

Graph Coverings with Few Eigenvalues or No Short Cycles

by

Maxwell Levit

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Combinatorics & Optimization

Waterloo, Ontario, Canada, 2023

© Maxwell Levit 2023

Examining Committee Membership

The following served on the Examining Committee for this thesis. The decision of the Examining Committee is by majority vote.

External Examiner: Matthew Fickus
Professor, Dept. of Mathematics and Statistics,
Air Force Institute of Technology

Supervisor: Chris Godsil
Professor Emeritus, Dept. of Combinatorics & Optimization,
University of Waterloo

Internal Member: Peter Nelson
Professor, Dept. of Combinatorics & Optimization,
University of Waterloo

Internal Member: Jon Yard
Professor, Dept. of Combinatorics & Optimization,
University of Waterloo

Internal-External Member: Jason Bell
Professor, Dept. of Pure Math, University of Waterloo

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis addresses the extent of the covering graph construction. How much must a cover X resemble the graph Y that it covers? How much can X deviate from Y ? The main statistics of X and Y which we will measure are their regularity, the spectra of their adjacency matrices, and the length of their shortest cycles. These statistics are highly interdependent and the main contribution of this thesis is to advance our understanding of this interdependence. We will see theorems that characterize the regularity of certain covering graphs in terms of the number of distinct eigenvalues of their adjacency matrices. We will see old examples of covers whose lack of short cycles is equivalent to the concentration of their spectra on few points, and new examples that indicate certain limits to this equivalence in a more general setting. We will see connections to many combinatorial objects such as regular maps, symmetric and divisible designs, equiangular lines, distance-regular graphs, perfect codes, and more. Our main tools will come from algebraic graph theory and representation theory. Additional motivation will come from topological graph theory, finite geometry, and algebraic topology.

Acknowledgements

Chris: For a seemingly limitless supply of encouragement, patience, and insight.

Sus and Bob: For a lifetime of support and the encouragement to pursue anything and everything of interest, even something as unfathomable as a Ph.D. in Mathematics.

Adina: For a reassuring presence through the darkest hours of this process, and for tolerating my relative absence in our recent windfall of brightness.

Bahir: For a clarity of purpose.

Dedication

To Bahir

Table of Contents

Examining Committee	ii
Author’s Declaration	iii
Abstract	iv
Acknowledgements	v
Dedication	vi
List of Figures	xi
1 Introduction	1
1.1 History of Covering Graphs	3
1.2 Regularity, Spectrum, and the Lack of Short Cycles	4
2 Preliminary Notions	7
2.1 Linear Algebra	8
2.2 Association Schemes	10
2.3 Equitable Partitions	11
2.4 Cayley Graphs	12
2.5 Representation Theory	12

2.6	Covering Graphs	13
2.7	Arc Functions	15
2.8	Spectra and Arc Functions	17
2.9	Equivalence of Arc Functions	17
2.10	The Groups of an Arc Function	18
3	Topological Context for Covers of Graphs	20
3.1	Closed Surfaces	21
3.2	Maps on Surfaces	21
3.3	Covers of Surfaces and their Homeomorphisms	23
3.4	Regular Maps and Automorphisms of Riemann Surfaces	27
4	Covers of Hypercubes	30
4.1	Four-cycle-free Covers of Hypercubes	30
4.2	Induced Subgraphs and Sensitivity	32
4.3	Twisted Convolution and Heisenberg Groups	34
4.4	A Cayley Graph Perspective	35
4.5	The O’Keefe–Wong Graph	37
4.6	Future Work	37
5	Extra-special Cayley Graphs	39
5.1	Cartesian Products as Cayley Graphs	40
5.2	Central Extensions and Cayley Graphs	41
5.3	Extra-special p -groups	43
5.4	Four-cycle-free Covers of Cartesian Products of p -cycles	46
5.5	Some Properties of our Covers	49
5.5.1	3_+^{1+2}	50
5.5.2	3_-^{1+2}	50

5.5.3	5_+^{1+2}	51
5.5.4	3_+^{1+4}	51
5.5.5	Graphical Regular Representations	51
5.5.6	Spectra and Induced Subgraphs	52
5.6	Future Work	52
6	Distance-regular Covers	54
6.1	Distance-regular Graphs	55
6.2	Imprimitivity	58
6.3	Drackns	59
6.3.1	Thas–Somma Drackns	60
6.3.2	Mathon Drackns	61
6.3.3	Drackns from Generalized Quadrangles	62
6.4	Parameters and Eigenvalues	63
6.5	Triangle-free Drackns	68
6.6	Future Work	72
7	Equivalence to Drackns Beyond Index Two	73
7.1	Seidel Matrices	74
7.2	Equiangular Lines	75
7.3	Strongly-regular Graphs	76
7.4	Complex Equiangular Lines	77
7.5	Arc Functions from Association Schemes	82
7.6	Local Properties	86
7.7	The Spectrum of Second-neighborhood	89
7.8	Future Work	93

8 Two-Eigenvalue Covers	95
8.1 Consequences of the 2-ev Property	96
8.2 Walk-Regular 2-ev Covers	97
8.3 Distance-regular 2-ev Covers	100
8.4 Examples of 2-ev Covers	104
8.5 Future Work	107
References	108
Index	120

List of Figures

2.1	The Line Graph of the Petersen Graph	14
3.1	A Cube Embedded in a Torus	25
3.2	A Cover of the Cube Embedded in the 2-Torus	26
3.3	A Cover of K_8 Embedded in a Surface of Genus 3	29
6.1	The Distance Distribution of the Icosahedron	55
6.2	The 1-Factorizations of K_6	70
7.1	An Association Scheme on 8 Points	83

Chapter 1

Introduction

A cover of a graph Y is a graph X along with a surjective graph homomorphism $\pi : X \rightarrow Y$ that maps the neighborhood of each vertex x bijectively onto the neighborhood of $\pi(x)$. So X is a graph that looks locally to be Y but exhibits some more complicated global structure. There are several other options for a formal definition of a cover. We can treat Y as a topological space and employ the definition of a covering space from algebraic topology. We can explicitly define group valued functions on the edges of Y and construct covers from permutation representations of the group. Or we can define a cover of Y to be any permutation representation of its fundamental group. We will encounter all of these perspectives.

Broadly, we study the question of how much a cover X must resemble the graph Y that it covers. The main statistics of X and Y which we will compare are their regularity, the spectra of their adjacency matrices, and the length of their shortest cycles. Let us clarify that “regularity” will usually mean distance-regularity as in Chapters 6 and 7, but we will also study a weaker regularity condition in Chapter 8. These statistics are highly interdependent, and one of the main aims of this thesis is to gain a better understanding of this interdependence. With this goal in mind, we outline what follows.

- Chapter 2 introduces the various preliminary material and notation we will need.
- Chapter 3 highlights some of the topological history and context surrounding covering graphs.
- In Chapter 4 we give a new, group-theoretic interpretation of known examples of covers whose lack of short cycles is equivalent to having few eigenvalues. We explain

how these examples are related to generalized hexagons, the sensitivity of Boolean functions, and the Heisenberg group in characteristic 2.

- In Chapter 5 we construct new examples generalizing those in Chapter 4. These examples hint at another connection between regularity and the lack of short cycles, but do not have well-behaved spectra. They also highlight the analogy between covering graphs and group extensions, and involve a detailed discussion of the extra-special p -groups.
- In Chapters 6 and 7 we study distance-regular antipodal covers of complete graphs (drackns). As with all distance-regular graphs, the adjacency matrices of drackns have very well-behaved eigenvalues, so we have a large supply of covers in which regularity and spectral considerations are closely linked. Chapter 6 is largely a survey of known results and examples, but Section 6.5 contains a new result on the connectivity of the second-neighborhood in certain distance-regular covers without short cycles.
- In Chapter 7 we begin with the well-known equivalence of index-2 drackns to several other well-studied combinatorial objects. Our main question is the extent to which these equivalences persist for drackns of higher index. This leads us in three directions each culminating in something new:
 - New examples of “almost distance-regular” covers with 5 distinct eigenvalues, derived from known equiangular tight frames. See Section 7.4.
 - A proof that the “association scheme equivalence” in the index 2 setting does not hold in general. See Section 7.5.
 - A determination of the spectrum of the second-neighborhood of a triangle-free drackn which is reminiscent of the situation for both index-2 drackns and triangle-free strongly-regular graphs. See Section 7.7.
- In Chapter 8 we study covers with two “new eigenvalues” a generalization of the key features of the examples from Chapters 4 and 6. We prove several new theorems that characterize the regularity of these covers in terms of their spectra and the regularity of the underlying base graphs.

In the remainder of the current chapter, we briefly discuss the history of covering graphs and then turn to a short discussion of our interest in regularity, spectra, and the lack of short cycles.

1.1 History of Covering Graphs

The theory of covering graphs has roots in several topological investigations from the 1800s. In 1851, Riemann introduced spaces suitable for analyzing complex functions which would later be formalized as the first covers of topological surfaces [110]. By the early 1920s it was known that these spaces had close connections to highly symmetric embeddings of graphs in surfaces, whose universal covers are tessellations of the hyperbolic plane. This relates covers of surfaces to regular maps. Running parallel to this, the 4-color conjecture rose to prominence and withstood several attempted proofs. This eventually lead Heawood to conjecture the correct bound for the number of colors needed to color graphs embedded in any surface [74]. Many authors worked to resolve Heawood’s conjecture, starting with Heffter in the late 1800s and ending with Ringle and Youngs’ complete resolution of the conjecture in the late 1960s [111]. Much of this work incorporated a “modern” treatment of covering graphs. We will expand on this portion of the history in Chapter 3.

Two “postmodern” treatments of covering graphs appeared in 1974. Gross gave a formalization of the covering graph techniques present in the resolution of the Heawood conjecture, and with Tucker extended this to the fully general “permutation voltage” formulation which allows one to study any branched covering of a surface in combinatorial terms [64]. Covers, or “voltage graphs” have been a mainstay in topological graph theory since then, and have assisted in the construction of regular maps on surfaces and other highly symmetric covering graphs [67], [102], [4], [94].

Also in 1974, following Conway’s covering graph construction of infinitely many connected cubic 5-transitive graphs, Biggs devoted a section of his book *Algebraic Graph Theory* to setting down a definition [10]. Around this time Biggs and collaborators Gardiner and D.H. Smith began to study distance-transitive graphs. Smith proved that every distance-transitive graph is either primitive, bipartite, or antipodal (a cover), and Gardiner noted that the same idea applies to all distance-regular graphs. Biggs and Gardiner began to study antipodal covers of distance-regular graphs and Gardiner published three papers on the topic which give a solid picture of the situation [46], [47], [48]. See Chapter 6 for further discussion of these ideas.

In the late 80s, Godsil began to study antipodal covers of distance-regular graphs with an emphasis on base graphs of low diameter. His students Hensel, Jurišić, and Schade considered antipodal covers of complete graphs and strongly-regular graphs. The paper [54] by Godsil and Hensel remains the primary source on drackns, and provides the technical foundation for much of Chapter 6.

Since the 70s it had been known that drackns of index 2 are equivalent to other interest-

ing mathematical objects such as sets of real equiangular lines and certain strongly-regular graphs. In 2015, Coutinho et al. extended this equivalence to certain drackns of larger index and systems of equiangular lines in complex space [27]. As a result the interplay between equiangular lines and drackns has become a subject of interest in the past decade. In fact, the most recently discovered examples of drackns come from a construction of complex equiangular lines due to Fickus et al. [41]. This parallel portion of the literature is discussed in detail in Chapter 7.

1.2 Regularity, Spectrum, and the Lack of Short Cycles

To conclude our introduction we would like to say something about our interest in regularity, graph spectra, and graphs without short cycles.

First of all, we appeal to an inherent interest in symmetry and note that regularity is the combinatorial analog of symmetry. For example, if we consider a graph whose automorphism group acts transitively on the vertices, it is necessary that all vertices of the graph have the same degree. A graph in which all vertices have the same degree is called **regular**, and this is the weakest regularity property we will be interested in. In fact almost every graph we consider throughout will be regular.

Suppose we are more ambitious and consider graphs whose automorphism group acts transitively on all subsets $\Gamma_i(v)$ of vertices at distance i from some fixed vertex v . This ambition places stronger combinatorial requirements on any candidate graph, and gives rise to one definition of distance-regularity. There are many interesting examples of distance-regular graphs. See Chapters 6 and 7 for a selection, and the references therein for a multitude.

Section 6.1 in particular gives the precise definition of distance-regularity and addresses a great deal of the interplay between regularity and the spectrum of the adjacency matrix. We advertise this interplay here by stating a classic result from algebraic graph theory: The adjacency matrix of a connected graph has exactly three distinct eigenvalues if and only if it has diameter 2 and is distance-regular. This result allows one to study distance-regularity in an algebraic framework, which leads to strong restrictions on the possible distance-regular graphs.

An important example of the utility of these restrictions comes from the Moore graphs, which we discuss next. This example also leads us to our third area of interest: The lack of short cycles in graphs.

The **girth** of a graph is the length of its shortest cycle. And the study of graphs with prescribed girth (i.e. without short cycles) is as old as algebraic graph theory itself: If Y is a k -regular graph with girth g , then by constructing a spanning tree “breadth first” it is easy to count that the number of vertices in Y is at least

$$n_0(k, g) := \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{d-1} & \text{if } g = 2d + 1 \\ 1 + k + k(k-1) + \cdots + k(k-1)^{d-2} + (k-1)^{d-1} & \text{if } g = 2d. \end{cases}$$

One could say that algebraic graph theory began when Hoffman and Singleton studied graphs that meet this so called “Moore bound” with equality [76].

Complete graphs and complete bipartite graphs give the unique examples for $g = 3, 4$ and all values of k , but for $g > 4$ graphs meeting the bound are quite rare. It turns out that any graph meeting the Moore bound is distance-regular of diameter d . This is where the spectral correspondence first proved its worth, allowing Hoffman and Singleton to show that a graph with $g = 5$ that meets this bound with equality must have $k \in \{2, 3, 7, 57\}$. These have become known as the **Moore Graphs**. The examples are the 5-cycle, the Petersen graph, and the Hoffman–Singleton graph, which realize the bound for the first three values of k . The existence of a Moore graph of valency 57 is the longest standing open problem in algebraic graph theory. The equality case of the Moore bound for graphs of odd girth greater than 5 was settled by Damerell, Bannai, and Ito, who used spectral graph theory to show that there are no graphs of valency $k > 2$ meeting the bound in this case [31] [7].

On the other hand, in the even girth case there are infinitely many non-trivial examples. They are also distance-regular, and are precisely the incidence graphs of incidence structures called generalized polygons which Tits developed in the course of his study of finite simple groups [137]. In particular, the examples of girth 6 are the incidence graphs of projective planes. We will encounter the graph-theoretic interpretation of generalized polygons at several points throughout this thesis.

Feit and Higman showed that the diameter d of the incidence graph of a generalized polygon must lie in $\{3, 4, 6, 8\}$ [40]. So, although we have infinitely many examples of graphs meeting the Moore bound in this case, there are many pairs of parameters (k, g) for which no graph with $n_0(k, g)$ vertices exists. A (k, g) -**cage** is a k -regular graph with girth g and the minimum number of vertices among all such graphs with this property. It is difficult in general to determine the correct number of vertices in a (k, g) -cage: See Exoo and Jajcay for a modern survey [39]. In many cases, the best upper bounds we possess come from covering graph constructions in which the short cycles of some base graph are “broken” [38].

The prospect of constructing cages as covers is particularly compelling when studying k -regular graphs of girth six which have $n_0(k, 6) + 2$ vertices. Biggs and Ito showed that such graphs are necessarily λ -fold covers of the incidence graphs of symmetric (v, k, λ) -designs [11]. One example is the 4-cycle-free cover of the 3-cube. Another example is the unique $(7, 6)$ -cage on 90 vertices, which Ito showed is a 3-fold cover of the point-plane incidence graph of 3-dimensional projective space over $GF(2)$ [105], [82]. See Section 4.5 for further discussion of this example.

We will discuss some topological motivation for breaking short-cycles in Chapter 3.

Chapter 2

Preliminary Notions

We begin, as we must, with the preliminary materials that we will need throughout.

The first five sections contain standard background material from algebraic graph theory and representation theory, while the last five set up the definitions, notations, and conventions surrounding covering graphs which we will use throughout.

We also note some common conventions, notations, and abuses which we will be using.

- $P := Q$ is read as “ P is defined to be Q .”
- We define $[n] := [1, \dots, n]$ for any positive integer n .
- G will always refer to a group. A generic graph is always called Y with X reserved for a cover of Y . Unless explicitly stated, all graphs are simple: They do not contain loops or multiple edges between the same two vertices.
- The “eigenvalues of a graph” are always the eigenvalues of that graph’s adjacency matrix. The spectrum of a graph is the multiset consisting of its eigenvalues counted with multiplicity.
- A 3-cycle in a graph will often be called a triangle.
- In a graph the **degree** of a vertex is the number of neighbors of that vertex. A graph in which all vertices have the same degree k is k -regular, or a graph of **valency** k .
- I_n is the $n \times n$ identity matrix. J_n and $J_{n,m}$ are the $n \times n$ and $n \times m$ matrices of all ones. When the context is clear, the subscripts are sometimes omitted. When found within an equation of matrices, 0 denotes the matrix of all 0’s.

- p is a prime, q is a prime power, and $GF(q)$ denotes the field of order q . $PG(n, q)$ denotes the standard projective space whose points are the 1-dimensional subspaces of an $n + 1$ -dimensional vector space over $GF(q)$.

2.1 Linear Algebra

The **adjacency matrix** of a graph Y is the 01 matrix with rows and columns indexed by the vertices of Y whose i, j entry is 1 if and only if i and j are adjacent. We denote it by $A(Y)$, or just A when the graph in question is either clear or arbitrary.

The i, j entry of A^2 is the inner product of A 's i th row and j th column, so it is the sum of the indices equal to 1 in both row i and column j . In other words, it is the number of common neighbors of i and j , or the number of walks from i to j of length 2. This observation and induction give a fundamental interpretation of powers of the adjacency matrix.

2.1.1 Lemma. *The (i, j) entry of the k th power of the adjacency matrix of Y is the number of walks in Y of length k which start at i and end at j .* \square

It follows that the trace of A^k possesses information about the number of closed walks in Y . For $k = 2, 3$ the only closed walks are, respectively, those which traverse an edge, and those which traverse a triangle. This implies the following formulas.

2.1.2 Lemma. *Let A be the adjacency matrix of a graph Y with e edges and t triangles.*

$$(a) \quad \text{tr}(A^2) = 2e.$$

$$(b) \quad \text{tr}(A^3) = 6t. \quad \square$$

Since the trace of a matrix is also equal to the sum of its eigenvalues, this lemma gives our first indication of the connection between a graph's spectrum and its structure. Or at least it will, once we pin down the spectral theory of adjacency matrices.

If A is a Hermitian matrix, such as the adjacency matrix of a graph, then for each eigenvalue θ of A , let U_θ denote a matrix whose columns form an orthonormal basis for the eigenspace of θ . The matrices $E_\theta := U_\theta U_\theta^*$ are called the **principal idempotents** of A . They define orthogonal projections onto the eigenspaces. From this description and a bit more work we obtain the spectral decomposition of A .

2.1.3 Theorem (Folklore, c.f. [52]). *Let A be a Hermitian matrix with distinct eigenvalues $\text{ev}(A)$ and with principal idempotents E_θ for $\theta \in \text{ev}(A)$. We have the following:*

(a) $E_\theta^2 = E_\theta$ and $E_\theta E_\tau = 0$ when $\theta \neq \tau$.

(b) $AE_\theta = \theta E_\theta$.

(c) $\sum_\theta E_\theta = I$.

(d) $A = \sum_{\theta \in \text{ev}(A)} \theta E_\theta$. □

Since the matrices E_θ are idempotent, they have eigenvalues 0 and 1, so $\text{tr}(E_\theta)$ is equal to the multiplicity of θ as an eigenvalue of A . We denote these multiplicities m_θ . Taking the trace of A^2 and A^3 and applying Theorem 2.1.3 and Lemma 2.1.2 we obtain:

2.1.4 Lemma. *If A is the adjacency matrix of a graph Y with e edges and t triangles, and $\text{ev}(A)$ are the distinct eigenvalues of A , then*

(a) $2e = \sum_{\theta \in \text{ev}(A)} m_\theta \theta^2$.

(b) $6t = \sum_{\theta \in \text{ev}(A)} m_\theta \theta^3$. □

Aside from the spectral decomposition, the most important result from matrix theory for the purposes of algebraic graph theory is the Perron-Frobenius theorem. This result gives useful information about the eigenvalues of maximum modulus in real matrices with non-negative entries. See [57, Chapter 8] or [79, Chapter 8] for a full statement and proof. We record only the relevant graph theoretic consequences.

2.1.5 Theorem. *Let A be the adjacency matrix of a graph Y .*

(a) *If Y is k -regular and connected then k is an eigenvalue of A with multiplicity 1 whose eigenspace is the span of the all-ones vector.*

(b) *Y is bipartite if and only if for each eigenvalue θ of A , the number $-\theta$ is also an eigenvalue with the same multiplicity.* □

2.2 Association Schemes

The theory of association schemes is rich and full of interesting connections to group theory, coding theory, design theory, and algebraic graph theory. The classic resource is Delsarte's thesis [35], and the book by Bannai et al. gives a more recent treatment [6]. All we will need is the basic definitions and a discussion of how one can construct association schemes from group actions on graphs.

A d -class **association scheme** of order n is a set $\mathcal{A} = \{A_0, A_1, \dots, A_d\}$ of $n \times n$ matrices with entries in $\{0, 1\}$ satisfying the properties

- (a) $A_0 = I$ and $\sum_{i=0}^d A_i = J$.
- (b) $A \in \mathcal{A} \implies A^T \in \mathcal{A}$.
- (c) $A_i A_j = A_j A_i$ for $i, j \in \{0, \dots, d\}$.
- (d) For $i, j, k \in \{0, \dots, d\}$ there exist constants $p_{i,j}^k$ so that $A_i A_j = \sum_{k=0}^d p_{i,j}^k A_k$.

We say that d is **symmetric** if all the matrices A_i are symmetric, and **asymmetric** otherwise. Properties (a) and (d) ensure that the A_i generate an algebra of dimension $d + 1$. And property (c) implies that this algebra is commutative. If only properties (a), (b), and (d) are satisfied then \mathcal{A} is a (homogeneous) **coherent configuration**.

If \mathcal{A} is a d -class association scheme and \mathcal{B} is an e -class association scheme of the same order with $e > d$ and \mathcal{A} can be obtained from \mathcal{B} by merging together (summing) certain classes from \mathcal{B} , then \mathcal{A} is a **fusion** of \mathcal{B} , and \mathcal{B} is a **fission** of \mathcal{A} .

The primary source of association schemes in this thesis will be distance-regular graphs, which define particularly nice instances. See [18, Chapter 2], for this theory.

We will also derive association schemes and coherent configurations from the automorphism groups of graphs via the following procedure: If we have any group G acting on a set V , there is an induced action on the set $V \times V$. The orbits of this action are called the **orbitals** of the action. If G acts transitively on V then the orbitals determine a coherent configuration with classes parameterized by the orbitals Ω and defined by

$$(A_\Omega)_{i,j} = \begin{cases} 1 & \text{if } (i, j) \in \Omega \\ 0 & \text{otherwise.} \end{cases}$$

We say that the action of G on V is **generously transitive** if for each pair of points $u, v \in V$ there is some g in G which exchanges u and v . The orbitals of a generously

transitive permutation action give rise to symmetric matrices A_Ω . If all the orbitals are symmetric, then each product $A_i A_j$ is symmetric, and

$$A_i A_j = (A_i A_j)^T = A_j^T A_i^T = A_j A_i.$$

So the resulting coherent configuration is in fact an association scheme.

As an example, define the d -cube as a graph with vertices \mathbb{Z}_2^d , which are adjacent if their difference is one of the standard basis vectors. For each pair of vertices u, v the map $x \mapsto x + u + v$ is an automorphism which exchanges u and v , so the automorphism group of the d -cube is generously transitive and determines an association scheme in which the adjacency matrix of the d -cube is one of the classes.

2.3 Equitable Partitions

A **partition** of a finite set V is a collection $\pi = \{\pi_0, \dots, \pi_d\}$ of disjoint non-empty subsets of V whose union is V . The **characteristic matrix** P of π is the $\pi \times V$ matrix defined by

$$P_{i,j} = \begin{cases} 1 & \text{if } j \in \pi_i \\ 0 & \text{otherwise.} \end{cases}$$

Let X be a graph, and $\pi = \{\pi_0, \dots, \pi_d\}$ a **partition** of $V(X)$. The partition π is **equitable** if for each pair $(i, j) \in \{0, \dots, d\} \times \{0, \dots, d\}$, and $u \in \pi_i$, the number of neighbors $C_{i,j}$ of u in π_j depends only on i and j and not on the choice of u . If π is an equitable partition we may define the **quotient graph** X/π whose vertices are the cells of π with $C_{i,j}$ arcs from π_i to π_j . The adjacency matrix of this directed multigraph is the **quotient matrix** $A(X)/\pi$.

2.3.1 Lemma. [52, Lemma 5.2.1] *Let π be a partition of the vertex set of X with characteristic matrix P and set $A := A(X)$. If π is equitable, then $AP = P(A/\pi)$. If there is a matrix B such that $AP = PB$ then π is equitable.* \square

The important point is that if we have an equitable partition of the graph X , then the spectrum of the quotient is a subset of the spectrum of X .

2.3.2 Lemma. [52, Lemma 5.2.2] *If π is an equitable partition of the graph X and B is the adjacency matrix of X/π , then the characteristic polynomial of B divides the characteristic polynomial of $A(X)$.* \square

2.4 Cayley Graphs

There is a very natural family of graphs which can be associated to any group G . These are the **Cayley graphs** for the group G , denoted $\text{Cay}(G, S)$ and defined as follows. Let S be an inverse-closed subset of G which does not contain the identity. The vertex set of $\text{Cay}(G, S)$ is G , and two elements g, h in G are adjacent if there exists an $s \in S$ so that $gh^{-1} = s$.

The graph $\text{Cay}(G, S)$ is undirected and loopless because S is inverse-closed and does not contain the identity. These are convenient graph theoretic assumptions for our purposes, but not required in general.

For each $g \in G$, left multiplication by g induces an automorphism on $\text{Cay}(G, S)$, so G is a subgroup of the automorphism group of $\text{Cay}(G, S)$ that acts regularly on the vertices. The converse is true as well.

2.4.1 Theorem. [114] *The graph Y is a Cayley graph for the group G if and only if the automorphism group of Y contains a regular subgroup isomorphic to G .* \square

2.5 Representation Theory

A (complex) **representation** ϕ of a group G is a homomorphism into $GL(r, \mathbb{C})$. We say that r is the **degree** of the representation. For each $A \in GL(r, \mathbb{C})$ the map $\phi^A(g)$ defined by

$$\phi^A(g) = A\phi(g)A^{-1}$$

is also a representation of G , and is said to be **equivalent** to ϕ . The **trivial** representation is the map from G onto the identity matrix I_r . If ϕ and ψ are representations of G with degrees r and s respectively then the map $\phi + \psi$ defined by

$$g \mapsto \begin{pmatrix} \phi(g) & 0 \\ 0 & \psi(g) \end{pmatrix}$$

is a representation of degree $r + s$. We say that $\phi + \psi$ is the **sum** of ϕ and ψ . The set of representations of G is closed under non-negative integral linear combinations. A representation ϕ is **irreducible** if there is no nontrivial subspace of \mathbb{C}^r that is invariant under $\phi(g)$ for all $g \in G$. It can be shown that a representation of a finite group G is reducible if and only if it is equivalent to a nontrivial positive integral linear combination

of irreducible representations of G . The irreducible representations present in this linear combination are the **constituents** of the representation.

Define a vector space \mathcal{V} which is the span over \mathbb{C} of the elements of G . So the elements of G form a basis for \mathcal{V} . Now define a G -action on \mathcal{V} where each $g \in G$ acts on the basis vectors by left multiplication and the action is extended linearly. In the basis of group elements of G , each g acts as a permutation matrix $\rho(g)$. The map $\rho : G \rightarrow GL(|G|, \mathbb{C})$ is called the (left) **regular representation** of G .

The following theorem collects the basic results from representation theory which we will need throughout. For proofs, see any introductory text on representation theory, say [84].

2.5.1 Theorem. *Let G be a finite group. Then*

- (a) G has finitely many inequivalent irreducible representations ϕ_i for $(i = 0, 1, \dots, k)$.
- (b) If r_i denotes the degree of ϕ_i and ρ is the regular representation of G then ρ is equivalent to $\sum_i r_i \phi_i$.
- (c) If $G = H_1 \times H_2$, then the regular representation of G is the tensor product of the regular representations of H_1 and H_2 .
- (d) If G is abelian then each $r_i = 1$ and $\phi_i(g)$ is a (not necessarily primitive) m th root of unity where m is the order of G . □

2.6 Covering Graphs

We come to the main objects studied in this thesis. A **cover** of the graph Y is a pair (X, π) consisting of a graph X along with a graph homomorphism $\pi : V(X) \rightarrow V(Y)$ called the **covering projection**, which maps the neighborhood of each vertex $x \in V(X)$ bijectively onto the neighborhood of $\pi(x)$. Explicit conditions equivalent to this definition are as follows.

- (a) For each $v \in V(Y)$, the subgraph of X induced by $\pi^{-1}(v)$ is an independent set.
- (b) For each edge $(u, v) \in E(Y)$, the subgraph of X induced by $\pi^{-1}(v) \cup \pi^{-1}(u)$ is a perfect matching.

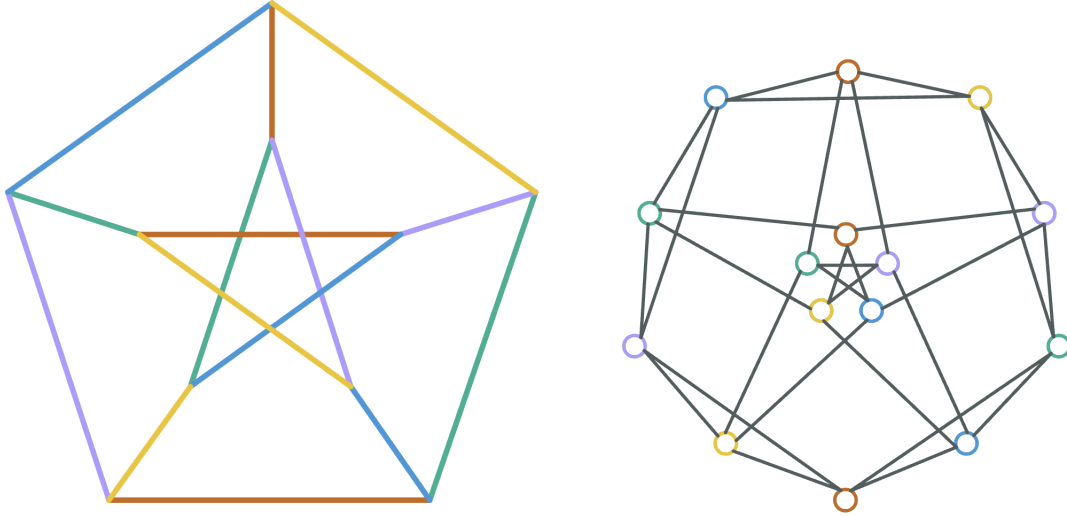


Figure 2.1: The Petersen graph and its line graph. This line graph is an index-3 cover of K_5 . The coloring determines the partition into fibers of the covering projection.

- (c) For each non-edge $(u, v) \notin E(Y)$ the subgraph of X induced by $\pi^{-1}(v) \cup \pi^{-1}(u)$ is an independent set.

We say that Y is the **base graph** of the cover (X, π) . Often we will use “ X is a cover of Y ” or “ X covers Y ” to indicate the existence of a cover (X, π) without specifying the projection π .

Informally, X covers Y if we may obtain X from Y by “blowing up” each vertex of the base graph into an independent set, and “blow up” each edge between two vertices into a perfect matching between the associated independent sets. These independent set $\pi^{-1}(v)$ are called the **fibers** of the cover.

If Y is connected, then the fibers of any cover X all have the same size which is called the **index** of the cover. In general we denote this size by r and will refer to X as an **r -fold cover** of Y , and index- r cover, or a cover of index r .

One pertinent small example is the cube, which is an index-2 cover of K_4 whose covering projection is the map which identifies opposite corners. A nice example of index 3 is the line graph of the Petersen graph, which is a 3-fold cover of K_5 . See Figure 2.1.

There is a simple infinite family of covers which “break” all the odd cycles of the graphs they cover. The **bipartite double cover** of a graph Y is the graph X obtained from Y by

replacing each vertex $v \in V(Y)$ with a pair of vertices v_1, v_2 , and each edge $(u, v) \in E(Y)$ with a pair of edges v_1u_2 and v_2u_1 . X is always bipartite since each odd cycle in Y is replaced with a cycle of twice the length. X is connected if and only if Y is not bipartite. The map $\gamma : V(X) \rightarrow V(Y)$ defined by $\gamma(w_i) = w$ is the covering projection.

2.7 Arc Functions

We have just given a combinatorial definition of a cover of a graph Y . This gives a good picture of these objects, but a different formulation will repeatedly prove its value when studying them. To that end, we identify the simple graph Y with the digraph Y on $V(Y)$ with arcs

$$\text{arc}(Y) = \{(a, b) : \{a, b\} \in E(Y)\} \cup \{(b, a) : \{a, b\} \in E(Y)\}.$$

Now let G be any finite group, a **G -arc function** on a graph Y is a map from pairs of vertices into the \mathbb{C} -group algebra of G satisfying the conditions

- (a) $f(v, u) = 0$ if and only if $(u, v) \notin \text{arc}(Y)$.
- (b) $f(v, u) = f(u, v)^{-1}$ for all $(u, v) \in \text{arc}(Y)$.

We abuse notation and denote a G -arc-function by the map $f : Y \rightarrow G$. The **arc matrix** of an arc function $f : Y \rightarrow G$ is the $|Y| \times |Y|$ matrix $A(f)$ with

$$A(f)_{i,j} = f(i, j).$$

The terms “arc function” and “arc matrix” will be used interchangeably. The arc matrix, along with a permutation action of G , determines a cover of Y . The next theorem is essentially a definition of the notation $A(f)^\rho$.

2.7.1 Theorem. *Let G be a finite group acting on the set $[r]$ and let ρ be the permutation representation of this action. Let $A(f)^\rho$ denote the 01 matrix obtained from the arc matrix $A(f)$ by replacing each element $g = A(f)_{i,j}$ with $\rho(g)$ and each 0 with an $r \times r$ block of zeros. $A(f)^\rho$ is the adjacency matrix of a cover $Y^{\rho(f)}$ of Y . \square*

This is a complete characterization of covers of Y .

2.7.2 Theorem. *If (X, π) is any cover of Y of index r , there is an arc function $f : Y \rightarrow G$ and a permutation action of G on $[r]$ with representation ρ so that $A(X) = Y^{\rho(f)}$.*

Proof. For each vertex $v \in Y$, arbitrarily label the vertices of $\pi^{-1}(v)$ as $(v, 1)$ through (v, r) . For each $ab \in E(Y)$ the perfect matching induced by $\pi^{-1}(a) \cup \pi^{-1}(b)$ defines a map from the copy of $[r]$ labeling the fiber over a to the copy of $[r]$ labeling the fiber over b , i.e. a permutation σ of $[r]$. We set $f(a, b) := \sigma$. This same perfect matching defines the inverse permutation from the labels of the fiber over b to the labels of the fiber over a , and so we set $f(b, a) := \sigma^{-1}$. \square

It is worth noting that most of the covers we will consider fall into one of two categories:

1. ρ is the regular representation of G .
2. G is the symmetric group $\text{Sym}(r)$ and ρ is the standard permutation representation of degree n .

The single notable exception to these cases occurs in Section 7.4, where we show how to construct some interesting covers of complete graphs using a certain degree p representation of the cyclic group of order $2p$. Because this is the only notable exception, we will typically omit the action ρ which is assumed to be regular unless G is $\text{Sym}(r)$ in which case it is assumed to be standard. Moreover, we follow [54] in saying that a cover is **cyclic** or **abelian** if the connection group is cyclic or abelian. This could become misleading if we were studying more exotic groups with simple representations, but we will not be doing so.

The main reason we have gone to the trouble of recasting covers as arc matrices is that the decomposition of the representation ρ into irreducibles allows us to study the spectra of covers quite effectively. We will see this momentarily. A second reason is that other combinatorial objects such as Gram matrices of equiangular lines (Sections 7.4) and weightings of association schemes (Section 7.5) naturally give rise to arc matrices which we can now interpret as covering graphs.

Now is also a good time to note that there is a large literature on arc functions which does not pay very much attention to the associated covering graphs. In our language, a **signed graph** is exactly a \mathbb{Z}_2 -arc function on a graph. Harary initiated the study of signed graphs in the 1950s [72], and they have remained an active area of research since then. See, for instance, the recent survey by Belardo et al. [8].

Zaslavsky coined the term “gain graphs” for the generalization of signed graphs in which the target group is arbitrary, and studied both signed and gain graphs extensively with an emphasis on matroids defined on their cycles [146]. Zaslavsky’s work has inspired many further investigations into gain graphs which are recorded in the bibliography [147]. We will not spend very much time on this literature, although in Section 8.4 we will discuss several examples which were developed as signed graphs and gain graphs.

2.8 Spectra and Arc Functions

Let $f : Y \rightarrow G$ be an arc function with arc matrix S and representation ρ . By definition, the adjacency matrix of the cover $Y^{\rho(f)}$ is $A(f)^\rho$. The representation ρ decomposes into a direct sum

$$\bigoplus_i \rho_i$$

of irreducible representations, which means there is some $[r] \times [r]$ matrix L so that $L\rho(g)L^{-1}$ is block diagonal with blocks of size the dimensions of ρ_i . So $A(f)^\rho$ is similar to the matrix

$$(L \otimes I_{|Y|}) A(f)^\rho (L^{-1} \otimes I_{|Y|})$$

which in turn is permutation equivalent to the block diagonal matrix

$$\bigoplus_i A(f)^{\rho_i}.$$

We record this for later use.

2.8.1 Theorem. [54, Section 8] *Let $f : Y \rightarrow G$ be an arc function with arc matrix $A(f)$ and action ρ . The adjacency matrix of the cover $Y^{\rho(f)}$ is similar to the block diagonal matrix*

$$\bigoplus_i A(f)^{\rho_i}.$$

Hence the spectrum of $Y^{\rho(f)}$ is the union of the spectra of the $A(f)^{\rho_i}$. \square

This gives us a great deal of information about the spectrum of the cover $Y^{\rho(f)}$. For instance, the sum in the group algebra $\mathbb{C}G$ of all elements of g spans an irreducible G -module for any action ρ , so the trivial representation ρ_0 is always a constituent of the permutation representation defining a cover. It follows that $A(f)^{\rho_0} = A(Y)$ is a block of the matrix in Theorem 2.8.1, so we recover the fact that all eigenvalues of Y are eigenvalues of X as well.

2.9 Equivalence of Arc Functions

Two distinct G -arc functions can yield isomorphic covering graphs. In particular, permuting the vertices of each fiber $\gamma^{-1}(u)$ of Y^f by some $\sigma_u \in G$ gives rise to an isomorphic cover $Y^{f'}$ with arc function $f'(i, j) = \sigma_i^{-1} f(i, j) \sigma_j$.

This gives rise to an equivalence relation on arc functions in which members of the same equivalence class give isomorphic graphs. We will not need the formal definition of this, but include it for posterity: For fixed G and Y , and any $v \in Y$ and $g \in G$ we define operations $S_{y,g}$ on the set of G -arc functions by:

$$S_{y,\sigma}(f(i, j)) = \begin{cases} \sigma_i^{-1} f(i, j) & \text{if } y = i, \\ f(i, j) \sigma_j & \text{if } y = j, \\ f(i, j) & \text{otherwise.} \end{cases}$$

We call these $S_{y,\sigma}$ the **switching** operations and say that two arc functions are **switching equivalent** if one can be obtained from the other by a sequence of switchings. The covers derived from switching equivalent arc functions are isomorphic. A reader trying to make sense of these definitions is encouraged to look at the case when $G = \mathbb{Z}_2$ and then extrapolate. This case is discussed extensively in several different ways in the first four sections of Chapter 7. Note that in this case, we use additive notation for \mathbb{Z}_2 and set $G = \{1, -1\}$.

The important point about switching is that it gives us a fair amount of control over our arc functions: For any arc function and any arc, we can find a switching equivalent arc function whose value on that arc (and its opposite arc) is the identity. In fact, if F is the set of edges of any forest in Y we may switch all arcs supported on F to be the identity simultaneously. Simply start with the vertices whose degree in the forest is 1, switch the edges incident to them to the identity, and proceed inductively.

So for any G -arc function f on Y and any spanning tree T of Y , there is an arc function switching equivalent to f with $f(i, j) = \text{id}$ for all $ij \in E(T)$. We say that an arc function with this property is **normalized**. Denote by $\langle f \rangle$ the group generated by the values which f takes on all arcs of Y . When Y possesses a vertex s adjacent to all other vertices, we may normalize on the spanning tree isomorphic to $K_{1,|Y|-1}$ with s the vertex of high degree. We will find this normalization very convenient when discussing covers of complete graphs.

2.10 The Groups of an Arc Function

Let Y be a connected graph with arc function f normalized on any tree. Let $X = Y^{\rho(f)}$ be the associated cover. There are two important groups associated with this cover. The first is the group of automorphisms of X which fix each fiber set-wise. We call this the **covering group** of the arc function, and denote it by $\text{cov}(\rho, f)$ or simply $\text{cov}(f)$ when the

action of f is clear. If an element of $\text{cov}(f)$ fixes a vertex in a given fiber, then it must fix all neighbors of that vertex as well, but then since Y is connected, this automorphism fixes each vertex of X , and is the identity. So $\text{cov}(f)$ acts semi-regularly on each fiber and has size at most r .

The second important group is the one generated by all permutations which are actually produced by the representation of the arc function. The **connection group** of the cover is

$$\text{con}(\rho, f) := \langle \rho(f(a)) : a \in A(Y) \rangle.$$

The connection group also acts on the fibers, and has a close “inverse” relationship with the covering group. This is detailed in the paper by Godsil and Hensel [54].

2.10.1 Lemma. [54, Lemma 7.2] *If f is a normalized arc function of index r defined on the connected graph Y , then $\text{cov}(\rho, f)$ is isomorphic to the centralizer of $\text{con}(\rho, f)$ in $\text{Sym}(r)$.* \square

Even more is true in the extreme case.

2.10.2 Lemma. [54, Lemma 7.3] *Let f be a normalized arc function of index r , and suppose Y^f is connected. $\text{cov}(f)$ acts regularly on any (and hence every) fiber if and only if $\text{con}(f)$ acts regularly on $[r]$. In this case, $\text{con}(f)$ is isomorphic to $\text{cov}(f)$.* \square

The opposite extreme is that $\text{cov}(f)$ is trivial and $\text{con}(f)$ is $\text{Sym}(r)$. For example, consider again the line-graph of the Petersen graph. The connection group for this cover is $\text{Sym}(3)$, so Lemma 2.10.1 implies that the covering group is trivial. Note that although there are automorphisms which fix any given fiber in the line graph of the Petersen graph, any such automorphism exchanges the other four fibers in pairs, and is not an element of the covering group.

Note also that it is necessary for f to be normalized in the above lemmas: Consider $Y = K_3$ with arc function

$$f(a, b) = f(b, c) = \text{id}, f(a, c) = (1, 2, 3).$$

This is switching equivalent to the unnormalized arc function

$$f'(a, b) = \text{id}, f'(b, c) = (1, 2), f'(c, a) = (2, 3).$$

We see that $\text{con}(f)$ is cyclic and $\text{con}(f')$ is $\text{Sym}(3)$, but $Y^f = Y^{f'}$ is the 9-cycle, which admits non-trivial fiber-fixing automorphisms.

Chapter 3

Topological Context for Covers of Graphs

In this short chapter, we give an overview of some topological context for studying covering graphs. We do not strictly require this context, but we include it because it provides motivation for cycle-breaking and an interesting connection between Riemann surfaces and distance-regular covers of complete graphs.

The motivation for cycle-breaking comes from the fact that a cover of an embedded graph gives rise to an embedding of the covering graph in a covering surface. If the base graph is embedded so that its facial cycles are all short, and the cover breaks those cycles, then there are relatively few faces in the induced embedding of the covering graph. We will discuss this more in Section 3.3 and give a few nice examples.

The connection to Riemann surfaces comes from the fact that the smallest maximally symmetric Riemann surface, the Klein Quartic, has a triangulation whose underlying graph is the unique distance-regular 3-fold cover of K_8 . This is part of a much larger story about regular maps on surfaces which we will sketch in Section 3.4.

Since we will not use any of this material explicitly in later chapters, we will avoid most of the details. However we do attempt to provide ample references for a reader interested in the full picture.

3.1 Closed Surfaces

We are interested in embedding graphs, so we need something into which they can embed. In the language of topology, a **surface** (without boundary) is a closed connected 2-manifold. That is, a surface is a connected topological space in which each point possesses an open neighborhood homeomorphic to the open disk in \mathbb{R}^2 . Recall that a homeomorphism is a continuous bijection with continuous inverse between topological spaces. These, and subsequent, topological definitions can be further unpacked, say by consulting [100]. We will not dwell on them, for an entirely combinatorial definition of surfaces captures all the same information.

Dehn and Heegard gave one such combinatorial definition of a closed surface as a “gluing together” of a finite collection of triangles satisfying two compatibility conditions: Each edge must be glued exactly once, so that it belongs to exactly two of the triangles; and each vertex must admit a cyclic ordering of the triangles so that subsequent triangles are glued along an edge [34]. One can extend easily to an equivalent theory in which polygons of larger size are glued together, which can be seen as a genesis of the graph embeddings we will discuss in the next section.

In 1927, Brauer demonstrated that Dehn and Heegard’s polygonal complexes can be reduced to equivalent diagrams containing only a single polygon in which all edges are identified in pairs, [15]. The simplest, and now well-known, examples are the two identifications of the edges of a square giving rise to the torus and the Klein bottle. In fact, Brauer showed that any representation of a surface other than the sphere generalizes either the torus or Klein bottle picture. This gives a combinatorial proof of the important classification of surfaces: Any surface is either a sphere with some number of “handles” or a sphere with some number of “crosscaps.” This number of handles or crosscaps is known as the **genus** of the surface, and this dichotomy is equivalent to the notion of orientability: A surface is orientable if and only if it is equivalent to a sphere with handles. We refer the reader to Section 1.3 in Stillwell’s book [131] for the details of this and the previous paragraph.

3.2 Maps on Surfaces

Let Y be a graph and let S be a surface. An **embedding** of Y in S consists of an injection $i : V(Y) \rightarrow S$ along with a map $i_e : E(Y) \rightarrow S$ in which each edge (u, v) is mapped to a subset of S homeomorphic to an interval $[0, 1]$ whose ends are u and v , so that the interior

of the image of each edge is disjoint from all other edges and all vertices. A helpful way to think of this is that Y is drawn on S so that edges do not cross. If Y is a 2-connected graph embedded in the surface S , then $S - Y$ is a collection of disconnected pieces of S each homeomorphic to a disk. The closures of these disks are called the **faces** of the embedded graph and the cycles of Y that bound them are the **facial cycles**.

There is an enormous amount of combinatorics associated with the question of which graphs embed in which surfaces. One branch of this story, which we only mention in passing, begins with the characterization of planar graphs as those that exclude K_5 and $K_{3,3}$ as minors. See Chapter 4 in Diestel’s book [36]. This branch continues in many directions, [112], [98], [24]. Another branch begins with the 4-color conjecture and the Heawood map-coloring problem. In the remainder of this section, we sketch that history. A reader who already knows this, or does not wish to, can safely skip to the next section.

In 1852, the Guthrie brothers, Frederick and Francis, asked their supervisor Augustus De Morgan for a proof of the “fact” that a map, colored so that two adjacent regions receive distinct colors, never requires more than four colors in total. Many mathematicians spent many years trying to prove this fact. Among them were Kempe and Tait, who in 1889 and 1890 gave two “proofs” that placated the mathematical community. But this happy resolution was short lived, for Petersen demonstrated a flaw in Tait’s argument in 1891 by exhibiting the graph which has come to bear his name. This came only a year after Heawood pointed out the flaw in Kempe’s argument [74].

Heawood’s 1890 paper dismantled Kempe’s proof but was able to salvage from its wreckage an argument that five colors are sufficient for any map. Moreover, Heawood extended the question to maps drawn on other surfaces and gave a strong upper bound on the necessary number of colors for maps drawn on surfaces other than the sphere. Letting $c(S)$ denote the Euler characteristic of a surface, Heawood’s bound is as follows:

3.2.1 Theorem. *The chromatic number of a simplicial graph embedded in a surface S other than the sphere, does not exceed*

$$H_S := \left\lceil 7 + \frac{\sqrt{49 - 24c(S)}}{2} \right\rceil. \quad \square$$

If $c(S) = 2$ then S is the sphere and H_S is 4, and this is the 4-color theorem which Heawood could not prove. That first and final case was only settled 86 years later by Appel and Haken [3]. Notably, their proof still used the ideas from Kempe’s attempt, along with additional machinery and a large amount of case work carried out by a computer. The

last doubts finally dissipated in 2005 when the computational aspects of the proof were formally verified by Gonthier, [59].

Heawood conjectured that his bound was tight: For each surface with Euler characteristic $c(S) \neq 2$, there is some graph embeddable in S whose chromatic number is H_S . We now know, and have, since 1968, that this conjecture is true with a single exception: the Klein bottle, for which $H_S - 1$ is the correct bound. The theorem was proven mostly by Ringle and Youngs, with contributions by Terry, Welsh, Gustin, and Meyer. Its proof spans about 300 pages across numerous journal articles, and we will not be reviewing it here. See Chapter 5 of the book by Gross and Tucker [65] for details. We do note that some of the cases are resolved by constructing graphs of chromatic number H_S as covers of smaller, more tractable looped multi-graphs. This use of covering graphs prompted Gross's formal development of covering graphs that has become standard in modern topological graph theory [63].

3.3 Covers of Surfaces and their Homeomorphisms

We have said a great deal about the combinatorial perspective on surfaces. On the other hand, working with continuous functions gives us a natural way to define a notion of covering spaces that is general enough to apply to surfaces. A **cover** of a surface S is a continuous map $\pi : T \rightarrow S$ between topological spaces with the property that each point $s \in S$ has some open neighborhood U whose preimage $\pi^{-1}(U)$ is a disjoint union of open sets in T , each mapped homeomorphically onto U by π . These disjoint open sets in T are called **sheets** of the cover. Graphs, viewed as simplicial complexes, are among the simplest topological spaces one might consider, and their topological covers are exactly the graph theoretic covers we will be studying throughout. When considering covers of surfaces it is usually best to relax the definition to allow for some finite set of points in S where the condition on preimages fails. To see why, let us consider the very first example of a cover of the sphere which comes to us all the way from Riemann's Ph.D. thesis in 1851 [110].

Consider the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and the effect of the function $f(z) = z^2$. For any point w the distinct square-roots $\pm\sqrt{w}$ are both mapped onto w by f , so f almost defines a 2-to-1 mapping of the sphere onto itself. This “almost” comes from the points 0 and ∞ , which have only one preimage. Until the 1850s, a mathematician who wanted to work with the “function” $g(z) = \sqrt{z}$ had to make a choice as to whether the result was the positive or negative square root. Riemann thought to circumvent this issue by viewing f as a bijective map from some new space \tilde{S} into the sphere, which allows one to study its

inverse $g : \tilde{S} \rightarrow S$ as an actual function. This is the primary purpose of Riemann surfaces, of which \tilde{S} is the first example.

The space \tilde{S} is almost a cover of the sphere in the sense described above. The issue is that the points 0 and ∞ have only one preimage under the covering projection f . Rather than try to fix this “issue” one defines a **branched cover** from the surface T to the surface S to be a homeomorphism $\pi : T \rightarrow S$ which is a covering map except on some finite set of points called the **branch points**.

We now come to the key point: Suppose a graph Y has both a cover $\pi : X \rightarrow Y$ and an embedding in S . Then there is a surface T that covers S so that X embeds in T and the restriction of the topological covering projection from T to S is π . Moreover, every branched cover of the surface S can be constructed in this way. The precise results are due to Gross and Tucker from 1977 [64], and we refer the reader to Sections 4.2 and 4.3 of their book [65] for the details.

An important application of this idea pertains to studying group actions on surfaces. A group G acts on a surface S if there is a homomorphism from G into the group of homeomorphisms from S to itself. The full group of homeomorphisms of a surface is a highly intractable object, so the small supply of homeomorphisms provided by the image of a group action is often novel information. Moreover, if we have an action on a surface with only finitely many fixed points, then the quotient of the surface by that action, in which points from the same orbit are identified, is itself a surface. See Section 4.3.3 of [64]. We now present a couple of examples that illustrate this phenomenon.

Example: The cube has a nice embedding in the torus with four hexagonal facial cycles, see Figure 3.1. Since the cube is a 2-fold cover of K_4 , we anticipate the existence of a branched 2-fold cover of the sphere by the torus which carries our embedded cube to a typical tetrahedral embedding of K_4 . Indeed, this can be realized by taking the toroidal drawing from Figure 3.1 and rotating it 180 degrees about the line through the center of all four hexagonal faces so that the top and bottom portions are switched. Identifying pairs of points that belong to the same orbit of this rotation gives the desired covering map with four branch points, one at the center of each hexagonal face. We refer the reader to the front cover of the book by Francis [44] for an excellent illustration of the quotient by this rotation.

Example: We begin with the unique 4-cycle-free cover of the cube of index 2 (c.f. Chapter 4). This graph has an embedding in the 2-torus which has been known at least since Coxeter and Moser’s book [30]. See Figure 3.2. Naturally this graph has the cube as a quotient, and so this embedding of the cover must somehow collapse onto an embedding

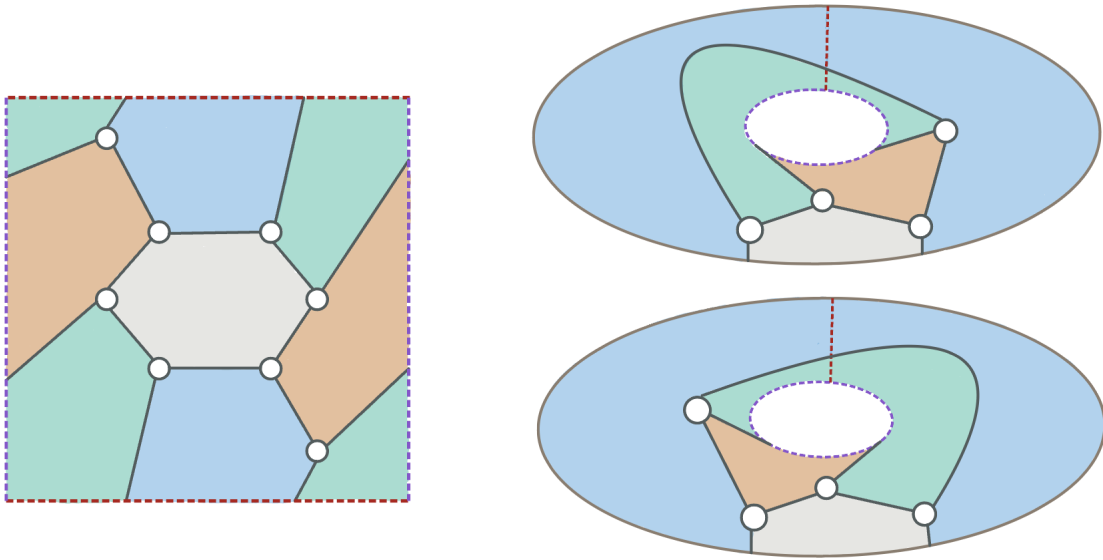


Figure 3.1: A cube embedded on a torus with four hexagonal faces. The right-hand figure shows the top and bottom of the embedding on the torus once it has been sliced “bagel-wise.”

of the cube on the sphere in which the six 8-cycles fold over themselves to become six 4-cycles. Our drawing of the covering graph embedded on the 2-torus demonstrates that this quotient is also realized by the action of a 180 degree rotation of the 2-torus whose axis is the center of the six octagonal faces. Without this combinatorial “skeleton” it would not be clear (at least to the author) that the sphere is the quotient of the 2-torus by this rotation.

In Section 1.2 we outlined an interest in covers that break the short cycles of a graph. The two examples above highlight some topological motivation for this interest: If we have an embedding of the graph Y in the surface S in which all facial cycles are short and X is a cover of Y in which all the short cycles are broken, then X has an embedding in a cover of S with the same number of faces. In this setting we have a chance to see how the surface should be drawn in order to highlight a group action whose quotient extends the covering map from X to Y .

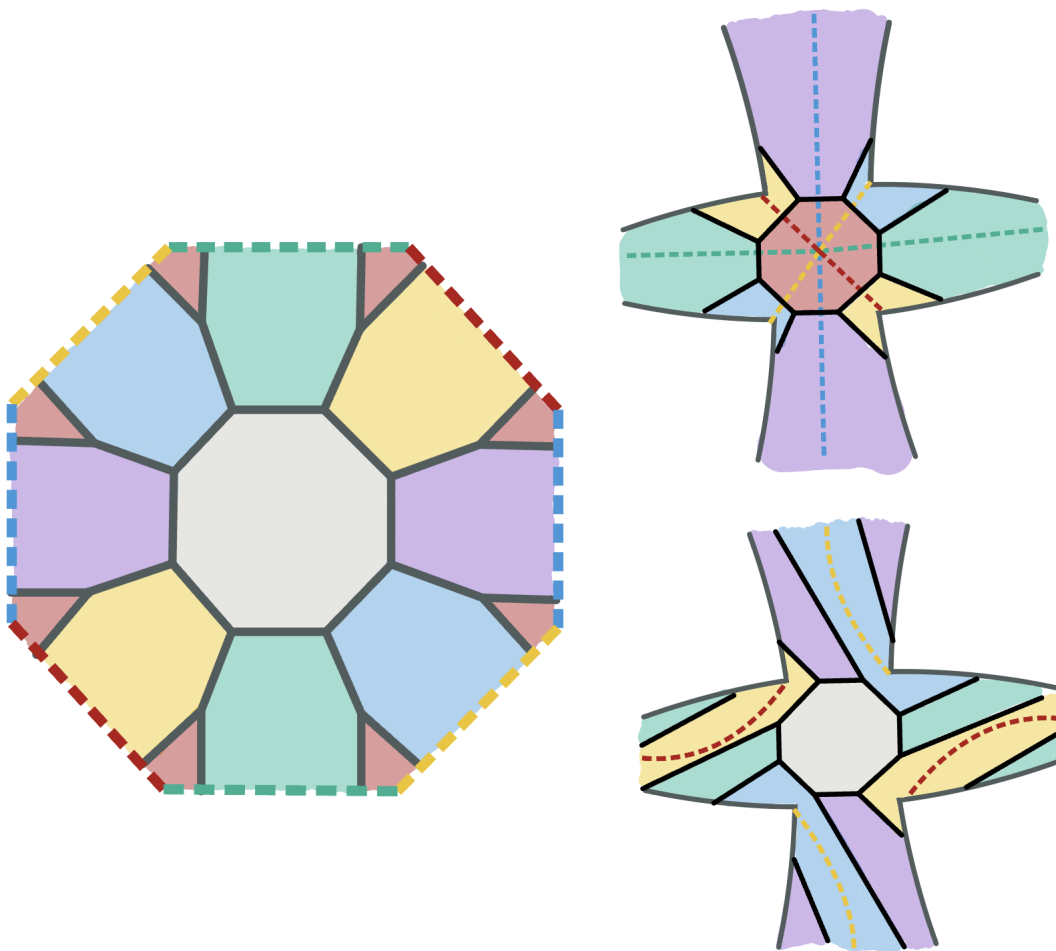


Figure 3.2: The 4-cycle-free 2-fold cover of the cube embedded on a 2-torus. In either of the right-hand figures one should imagine the north-south portions of the crosses bending up out of the page and meeting to form one handle. The portions from the top figure represent the inside of this handle and the portions from the bottom figure represent the outside. Conversely, the east-west portions bend down into the page to form the other handle, and here the top portions represent the outside while the bottom portions represent the inside.

3.4 Regular Maps and Automorphisms of Riemann Surfaces

Graph embeddings on surfaces that exhibit a large amount of symmetry have captured mathematicians’ attention for centuries. The strongest symmetry condition one usually considers is that some subgroup of the automorphism group of the graph acts transitively on triples (v, e, f) of a vertex v , an edge e containing v , and a face f incident with e . Such embeddings are called **regular maps**, which is somewhat unfortunate because both “regular” and “map” have multiple mathematical meanings. We will not use one word without the other in this context, so hopefully we are safe from confusion.

The platonic solids embedded in the sphere are the first examples of regular maps. Figures 3.1 and 3.2 provide additional examples. For many more examples and a classical introduction to the topic see Coxeter and Moser’s book [30, Chapter 8]. Alternatively, the paper of Conder and Dobcsányi provides a modern computational approach and complete lists of parameters of regular maps on surfaces of genus 2 through 15 [26].

The intrinsic beauty of regular maps is already a good motivation for their study. However, the truly remarkable mathematical feature is that any regular map has, as an infinite cover, a tessellation of the unit disk model of the hyperbolic plane. Such a tessellation determines a discrete group of homeomorphisms of the hyperbolic plane that preserve angles, and thus yields information about the automorphisms of an associated Riemann surface.

A **Riemann surface** is a 1-dimensional complex manifold, or equivalently, an orientable surface equipped with a metric that allows one to measure angles. The classical introduction is [128], and the history of the relationship to the covering graph perspective, namely uniformization, is addressed in [33]. The first example is the cover of the Riemann sphere which we remarked on in Section 3.3. This example illustrates the importance of Riemann surfaces in analysis: They provide suitable domains for studying complex functions analytically.

A homeomorphism of a Riemann surface that preserves the metric is called an **automorphism**. For Riemann surfaces of genus $g > 1$, Hurwitz showed that the automorphism group has size at most $84(g - 1)$ [81]. This is closely related to the fact that the hyperbolic plane is the universal cover of such a surface, and the discrete groups of automorphisms of this cover are determined by its tessellations, i.e. by the combinatorics of regular maps. The analogous fact for regular maps can be proven in entirely combinatorial terms, see for instance, [12, Chapter 5]. A good reference for synthesis between the analytic and combinatorial perspectives is [85]. We also refer the reader to the volume [145] for a survey

of the contemporary questions surrounding automorphisms of Riemann surfaces, many of which are combinatorial.

Riemann surfaces, or regular maps, with exactly $84(g - 1)$ automorphisms are called **Hurwitz surfaces** and are of particular interest. The smallest example, first found by Klein, can be defined as a regular map of genus 3 with 24 vertices and 56 triangular faces. The associated surface has so many remarkable features and descriptions that it is the subject of an entire book and accompanying sculpture at the Mathematical Sciences Research Institute in Berkeley [92]. There is also a strong connection to the covers of graphs we will be studying: The graph underlying this regular map is the $(8, 3, 2)$ -drackn (c.f. Section 6.3 and Figure 3.3).

More generally, Singerman and Strudwick studied quotients of hyperbolic tessellations related to the classical modular group [122]. The resulting regular maps include Klein's as well as several other Hurwitz surfaces. Stanier showed that all of these examples are p -fold covers of complete graphs on $p + 1$ vertices with exactly 4 eigenvalues [129]. In fact the graphs underlying these quotients are distance-regular covering graphs which were discovered Mathon in an entirely different context [97]. See also Section 6.3.2.

The discussion in the previous two paragraphs demonstrates that certain covers of algebraic and combinatorial interest sit at the heart of some of the nicest topologically motivated examples. This raises the question of whether other interesting distance-regular covers of graphs have regular embeddings related to Riemann surfaces. This is a question for future work.

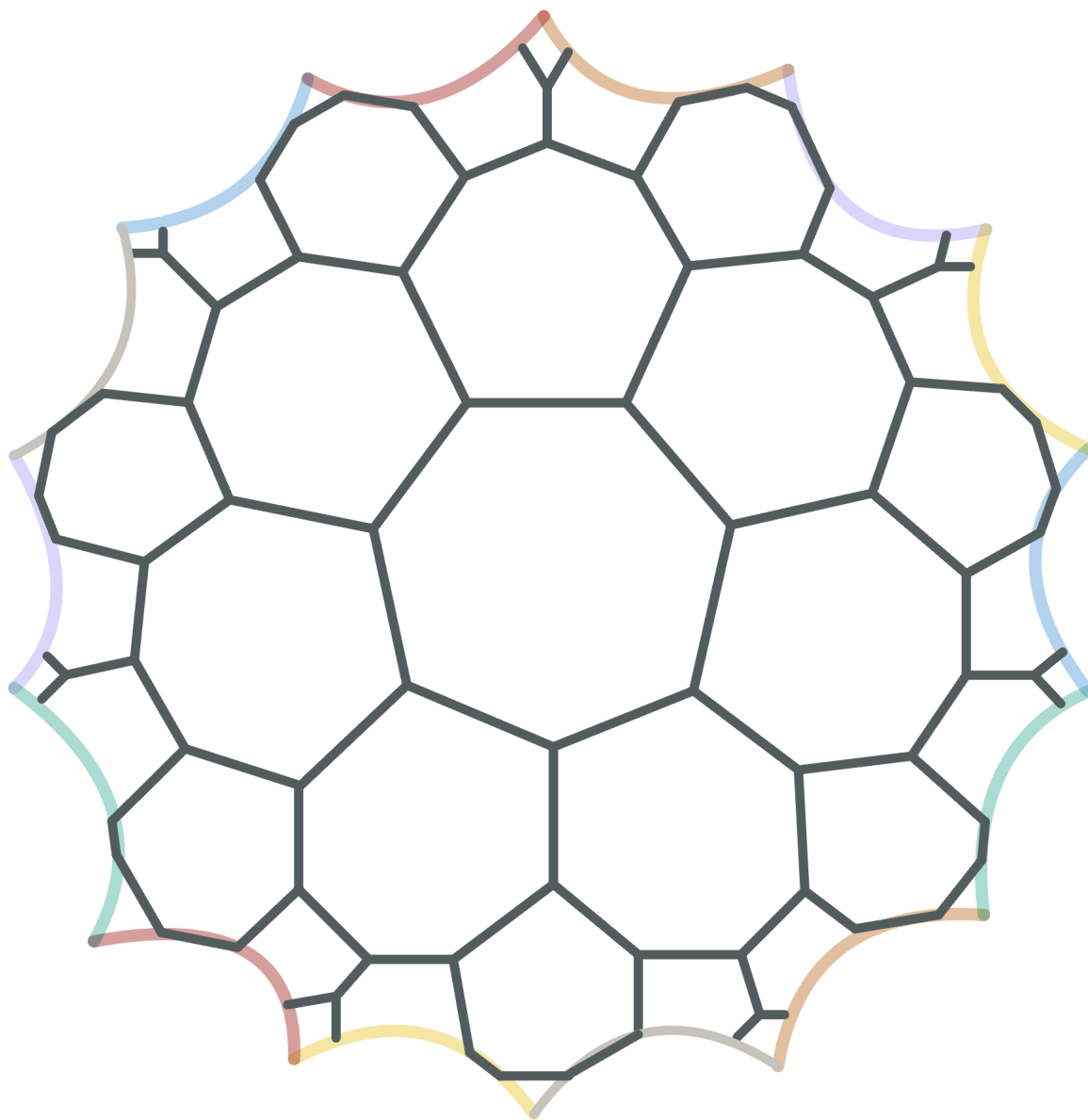


Figure 3.3: The dual of the unique distance-regular antipodal cover of K_8 , embedded in the orientable surface of genus 3. This figure is obtained from a quotient of a tessellation of the hyperbolic plane. The boundary regions are to be identified in pairs according to their colors.

Chapter 4

Covers of Hypercubes

For each hypercube Q_n there is a unique index-2 cover \tilde{Q}_n which contains no 4-cycles. These graphs were discovered by Cohen and Tits in 1984, and are the subject of this chapter. This is mostly an exposition of known work, but in Section 4.4 we give a new construction of these covers as Cayley graphs for the Heisenberg group in characteristic 2. This new interpretation suggests a natural generalization which we pursue in Chapter 5, and the covers themselves provide some of our motivation for defining the “two-eigenvalue” covers in Chapter 8.

In Section 4.1 we give a short elementary proof of the existence and uniqueness of these covers, and then discuss the graphs briefly. In Section 4.2 we turn to a conjecture from 1989 on the sensitivity of Boolean functions and describe how Huang’s sensational resolution to this conjecture in 2019 is inexorably tied to the covers \tilde{Q}_n . In Section 4.3 we recount Tao’s perspective on Huang’s proof which links Huang’s result to a group convolution operation on a Heisenberg group. We recast this perspective in Section 4.4 as a simple but novel result in algebraic graph theory: The covers \tilde{Q}_n are Cayley graphs for the Heisenberg groups. We explain how this motivates our work in Chapter 5. In Section 4.5 we point out some similarities between the covers \tilde{Q}_n and the O’Keefe–Wong graph, which Ito showed to be a cover of the point-plane incidence graph of $PG(3, 2)$.

4.1 Four-cycle-free Covers of Hypercubes

We construct the subjects of this chapter and then discuss some of their interesting combinatorial properties and occurrences. Cohen and Tits’ original existence proof in [25]

is interesting, but non-constructive and requires substantial topological machinery. Fortunately, there is also a simple combinatorial construction. This proof is probably best attributed as folklore.

4.1.1 Theorem. *For each hypercube Q_n , $n \geq 2$ there is an index-2 cover of Q_n with no 4-cycles. This graph is unique up to isomorphism.*

Proof. We construct a suitable arc function $f_n : Q_n \rightarrow \mathbb{Z}_2$ with arc matrix A_n defined inductively. Let

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and define

$$A_n = \begin{pmatrix} A_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & -A_{n-1} \end{pmatrix}.$$

Ignoring the signs of the entries, this is the inductive construction of the hypercubes: take two copies of the $n - 1$ dimensional hypercube and join them by a perfect matching. So A_n is an arc matrix for Q_n . We claim that $A_n^2 = nI_{2^n}$. This is certainly true for A_1 , and by induction

$$A_n^2 = \begin{pmatrix} (A_{n-1})^2 + I_{2^{n-1}} & 0 \\ 0 & (A_{n-1})^2 + I_{2^{n-1}} \end{pmatrix} = nI_{2^n}.$$

Now let C be any 4-cycle in Q_n , with i, j two antipodal points on C . The i, j entry of A_n^2 is the sum

$$f(i, x)f(x, j) + f(i, y)f(y, j)$$

of the values of the arc function along the two paths in Q_n from i to j . By our calculation above, this value is zero, so exactly 1 or 3 of the edges of C have sign -1 . So the value of the arc function around C is -1 and C lifts to an 8-cycle in the cover $Q_n^{f_n}$, so this cover contains no 4-cycles.

We also prove uniqueness by induction. Clearly the 8-cycle is the only 4-cycle-free index-2 cover of the 4-cycle. Let f be an arc function on Q_n for a 4-cycle-free cover. Via switching equivalence, we may assume that f is the identity on any spanning forest of Q_n without changing the isomorphism class of Q_n^f . So we assume that f is the identity on a perfect matching joining two subgraphs A and B isomorphic to Q_{n-1} . By induction, the restriction of f to A gives the unique index-2 cover of A with no 4-cycles. And by our choice that f is the identity on the perfect matching from A to B , the values of f on B are determined by their values on A : Anywhere that f is -1 on an edge of A , it must be 1 on the corresponding edge of B and vice-versa. So the arc function is determined up to switching equivalence, and the cover is determined up to isomorphism. \square

We conclude this section with a discussion of some particular details about the small members of this family.

The cover \tilde{Q}_2 is the 8-cycle. The cover \tilde{Q}_3 was first discovered by Kantor in 1882 [87]. Years earlier, Möbius had asked if it was possible to inscribe two n -gons, one inside the other so that the midpoints of the edges of one were the vertices of the other, and vice-versa [101]. Kantor showed that this is possible for a pair of 4-gons in the complex plane, see Page 428 in Coxeter [28] for an explicit description in English. The incidence graph of the edges and vertices of these mutually inscribed polygons is \tilde{Q}_3 . Coxeter also studied this graph in the context of regular maps on surfaces. See Figure 3.2 for a regular embedding of \tilde{Q}_3 . The cover of \tilde{Q}_3 occurs as the subgraphs induced on the 5th and 6th neighborhoods of certain vertices in the incidence graph of the smallest 3-regular generalized hexagon. This occurrence is what prompted Cohen and Tits to study these graphs in the first place [25].

The cover \tilde{Q}_4 a distance-regular 4-fold cover of $K_{4,4}$. In fact the cover \tilde{Q}_n is distance-regular only for $n = 2$ and $n = 4$, but in all other cases, the covers are quite close to distance-regular in several ways:

- The covers \tilde{Q}_n are all **walk-regular**, which means that for any vertex v and positive integer k , the number of closed walks of length k starting from (and ending at) vertex v is independent of the chosen vertex v . We will discuss this property and prove this fact in Chapter 8.
- The automorphism group of \tilde{Q}_n is generously transitive on the vertices and so determines an association scheme containing the adjacency matrix of \tilde{Q}_n (c.f. Section 2.2). This scheme is nearly distance-regular in the sense that its distance distribution is almost linear. (See [18, Chapter 9.2]).

4.2 Induced Subgraphs and Sensitivity

This part of the story begins with a conjecture by Nisan and Szegedy on the sensitivity of Boolean functions [103]. In the language of graph theory, a **Boolean function** is a (not necessarily proper) 2-coloring of the vertices of the hypercube. The **sensitivity** $s(f)$ of the Boolean function f is the maximum, over all vertices $x \in Q_n$, of the number of neighbors y of x with $f(x) \neq f(y)$. Nisan introduced another measure of sensitivity called “block sensitivity” denoted $bs(f)$ and gave a characterization, in terms of $bs(f)$, of the time required to compute f on an unlimited number of parallel processors using the

CREW-PRAM model, Nisan [104]. It is clear from the definition, (found in [104]) that $bs(f) \geq s(f)$, but it is not clear how much smaller $s(f)$ can be than $bs(f)$.

In 1989, Nisan and Szegedy conjectured that there is some absolute constant C so that $bs(f)$ is bounded above by $s(f)^C$, and this became known as the **sensitivity conjecture**. Also in 1989, Gotsman and Linial gave an alternate interpretation of the sensitivity conjecture in graph theoretic language, [61]. From their result it follows that the sensitivity conjecture is true provided the maximum degree of an induced subgraph of Q_n on $2^{n-1} + 1$ vertices is at least \sqrt{n} . However, a proof of this graph theoretic statement remained elusive for 30 years. It was recently given by Huang [80], and so the sensitivity conjecture is true.

Huang's proof consists of three short lemmas, each of which would be at home in an undergraduate course on algebraic graph theory. The first is Cauchy's interlacing theorem, see Godsil and Royle [57, Chapter 9].

4.2.1 Lemma. *Let $\theta_i(M)$ denote the i th largest eigenvalue of a symmetric matrix M . If B is an $m \times m$ induced submatrix of the symmetric $n \times n$ matrix A , then for all $1 \leq i \leq m$ the eigenvalues of A and B satisfy*

$$\theta_i(A) \geq \theta_i(B) \geq \theta_{i+n-m}(A). \quad \square$$

Huang's second lemma is equivalent to the construction of the arc matrices A_n we defined in the proof of Theorem 4.1.1 given above along with the calculation of their spectrum. This was pointed out by Godsil, Silina, and the author in [55].

4.2.2 Lemma. *The matrices A_n defined in the proof of Theorem 4.1.1 exist and have eigenvalues $\sqrt{n}, -\sqrt{n}$, each with multiplicity 2^{n-1} .* \square

The third lemma is a generalization of the well-known fact that the maximum degree of a graph upper bounds the largest eigenvalue of its adjacency matrix.

4.2.3 Theorem. [80] *Let Y be a graph with adjacency matrix A , and let A' be obtained from A by setting $A'_{i,j} = A'_{j,i} = -1$ for some subset of the edges $\{i, j\} \in E(Y)$. Then*

$$\Delta(H) \geq \theta_1(A'). \quad \square$$

4.2.4 Theorem. [80] *Any induced subgraph of Q_n on $(2^{n-1} + 1)$ vertices has maximum degree at least \sqrt{n} .*

Proof. Take any induced subgraph H of Q_n on $2^{n-1} + 1$ vertices and let A_H be the submatrix of A_n induced by the vertices of H . The first two lemmas imply $\Delta(H) \geq \theta_1(A_H)$. The second two lemmas imply $\theta_1(A_H) \geq \theta_{2^{n-1}}(A) = \sqrt{n}$. \square

4.3 Twisted Convolution and Heisenberg Groups

Huang's sudden resolution to the sensitivity conjecture provoked a great deal of immediate response. Some authors considered extensions and analogs of induced subgraphs of other graphs, such as Cartesian products of bipartite graphs [78], products of oriented cycles [136], halved cubes and their products with cubes [2], and abelian Cayley graphs [106]. Other authors related the ± 1 signed matrix used by Huang to other known objects of interest such as the 4-cycle-free covers of the hypercubes [55], an operator on the exterior algebra of an n -dimensional vector space [88], a certain element of a Clifford Algebra [95], Jordan-Wigner Transforms and Majorana Fermions [66], and a convolution operator on the Heisenberg group in characteristic 2 [133]. This last interpretation by Tao [133] is the subject of this section.

Let \mathbb{Z}_2^n denote the additive group of an n -dimensional vector space over the two element field. Let $\beta : \mathbb{Z}_2^n \times \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2$ be the bilinear form given by

$$\beta(x, y) := \sum_{1 \leq i < j \leq n} x_i y_j.$$

The **Heisenberg extension** of \mathbb{Z}_2^n is the group H_n is defined on the set $\mathbb{Z}_2^n \times \mathbb{Z}_2$ with multiplication

$$(x, t)(y, s) = (x + y, s + t + \beta(x, y)).$$

Tao observed the following connection between H_n and the signed matrices A_n used in Huang's proof of the sensitivity conjecture, [133]. Let G be a finite group, and let $\ell^2(G)$ be the space of functions from G to \mathbb{C} . For $f, g \in \ell^2(G)$, the **convolution** $\star_G : \ell^2(G) \times \ell^2(G) \rightarrow \ell^2(G)$ is the function defined by

$$f \star_G g(x) = \sum_{y \in G} f(y)g(y^{-1}x).$$

The action of the adjacency matrix of Q_n on $f \in \ell^2(\mathbb{Z}_2^n)$ is a convolution

$$Af = f \star_{\mathbb{Z}_2^n} \mu$$

with $\mu(x) = 1$ if x is any of the standard basis vectors e_1, \dots, e_n , and $\mu(x) = 0$ otherwise. The **twisted convolution** $\star_\beta : \ell^2(\mathbb{Z}_2^n) \times \ell^2(\mathbb{Z}_2^n) \rightarrow \ell^2(\mathbb{Z}_2^n)$ is given by

$$f \star_\beta g(x) = \sum_{y \in \mathbb{Z}_2^n} (-1)^{\beta(y, y^{-1}x)} f(y)g(y^{-1}x),$$

and there is a corresponding operator A_β whose action on f is defined by

$$A_\beta f = f \star_\beta \mu.$$

The operator A_β has the same spectrum as A_n , and is equal to A_n for a suitable choice of basis.

Tao observed that, rather than work with a twisted convolution on \mathbb{Z}_2^n , one can instead work with a regular convolution on its Heisenberg extension. Consider the map $L : \ell^2(\mathbb{Z}_2^n) \rightarrow \ell^2(H_n)$, defined by $f \mapsto L(f)$ with

$$L(f)(x, t) = (-1)^t f(x).$$

One can show that for $f, g \in \ell^2(\mathbb{Z}_2^n)$,

$$L(f) \star_{H_n} L(g) = L(f \star_\beta g)$$

which relates Huang's matrix (i.e. the arc matrix of \tilde{Q}_n) to H_n via the lift L .

4.4 The Cayley Graph Perspective: A Second Construction

In this section we give a different perspective on the relationship between the Heisenberg extension and the covers \tilde{Q}_n which is simpler from the point of view of algebraic graph theory. The result is new, although it is essentially a reformulation of Tao's perspective from the previous section. Also the cover \tilde{Q}_3 was certainly already known to be a Cayley graph for a group isomorphic to H_3 . The result appears in the preprint by the author [91].

4.4.1 Theorem. [91] *Let $B_n = \{(e_1, 0), (e_2, 0), \dots, (e_n, 0)\}$. The cover \tilde{Q}_n is isomorphic to $\text{Cay}(H_n, B_n)$.*

Proof. First note that the connection set B_n is inverse-closed and does not contain the identity, so our Cayley graph is undirected and loopless. The projection $\pi : H_n \rightarrow \mathbb{Z}_2^n$ defined by $\pi(g, s) = g$ serves as a 2-fold covering map from $\text{Cay}(H_n, B_n)$ onto $\text{Cay}(\mathbb{Z}_2^n, \pi(B_n)) \cong Q_n$. Now suppose $\text{Cay}(H_n, B_n)$ contains a 4-cycle C . This cycle's image $\pi(C)$ must be a 4-cycle in $\text{Cay}(\mathbb{Z}_2^n, \pi(B_n))$. So there is some $g \in \mathbb{Z}_2^n$ and $i, j \in \{1, \dots, n\}$, with $i < j$, so that

$$V(\pi(C)) = \{g, g + e_i, g + e_i + e_j, g + e_j\}.$$

This implies the existence of $t \in \{0, 1\}$ so that

$$V(C) = \{(g, t), (g, t)(e_i, 0), (g, t)(e_i, 0)(e_j, 0), (g, t)(e_i, 0)(e_j, 0)(e_i, 0)^{-1}\}.$$

But since C is a 4-cycle we have

$$(e_i, 0)(e_j, 0)(e_i, 0)^{-1}(e_j, 0)^{-1} = (\bar{0}, 0).$$

This contradicts the following direct calculation from the definition of H_n .

$$\begin{aligned} (e_i, 0)(e_j, 0)(e_i, 0)^{-1}(e_j, 0)^{-1} &= (e_i + e_j, \beta(e_i, e_j))(e_i, 0)(e_j, 0) \\ &= (e_i, \beta(e_i, e_j) + \beta(e_i + e_j, e_i))(e_i, 0) \\ &= (\bar{0}, \beta(e_i, e_j) + \beta(e_i + e_j, e_i) + \beta(e_i, e_i)) \\ &= (\bar{0}, 1 + 0 + 0) \\ &= (\bar{0}, 1). \end{aligned}$$

Since \tilde{Q}_n is the unique 4-cycle-free cover of Q_n of index 2, the result follows. \square

We conclude this section with a few remarks on our result.

1. We find it slightly surprising that it suffices to choose as connection set the vectors $(e_i, 0)$. This suggests that the presentation given for H_n is somehow the “correct” one. This is probably related to the geometry of the bilinear form β , but we do not have anything concrete to say about this. In a similar vein, it would be interesting to have a characterization of the n -tuples (x_1, \dots, x_n) so that the Cayley graph for the Heisenberg extension of \mathbb{Z}_2^n with connection set

$$\{(e_1, x_1), \dots, (e_n, x_n)\}$$

is isomorphic to \tilde{Q}_n .

2. Tao’s map L from the previous section behaves nicely in the Cayley graph context. We can construct an eigenbasis for the adjacency matrix of \tilde{Q}_n by taking a full collection of the eigenfunctions for $A(Q_n)$ and “doubling them up” into functions which are constant on the fibers of the cover. This accounts for half the eigenvalues of the cover. The other half belong to eigenspaces for the “new” eigenvalues \sqrt{n} and $-\sqrt{n}$. These new eigenfunctions are orthogonal to the old ones, so for any such function f and fiber $\{v, v'\}$ we have $f(v) = -f(v')$. It follows that the map L lifts any function on Q_d to a function on the cover within the support of the \sqrt{n} and $-\sqrt{n}$ eigenspaces.

3. Theorem 4.4.1 demonstrates that, in the context of Cayley graphs, there is a nice analogy between covers of graphs and extension of groups. We will study this analogy further in Chapter 5, where we develop an analog of Theorem 4.4.1 in odd characteristic by using well-known extensions of p -groups to construct 4-cycle-free covers of Cartesian products of p -cycles.

4.5 The O’Keefe–Wong Graph: A Curious Cousin

We conclude with a short discussion of another cover which is curiously similar to the Cohen-Tits covers in several ways. The **O’Keefe–Wong graph** is the unique $(7, 6)$ -cage (c.f. Section 1.2). O’Keefe and Wong discovered this graph and other cages via computer search. Ito gave an alternative construction pertinent to our interests [82].

4.5.1 Theorem. [82] *There is a unique graph on 90 vertices with valency 7 and girth 6. It is a cyclic, antipodal 3-fold cover of the point-plane incidence graph of $PG(3, 2)$.* \square

Ito begins with the incidence graph Y of the points and planes in $PG(3, 2)$, and then explicitly constructs a \mathbb{Z}_3 -arc matrix for this graph which squares to $7I$. Moreover, Ito shows that in the cover produced by this arc matrix, all of the 4-cycles of Y are broken, so the graph has girth 6. Ito uses the fact that there are no 4-cycles to prove that the fibers of the cover are a system of imprimitivity for the automorphism group, a key step in his determination of this automorphism group.

The automorphism group of Y contains the alternating group A_7 and Ito shows that the automorphism group of X contains a non-split central extension of A_7 by C_3 (c.f. Section 5.2). Schur showed that among all the alternating groups, only A_6 and A_7 possess non-split central extensions by C_3 [115]. This suggests a sense in which the O’Keefe–Wong graph is quite exceptional.

4.6 Future Work

1. Characterize the n -tuples (x_1, \dots, x_n) so that the Cayley graph for the Heisenberg extension of \mathbb{Z}_2^n with connection set

$$\{(e_1, x_1), \dots, (e_n, x_n)\}$$

is isomorphic to \tilde{Q}_n .

2. Suppose we have a graph Y and subgroup of the additive group of its cycle-space of index r which contains no short cycles. Is this sufficient to produce a short-cycle-free cover of Y of index r ?

The motivation for this question comes from Cohen and Tits' proof of Theorem 4.1.1. They proceed by claiming (without proof!) the existence of a 4-cycle-free subgroup of the cycle-space of the hypercube. They then lift this to a 4-cycle-free subgroup of fundamental group which in turn corresponds to a connected 4-cycle-free cover of index 2 via a well-known result in algebraic topology. We refer the reader to [73, Section 1.3] for the details of this correspondence.

Showing the existence of the 4-cycle-free subgroup of the cycle-space is a simple enough combinatorial exercise. It would be very interesting to find a combinatorial method to circumvent the next few topological steps.

3. Are there graphs Y whose automorphism groups contain A_6 which admit covers whose automorphism groups contain the non-split central extension of A_6 ? If so, is the combinatorics of these examples similar to the graphs discussed in this chapter?
4. What other cages can be constructed as covers by breaking the 4-cycles of incidence graphs?
5. The cover \tilde{Q}_3 and the O'Keefe–Wong graph are both λ -fold covers of symmetric (v, k, λ) -designs, as studied by Biggs and Ito [11]. Do other λ -fold covers of symmetric (v, k, λ) -designs exist?
6. Nedela and Škovič have constructed regular embeddings of the hypercube [102]. Certain of these embeddings have the 4-cycles as their facial cycles and so induce embeddings of \tilde{Q}_n whose facial cycles are all octagons. Are these embeddings regular?

Chapter 5

Extra-special Cayley Graphs

The covers of a graph Y are graphs admitting a certain type of surjective graph homomorphism onto Y . The extensions of a group G are the groups admitting a certain type of surjective group homomorphism onto G . The coincidence of these descriptions suggests that we may be able to construct covers of graphs from extensions of groups, or conversely, provided we have some reasonable translation from one to the other. Cayley graphs provide a natural setting for producing groups from graphs, and indeed, we will see that it is natural to view Cayley graphs for a group extension as covers of Cayley graphs for the base of the extension.

This correspondence between covers and group extensions is already well-known. For instance it occurs implicitly in Malnič, Nedela, and Škoviera's study of the automorphism lifting in voltage graphs [94].

We study this phenomenon with an eye towards cycle-breaking, and other interesting combinatorial properties of the covers that can be derived from extension groups. We have already seen a first family of examples: The hypercubes are Cayley graphs for \mathbb{Z}_2^d , and their 4-cycle-free covers, discussed in Chapter 4, are Cayley graphs for a central extension of \mathbb{Z}_2^d .

In this chapter we will construct 4-cycle-free covers of Cartesian products of prime length cycles as Cayley graphs for the extra-special p -groups. These groups are central extensions of elementary abelian p -groups, so this is a direct generalization of Theorem 4.4.1. This construction is the subject of the preprint [91] by the author.

As for the interesting combinatorial properties of these covers, there are more than a few. First, we find that, along with the short cycles, much of the symmetry is broken in

our newly constructed covers, and it appears that an infinite subfamily have automorphism groups of minimal size. See Section 5.5.5. Secondly, we find an interesting connection to a known family of drackns: Our covers are subgraphs of the Thas–Somma drackns. Moreover, in all the computationally accessible cases the coherent configuration generated by the orbits of a vertex stabilizer has classes which fuse together to give interesting partitions of these drackns.

5.1 Cartesian Products as Cayley Graphs

To begin, we will need some graphs and some groups. The **Cartesian product** $Y_1 \square Y_2$ of the graphs $Y_1 = (V_1, E_1)$ and $Y_2 = (V_2, E_2)$ is the graph with vertex set $V_1 \times V_2$ and (v_1, v_2) adjacent to (u_1, u_2) if either

$$v_1 = u_1 \text{ and } \{v_2, u_2\} \in E_2$$

or

$$v_2 = u_2 \text{ and } \{v_1, u_1\} \in E_1.$$

The first example is the four-cycle $K_2 \square K_2$. The next example is $K_2 \square K_2 \square K_2$, the 3-cube. And indeed, the n th example is the n -cube $K_2 \square \dots \square K_2$. Already we would like some notation for iterated Cartesian products, so we let

$$Y^d := Y \square Y \square \dots Y$$

denote the Cartesian product of d copies of Y . A more general family of examples including the hypercubes is the **Hamming graphs**

$$H(n, d) := K_n^d.$$

An equivalent definition of $H(n, d)$ is the graph on $[n]^d$ with two d -tuples adjacent if they differ in exactly one coordinate.

Throughout this chapter, we let C_n denote the cycle of length n . For $n > 3$, the graph C_n^d is a proper spanning subgraph of K_n^d , but still a reasonable generalization of K_2^d provided we interpret $K_2 = C_2$ as a “2-cycle”. When n is prime, Cayley graphs for \mathbb{Z}_p^d are a natural place to make such an interpretation. It is probably safe to attribute the next result as folklore, but [144] is an explicit citation.

5.1.1 Lemma. *Let $G = \mathbb{Z}_p^d$ be the additive group of a d -dimensional vector space \mathcal{V} and let S be a basis for \mathcal{V} . Let $S^{-1} = \{s^{-1} : s \in S\}$. The Cayley graph $\text{Cay}(G, S \cup S^{-1})$ is isomorphic to C_p^d . \square*

So we have a group \mathbb{Z}_p^d and a graph C_p^d , and we would like to construct a cover of the later from an extension of the former. It remains to consider some interesting extensions of \mathbb{Z}_p^d .

5.2 Central Extensions and Cayley Graphs

Put plainly, the group E is an **extension** of the group G by the subgroup N if the quotient of E by N is isomorphic to G . To be a bit more precise, a **short exact sequence** of groups is a sequence of group homomorphisms

$$1 \rightarrow N \xrightarrow{i} E \xrightarrow{q} G \rightarrow 1$$

in which the image of each homomorphism is the kernel of the next. The presence of the trivial group at either end of the sequence imply that $i : N \rightarrow E$ is injective and $q : E \rightarrow G$ is surjective. The conditions on q and i imply that the quotient of E by the image of N is isomorphic to G . When such a sequence exists, we say that E is an **extension** of G by N . For our purposes it suffices to assume that N is abelian, write $N = A$, and use additive notation for its multiplication. In this case, The conjugation action of E on itself fixes $i(A)$ *elementwise*, which implies that $E/A = G$ acts on A . We will end up being interested in only the simplest case: when G acts trivially on A . This happens if and only if $i(A)$ is central in E , in which case we say that E is a **central extension**. Any central extension by an abelian group A can defined explicitly as a group on the set $G \times A$ with a multiplication

$$(g, a)(h, b) := (gh, a + b + \kappa(g, h))$$

where $\kappa : G \times G \rightarrow A$ satisfies the property

$$\kappa(ab, c) + \kappa(a, b) = \kappa(a, bc) + \kappa(b, c).$$

Such a map κ is called a **2-cocycle**, and this condition is equivalent to the associativity of the multiplication we have just defined. We may always assume that

$$\kappa(g, id) = \kappa(id, g) = 0,$$

in which case we say that the 2-cocycle is **normalized**. In fact, the central extensions are characterized by normalized 2-cocycles, and a natural notion of equivalence of extensions corresponds to a difference in the corresponding 2-cocycles by a map called a 2-coboundary. This is the correspondence between central extensions and the second cohomology group $H^2(G, N)$. We refer the reader to [20] for further details of this case, as well as the more general setting when the action of G on A need not be trivial.

Cayley graphs provide a nice translation from groups to graphs. We can now formalize the sense in which this translation “commutes” with the constructions of covering graphs and group extensions. This is not a new idea, but this explicit presentation is the author’s.

5.2.1 Theorem. *Let E be an extension of G by A , and let S be an inverse-closed subset of E . Suppose $S \cap i(A) = \emptyset$ and $q|_S$ is injective. Then $\text{Cay}(E, S)$ is a cover of $\text{Cay}(G, q(S))$ with covering projection q .*

Proof. Since $S \cap i(A) = \emptyset$, the subsets $F_g := \{(g, a) : a \in A\}$ induce independent sets in X . If X is edgeless there is nothing left to prove, so let (g, a) and (h, b) be adjacent vertices of X . Then there exists $(s_G, s_A) \in S$ with

$$(g, a) = (s_G, s_A)(h, b).$$

Thus

$$(s_G, s_A) = (gh^{-1}, a - b + \kappa(g, h^{-1}) - \kappa(h, h^{-1}))$$

so for $(h, b') \in F_h$, we have

$$(s_G, s_A)(h, b') = (g, a - b + b' + \kappa(g, h^{-1}) - \kappa(h, h^{-1})) + \kappa(gh^{-1}, h).$$

Applying the 2-cocycle identity to $\kappa(gh^{-1}, h)$ this reduces to

$$(s_G, s_A)(h, b') = (g, a - b + b' + \kappa(g, hh^{-1})) = (g, a - b + b'),$$

so F_g and F_h induce a subgraph containing a perfect matching. Since q is injective on s , no further element of s is responsible for edges joining F_g to F_h , and this subgraph is exactly a perfect matching. Moreover, $s_G \in q(S)$ and so this perfect matching covers an edge of $\text{Cay}(G, q(S))$. \square

We will explore this idea concretely with a certain well-known family of extension groups.

5.3 Extra-special p -groups

Our definitions follow Gorenstein [60]. As always, p is a prime. A group G is a **p -group** if the order of each element is divisible by p . If G is abelian and the order of each non-identity element of G is exactly p , then G is an **elementary abelian p -group**. Any elementary abelian p -group is isomorphic to \mathbb{Z}_p^d for some d [60].

There is just one p -group of order p , and exactly two of order p^2 , all three of these groups are abelian. For order p^3 there are three abelian p -groups and two non-abelian ones [21]. These non-abelian groups of order p^3 are the smallest examples of extra-special p -groups. A p -group G is **extra-special** if its center $Z = Z(G)$ is a cyclic group of order p and G/Z is an elementary abelian p -group of order greater than 1.

For each nonnegative integer d there are exactly two non-isomorphic extra-special p -groups of order p^{1+2d} and none of order p^{2d} . See [60, Section 5.2]. One of these groups has exponent p , the other has exponent p^2 . We follow the convention to denote these groups by p_+^{1+2d} and p_-^{1+2d} , respectively. Gorenstein shows how to construct these groups from copies of the two extra-special groups of order p^3 using a certain product operation which identifies the centers of two groups. We will use an equivalent description which highlights the fact that these extra-special groups are central extensions of the elementary abelian groups \mathbb{Z}_p^{2d} .

From the definition of the extra-special p -groups we have short exact sequences

$$1 \rightarrow \mathbb{Z}_p \rightarrow p_{\pm}^{1+2d} \rightarrow \mathbb{Z}_p^{2d} \rightarrow 1$$

in which \mathbb{Z}_p is identified with $Z := Z(p_{\pm}^{1+2d})$. The next two lemmas detail 2-cocycles which give rise to the two isomorphism classes of extra-special p -groups. We will identify \mathbb{Z}_p^{2d} with $\mathbb{Z}_p^d \times \mathbb{Z}_p^d$ in order to make the notation clearer. We note that these calculations are standard, but we include them because we do not know of a suitable reference.

5.3.1 Lemma. *The map $\kappa_+ : (\mathbb{Z}_p^d \times \mathbb{Z}_p^d) \times (\mathbb{Z}_p^d \times \mathbb{Z}_p^d) \rightarrow \mathbb{Z}_p$ defined by*

$$((\bar{a}, \bar{b}), (\bar{c}, \bar{d})) \mapsto \bar{b} \cdot \bar{c}$$

is a 2-cocycle. It determines the extra-special p -group p_+^{1+2d} of exponent p on the set $\mathbb{Z}_p^d \times \mathbb{Z}_p^d \times \mathbb{Z}_p$.

Proof. The cocycle condition follows from the linearity of the dot product. The induced group multiplication is

$$(\bar{a}, \bar{b}, z)(\bar{c}, \bar{d}, w) = (\bar{a} + \bar{c}, \bar{b} + \bar{d}, z + w + \bar{b} \cdot \bar{c}). \quad (5.1)$$

Associativity of this multiplication follows from the cocycle condition. The identity element is $(\bar{0}, \bar{0}, 0)$ and the inverse is given by

$$(\bar{a}, \bar{b}, z)^{-1} = (-\bar{a}, -\bar{b}, -z + \bar{a} \cdot \bar{b}).$$

To verify that the group is extra-special, we calculate the commutator of a generic pair of elements.

$$\begin{aligned} [(\bar{a}, \bar{b}, z), (\bar{c}, \bar{d}, w)] &= (\bar{a}, \bar{b}, z)^{-1}(\bar{c}, \bar{d}, w)^{-1}(\bar{a}, \bar{b}, z)(\bar{c}, \bar{d}, w) \\ &= (-\bar{a}, -\bar{b}, -z + \bar{a} \cdot \bar{b})(-\bar{c}, -\bar{d}, -w + \bar{c} \cdot \bar{d})(\bar{a}, \bar{b}, z)(\bar{c}, \bar{d}, w) \\ &= (0, 0, \bar{a} \cdot \bar{b} + \bar{c} \cdot \bar{d} + (-\bar{b}) \cdot (-\bar{c}) + (-\bar{b} - \bar{d}) \cdot \bar{a} + (-\bar{d}) \cdot \bar{c}) \\ &= (0, 0, \bar{b} \cdot \bar{c} - \bar{a} \cdot \bar{d}). \end{aligned}$$

It follows that a fixed element (\bar{a}, \bar{b}, z) is in the center $Z = Z(G)$ if and only if $\bar{b} \cdot \bar{x} - \bar{y} \cdot \bar{a}$ is identically zero as (\bar{x}, \bar{y}) ranges over $\mathbb{Z}_p^d \times \mathbb{Z}_p^d$. This happens if and only if $\bar{a} = \bar{b} = \bar{0}$, hence

$$Z = \{(\bar{0}, \bar{0}, z) : z \in \mathbb{Z}_p\}.$$

Next, note that the cosets gZ in G depend only on the first $2d$ coordinates of the vector g . After we restrict to these coordinates the multiplication rule for the group degenerates to vector addition, so G/Z is isomorphic to \mathbb{Z}_p^{2d} .

Finally, we calculate

$$(\bar{a}, \bar{b}, x)^p = (\bar{0}, \bar{0}, \bar{b} \cdot \bar{a} + 2\bar{b} \cdot \bar{a} + \cdots + (p-1)\bar{b} \cdot \bar{a}) = (\bar{0}, \bar{0}, \binom{p}{2}\bar{b} \cdot \bar{a}) = (\bar{0}, \bar{0}, 0).$$

So the exponent is p and G is p_+^{1+2d} . □

Next we give a similar lemma describing the other extra-special group of order p^{1+2d} over the same set. This requires a less natural multiplication. Let ι be the inclusion $\mathbb{Z}_p \hookrightarrow \mathbb{Z}$. Define a map $\phi : \mathbb{Z}_p \times \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ by

$$\phi(a, b) = \begin{cases} 0, & \text{if } \iota(a) + \iota(b) < p \\ 1, & \text{if } \iota(a) + \iota(b) \geq p. \end{cases}$$

Let \bar{a}, \bar{c} be vectors in \mathbb{Z}_p^d , and let a_1, c_1 denote their first coordinates. We use the function ϕ to “carry” the data from this first coordinate to the coordinate of the commutator.

5.3.2 Lemma. *The map $\kappa_- : (\mathbb{Z}_p^d \times \mathbb{Z}_p^d) \times (\mathbb{Z}_p^d \times \mathbb{Z}_p^d) \rightarrow \mathbb{Z}_p$ defined by*

$$((\bar{a}, \bar{b}), (\bar{c}, \bar{d})) \mapsto \bar{b} \cdot \bar{c} + \phi(a_1, c_1)$$

is a 2-cocycle. It determines the extra-special p -group, p_-^{1+2d} of exponent p^2 on the set $\mathbb{Z}_p^d \times \mathbb{Z}_p^d \times \mathbb{Z}_p$.

Proof. As in the previous lemma, the first summand in the expression for κ_- will satisfy the cocycle condition since the dot product is linear. For the second summand we must show that

$$\phi(a+b, c) + \phi(a, b) = \phi(a, b+c) + \phi(b, c).$$

If $\iota(a) + \iota(b) + \iota(c) < p$ then all four terms in this equation are 0. If $2p < \iota(a) + \iota(b) + \iota(c)$ then all four terms are 1. Now suppose $p < \iota(a) + \iota(b) + \iota(c) < 2p$. If $\iota(a) + \iota(b) < p$ then $\iota(a+b) + \iota(c) \geq p$, and the left side of the equation is $1 + 0$. If $\iota(a) + \iota(b) \geq p$ then $\iota(a+b) + \iota(c) < p$, and the left side of the equation is $0 + 1$. Hence the left side is identically 1. The same argument applied to $\iota(b) + \iota(c)$ shows that the right side is also identically 1. So κ_- is a 2-cocycle. The induced multiplication is

$$(\bar{a}, \bar{b}, z)(\bar{c}, \bar{d}, w) = (\bar{a} + \bar{c}, \bar{b} + \bar{d}, z + w + \bar{b} \cdot \bar{c} + \phi(a_1, c_1)). \quad (5.2)$$

Associativity of this multiplication follows from the cocycle condition. The identity element is $(\bar{0}, \bar{0}, 0)$ and the inverse is given by

$$(\bar{a}, \bar{b}, z)^{-1} = (-\bar{a}, -\bar{b}, -z + \bar{a} \cdot \bar{b} - \phi(a_1, -a_1)).$$

Denote this group by G . To verify that G is extra-special, we calculate the commutator of a generic pair of elements.

$$\begin{aligned} & [(\bar{a}, \bar{b}, z), (\bar{c}, \bar{d}, z)] \\ &= (\bar{a}, \bar{b}, z)^{-1}(\bar{c}, \bar{d}, z)^{-1}(\bar{a}, \bar{b}, z)(\bar{c}, \bar{d}, z) \\ &= (-\bar{a}, -\bar{b}, -z + \bar{a} \cdot \bar{b} - \phi(-a_1, -a_1))(-\bar{c}, -\bar{d}, -z + \bar{c} \cdot \bar{d} - \phi(c_1, -c_1))(\bar{a}, \bar{b}, z)(\bar{c}, \bar{d}, z) \\ &= (0, 0, \bar{b} \cdot \bar{c} - \bar{a} \cdot \bar{d} - \phi(c_1, -c_1) - \phi(-a_1, a_1) + \phi(-a_1, -c_1) + \phi(-a_1 - c_1, a_1) + \phi(-c_1, c_1)) \\ &= (0, 0, \bar{b} \cdot \bar{c} - \bar{a} \cdot \bar{d} - \phi(-a_1, a_1) + \phi(-a_1, -c_1) + \phi(-a_1 - c_1, a_1)). \end{aligned}$$

We claim that

$$-\phi(-a_1, a_1) + \phi(-a_1, -c_1) + \phi(-a_1 - c_1, a_1) = 0$$

for all values of a_1 and c_1 . This is immediate when $a_1 = 0$ since all three summands are 0. When $a_1 \neq 0$, we have $-\phi(-a_1, a_1) = -1$ and it remains to verify that $\phi(-a_1, -c_1)$, and $\phi(-a_1 - c_1, a_1)$ take different values.

Suppose $\phi(-a_1, -c_1) = 1$. Then $\iota(-a) + \iota(-c) \geq p$ which implies $\iota(-a - c) = \iota(-a) + \iota(-c) - \iota(p)$, hence $\iota(-a - c) + \iota(a) = \iota(c) + \iota(-a) + \iota(a) - \iota(p) = \iota(c) < p$, and $\phi(-a - c, a) = 0$.

Suppose $\phi(-a_1, -c_1) = 0$. Then $\iota(-a) + \iota(-c) < p$ which implies $\iota(-a - c) = \iota(-a) + \iota(-c)$, hence $\iota(-a - c) + \iota(a) = \iota(c) + \iota(-a) + \iota(a) = p + \iota(c) \geq p$, and $\phi(-a - c, a) = 1$. This proves the claim, and reduces our commutator to the familiar formula

$$[(\bar{a}, \bar{b}, z), (\bar{c}, \bar{d}, z)] = (0, 0, \bar{b} \cdot \bar{c} - \bar{a} \cdot \bar{d}).$$

As in the proof of Lemma 5.3.1 this implies that

$$Z(G) = \{(\bar{0}, \bar{0}, z) : z \in \mathbb{Z}_p\}.$$

This in turn implies that G/Z is elementary abelian.

Finally, let \bar{v} be the vector $(1, 0, \dots, 0)$ in \mathbb{Z}_p^d . We calculate $(\bar{v}, \bar{0}, 0)^p$. Note that for $k < p$, $\phi(k - 1, 1) = 0$, and so $(\bar{v}, \bar{0}, 0)^k = (k\bar{v}, \bar{0}, 0)$. Hence

$$(\bar{v}, \bar{0}, 0)^p = ((p - 1)\bar{v}, \bar{0}, 0)(\bar{v}, \bar{0}, 0) = (\bar{0}, \bar{0}, 1).$$

So $(\bar{v}, \bar{0}, 0)$ has order p^2 . It follows that G has exponent p^2 , and is p_-^{1+2d} . \square

As a nice byproduct of this presentation of the extra-special p -groups, we see an elementary example of a definition due to Philip Hall used in the classification of p -groups of small order [69]. Let $[G, G]$ denote the group generated by all commutators of G . Two groups G_1 and G_2 are **isoclinic** if there are isomorphisms $\theta : G_1/Z(G_1) \rightarrow G_2/Z(G_2)$ and $\psi : [G_1, G_1] \rightarrow [G_2, G_2]$ so that for all $g, h \in G_1/Z(G_1)$,

$$\psi([g, h]) = [\theta(g), \theta(h)].$$

By expressing p_+^{1+2d} and p_-^{1+2d} as groups on the same set with identical commutators, we have shown that these groups are isoclinic with θ and ψ both chosen to be identity maps.

5.4 Four-cycle-free Covers of Cartesian Products of p -cycles

We are ready to construct 4-cycle free covers of C_p^{2d} as Cayley graphs for the groups p_\pm^{1+2d} . So we must construct connection sets S_\pm whose images under the quotient map are

connection sets for C_p^{2d} as a Cayley graph of \mathbb{Z}_p^d . As we noted above, we may use any basis of \mathbb{Z}_p^d for this purpose. Let $e_1, \dots, e_d, f_1, \dots, f_d$ denote the standard basis for $\mathbb{Z}_p^d \times \mathbb{Z}_p^d$, and let $[n] = \{1, \dots, n\}$. Define sets

$$A = \left\{ \sum_{i \in [k]} e_i + \sum_{j \in [k-1]} f_j : k \in [d] \right\}, \quad B = \left\{ \sum_{i \in [k-1]} e_i + 2e_k + \sum_{j \in [k]} f_j : k \in [d] \right\}$$

and

$$S = A \cup B.$$

For example, if $d = 2$, S consists of:

$$\begin{aligned} (1, 0) \times (0, 0) &\in A, \\ (2, 0) \times (1, 0) &\in B, \\ (1, 1) \times (1, 0) &\in A, \\ (1, 2) \times (1, 1) &\in B. \end{aligned}$$

Recall from the previous section that we may consider $\mathbb{Z}_p^d \times \mathbb{Z}_p^d \times \mathbb{Z}_p$ as the set on which both extra-special p -groups p_{\pm}^{1+2d} are defined. We will use the set S to build a connection set for Cayley graphs on these groups. To this end, let $\epsilon : \mathbb{Z}_p^d \times \mathbb{Z}_p^d \hookrightarrow \mathbb{Z}_p^d \times \mathbb{Z}_p^d \times \mathbb{Z}_p$ be the natural inclusion defined by $\epsilon(\hat{a}, \hat{b}) = (\hat{a}, \hat{b}, 0)$.

The next lemma will ensure that the set $\epsilon(S)$ has the necessary properties to produce covers of C_p^{2d} that contain no 4-cycles.

5.4.1 Lemma. *Let $S = A \cup B$ and ϵ be defined as above.*

- (a) *S is a basis for the vector space \mathbb{Z}_p^{2d} .*
- (b) *No two elements of $\epsilon(S)$ commute with respect to the multiplication (5.1) or (5.2).*

Proof. Write the elements of S as the rows of a $2d \times 2d$ matrix. Move all the even indexed columns to the end while preserving the relative order of these columns. In other words, apply the permutation of column indices given by

$$2i \mapsto d + i, \quad 2d - i \mapsto d - i$$

for $i \in \{1, \dots, d\}$. It is immediate that this matrix is lower triangular with non-zero diagonal entries, hence its rows are a basis for \mathbb{Z}_p^{2d} . This proves (a).

We have noted in the previous section that the commutator maps for p_-^{1+2d} and p_+^{1+2d} are identical. To prove (b) it suffices to verify that this commutator is nonzero for distinct $g, h \in \epsilon(S)$. Indeed, we calculate

$$[g, h] = \begin{cases} (0, 0, 1), & g, h \in A \\ (0, 0, -1), & g, h \in B \\ (0, 0, 1), & g \in A, h \in B. \end{cases} \quad \square$$

The final step in our construction is to make $\epsilon(S)$ into a connection set for a Cayley graph of p_{\pm}^{1+2d} . To do this we must take the set's closure under inverse with respect to the appropriate multiplication. Define S_+ , respectively S_- , by

$$S_{\pm} = \epsilon(S) \cup \epsilon(S)^{-1}$$

where $\epsilon(S)^{-1}$ consists of the inverses of elements of $\epsilon(S)$ with respect to the multiplication (5.1), respectively (5.2).

5.4.2 Theorem. *Let p be an odd prime and d a non-negative integer. Let S, ϵ, S_{\pm} be defined as above. The Cayley graphs $\text{Cay}(p_+^{1+2d}, S_+)$ and $\text{Cay}(p_-^{1+2d}, S_-)$ are non-isomorphic 4-cycle-free p -fold covers of C_p^{2d} .*

Proof. Let $T = \{e_1, \dots, e_{2d}\}$ be the standard basis for \mathbb{Z}_p^{2d} . Recall that C_p^{2d} is a Cayley graph for \mathbb{Z}_p^{2d} with connection set

$$T \cup T^{-1} = \{\pm e_1, \dots, \pm e_d\}.$$

T is a basis for \mathbb{Z}_p^{2d} and by Lemma 5.4.1 (a) the set S is as well. Choose a bijection from T to S , and let α be its extension to a linear map. It follows that α , considered as a map between vertex sets, is an isomorphism from $\text{Cay}(\mathbb{Z}_p^{2d}, T \cup T^{-1})$ to $\text{Cay}(\mathbb{Z}_p^{2d}, \alpha(T) \cup \alpha(T)^{-1})$. Since $\alpha(T) \cup \alpha(T)^{-1}$ is precisely the image of S_{\pm} under the quotient of p_{\pm}^{1+2d} by $Z(p_{\pm}^{1+2d})$, the Cayley graphs $\text{Cay}(p_+^{1+2d}, S_+)$ and $\text{Cay}(p_-^{1+2d}, S_-)$ are covers of C_p^{2d} whose fibers are the cosets of Z in p_{\pm}^{1+2d} .

Now suppose there is some 4-cycle C in $\text{Cay}(p_{\pm}^{1+2d}, S_{\pm})$. From the definition of a covering graph, the vertices of this 4-cycle must lie in four different fibers, hence the

image $\gamma(C)$ under the covering map is also a 4-cycle. The vertex set of any 4-cycle in $\text{Cay}(\mathbb{Z}_p^{2d}, \alpha(T) \cup \alpha(T)^{-1})$ is of the form

$$\{v, g + v, h + g + v, -g + h + g + v\}$$

for some $v \in \mathbb{Z}_p^{2d}$, $g, h \in \alpha(T)$ with $g \notin \{h, h^{-1}\}$. This implies that the vertex set of C is

$$\{\tilde{v}, \tilde{g}\tilde{v}, \tilde{h}\tilde{g}\tilde{v}, \tilde{g}^{-1}\tilde{h}\tilde{g}\tilde{v}\}$$

for some $\tilde{v} \in \gamma^{-1}(v)$, $\tilde{g}, \tilde{h} \in S_{\pm}$ with $\tilde{g} \notin \{\tilde{h}, \tilde{h}^{-1}\}$. Since C is a 4-cycle

$$\tilde{v} = \tilde{h}^{-1}\tilde{g}^{-1}\tilde{h}\tilde{g}\tilde{v},$$

so \tilde{g} and \tilde{h} commute, contradicting Lemma 5.4.1 (b).

To conclude, we verify that the two covers are not isomorphic. If a cover of C_p^{2d} contains a p -cycle, the image under the covering map of this p -cycle must be a cycle of order divisible by p , thus a p -cycle. The only such p -cycles in $\text{Cay}(C_p^{2d}, T \cup T^{-1})$ have as vertices the powers of an element of T . However, with respect to the multiplication (5.2) each element $t \in S_-$ satisfies $t^p = (0, \dots, 0, 1)$ if it came from the set A , or $t^p = (0, \dots, 0, 2)$ if it came from the set B . So S_- consists only of elements of order p^2 . It follows that $\text{Cay}(p_-^{1+2d}, S_-)$ contains no p -cycles.

On the other hand, since each $s \in S_+$ has order p , $\text{Cay}(p_+^{1+2d}, S_+)$ does contain p -cycles with vertices

$$g, sg, s^2g, \dots, s^{p-1}g$$

for each $s \in S_+$ and $g \in G$. So the two covers are not isomorphic. \square

Now it is easy to construct covers for C_p^{2d-1} as induced subgraphs.

5.4.3 Corollary. *Consider C_p^{2d} as the image of the covering maps associated with the two covers given in the previous theorem. If V is the vertex set of an induced C_p^{2d-1} in C_p^{2d} , then the preimages of V under the covering maps are the vertex sets of induced non-isomorphic 4-cycle free covers of C_p^{2d-1} .* \square

5.5 Some Properties of our Covers

We note some interesting properties of some of the graphs we have constructed. The results here are computational, and the association schemes discussed were determined

via the following procedure: We use SAGE to build the graph in question, calculate its automorphism group and then determine the association scheme from the orbitals of the action of the automorphism group (c.f. Section 2.2.)

In this section, we denote the 4-cycle-free covers by the groups we used to construct them.

5.5.1 3_+^{1+2}

The cover of $C_3 \square C_3$ which comes from the group of exponent p is rather remarkable: It sits inside a symmetric association scheme

$$\{A_0, A_1, A_2, A_3, A_4, A_5\}$$

in which

- A_1 and A_2 are the adjacency matrices of isomorphic copies of 3_+^{1+2} .
- $A_1 + A_2$ is the adjacency matrix of the cyclic $(9, 3, 3)$ -drackn (c.f. Section 6.3).
- $\{A_0, A_1 + A_2, A_3 + A_4, A_5\}$ is the symmetric 3-class association scheme generated by the $(9, 3, 3)$ -drackn.

This appears to be our only new example where the coherent configuration generated by the orbitals is commutative. Nevertheless, certain classes in the non-commutative coherent configurations that contain the other covers also sum together to form the adjacency matrices of certain distance-regular graphs called the Thas–Somma drackns (c.f. 6.3.1).

5.5.2 3_-^{1+2}

The coherent configuration determined by the orbitals in this case has 15 classes, many of which are not symmetric. Fusing all asymmetric classes with their transposes we have a decomposition of K_{27} into 9 subgraphs:

- Two different subgraphs isomorphic to $9C_3$.
- A subgraph isomorphic to $3(C_3 \square C_3)$.
- Three different subgraphs isomorphic to 3_-^{1+2} .

- Three different subgraphs isomorphic to $3C_9$.

If we take the union of one set of 9 triangles, one copy of 3_-^{1+2} and one set of three 9-cycles, we obtain the cyclic $(9, 3, 3)$ -drackn again. If we replace the set of 9 triangles with the other set of 9 triangles, we get the non-cyclic $(9, 3, 3)$ -drackn. This last transformation is an instance of a result of Jurišić on constructing new drackns from the Thas–Somma drackns by “switching on a spread” in the underlying generalized quadrangle [86].

5.5.3 5_+^{1+2}

This example sits in a 19 class coherent configuration with 4 asymmetric classes. Among the symmetric classes there are four of valency 4, nine of valency 8, and two of valency 2. The sum of the 4 classes of valency 4 and a unique class of valency 8 is the Thas–Somma $(25, 5, 5)$ -drackn.

5.5.4 3_+^{1+4}

This example sits in a 20 class coherent configuration in which all but two classes are symmetric. Among the symmetric classes, there is one of valency 2, 8 of valency 8, and 9 of valency 16. Four of the valency 8 classes are isomorphic to the cover 3_+^{1+4} the other four are not. If we take the union of all 8 classes of valency 8, along with a unique class of valency 16, the resulting graph is the Thas–Somma $(81, 3, 27)$ -drackn. It is notable that a drackn is still appearing in this “higher dimensional” example.

5.5.5 Graphical Regular Representations

If X is a Cayley graph for the group G , then G acts on X as a group of automorphisms. If G is the whole automorphism group of X then X is a **graphical regular representation**. Cayley graphs with this property have been extensively studied since at least the 1970s, and are still an active research area today [51], [127], [37].

Each of our examples 5_-^{1+2} , 7_-^{1+2} , 11_-^{1+2} , 13_-^{1+2} is a graphical regular representation. So we wonder if all of our examples, p_-^{1+2} , $p > 3$ are graphical regular representations. This shows a setting in which breaking all the short cycles is linked to breaking symmetry.

5.5.6 Spectra and Induced Subgraphs

We have found the covers above by generalizing and synthesizing ideas surrounding Huang's proof of the sensitivity conjecture. So it is very natural to wonder if the existence of these covering graphs allows us to derive meaningful bounds on the maximum degree of an induced subgraph of C_p^d . The empirical evidence is not terribly promising.

Let A be the p th-root-of-unity valued adjacency matrix for C_p^{2d} associated with the cover p_{\pm}^{1+2d} . The major obstacle in applying interlacing is that the size of the largest eigenvalue of A appears to grow very slowly as a function of d . Moreover it has relatively low multiplicity. This diminishes the quality of any bound on maximum degree, as well as the size of subgraph to which such a bound can be applied. For instance: Using Huang's method and our covers, we can show that every induced subgraph of C_3^2 on at least 4 vertices contains a vertex of degree at least 3, and any induced subgraph of C_3^4 on at least 46 vertices has a vertex of degree at least 4. On the one hand, at least for these small examples, our bounds are better than those obtained by Tikaradze [136], who considered induced subgraphs of Cartesian products of directed cycles. On the other hand, C_3^4 is 3-colorable, so it seems that the most interesting problem is to obtain a bound on the maximum degree of an induced subgraph on $81/3 + 1 = 28$ vertices, far outside the reach of our graphs' spectra.

If nothing else, the apparent limitations in this method serve as a reminder of just how perfectly Huang's proof fits its intended purpose. Huang's matrix, or the 4-cycle-free covers of Q_n , have exactly the correct spectrum to give exactly the right bound on induced subgraphs of exactly the pertinent size. It appears that this is too much to hope for from our generalization.

5.6 Future Work

1. How many different isomorphism classes of 4-cycle-free covers of C_p^d are there? We have exhibited two for each p and d . One can find more by replacing any subset of the p^2 cycles in p_{-}^{1+2d} with p disjoint p -cycles, but these will not necessarily be Cayley graphs. Are our examples the only Cayley examples?
2. Can we give a concrete description of the relationship between our newly constructed covers and the Thas-Somma drackns?
3. Is it always the case that for $p > 3$ our covers p_{-}^{1+2} have minimal automorphism groups (of order p^3)?

4. What is the correct upper bound on the maximum degree of an induced subgraph of C_p^d . How does the correct bound compare to the lower bounds which can be extracted from our covers?
5. Do our 4-cycle-free covers give rise to any interesting embeddings? For example, $C_3 \square C_3$ has a regular embedding on the torus with faces of length 4. It follows that each of our covers of $C_3 \square C_3$ has an embedding in a surface of genus 10 with all faces bounded by 12-cycles. Are either of these embeddings regular? Another example: Mohar et al. demonstrated that the Cartesian product of three copies of C_3 has an embedding in a surface of genus 7 with many faces that are triangles and 4-cycles [99]. What do the derived embeddings of our covers look like?

Chapter 6

Distance-Regular Antipodal Covers of Complete Graphs

This mostly expository chapter exhibits one of the main themes of algebraic graph theory: If a matrix algebra is defined by a suitably nice set of combinatorial objects, the eigenvalues of a canonical basis for the algebra can be used to calculate its structure coefficients and conversely.

The matrix algebras we will study are generated by the distance matrices of distance-regular graphs. Predominantly we are interested in distance-regular graphs of diameter $d = 3$ which are antipodal. This means that the relation on vertices of “being at distance 0 or d ” is an equivalence relation. We will see that a graph with these properties is necessarily a cover of the complete graph K_n for some n . So we are studying **Distance-Regular Antipodal Covers of K_n** , or **drackns**.

We begin with a short introduction to distance-regular graphs in general in Section 6.1. In Section 6.2 we state a result of Smith which partitions the study of distance regular graphs in such a way that the drackns are seen to be a natural subclass. We begin to discuss drackns in Section 6.3, and then continue with a discussion of their parameters in Section 6.4. We then conclude with Section 6.5 where we discuss triangle-free drackns and prove a new result: The subgraph induced on the second-neighborhood of a triangle-free drackn must be connected, except for possibly the second-neighborhood of the second-neighborhood of a vertex in a Moore graph of valency 57.

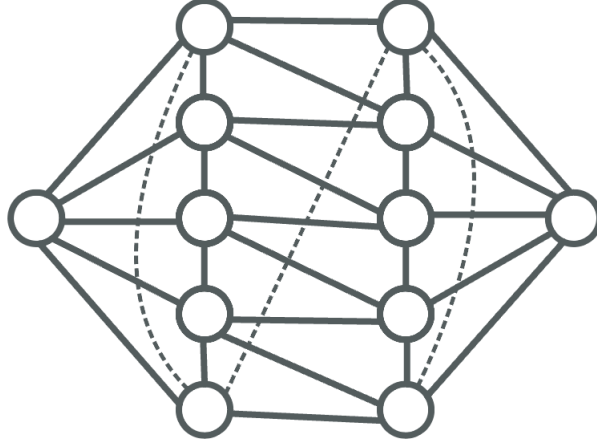


Figure 6.1: The icosahedron, with vertices organized according to their distance from the left-most vertex.

6.1 Distance-regular Graphs

The definition of distance-regularity was given (implicitly) by Biggs in [9], but the first examples are the graphs associated with the Platonic solids which date back to antiquity.

To motivate the definition, let us consider the icosahedron explicitly. See Figure 6.1. Pick an arbitrary vertex v and let v' be the unique vertex at distance 3 from v . The standard 3-dimensional representation of the icosahedron has a rotational symmetry of order 5 fixing the line through v and v' . This geometric symmetry has a corresponding graph-theoretic symmetry: a collection of automorphisms which fixes v and v' and acts transitively on the neighborhood of each. It follows that there are constants a, b, c so that each vertex at distance 2 from v has a neighbors at distance 2 from v , b neighbors at distance 1 from v and c neighbors at distance 3 from v . In this example, it is easy to see that $(a, b, c) = (1, 2, 1)$.

More generally, let X be a connected regular graph of diameter d with $v \in V(X)$. The **distance partition** of X with respect to v is a partition of $V(X)$ into cells

$$\{\Gamma_0(v), \Gamma_1(v), \dots, \Gamma_d(v)\}$$

where $\Gamma_i(v)$ consists of the vertices at distance i from v . Constants reminiscent of a, b, c above must exist whenever we have a graph whose automorphism group acts transitively on the cells of a distance partition.

Recall that a partition $\pi = (\pi_1, \dots, \pi_k)$ of $V(X)$ is **equitable** if for each pair $(i, j) \in \{0, \dots, d\} \times \{0, \dots, d\}$, and $u \in \pi_i$, the number of neighbors of u in π_j depends only on i and j .

Suppose we have a graph X of diameter d with an equitable distance partition with respect to some vertex v_0 , then this graph has an **intersection array**

$$\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$$

where b_i is the number of neighbors of any $u \in \Gamma_i(v_0)$ that are at distance $i+1$ from v_0 , and c_i is the number of neighbors of any $u \in \Gamma_i(v_0)$ that are at distance $i-1$ from v_0 . If this intersection array is the same for each choice of v_0 then X is said to be **distance-regular**. A distance-regular graph of diameter 2 is called **strongly-regular**.

There are also well-defined constants a_i equal to the number of neighbors of any $u \in \Gamma_i(v_0)$ that are at distance i from v_0 , but they are determined by the intersection array. Indeed, note that the parameter b_0 is the valency of the vertex v_0 , so a distance-regular graph is b_0 -regular. Each vertex u at distance i from v_0 cannot have neighbors w at distance $\ell < i-1$ (or $\ell' > i+1$) from v , since a shortest path from v_0 to w to u (or v_0 to u to w) would contradict the definition of Γ_ℓ (or $\Gamma_{\ell'}$). It follows that $b_0 = a_i + b_i + c_i$.

Distance-regularity is a reasonably simple combinatorial condition which is satisfied by a great number of interesting graphs. The definitive source on these graphs is the book by Brouwer, Cohen and Neumaier [18]. We pause the discussion to give a few examples.

1. The **Hamming graphs** $H(n, d)$ whose vertices are d -tuples from an alphabet of size n with tuples adjacent if they differ in exactly one coordinate position.
2. The **Shrikhande graph** on 16 vertices with the same parameters a_i, b_i, c_i as the Hamming graph $H(4, 2)$.
3. The **Johnson graphs** $J(n, k)$ whose vertices are the k element subsets of $[n]$ with subsets adjacent if their intersection has size $k-1$.
4. The **Grassmann graphs** $Gr_q(n, k)$ whose vertices are the k -dimensional subspaces of an n -dimensional vector space over $GF(q)$ with subspaces adjacent if their intersection has dimension $k-1$.
5. The **Moore graphs**. Regular graphs of diameter 2 and girth 5. They consist of the 5-cycle, the Petersen Graph, the Hoffman–Singleton Graph, and perhaps some highly sought-after graphs of valency 57.

6. The **generalized polygons**. Regular bipartite graphs of diameter d and girth $2d$. They are the incidence graphs of certain finite geometries introduced by Tits in the study of the finite simple groups. The incidence graphs of projective planes provide an infinite family of examples.
7. The **Higman–Sims graph**, whose automorphism group contains, as an index 2 subgroup, the sporadic finite simple group of order 44352000 by the same name.
8. The **drackns**. Antipodal covers of the complete graph with diameter 3. The topic of this chapter.

Many distance-regular graphs admit a group of automorphisms that act transitively on the cells of the distance-partition. However this is not a requirement. The Shrikhande graph is the smallest distance-regular graph with no such group of automorphisms.

Aside from intrinsic interest in the examples, the main appeal of distance-regularity is the stringent conditions it imposes on the algebra generated by the adjacency matrix of a graph. To study this, we must articulate the distance-regularity condition algebraically.

The following results on distance-regular graphs are well known. Our treatment follows [18, Section 4.1]. Let $d(u, v)$ denote the distance between two vertices in a graph. The i th-**distance matrix** of a graph is denoted A_i and defined to be the $V \times V$ matrix with

$$(A_i)_{u,v} = \begin{cases} 1 & \text{if } d(u, v) = i \\ 0 & \text{otherwise.} \end{cases}$$

In particular, A_1 is the adjacency matrix of the graph. If the graph in question has diameter d we also set $A_{-1} = A_{d+1} = 0$ so that the following recurrence is well-defined.

6.1.1 Theorem. *If X is a distance-regular graph and A_i is the i th distance matrix of A then*

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}.$$

Proof. The u, v entry of $A_1 A_i$ is the number of vertices at distance 1 from u and distance i from v . This number is zero unless $d(u, v) \in \{i-1, i, i+1\}$. If $d(u, v) = i-1$ this number is b_i and the u, v entry of A_{i-1} is 1. Similar statements hold for the other possible distances between u and v . \square

Now we are prepared to consider the algebra generated by A_1 .

6.1.2 Theorem. *Let A be the adjacency matrix of a distance-regular graph of diameter d . The algebra generated by A has dimension $d + 1$ and is spanned by the distance matrices A_i for $i \in [0, \dots, d]$.*

Proof. The three term recurrence from Theorem 6.1.1 implies that each power A^ℓ for $\ell > d$ can be written as a sum of smaller powers. So any element of the algebra may be written as a linear combination of A_0, \dots, A_d . By definition, the u, v entry of A_i is nonzero for exactly one $i \in \{0, \dots, d\}$, and so these A_i are linearly independent. \square

Another interpretation of Theorem 6.1.1 is the existence of a sequence of polynomials $p_i(x)$ so that $A_i = p_i(A_1)$. They can be defined recursively by

$$p_{-1}(x) = 0, \quad p_0(x) = 1, \quad p_1(x) = x, \quad c_{i+1}p_{i+1}(x) = (x - a_i)p_i(x) - b_{i-1}p_{i-1}(x).$$

The utility of this perspective is evident from the proof of the following theorem.

6.1.3 Theorem. *A distance-regular graph of diameter d has exactly $d + 1$ distinct eigenvalues.*

Proof. The distance matrices A_i for $i \in \{0, \dots, d\}$ are linearly independent, so the minimal polynomial of A has degree at least $d + 1$. Because $p_{d+1}(A) = A_{d+1} = 0$ the minimal polynomial of A has degree at most $d + 1$. Each root of this minimal polynomial is a root of A , and because A is real and symmetric, it is diagonalizable, so the minimal polynomial has distinct roots. These $d + 1$ roots are the distinct eigenvalues of A . \square

6.2 Imprimitivity

For $i > 0$ the distance matrices A_i of a distance-regular graph X are symmetric 01 matrices with 0 diagonal. In other words, they are adjacency matrices of graphs that we call the **distance graphs** and denote X_i . If the graphs X_i are connected for all $i \in \{1, \dots, d\}$ we say that X is **primitive**. If some X_i is disconnected X is **imprimitive**. Any bipartite graph is imprimitive with X_2 disconnected since two vertices from different color classes must be at odd distance.

Somewhat surprisingly, X_d is the only other distance graph which can be disconnected and there is only one way in which it can be disconnected. If X_d is a disjoint union of cliques we say that X is **antipodal**. The following theorem was proven by Smith, [123] for distance-transitive graphs, and extends easily to all distance-regular graphs, as noted by Gardiner in an unpublished manuscript.

6.2.1 Theorem (Smith [123]). *An imprimitive distance-regular graph with valency $k > 2$ is either bipartite, antipodal, or both.* \square

6.3 Drackns

If the reader has accepted that distance-regular graphs are interesting and worth studying, Smith's result does a good job of extending that acceptance to the study of drackns: The distance regular graphs of diameter 1 are the complete graphs. Distance-regular graphs of diameter 2 are the strongly-regular graphs, about which much is known, see [19] and the 751 references therein. The next case to investigate is distance-regular graphs of diameter 3. Smith's result further partitions our effort into the study of primitive, antipodal, bipartite, and antipodal and bipartite distance-regular graphs. Indeed, this is the taxonomy adopted by Brouwer, Cohen and Neumaier in [18], the primary resource on the topic of distance-regular graphs.

We shall see very soon that every antipodal distance-regular graph of diameter 3 is a drackn, so we are studying one of the main subclasses of distance-regular graphs of small diameter. Moreover, the theory of drackns quite often runs parallel to the theory of strongly-regular graphs. We offer two informal explanations for this.

The first explanation is that the distance partition of a drackn looks very much like the distance partition of a strongly-regular graph. In both cases we pick a vertex v , which is adjacent to a few of the vertices, and at distance 2 from most of the vertices. The difference is that a drackn also has a very small set of vertices at distance 3 from v , but they are joined onto the vertices at distance 2 in a very predictable way, so we have not introduced too much complexity over the strongly-regular setting.

The second explanation is that two of the four distinct eigenvalues of a drackn are $n - 1$ and -1 , this leaves two eigenvalues, θ and τ which parameterize the structure coefficients of the algebra generated by the adjacency matrix of the cover. Compare this to strongly-regular graphs, where the valency is always an eigenvalue and two other eigenvalues parameterize the structure coefficients. These eigenvalues satisfy similar identities as θ and τ , so many spectral results for strongly-regular graphs have drackn analogs. Theorems 6.4.3 and 6.4.6 are examples of this.

Before we move onto the theory, we give a few small examples of drackns, and a few infinite families which we shall have reason to mention later.

1. The smallest examples of drackns are the 6-cycle covering K_3 and the cube covering

K_4 . Both of these belong to the infinite family of bipartite double-covers of K_n , and are isomorphic to $K_{n,n}$ “minus a matching.”

2. The icosahedron is a 2-fold cover of K_6 .
3. The line graph of the Petersen graph is the smallest drackn of index 3, see Figure 2.1. It is the smallest member in both the infinite families of drackns from Sections 6.3.2 and 6.3.3 below.
4. The next smallest drackn of index 3 is a 3-fold cover of K_8 which can be embedded on a surface of genus 3, See Figure 3.3 This embedding is sometimes known as the Klein Quartic due to its relation to the smallest Hurwitz surface (c.f. 3.4). See also Section 7.5 for a construction.
5. There is a cyclic index 3 drackn covering K_9 that we will see several times throughout this thesis: A new construction is described in Section 7.4. An old construction is described in Section 7.5 and pictured in Figure 7.1. This graph is also a member of the two infinite families of drackns from Sections 6.3.1 and 6.3.3 below. We have described a partition of this graph into two isomorphic edge-disjoint subgraphs in Section 5.5.

We conclude the section with three infinite families of drackns. We note that these are not the only infinite families known, see [89] for a list which was complete in 2011 and [139] and [41] for the new examples found since then.

We will use the notation of an (n, r, c) -drackn which we define in Section 6.4.

6.3.1 Thas–Somma Drackns

For each even integer $2k$ and prime power q there exists a cyclic (q^{2k}, q, q^{2k-1}) -drackn. The examples for q a power of 2 were first found by J. Thas [135], and for odd q by Somma [126].

The following construction is from [18, Section 12.5.3]. Let V be a $2n$ dimensional vector space over the finite field F_q of order $q = p^k$. Let B be a non-degenerate **symplectic form** on B , i.e. a bilinear map $B : V \times V \rightarrow F_q$ satisfying

- $B(u, u) = 0 \quad \forall u \in V$.
- $B(u, v) = 0 \quad \forall u \in V$ implies $v = 0$.

Define a graph Γ on the vertex set $V \times F_q$ with (u, α) adjacent to (v, β) if and only if $B(u, v) = \alpha - \beta$.

To show that this graph is a drackn, it is useful to employ Lemma 6.4.2 from the next section. This proof follows Hensel [75].

6.3.1 Theorem. *The graph Γ defined above is a $(q^{2n}, q, q^{2n} - 1)$ -drackn.*

Proof. The sets $F_u := \{(u, \alpha) : \alpha \in F_q\}$ partition $V(\Gamma)$ into $|V| = q^2$ independent sets of size q since $0 = B(u, u) = \alpha - \beta$ only if $\alpha = \beta$. We claim these are the fibers of a cover. Indeed, any vertex (u, α) in F_u is adjacent to exactly one vertex in F_v , namely $(v, \alpha - B(u, v))$. We use Lemma 6.4.2 to show that the cover is distance regular by counting the number of common neighbors of a pair of non-adjacent vertices $(u, \alpha), (v, \beta)$ from different fibers.

That is, we count the size of the set

$$\{(w, \gamma) : w \in V - \{u, v\}, B(u, w) = \alpha - \gamma, B(v, w) = \beta - \gamma\}.$$

This set contains (w, γ) if and only if $\alpha - \beta = B(u, w) - B(v, w) = B(u - v, w)$. Since this quantity is independent of γ , we may restrict our attention to counting vectors $w \in V$ with $B(u - v, w) = \alpha - \beta$. Since B is non-degenerate and $u \neq v$ there is some x so that $B(u - v, x) \neq 0$. So there exists

$$x' := \frac{\alpha - \beta}{B(u - v, x)}$$

with $B(u - v, x') = \alpha - \beta$. The $w \in V$ we wish to count are exactly those w so that $B(u - v, x' - w) = 0$, i.e. the elements of the $2n - 1$ dimensional subspace of V orthogonal to $u - v$ with respect to B . There are q^{2n-1} such vectors. \square

In practice, verifying the condition from Lemma 6.4.2 is usually the simplest way to show that a given construction produces a drackn. All of the infinite families discussed in the rest of this chapter have proofs of this kind, and the details are similar to the above, so we omit them.

6.3.2 Mathon Drackns

Mathon constructed the following infinite family of drackns in [97]. Again we give a construction from [18, Section 12.5.3] and again this construction is related to a non-degenerate symplectic form B on a 2-dimensional vector space V over $GF(q)$.

Let $q = rc + 1$ be a prime power and K a subgroup of order r in the multiplicative group of $GF(q)$. Let Γ be a graph with vertices $\{Kv : v \in V \setminus \{0\}\}$ so that Kv and Ku are adjacent if $B(u, v) \in K$. Γ is a distance-regular cover of K_{q+1} of index r .

The example with $q = 7$ and $r = 3$ embeds on a surface of genus 3 known as the Klein Quartic (c.f. 3.4.) More generally, the examples with q prime and $r = (q - 1)/2$ have triangular embeddings related to the so called “Farey tessellations” of the hyperbolic plane. See [122] and [130].

6.3.3 Drackns from Generalized Quadrangles

Succinctly, a **generalized quadrangle** of order (s, t) is a bi-regular bipartite graph with valencies $s + 1$ and $t + 1$ of girth 8 and diameter 4. The part of the partition containing vertices of degree $s + 1$ is called the lines, and the part containing the vertices of degree $t + 1$ is called the points. The **point graph** of a generalized quadrangle has the points as its vertices, two of which are adjacent if they are adjacent to the same line of the generalized quadrangle.

A great many facts about generalized quadrangles are well known, (see for instance, [57, Chapter 5]): Generalized quadrangles are distance-regular and their point graphs are strongly-regular; examples are known of orders $(q, 1)$, (q, q) , (q, q^2) , (q^2, q^3) , $(q - 1, q + 1)$ for q a prime power, and $(q + 1, q)$ for q an even prime power greater than 2; and of course, these all arise as the incidence graphs of certain finite geometries, which explains the very suggestive names given to their vertices.

A **spread** in a generalized quadrangle is a set of lines whose neighborhoods partition the points. Hence the point graph of a generalized quadrangle with a spread admits a partition of its vertices into cliques of size $s + 1$. Brouwer showed that the graph obtained from the point graph by deleting the edges of these cliques is an $(st + 1, s + 1, t - 1)$ -drackn [17].

For all the orders above except (q^2, q^3) , we know of examples of generalized quadrangles with spreads, so Brouwer’s construction yields drackns with parameters $(q^2 + 1, q + 1, q - 1)$, $(q^3 + 1, q + 1, q^2 - 1)$, (q^2, q, q) for all q and $(q^2, q + 2, q - 2)$ for q an even prime power greater than 2.

Brouwer’s construction is actually a bit stronger than we have stated. It allows one to construct a drackn from any strongly-regular graph with the same parameters as the point graph of a generalized quadrangle. Very recently, Guo and van Dam have constructed a very large number of strongly-regular graphs with the same parameters as the point graphs

of the generalized quadrangles of order (q, q) and (q, q^2) [140]. This in turn gives many new examples of drackns via Brouwer's result. Drackns with the same parameters have also been constructed in [42].

6.4 Parameters and Eigenvalues

With examples in hand, we move onto the algebraic theory of drackns. Since a drackn X is distance-regular of diameter 3 it possesses an intersection array

$$\{b_0, b_1, b_2; c_1, c_2, c_3\}$$

which determines the structure of the algebra generated by the distance matrices of X . In this section we will study the relationship between these parameters as well as restrictions on their possible values. It turns out that a smaller set of nicer parameters determine the intersection array of a drackn.

6.4.1 Theorem. [46] *Suppose that X is distance-regular antipodal cover of K_n of index r . Then the intersection array of X is determined by the parameters n, r , and c_2 .*

Proof. Pick a vertex v and construct the distance partition $\{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3\}$. Note that b_0 , the valency of X , is $n - 1$. So $|\Gamma_1| = n - 1$. Since X is antipodal, $|\Gamma_3| = r - 1$, so

$$|\Gamma_2| = nr - (n - 1) - (r - 1) - 1 = (n - 1)(r - 1).$$

Each vertex in Γ_3 has no neighbors in Γ_3 , and so $c_3 = n - 1$. We count, in two ways, the edges in the bipartite subgraphs induced by the partitions $[\Gamma_i, \Gamma_{i+1}]$, $i \in \{0, 1, 2\}$. This yields the equations

$$1b_0 = (n - 1)c_1, \quad b_1(n - 1) = c_2(n - 1)(r - 1), \quad c_3(r - 1) = b_2(n - 1)(r - 1),$$

from which it follows that the intersection array of X is

$$\{n - 1, c_2(r - 1), 1; 1, c_2, n - 1\}.$$

□

We say that n, r, c_2 are the **parameters** of the drackn, and use the notation (n, r, c_2) -drackn as a shorthand for “a drackn with parameters n, r, c_2 ”. If it is possible that a (a, b, c) -drackn exists, we say that (a, b, c) is a **feasible parameter set**.

Note that c_2 is a particularly good choice to parameterize drackns, as illustrated by the following lemma due to Gardiner, [46].

6.4.2 Lemma. [46] *Let X be any arbitrary cover of K_n of index r , and let c_2 be a positive integer. X is distance-regular and antipodal if and only if every pair of non-adjacent vertices from different fibers have c_2 common neighbors.*

Proof. Let X be an (n, r, c_2) -drackn. Then X has diameter 3, so any non-adjacent vertices from different fibers, are at distance 2, and have c_2 common neighbors by definition.

Conversely, suppose each pair of non-adjacent vertices from different fibers are at distance 2. Let u_1, u_2 be vertices in the same fiber, and choose some other fiber F_v . Since X is a cover, each of u_1, u_2 has a unique neighbor $v_1, v_2 \in F_v$. So u_1, v_2 are non-adjacent vertices from distinct fibers, and thus have $c_2 > 0$ common neighbors. Let w be one such common neighbor. (u_1, w, v_2, u_2) is a path of length 3 in X , so X has diameter at most 3. In fact, the diameter must be exactly 3 because the definition of a cover implies that u_1, u_2 cannot be adjacent or have a common neighbor.

Now we check that the intersection array is well-defined: Pick a vertex w and consider the distance partition $\{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3\}$ with respect to w . $c_1 = 1$ and $b_0 = n - 1$ is the degree of w .

Because X is antipodal with $\Gamma_0 \cup \Gamma_3$ a fiber, each vertex in Γ_3 is adjacent only to vertices in Γ_2 , so c_3 is equal to $n - 1$. Also because $\Gamma_0 \cup \Gamma_3$ is a fiber each vertex in Γ_2 has at most one neighbor in Γ_3 , so the neighborhoods of vertices in Γ_3 must partition Γ_2 , and b_2 is equal to 1.

Since Γ_2 is precisely the vertices from different fibers than v which are not adjacent to v , c_2 is defined by assumption. Finally, pick some neighbor u of v and let F be the fiber containing u . Each of the $r - 1$ other vertices in F has c_2 common neighbors with u . These $c_2(r - 1)$ common neighbors must all be distinct, since the vertices of F are at distance 3 from one another, so b_1 is equal to $c_2(r - 1)$.

So for any choice of w we see that X has intersection array

$$\{n - 1, c_2(r - 1), 1; 1, c_2, n - 1\}.$$

□

Next we determine the eigenvalues of a drackn in terms of the parameters (n, r, c_2) . Again this result is originally due to Gardiner [46].

6.4.3 Theorem. [46] *Set $\delta = a_1 - c_2$. The eigenvalues of a (n, r, c_2) -drackn X are*

$$n - 1, \quad -1, \quad \theta := \frac{\delta + \sqrt{\delta^2 + 4(n - 1)}}{2}, \quad \tau := \frac{\delta - \sqrt{\delta^2 + 4(n - 1)}}{2}$$

with respective multiplicities

$$1, \quad n-1, \quad m_\theta = \frac{n(r-1)\tau}{\tau-\theta}, \quad m_\tau = \frac{n(r-1)\theta}{\theta-\tau}.$$

Proof. X is connected of valency $n-1$, so $n-1$ is an eigenvalue of multiplicity 1, via Theorem 2.1.5.

For each eigenvector in the -1 eigenspace of K_n , we create a new vector by copying the value on vertex v to all vertices in the fiber over v . Since the equations defining an eigenvector of a graph are local to each vertex, these “blown up” vectors must be eigenvectors for X with eigenvalue -1 .

Formally, let π be the covering projection from X to K_n , let U be a basis for the -1 eigenspace of K_n , and consider the vectors

$$\{u \otimes J_{1,r} : u \in U\}$$

which span a space of dimension $n-1$. For each such vector and each vertex $a \in V(X)$ we find

$$(A(X)(u \otimes J_{1,r}))_a = \sum_{b \sim a} (u \otimes J_{1,r})_{e_b} = \sum_{\pi(b) \sim \pi(a)} u e_{\pi(b)} = -1 u_{\pi(a)} = -1 (u \otimes J_{1,r})_a.$$

So -1 is an eigenvalue of multiplicity at least $n-1$.

Now we consider the distance partition of X with respect to an arbitrary vertex. Since X is distance-regular this partition is equitable (c.f. Section 2.3) with quotient matrix

$$\begin{pmatrix} a_0 & b_0 & 0 & 0 \\ c_1 & a_1 & b_1 & 0 \\ 0 & c_2 & a_2 & b_2 \\ 0 & 0 & c_3 & a_3 \end{pmatrix} = \begin{pmatrix} 0 & n-1 & 0 & 0 \\ 1 & n-2-c_2(r-1) & c_2(r-1) & 0 \\ 0 & c_2 & n-2-c_2 & 1 \\ 0 & 0 & n-1 & 0 \end{pmatrix}.$$

Set $\delta := a_1 - c_2$. It is elementary to calculate that the eigenvalues of the above matrix are $n-1$, -1 and

$$\theta = \frac{\delta + \sqrt{\delta^2 + 4(n-1)}}{2}, \quad \tau = \frac{\delta - \sqrt{\delta^2 + 4(n-1)}}{2}.$$

From Lemma 2.3.2, these four eigenvalues must be eigenvalues of X . Since X has diameter 3, Theorem 6.1.3 implies these are the only eigenvalues of X . It remains to calculate their multiplicities.

Using Lemma 2.1.4 applied to the 0th, 1st, and 2nd powers of the adjacency matrix of X we obtain a system of 3 equations in the variables m_{-1}, m_θ, m_τ .

$$\begin{aligned} rn &= 1 + m_{-1} + m_\theta + m_\tau \\ 0 &= (n-1)1 + (-1)m_{-1} + \theta m_\theta + \tau m_\tau \\ rn(n-1) &= (n-1)^2 + (-1)^2 m_{-1} + \theta^2 m_\theta + \tau^2 m_\tau \end{aligned}$$

Solving this system we obtain

$$m_{-1} = n-1, \quad m_\theta = \frac{n(r-1)\tau}{\tau-\theta}, \quad m_\tau = \frac{n(r-1)\theta}{\theta-\tau}. \quad \square$$

It is now easy to show that our supply of bipartite drackns is very limited.

6.4.4 Lemma. *Let X be an (n, r, c) drackn. If X is bipartite then $r = 2$ and X is isomorphic to the bipartite double cover of K_n .*

Proof. X has four eigenvalues $n-1, -1, \theta, \tau$ with $\theta > \tau$. Thus by Theorem 2.1.5 we have $\theta = 1, \tau = 1-n$ and τ must have multiplicity 1. From Theorem 6.4.3 we find that

$$1 = m_\tau = \frac{n(r-1)(1)}{1+n-1} = r-1.$$

Let f be an arc function for X which we may assume is normalized on a star T , a tree with a vertex of degree $n-2$. Since $r = 2$ and X is bipartite f must take value -1 on all arcs outside of T . Switching on the vertex of degree $n-2$ in T yields the canonical description of the bipartite double cover of K_n . \square

Another immediate consequence of Theorem 6.4.3 is that $\frac{n(r-1)\tau}{\tau-\theta}$ and $\frac{n(r-1)\theta}{\theta-\tau}$ are integers, this gives some restriction on the possible values of n, r, c_2 for which a drackn can exist. We now mention a great many such parametric restrictions for drackns, compiled in [54]. We are not going to use very many of these restrictions explicitly, but we mention them because they give structure to the otherwise amorphous notion of a feasible parameter set that we defined above. We can now say that a tuple (n, r, c_2) is a **feasible parameter set** if it satisfies all of the conditions of the following theorem.

6.4.5 Theorem. [54] *Suppose that X is an n, r, c_2 drackn with eigenvalues $n-1, -1, \theta, \tau$ and eigenvalue multiplicities $1, n-1, m_\theta, m_\tau$ as defined above. Let $\delta = a_1 - c_2$.*

- (a) n, r , and c_2 are all integers with $1 \leq (r-1)c_2 \leq n-2$.
- (b) If $\delta = 0$, then $\theta = -\tau = \sqrt{n-1}$. If $\delta \neq 0$ then θ and τ are integers.
- (c) $m_\theta = \frac{n(r-1)\tau}{\tau-\theta}$ and $m_\tau = \frac{n(r-1)\theta}{\theta-\tau}$ are integers.
- (d) If n is even, then c_2 is even.
- (e) $n \leq c_2(2r-1)$ with equality only if X is the icosahedron or the line graph of the Petersen graph.
- (f) $\delta = \pm 2$ implies n is a square.
- (g) $c_2 = 1$ implies
 - i) n is odd.
 - ii) $(n-r)|(n-1)$
 - iii) $(n-r)(n-r+1)|rn(n-1)$
 - iv) $(n-r)^2 \leq n-1$
- (h) $r > 2$ implies $\theta^3 \geq n-1$.
- (i) For $\beta \in \{\theta, \tau\}$, if $n > m_\beta - r + 3$ then $\beta + 1$ divides c_2 . □

Items (a) through (e) are due to Gardiner [46], and items (f) through (i) were derived by Godsil and Hensel as applications of results from [18], and unpublished work of Hoare (c.f. [54]). We note that in [54], item (e) incorrectly omits the line graph of the Petersen graph from the characterization of equality.

A further consequence of our eigenvalue formulas is that the parameter $\delta := a_1 - c_2$ translates quite succinctly between the eigenvalues and structure coefficients of a drackn: Adding together the expressions for θ and τ found in Theorem 6.4.3, we obtain

$$a_1 - c_2 = \delta = \theta + \tau.$$

Further still, when the numerator of the expression

$$m_\theta - m_\tau = \frac{n(r-1)(\theta + \tau)}{\sqrt{\delta^2 + 4(n-1)}}$$

is non-zero, δ is non-zero, so θ and τ are integers. This implies $\sqrt{\delta^2 + 4(n-1)}$ is an integer, which in turn can be cajoled into the following:

6.4.6 Theorem. [54] *For fixed r and δ , there are only finitely many feasible parameter sets for (n, r, c) drackns, unless $\delta \in \{-2, 0, 2\}$.* \square

The cases $\delta = 0$ and $\delta = -2$ can be further distinguished by the equivalence of such drackns with incidence structures which are in essence, covers of symmetric designs. See [54, Section 5] for details.

6.5 Triangle-free Drackns

We conclude with a discussion of triangle-free drackns, i.e. those with $a_1 = 0$. These are our next example of covers which break short cycles.

Insofar as drackns are a natural generalization of strongly-regular graphs, the existence of triangle-free drackns is an interesting topic: We know of only seven triangle-free strongly-regular graphs. Three of them are the Moore graphs, the 5-cycle, the Petersen Graph, and the Hoffman–Singleton graph. The largest is the Higman–Sims graph on 100 vertices, and in fact, all seven are subgraphs of this graph, see [53] for details. The existence of an 8th triangle-free strongly-regular graph is one of the larger open problems in algebraic graph theory. The existence of a Moore graph of valency 57 would provide an answer to this problem, but there could be many other triangle-free strongly-regular graphs as well.

In fact, Gardiner proved that any Moore graph gives rise to a triangle-free drackn.

6.5.1 Theorem. [46] *The subgraph induced on the second-neighborhood of any vertex in a Moore graph of valency $n - 1$ is a $(n, n - 1, 1)$ -drackn.* \square

Moreover, Gardiner showed that the subgraph induced on the second-neighborhoods of Moore graphs are the only drackns with $n = r + 1$.

6.5.2 Theorem. [46] *If X is a $(n, n - 1, c)$ -drackn then $c = 1$, and $n \in \{3, 7, 57\}$.* \square

The second-neighborhood of the Petersen graph is a 6-cycle, about which there is little to say. The $(7, 6, 1)$ -drackn induced on any second-neighborhood of the Hoffman–Singleton graph is a particularly beautiful example which we pause to discuss. We follow Cameron and Van Lint’s treatment in [23, Chapter 6], although the ideas go back to Hoffman and Singleton’s original study of the Moore graphs [76].

Example: For any group G and any element g , the conjugation map $x \mapsto gxg^{-1}$ is an automorphism of G . These are known as the inner automorphisms, and for almost every symmetric group $\text{Sym}(n)$, they are the only automorphisms. The single exception is the group $\text{Sym}(6)$ in which the inner automorphisms form a subgroup of index 2.

There is a very good combinatorial description of this extra symmetry of $\text{Sym}(6)$. See Figure 6.2: The complete graph K_6 has 15 edges E and 15 perfect matchings F . There are six collections of five pairwise coincident edges (called the vertices, V) and there are also six collections of five pairwise disjoint perfect matchings, called the one-factorizations, Z . Moreover, any two vertices are contained in exactly one edge, and any two one-factorizations intersect in exactly one perfect matching. So any bijection from V to Z induces a bijection between E and F . If we identify E with the transpositions in $\text{Sym}(n)$ and F with the products of three disjoint transpositions, then the bijections from E to F induce automorphisms of $\text{Sym}(6)$ under this identification.

We can also use the combinatorics of this setting to describe the $(7, 6, 1)$ -drackn. We would like to construct an arc function for a 6-fold cover of K_7 . We may assume we have normalized on a vertex, so we must determine permutations in $\text{Sym}(6)$ for each of the remaining 15 edges of K_6 . Identifying each edge with a product of three transpositions as above gives the desired distance-regular cover. This graph is the second-neighborhood of the Hoffman–Singleton graph.

Unlike the strongly-regular setting, we are also lucky enough to have an infinite family of examples of triangle-free drackns. These graphs have parameters $(2^{2t}, 2^{2t-1}, 2)$ for all t and were constructed by DeCaen, Mathon, and Moorehouse, [32]. Their construction is succinct, but not terribly enlightening: Set $q = 2^{2t-1}$, and define a graph on the vertex set $GF(q) \times GF(2) \times GF(q)$ with distinct triples (a, i, α) and (b, j, β) adjacent if

$$\alpha + \beta = a^2b + ab^2 + (i + j)(a^3 + b^3).$$

We also have the following result from Gardiner, which immediately implies the index of a triangle-free drackn must be at least 4.

6.5.3 Theorem. [48] *If X is an (n, r, c) -drackn of girth at least 4, then either*

(a) $r = 2$ and X is the bipartite double cover of K_n .

(b) $r - 2 > c^{1/2}$. □

In the same paper Gardiner pushed this a little further and showed that any drackn of index 4 must also contain triangles [48].

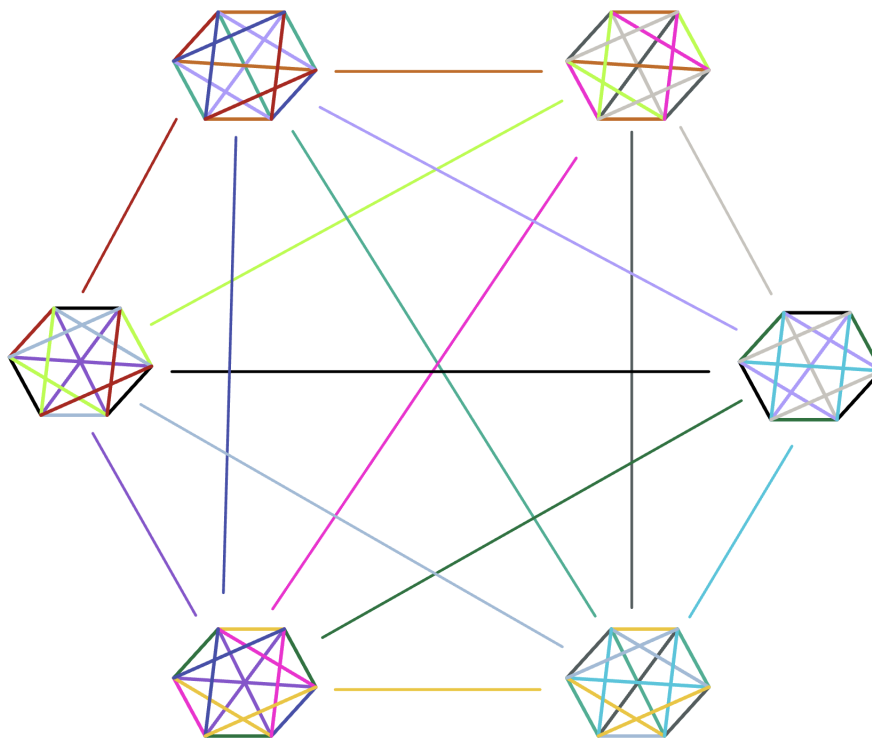


Figure 6.2: The six 1-factorizations of K_6 , drawn in bijection with the vertices of K_6 . This induces a bijection between the edges of K_6 and its perfect matchings.

6.5.4 Theorem. [48] *If X is an (n, r, c) -drackn with $r = 3$ or $r = 4$, then X contains triangles.* \square

This was about the sum total of our knowledge on triangle-free drackns until now.

In Section 7.7 we will calculate the eigenvalues of the subgraph induced on any second-neighborhood of a triangle-free drackn. In order to do this, it is helpful to know that such a subgraph is connected, a new result which we will prove now.

6.5.5 Theorem. *Suppose X is a triangle-free (n, r, c_2) -drackn with $c_2 > 1$ and $r > 2$. Then for any vertex v , the subgraph of X induced on the second-neighborhood of v is connected.*

Proof. Pick v arbitrarily and let B be the subgraph of X induced on $\Gamma_2(v)$. For non-adjacent vertices x, y from different fibers, let $C(x, y)$ denote the c_2 common neighbors of x and y .

Suppose for sake of contradiction that B has at least two distinct components U_1, U_2 . Each component is regular of valency

$$a_2 = n - 1 - b_2 - c_2 = n - 2 - c_2.$$

Moreover, each component is triangle-free and any pair of vertices at distance 2 in B have at most c_2 common neighbors. It follows that

$$|U_i| \geq 1 + a_2 + a_2(a_2 - 1)/c_2.$$

First note that Theorem 6.4.5 (a) implies $n \geq 6$. From Theorem 6.4.5 (a) and Theorem 6.5.3 (b) we see that $\frac{n-2}{3} \geq c_2$. So

$$a_2 \geq \frac{2}{3}(n - 2)$$

which implies

$$1 + a_2 + a_2(a_2 - 1)/c_2 > \frac{4}{3}(n - 2) > n - 1.$$

For the moment we assume $c_2 > 2$. We will deal with the other case separately. Suppose a component, say U_1 , contains two vertices a, b from the same fiber. U_2 must contain a vertex x from a different fiber, since it contains at least r vertices and is connected. But $C(a, x)$ and $C(b, x)$ each contain at most one vertex of $\Gamma_3(v)$ and no vertices of $\Gamma_2(v)$. So

$C(a, x) \cap \Gamma_1(v)$ and $C(b, x) \cap \Gamma_1(v)$ are each subsets of $C(v, x)$ of size at least $c_2 - 1$. Since $c_2 > 2$, this implies $C(a, x)$ and $C(b, x)$ intersect, contradicting the assumption that a and b were from the same fiber. It follows that each component contains at most one vertex from each fiber, so the components have size at most $n - 1$, a contradiction.

Now assume $c_2 = 2$ and suppose U_1 contains three vertices a, b, c from the same fiber. Again, U_2 must contain a vertex x from a different fiber, and now the sets $C(a, x)$, $C(b, x)$, $C(c, x)$ each have nontrivial intersection with the two element set $C(v, x)$. So two of a, b, c have a common neighbor, a contradiction. So each U_i contains at most two vertices from each fiber, and $|U_i| \leq 2(n - 1)$. But we saw that $|U_i| \geq 1 + a_2(a_2 - 1)/c_2$ which in this case is equal to $\frac{1}{2}(n^2 - 7n + 14)$. Whenever $n \geq 10$ we have $\frac{1}{2}(n^2 - 7n + 14) > 2(n - 1)$ and we obtain the desired contradiction. Theorem 6.5.4 implies that $r > 4$, so

$$n - 2 \geq c_2(r - 1) = 2(r - 1) \geq 8$$

and we are done. □

What happens in the above theorem if we allow $c_2 = 1$? The identity

$$a_1 = n - 2 - (r - 1)c_2$$

implies $r = n - 1$. So Theorem 6.5.2 implies that the only candidate drackns with $r > 2$, $a = 0$, $c = 1$ are the $(7, 6, 1)$ -drackn and the putative $(57, 56, 1)$ -drackns. We have checked that the subgraph induced on any second-neighborhood in the $(7, 6, 1)$ -drackn is connected, but we do not see an easy way to extend the above argument to give information about the second-neighborhood of the second-neighborhood of a Moore graph of valency 57. This is certainly an interesting open question.

6.6 Future Work

1. Does Theorem 6.5.5 hold when $c_2 = 1$? In other words, is the subgraph induced on the second-neighborhood of the second-neighborhood of a vertex in a Moore graph of valency 57 always connected?
2. Is the triangle-free hypothesis really necessary in Theorem 6.5.5? In every drackn we have checked the subgraph induced on the second-neighborhood of each vertex is connected, so the result may hold in general. This would be analogous to the fact that the subgraph induced on the second-neighborhood of a vertex in a strongly-regular graph is connected unless the graph is complete multipartite [49].

Chapter 7

Equivalence to Drackns Beyond Index Two

Drackns of index 2 are the simplest and best understood examples we have. This is not surprising since $\text{Sym}(2)$ is a very small group whose combinatorics and representation theory are minimally involved. Moreover, the drackns with $r = 2$ are closely related to a number of other combinatorial and geometric concepts including real equiangular lines, and strongly-regular graphs with parameters $(n, 2c, a, c)$. We will describe these connections in Sections 7.1 through 7.3.

To what extent are these connections preserved for drackns of index greater than 2?

This is the main question of this chapter and will lead us in three different directions. One direction stems from equiangular lines, and the other two from strongly-regular graphs.

Section 7.4 discusses the relationship between drackns of index $r > 2$ and equiangular lines in complex space. This relationship has been developed by Coutinho et al. in [27], extended by Iverson and Mixon in [83], and used by Fickus et al. to construct new drackns in [41]. After discussing all of this, we will use ideas from [27] and examples of equiangular tight frames from [43] to give a new construction of the cyclic $(9, 3, 3)$ drackn, and two new antipodal covers of K_{65} and K_{217} which are “nearly” distance-regular in the sense that they have exactly five eigenvalues. It seems that these are the first examples of five-eigenvalue antipodal covers of K_n and indicate a new family of graphs worth investigating.

Section 7.5 discusses the prospect, first raised in [18, Chapter 12], of building arc functions for drackns of index greater than 2 from association schemes. We demonstrate

a limitation to any such theory by showing that the non-cyclic $(9, 3, 3)$ -drackn cannot be built from an association scheme in a fashion reminiscent of the $r = 2$ setting. This involves a novel analysis of the representations of $\text{Sym}(3)$ -arc functions.

Finally, in Sections 7.6 and 7.7 we calculate the spectrum of the subgraphs induced on the second neighborhood of any vertex in a triangle-free drackn. It turns out that these subgraphs always have 6 eigenvalues which are easy to describe in terms of the parameters of the drackn. This gives a spectral analog to the fact that the distance neighborhoods of an $(n, 2, c)$ -drackn are strongly-regular graphs.

7.1 Seidel Matrices

A **Seidel matrix** is a symmetric matrix with zero diagonal and off-diagonal entries in $\{1, -1\}$. In other words, Seidel matrices are arc matrices for \mathbb{Z}_2 -valued arc functions on K_n , which are in turn equivalent to covers of K_n of index 2. In the previous chapter we studied covers of K_n which are distance-regular and antipodal. These covers have particularly nice Seidel matrices.

7.1.1 Theorem. *Let S be the Seidel matrix of an index-2 cover X of K_n . X is a drackn if and only if S has etwo distinct eigenvalues.*

Proof. Suppose X is an $(n, 2, c)$ -drackn. Theorem 6.4.3 implies that X has 4 distinct eigenvalues, including $n - 1$ with multiplicity 1 and -1 with multiplicity $n - 1$. From Lemma 2.8.1 we see that the remaining eigenvalues are the eigenvalues of the arc matrix S . The converse also follows from the considerations in Section 2.8. We will prove a stronger version of it in Theorem 8.3.2 from the next chapter. \square

In the next two sections we recount two other well known interpretations of Seidel matrices: real equiangular lines and “switching classes” of graphs.

Here too the Seidel matrices with exactly two distinct eigenvalues give rise to partcularly nice subfamilies: Sets of real equiangular lines for which the relative bound is tight, and strongly-regular graphs with parameters $(n - 1, 2c, a, c)$.

These equivalences were already known to J.J. Seidel, who studied the situation extensively throughout the 1970s, often working with collaborators such as Goethels, Lemmens, and Taylor [117],[58], [90],[120],[119]. See [118] and [120] for two surveys on these topics, and [141] for a more recent survey with an emphasis on the real equiangular lines side of the story.

We note that, as of this writing, the most recently discovered strongly-regular graph has parameters $(65, 32, 15, 16)$ and therefore comes attached to a set of real equiangular lines, a regular 2-graph, and drackn of index 2 [62].

An equivalence relation on Seidel matrices will be important in both of the examples we consider. Let $d_i := I - 2e_i^T e_i$ be the diagonal matrix whose entries are 1 except for the i th, which is -1 . The matrix group $D = \langle d_1, \dots, d_n \rangle$ acts by conjugation on the space of all real $n \times n$ matrices, and in particular, on Seidel matrices. In fact, the orbit of a given Seidel matrix S under D 's conjugation action consists entirely of Seidel matrices. Such an orbit is called the **switching class** of S .

7.2 Equiangular Lines

A set of n lines through the origin in \mathbb{R}^d is **equiangular** if the angle between any two lines is the same. To make this precise, choose unit vectors x_1, \dots, x_n spanning the lines, and let U be the $d \times n$ matrix whose columns are the x_i . The set of lines is **equiangular** if there is a constant α , and a symmetric matrix with zero diagonal and off-diagonal entries ± 1 so that

$$U^T U = I + \alpha S.$$

Here $\pm\alpha$ is the cosine of the angle between the vectors given by the columns of U , and S is a Seidel matrix.

Notice that choosing the opposite unit norm representative $-x_i$ for one of the lines amounts to scaling a column of U by -1 , which in turn scales a row and column of S by -1 . So the switching class of S represents exactly those sets of equiangular lines obtainable from one another via changing our choice of unit norm representatives.

The first question one might ask about equiangular lines is how large a set can occur in a given dimension. An accessible bound is due to Gerzon, who communicated it to Lemmens and Seidel, [90].

7.2.1 Theorem (The Absolute Bound). [90] *There are at most $\binom{d+1}{2}$ equiangular lines in \mathbb{R}^d .* □

There are only four known examples of sets of real lines where the absolute bound is tight: $d = 2, 3, 7, 23$. They are related to the hexagon, the icosahedron, the 27 lines on a generic cubic surface, and a doubly transitive action of Conway's third sporadic simple group, respectively. See [57, Chapter 11] for further discussion of these examples.

One can also give a bound on n which depends on both d and α , the cosine of common angle between pairs of lines.

7.2.2 Theorem (The Relative Bound). [90] *Suppose there is a set of n equiangular lines in \mathbb{R}^d and that the cosine of the pairwise angle between any two of them is $\pm\alpha$. If $\alpha^{-2} > d$ then*

$$n \leq \frac{d - d\alpha^2}{1 - d\alpha^2}.$$

Equality in this bound holds if and only if the projections X_i onto these lines satisfy $\sum_{i=1}^d X_i = (n/d)I$. \square

Fortunately, there are many more examples of sets of lines meeting this bound with equality. All examples with at most 36 lines have been computed and tabulated, see [22], [113].

There is intrinsic interest in sets of lines meeting either of the above bounds with equality. For us, they are particularly pertinent because they are equivalent to drackns.

7.2.3 Theorem. [90] *A set of equiangular lines in \mathbb{R}^d meets the relative bound with equality if and only if its Seidel matrix has exactly two distinct eigenvalues.* \square

This result combined with Theorem 7.1.1 gives the stated equivalence of drackns and real equiangular lines meeting the relative bound.

7.3 Strongly-regular Graphs

We turn to the equivalence of drackns with certain strongly-regular graphs. Here it is convenient to interpret the Seidel matrix S as a sort of “alternative adjacency matrix” in which -1 signifies adjacency and 1 and 0 signify non-adjacency. In other words, if S is a Seidel matrix, we consider the graph X_S with adjacency matrix

$$A_S := \frac{1}{2}(J - I - S).$$

Through this correspondence, we can also interpret switching as an operation on simple graphs. Two graphs X and X' are **switching equivalent** if and only if there is some subset of the vertices of X so that X' is obtained from X by complementing the bipartite subgraph induced by the partition $(U, V(X) - U)$. Switching a Seidel matrix so that the first row and column are non-negative translates to switching the graph X_S to a representative that contains an isolated vertex. The Seidel matrix has two eigenvalues when the non-isolated part of such a graph is particularly nice.

7.3.1 Theorem. [90], (c.f. [57] Chapter 11.) Let S be a Seidel matrix, and X_S the graph with adjacency matrix $\frac{1}{2}(J - I - S)$. S has two distinct eigenvalues if and only if X_S is switching equivalent to a graph $K_1 \cup Z$ with Z a strongly-regular graph with parameters $(n - 1, 2c, a, c)$.

There are two consequences of this description that will be relevant for us.

7.3.2 Corollary. Let X be an $(n, 2, c)$ -drackn with Seidel matrix S and suppose S is normalized so that the off-diagonal entries of the first row and column are all positive.

- (a) Let S' be the matrix obtained from S by deleting the first row and column. The matrices

$$\{I_{n-1}, \frac{1}{2}(J_{n-1} - S'), \frac{1}{2}(J_{n-1} + S')\}$$

form a symmetric association scheme.

- (b) For any vertex $v \in X$ the subgraphs induced on the first and second neighborhoods of v are isomorphic strongly-regular graphs. \square

These conditions hold and are equivalent for drackns of index 2, but this is far from true for drackns in general. Still, some vestiges of these conditions persist in the higher index setting. They will be the topics of Sections 7.5, 7.6 and 7.7.

7.4 Complex Equiangular Lines

We come to the first answer to the main question of this chapter: Certain drackns of index greater than 2 correspond to sets of equiangular lines in complex space. We discuss this perspective and then give some new examples of antipodal covers derived from systems of complex lines which are “nearly” distance-regular in the sense that they have exactly 5 distinct eigenvalues.

The setup for studying systems of equiangular lines in complex Euclidean space is much the same as in the real setting. A set of lines through the origin in \mathbb{C}^d spanned by vectors x_1, \dots, x_n is equiangular if there is a real number α so that for all i and j ,

$$|\langle x_i, x_j \rangle| = \alpha.$$

Here $\langle \cdot, \cdot \rangle$ is the usual Hermitian inner product, and we may still form a matrix U whose columns are the vectors x_i for which U^*U is the Gram matrix of the set of lines. However,

the values of the inner products $\langle x_i, x_j \rangle$ are now free to roam around the unit circle, so the decomposition

$$UU^* = I + \alpha T$$

involves a Hermitian matrix T with zero diagonal and unit modulus off-diagonal entries. It is not obvious, in general, what combinatorics one can attach to such a matrix T . However, if we restrict to the subclass of matrices whose off-diagonal entries are r th roots of unity, the combinatorics presents itself and we can show the equivalence of certain sets of complex lines with certain (n, r, c_2) -drackns.

As in the real case, the sets of lines of the most interest are those meeting the absolute or relative bound. The complex absolute bound is quite similar in spirit to the real version, although we must replace $\binom{d+1}{2}$, the dimension of the space of real symmetric $d \times d$ matrices, with d^2 , the (real) dimension of the space of Hermitian $d \times d$ matrices.

7.4.1 Theorem. (Gerzon, see [90]) *If there is a set of n equiangular lines in \mathbb{C}^d then*

$$n \leq d^2. \quad \square$$

Sets of lines meeting this bound are of great interest in quantum information where they are known as symmetric informationally complete positive operator valued measures, or **SIC-POVMs**.

Zauner conjectured that, in sharp contrast to the real case, there is always a set of lines meeting the complex absolute bound [148]. Zauner’s conjecture goes further, predicting that a set of d^2 lines can always be obtained as the orbit of a single line under the action of a group known as a “Weyl-Heisenberg” group. The conjecture is widely believed to be true and has been verified computationally up to dimension 53 [116], [45].

The relative bound of Theorem 7.2.2 goes through in the complex case without any modifications, and the equality case has exactly the characterization one would expect.

7.4.2 Theorem (Lemmens and Seidel [90]). *If G is the Gram matrix of a set of equiangular lines in \mathbb{C}^d with angle α then the matrix $S = \frac{1}{\alpha}(G - I)$ has exactly two distinct eigenvalues if and only if this set of lines meets the relative bound with equality.* \square

A set of lines meeting the relative bound in either the real or complex case is called an **equiangular tight frame**. These objects have garnered much interest due to their applications in signal processing [142], [132].

Coutinho et al. used the above characterization of tightness in the relative bound to derive complex equiangular lines from drackns in [27].

7.4.3 Theorem. [27] If X is an abelian (n, r, c) -drackn with eigenvalues

$$n, -1, \theta, \tau,$$

then there are

- (a) n complex lines in dimension $(n - \frac{m_\theta}{r-1})$ meeting the relative bound.
- (b) n complex lines in dimension $(n - \frac{m_\tau}{r-1})$ meeting the relative bound.

Coutinho et al. also give a partial converse to this statement. For this direction an extra condition is required on the off diagonal entries of the Gram matrix.

7.4.4 Theorem. [27] Suppose x_1, \dots, x_n is a set of complex equiangular lines in \mathbb{C}^d with angle α meeting the relative bound. Let G be the Gram matrix of this set of lines. If each off-diagonal entry of the matrix

$$S = \frac{1}{\alpha}(G - I)$$

is an r th root of unity with r a prime, then there exists an (n, r, c) drackn with

$$c = \frac{1}{r} \left((n-2) + \frac{2d-n}{\alpha d} \right).$$

We include a sketch of the proof given in Coutinho et al. since it will be relevant to our discussion below.

Proof. [(Sketch), See [27]] Let C_r denote the cyclic group of order r . We use the matrix S defined in the theorem statement as a C_r -valued arc function on K_n and show that the resulting cover has exactly 4 distinct eigenvalues. By Theorem 2.8.1, the spectrum of this cover contains n with multiplicity 1, -1 with multiplicity $n-1$ and the union of the eigenvalues of S^{ρ_i} where ρ_i runs over the non-trivial irreducible representations of the cyclic group C_r . Since r is prime, these S^{ρ_i} all have the same minimal polynomial: If $m(x)$ is the minimal polynomial of some S^{ρ_i} then $m(S^{\rho_j}) = 0$ since the effect of ρ_j on the entries of S is to permute the r th roots of unity, and all such roots of unity are roots of the r th cyclotomic polynomial $\Phi_r(t) = t^{n-1} + t^{n-2} + \dots + t + 1$. So the S^{ρ_j} have the same eigenvalues which are the two distinct eigenvalues of S .

From here one can use the exact values of the eigenvalues of S to calculate the parameters of the associated drackn and prove that it is in fact distance-regular. See [27] or Section 8.3 for these details. \square

Let us refer to $\frac{1}{\alpha}(G - I)$ as the normalized Gram matrix of a set of lines. Unfortunately, Coutinho et al. were not able to find any equiangular tight frames with this additional property on the off-diagonal entries of the normalized Gram matrix, except of course for those coming from a known drackn via Theorem 7.4.3. They do mention that the Gram matrices coming from the “Steiner” equiangular tight frames of Fickus et al. [43] are almost suitable candidates to apply Theorem 7.4.4.

These “Steiner” equiangular tight frames are so named because they are constructed from the incidence matrices of Steiner systems. A **Steiner system** (of strength 2), is a block design with parameters $2-(v, k, 1)$. i.e. a set V called the **points** of the design with $|V| = v$ and a collection of k -element subsets of V called **blocks** with the property that any pair of points is contained in exactly 1 block.

The construction of equiangular lines from [43] takes as input the $\frac{v(v-1)}{k(k-1)} \times v$ incidence matrix A of a $2-(v, k, 1)$ design, along with a $(1 + \frac{v-1}{k-1}) \times (1 + \frac{v-1}{k-1})$ matrix H with orthogonal rows and unimodular entries. In practice, H is usually chosen to be some variant on a Hadamard matrix, possibly with complex entries. By replacing the 1’s in A with suitably chosen rows of H , and the 0’s with $1 \times (1 + \frac{v-1}{k-1})$ blocks of 0’s, one obtains a matrix whose $N := v(1 + \frac{v-1}{k-1})$ columns are unit norm representatives of equiangular lines in $M := \frac{v(v-1)}{k(k-1)}$ dimensional space. See [43, Theorem 1] for the remaining details.

Certain Steiner equiangular tight frames have normalized Gram matrices that almost satisfy the condition of Theorem 7.4.4. In these otherwise suitable equiangular tight frames, the off-diagonal entries of the normalized Gram matrix are either equal to a p th root of unity, or to -1 (up to scaling by the angle α). However, for any prime $p > 2$, the regular representation of C_r , used in the proof of Theorem 7.4.4, cannot contend with these -1 entries. In the remainder of this section, we discuss our attempt to circumvent this obstruction, and some interesting new examples that have come from this attempt.

Consider the action of $C_{2r} = \langle z \rangle$ on $[r]$ in which z acts as $(12 \dots r)$. The permutation representation ρ of this action has degree r and kernel $\{1, -1\}$. Let us call this the **halved representation**. We apply the halved representation of C_{2p} to the matrix S from Theorem 7.4.4 rather than the regular representation of C_p . A priori it is not clear that this is a good idea, but the examples are promising:

1. Take A to be the incidence matrix of the $2-(3, 2, 1)$ design whose points are the vertices of a triangle and blocks are the edges. Let H be the 3×3 complex Hadamard matrix, i.e. the character table of C_3 . The resulting Steiner equiangular tight frame of 9 lines in \mathbb{C}^3 has normalized Gram matrix with off-diagonal entries in $\{1, -1, \omega, \omega^2\}$, ω a third root of unity. Applying the halved representation gives the adjacency

matrix of the cyclic $(9, 3, 3)$ -drackn. This is the only Steiner equiangular tight frame known which meets the absolute bound [43], suggesting that this example may be exceptional.

2. Take A to be the incidence matrix of the 2 -($13, 4, 1$) design that comes from the points and lines of the projective plane $PG(2, 3)$. Let H be the character table of C_5 . The off-diagonal entries of the normalized Gram matrix of the resulting Steiner equiangular tight frame are either -1 or 5th roots of unity, and there is a feasible parameter set for a $(65, 5, 15)$ -drackn which could be derived from a degree 5 representation of C_{10} . However, when we apply the halved representation, we find something quite surprising: The resulting graph is an antipodal 5-fold cover of K_{65} with spectrum

$$64^{(1)}, \quad -1^{64}, \quad 14^{(52)}, \quad 4^{(52)} \quad -6^{(156)}.$$

This is a cover of K_n with three “new” eigenvalues, in the sense of Chapter 8.

3. Much the same as the previous example, take A to be the incidence matrix of the 2 -($31, 6, 1$) design coming from the projective plane $PG(2, 5)$. Let H be the character table of C_7 . Again, there is a feasible parameter set $(217, 7, 35)$, and again, applying the halved representation of C_{14} we instead obtain an antipodal cover of K_{217} with three new eigenvalues.

The reason this works is that the constituents of the halved representation of C_{2p} are exactly the same as the constituents of the regular representation of C_p . The proof of Theorem 7.4.4 ensures that these constituents have identical eigenvalues. So we are, in essence, working with the regular representation and “pretending” that the -1 entries in the matrix S are $+1$ instead. The miraculous fact, for which we do not yet have an explanation, is that this augmented matrix has 2 eigenvalues in our first example, and 3 eigenvalues in the second and third example.

It seems that these later two graphs are the first examples of antipodal covers of complete graphs with 3 new eigenvalues. Because of this novelty, we know almost nothing about such graphs in general. Are there infinitely many such covers? Do they have other combinatorial properties similar to their distance-regular cousins? Can we derive examples from the incidence matrix of any projective plane? The list goes on.

These examples are also interesting in that they provide, to the best of our knowledge, the first instances of covering graphs derived from arc functions using unfaithful permutation representations. The prospect of novel constructions of interesting covers associated

with unfaithful representations means we have far more freedom when designing covering graphs based on arc functions.

We note that Fickus et al. have recently succeeded in deriving new drackns from complex equiangular lines. In [41], they construct new equiangular tight frames by augmenting the incidence matrices of Steiner systems with additional vectors. When these augmented frames are derived using matrices populated by prime roots of unity, the entries of the associated Gram matrices are prime roots of unity as well, and give rise to the most recently discovered infinite family of drackns. The smallest new examples have parameters $(55, 5, 10)$ and $(105, 7, 14)$.

7.5 Arc Functions from Association Schemes

If X is any cover of K_n determined by an arc function normalized on a star centered at z , then for $g \in \text{con}(f)$ we define a matrix A_g with rows and columns indexed by $V(K_n) - z$ whose (i, j) entry is 1 if and only if $f(i, j) = g$, and zero otherwise. The set $\{A_g : g \in \text{con}(f)\}$ is closed under transpose and sums to $J - I$, so for certain special arc functions f it may be the case that

$$\{I_{n-1}\} \cup \{A_g : g \in \text{con}(f)\}$$

is an association scheme. In this case we say that X is **scheme developed**, and that the matrix A_g **supports** the element g .

For example, we can construct the $(8, 3, 2)$ -drackn as follows: Decompose K_7 into 3 disjoint 7-cycles and construct an arc function f on K_7 whose values on these three cycles are the three involutions in the symmetric group $\text{Sym}(3)$. Extend f to an arc function on K_8 normalized on the star centered at the additional vertex. The resulting map is an arc function for the $(8, 3, 2)$ -drackn. Setting A_i to be the adjacency matrices of these 7-cycles for $i \in \{1, 2, 3\}$ one can verify that

$$A_1 A_2 = A_2 A_1 = A_1 + A_3, \quad A_2 A_3 = A_3 A_2 = A_1 + A_2, \quad A_1 A_3 = A_3 A_1 = A_2 + A_3$$

and

$$A_1^2 = A_2 + 2I, \quad A_2^2 = A_3 + 2I, \quad A_3^2 = A_1 + 2I,$$

so this construction shows that the $(8, 3, 2)$ -drackn is scheme developed.

In this section we are concerned with the question of which drackns are scheme developed. Our motivation comes from Corollary 7.3.2 (a), where we saw that an index-2

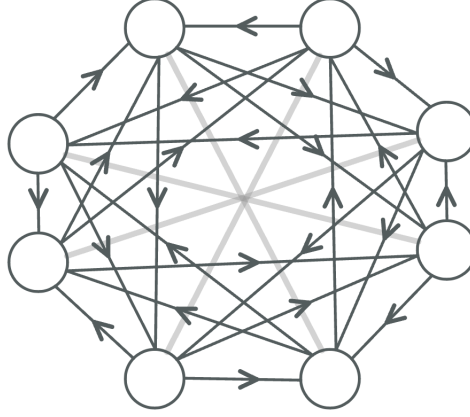


Figure 7.1: We consider the above orientation of K_8 minus a matching. Let A denote the adjacency matrix of this oriented graph. One can check that A , A^T and the adjacency matrix of the missing matching form an association scheme. This scheme develops the cyclic $(9, 3, 3)$ if we associate the two 3-cycles in \mathbb{Z}_3 with A and A^T and the identity with the remaining class.

cover of K_n is a drackn if and only if it is developed from a strongly-regular graph, i.e. a symmetric 2-class association scheme. To what extent does this result hold when $r > 2$?

It seems that there is only one result in the literature that addresses this question: A proposition in [18, Chapter 12.7] that characterizes (in somewhat grizzly fashion) the drackns developed from symmetric association schemes. That result addresses the closest analog to the $r = 2$ case, but is quite limited because it requires the image of f to consist exclusively of involutions. Many drackns, such as cyclic drackns of odd index, have no arc function with this property. We refer the reader to Proposition 12.7.1 from [18] for the details of that characterization. The known examples addressed by that result are the covers from Section 6.3.2, including the $(8, 3, 2)$ -drackn above, and two sporadic examples due to Mathon [96] and Hollmann, [77].

The result above addresses only a small fraction of the possible drackns. In particular, it omits cyclic drackns of odd index. This is notable because the cyclic $(9, 3, 3)$ -drackn is scheme developed. See Figure 7.1. The observation that this drackn is scheme developed seems to be new.

So there is room to expand the theory of scheme developed drackns in the cyclic case. On the other hand, we will now demonstrate a notable limitation to scheme developed

drackns: We show that a small non-cyclic drackn cannot be scheme developed. In particular, Jurišić showed that there are exactly two drackns with parameters $(9, 3, 3)$, the cyclic drackn mentioned above, and a second non-cyclic example. Each can be obtained from the point graph of the generalized quadrangle $GQ(2, 4)$ by deleting one of two non-isomorphic spreads. See Section 6.3.3 and [86].

We will show that this non-cyclic $(9, 3, 3)$ -drackn is not scheme developed using a simple strategy. The association schemes on 8 points are tabulated in [70], (c.f. [71]). We check that no assignment of the elements of $\text{Sym}(3)$ to the classes of these schemes determines the arc function of a drackn. Rather than take a fully “brute force” approach, we prove a lemma on the structure of $\text{Sym}(3)$ -arc functions, allowing us to rule out 20 of the 21 schemes on 8 points by hand. This lemma may be of independent interest: It looks to be the first evidence that the decomposition of the standard representation into irreducibles is useful in studying arc functions.

7.5.1 Lemma. *Let X be an $(n, 3, c)$ -cover with arc function $f : K_n \rightarrow \text{Sym}(3)$ with connection group $\text{Sym}(3)$. Suppose without loss of generality that f is normalized on a star with center z . For each vertex v in $V = V(Y)$, let $f_{\text{out}}^v := \{f(v, u) : u \in V - v\}$ be the multiset of values f takes on arcs out of v . For each $v \neq z$ there are constants v_1, v_2 so that*

- (a) *Each involution in $\text{Sym}(3)$ occurs in f_{out}^v with multiplicity v_1 .*
- (b) *Both elements of order 3 occur with multiplicity v_2 .*
- (c) *The identity occurs with multiplicity $v_1 + \delta$.*

Proof. From the arc matrix Y^f we obtain X by applying the permutation representation ρ elementwise. It follows from Theorem 2.8.1, that X is similar to the direct sum

$$\bigoplus_i Y^{\rho_i(f)}$$

where the summation runs over the irreducible representations of ρ . In this case, there are two such irreducible representations, the trivial representation ρ_0 and an irreducible representation ρ_1 of degree 2. X has eigenvalues $n - 1$ with multiplicity 1 and -1 with multiplicity $n - 1$, the same as Y^{ρ_0} , so Y^{ρ_1} must have eigenvalues θ and τ with multiplicities determined as in Section 6.4. Let $S = Y^{\rho_0}$. Since S has two eigenvalues, it satisfies the equation defined by its minimal polynomial

$$S^2 = \delta S + (n - 1)I.$$

Comparing the (z, j) blocks of the sides of this equation for $j \neq z$ we find that

$$\sum_{g \in f_{out}^j} \rho_1(g) = \delta I_2.$$

There is only one representation ρ_1 up to similarity, and the statements we are about to make are similarity invariant, so it suffices to consider a concrete instance of ρ_1 such as

$$\rho_1((1, 2)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_1((2, 3)) = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$$

where ω is a primitive third root of unity. Consider the linear span of $\{\rho_1(g), g \in \text{Sym}(3)\}$. This is isomorphic to the row space of

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ \omega & 0 & 0 & \omega^2 \\ \omega^2 & 0 & 0 & \omega \\ 0 & 1 & 1 & 0 \\ 0 & \omega^2 & \omega & 0 \\ 0 & \omega & \omega^2 & 0 \end{pmatrix}.$$

a matrix whose left nullspace is spanned by $(1, 1, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 1, 1)$. It follows that a sum of entries $A_{i,j}^{\rho_1(f)}$ is equal to the 2×2 matrix of zeros if and only if there are scalars v_1 and c_2 so that $\rho_1(\tau_1), \rho_1(\tau_2), \rho_1(\tau_3)$ occur v_1 times and $\rho_1(\text{id}), \rho_1(c_1), \rho_1(c_2)$ occur v_2 times. Moreover, the only sum of entries which can equal $-I_2$ is $\rho_1(c_1) + \rho_1(c_2)$. So the only way a sum $\sum_{g \in U} \rho_1(g)$ can equal $-kI_2$ is if U contains v_1 occurrences of each transposition, v_2 occurrences of each 3-cycle, and $v_2 - k$ occurrences of the identity. \square

This lemma will allow us to quickly rule out all but one of the possible association schemes on 8 vertices. The single remaining scheme is checked by computer.

7.5.2 Theorem. *The non-cyclic $(9, 3, 3)$ -drackn is not scheme developed.*

Proof. A $(9, 3, 3)$ -cover has $\delta = -2$. So Lemma 7.5.1 implies that in a scheme that develops this drackn we must have three classes of valency ℓ_1 supporting the three involutions in $\text{Sym}(3)$. Further, we must have two classes of valency ℓ_2 supporting the 3-cycles, and one class of valency $\ell_2 - 2$ supporting the identity.

Since the valencies sum to 7, the only possibilities are

$$\ell_1 = 0, \quad \ell_2 = 3$$

or

$$\ell_1 = 1, \quad \ell_2 = 2.$$

In the former case, each arc takes a value in $C_3 \subset \text{Sym}(3)$ and the arc function is cyclic, a contradiction. So we must be in the second case.

Consulting [70], and employing the labeling given there, we see that there are only four association schemes with the required valencies: In schemes 13 and 15, the classes of valency 2 are symmetric, so cannot support 3-cycles. And in scheme 16, two of the classes of valency 1 are asymmetric, and cannot support involutions. This leaves only scheme 14. We finish via computer, checking the 12 allowed assignments of group elements to scheme classes. In all cases, the resulting covers of K_9 are isomorphic, but the graph is not distance-regular. \square

We note another new computational result of the same flavor. $(25, 3, 7)$ is one of the smallest feasible parameter sets for which it is unknown if an (n, r, c) -drackn exists. Following the method in the proof above we have shown that a cover with these parameters cannot be scheme developed. In this case, Lemma 7.5.1 and analysis of the valencies of the symmetric classes rule out all but two of the 750 association schemes on 24 points. The remaining two association schemes were ruled out via computer. Thus we have the following result.

7.5.3 Theorem. *There is no scheme developed $(25, 3, 7)$ -drackn.* \square

7.6 Local Properties

In Corollary 7.3.2 (b) we saw that the subgraphs induced on the first and second neighborhoods of an index-2 drackn are both strongly-regular graphs with parameters determined by n and c . There is also a well-known relationship between the eigenvalues of a strongly-regular graph and the eigenvalues of its second-neighborhood. This relationship gives the most information about the neighbourhoods when the strongly-regular graph is triangle-free.

Over the next two sections we will investigate a common analog of these ideas: We determine, as a function of (n, r, c_2) , the eigenvalues of the second neighborhood of a triangle-free (n, r, c_2) -drackn with $r > 2$. We find, in analogy with both of the results above, that the spectrum of the second neighborhood is completely determined by the parameters of the (n, r, c_2) -drackn. As a consequence, we find that the smallest, and second-largest eigenvalues of such a second-neighborhood are identical to the smallest and second-largest eigenvalues of the drackn itself.

The most natural way to study the local structure of a distance-regular graph is to study a Terwilliger algebra, or subconstituent algebra, of the graph, [134]. Tomiyama and Yamazaki [138] gave a complete description of the Terwilliger algebras of strongly-regular graphs. This can be used to determine the spectra of the subgraphs induced on the first and second neighborhoods.

Many of the salient properties of the Terwilliger algebra of a drackn were investigated by Shirazi in his Ph.D. thesis [121], but the work there does not give a description of the spectrum of the second neighborhood of a triangle-free drackn. Our result follows the same basic structure as [121]: Study a pair of partitions that arise in any drackn, then derive useful identities relating these partitions to the adjacency matrices of the subgraphs induced on the first and second neighborhoods.

Now let X be a drackn and let v be any vertex. As in Chapter 6 let $\Gamma_i := \Gamma_i(v)$ denote the set of vertices at distance i from v in the drackn X . For any vertex v in a drackn X , we specify two useful partitions of the set $\Gamma_2(v)$.

Since X is a cover, it comes equipped with a surjective homomorphism π onto $V(K_n)$, and the preimages $\pi^{-1}(y)$ of vertices y of K_n partition $V(X)$ into n fibers of size r . The **row partition** is the restriction of the partition $\{\pi^{-1}(y) : y \in V(K_n)\}$ to $\Gamma_2(v)$. Because X is antipodal, the fiber containing our specified vertex v is $\Gamma_0(v) \cup \Gamma_3(v)$, and because v has exactly one neighbor in each other fiber of π^{-1} , the row partition is a partition into $(n - 1)$ sets of size $(r - 1)$, each of which induces a coclique in X .

Each of the $r - 1$ vertices in $\Gamma_3(v)$ has $n - 1$ neighbors in $\Gamma_2(v)$. The **column partition** of $\Gamma_3(v)$ is the partition $\{N(y) : y \in \Gamma_3(v) \setminus \{v\}\}$.

To motivate the names of these partitions, note that for any pair (y, z) with $y \in \Gamma_1(v)$ and $z \in \Gamma_3(v)$, there is a unique neighbor of z in the fiber containing y . It follows that each part of the row partition intersects each part of the column partition in exactly one vertex, so the whole of $\Gamma_2(v)$ can be drawn in a rectangular grid whose rows and columns are the parts of the corresponding partitions.

Throughout this and the following section, we let R and C denote the transposes of the characteristic matrices of the row and column partitions respectively. These partitions will be important, so let us make them explicit.

R is an $(n - 1)(r - 1) \times (n - 1)$ matrix whose rows are indexed by the vertices of $\Gamma_2(v)$ and whose columns are indexed by the fibers of X that do not contain v , or equivalently by the neighbors of v . The u, w entry of R is 1 if the vertex u is at distance 3 from w and 0 otherwise.

C is an $(n-1)(r-1) \times (r-1)$ matrix with rows indexed by the vertices of $\Gamma_2(v)$ and columns indexed by the vertices in the fiber over v , other than v itself. The (u, w) entry of C is 1 if u is adjacent to w and 0 otherwise.

Indexing rows and columns of the adjacency matrix of X so that all vertices in Γ_i occur before all vertices of Γ_j whenever $i < j$ we find the block structure

$$A_1 = \begin{pmatrix} 0 & J & 0 & 0 \\ J & B_1 & N^T & 0 \\ 0 & N & B_2 & C \\ 0 & 0 & C^T & 0 \end{pmatrix}$$

where the diagonal blocks are of orders

$$1, n-1, (n-1)(r-1), (r-1),$$

B_1 and B_2 are adjacency matrices for the subgraphs induced on Γ_1 and Γ_2 , and

$$\begin{pmatrix} 0 & N^T \\ N & 0 \end{pmatrix}$$

is the subgraph induced by the bipartition (Γ_1, Γ_2) . Note that J and 0 denote the all ones and all zeros matrices of various sizes. The appropriate dimensions can of course be recovered from the dimensions of the diagonal blocks.

Using the same ordering of vertices, it is elementary to find the block decomposition for the 3rd distance matrix of X .

We have

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & J \\ 0 & 0 & R^T & 0 \\ 0 & R & RR^T - I & 0 \\ J & 0 & 0 & J - I \end{pmatrix}$$

where the diagonal blocks are sized as above.

Since X has diameter 3 it follows that

$$A_2 = J - I - A_1 - A_3 = \begin{pmatrix} 0 & 0 & J & 0 \\ 0 & J - I - B_1 & J - R^T & J \\ J & J - R & J - RR^T - B_2 & J - C \\ 0 & J & J - C^T & 0 \end{pmatrix}.$$

Next we consider the implications of this block decomposition in the presence of the three term recurrence of Theorem 6.1.1:

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}.$$

For $i \in \{1, 2, 3\}$ and $j, k \in \{1, 2, 3, 4\}$ this recurrence restricts to an equation comparing the (j, k) blocks of the left and right side of the equations for the product of A_1 with A_i . The following table lists a subset of these equations that we will need in the next section.

(i, j, k)	
$(1, 2, 3)$	$NB_1 + B_2N = (a_1 - c_2)N + c_2J - c_2R$
$(1, 3, 3)$	$NN^T + B_2^2 + CC^T + c_2RR^T = (a_1 - c_2)B_2 + c_2J + (n - 1)I$
$(1, 3, 4)$	$C^TB_2 = (a_1 - c_2)C^T + c_2J$
$(3, 3, 2)$	$B_2R = J - N - R$

We will also need the identity

$$R^T R = (r - 1)I$$

which follows from the fact that the rows of R^T correspond to cells of the row partition.

7.7 The Spectrum of the Second-neighborhood of a Triangle-free Drackn

We are ready to determine the spectrum of the second neighborhood of all triangle-free drackns, except possibly those which come from Moore graphs of valency 57.

7.7.1 Theorem. *Suppose X is a triangle-free (n, r, c) -drackn, with $r > 2$, $c > 1$, and spectrum*

$$(n - 1)^{(1)}, -1^{(n-1)}, \theta^{(m_\theta)}, \tau^{(m_\tau)}.$$

Then the subgraph induced by the second-neighborhood of any vertex has spectrum

$$(n - 2 - c_2)^{(1)}, \theta^{(m_\theta - n - r + 3)}, 0^{(n-2)}, -c_2^{(r-2)}, -(c_2 + 1)^{(n-2)}, \tau^{(m_\tau - n - r + 3)}.$$

Note that the theorem as stated does not apply to the unique $(7, 6, 1)$ -drackn, but the conclusion holds for that graph as well, leaving only the second neighborhoods of putative Moore graphs of valency 57 unaccounted for (c.f. Theorem 6.5.5 and the discussion following it.)

Proof. Throughout we assume X is as in the hypothesis of the theorem. Let v be an arbitrary vertex of X and let B_1, B_2, N, R, C, J be the matrices defined in Section 7.6. Recall that $a_2 = n - 2 - c_2$ and $\delta = a_1 - c_2 = -c_2$. We utilize the following decomposition of the eigenspaces of B_2 :

- (a) $E_{a_2} := \text{col}(J)$
- (b) $E_{-c_2} := \ker(J^T) \cap \text{col}(C)$
- (c) $E_{-c_2-1} := \ker(J^T) \cap \ker(C^T) \cap \text{col}(N + R)$
- (d) $E_0 := \ker(J^T) \cap \ker(C^T) \cap \text{col}(N - c_2 R)$
- (e) $E_{\{\theta, \tau\}} := \ker(J^T) \cap \ker(C^T) \cap \ker(R^T) \cap \ker(N^T)$

At present the E_λ are just names, but we will show that in fact they are the projections onto the λ eigenspaces, with the exception of $E_{\{\theta, \tau\}}$, the projection onto the sum of two eigenspaces. First we verify that these spaces partition the B_2 eigenspace. We will repeatedly use the fact that for any $n \times m$ matrix M , $\text{col}(M) \oplus \ker(M^T)$ is an orthogonal decomposition of an n -dimensional vector space. From this fact it is immediate that E_{a_2} and E_δ have trivial intersection with each other and with the other three spaces. For a vector

$$u \in \text{col}(N + R) \cap \text{col}(N - c_2 R)$$

there is a vector v so that

$$Nv + Rv = u = Nv - c_2 Rv$$

which implies $Rv = 0$ since $c_2 > 0$. The columns of R have disjoint supports and are thus linearly independent, so $v = 0$, and $E_{\delta-1}$ and E_0 have trivial intersection. Now, any eigenvector u of B_2 with

$$u \in \ker(N^T + R^T) \cap \ker(N^T - c_2 R^T)$$

satisfies

$$N^T u + R^T u = N^T u - c_2 R^T u$$

so $R^T u = 0$, since $c_2 > 0$. Hence $u \in \ker(R^T)$. From the transpose of Equation (3, 3, 2) we find that

$$R^T B_2 u = J - N^T u - R^T u$$

so

$$0 = -N^T u$$

for any B_2 eigenvector $u \in \ker(R^T) \cap \ker(J)$. It follows that all the remaining eigenvectors are elements of $E_{\theta, \tau}$.

Now we show that the E_λ are in fact λ eigenspaces or a sum of two eigenspaces in the final case.

E_{a_2} : By definition the subgraph of X induced on Γ_2 is a_2 regular. So B_2 has constant row sum $a_2 = n - 2 - c_2$, and $J_{1, (n-1)(r-1)}$ is an eigenvector with eigenvalue a_2 . By Theorem 6.5.5, this subgraph is connected, so the eigenvalue a_1 has multiplicity 1, and $J_{1, (n-1)(r-1)}$ is the entire eigenspace.

E_{-c_2} : From Equation (1,3,4) we have

$$B_2 C = \delta C + c_2 J.$$

So any vector in $\text{col}(C) \cap \ker(J_{1, r-1})$ is an eigenvector of B_2 with eigenvalue δ . The supports of the columns of C are disjoint, so C has rank $r - 1$, hence $\ker(J^T) \cap \text{col}(C)$ is a $\delta = -c_2$ eigenspace of dimension $r - 2$.

E_{-c_2-1} : From Equation (1,2,3), and the fact that B_1 is the zero matrix, we have $B_2 N = c_2(J - N - R)$, and adding $(c_2 + 1)N$ to both sides we obtain

$$(B_2 + (c + 1)I)N = c_2 J + N - c_2 R.$$

Adding $(c_2 + 1)R$ to both sides of Equation (3,3,2) we obtain

$$(B_2 + (c + 1)I)R = J - (N - c_2 R),$$

or

$$(B_2 + (c + 1)I)(N + R) = (c_2 + 1)J.$$

So any vector in

$$\ker(J^T) \cap \text{col}(N + R)$$

is an eigenvector for B_2 with eigenvalue $-(c_2 + 1)$.

E_0 : From Equation (1,2,3), Equation (3,3,2), and the fact that B_1 is the zero matrix, we have

$$B_2 N = c_2(J - R - N) = c_2 B_2 R.$$

So $B_2(N - c_2 R) = 0$, and any vector in $\ker(J^T) \cap \text{col}(N - c_2 R)$ is an eigenvector of B_2 with eigenvalue 0.

$E_{\{\theta, \tau\}}$: Applying Equation (1,3,3) to any vector $u \in E_{\{\theta, \tau\}}$, we find that

$$B_2^2 u + \delta B_2 u - (n-1)u = 0$$

so the eigenvalue corresponding to u is a root of $x^2 + \delta x - (n-1)$. i.e., it is θ or τ .

Finally, we calculate the multiplicities of the eigenvalues. We have seen that $m_{a_2} = 1$ and $m_{-c_2} \geq r-2$. In fact, since all eigenvectors of B_2 outside of $\text{col}(C)$ have an eigenvalue other than c_2 we have $m_{-c_2} = r-2$. Similar to the derivation of the eigenvalues of a drackn (c.f. Theorem 6.4.3) we now use Lemma 2.1.4 to calculate the trace of the 1st, 2nd, and 3rd powers of the adjacency matrix of B_2 . Since we know two of the six multiplicities of eigenvalues of B_2 , and since one of our unknown multiplicities is attached to the eigenvalue 0, this gives us three equations in the variables $m'_\theta, m'_\tau, m_{\delta-1}$. Here m'_θ and m'_τ are the multiplicities of θ and τ in B_2 , and m_θ and m_τ are their multiplicities in X .

$$\begin{aligned} 0 &= a_2 + \delta(r-2) + \theta m'_\theta + \tau m'_\tau + (\delta-1)m_{\delta-1} \\ (n-1)(r-1)a_2 &= a_2^2 + \delta^2(r-2) + \theta^2 m'_\theta + \tau^2 m'_\tau + (\delta-1)^2 m_{\delta-1} \\ 0 &= a_2^3 + \delta^3(r-2) + \theta^3 m'_\theta + \tau^3 m'_\tau + (\delta-1)^3 m_{\delta-1} \end{aligned}$$

We rewrite these equations as

$$\begin{pmatrix} \theta & \tau & \delta-1 \\ \theta^2 & \tau^2 & (\delta-1)^2 \\ \theta^3 & \tau^3 & (\delta-1)^3 \end{pmatrix} \begin{pmatrix} m'_\theta \\ m'_\tau \\ m_{\delta-1} \end{pmatrix} = \begin{pmatrix} -a_2 - (r-2)\delta \\ (n-1)(r-1)a_2 - a_2^2 - (r-2)\delta \\ -a_2^3 - (r-2)\delta^3 \end{pmatrix}.$$

Since $\delta = \theta + \tau - 1$, one can calculate that the determinant of the 3×3 matrix is

$$-(\theta-1)\theta(\tau-1)\tau(\theta-\tau)(\theta+\tau-1),$$

so this matrix is invertible unless $\theta = \tau$, $\theta + \tau = 1$ or $\theta = 1$. All are impossible: $\theta = \tau$ implies $c^2 = -4(n-1)$ via the expressions for θ and τ in terms of n, r, c ; $\theta + \tau = \delta = -c$ must be negative; and $\theta = 1$ implies $\tau = -1 - c$, so $n-1 = -\theta\tau = c+1$ and $0 = a_1 = (n-2) - (r-1)(n-2)$, and $r = 2$. So the matrix is invertible and there is a unique solution for the multiplicities of θ, τ , and $\delta-1$ in B_2 . We omit the calculation of the inverse of this

matrix, but note that the result is easily written in terms of the multiplicities of θ and τ in X . We have

$$m'_\theta = m_\theta - n - r + 3, \quad m'_\tau = m_\tau - n - r + 3, \quad m_{\delta-1} = n - 2.$$

Alternatively, one can verify that this is the correct solution by substituting these values into the system of equations above and applying the identities

$$\theta m_\theta + \tau m_\tau = 0, \quad \theta^2 m_\theta + \tau^2 m_\tau = rn(n-1) - (n-1)^2 - (n-1),$$

which come from the equations in the proof of Theorem 6.4.3, along with a third identity for the third powers. Again, we omit the calculation.

We now know the multiplicities of all eigenvalues of B_2 except for m_0 . Since the sum of the multiplicities of all the eigenvalues is equal to $(n-1)(r-1)$ we find that $m_0 = n-2$ and we are done. \square

A nice consequence of this result is that we know that θ is the second largest eigenvalue of B_2 , and τ is the smallest. These are often the most pertinent eigenvalues for spectral characterizations of graphs such as those involving the size of maximal independent sets, [68], expansion properties [1], or deviation from being bipartite [16], [5].

7.8 Future Work

First of all, it will be interesting to study the antipodal covers of K_{65} and K_{217} that we have constructed from Steiner ETFs in Section 7.4. This is very recent work, so there are lots of questions that there has not yet been time to try to answer. We list just a few.

1. Other than antipodality and sparsity of spectrum, do our newly constructed examples have other nice graph theoretic properties? Are they walk-regular? Do they have nice automorphism groups?
2. Are there more Steiner ETFs whose Gram matrices entries are either prime roots of unity or -1 ? Does the same trick of using the unfaithful representation of C_{2r} give antipodal 5-eigenvalue covers? In particular, we expect that this procedure always works for ETFs arising from the incidence matrices of prime order projective planes.
3. Are there other unfaithful representations we can use to generate covers with few eigenvalues from interesting arc matrices?

Second, we wonder which association schemes can develop drackns.

4. Is every cyclic drackn of index 3 scheme developed?
5. The graph we obtain from scheme 14 in the proof of Theorem 7.5.2 has spectrum

$$8^{(1)}, -1^{(8)}, 4^{(2)}, -2^{(4)}, -4^{(4)}, 2^{(8)}.$$

Compare this to the spectrum of the $(9, 3, 3)$ -drackn:

$$8^{(1)}, -1^{(8)}, 2^{(12)}, -4^{(6)}.$$

We see that the graph from scheme 14 has “almost” the correct spectrum, with only a few eigenvalues having the wrong sign. Is there a way to correct this?

Third, we address questions surrounding Theorem 7.7.1.

6. Can we leverage the spectrum of the second-neighborhood of a triangle-free drackn to rule out any more feasible parameter sets for drackns?
7. Does Theorem 7.7.1 hold for a Moore graph of valency 57? Most of our proof should go through with no trouble, but we would need to verify that the subgraph induced on the second neighborhood of the second neighborhood of any vertex in such a Moore graph is connected.
8. Is the second neighborhood of a triangle-free drackn ever distance-regular? Our result implies that this is only possible if this second neighborhood has diameter 5, which is not true for any of our known triangle-free drackns.
9. What other triangle-free drackns exist?

Chapter 8

Two-Eigenvalue Covers

Let $f : Y \rightarrow G$ be an arc function with arc matrix S and permutation representation ρ . We have seen in Section 2.8 that the adjacency matrix of $Y^{\rho(f)}$ is similar to a direct sum of the matrices S^{ρ_i} running over the irreducible constituents ρ_i of ρ counted with multiplicity. The trivial representation is always a constituent of ρ and so the eigenvalues of $S^{\rho_{triv}} = Y$ are eigenvalues of $Y^{\rho(f)}$. These are the **old eigenvalues** of $Y^{\rho(f)}$, and all other eigenvalues are the **new eigenvalues** of $Y^{\rho(f)}$. In this chapter we study covers with exactly two distinct new eigenvalues, which we will call **2-ev** covers.

It is worth noting that the only symmetric or Hermitian matrices with exactly one eigenvalue are scalar multiples of the identity matrix. Consequently, there is no analogous notion of a “1-ev cover,” so we are studying the smallest “jump” in the number of eigenvalues that the covering graph construction can produce.

Our motivation comes from two results we have encountered in earlier chapters. First, the covers \tilde{Q}_n from Chapter 4 are 2-ev covers, and the fact that their arc matrices have 2 eigenvalues is precisely what makes them so useful in Huang’s proof of the sensitivity conjecture (c.f. Section 4.2). Second, every drackn is a 2-ev cover, and this property is what connects them so closely to equiangular lines.

Motivated by these examples, we study the extent to which 2-ev covers preserve the nice properties of the graphs they cover. We offer several results on the topic when the action of the arc function of the cover is particularly well-behaved. In Section 8.2 we show that any abelian 2-ev cover of a walk-regular graph is itself walk-regular. In particular this implies that the 4-cycle-free covers of the hypercubes from Chapter 4 are walk-regular. Then in Section 8.3 we determine the necessary and sufficient conditions under which a

cyclic 2-ev cover of a strongly-regular graph is distance-regular. The results from Sections 8.2 and 8.3 are joint work with Godsil and Silina, which appeared in [55].

We note that the “well-behavedness” conditions we impose to prove our theorems are not so much of a concession when we consider the known examples of 2-ev covers of walk-regular graphs in Section, 8.4 all of which are appropriately well-behaved.

8.1 Consequences of the 2-ev Property

From the results of Section 2.8, we know that the spectrum of a cover is closely related to the irreducible representations of the permutation action of its arc function. We explain this relationship when X is a 2-ev cover.

8.1.1 Lemma. *Let X be a 2-ev cover of Y with arc matrix S action ρ and new eigenvalues θ and τ . Let ρ_i be any non-trivial irreducible constituent of ρ and set $S_i := S^{\rho_i}$. Then*

$$S_i^2 = (\theta + \tau)S_i - \theta\tau I_{\deg(\rho_i)|Y|}.$$

Proof. By Theorem 2.8.1, the adjacency matrix of X is similar to the direct sum of the S_i where i runs over the irreducible constituents of ρ . So the spectrum of X is the union of the spectra of these S_i . The trivial representation is a constituent which accounts for all of the old eigenvalues of X . Since X is a 2-ev cover, each of the remaining S_i has spectrum a subset of $\{\theta, \tau\}$.

Now we have

$$\text{tr}(A(X)) = \text{tr}((A(Y))) = 0,$$

so if one of θ or τ was 0, the other would need to be zero as well, a contradiction. So neither is zero, and in fact one must be positive and the other negative. Now S_i has zero diagonal, so $\text{tr}(S^{\rho_i}) = 0$, and each of θ and τ must occur in the spectrum of S^{ρ_i} . This implies that the minimal polynomial of S_i is

$$(x - \theta)(x - \tau) = 0$$

from which we obtain

$$S_i^2 = (\theta + \tau)S_i - \theta\tau I_{\deg(\rho_i)|Y|}.$$

□

This already gives certain restrictions on the graphs Y which can admit 2-ev covers.

8.1.2 Corollary. *If X is a 2-ev cover of Y with new eigenvalues θ and τ , then Y is regular of valency $-\theta\tau$.*

Proof. If the vertex v of Y has valency k then the (v, v) block of S_i^2 is

$$\sum_{u \sim v} \rho_i(f(v, u)) \rho_i(f(u, v)) = \sum_{u \sim v} \rho_i(id) = k I_{\deg(\rho_i)}.$$

From Lemma 8.1.1, this block is also equal to $-\theta\tau I_{\deg(\rho_i)}$. □

8.2 Walk-Regular 2-ev Covers

Our main question in this chapter is how the nice “regularity” of a graph Y behaves when we pass to a 2-ev cover of Y . Our first answer is that walk-regularity is always preserved by 2-ev covers with regular abelian actions. Recall that a graph is **walk-regular** if, for each integer $k > 0$, the number of closed walks of length k starting from (and ending at) vertex v is independent of the chosen vertex v . Equivalently, a graph is walk-regular if the k th power of its adjacency matrix has constant diagonal for all positive integers k . Walk-regular graphs have been studied due to their “spectral regularity”: In a walk-regular graph Y , the characteristic polynomial of the subgraph obtained by deleting v and all incident edges is independent of the chosen vertex. See the paper by Godsil and McKay [56].

The new result of this section is that walk-regularity of a base graph Y is inherited by any sufficiently nice cover.

8.2.1 Theorem. *If Y is a walk-regular graph and X is a cyclic 2-ev cover of Y then X is walk-regular.*

Proof. Let S denote the arc matrix of X , with values in the cyclic group $C_r = \{0, \dots, r-1\}$ written additively. By assumption, the representation of this arc function is the regular representation of C_r with irreducible constituents $\rho_0, \dots, \rho_{r-1}$. Set $S_i := S^{\rho_i(f)}$, and for $g \in C_r$ let A_g denote the matrix

$$(A_g)_{u,v} = \begin{cases} 1 & \text{if } f(u, v) = g \\ 0 & \text{otherwise.} \end{cases}$$

Thus we may write

$$S_1 = \sum_{i=0}^{r-1} \omega^i A_i$$

where ω is a primitive r th root of unity. The matrices S_j for $j \in \{2, \dots, r-1\}$ are given by

$$S_j = \sum_{i=0}^{r-1} \omega^{j \cdot i} A_i.$$

Let \circ denote the entry-wise matrix product, and let ℓ be a non-negative integer.

Claim.

$$A(X)^\ell \circ I_{r|Y|} = \left(\frac{1}{r} \left(\sum_{j=0}^{r-1} S_j^\ell \right) \otimes I_r \right) \circ I_{r|Y|}.$$

Once we prove the claim, the result follows quickly: For any $j \neq 0$, Theorem 8.1.1 implies that S_j^2 is a polynomial in $\{I, S_j\}$, so any power of S_j is a polynomial in $\{I, S_j\}$ and must have constant diagonal. Since Y is walk-regular, $S_0 = A(Y)^\ell$ has constant diagonal as well, so our claim implies the diagonal of $A(X)^\ell$ is the diagonal of a sum of matrices with constant diagonal and we are done.

Proof of Claim. By definition we have

$$A(X) = \sum_{i=0}^{r-1} A_i \otimes \rho(\omega^i).$$

So $A(X)^\ell$ is a sum of products of the form

$$(A_{m_1} A_{m_2} \dots A_{m_\ell}) \otimes \rho(\omega^{\sum_k m_k})$$

where $m_i \in \{0, \dots, r-1\}$. Since ρ is regular, the only summands that have non-zero diagonal entries are those for which the right-hand tensor factor is the identity. These are the terms where $\sum m_k = 0 \pmod{r}$. Let \mathcal{M} be the set of ℓ -tuples (m_1, \dots, m_ℓ) of elements of C_r , and let \mathcal{M}_0 the subset of \mathcal{M} for which $\sum m_k = 0 \pmod{r}$. The previous remark shows that $A(X)^\ell$ has the same diagonal as

$$\sum_{M \in \mathcal{M}_0} \left(\prod_{m_i \in M} A_{m_i} \right) \otimes I_r = \left(\sum_{M \in \mathcal{M}_0} \prod_{m_i \in M} A_{m_i} \right) \otimes I_r. \quad (8.1)$$

On the other hand, notice that

$$S_j^\ell = \left(\sum_{i=0}^{r-1} \omega^{j \cdot i} A_i \right)^\ell = \sum_{M \in \mathcal{M}} \prod_{m_i \in M} \omega^{j \cdot m_i} A_{m_i}.$$

Now, summing over all such S_j we have

$$\sum_{j=0}^{r-1} S_j^\ell = \sum_{j=0}^{r-1} \sum_{M \in \mathcal{M}} \prod_{m_i \in M} \omega^{j \cdot \sum m_i} A_{m_i} = \sum_{M \in \mathcal{M}} \sum_{j=0}^{r-1} \prod_{m_i \in M} \omega^{j \cdot \sum m_i} A_{m_i}.$$

For any $M \in \mathcal{M} \setminus \mathcal{M}_0$ the ω^j term is independent of the product, hence

$$\sum_{j=0}^{r-1} \prod_{m_i \in M} \omega^{j \cdot \sum m_i} A_{m_i} = \sum_{j=0}^{r-1} \omega^j \prod_{m_i \in M} \omega^{\sum m_i} A_{m_i} = 0.$$

And so

$$\begin{aligned} \sum_{j=0}^{r-1} S_j^\ell &= \sum_{M \in \mathcal{M}_0} \sum_{j=0}^{r-1} \prod_{m_i \in M} \omega^{j \cdot \sum m_i} A_{m_i} \\ &= \sum_{M \in \mathcal{M}_0} \sum_{j=0}^{r-1} \prod_{m_i \in M} A_{m_i} \\ &= r \sum_{M \in \mathcal{M}_0} \prod_{m_i \in M} A_{m_i}. \end{aligned}$$

Together with Equation 8.1 this proves the claim. \square

It is easy to extend this argument to covers with regular actions by any abelian group.

8.2.2 Corollary. *If Y is a walk-regular graph and X is an abelian 2-ev cover with arc matrix S then X is walk-regular.*

Proof. The regular representation of the abelian group $G = H_1 \times H_2$ is the tensor product of the regular representations of H_1 and H_2 . Hence for each $g \in G$ we have $\rho(g) = \rho_{H_1}(h_1) \otimes \rho_{H_2}(h_2)$, and $\rho(g)$ still has zero diagonal unless $g = \text{id}$. So the only terms in the expansion of $(\sum_{g \in G} A_g \otimes \rho(g))^\ell$ that contribute to the diagonal are those whose first tensor factor has subscripts whose sum is zero mod $|G|$. The irreducible representations of ρ are still of degree one and the argument proceeds as above. \square

8.3 Distance-regular 2-ev Covers

Strengthening the assumption from the previous section we determine when a cyclic 2-ev cover of a connected strongly-regular graph is distance-regular.

A regular graph is **amply-regular** if any pair of adjacent vertices have a common neighbors and any pair of vertices at distance 2 have c common neighbors. We begin with a key lemma showing that the 2-ev condition forces an arc function on an amply-regular graph to take a very particular form. In practice, we will only be using base graphs which are complete or strongly-regular.

8.3.1 Lemma. *Let Y be a connected amply-regular graph with parameters a and c . Suppose X is a cyclic 2-ev cover of Y with arc matrix S . Let ϕ be a non-trivial irreducible representation of the cyclic group of order r , and let $M := S^\phi$ with minimal polynomial $x^2 - \lambda x - \mu$.*

Let $\mathbf{1}$ and $\mathbf{0}$ denote vectors of all ones and zeros respectively and assume that the arc function is normalized so that M is of the form

$$M = \begin{pmatrix} 0 & \mathbf{1} & \mathbf{0} & \dots \\ \mathbf{1}^T & B_1 & N_1 & \\ \mathbf{0}^T & N_2 & B_2 & \\ \vdots & & & \ddots \end{pmatrix}.$$

There exist constants

$$t = \frac{a - \lambda}{r}, \quad s = \frac{c}{r}$$

so that

- *For each $i \in \{1, \dots, r-1\}$, each column of the submatrix B_1 contains exactly t entries equal to ω^i .*
- *For each $j \in \{0, \dots, r-1\}$ each column of the submatrix N_1 contains exactly s entries equal to ω^j .*

Proof. For a submatrix M' of M , let $C_m(M')$ denote the sum of the entries of column m of M' . Matrix multiplication shows that the first row of M^2 is

$$(k \quad C_1(B_1) \quad \dots \quad C_k(B_1) \quad C_1(N_1) \quad \dots \quad C_\ell(N_1) \quad 0 \quad \dots \quad 0).$$

On the other hand, $M^2 = \lambda M + \mu I$ so the first row of M^2 is

$$(\mu \quad \lambda \quad \dots \quad \lambda \quad 0 \quad \dots \quad 0).$$

We have three immediate consequences.

- $\mu = k$.
- For each $1 \leq m \leq k$, $C_m(B_1) = \lambda$.
- For each $1 \leq n \leq \ell$, $C_n(N_1) = 0$.

For some fixed column m' of B_1 , let t_i denote the number of occurrences of ω^i in that column. We may write $C'_{m'}(B_1) = \sum_{i=0}^{r-1} t_i \omega^i$. Note that M is Hermitian and that λ is (minus) the sum of the two distinct eigenvalues of M , hence λ is real. So the above expression for λ must be symmetric under any permutation of the r th roots of unity which fixes the rationals. It follows that $t_1 = t_2 = \dots = t_{r-1}$. Denote this common value by t . We have

$$\lambda = C_{m'}(B_1) = t_0 + t \left(\sum_{i=1}^{r-1} \omega^i \right) = t_0 - t.$$

Since Y is amply-regular, we have $\sum_{i=0}^{r-1} t_i = a$ which we may now write as

$$t_0 + t(r-1) = a.$$

Combining our equations we obtain

$$t = \frac{a - \lambda}{r}.$$

Noting that t is independent of the chosen column m' proves the first claim.

For some fixed column n' of N_1 , let s_j denote the number of occurrences of ω^j in that column. We proceed as above with the equations

$$C_{n'}(N_1) = \sum_{j=0}^{r-1} s_j \omega^j = 0, \quad \sum_{j=0}^{r-1} s_j = c$$

and find that

$$s := s_0 = s_1 = \dots = s_{r-1} = \frac{c}{r}.$$

is independent of n' . This proves the second claim. □

Before turning to covers of strongly-regular graphs, we use this lemma to generalize the fact that a Seidel matrix with two distinct eigenvalues determines a $(n, 2, c)$ -drackn.

8.3.2 Theorem. *If Y is the complete graph on n vertices and X is a connected cyclic 2ev-cover of Y , then X is a drackn.*

Proof. First we consider the arc matrix M which defines X and apply Lemma 8.3.1. We carry over the notation from Lemma 8.3.1, in particular the parameter t . Note that $t > 0$, since $t = 0$ implies $Y^{\rho(f)}$ is the disjoint union of r copies of K_n .

Label the vertices of Y as v_0, \dots, v_{n-1} , and let γ be the covering map from $Y^{\rho(f)}$ to Y . For each fiber $\gamma^{-1}(v_i)$ label its vertices as $(v_i, 0), \dots, (v_i, r-1)$. Let v_0 be the vertex of Y indexing the first row and column of S , and choose (arbitrarily) some vertex of $\gamma^{-1}(v_0)$ to be $(v_0, 0)$. We consider the distance partition of $Y^{\rho(f)}$ with respect to $(v_0, 0)$.

$Y^{\rho(f)}$ has at most 4 distinct eigenvalues: $n-1$, -1 , and the eigenvalues of S , hence it has diameter at most 3. Any two vertices of $Y^{\rho(f)}$ from the same fiber are not adjacent and do not have any common neighbors, hence the diameter of Y^f is exactly 3. The assumption that S is normalized along the first row and column is equivalent to the claim that

$$\Gamma_1(v_0, 0) = \{(v_i, 0) : i \in \{1, \dots, n-1\}\}.$$

For any $j \neq 0$ and (v_j, q) , the result of Lemma 8.3.1 and the fact that $t > 0$ imply that the j th column of S contains some entry $S_{i,j} = \omega^q, i \neq 0$. Hence $(v_j, q) \sim (v_i, 0)$ and the distance between $(v_0, 0)$ and (v_j, q) is at most 2, and so exactly 2. It follows that each vertex that is neither a neighbor of $(v_0, 0)$ nor in $\gamma^{-1}(v_0)$ is at distance two from $(v_0, 0)$. Moreover for any such vertex (v_j, q) there are exactly t entries of the j th column equal to ω^q . So $(v_0, 0)$ and (v_j, q) have exactly t common neighbors. It now follows from Lemma 6.4.2 that X is an (n, r, t) -drackn. \square

We come to the main theorem of this section. We assume Y is strongly-regular and characterize the cyclic distance-regular 2-ev covers of Y . The salient condition ends up being that the new eigenvalues of the cover sum to the number of triangles on an edge in Y .

8.3.3 Theorem. *Let Y be a connected strongly-regular graph with parameters $a := a_1$ and $c = c_2$ and X a cyclic 2-ev cover of Y with new eigenvalues θ and τ . X is distance-regular if and only if $a = \theta + \tau$.*

Proof. Let M be the arc matrix of the cover. By Lemma 8.1.1 the minimal polynomial of M is

$$x^2 - \lambda x - \mu = (x - \theta)(x - \tau).$$

Suppose $a = \lambda$. Applying Lemma 8.3.1 we find that $t = 0$ and B_1 is a 0-1 matrix. We construct the distance partition of $Y^{\rho(f)}$ with respect to $(v_0, 0)$ and find that the neighborhood of $(v_0, 0)$ induces a subgraph isomorphic to $Y[B_1]$. Call this subgraph H_0 , and note that, since the arc function is trivial on B_1 , there are $r - 1$ other subgraphs H_1, \dots, H_{r-1} contained in $Y^{\rho(f)}$, each of which is the neighborhood of some (v_0, j) for $j \in \{1, \dots, r - 1\}$. Moreover, there are no edges between any of the distinct H_i since the arc function is trivial on B_1 .

Since Y is connected, we have $c > 0$. By Lemma 8.3.1, each column of N_1 contains $\frac{c}{r}$ entries equal to each ω^i . Hence each vertex (v_j, i) for $v_j \in B_2$ is adjacent to $\frac{c}{r}$ vertices of each H_i . This shows that $\Gamma_2(v_0, 0)$ is the union of fibers over vertices indexing B_2 , and $\Gamma_3(v_0, 0)$ is the union of H_i for $i > 0$. Finally, $\Gamma_4(v_0, 0)$ is the remaining $r - 1$ vertices of the v_0 fiber. It follows that the distance partition is equitable, with intersection array

$$\{k, k - a - 1, \frac{c(r - 1)}{r}, 1; 1, \frac{c}{r}, k - a - 1, k\}.$$

Now suppose $a \neq \lambda$. So $t > 0$ and each column of B_1 other than the first contains each ω^i for $i \in \{0, \dots, r - 1\}$. Since Y is connected, $c > 0$, and each column of N_1 also contains at least one entry equal to each ω^i .

As in the proof of Theorem 8.3.2, this means that when we construct the distance partition of $Y^{\rho(f)}$ with respect to $(v_0, 0)$, we have

$$\Gamma_1(v_0, 0) = \{(v_j, 0) : j \in \{1, \dots, n - 1\}\}$$

and

$$\Gamma_3(v_0, 0) \supseteq \{(v_0, j) : j \in \{1, \dots, r - 1\}\}.$$

There are two types of vertices remaining. Those of the form (v_α, i) for v_α a neighbor of v_0 and those of the form (v_β, i) for v_β at distance 2 from v_0 . By Lemma 8.3.1, each v_α is adjacent to $t > 0$ neighbors of v_0 for which the edge joining them is assigned the value ω^j . Since this holds for all $j \in \{1, \dots, r - 1\}$ each (v_α, i) for $i > 0$ is at distance 2 from $(v_0, 0)$. Similarly, by Lemma 8.3.1 all vertices of the fiber $\gamma^{-1}(v_\beta)$ are at distance two from $(v_0, 0)$, and none of these vertices are incident with any vertex of $\gamma^{-1}(v_0)$, so some vertices in Γ_2 have neighbors in Γ_3 while others do not, and X is not distance-regular. \square

We note that Paulsen and Bodmann used similar techniques as ours to characterize sets of equiangular lines arising from C_3 -valued arc matrices, [14]. Bodmann and Elwood then extended this to a characterization for all cyclic arc matrices [13]. We only learned of this work several years after the publication of our paper [55]. It is possible that one can show

their results are equivalent to ours, using the translation from lines to covers discussed in Section 7.4. But we do not know if this translation would be easier than proving either result directly.

As a special case of Theorem 8.3.3, we show that every cyclic 2-ev cover of a complete bipartite graph is distance-regular.

8.3.4 Corollary. *Suppose Y is complete bipartite and X is a cyclic r -fold 2-ev cover of Y . Then $Y = K_{n,n}$ with $r \mid n$, and X is bipartite and distance-regular of diameter 4.*

Proof. It follows from Lemma 8.1.1 that Y is regular. Hence Y is strongly-regular with parameters $(2n, n, 0, n)$ for some n . Normalizing as in Lemma 8.3.1 and noting that $a = 0$ we have

$$S = \begin{pmatrix} 0 & \mathbf{1} & \mathbf{0} \\ \mathbf{1}^T & \mathbf{0}_{n \times n} & N_1 \\ \mathbf{0}^T & N_1^T & B_2 \end{pmatrix}.$$

Hence $\lambda = 0$ as well, and X is distance-regular. Moreover, each column of N_1 contains $\frac{c}{r} = \frac{n}{r}$ entries equal to ω^i , hence $r \mid n$. By Theorem 2.1.5 a graph is bipartite if and only if its adjacency spectrum is symmetric around zero. Y is bipartite, and M has spectrum symmetric around zero. Hence X has spectrum symmetric around zero and is bipartite. \square

8.4 Examples of 2-ev Covers

We conclude with a discussion of some examples of 2-ev covers.

First there are the antipodal distance-regular covers of distance-regular graphs. A theorem due to Gardiner shows that there are only a few possible diameters for which examples of this kind can exist.

8.4.1 Theorem. [46] *If Y is a distance-regular graph of diameter d and X is an antipodal distance-regular cover of Y , then X has diameter $2d$ or $2d + 1$.* \square

From this theorem and the fact that a distance-regular graph of diameter d has $d + 1$ distinct eigenvalues, we see that the only antipodal distance-regular examples of 2-ev covers have diameter 3 or 4. The diameter 3 case is exactly the drackns, about which much is said elsewhere in this thesis.

In the diameter 4 case we have the several infinite families coming from covers of the complete bipartite graphs:

- The 2-fold covers of $K_{n,n}$ derived from Hadamard matrices and known as the Hadamard graphs, see [18, Section 1.9].
- r -fold covers of $K_{q,q}$ for q a prime power and certain values of r . In particular, when $r = q$ the graphs with vertex set $GF(q)^2 \times GF(q)^2$ with (y, x) adjacent to (a, b) if $y = ax + b$ are distance-regular 2-ev covers, see Gardiner [47].

Any other diameter 4 example must be a distance-regular cover of strongly-regular graph, and there are only twelve examples known. They were surveyed by Jurišić in his 1995 Ph.D. thesis [86], and no further examples have arisen since then. In brief, they are as follows:

- Index 2 covers of the folded 5-cube, the folded and halved 8-cube, and the Johnson graph $J(8, 4)$. See [18, Section 4.2]
- A cover of index 3 derived from the ternary Golay code. See [18, Section 11.3].
- Three covers of index 3 found by Soicher in [125].
- Five covers of index 3 characterized by a group theoretic condition studied by Smith [124]. See [18] Section 13.2 for details.

When we turn to more general examples of 2-ev covers we immediately face the opposite problem: the examples are plentiful and poorly documented. This is because examples of index 2 are equivalent to signed graphs with two eigenvalues. These signed graphs have attracted a great deal of attention in the past decade, for instance, see [50], [107], [8], [129], [108]. Moreover, the natural generalization from signed graphs to “complex unit gain graphs” with two eigenvalues is gaining popularity, [109], [93], [143]. Any two-eigenvalue gain-graph whose gains are p -th roots of unity is equivalent to cyclic 2-ev covers of index p .

We compromise with the modest list of examples of 2-ev covers where the base graph is walk-regular but the cover is not distance-regular. In all such examples that we know of, the covers are cyclic and Theorem 8.2.1 applies.

1. **The 4-cycle-free covers of the hypercubes.** See Chapter 4 for an extensive discussion.

2. **The O’Keefe–Wong graph.** O’Keefe and Wong found the smallest graph of girth 6 and valency 7 via computer search [105]. It showed that this graph is a 3-fold cover of the point-plane incidence graph of $PG(3, 2)$. [82]. In fact this is a 2-ev cover with new eigenvalues $\pm\sqrt{7}$. See Section 4.5 for several connections between this and the previous example.
3. **Positive roots in A_n .** Ramezani gave the following construction of 2-ev covers of the line graphs of complete graphs [107]. Denote the vertices of $L(K_n)$ by *ordered* pairs (a, b) with $a < b$. The arc function

$$f((a, b), (c, d)) = \begin{cases} 1 & \text{if } a = c \text{ or } b = d \\ -1 & \text{if } a = d \text{ or } b = c \end{cases}$$

determines a 2-ev cover with new eigenvalues $n - 2$ and -2 . Since adjacent vertices in $L(K_n)$ have $n - 2$ common neighbors Theorem 8.3.3 implies that, for $n \geq 4$, these covers are not distance-regular.

Wissing and van Dam point out that this construction is equivalent to taking the Gram matrix of the sets of vectors $e_i - e_j$ for $i < j$, in \mathbb{R}^n [143].

4. **Positive roots in D_n .** Here we have a new example, although it is very much inspired by the previous one. Take the vectors $e_i \pm e_j$ for $i < j$ in \mathbb{R}^n . The Gram matrix of these vectors is \mathbb{Z}_2 -arc matrix with two eigenvalues

$$(2n - 4)^{(n)}, (-2)^{(n^2 - 2n)}$$

and so defines a 2-ev cover. The base graph is not distance-regular, but it is walk-regular, so the covers are as well.

We conclude with an interesting collection of non-examples due to Wissing and van Dam. Motivated in part by their interpretation of the third example in our list, Wissing and van Dam studied arc matrices of 2-ev covers coming from the Gram matrices of systems of lines with pairwise angles in the set $\{0, \alpha\}$ [143]. The first example they consider gives an arc matrix on $GQ(2, 2)$ which determines the Foster graph, the smallest of the five antipodal 3-covers discussed in [18, Section 13.2].

Their next example comes from the Witting polytope, so named by Coxeter, [29]. This collection of 240 points in \mathbb{C}^4 are situated in forty 1-dimensional subspaces. The Gram matrix of lines spanning these spaces gives the arc matrix of a cyclic 6-fold cover of the complement of the symplectic generalized quadrangle $GQ(3, 3)$. Unfortunately, we cannot

conclude that these are all the new eigenvalues since the real irreducible representation of C_6 may induce two further eigenvalues.

Wissing and van Dam also consider two additional 2-ev arc matrices distilled from sets of lines in complex space. One yields a cover of the point graph of the generalized quadrangle $GQ(4, 2)$, which cannot be distance-regular. The other yields the arc matrix of the third of five antipodal 3-covers from [18, Section 13.2].

8.5 Future Work

1. It is possible that Theorem 8.2.1 holds for all 2-ev covers, not just abelian ones, but new proof techniques would certainly be needed.
2. We expect that the hypothesis of Theorem 8.3.3 can be weakened to abelian 2-ev covers. Possibly it can be weakened even further to all 2-ev covers, but again, this would require new techniques.
3. In order for either of the previous two directions to be meaningful, we need examples of non-abelian 2-ev covers. The only examples currently known are those coming from non-abelian drackns. In this case distance-regularity is already a given. We expect that there are more general constructions of non-abelian 2-ev covers, and hope that researchers will start to search for them when the abelian setting is more fully understood.

References

- [1] Noga Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986. [93](#)
- [2] Noga Alon and Kai Zheng. Unitary signings and induced subgraphs of Cayley graphs of \mathbb{Z}_2^n . *arXiv preprint arXiv:2003.04926*, 2020. [34](#)
- [3] Kenneth Appel and Wolfgang Haken. *Every planar map is four colorable*, volume 98. American Mathematical Soc., 1989. [22](#)
- [4] Dan Archdeacon, Pavol Gvozďjak, and Jozef Širáň. Constructing and forbidding automorphisms in lifted maps. *Mathematica Slovaca*, 47(2):113–129, 1997. [3](#)
- [5] József Balogh, Felix Christian Clemen, and Bernard Lidický. The spectrum of triangle-free graphs. *arXiv preprint arXiv:2204.00093*, 2022. [93](#)
- [6] Eiichi Bannai, Etsuko Bannai, Tatsuro Ito, and Rie Tanaka. *Algebraic Combinatorics*. Berlin, Boston: De Gruyter, 2021. [10](#)
- [7] Eiichi Bannai and Tatsuro Ito. On finite moore graphs. *J. Fac. Sci. Tokyo Univ*, 20(191-208):80, 1973. [5](#)
- [8] Francesco Belardo, Sebastian M Cioabă, Jack H Koolen, and Jianfeng Wang. Open problems in the spectral theory of signed graphs. *arXiv preprint arXiv:1907.04349*, 2019. [16](#), [105](#)
- [9] Norman Biggs. Intersection matrices for linear graphs. In D.J.A Welsh, editor, *Combinatorial Mathematics and its applications*, pages 15–23. Acad. Pres, London, 1969. [55](#)
- [10] Norman Biggs. *Algebraic Graph Theory, 2nd Edition*. Cambridge Mathematical Library. Cambridge University Press, 1974. [3](#)

- [11] Norman Biggs and Tatsuro Ito. Graphs with even girth and small excess. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 88, pages 1–10. Cambridge University Press, 1980. [6](#), [38](#)
- [12] Norman L. Biggs and Arthur T. White. *Permutation groups and combinatorial structures*, volume 33. Cambridge University Press, 1979. [27](#)
- [13] Bernhard Bodmann and Helen Elwood. Complex equiangular parseval frames and seidel matrices containing p th roots of unity. *Proceedings of the American Mathematical Society*, 138(12):4387–4404, 2010. [103](#)
- [14] Bernhard G. Bodmann, Vern I. Paulsen, and Mark Tomforde. Equiangular tight frames from complex Seidel matrices containing cube roots of unity. *Linear Algebra and its Applications*, 430(1):396–417, 2009. [103](#)
- [15] H.R. Brahana. Regular maps and their groups. *American Journal of Mathematics*, 49(2):268–284, 1927. [21](#)
- [16] Stephan Brandt. The local density of triangle-free graphs. *Discrete mathematics*, 183(1-3):17–25, 1998. [93](#)
- [17] Andries Brouwer. Distance regular graphs of diameter 3 and strongly regular graphs. *Discrete Mathematics*, 49(1):101–103, 1984. [62](#)
- [18] Andries Brouwer, Arjeh Cohen, and Arnold Neumaier. *Distance-Regular Graphs*. Springer-Verlag Berlin Heidelberg, 1989. [10](#), [32](#), [56](#), [57](#), [59](#), [60](#), [61](#), [67](#), [73](#), [83](#), [105](#), [106](#), [107](#)
- [19] Andries Brouwer and H. Van Maldeghem. *Strongly Regular Graphs*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2022. [59](#)
- [20] Kenneth S Brown. *Cohomology of Groups*, volume 87. Springer Science & Business Media, 2012. [42](#)
- [21] William Burnside. *Theory of Groups of Finite Order*. University Press, 1911. [43](#)
- [22] F. C. Bussemaker, R. A. Mathon, and J. J. Seidel. Tables of two-graphs. In Sid-dani Bhaskara Rao, editor, *Combinatorics and Graph Theory*, pages 70–112, Berlin, Heidelberg, 1981. Springer Berlin Heidelberg. [76](#)
- [23] P.J. Cameron and J.H. Van Lint. *Designs, Graphs, Codes and their Links*, volume 3. Cambridge University Press Cambridge, 1991. [68](#)

- [24] Johannes Carmesin. Embedding simply connected 2-complexes in 3-space–i. a Kuratowski-type characterisation. *arXiv preprint arXiv:1709.04642*, 2017. [22](#)
- [25] Arjeh Cohen and Jacques Tits. On generalized hexagons and a near octagon whose lines have three points. *European Journal of Combinatorics*, **6**(1):13–27, 1985. [30](#), [32](#)
- [26] Marston Conder and Peter Dobcsányi. Determination of all regular maps of small genus. *Journal of Combinatorial Theory, Series B*, 81(2):224–242, 2001. [27](#)
- [27] Gabriel Coutinho, Chris Godsil, Mirhamed Shirazi, and Harmony Zhan. Equiangular lines and covers of the complete graph. *Linear Algebra and its Applications*, **488**:264 – 283, 2016. [4](#), [73](#), [78](#), [79](#)
- [28] H.S.M. Coxeter. Self-dual configurations and regular graphs. *Bulletin of the American Mathematical Society*, 56(5):413–455, 1950. [32](#)
- [29] H.S.M. Coxeter. *Regular Complex Polytopes*. Cambridge University Press, 1975. [106](#)
- [30] H.S.M. Coxeter and W.O.J. Moser. *Generators and relations for discrete groups*, volume 14. Springer Science & Business Media, 2013. [24](#), [27](#)
- [31] R. Mark Damerell. On Moore graphs. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 74, pages 227–236. Cambridge University Press, 1973. [5](#)
- [32] Dom de Caen, Rudolph Mathon, and G. Eric Moorhouse. A family of antipodal distance-regular graphs related to the classical preparata codes. *Journal of Algebraic Combinatorics*, 4:317–327, 1995. [69](#)
- [33] Henri Paul de Saint-Gervais. *Uniformization of Riemann Surfaces*. EMS Press, January 2016. [27](#)
- [34] Max Dehn and Poul Heegaard. *Analysis Situs*. Springer, 1910. [21](#)
- [35] Philippe Delsarte. An algebraic approach to the association schemes of coding theory. *Philips Res. Rep. Suppl.*, 10:vi+–97, 1973. [10](#)
- [36] Reinhard Diestel. *Graph Theory. 5th ed.*, volume GTM 173. Springer, 2016. [22](#)
- [37] Jia-Li Du, Yan-Quan Feng, and Pablo Spiga. A conjecture on bipartite graphical regular representations. *Discrete Mathematics*, 343(8):111913, 2020. [51](#)

- [38] Geoffrey Exoo. Voltage graphs, group presentations and cages. *the electronic journal of combinatorics*, 11(1):N2, 2004. [5](#)
- [39] Geoffrey Exoo and Robert Jajcay. Dynamic cage survey. *The electronic journal of combinatorics*, pages DS16–Jul, 2012. [5](#)
- [40] Walter Feit and Graham Higman. The nonexistence of certain generalized polygons. *Journal of algebra*, 1(2):114–131, 1964. [5](#)
- [41] Matthew Fickus, John Jasper, Dustin G. Mixon, and Jesse Peterson. Tremain equiangular tight frames. *Journal of Combinatorial Theory, Series A*, 153:54–66, 2018. [4](#), [60](#), [73](#), [82](#)
- [42] Matthew Fickus, John Jasper, Dustin G Mixon, Jesse D Peterson, and Cody E Watson. Polyphase equiangular tight frames and abelian generalized quadrangles. *Applied and Computational Harmonic Analysis*, 47(3):628–661, 2019. [63](#)
- [43] Matthew Fickus, Dustin G. Mixon, and Janet C. Tremain. Steiner equiangular tight frames. *Linear algebra and its applications*, 436(5):1014–1027, 2012. [73](#), [80](#), [81](#)
- [44] George K Francis. *A Topological Picturebook*, volume 2. Springer. [24](#)
- [45] Christopher A. Fuchs, Michael C. Hoang, and Blake C. Stacey. The sic question: History and state of play. *Axioms*, 6(3):21, 2017. [78](#)
- [46] Anthony Gardiner. Antipodal covering graphs. *Journal of Combinatorial Theory, Series B*, 16(3):255–273, 1974. [3](#), [63](#), [64](#), [67](#), [68](#), [104](#)
- [47] Anthony Gardiner. Imprimitve distance-regular graphs and projective planes. *Journal of Combinatorial Theory, Series B*, 16(3):274–281, 1974. [3](#), [105](#)
- [48] Anthony Gardiner. Antipodal graphs of diameter three. *Linear Algebra and its Applications*, 46:215–219, 1982. [3](#), [69](#), [71](#)
- [49] Anthony Gardiner, Chris Godsil, Andrew Hensel, and Gordon Royle. Second neighbourhoods of strongly regular graphs. *Discrete mathematics*, 103(2):161–170, 1992. [72](#)
- [50] E. Ghasemian and G.H. Fath-Tabar. On signed graphs with two distinct eigenvalues. *Filomat*, 31(20):6393–6400, 2017. [105](#)

- [51] Chris Godsil. On the full automorphism group of a graph. *Combinatorica*, 1:243–256, 1981. [51](#)
- [52] Chris Godsil. *Algebraic Combinatorics*, volume 6. CRC Press, 1993. [9](#), [11](#)
- [53] Chris Godsil. Problems in algebraic combinatorics. *the electronic journal of combinatorics*, pages F1–F1, 1995. [68](#)
- [54] Chris Godsil and Andrew Hensel. Distance regular covers of the complete graph. *J. Comb. Theory, Ser. B*, **56**(2):205–238, 1992. [3](#), [16](#), [17](#), [19](#), [66](#), [67](#), [68](#)
- [55] Chris Godsil, Maxwell Levit, and Olha Silina. Graph covers with two new eigenvalues. *European Journal of Combinatorics*, 93:103280, 2021. [33](#), [34](#), [96](#), [103](#)
- [56] Chris Godsil and Brendan McKay. Feasibility conditions for the existence of walk-regular graphs. *Linear Algebra and its Applications*, **30**, 1980. [97](#)
- [57] Chris Godsil and Gordon F Royle. *Algebraic graph theory*, volume 207. Springer Science & Business Media, 2001. [9](#), [33](#), [62](#), [75](#), [77](#)
- [58] Jean-Marie Goethals and J.J. Seidel. The regular two-graph on 276 vertices. In *Geometry and Combinatorics*, pages 177–192. Elsevier, 1991. [74](#)
- [59] Georges Gonthier et al. Formal proof—the four-color theorem. *Notices of the AMS*, 55(11):1382–1393, 2008. [23](#)
- [60] Daniel Gorenstein. *Finite Groups*. AMS Chelsea Publishing Series. American Mathematical Society, 2007. [43](#)
- [61] C. Gotsman and N. Linial. The equivalence of two problems on the cube. *Journal of Combinatorial Theory, Series A*, **61**(1):142 – 146, 1992. [33](#)
- [62] Oleg Gritsenko. On strongly regular graph with parameters (65; 32; 15; 16). *arXiv preprint arXiv:2102.05432*, 2021. [75](#)
- [63] Jonathan L. Gross. Voltage graphs. *Discrete mathematics*, 9(3):239–246, 1974. [23](#)
- [64] Jonathan L. Gross and Thomas W. Tucker. Generating all graph coverings by permutation voltage assignments. *Discrete Mathematics*, 18(3):273–283, 1977. [3](#), [24](#)
- [65] Jonathan L. Gross and Thomas W. Tucker. *Topological graph theory*. Courier Corporation, 2001. [23](#), [24](#)

- [66] Yingfei Gu and Xiao-Liang Qi. Majorana fermions and the sensitivity conjecture. *arXiv preprint arXiv:1908.06322*, 2019. [34](#)
- [67] Pavol Gvozďjak and Jozef Sirán. Regular maps from voltage assignments. *Contemporary Mathematics*, 147:441–441, 1993. [3](#)
- [68] Willem H. Haemers. Hoffman’s ratio bound. *Linear Algebra and its Applications*, 617:215–219, 2021. [93](#)
- [69] Phillip Hall. The classification of prime-power groups. *Journal für die reine und angewandte Mathematik*, 181:130–141, 1940. [46](#)
- [70] Akihide Hanaki and Izumi Miyamoto. Classification of association schemes of small order, online catalogue. URL <http://math.shinshu-u.ac.jp/~hanaki/as>. [84](#), [86](#)
- [71] Akihide Hanaki and Izumi Miyamoto. Classification of association schemes of small order. *Discrete Mathematics*, 264(1-3):75–80, 2003. [84](#)
- [72] Frank Harary. On the notion of balance of a signed graph. *Michigan Mathematical Journal*, 2(2):143–146, 1953. [16](#)
- [73] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2005. [38](#)
- [74] Percy John Heawood. Map color theorems. *Quant. J. Math.*, 24:332–338, 1890. [3](#), [22](#)
- [75] Andrew Hensel. *Antipodal distance regular graphs*. Master’s Thesis, University of Waterloo, 1989. [61](#)
- [76] Alan J. Hoffman and Robert R. Singleton. On Moore graphs with diameters 2 and 3. *IBM Journal of Research and Development*, 4(5):497–504, 1960. [5](#), [68](#)
- [77] Henk Hollmann. Association schemes. *Master’s Thesis, Eindhoven University of Technology*, 1982. [83](#)
- [78] Zhen-Mu Hong, Hong-Jian Lai, and Jianbing Liu. Induced subgraphs of product graphs and a generalization of Huang’s theorem. *Journal of Graph Theory*, 98(2):285–308, 2021. [34](#)
- [79] Roger A. Horn and Charles R. Johnson. *Matrix analysis*. Cambridge university press, 2012. [9](#)

- [80] Hao Huang. Induced subgraphs of hypercubes and a proof of the Sensitivity Conjecture. *Annals of Mathematics*, **190**:949–955, 2019. [33](#)
- [81] Adolf Hurwitz. Über algebraische Gebilde mit eindeutigen Transformationen in sich. *Mathematische Annalen*, 41(3):403–442, 1892. [27](#)
- [82] Tatsuro Ito. On a graph of O’Keefe and Wong. *Journal of Graph Theory*, 5(1):87–94, 1981. [6](#), [37](#), [106](#)
- [83] J. Iverson and D. Mixon. Doubly transitive lines I: Higman pairs and roux. *CoRR*, abs/1806.09037, 2018. [73](#)
- [84] Gordon James and Martin Liebeck. *Representations and Characters of Groups*. Cambridge University Press, 2 edition, 2001. [13](#)
- [85] Gareth A. Jones and David Singerman. Theory of maps on orientable surfaces. *Proceedings of the London Mathematical Society*, 3(2):273–307, 1978. [27](#)
- [86] Aleksandar Jurišić. *Antipodal Covers*. PhD thesis, University of Waterloo, 1995. [51](#), [84](#), [105](#)
- [87] Seligmann Kantor. *Ueber die Configurationen (3, 3) mit den Indices 8, 9 und ihren Zusammenhang mit den Curven dritter Ordnung*. 1882. [32](#)
- [88] Roman Karasev. Huang’s theorem and the exterior algebra. *arXiv preprint arXiv:1907.11175*, 2019. [34](#)
- [89] Mikhail Klin and Christian Pech. A new construction of antipodal distance regular covers of complete graphs through the use of godsil-hensel matrices. *Ars Mathematica Contemporanea*, 4(2):205–243, 2011. [60](#)
- [90] P.W.H. Lemmens and J.J. Seidel. Equiangular lines. *Journal of Algebra*, **24**(3):494 – 512, 1973. [74](#), [75](#), [76](#), [77](#), [78](#)
- [91] Maxwell Levit. Cayley graphs for extraspecial p-groups and a covering graph perspective on Huang’s theorem. *arXiv preprint arXiv:2010.14634*, 2020. [35](#), [39](#)
- [92] Silvio Levy, editor. *The Eightfold Way: The Beauty of Klein’s Quartic Curve*, volume 35. Cambridge university Press, 2001. [28](#)
- [93] Lu Lu, Jianfeng Wang, and Qiongxiang Huang. Complex unit gain graphs with exactly one positive eigenvalue. *Linear Algebra and its Applications*, 608:270–281, 2021. [105](#)

- [94] Aleksander Malnič, Roman Nedela, and Martin Škoviera. Lifting graph automorphisms by voltage assignments. *European Journal of Combinatorics*, 21(7):927–947, 2000. [3](#), [39](#)
- [95] Daniel V. Mathews. The sensitivity conjecture, induced subgraphs of cubes, and clifford algebras. *Journal of the European Mathematical Society*, 24(12):4201–4205, 2021. [34](#)
- [96] Rudolf Mathon. 3-class association schemes. In *Proceedings of the Conference on Algebraic Aspects of Combinatorics (Univ. Toronto, Toronto, Ont., 1975)*, pages 123–155, 1975. [83](#)
- [97] Rudolf Mathon. Lower bounds for Ramsey numbers and association schemes. *Journal of Combinatorial Theory, Series B*, 42(1):122–127, 1987. [28](#), [61](#)
- [98] Bojan Mohar. Graph minors and graphs on surfaces. *London Mathematical Society Lecture Note Series*, pages 145–163, 2001. [22](#)
- [99] Bojan Mohar, Tomaz Pisanski, Martin Skoviera, and Arthur White. The cartesian product of three triangles can be embedded into a surface of genus 7. *Discrete mathematics*, 56(1):87–89, 1985. [53](#)
- [100] James Munkres. *Topology. 2nd ed.* Prentice Hall, 2000. [21](#)
- [101] A.F. Möbius. Kann von zwei dreiseitigen Pyramiden eine jede in bezug auf die andere um- und eingeschrieben zugleich heißen? *Journal für die reine und angewandte Mathematik*, 1828(3):273–278, 1828. [32](#)
- [102] Roman Nedela and Martin Škoviera. Regular maps from voltage assignments and exponent groups. *European Journal of Combinatorics*, 18(7):807–823, 1997. [3](#), [38](#)
- [103] N. Nisan and M. Szegedy. On the degree of boolean functions as real polynomials. *Comput. Complex.*, 4:301–313, 1994. [32](#)
- [104] Noam Nisan. CREW PRAMs and decision trees. In *Proceedings of the twenty-first annual ACM symposium on Theory of computing*, pages 327–335, 1989. [33](#)
- [105] M. O’Keefe and P.K. Wong. The smallest graph of girth 6 and valency 7. *Journal of Graph theory*, 5(1):79–85, 1981. [6](#), [106](#)
- [106] Aaron Potechin and Hing Yin Tsang. A conjecture on induced subgraphs of cayley graphs. *arXiv preprint arXiv:2003.13166*, 2020. [34](#)

- [107] Farzaneh Ramezani. Constructing signed strongly regular graphs via star complement technique. *Mathematical Sciences*, 12:157–161, 2018. [105](#), [106](#)
- [108] Farzaneh Ramezani. Some regular signed graphs with only two distinct eigenvalues. *Linear and Multilinear Algebra*, 70(3):517–530, 2022. [105](#)
- [109] Nathan Reff. Spectral properties of complex unit gain graphs. *Linear algebra and its applications*, 436(9):3165–3176, 2012. [105](#)
- [110] Bernhard Riemann. *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*. Huth, 1851. [3](#), [23](#)
- [111] Gerhard Ringel and John W.T. Youngs. Solution of the Heawood map-coloring problem. *Proceedings of the National Academy of Sciences*, 60(2):438–445, 1968. [3](#)
- [112] Neil Robertson and Paul D. Seymour. Graph minors. I. excluding a forest. *Journal of Combinatorial Theory, Series B*, 35(1):39–61, 1983. [22](#)
- [113] Gordon Royle and Edward Spence. Classification of regular two-graphs on 36 and 38 vertices. *The Australasian Journal of Combinatorics [electronic only]*, 24, 01 2001. [76](#)
- [114] Gert Sabidussi. On a class of fixed-point-free graphs. *Proceedings of the American Mathematical Society*, 9(5):800–804, 1958. [12](#)
- [115] Issai Schur. Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare substitutionen. 1911. [37](#)
- [116] Andrew James Scott and Markus Grassl. Symmetric informationally complete positive-operator-valued measures: A new computer study. *Journal of Mathematical Physics*, 51(4):042203, 2010. [78](#)
- [117] J.J. Seidel. Strongly regular graphs with $(-1, 1, 0)$ adjacency matrix having eigenvalue 3. *Linear Algebra and its Applications*, 1:281–298, 1968. [74](#)
- [118] J.J. Seidel. A survey of two-graphs. *Geometry and Combinatorics*, pages 146–176, 1991. [74](#)
- [119] J.J. Seidel, D.G. Corneil, and R.A. Mathon. *Geometry and Combinatorics: Selected Works of J.J. Seidel*. Academic Press, 1991. [74](#)

- [120] J.J. Seidel and D.E. Taylor. Two-graphs, a second survey. In *Geometry and Combinatorics*, pages 231–254. Elsevier, 1991. [74](#)
- [121] Mirhamed Mirjalalieh Shirazi. *Equiangular Lines and Antipodal Covers*. PhD thesis, University of Waterloo, 2010. [87](#)
- [122] David Singerman and James Strudwick. The Farey maps modulo n . *Acta Mathematica Universitatis Comenianae*, 89(1), 2020. [28](#), [62](#)
- [123] D. H. Smith. Primitive and imprimitive graphs. *The Quarterly Journal of Mathematics*, **22**(4):551–557, 12 1971. [58](#), [59](#)
- [124] Stephen D. Smith. Nonassociative commutative algebras for triple covers of 3-transposition groups. *Michigan Mathematical Journal*, 24(3):273–287, 1977. [105](#)
- [125] Leonard H. Soicher. Three new distance-regular graphs. *European journal of combinatorics*, 14(5):501–505, 1993. [105](#)
- [126] Clelia Somma. An infinite family of perfect codes in antipodal graphs. *Rend. Mat. Appl.*(7), 3:465–474, 1983. [60](#)
- [127] Pablo Spiga. Cubic graphical regular representations of finite non-abelian simple groups. *Communications in Algebra*, 46(6):2440–2450, 2018. [51](#)
- [128] George Springer. *Introduction to Riemann surfaces*. American Mathematical Society, 2008. [27](#)
- [129] Zoran Stanić. Spectra of signed graphs with two eigenvalues. *Applied Mathematics and Computation*, 364:124627, 2020. [28](#), [105](#)
- [130] Margaret Rosemary Stanier. Regular coverings and parallel products of Farey maps. *Acta Mathematica Universitatis Comenianae*, 91(1):1–18, 2022. [62](#)
- [131] John Stillwell. *Classical Topology and Combinatorial Group Theory*, volume 72. Springer Science & Business Media, 2012. [21](#)
- [132] Thomas Strohmer and Robert W Heath Jr. Grassmannian frames with applications to coding and communication. *Applied and computational harmonic analysis*, 14(3):257–275, 2003. [78](#)
- [133] Terence Tao. Twisted convolution and the sensitivity conjecture, 2019. <https://terrytao.wordpress.com/2019/07/26/twisted-convolution-and-the-sensitivity-conjecture/>. [34](#)

- [134] Paul Terwilliger. The subconstituent algebra of an association scheme, (Part i). *Journal of Algebraic Combinatorics*, 1(4):363–388, 1992. [87](#)
- [135] Joseph A. Thas. Two infinite classes of perfect codes in metrically regular graphs. *Journal of Combinatorial Theory, Series B*, 23(2-3):236–238, 1977. [60](#)
- [136] Akaki Tikaradze. Induced subgraphs of powers of oriented cycles. *Linear and Multilinear Algebra*, 70(17):3301–3303, 2022. [34](#), [52](#)
- [137] Jacques Tits. On triality and certain groups that deduce from it. *Mathematical Publications of the IHÉS*, 2:13–60, 1959. [5](#)
- [138] Masato Tomiyama and Norio Yamazaki. The subconstituent algebra of a strongly regular graph. *Kyushu Journal of Mathematics*, 48(2):323–334, 1994. [87](#)
- [139] Lyudmila Tsiovkina. Arc-transitive antipodal distance-regular covers of complete graphs related to $su_3(q)$. *Discrete Mathematics*, 340(2):63–71, 2017. [60](#)
- [140] Edwin van Dam and Krystal Guo. Pseudo-geometric strongly regular graphs with a regular point. *arXiv preprint arXiv:2204.04755*, 2022. [63](#)
- [141] Shayne Waldron. On the construction of equiangular frames from graphs. *Linear Algebra and its applications*, 431(11):2228–2242, 2009. [74](#)
- [142] Lloyd Welch. Lower bounds on the maximum cross correlation of signals (corresp.). *IEEE Transactions on Information theory*, 20(3):397–399, 1974. [78](#)
- [143] Pepijn Wissing and Edwin R van Dam. Unit gain graphs with two distinct eigenvalues and systems of lines in complex space. *Discrete Mathematics*, 345(6):112827, 2022. [105](#), [106](#)
- [144] David Witte, Gail Letzter, and Joseph Gallian. On hamiltonian circuits in cartesian products of Cayley digraphs. *Discrete Mathematics*, **43**(2):297 – 307, 1983. [40](#)
- [145] Aaron Wootton, S Allen Broughton, and Jennifer Paulhus. *Automorphisms of Riemann Surfaces, Subgroups of Mapping Class Groups and Related Topics*, volume 776. American Mathematical Society, 2022. [27](#)
- [146] Thomas Zaslavsky. Signed graphs. *Discrete Applied Mathematics*, 4(1):47–74, 1982. [16](#)

- [147] Thomas Zaslavsky. A mathematical bibliography of signed and gain graphs and allied areas. *The Electronic Journal of Combinatorics*, 2012. [16](#)
- [148] Gerhard Zauner. Quantum designs: Foundations of a noncommutative design theory. *International Journal of Quantum Information*, 9(01):445–507, 2011. [78](#)

Index

(8, 3, 2)-drackn, [28](#), [60](#), [62](#), [82](#)

(9, 3, 3)-drackn, [50](#), [60](#), [62](#), [84](#)

G -arc function, [15](#)

p -group, [43](#)

2-cocycle, [41](#)

2-ev, [95](#)

abelian cover, [16](#)

adjacency matrix, [8](#)

amply-regular, [100](#)

antipodal, [58](#)

arc matrix, [15](#)

association scheme, [10](#)

asymmetric association scheme, [10](#)

base graph, [14](#)

bipartite double cover, [14](#), [60](#)

blocks, [80](#)

Boolean function, [32](#)

branch points, [24](#)

branched cover, [24](#)

cage, [5](#)

Cartesian product, [40](#)

Cayley graphs, [12](#)

central extension, [41](#)

characteristic matrix, [11](#)

coherent configuration, [10](#)

column partition, [87](#)

connection group, [19](#)

constituents, [13](#)

convolution, [34](#)

cover, [13](#), [23](#)

covering group, [18](#)

covering projection, [13](#)

cyclic cover, [16](#)

degree, [7](#)

distance graphs, [58](#)

distance matrix, [57](#)

distance partition, [55](#)

elementary abelian, [43](#)

embedding, [21](#)

equiangular lines, [75](#), [77](#)

equiangular tight frame, [75](#), [78](#)

equitable, [11](#)

extension, [41](#)

faces, [22](#)

facial cycles, [22](#)

feasible parameter set, [63](#), [66](#)

fibers, [14](#)

fission, [10](#)

fusion, [10](#)

generalized polygons, [5](#), [57](#)

generalized quadrangle, [62](#)

generously transitive, 10
 genus, 21
 girth, 5
 graphical regular representation, 51
 Grassmann graphs, 56

 halved representation, 80
 Hamming graphs, 40, 56
 Higman–Sims graph, 57
 Hurwitz surfaces, 28

 imprimitive, 58
 index, 14
 isoclinic, 46

 Johnson graphs, 56

 Moore graph, 5, 56, 68, 69, 89

 new eigenvalues, 95
 normalized 2-cocycle, 42
 normalized arc function, 18

 O’Keefe–Wong graph, 6, 37
 old eigenvalues, 95
 orbitals, 10

 parameters of a drackn, 63
 partition, 11
 point graph, 62
 points, 80
 primitive, 58
 principal idempotents, 8

 quotient graph, 11
 quotient matrix, 11

 r-fold cover, 14
 regular, 4
 regular maps, 27
 Riemann surface, 27
 Riemann surface automorphism, 28
 row partition, 87

 scheme developed, 82
 Seidel matrix, 74
 sensitivity, 32
 sensitivity conjecture, 33
 sheets, 23
 short exact sequence, 41
 Shrikhande graph, 56
 SIC-POVM, 78
 signed graph, 16
 spread, 62
 Steiner system, 80
 strongly-regular, 56
 sum, 12
 supports, 82
 surface, 21
 switching, 18
 switching class, 75
 switching equivalent, 18, 76
 symmetric association scheme, 10
 symplectic form, 60

 Thas–Somma drackns, 50, 60

 valency, 7

 walk-regular, 32, 97