully devoted to the case for generalized LQ optimal control problem with bounded control. In our thesis, the quadratic character of the (generalized) LQ optimal control problem has enabled us to do a more explicit analysis of the properties of the system and its optimal solution. This will become apparent in the following sections.

1.2 Turnpike property

The turnpike phenomena have been reported and studied in the context of mathematical economics in the early works by Von Neumann [33] in 1945 regarding the approximation property of optimal growing economy to a balanced equilibrium path. In the book by Dorfman, Samuelson and Solow [36] in 1958, the term turnpike property was first coined. In general, the turnpike property describes the long-time behavior of the optimally controlled systems whose optimal trajectories and controls over a sufficiently large time horizon stay for most of the time close to a prescribed steady state of the system. Since 1960s, the turnpike phenomena has received continuous interest in economy because of the structural insights they allow on the structure of the optimal solutions. See, e.g., McKenzie [32]. Following these results, turnpike phenomena also have been widely observed and investigated in the context of mathematical biology [26] and chemical processes [1] among other applications. In the last decade, various turnpike properties have been defined and extensively studied by the mathematical community in the context of optimal control, also in connection with the stability of Model Predictive Control (MPC) schemes (see, e.g., [23]) and the qualitative properties of the control systems (see, e.g., [35]). Roughly speaking, the turnpike property helps to extend the stability results [3] obtained by constructing an appropriate Lyapunov function using strict dissipativity for the MPC closed loop to larger classes of MPC schemes, and on the other hand, the occurrence of turnpike property is known to be closely linked to the controllability and observability of the system. The turnpike property can also be used to synthesize long-term optimal trajectories. See, e.g., [2, 24]. The monographs [47, 48, 49] present a complete overview on turnpike properties in various optimal control and variational problems.

There is no united definition of the turnpike property, but the measure turnpike and the exponential turnpike are two notions of particular importance since they each correspond to a major method to the characterization of turnpike property. The kind of method suitable to prove the exponential turnpike property takes advantage of the hyperbolicity feature around the steady state of the optimality system resulting from the Pontryagin's maximum principle. See, e.g., [35, 39, 40]. The method suitable to deduce the measure turnpike property exploits the connection between turnpike and dissipativity properties of the control system. In most cases, the prescribed trajectory and control which is approximated by the optimal pair is the minimizer of the corresponding optimal steady state problem, i.e., the optimal steady state of the system. However, turnpike phenomena have been observed towards different target states, for example suitable periodic orbits [38, 39, 46].

Many interesting sufficient conditions (most are based on detectability and stabilizability) for turnpike properties have been found in both the linear and nonlinear setting. Good references are, e.g., [13, 19] for discrete-time systems, [35, 40] for finite dimensional continuous-time systems, [9] for Delay differential equations, and [8, 39] for infinite dimensional systems with control inside the domain. However, few necessary conditions for turnpike properties are found in the past, even for finite dimensional case. In recent years, the necessary conditions for measure turnpike property have been studied in full details in [20, 21] for finite dimensional generalized LQ optimal control problems in discrete-time and continuous-time setting respectively. We point out that our results in section 3.4 which eventually yield a surprising necessary and sufficient condition for the turnpike property in finite dimensional setting are motivated by their work.

1.3 Organization

Our thesis is organized as follows. In chapter 2, we first introduce all the necessary basics about strongly continuous semigroups of operators on Hilbert spaces in section 2.1, then we introduce the optimal control problem in infinite dimensional setting and formulate the generalized LQ optimal control problem in section 2.2. The standard results of Pontryagin's maximum principle and Riccati equations are recalled in subsection 2.2.1 and subsection 2.2.2, respectively.

In chapter 3, we present all our results on the turnpike property for generalized LQ optimal control problem. We start with proving the existence and uniqueness of the optimal pair for generalized LQ optimal control problem in section 3.1. In section 3.2, we introduce two notions of turnpike of particular interest. In section 3.3 we discuss a well-known sufficient condition for turnpike property based on the detectability and stabilizablilty of the control system. In section 3.4, we prove several necessary conditions of turnpike property in terms of the detectability, stabilizablilty and the turnpike reference of the control system. Combining the previous results, we derive a necessary and sufficient condition of the turnpike property for finite dimensional case and point spectrum case in section 3.5. We also show that the exponential turnpike property for the generalized LQ optimal control problem is equivalent to the exponential turnpike property for the LQ optimal control problem plus an algebraic condition. In section 3.6, some illustrative examples are discussed to show the potential applications of our results.

In chapter 4, we conclude our thesis and discuss several open problems.

Chapter 2

Mathematical background

In the remaining part of this thesis, we assume that the standard concepts and results of functional analysis are known to the reader. These includes space completion, Riesz's representation theorem, uniform boundedness principle, closed range theorem and some properties of non-negative operators, Bochner integral and Hilbert space valued L^2 functions and so on. Good references are [7, 14, 25, 34, 37, 45, 41]. We use $\|\cdot\|$ (resp. $\langle\cdot,\cdot\rangle$) to denote the norm (resp. inner product) on all complex Hilbert spaces. However, sometimes we write $\|\cdot\|_{L^2}$ (resp. $\langle\cdot,\cdot\rangle_{L^2}$) to emphasise the norm (resp. inner product) is considered on the corresponding L^2 space. In the remaining part of this thesis, unless otherwise stated, we assume $\mathcal{H},\mathcal{U}$ and $\mathcal{Y}$ are all complex Hilbert spaces.

2.1 Operator semigroups

In this subsection, we collect some standard results on operator semigroups. For more details on this section, we refer [41, Chapter 1-3].

For $A \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}$, the matrix exponential e^{tA} is defined by

$$e^{tA} := I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots$$

The absolute convergence of the above series follows easily from basic linear algebra. In finite dimensional systems, the matrix exponentials are used to describe the evolution of the state of a linear system in the absence of an input. Considering the following linear dynamics:

$$\begin{cases} \dfrac{dx}{dt} = Ax + Bu, \\ x(0) = x_0 \in \mathbb{R}^n. \end{cases}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $u \in L^2_{loc}((0,\infty),\mathbb{R}^m)$. The formula

$$x(t) := e^{tA}x_0 + \int_0^t e^{(t-s)A}Bu(s)ds$$

defines the unique absolute continuous function $x(t)$ on $[0,\infty)$ that verifies the above equation almost everywhere. Strongly continuous operator semigroups are the natural generalization of matrix exponentials to infinite dimensional systems. This section is devoted to introduce the basics of strongly continuous operator semigroups on (complex) Hilbert spaces.

Definition 2.1.1. *We call a family $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ of operators in $\mathcal{L}(\mathcal{H})$ a strongly continuous operator semigroup (or a C_0-semigroup) on $\mathcal{H}$ if*

(a) $\mathcal{T}_0 = I$

(b) $\mathcal{T}_{t+s} = \mathcal{T}_t\mathcal{T}_s$ *for every* $t, s \geq 0$

(c) $\lim_{t>0,t\to 0} \mathcal{T}_t x = x$ *for every* $x \in \mathcal{H}$.

The growth bound $w_0(\mathcal{T})$ of $\mathcal{T}$ is defined by

$$w_0(\mathcal{T}) := \inf_{t\in(0,\infty)} \frac{1}{t}\log\|\mathcal{T}_t\|.$$

Clearly, $w_0(\mathcal{T}) \in [-\infty,\infty)$.

The infinitesimal generator (or just the generator) $A : D(A) \to \mathcal{H}$ of $\mathcal{T}$ is defined by

$$D(A) := \left\{ x \in \mathcal{H} \mid \lim_{t>0,t\to 0} \frac{\mathcal{T}_t x - x}{t} \textit{ exists} \right\},$$
$$Ax := \lim_{t>0,t\to 0} \frac{\mathcal{T}_t x - x}{t}, \quad \forall x \in D(A).$$

We collect some useful facts about C_0-semigroups in the following proposition.

Proposition 2.1.2. *For a C_0-semigroup $\mathcal{T}$, its growth bound $w_0(\mathcal{T})$ and generator A, the following statements hold:*

(a) For any $w > w_0(\mathcal{T})$, there exists some $M_w > 0$ such that

$$\|\mathcal{T}_t\| \leq M_w e^{wt}, \quad \forall t \geq 0.$$

(b) *The function* $\psi : [0,\infty) \times \mathcal{H} \to \mathcal{H}$ *defined by* $\psi(t,x) := \mathcal{T}_t x$ *is continuous with respect to the induced product topology.*

(c) A *is a closed operator, i.e., the graph set* $G(A)$ *of* A*, defined by*

$$G(A) := \left\{ \begin{bmatrix} f \\ Af \end{bmatrix} \;\middle|\; f \in D(A) \right\}$$

is closed in $\mathcal{H} \times \mathcal{H}$*.*

(d) $D(A)$ *is dense in* $\mathcal{H}$*.*

(e) *For any* $s \in \mathbb{C}$ *with* $\operatorname{Re} s > w_0(T)$*,* $s \in \rho(A)$*.*

(f) *For any* $x \in D(A)$ *and* $t \geq 0$*,* $\mathcal{T}_t x \in D(A)$ *and*

$$\frac{d\mathcal{T}_t x}{dt} = A\mathcal{T}_t x = \mathcal{T}_t A x.$$

Assume $A_0 : D(A_0) \to \mathcal{H}$ is a densely defined closed operator on a Hilbert space $\mathcal{H}$, then its adjoint operator, denoted A_0^* is defined on the set

$$D(A^*) = \left\{ y \in \mathcal{H} \;\middle|\; \sup_{x \in D(A), x \neq 0} \frac{\langle Ax, y\rangle}{\|x\|} < \infty \right\}.$$

By Riesz's representation theorem, for each $y \in D(A^*)$, there exists a unique $w \in \mathcal{H}$ such that $\langle Ax, y\rangle = \langle x, w\rangle$ holds for any $x \in D(A)$. Then we define $A^* y = w$, so that

$$\langle Ax, y\rangle = \langle x, A^* y\rangle, \quad \forall x \in D(A), y \in D(A^*).$$

It can be shown that the adjoint operator A_0^* as defined above is also densely defined and closed. Moreover, if $s \in \rho(A)$, then $\overline{s} \in \rho(A^*)$. When A is further assumed to be the generator of a C_0-semigroup, we have the following proposition.

Proposition 2.1.3. *Let* $\mathcal{T}$ *be a* C_0*-semigroup on* $\mathcal{H}$ *and* A *be its generator, then* $\mathcal{T}^* := (\mathcal{T}_t^*)_{t\geq 0}$ *is also a* C_0*-semigroup, with generator* A^**. Moreover,* $w_0(\mathcal{T}) = w_0(\mathcal{T}^*)$*.*

The semigroup $\mathcal{T}^*$ is called the adjoint semigroup of $\mathcal{T}$.

We now introduce two very important spaces, named $\mathcal{H}_1$ and $\mathcal{H}_{-1}$, as well as their dual with respect to the pivot space $\mathcal{H}$, $\mathcal{H}_{-1}^d$ and $\mathcal{H}_1^d$.

Proposition 2.1.4. *Let A be the generator of a C_0-semigroup $\mathcal{T}$ on $\mathcal{H}$ and $s \in \rho(A)$. The space $D(A)$ endowed with the inner product*

$$\langle x_1, x_2 \rangle_{\mathcal{H}_1} := \langle (sI - A)x_1, (sI - A)x_2 \rangle, \quad \forall x_1, x_2 \in D(A),$$

and the space $D(A^*)$ endowed with the inner product

$$\langle y_1, y_2 \rangle_{\mathcal{H}_1^d} := \langle (\bar{s}I - A^*)y_1, (\bar{s}I - A^*)y_2 \rangle, \quad \forall y_1, y_2 \in D(A^*)$$

are all Hilbert spaces, denoted $\mathcal{H}_1$ and $\mathcal{H}_1^d$, respectively.

Let $\mathcal{H}_{-1}$ be the space completion of $\mathcal{H}$ with respect to the inner product

$$\langle x_1, x_2 \rangle_{\mathcal{H}_{-1}} := \langle (sI - A)^{-1}x_1, (sI - A)^{-1}x_2 \rangle, \quad \forall x_1, x_2 \in \mathcal{H},$$

and $\mathcal{H}_{-1}^d$ be the space completion of $\mathcal{H}$ with respect to the inner product

$$\langle y_1, y_2 \rangle_{\mathcal{H}_{-1}^d} := \langle (\bar{s}I - A^*)^{-1}y_1, (\bar{s}I - A^*)^{-1}y_2 \rangle, \quad \forall y_1, y_2 \in \mathcal{H},$$

then $\mathcal{H}_{-1}$ and $\mathcal{H}_{-1}^d$ are all Hilbert spaces.

In fact, the induced norm on $\mathcal{H}_1$, $\mathcal{H}_1^d$, $\mathcal{H}_{-1}$ or $\mathcal{H}_{-1}^d$ is always equivalent to the corresponding induced norm of the same space defined with respect to a different $s \in \rho(A)$. So from the topology point of view, there is no need to specific the corresponding $s \in \rho(A)$.

Concerning the properties of these spaces, we have following proposition.

Proposition 2.1.5. *Under the assumptions of Proposition 2.1.4, the following statements hold:*

(a) The space $\mathcal{H}_{-1}^d$ is isomorphic to the space $(\mathcal{H}_1)'$, i.e., there exists a bijective linear map $J : \mathcal{H}_{-1}^d \to (\mathcal{H}_1)'$ such that

$$\|y\|_{\mathcal{H}_{-1}^d} = \sup_{x \in \mathcal{H}_1, \|x\|_{\mathcal{H}_1} = 1} |Jy(x)|.$$

(b) The space $\mathcal{H}_{-1}$ is isomorphic to the space $(\mathcal{H}_1^d)'$.

(c) The identity maps from $\mathcal{H}_1$ to $\mathcal{H}$, $\mathcal{H}_1^d$ to $\mathcal{H}$, $\mathcal{H}$ to $\mathcal{H}_{-1}$ and $\mathcal{H}$ to $\mathcal{H}_{-1}^d$ are all continuous.

(c) $A \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$, and has a unique extension in $\mathcal{L}(\mathcal{H}, \mathcal{H}_{-1})$.

(d) $A^* \in \mathcal{L}(\mathcal{H}_1^d, \mathcal{H})$, and has a unique extension in $\mathcal{L}(\mathcal{H}, \mathcal{H}_{-1}^d)$.

In the sequel, we will use A and A^* again to denote the extension of A and A^* in $L(\mathcal{H}, \mathcal{H}_{-1})$ and $L(\mathcal{H}, \mathcal{H}_{-1})$, respectively. And we will identify $\mathcal{H}^d_{-1}$ with $(\mathcal{H}_1)'$ by not distinguishing y and Jy as in Proposition 2.1.5. Concerning the duality relations, we define the pairing $\langle \cdot, \cdot \rangle_{\mathcal{H}_1, \mathcal{H}^d_{-1}}$ by setting

$$\langle x, y \rangle_{\mathcal{H}_1, \mathcal{H}^d_{-1}} := Jy(x), \ \forall x \in \mathcal{H}_1, y \in \mathcal{H}^d_{-1},$$

and the pairing $\langle \cdot, \cdot \rangle_{\mathcal{H}^d_{-1}, \mathcal{H}_1}$ by setting

$$\langle y, x \rangle_{\mathcal{H}^d_{-1}, \mathcal{H}_1} := \overline{Jy(x)}, \ \forall x \in \mathcal{H}_1, y \in \mathcal{H}^d_{-1}.$$

Similarly, we will also identify $\mathcal{H}_{-1}$ with $(\mathcal{H}^d_1)'$. The pairings $\langle \cdot, \cdot \rangle_{\mathcal{H}^d_1, \mathcal{H}_{-1}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}^d_1, \mathcal{H}_{-1}}$ are defined analogously.

Our next proposition shows that if A generates a semigroup and $P \in L(\mathcal{H})$, then $A + P$ still generates a semigroup.

Proposition 2.1.6. *If A generates a C_0-semigroup $\mathcal{T}$ on $\mathcal{H}$ and $P \in \mathcal{L}(\mathcal{H})$, then $A+P : D(A) \to \mathcal{H}$ is the generator of a C_0-semigroup $\widetilde{\mathcal{T}}$ on $\mathcal{H}$. Moreover, if $\mathcal{T}$ satisfies the estimate*

$$\|\mathcal{T}_t\| \leq Me^{wt}, \ \forall t \geq 0$$

for some $w \in \mathbb{R}$ and $M \geq 1$, then $\widetilde{\mathcal{T}}$ satisfies the estimate

$$\|\widetilde{\mathcal{T}}_t\| \leq Me^{(w+M\|P\|)t}, \ \forall t \geq 0.$$

From now on, we will assume that A is the generator of some C_0-semigroup $\mathcal{T}$ on $\mathcal{H}$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ and $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$. We now introduce the input map and output map.

Definition 2.1.7. *For each $t \geq 0$, the input map $\Phi_t : L^2((0,\infty), \mathcal{U}) \to \mathcal{H}$ is defined by*

$$\Phi_t u := \int_0^t \mathcal{T}_{t-s} Bu(s) ds,$$

and the output map $\Psi_t : \mathcal{H} \to L^2((0,\infty), \mathcal{Y})$ is defined by

$$(\Psi_t x)(s) := \begin{cases} C\mathcal{T}_s x, & s \in [0,t], \\ 0, & s > t. \end{cases}$$

Obviously $\Phi_t \in \mathcal{L}(L^2((0,\infty), \mathcal{U}), \mathcal{H})$ and $\Psi_t \in \mathcal{L}(\mathcal{H}, L^2((0,\infty), \mathcal{Y}))$ for any $t \geq 0$. Now we complement some properties of the input and output operator.

Proposition 2.1.8. *Let $w \in \mathbb{R}$ and $M \geq 1$ be such that $\|\mathcal{T}_t\| \leq Me^{wt}$, for all $t \geq 0$.*

(a) *If $w > 0$, there exists $\widetilde{M} \geq 0$ such that $\|\Phi_t\|, \|\Psi_t\| \leq \widetilde{M}e^{wt}$, $\forall t \geq 0$.*

(b) *If $w = 0$, there exists $\widetilde{M} \geq 0$ such that $\|\Phi_t\|, \|\Psi_t\| \leq \widetilde{M}(1+t)^{\frac{1}{2}}$, $\forall t \geq 0$.*

(c) *If $w < 0$, there exists $\widetilde{M} \geq 0$ such that $\|\Phi_t\|, \|\Psi_t\| \leq \widetilde{M}$, $\forall t \geq 0$.*

Proposition 2.1.9. *Let the function $\psi : [0,\infty) \times L^2((0,\infty),\mathcal{U}) \to \mathcal{H}$ be defined by*

$$\psi(t,u) := \Phi_t u.$$

Then ψ is continuous with respect to the product topology on $[0,\infty) \times L^2((0,\infty),\mathcal{U})$.

The next two proposition provides the regularity result of the solution of certain systems.

Proposition 2.1.10. *For any $x_0 \in \mathcal{H}$ and $u \in L^2_{loc}((0,\infty),\mathcal{U})$, the problem*

$$\begin{cases} \dot{x} = Ax + Bu, \\ x(0) = x_0 \end{cases}$$

admits a unique solution $x \in C([0,\infty),\mathcal{H}) \cap H^1_{loc}((0,\infty),\mathcal{H}_{-1})$ in the sense that

$$x(t) - x_0 = \int_0^t Ax(s) + Bu(s)ds, \quad \forall t \geq 0$$

with the integration carried out in $\mathcal{H}_{-1}$. Moreover, this solution x is given by

$$x(t) = \mathcal{T}_t x_0 + \Phi_t u, \ \forall t \geq 0.$$

If, in addition, $u \in H^1_{loc}((0,\infty),\mathcal{U})$ and $x_0 \in D(A)$, then the solution is in fact a strict one, i.e., $x \in C^1([0,\infty),\mathcal{H}) \cap C([0,\infty),\mathcal{H}_1)$.

Proposition 2.1.11. *Let $x_0 \in \mathcal{H}$, $F \in C_s([0,\infty),\mathcal{L}(\mathcal{H}))$ and $f \in L^2_{loc}((0,\infty),\mathcal{H})$. The problem*

$$\dot{x}(t) = Ax(t) + F(t)x(t) + f(t), \quad x(0) = x_0 \in \mathcal{H}$$

admits a unique solution $x \in C([0,\infty),\mathcal{H}) \cap H^1_{loc}((0,\infty),\mathcal{H}_{-1})$ in the sense that

$$x(t) - x_0 = \int_0^t Ax(s) + F(s)x(s) + f(s)ds, \quad \forall t \geq 0$$

with the integration carried out in $\mathcal{H}_{-1}$.

If, in addition, $F \in C^1_s([0,\infty),\mathcal{L}(\mathcal{H}))$, $f \in H^1_{loc}((0,\infty),\mathcal{H})$ and $x_0 \in D(A)$, then the solution is in fact a strict one, i.e., $x \in C^1([0,\infty),\mathcal{H}) \cap C([0,\infty),\mathcal{H}_1)$.

2.2 Optimal control

Roughly speaking, the (unconstrained) optimal control problem for infinite dimensional systems aims to find the optimal pair $(x^*(\cdot), u^*(\cdot))$ which minimizes a functional in a form of

$$J_T(x(\cdot), u(\cdot)) := \int_0^T f_0(t, x(t), u(t))dt,$$

among all admissible pairs $(x(\cdot), u(\cdot))$.

Here x is considered to be the (mild) solution of the evolution system

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), u(t)), \quad t \in [0, T], \\ x(0) = x_0. \end{cases}$$

In this thesis, we are concerned with the following generalized LQ optimal control problem corresponding to time horizon $T > 0$:

The generalized LQ optimal control problem $(GLQ)_T$. Find the optimal pair

$$(x^*(\cdot), u^*(\cdot)) \in L^2((0, T), \mathcal{H}) \times L^2((0, T), \mathcal{U})$$

which minimizes the cost functional

$$J_T(x_0, u) := \int_0^T \ell(x(t), u(t))dt, \tag{2.1}$$

where the running cost $\ell : \mathcal{H} \times \mathcal{U} \to \mathbb{R}$ is defined by

$$\ell(x, u) := \|Cx\|^2 + \|Ku\|^2 + 2\operatorname{Re}\langle z, x\rangle + 2\operatorname{Re}\langle v, u\rangle, \tag{2.2}$$

and the associated dynamical system is

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = x_0 \in \mathcal{H}. \end{cases} \tag{2.3}$$

Here, we assume that $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, $K \in \mathcal{L}(\mathcal{U})$, $z \in \mathcal{H}$ and $v \in \mathcal{U}$. K is further assumed to be coercive, i.e., there exists a constant $m > 0$ such that $\langle K^*Ku, u\rangle \geq m\|u\|^2$ for any $u \in \mathcal{U}$. Regarding the dynamical system, we assume that A is the generator of a strongly continuous semigroup $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ on $\mathcal{H}$ and $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$.

In particular, if $z = 0$ and $v = 0$, we shall call it the LQ optimal control problem, which is abbreviated to $(LQ)_T$.

In the following subsections, we will introduce the necessary background about Pontryagin's maximum principle and the Riccati equations. The maximum principle gives necessary condition for the optimality system, and the Riccati equations help to establish the optimal linear state feedback control of LQ optimal control problem.

2.2.1 Maximum principle

Consider the general optimal control problem mentioned at the beginning of this section.

Now assume that $\mathcal{H}, \mathcal{U}, \mathcal{Y}$ are all finite dimensional Hilbert spaces and $A = 0$. Define the Hamiltonian $H : [0, T] \times \mathcal{H} \times \mathcal{U} \times \mathcal{H} \times \mathbb{R}$ by

$$H(t, x, u, p, p_0) := p_0 f_0(t, x, u) + \langle p, f(t, x, u) \rangle.$$

Pontryagin's maximum principle claims that, under some regularity conditions, if $(x^*(\cdot), u^*(\cdot))$ is an optimal pair for the above optimal control problem, then there exists a nontrivial pair $(p_0, p(\cdot)) \in \mathbb{R} \times C([0, T], \mathcal{H})$ such that

$$\begin{cases} p_0 \leq 0, \\ \dot{x}^*(t) = f(t, x^*(t), u^*(t)), \text{ a.e. } t \in [0, T], \\ x^*(0) = x_0, \\ \dot{p}(t) = -f_x(t, x^*(t), u^*(t))^* p(t) - (f_0)_x(t, x^*(t), u^*(t)) p_0, \text{ a.e. } t \in [0, T], \\ p(T) = 0, \\ H(t, x^*(t), u^*(t), p(t), p_0) = \max_{u \in \mathcal{U}} H(t, x^*(t), u, p(t), p_0), \text{ a.e. } t \in [0, T]. \end{cases}$$

In the above, we refer to p as the adjoint state. Several generalizations of Pontryagin's maximum principle to infinite dimensional case are available. See, e.g., [29, 15] for more details.

In particular, Pontryagin's maximum principle has been extensively studied in [29] for LQ optimal control problem, considering both bounded and unbounded input. Roughly speaking, Pontryagin's maximum principle holds at $x_0 \in \mathcal{H}$ if and only if the LQ optimal control problem (possibly with unbounded input) is solvable at $x_0 \in \mathcal{H}$, i.e., an optimal pair exists. In fact, we will prove in section 3.1 that the optimal pair exists and is unique for our optimal control problem $(GLQ)_T$ (thus also $(LQ)_T$) with any initial condition $x_0 \in \mathcal{H}$. This explains the motivation of the Hamiltonian systems appearing in our proof.

However, to the sake of simplicity, we will directly verify these Hamiltonian systems in our proof instead of explaining in details the statement (which is a little bit technical) of Pontryagin's maximum principle in infinite dimensional setting and deriving these Hamiltonian systems. Based on our result, we can eventually show that if turnpike property is satisfied for $(GLQ)_T$, then Pontryagin's maximum principle holds with $p_0 = -1$ (by the linearity of the Hamiltonian system, this is equivalent to $p_0 \neq 0$).

2.2.2 Riccati equations

In this subsection, we introduce some well-known results on differential and algebraic Riccati equations. This material can be found in many books, of which we mention [6, Part IV, Section 1] and [6, Part V, Section 1]. In this subsection, we assume that all the assumptions of $(GLQ)_T$ are verified.

Consider the differential Riccati equation:

$$\begin{cases} \dfrac{dP}{dt} - A^*P - PA + PB(K^*K)^{-1}B^*P - C^*C = 0 \\ P(0) = P_0 \in \Sigma^+(\mathcal{H}). \end{cases} \tag{2.4}$$

Since A is unbounded, it is not clear at this moment what we mean by a solution of (2.4). We now define the notion of mild solution and weak solution.

Definition 2.2.1. *A function $P \in C_s([0, \infty), \Sigma(\mathcal{H}))$ is called a mild solution of problem (2.4) if it verifies*

$$P(t)x_0 = \mathcal{T}_t^* P_0 \mathcal{T}_t x_0 + \int_0^t \mathcal{T}_s^* C^* C \mathcal{T}_s x_0 ds - \int_0^t \mathcal{T}_{t-s}^* P(s) B(K^*K)^{-1} B^* P(s) \mathcal{T}_{t-s} x_0 ds,$$

for any $x_0 \in \mathcal{H}$ and $t \geq 0$.

A function $P \in C_s([0, \infty), \Sigma(\mathcal{H}))$ is called a weak solution of problem (2.4) if $P(0) = P_0$, and for any $x, y \in D(A)$, $\langle P(\cdot)x, y\rangle$ is a differentiable function such that

$$\begin{aligned} \frac{d\langle P(t)x, y\rangle}{dt} &= \langle P(t)x, Ay\rangle + \langle P(t)Ax, y\rangle \\ &\quad - \langle (K^*K)^{-1}B^*P(t)x, B^*P(t)y\rangle + \langle Cx, Cy\rangle, \quad \forall t \geq 0. \end{aligned}$$

Concerning the mild and weak solution of (2.4), we have the following result.

Proposition 2.2.2. *P is a mild solution of problem (2.4) if and only if P is a weak solution of problem (2.4). For any $P_0 \in \Sigma^+(\mathcal{H})$, there exists a unique mild solution $P \in C_s([0, \infty), \Sigma^+(\mathcal{H}))$ of the differential Riccati equation (2.4).*

The following well-known transform (see, e.g., [6, Part IV, Chapter 1, Proposition 6.2]) shows the cost functional J_T from $(GLQ)_T$ can be rewritten as:

$$J_T(x_0, u) = \int_0^T \|K(u(t) + (K^*K)^{-1}B^*P(T-t)x(t))\|^2 dt + \langle P(T)x_0, x_0\rangle, \tag{2.5}$$

where x is the solution of (2.3). So, the optimal control for the LQ optimal control problem can be given in a feedback form:

$$u(t) = -(K^*K)^{-1}B^*P(T-t)x(t),\ t \in [0,T].$$

By imposing some restriction on P_0, we can ensure P to have stronger regularity. Now for any $F \in \Sigma(\mathcal{H})$, we introduce the bilinear map

$$\varphi_F(x,y) := \langle Fx, Ay\rangle + \langle Ax, Fy\rangle, \quad \forall x,y \in D(A).$$

Let $D(\mathcal{A}) := \{F \in \Sigma(\mathcal{H}) \,|\, \varphi_F \text{ has a continuous extension to } \mathcal{H} \times \mathcal{H}\}$, then for any $F \in D(\mathcal{A})$, there exists a unique bounded linear operator $\mathcal{A}(F) : \mathcal{H} \to \mathcal{H}$ satisfying

$$\langle \mathcal{A}(F)x, y\rangle = \langle Fx, Ay\rangle + \langle Ax, Fy\rangle, \quad \forall x,y \in D(A).$$

Definition 2.2.3. *A strict solution of* (2.4) *is a function* $P \in C_s([0,\infty), \Sigma(\mathcal{H}))$ *such that*

(a) $P(\cdot) \in C_s^1([0,\infty), \Sigma(\mathcal{H}))$.

(b) $P(t) \in D(\mathcal{A}),\ \forall\, t \geq 0$ *and* $P(0) = P_0$.

(c) $\mathcal{A}(P(\cdot)) \in C_s([0,\infty), \Sigma(\mathcal{H}))$ *and* $P' = \mathcal{A}(P) - PB(K^*K)^{-1}B^*P + C^*C$.

Concerning the strict solutions of problem (2.4), we have the following result.

Proposition 2.2.4. *If P is a strict solution of problem* (2.4)*, then P is also a mild solution of problem* (2.4)*. Moreover, if $P_0 \in \Sigma^+(\mathcal{H})$ and $P_0 \in D(\mathcal{A})$, then there exists a unique strict solution $P \in C_s([0,\infty), \Sigma^+(\mathcal{H}))$ of the differential Riccati equation* (2.4).

Notice that $0 \in D(\mathcal{A})$, so the unique solution P of problem (2.4) with initial condition $P_0 = 0$ is actually a strict one.

Now let us consider the algebraic Riccati equation:

$$A^*P + PA - PB(K^*K)^{-1}B^*P + C^*C = 0. \tag{2.6}$$

Definition 2.2.5. *A solution of the algebraic Riccati equation* (2.6) *is a function* $P \in \Sigma(\mathcal{H})$ *such that*

$$\langle Px, Ay\rangle + \langle PAx, y\rangle - \langle (K^*K)^{-1}B^*Px, B^*Py\rangle + \langle Cx, Cy\rangle = 0,\ \forall x,y \in D(A).$$

The next proposition collects some facts about the solution of (2.6).

Proposition 2.2.6. *Define the cost functional* $J_\infty : \mathcal{H} \times L^2((0,\infty),\mathcal{U}) \to \overline{\mathbb{R}}$ *by*

$$J_\infty(x_0, u) := \int_0^\infty \|Cx(t)\|^2 + \|Ku(t)\|^2 dt,$$

where x is the solution of (2.3). Then the following statements hold:

(a) A non-negative solution of the algebraic Riccati equation (2.6) exists if and only if the pair (A, B) is C-stabilizable, i.e., for any $x_0 \in \mathcal{H}$,

$$\inf_{u \in L^2((0,\infty),\mathcal{H})} J_\infty(x_0, u) < \infty. \tag{2.7}$$

Notice (2.7) does not rely on the selection of coercive $K \in \mathcal{L}(\mathcal{U})$.

(b) If (A, B) is C-stabilizable, then there exists a minimal non-negative solution of the algebraic Riccati equation (2.6), denoted $P_{\min}$, i.e., $P_{\min}$ is a non-negative solution of (2.6), and for any non-negative solution P of (2.6),

$$\langle P_{\min} x, x\rangle \le \langle Px, x\rangle, \ \forall x \in \mathcal{H}.$$

(c) If (A, B) is C-stabilizable, then for any $x_0 \in \mathcal{H}$, the optimal control to minimize $J_\infty(x_0, \cdot)$ is given in a feedback form

$$u(t) = -(K^*K)^{-1} B^* P_{\min} x(t), \ \forall t \ge 0.$$

Moreover,

$$\inf_{u \in L^2((0,\infty),\mathcal{U})} J_\infty(x_0, u) = \langle P_{\min} x_0, x_0\rangle.$$

(d) If (A, B) is C-stabilizable and (A, C) is exponentially detectable, i.e., the pair (A^*, C^*) is I-stabilizable, then the C_0-semigroup generated by $A - B(K^*K)^{-1}B^*P_{\min}$ is exponentially stable.

In the remaining part of this thesis, we will use $P_{\min}$ to denote the minimal non-negative solution of the algebraic Riccati equation (2.6).

The next proposition concerns the convergence property of P to $P_{\min}$. In section 3.3 we will generalize the exponential convergence result to infinite dimensional setting based on our stabilizability and detectability assumptions.

Proposition 2.2.7. *Suppose (A, B) is C-stabilizable. Let $P(\cdot)$ be the solution of (2.4) with initial condition $P_0 \in \Sigma^+(\mathcal{H})$ and $P_{\min}$ be the minimal non-negative solution of (2.6), then $P(\cdot)$ converges to $P_{\min}$ in strong operator topology.*

Additionally, if $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{Y}$ are all finite dimensional spaces, then $P(\cdot)$ converges to $P_{\min}$ exponentially in norm topology, i.e., there exist constants $M, k > 0$ such that

$$\|P(t) - P_{\min}\| \leq Me^{-kt}, \ \forall t \geq 0.$$

Remark 2.2.8. *The initial condition $P_0 \in \Sigma^+(\mathcal{H})$ is linked to the terminal cost of the LQ optimal control problem. In our setting, the terminal term is 0, so in the reminder of this thesis, unless otherwise stated, we will use P to denote the unique solution of problem (2.4) with initial condition $P_0 = 0$.*

Chapter 3

Turnpike property for generalized linear-quadratic optimal control problem

3.1 Existence and uniqueness of the optimal control

In this section, we prove that for any $T > 0$, the optimal pair of the optimal control problem $(GLQ)_T$ exists and is unique.

Theorem 3.1.1. *For any $T > 0$, there exists a unique optimal pair*

$$(x^*(\cdot), u^*(\cdot)) \in L^2((0,T),\mathcal{H}) \times L^2((0,T),\mathcal{U})$$

of the optimal control problem $(GLQ)_T$.

Proof. Fix some $T > 0$ and $x_0 \in \mathcal{H}$. Define

$$m := \inf_{u \in L^2((0,T),\mathcal{H})} J_T(x_0, u).$$

Obviously $m > -\infty$. Now let $(u_n)_{n\in\mathbb{N}}$ be a sequence of controls in $L^2((0,T),\mathcal{U})$ such that

$$\lim_{n\to\infty} J_T(x_0, u_n) = m. \tag{3.1}$$

We claim that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence. In fact, since K is coercive, there exists $m_K > 0$

such that $K^*K \geq m_K I$. Notice that for any $u \in L^2((0,T),\mathcal{U})$,

$$J_T(x_0,u) = \int_0^T \|Cx(t)\|^2 + \|Ku(t)\|^2 + 2\,\mathrm{Re}\langle z, x(t)\rangle + 2\,\mathrm{Re}\langle v, u(t)\rangle dt$$
$$\geq m_K\|u\|_{L^2}^2 - \int_0^T 2\|z\|\|x(t)\| + 2\|v\|\|u(t)\|dt.$$

Since $x(t) = \mathcal{T}_t x_0 + \Phi_t u$, there exists $m_1, m_2 > 0$ such that

$$\int_0^T 2\|z\|\|x(t)\|dt \leq \int_0^T m_1 + m_2\|u\|_{L^2}dt = T(m_1 + m_2\|u\|_{L^2}).$$

By Hölder's inequality, there exists $m_3 > 0$ such that

$$\int_0^T 2\|v\|\|u(t)\|dt = T\|v\|\|u\|_{L^1} \leq m_3\|u\|_{L^2}.$$

Combining the above estimates, we obtain that

$$J_T(u,x_0) \geq m_K\|u\|_{L^2}^2 - (m_2T + m_3)\|u\|_{L^2} - m_1T, \;\; \forall u \in L^2((0,T),\mathcal{U}).$$

This, together with (3.1) implies that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded. So, there exists a subsequence, denoted again $(u_n)_{n\in\mathbb{N}}$ and $u^* \in L^2((0,T),\mathcal{H})$ such that $u_n \rightharpoonup u^*$ as $n \to \infty$.

Denote by x_n, $n \in \mathbb{N}$ the trajectory corresponding to initial condition x_0 and input u_n and x^* the trajectory corresponding to initial condition x_0 and input u^*. Notice that, for any $n \in \mathbb{N}$,

$$\|Ku^*\|_{L^2}^2 = \|K(u^* - u_n)\|_{L^2}^2 + 2\,\mathrm{Re}\langle K^*Ku^*, u_n\rangle_{L^2} - \|Ku_n\|_{L^2}^2.$$

Since $u_n \rightharpoonup u^*$ as $n \to \infty$, we obtain that, for any $n \in \mathbb{N}$,

$$\|Ku^*\|_{L^2}^2 \geq \limsup_{n\to\infty}(2\,\mathrm{Re}\langle K^*Ku^*, u_n\rangle_{L^2} - \|Ku_n\|_{L^2}^2) = 2\|Ku^*\|_{L^2}^2 - \liminf_{n\to\infty}\|Ku_n\|_{L^2}^2.$$

A simple calculation shows that for any $n \in \mathbb{N}$,

$$\|Ku^*\|_{L^2}^2 + 2\,\mathrm{Re}\langle v, u^*\rangle_{L^2} \leq \liminf_{n\to\infty}(\|Ku_n\|_{L^2}^2 + 2\,\mathrm{Re}\langle v, u_n\rangle_{L^2}). \tag{3.2}$$

Besides, since $\Phi_\cdot \in \mathcal{L}(L^2((0,T),\mathcal{U}), L^2((0,T),\mathcal{H}))$, $\Phi_\cdot$ is a weak-weak continuous linear operator (this follows easily from the definition of weak convergence), which implies that

$$x_n(\cdot) = \mathcal{T}_\cdot x_0 + \Phi_\cdot u_n \rightharpoonup \mathcal{T}_\cdot x_0 + \Phi_\cdot u^* = x^*(\cdot) \;\text{ in }\; L^2((0,T),\mathcal{H}).$$

Now following the same manner as the proof of (3.2), we deduce that

$$\|Cx^*\|_{L^2}^2 + 2\,\mathrm{Re}\langle z, x^*\rangle_{L^2} \leq \liminf_{n\to\infty}(\|Cx_n\|_{L^2}^2 + 2\,\mathrm{Re}\langle z, x_n\rangle_{L^2}). \tag{3.3}$$

This, together with (3.2) implies that

$$J_T(x_0, u^*) \leq \lim_{n\to\infty} J_T(x_0, u_n) = \inf_{u\in L^2((0,T),\mathcal{H})} J_T(x_0, u).$$

So, (x^*, u^*) is an optimal pair of the problem $(GLQ)_T$.

It only remains to prove the uniqueness of the optimal pair. Assume that (x_1^*, u_1^*) and (x_2^*, u_2^*) are two distinct optimal pairs of the problem $(GLQ)_T$. Simple calculations show that

$$\frac{J(x_0, u_1^*) + J(x_0, u_2^*)}{2} - J\left(x_0, \frac{u_1^* + u_2^*}{2}\right) = \frac{1}{4}\int_0^T \|C(x_1^*(t) - x_2^*(t))\|^2 + \|K(u_1^*(t) - u_2^*(t))\| dt.$$

Since obiviously $u_1^* \neq u_2^*$ in $L^2((0,T),\mathcal{U})$ and K is coercive, we obtain that

$$J\left(\frac{u_1^* + u_2^*}{2}, x_0\right) < \frac{J(u_1^*, x_0) + J(u_2^*, x_0)}{2} = \inf_{u\in L^2((0,T),\mathcal{H})} J_T(u, x_0),$$

which gives a contradiction. $\square$

3.2 Some notions of turnpike

In the remaining part of this thesis, we denote by $x_T^*(\cdot, x_0)$ and $u_T^*(\cdot, x_0)$, or simply by x^* and u^* when x_0 and T are clear from the context, the optimal trajectory and optimal control of problem $(GLQ)_T$ (or $(LQ)_T$) corresponding to initial condition $x_0 \in \mathcal{H}$ and time horizon $T > 0$.

Now, we are well-prepared to define the measure turnpike property and the exponential turnpike property at some steady state (x_e, u_e). We recall that a steady state of system (2.3) is a pair $(x_e, u_e) \in \mathcal{H} \times \mathcal{U}$ such that $Ax_e + Bu_e = 0$.

Remark 3.2.1. *Since B is bounded, if (x_e, u_e) is a steady state, then $Ax_e = -Bu_e \in \mathcal{H}$. This implies x_e must be an element of $D(A)$.*

Definition 3.2.2. *We say that the optimal control problem $(GLQ)_T$ satisfies the measure turnpike property at some steady state (x_e, u_e) if, for any bounded neighborhood $\mathcal{N}$ of x_e and $\varepsilon > 0$,* *there exists a constant $M_{\mathcal{N},\varepsilon} > 0$ such that for all $x_0 \in \mathcal{N}$ and time horizon $T > 0$, the optimal trajectory $x_T^*(\cdot, x_0)$ and optimal control $u_T^*(\cdot, x_0)$ of problem $(GLQ)_T$ satisfy that*

$$\mu \left\{ t \in [0,T] \;\middle|\; \|x_T^*(t,x_0) - x_e\| + \|u_T^*(t,x_0) - u_e\| > \varepsilon \right\} \leq M_{\mathcal{N},\varepsilon},$$

where μ denotes the Lebesgue measure on $\mathbb{R}$.

We say that the optimal control problem $(GLQ)_T$ satisfies the exponential turnpike property at some steady state (x_e, u_e) if, for any bounded neighborhood $\mathcal{N}$ of x_e, there exists some positive constants $M_{\mathcal{N}}$ and k such that for all $x_0 \in \mathcal{N}$ and time horizon $T > 0$, the optimal trajectory $x_T^*(\cdot, x_0)$ and optimal control $u_T^*(\cdot, x_0)$ of problem $(GLQ)_T$ satisfy that

$$\|x_T^*(t,x_0) - x_e\| + \|u_T^*(t,x_0) - u_e\| \leq M_{\mathcal{N}}(e^{-kt} + e^{-k(T-t)}),\ \forall t \in [0,T].$$

Remark 3.2.3. *It is clear that exponential turnpike property is stronger than the measure turnpike property. So, any sufficient condition for exponential turnpike property is automatically a sufficient condition for measure turnpike property, and conversely, any necessary condition for measure turnpike property is also a necessary condition for exponential turnpike property.*

To study the turnpike property, it is useful to analyse the optimal steady state corresponding to the running cost ℓ. The optimal steady state problem is defined as:

$$\inf_{x \in D(A),\, u \in \mathcal{U}} \ell(x,u) \quad \text{s.t. } Ax + Bu = 0. \tag{3.4}$$

If (x_e, u_e) is a minimizer of equation (3.4), we say (x_e, u_e) is an optimal steady state.

We will also need several structural-theoretical properties of the control system under consideration, which we introduce in the following.

Definition 3.2.4. *The pair (A, B) is called exponentially stabilizable if (A, B) is I-stabilizable. The pair (A, C) is called exponentially detectable if (A^*, C^*) is I-stabilizable.*

Remark 3.2.5. *It can be proved that (A, B) is exponentially stabilizable if and only if there exists some $F \in \mathcal{L}(\mathcal{Y}, \mathcal{H})$ such that $A + BF$ generates an exponentially stable semigroup. Similarly, (A, C) is exponentially detectable if and only if there exists some $L \in \mathcal{L}(\mathcal{H}, \mathcal{U})$ such that $A + LC$ generates an exponentially stable semigroup. See, e.g., [6, Part V, Chapter 1, Remark 3.2].*

Let the operator $[A\ B] : D(A) \times \mathcal{U} \to \mathcal{H}$ be defined by

$$[A\ B] \begin{bmatrix} x \\ u \end{bmatrix} := Ax + Bu.$$

Notice that, since A and B are both closed operators, $\ker[A\ B]$ is closed in $\mathcal{H} \times \mathcal{U}$.

We denote by $\begin{bmatrix} A^* \\ B^* \end{bmatrix}$ the adjoint operator of $[A\ B]$. It's easy to check that the domain of $\begin{bmatrix} A^* \\ B^* \end{bmatrix}$ is $D(A^*)$ and we have

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} w = \begin{bmatrix} A^* w \\ B^* w \end{bmatrix} \in \mathcal{H} \times \mathcal{U} , \quad \forall w \in D(A^*).$$

3.3 Sufficient condition for the turnpike property

Our main result of this section is the following one:

Theorem 3.3.1. *If the pair (A, B) is exponentially stabilizable and the pair (A, C) is exponentially detectable, then problem $(GLQ)_T$ shows exponential turnpike property (thus also measure turnpike property) at some steady state (x_e, u_e).*

The proof is lengthy and thus divided into several steps.

Step 1: Exponential convergence of P to $P_{\min}$.

As the first step, we will show that $P(\cdot)$ converges exponentially to $P_{\min}$ in norm based on our stabilizability and detectability assumption.

Step 2: Range condition and existence of the optimal adjoint state

In this step, we will prove that $\operatorname{ran}[A\ \ B] = \mathcal{H}$, which helps us to establish the existence and uniqueness of the optimal steady state and the corresponding optimal adjoint state.

Step 3: Several estimates

In this step, we will make some simplifications to $(GLQ)_T$ and prove two crucial estimates regarding the optimally controlled system of $(LQ)_T$.

Step 4: Explicit solution of the optimal control

We will derive the optimal control of $(GLQ)_T$ in a closed form. The optimal trajectory can then be solved as the solution of a corresponding evolution problem.

Step 5: Main proof

Finally, we will complete the proof of Theorem 3.3.1 by means of a comparison between the cost corresponding to the optimal control and a perturbed control.

Each of the subsections that follow corresponds to one step in the proof. Until the end of this section, we assume that the pair (A, B) is exponentially stabilizable and the pair (A, C) is exponentially detectable.

3.3.1 Exponential convergence of P to $P_{\min}$

The proof of our main result crucially relies on the exponential convergence in norm of $P(\cdot)$ to $P_{\min}$. Different from the finite dimensional case, this convergence result turns out to be a consequence of both the stabilizability and detectability of the system. This section is devoted to the proof of this result, preceded by two preliminaries lemmas.

Lemma 3.3.2. *There exists a constant $M > 0$ such that for every $T > 0$, $u \in L^2((0,T),\mathcal{U})$ and $x_0 \in \mathcal{H}$, we have*

$$\|x(T)\|^2 \leq M\left(\int_0^T \|Cx(t)\|^2 + \|u(t)\|^2 dt + \|x_0\|^2\right), \tag{3.5}$$

where x is the solution of system (2.3).

Proof. Since the pair (A,C) is exponentially detectable, there exists some $F \in \mathcal{L}(\mathcal{Y},\mathcal{H})$ such that $A + FC$ is exponentially stable. Let $\phi_0 \in D(A^*)$ and ϕ be the solution of problem

$$\begin{cases} \dfrac{d\phi}{dt} = (A^* + C^*F^*)\phi, \\ \phi(0) = \phi_0. \end{cases}$$

Owing to the exponential stability of $A^* + C^*F^*$, there exists some constant $M_0 > 0$ such that for any $T > 0$, we have

$$\|\phi(T)\| \leq M_0\|\phi_0\|$$

and

$$\|\phi(\cdot)\|_{L^2((0,T),\mathcal{H})} \leq M_0\|\phi_0\|.$$

Notice that

$$\begin{aligned} \frac{d\left\langle x(t), \phi(T-t)\right\rangle}{dt} &= \langle x(t), -(A^* + C^*F^*)\phi(T-t)\rangle + \langle Ax(t) + Bu(t), \phi(T-t)\rangle_{\mathcal{H}_{-1},\mathcal{H}_1^d} \\ &= -\langle Cx(t), F^*\phi(T-t)\rangle + \langle Bu(t), \phi(T-t)\rangle. \end{aligned}$$

Applying Hölder's inequality, we obtain

$$\begin{aligned} \langle x(T), \phi_0\rangle =& \langle x_0, \phi(T)\rangle + \int_0^T -\langle Cx(t), F^*\phi(T-t)\rangle + \langle Bu(t), \phi(T-t)\rangle\, dt \\ \leq& \|x_0\|\,\|\phi(T)\| + \|Cx\|_{L^2}\,\|F^*\phi\|_{L^2} + \|Bu\|_{L^2}\|\phi\|_{L^2} \\ \leq& M_1\|x_0\|\|\phi_0\| + M_2\|Cx\|_{L^2}\|\phi_0\| + M_3\|u\|_{L^2}\|\phi_0\| \\ \leq& \sqrt{M}\|\phi_0\|\left(\|Cx\|_{L^2}^2 + \|u\|_{L^2}^2 + \|x_0\|^2\right)^{\frac{1}{2}} \end{aligned}$$

for some constants M_1, M_2, M_3 and $M > 0$.

Now, by letting $\phi_0 \to x(T)$ in $\mathcal{H}$, we get the estimate (3.5). $\square$

Lemma 3.3.3. *There exists a constant $M > 0$ such that for every $T > 0$, $f \in L^2((0,T),\mathcal{Y})$ and $p_0 \in \mathcal{H}$, we have*

$$\|p(T)\|^2 \le M\left(\int_0^T \|B^*p(t)\|^2 + \|f(t)\|^2 dt + \|p_0\|^2\right), \tag{3.6}$$

where p is the solution of

$$\begin{cases} \dfrac{dp}{dt} = A^*p + C^*f, \\ p(0) = p_0. \end{cases}$$

This is just the dual version of Lemma 3.3.2, so we skip the proof.

Lemma 3.3.4 (Exponential convergence rate of P)**.** *There exist constants $M, \beta > 0$ such that*

$$\|P_{\min} - P(t)\| \le Me^{-\beta t}, \quad \forall t \ge 0. \tag{3.7}$$

Proof. Fix some $T > 0$ and $x_0 \in \mathcal{H}$. We denote by x the solution of

$$\begin{cases} \dfrac{dx}{dt} = (A - BB^*P_{\min})x \quad \text{in } [0,T], \\ x(0) = x_0. \end{cases} \tag{3.8}$$

Define $p(t) := P_{\min}x(T-t)$ on $[0,T]$. We claim that p coincides with the solution of

$$\begin{cases} \dfrac{dy(t)}{dt} = A^*y(t) + C^*Cx(T-t) \quad \text{in } [0,T], \\ y(0) = P_{\min}x(T). \end{cases} \tag{3.9}$$

In fact, suppose $x_0 \in D(A)$, then for any $w \in D(A)$, we have

$$\begin{aligned} \frac{d\langle p(t), w\rangle}{dt} =& -\langle P_{\min}Ax(T-t), w\rangle + \langle P_{\min}BB^*P_{\min}x(T-t), w\rangle \\ =& \langle A^*P_{\min}x(T-t) + C^*Cx(T-t), w\rangle_{\mathcal{H}^d_{-1},\mathcal{H}_1} \\ =& \langle A^*p(t) + C^*Cx(T-t), w\rangle_{\mathcal{H}^d_{-1},\mathcal{H}_1}. \end{aligned}$$

This implies that

$$\begin{aligned}\langle p(t)-p(0), w\rangle_{\mathcal{H}_{-1}^d,\mathcal{H}_1} &= \int_0^t \langle A^*p(s)+C^*Cx(T-s), w\rangle_{\mathcal{H}_{-1}^d,\mathcal{H}_1} ds \\ &= \left\langle \int_0^t A^*p(s)+C^*Cx(T-s)ds, w \right\rangle_{\mathcal{H}_{-1}^d,\mathcal{H}_1}\end{aligned}$$

holds for any $t \in [0,T]$ and $w \in \mathcal{H}_1$. So,

$$p(t)-p(0) = \int_0^t A^*p(s)+C^*Cx(T-s)ds \ \text{ in } \ \mathcal{H}_{-1}^d, \ \ \forall t \in [0,T].$$

By the continuity of $A^*p(\cdot)+C^*Cx(T-\cdot)$ in $\mathcal{H}_{-1}^d$, we have

$$\frac{dp(t)}{dt} = A^*p(t)+C^*Cx(T-t) \ \text{ in } \ \mathcal{H}_{-1}^d, \ \ \forall t \in [0,T].$$

Since $p(0)=y(0)$, we conclude that $p=y$ on $[0,T]$.

Now, for any $x_0 \in \mathcal{H}$, we let $(z_n)_{n\in\mathbb{N}} \subset D(A)$ be a sequence such that $\lim_{n\to\infty} z_n = x_0$ in $\mathcal{H}$. Let x_n and y_n denote the solution of (3.8) and (3.9) with x_0 replaced by z_n, then $x_n \to x$ in L^2 norm and $P_{\min}x_n(T) \to P_{\min}x(T)$ as $n\to\infty$. So, $y_n(\cdot)$ converges to $y(\cdot)$ pointwisely on $[0,T]$. It is also clear that $y_n(\cdot) = P_{\min}x_n(T-\cdot) \to P_{\min}x(T-\cdot) = p(\cdot)$ pointwisely on $[0,T]$. So, we must have that $p=y$ on $[0,T]$.

Consider the following evolution problem

$$\begin{cases} \dfrac{d\widetilde{x}(t)}{dt} = (A-BB^*P(T-t))\widetilde{x}(t) \ \text{ in } [0,T], \\ \widetilde{x}(0)=x_0. \end{cases} \tag{3.10}$$

Notice that $P(T-\cdot) \in C_s([0,T], L(\mathcal{H}))$. Thanks to Proposition 2.1.11, problem (3.10) admits a unique solution $\widetilde{x} \in H^1((0,T),\mathcal{H}^{-1}) \cap C([0,T],\mathcal{H})$. Define $\widetilde{p}(t) := P(t)\widetilde{x}(T-t)$ on $[0,T]$. Similarly, we claim that $\widetilde{p}$ coincides with the solution of

$$\begin{cases} \dfrac{d\widetilde{y}(t)}{dt} = A^*\widetilde{y}(t)+C^*C\widetilde{x}(T-t) \ \text{ in } [0,T], \\ \widetilde{y}(0)=0. \end{cases} \tag{3.11}$$

In fact, if $x_0 \in D(A)$, then by Proposition 2.1.11 and Proposition 2.2.4, we have

$$\widetilde{x} \in C^1([0,T],\mathcal{H}) \cap C([0,T],D(A)). \tag{3.12}$$

Let $t \in [0,T]$ and $h \in \mathbb{R}$ be sufficiently small. Observe that

$$\frac{\widetilde{p}(t+h)-\widetilde{p}(t)}{h} = P(t+h)\left(\frac{\widetilde{x}\,(T-(t+h))-\widetilde{x}(T-t)}{h} + (A-BB^*P(T-t))\widetilde{x}(T-t)\right) \\ -P(t+h)(A-BB^*P(T-t))\widetilde{x}(T-t) + \frac{P(t+h)-P(t)}{h}\widetilde{x}(T-t).$$

By uniform boundedness principle applied to P, (3.10), (3.12) and Proposition 2.2.4, taking $h \to 0$, we then obtain that

$$\begin{aligned}\frac{d\widetilde{p}(t)}{dt} &= -P(t)(A-BB^*P(t))\widetilde{x}(T-t) \\ &\qquad + (\mathcal{A}(P(t)) - P(t)BB^*P(t) + C^*C)\widetilde{x}(T-t) \\ &= (\mathcal{A}(P(t)) - P(t)A)\widetilde{x}(T-t) + C^*C\widetilde{x}(T-t) \\ &= A^*P(t)\widetilde{x}(T-t) + C^*C\widetilde{x}(T-t) \\ &= A^*\widetilde{p}(t) + C^*C\widetilde{x}(T-t)\end{aligned}$$

holds in $\mathcal{H}_d^{-1}$. Since $\widetilde{p}(0)=0$, this proves our claim for the case $x_0 \in D(A)$.

Now, for any $x_0 \in \mathcal{H}$, we let $(\widetilde{z}_n)_{n\in\mathbb{N}} \subset D(A)$ be a sequence such that $\widetilde{z}_n \to x_0$ in $\mathcal{H}$ as $n \to \infty$. We use $\widetilde{x}_n$ and $\widetilde{y}_n$ to denote the solution of (3.10) and (3.11) with x_0 replaced by $\widetilde{z}_n$. By [6, Proposition 3.6], $\widetilde{x}_n \to \widetilde{x}$ both uniformly and in L^2 norm, so $\widetilde{y}_n(\cdot)$ converges to $y(\cdot)$ pointwisely on $[0,T]$. On the other hand, it is also clear that $\widetilde{y}_n(\cdot)$ converges to $\widetilde{p}(\cdot)$ pointwisely on $[0,T]$. So we conclude that $\widetilde{y} = \widetilde{p}$ on $[0,T]$.

Now we claim that the following inequality holds with any $x_0 \in \mathcal{H}$:

$$\int_0^T \|B^*(p(t)-\widetilde{p}(t))\|^2 + \|C(x(t)-\widetilde{x}(t))\|^2 dt \leq \|p(0)\|\|\widetilde{x}(T)-x(T)\|\,, \tag{3.13}$$

where $x, p, \widetilde{x}$ and $\widetilde{p}$ is the solution of (3.8), (3.9), (3.10) and (3.11), respectively. To prove this, we first assume that $x_0 \in D(A)$. Since $x,\ \widetilde{x} \in C^1([0,T],\mathcal{H}) \cap C([0,T],D(A))$, we have

$$\begin{aligned}&\frac{d\langle \widetilde{p}(t)-p(t), \widetilde{x}(T-t)-x(T-t)\rangle}{dt} \\ &\quad= -\langle \widetilde{p}(t)-p(t), A(\widetilde{x}(T-t)-x(T-t))\rangle + \langle \widetilde{p}(t)-p(t), BB^*(\widetilde{p}(t)-p(t))\rangle \\ &\qquad\quad + \langle A^*(\widetilde{p}(t)-p(t)), \widetilde{x}(T-t)-x(T-t)\rangle_{\mathcal{H}^d_{-1},\mathcal{H}_1} \\ &\qquad\quad + \langle C^*C(\widetilde{x}(T-t)-x(T-t)), \widetilde{x}(T-t)-x(T-t)\rangle \\ &\quad= \|C(x(T-t)-\widetilde{x}(T-t))\|^2 + \|B^*(p(t)-\widetilde{p}(t))\|^2, \qquad \forall t \in [0,T].\end{aligned}$$

So, we deduce that

$$\begin{aligned}\int_0^T & \|B^*(p(t)-\widetilde{p}(t))\|^2+\|C(x(t)-\widetilde{x}(t))\|^2dt \\ &= \langle \widetilde{p}(T)-p(T),\widetilde{x}(0)-x(0)\rangle - \langle \widetilde{p}(0)-p(0),\widetilde{x}(T)-x(T)\rangle \\ &= \langle p(0),\widetilde{x}(T)-x(T)\rangle \\ &\leq \|p(0)\|\|\widetilde{x}(T)-x(T)\| \ .\end{aligned} \tag{3.14}$$

Since for any $x_0 \in \mathcal{H}$, there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset D(A)$ such that $z_n \to x_0$ in $\mathcal{H}$ as $n \to \infty$, and the resulting trajectories $x_n, p_n, \widetilde{x}_n$ and $\widetilde{p}_n$ will converge to $x, p, \widetilde{x}$ and $\widetilde{p}$ both pointwisely and in L^2 norm, we conclude that equation (3.13) holds for any $x_0 \in \mathcal{H}$.

Notice that

$$\frac{d(\widetilde{x}(t)-x(t))}{dt} = A(\widetilde{x}(t)-x(t)) - BB^*(\widetilde{p}(T-t)-p(T-t)) \ \ \forall t\in[0,T].$$

By Lemma 3.3.2, there exists some constant $M_1 > 0$ such that

$$\|\widetilde{x}(T)-x(T)\|^2 \leq M_1\int_0^T \|B^*(p(t)-\widetilde{p}(t))\|^2+\|C(x(t)-\widetilde{x}(t))\|^2dt.$$

Combining this with (3.14), we obtain

$$\int_0^T \|B^*(p(t)-\widetilde{p}(t))\|^2+\|C(x(t)-\widetilde{x}(t))\|^2dt \leq M\|p(0)\|^2. \tag{3.15}$$

Similarly, notice that

$$\frac{d(\widetilde{p}(t)-p(t))}{dt} = A^*(\widetilde{p}(t)-p(t)) - C^*C(\widetilde{x}(T-t)-x(T-t)), \ \ \forall t\in[0,T].$$

By Lemma 3.3.3 there exists some $M_2 > 0$ such that

$$\|\widetilde{p}(T)-p(T)\|^2 \leq M_2\left(\int_0^T \|B^*(p(t)-\widetilde{p}(t))\|^2+\|C(x(t)-\widetilde{x}(t))\|^2dt+\|p(0)\|^2\right). \tag{3.16}$$

Since (A,C) is exponentially detectable, $A - BB^*P_{\min}$ generates an exponentially stable semigroup. So, there exists $M_3, \beta > 0$ such that

$$\|p(0)\| = \|P_{\min}x(T)\| \leq M_3 e^{-\beta T}\|x_0\|. \tag{3.17}$$

Substituting (3.15) and (3.17) into (3.16), it follows that there exists some $M > 0$ (independent of T) such that

$$\|\widetilde{p}(T)-p(T)\|^2 \leq Me^{-2\beta T}\|x_0\|^2, \ \ \forall x_0\in\mathcal{H}, T>0.$$

Finally, since $\|\widetilde{p}(T)-p(T)\| = \|\left(P_{\min}-P(T)\right)x_0\|$, this lemma then follows. $\square$

3.3.2 Range condition and existence of the optimal adjoint state

In this subsection, we introduce a closed range condition to show the existence and uniqueness of the optimal steady state and its corresponding optimal adjoint state.

Lemma 3.3.5 (Range condition)**.** $\operatorname{ran}[A\ B] = \mathcal{H}$.

Proof. Since the pair (A, B) is exponentially stabilizable, there exists $F \in \mathcal{L}(\mathcal{H},\mathcal{U})$ such that $A+BF$ generates an exponentially stabilizable semigroup $\widetilde{\mathcal{T}}$ with $D(A+BF) = D(A)$. Notice that for any $x_0 \in \mathcal{H}$ and $t \in [0, \infty)$,

$$\widetilde{\mathcal{T}}_t x_0 - x_0 = (A + BF)\int_0^t \widetilde{\mathcal{T}}_s x_0 ds.$$

Taking $t \to \infty$, we obtain

$$-x_0 = (A + BF)\int_0^\infty \widetilde{\mathcal{T}}_s x_0 ds = A\int_0^\infty \widetilde{\mathcal{T}}_s x_0 ds + B\int_0^\infty F\widetilde{\mathcal{T}}_s x_0 ds.$$

Notice that since $\widetilde{\mathcal{T}}$ is exponentially stable, the term $\int_0^\infty \widetilde{\mathcal{T}}_s x_0 ds$ is well defined. From the above equation, we easily deduce that $-x_0 \in \operatorname{ran}[A\ B]$. Now, since $x_0 \in \mathcal{H}$ can be chosen arbitrarily, we conclude that $\operatorname{ran}[A\ B] = \mathcal{H}$. $\square$

Lemma 3.3.6 (Existence of the optimal steady state and adjoint state)**.** *The optimal steady state problem* (3.4) *admits a unique minimizer* (x_e, u_e)*. Moreover, there exists a* $w \in D(A^*)$ *such that*

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} w = \begin{bmatrix} z + C^*Cx_e \\ v + K^*Ku_e \end{bmatrix}. \tag{3.18}$$

Proof. Let $V = \ker[A, B]$. Since A and B are both closed operator, V is a closed subspace of $\mathcal{H} \times \mathcal{U}$. Observe that if $(x, u) \in V$, then

$$\ell\,(x, u) = \left\langle \mathbb{P}_V \begin{bmatrix} C^*C & 0 \\ 0 & K^*K \end{bmatrix}\Bigg|_V \begin{bmatrix} x \\ u \end{bmatrix} + 2\mathbb{P}_V \begin{bmatrix} z \\ v \end{bmatrix}, \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle.$$

We claim that the non-negative operator $\mathcal{P} \in \mathcal{L}(V)$ defined by

$$\mathcal{P} := \mathbb{P}_V \begin{bmatrix} C^*C & 0 \\ 0 & K^*K \end{bmatrix}\Bigg|_V$$

is strictly positive. In fact, if $\mathcal{P}$ is not strictly positive, then there exists a sequence

$$\left(\begin{bmatrix} x_n \\ u_n \end{bmatrix}\right)_{n\in\mathbb{N}} \subset V$$

such that $\|x_n\|^2 + \|u_n\|^2 = 1$ for all $n \in \mathbb{N}$ and

$$\left\langle \mathcal{P}\begin{bmatrix} x_n \\ u_n \end{bmatrix}, \begin{bmatrix} x_n \\ u_n \end{bmatrix}\right\rangle = \|Cx_n\|^2 + \|Ku_n\|^2 \to 0 \text{ as } n \to \infty.$$

This implies that $Cx_n \to 0$ and $Ku_n \to 0$. Since K is coercive, we obtain $u_n \to 0$, and thus $Ax_n = -Bu_n \to 0$.

Recall that if (A, C) is exponentially detectable, then there exists $F \in \mathcal{L}(\mathcal{Y},\mathcal{U})$ such that $A + FC$ generates an exponentially stable semigroup. By [11, Theorem 5.1.3], there exists a $P \in \Sigma^+(H)$ such that

$$2\operatorname{Re}\langle (A + FC)x, Px\rangle \le -\|x\|^2,\ \forall x \in D(A).$$

Substituting x_n into the above estimate, simple calculation shows

$$\|x_n\| \le 2\|P\|\|(A + FC)x_n\|,\ \forall n \in \mathbb{N}.$$

Letting $n \to \infty$, since $(A + FC)x_n \to 0$, we obtain that $x_n \to 0$. However, this contradicts the fact that $u_n \to 0$ and $\|x_n\|^2 + \|u_n\|^2 = 1$.

On the other hand, since $\mathcal{P}$ is strictly positive, it follows that for any $(x, u) \in V$,

$$\ell(x,u) = \left\langle \mathcal{P}\begin{bmatrix} x \\ u \end{bmatrix} + 2\mathbb{P}_V\begin{bmatrix} z \\ v \end{bmatrix}, \begin{bmatrix} x \\ u \end{bmatrix}\right\rangle = \left\|\mathcal{P}^{-\frac{1}{2}}\left(\mathcal{P}\begin{bmatrix} x \\ u \end{bmatrix} + \mathbb{P}_V\begin{bmatrix} z \\ v \end{bmatrix}\right)\right\|^2 - \left\|\mathcal{P}^{-\frac{1}{2}}\mathbb{P}_V\begin{bmatrix} z \\ v \end{bmatrix}\right\|^2.$$

So, the unique minimizer (x_e, u_e) of the optimal steady state problem (3.4) is characterized by

$$\mathcal{P}\begin{bmatrix} x_e \\ u_e \end{bmatrix} + \mathbb{P}_V\begin{bmatrix} z \\ v \end{bmatrix} = \mathbb{P}_V\left(\begin{bmatrix} C^*C & 0 \\ 0 & K^*K \end{bmatrix}\Big|_V \begin{bmatrix} x_e \\ u_e \end{bmatrix} + \begin{bmatrix} z \\ v \end{bmatrix}\right) = 0.$$

This is further equivalent to

$$\begin{bmatrix} z + C^*Cx_e \\ v + K^*Ku_e \end{bmatrix} = \begin{bmatrix} C^*C & 0 \\ 0 & K^*K \end{bmatrix}\Big|_V \begin{bmatrix} x_e \\ u_e \end{bmatrix} + \begin{bmatrix} z \\ v \end{bmatrix} \in V^\perp = \overline{\operatorname{ran}}\begin{bmatrix} A^* \\ B^* \end{bmatrix}.$$

Finally, by Lemma 3.3.5 and closed range theorem,

$$\begin{bmatrix} z + C^*Cx_e \\ v + K^*Ku_e \end{bmatrix} \in \overline{\operatorname{ran}}\begin{bmatrix} A^* \\ B^* \end{bmatrix} = \operatorname{ran}\begin{bmatrix} A^* \\ B^* \end{bmatrix},\ \ \ker\begin{bmatrix} A^* \\ B^* \end{bmatrix} = \operatorname{ran}[A\ B]^\perp = \{0\}.$$

Hence we conclude that there exists a unique $w \in D(A^*)$ such that

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} w = \begin{bmatrix} z + C^*Cx_e \\ v + K^*Ku_e \end{bmatrix}.$$

□

Remark 3.3.7. *The vector w defined in the above lemma is called the optimal adjoint state of problem $(GLQ)_T$ since it can be seen as the infinite dimensional analogy of the Lagrange multiplier of the optimal steady state problem* (3.4)*. This vector can be utilized to obtain the following transform: Recall if x is the solution of* (2.3) *corresponding to input $u \in L^2((0,T),\mathcal{U})$ and initial condition $x_0 \in \mathcal{H}$, then*

$$\begin{aligned} &\int_0^T \langle z, x(t)\rangle + \langle v, u(t)\rangle dt \\ &\qquad = \int_0^T \langle w, Ax(t) + B^*u(t)\rangle_{\mathcal{H}_1^d,\mathcal{H}_{-1}} - \langle Cx_e, Cx(t)\rangle - \langle Ku_e, Ku(t)\rangle dt \\ &\qquad = \langle w, x(T) - x_0\rangle + \int_0^T -\langle Cx_e, Cx(t)\rangle - \langle Ku_e, Ku\rangle dt. \end{aligned}$$

Now combining this with (2.5) *gives that*

$$\begin{aligned} J_T(x_0, u) = &\int_0^T \|K\{(u(t) - u_e) + (K^*K)^{-1}B^*P(T-t)(x(t) - x_e)\}\|^2 dt \\ &- J_T(x_e, u_e) + \langle P(T)(x_0 - x_e), (x_0 - x_e)\rangle + 2\,\mathrm{Re}\langle w, x(T) - x_0\rangle. \end{aligned} \tag{3.19}$$

3.3.3 Several estimates

This subsection is mainly devoted to the proof of two crucial estimates. However, before we introduce the estimates, we will first make some simplifications to problem $(GLQ)_T$, and these simplifications will be adopted in the remaining part of section 3.3.

First, observe that, without loss of generality, we can assume that $K = I$. Otherwise, we may define a new inner product $\langle\cdot,\cdot\rangle_{\text{new}}$ on $\mathcal{U}$ by

$$\langle u_1, u_2\rangle_{\text{new}} = \langle (K^*K)^{\frac{1}{2}}u_1, (K^*K)^{\frac{1}{2}}u_2\rangle,\ \forall u_1, u_2 \in \mathcal{U}.$$

We now endow $\mathcal{U}$ with the new inner product $\langle\cdot,\cdot\rangle_{\text{new}}$. Since the norm induced by this inner product is equivalent to the standard norm in $\mathcal{U}$, we still have that $B \in \mathcal{L}(\mathcal{U},\mathcal{H})$ and the pair (A, B) is exponentially stabilizable. The running cost $\ell : \mathcal{H} \times \mathcal{U} \to \mathbb{R}$ shall now be recast as

$$\ell(x, u) := \|Cx\|^2 + \|u\|^2_{\text{new}} + 2\,\mathrm{Re}\langle z, x\rangle + 2\,\mathrm{Re}\langle (K^*K)^{-1}v, u\rangle_{\text{new}}$$

where $\|\cdot\|_{\text{new}}$ is the norm on $\mathcal{U}$ induced by $\langle\cdot,\cdot\rangle_{\text{new}}$.

We will also assume that the unique optimal steady state $(x_e, u_e) = (0,0)$ in the following subsections, thanks to the next result.

Proposition 3.3.8. *$(x_e, u_e) \in D(A)\times\mathcal{U}$ is the unique optimal steady state of the optimal steady state problem (3.4) if and only if $(0,0)$ is the unique optimal steady state of the modified optimal steady state problem*

$$\min_{x\in D(A),\, u\in\mathcal{U}} \widetilde{\ell}(x,u) \quad \textit{s.t.} \quad Ax + Bu = 0, \tag{3.20}$$

where $\widetilde{\ell} : \mathcal{H}\times\mathcal{U}\to\mathbb{R}$ is defined by

$$\widetilde{\ell}(x,u) := \|Cx\|^2 + \|Ku\|^2 + 2\operatorname{Re}\langle z + C^*Cx_e, x\rangle + 2\operatorname{Re}\langle v + K^*Ku_e, u\rangle\ .$$

In this case, problem $(GLQ)_T$ satisfies the (measure or exponential) turnpike property at (x_e, u_e) if and only if the problem

$$\min_{u\in L^2((0,T),\mathcal{U})} \int_0^T \widetilde{\ell}(x(t),u(t))\,dt \quad \text{s.t. } \dot{x} = Ax + Bu, \quad x(0) = x_0 \in \mathcal{H}$$

satisfies the (measure or exponential) turnpike property at $(0,0)$.

Proof. For any $x\in\mathcal{H}$ and $u\in\mathcal{U}$, set $\widetilde{x} := x - x_e$ and $\widetilde{u} := u - u_e$. If (x_e,u_e) is the unique minimizer of the optimal steady state problem (3.4), we easily see that $(0,0)$ is the minimizer of

$$\min_{\widetilde{x}\in D(A),\,\widetilde{u}\in\mathcal{U}} \ell(\widetilde{x}+x_e, \widetilde{u}+u_e) \qquad \text{s.t. } A\widetilde{x} + B\widetilde{u} = 0$$

and vice versa. Besides, it's not hard to see that $\ell(\widetilde{x}+x_e,\widetilde{u}+u_e)$ only differs from $\widetilde{\ell}(\widetilde{x},\widetilde{u})$ by a constant term. So, $(0,0)$ is the unique minimizer of problem (3.20).

Now, let (x^*,u^*) denote the optimal pair for problem $(GLQ)_T$. Observe that

$$\frac{d\widetilde{x}}{dt} = \frac{d(x-x_e)}{dt} = Ax + Bu = A\widetilde{x} + B\widetilde{u}.$$

So, $(\widetilde{x}^*,\widetilde{u}^*) := (x^*-x_e, u^*-u_e)$ is the unique optimal pair of problem

$$\min_{\widetilde{u}\in L^2((0,T),\mathcal{U})} \int_0^T \ell(\widetilde{x}(t)+x_e,\widetilde{u}(t)+u_e)\,dt \ \text{ s.t. } \dot{\widetilde{x}} = A\widetilde{x} + B\widetilde{u}, \quad \widetilde{x}(0) = \widetilde{x}_0 := x_0 - x_e. \tag{3.21}$$

As mentioned before, $\ell(\widetilde{x}+x_e,\widetilde{u}+u_e)$ only differs from $\widetilde{\ell}(\widetilde{x},\widetilde{u})$ by a constant term. So, if (x^*,u^*) is the optimal pair for problem $(GLQ)_T$, then (x^*-x_e,u^*-u_e) is the optimal pair for problem (3.21) and vice versa. Therefore, we deduce that problem $(GLQ)_T$ satisfies the (measure or exponential) turnpike property at (x_e,u_e) if and only if problem (3.21) satisfies the (measure or exponential) turnpike property at $(0,0)$. $\square$

We now introduce a class of evolution operators that is of particular interest.

Assume that τ and T satisfies $0 \le \tau < T$. Let $U_T(\cdot, \tau) : [\tau, T] \to \mathcal{L}(\mathcal{H})$ denote the evolution operator of the following problem:

$$\dot{x}(t) = (A - BB^*P(T-t))x(t), \quad x(\tau) = x_0 \in \mathcal{H}, \quad t \in [\tau, T].$$

That is, $U_T(\cdot, \tau)$ is defined by setting

$$U_T(t, \tau)x_0 := x(t), \quad \forall x_0 \in \mathcal{H}, t \in [\tau, T],$$

where x is the solution of the above evolution problem.

Next, we prove two crucial properties of the evolution operator U_T.

Lemma 3.3.9 (Exponential convergence of U_T)**.** *There exist some positive constants M and k such that*

$$\|U_T(t, \tau)\| \le Me^{-k(t-\tau)}, \ \ \forall t \in [\tau, T] \tag{3.22}$$

holds for any τ and T satisfying $0 \le \tau < T$.

Proof. Suppose $\tau, T \in \mathbb{R}$ satisfies that $0 \le \tau < T$. By Proposition 2.2.7, there exist some positive constants M_0 and k_0 such that

$$\|P(t) - P_{\min}\| \le M_0 e^{-k_0 t}, \ \ \forall t \in [0, \infty).$$

Fix some $x_0 \in \mathcal{H}$. Let $y : [\tau, \infty) \to \mathcal{H}$ be defined by

$$y(t) := e^{k_1(t-\tau)}U_T(t, \tau)x_0, \ \ t \in [\tau, \infty)$$

with some sufficiently small $k_1 > 0$ so that $A - BB^*P_{\min} + k_1 I$ still generates an exponentially stable semigroup, denoted $\widetilde{\mathcal{T}}$.

For any $t \in [\tau, T]$, a straightforward computation leads to

$$\begin{aligned}\dot{y}(t) &= (A - BB^*P_{\min} + k_1 I)y(t) + (BB^*P_{\min} - BB^*P(T-t))y(t)\\ y(t) &= \widetilde{\mathcal{T}}_{t-\tau}y(\tau) + \int_\tau^t \widetilde{\mathcal{T}}_{t-s}(BB^*P_{\min} - BB^*P(T-s))y(s)ds.\end{aligned}$$

So, there exist constants M_1 and $M_2 > 0$ such that,

$$\begin{aligned}\|y(t)\| &\le M_1\|y(\tau)\| + \int_\tau^t M_2 e^{-k_0(T-s)}\|y(s)\|ds\\ &\le M_1\|x_0\| + M_2 e^{-k_0(T-t)}\int_\tau^t \|y(s)\|ds.\end{aligned} \tag{3.23}$$

Now fix some constant $S > 0$ such that $M_2 e^{-k_0 S} < k_1$.

We first discuss the case $\tau \geq T - S$. Referring to (3.23), we obtain that

$$\|y(t)\| \leq M_1 \|x_0\| + M_2 \int_\tau^t \|y(s)\| ds.$$

Since $t - \tau \leq S$, applying Grönwall's inequality to the above equation, we get

$$\|y(t)\| \leq M_1 \|x_0\| e^{M_2(t-\tau)} \leq M_1 e^{M_2 S} \|x_0\|.$$

It follows that

$$\|U_T(t,\tau)x_0\| \leq \|y(t)\| \leq M_1 e^{(M_2+1)S} e^{-(t-\tau)} \|x_0\|,$$

which proves condition (3.22) with suitable coefficients.

We now consider the case $\tau < T - S$ and $t \in [\tau, T - S]$. From (3.23) we obtain

$$\|y(t)\| \leq M_1 \|x_0\| + M_2 e^{-k_0 S} \int_\tau^t \|y(s)\| ds.$$

Applying Grönwall's inequality to the above equation, we get

$$\|y(t)\| \leq M_1 \|x_0\| e^{M_2 e^{-k_0 S}(t-\tau)}.$$

Thus

$$\|U_T(t,\tau)x_0\| = e^{-k_1(t-\tau)} \|y(t)\| \leq M_1 \|x_0\| e^{(M_2 e^{-k_0 S} - k_1)(t-\tau)},$$

where $M_2 e^{-k_0 S} - k_1 < 0$. Hence we conclude (3.22) is satisfied with suitable coefficients also in this case.

It remains to prove (3.22) for the case $\tau < T - S$ and $t \in (T - S, T]$. From the definition of U_T, we have

$$U_T(t,\tau) = U_{T-\tau}(t-\tau, 0) = U_S(S - T + t, 0) U_{T-\tau}(T - S - \tau, 0). \tag{3.24}$$

Notice that $U_S(S - T + t, 0)$ satisfies the estimate for the first case, so we have

$$\|U_S(S - T + t, 0)\| \leq M_1 e^{(M_2+1)S} e^{-(S-T+t)} \|x_0\| \leq M_1 e^{(M_2+1)S} \|x_0\|.$$

It is also clear $U_{T-\tau}(T - S - \tau, 0)$ satisfies the estimate for the second case, so

$$\|U_{T-\tau}(T - S - \tau, 0)\| \leq M_1 e^{(M_2 e^{-k_0 S} - k_1)(T-S-\tau)} \|x_0\| \leq M_1 e^{(M_2 e^{-k_0 S} - k_1)(t-S-\tau)} \|x_0\|.$$

Combining the above two estimates with (3.24), we deduce that (3.22) is satisfied also in this case with suitable coefficients M and k (independent of T).

Since $t \in [\tau, T]$ is arbitrary, this lemma follows by choosing the largest M and the smallest k among all the three cases. $\square$

Let the operator $\widetilde{\Phi}_t^T : L^2((0,t),\mathcal{U}) \to \mathcal{H}$ be defined by

$$\widetilde{\Phi}_t^T \widetilde{u} := \int_0^t U_T(t,s)B\widetilde{u}(s)ds, \quad \forall T > 0, 0 \le t < T.$$

It is clear $\widetilde{\Phi}_t^T \in \mathcal{L}(L^2((0,t),\mathcal{U}),\mathcal{H})$.

Lemma 3.3.10 (Uniform boundedness of $\widetilde{\Phi}_t^T$)**.** *The operator $\widetilde{\Phi}_t^T$ is uniformly bounded in norm for all any t and T satisfying $0 \le t < T$.*

Proof. By Lemma 3.3.9 and Hölder's inequality, we obtain

$$\begin{aligned}
\|\widetilde{\Phi}_t^T \widetilde{u}\| &\le \int_0^t \|U_T(t,s)\| \, \|B\widetilde{u}(s)\| \, ds \\
&\le \left(\int_0^t M^2 e^{-2k(t-s)} ds \right)^{\frac{1}{2}} \left(\int_0^t \|B\|^2 \|\widetilde{u}(s)\|^2 ds \right)^{\frac{1}{2}} \\
&\le M\|B\| \left(\frac{1 - e^{-2kt}}{2k} \right)^{\frac{1}{2}} \|\widetilde{u}\| \le M\|B\| \left(\frac{1}{2k} \right)^{\frac{1}{2}} \|\widetilde{u}\|
\end{aligned}$$

where M and k are the coefficients in Lemma 3.3.9. Simple considerations to this equation show that $\widetilde{\Phi}_t^T$ is uniformly bounded in norm for any t and T satisfying $0 \le t \le T$. $\square$

3.3.4 Explicit solution of the optimal control

In this subsection, the optimal control of problem $(GLQ)_T$ is solved in a closed form.

Lemma 3.3.11. *For any $T > 0$ and $x_0 \in \mathcal{H}$, the optimal pair $(x_T^*(\cdot, x_0), u_T^*(\cdot, x_0))$ of problem $(GLQ)_T$ satisfies*

$$\begin{aligned}
u_T^*(t,x_0) - u_e = &-(K^*K)^{-1}B^*P(T-t)(x_T^*(t,x_0) - x_e) \\
&- (K^*K)^{-1}B^*(U_{T-t}(T-t,0))^* w, \quad \forall t \in [0,T],
\end{aligned} \tag{3.25}$$

where w is the optimal adjoint state.

Proof. For any $T > 0$, we claim that $(U_T(T,0))^*$ is the evolution operator at time T of the following evolution system:

$$\begin{cases} \dfrac{dp(t)}{dt} = (A^* - P(t)B(K^*K)^{-1}B^*)p(t), \\ p(0) = p_0 \in \mathcal{H}. \end{cases}$$

In other words, we have $(U_T(T,0))^* p_0 = p(T)$.

The proof is through Yosida approximation. Let $A_n := nA(nI - A)^{-1} \in \mathcal{L}(\mathcal{H})$ denote the Yosida approximation of A for sufficiently large $n \in \mathbb{N}$. Assume that $x_0, p_0 \in \mathcal{H}$. For each n, we let x_n denote the solution of problem

$$\begin{cases} \dfrac{dx_n(t)}{dt} = (A_n - B(K^*K)^{-1}B^*P(T-t))x_n(t), \quad t \in [0,T] \\ x_n(0) = x_0, \end{cases}$$

and p_n denote the solution of problem

$$\begin{cases} \dfrac{dp_n(t)}{dt} = (A_n^* - P(t)B(K^*K)^{-1}B^*)p_n(t), \quad t \in [0,T] \\ p_n(0) = p_0. \end{cases}$$

Since A_n is bounded, we can easily verify (by showing the derivative is 0) that

$$\langle x_0, p_n(T) \rangle = \langle x_n(T), p_0 \rangle.$$

By [6, Part II, Chapter 1, Proposition 3.4], $p_n(T) \to p(T)$ and $x_n(T) \to U_T(T,0)x_0$ as $n \to \infty$. This further implies

$$\langle x_0, p(T) \rangle = \langle U_T(T,0)x_0, p_0 \rangle = \langle x_0, (U_T(T,0))^* p_0 \rangle.$$

Since $x_0, p_0 \in \mathcal{H}$ can be chosen arbitrarily, our claim then follows.

Now fix some $T > 0$ and define $p(\cdot) = (U_\cdot(\cdot, 0))^* w$ on $[0,T]$. Notice that $w \in D(A^*)$, so by Proposition 2.1.11,

$$p(\cdot) \in C([0,T], D(A^*)) \cap C^1([0,T], \mathcal{H}).$$

We then deduce that

$$\frac{dp(T-t)}{dt} = -(A^* - P(T-t)B(K^*K)^{-1}B^*)p(T-t) \;\text{ in }\; \mathcal{H}, \;\; \forall t \in [0,T].$$

Let x be the solution of problem (2.3) corresponding to input $u \in L^2((0,T),\mathcal{U})$ and initial condition $x_0 \in \mathcal{H}$, then

$$\begin{aligned}
&2\operatorname{Re}\langle x(T), w\rangle - 2\operatorname{Re}\langle x_0, p(T)\rangle \\
&\quad = \operatorname{Re}\int_0^T 2\left\langle Ax(t) + Bu(t), p(T-t)\right\rangle_{H_{-1},H_1^d} \\
&\qquad\qquad + 2\left\langle x(t), -(A^* - P(T-t)B(K^*K)^{-1}B^*)p(T-t)\right\rangle dt \\
&\quad = \operatorname{Re}\int_0^T 2\left\langle u(t), B^*p(T-t)\right\rangle \\
&\qquad\qquad + 2\left\langle (K^*K)^{-1}B^*P(T-t)x(t), B^*p(T-t)\right\rangle dt.
\end{aligned}$$

Combining the above equation with (3.19), we obtain

$$\begin{aligned}
J_T(x_0,u) &= \int_0^T \|K\{(u(t)-u_e) + (K^*K)^{-1}B^*P(T-t)(x(t)-x_e)\}\|^2 dt \\
&\qquad + 2\operatorname{Re}\langle w, x(T)\rangle + M_0 \\
&= \int_0^T \|K\{(u(t)-u_e) + (K^*K)^{-1}B^*P(T-t)(x(t)-x_e)\}\|^2 \\
&\qquad + 2\operatorname{Re}\left\langle (K^*K)^{-1}B^*P(T-t)(x(t)-x_e), B^*p(T-t)\right\rangle \\
&\qquad + 2\operatorname{Re}\left\langle (u(t)-u_e), B^*p(T-t)\right\rangle dt + M_1 \\
&= \int_0^T \left\|K\left\{(u(t)-u_e) + (K^*K)^{-1}B^*\left[P(T-t)(x(t)-x_e) + p(T-t)\right]\right\}\right\|^2 dt \\
&\qquad + M_2
\end{aligned}$$

where M_0, M_1 and $M_2 \in \mathbb{R}$ are constants independent of u.

This implies that, if the feedback law

$$\begin{aligned}
u(t) - u_e = &-(K^*K)^{-1}B^*P(T-t)(x(t)-x_e) \\
&- (K^*K)^{-1}B^*(U_{T-t}(T-t,0))^*w, \quad \forall t \in [0,T],
\end{aligned}$$

admits a solution in $L^2((0,T),\mathcal{H})$, then this solution is optimal.

By Proposition 2.1.11, this problem does admit such a solution $x \in L^2((0,T),\mathcal{H})$. So, the optimal pair verifies equation (3.25). □

Remark 3.3.12. *If $K = I$ and the optimal steady state $(x_e, u_e) = (0,0)$, then equation (3.25) can be simplified to*

$$u_T^*(t,x_0) = -B^*P(T-t)x_T^*(t,x_0) - B^*(U_{T-t}(T-t,0))^*w, \quad \forall t \in [0,T].$$

3.3.5 Main proof

We are now in position to prove Theorem 3.3.1.

Proof. Without loss of generality, we assume that $K = I$ and the unique optimal steady state $(x_e, u_e) = (0, 0)$. By Lemma 3.3.6, there exists some $w \in D(A)$ such that

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} w = \begin{bmatrix} z \\ v \end{bmatrix}.$$

Fix $\mathcal{N} \subset \mathcal{H}$ to be some bounded neighborhood of $x_e = 0$, $x_0 \in \mathcal{N}$ and $T > 0$.

For any $T_0 \in [0, T)$, consider the input function defined by:

$$u_{T_0}(t) := \begin{cases} -B^* P(T-t) x_{T_0}(t), & t \in [0, T_0), \\ -B^* P(T-t) x_{T_0}(t) - (B^* U_{T-t}(T-t, 0))^* w, & t \in [T_0, T], \end{cases}$$

where x_{T_0} is the solution of problem (2.3) corresponding to input u_{T_0} and initial condition x_0. Clearly the above definition gives a unique $u_{T_0} \in L^2((0,T), \mathcal{U})$.

Define that $p(\cdot) = (U_\cdot(\cdot, 0))^* w$ on $[0, T]$. By equation (3.19) and Remark 3.3.12, the cost functional is given by

$$\begin{aligned} J_T(x_0, u_{T_0}) &= \int_0^T \|u_{T_0}(t) + B^* P(T-t) x_{T_0}(t)\|^2 dt + \langle P(T) x_0, x_0 \rangle + 2 \operatorname{Re}\langle w, x_{T_0}(T) - x_0 \rangle \\ &= \int_{T_0}^T \|B^* p(T-t)\|^2 dt + \langle P(T) x_0, x_0 \rangle + 2 \operatorname{Re}\langle w, x_{T_0}(T) - x_0 \rangle. \end{aligned}$$

Let (x^*, u^*) be the optimal pair of $(GLQ)_T$. Observe that

$$J_T(x_0, u^*) = \int_0^T \|B^* p(T-t)\|^2 dt + \langle P(T) x_0, x_0 \rangle + 2 \operatorname{Re}\langle w, x^*(T) - x_0 \rangle.$$

Since $J_T(x_0, u_{T_0}) > J_T(x_0, u^*)$, the difference of the two equations gives

$$\|B^* p(T - \cdot)\|^2_{L^2((0,T_0),\mathcal{U})} = \int_0^{T_0} \|B^* p(T-t)\|^2 dt < 2 \operatorname{Re}\langle w, x_{T_0}(T) - x^*(T) \rangle.$$

Simple considerations about the corresponding evolution systems of x_{T_0} and x^* show that

$$x_{T_0}(T_0) - x^*(T_0) = U_T(T_0, 0) x_0 - [U_T(T_0, 0) x_0 + \widetilde{\Phi}^T_{T_0}(-B^* p(T - \cdot))] = \widetilde{\Phi}^T_{T_0}(B^* p(T - \cdot)),$$

and thus

$$x_{T_0}(T) - x^*(T) = U_T(T, T_0)(x_{T_0}(T_0) - x^*(T_0)) = U_T(T, T_0)\widetilde{\Phi}_{T_0}^T(B^*p(T-\cdot)).$$

Following from Lemma 3.3.9 and Lemma 3.3.10, we deduce that there exist some positive constants M_0 and k such that

$$\begin{aligned}\|B^*p(T-\cdot)\|^2_{L^2((0,T_0),\mathcal{U})} &< 2\,\mathrm{Re}\left\langle w, U_T(T,T_0)\widetilde{\Phi}_{T_0}^T(B^*p(T-\cdot))\right\rangle \\ &\leq M_0 e^{-k(T-T_0)}\|B^*p(T-\cdot)\|_{L^2((0,T_0),\mathcal{U})}.\end{aligned}$$

That is, $\|B^*p(T-\cdot)\|_{L^2((0,T_0),\mathcal{U})} < M_0 e^{-k(T-T_0)}$. Since $x^*(T_0) = U_T(T_0, 0)x_0 - \widetilde{\Phi}_{T_0}^T(B^*p(T-\cdot))$, applying Lemma 3.3.9 and Lemma 3.3.10 again, we deduce that there exists some $M_1 > 0$ such that

$$\|x^*(T_0)\| \leq M_1(e^{-kT_0}\|x_0\| + e^{-k(T-T_0)}). \tag{3.26}$$

Finally, recall from Remark 3.3.12 that

$$u^*(T_0) = -B^*P(T-T_0)x^*(t) - B^*p(T-T_0).$$

Now Lemma 3.3.9 implies there exists $M_2 > 0$ such that

$$p(T-T_0) = (U_{T_0}(T_0, 0))^*w \leq M_2 e^{-kT_0}.$$

By uniform boundedness principle and (3.26), a straightforward calculation shows

$$\|u^*(T_0)\| \leq M_3(e^{-kT_0}\|x_0\| + e^{-k(T-T_0)}). \tag{3.27}$$

holds for some $M_3 > 0$ independent of the selection of T, T_0.

Since $\mathcal{N}$ is bounded and T_0 can be arbitrarily chosen in $[0, T]$, now equation (3.26) and (3.27) together implies the exponential (and also the measure) turnpike property at $(0, 0)$.

□

3.4 Necessary condition for the turnpike property

In this section, we deduce several necessary conditions for turnpike property. The following subsections are devoted to the proof of these results.

Definition 3.4.1. *The unobservable subspace U^∞ of a C_0-semigroup $\mathcal{T}$ is defined as*

$$U^\infty := \{x \in \mathcal{H} \mid C\mathcal{T}_t x = 0 \ \textit{for any} \ \ t \in [0,\infty)\}.$$

Remark 3.4.2. *Obviously U^∞ is a closed subspace of $\mathcal{H}$ and is invariant under $\mathcal{T}$, so the restriction of $\mathcal{T}$ to U^∞ is a C_0-semigroup on U^∞, and the corresponding generator is just $A|_{D(A)\cap U^\infty}$. See, e.g., [41, Proposition 2.4.3].*

Our first result provides several necessary conditions for the turnpike property in terms of the turnpike reference, stabilizability and detecability of the system.

Theorem 3.4.3. *If the problem $(GLQ)_T$ satisfies the measure or exponential turnpike property at some steady state (x_e, u_e), then following statements hold:*

(a) *$\mathcal{T}$ is exponentially stable on U^∞.*

(b) *(x_e, u_e) is the unique optimal steady state of problem (3.4).*

(c) *The pair (A, B) is exponentially stabilizable.*

The next result shows that when the turnpike property holds, the optimal control can be explicitly solved as in subsection 3.3.4.

Corollary 3.4.4. *If problem $(GLQ)_T$ satisfies the measure or exponential turnpike property at some controlled equilibrium (x_e, u_e), then there exists a unique $w \in D(A^*)$ such that*

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} w = \begin{bmatrix} z + C^*Cx_e \\ v + K^*Ku_e \end{bmatrix}.$$

Moreover, the optimal control of problem $(GLQ)_T$ is given in a feedback law form by

$$\begin{aligned} u_T^*(t, x_0) - u_e = &-(K^*K)^{-1}B^*P(T-t)(x_T^*(t, x_0) - x_e) \\ &- (K^*K)^{-1}B^*(U_{T-t}(T-t, 0))^*w, \quad \forall t \in [0, T]. \end{aligned}$$

3.4.1 Proof of Theorem 3.4.3 (a)

Lemma 3.4.5. *For any time horizon $T > 0$ and $x_0 \in U^\infty$,*

$$u_T^*(\cdot, x_0) = u_T^*(\cdot, \gamma x_0), \quad \forall \gamma \in \mathbb{C}.$$

Proof. Suppose that $T > 0$ and $x_0 \in U^\infty$. For any $\gamma \in \mathbb{C}$, let x_γ be the solution of system (2.3) on $[0, T]$ corresponding to initial condition γx_0 and input $u \in L^2((0,T), \mathcal{U})$, then

$$x_\gamma(t) = \mathcal{T}_t \gamma x_0 + \Phi_t(u), \quad \forall t \in [0, T].$$

Since $x_0 \in U^\infty$, notice that for any $t \in [0, T]$,

$$\|Cx_\gamma(t)\|^2 = \|C\mathcal{T}_t\gamma x_0\|^2 + 2\operatorname{Re}\langle C\mathcal{T}_t\gamma x_0, C\Phi_t(u)\rangle + \|C\Phi_t(u)\|^2 = \|C\Phi_t(u)\|^2,$$

and

$$2\operatorname{Re}\langle z, x_\gamma(t)\rangle = 2\operatorname{Re}\langle z, \mathcal{T}_t\gamma x_0\rangle + 2\operatorname{Re}\langle z, \Phi_t(u)\rangle.$$

So, we have

$$\begin{aligned} J_T(\gamma x_0, u) &= \int_0^T \|Cx_\gamma(t)\|^2 + \|Ku(t)\|^2 + 2\operatorname{Re}\langle z, x(t)\rangle + 2\operatorname{Re}\langle v, u(t)\rangle dt \\ &= \int_0^T \|C\Phi_t(u)\|^2 + 2\operatorname{Re}\langle z, \mathcal{T}_t\gamma x_0 + \Phi_t(u)\rangle + \|Ku(t)\|^2 + 2\operatorname{Re}\langle v, u(t)\rangle dt. \end{aligned}$$

Clearly for any $\gamma \in \mathbb{C}$, the terms concerning u in $J_T(\gamma x_0, u)$ are all the same. Hence we conclude that $u_T^*(\cdot, \gamma x_0) = u_T^*(\cdot, x_0)$ for any $\gamma \in \mathbb{C}$. □

Proof of Theorem 3.4.3 (a). Without loss of generality, assume that the measure turnpike property is satisfied at $(x_e, u_e) = (0, 0)$. Let $\mathcal{N}$ be the closed unit ball with center 0 in $\mathcal{H}$.

The proof is by contradiction. Observe that there exists a sufficiently large constant $M > 0$ such that, if $x_0 \in \mathcal{H}$ and $t_0 \geq 2M_{\mathcal{N},\varepsilon} + 1$ where$M_{\mathcal{N},\varepsilon}$ is defined as in Definition 3.2.2, then

$$\|\mathcal{T}_{t_0}x_0\| \geq M \implies \|\mathcal{T}_t x_0\| > 2\varepsilon, \quad \forall t \in [t_0 - 2M_{\mathcal{N},\varepsilon} - 1, t_0].$$

In fact, by Proposition 2.1.2, there exist positive constants k and M_k such that

$$M \leq \|\mathcal{T}_{t_0}x_0\| \leq \|\mathcal{T}_{t_0-t}\|\|\mathcal{T}_t x_0\| \leq M_k e^{k(t_0-t)}\|\mathcal{T}_t x_0\|, \quad \forall t \in [t_0 - 2M_{\mathcal{N},\varepsilon} - 1, t_0].$$

Now it is clear if $M > 2\varepsilon M_k e^{k(2M_{\mathcal{N},\varepsilon}+1)}$, then $\|\mathcal{T}_t x_0\| > 2\varepsilon$.

On the other hand, if $\mathcal{T}$ is not exponentially stable on U^∞, then according to Proposition V1.2 in [14], the restriction of $\mathcal{T}$ on U^∞ does not converge to 0 in operator norm as $t \to \infty$. In other words, there exists some $\varepsilon > 0$ such that

$$\limsup_{t \geq 0} \|\mathcal{T}_t|_{U^\infty}\| > \varepsilon.$$

As a consequence, we are able to find some $t_r > 2M_{\mathcal{N},\frac{\varepsilon}{4M}} + 2M_{\mathcal{N},\varepsilon} + 2$ and $x_0 \in \mathcal{H}$ satisfying $\|x_0\| \leq 1$ such that $\|\mathcal{T}_{t_r} x_0\| \geq \varepsilon$.

By measure turnpike property, there exists a point $t_l \in [0, 2M_{\mathcal{N},\frac{\varepsilon}{4M}} + 1]$ such that

$$\left\|x^*_{t_r}(t_l, x_0)\right\| \leq \frac{\varepsilon}{4M} \quad \text{and} \quad \left\|x^*_{t_r}\left(t_l, \frac{x_0}{2}\right)\right\| \leq \frac{\varepsilon}{4M}.$$

Since $x_0 \in U^\infty$, following from Lemma 3.4.5, we obtain that

$$\frac{\|\mathcal{T}_{t_l} x_0\|}{2} = \left\|x^*_{t_r}(t_l, x_0) - x^*_{t_r}\left(t_l, \frac{x_0}{2}\right)\right\| \leq \left\|x^*_{t_r}(t_l, x_0)\right\| + \left\|x^*_{t_r}\left(t_l, \frac{x_0}{2}\right)\right\| \leq \frac{\varepsilon}{2M}.$$

This implies that $\|\frac{M}{\varepsilon}\mathcal{T}_{t_l} x_0\| \leq 1$. Thus $\frac{M}{\varepsilon}\mathcal{T}_{t_l} x_0 \in \mathcal{N}$.

Also notice

$$\left\|\mathcal{T}_{t_r - t_l}\frac{M}{\varepsilon}\mathcal{T}_{t_l} x_0\right\| = \left\|\frac{M}{\varepsilon}\mathcal{T}_{t_r} x_0\right\| > \frac{M}{\varepsilon}\varepsilon \geq M \quad \text{and} \quad t_r - t_l > 2M_{\mathcal{N},\varepsilon} + 1,$$

so

$$\left\|\mathcal{T}_t \frac{M}{\varepsilon}\mathcal{T}_{t_l} x_0\right\| > 2\varepsilon, \quad \forall t \in [t_r - t_l - 2M_{\mathcal{N},\frac{\varepsilon}{2}} - 1, t_r - t_l].$$

From Lemma 3.4.5, we obtain that

$$\left\|x^*_{t_r}\left(t, \frac{M}{\varepsilon}\mathcal{T}_{t_l} x_0\right) - x^*_{t_r}\left(t, \frac{1}{2}\frac{M}{\varepsilon}\mathcal{T}_{t_l} x_0\right)\right\| = \left\|\frac{1}{2}\mathcal{T}_t \frac{M}{\varepsilon}\mathcal{T}_{t_l} x_0\right\| > \varepsilon$$

holds for any $t \in [t_r - t_l - 2M_{\mathcal{N},\frac{\varepsilon}{2}} - 1, t_r - t_l]$. However, this gives a contradiction since the length of this interval is greater than $2M_{\mathcal{N},\frac{\varepsilon}{2}} + 1$, and by the measure turnpike property, there must exist a $t \in [t_r - r_l - 2M_{\mathcal{N},\frac{\varepsilon}{2}} - 1, t_r - t_l]$ so that

$$\left\|x^*_{t_r}\left(t, \frac{M}{\varepsilon}\mathcal{T}_{t_l} x_0\right)\right\| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \left\|x^*_{t_r}\left(t, \frac{1}{2}\frac{M}{\varepsilon}\mathcal{T}_{t_l} x_0\right)\right\| \leq \frac{\varepsilon}{2},$$

which further implies

$$\left\|x^*_{t_r}\left(t, \frac{M}{\varepsilon}\mathcal{T}_{t_l} x_0\right) - x^*_{t_r}\left(t, \frac{1}{2}\frac{M}{\varepsilon}\mathcal{T}_{t_l} x_0\right)\right\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.$$

□

3.4.2 Proof of Theorem 3.4.3 (b)

We first prove two preliminary lemmas.

Lemma 3.4.6 (Characterization of the optimal steady state)**.** *$(x_e, u_e) \in D(A) \times \mathcal{U}$ is an optimal steady state of the optimal steady state problem* (3.4) *if and only if*

$$\begin{bmatrix} z + C^*Cx_e \\ v + K^*Ku_e \end{bmatrix} \in \overline{\operatorname{ran}} \begin{bmatrix} A^* \\ B^* \end{bmatrix}.$$

Moreover, the optimal steady state is unique if and only if

$$\ker A \cap \ker C = \{0\}.$$

Proof. Let $V = \ker[A, B]$. Recall that

$$\ell(x, u) = \left\langle \mathbb{P}_V \begin{bmatrix} C^*C & 0 \\ 0 & K^*K \end{bmatrix} \Bigg|_V \begin{bmatrix} x \\ u \end{bmatrix} + 2\mathbb{P}_V \begin{bmatrix} z \\ v \end{bmatrix}, \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle.$$

Let the non-negative operator $\mathcal{P} : V \to V$ be defined by

$$\mathcal{P} := \mathbb{P}_V \begin{bmatrix} C^*C & 0 \\ 0 & K^*K \end{bmatrix} \Bigg|_V.$$

Notice that $\mathcal{P} \in L(V)$ and $\mathbb{P}_V \begin{bmatrix} z \\ v \end{bmatrix} \in V$. So, by [22, Lemma 4], $(x_e, u_e) \in V$ is a minimizer of problem (3.4) if and only if

$$\mathbb{P}_V \begin{bmatrix} C^*C & 0 \\ 0 & K^*K \end{bmatrix} \Bigg|_V \begin{bmatrix} x_e \\ u_e \end{bmatrix} = -\mathbb{P}_V \begin{bmatrix} z \\ v \end{bmatrix}. \tag{3.28}$$

Equivalently, $(x_e, u_e) \in V$ satisfies

$$\begin{bmatrix} z + C^*Cx_e \\ v + K^*Ku_e \end{bmatrix} = \begin{bmatrix} C^*C & 0 \\ 0 & K^*K \end{bmatrix} \Bigg|_V \begin{bmatrix} x_e \\ u_e \end{bmatrix} + \begin{bmatrix} z \\ v \end{bmatrix} \in V^\perp = \overline{\operatorname{ran}} \begin{bmatrix} A^* \\ B^* \end{bmatrix}.$$

This proves our first claim.

On the other hand, simple considerations show that the uniqueness of $(x_e, u_e) \in V$ verifying equation (3.28) is equivalent to $\ker \mathcal{P} = \{(0, 0)\}$. Now let us show that $\ker \mathcal{P} = \{(0, 0)\}$ is further equivalent to $\ker A \cap \ker C = \{0\}$.

Suppose that $\ker A \cap \ker C = \{0\}$. If $(x_0, u_0) \in \ker \mathcal{P}$, then we have

$$\left\langle \mathcal{P}\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} C^*Cx_0 \\ K^*Ku_0 \end{bmatrix}, \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\rangle = \|Cx_0\|^2 + \|Ku_0\|^2 = 0.$$

Since K is coercive, we deduce that $u_0 = 0$ and $Cx_0 = 0$. Besides, since $(x_0, u_0) \in V$, we have $Ax_0 = 0 - Bu_0 = 0$. This implies $x_0 \in \ker A \cap \ker C$. Thus $x_0 = 0$ and $\ker \mathcal{P} = \{(0, 0)\}$.

Conversely, if there exists some $x_0 \in \ker A \cap \ker C$ and $x_0 \neq 0$, we can easily verify that $(x_0, 0) \in \ker \mathcal{P} \subset V$. So, $\ker \mathcal{P} \neq \{(0, 0)\}$.

□

The following technical lemma provides a very conservative estimate for the lower bound of the cost functional.

Lemma 3.4.7. *Suppose $\mathcal{N}$ is a bounded subset of $\mathcal{H}$. Then for any $T \in (0, \infty)$ and $x_0 \in \mathcal{N}$,*

$$\min_{u \in L^2((0,T),\mathcal{U})} J(x_0, u) \geq -M(e^{2kT} - 1)$$

for some constants $M, k > 0$.

Proof. Since K is coercive, K^*K is bounded below and thus invertible. Let $u_0 := -(K^*K)^{-1}v$, then there exists $M_K > 0$ such that

$$\|Ku\|^2 + 2\operatorname{Re}\langle v, u\rangle = \|K(u - u_0)\|^2 - \|Ku_0\|^2 \geq M_K\|u - u_0\|^2 - \|Ku_0\|^2, \quad \forall u \in \mathcal{U}. \tag{3.29}$$

Besides, by Proposition 2.1.2 and Proposition 2.1.8, there exist positive constants k, M_1 and M_2 such that

$$\|\Phi_t\| \leq M_1 e^{kt}, \quad \text{and} \quad \|\mathcal{T}_t\| \leq M_2 e^{kt}, \quad \forall t \geq 0$$

Assume that $T > 0$, $u \in L^2((0, T), \mathcal{U})$ and x is the solution of (2.3) corresponding to initial condition $x_0 \in \mathcal{N}$ and input u, then for any $t \in [0, T]$,

$$\begin{aligned} 2\operatorname{Re}\langle z, x(t)\rangle &\geq -2\|z\| \left(\|\mathcal{T}_t x_0\| + \|\Phi_t(u - u_0)\| + \|\Phi_t u_0\|\right) \\ &\geq -2\|z\|(M_2 e^{kt}\|x_0\| + M_1 e^{kt}\|u - u_0\|_{L^2} + \|\Phi_t u_0\|). \end{aligned} \tag{3.30}$$

Concerning the term $\Phi_t u_0$, we have

$$\begin{aligned} \|\Phi_t u_0\| &= \left\| \int_0^t \mathcal{T}_{t-s} B u_0 ds \right\| \\ &\leq \int_0^t M_2 e^{k(t-s)} \|Bu_0\| ds \\ &\leq \frac{e^{kt}}{k} M_2 \|Bu_0\|. \end{aligned} \tag{3.31}$$

Now (3.29), (3.30) and (3.31) together implies there exist some positive constants M_3 and M_4 such that

$$\ell(x(t), u(t)) \geq -M_3 e^{kt} - M_4 e^{kt}\|u - u_0\|_{L^2} + M_K\|u(t) - u_0\|^2 - \|Ku_0\|^2, \;\; \forall t \in [0, T].$$

Integrating ℓ over $[0, T]$, a straightforward calculation gives

$$J_T(x_0, u) \geq -\frac{M_4}{k}(e^{kT} - 1)\|u - u_0\|_{L^2} - \frac{M_3}{k}(e^{kT} - 1) + M_K\|u - u_0\|_{L^2}^2 - \|Ku_0\|^2 T.$$

Also notice, as a quadratic function with respect to $\|u - u_0\|_{L^2}$,

$$M_K\|u - u_0\|_{L^2}^2 - \frac{M_4}{k}(e^{kT} - 1)\|u - u_0\|_{L^2} \geq -\frac{M_4^2(e^{kT} - 1)^2}{4M_0 k^2}.$$

Simple considerations about the above two inequalities show that

$$J_T(x_0, u) \geq -M_5(e^{2kT} - 1) - M_6(e^{kT} - 1) - M_7 T$$

holds for some positive constants M_5, M_6 and M_7. Since the dominant term of the above estimate is $e^{2kT} - 1$, there exists some $M > 0$ (independent of T) such that

$$J_T(x_0, u) \geq -M(e^{2kT} - 1).$$

The lemma then follows from the fact that $u \in L^2((0, T), \mathcal{U})$ can be arbitrarily chosen.

□

Now we are in the position to prove Theorem 3.4.3 (b).

Proof of Theorem 3.4.3 (b). We first show that if the measure turnpike property holds at some steady state (x_e, u_e), then (x_e, u_e) is necessarily an optimal steady state.

Otherwise, there exists some steady state $(\widetilde{x}_e, \widetilde{u}_e)$ such that $\ell(\widetilde{x}_e, \widetilde{u}_e) < \ell(x_e, u_e)$. Define $(dx, de) := (\widetilde{x}_e, \widetilde{u}_e) - (x_e, u_e)$. Fix some bounded neighborhood $\mathcal{N}$ of x_e, and some sufficiently small $\lambda \in (0, 1]$ such that $x_e + \lambda dx \in \mathcal{N}$, then by the convexity of ℓ,

$$\ell(x_e + \lambda dx, u_e + \lambda du) < \ell(x_e, u_e).$$

Let us denote the equilibrium point $(x_e + \lambda dx, u_e + \lambda du)$ by $(\widetilde{x}_e, \widetilde{u}_e)$ again.

Since ℓ is continuous, there exist some sufficiently small ε and $\delta > 0$ so that, for any $(x_0, u_0) \in \mathcal{H} \times \mathcal{U}$,

$$\|x_0 - x_e\| + \|u_0 - u_e\| \leq \delta \implies \ell(\widetilde{x}_e, \widetilde{u}_e) + \varepsilon < \ell(x_0, u_0).$$

Now fix some $T > 0$. Then we know that the set

$$A := \{t \in [0, T] \mid \|u_T^*(t, \widetilde{x}_e) - u_e\| + \|x_T^*(t, \widetilde{x}_e) - x_e\| > \delta\}$$

is open, and its Lebesgue measure is smaller than $M_{\mathcal{N},\delta}$ where $M_{\mathcal{N},\delta}$ is defined as in Definition 3.2.2.

Without loss of generality, assume that $A = \bigcup_{j=1}^{\infty}(t_{j,l}, t_{j,r})$ where $((t_{j,l}, t_{j,r}))_{j\in\mathbb{N}}$ are disjoint open intervals (the number of such intervals may be finite, and there may be at most two intervals that contain an endpoint 0 or T, but the proof is basically the same).

Observe that at each left endpoint $t_{j,l}$, $j \in \mathbb{N}$, of these intervals, we must have

$$\|x_T^*(t_{j,l}, \widetilde{x}_e) - \widetilde{x}_e\| \leq \|x_T^*(t_{j,l}, \widetilde{x}_e) - \widetilde{x}_e\| + \|u_T^*(t_{j,l}, \widetilde{x}_e) - \widetilde{u}_e\| = \delta.$$

From Lemma 3.4.7, there exist some $M, k > 0$ such that

$$-M(e^{k(t_{j,r}-t_{j,l})} - 1) \leq \int_{t_{j,l}}^{t_{j,r}} \ell(x_T^*(t, \widetilde{x}_e), u_T^*(t, \widetilde{x}_e))dt.$$

Owning to the countable additivity of Lebesgue integral, we have

$$-M\sum_{j=1}^{\infty}(e^{k(t_{j,r}-t_{j,l})} - 1) \leq \int_A \ell(x_T^*(t, \widetilde{x}_e), u_T^*(t, \widetilde{x}_e))dt.$$

Now simple calculation shows for any t_1 and $t_2 > 0$,

$$-M(e^{k(t_1+t_2)} - 1) \leq -M(e^{kt_1} - 1) - M(e^{kt_2} - 1).$$

Combining the two inequalities, we obtain that

$$-M(e^{kM_{\mathcal{N},\delta}} - 1) \leq -M(e^{k\mu(A)} - 1) \leq \int_A \ell(x_T^*(t, \widetilde{x}_e), u_T^*(t, \widetilde{x}_e))dt. \tag{3.32}$$

Meanwhile, for any $t \in [0, T] \setminus A$, since $\|x_T^*(t, \widetilde{x}_e) - \widetilde{x}_e\| + \|u_T^*(t, \widetilde{x}_e) - \widetilde{u}_e\| \leq \delta$, we have

$$\ell(\widetilde{x}_e, \widetilde{u}_e) + \varepsilon < \ell(x_T^*(t, \widetilde{x}_e), u_T^*(t, \widetilde{x}_e)),$$

so

$$(T - M_{\mathcal{N},\delta})(\ell(\widetilde{x}_e, \widetilde{u}_e) + \varepsilon) < \int_{[0,T]\setminus A} \ell(x_T^*(t, \widetilde{x}_e), u_T^*(t, \widetilde{x}_e))dt. \tag{3.33}$$

Finally, since $(\widetilde{x}_e, \widetilde{u}_e)$ is a steady state,

$$\int_0^T \ell(x_T^*(t, \widetilde{x}_e), u_T^*(t, \widetilde{x}_e))dt \leq J_T(x_e, u_e) = T\ell(\widetilde{x}_e, \widetilde{u}_e). \tag{3.34}$$

Combining (3.32), (3.33) and (3.34), simple calculation shows

$$T\varepsilon < M(e^{kM_{\mathcal{N},\delta}} - 1) + M_{\mathcal{N},\delta}(\ell(\widetilde{x}_e, \widetilde{u}_e) + \varepsilon).$$

This leads to a contradiction when T is taken sufficiently large.

Recall from Lemma 3.4.6 that the uniqueness of the optimal steady state is characterized by

$$\ker A \cap \ker C = \{0\}.$$

In fact, if $x \in \ker A \cap \ker C$, then $C\mathcal{T}_t x = Cx = 0$ for any $t \geq 0$. So, x belongs to the unobservable subspace U^∞ of $\mathcal{T}$. Since we have prove in Theorem 3.4.3 (a) that $\mathcal{T}$ is exponentially stable on U^∞, we know that $\|\mathcal{T}_t x\| = \|x\| \to 0$ as $t \to \infty$. Thus $x = 0$. □

3.4.3 Proof of Theorem 3.4.3 (c)

The following lemma provides a conservative estimate on the upper bound of the L^2-norm of the optimal control when the initial point lies in a bounded set.

Lemma 3.4.8. *Assume that $\mathcal{N}$ is a bounded set in $\mathcal{H}$. Then for any $T > 0$, there exists some $M_u > 0$ (dependent on T) such that*

$$\sup_{x_0 \in \mathcal{N}} \|u_T^*(\cdot, x_0)\|_{L^2} \leq M_u.$$

Proof. Fix some $T > 0$. We first prove there exists some $M_0 > 0$ such that

$$\sup_{x_0 \in \mathcal{N}} \int_0^T \ell(x_T^*(t, x_0), u_T^*(t, x_0))dt \leq M_0. \tag{3.35}$$

To see this, consider the case for input $u \equiv 0$, then we obtain

$$\int_0^T \ell(x_T^*(t, x_0), u_T^*(t, x_0))dt \leq \int_0^T \|C\mathcal{T}_t x_0\|^2 + 2\,\mathrm{Re}\langle z, \mathcal{T}_t x_0\rangle dt, \quad \forall x_0 \in \mathcal{N}.$$

Since $\mathcal{T}_t x_0$ is bounded in norm on $[0,T]$ for any $x_0 \in \mathcal{N}$, simple considerations show that the positive constant M_0 exists.

Now notice, for any $x_0 \in \mathcal{N}$ and $t \in [0,T]$,

$$2\operatorname{Re}\langle z, x_T^*(t,x_0)\rangle \geq -2\|z\|\|\mathcal{T}_t x_0 + \Phi_t u_T^*(\cdot,x_0)\| \geq -M_1 - M_2\|u_T^*(\cdot,x_0)\|_{L^2},$$

holds with some M_1, $M_2 > 0$, and thus

$$\begin{aligned}\int_0^T \ell(x_T^*(t,x_0), u_T^*(t,x_0))dt \geq \int_0^T & -M_1 - M_2\|u_T^*(\cdot,x_0)\|_{L^2} \\ & + M_K\|u_T^*(t,x_0)\|^2 - 2\|v\|\|u_T^*(t,x_0)\|dt \\ \geq & -M_1T - M_2T\|u_T^*(\cdot,x_0)\|_{L^2} + M_K\|u_T^*(\cdot,x_0)\|_{L^2}^2 \\ & - 2\|v\|\|u_T^*(\cdot,x_0)\|_{L^1}\end{aligned}$$

holds with some $M_K, M_3 > 0$, where $\|\cdot\|_{L^1}$ denotes the L^1-norm on $L^1((0,T),\mathcal{U})$. By Hölder's inequality, the L^1-norm is dominated by the L^2-norm, so there exists $M_4 > 0$ such that

$$\int_0^T \ell(x_T^*(t,x_0), u_T^*(t,x_0))dt \geq -M_1T - M_4\|u_T^*(\cdot,x_0)\|_{L^2} + M_K\|u_T^*(\cdot,x_0)\|_{L^2}^2.$$

Now simple considerations about equation (3.35) and the above two inequalities show that the L^2-norm of $u_T^*(\cdot,x_0)$ is uniformly bounded above for all $x_0 \in \mathcal{N}$, i.e., there exists some positive constant M_u (dependent on T) such that

$$\sup_{x_0\in\mathcal{N}} \|u_T^*(\cdot,x_0)\|_{L^2} \leq M_u.$$

□

Proof of Theorem 3.4.3 (c). Assume the measure turnpike property is satisfied at (x_e,u_e). Without loss of generality, let $\mathcal{N}$ be the closed unit ball in $\mathcal{H}$ with center x_e.

We claim that, for any $x_0 \in \mathcal{H}$, there exists some $u \in L^2((0,\infty),\mathcal{U})$ such that the corresponding trajectory x with input u and initial condition x_0 is L^2-integrable (on $(0,\infty)$). The case for $x_0 = 0$ is trivial, so let us assume that $x_0 \neq 0$.

Fix some $T > M_{\mathcal{N},\frac{1}{2}} + 2$. Define $t_0 := 0$ and $\widetilde{x}_0 := \frac{x_0}{\|x_0\|} + x_e \in \mathcal{N}$. Notice

$$\mu\{t\in[0,T] \mid \|x_T^*(t,\widetilde{x}_0) - x_e\| + \|u_T^*(t,\widetilde{x}_0) - u_e\| > 1/2\} \leq M_{\mathcal{N},\frac{1}{2}}.$$

Since $T > M_{\mathcal{N},\frac{1}{2}} + 2$, there exists a $t_1 > 1$ satisfying

$$\|x_T^*(t_1, \widetilde{x}_0) - x_e\| \leq \frac{1}{2}.$$

We define $u(t)$ on $[t_0, t_1)$ by setting

$$u(t) := \|x_0\| \left(u_T^*(t, \widetilde{x}_0) - u_e\right), \ t \in [t_0, t_1).$$

We claim that the solution of system (2.3) on $[t_0, t_1]$ corresponding to initial condition x_0 and input u, denoted x, is given by

$$x(t) = \|x_0\| \left(x_T^*(t, \widetilde{x}_0) - x_e\right), \quad \forall t \in [t_0, t_1].$$

In fact, we can verify this by noticing that $x(0) = x_0$ and

$$\begin{aligned}\dot{x}(t) &= \|x_0\|(Ax_T^*(t, \widetilde{x}_0) + Bu_T^*(t, \widetilde{x}_0)) \\ &= Ax(t) + A\|x_0\|x_e + Bu(t) + B\|x_0\|u_e \\ &= Ax(t) + Bu(t).\end{aligned}$$

So, we have

$$\|x(t_1)\| \leq \frac{1}{2}\|x_0\|.$$

On the other hand, notice

$$\begin{aligned}\int_{t_0}^{t_1} \|x(t)\|^2 + \|u(t)\|^2 dt \leq \|x_0\|^2 (T\|x_T^*(\cdot, \widetilde{x}_0)\|_{L^\infty}^2 + 2T\|x_e\|\|x_T^*(\cdot, \widetilde{x}_0)\|_{L^\infty} + T\|x_e\|^2 \\ + \|u_T^*(\cdot, \widetilde{x}_0)\|_{L^2}^2 + 2\|u_e\|\|u_T^*(\cdot, \widetilde{x}_0)\|_{L^1} + T\|u_e^2\|),\end{aligned}$$

where the L^1, L^2 and L^∞-norms here are considered on the domain $[t_0, t_1]$. Recall from Lemma 3.4.8 that there exists some $M_1 > 0$ such that

$$\|u_T^*(\cdot, \widetilde{x}_0)\|_{L^2} < M_1, \quad \forall \widetilde{x}_0 \in \mathcal{N}.$$

By Hölder's inequality, the L^2-norm dominates the L^1-norm, so there exists some $M_2 > 0$ such that

$$\|u_T^*(\cdot, \widetilde{x}_0)\|_{L^1} \leq M_2 \ \ \forall \widetilde{x}_0 \in \mathcal{N}.$$

Finally, since

$$\|x_T^*(t, \widetilde{x}_0)\| \leq \|\mathcal{T}_t \widetilde{x}_0\| + \|\Phi_t\|\|u_T^*(\cdot, \widetilde{x}_0)\|_{L^2},$$

$x_T^*(t, \widetilde{x}_0)$ is uniformly bounded in norm for any $\widetilde{x}_0 \in \mathcal{N}$ and $t \in [t_0, t_1]$, i.e., there exists some positive constant M_3 such that

$$\|x_T^*(t, \widetilde{x}_0)\| \leq M_3, \quad \forall \widetilde{x}_0 \in \mathcal{N}, t \in [t_0, t_1].$$

Combining all these, we deduce that there exists $M > 0$ such that

$$\int_{t_0}^{t_1} \|x(t)\|^2 + \|u(t)\|^2 dt < \|x_0\|^2 M, \quad \forall \widetilde{x}_0 \in \mathcal{N}.$$

Now we can repeat the above argument with $x(t_1)$ in place of $x(t_0) = x_0$. By induction, we can find a sequence $(t_i)_{i \in \mathbb{N}}$ satisfying $t_{i+1} - t_i > 1$, and let u be defined inductively on each interval $[t_i, t_{i+1})$ by setting

$$u|_{[t_i, t_{i+1})}(t) := \|x(t_i)\| \left(u_T^*(t, \widetilde{x}_i) - u_e\right)$$

where $\widetilde{x}_i := \frac{x(t_i)}{\|x(t_i)\|} + x_e$, $i = 0, 1, 2, \ldots$. In particular, our construction of t_i gives

$$\|x(t_i)\| \leq \left(\frac{1}{2}\right)^i \|x_0\|, \quad \forall i \in \mathbb{N}.$$

Finally, since for each $i \in \mathbb{N}$, $\widetilde{x}_i$ belongs to $\mathcal{N}$, we have

$$\int_{t_i}^{t_{i+1}} \|x(t)\|^2 + \|u(t)\|^2 dt < \|x(t_i)\|^2 M \leq \left(\frac{1}{4}\right)^i \|x_0\|^2 M.$$

So,

$$\int_0^\infty \|x(t)\|^2 + \|u(t)\|^2 dt \leq \frac{4}{3} \|x_0\|^2 M.$$

Theorem 3.4.3 (c) now follows easily from Definition 3.2.4. □

3.4.4 Proof of Corollary 3.4.4

Proof. Assume that problem $(GLQ)_T$ satisfies the measure turnpike property at some steady state (x_e, u_e), then by Lemma 3.4.6 and Theorem 3.4.3 (b), we have

$$\begin{bmatrix} z + C^* C x_e \\ v + K^* K u_e \end{bmatrix} \in \overline{\operatorname{ran}} \begin{bmatrix} A^* \\ B^* \end{bmatrix}.$$

Besides, by Theorem 3.4.3 (c) and Lemma 3.3.5, ran $\begin{bmatrix} A^* \\ B^* \end{bmatrix}$ is closed and

$$\ker \begin{bmatrix} A^* \\ B^* \end{bmatrix} = (\mathrm{ran}[A\ B])^{\perp} = \mathcal{H}^{\perp} = \{0\}.$$

So, there exists a unique $w \in D(A^*)$ satisfying

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} w = \begin{bmatrix} z + C^*Cx_e \\ v + K^*Ku_e \end{bmatrix}.$$

Following line by line the proof of Lemma 3.3.11, we can show that the optimal control of problem $(GLQ)_T$ is given in a feedback law form by

$$\begin{aligned} u_T^*(t, x_0) - u_e = &-(K^*K)^{-1}B^*P(T-t)(x_T^*(t, x_0) - x_e) \\ &- (K^*K)^{-1}B^*(U_{T-t}(T-t, 0))^*w, \quad \forall t \in [0, T]. \end{aligned}$$

□

3.5 If and only if characterization

Based on our previous results, we can further obtain some necessary and sufficient conditions for the turnpike property. This part includes:

1. The if and only if characterization of both notions of the turnpike property in terms of the stabilizability and detectability for finite dimensional case and point spectrum case.
2. The equivalence between the exponential turnpike property of problem $(GLQ)_T$ and the exponential turnpike property of problem $(LQ)_T$ plus an algebraic condition.

Following subsections are devoted to the proof of these results.

Now let us denote by $\sigma^-(A)$, $\sigma^0(A)$ and $\sigma^+(A)$ the set of all the elements in $\sigma(A)$ with negative, zero and positive real part, respectively. In the same manner as [6], we assume

$$\begin{cases} (a) & \text{the set } \sigma^+(A) \text{ consists of a finite set of} \\ & \text{eigenvalues of finite algebraic multiplicity,} \\ (b) & \text{there exists } \varepsilon > 0, N_A > 0 \text{ such that} \\ & \sup_{s\in\sigma^-(A)} \mathrm{Re}\, s < -\varepsilon,\ \|\mathcal{T}_t\mathbb{P}_A^-\| \le N_A e^{-\varepsilon t},\ \forall t \ge 0. \end{cases} \qquad (\mathcal{PS})$$

Here $\mathbb{P}_A^-$ represents the projector on $\sigma^-(A)$ defined by

$$\mathbb{P}_A^- := \frac{1}{2\pi i}\int_{\gamma^-}(sI - A)^{-1}ds$$

where γ^- is a simple Jordan curve around $\sigma^-(A)$. The projectors $\mathbb{P}_A^0$ and $\mathbb{P}_A^+$ are defined analogously. Then $\mathbb{P}_A^-(\mathcal{H})$, $\mathbb{P}_A^0(\mathcal{H})$ and $\mathbb{P}_A^+(\mathcal{H})$ are invariant subspace for $\mathcal{T}$.

Remark 3.5.1. *By [6, Part V, Chapter 1, Remark 3.5], assumptions ($\mathcal{PS}$) are verified in each of the following cases:*

(a) *$\mathcal{H}$ is finite dimensional.*

(b) *$\mathcal{T}_t$ is compact for any $t > 0$.*

Our first result provides a complete characterization of the turnpike property in terms of the stabilizability and detectability of the system in a special case (point spectrum case).

Theorem 3.5.2. *If A fulfills assumptions ($\mathcal{PS}$), then the following statements are equivalent:*

(a) *Problem $(GLQ)_T$ satisfies the exponential turnpike property at some steady state.*

(b) *Problem $(GLQ)_T$ satisfies the measure turnpike property at some steady state.*

(c) *The pair (A, B) is exponentially stabilizable and the pair (A, C) is exponentially detectable.*

The following corollary for the finite dimensional case is a direct consequence of the above theorem.

Corollary 3.5.3. *If $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{Y}$ are all finite dimensional spaces, then the following statements are equivalent:*

(a) *Problem $(GLQ)_T$ satisfies the exponential turnpike property at some steady state.*

(b) *Problem $(GLQ)_T$ satisfies the measure turnpike property at some steady state.*

(c) The pair (A, B) is stabilizable and the pair (A, C) is detectable.

Our last result shows the exponential turnpike property of the generalized LQ optimal control problem is equivalent to the exponential turnpike property of the LQ optimal control problem plus an algebraic condition.

Theorem 3.5.4. *Problem $(GLQ)_T$ satisfies the exponential turnpike property at some steady state (x_e, u_e) if and only if problem $(LQ)_T$ satisfies the exponential turnpike property at $(0,0)$ and there exists a vector $w \in D(A^*)$ such that*

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} w = \begin{bmatrix} z + C^*Cx_e \\ v + K^*Ku_e \end{bmatrix}.$$

3.5.1 Proof of Theorem 3.5.2

The following lemma provides a Hautus type condition for detectability.

Lemma 3.5.5. *If A fulfills assumptions ($\mathcal{PS}$), then the following conditions are equivalent:*

(a) *The pair (A, C) is exponentially detectable.*

(b) $\ker(sI - A) \cap \ker C = \{0\}$, $\forall s \in \sigma^0(A) \cup \sigma^+(A)$.

We refer to [6, Part V, Chapter 1, Proposition 3.3] for a proof of this lemma. Next, let us provide a proof of Theorem 3.5.2.

Proof of Theorem 3.5.2. $(a) \Rightarrow (b)$: This is trivial.

$(c) \Rightarrow (a)$: This has been proved in Theorem 3.3.1.

So, we only need to verify that $(b) \Rightarrow (c)$. Suppose now problem $(GLQ)_T$ satisfies the measure turnpike property at some steady state, then it follows from Theorem 3.4.3 (a) that $\mathcal{T}$ is exponentially stable on U^∞.

Assume that $s \in \sigma^-(A) \cup \sigma^0(A)$ and $x \in \ker(sI - A) \cap \ker C$, then $C\mathcal{T}_t x = e^{st}Cx = 0$ for any $t \in [0, \infty)$. This implies $x \in U^\infty$ and $T_t x \to 0$ as $t \to \infty$. Moreover, since $\mathrm{Re}\, s \geq 0$,

$$\|T_t x\| = |e^{st}|\|x\| \geq \|x\|, \quad \forall t \in [0, \infty).$$

Thus $x = 0$. This further implies that for any $s \in \sigma^-(A) \cup \sigma^0(A)$, we have

$$\ker(sI - A) \cap \ker C = \{0\}.$$

Now $(b) \Rightarrow (c)$ follows easily from Lemma 3.5.5. □

3.5.2 Proof of Theorem 3.5.4

Proof of Theorem 3.5.4. Suppose problem $(GLQ)_T$ satisfies the exponential turnpike property at some steady state (x_e, u_e) and $\mathcal{N}$ is some bounded neighborhood of 0 in $\mathcal{H}$.

By the exponential turnpike property, there exists some positive constants M_1 and $k > 0$ such that for any $y \in 2\mathcal{N}$ and $t \in [0, T]$,

$$\|x_T^*(t, x_e + y) - x_e\| + \|u_T^*(t, x_e + y) - u_e\| \leq M_1(e^{-kt} + e^{-k(T-t)}). \tag{3.36}$$

Observe that for any $T > 0$ and $x_0 \in \mathcal{H}$, the trajectory

$$\widetilde{x}_T^*(\cdot, x_0) := x_T^*(\cdot, x_e + 2x_0) - x_T^*(\cdot, x_e + x_0)$$

defined on $[0, T]$ satisfies $x_T^*(0, x_0) = x_0$ and

$$\begin{aligned} \frac{d\widetilde{x}_T^*(t, x_0)}{dt} =& A\widetilde{x}_T^*(t, x_0) + B(u_T^*(t, x_e + 2x_0) - u_T^*(t, x_e + x_0)) \\ =& (A - B(K^*K)^{-1}B^*P(T-t))\widetilde{x}_T^*(t, x_0) \end{aligned}$$

for any $t \in [0, T]$ (see Corollary 3.4.4). So, $\widetilde{x}_T^*(\cdot, x_0)$ is the optimal trajectory of problem $(LQ)_T$ corresponding to time horizon T and initical condition x_0. Similarly, define

$$\widetilde{u}_T^*(\cdot, x_0) := u_T^*(\cdot, x_e + 2x_0) - u_T^*(\cdot, x_e + x_0)$$

on $[0, T]$. Since for any $t \in [0, T]$,

$$u_T^*(t, x_e + 2x_0) - u_T^*(t, x_e + x_0) = -(K^*K)^{-1}B^*P(T-t)\widetilde{x}_T^*(t, x_0),$$

we deduce that $\widetilde{u}_T^*(\cdot, x_0)$ is the optimal control of problem $(LQ)_T$.

Now, notice that for any $t \in [0, T]$,

$$\begin{aligned} \|\widetilde{x}_T^*(t, x_0)\| &= \|x_T^*(t, x_e + 2x_0) - x_T^*(t, x_e + x_0)\| \\ &\leq \|x_T^*(t, x_e + 2x_0) - x_e\| + \|x_T^*(t, x_e + x_0) - x_e\|, \end{aligned}$$

and

$$\begin{aligned} \|\widetilde{u}_T^*(t, x_0)\| &= \|u_T^*(t, x_e + 2x_0) - u_T^*(t, x_e + x_0)\| \\ &\leq \|u_T^*(t, x_e + 2x_0) - u_e\| + \|u_T^*(t, x_e + x_0) - u_e\|. \end{aligned}$$

Combining these two estimates with equation (3.36), we obtain

$$\|\widetilde{x}_T^*(t,x_0)\| + \|\widetilde{u}_T^*(t,x_0)\| \le 2M_1(e^{-kt} + e^{-k(T-t)}), \quad \forall t \in [0,T].$$

Since $\mathcal{N}, T$ and x_0 can all be arbitrarily chosen, the exponential turnpike property is satisfied for problem $(LQ)_T$ at $(0,0)$. Now by Corollary 3.4.4, there exists a vector $w \in D(A^*)$ satisfying the algebraic condition

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} w = \begin{bmatrix} z + C^*Cx_e \\ v + K^*Ku_e \end{bmatrix}.$$

Conversely, assume that problem $(LQ)_T$ satisfies the exponential turnpike property at $(0,0)$ and there exists a vector $w \in D(A^*)$ satisfying the above algebraic condition. Recall the optimal trajectory x^* of problem $(LQ)_T$ corresponding to time horizon $T > 0$ and initial condition $x_0 \in \mathcal{H}$ is given by

$$x^*(t) = U_T(t,0)x_0, \quad \forall t \in [0,T].$$

Let $\mathcal{N}$ be the closed unit ball in $\mathcal{H}$ with center 0, then the exponential turnpike property implies there exist some positive constants M_0 and k_0 such that

$$\|U_T(t,0)x_0\| \le M_0(e^{-k_0 t} + e^{-k_0(T-t)}), \quad \forall x_0 \in \mathcal{N}, T > 0, t \in [0,T].$$

Equivalently,

$$\|U_T(t,0)\| \le M_0(e^{-k_0 t} + e^{-k_0(T-t)}), \quad \forall T > 0, t \in [0,T]. \tag{3.37}$$

Also notice, for any $T > 0$

$$U_T(T,0) = U_T\left(T, \frac{T}{2}\right) U_T\left(\frac{T}{2}, 0\right) = U_{\frac{T}{2}}\left(\frac{T}{2}, 0\right) U_T\left(\frac{T}{2}, 0\right),$$

so

$$\|U_T(T,0)\| \le M_0(e^{-k_0\frac{T}{2}} + 1)2M_0 e^{-k_0\frac{T}{2}} \le 4M_0^2 e^{-\frac{k_0}{2}T}. \tag{3.38}$$

On the other hand, Theorem 3.4.3 (c) and Lemma 3.3.5 implies $\ker \begin{bmatrix} A^* \\ B^* \end{bmatrix} = \{0\}$, so w is the unique vector in $D(A^*)$ satisfying

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} w = \begin{bmatrix} z + C^*Cx_e \\ v + K^*Ku_e \end{bmatrix}.$$

Following line by line the proof of Lemma 3.3.11, we deduce that the optimal control of problem $(GLQ)_T$ is given in a feedback law form by

$$\begin{aligned} u_T^*(t,x_0) - u_e = {} & -(K^*K)^{-1}B^*P(T-t)(x_T^*(t,x_0) - x_e) \\ & - (K^*K)^{-1}B^*(U_{T-t}(T-t,0))^*w, \quad \forall t \in [0,T]. \end{aligned}$$

Then it follows for any $t \in [0,T]$,

$$x_T^*(t,x_0) - x_e = U_T(t,0)(x_0 - x_e) \\ - \int_0^t U_{T-s}(t-s,0)B(K^*K)^{-1}B^*(U_{T-s}(T-s,0))^* w ds.$$

Simple considerations to (3.37), (3.38) and the above equation show that there exists some positive constant M_1 such that for any $t \in [0,T]$,

$$\|x_T^*(t,x_0) - x_e\| \leq M_0(e^{-k_0 t} + e^{-k_0(T-t)})\|x_0 - x_e\| \\ + \int_0^t M_1(e^{-k_0(t-s)} + e^{-k_0(T-t)})e^{-\frac{k_0}{2}(T-s)} ds.$$

A straightforward computation shows

$$\|x_T^*(t,x_0) - x_e\| \leq M_0(e^{-k_0 t} + e^{-k_0(T-t)})\|x_0 - x_e\| + M_2 e^{-\frac{k_0}{2}(T-t)}, \quad \forall t \in [0,T] \tag{3.39}$$

holds with some constant $M_2 > 0$ (independent of T).

Finally, recall

$$u_T^*(t,x_0) - u_e = -(K^*K)^{-1}B^*P(T-t)(x_T^*(t,x_0) - x_e) \\ - (K^*K)^{-1}B^*(U_{T-t}(T-t,0))^* w, \quad \forall t \in [0,T].$$

By uniform boundedness principle, (3.38) and (3.39), there exist some $M_3, M_4 > 0$ (independent of T) such that

$$\|u_T^*(t,x_0) - u_e\| \leq M_3(e^{-k_0 t} + e^{-k_0(T-t)})\|x_0 - x_e\| + M_4 e^{-\frac{k_0}{2}(T-t)}, \quad \forall t \in [0,T]. \tag{3.40}$$

Now it is trivial to see from (3.39) and (3.40) that problem $(GLQ)_T$ satisfies the exponential turnpike property at (x_e, u_e). $\square$

3.6 Examples

In this section, we will discuss some applications of our results.

3.6.1 Parabolic equations

Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary $\partial\Omega$ of C^2. Let $\mathcal{H} = \mathcal{U} = \mathcal{Y} = L^2(\Omega)$, and B, C and K be arbitrary operators in $\mathcal{L}(L^2(\Omega))$. We define $A : D(A) \to \mathcal{H}$ by

$$\begin{aligned} &D(A) = H^2(\Omega) \cap H_0^1(\Omega), \\ &Ah = (\Delta + cI)h, \quad \forall h \in D(A). \end{aligned}$$

where $c \in \mathbb{R}$. By [6, Paty IV, Chapter 1, Section 8.1], A generates a strongly continuous (in fact, analytic) semigroup on $\mathcal{H}$. Now the above functional framework is well suited to describe the distributed control of the following parabolic equation with Dirichlet boundary conditions:

$$\begin{cases} \dfrac{\partial h}{\partial t}(x,t) = (\Delta_x + c)h(x,t) + B(u(\cdot,t))(x), \ \text{ in } \ \Omega \times [0,T], \\ h(\cdot, 0) = h_0 \in L^2(\Omega), \\ h(t,x) = 0, \ \text{ on } \ \partial\Omega \times [0,T]. \end{cases}$$

Let $z, v \in L^2(\Omega)$. Consider the optimal control problem: To minimize

$$J_T(h_0, u) = \int_0^T \int_\Omega |(Ch(\cdot,t))(x) - z(x)|^2 + |(Ku(\cdot,t))(x) - v(x)|^2 dxdt$$

over all $u \in L^2((0,T), L^2(\Omega))$. Obviously this cost functional only differs from the one given in $(GLQ)_T$ by a constant, so our results on turnpike property is valid for this system.

By [41, Remark 3.6.4], $-\Delta$ is a strictly positive operator with compact resolvents, so by spectral theorem, A is diagonalizable with a orthonormal basis $(\psi_k)_{k\in\mathbb{N}}$ of eigenvectors such that the corresponding sequence of eigenvalues $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ is decreasing and satisfies $\lambda_k \to -\infty$ as $k \to \infty$. Since (see [41, Section 2.6])

$$\mathcal{T}_t x = \sum_{k=0}^{\infty} e^{\lambda_k t}\psi_k, \ \ \forall x \in \mathcal{H}, t \in [0, \infty),$$

$\mathcal{T}_t$ is compact for any $t > 0$. So by Remark 3.5.1, this problem fulfills assumptions ($\mathcal{PS}$).

If $\lambda_0 < 0$, then $\mathcal{T}$ is exponentially stable, so the pair (A, B) and (A, C) are exponentially stabilizable and exponentially detectable for any $B, C \in \mathcal{L}(L^2(\Omega))$, respectively. If there exists $m \in \mathbb{N}$ such that $\lambda_m \geq 0$ and $\lambda_{m+1} < 0$, then by Lemma 3.5.5, the pair (A, C) (resp. (A, B)) is exponentially detectable (resp. exponentially stabilizable) if and only if

$$\ker(\lambda_i I - A) \cap \ker C = \{0\} \ \ (\text{resp. } \ker(\lambda_i I - A^*) \cap \ker B^* = \{0\}), \ \ \forall i = 0, 1, \ldots, m.$$

As a consequence, the exponential turnpike property is equivalent to the measure turnpike property in this case, and it holds if and only if the above Hautus type conditions hold.

3.6.2 Wave equations

Let Ω be as in last example. Let $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$ and $\mathcal{U} = \mathcal{Y} = L^2(\Omega)$. Then $\mathcal{H}$ is a Hilbert space endowed with the inner product

$$\left\langle \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \right\rangle := \int_\Omega \frac{df_1}{dx}(s)\overline{\frac{df_2}{dx}}(s) + g_1(s)\overline{g_2(s)}ds, \;\; \forall \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}, \begin{bmatrix} f_2 \\ g_2 \end{bmatrix} \in \mathcal{H}.$$

We define $A : D(A) \to \mathcal{H}$ and $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ by

$$\begin{aligned} &D(A) := \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega), \\ &A \begin{bmatrix} f \\ g \end{bmatrix} := \begin{bmatrix} g \\ \Delta f \end{bmatrix}, \quad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in D(A), \\ &Bu := \begin{bmatrix} 0 \\ u \end{bmatrix}, \; \forall u \in L^2(\Omega). \end{aligned}$$

By [41, Proposition 3.7.6] and Stone's theorem, A generates a unitary semigroup, that is, A generates a C_0-semigroup $\mathcal{T}$ satisfying

$$\|\mathcal{T}_t x\| = \|x\|, \;\; \forall x \in \mathcal{H}, t \geq 0.$$

By [41, Example 11.2.2], the pair (A, B) is exactly controllable, thus also exponentially stabilizable. Next, we let $K = I \in \mathcal{L}(\mathcal{U})$ and denote

$$h_0(\cdot) = \begin{bmatrix} h_{0,1}(\cdot) \\ h_{0,2}(\cdot) \end{bmatrix} \in \mathcal{H}.$$

Then the above functional framework is well suited to describe the distributed control of the following wave equation:

$$\begin{cases} \dfrac{\partial^2 h}{\partial t^2}(x,t) = \Delta_x h(x,t) + u(x,t), \;\; \text{in} \;\; \Omega \times [0,T], \\ h(\cdot,0) = h_{0,1}(\cdot) \in H_0^1(\Omega), \;\; h_t(\cdot,0) = h_{0,2}(\cdot) \in L^2(\Omega), \\ h(x,t) = 0, \;\; \text{on} \;\; \partial\Omega \times [0,T]. \end{cases} \tag{3.41}$$

Now, We denote

$$z(\cdot) = \begin{bmatrix} z_1(\cdot) \\ z_2(\cdot) \end{bmatrix} \in \mathcal{H}, \;\; v = v(\cdot) \in \mathcal{U}.$$

and define $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$ by

$$C \begin{bmatrix} f \\ g \end{bmatrix} = g, \quad \forall \begin{bmatrix} f \\ g \end{bmatrix} \in \mathcal{H}.$$

Consider the optimal control problem $(GLQ)_T$: To minimize

$$J_T(h_0(\cdot), u) = \operatorname{Re} \int_0^T \int_\Omega |h_t(x,t)|^2 + 2\nabla_x h(x,t) \cdot \overline{\nabla z_1(x)} + 2h_t(x,t)\overline{z_2(x)} \\ + |u(x,t)|^2 + 2u(x,t)\overline{v(x)} dx dt.$$

Thanks to [41, Theorem 7.4.1], the pair (A, C) is exactly observable, thus also exponentially detectable, so by Theorem 3.3.1, the turnpike property is satisfied for problem $(GLQ)_T$.

Notice for any $\begin{bmatrix} f \\ g \end{bmatrix} \in D(A)$ and $u \in \mathcal{U}$,

$$A \begin{bmatrix} f \\ g \end{bmatrix} + Bu = 0 \Longleftrightarrow g = 0 \text{ and } \Delta f + u = 0.$$

and if $\left(\begin{bmatrix} f \\ g \end{bmatrix}, u \right)$ is a steady state, $g = 0$ implies

$$\ell \left(\begin{bmatrix} f \\ g \end{bmatrix}, u \right) = \operatorname{Re} \int_\Omega 2\nabla f(x) \cdot \overline{\nabla z_1(x)} + |\Delta f(x)|^2 - 2\Delta f(x)\overline{v(x)} dx.$$

Since $z_1 \in H_0^1(\Omega)$, $Dz_1(\cdot) = 0$ on $\partial\Omega$, where $Dz_1(\cdot)$ is the Dirichlet trace of z_1. So,

$$\int_\Omega \nabla f(x) \cdot \overline{\nabla z_1(x)} dx = \int_{\partial\Omega} Nf(x)\overline{Dz_1(x)} dx - \int_\Omega \Delta f(x)\overline{z_1(x)} dx \\ = -\int_\Omega \Delta f(x)\overline{z_1(x)} dx,$$

where $Nf(\cdot)$ is the Neumann trace of f on $\partial\Omega$.

Combining the two equations, we easily see that the optimal steady state $\left(\begin{bmatrix} f_e \\ g_e \end{bmatrix}, u_e \right)$ is uniquely characterized by

$$g_e = 0 \text{ and } \Delta f_e = u_e = z_1 + v.$$

Let u^* denote the optimal control and h^* denote the solution of PDE (3.41) corresponding to control u^*. In this case, the turnpike property ensures that for sufficiently large $T > 0$, $h^*(\cdot, t)$,

$\nabla_x h^*(\cdot,t)$ $h_t^*(\cdot,t)$ and $u^*(\cdot,t)$ will stay, for most of the time horizon, close to $f_e(\cdot)$, $\nabla f_e(\cdot)$, 0 and $(z_1+v)(\cdot)$ in L^2-sense respectively (the argument with respect to $h^*(\cdot,t)$ and $f_e(\cdot)$ is a straight consequence of Poincare's inequality).

From the definition of exponential turnpike property, it is trivial to show that (under the setting of Definition 3.2.2) there exist some constants $M, w > 0$ such that for any $t \in [0,T]$,

$$\|\nabla h^*(\cdot,t) - f_e(\cdot)\| + \|h_t^*(\cdot,t)\| + \|u^*(\cdot,t) - (z_1+v)(\cdot)\| \le M(e^{-wt} + e^{-w(T-t)}).$$

3.6.3 Delay equations

Consider the retarded differential equations of the following type:

$$\begin{cases} \dfrac{dh}{dt}(t) = A_0 h(t) + \displaystyle\sum_{i=1}^{N} A_i h(t-t_i) + B_0 u, \ t \ge 0 \\ h(0) = h_0 \\ h(\theta) = f(\theta), \ \text{a.e.}, \ \theta \in [-t_N, 0] \end{cases}$$

where $0 < t_1 < t_2 < \cdots < t_N$ represent the point delays, $A_i \in \mathcal{L}(\mathbb{C}^n)$, $i = 0, 1, \cdots, N$, $B_0 \in \mathcal{L}(\mathbb{C}^m, \mathbb{C}^n)$, $h_0 \in \mathbb{C}^n$ and $f \in L^2((-t_N, 0), \mathbb{C}^n)$.

We are interested in the following optimal control problem: To minimize

$$J_T((h_0, f(\cdot)), u) = \int_0^T \|C_0 h(t) - z_0\|^2 + \|Ku(t) - v_0\|^2 dt \tag{3.42}$$

over all $u \in L^2((0,T), \mathbb{C}^m)$, where $C_0 \in L(\mathbb{C}^n, \mathbb{C}^r)$ and $K \in \mathcal{L}(\mathcal{U})$.

We refer to [11, Section 3.3] for the following semigroup framework used to analyse the above system: Let $\mathcal{H} = \mathbb{C}^n \times L^2((-t_N, 0), \mathbb{C}^n)$ and $\mathcal{U} = \mathbb{C}^m$. Define $A : D(A) \to \mathcal{H}$ by

$$\begin{aligned} D(A) &= \left\{ \begin{bmatrix} x \\ g \end{bmatrix} \in \mathbb{C}^n \times H^1((-t_N, 0), \mathbb{C}^n) \ \middle| \ g(0) = x \right\}, \\ A \begin{bmatrix} x \\ g \end{bmatrix} &:= \begin{bmatrix} A_0 x + \sum_{i=1}^N A_i g(-t_i) \\ \dfrac{dg}{dt} \end{bmatrix}, \quad \forall \begin{bmatrix} x \\ g \end{bmatrix} \in D(A), \\ Bu &:= \begin{bmatrix} B_0 u \\ 0 \end{bmatrix}, \ \forall u \in \mathcal{U}. \end{aligned}$$

Let $\mathcal{Y} = \mathbb{C}^r$, and $C \in L(\mathcal{H}, \mathcal{Y})$ be defined by

$$C \begin{bmatrix} x \\ g \end{bmatrix} = C_0 x, \ \forall \begin{bmatrix} x \\ g \end{bmatrix} \in \mathcal{H}.$$

If we denote

$$z = \begin{bmatrix} -C_0^* z_0 \\ 0 \end{bmatrix}, \ v = -K^* v_0,$$

then it is easy to verify that problem $(GLQ)_T$ is equivalent to problem (3.42).

By [11, Theorem 8.2.5], the pair (A, B) is exponentially stabilizable if and only if

$$\operatorname{ran}(sI - A_0 - \sum_{i=1}^{N} A_i e^{-st_i}, B_0) = \mathbb{C}^n, \ \forall s \in \overline{\mathbb{C}^+},$$

where $\overline{\mathbb{C}^+}$ is the set of complex numbers with non-negative real part. Similarly, the pair (A, C) is exponentially stabilizable if and only if

$$\ker \begin{bmatrix} sI - A_0 - \sum_{i=1}^{N} A_i e^{-st_i} \\ C_0 \end{bmatrix} = \{0\}, \ \forall s \in \overline{\mathbb{C}^+}.$$

So, we have some simple conditions to determine the (exponential) stabilizability and detectability, thus also the turnpike property. For this problem, if the (exponential) stabilizability and detectability are verified, then the turnpike property will be satisfied for problem $(OCP)_T$. In particular, (under the setting of Definition 3.2.2) the following estimate can be easily derived:

$$|h^*(t) - x_e| + |u^*(t) - u_e| \leq M(e^{-wt} + e^{-w(T-t)}), \quad \forall t \in [0, T],$$

where u^* is the optimal control, h^* is the corresponding solution and $\left(\begin{bmatrix} x_e \\ f_e \end{bmatrix}, u_e \right)$ is the unique optimal steady state.

3.6.4 Model predictive control (MPC)

MPC is a controller used for infinite time horizon (or very large finite time horizon) optimal control problems. The idea of MPC is to repeatedly solve the optimal control problem on a shorter time horizon and then apply the optimal control to a small time slot. Compared to LQR, the advantage of MPC is that it is more flexible when the system varies with time (e.g. automated driving system) since MPC optimizes the current time slot, while also taking future time slots into account.

In this subsection, we prove that the turnpike property for problem $(LQ)_T$ implies the effectiveness of the continuous-time MPC. So if the turnpike property is observed in some real-world implementation, it is reasonable to consider MPC as a controller which balances the flexibility of the time varying system and the long-term operating cost. Notice that in practical applications, the term 'MPC' is usually referred to the discrete-time MPC because the optimal control of a continuous-time system is usually not available, especially for nonlinear systems. For a continuous-time system, people often first discretize the system to a discrete-time one, then use the optimal control of the corresponding finite time horizon discrete-time problem (which can be more easily solved) to approximate the optimal control in continuous-time setting.

Theorem 3.6.1. *Suppose that all the assumptions of $(LQ)_T$ are verified. If the exponential turnpike property holds, then for any $\varepsilon > 0$ and bounded subset $\mathcal{N}$ of $\mathcal{H}$, there exist some $T > 0$ and $\tau \in (0, T)$ such that the trajectory x, with arbitrary initial condition $x_0 \in \mathcal{N}$, defined recursively by*

$$x(t) = \begin{cases} x_0, & t = 0, \\ x_T^*(t - n\tau, x(n\tau)), & t \in (n\tau, (n+1)\tau], \quad n \in \mathbb{N}, \end{cases} \tag{3.43}$$

and the corresponding control u, x given by

$$u(t) = u_T^*(t - n\tau, x(n\tau)), \quad t \in [n\tau, (n+1)\tau), \quad n \in \mathbb{N} \tag{3.44}$$

satisfies that

$$\int_0^\infty \ell(x(t), u(t))dt < \langle P_{\min} x_0, x_0 \rangle + \varepsilon. \tag{3.45}$$

If the measure turnpike property holds, then for any $\varepsilon > 0$ and $x_0 \in \mathcal{N}$ and bounded subset $\mathcal{N}$ of $\mathcal{H}$, there exist $T > 0$ and a sequence $(\tau_n)_{n\in\mathbb{N}}$ satisfying $\tau_0 = 0$ and $0 < \tau_{n+1} - \tau_n < T$, $n \in \mathbb{N}$ such that the trajectory x, with arbitrary initial condition $x_0 \in \mathcal{N}$, defined recursively by

$$x(t) = \begin{cases} x_0, & t = 0, \\ x_T^*(t - \tau_n, x(\tau_n)), & t \in (\tau_n, \tau_{n+1}], \quad n \in \mathbb{N}, \end{cases} \tag{3.46}$$

and the corresponding control u, given by

$$u(t) = u_T^*(t - \tau_n, x(\tau_n)), \quad t \in [\tau_n, \tau_{n+1}), \quad n \in \mathbb{N}$$

satisfies equation (3.45).

Notice that x in the first (resp. second) case is the trajectory generated by applying the MPC controller which optimizes the system over a time horizon T at each step and applies the optimal control to a time slot τ (resp. $\tau_{n+1} - \tau_n$, $n \in \mathbb{N}$). Also recall that the term $\langle P_{\min} x_0, x_0 \rangle$ is the minimum cost of the infinite time horizon LQ optimal control problem. See Proposition 2.2.6 (c).

Proof. Assume the exponential turnpike property is satisfied. Since for problem $(LQ)_T$,

$$x_T^*(t, \lambda x_0) = \lambda x_T^*(t, x_0), \ \forall x_0 \in \mathcal{H}, \ \lambda \in \mathbb{C}, \ 0 \le t \le T, \tag{3.47}$$

simple consideration shows that there exist constants $M, w > 0$ such that

$$\|x_T^*(t, x_0)\| \le M \|x_0\| (e^{-wt} + e^{-w(T-t)}), \ \forall x_0 \in \mathcal{H}, \ 0 \le t \le T.$$

Fix some $x_0 \in \mathcal{H}$ and $0 < \tau < T$. Let x and u be defined as (3.43) and (3.44). By induction,

$$\|x(n\tau)\| \le M^n \|x_0\| (e^{-w\tau} + e^{-w(T-\tau)})^n, \ \ \forall n \in \mathbb{N}.$$

Notice that for any $n \in \mathbb{N}$,

$$\int_{n\tau}^{(n+1)\tau} \ell(x(t), u(t)) = \langle P(T) x(n\tau), x(n\tau) \rangle - \langle P(T-\tau) x((n+1)\tau), x((n+1)\tau) \rangle,$$

so if $M(e^{-w\tau} + e^{-w(T-\tau)}) < 1$,

$$\begin{aligned}
\int_0^\infty \ell(x(t), u(t)) &= \langle P(T) x_0, x_0 \rangle + \sum_{n=1}^\infty (\langle P(T) x(n\tau), x(n\tau) \rangle - \langle P(T-\tau) x(n\tau), x(n\tau) \rangle) \\
&\le \langle P_{\min} x_0, x_0 \rangle + \sum_{n=1}^\infty \|P(T)\| M^{2n} \|x_0\|^2 (e^{-w\tau} + e^{-w(T-\tau)})^{2n} \\
&= \langle P_{\min} x_0, x_0 \rangle + \|P(T)\| \|x_0\|^2 \frac{M^2 (e^{-w\tau} + e^{-w(T-\tau)})^2}{1 - M^2 (e^{-w\tau} + e^{-w(T-\tau)})^2}.
\end{aligned}$$

Now it is trivial to see that (3.45) holds for some suitable $0 < \tau < T$.

Now assume the measure turnpike property is satisfied. Similar to the first case, simple consideration shows that for any $\delta > 0$, there exists some $M_\delta > 0$ such that for any $T > 0$,

$$\mu \left\{ t \in [0, T] \ \middle| \ \|x_T^*(t, x_0)\| + \|u_T^*(t, x_0)\| > \delta \|x_0\| \right\} \le M_\delta, \ \forall x_0 \in \mathcal{H}, \ 0 \le t \le T.$$

So, if we set $\tau_0 = 0$ and $T > M_\delta$, then there exists a point $\tau_1 \in [0, T]$ such that

$$\|x_T^*(\tau_1, x_0)\| \le \delta \|x_0\|.$$

By repeating this step recursively, we can define x and u as (3.46) and (3.47) on $[0, \infty)$ and obtain a sequence $(\tau_n)_{n\in\mathbb{N}}$ such that $0 < \tau_{n+1} - \tau_n < T$ and $\|x(\tau_{n+1})\| \le \delta\|x(\tau_n)\|$, $n \in \mathbb{N}$.

Similar to the first case, we have

$$\begin{aligned}\int_0^\infty \ell(x(t), u(t)) &= \langle P(T)x_0, x_0\rangle + \sum_{n=1}^\infty (\langle P(T)x(n\tau), x(n\tau)\rangle - \langle P(T-\tau)x(n\tau), x(n\tau)\rangle) \\ &\le \langle P_{\min}x_0, x_0\rangle + \sum_{n=1}^\infty \|P(T)\|\delta^{2n}\|x_0\|^2 \\ &\le \langle P_{\min}x_0, x_0\rangle + \|P(T)\|\|x_0\|^2 \frac{\delta^2}{1-\delta^2}.\end{aligned}$$

Now the theorem follows easily by choosing a sufficiently small δ.

□

Chapter 4

Conclusions and open problems

In this thesis, we worked on the characterization of turnpike property for infinite dimensional generalized linear quadratic optimal control problem. Chapter 1 is devoted to the introduction of this problem. In section 1.1, we presented the history and important results of the optimal control theory for infinite dimensional systems. The LQ optimal control problem is of particular interest since it admits a closed form solution of the optimal control, so we briefly discussed the existing results on LQ optimal control problem in various infinite dimensional settings. Finally, we introduced the framework of the generalized LQ optimal control problem. In section 1.2, we recalled the development of the theory of turnpike property. We also gave a list of references on the existing results of turnpike property in different settings. Section 1.3 is devoted to the organization of this paper.

In chapter 2, we aimed to introduce all the necessary background on infinite dimensional generalized LQ optimal control problem. In section 2.1, we recalled the definition and properties of operator semigroups, which is the basis of infinite dimensional evolution problem. In section 2.2.1, some results concerning Pontryagin's Maximum principle for infinite dimensional systems were given in order to help understand the Hamiltonian systems, optimal adjoint state and algebraic condition appearing in later literature. In section 2.2.2, we introduced the standard results on the solution of the differential and algebraic Riccati equation. Riccati equation is a basic tool in our analysis. It has allowed us to solve the explicit formula for the optimal control and establish the double-sided exponential estimate of the optimally controlled system.

We presented our main results in chapter 3. In section 3.1, we first proved the existence and uniqueness of the optimal pair for infinite dimensional generalized LQ optimal control problem. Then in section 3.2, two important notions of turnpike property, the measure and the exponential turnpike property, together with the methodologies to deduce them were discussed. In section 3.3, it was proved that the (exponential) stabilizability and detectability of the system is a sufficient condition for the exponential turnpike property. While the necessary condition for the turnpike property seems to be multifold. In section 3.4, we proved several necessary conditions of turnpike property in terms of the detectability, stabilizablilty and the turnpike reference of the control system. Based on these results, we showed in section 3.5 that the turnpike property can be completely characterized by the exponential stabilizability and detectability for the finite dimensional case and point spectrum case. We also provided an equivalence result on the turnpike property for generalized LQ optimal control problem and the turnpike property for the LQ optimal control problem plus an algebraic condition. After this, we discussed several applications of our theoretical results. The parabolic equations, wave equations, delay equations and model predictive control (MPC) were investigated in subsection 3.6.1, 3.6.2, 3.6.3 and 3.6.4 respectively.

In the following subsections, we take a glance at some open problems for future research.

4.1 Regarding unbounded input

The motivation behind unbounded input mainly comes from various boundary control systems. The term "boundary" here refers to that the actuation and sensing are through the boundary conditions of the system. This feature is very practical and important in real applications.

Roughly speaking, the systems described by PDEs with non-homogeneous boundary conditions appear in the following form:

$$\frac{dx}{dt}(t) = Lx(t), \quad Gz(t) = u(t),$$

where L is some differential operator and G is some boundary trace operator. With some basic assumptions, the above equation can be reformulated into an abstract evolution problem described by $\dot{x} = Ax + Bu$. Though in this thesis we have considered this standard form of infinite dimensional control systems, the control operator B is always assumed to be bounded. It can be seen from our PDE examples in section 3.6.1 and 3.6.2 that, as our B is bounded, the control is distributed and inside the domain. However, the control on the boundary is more realistic in physical world, and that will often lead us to some unbounded control operator. Here we refer to [41, Chapter 10] and the references therein for the definition of well-posed boundary control systems and how to translate such a system into our familiar form $\dot{x} = Ax + Bu$.

Unfortunately, systems with an unbounded control operator are often far more technical than those with a bounded control operator. In fact, unless additional assumptions are introduced, the mild solution may not necessarily take value in $\mathcal{H}$, but perhaps in $\mathcal{H}^{-1}$, which sometimes is not even a real function space, but a space of distributions, or perhaps can only be defined in L^2-sense, and is not continuous in $\mathcal{H}$.

The most popular setting which retains a broad level of generality is to assume B is an admissible control operator. Unfortunately, to the best of our knowledge, it is not possible to perfectly extend the theory for LQ optimal control problem with bounded input to unbounded setting. As noted in [42]: 'In the formula linking the optimal feedback operator to the optimal cost operator, as well as in the Riccati equation, the weighting operator of the input has to be replaced by another operator, which can be derived from the spectral factorization of the Popov function', 'Despite its simplicity, this is a "nasty" problem when we want to reconcile it with the existing LQ optimal control theory: various unbounded operators pop up and their domains do not match'. Relevant work can be found in, e.g., [16, 42].

The unbounded input case is also studied under another class of assumptions that have received wide attention. More precisely, A is assumed to be the generator of some analytic semi-group, and the unboundedness of B is restricted by the condition

$$D = (sI - A)^{-1}B, \quad D \in L(\mathcal{U}, D(A^\alpha)).$$

Since the same formulas relating Riccati equations to state feedback operator still hold in this case, it is conceivable that similar results as in our thesis can also be proved in this setting, but perhaps with suitable extensions or changes. Due to the technical difficulty of this problem, we have not attempted to provide complete explanation to all the details in this thesis. Instead, we refer to [6, Part IV and Part V, Chapter 2] and [29, Chapter 9] for further discussions.

4.2 Regarding constrained case

Similar to the unbounded input case, systems with state and input constraints are also more practical in real applications, especially for finite dimensional systems. However, it is very difficult from both the mathematics and engineering point of view to seek a closed form solution for a constrained optimal control problem. To the best of our knowledge, there is no perfect extension of the Riccati theory for the feedback controls in the constrained case.

However, the method which exploits the connection between turnpike and dissipativity properties of the control system can be utilized to find the sufficient condition for turnpike property even with the presence of state and input constraints. As first introduced by Willems in [43, 44], dissipativity describes the abstract energy balance of a dynamical system in terms of the stored and the supplied energy. This kind of method has motivated the definition of measure turnpike property.

Since the dissipativity-based method does not rely on the structure of the optimality system, instead of providing a certain quantitative estimate, the measure turnpike property only guarantees that when T is large enough, the distance between the optimal pair and optimal steady state can be arbitrarily small for all time horizon but on a set with measure no more than a constant.

Strict dissipativity and strict pre-dissipativity are two notions useful in the characterization of turnpike properties for generalized LQ optimal control problem. The following definition of strict dissipativity and strict pre-dissipativity is cited from [22].

Definition 4.2.1. *Let $\mathcal{K}$ denote the set of dissipation rates:*

$$\mathcal{K} := \big\{\alpha : [0,\infty) \to [0,\infty) \,|\, \alpha \textit{ is continuous} \\ \textit{and strictly increasing with } \alpha(0) = 0\big\}.$$

A storage function on $\mathcal{H}$ is a continuously (Fréchet) differentiable function $V : \mathcal{H} \to \mathbb{R}$.

We say that problem $(GLQ)_T$ is strictly pre-dissipative at some steady state (x_e, u_e) if there exists a storage function V on $\mathcal{H}$ and a dissipation rate $\alpha \in \mathcal{K}$ such that for all $x \in D(A)$ and all $u \in \mathcal{U}$ we have

$$V'(x)(Ax + Bu) \;\leq\; \ell(x, u) - \ell(x_e, u_e) - \alpha(\|x - x_e\|).$$

If the storage function V is bounded from below, we say problem $(GLQ)_T$ is strictly dissipative at (x_e, u_e).

Their connection between dissipativity notions and the measure turnpike property has been studied with full details in [17, 18] for finite dimensional generalized LQ optimal control problems in discrete and continuous time respectively, even in the presence of state and control constraints. In particular, in [17, 18], the necessary condition of the measure turnpike property is first deduced for problem $(GLQ)_T$, which has motivated our results in section 3.4.

Now let us discuss the setting for unconstrained case. We use X and U to denote the set of state and input constraints, i.e., we require $x(t) \in X$ and $u(t) \in U$ almost everywhere on $[0, T]$. We assume X and U are both closed convex sets. The measure turnpike property for constrained generalized LQ optimal control problem is still defined in the same way, but now with bounded $\mathcal{N} \subset X$. We illustrate the results in [18] for constrained case by the following block diagram:

every eigenvalue of $A|_{U^\infty}$ has nonzero real part, and the unique optimal steady state (which does exist) $(x_e, u_e) \in \text{int}\,(X \times U)$

$$\Updownarrow$$

strict dissipativity at $(x^e, u^e) \in \text{int}\,(X \times U)$

$$(A, B) \text{ stabilizable} \Downarrow$$

measure turnpike property at $(x^e, u^e) \in \text{int}\,(X \times U)$

Figure 4.1: Block diagram for unconstrained case

Note that even for finite dimensional case, one does not know if the measure turnpike property for constrained case implies the real part of any unobservable eigenvalue of A is nonzero (but it can be verified that the pair (A, B) is stabilizable since (x^e, u^e) is a steady state in the interior of $\mathcal{X} \times \mathcal{U}$). We believe that this statement is true, but currently cannot prove it.

Another interesting question is that: Suppose (A, B) is stabilizable and the real part of any unobservable eigenvalue of A is nonzero, but the global optimal steady state is not in the interior of $X \times U$, then if turnpike property will be satisfied at the constrained optimal steady state on the boundary of $X \times U$, i.e., the unique minimizer of the following problem

$$\inf_{x \in X, u \in U} \ell(x, u) \quad \text{s.t.} \;\; Ax + Bu = 0.$$

In fact, it is not hard to verify that ℓ is bounded below and tends to $+\infty$ as $\|x\| + \|u\| \to \infty$, so such a minimizer does exist. Moreover, suppose that (x_1, u_1) and (x_2, u_2) are two minimizers, then we can prove ℓ admits a strictly smaller value at the middle point of the two points. This implies the uniqueness of the minimizer. Besides, the constrained optimal steady state must occur at the boundary since if it is an interior point, by the convexity of this problem, it is an global minimizer, which contradicts to our assumption.

The author believes that if (A, B) is stabilizable and there exists a unique constrained optimal steady state, then the following alternative for long time behavior of optimally controlled system will hold: the optimal pair either stays close to the unique constrained optimal steady state, or a periodic orbit which is nonobservable for most of the time horizon. If we further assume the real part of any unobservable eigenvalue of A is nonzero, then such orbit does not exist. The following illustrative example shows the existence of such a periodic orbit.

Example 4.2.2. *Let*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is easy to verify (A, B) is stabilizable. Let $C = 0$, $K = I$, $z = 0$ and $v = 0$, then the unique global optimal steady state is obviously $(0, 0)$. If we let X and U be the closed unit ball in $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively, then obviously for any $x_0 \in X$ and $T > 0$, the corresponding optimal control is $u \equiv 0$, so the optimal trajectory $x_T(\cdot, x_0)$ will be

$$x_T(t, x_0) = \mathcal{T}_t x_0 = \begin{bmatrix} cos(t)(x_0)_1 + sin(t)(x_0)_2 \\ -sin(t)(x_0)_1 + cos(t)(x_0)_2 \\ e^{-t}(x_0)_3 \end{bmatrix}, \quad \forall t \in [0, T].$$

Now it is clear that the optimal pair will stay close to the orbit

$$\begin{bmatrix} x_p(t) \\ u_p(t) \end{bmatrix} := \begin{bmatrix} cos(t)(x_0)_1 + sin(t)(x_0)_2 \\ -sin(t)(x_0)_1 + cos(t)(x_0)_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \forall t \geq 0$$

for most of the time horizon.

However, instead of letting $C = 0$, if we assume that

$$C \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$$

(equivalently, every nonobservable eigenvalue has a nonzero real part), then by Theorem 8.4 in [18], the measure turnpike property is satisfied at $(\vec{0}, \vec{0})$, so the optimal pair will stay close to $(\vec{0}, \vec{0})$ for most of the time horizon.

For the case that the global optimal steady state does not belong to the interior of $\mathcal{H} \times \mathcal{U}$, we recall [18, Example 8.9] in the following:

Example 4.2.3. *Set $\dot{x}(t) = -2x(t) + u(t)$, $\ell(x,u) = u^2$, $X = [a,b]$, where $0 < a < b$ and $U = \mathbb{R}$. Observe that the unique constrained optimal steady state is $(a, 2a)$, and the optimal control is made up of two stage: 1. $u \equiv 0$ and x decreases to a. 2. $u \equiv 2a$ and $x \equiv a$. So, the measure turnpike property is satisfied at $(a, 2a)$.*

Notice that the turnpike reference of this problem is the constrained optimal steady state.

It is still an open question whether these problems can be addressed by dissipativity-based method. These questions will be investigated in future research.

4.3 Regarding unbounded observation and terminal cost

In this subsection, we discuss the possibility to extend our results to the case that C is an admissible observation operator and there is some terminal cost $P_0 \in \Sigma^+(\mathcal{H})$.

Definition 4.3.1. *We say that $C : D(A) \to \mathcal{Y}$ is an admissible observation operator for $\mathcal{T}$ if there exists some $T \geq 0$ such that the map $\Psi_T : D(A) \to L^2((0,T), \mathcal{Y})$ defined by*

$$(\Psi_T x)(\cdot) = C\mathcal{T}.x, \ \ \forall x \in D(A) \tag{4.1}$$

admits a (unique) bounded extension to $\mathcal{H}$. Equivalently, C is an admissible observation operator if and only if there exists a $T > 0$ and $M_T \geq 0$ such that

$$\|\Psi_T x\| \leq M_T \|x\|, \ \ \forall x \in D(A).$$

The admissibility is a very standard assumption which restricts the unboundedness of observation. The following proposition owns to [41, Proposition 4.3.2].

Proposition 4.3.2. *If C is an admissible observation operator for $\mathcal{T}$, then there exists a function $M : [0, \infty) \to \mathbb{R}^+$ such that*

$$\|\Psi_t x\| \leq M(t)\|x\|, \ \ \forall t \in [0, \infty), x \in D(A).$$

If C is admissible, then we denote the extension of Ψ_t, $\forall t \geq 0$ to $\mathcal{H}$ by the same symbol. In the following proposition, we let $T > 0$ and $L^2((0,T), D(A))$ be endowed with the inner product of $L^2((0,T), \mathcal{H})$. Then $D(A) \times L^2((0,T), D(A))$ is dense in $\mathcal{H} \times L^2((0,T), \mathcal{H})$. The density follows easily from the definition of Bochner integrability. We refer to [41, Section 12.5] for further details.

Proposition 4.3.3. *Suppose C is an admissible observation operator and $T > 0$. Let the operator $\mathcal{F} : D(A) \times L^2((0,T), D(A)) \to L^2((0,T), \mathcal{Y})$ be defined by*

$$\mathcal{F}(x_0, f)(t) := \Psi_t x_0 + \int_0^t C\mathcal{T}_{t-s} f(s)ds, \quad \forall t \in [0,T], (x_0, f) \in D(A) \times L^2((0,T), D(A)).$$

Then $\mathcal{F}$ has a continuous extension to $\mathcal{H} \times L^2((0,T), \mathcal{H})$. In other words, there exists $M > 0$ such that

$$\|\mathcal{F}(x_0, f)\|_{L^2} \leq M(\|x_0\| + \|f\|_{L^2}).$$

We will still use $\mathcal{F}$ to denote the extension of $\mathcal{F}$ to $\mathcal{H} \times L^2((0,T), \mathcal{H})$. Let x denote the solution of problem (2.3) corresponding to initial condition $x_0 \in \mathcal{H}$ and input $u \in L^2((0,T), \mathcal{U})$, then $\mathcal{F}(x_0, Bu)$ is the generalization of Cx in $L^2((0,T), \mathcal{H})$. For a proof of this proposition, we refer to [6, Part IV, Chapter 1, Lemma 6.1].

At this moment, the cost functional J_T is not a-priori defined for admissible C since the term $\int_0^T \|Cx(t)\|^2 dt$ requires $x(t) \in D(A)$ for any t, which is generally not true. However, the above proposition shows that $\mathcal{F}(x_0, Bu)$ is well-defined for any $x_0 \in \mathcal{H}$ and $u \in L^2((0,T), \mathcal{U})$. So, now we are able to consider the optimal control problem of minimizing the following cost functional J_T defined by

$$\begin{aligned} J_T(u, x_0) := \|\mathcal{F}(x_0, Bu)\|_{L^2}^2 + \int_0^T & \|Ku(t)\|^2 + 2\operatorname{Re}\langle z, x(t)\rangle \\ & + 2\operatorname{Re}\langle v, u(t)\rangle dt + \langle P_0 x(T) + r, x(T)\rangle, \end{aligned}$$

where $x_0 \in \mathcal{H}$ is the initial condition and $P_0 \in \Sigma^+(\mathcal{H}), r \in \mathcal{H}$ is the quadratic and linear term of the terminal cost, respectively. In particular, when C is bounded, $P_0 = 0$ and $r = 0$, this definition of J_T coincides with (2.1).

We believe that all the results in this thesis can be generalized to the case of admissible observation and terminal cost. In fact, we have already sketched a proof for Theorem 3.3.1 (the sufficient condition for the turnpike property) in this new setting, but since it would be too lengthy to write a detailed proof here, we will only sketch the idea of some difficult steps.

Basically, to prove Theorem 3.3.1, we only need to generalize the results in each subsection of section 3.3 to the new case. Most of the steps are pretty straightforward and can be easily justified. For instance, since the proof of Lemma 3.3.2, Lemma 3.3.3 and Lemma 3.3.5 does not rely on the boundedness of C and the choice of $P(0)$, we only need to substitute Cx with $\mathcal{F}(x_0, Bu)$ and set $P(0) = P_0$.

The major difficulty is from the proof of Lemma 3.3.4. First, we need to extend the related notions of the solution of differential and algebraic Riccati equation to the admissible observation and terminal cost case. Then, we need to prove the convergence of $P(t)$ to $P_{\min}$ in operator norm. In this thesis, this convergence is proved by utilizing the property of the strong solution of differential Riccati equation to verify the Hamiltonian system arising from Pontryagin's maximum principle. More precisely, we need to prove that $p(t) := P_{\min}x(T-t)$ coincides with the solution of

$$\begin{cases} \dfrac{dx}{dt} = (A - BB^*P_{\min})x \quad \text{in } [0,T], \\ x(0) = x_0 \in \mathcal{H}, \\ \dfrac{dp(t)}{dt} = A^*p(t) + C^*Cx(T-t) \quad \text{in } [0,T], \\ p(0) = P_{\min}x(T), \end{cases}$$

and $\widetilde{p}(t) := P(t)\widetilde{x}(T-t)$ coincides with the solution of

$$\begin{cases} \dfrac{d\widetilde{x}(t)}{dt} = (A - BB^*P(T-t))\widetilde{x}(t) \quad \text{in } [0,T], \\ \widetilde{x}(0) = x_0, \\ \dfrac{d\widetilde{p}(t)}{dt} = A^*\widetilde{p}(t) + C^*C\widetilde{x}(T-t) \quad \text{in } [0,T], \\ \widetilde{p}(0) = P_0\widetilde{x}(T). \end{cases}$$

To show $\widetilde{p}(\cdot) = P(\cdot)x(T-\cdot)$, we need to use the property of strict solution (which says x is differentiable if $x_0 \in D(A)$). This is no longer possible since for admissible C, Cx is only defined in L^2 sense, so the differential should be understood as distributional derivative.

To resolve this problem, we first assume $P_0 = 0$ and show that the cost functional $J_T(B^*\cdot, \cdot)$ as a function of $\widetilde{p} \in L^2((0,T),\mathcal{H})$ and $x_0 \in \mathcal{H}$ can be written as

$$J_T(B^*\widetilde{p}, x_0) = \langle S_1\widetilde{p}, \widetilde{p}\rangle + 2\operatorname{Re}\langle \widetilde{p}, S_2x_0\rangle + \langle S_3x_0, x_0\rangle$$

where S_1 is a strictly positive operator in $\mathcal{L}(L^2((0,T),\mathcal{H}))$, $S_2 \in \mathcal{L}(\mathcal{H}, L^2((0,T),\mathcal{H}))$ and $S_3 \in \Sigma^+(\mathcal{H})$. So, when $x_0 \in \mathcal{H}$ is fixed, $\widetilde{p} = -(S_1)^{-1}S_2x_0$ is the unique minimizer of $J_T(B^*\cdot, x_0)$. Since S_1, S_2, S_3 are known, after some calculation, we can verify that $\widetilde{p}$ coincides with the solution of the second Hamiltonian system.

On the other hand, we know that $P(\cdot)x(T-\cdot)$ is also the minimizer of $J_T(B^*\cdot, x_0)$, by the uniqueness of the minimizer, $\widetilde{p}(\cdot) = P(\cdot)x(T-\cdot)$. Now we can follow the same steps to show that $P(t)$, with initial condition $P_0 = 0$, converges to $P_{\min}$ in norm. Then by the monotonicity and attractivity properties of P (see [6, Part V, Chapter 1, Section 4.2]), $P(t)$ with arbitrary $P_0 \in \Sigma^+(\mathcal{H})$ will also converge to $P_{\min}$ in norm.

An alternative way to prove this is to show the convergence relation as in our Lemma 3.3.4 through Yosida approximation and contraction mapping theorem. The steps will be basically similar to [6, Part II, Chapter 1, Proposition 3.4].

In general, although the proof for the sufficient condition of the turnpike property will be more technical and lengthy if C is unbounded and $P_0 \neq 0$, the idea remains almost the same. The generalization of the necessary conditions for turnpike property in this setting will be investigated in our future research.

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