(Non)-Invertible Topology in Quantum Field Theory

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapter 2 is based off [252] and [254]
Chapter 3 is based off [66], written in collaboration with Arun Debray. Both authors contributed equally.
Chapter 4 is based off [152, 153], both written in collaboration with Theo Johnson-Freyd. Both authors contributed equally.
Chapter 4 is based off [251], and [78], the latter written in collaboration with Thibault Décoppet. Both authors contributed equally.

Abstract

This thesis aims to highlight aspects of mathematics and physics that arise in topological field theories. We will consider invertible and noninvertible topological theories. In the former case, we compute the classification of these invertible theories which arise as the trivial bulk of some anomalous theory one dimension lower. The computation tools used here were conceived in algebraic topology and this work aims to develop these techniques for applications to physical theories. In the latter case, to study such theories in low dimensions we develop part of the theory of fusion 2-categories. Using techniques here allow us to classify noninvertible phases up to equivalence.

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Dedication

To my parents Yanyun Liu, and Lu Yu.

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Chapter 1

Introduction and summary

Understanding the space of quantum field theories (QFT) includes answering questions regarding the dynamics of theories, and the phases that theories can flow to upon adding deformations. This thesis focuses on the subsector of QFT known as topological quantum field theories (TQFT). These theories arise as the low energy description of QCD-like theories and in condensed matter systems where the interactions are strong. The fact that these theories are topological means that the local propagating degrees of freedom have been gapped out, and only very coarse grained information about the theory remains. The theory is not completely trivial in many cases, however, and the topological aspects allow for mathematical tools to be used to give them classifications. TQFTs are therefore naturally a bridge between physics and mathematics. In particular, they can be considered as the "dual" to manifolds and nontrivial properties regarding the topology of manifolds is revealed in studying TQFTs.

From the perspective of physics, since TQFTs commonly arise in the nonperturbative regimes of field theories, extracting as much information as possible from the topological side can greatly improve our understanding of strong coupling does to the dynamics. The first type of topological theory that arises is the *symmetry protected topological phase* (SPT phase), and is an invertible TQFT. These phases are also related to the 't Hooft anomaly of a global symmetry and computable via techniques in algebraic topology. The second type of topological theory is the *topological order*, which are noninvertible theories. In this setting, extended operators of different dimensions form the structure of a higher category, and in some cases one can even classify these theories using categorical techniques. The remainder of this thesis summarizes the advances in mathematics and physics inspired by the ideas of invertible and noninvertible topological field theory.

1.0.1 Chapter 2

In the first part of chapter 2 we discuss anomalies, specifically 't Hooft anomalies. For a d-dimensional theory on a manifold of tangential structure H and global symmetry G, the anomaly is classified by a cobordism group in degree $d+1$ [158]. In low degree, this is well approximated by group cohomology, and for fermionic theories which depend on a choice of spin structure, by supercohomology. In [252] I looked at symmetries and anomalies of (1+1)d theories. In this dimension, the only interesting higher form symmetry is the one-form symmetry $\mathcal{A}_{[1]}$, so we can completely understand the interaction between it and the zero-form symmetry G. The result is a 2-group $\mathbb{G}$ such that

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}_{[1]} \arrow[r, "i"] & \mathbb{G} \arrow[r, "\pi"] & G \arrow[l, bend left, "\varphi"] \arrow[r] & 0 \end{tikzcd},$$

where isomorphism classes of φ determine the Postnokov $\beta_{\mathbb{G}}$ as a class in the cohomology $\mathrm{H}^3(G;A)$, or a "split" 2-group, which also goes by the name symmetry fractionalization [14]. In the latter case, the splitting map φ gives the trivial class for $\beta_{\mathbb{G}}$, and we have an extension $\mathcal{A}_{[1]} \rtimes G$.

The main computational tool used was the Atiyah-Hirzebruch spectral sequence (AHSS), which allowed for computations in the fermionic theories. This spectral sequence had signature

$$E^{p,q} = \mathrm{H}^p(X, \mathrm{SH}^q) \Rightarrow \mathrm{SH}^{p+q}(X), \tag{1.0.1}$$

where X is a space and SH is supercohomology.

By using the AHSS in the bosonic case, I compute the anomalies for 2-groups and split 2-groups in the case where the zero-form and one-form symmetry are discrete abelian groups. The main theorems showed that for the 2-group, the cohomology $\mathrm{H}^3(\mathbb{G};\mathrm{U}(1))$ consists of pairs (α,γ) where $\alpha : G \to \widehat{A}$ is a homomorphism, with $\widehat{A}$ the Pontryagin dual of the one-form group, and $\gamma \in \mathrm{C}^3(G;\mathrm{U}(1))$, such that

$$d\gamma = \alpha \cup \beta_{\mathbb{G}}. \tag{1.0.2}$$

A special feature about $2d$ theories is that the one form symmetry leads to a decomposition structure, in which the theory breaks up into direct sums. When the one-form symmetry is nonanomalous, one can also gauge this symmetry, which leads to a dual (-1)-form symmetry. The operators associated to this are codimension zero, thus giving the interpretation of spacefilling operator, which projects to direct sums.

The second part of the paper focuses on symmetry fractionalization. The data required to define symmetry fractionalization starts with a class of maps $[\rho] : G \to \mathrm{Aut}(\mathcal{A}_{[1]})$, that describe how the symmetry acts on the set of one-form symmetries. To fractionalize the global symmetry means specifying a local projective symmetry action that is compatible with the action on the one-form symmetry. The projective symmetry action can be characterized by a set of phases $\eta_a(g,h) \in U(1)$ for each object $a \in \mathcal{A}_{[1]}$, where g, h are elements of the bosonic symmetry group G. This characterizes the difference in phase obtained when acting on a separately by g and h, versus the composite gh. Modding out by gauge redundancies, we can characterize the symmetry fractionalization classes which correspond to equivalence classes of η. The different symmetry fractionalization classes depend on ρ and they moreover form a torsor over $\mathrm{H}^2(BG;\mathcal{A})$. For the case of the split two group, there is an isomorphism

$$\mathrm{H}^3(\mathcal{A}_{[1]} \rtimes G\,;\mathrm{U}(1)) \cong \mathrm{H}^3(G\,;\mathrm{U}(1)) \oplus \mathrm{H}^1(G\,;\mathrm{H}^2(\mathcal{A}_{[1]};\mathrm{U}(1)))\,. \tag{1.0.3}$$

The second part of chapter 2 focuses on a question about detecting the value of an anomaly. We introduce the notion of "genus-one data" for theories in (1+1)-dimensions with an anomalous finite group global symmetry. We outline the groups for which genus-one data is effective in detecting the anomaly, and also show that genus-one data is insufficient to detect the anomaly for dicyclic groups. Detecting the values of the anomaly that a specific theory realizes can be a tough problem in general, even though one may have a classification of the anomaly through cohomology techniques. This task is especially difficult in topological theories, but tractable in certain situations when the theory is free and the matter content is known. This part of the thesis inspired work in progress on *anomaly indicators* which evaluates a partition function for a topological theory in one dimension higher.

1.0.2 Chapter 3

In chapter 3 we utilize the Adams spectral sequence and shearing arguments for twisted spin structure to compute the anomalies of U-duality. The theory with the U-duality group arises from reducing $10d$ string theory to $4d$. The U-duality group $E_{7(7)}$ mixes with fermion parity, and the resulting group has SU(8) as the maximal compact subgroup. The resulting bordism group for the anomaly is $\Omega_5^{\mathrm{Spin}\times_{\{\pm1\}}\mathrm{SU}(8)} = \Omega_5^{\mathrm{Spin}}(B\mathrm{SU}(8)/\{\pm1\})$. We compute the first five bordism groups and show that

$$\Omega_5^{\mathrm{Spin}}(B\mathrm{SU}(8)/\{\pm1\}) \cong \mathbb{Z}_2\,, \tag{1.0.4}$$

and the generating manifold is the Wu manifold $W := \mathrm{SU}(3)/\mathrm{SO}(3)$. With the knowledge of the anomaly group and the generating manifolds, we can compute the value of the anomaly on this manifold to show that it vanishes.

Anomalies, otherwise known as SPTs, are special topological field theories. They belong to a space of invertible reflection positivity extended field theories and classified by the homotopy mapping groups $[MT\xi, \Sigma^{n+1}I\mathbb{Z}]$. ξ is a map $\xi : BG \to B\mathrm{O}$ denoted as a symmetry type, giving a manifold M has a BG-structure. This means we have a lift α:

$$\begin{tikzcd} & BG \arrow[d, "\xi"] \\ M \arrow[ur, "\alpha"] \arrow[r, "\tau_M"] & BO\,, \end{tikzcd} \tag{1.0.5}$$

where τ classifies the tangent bundle for M. The symmetry types encoding the topological information needed to define a field theory that can have particlar matter content. A fermionic theory with spinors has to be formulated on manifolds with spin structure, or perhaps a variant thereof. A theory with time-reversal symmetry can be put on unoriented manifolds. Determining the symmetry type is an important first step in formulating a mathematical question about field theory.

With the introduction of symmetry type, we will define an equivalence relation on the set of closed, d-dimensional ξ-manifolds. We say that two manifolds $M \sim N$ if there is a $(d+1)$-dimensional ξ-manifold X such that there is a diffeomorphism $\partial X \cong M \sqcup N$. The manifold X is a bordism between the two manifolds M and N, and M and N fit into the same equivalence class Ω^{ξ}_{d}, which has the structure of a group. Computing such groups is done using techniques of homotopy theory; in order to implement homotopical techniques we introduce Thom spectra.

We begin with the construction of a Thom space. Consider a vector bundle $V \to X$, the Thom space $\mathrm{Th}(X,V)$ is defined as the quotient of the unit disc bundle by the unit sphere bundle of V: $D(V)/S(V)$. One can also use virtual vector bundles, aka formal difference of real vector bundles, to define Thom spaces. Recall that BO is the classifying space for virtual stable vector bundles, so we can regard BO as the classifying space for rank-zero virtual vector bundles. Now given $V \to X$ and $\xi : BG \to BO$ the classifying map for V, we can define $BG_n = BO_n \times_{BO} BG$. Furthermore we define $V_n \to BG_n$ to be pulled back from V along the map $BG_n \to BG$. The Thom spectrum of V is a space whose homotopy in degree n is $\mathrm{Th}(BG_n, V_n)$, and denoted as BG^V. There are a special type of Thom spectra called Madsen-Tillman spectra and are of the form $\xi^*(-V) \to BG$

This section uses the Adams spectral sequence for computing the homotopy groups of Thom spectra. A theorem of Pontryagin Thom gives an isomorphism between these homotopy groups and bordism groups. When applied to Madsen-Tillman spectra $MT\xi$, corresponding bordism groups are precisely Ω^{ξ}_{d}.

In many instances of physical interest, a symmetry type ξ can be recast as a twist of a more familiar symmetry type. These include SO, Spin, Spin^c, and String, and they are relevant for many physical theories. In many of the known examples in the literature, these twists were done via a virtual vector bundle. One important aspect of using vector bundles twists is to turn a more exotic symmetry type ξ into a twisted version of one that we know, i.e. a "twisted ξ'-structure" where ξ' is one of the Spin, Spin^c, or String. In other words, an H-structure on a vector bundle $E \to M$ is equivalent data to a vector bundle $V \to M$

and a H' structure on $E \oplus V$. In this case, we can split the Thom spectrum as

$$MTH \xrightarrow{\simeq} MTH' \wedge X^V \,, \tag{1.0.6}$$

for some Thom space X^V.

We discover in [66] that the twist of spin bordism that we use does not arise from a vector bundle, but we noticed that if we ignored this fact and pretend that there was a fake vector bundle twist, we still arrive at the correct answer. This part of the thesis inspired ongoing work to uncover the origin of this fact.

1.0.3 Chapter 4

In chapter 4 we aim to give a classification of topological orders in (4+1)d. Recent developments in our understanding of symmetries have led to higher form symmetries [113], which are implemented by topological invertible operators of lower codimension. In general, a (d+1)-dimensional theory can also have noninvertible operators, that interact in a complicated manner which is captured by a monoidal d-category. A full topological order accommodates extra structure of Karoubi completeness, rigidity, and remote detectability to the d-category [149]. In [153] we classified topological orders in (4+1)-dimensions. The nontrivial surface operators have three ambient dimensions in which they can compose; the 2-category of surface operators is therefore three-monoidal, aka sylleptic. A crucial step in the classification was a theorem proved in [152] that constrained the surface operators to only have grouplike fusion rules, if we can remove all the line operators. That is, if the endomorphism 1-category is trivial. Proving this theorem about 2-categories is the goal of the first part of this section.

For the second part, we will show that:

$$\{\text{super (4+1)d topological orders with no lines}\} = \{\text{symplectic finite Abelian groups}\}$$
$$\{\text{bosonic (4+1)d topological orders}\}/\text{Morita equivalence} \cong \mathbb{Z}_2^\infty \,.$$

A feature of the classification for topological orders is that it is given up to *Morita equivalence*. Two theories are Morita equivalent if they can be separated by a gapped interface. In the case of bosonic (4+1)d topological orders, this means that there are infinitely many phases, which are not related by a gapped boundary. A way of establishing a Morita equivalence is through a procedure of *condensation*, which will be discussed in more detail in chapter 5.

1.0.4 Chapter 5

In chapter 5 we use the ideas of condensation developed in [109] in (2+1)d topological theories and for higher dimensions where theories are described by fusion 2-categories. We start in (2+1)d where the topological theory is described by a fusion 1-category; this is a finite semisim- ple monoidal 1-category with duals and simple monoidal unit. What we do from a physical point of view is to start with a parent theory and adiabatically tune a set of parameters. This leads to a child phase, where the topological content of the parent is nontrivially manipulated. With condensation, we can build new operators from a network of lower dimensional operators, as well as gapped phases starting from the vacuum. The particular condensation that I have focused on is in (2+1)d, which goes by the name of anyon condensation [251], which is a manipulation on the line operators of a theory. A further general form of condensation for higher dimensions has been proposed in [109]. Mathematically, to perform condensation in (2+1)d one chooses to condense a set of lines that gives a commutative Frobenius algebra. I worked to enlarge the notion of what is "condensable". By relaxing the commutative condition, I was able to define a way of condensing anyons that are not just bosons, or abelian. This led to an understanding of how to condense categorical symmetries, that are implemented by nonabelian anyons and thus noninvertible. While the inspiring physical interpretation of condensation applies to nonabelian anyons, I build upon this by giving a systematic procedure of computing the new spectrum using the condensation algebra and the data of the lines. In particular the method used to project out lines is a form of condensation that can be performed due to the existence of a *fiber functor*, which maps the lines to the vacuum. The tool of anyon condensation can be used to verify conformal embeddings, generalizations of modular invariants, and the branching functions associated to the topological coset theories.

In the next part of this chapter I discuss condensations in fusion 2-categories and generalizing the idea of anomalies for regular symmetries to noninvertible symmetries. These are symmetries that are enacted by topological operators that do not have an inverse, in particular, two such topological operators may fuse into a multitude of other operators. This is the heart of the non-invertible part of the thesis. In recent years the proposal of the cobordism hypothesis with regards to the Swampland program has allowed for many nontrivial statements of quantum gravity theories. At the heart of this hypothesis is the statement that there are no global symmetries in a theory of quantum gravity. Therefore all global symmetries, invertible or not, that are not broken in the UV must be gauged. This naturally raises the question of what settings do we have good control over the obstructions to gauging noninvertible symmetries.

We introduce fusion 2-categories because they encapsulate the interactions of surface and line operators, and are therefore useful in capturing the topological content of theories in low dimensions. The success that fusion 1-categories found in many applications to low dimensional topology and physics furthermore entices us to study the natural extension to higher category number. Fusion 2-categories were introduced in [85] as a result of categorifying the notion of a fusion 1-category over an algebraically closed field of characteristic zero. Fusion 2-categories are by definition finite semisimple monoidal 2-categories with duals and simple monoidal unit. One of the main mathematical results of [78] proves a theorem about fermionic symmetric fusion 2-categories. These are fusion 2-

categories enriched over 2**SVec** with extra levels of monoidality on the objects. The theorem generalizes a theorem of Deligne's for 1-categories [79]. We also generalize the notion of an anomaly for a symmetry to a noninvertible symmetry enacted by surface operators using the framework of condensation in 2-categories. Given a multifusion 2-category, potentially with some additional levels of monoidality, we prove theorems about the structure of the 2-category obtained by condensing a suitable algebra object. We give examples where the resulting category displays grouplike fusion rules and through a cohomology computation, find the obstruction to condensing further to the vacuum theory. As a consequence, we show that every symmetric fusion 2-category admits a fibre 2-functor to 2**SVec**.

Chapter 2

Symmetries in 1+1d and anomalies

Abstract. We investigate the interactions of discrete zero-form and one-form global symmetries in $(1{+}1)d$ theories. Focus is put on the interactions that the symmetries can have on each other, which in this low dimension result in 2-group symmetries or symmetry fractionalization. A large part of the discussion will be to understand a major feature in $(1{+}1)d$: the multiple sectors into which a theory decomposes. We perform gauging of the one-form symmetry, and remark on the effects this has on our theories, especially in the case when there is a global 2-group symmetry. We also implement the spectral sequence to calculate anomalies for the 2-group theories and symmetry fractionalized theory in the bosonic and fermionic cases. Lastly, we discuss topological manipulations on the operators which implement the symmetries, and draw insights on the $(1{+}1)d$ effects of such manipulations by coupling to a bulk $(2{+}1)d$ theory.

2.1. Introduction

Understanding symmetries is key to revealing many nontrivial features of quantum field theories. In different dimensions, various higher form symmetries may exist in a theory. These symmetries can be encoded in the higher codimension operators, and in many cases have the structure of a higher category. Global symmetries do not need to stay stagnant either: one can promote the symmetry to a dynamical symmetry under the assumption of the symmetry having no t'Hooft anomaly. This is done by coupling to a background connection and gauging the symmetry. For finite symmetries, another modern point of view of gauging is performing a categorical condensation [109]. Furthermore, the symmetries can interact with each other in nontrivial ways to give higher n-groups. It has therefore become increasingly necessary to implement techniques from category theory and topology to consolidate information about symmetries. This includes, the possible 't Hooft anomalies, the algebraic structure of higher codimensional defects, and the groupoid of ways in which theories are related to each other by gauging [114, 112, 137].

The purpose of this paper is to study theories in (1+1)d that exhibit a discrete global zero-form and one-form symmetry. We explore two possibilities: the first is when the zero-form symmetry is nontrivially extended by the one-form symmetry leading to a 2-group, and the other is when the extension is trivialized leading to symmetry fractionalization. Theories in (1+1)d are interesting from this point of view because the higher form symmetries are restricted only to one-forms, and it is possible to keep track of the interactions between the line operators that implement the one-form symmetry and the point operators that implement the zero-form symmetry. It is also possible to calculate the anomalies of theories with 2-group symmetries and theories with symmetry fractionalization rigorously in this low dimension. We present explicit calculations and give formulas for the cohomology that classify the anomalies.

An aspect that we will emphasize is the notion of having multiple ground states, or disjoint sectors, in a theory. In (1+1)d theories, the one-form symmetry gives information about the number of local ground states. Not only is there information about the multiple subsectors in a theory, but there is also information contained in moving between the sectors. We address these two points by laying down some theoretical framework and also giving examples. Furthermore, we discuss gauging one-form symmetries and explaining the dual (-1)-form symmetry from a physical and mathematical viewpoint. We find that if we gauge the one-form group in a 2-group, the extension in the 2-group becomes a mixed anomaly that restricts to each subsector. It is also natural to apply this knowledge of subsectors on the side of symmetry fractionalization, from which it is possible to make relations with discrete torsion.

The layout of the paper is as follows: in §2.2 we begin by reviewing generalized symmetries and give a precise mathematical definition of one-form symmetry and of 2-groups. We then focus on gauging in a theory with 2-group symmetry, outlining properties of one-form symmetries as also discussed in [218]. We also track the relationship between mixed anomalies and extensions, given by the Serre spectral sequence, at the level of partition functions. In §2.2.7 and §2.2.8 we employ spectral sequence techniques to calculate anomalies for (1+1)d bosonic and fermionic theories, respectively. We discuss symmetry fractionalization in §2.3 and relate it to discrete torsion for the zero-form symmetry, a special feature that exists in (1+1)-dimensions. We then give the anomalies for (1+1)d theories exhibiting symmetry fractionalization. Finally, we finish off by discussing manipulations regarding the topological sectors of (1+1)d theories and how they can be recovered by coupling to a bulk topological field theory (TFT) [159].

2.2. 2-group Global Symmetry in (1+1)d Theories

Given a d-dimensional quantum field theory with a p-form global symmetry, one is also provided with a particular set of topological codimension $p+1$ operators: the charged operators for the p-form symmetry, which implement the group action of the p-form symmetry upon crossing these operators [113]. By virtue of their group-structure, these symmetry defects form a subset of all invertible topological operators in the theory. Another set which exists is the collection of non-invertible defects [40], which we will not consider for our purposes. The fact that the group like property of higher form symmetries are encoded in a collection of invertible defects of various codimensions implies that in (1+1)-dimensions the only higher form symmetry that is possible is a one-form symmetry. While zero-form symmetries of a theory are well understood to be described by ordinary groups, a generalization is required to talk precisely about the group structure of one-form symmetries and 2-groups; this will be the focus of the following subsections.

2.2.1. Defining a 2-group: 1-forms

We will build to the definition of a 2-group by introducing some formal definitions from category theory and homotopy theory required to sharply define a 2-group. We will also illustrate how 2-group symmetries are a group of one-form symmetries and zero-form symmetries "interacting" with each other. Along the way we will have properly defined the notion of one-form symmetry, and the analogue applies for higher form symmetries in higher dimensions.

We start off with the notion of a *group object*, which is a generalization of the structure of groups to categories other than **Set**, the category of sets. That is to say, the underlying set of elements for the group, which is typically an object of **Set**, is replaced by another object from some other category. More formally, if X is a group object in the category $\mathcal{C}$, then there are maps:

$$m : X \times X \to X\,, \quad e : 1 \to X\,, \quad \text{inv} : X \to X, \tag{2.2.1}$$

where m is an associative multiplication, e is a map from the terminal object $1 \in \mathcal{C}$ which is a two sided unit of m, and inv is the two sided inverse of m [182]. We see from the above requirement that one recovers the traditional notion of a group if $\mathcal{C} = \textbf{Set}$ and X is a group object in **Set**. In this case we define m such that it takes the form of group multiplication for the underlying set, e such that it selects the identity of X, and inv such that it assigns to all group elements its inverse. Hence, one could say the collection of zero-form symmetries of a theory is described by a group object in the category **Set**.

We now turn to a generalization of groups: *groupoids*. A groupoid has the features of a group in which any two elements may not be meaningfully composed. In particular, a groupoid is a (small) category where every morphism is an isomorphism. The category of all groupoids is **Grpd**. The objects in this category are groupoids, which are categories themselves, and the morphisms between objects are actually functors of groupoids. We can simplify this generalization to once again recover the traditional notion of a group, which is given by the *morphisms in a groupoid*.

Suppose that a groupoid only has one object, and consider a group G. We could have a morphism from that one object to itself, with each morphism given by an element $g \in G$. Any two morphisms given by $g_1, g_2 \in G$ may be composed, just as in the group. Associativity of morphisms holds because of the associativity of the group operation, and the identity of G is the identity morphism for the object of the groupoid. Hence, the morphisms of a groupoid with only one object form a group under composition. This is a notion of *delooping* applied to a group, which categorifies it into a groupoid.

Returning to physics, we say a "group of 1-form symmetries", denoted $\mathcal{A}_{[1]}$, is a group object in the category of groupoids with only one object. To reflect this, we introduce the following notation:

$$
\begin{aligned}
\pi_0\, \mathcal{A}_{[1]} &:= \mathrm{ob}(\mathcal{A}_{[1]}) = \{*\} \\
\pi_1\, \mathcal{A}_{[1]} &:= \mathrm{Hom}(*, *) = A\,,
\end{aligned}
\tag{2.2.2}
$$

where $\{*\}$ denotes the single object of the groupoid. Here the group of morphisms $\pi_1\, \mathcal{A}_{[1]}$ is what is meant when a said theory possesses an "A group one-form symmetry". The composition of maps also endows $\mathcal{A}_{[1]}$ with a unique group law. Notably, as a groupoid with group law, $\mathcal{A}_{[1]}$ contains a multiplication $m : \mathcal{A}_{[1]} \times \mathcal{A}_{[1]} \to \mathcal{A}_{[1]}$, and an associator β. The associator is a 3 cocycle such that for $a_1, a_2, a_3 \in \mathrm{ob}(\mathcal{A}_{[1]})$, $\beta(a_1, a_2, a_3)$ gives the isomorphism $m(m(a_1, a_2), a_3) \simeq m(a_1, m(a_2, a_3))$ for the associativity of m. An important point to make is that in order for this group law to be unique, we require A to be abelian. Henceforth, when we mention the specific one-form symmetry group, what one should think of is the underlying A. This is what is really meant when thinking of one-form symmetries as groups in the regular sense, such as the center $\mathbb{Z}_N$ one-form symmetry in $4d$ $SU(N)$ pure Yang-Mills theory.

For the purpose of this paper we will take A to be a finite group, and in later sections we will consider more specific cases for this finite group. In the subsequent sections, we could also refer to the one-form symmetry as BA. Analogously, BG where G is treated as a $(p-1)$-form group can be thought of as $K(G, p)$, the p-th Eilenberg-Maclane space of G. When computing group (super)cohomology in the later sections we are computing (super)cohomology on the classifying space of the group, rather than cohomology of the topological space of the group. In our case, the latter would be a finite set of points, which does not have interesting cohomology. In order to conserve on notation we will write $H^\bullet(G) = H^\bullet_{group}(G) = H^\bullet(BG)$, to reflect this fact for G the zero-form symmetry. For the one-form symmetry, we will denote the cohomology by $H^\bullet(\mathcal{A}_{[1]})$ or $H^\bullet(BA)$, to mean the cohomology of the space BA. We follow the usual slight abuse of notation, taking the space BA as the space that carries a unique group structure that is the underlying group for a 1-form symmetry, and also taking BG to be the delooped space which gives the group cohomology of G.

2.2.2. Defining a 2-group: Including 0-form

In general, it is possible to have zero-form symmetries along with one-form symmetries; we will take the zero-form symmetries to be the group $G_{[0]} = G$ which is a finite group. Furthermore, there could be a nontrivial action of the zero-form on the one-form symmetry. Our goal will be to study groupoids with group law, $\mathbb{G}$, that fit in a sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}_{[1]} \arrow[r, "i"] & \mathbb{G} \arrow[r, "\pi"] & G \arrow[r] \arrow[l, bend left, "\varphi"] & 0\,, \end{tikzcd}$$

where we arbitrarily choose a splitting φ at the level of groupoids. The complete data of φ is that for all $g \in G$, $\varphi(g)$ is an object in $\mathbb{G}$ and thus $\mathbb{G} \cong \mathcal{A}_{[1]} \rtimes_\varphi G$. Such $\mathbb{G}$ is a groupoid with group law and $(*, g) \in \text{ob}(\mathbb{G})$; this is known as a 2-group. The associator $\beta_\mathbb{G}$ in $\mathbb{G}$ is a 3-cocyle valued in the group A and gives the isomorphism

$$m_\mathbb{G}[m_\mathbb{G}[(*, g_1), (*, g_2)], (*, g_3))] \simeq m_\mathbb{G}[(*, g_1), m_\mathbb{G}[(*, g_2), (*, g_3))]] \tag{2.2.3}$$

where $m_\mathbb{G}$ is the multiplication of objects in the groupoid. A different choice φ' of splitting which is isomorphic to φ changes $\beta_\mathbb{G}$ by an exact term. Therefore, isomorphism classes of φ determine $\beta_\mathbb{G}$ as a class in the cohomology $H^3(G; A)$. In the literature, this class is known as the Postnikov class and is said to take values in the one-form symmetry, viewed in its full form as a groupoid. This is convenient because it allows us to view the associator as a topological defect that implements the one-form symmetry; in (1+1)d this is a codimension two operator.

We will come back to how this one form symmetry defect can be related to the lack of associativity of composing zero-form symmetry defects in a later subsection. We furthermore point out that, as can be seen from the above short exact sequence, $\mathcal{A}_{[1]}$ is a subgroupoid of $\mathbb{G}$, but G need not be. Therefore, one could always ask the question of gauging the one-form symmetry in a 2-group, but not necessarily the zero-form.

A more topological viewpoint of 2-groups is from the point of view of its classifying space. The classifying space of a two group is a *homotopy 2-type* [1] [64], meaning it only has two nontrivial homotopy groups. Consider a connected space $B\mathbb{G}$ such that:

$$\pi_1(B\mathbb{G}) \cong G, \quad \pi_2(B\mathbb{G}) \cong A. \tag{2.2.4}$$

In order to have the two symmetries G and A mix, take $B\mathbb{G}$ as a fiber bundle over BG with fiber $K(A, 2)$. We can specify the bundle $p : B\mathbb{G} \to BG$ by its homotopy cofiber i.e., a map $k : BG \to \Sigma K(A, 2) \cong K(A, 3)$ known as the *k-invariant* of the space $B\mathbb{G}$. The associator $\beta_\mathbb{G}$ was a class in the third cohomology, which can be represented by maps into

[1]This is only true for geometrically discrete 2-groups that we are considering, and not generally for 2-group analogues of Lie groups.

Eilenberg-Maclane spaces. This is to say $H^3(G;\mathcal{A}_{[1]}) \cong [K(G,1), K(A,3)]$, which sends the associator of $\mathbb{G}$ to the k-invariant of $B\mathbb{G}$. Moreover, every homotopy 2-type $B\mathbb{G}$ is the classifying space of some 2-group $\mathbb{G}$.

2.2.3. Symmetry Defects and 2-group Structure

We now study a physical system with 2-group symmetry at the level of its symmetry defects. We choose a triangulation of the space with a defect network of lines corresponding to coupling the zero-form symmetry to background connection [221]. The line defects in a (1+1)d theory form a 1-monoidal 1-category where the trivalent junction of line defects is codimension two and serve as morphisms between codimension one objects. In general, a p-monoidal q-category contains objects of dimension q for which there are p ambient dimensions to compose object. Another way to understand this type of category is to envision it as a way of encapsulating the ways to "compose" $(q-1)$-branes in $(p+q)$-dimensions of spacetime.

In an n-dimensional theory, the natural object to study is an"n-group" in which one considers the topological operators that implement k-form symmetries for $k \leq (n-1)$. In analogy with the case in (1+1)d, the objects of codimension one have morphisms that are codimension two, and the codimension two objects have morphisms that are codimension three, etc. This results in the notion of a weak n-category, where the compositions are only associative up to higher coherence relations. In physical systems of interest we must also implement the condition that the n-category has trivial center in the sense that no operator can "commute" in a higher categorical sense with all other operators in the theory. If there is a nontrivial center in the category, then we deem that there is a gravitational anomaly and therefore obstructs the consistent realization of the category as a physical system purely in n-dimensions [171].

In (1+1)d the zero-form symmetry operators of $\mathbf{g}, \mathbf{h}, \mathbf{k}$ can be composed in two ways, and going between the two is a matter of applying an F-move. As depicted in figure 5.2, a point operator $\beta_{\mathbb{G}}(\mathbf{g}, \mathbf{h}, \mathbf{k})$ valued in the one-form symmetries can be created due to this move, signifying a nontrivial interaction between the two types of symmetries and therefore a two group structure. The F-move involving the zero-form symmetry defect could furthermore generate a phase $\omega(\mathbf{g}, \mathbf{h}, \mathbf{k}) \in \mathrm{C}^3(G, \mathrm{U}(1))$ that is purely attributed to the fact that the Hilbert space is in a projective representation of the zero-form symmetry. We will later explicity calculate anomalies of this type for full 2-group theories using techniques in group cohomology.

Another feature in (1+1)-dimensions is that point operators, which are charged under the zero-form global symmetry and pass through a symmetry line defect, are acted upon by the line, see figure 2.2. Therefore, the point operator labeled by $\mathbf{a}$ becomes $\mathbf{a}_{\mathbf{g}}$ after passing through a $\mathbf{g}$-defect. If $\mathbf{a}$ was the only unique point operator that existed in a theory,

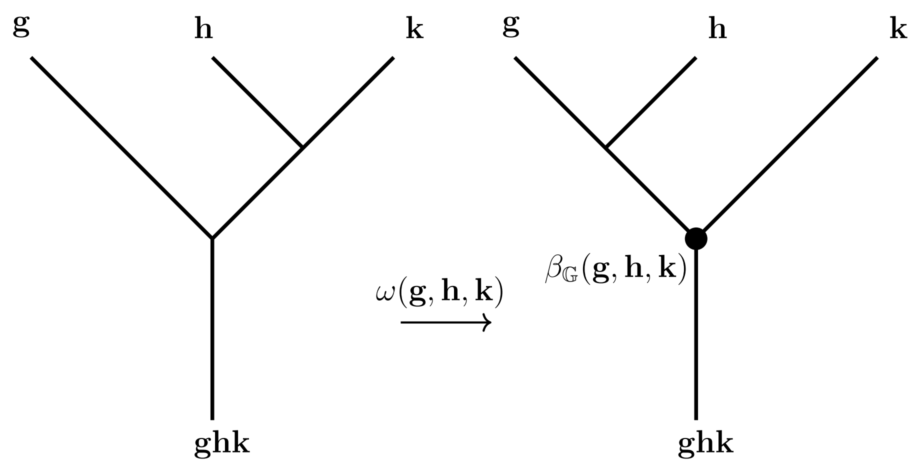


Figure 2.1: F-move

then $\mathbf{a_g} = \mathbf{a}$ which implies that the nontrivial $\mathbf{g}$-defect could not even be detected[2]. This means that in order to consider nontrivial zero-form operators, we must have at least two independent point operators. In theories living in (2+1)-dimensions, a single unique point operator would suffice because lines can detect other lines through their braiding, and lines can detect surfaces by puncturing at points, see figure 2.4.

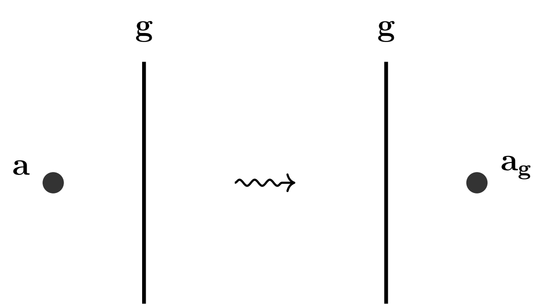


Figure 2.2: The point operator labeled by $\mathbf{a}$ receives a $\mathbf{g}$ action upon passing through the defect line.

The local point operators determine the ground states of our system, which means that theories in (1+1)d are most interesting to study in the presence of multiple ground states, or vacua. For the remainder of this note we will use the terms "ground state" and "vacua" interchangeably. Going back again to (2+1)-dimensions, one could also consider multiple ground states, and in general there will exist a modular tensor category (MTC) describing the information and interactions of anyons in the theory around each ground states. The theory with respect to a single ground state is what is usually referred to in the traditional

[2]The fact that all operators need to be detectable goes by the name *remote detectability*. In topological orders such a condition is necessary in order to have a consistent anomaly free theory.

discussion of topological order, in which case it is described by a *fusion n-category*. We see that in order to give a more complete treatment of topological orders, and general theories in (1+1)-dimensions, one needs to modify the definition to take into account the possibility of multiple ground states and a decomposition into subsectors. A more complete definition of topological orders in $(n+1)$-dimensions with multiple ground states is as a *multifusion n-category* [149].

Suppose we are presented with a theory $\mathcal{T}$ in (1+1)-dimensions with some global zero-form symmetry group G, and that $\mathcal{T}$ has multiple vacua when considered at some finite energy. If the system is at any particular vacua, then at finite energy there could be instantons that tunnel between different vacua. We therefore do not say that these vacua can be regarded independently, but rather should be thought of as a family of vacua. By flowing to the deep infrared (IR), all the massive instantons are integrated out, and it is sensible to claim that the system indeed sits at a particular vacuum. Due to the potential between the vacua becoming arbitrarily high in the IR, we regard each point operator as also labeling a subsector of our theory where each subsector is decoupled and truly independent of the others.

This vacuum at which our system sits is labeled by a point operator in the set of $\mathcal{S} = \{\widehat{1}, \widehat{2}, \ldots, \widehat{M}\}$ where M is the cardinality of $\widehat{A} = \hom(A, \mathrm{U}(1))$, the Pontrjagin dual group of A. We see that the one form symmetry is an emergent symmetry of $\mathcal{T}$ in the deep IR. The algebra of point operators is a finite dimensional commutative and separable algebra and is isomorphic to a direct sum indexed by $\mathrm{Spec}(\mathcal{A}_{[1]})(\mathbb{C})$. Thus, we can write $\mathcal{T}$ in the deep IR as a direct sum of its subsectors $\mathcal{T} = \mathcal{T}_{\widehat{1}} \oplus \mathcal{T}_{\widehat{2}} \oplus \ldots \oplus \mathcal{T}_{\widehat{M}}$. The theories we will be most interested in for this paper are those that exhibit the properties of $\mathcal{T}$ in the far IR. We note in passing that in (0+1)-dimensions, a zero-form symmetry G alone will split theory $\mathcal{Q} = \bigoplus_{\hat{\alpha} \in \widehat{G}} \mathcal{Q}_{\hat{\alpha}}$, where $\widehat{G}$ is the Pontryagin dual to G.

In a local patch of the theory with two insertions of point operators, we could imagine composing the two points and producing another point operator, therefore endowing this set of operators with a multiplication structure. To better illustrate composing one-form operators and the relation with degenerate ground states, we give an example again in the $(2+1)d$ world but consider one of the spatial dimensions compactified on a circle. Along this compactified direction we may wrap a number of anyons of which some may generate a one-form symmetry. Each wrapping of an anyon labels a ground state. We may decide to bring two anyons labeled by a and b close together and fuse them using the rule $a \times b = \sum_c N_{ab}^c\, c$. For an abelian anyon a, this fusion gives just a single anyon; we see that the multiple vacua give a representation of the fusion of anyons.

In a general $(n+1)$-dimensional theory, the n-dimensional operators do not only have a fusion, or monoidal n-category structure, but it is also possible to produce composite operators by taking direct sums. Therefore, it is also possible to consider the situation in which a complex linear combination of multiple operators in the set $\mathcal{S}$ as solely operators that do not interact with each other in the way of fusion, giving the addition structure.

We denote $\mathbb{C}[\mathcal{A}_{[1]}]^{\times}$ as the ring of invertible one form operators with the ring structure as previously laid out.

Examples of multiple ground states

To get a better handle on how to understand multiple ground states and the subtleties which arise from the decomposition into subsectors we consider some examples. Let us consider a (2+1)d topological quantum field theory (TQFT) described by a MTC $\mathcal{C}$, such that the category $\mathcal{C}$ has as its boundary a rational conformal field theory (RCFT) that consists of a pair of unitary vertex operator algebras (VOA) V, W and an equivalence of categories $\Phi : \mathrm{Rep}(V) \xrightarrow{\sim} \mathrm{Rep}(W)^{\mathrm{op}}$. The (1+1)$d$ operators that arise from a (2+1)d object can be lines, obtained by running an operator of codimension 1 parallel to the top and bottom of the slab, or local point operators by letting a line in bulk end on the sides of the slab, as in figure 2.3. In particular, an object in $\mathcal{C}$ contains a vector space of ways to end, which is in fact a module for the VOAs. For $X \in \mathcal{C}$ traversing between the left

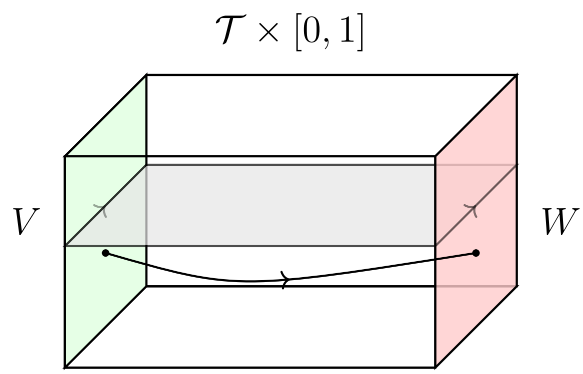


Figure 2.3: Line traversing the bulk.

and right face, let $V(X)$ be the corresponding V-module, and $W(X)$ for the corresponding W-module. The full algebra of local operators is:

$$\mathcal{A} = \bigoplus_{X \in I} V(X) \otimes W(X) \tag{2.2.5}$$

where I is a set of representatives of simple objects. Each $V(X) \otimes W(X)$ has an action of the left and right Virasoro algebra $\mathrm{Vir}_L \otimes \mathrm{Vir}_R$, and so of L_0, $\bar{L}_0$; the only state with $L_0 = \bar{L}_0 = 0$ is $|0\rangle \otimes \overline{|0\rangle}$ which is the local ground state, in this case also the vacuum.

This axiomatization describes the local operators of the CFT, but we note that there was no mention of a Hilbert space of states on the interval. Due to the lack of that extra information, automorphisms of an RCFT (V, W, Φ) can have 't Hooft anomalies, and one must consider the possibility of other "anomaly" type data. If we have an RCFT with two local ground states we can take their direct sum, then we should have data (V_1, W_1, Φ_1) for the first ground state, and (V_2, W_2, Φ_2) for the second. We need some extra information which comes in the form of some anomaly trivializing information, consisting of a choice of "category of walls" between the two ground states. Looking at the full operator content, the two vacua turn the data into a 2×2 block matrix:

$$\left(\begin{array}{c|c} (V_1, W_1, \Phi_1) & \text{extra data} \\ \hline (\text{extra data})^T & (V_2, W_2, \Phi_2) \end{array}\right) \tag{2.2.6}$$

where the "extra data" encodes the passage from one ground state to the other. Another way to view this direct sum is to consider instead of working purely in $2d$, working with a pair consisting of

$$\{(1{+}1)d \text{ boundary condition, } (2{+}1)d \text{ theory}\}.$$

The operator content as well as the anomalous data of the $2d$ theory determines an absolute $2d$-$3d$ theory i.e. the data (V_1, W_1, Φ_1) leads to data about a cobordism invariant of 3-manifolds coming from the central charges of (V_1, W_1, Φ_1) where the operators in $3d$ are the center of the operators in $2d$. We can take direct sums of absolute $2d$-$3d$ theories, by summing in both $2d$ and $3d$.

A concrete realization of the extra data that can appear between two ground states can be observed in anyon condensation. This gives a way of interpolating between going between two different $(2{+}1)d$ topological orders by deforming adiabatically. In the categorical language we start off with a MTC $\mathcal{C}$ and consider some condensible algebra α, built by condensing an abelian line $a \in \mathcal{C}$ of integer spin that generates the one-form symmetry with the vaccuum [138, 169]. Performing the condensation involves projecting out lines that have nontrivial monodromy charge with a. We then land in a new MTC $\mathcal{C}'$ separated from $\mathcal{C}$ by a gapped domain wall. The domain wall excitations are given by the category of α-modules in $\mathcal{C}$ which can fuse but not braid with each other.

Let us furthermore consider the case in which a G-symmetry acts on the local ground states. In special cases, the G-symmetry may protect a degeneracy in ground states, by forbidding any small deformation in the form of local operators that can distinguish between ground states. By turning off the symmetry we might expect a decomposition into a direct sum of subtheories, but more information is necessary. Namely, the data regarding the relative phases between the ground states. At a particular ground state, it is possible to stack with a trivial, or invertible theory. Such theories are invertible from the point of view of field theory, which says we have a map $\alpha : \mathbf{Bord}_3 \to \mathbf{Vec}_{\mathbb{C}}$ from three dimensional bordisms of two manifolds to specifically the subset of *invertible vector spaces*. Invertibility of vector spaces is given by invertibility under the tensor product, and a vector space is only invertible under tensor product if it is one dimensional. Hence, we require $\alpha(Y^2) \in \mathbf{Vec}_{\mathbb{C}}$ to be a line. Invertible field theories are conjectured to describe the low energy limits of symmetry protect topological (SPT) phases, where a symmetry G protects the phase from being gapless. We will associate the language of "stacking with SPT" in order to refer to the relative information that can exist between ground states. In (1+1)d, these SPTs are classified by $H^2(G\,;\mathrm{U}(1))$. If however, as in many cases, G has an action on the coefficients, then this *twisted cohomology* may be nontrivial. More precisely, for any module $\rho : \mathbb{Z}[G] \to \mathrm{Aut}(A)$ over $G = C_N = \langle t\rangle$ cyclic, we have:

$$H^\bullet(G;A) = \begin{cases} A^G \text{ fixed points}, & \bullet = 0, \\ \ker(\rho(N))/\operatorname{im}(\rho(t)-1), & \bullet = \text{odd}, \\ \ker(\rho(t)-1)/\operatorname{im}(\rho(N)), & \bullet = \text{even} > 0. \end{cases} \tag{2.2.7}$$

Here $N = 1 + t + t^2 + \ldots + t^{n-1}$. As an explicit example, take $G = \mathbb{Z}_2^T$ and $A = \mathrm{U}(1)$, where $\mathbb{Z}_2^T$ acts by complex conjugation on $\mathrm{U}(1)$. Then $H^2(\mathbb{Z}_2^T\,;\mathrm{U}(1)) = \mathbb{Z}_2$, which means we have two choices of SPT that can manifest as information of a relative phases between the ground states.

2.2.4. 2-group Background Gauge Fields

The decomposition structure of theories in (1+1)d implies that we must modify our 2-group ingredients to take this into account. The Postnikov $\beta_{\mathbb{G}}$, which is a class in $H^3(G\,;\mathcal{A}_{[1]})$, must now be modified to encompass the statement that the lack of associativity in G-symmetry defects can manifest as a complex linear combination of invertible one-form operators at a trivalent junction of G-defects. There exists a group homomorphism $f : \mathcal{A}_{[1]} \to \mathbb{C}[\mathcal{A}_{[1]}]^\times$ that acts functorially on cohomology. Therefore, we can build $[f(\beta_{\mathbb{G}})] \in H^3(G\,;\mathbb{C}[\mathcal{A}_{[1]}]^\times)$. We now use the fact that there is a ring isomorphism of

$$\mathbb{C}[\mathcal{A}_{[1]}]^\times = \bigoplus_{\widehat{A}} \mathbb{C}^\times\,, \tag{2.2.8}$$

to express $[f(\beta_{\mathbb{G}})]$ as a class

$$[f(\beta_{\mathbb{G}})] \in \bigoplus_{\widehat{A}} H^3(G\,;\mathbb{C}^\times) = \bigoplus_{\widehat{A}} H^3(G\,;\mathrm{U}(1)). \tag{2.2.9}$$

We will denote a 2-group theory $\mathcal{T}$ with partition function $\mathcal{Z}^{\mathcal{T}}_{A^{(1)};\,B^{(2)}}$, where $A^{(1)}$ and $B^{(2)}$ are the background gauge fields that couple to the zero-form symmetry, and one-form symmetry respectively. The upper indices denote the fact that an n-form symmetry couples to an $(n{+}1)$-form background gauge field. Under gauge transformations the fields transform as [19, 211, 96]

$$A^{(1)} \to A^{(1)} + \frac{1}{N} d\lambda^{(0)},\;\; B^{(2)} \to B^{(2)} + \frac{1}{M} d\Lambda^{(1)} + \frac{\beta_{\mathbb{G}}}{NM}\lambda^{(0)} \cup dA^{(1)}\,, \tag{2.2.10}$$

where $\lambda^{(0)}$ and $\Lambda^{(1)}$ are zero and one cochains for the background G and $\mathcal{A}_{[1]}$ symmetries respectively. The transformation in $B^{(2)}$ involving a mixture between the two symmetries is characteristic of a 2-group theory, and is parametrized by the Postnikov. The cup product is used in the way which is described in [95], for cup product Chern-Simons theories.

While the Postnikov was touted as a topological defect in §2.2.3 valued in groupoids, here in the gauge transformations $\beta_{\mathbb{G}}$ takes a numerical value as an integer modulo M representing the class $[\beta_{\mathbb{G}}] \in H^3(G\,;A)$[3]. For convenience, from now on we normalize all the background connections for discrete symmetry groups so that there is no need to divide by the order of the group. Any integral over a gauge field is understood to be analogous to a discrete Fourier transformation. The partition function $\mathcal{Z}^{\mathcal{T}}$ attains a term under gauge transformation which takes the form

$$\exp\left(i\,\beta_{\mathbb{G}}\int \lambda^{(0)}\cup dA^{(1)}\right). \tag{2.2.11}$$

But, because the Postnikov class should now more precisely be thought of as the class $[f(\beta_{\mathbb{G}})]$, the gauge field $B^{(2)}$ in principle could have different 2-group gauge transformation on each sector of the original theory

$$A^{(1)}_{\widehat{\alpha}} \to A^{(1)}_{\widehat{\alpha}} + d\lambda^{(0)}_{\widehat{\alpha}},\;\; B^{(2)}_{\widehat{\alpha}} \to B^{(2)}_{\widehat{\alpha}} + d\Lambda^{(1)}_{\widehat{\alpha}} + f(\beta_{\mathbb{G}})_{\widehat{\alpha}}\,\lambda^{(0)}_{\widehat{\alpha}}\cup dA^{(1)}_{\widehat{\alpha}}, \tag{2.2.12}$$

and therefore the partition function also attains a term on each sector of the form

$$\bigoplus_{\widehat{\alpha}} \mathcal{Z}^{\mathcal{T}}_{A^{(1)}_{\widehat{\alpha}};\,B^{(2)}_{\widehat{\alpha}}}\,\exp\left(i\,f(\beta_{\mathbb{G}})_{\widehat{\alpha}}\int \lambda^{(0)}_{\widehat{\alpha}}\cup dA^{(1)}_{\widehat{\alpha}}\right). \tag{2.2.13}$$

We remark that even if $f(\beta_{\mathbb{G}})_{\widehat{\alpha}}$ does not vanish, this phase should not be considered an anomaly in the usual sense of 't Hooft anomalies. Since we are strictly speaking dealing with a zero-form symmetries extended by one-form symmetries, the change in the partition function under gauge transformation is controlled by the extension, which manifests as a one form operator.

2.2.5. Gauging in a 2-group theory

Having established the formalism for the partition function and gauge transformations, in this section we begin to manipulate the theory at the level of its partition function. For $\mathcal{T}$ in which $\mathcal{A}_{[1]}$ acts nonanomalously, we can ask to gauge this symmetry. After this we land on a theory denoted $\mathcal{T}//\mathcal{A}_{[1]}$. We will see that upon gauging the one-form symmetry, $\mathcal{T}//\mathcal{A}_{[1]}$ will have a mixed anomaly between the zero-form symmetry and the "dual" symmetry,

[3]It is important to keep track of what category $\beta_{\mathbb{G}}$ is valued in so we can distinguish between (mixed) anomalies and 2-groups.

which is a (-1)-form symmetry, $\widehat{\mathcal{A}}_{[-1]}$, controlled by the Postnikov, from the ungauged 2-group. To tell this story in a more familiar way, we outline the gauging procedure with zero-form symmetries; the logic carries over to one-form symmetries upon shifting some indices. Suppose we have a theory $\mathcal{T}'$ in d-dimensions with an action of $\widetilde{G} = G_{[0]} \rtimes_\beta A_{[0]}$ where $\beta \in H^2(G_{[0]}; A_{[0]})$, so that $\widetilde{G}$ is an extension by two zero-form symmetries. In gauging $A_{[0]}$ in d-dimensions we expect to get a dual group that is a $\widehat{A}_{[d-2]}$ in $\mathcal{T}'//A_{[0]}$ and there could be a purely mixed anomaly between $G_{[0]} \times \widehat{A}_{[d-2]} = \widehat{G}$ which is a class in $H^{d+1}(\widehat{G}\,; \mathrm{U}(1))$. This cohomology can be calculated by a spectral sequence, in which on the E_2 page we have $H^p(G; H^q(\widehat{A}_{[d-2]}\,; \mathrm{U}(1)))$, that converges to $H^{p+q}(\widehat{G}; \mathrm{U}(1))$. We have that $H^{d-1}(\widehat{A}_{[d-2]}\,; \mathrm{U}(1)) = \mathrm{Hom}(\widehat{A}_{[d-2]}\,, \mathrm{U}(1)) = A_{[0]}$, and $H^2(G; H^{d-1}(\widehat{A}_{[d-2]}\,; \mathrm{U}(1)))$ converges to $H^{d+1}(\widehat{G}; \mathrm{U}(1))$. This is interpreted to mean that the mixed anomaly is given by the extension β from the original $\mathcal{T}'$ theory, by cupping with a $(d-1)$-cochain valued in $\widehat{A}_{[d-2]}$.

One can also tell this story in reverse. Suppose that a theory $\widetilde{\mathcal{T}}'$ is acted upon by the group $G \times \widehat{A}_{[d-2]}$ and there exists a purely mixed anomaly $\alpha \in H^2(G\,; H^{d-1}(\widehat{A}_{[d-2]}\,; \mathrm{U}(1)))$. In gauging $\widehat{A}_{[d-2]}$, we expect a $A_{[0]}$ symmetry in $\widetilde{\mathcal{T}}'//\widehat{A}_{[d-2]}$. The analogue of 2-groups involving G and $A_{[0]}$ is controlled by an extension in $H^2(G\,; A_{[0]}) =$ $H^2(G\,; H^{d-1}(\widehat{A}_{[d-2]}\,; \mathrm{U}(1)))$, which is α. We see through this spectral sequence argument that a theory with a mixed anomaly, when gauged, becomes a theory with 2-group symmetry, where the class of the anomaly becomes the extension defining the 2-group.

When we focus specifically for our theory $\mathcal{T}$ in (1+1)-dimensions, then our extension is valued in $H^3(G\,; \mathbb{C}[\mathcal{A}_{[1]}]^\times)$. Due to the functorality of the spectral sequence, there is a homomorphism from the spectral sequence of $\mathbb{C}[\mathcal{A}_{[1]}]^\times.\,G$ to the spectral sequence of $\mathcal{A}_{[1]}.G$, here as $X.Y$ denotes the extension of Y by X. The homomorphism is given on the E_2 page by

$$H^\bullet(G; f^\bullet) : H^\bullet(G; H^\bullet(\mathbb{C}[\mathcal{A}_{[1]}]^\times; \mathrm{U}(1))) \to H^\bullet(G; H^\bullet(\mathcal{A}_{[1]}; \mathrm{U}(1)))\,, \tag{2.2.14}$$

where

$$f^\bullet : H^\bullet(\mathbb{C}[\mathcal{A}_{[1]}]^\times; \mathrm{U}(1)) \to H^\bullet(\mathcal{A}_{[1]}; \mathrm{U}(1)) \tag{2.2.15}$$

is the pull back of the homomorphism $f : \mathcal{A}_{[1]} \to \mathbb{C}[\mathcal{A}_{[1]}]^\times$. This means that upon gauging, it makes sense to take the restriction of $f(\beta_{\mathbb{G}})$ to each element $\widehat{\alpha} \in \widehat{\mathcal{A}}_{[1]}$, and therefore there is a notion by which the mixed anomaly can be restricted over any particular subsector into which a theory decomposes.

We now perform the gauging explicitly at the level of partition function. In the following, we use lower case letters to denote that the background gauge field is being integrated over in the path integral. In order to gauge the one form symmetry in $\mathcal{Z}^{\mathcal{T}}$ we integrate over the two-form gauge field [49]

$$\mathcal{Z}^{\mathcal{T}//\mathcal{A}_{[1]}}_{A^{(1)};\,C^{(0)}} = \int Db^{(2)}\, \mathcal{Z}^{\mathcal{T}}_{A^{(1)};\,b^{(2)}} \exp\left(-i \int C^{(0)} \cup b^{(2)}\right), \tag{2.2.16}$$

where $C^{(0)}$ is the background gauge field for the dual (-1)-form symmetry [50] in $\mathcal{T}//\mathcal{A}_{[1]}$. One can furthermore gauge the (-1)-form symmetry by summing over the zero-form gauge field. This takes us back to $\mathcal{T}$:

$$
\begin{aligned}
&\int Dc^{(0)}\, \mathcal{Z}^{\mathcal{T}//\mathcal{A}_{[1]}}_{A^{(1)};\,c^{(0)}} \exp\left(i\int B^{(2)}\cup c^{(0)}\right)\\
&= \int Dc^{(0)} Db^{(2)}\, \mathcal{Z}^{\mathcal{T}}_{A^{(1)};\,b^{(2)}} \exp\left(-i\int c^{(0)}\cup b^{(2)}\right)\exp\left(i\int B^{(2)}\cup c^{(0)}\right)\\
&= \mathcal{Z}^{\mathcal{T}}_{A^{(1)};\,B^{(2)}}.
\end{aligned}
\tag{2.2.17}
$$

We now move on to the case of a two group with $\beta_{\mathbb{G}}$, and again gauge the one form symmetry in the manner of (2.2.16). If we implement the 2-group transformation in (2.2.10) we get

$$
\begin{aligned}
\mathcal{Z}^{\mathcal{T}//\mathcal{A}_{[1]}}_{A^{(1)};C^{(0)}} &= \int Db^{(2)}\, \mathcal{Z}^{\mathcal{T}}_{A^{(1)};\,b^{(2)}} \exp\left(-i\int C^{(0)}\cup b^{(2)}\right)\\
&\quad\times \exp\left(-i\int C^{(0)}\cup d\Lambda^{(1)}\right)\exp\left(-i\int C^{(0)}\cup\left(\beta_{\mathbb{G}}\cup\lambda^{(0)}\cup dA^{(1)}\right)\right),
\end{aligned}
\tag{2.2.18}
$$

where $C^{(0)}$ and $\lambda^{(0)}\cup dA^{(1)}$ take value in $\widehat{\mathcal{A}}_{[-1]}$ and G and the Postnikov we take to be valued in $\mathcal{A}_{[1]}$ instead of its c-number value as in (2.2.10). We claim that the (-1)-form symmetry is best thought of as $\mathrm{Spec}(\mathcal{A}_{[1]})(\mathbb{C}) = \hom(\mathcal{A}_{[1]},\mathbb{C})$, and presently justify this proposition. Inside the exponential, cupping $C^{(0)}$ with $\beta_{\mathbb{G}}$ is this map and gives a c-number contribution multiplying the curvature of the background G gauge field. If we consider attaching a term in the above partition function of the form

$$
\exp\left(i\,\left(C^{(0)}\cup\beta_{\mathbb{G}}\right)\int\lambda^{(0)}\cup dA^{(1)}\right),
\tag{2.2.19}
$$

this will exactly cancel out the exponential on the right hand side in (2.2.18). This factor is a mixed anomaly in the gauged theory, which is controlled by the extension $\beta_{\mathbb{G}}$ we started off with in the 2-group ungauged theory. Going in reverse, consider a mixed anomaly classified by $\widehat{\beta}_{\widetilde{\mathbb{G}}}\in H^3(G\,,\widehat{\mathcal{A}}_{[-1]})$ in the theory $\mathcal{T}//\mathcal{A}_{[1]}$ and gauge the $\widehat{\mathcal{A}}_{[-1]}$ symmetry by integrating over $C^{(0)}$. The partition function takes the form

$$
\begin{aligned}
&\mathcal{Z}^{\mathcal{T}//\mathcal{A}_{[1]}}_{A^{(1)};C^{(0)}}\ \exp\left(i\widehat{\beta}_{\widetilde{\mathbb{G}}}\int\lambda^{(0)}\cup \mathrm{d}A^{(1)}\cup C^{(0)}\right)\\
&\xrightarrow{\text{gauge}}\int Dc^{(0)}\mathcal{Z}^{(\mathcal{T}//\mathcal{A}_{[1]})//\widehat{\mathcal{A}}_{[-1]}}_{A^{(1)};\,c^{(0)}}\exp\left(i\widehat{\beta}_{\widetilde{\mathbb{G}}}\int\lambda^{(0)}\cup \mathrm{d}A^{(1)}\cup c^{(0)}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\exp\left(i\int B^{(2)}\cup c^{(0)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \int Dc^{(0)} Db^{(2)} \mathcal{Z}^{\mathcal{T}}_{A^{(1)};\, b^{(2)}} \exp\left(i\widehat{\beta}_{\widetilde{\mathbb{G}}} \int \lambda^{(0)} \cup \mathrm{d}A^{(1)} \cup c^{(0)} \right) \\
&\qquad\qquad\qquad\qquad \times \exp\left(-i \int c^{(0)} \cup b^{(2)} \right) \exp\left(i \int B^{(2)} \cup c^{(0)} \right) \\
&= \mathcal{Z}^{\mathcal{T}}_{A^{(1)};\, B^{(2)} + \widehat{\beta}_{\widetilde{\mathbb{G}}} \int \lambda^{(0)} \cup dA^{(1)}} \\
&= \mathcal{Z}^{\mathcal{T}}_{A^{(1)};\, B^{(2)}} \exp\left(i\widehat{\beta}_{\widetilde{\mathbb{G}}} \int \lambda^{(0)} \cup dA^{(1)} \right),
\end{aligned} \tag{2.2.20}
$$

which is exactly the transformation in a two group theory with Postnikov $\widehat{\beta}_{\widetilde{\mathbb{G}}}$. Note here that $\left(i\widehat{\beta}_{\widetilde{\mathbb{G}}} \int \lambda^{(0)} \cup dA^{(1)} \right)$ plays a different role than in (2.2.19), where the Postnikov was cupped with C^0 because $\widehat{\beta}_{\widetilde{\mathbb{G}}}$ is strictly speaking valued in $\widehat{\mathcal{A}}_{[-1]}$, and there is no sense in which it can be canceled in the same way a phase can be.

We return back to considering the Postnikov under the homomorphism f as in (2.2.9) and apply this to the partition function of $\mathcal{Z}_{\mathcal{T}}$. By conducting the gauging procedure of (2.2.18) along with the functorial property of f we see that there are mixed anomalies in each subsector labeled by $\widehat{\alpha}$, and the partition function is given by

$$
\mathcal{Z}^{\mathcal{T}//\mathcal{A}_{[1]}}_{A^{(1)}_{\widehat{\alpha}};\, C^{(0)}_{\widehat{\alpha}}} = \int Db^{(2)}_{\widehat{\alpha}} \mathcal{Z}^{\mathcal{T}}_{A^{(1)}_{\widehat{\alpha}};\, b^{(2)}_{\widehat{\alpha}}} \exp\left(i \left(C^{(0)}_{\widehat{\alpha}} \cup f(\beta_{\mathbb{G}})_{\widehat{\alpha}} \right) \int \lambda^{(0)}_{\widehat{\alpha}} \cup dA^{(1)}_{\widehat{\alpha}} \right), \tag{2.2.21}
$$

where the subscript $\widehat{\alpha}$ denotes the restriction of the gauge field to the subsector $\widehat{\alpha}$ and the full gauge field is written as $C^{(0)} = \bigoplus_{\widehat{\alpha}} C^{(0)}_{\widehat{\alpha}}$. Here, $\langle C^{(0)}_{\widehat{\alpha}} \cup - \rangle : \hom(\mathcal{A}_{[1]}, \mathbb{C})\big|_{\widehat{\alpha}}$ and the term $\left(i\, f(\beta_{\mathbb{G}})_{\widehat{\alpha}} \cup C^{(0)}_{\widehat{\alpha}} \right)$ makes sense as a complex number. This makes $\left(i\, f(\beta_{\mathbb{G}})_{\widehat{\alpha}} \cup C^{(0)}_{\widehat{\alpha}} \right) \int \lambda^{(0)}_{\widehat{\alpha}} \cup dA^{(1)}_{\widehat{\alpha}}$ a mixed anomaly.

2.2.6. Space of (-1)-form symmetries

In order to fit in line with the definition of symmetry defects given in §2.2, a (-1)-form symmetry for a d-dimensional theory must be implemented by codimension zero defects, or "spacefilling defects", which are theories themselves. This says that in gauging a one-form symmetry, we have effectively projected ourselves onto a subtheory of the original $\mathcal{T}$ existing in the direct sum. Furthermore, gauging this (-1)-form symmetry should give us back a family of theories as can be seen in the following way. For an element $\widehat{\alpha}$ of $\widehat{\mathcal{A}}_{[-1]}$ we can build the term $\exp\left(i \int B^{(2)} \cup C^{(0)}_{\widehat{\alpha}} \right)$ to be inserted into the path integral along with the partition function $\mathcal{Z}^{\mathcal{T}//\mathcal{A}_{[1]}}$. We then take a direct sum over $\widehat{\alpha}$ while integrating over gauge field corresponding to $\widehat{\mathcal{A}}_{[-1]}$,

$$
\begin{aligned}
Z^{\mathcal{T}//\mathcal{A}_{[1]}//\widehat{\mathcal{A}}_{[-1]}} &= \bigoplus_{\widehat{\alpha}} \int Dc^{(0)}_{\widehat{\alpha}} \int Db^{(2)} \mathcal{Z}^{\mathcal{T}}_{A^{(1)};\,b^{(2)}} \exp\left(-i\int b^{(2)} \cup c^{(0)}_{\widehat{\alpha}}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \exp\left(i\int B^{(2)} \cup c^{(0)}_{\widehat{\alpha}}\right) \\
&= \mathcal{Z}^{\mathcal{T}}_{A^{(1)};\,B^{(2)}} \,,
\end{aligned} \tag{2.2.22}
$$

which takes us back to the original family of theories with a one-form symmetry.

From the point of view of a defect that selects a particular subsector of a theory, we see that there is no such p-form symmetry for $p < (-1)$. As a brief note for completeness we give a mathematical way to understand other negative form symmetries. We explained in §2.2 that one-form symmetries are one-to-one with group objects in groupoids, such that only π_1 was nontrivial. We can define an n-groupoid as a category in which objects support $\pi_0, \pi_1, \ldots, \pi_n$ homotopies. Given G, a group object in n-groupoids, it is possible to form what we will call BG, which is an $n+1$-groupoid. Here, the group law of the n-groupoid becomes the composition law in the $n+1$-groupoid. Furthermore, $\pi_{i-1}G = \pi_i BG$, and thus $\pi_{-1}G = \pi_0 BG$, the right-hand-side of the equality being well defined. This gives a view of (-1)-form symmetry in terms of homotopy if only $\pi_0 BG$ is nontrivial for G is a group object in (-1)-groupoids. Instead of n-groupoids being defined with only a single group law, it is possible to include multiple group laws. In essence, this means that associativity can be given multiple ways of being isomorphic. Starting with an n-groupoid and permitting two group laws, it is possible to form an $(n+1)$-groupoid with a single group law, and subsequently an $(n+2)$-groupoid. This provides a mathematical way to define π_{-n} and therefore $(-n)$-form symmetries if one is willing to consider multiple group laws.

2.2.7. 2-group Anomalies

In (1+1)d a theory with 2-group global symmetry can itself have an anomaly that is a class in $H^3(\mathbb{G}\,, \mathrm{U}(1))$. This amounts to asking whether the entire 2-group can be gauged, or if there is an obstruction to doing so. As was pointed out in [19], the fact that this anomaly is a class in the third cohomology is a facet of the dimension of the theory we are considering, and should not be confounded with the Postnikov, which is strictly speaking valued in $\mathcal{A}_{[1]}$ and is not a bona fide anomaly. We will study this 2-group anomaly by using the Serre spectral sequence and by using the convergence of $H^\bullet(G; H^\bullet(\mathcal{A}_{[1]}; \mathrm{U}(1))) \Rightarrow H^\bullet(\mathbb{G}; \mathrm{U}(1))$; the group of one-form symmetry, in this case, we take to be cyclic of odd order. The zero-form symmetry we still leave to be a general finite group. For this and other subsequent calculations, we will only focus on the low degree cohomology.

The E_2 page has $d_2 = 0$, and the next differential $d_3 = \langle \beta_{\mathbb{G}}, - \rangle$. If

$\omega \in H^\bullet(G; H^\bullet(\mathcal{A}_{[1]}; \mathrm{U}(1)))$, then

$$d_3(\omega) = \beta_{\mathbb{G}} \cup \omega \in H^{\bullet+3}(G; A \otimes H^\bullet(\mathcal{A}_{[1]}\,; \mathrm{U}(1))). \tag{2.2.23}$$

To see that this makes sense we note that since $\mathcal{A}_{[1]}$ is a one-form symmetry, then for the underlying group we have $A = H_2(\mathcal{A}_{[1]}\,; \mathbb{Z}) = H_2(K(A,2)\,; \mathbb{Z})$. Thus this slant product is the map $H_2 \otimes H^\bullet \to H^{\bullet-2}$. This shows that our claim for d_3 is a map $H^\bullet(G; H^\bullet(\mathcal{A}_{[1]}; \mathrm{U}(1))) \to H^{\bullet+3}(G; H^{\bullet-2}(\mathcal{A}_{[1]}; \mathrm{U}(1)))$, as it should be. The E_2 page in degree ≤ 4 looks like

$$\begin{array}{c|ccccc}
S^2\widehat{A} & S^2\widehat{A} & H^1(G; S^2\widehat{A}) & H^2(G; S^2\widehat{A}) & H^3(G; S^2\widehat{A}) & H^4(G; S^2\widehat{A}) \\
0 & 0 & 0 & 0 & 0 & 0 \\
\widehat{A} & \widehat{A} & H^1(G; \widehat{A}) & H^2(G; \widehat{A}) & H^3(G; \widehat{A}) & H^4(G; \widehat{A}) \\
0 & 0 & 0 & 0 & 0 & 0 \\
\mathrm{U}(1) & \mathrm{U}(1) & H^1(G; \mathrm{U}(1)) & H^2(G; \mathrm{U}(1)) & H^3(G; \mathrm{U}(1)) & H^4(G; \mathrm{U}(1)) \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}\;, \tag{2.2.24}$$

where $\widehat{A} = H^2(\mathcal{A}_{[1]}; \mathrm{U}(1))$, and $S^2\widehat{A} = \mathrm{Sym}^2\,\widehat{A}$ which are the quadratic forms on $\widehat{A}$. In total degree 3, we have $\bigoplus_{i=0}^3 H^i(G\,; H^{3-i}(\mathcal{A}_{[1]}\,, \mathrm{U}(1))$ and so we consider if any of these elements can support, or receive a differential d_3. By the property of differential

$$H^i(G; H^{2-i}(\mathcal{A}_{[1]}\,; \mathrm{U}(1))) \xrightarrow{d_3} H^{i+3}(G; H^{-i}(\mathcal{A}_{[1]}\,; \mathrm{U}(1))) = 0, \text{ if } i \neq 0, \tag{2.2.25}$$

and therefore the only d_3 in this case is from $H^0(G; H^2(\mathcal{A}_{[1]}\,; \mathrm{U}(1))) = \widehat{A}$, which lands in $H^3(G; H^0(\mathcal{A}_{[1]}\,; \mathrm{U}(1))) = H^3(G; \mathrm{U}(1))$. Furthermore we have

$$H^i(G; H^{3-i}(\mathcal{A}_{[1]}\,; \mathrm{U}(1))) \xrightarrow{d_3} H^{i+3}(G; H^{1-i}(\mathcal{A}_{[1]}\,; \mathrm{U}(1))) = 0, \text{ if } i \neq 0, 1\,, \tag{2.2.26}$$

but the coefficient $H^3(\mathcal{A}_{[1]}\,; \mathrm{U}(1)) = 0$. Therefore the only d_3 here is from $H^1(G\,, H^2(\mathcal{A}_{[1]}\,; \mathrm{U}(1))$ to $H^4(G\,; \mathrm{U}(1))$. The E_∞ page in total degree 3 contains two entries, which are

$$\ker(d_3 = \langle - \cup \beta\rangle : H^1(G; \widehat{A}) \to H^4(G; \mathrm{U}(1))) \text{ and } \operatorname{coker}(d_3 : \widehat{A} \to H^3(G; \mathrm{U}(1)))\,. \tag{2.2.27}$$

There is an extension problem to solve here, with $H^3(\mathbb{G}\,; \mathrm{U}(1))$ fitting in the short exact sequence

$$\operatorname{coker}(d_3 : \widehat{A} \to H^3(G; \mathrm{U}(1))) \to H^3(\mathbb{G}\,, \mathrm{U}(1)) \to \ker(d_3 = \langle - \cup \beta_{\mathbb{G}}\rangle : H^1(G; \widehat{A}) \to H^4(G; \mathrm{U}(1))). \tag{2.2.28}$$

There is an image of $\omega \in H^3(\mathbb{G}; \mathrm{U}(1))$, which we call α, is such that $d\alpha = 0$ and is in the kernel of d_3, i.e., is zero in cohomology. Since $\beta_{\mathbb{G}}$ was chosen to be a cocycle, then $\alpha \cup \beta_{\mathbb{G}}$ is a cocycle; we claim that it is $d\gamma$ for some 3-cochain γ coming from the cokernal set, that witnesses α being in the kernal of d_3. This means that the cohomology $H^3(\mathbb{G}; \mathrm{U}(1))$ consists of pairs (α, γ) where $\alpha : G \to \widehat{A}$ is a homomorphism, and $\gamma \in \mathrm{C}^3(G; \mathrm{U}(1))$, such that

$$d\gamma = \alpha \cup \beta_{\mathbb{G}}. \tag{2.2.29}$$

If we go to a specific case where $G = \mathbb{Z}_2$, then $\alpha = 0$ since A was a finite group of odd order, and then γ is in fact a cocycle. In this case the group the cohomology would just be given by $\gamma \in H^3(G, \mathrm{U}(1)) = G = \mathbb{Z}_2$. This is also true in general whenever G is finite cyclic, and $\gcd(|G|, |A|) = 1$.

2.2.8. 2-groups and Supercohomology

We calculate the anomalies in $(1{+}1)d$ fermionic theories with 2-group global symmetry; these anomalies live in supercohomology. We will more specifically consider what is known as extended supercohomology [239] in the following. Bosonic anomalies that are $\frac{1}{2}\,(\mathrm{mod}\ 1)$, when restricted to a one-form symmetry subgroup of $\mathbb{G}$, become trivialized in supercohomology.

We take $\mathrm{fGP}^\times$ to be the spectrum of fermionic phases. This is a sequence of of topological spaces, namely fermionic invertible gapped systems $\ldots, \mathrm{fGP}^\times_{-1}, \mathrm{fGP}^\times_0, \mathrm{fGP}^\times_1,$ $\mathrm{fGP}^\times_2, \ldots$ where the subscript denotes the spacetime dimension. This sequence comes with homotopy equivalences $\mathrm{fGP}^\times_{n-1} \xrightarrow{\sim} \Omega\, \mathrm{fGP}^\times_n$, and since $\pi_n \Omega\, \mathrm{fGP}^\times_n = \pi_{n+1}\, \mathrm{fGP}^\times_n$, we define the homotopy groups of $\mathrm{fGP}^\times$ by $\pi_n\, \mathrm{fGP}^\times = \pi_0\, \mathrm{fGP}^\times_{-n}$ and $\pi_{-n}\, \mathrm{fGP}^\times = \pi_0\, \mathrm{fGP}^\times_n$. Calculating these groups gives the n-dimensional fermionic phases with abelian group structure $\pi_0\, \mathrm{fGP}^\times_{-n}$, where the group composition is by stacking. In what follows we will only focus on the low dimensional homotopy groups of this spectrum, the well established ones are [110]:

$$\pi_0\, \mathrm{fGP}^\times = \mathrm{U}(1), \quad \pi_{-1}\, \mathrm{fGP}^\times = \mathbb{Z}_2, \quad \pi_{-2}\, \mathrm{fGP}^\times = \mathbb{Z}_2. \tag{2.2.30}$$

The only nontrivial degree-2 stable cohomology operation from $\mathbb{Z}_2$ to $\mathrm{U}(1)$ is $(-1)^{\mathrm{Sq}^2}$, as $\mathrm{Sq}^2 : H^\bullet(-\,; \mathbb{Z}_2) \to H^{\bullet+2}(-\,; \mathbb{Z}_2)$ and $(-1)^x : H^\bullet(-\,; \mathbb{Z}_2) \to H^\bullet(-\,; \mathrm{U}(1))$. The nontrivial degree-2 stable cohomology operation connecting $\mathbb{Z}_2$ to $\mathbb{Z}_2$ is just Sq^2.

An n-cocycle in supercohomology (SH^n) consists of a triple (α, β, γ) where α is a degree-n $\mathrm{U}(1)$-cochain, β is a degree-$(n-1)$ $\mathbb{Z}_2$-cochain, and γ is a degree-$(n-2)$ cochain and they solve:

$$d\gamma = 0, \qquad d\beta = \mathrm{Sq}^2\gamma, \qquad d\alpha = (-1)^{\mathrm{Sq}^2\beta} + f(\gamma). \tag{2.2.31}$$

In the literature, γ is referred to as the "Majorana layer" when in degree one, and β is the "Gu-Wen" layer when in degree two, and α is a 't Hooft anomaly in the bosonic sense. We want to calculate the supercohomology $\mathrm{SH}^3(\mathbb{G})$ for $\mathbb{G} = \mathcal{A}_{[1]} \rtimes_{\beta_{\mathbb{G}}} G$, but now G and $\mathcal{A}_{[1]}$ are the group $\mathbb{Z}_2$ and $B\mathbb{Z}_2$ respectively, for otherwise supercohomology reduces to standard cohomology.

For completeness we present this calculation in pieces, where we also calculate the supercohomology of $\mathbb{G} = \mathbb{Z}_2$ and $\mathbb{G} = B\mathbb{Z}_2$. The supercohomology as a generalized cohomology theory takes value in a spectrum. By using the Atiyah-Hirzebruch spectral sequence we have $\mathrm{SH}^\bullet(\mathbb{Z}_2) \Leftarrow H^\bullet(\mathbb{Z}_2, \mathrm{SH}^\bullet(\mathrm{pt}))$ where $\mathrm{SH}^\bullet(\mathrm{pt}) = \pi_{-\bullet}(\mathrm{pt})$. In the extended cohomology case we need:

$$\mathrm{SH}^0(\mathrm{pt}) = \mathrm{U}(1)\,, \quad \mathrm{SH}^1(\mathrm{pt}) = \mathbb{Z}_2\,, \quad \mathrm{SH}^2(\mathrm{pt}) = \mathbb{Z}_2\,, \tag{2.2.32}$$

and the homotopy in higher nonnegative cohomological degree vanishes. The ring $H^\bullet(\mathbb{Z}_2; \mathbb{Z}_2) = \mathbb{Z}_2[t]$ with t in degree one, and except in degree 0, the map $(-1)^x : \mathbb{Z}_2 \to \mathrm{U}(1)$ is a surjection on $H^\bullet(\mathbb{Z}_2; \mathbb{Z}_2) \to H^\bullet(\mathbb{Z}_2; \mathrm{U}(1))$. This gives the E_2 in low degree as

$$\begin{array}{c|cccccc} \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 t & \mathbb{Z}_2 t^2 & \mathbb{Z}_2 t^3 & \mathbb{Z}_2 t^4 \\ \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 t & \mathbb{Z}_2 t^2 & \mathbb{Z}_2 t^3 & \mathbb{Z}_2 t^4 \\ \mathrm{U}(1) & \mathrm{U}(1) & (-1)^{\mathbb{Z}_2 t} & 0 & (-1)^{\mathbb{Z}_2 t^3} & 0 \\ \hline & 0 & 1 & 2 & 3 & 4 & . \end{array} \tag{2.2.33}$$

In what follows we will strip off the $\mathbb{Z}_2$ for simplicity and only leave the generator. The d_2 differential in the Atiyah-Hirzebruch spectral sequence is the k-invariants of the spectrum, these are $\mathrm{Sq}^2 : E_2^{\bullet,2} \to E_2^{\bullet+2,1}$ and $(-1)^{\mathrm{Sq}^2} : E_2^{\bullet,1} \to E_2^{\bullet+2,0}$. On generators t, Sq^2 acts as a second order operator by differentiation. Namely, $\mathrm{Sq}^2 = t^4 \frac{1}{2}\frac{\partial^2}{\partial^2 t} : t^i \mapsto \binom{i}{2} t^{i+2}$. The d_2 mapping out of generators in $i \equiv 0, 1 \mod 4$ therefore vanish, and the d_2 mapping out of generators in $i \equiv 2, 3 \mod 4$ are isomorphisms. The E_3 page is

$$\begin{array}{c|cccccc} \mathbb{Z}_2 & 1 & t & 0 & 0 & t^4 \\ \mathbb{Z}_2 & 1 & t & t^2 & 0 & 0 \\ \mathrm{U}(1) & \mathrm{U}(1) & t & 0 & t^3 & 0 \\ \hline & 0 & 1 & 2 & 3 & 4 & . \end{array} \tag{2.2.34}$$

In total degree ≤ 4, the above E_3 page is the E_∞ page, and so in total degree three $\mathrm{SH}^3(\mathbb{Z}_2) = \mathbb{Z}_2.\mathbb{Z}_2.\mathbb{Z}_2$; now we are left with an extension problem to solve.

We first look at the extension $\mathbb{Z}_2.\mathbb{Z}_2$ between the top and middle row. The extension gives information about the failure of Sq^2 to act linearly on cocycles. For $a, b \in E_\infty^{1,2}$, $\mathrm{Sq}^2(a + b) = (a + b) \cup (a + b) = a^2 + b^2$ only if $a \cup b = b \cup a$, i.e., that the cup product is commutative. The lack of commutativity gives an element $a \cup b \in E_\infty^{2,1}$, so the extension is nontrivial and we get a $\mathbb{Z}_4$ from the top and middle rows. To understand the extension of $\mathbb{Z}_4.\mathbb{Z}_2$ we consider the image of $(-1)^{\mathrm{Sq}^2(t^2)}$ which as shown on the E_3 page is zero in cohomology, implying that it is a coboundary $d\lambda$ for $\lambda \in H^3(\mathbb{Z}_2; \mathrm{U}(1)) = E_\infty^{3,0}$. The extension information is therefore embedded in λ, much like how the data for the first extension was embedded in $a \cup b$, and so the extension is nontrivial. We see that $\mathrm{SH}^3(\mathbb{Z}_2) = \mathbb{Z}_8$. It is also possible to arrive at this conclusion on a more physical level; the spectral sequence reveals that in total degree three, whatever group this is, must have order eight. The supercohomology $\mathrm{SH}^\bullet(K(\mathbb{Z}_2,1))$ classifies the superfusion categories with $\mathbb{Z}_2$ fusion rules. The bosonic shadow must then have an object akin to a "fermion" with $\mathbb{Z}_2$ fusion rules, and there are eight categories: four with Ising fusion rules, two with $\mathbb{Z}_4$ fusion rules, one with $\mathbb{Z}_2^2$ fusion rules and nontrivial associator, and one with $\mathbb{Z}_2^2$ fusion rules and trivial associator, which are part of the $\mathrm{Spin}(N)_1$ monoidal categories. Recognizing them as such allows us to recognize the group structure as a $\mathbb{Z}_8$, and furthermore allows for the identification of the $\mathbb{Z}_8$ in supercohomology with the $\mathbb{Z}_8$ of Bott periodicity.

We now compute $\mathrm{SH}^\bullet(B\mathbb{Z}_2)$ by converging from $H^\bullet(B\mathbb{Z}_2\,;\mathrm{SH}^\bullet(\mathrm{pt}))$. The cohomology ring of $B\mathbb{Z}_2$ with coefficients in $\mathbb{Z}_2$ is generated over the Steenrod algebra with a generator T in degree two [133]:

$$H^\bullet(B\mathbb{Z}_2\,,\mathbb{Z}_2)=\mathbb{Z}_2[T\,,\,\mathrm{Sq}^1(T)\,,\,\mathrm{Sq}^2(T)\,,\,\mathrm{Sq}^2\mathrm{Sq}^1(T)\,,\,\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1(T)\,,\ldots]\,. \tag{2.2.35}$$

The E_2 page is

$$\begin{array}{c|ccccc}
\mathbb{Z}_2 & 1 & 0 & T & \mathrm{Sq}^1(T) & \mathrm{Sq}^2(T)\\
\mathbb{Z}_2 & 1 & 0 & T & \mathrm{Sq}^1(T) & \mathrm{Sq}^2(T)\\
\mathrm{U}(1) & \mathrm{U}(1) & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_4\\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}\,. \tag{2.2.36}$$

We want to determine if the d_2 mapping out from T is nonzero. The Universal Coefficient theorem gives the cohomology $H^\bullet(A,\mathrm{U}(1))$ as an extension of a hom class and an Ext class in homology with $\mathbb{Z}$ coefficients. This reveals that row zero of the E_2 page consists solely of hom classes, because $\mathrm{Ext}(A,\mathrm{U}(1))=0$ for any abelian group A. The first row consists of hom and Ext classes; applying Sq^2 to T in $E_2^{2,1}$ gives $\mathrm{Sq}^2(T)$ but there is no Ext class in that degree, which mean $\mathrm{Sq}^2(T)$ must be a hom class. Since the map $(-1)^x:\mathbb{Z}_2\to\mathrm{U}(1)$ is injective, the map on cohomology must be injective on hom classes. Therefore $(-1)^{\mathrm{Sq}^2(T)}\neq 0$, and because this map is injective, we find that T is killed by this differential. Thus, $\mathrm{SH}^3(B\mathbb{Z}_2)=0$ as nothing survives in that degree.

At this point we can compute $\mathrm{SH}^\bullet(B\mathbb{Z}_2\rtimes_\beta\mathbb{Z}_2)$, on the E_2 page for the Atiyah-Hirzebruch spectral sequence we will need $H^\bullet(B\mathbb{Z}_2\rtimes_\beta\mathbb{Z}_2\,;\mathrm{U}(1))$ and $H^\bullet(B\mathbb{Z}_2\rtimes_\beta\mathbb{Z}_2\,;\mathbb{Z}_2)$. We build up the E_2 page for $\mathrm{SH}^\bullet(B\mathbb{Z}_2\rtimes_\beta\mathbb{Z}_2)$ in steps, first starting by obtaining $H^\bullet(B\mathbb{Z}_2\rtimes_\beta\mathbb{Z}_2\,;\mathrm{U}(1))$ and $H^\bullet(B\mathbb{Z}_2\rtimes_\beta\mathbb{Z}_2\,;\mathbb{Z}_2)$ with the Serre spectral sequence. The E_2 page of $H^\bullet(\mathbb{Z}_2\,;H^\bullet(B\mathbb{Z}_2\,;\mathrm{U}(1)))$ is

$$\begin{array}{c|cccccc}
\mathbb{Z}_4 & \mathbb{Z}_4 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2\\
0 & 0 & 0 & 0 & 0 & 0\\
\mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2\\
0 & 0 & 0 & 0 & 0 & 0\\
\mathrm{U}(1) & \mathrm{U}(1) & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0\\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}\,. \tag{2.2.37}$$

The d_2 differential vanishes for degree reasons and the d_3 differential is given by information of the extension, $\beta\in H^3(\mathbb{Z}_2\,;B\mathbb{Z}_2)$. The row in degree two represents $\widehat{\mathbb{Z}}_2=\hom(B\mathbb{Z}_2,\mathrm{U}(1))$ by the Hurewicz theorem, which gives $d_3=(-1)^{\langle-\cup\beta\rangle}:H^\bullet(\mathbb{Z}_2\,;\mathbb{Z}_2)\to H^\bullet(\mathbb{Z}_2\,;\mathrm{U}(1))$. We now consider the d_3 map in the row of degree four, where the $\mathbb{Z}_4$ denotes the space of quadratic forms on $\mathbb{Z}_2$. We must therefore have a map $\{\text{quadratic forms}\} \otimes \mathbb{Z}_2 \to \widehat{\mathbb{Z}}_2$, which in this case is given by the mod 2 reduction, and we find

$$\begin{aligned} d_3 :\ H^\bullet(\mathbb{Z}_2\,;\mathbb{Z}_4) &\to H^\bullet(\mathbb{Z}_2\,;\mathbb{Z}_2) \\ x &\mapsto xt^3 \quad \bmod 2. \end{aligned} \tag{2.2.38}$$

The E_4 page converges to the E_∞ page in low degree and is

$$\begin{array}{c|ccccc} \mathbb{Z}_4 & \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{U}(1) & \mathrm{U}(1) & \mathbb{Z}_2 & 0 & 0 & 0 \\ \hline & 0 & 1 & 2 & 3 & 4 \end{array} \tag{2.2.39}$$

with $H^\bullet(B\mathbb{Z}_2 \rtimes_\beta \mathbb{Z}_2\,;\mathrm{U}(1)) = \mathrm{U}(1)\,,\mathbb{Z}_2\,,0\,,\mathbb{Z}_2\,,\mathbb{Z}_2\,,\ldots$

For the calculation of $H^\bullet(B\mathbb{Z}_2 \rtimes_\beta \mathbb{Z}_2\,;\mathbb{Z}_2)$, on the E_2 page we drop the $\mathbb{Z}_2$ everywhere, and only present the generators at that total degree. This gives

$$\begin{array}{c|ccccc} \mathrm{Sq}^2T & \mathrm{Sq}^2T & t\mathrm{Sq}^2T & t^2\mathrm{Sq}^2T & t^3\mathrm{Sq}^2T & t^4\mathrm{Sq}^2T \\ \mathrm{Sq}^1T & \mathrm{Sq}^1T & t\mathrm{Sq}^1T & t^2\mathrm{Sq}^1T & t^3\mathrm{Sq}^1T & t^4\mathrm{Sq}^1T \\ T & T & tT & t^2T & t^3T & t^4T \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & t & t^2 & t^3 & t^4 \\ \hline & 0 & 1 & 2 & 3 & 4 \end{array}\ , \tag{2.2.40}$$

with $d_3 = t^3 \frac{d}{dT}$. In low degree we have $H^\bullet(B\mathbb{Z}_2 \rtimes_\beta \mathbb{Z}_2\,;\mathbb{Z}_2) = \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2^2, \ldots$. This is consistent with $H^\bullet(B\mathbb{Z}_2 \rtimes_\beta \mathbb{Z}_2\,;\mathrm{U}(1)) = \mathrm{U}(1)\,,\mathbb{Z}_2\,,0\,,\mathbb{Z}_2\,,\mathbb{Z}_2\,,\ldots$, where by the Universal Coefficient theorem, each $\mathbb{Z}_2$ is a class of the same degree, and a class in one degree lower. On E_∞, a basis for $H^\bullet(B\mathbb{Z}_2 \rtimes_\beta \mathbb{Z}_2\,;\mathbb{Z}_2)$ is $\{1, t, t^2, \mathrm{Sq}^1T, T^2, t\mathrm{Sq}^1T, \ldots\}$, and the ring structure is $\mathbb{Z}_2[t, \mathrm{Sq}^1T, T^2, \ldots]/(t^3 = 0, \ldots)$. We assemble now the E_2 page for supercohomology

$$\begin{array}{c|ccccc} \mathbb{Z}_2 & \mathbb{Z}_2 1 & \mathbb{Z}_2 t & \mathbb{Z}_2 t^2 & \mathbb{Z}_2\mathrm{Sq}^1T & \mathbb{Z}_2.\mathbb{Z}_2\,T^2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 1 & \mathbb{Z}_2 t & \mathbb{Z}_2 t^2 & \mathbb{Z}_2\mathrm{Sq}^1T & \mathbb{Z}_2.\mathbb{Z}_2\,T^2 \\ \mathrm{U}(1) & \mathrm{U}(1) & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ \hline & 0 & 1 & 2 & 3 & 4 \end{array}\ . \tag{2.2.41}$$

We notice that the bottom row on E_∞ is the image of $H^\bullet(\mathbb{Z}_2\,;\mathrm{U}(1)) \to H^\bullet(B\mathbb{Z}_2 \rtimes_\beta \mathbb{Z}_2\,;\mathrm{U}(1))$, therefore nothing on that row survives beyond degree higher than three, and the d_2 differential, Sq^2 kills the t^2 in degree $E_2^{2,1}$ as $\mathrm{Sq}^2(t^2) = t^4 = 0$. Finally, t in $E_2^{1,2}$ survives because Sq^2 vanishes there. Altogether, this suggest that the anomaly for a fermionic theory in (1+1)-dimensions with 2-group global symmetry is $\mathbb{Z}_4$. [4]

2.3. Split 2-groups and Symmetry Fractionalization

2.3.1. Review of Symmetry Fractionalization

We now consider the case in which the functorial obstruction class $\beta_{\mathbb{G}}$ to the extension of G by $\mathcal{A}_{[1]}$ is trivial in $H^3(G;\mathcal{A}_{[1]})$. Since this is a special case of a 2-group we will call this case a "split 2-group". With an action $Ho : G \to \mathrm{Aut}(\mathcal{A}_{[1]})$ and trivial Postnikov, then the extension $\mathbb{E}$ of G by $\mathcal{A}_{[1]}$ inducing Ho is in bijection with classes $\nu(\mathbf{g},\mathbf{h}) \in H^2_{Ho}(G;\mathcal{A}_{[1]})$. Our split 2-group is known in the literature as a symmetry fractionalized phase, which is specified by ν [139, 14, 41]. For (2+1)d phases, one can understand symmetry fractionalization as the difference between an anyon a being acted upon by $\mathbf{g}$ and $\mathbf{h}$ defects separately, versus being acted upon by the composite $\mathbf{gh}$ defect. The relationship between different symmetry fractionalizations also becomes clear from a physical point of view when we consider modifying the junction between three zero-form defects to include an anyon α.

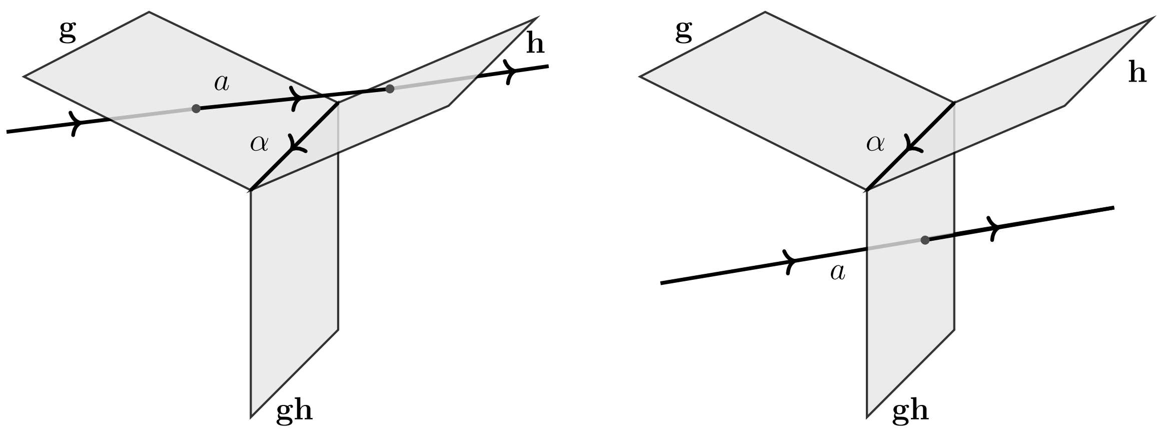


Figure 2.4: The line a is acted on by the zero-form symmetry as it punctures the surfaces, and passing the line a by α results in a braiding.

A line operator implementing the one-form symmetry when passing from above α to below in figure 2.4, picks up a braiding [15]. If we further choose the class of $[0] \in H^2(G;\mathcal{A}_{[1]})$ then this means the symmetries of the phase is simply $\mathcal{A}_{[1]} \times G$, i.e, there is no action of the zero-form symmetry on the one-form symmetry. The above information can also be

[4]It is also possible to calculate this supercohomology by converging to it with $H^\bullet(G\,,\mathrm{SH}^\bullet(B\mathbb{Z}_2))$.

presented by considering the following exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A}_{[1]} \arrow[r] & \mathbb{E} \arrow[r] & G \arrow[l, bend left, "\varphi"] \arrow[r] & 0 \end{tikzcd},$$

where the splitting map φ gives the trivial class $\beta_{\mathbb{G}}$, with $\mathbb{E} = \mathcal{A}_{[1]} \rtimes G$. There are still $H^2(G\,;\mathcal{A}_{[1]})$ choices of conjugacy classes in how we trivialize, for if φ_1 and φ_2 are two such splittings, then there could exist $m \in \mathcal{A}_{[1]}$ such that $\varphi_2(g) = (1,m) \circ \varphi_1(g) \circ (1,m)^{-1}$ in $\mathbb{E}$. In (1+1)d it makes sense to consider $\nu \in H^2(G\,;\mathbb{C}[\mathcal{A}_{[1]}]^\times)$, just as we did for the case of the 2-group. By the same argument as we gave for the Postinikov, we can write $f(\nu) \in \bigoplus_{\hat{A}} H^2(G\,;\mathrm{U}(1))$. The symmetry fractionalization thus appears as if we are choosing to assign a (1+1)d zero-form discrete torsion [229] for the different disjoint theories.

2.3.2. Anomalies for the Split 2-group

Anomalies in symmetry fractionalized theories in (1+1)d are classified by a class in $H^3(\mathcal{A}_{[1]} \rtimes G\,;\mathrm{U}(1))$. We first remark that if $\nu = [0]$, then $Ho : G \to \mathrm{Aut}(\mathcal{A}_{[1]})$ is trivial and the cohomology $H^3_Ho(\mathcal{A}_{[1]} \times G\,;\mathrm{U}(1))$ is calculated with the Kunneth formula. That is to say that all higher differentials in the spectral sequence vanish and $E_2 = E_\infty$. If ν is not trivial, then the d_2 differential is given by cupping with ν, i.e. $\langle \nu \cup - \rangle$. The E_2 page of the spectral sequence we run in order to calculate the anomalies takes the form of (2.2.24), and has no elements that can receive a d_2 differential. The cohomology $H^3(\mathcal{A}_{[1]} \rtimes G\,;\mathrm{U}(1))$ fits in a short exact sequence

$$H^3(G\,;\mathrm{U}(1)) \to H^3(\mathcal{A}_{[1]} \rtimes G\,;\mathrm{U}(1)) \to H^1(G\,;\widehat{A}). \tag{2.3.1}$$

The first map is given by the pullback along the map $\mathbb{E} \to G$ (given by projection). The second map can be built in the following way: we begin by considering a function from G to 2-cochains, δ, on $\mathcal{A}_{[1]}$ that goes

$$\Delta(g) = \delta g - \delta, \text{ for } d\delta \in H^3(\mathcal{A}_{[1]}\,;\mathrm{U}(1))\,. \tag{2.3.2}$$

Next, we prove properties of Δ. We note that $d\delta$ can be thought of as the restriction of a class $\omega \in H^3(\mathcal{A}_{[1]} \rtimes G\,;\mathrm{U}(1))$ to $H^3(\mathcal{A}_{[1]}\,;\mathrm{U}(1))$ because away from the prime 2, $H^3(\mathcal{A}_{[1]}\,;\mathrm{U}(1)) = 0$. By taking the differential in $\mathcal{A}_{[1]}$ of Δ we see that

$$d_{\mathcal{A}_{[1]}}\Delta(g) = d\delta\, g - d\delta = 0\,, \tag{2.3.3}$$

and by taking the twisted differential in G we get

$$\begin{aligned} d_G\Delta(g,h) &= \Delta(g)h - \Delta(gh) + \Delta(h) \\ &= (\delta g - \delta)h - (\delta gh - \delta) + \delta h - \delta \end{aligned}$$

$$= 0\,. \tag{2.3.4}$$

This shows that $\Delta(g)$ is actually a 2 cocycle in $\mathcal{A}_{[1]}$ and a one cocycle in G, i.e $\Delta \in H^1(G\,; H^2(\mathcal{A}_{[1]}; \mathrm{U}(1))) = H^1(G\,; \widehat{\mathcal{A}}_{[-1]})$. A different choice of δ, differing by another cochain $\chi \in H^2(\mathcal{A}_{[1]}\,; \mathrm{U}(1))$, changes $\Delta(g)$ to $\Delta(g) + \chi\, g - \chi$. The difference $\chi\, g - \chi = d_G\chi$, is the differential of a zero-cochain valued in $H^1(G; \widehat{\mathcal{A}}_{[1]})$. We see that the short exact sequence in (2.3.1) does not depend on splitting of $\mathbb{E} \to G$ and therefore for each splitting, there is an isomorphism

$$H^3(\mathcal{A}_{[1]} \rtimes G\,; \mathrm{U}(1)) \cong H^3(G\,; \mathrm{U}(1)) \oplus H^1(G\,; \widehat{\mathcal{A}}_{[-1]})\,, \tag{2.3.5}$$

where the direct sum on the right hand side can be evaluated simply in our case, yielding the anomaly. Since two splittings differ by ν, a change in a splitting by ν changes the direct sum side of the isomorphism by

$$\begin{pmatrix} 1 & \langle \nu \cup - \rangle \\ 0 & 1 \end{pmatrix}, \tag{2.3.6}$$

where explicitly $\langle \nu \cup - \rangle$ is a map from $H^1(G; \widehat{\mathcal{A}}_{[-1]}) \to H^3(G\,; \mathrm{U}(1))$.

2.3.3. Supercohomology for the Split 2-group

In the case of calculating supercohomology for the group $\mathbb{E} = \mathcal{A}_{[1]} \rtimes G$, this semi-direct product reduces to a product because there are no automorphism of $\mathbb{Z}_2$. As in the previous section, away from the prime 2 this is just regular cohomology. The spectral sequence in this case gives $\mathrm{SH}^\bullet(\mathcal{A}_{[1]} \times G) \Leftarrow H^\bullet(G\,; \mathrm{SH}^\bullet(\mathcal{A}_{[1]})) \cong H^\bullet(G\,; \mathbb{Z}) \otimes^{\mathbb{L}} \mathrm{SH}^\bullet(\mathcal{A}_{[1]})$ [5], where $\mathbb{L}$ denotes the left derived tensor product. The above isomorphism is given by the Universal Coefficient theorem. This implies $\mathrm{SH}^3(B\mathbb{Z}_2 \times \mathbb{Z}_2) \Leftarrow \bigoplus_{i+j=3} H^i(\mathbb{Z}_2\,; \mathbb{Z}) \otimes^{\mathbb{L}} \mathrm{SH}^j(B\mathbb{Z}_2)$. In order to evaluate the Tor group up to this degree, we in principle need $\mathrm{SH}^4(B\mathbb{Z}_2)$. One could bypass this calculation due to the fact that $H^0(\mathbb{Z}_2\,; \mathbb{Z}) = \mathbb{Z}$ and $\mathrm{Tor}(A\,, \mathbb{Z}) = 0$ for any finite abelian group A. We nonetheless present this calculation because this allows us to say some facts about braided fusion supercategories with $\mathbb{Z}_2$ fusion rules, since these are parametrized by $\mathrm{SH}^4(B\mathbb{Z}_2)$. We have the E_2 page of $\mathrm{SH}^\bullet(B\mathbb{Z}_2)$ in low degrees as

$$\begin{array}{c|ccccc} \mathbb{Z}_2 & 1 & 0 & 0 & \mathrm{Sq}^1(T) & \mathrm{Sq}^2(T) \\ \mathbb{Z}_2 & 1 & 0 & 0 & \mathrm{Sq}^1(T) & \mathrm{Sq}^2(T) \\ \mathrm{U}(1) & \mathrm{U}(1) & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\ \hline & 0 & 1 & 2 & 3 & 4 \end{array}\,, \tag{2.3.7}$$

[5] The reader is reminded that the notation

$$\bigoplus_{i+j=n} H^i(G\,; M) \otimes^{\mathbb{L}} H^j(G'\,; M) = \bigoplus_{i+j=n} H^i(G\,; M) \otimes H^j(G'\,; M) \bigoplus_{i+j=n+1} \mathrm{Tor}(H^i(G\,; M)\,, H^j(G'\,; M))$$

where we have killed the generator T in degree $(2,1)$ and $(2,2)$ as per the discussion after (2.2.36). We must determine what happens to $\mathrm{Sq}^1(T)$ in $(3,1)$ under $(-1)^{\mathrm{Sq}^2}$. This amounts to identifying whether $\mathrm{Sq}^2\mathrm{Sq}^1(T)$ is in the kernel of $(-1)^x : H^5(B\mathbb{Z}_2\,;\mathbb{Z}_2) \to H^5(B\mathbb{Z}_2\,;\mathrm{U}(1))$, where the generators of $H^5(B\mathbb{Z}_2\,;\mathbb{Z}_2)$ are $\{\,\mathrm{Sq}^2\mathrm{Sq}^1(T)\,, T\mathrm{Sq}^1(T)\,,$
$\mathrm{Sq}^2\mathrm{Sq}^1(T) + T\mathrm{Sq}^1(T)\,\}$. To discern this we consider the short exact sequence,

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{(-1)^x} \mathrm{U}(1) \xrightarrow{x^2} \mathrm{U}(1) \longrightarrow 0\,. \tag{2.3.8}$$

Let $\square$ be the Bockstein of this sequence in cohomology such that $\square : H^n(B\mathbb{Z}_2\,;\mathrm{U}(1)) \to H^{n+1}(B\mathbb{Z}_2\,;\mathbb{Z}_2)$. Then $\square(-1)^x = \mathrm{Sq}^1(x)$ and if $\mathrm{Sq}^1(x) \neq 0$, that implies $(-1)^x \neq 0$. Applying Sq^1 to the generators above we have $\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1(T) = \mathrm{Sq}^3\mathrm{Sq}^1(T)$ by an Adem relation, and $\mathrm{Sq}^3\mathrm{Sq}^1(T) = (\mathrm{Sq}^1(T))^2 \neq 0$ because $\mathrm{Sq}^1(T)$ is in degree three. Next,

$$\mathrm{Sq}^1(T\mathrm{Sq}^1(T)) = \mathrm{Sq}^1(T)\mathrm{Sq}^1(T) + \mathrm{Sq}^1\mathrm{Sq}^1(T) = (\mathrm{Sq}^1(T))^2. \tag{2.3.9}$$

Where we used the fact that Sq^1 acts as a derivation, and $\mathrm{Sq}^1\mathrm{Sq}^1(T) = 0$. Finally, $\mathrm{Sq}^1(\mathrm{Sq}^2\mathrm{Sq}^1(T) + T\mathrm{Sq}^1(T)) = 0$. We see that $\mathrm{Sq}^2\mathrm{Sq}^1(T)$ is not in the kernel of $(-1)^x$, so that d_2 differential is nontrivial. We find that the $\mathrm{SH}^4(B\mathbb{Z}_2) = \mathbb{Z}_2$.

We now remark that the $\mathbb{Z}_4 = H^4(B\mathbb{Z}_2\,;\mathrm{U}(1))$ were the four bosonic braided fusion categories with $\mathbb{Z}_2$ fusion rules. We know them explicitly as $\mathrm{Vec}_{\mathbb{Z}_2}$, Semion, anti-Semion, and SVec. The map to $\mathrm{SH}^4(B\mathbb{Z}_2)$ takes a braided fusion category and tensors it with SVec to produce a braided fusion supercategory. What we find is that, after this tensor product, SVec becomes equivalent to $\mathrm{Vec}_{\mathbb{Z}_2}$, and the two Semions become equivalent. This is the statement that $d_2 : H^3(B\mathbb{Z}_2\,;\mathbb{Z}_2) \to H^5(B\mathbb{Z}_2\,;\mathrm{U}(1))$ was nonzero. The above calculation also says that we do not get any new braided fusion supercategories with $\mathbb{Z}_2$ fusion rules.

Another possible way to convince oneself of this is as follows: suppose that there were a class in $\mathrm{SH}^4(B\mathbb{Z}_2)$ with a nontrivial Majorana layer, which is the image of the class on the top row, $\hom(\mathbb{Z}_2, \mathbb{Z}_2)$, by the universal coefficient theorem. This data would imply that the supercategory had a Majorana object, meaning one with endomorphisms $\mathrm{Cliff}_{\mathbb{C}}(1)$ rather than $\mathbb{C}$. The underlying category of our putative braided supercategory is a braided non-super fusion category. Each ordinary object becomes two objects, each Majorana object becomes one object, and the vacuum becomes a boson and a fermion. That fermion must have fermionic statistics, and also needs to braid trivially with all other objects. Hence, it is an invisible fermion. If there were a Majorana layer, then the underlying non-super category would have Ising fusion rules. This is inconsistent with an invisible fermion and so can not happen.

One could also ponder the existence of a nontrivial Gu-Wen layer. If the Gu-Wen layer is nontrivial, then its value is $\mathrm{Sq}^1 t$, where t is the generator of $H^\bullet(B\mathbb{Z}_2\,;\mathbb{F}_2)$ of degree two. Since Sq^1 is a stable cohomology operator, it commutes with the loop map

$H^n(B\mathbb{Z}_2\,;\mathbb{F}_2) \to H^{n-1}(\Omega B\mathbb{Z}_2\,;\mathbb{F}_2)$. In our case, $\Omega B\mathbb{Z}_2 = \mathbb{Z}_2$ with Ω taking a braided fusion object to its underlying monoidal object. Let us take $s = \Omega t$ to be the generator of $H^\bullet(\Omega B\mathbb{Z}_2\,;\mathbb{Z}_2)$, we see $\mathrm{Sq}^1(s) = s^2 \neq 0$. Thus, if there is a Gu-Wen layer, then it is possible to observe this already on the underlying non-braided category as an effect on the fusion coefficients. In the presence of a Gu-Wen layer, the underlying non-super category would have $\mathbb{Z}_4$ fusion rules. Like the case for the Majorana layer, this is inconsistent with an invisible fermion, and as such can not exist.

2.3.4. Subtheories with SPT

We return to the last part of section §2.3.1 where the assignment of symmetry fractionalization appears as a choice of zero-form discrete torsion, which we will just call "discrete torsion", on each subsector of the theory $\mathcal{T}$. The discrete group $\mathbb{Z}_p$ that we took in §2.2.1 does not admit discrete torsion, so for this section we will work with the specific example where $G = \mathbb{Z}_p \times \mathbb{Z}_p$.

We denote the (1+1)d partition function of $\mathcal{T}$ with zero-form and one-form global symmetry placed on a torus as $\mathcal{Z}_{A_a,A_b,B_a,B_b;\chi^{(2)}}$. Here, we drop the superscripts on A and B which are background gauge fields for the G symmetry with $(A_a, B_a) \in \mathbb{Z}_p \times \mathbb{Z}_p$. The subscripts indicate the cycle on the torus which the gauge field wraps.

The symmetry fractionalization is classified by a class in $H^2(G;\mathbb{C}[\mathcal{A}_{[1]}]^\times)$ and is part of the information assigned to $\mathcal{T}$. Due to the fact that $\mathcal{T}$ decomposes into subsectors, the symmetry fractionalization on the full theory was equivalently seen as choices of discrete torsion on the subsectors, as given by the homomorphism f at the end of §2.3.1. We want to understand how the action of discrete torsion on the subsectors presents itself in the full theory, thereby understanding the action of symmetry fractionalization on $\mathcal{T}$. However, it is expected that acting simply on the subtheories will re-mix in the full theory in a nontrivial way. In order to understand how discrete torsion acts on the partition function for the full theory, we consider writing the partition function in a basis as

$$\begin{aligned}\mathcal{Z}^{\mathcal{T}}_{A_a,X,B_a,Y;\chi^{(2)}} = \int DA_a DB_b D\xi^{(0)}_{\widehat{\kappa}} \exp\left(i\int \chi^{(2)} \cup \xi^{(0)}_{\widehat{\kappa}}\right) \\ \times \exp\left(i\int (A_b \cup X + B_b \cup Y)\right) \mathcal{Z}^{\mathcal{T}//\mathcal{A}_{[1]}}_{A_a,A_b,B_a,B_b;\xi^{(0)}_{\widehat{\kappa}}}\,,\end{aligned} \tag{2.3.10}$$

where X and Y denote some other background gauge fields for the zero-form symmetry. The action of discrete torsion by the operator $S_{\widehat{\kappa}}$ on the partition function of the full theory is implemented in this basis to act on the subtheories. We have

$$S_{\widehat{\kappa}}\, \mathcal{Z}^{\mathcal{T}}_{A_a,X,B_a,Y;\chi^{(2)}} = \int DA_a DB_b D\xi^{(0)}_{\widehat{\kappa}} \exp\left(i\int \chi^{(2)} \cup \xi^{(0)}_{\widehat{\kappa}}\right) \exp\left(i\int (A_b \cup X + B_b \cup Y)\right)$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times S_{\widehat{\kappa}}\, \mathcal{Z}^{\mathcal{T}//\mathcal{A}_{[1]}}_{A_a,A_b,B_a,B_b;\xi^{(0)}_{\widehat{\kappa}}} \\
&= \int DA_a DB_b D\xi^{(0)}_{\widehat{\kappa}} \exp\left(i\int \chi^{(2)}\cup\xi^{(0)}_{\widehat{\kappa}}\right)\exp\left(i\int (A_b\cup X + B_b\cup Y)\right) \\
&\qquad\qquad \times \exp\left(i\,\ell_{\xi^{(0)}_{\widehat{\kappa}}}\int (A_a\cup B_b - A_b\cup B_a)\right)\mathcal{Z}^{\mathcal{T}//\mathcal{A}_{[1]}}_{A_a,A_b,B_a,B_b;\xi^{(0)}_{\widehat{\kappa}}} \\
&= \int DA_a DB_b D\xi^{(0)}_{\widehat{\kappa}} \exp\left(i\int \chi^{(2)}\cup\xi^{(0)}_{\widehat{\kappa}}\right) \\
&\quad \times \exp\left(i\int \left(A_b\cup\left(X-\ell_{\xi^{(0)}_{\widehat{\kappa}}}B_a\right)+B_b\cup\left(Y-\ell_{\xi^{(0)}_{\widehat{\kappa}}}A_a\right)\right)\right)\mathcal{Z}^{\mathcal{T}//\mathcal{A}_{[1]}}_{A_a,A_b,B_a,B_b;\xi^{(0)}_{\widehat{\kappa}}} \\
&= \int DA_a DB_b \exp\left(i\int \left(A_b\cup\left(X-\ell_{\chi^{(2)}_{\widehat{\kappa}}}B_a\right)+B_b\cup\left(Y-\ell_{\chi^{(2)}_{\widehat{\kappa}}}A_a\right)\right)\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \mathcal{Z}^{\mathcal{T}}_{A_a,A_b,B_a,B_b;\chi^{(2)}} \\
&= \mathcal{Z}^{\mathcal{T}}_{A_a,X-\ell_{\chi_{\widehat{\kappa}}}B_a,B_a,Y-\ell_{\chi_{\widehat{\kappa}}}A_a;\chi^{(2)}}\,, \qquad (2.3.11)
\end{aligned}
$$

where, $\ell_{\xi^{(0)}_{\widehat{\kappa}}}$ is a natural number modulo the order of the one-form symmetry group.

We see that a choice of discrete torsion, when we write $\mathcal{Z}^{\mathcal{T}}$ in the basis of (2.3.10), acts as a permutation matrix if we regard $\mathcal{Z}^{\mathcal{T}}$ was a vector labeled by its indices. More precisely, $S_{\widehat{\kappa}}$ acts as a permutation matrix on the G background fields. This fact is nontrivial when we solely view implementing discrete torsion as a manipulation in $2d$; however, this becomes clear after coupling to a bulk (2+1)d TFT and interpreting the *topological manipulations* in $2d$ as permutation of the bulk topological defects [114]. By topological manipulations we mean actions which leave any local dynamics the same, but can change the correlators in systems with nontrivial topological sectors. Our discussion demonstrates that symmetry fractionalization can equivalently be understood from this point of view, with different choices of symmetry fractionalization acting as different permutations of the background zero-form gauge fields.

2.3.5. Discrete Torsion and the One-Form Symmetry

In this subsection we comment on the manipulations within the topological sector of our (1+1)d theory involving the one-form symmetry. In the spirit of the previous section, manipulations such as applying "discrete one-form torsion", gauging the one-form symmetry, and permuting the local ground states can be understood by coupling to a bulk (2+1)d TFT where the one-form symmetry is a dynamical gerbe [160]. In particular, the topological boundary conditions of the defects in the the bulk will give the possible manipulations regarding the topological sectors of the boundary theory.

As a way to interpret the one-form torsion, we can consider taking the subtheories labeled by Spec($\mathcal{A}_{[1]}$)($\mathbb{C}$) of the discrete (-1)-form symmetry and fiber them over a circle

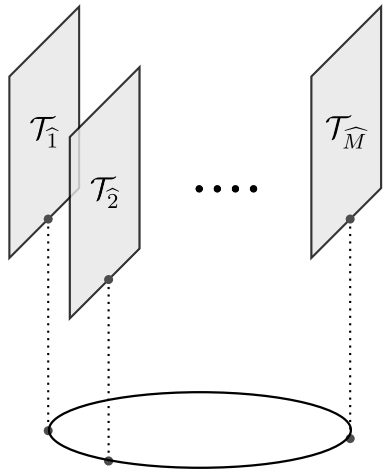


Figure 2.5: Theories fibered over a circle.

so that at discrete points over the circle lives a $(1{+}1)d$ theory, see figure 2.5. Arranging the families of theories in this way moves us one dimension higher, where the extra dimension involves making the parametrization of the circle into a physical spacetime. At an energy scale much above the deep IR, operators can act on the spacetime of some subsector and move us between the different sectors. If however we gauge the one-form symmetry in $\mathcal{T}$ by choosing a projector $\widehat{\kappa}$ in $\widehat{\mathcal{A}}_{[-1]}$, then flowing to the far IR eliminates the possibility of tunneling out of the subsector labeled by $\widehat{\kappa}$ and then the original theory $\mathcal{T}$ is decomposed into a disjoint union. Suppose now we act by $\widetilde{S}_{\xi^{(0)}}$, the one-form discrete torsion operator, on the partition function of $\mathcal{Z}^{\mathcal{T}}$, and then gauge the one-form symmetry with respect to the projector $\widehat{\kappa}$. The operator $\widetilde{S}_{\xi^{(0)}}$ cups $\xi^{(0)}$ to the connection of the one-form symmetry, and multiplies the partition function by a "phase". We give the action of the one-form discrete torsion on the partition function where we suppress the zero-form gauge field indices from (2.3.10),

$$\begin{aligned}
&\int D\chi^{(2)} \exp\left(-i\int \chi^{(2)} \cup \delta^{(0)}_{\widehat{\kappa}}\right) \widetilde{S}_{\xi^{(0)}}\, \mathcal{Z}^{\mathcal{T}}_{\chi^{(2)}} \\
&\quad = \int D\chi^{(2)} \exp\left(-i\int \chi^{(2)} \cup \delta^{(0)}_{\widehat{\kappa}}\right) \exp\left(i\int \chi^{(2)} \cup \xi^{(0)}\right) \mathcal{Z}^{\mathcal{T}}_{\chi^{(2)}} \\
&\quad = \mathcal{Z}^{\mathcal{T}//\mathcal{A}_{[1]}}_{\xi^{(0)}-\delta^{(0)}_{\widehat{\kappa}}}.
\end{aligned} \tag{2.3.12}$$

By acting with $\widetilde{S}_{\xi^{(0)}}$ on $\mathcal{Z}^{\mathcal{T}}$ we have shifted the theory to be in the vacuum labeled by $\xi^{(0)} - \delta^{(0)}_{\widehat{\kappa}}$, instead of the vacuum labeled by $\delta^{(0)}_{\widehat{\kappa}}$. We see the one-form discrete torsion acts analogously to the zero-form discrete torsion as a permutation of the (-1)-form index, to switch between subsectors. The subtlety to note is that the permutation of subsectors is not an effect that takes place in the IR, but rather by applying discrete one-form torsion we have modified the projector which gauges the one-form symmetry.

So far we have performed gauging as a topological manipulation done purely in (1+1)d, we can however regard this action as being implemented by a defect in the bulk TFT which filters the family of theories to a particular one. Imposing a Dirichlet boundary condition on the bulk gerbe fields results in the boundary having subsectors, as the one-form symmetry becomes a global symmetry on the boundary. We can denote the subsectors as states on the boundary, which can be written as $|\mathcal{T}_{\widehat{i}}\rangle$.

We then place the filter defect such that passing this defect changes the gerbe fields in the bulk to their dual fields, see figure 2.6. By composing this defect with the boundary

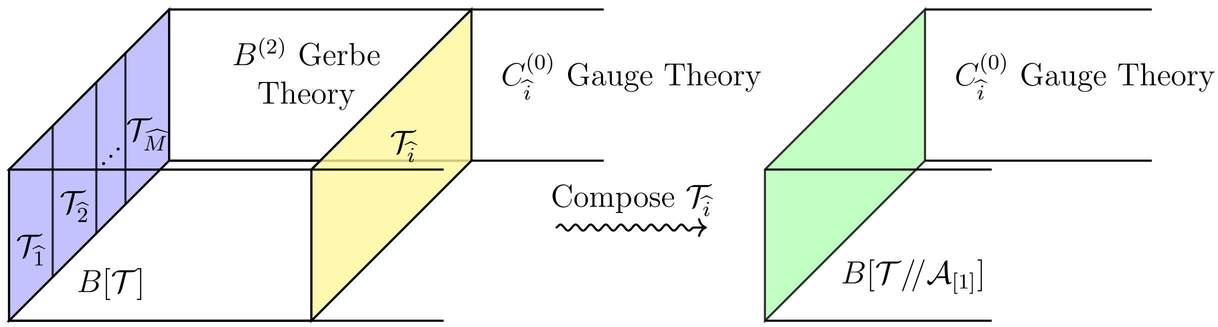


Figure 2.6: The boundary of the bulk theory is denoted $B[\mathcal{T}]$. The resulting theories $\mathcal{T}_{\widehat{i}}$ are separated by walls. When we compose the filter defect in with the boundary, the new boundary becomes that of the bulk theory $\mathcal{T}//\mathcal{A}_{[1]}$.

theory, we obtain a composite boundary condition. Therefore, gauging a one-form symmetry can be treated as implementing this composite boundary condition on the bulk fields of the (2+1)d theory with $C^{(0)}_{\widehat{i}}$ connection.

Another obvious topological manipulation one can also perform is to permute the sectors of the boundary theory, since they are labeled by the one-form symmetry. From the bulk point of view, this can be seen as a permutation of topological codimension one defects, which end as lines on the boundary. The boundary conditions of these defects can be seen

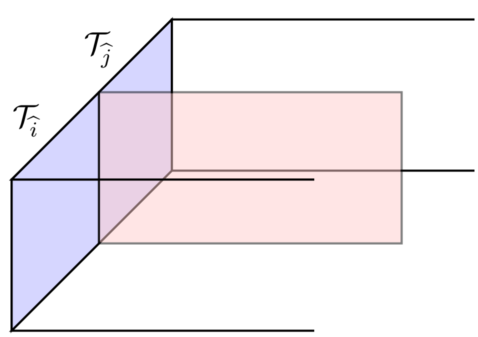


Figure 2.7: A single defect ending on the boundary separates two subsectors by a line.

as what separates two subsectors of the (1+1)d theory, and thereby crossing a line amounts to traversing between theory $\mathcal{T}_{\widehat{i}}$ and $\mathcal{T}_{\widehat{j}}$, as displayed in figure 2.7. The set of ways for these bulk defects to end, and therefore all the ways to traverse between theories, should therefore account for all the ways to permute the boundary theories. An example of a theory that exhibits the property of being able to move between subsectors as previously described, is a (1+1)d U(1) gauge theory with a charge q massless Dirac fermion [168]. One can build a topological local operator $V_k = e^{\frac{2\pi i k}{q}\frac{F_{01}}{e^2}}$ as the symmetry operator of the one-form $\mathbb{Z}_q$ symmetry. We can diagonalize this operator such that it acts on a ground state $|a_1\rangle$ with eigenvalue $e^{\frac{2\pi i k a_1}{q}}$. If two states $|a_1\rangle$ and $|a_2\rangle$ are such that $a_1 \not\equiv -a_2 \mod q$ then the following inner product for the overlap between the two states obeys the equality:

$$\langle a_1|V_k U(t)|a_2\rangle = e^{\frac{-2\pi i k a_1}{q}}\langle a_1|U(t)|a_2\rangle = e^{\frac{2\pi i k a_2}{q}}\langle a_1|U(t)|a_2\rangle\,, \tag{2.3.13}$$

which implies that $\langle a_1|U(t)|a_2\rangle = 0$. Here, $U(t)$ is a unitary operator which implements time evolution. There is no mixing between the subsectors $|a_1\rangle$ and $|a_2\rangle$, which means the domain walls separating the two sectors have infinite tension. Another object which we consider is the Wilson line made by a massive probe particle of charge $p \not\equiv 0 \mod q$ which is charged under the one-form symmetry; $W_p = e^{2\pi i p \oint A}$ with $V_k W_p = e^{\frac{2\pi i k p}{q}} W_p V_k$. We may therefore allow the Wilson line to surround a subsector specified by a local ground state and calculate

$$\begin{aligned} V_k W_p|a_1\rangle &= e^{\frac{2\pi i k p}{q}} W_p V_k|a_1\rangle \\ &= e^{\frac{2\pi i k (p+a_1)}{q}} W_p|a_1\rangle\,, \end{aligned} \tag{2.3.14}$$

which means that the Wilson line separates the different subsectors $|a_1\rangle$ and $|p+a_1\rangle$, because V_k acting on sector $|a_1\rangle$ wrapped with a Wilson line takes us to a different sector $|p+a_1\rangle$. The Wilson line in this example takes exactly the interpretation as the ending of the bulk defect at the boundary.

2.4. Genus-One Data and Anomaly detection

Theories in d spacetime dimensions with a global symmetry group G can have obstructions to promoting the global symmetry to a gauge symmetry. In field theory, one way to work with a global symmetry is to couple it to a background gauge field. Promoting the symmetry to a gauge symmetry is the same as asking whether it is possible to integrate over these background fields in the path integral, a process known as *gauging* or in other contexts *orbifolding* [115]. When gauging is not possible for a certain symmetry, then we say that the theory has an 't Hooft anomaly, that is, an obstruction classified by a class in $\mathrm{H}^{d+1}(G\,;\mathrm{U}(1))$ when the dimension is low. This means that anomalies are inherently topological in nature, and are moreover robust to deformations by local operators. These deformations may flow the theory to be in a strongly coupled regime, which makes the dynamics hard to discern. Information about the anomalies puts constraints on the dynamics, enough so that we are able to make conjectures about the strongly coupled phases. The anomaly is always present along the renormalization group flow, so whatever value the anomaly takes in, say, a weakly coupled regime, must be matched in the strongly coupled regime.

If the symmetry has an anomaly, then detecting the anomaly, i.e. determining what value the anomaly takes is often not a very systematic process and depends on the symmetry at hand. One such way of detecting the anomaly is to study the Hilbert space of the theory on some manifold, such as the torus [82]. However, there is no guarantee that one can detect all such anomalies for any symmetry simply by applying one particular method. It was shown in [189] how to detect anomalies of $\mathbb{Z}_N$ global symmetry in (1+1)d unitary conformal field theory, but not much attention has been given to anomalies of nonabelian global symmetry. We will be interested in a method of detecting anomalies in (1+1)d theories by constructing a stack $\mathcal{M}^G$. This stack will contain information of the theory when placed on a torus with a G-bundle, for G a finite group. In the full construction of $\mathcal{M}^G$ we will have to quotient by automorphisms of the torus and trivializations of the G-bundle. We refer to this stack and automorphism information as *genus-one* data. Over each point of this stack is a torus bundle, and by integrating the anomaly $\alpha \in \mathrm{H}^3(BG\,;\mathrm{U}(1))$ over the torus bundle, we see that $\mathcal{M}^G$ furnishes a line bundle. The kernel of this integration map is precisely the failure to detect α.

We remark in passing an application of this line bundle. In problems involving Moonshine there is a connection between "analytic" data involving modularity and growth rates of certain functions, with representations of finite groups. The modularity is particularly important because this combines with the finite groups into holomorphic sections of a line bundle on $\mathcal{M}^G$ [43]. This line bundle is the integral of α (in all computed examples), i.e., it is the image of an anomaly. If one is interested in the physical reason which unites the two separated pieces of data given in the Moonshine, it is useful to search for this anomaly itself for this information.

The goal of this paper is to show that

Proposition 2.4.1. The genus-one data applied to detect anomalies for the symmetry given by the dicyclic group of order $4N$, Dic_N, has an undetectable $\mathbb{Z}_2$ kernel.

The structure of the paper is as follows: in section 2.5 we spell out the conditions that are specific to genus-one data, along with the construction of the stack $\mathcal{M}^G$. We also explain how to break down the question from a general finite group to studying p-groups. In section 2.6 we recast the method of detecting anomalies associated to a line bundle over $\mathcal{M}^G$, to finding phases of 2d partition functions which are eigenvalues of acting with modular transformations. We investigate how genus-one constraints affect our ability to detect anomalies of dicyclic groups and show Proposition 2.4.1. Section 2.7 contains an example where we apply the techniques of manipulating partition functions to see if we can fully detect the anomaly for $\widehat{\mathrm{SU}}(2)_k$ WZW model with quaternion symmetry.

2.5. Genus-One Data

Consider a theory in (1+1)d which enjoys a global symmetry G. We start with a stack $\mathcal{M}^G = (E, P)$ where E is oriented and there exists an isomorphism $E \simeq \mathbb{T}^2$. Furthermore, we equip E with a G-bundle where $P : E \to BG$. This stack has a standard presentation as follows: for any choice of isomorphism $f : \mathbb{T}^2 \to E$, the map $P \circ f$ is a G-bundle on the standard torus that has holonomies along the two cycles. We also choose a trivialization, φ, of $P \circ f$ at some basepoint that we will take to be the origin of $\mathbb{T}^2$. The stack $\mathcal{M}^G$ is a quotient under the automorphisms of these two choices extra choices. We can therefore write a stack $\widetilde{\mathcal{M}}^G$ that is a covering stack of $\mathcal{M}^G$, more specifically, $\widetilde{\mathcal{M}}^G = \{E, P, f, \varphi\}$. Once we have chosen f then E is no more data, so we are now talking about the space of bundles of the standard torus trivalized at the origin. This is the same as the set of maps

$$\hom(\pi_1 \mathbb{T}^2|_{\text{origin}}\,, G) = \{(x, y) \in G \times G \,|\, [x, y] = 1\}, \tag{2.5.1}$$

i.e. $\widetilde{\mathcal{M}}^G$ is the set of commuting pairs in G. The map from $\widetilde{\mathcal{M}}^G \to \mathcal{M}^G$ presents $\mathcal{M}^G$ as a quotient groupoid of $\widetilde{\mathcal{M}}^G$ by forgetting the data of f and φ. We note that G-bundles at a point are always trivalizable and there are $|G|$ many trivialization, so in order to forget φ we quotient $\hom(\mathbb{Z}^2\,, G)$ by "changes of trivialization". This gives $\hom(\mathbb{Z}^2\,, G)//G$, where the G action is by conjugation on the holonomies. In other words, the action on g on (x, y) is given by

$$(x, y) \triangleleft g := (g^{-1}xg\,, g^{-1}yg)\,. \tag{2.5.2}$$

To forget the data of f, we use the fact that any two isomorphism differ by an automorphism of the standard two-dimensional torus. We therefore also left-quotient $\hom(\mathbb{Z}^2, G)$ by the group $\mathrm{SL}(2, \mathbb{Z})$. An element $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ acts on $(x, y)^\mathsf{T}$, where T denotes the transpose of the row vector, by

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\gamma} \triangleright (x, y)^\mathsf{T} = (x^a y^b, x^c y^d)^\mathsf{T}\,. \tag{2.5.3}$$

Remark 2.5.4. We are using the fact that x and y commute so that the above formula gives an action. The two actions by γ and g also commute with each other. We see that as a groupoid

$$\mathcal{M}^G = \mathrm{SL}(2, \mathbb{Z})\backslash\backslash \hom(\mathbb{Z}^2, G)//G. \tag{2.5.5}$$

Over each point in $\mathcal{M}^G$ lives a torus bundle $\mathcal{E}^G = \{E, P, z \in E\}$, where z is a point in the torus E and the fibers of the map to $\mathcal{M}^G$ are oriented 2-tori which are the "points" E themselves in $\mathcal{M}^G$. The map P now takes $\mathcal{E}^G \to BG$ by mapping $(E, P, z) \mapsto P(z)$. If the theory has an anomaly $\alpha \in \mathrm{H}^3(G; \mathrm{U}(1))$ which maps $BG \to \mathrm{U}(1)[3]$, then we can use the composed maps $P^*\alpha$ as a map from $\mathcal{E}^G \to \mathrm{U}(1)[3]$. Here, the brackets denote the degree of suspension for the regular group $\mathrm{U}(1)$. Therefore, $\mathcal{M}^G$ carries a line bundle which are the maps $(E, P) \mapsto \int_E P^*\alpha$; a line bundle over a groupoid is the same data as associating to

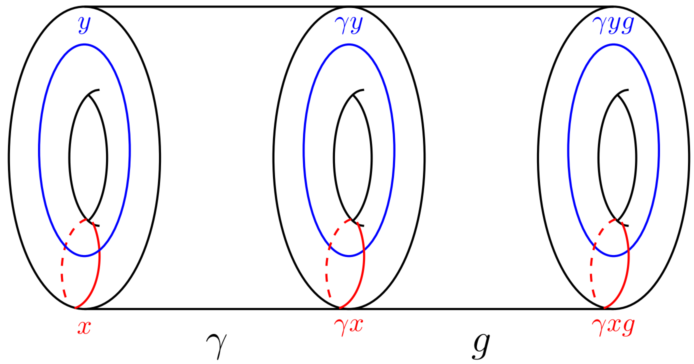


Figure 2.8: Each two-torus has, wrapped along its cycles, a commuting pair of elements $x, y \in G$. In the third direction we draw the mapping cylinder first acting by γ and then by g between two-tori, with the ends identified.

every automorphism in the groupoid a U(1) number. In particular, a typical object of $\mathcal{M}^G$ given by $(x,y)^\mathsf{T}$ and a typical automorphism of this object is given by (γ, g) so that

$$\gamma \triangleright (x,y)^\mathsf{T} = (x,y)^\mathsf{T} \triangleleft g\,. \tag{2.5.6}$$

Thus, in order to give the information about the line bundle, we need to assign for each point (x,y) a group homomorphism, which is $\int_E P^*\alpha = \int \alpha : (\gamma, g) \to \mathrm{U}(1)$. We do this in the following way. We start with a standard two-torus and wrap along the a and b cycle the elements x and y, which attaches a G-bundle to this torus. We now take the cylinder on the G-bundle, but apply a twist γ to the two cycles. Then, we take g to change the trivialization of the G-bundle to return to a configuration that matches what we started with, and lastly identify the starting and ending tori. This procedure is depicted in Figure 5.4. This gives a closed 3-manifold with a G-bundle that we can integrate α over. A form of this construction was given by [117] [6].

The overall question can now be phrased in terms of the line bundle as follows: given $\int \alpha \in \mathrm{H}^1(\mathcal{M}^G; \mathrm{U}(1))$, with α an anomaly in $\mathrm{H}^3(BG; \mathrm{U}(1))$, then is it possible to determine the value of α? The kernel of the map $\mathrm{H}^3(BG; \mathrm{U}(1)) \xrightarrow{\int} \mathrm{H}^1(\mathcal{M}^G; \mathrm{U}(1))$ is exactly our failure to be able to detect the anomaly. For any G, we can choose to restrict to a p-Sylow subgroup, i.e. a maximal p-group where every element is a power of p, denoted by S; we can do this prime by prime. It is therefore possible to restrict the cohomology $\mathrm{H}^3(BG; \mathrm{U}(1))$ along the S subgroup,

[6] While this reference constructs the analogue of $\mathcal{M}^G$ with a line bundle, an action by any automorphism of the G-bundle on the line bundle is given by multipication with the Chern-Simons invariant of the glued mapping cylinder.

$$\begin{tikzcd} \mathrm{H}^3(BG;\mathrm{U}(1))_{(p)} \arrow[r] \arrow[d, hook] & \mathrm{H}^1(\mathcal{M}^G;\mathrm{U}(1)) \arrow[d] \\ \mathrm{H}^3(BS;\mathrm{U}(1))_{(p)} \arrow[r] & \mathrm{H}^1(\mathcal{M}^S;\mathrm{U}(1))\,, \end{tikzcd}$$

where the subscript p denotes p-local cohomology. The map from $\mathrm{H}^3(BG;\mathrm{U}(1))_{(p)}$ to $\mathrm{H}^3(BS;\mathrm{U}(1))_{(p)}$ is a p-local injection [39, §XII.8], but not an injection on the full cohomology, unless one takes a product over all p. If there exists G so that integration is not an injection, then there must be an S such that integration is not an injection. To study this question on all groups we therefore focus on the p-groups.

Given that S is a p-group, we can use a fundamental fact of p-groups which states that for any p-group there exists a central order-p element, thus we have $S = \mathbb{Z}_p\,.S'$. To break down the problem even further, we can temporarily restrict α to the $\mathbb{Z}_p$ subgroup, then by naturality we have $\int \alpha\big|_{\mathbb{Z}_p} = \left(\int \alpha\right)\big|_{\mathcal{M}^{\mathbb{Z}_p}}$. We see that the central element does not contribute to the kernel by using the fact that:

Lemma 2.5.1. [108, §3.3] The map $\int \alpha|_{\mathbb{Z}_p} : \mathrm{H}^3(B\mathbb{Z}_p;\mathrm{U}(1)) \to \mathrm{H}^1(\mathcal{M}^{\mathbb{Z}_p};\mathrm{U}(1))$ is injective, i.e., if $\int \alpha|_{\mathbb{Z}_p} = 1$ then $\alpha|_{\mathbb{Z}_p} = 1$.

Remark 2.5.7. The question can also be phrased in another form that is in terms of extensions rather than anomalies. For concreteness, suppose that a theory has as its symmetry group, $G = \mathbb{Z}_p \times G'$ where G' is a finite p-group. The only anomaly is mixed, living in $\mathrm{H}^2(G';\mathrm{H}^1(\mathbb{Z}_p;\mathrm{U}(1))) = \mathrm{H}^2(G';\widehat{\mathbb{Z}}_p)$. If we gauge the $\mathbb{Z}_p$ symmetry as in [25] we get a central extension $\mathbb{Z}_p.G'$ symmetry action for the gauged theory, where the extension data is the mixed anomaly [222]. The question is therefore equivalent to asking: can one work out which extension using only genus-one data?

From a categorical point of view "modular data" of a modular tensor category means looking at its corresponding $\mathrm{SL}(2,\mathbb{Z})$ representation. The modular tensor category $\mathcal{Z}(\mathbf{Vec}^\alpha[G])$ has modular data, and it was shown in [195] that is is insufficient to determine α. However, it was shown by Kirillov Jr. [165] that $\mathcal{Z}(\mathbf{Vec}^\alpha[G])$ along with the full data of the subcategory $\mathbf{Rep}(G)$ was sufficient to determine α. Our current problem is an intermediate of these two situations. On the one hand we have more than modular data because we also incorporate data of the group that the modular tensor category came from, hence the fact that we can conjugation elements of $\mathcal{M}^G$ by group elements. On the other hand, we do not have the full category $\mathbf{Rep}(G)$ to apply the Kirillov Jr. construction.

2.6. Partition Functions

Performing the integral over the mapping cylinder is in general hard to do and involves knowledge of how to triangulate the manifold, however, there are instances when this can be done. We can consider the case in which the mapping cylinder in Figure 5.4 is G-equivariantly cobordant to the Lens space $L(N,1)$, or when the twists applied in the third direction is trivial, yielding a 3-torus $\mathbb{T}^3$. This is the case when we are only concerned with $\mathbb{Z}_N^k$ groups and the anomaly $\alpha \in \mathrm{H}^3(\mathbb{Z}_N^k;\mathrm{U}(1))$. The third cohomology evaluates to $\mathbb{Z}_N^{[\binom{k}{1}+\binom{k}{2}+\binom{k}{3}]}$ and the cocyles are of the following three forms:

$$\alpha^I(a,b,c) = \exp\left(\frac{2\pi i q^I}{N^2} a^I(b^I + c^I - [b^I + c^I])\right), \tag{2.6.1a}$$

$$\alpha^{IJ}(a,b,c) = \exp\left(\frac{2\pi i q^{IJ}}{N^2} a^I(b^J + c^J - [b^J + c^J])\right), \tag{2.6.1b}$$

$$\alpha^{IJK}(a,b,c) = \exp\left(\frac{2\pi i q^{IJK}}{N} a^I b^J c^K\right), \tag{2.6.1c}$$

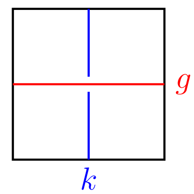
where the superscript indices take values in $\{1,\dots,k\}$, and $a,b,c \in \mathbb{Z}_N^k$ [63]. We denote $[b^I + c^I] := b^I + c^I \mod N$, and q^I, q^{IJ}, q^{IJK} takes values mod N, meant as a representative of the cocycle. To argue why there are $\binom{k}{2}$ many cocycles of the form in (2.6.1b) we note that the 3-cocycles α^{IJ} and α^{JI} are equivalent, since they differ by a coboundary. A similar argument holds for cocycles of the third type in (2.6.1c), and therefore there are only $\binom{k}{3}$ many, as permutations of the labels I,J,K give equivalent cocycles up to coboundary. These three types of cocycles correspond to the generators of $\mathrm{H}^3(\mathbb{Z}_N^k;\mathrm{U}(1))$, which at the level of gauge fields corresponds to self coupling of the gauge fields, pairwise couplings of the gauge fields, or coupling each of the three distinct fluxes together. Each of these cocycles corresponds to a theory in (2+1)d and is the action of a G-SPT. By the anomaly inflow mechanism, we can think of our (1+1)d theory with anomaly α as the boundary of this bulk G-SPT. While the boundary theory is anomalous, the entire bulk boundary set up is non-anomalous, thus the SPT exactly captures the anomaly data in its action. The partition function for the G-SPT when placed on $L(N,1)$ is sufficient to detect the first two types of cocycles, while the last is detectable when placed on $\mathbb{T}^3$ [227]. In particular, the partition function for each of the SPTs is a $\mathrm{U}(1)$ valued topological invariant used to distinguish the phase. The partition functions are built out of a *response function*, which treats the symmetry G as a flat background connection; these functions can be shown to match the expression for the group cocycles in (2.6.1). Evaluating the partition function, i.e. integrating over $L(N,1)$, amounts to integrating the response function over a homology 1-cycle that generates $\mathrm{H}_1(L(N,1),\mathbb{Z})$. The set of invaraints for the three cycles in (2.6.1) is given by

$$\left\{ \exp\left(\frac{2\pi i\, q_I}{N} a_I^2\right), \quad \exp\left(\frac{2\pi i\, q_{IJ}}{N} a_I a_J\right), \quad \exp\left(\frac{2\pi i q_{IJK}}{N}\epsilon^{ijk} a_{I,i}\, b_{J,j}\, c_{K,k}\right) \right\}, \tag{2.6.2}$$

were the indices i,j,k on the last factor indicate the cycles on $\mathbb{T}^3$. We can also consider general discrete Abelian groups which are always isomorphic to $\prod_{I=1}^k \mathbb{Z}_{N^I}$; the SPTs can be detected on $L(N^I, 1), L(\gcd(N^I, N^J), 1)$ and $\mathbb{T}^3$.

We can convert the problem involving integrating over the mapping cylinder into the language of partition functions. In (1+1)d, these are objects which transform as a modular form with respect to τ on the moduli space of flat 2-tori. If our theory enjoys a symmetry G, then the torus base manifold of our theory is equipped with a G-bundle and the map $P : \mathbb{T}^2 \to BG$ is a pair of commuting elements (up to conjugation) $g, k \in G$ each wrapping one of the cycles of the torus. We define the partition function, with $q = \exp 2\pi i\tau$ and $\overline{q} = \exp -2\pi i\overline{\tau}$, as

$$Z_{g,k}(\tau, \overline{\tau}) = \mathrm{Tr}_{\mathcal{H}_k}\left(g\, q^{h-\frac{c}{24}}\, \overline{q}^{\bar{h}-\frac{\bar{c}}{24}}\right), \tag{2.6.3}$$



which is a configuration that is *twisted* by g in the spatial direction, and *twined* by k in the time direction. The trace is over the *defect Hilbert space*, this is from the k-defect intersecting the spatial circle and implements a twisted periodic boundary condition [40]. These partition functions are precisely the sections of the line bundle defined by $\int \alpha$ over the stack $\mathcal{M}^G$. An anomaly then has to do with an obstruction to this line bundle being trivializable. For a special case where $G = \mathbb{Z}_N$ we can consider the component (g, e), where e is the identity, of $\mathcal{M}^G$. A modular S transformation on the partition function exchanges the two cycles of the torus so the g defect now acts at a fixed time and the partition function is

$$Z_{e,g}(\tau, \overline{\tau}) = \mathrm{Tr}_{\mathcal{H}_g}\left(q^{h-\frac{c}{24}}\, \overline{q}^{\bar{h}-\frac{\bar{c}}{24}}\right) = Z_{g,e}\left(-\tfrac{1}{\tau}, -\tfrac{1}{\overline{\tau}}\right). \tag{2.6.4}$$

Under the T transformation, which maps $\tau \to \tau + 1$, we see that this partition function is modular up to a multiplier of a phase which records the line bundle. To compute the phase we note that the spins $h - \overline{h}$ of the states in the defect Hilbert space takes value in $\frac{\ell}{N^2} + \frac{\mathbb{Z}}{N}$, where ℓ is an integer modulo N [189], where it is referred to as a *spin selection rule*. This implies the following, which was also mentioned in [108]:

Proposition 2.6.1. If an anomaly of the G action is given by $\ell \in \mathrm{H}^3(\mathbb{Z}_N; \mathrm{U}(1))$, $T^N = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$ acts on $Z_{e,g}$ with multiplier $\exp\left(\frac{2\pi i \ell}{N}\right)$.

An immediate corollary is that knowledge of the partition function is sufficient to determine the anomaly for $\mathbb{Z}_N$ groups. Going back to our picture using genus-one data and the mapping cylinder, this example for $\mathbb{Z}_N$ groups would be what happens if we wrap e, g along the cycles labeled by x, y in Figure 5.4 and apply $\gamma = T^N$ along the third direction giving the entire mapping cylinder the structure of a Lens space $L(N, 1)$. Since the manifold used to detect the 3-cocycles for the case of a general discrete Abelian group is also a Lens space, or a 3-torus, then genus-one data is sufficient to detect anomalies of Abelian groups.

Furthermore, it is sufficient to detect the anomaly for S a p-group as in §2.5, which has a restriction to an Abelian S'.

Definition 2.6.2. A subgroup $S \subseteq G$ is a *categorical Schur detector (CSD)* at p if the restriction map $\mathrm{H}^3(G;\mathrm{U}(1)) \to \mathrm{H}^3(S;\mathrm{U}(1))$ on the p parts is injective. More generally, a set of subgroups $S \subseteq G$ is a *joint categorical Schur detector* at p if the total restriction map $\mathrm{H}^3(G;\mathrm{U}(1)) \to \prod_S \mathrm{H}^3(S;\mathrm{U}(1))$ on the p parts is injective.

If the group G has an Abelian joint CSD, i.e. one where all S in the set are Abelian, then we would be able to detect the anomaly by our ability to integrate over Lens spaces for any Abelian group.

Example. The notion of CSD was also used in [151] for cohomology in degree four where it was shown that for $G = \mathrm{Co}_0$, the linear isometry group of the Leech lattice, the restriction map $\mathrm{H}^4(\mathrm{Co}_0;\mathbb{Z}) \to \mathrm{H}^4(S;\mathbb{Z})$ is injective, where S is isomorphic to the product of the cyclic group of order 3 and the binary dihedral or group of order 16. We see that a p-Sylow group of G is also an example of a CSD at p but the case in which G is the extraspecial group p^{1+2}_{+} does not have an individual CSD. The lack of a CSD comes simply from the fact that $\mathrm{H}^3(G;\mathrm{U}(1))$ has dimension 4 in p while $\mathrm{H}^3(S;\mathrm{U}(1))$ only has dimension 3, so there is no injection. Take the case of $p = 3$, it was shown in [198] that for G an extraspecial p-group of order 27 with exponent 3 has no *essential cohomology* in any degree. Essential cohomology is the $\mathbb{Z}_p$-cohomology that gives the common kernel of the restrictions to all proper subgroups of G as in definition 2.6.2, i.e. the cohomology fits in the exact sequence

$$\mathrm{H}^\bullet_{\mathrm{Ess}}(G\,;\mathbb{Z}_p) \to \mathrm{H}^\bullet(G\,;\mathbb{Z}_p) \to \prod_{S\subset G} \mathrm{H}^\bullet(S\,;\mathbb{Z}_p)\,. \tag{2.6.5}$$

When we restrict to degree three, specifically with $\mathrm{U}(1)$ coefficients by the standard long exact sequence, this measures the failure for there to be a joint CSD, so vanishing essential cohomology indicates there is a joint CSD in this case.

An important and natural question is how to classify p-groups with non-zero essential cohomology. Let G be an elementary Abelian p-group with rank $i > 0$, the cohomology ring of G is standard and given by

$$\mathrm{H}^\bullet(G\,;\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p[x_1,\, x_2\,\ldots,\, x_i] & p=2, \quad \deg(x_i)=1 \\ \mathbb{Z}_p[x_1,\, x_2\,\ldots,\, x_i] \otimes \bigwedge(y_1,\, y_2, \ldots,\, y_i) & p>2, \quad 2\deg(y_i) = \deg(x_i) = 2\,. \end{cases} \tag{2.6.6}$$

For $p = 2$, $\mathrm{H}^\bullet(G\,;\mathbb{Z}_p) \neq 0$, and for $p > 0$ the essential cohomology is the Steenrod Closure of the product of $y_1 \cdots y_i$ [6]. It was conjectured in [35] that the essential cohomology of an arbitrary p-group is free and finitely generated over a certain polynomial subalgebra in $\mathrm{H}^\bullet(G\,;\mathbb{Z}_p)$; this conjecture holds for elementary p-groups.

Let us move to the case in which the global symmetry is a group that has even order by considering the dicyclic, or binary dihedral, group Dic_N. A special case is Q_8 which is also an extraspecial group of order 8. We will show that:

Proposition 2.6.3. Let G be a subgroup of SU(2), then G has no joint CSD.

Recall that the dihedral group Dih_N, a group of order $2N$ is the group of symmetries of a N-gon and lives as a subgroup of SO(3), where the reflection is implemented as a 180 degree rotation in 3d. A 180 degree rotation lifts with order four to the double cover $\mathrm{Spin}(3) = \mathrm{SU}(2)$. The restriction of Spin(3) along the map from $\mathrm{O}(2) \to \mathrm{SO}(3)$ leads to the group $\mathrm{Pin}^-(2)$, where reflections square to -1. The further restriction of $\mathrm{Pin}^-(2)$ along the map from $\mathrm{Dih}_N \to \mathrm{O}(2)$ leads to Dic_N; the bindary dihedral groups are the "discrete" versions of $\mathrm{Pin}^-(2)$. This is summarized in the diagram below

$$\begin{tikzcd} \mathrm{Dic}_N \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \mathrm{Dih}_N \arrow[d, two heads] \\ \mathrm{Pin}^-(2) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & \mathrm{O}(2) \arrow[d, two heads] \\ \mathrm{Spin}(3) \arrow[r] & \mathrm{SO}(3)\,. \end{tikzcd} \tag{2.6.7}$$

Proof of Proposition 2.6.3. The bindary dihedral group G acts faithfully on Spin(3) which has the topology of a three-sphere. There is a fibration

$$\begin{tikzcd} S^3/G \arrow[r] & S^3 \arrow[d] \\ & BG \end{tikzcd} \tag{2.6.8}$$

where S^3/G is an oriented three-manifold, so has cohomology in degree three and below. From the fibration one can compute the group cohomology of BG. It is known that for any finite subgroup G of the three-sphere that (see for example, [89]):

$$\mathrm{H}^i(BG;\mathrm{U}(1)) = \begin{cases} \mathrm{U}(1) & i = 0\,, \\ G^{ab} & i \equiv 1 \mod 4\,, \\ \mathbb{Z}_{|G|} & i \equiv 3 \mod 4\,, \\ 0 & i > 0 \text{ and even}\,, \end{cases} \tag{2.6.9}$$

where G^{ab} denotes the abelianization of G and $\mathbb{Z}_{|G|}$ denotes the group of complex $|G|$-th roots of unity. If S is a subgroup of G then the restriction $\mathrm{H}^3(G;\mathrm{U}(1)) \to \mathrm{H}^3(S;\mathrm{U}(1))$ is a surjection and loses information about which subgroup of $\mathrm{H}^3(G;\mathrm{U}(1))$ it is, as it only dependents on the order of S. $\square$

We now see if genus-one data can still detect the anomaly. Let $G = \mathrm{Dic}_N$ and consider wrapping the commuting pair of an element $g \in G$ and the identity e around the cycles of the two-torus. As per Figure 5.4, we will let the group element which runs along the third direction of the mapping cylinder be h. The elements g (and h) could be rotations or reflections i.e. $g^N = c$ or $g^2 = c$ where c is the central element. By (2.5.6) it must be that

$$\gamma \begin{pmatrix} e \\ g \end{pmatrix} = \begin{pmatrix} e \\ h\, g\, h^{-1} \end{pmatrix}, \tag{2.6.10}$$

and furthermore this is the most complicated configuration for the binary dihedral group that the constraints of genus-one will allow. Take g to be rotation, and h to be a reflection, then hgh^{-1} is g^{-1}. So what are the possible γ's? The second component of the vector after acting by γ is $e^c g^d$ which must equal g^{-1}, thus $d = 2N - 1$ and c is free to be anything. The first component is $e^a g^b = 1$. So a is free but $g^b = 1$, so $b = b'(2N)$. In this case

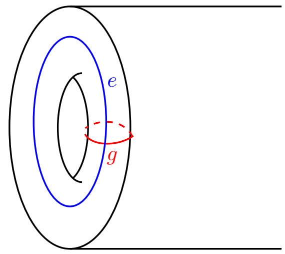
$$\gamma = \begin{pmatrix} a & b'(2N) \\ c & 2N - 1 \end{pmatrix}. \tag{2.6.11}$$

We will take the matrix entries of γ modulo $2N$ since the rotations is a cyclic group of order $2N$, and use the fact that $\det \gamma = 1$. But because b is zero mod $2N$ the two valid matrices are

$$\begin{pmatrix} -1 & 0 \\ -c & -1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & -2N \\ 0 & -1 \end{pmatrix}, \tag{2.6.12}$$

note that since c was free to take any value mod $2N$, we write it as $-c$ in the matrix. We now take h to be a rotation, and g to be a reflection. Then, in order for $hgh^{-1} = g^d$, it must be that $hg = g^d h$, which implies $g^{-1}hg = g^{d-1}h$ and so $h^{-1} = g^{d-1}h$. But $h^{-2} = g^{d-1}$ has no solutions in general if h is a generator of rotations. For example, in the case of a 2-gon or 4-gon, it is possible to satisfy the equality. If h and g are both reflections then on the one hand hgh^{-1} is given by taking g and reflecting about the h axis. The value of hgh^{-1} is g or $-g$ if h and g are the same reflection or off by 90 degrees, respectively. On the other hand, when acting by γ we have $e^c g^d = \pm g$ depending on whether d is even or odd. The case where $g = h$ is uninteresting as (2.6.10) would only be satisfied for γ equal to the identity matrix and adds nothing new in the third direction to help detect the anomaly. Thus we see that the set of γ is spanned by $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$. If h and g were both rotations and thus cyclic subgroups, we know that restriction to any subgroup is not injective, so that will in general not be optimal for allowing us to detect the anomaly. We conclude that:

Proposition 2.6.4. For the dicyclic group Dic_N where the anomaly is a $4N$-th root of unity, genus-one data contains the most information is when the whole group can be generated, i.e. when g is a generator of rotation, and h is a reflection.



γ, h

This forces γ to be in the coset of the matrices

$$\begin{pmatrix} -1 & 0 \\ -c & -1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & -2N \\ 0 & -1 \end{pmatrix}. \tag{2.6.13}$$

Recall that when G is just a cyclic group and h is trivial, the choice of acting on the partition function by $\begin{pmatrix} 1 & |G| \\ 0 & 1 \end{pmatrix}$ extracted a nontrivial G-th root of unity eigenvalue. Acting by the first matrix in (2.6.13) shifts the modulus from $\tau \mapsto \frac{\tau}{\tau+c}$ and amounts to applying T^c and then S transformations to the partition function $\mathrm{Tr}_{\mathcal{H}}\left(g\, q^{h-\frac{c}{24}}\, \overline{q}^{\bar{h}-\frac{\bar{c}}{24}}\right)$, where g is wrapped in the spatial direction. By the spin selection rule, applying T^c for $c \mod |G|$ will not produce a $|G|$-th root of unity. We therefore expect that the second matrix in (2.6.13) will detect the anomaly. We can test this on a theory which has as its symmetry a general dicyclic group, and defer the computation of the anomaly for a specific partition function and symmetry group to the next section. Let $\mathcal{T}_g = S\left[\, \mathrm{Tr}_{\mathcal{H}}\left(g\, q^{h-\frac{c}{24}}\, \overline{q}^{\bar{h}-\frac{\bar{c}}{24}}\right)\right]$, then acting by $\begin{pmatrix} -1 & -2N \\ 0 & -1 \end{pmatrix}$ gives

$$\mathcal{T}_g \xrightarrow{-(T^{2N})} \exp\left(\frac{\pi i \ell}{N}\right) \mathcal{T}_{g^{-1}}\,, \tag{2.6.14}$$

where we have used that fact that $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathcal{T}_g = \mathcal{T}_{g^{-1}}$. But $\mathcal{T}_{g^{-1}} = \mathcal{T}_{hgh^{-1}}$, since g is a rotation and h is a reflection, and $\mathcal{T}_{hgh^{-1}} = \mathcal{T}_g$ by cyclicity of the trace. At best we are able to detect only a $2N$-th root of unity. The map $\mathrm{H}^3(BG; \mathrm{U}(1)) \xrightarrow{\int} \mathrm{H}^1(\mathcal{M}^G; \mathrm{U}(1))$ therefore has a kernel that is at least of order 2.

The restriction to the cyclic subgroup of rotations gives $\{0, 2N\} \mod 4N$ as the elements of the $\mathbb{Z}_2$ kernel. One could hope to detect an anomaly $\alpha \in \{0, 2N\}$. A common strategy when faced with anomalies and extensions of a group is to gauge some symmetry subgroup. The group $\mathrm{Dic}_N = C.\mathbb{Z}_2$ as a nonsplit, noncentral extension with $C \cong \mathbb{Z}_{2N}$ a normal subgroup. By the Serre spectral sequence we have $\mathrm{H}^\bullet(\mathrm{Dic}_N\,; \mathrm{U}(1)) \Leftarrow \mathrm{H}^\bullet(\mathbb{Z}_2\,; \mathrm{H}^\bullet(C\,; \mathrm{U}(1)))$, with the E_2 page:

$$E_2^{ij} = \begin{array}{c|cccccc} j & & & & & & \\ & & & & & & \\ \mathrm{Sym}^2\,\widehat{C} & \mathrm{Sym}^2\,\widehat{C} & & & & & \\ 0 & 0 & 0 & \dots & & & \\ \widehat{C} & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \dots & & \\ \mathrm{U}(1) & \mathrm{U}(1) & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 & \\ \hline & 0 & 1 & 2 & 3 & 4 & i\,. \end{array} \tag{2.6.15}$$

Where $\widehat{C}$ denotes the Pontryagin dual of C, and is the dual symmetry after gauging C. The entry $\mathrm{Sym}^2\,\widehat{C}$ survives on the E_∞ page because it is the image of the restriction map $\mathrm{H}^3(\mathrm{Dic}_N\,;\mathrm{U}(1)) \twoheadrightarrow \mathrm{H}^3(C\,;\mathrm{U}(1))$. The $\mathbb{Z}_2$ in bidegree $(2,1)$ survives on E_∞ for degree reasons; along with $\mathrm{Sym}^2\,\widehat{C}$, these two contribute an order of already $4N$, and so the $d_2 : E_2^{1,1} \to E_2^{3,0}$ must be an isomorphism. The data of α living purely over $\mathbb{Z}_2$ in $(2,1)$ now becomes the extension of the groups $\widehat{C}.\mathbb{Z}_2$ for the gauged theory. In particular, this group is dihedral if $\alpha = 0$ and again dicyclic if $\alpha = 2N$. Reflections lift with order 2 in former case, and order 4 in the latter. A reflection h in the ungauged theory squares to -1 in the group, and lives on in $\mathbb{Z}_2$ part of the gauged theory. However, this is insufficient to tell if this h is -1 in the $\mathbb{Z}_2$ action, and thus in conclusion we are unable to distinguish the elements in the kernel.

2.7. WZW Example

In this section we present an example of attempting to detect the anomaly in a WZW theory with symmetry $G = Q_8$, and failing to fully capture all possible values of the anomaly. The quaternion group not only fits the bill for Proposition 2.6.3 but from the point of view of essential cohomology, it was shown in [5] that a p-group has essential cohomology if all its elements of order p are central. Q_8 is the unique group in which every element of order 2 is central. We consider the WZW theory $\widehat{\mathrm{SU}}(2)_k$ which has $\frac{\mathrm{SU}(2)_\mathrm{L}\times\mathrm{SU}(2)_\mathrm{R}}{\mathbb{Z}_2}$ symmetry (see [42] for a summary of symmetries for WZW CFTs), and anomaly $(k,-k)$. We can consider the $\mathrm{SU}(2)_\mathrm{L}$ symmetry, to which $Q_8 \subset \mathrm{SU}(2)_\mathrm{L}$. From computing $\mathrm{H}^3(BQ_8\,;\mathrm{U}(1))$, we know that this group should admit an anomaly that is mod 8 and therefore we take k also mod 8. We deem that the anomaly is detectable if we can extract the full $\mathbb{Z}_8$ group for the range of k. The generator g of the $\mathbb{Z}_4$ group of rotation is placed on one cycle of the torus, and the identity e is placed on the other due to the fact that the pair must commute. The characters of $\widehat{\mathrm{SU}}(2)_k$ are given by the Weyl-Kac character formula and take the form [26, Section 11]:

$$\chi_\ell^k(\tau,z) = \frac{\Theta_{\ell+1,k+2}(\tau,z) - \Theta_{-\ell-1,k+2}(\tau,z)}{\Theta_{1,2}(\tau,z) - \Theta_{-1,2}(\tau,z)}\,, \tag{2.7.1}$$

with $0 \leq \ell < k$ and the generalized SU(2) Θ-functions defined as

$$\Theta_{\ell,k}(\tau, z) = \sum_{n \in \mathbb{Z} + \frac{\ell}{2k}} q^{kn^2} e^{-2\pi i n k z} \,. \tag{2.7.2}$$

The partition function is defined by $Z(\tau, \overline{\tau}, z, \overline{z}) = \sum_{j=1}^{k} \chi_j^k \, \overline{\chi}_j^k$. When twisted in the spatial direction by g, this gives

$$Z_{g,e} = \mathrm{Tr}_{\mathcal{H}} \left(g \, q^{h - \frac{c}{24}} \, \overline{q}^{\bar{h} - \frac{\bar{c}}{24}} \, e^{-2\pi i z \hat{j}^3} \, e^{2\pi i \overline{z} \hat{j}^3} \right) , \tag{2.7.3}$$

where z is the chemical potential for the U(1)-charge and $\hat{j}^3$ plays the role of the operator which has as its eigenvalue the $\frac{\mathbb{Z}}{2}$ representation of SU(2) in the usual angular momentum algebra. When g wraps in the time direction, after applying an S transformation to (2.7.3), we see that by conjugation we can take g to $\hat{j}^3$ and thus giving

$$\mathrm{Tr}_{\mathcal{H}_g} \left(q^{h - \frac{c}{24}} \, \overline{q}^{\bar{h} - \frac{\bar{c}}{24}} \, e^{-2\pi i (z + \frac{1}{4}) \hat{j}^3} \, e^{2\pi i \overline{z} \hat{j}^3} \right) . \tag{2.7.4}$$

This is because any element $\kappa \in Q_8$ can be written as $i\sigma_i$, for some Pauli matrix σ_i, in the Lie algebra of SU(2) and all SU(2) elements are conjugate to each other. When g is applied in the spatial direction, unless g is the central element, this breaks the global symmetry to U(1) which is the centralizer of g. Any meaningful partition functions could then only have U(1) elements wrapping the time direction, in particular, a U(1) group spanned by $\exp\left(-2\pi i z \hat{j}^3\right)$. Applying the S transformation to (2.7.3) so that g is wrapped in the time direction essentially amounts to shifting $z \mapsto z + \frac{1}{4}$. Then applying $-(T^{2N})$, for $N = 2$, on the partition function and using the spin selection rule gives a phase $\exp\left(2\pi i \,(2N) \left(\frac{\ell}{(2N)^2} + \frac{\mathbb{Z}}{2N}\right)\right) = \exp\left(\frac{\pi i \ell}{N}\right)$, which is only a fourth-root of unity.

Remark 2.7.5. There is another analogous computation we can conduct with free fermions. The fact that the dicyclic group is a subgroup of SU(2) means it acts on $\mathbb{C}^2$. It therefore also acts on the vertex algebra of two complex fermions. The generators of rotations and reflections in this case are respectively

$$g \mapsto \begin{pmatrix} e^{\pi i/N} & 0 \\ 0 & e^{-\pi i/N} \end{pmatrix} , \quad h \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{2.7.6}$$

We can compute twisted-twining genera for this vertex algebra, and just see how these modular functions transform under $\mathrm{SL}(2, \mathbb{Z})$. One subtlety to note is that this method uses a fermionic theory whose anomalies are classified differently than in the bosonic case. One should then take the appropriate restriction to the bosonic part of the full anomaly.

Chapter 3

U-duality Anomaly

Abstract. We perform a bordism computation to show that the $E_{7(7)}(\mathbb{R})$ U-duality symmetry of 4d $\mathcal{N} = 8$ supergravity could have an anomaly invisible to perturbative methods; then we show that this anomaly is trivial. We compute the relevant bordism group using the Adams and Atiyah-Hirzebruch spectral sequences, and we show the anomaly vanishes by computing η-invariants on the Wu manifold, which generates the bordism group.

3.1. Introduction

One of the most surprising discoveries in the field of string theory is the existence of duality symmetries. These symmetries show that the same theory can be described in superficially different ways. In some cases, this can be seen via a transformation of the parameters of the theory, or even the spacetime itself. One such symmetry is U-duality, given by the group $E_{n(n)}(\mathbb{Z})$. By starting with an 11-dimensional theory which encompasses the type IIA string theory, and compactifying on an n-torus, we gain an $\mathrm{SL}_n(\mathbb{Z})$ symmetry from the mapping class group on the n-torus. We arrive at the same theory by compactifying 10d type IIB on a $n-1$-torus, and obtain an $\mathrm{O}_{n-1,n-1}(\mathbb{Z})$ symmetry related to T-duality. The group $E_{n(n)}(\mathbb{Z})$ is then generated by the two aforementioned groups.

In the low energy regime of the 11d theory, which is 11d supergravity, we have an embedding of $E_{n(n)}(\mathbb{Z}) \hookrightarrow E_{n(n)}$ upon applying the torus compactification procedure. The latter group is the U-duality of supergravity. One finds a maximally noncompact form of E_n after the compactification, and this is denoted $E_{n(n)}(\mathbb{R})$. The maximally noncompact form of a Lie group of rank n contains n more noncompact generators than compact generators. For the purpose of this paper, we reduce 11-dimensional supergravity on a 7-dimensional torus. This gives a maximal supergravity theory, i.e. 4d $\mathcal{N} = 8$ supergravity, with an E_7 symmetry.[1] The noncompact form is $E_{7(7)}$ which is 133-dimensional and is diffeomorphic, but not isomorphic, to $\mathrm{SU}_8/\{\pm 1\} \times \mathbb{R}^{70}$.

Because this is a symmetry of the theory, one can ask if it is anomalous, and in particular if there are any global anomalies. Since 4d $\mathcal{N} = 8$ supergravity arises as the low energy effective theory of string theory, then a strong theorem of quantum gravity saying that there are no global symmetries implies that the U-duality symmetry must be gaugeable. Therefore, the existence of any global anomaly would require a mechanism for its cancellation. It would therefore be an interesting question to consider if additional topological terms need to be added to cancel the nonperturbative anomaly as in [67], but we show that with the matter content of 4d maximal supergravity is sufficient to cancel the anomaly on the nose.

The purpose of this paper is to answer:

Question 3.1.1. Can 4d $\mathcal{N} = 8$ supergravity with an $E_{7(7)}$ symmetry have a nontrivial anomaly topological field theory (TFT)? If it can, how do we show that the anomaly vanishes?

We find that theories with this symmetry type can have a nontrivial anomaly, so we have to check whether 4d $\mathcal{N} = 8$ supergravity carries this nontrivial anomaly.

Theorem 3.1.2. *The group of deformation classes of 5d reflection-positive, invertible TFTs on spin-*SU_8 *manifolds is isomorphic to* $\mathbb{Z}/2$*. In this group, the anomaly field theory of 4d* $\mathcal{N} = 8$ *supergravity is trivial.*

The order of the global anomaly is equal to the order of a bordism group in degree 5 that can be computed from the Adams spectral sequence. We find that the global anomaly is $\mathbb{Z}/2$ valued, but nonetheless is trivial when we take into account the matter content of 4d $\mathcal{N} = 8$ supergravity. In order to see the cancellation we first find the manifold generator of the bordism group, which happens to be the Wu manifold, and compute η-invariants on it. Even if an anomaly is trivial, trivializing it is extra data, but our computation gives us a unique trivialization for free; see Remark 3.3.6 for more. This bordism computation is also mathematically intriguing because we find ourselves working over the entire Steenrod algebra, however the specific properties of the problem we are interested in make this tractable.

This work only focuses on U-duality as the group $E_{7(7)}$ rather than $E_{7(7)}(\mathbb{Z})$, because the cohomology of the discrete group that arises in string theory is not known, and a

[1]Dimensional reduction of IIB supergravity on an 6-dimensional torus also yields the same symmetry.

strategy we employ of taking the maximal compact subgroup will not work. But one could imagine running a similar Adams computation for the group $E_7(\mathbb{Z})$ and checking that the anomaly vanishes. There are also a plethora of dualities that arise from compactifying 11d supergravity that one can also compute anomalies of, among them are the U-dualities that arise from compactifying on lower dimensional tori. In upcoming work [70] we study the anomalies of T-duality in a setup where the group is small enough to be computable, but big enough to yield interesting anomalies.

The structure of the paper is as follows: in §3.2 we present the symmetries and tangential structure for the maximal 4d supergravity theory with U-duality symmetry and turn it into a bordism computation. We also give details on the field content of the theory and how it is compatible with the type of manifold we are considering. In §3.3 we review the possibility of global anomalies, and invertible field theories. In §3.4 we perform the spectral sequence computation and give the manifold generator for the bordism group in question. In §3.5 we show that the anomaly vanishes by considering the field content on the manifold generator.

3.2. Placing the U-duality symmetry on manifolds

In this section, we review how the $E_{7(7)}$ U-duality symmetry acts on the fields of 4d $\mathcal{N}=8$ supergravity; then we discuss what kinds of manifolds are valid backgrounds in the presence of this symmetry. We assume that we have already Wick-rotated into Euclidean signature. We determine a Lie group H_4 with a map $\rho_4\colon H_4 \to \mathrm{O}_4$ such that 4d $\mathcal{N}=8$ supergravity can be formulated on 4-manifolds M equipped with a metric and an H_4-connection $P, \Theta \to M$, such that $\rho_4(\Theta)$ is the Levi-Civita connection. As we review in §3.3, anomalies are classified in terms of bordism; once we know H_4 and ρ_4, Freed-Hopkins' work [100] tells us what bordism groups to compute.

The field content of 4d $\mathcal{N}=8$ supergravity coincides with the spectrum of type IIB closed string theory compactified on T^6 and consists of the following fields:

- 70 scalar fields,
- 56 gauginos (spin 1/2),
- 28 vector bosons (spin 1),
- 8 gravitinos (spin 3/2), and
- 1 graviton (spin 2).

Cremmer-Julia [47] exhibited an $\mathfrak{e}_{7(7)}$ symmetry of this theory, meaning an action on the fields for which the Lagrangian is invariant. Here, $\mathfrak{e}_{7(7)}$ is the Lie algebra of the real, noncompact Lie group $E_{7(7)}$, which is the split form of the complex Lie group $E_7(\mathbb{C})$. Cartan [38, §VIII] constructed $E_{7(7)}$ explicitly as follows: the 56-dimensional vector space

$$V := \Lambda^2(\mathbb{R}^8) \oplus \Lambda^2((\mathbb{R}^8)^*) \tag{3.2.1}$$

has a canonical symplectic form coming from the duality pairing. $E_{7(7)}$ is defined to be the subgroup of $\mathrm{Sp}(V)$ preserving the quartic form

$$q(x^{ab}, y_{cd}) = x^{ad}y_{bc}x^{cd}y_{da} - \frac{1}{4}x^{ab}y_{ab}x^{cd}y_{cd} + \frac{1}{96}\left(\epsilon_{abc\cdots h}x^{ab}x^{cd}x^{ef}x^{gh} + \epsilon^{abc\cdots h}y_{ab}y_{cd}y_{ef}y_{gh}\right). \tag{3.2.2}$$

Thus, by construction, $E_{7(7)}$ comes with a 56-dimensional representation, which we denote **56**.

$E_{7(7)}$ is noncompact; its maximal compact is $\mathrm{SU}_8/\{\pm 1\}$, giving us an embedding $\mathfrak{su}_8 \subset \mathfrak{e}_{7(7)}$. Thus $\pi_1(E_{7(7)}) \cong \mathbb{Z}/2$; let $\widetilde{E}_{7(7)}$ denote the universal cover, which is a double cover.

There is an action of $\mathfrak{e}_{7(7)}$ on the fields of 4d $\mathcal{N} = 8$ supergravity, but in this work we only need to know how $\mathfrak{su}_8 \subset \mathfrak{e}_{7(7)}$ acts: we will see in §3.3.2 that the anomaly calculation factors through the maximal compact subgroup of $E_{7(7)}$. For the full $\mathfrak{e}_{7(7)}$ story, see [93, §2]; the $\mathfrak{e}_{7(7)}$-action exponentiates to an $\widetilde{E}_{7(7)}$-action on the fields. The $\mathfrak{su}_8$-action is:

1. The 70 scalar fields can be repackaged into a single field valued in $E_{7(7)}/(\mathrm{SU}_8/\{\pm 1\})$ with trivial $\mathfrak{su}_8$-action.
2. The gauginos transform in the representation $\mathbf{56} := \Lambda^3(\mathbb{C}^8)$.
3. The vector bosons transform in the 28-dimensional representation $\Lambda^2(\mathbb{C}^8)$, which we call **28**.
4. The gravitinos transform in the defining representation of $\mathfrak{su}_8$, which we denote **8**.
5. The graviton transforms in the trivial representation.

The presence of fermions (the gauginos and gravitinos) means that we must have data of a spin structure, or something like it, to formulate this theory. In quantum physics, a strong form of G-symmetry is to couple to a G-connection, suggesting that we should formulate 4d $\mathcal{N} = 8$ supergravity on Riemannian spin 4-manifolds M together with an $\widetilde{E}_{7(7)}$-bundle

$P \to M$ and a connection on P. The spin of each field tells us which representation of Spin_4 it transforms as, and we just learned how the fields transform under the $\widetilde{E}_{7(7)}$-symmetry, so we can place this theory on manifolds M with a *geometric* $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$*-structure*, i.e. a metric and a principal $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$-bundle $P \to M$ with connection whose induced O_4-connection is the Levi-Civita connection. The fields are sections of associated bundles to P and the representations they transform in. The Lagrangian is invariant under the $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$-symmetry, so defines a functional on the space of fields, and we can study this field theory as usual.

However, we can do better! We will see that the representations above factor through a quotient H_4 of $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$, which we define below in (3.2.4), so the same procedure above works with H_4 in place of $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$. A lift of the structure group to H_4 is less data than a lift all the way to $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$, so we expect to be able to define 4d $\mathcal{N} = 8$ supergravity on more manifolds. This is similar to the way that the $\mathrm{SL}_2(\mathbb{Z})$ duality symmetry in type IIB string theory can be placed not just on manifolds with a $\mathrm{Spin}_{10} \times \mathrm{Mp}_2(\mathbb{Z})$-structure,[2] but on the larger class of manifolds with a $\mathrm{Spin}_{10} \times_{\{\pm 1\}} \mathrm{Mp}_2(\mathbb{Z})$-structure [207, §5], or how certain SU_2 gauge theories can be defined on manifolds with a $\mathrm{Spin}_n \times_{\{\pm 1\}} \mathrm{SU}_2$ structure [238].

Let $-1 \in \mathrm{Spin}_4$ be the nonidentity element of the kernel of $\mathrm{Spin}_4 \to \mathrm{SO}_4$ and let x be the nonidentity element of the kernel of $\widetilde{E}_{7(7)} \to E_{7(7)}$. The key fact allowing us to descend to a quotient is that -1 acts nontrivially on the representations of $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$ above, and x acts nontrivially, but on a given representation, these two elements both act by 1 or they both act by -1. We can check this even though we have not specified the entire $\mathfrak{e}_{7(7)}$-action on the fields, because $-1 \in \widetilde{E}_{7(7)}$ is contained in the copy of SU_8 in $\widetilde{E}_{7(7)}$, and we have specified the $\mathfrak{su}_8$-action. Therefore the $\mathbb{Z}/2$ subgroup of $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$ generated by $(-1, x)$ acts trivially, and we can form the quotient

$$H_4 \coloneqq \mathrm{Spin}_4 \times_{\{\pm 1\}} \widetilde{E}_{7(7)} = (\mathrm{Spin}_4 \times \widetilde{E}_{7(7)})/\langle(-1, x)\rangle. \tag{3.2.4}$$

The representations that the fields transform in all descend to representations of H_4, so following the procedure above, we can define 4d $\mathcal{N} = 8$ supergravity on manifolds M with a *geometric* H_4*-structure*, i.e. a metric, an H_4-bundle $P \to M$, and a connection on P whose induced O_4-connection is the Levi-Civita connection.

Remark 3.2.5. As a check to determine that we have the correct symmetry group, we can compare with other string dualities. The U-duality group contains the S-duality group for

[2]Here $\mathrm{Mp}_2(\mathbb{Z})$ is the *metaplectic group*, a central extension of $\mathrm{SL}_2(\mathbb{Z})$ of the form

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2(\mathbb{Z}) \longrightarrow 1. \tag{3.2.3}$$

type IIB string theory, which comes geometrically from the fact that 4d $\mathcal{N} = 8$ supergravity can be constructed by compactifying type IIB string theory on T^6. Therefore the ways in which the duality groups mix with the spin structure must be compatible. As explained by Pantev-Sharpe [207, §5], the $\mathrm{SL}_2(\mathbb{Z})$ duality symmetry of type IIB string theory mixes with the spin structure to form the group $\mathrm{Spin}_{10} \times_{\{\pm 1\}} \mathrm{Mp}_2(\mathbb{Z})$, where $\mathrm{Mp}_2(\mathbb{Z})$ is the metaplectic group from Footnote 2.

Therefore the way in which the U-duality group mixes with $\{\pm 1\} \subset \mathrm{Spin}_4$ must also be nontrivial. Extensions of a group G by $\{\pm 1\}$ are classified by $H^2(BG; \{\pm 1\})$. If G is connected, BG is simply connected and the Hurewicz and universal coefficient theorems together provide a natural identification

$$H^2(BG; \{\pm 1\}) \xrightarrow{\cong} \mathrm{Hom}(\pi_2(BG), \{\pm 1\}) = \mathrm{Hom}(\pi_1(G), \{\pm 1\}). \tag{3.2.6}$$

As $\pi_1(E_{7(7)}) \cong \mathbb{Z}/2$, there is only one nontrivial extension of $E_{7(7)}$ by $\{\pm 1\}$, namely the universal cover $\widetilde{E}_{7(7)} \to E_{7(7)}$. That is, by comparing with S-duality, we again obtain the group H_4, providing a useful double-check on our calculation above.

3.3. Anomalies, invertible field theories, and bordism

3.3.1. Generalities on anomalies and invertible field theories

It is sometimes said that in mathematical physics, if you ask four people what an anomaly is, you will get five answers. The goal of this section is to explain our perspective on anomalies, due to Freed-Teleman [105], and how to reduce the determination of the anomaly to a question in algebraic topology, an approach due to Freed-Hopkins-Teleman [101] and Freed-Hopkins [100].

Whatever an anomaly is, it signals a mild inconsistency in the definition of a quantum field theory. For example, if a quantum field theory Z is n-dimensional, one ought to be able to evaluate it on a closed n-manifold M, possibly equipped with some geometric structure, to obtain a complex number $Z(M)$, called the *partition function* of M. If Z has an anomaly, $Z(M)$ might only be defined after some additional choices, and in the absence of those choices $Z(M)$ is merely an element of a one-dimensional complex vector space $\alpha(M)$.

The theory Z is local in M, so $\alpha(M)$ should also be local in M. One way to express this locality is to ask that $\alpha(M)$ is the state space of M for some $(n+1)$-dimensional quantum field theory α, called the *anomaly field theory* α of Z. The condition that the state spaces of α are one-dimensional follows from the fact that α is an *invertible field theory* [102, Definition 5.7], meaning that there is some other field theory α^{-1} such that $\alpha \otimes \alpha^{-1}$ is isomorphic to the trivial field theory $\mathbf{1}$.[3,4] This approach to anomalies is due to Freed-Teleman [105]; see also Freed [97, 98].

We can therefore understand the possible anomalies associated to a given n-dimensional quantum field theory Z by classifying the $(n+1)$-dimensional invertible field theories with the same symmetry type as Z. The classification of invertible *topological* field theories is due to Freed-Hopkins-Teleman [101], who lift the question into stable homotopy theory; see Grady-Pavlov [122, §5] for a recent generalization to the nontopological setting.

Supergravity with its U-duality symmetry is a unitary quantum field theory, and therefore its anomaly theory satisfies the Wick-rotated analogue of unitarity: *reflection positivity.* Freed-Hopkins [100] classify reflection-positive invertible field theories, again using stable homotopy theory. Let $\mathrm{O} \coloneqq \lim_{n\to\infty} \mathrm{O}_n$ be the infinite orthogonal group.

Theorem 3.3.1 (Freed-Hopkins [100, Theorem 2.19])**.** *Let $n \geq 3$, H_n be a compact Lie group, and $\rho_n\colon H_n \to \mathrm{O}_n$ be a homomorphism whose image contains SO_n. Then there is canonical data of a topological group H and a continuous homomorphism $\rho\colon H \to \mathrm{O}$ such that the pullback of ρ along $\mathrm{O}_n \hookrightarrow \mathrm{O}$ is ρ_n.*

In other words, when the hypotheses of this theorem hold, we have more than just H_n-structures on n-manifolds; we can define H-structures on manifolds of any dimension, by asking for a lift of the classifying map of the stable tangent bundle $M \to B\mathrm{O}$ to BH; a manifold equipped with such a lift is called an *H-manifold*. Following Lashof [183], this allows us to define bordism groups Ω_k^H and a homotopy-theoretic object called the *Thom spectrum MTH*, whose homotopy groups are the H-bordism groups. See [18, §2] for more on the definition of MTH and its context in stable homotopy theory.

Theorem 3.3.2 (Freed-Hopkins [100])**.** *With H_n as in Theorem 3.3.1, the abelian group of deformation classes of n-dimensional reflection-positive invertible topological field theories on H_n-manifolds is naturally isomorphic to the torsion subgroup of $[MTH, \Sigma^{n+1} I_{\mathbb{Z}}]$.*

Freed-Hopkins then conjecture (*ibid.*, Conjecture 8.37) that the whole group $[MTH, \Sigma^{n+1} I_{\mathbb{Z}}]$ classifies all reflection-positive invertible field theories, topological or not.

[3] The relationship between invertibility and one-dimensional state spaces is that $\alpha \otimes \alpha^{-1} \simeq \mathbf{1}$ means that on any closed, n-manifold M, there is an isomorphism of complex vector spaces $\alpha(M) \otimes \alpha^{-1}(M) \cong \mathbf{1}(M) = \mathbb{C}$. This forces $\alpha(M)$ and $\alpha^{-1}(M)$ to be one-dimensional. Often the converse is also true: see Schommer-Pries [214].

[4] In some cases, we do not want to assume α extends to closed n-manifolds; see Freed-Teleman [105] for more information. But the U-duality anomaly we investigate in this work does extend.

The notation $[MTH, \Sigma^{n+1} I_{\mathbb{Z}}]$ means the abelian group of homotopy classes of maps between MTH and an object $\Sigma^{n+1} I_{\mathbb{Z}}$ belonging to stable homotopy theory; see [100, §6.1] for a brief introduction in a mathematical physics context. We mentioned MTH above; $I_{\mathbb{Z}}$ is the *Anderson dual of the sphere spectrum* [8, 250], characterized up to homotopy equivalence by its universal property, which says that there is a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\pi_{n-1}(E), \mathbb{Z}) \longrightarrow [E, \Sigma^n I_{\mathbb{Z}}] \longrightarrow \mathrm{Hom}(\pi_n(E), \mathbb{Z}) \longrightarrow 0. \tag{3.3.3}$$

Applying this when $E = MTH$, we obtain a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\Omega^H_{n+1}, \mathbb{Z}) \xrightarrow{\phi} [MTH, \Sigma^{n+2} I_{\mathbb{Z}}] \xrightarrow{\psi} \mathrm{Hom}(\Omega^H_{n+2}, \mathbb{Z}) \longrightarrow 0 \tag{3.3.4}$$

decomposing the group of possible anomalies of unitary QFTs on H_n-manifolds. These two factors admit interpretations in terms of anomalies.

1. The quotient $\mathrm{Hom}(\Omega^H_{n+2}, \mathbb{Z})$ is a free abelian group of degree-$(n+2)$ characteristic classes of H-manifolds. The map ψ sends an anomaly field theory to its anomaly polynomial. This is the part of the anomaly visible to perturbative methods, and sometimes is called the *local anomaly*.

2. The subgroup $\mathrm{Ext}(\Omega^H_{n+1}, \mathbb{Z})$ is isomorphic to the abelian group of torsion bordism invariants $f\colon \Omega^H_{n+1} \to \mathbb{C}^\times$. These classify the reflection-positive invertible *topological* field theories α_f: the correspondence is that the bordism invariant f is the partition function of α_f. This part of an anomaly field theory is generally invisible to perturbative methods and is called the *global anomaly*.

Work of Yamashita-Yonekura [248] and Yamashita [247] relates this short exact sequence to a differential generalized cohomology theory extending $\mathrm{Map}(MTH, \Sigma^{n+1} I_{\mathbb{Z}})$.

3.3.2. Specializing to the U-duality symmetry type

For us, $n = 4$ and the symmetry type is $H_4 = \mathrm{Spin} \times_{\{\pm 1\}} \widetilde{E}_{7(7)}$. This group is not compact, so Theorems 3.3.1 and 3.3.2 above do not apply. However, we can work around this obstacle: Marcus [193] proved that the anomaly polynomial of the $E_{7(7)}$ symmetry vanishes,[5] meaning

[5] Marcus' analysis does not discuss the question of H_4 versus $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)}$, but this does not matter: in many cases including the one we study, the anomaly polynomial for a d-dimensional field theory on G-manifolds is an element of $H^{d+2}(BG; \mathbb{Q})$, and rational cohomology is insensitive to finite covers such as $\mathrm{Spin}_4 \times \widetilde{E}_{7(7)} \to H_4$. Thus Marcus' computation applies in our case too.

that the anomaly field theory is a *topological* field theory. Thinking of topological field theories as symmetric monoidal functors $\mathbf{Bord}_n^{H_n} \to \mathbf{C}$, we can freely adjust the structure we put on manifolds in these theories as long as the induced map on bordism categories is an equivalence. We make two adjustments.

1. First, forget the metric and connection in the definition of a geometric H_4-structure. The space of such data is contractible and therefore can be ignored for topological field theories.

2. We can then replace H_4 with its maximal compact subgroup: for any Lie group G with $\pi_0(G)$ finite, inclusion of the maximal compact subgroup $K \hookrightarrow G$ is a homotopy equivalence [191, 142] and defines a natural equivalence of groupoids $\mathbf{Bun}_K(X) \xrightarrow{\simeq} \mathbf{Bun}_G(X)$ on spaces X, hence a symmetric monoidal equivalence of bordism categories of manifolds with these kinds of bundles.

Spin_4 is compact, and the maximal compact of $\widetilde{E}_{7(7)}$ is SU_8, so the maximal compact of H_4 is the group $\mathrm{Spin}_4 \times_{\{\pm 1\}} \mathrm{SU}_8$. Now Theorems 3.3.1 and 3.3.2 apply: the stabilization of $\mathrm{Spin}_4 \times_{\{\pm 1\}} \mathrm{SU}_8$ is $\mathrm{Spin}\text{-}\mathrm{SU}_8 \coloneqq \mathrm{Spin} \times_{\{\pm 1\}} \mathrm{SU}_8$, and the anomaly field theory is classified by the torsion subgroup of $[MT(\mathrm{Spin}\text{-}\mathrm{SU}_8), \Sigma^6 I_{\mathbb{Z}}]$, which is determined by $\Omega_5^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$.

In Theorem 3.4.26, we prove $\Omega_5^{\mathrm{Spin}\text{-}\mathrm{SU}_8} \cong \mathbb{Z}/2$, so there is potential for the anomaly field theory to be nontrivial.

Concretely, a manifold with a spin-SU_8 structure is an oriented manifold M with a principal $\mathrm{SU}_8/\{\pm 1\}$-bundle $P \to M$ and a trivialization of $w_2(M) + a(P)$, where a is the unique nonzero element of $H^2(B(\mathrm{SU}_8/\{\pm 1\}); \mathbb{Z}/2)$.

Remark 3.3.5. Computing bordism groups to determine whether an anomaly is trivial is a well-established technique in the mathematical physics literature: other papers taking this approach include [244, 164, 199, 32, 200, 127, 135, 215, 51, 119, 201, 225, 233, 232, 236, 246, 23, 55, 53, 54, 126, 132, 136, 146, 157, 226, 235, 234, 237, 99, 100, 67, 80, 124, 123, 167, 185, 184, 186, 224, 252, 240, 52, 66, 187, 223, 249].

Remark 3.3.6. Once we know an invertible field theory is trivializable, there is the question of what additional data is needed to trivialize it, and anomaly cancellation includes supplying this data for the anomaly field theory. In general there is more than one way to do so: a standard obstruction-theoretic argument in algebraic topology implies the set of trivializations of an n-dimensional reflection-positive invertible field theory on H-manifolds is a torsor over $[MTH, \Sigma^n I_{\mathbb{Z}}]$, i.e. the corresponding group of invertible field theories in one dimension lower. This has the physics implication that any two trivializations of an anomaly differ by a θ-angle.

For the U-duality symmetry, we do not need to worry about this, which we get essentially for free from our computation in §3.4: $\Omega_4^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$ is free and $\Omega_5^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$ is torsion, so $[MT(\mathrm{Spin}\text{-}\mathrm{SU}_8), \Sigma^5 I_{\mathbb{Z}}] = 0$, so there is a unique way to trivialize the U-duality anomaly field theory.

This point about the additional data of a trivialization was first raised by Freed-Moore [102], and includes what they refer to as "setting the quantum integrand;" see also Freed [98, §11.4].

3.4. Spectral sequence computation

The E_2 page for U-duality in the Adams spectral sequence is [3, Theorem 2.1, 2.2]

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(MT(\mathrm{Spin}\text{-}\mathrm{SU}_8);\mathbb{Z}/2),\mathbb{Z}/2) \Rightarrow \pi_{s-t}(MT(\mathrm{Spin}\text{-}\mathrm{SU}_8))_2^\wedge \cong (\Omega_{s-t}^{\mathrm{Spin}\text{-}\mathrm{SU}_8})_2^\wedge, \quad (3.4.1)$$

which converges to the 2-completion of the desired bordism group via the Pontrjagin-Thom construction.

Let $G_8 \coloneqq \mathrm{SU}_8/\{\pm 1\}$. The standard way to tackle Adams spectral sequence questions such as (3.4.1) would be to re-express a spin-SU_8 structure on a vector bundle $E \to M$ as data of a principal G_8-bundle $P \to M$ and a spin structure on $E \oplus \rho_P$, where ρ_P is the associated bundle to P and some representation ρ of G_8. Once this is done, one invokes a change-of-rings theorem that makes calculating the E_2-page of (3.4.1) much easier. For several great examples of this technique, see [32, 18].

Unfortunately, this strategy is not available for spin-SU_8 bordism. The reason is that ρ, thought of as a map $\rho\colon G_8 \to \mathrm{O}_n$ for some n, cannot lift to a map $G_8 \to \mathrm{Spin}_n$; if it does, a spin structure on $E \oplus \rho_P$ is equivalent to a spin structure on E by the two-out-of-three property. However, G_8 does not have any non-spin representations.

Theorem 3.4.2. *All representations $\rho\colon G_8 \to \mathrm{O}_n$ lift to Spin_n.*

The proof is given in [219].[6] Thus we cannot proceed via the usual change-of-rings simplification, and we must run the Adams spectral sequence over the entire mod 2 Steenrod

[6] Here is another proof using $H^*(BG_8;\mathbb{Z}/2)$, which we calculate in Theorem 3.4.4 in low degrees. Suppose a non-spin representation ρ of G_8 exists, and let $V \to BG_8$ be the associated vector bundle. Since $H^1(BG_8;\mathbb{Z}/2) = 0$ and $H^2(BG_8;\mathbb{Z}/2) = \mathbb{Z}/2\cdot a$, $w_1(V) = 0$ and $w_2(V) = a$. Using the Thom isomorphism and how it affects the $\mathcal{A}$-module structure on cohomology (see, e.g., [18, §3.3, §3.4]), we can compute that if U is the Thom class in the cohomology of the Thom spectrum $(BG_8)^V$, then $\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2 U = U(ab+d)$, $\mathrm{Sq}^4\mathrm{Sq}^1 U = 0$, and there is no class x with $\mathrm{Sq}^1(Ux) = U(ab+d)$. This is a contradiction because $\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2 = \mathrm{Sq}^4\mathrm{Sq}^1 + \mathrm{Sq}^1\mathrm{Sq}^4$.

algebra $\mathcal{A}$, which is harder. Similar problems occur in a few other places in the mathematical physics literature, including [99, 66]. It would be interesting to find more problems where similar complications occur when trying to work with twisted spin bordism.

In order to set up the Adams computation, a necessary step is to establish the two theorems in §3.4.2 with the goal to give the Steenrod actions on $H^*(B(\mathrm{Spin}\text{-}\mathrm{SU}_8);\mathbb{Z}/2)$. Applying the Thom isomorphism takes care of the rest. We also detail the simplifications that make working over the entire Steenrod algebra accessible. We refer the reader to [18] which highlights many of the computational details of the Adams spectral sequence, but mainly employs a change of rings to work over $\mathcal{A}(1)$. We start by showing that computing the 2-completion is sufficient for the tangential structure we are considering.

3.4.1. Nothing interesting at odd primes

We will show that the Adams spectral sequence computation that we run which only gives the two torsion part of the anomaly is sufficient for our purposes.

Proposition 3.4.3. $\Omega_*^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$ *has no p-torsion when p is an odd prime.*

Proof. The quotient $\mathrm{Spin}\times\mathrm{SU}_8 \to \mathrm{Spin}\text{-}\mathrm{SU}_8$ is a double cover, hence on classifying spaces is a fiber bundle with fiber $B\mathbb{Z}/2$. $H^*(B\mathbb{Z}/2;\mathbb{Z}/p) = \mathbb{Z}/p$ concentrated in degree 0, so $B(\mathrm{Spin}\times\mathrm{SU}_8) \to B(\mathrm{Spin}\text{-}\mathrm{SU}_8)$ is an isomorphism on $\mathbb{Z}/p$ cohomology (e.g. look at the Serre spectral sequence for this fiber bundle). The Thom isomorphism lifts this to an isomorphism of cohomology of the relevant Thom spectra, and then the stable Whitehead theorem implies that the forgetful map $\Omega_*^{\mathrm{Spin}}(B\mathrm{SU}_8) \to \Omega_*^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$ is an isomorphism on p-torsion.

The same argument applies to the double cover $\mathrm{Spin}\times\mathrm{SU}_8 \to \mathrm{SO}\times\mathrm{SU}_8$, so the p-torsion in $\Omega_*^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$ is isomorphic to the p-torsion in $\Omega_*^{\mathrm{SO}}(B\mathrm{SU}_8)$. Now apply the Atiyah-Hirzebruch spectral sequence. Averbuh [10] and Milnor [197, Theorem 5] prove there is no p-torsion in Ω_*^{SO}, and Borel [27, Proposition 29.2] shows there is no p-torsion in $H_*(B\mathrm{SU}_8;\mathbb{Z})$ and $H_*(B\mathrm{SU}_8;\mathbb{Z}/2)$. Therefore the only way to obtain p-torsion in $\Omega_*^{\mathrm{SO}}(B\mathrm{SU}_8)$ would be from a differential between free summands, but all free summands in Ω_*^{SO} and $H_*(B\mathrm{SU}_8;\mathbb{Z})$ are contained in even degrees, so there are no differentials between free summands, and therefore no p-torsion. ☐

3.4.2. Computing the cohomology of $B(\mathrm{Spin}\text{-}\mathrm{SU}_8)$

We first prove Theorem 3.4.4, where we compute $H^*(BG_8;\mathbb{Z}/2)$ and its $\mathcal{A}$-module structure in low degrees. We then use this to compute $H^*(B(\mathrm{Spin}\text{-}\mathrm{SU}_8);\mathbb{Z}/2)$ as an $\mathcal{A}$-module in low degrees in Theorem 3.4.22, allowing us to run the Adams spectral sequence in §3.4.3. Our computations make heavy use of the Serre spectral sequence; for more on the Serre spectral sequence and its application to physical problems see [119, 252, 187, 186, 55, 52]. See also Manjunath-Calvera-Barkeshli [192, §D.6], who perform a related Serre spectral sequence computation to determine some integral cohomology groups of BG_8.

Theorem 3.4.4. *$H^*(BG_8;\mathbb{Z}/2) \cong \mathbb{Z}/2[a,b,c,d,e,\dots]/(\dots)$ with $|a|=2$, $|b|=3$, $|c|=4$, $|d|=5$, and $|e|=6$, and there are no other generators or relations below degree 7. The Steenrod squares are*

$$\begin{aligned}
\mathrm{Sq}(a) &= a + b + a^2\\
\mathrm{Sq}(b) &= b + d + b^2\\
\mathrm{Sq}(c) &= c + e + \mathrm{Sq}^3(c) + c^2\\
\mathrm{Sq}(d) &= d + b^2 + \mathrm{Sq}^3(d) + \mathrm{Sq}^4(d) + d^2.
\end{aligned} \tag{3.4.5}$$

Proof. We first give the cohomology of BG_8 by using the Serre spectral sequence for the fibration $G_8 \to \mathrm{pt} \to BG_8$. The cohomology $H^*(G_8;\mathbb{Z}/2)$ is given in [17, Theorem 7.2] which we reproduce here:

$$H^*G_8;\mathbb{Z}/2) \cong \mathbb{Z}/2[z_1]/z_1^8 \otimes \bigwedge(z_2,\dots,z_7)\,, \quad \deg z_i = 2i-1\,. \tag{3.4.6}$$

The E_2-page

$$E_2^{p,q} = H^p(BG_8;H^q(G_8;\mathbb{Z}/2)) \Longrightarrow H^{p+q}(\mathrm{pt};\mathbb{Z}/2) \tag{3.4.7}$$

begins as follows:

$$\begin{array}{c|ccccccc}
8 & z_1^8,\, z_1^5z_2,\, z_1^4z_2,\, z_1^5z_3,\, z_1z_4,\, z_2z_3 & 0 & & & & & \\
7 & z_1^7,\, z_1^4z_2,\, z_1^2z_3,\, z_4 & 0 & & & & & \\
6 & z_1^6,\, z_1^3z_2,\, z_1z_3 & 0 & & & & & \\
5 & z_1^5,\, z_1^2z_2,\, z_3 & 0 & & & & & \\
4 & z_1^4,\, z_1z_2 & 0 & & & & & \\
3 & z_1^3,\, z_2 & 0 & & & & & \\
2 & y = z_1^2 & 0 & & & & & \\
1 & z_1 & 0 & & & & & \\
0 & 1 & 0 & a & b & (a^2,c) & (ab,d) & (a^3,b^2,e) \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6\,.
\end{array} \tag{3.4.8}$$

Since this spectral sequence converges to $H^*(\mathrm{pt})$, there must be a d_2 differential from z_1 to a, and a d_3 differential from $y = z_1^2$ to b. The new elements in the zeroth column that are not killed by lower differentials must all transgress, because there are no other elements in the spectral sequence that could kill them, so we infer the existence of the classes c, d, and e, such that d_4 maps z_2 to c, d_5 maps z_1^4 to d, and d_6 maps z_3 to e. With the generators in low degree at our disposal, we now give the Steenrod action on these generators. For this we consider the fibration $B\mathrm{SU}_8 \to BG_8 \to B^2\mathbb{Z}/2$; this allows us to determine the Steenrod squares of everything in the image of the pullback map $H^*(B^2\mathbb{Z}/2;\mathbb{Z}/2) \to H^*(BG_8;\mathbb{Z}/2)$. Serre [216, Théorème 2] showed that $H^*(B^2\mathbb{Z}/2;\mathbb{Z}/2) = \mathbb{Z}/2[T,\, y \coloneqq \mathrm{Sq}^1T,\, z \coloneqq \mathrm{Sq}^2\mathrm{Sq}^1T,\, \ldots]$, so in the Serre spectral sequence

$$E_2^{p,q} = H^p(B^2\mathbb{Z}/2; H^q(B\mathrm{SU}_8;\mathbb{Z}/2)) \Longrightarrow H^{p+q}(BG_8;\mathbb{Z}/2) \tag{3.4.9}$$

the E_2-page is given in low degrees by

$$\begin{array}{c|ccccccccc}
10 & c_2c_3 \\
9 & 0 \\
8 & c_2^2, c_4 \\
7 & 0 & 0 & 0 & 0 \\
6 & c_3 & 0 & c_3T & c_3y \\
5 & 0 & 0 & 0 & 0 \\
4 & c_2 & 0 & c_2T & c_2y & c_2T^2 & (c_2z, c_2Ty) \\
3 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & T & y & T^2 & (z, Ty) & (T^3, y^2) & (T^2y, Tz) & (T^4, Ty^2, yz) \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8\,,
\end{array} \tag{3.4.10}$$

where the c_i are the mod 2 reductions of the corresponding Chern classes in the cohomology of $B\mathrm{SU}_8$. We immediately see that the classes a and b are pulled back from T and $y = \mathrm{Sq}^1T$ respectively, since there are no differentials that hit these two generators. Furthermore c pulls back to c_2 in the cohomology of $B\mathrm{SU}_8$, and d is the pullback of $z \in H^5(B^2\mathbb{Z}/2;\mathbb{Z}/2)$. Thus

$$\begin{aligned}
\mathrm{Sq}^1 a &= b, & \mathrm{Sq}^2 a &= a^2,\\
\mathrm{Sq}^1 b &= 0, & \mathrm{Sq}^2 b &= d,\\
\mathrm{Sq}^1 d &= b^2, & \mathrm{Sq}^2 d &= 0\,.
\end{aligned} \tag{3.4.11}$$

Lastly, we need to determine the action of the Steenrod operators on c and e.

Lemma 3.4.12. *The classes c and e are in the image of the mod 2 reduction map*

$$r\colon H^*(BG_8;\mathbb{Z}) \to H^*(BG_8;\mathbb{Z}/2).$$

Corollary 3.4.13. $\mathrm{Sq}^1(c) = 0$ *and* $\mathrm{Sq}^1(e) = 0$.

Proof. Sq^1 is the Bockstein for the short exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$. Therefore if x is in the image of $r_4\colon H^*(-;\mathbb{Z}/4) \to H^*(-;\mathbb{Z}/2)$, then $\mathrm{Sq}^1(x) = 0$. And the mod 2 reduction map $\mathbb{Z} \to \mathbb{Z}/2$ factors through $\mathbb{Z}/4$. $\square$

Proof of Lemma 3.4.12. The map r induces a map of Serre spectral sequences for the fibration $B\mathbb{Z}/2 \to B\mathrm{SU}_8 \to BG_8$; we run the Serre spectral sequence with $\mathbb{Z}$ coefficients, which has signature

$$E_2^{*,*} = H^*(BG_8; H^*(B\mathbb{Z}/2;\mathbb{Z})) \Longrightarrow H^*(B\mathrm{SU}_8;\mathbb{Z}). \tag{3.4.14}$$

Since BG_8 is simply connected, we do not need to worry about local coefficients. We know that $H^*(B\mathbb{Z}/2;\mathbb{Z}) \cong \mathbb{Z}[z]/2z$, where $|z| = 2$, and Borel [28, §29] computed $H^*(B\mathrm{SU}_8;\mathbb{Z}) \cong \mathbb{Z}[c_2,\dots,c_8]$, with $|c_i| = 2i$, so we may run the spectral sequence in reverse. The E_2-page for (3.4.14) is:

$$\begin{array}{c|ccccccc}
6 & z^3 & 0 & 0 & \alpha z^3 & c_2z^3 & \beta z^3 & (c_3z^3, \alpha z^3)\\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
4 & z^2 & 0 & 0 & \alpha z^2 & c_2z^2 & \beta z^2 & (c_3z^2, \alpha^2 z^2)\\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
2 & z & 0 & 0 & \alpha z & c_2z & \beta z & (c_3z, \alpha^2 z)\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & \alpha & c_2 & \beta & (c_3, \alpha^2)\\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6\,.
\end{array} \tag{3.4.15}$$

As $H^2(B\mathrm{SU}_8;\mathbb{Z}) = 0$, $z \in E_2^{0,2} = H^2(B\mathbb{Z}/2;\mathbb{Z})$ admits a differential. The only option is a transgressing d_3; let $\alpha \coloneqq d_3(z)$. Since $2z = 0$, $2\alpha = 0$. The Leibniz rule (now with signs) tells us

$$d_2(z^2) = zd_2(z) + d_2(z)z = 2\alpha z = 0. \tag{3.4.16}$$

Therefore if z^2 admits a differential, the differential must be the transgressing $d_5\colon E_4^{0,4} \to E_4^{5,0}$, see (3.4.15). But z^2 does admit a differential. One way to see this is to compute the pullback $H^4(B\mathrm{SU}_8;\mathbb{Z}) \to H^4(B\mathbb{Z}/2;\mathbb{Z})$. Since $H^4(B\mathrm{SU}_8;\mathbb{Z})$ is generated by c_2 of the defining representation $\mathbb{C}^8$, we can restrict that representation to $\mathbb{Z}/2$ and compute its second Chern class to compute the pullback map. As a representation of $\mathbb{Z}/2$, $\mathbb{C}^8$ is a direct sum of 8 copies of the sign representation, so its total Chern class is $c(8\sigma) = (1+z)^8$ by the Whitney sum rule, and the z^2 term is $\binom{8}{2}z^2$, which is even. Since $2z^2 = 0$, this implies c_2 pulls back to 0. If z^2 did not support a differential, then it would be in the image of this pullback map, so we have discovered that z^2 admits a differential, specifically d_5. Let $\beta \coloneqq d_5(z^2)$. From the spectral sequence we see that $H^4(BG_8;\mathbb{Z})$ is isomorphic to $H^4(B\mathrm{SU}_8;\mathbb{Z})$, and c_2 is an element in this cohomology. By using the mod 2 reduction map from $H^4(B\mathrm{SU}_8;\mathbb{Z}) \to H^4(B\mathrm{SU}_8;\mathbb{Z}/2)$, and the pullback map induced from $\mathrm{SU}_8 \xrightarrow{f} G_8$ we see that c is the mod 2 reduction of c_2 in $H^4(BG_8;\mathbb{Z})$. This is summarized in the following diagram:

$$\begin{tikzcd} c_2 \in H^4(BG_8;\mathbb{Z}) \arrow[r, "\text{mod } 2"] \arrow[d, "\cong"] & H^4(BG_8;\mathbb{Z}/2) \arrow[d, "f^*"] \\ H^4(B\mathrm{SU}_8;\mathbb{Z}) \arrow[r] & H^4(B\mathrm{SU}_8;\mathbb{Z}/2)\,. \end{tikzcd} \tag{3.4.17}$$

□

We define $e \coloneqq \mathrm{Sq}^2(c)$. This is not parallel to the definition of c: we defined c as the mod 2 reduction of c_2, but we have not addressed whether $e = c_3$ mod 2. This choice of definition presents ambiguities in the action of the Steenrod squares on e and the relations in the cohomology ring, but these ambiguities are in too high of a degree to affect the computation at hand. □

Remark 3.4.18. Toda [228] uses another approach to compute $H^*(BG;\mathbb{Z}/2)$ when G is compact, simple, and not simply connected: the Eilenberg-Moore spectral sequence

$$E_2^{p,q} = \mathrm{coTor}^{p,q}_{H^*(B\pi_1(G);\mathbb{Z}/2)}(H^*(B\widetilde{G};\mathbb{Z}/2),\mathbb{Z}/2) \Longrightarrow H^{p+q}(BG;\mathbb{Z}/2), \tag{3.4.19}$$

where $\widetilde{G} \to G$ is the universal cover, the coalgebra structure on $H^*(B\pi_1(G);\mathbb{Z}/2)$ comes from multiplication on $\pi_1(G)$, and the comodule structure on $H^*(B\widetilde{G};\mathbb{Z}/2)$ comes from the inclusion $\pi_1(G) \hookrightarrow \widetilde{G}$ and multiplication in $\widetilde{G}$. If you apply this to $G = G_8$, however, the E_2-page of the Eilenberg-Moore spectral sequence is identical to the E_2-page of the Serre spectral sequence (3.4.9) in the range relevant to us.

We now compute $H^*(B(\mathrm{Spin}\text{-}\mathrm{SU}_8;\mathbb{Z}/2)$, which is what we actually need for U-duality. There is a central extension

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Spin}\text{-}\mathrm{SU}_8 \longrightarrow \mathrm{SO}\times G_8 \longrightarrow 0, \tag{3.4.20}$$

which we think of physically as "quotienting by fermion parity." Such extensions are classified by a class in $H^2(B(\mathrm{SO}\times G_8);\mathbb{Z}/2)$. (3.4.20) is classified by $w_2 + a$, which one can prove by pulling back along $\mathrm{SO} \to \mathrm{SO} \times G_8$ and $G_8 \to \mathrm{SO} \times G_8$ and observing that both pulled-back extensions are non-split.

Taking classifying spaces, we obtain a fibration

$$B\mathbb{Z}/2 \longrightarrow B(\mathrm{Spin}\text{-}\mathrm{SU}_8) \longrightarrow B(\mathrm{SO} \times G_8)\,, \tag{3.4.21}$$

and we apply the Serre spectral sequence to this fibration using knowledge of the cohomology of BG_8.

Theorem 3.4.22. *$H^*(B(\mathrm{Spin}\text{-}\mathrm{SU}_8); \mathbb{Z}/2) \cong \mathbb{Z}/2[a, b, c, w_4, d, e, \ldots]$ with $|a| = 2$, $|b| = 3$, $|c| = 4$, $|w_4| = 4$,$|d| = 5$, and $|e| = 6$. The map $\mathrm{Spin}\text{-}\mathrm{SU}_8 \to \mathrm{SO} \times G_8$ induces a quotient map on cohomology, and the Steenrod squares of a, b, c, d, and e are given in (3.4.5) along with*

$$\begin{aligned} \mathrm{Sq}^1 w_4 &= ab + d\,, \\ \mathrm{Sq}^2 w_4 &= aw_4 + \ldots\,. \end{aligned} \tag{3.4.23}$$

Proof. We run the Serre spectral sequence with signature

$$E_2^{*,*} = H^*(B(\mathrm{SO} \times G_8); H^*(B\mathbb{Z}/2; \mathbb{Z}/2)) \Longrightarrow H^*(B(\mathrm{Spin}\text{-}\mathrm{SU}_8); \mathbb{Z}/2)\,, \tag{3.4.24}$$

where the E_2-page is given by:

$$\begin{array}{c|cccccc}
5 & t^5 & 0 & & & & \\
4 & t^4 & 0 & & & & \\
3 & t^3 & 0 & & & & \\
2 & t^2 & 0 & (t^2a, t^2a + t^2w_2) & \ldots & & \\
1 & t & 0 & (ta, ta{+}tw_2) & (tb, tb{+}tw_3) & \ldots & \\
0 & 1 & 0 & (a, a{+}w_2) & (b, b{+}w_3) & \begin{pmatrix} a^2, c, w_4, a^2{+}w_2^2, \\ a(a+w_2) \end{pmatrix} & \begin{pmatrix} ab{+}d{+}w_5, (a{+}w_2)(b{+}w_3) \\ a(b+w_3), b(a+w_2) \end{pmatrix} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5\,.
\end{array} \tag{3.4.25}$$

The w_i are the Stiefel-Whitney classes of $B\mathrm{SO}$, and t is the generator of the cohomology $H^*(B\mathbb{Z}/2; \mathbb{Z}/2)$. The differential $d_2 : E_2^{0,1} \to E_2^{2,0}$ hits the class for the extension (3.4.21) that gives $\mathrm{Spin}\text{-}\mathrm{SU}_8$, which is $a + w_2$, and identifies $a = w_2$. Applying the Leibniz rule shows $d_2(t^{2n+1}) = t^{2n}a$, and that $d_2(t^{2n}) = 0$: something else must kill the even powers of t. We then use Kudo's transgression theorem [176], which says that Steenrod squares commute with transgression in the Serre spectral sequence. Therefore $d_3 : E_3^{0,2} \to E_3^{3,0}$ sends $t^2 \mapsto b + w_3$, since the transgressing d_2 sends $\mathrm{Sq}^1 t = t^2$ to $\mathrm{Sq}^1(a + w_1)$. In total degree

4, there is a likewise d_4 differential that takes t^4 to $ab + d + w_5$, i.e. this differential takes $\mathrm{Sq}^2 t^2$ to $\mathrm{Sq}^2(b + w_3)$.[7] We see that there is a new class w_4 which pulled back from $B\mathrm{SO}$. Applying the Wu formula then establishes (3.4.23). □

3.4.3. The Adams Computation

In this section, we run the Adams spectral sequence for Spin-SU_8 bordism.

Theorem 3.4.26. *Up to degree 5, the first few groups of* Spin-SU_8 *bordism are*

$$\begin{aligned}
\Omega_0^{\mathrm{Spin}\text{-}\mathrm{SU}_8} &\cong \mathbb{Z}\\
\Omega_1^{\mathrm{Spin}\text{-}\mathrm{SU}_8} &\cong 0\\
\Omega_2^{\mathrm{Spin}\text{-}\mathrm{SU}_8} &\cong 0\\
\Omega_3^{\mathrm{Spin}\text{-}\mathrm{SU}_8} &\cong 0\\
\Omega_4^{\mathrm{Spin}\text{-}\mathrm{SU}_8} &\cong \mathbb{Z}^2\\
\Omega_5^{\mathrm{Spin}\text{-}\mathrm{SU}_8} &\cong \mathbb{Z}/2.
\end{aligned} \tag{3.4.27}$$

Treating $d \in H^5(BG_8; \mathbb{Z}/2)$ as a characteristic class, the bordism invariant $(M, P) \mapsto \int_M d(P) \in \mathbb{Z}/2$ realizes the isomorphism $\Omega_5^{\mathrm{Spin}\text{-}\mathrm{SU}_8} \to \mathbb{Z}/2$.

Proof. The first simplification to working with the entire Steenrod algebra is that the only higher Steenrod operator beyond Sq^2 in $\mathcal{A}$ that we must incorporate for the purpose of working up to degree 5 is Sq^4. As input, we need the $\mathcal{A}$-module structure on $H^*(MT(\mathrm{Spin}\text{-}\mathrm{SU}_8); \mathbb{Z}/2)$, which by the Thom isomorphism is given by $\mathbb{Z}/2[a, b, c, w_4, d, e, \ldots]\{U\}$, where $U \in H^0(MTSO; \mathbb{Z}/2)$ is the Thom class coming from the tautological bundle over $B\mathrm{SO}$. For any cohomology class x coming from $B\mathrm{SO}$, we can get the Steenrod squares of Ux from the $\mathcal{A}$-module structure on $MTSO$. We have also previously determined the action of Steenrod squares on elements of the cohomology of BG_8, and therefore we know the Steenrod action on all elements in $H^*(MT(\mathrm{Spin}\text{-}\mathrm{SU}_8); \mathbb{Z}/2)$. We thus have [18, Remark 3.3.5]

$$\mathrm{Sq}^k(Ux) = \sum_{i=0}^{k} \mathrm{Sq}^i(U)\mathrm{Sq}^{k-i}(x) = \sum_{i=0}^{k} U w_i\, \mathrm{Sq}^{k-i}(x), \tag{3.4.28}$$

[7] We slightly change the basis for the degree 5 generators here so that the d_4 differential identifies w_5 with $ab + d$ and therefore $\mathrm{Sq}^1(w_4 U) = w_5 U = (ab + d)U$ agrees with $\mathrm{Sq}^2(bU)$. This is necessary in order to have a valid $\mathcal{A}$ module. We point to [4] as a reference for the fact that M_n does not lift from an $\mathcal{A}(1)$ module to an $\mathcal{A}$ module for any finite n. This means in the degree we are considering, there must be a node in degree 4 that is joined with $(ab + d)U$ upon acting by Sq^1.

where $w_1 = 0$ when pulled back from $MTSO$ and $w_2 = a, w_3 = b, w_5 = ab + d$ by the proof of Theorem 3.4.22. After localizing at $p = 2$, $MTSO$ is a direct sum of Eilenberg-MacLane spectra, which in low degree is

$$H^*(MTSO; \mathbb{Z}/2) \cong H^*(H\mathbb{Z}) \oplus \Sigma^4 H^*(H\mathbb{Z}) \oplus \Sigma^5 H^*(H\mathbb{Z}/2) \oplus \dots . \tag{3.4.29}$$

Under the quotient map in cohomology

$$H^*(MTSO \wedge (BG_8)_+; \mathbb{Z}/2) \to H^*(MT(\text{Spin-SU}_8); \mathbb{Z}/2); \mathbb{Z}/2), \tag{3.4.30}$$

the three summands in (3.4.29) survive, and in addition we pick up a new summand M in $H^*(B(\text{Spin-SU}_8; \mathbb{Z}/2)$ containing Uc which came from the cohomology of BG_8. We have not fully determined the $\mathcal{A}$-module structure of M, but if we quotient M by the submodule of all elements of degrees seven and above, we obtain the $\mathcal{A}$-module $\Sigma^4 C\eta$, where $C\eta$ consists of two $\mathbb{Z}/2$ summands in degrees 0 and 2 joined by a Sq^2, and $\Sigma^k C\eta$ denotes the shift of $C\eta$ in which the grading of every element is increased by k. Thus, if we quotient by all elements in degrees 7 and above, there is an isomorphism of $\mathcal{A}$-modules

$$H^*(MT(\text{Spin-SU}_8); \mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2 \oplus \Sigma^4(\mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2) \oplus \Sigma^4 C\eta \oplus \Sigma^5 \mathcal{A} \oplus P, \tag{3.4.31}$$

where P contains no nonzero elements in degrees 5 and below, and we use the fact that $H^*(H\mathbb{Z}) = \mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2$ and $H^*(H\mathbb{Z}/2) = \mathcal{A}$, which follows from computations of Serre [216]. The red summand $\mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2$ is generated by U, and is worked out in Figure 3.1 by using (3.4.28). The green summand is generated by Ua^2, and the purple summand is generated by Ud. Quotienting by high-degree elements does not affect the Ext groups in the degrees we need for the theorem.

To compute the E_2-page of the Adams spectral sequence we need to know Ext of each summand in (3.4.31) (Ext(–) means $\mathrm{Ext}_{\mathcal{A}}^{*,*}(-; \mathbb{Z}/2)$.) By using the change of rings theorem [18, Section 4.5], we get $\mathrm{Ext}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{Z}/2, \mathbb{Z}/2) = \mathrm{Ext}_{\mathcal{A}(0)}(\mathbb{Z}/2, \mathbb{Z}/2)$, and since $\mathcal{A}(0)$ only includes Sq^1, this just gives $\mathbb{Z}/2[h_0]$ [18, Remark 4.5.4], where $h_0 \in \mathrm{Ext}^{1,1}$. The same logic applies for the Ext of the green summand, and the Ext of the purple summand contributes a $\mathbb{Z}/2$ in degree 5.

The last ingredient we need is $\mathrm{Ext}_{\mathcal{A}}(C\eta)$, at least in low degrees.

Lemma 3.4.32. *$\mathrm{Ext}_{\mathcal{A}}(C\eta)$ is isomorphic to $\mathbb{Z}/2[h_0]$ in topological degree 0 and vanishes in topological degree 1.*

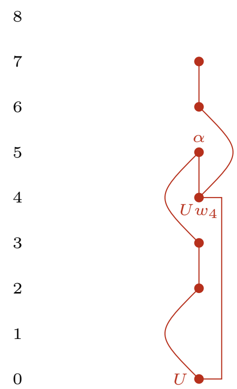


Figure 3.1: The only relevant higher Steenrod operation in this degree is Sq^4, which acts on U to give Uw_4. This is connected to $\alpha = (ab + d)U$ by Sq^1.

Proof. We use a standard technique: $C\eta$ is part of a short exact sequence of $\mathcal{A}$-modules

$$0 \longrightarrow \Sigma^2\mathbb{Z}/2 \longrightarrow C\eta \longrightarrow \mathbb{Z}/2 \longrightarrow 0, \tag{3.4.33}$$

and a short exact sequence of $\mathcal{A}$-modules induces a long exact sequence of Ext groups. It is conventional to draw this as if on the E_1-page of an Adams-graded spectral sequence; see [18, §4.6] for more information and some additional examples. We draw the short exact sequence (3.4.33) in Figure 3.2, left, and we draw the induced long exact sequence in Ext in Figure 3.2, right. Looking at this long exact sequence, there are three boundary maps that could be nonzero in the range displayed; because boundary maps commute with the $\mathrm{Ext}_{\mathcal{A}}(\mathbb{Z}/2)$-action, these boundary maps are all determined by

$$\partial\colon \mathrm{Ext}_{\mathcal{A}}^{0,2}(\Sigma^2\mathbb{Z}/2) \to \mathrm{Ext}_{\mathcal{A}}^{1,2}(\mathbb{Z}/2). \tag{3.4.34}$$

This boundary map is either 0 or an isomorphism, and it must be an isomorphism, because

$$\mathrm{Ext}_{\mathcal{A}}^{0,2}(C\eta) = \mathrm{Hom}_{\mathcal{A}}(C\eta, \Sigma^2\mathbb{Z}/2) = 0, \tag{3.4.35}$$

and if the boundary map vanished, we would obtain $\mathbb{Z}/2$ for this Ext group. Thus we know $\mathrm{Ext}_{\mathcal{A}}(C\eta)$ in the range we need. $\square$

Compiling the information of Ext on (3.4.31) we draw the E_2-page of the Adams spectral sequence through topological degree 5 in Figure 3.3.[8]

In this range, the only differentials that could be nonzero go from the 5-line to the

[8]The modules in red, blue, and purple are pulled back from $MTSO$.

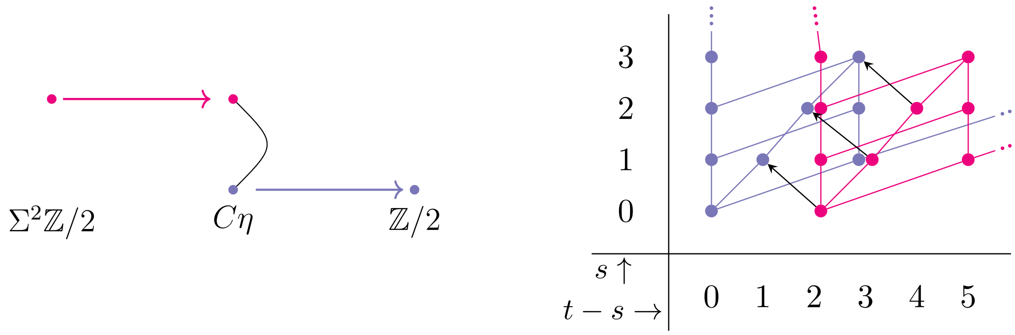


Figure 3.2: Left: the short exact sequence (3.4.33). Right: the induced long exact sequence in Ext groups. These diagrams are part of the proof of Lemma 3.4.32.

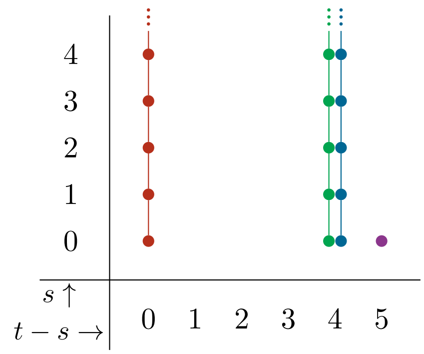


Figure 3.3: The E_2-page of the Adams spectral sequence computing $\Omega_*^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$.

4-line. Usually we would need to know the 6-line in order to determine if there are any differentials from the 6-line to the 5-line, so that we could evaluate $\Omega_5^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$, but the 5-line is concentrated in filtration zero, and all Adams differentials land in filtration 2 or higher, so what we have computed is good enough.

Returning to the differentials from the 5-line to the 4-line: Adams differentials must commute with the action of h_0 on the E_r-page, and h_0 acts by 0 on the 5-line but injectively on the 4-line, so these differentials must also vanish. Thus the spectral sequence collapses giving the bordism groups in the theorem statement. The fact that $\Omega_5^{\mathrm{Spin}\text{-}\mathrm{SU}_8} \cong \mathbb{Z}/2$ is detected by $\int d$ follows from the fact that its image in the E_∞-page is in Adams filtration zero, corresponding to Ext of the free $\Sigma^5\mathcal{A}$ summand generated by Ud; see [99, §8.4].[9]

$\square$

[9] While we do not draw the $\mathcal{A}(1)$ modules up to degree 6, there is a way to access information in this degree. We know that if we replace the spin bordism of BG_8 with the oriented bordism of $B\mathrm{SU}_8$, then the Atiyah-Hirzebruch spectral sequence for oriented bordism tensored with $\mathbb{Q}$ tells us in degree 6, there should be one $\mathbb{Q}$ summand that is detected by c_3 of the SU_8-bundle.

3.4.4. Determining the Manifold Generator

We now determine the generator of $\Omega_5^{\mathrm{Spin}\text{-}\mathrm{SU}_8} \cong \mathbb{Z}/2$. We start by considering a map $\widetilde{\Phi}\colon \mathrm{SU}_2 \to \mathrm{SU}_8$ sending a matrix A to its fourfold block sum $A \oplus A \oplus A \oplus A$. This sends $-1 \mapsto -1$, so $\widetilde{\Phi}$ descends to a map

$$\Phi\colon \mathrm{SO}_3 = \mathrm{SU}_2/\{\pm 1\} \longrightarrow \mathrm{SU}_8/\{\pm 1\} = G_8. \tag{3.4.36}$$

Recall that $H^*(B\mathrm{SO}_3; \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3]$ and that there are three classes a, b, and d in $H^*(B(\mathrm{SU}_8/\{\pm 1\}); \mathbb{Z}/2)$.

Lemma 3.4.37. *$\Phi^*(a) = w_2$, $\Phi^*(b) = w_3$, and $\Phi^*(d) = w_2 w_3$.*

This will imply that to find a generator, all we have to do is find a closed, oriented 5-manifold M with a principal SO_3-bundle $P \to M$ with $w_2(M) = w_2(P)$ and $w_2(P)w_3(P) \neq 0$. This is easier than directly working with G_8!

Proof of Lemma 3.4.37. Once we show $\Phi^*(a) = w_2$, we're done:

$$\Phi^*(b) = \Phi^*(\mathrm{Sq}^1(a)) = \mathrm{Sq}^1(\Phi^*(a)) = \mathrm{Sq}^1(w_2) = w_3, \tag{3.4.38a}$$

where the last equal sign follows by the Wu formula. In a similar way

$$\Phi^*(d) = \Phi^*(\mathrm{Sq}^2(b)) = \mathrm{Sq}^2(\Phi^*(b)) = \mathrm{Sq}^2(w_3) = w_2 w_3, \tag{3.4.38b}$$

again using the Wu formula. So all we have to do is pull back a.

Consider the commutative diagram of short exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & \mathrm{SU}_2 \arrow[r] \arrow[d, "\widetilde{\Phi}"] & \mathrm{SO}_3 \arrow[r] \arrow[d, "\Phi"] & 1 \\
1 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathrm{SU}_8 \arrow[r] & G_8 \arrow[r] & 1.
\end{tikzcd} \tag{3.4.39}$$

Taking classifying spaces, this shows that the pullback of the fiber bundle $B\mathbb{Z}/2 \to B\mathrm{SU}_8 \to BG_8$ along the map $\Phi\colon B\mathrm{SO}_3 \to BG_8$ is the fiber bundle $B\mathbb{Z}/2 \to B\mathrm{SU}_2 \to B\mathrm{SO}_3$. We therefore obtain a map between the Serre spectral sequences computing the mod 2 cohomology rings of $B\mathrm{SU}_2$ and $B\mathrm{SU}_8$, and it is an isomorphism on $E_2^{0,*}$, i.e. on the cohomology of the fiber.

Both $B\mathrm{SU}_2$ and $B\mathrm{SU}_8$ are simply connected, so $H^1(-; \mathbb{Z}/2)$ vanishes for both spaces. Therefore in both of these Serre spectral sequences, the generator x of $E_2^{0,1} = H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$

must admit a differential. The only differential that can be nonzero is the transgressing $d_2\colon E_2^{0,1} \to E_2^{2,0}$; in $E_2(\mathrm{SU}_8)$, we saw in (3.4.8) that $d_2(x) = a$, and in $E_2(\mathrm{SU}_2)$, $d_2(x) = w_2$, because w_2 is the only nonzero element of $E_2^{2,0} = H^2(B\mathrm{SO}_3;\mathbb{Z}/2)$. Since the pullback map of spectral sequences commutes with differentials, this means $\Phi^*(a) = w_2$ as desired. □

Now let $W \coloneqq \mathrm{SU}_3/\mathrm{SO}_3$, which is a closed, oriented 5-manifold called the *Wu manifold*, and let $P \to W$ be the quotient $\mathrm{SU}_3 \to \mathrm{SU}_3/\mathrm{SO}_3$, which is a principal SO_3-bundle. For completeness we prove the following proposition about the cohomology of the Wu manifold.

Proposition 3.4.40. *$H^*(W;\mathbb{Z}/2) \cong \mathbb{Z}/2[z_2, z_3]/(z_2^2, z_3^2)$ with $|z_2| = 2$ and $|z_3| = 3$. The Steenrod squares are*

$$\begin{aligned} \mathrm{Sq}(z_2) &= z_2 + z_3 \\ \mathrm{Sq}(z_3) &= z_3 + z_2 z_3, \end{aligned} \tag{3.4.41}$$

and the Stiefel-Whitney class is $w(W) = 1 + z_2 + z_3$. Moreover, $w(P) = 1 + z_2 + z_3$. Thus $w_2(M) = w_2(P)$, so W with G_8-bundle induced from P has a spin-SU_8 structure, and $w_2(P)w_3(P) \neq 0$, meaning (W, P) is our sought-after generator of $\Omega_5^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$.

Proof. Once we know the cohomology ring and the Steenrod squares are as claimed, the total Stiefel-Whitney class of W follows from Wu's theorem as follows. The second Wu class v_2 is defined to be the Poincaré dual of the map

$$x \mapsto \int_W \mathrm{Sq}^2(x)\colon H^3(W;\mathbb{Z}/2) \to H^5(W;\mathbb{Z}/2) \to \mathbb{Z}/2 \tag{3.4.42}$$

via the Poincaré duality identification $H^2(W;\mathbb{Z}/2) \cong (H^3(W;\mathbb{Z}/2))^\vee$. Wu's theorem shows that $v_2 = w_2 + w_1^2$, so since $H^1(W;\mathbb{Z}/2) = 0$, $w_1 = 0$ and $w_2 = v_2$. Since $\mathrm{Sq}^2(z_3) = z_2 z_3$, $w_2 \neq 0$, so it must be z_2. Then $w_3 = \mathrm{Sq}^1(w_2) = z_3$; w_4 is trivial for degree reasons; and $w_5 = 0$ follows from the Wu formula calculating $\mathrm{Sq}^1(w_4)$.

So we need to compute the cohomology ring. Consider the Serre spectral sequence for the fiber bundle

$$\begin{tikzcd} \mathrm{SO}_3 \arrow[r] & \mathrm{SU}_3 \arrow[d] \\ & W, \end{tikzcd} \tag{3.4.43}$$

which has signature

$$E_2^{*,*} = H^*(W; H^*(\mathrm{SO}_3;\mathbb{Z}/2)) \Longrightarrow H^*(\mathrm{SU}_3;\mathbb{Z}/2). \tag{3.4.44}$$

A priori we must account for the action of $\pi_1(W)$ on $H^*(\mathrm{SO}_3;\mathbb{Z}/2)$, but using the long exact sequence in homotopy groups associated to a fiber bundle one deduces that W is simply connected because SU_3 is; therefore we do not have to worry about this. Moreover, because W is simply connected, the universal coefficient theorem tells us $H^1(W;\mathbb{Z}/2) = 0$.

As manifolds, $\mathrm{SO}_3 \cong \mathbb{RP}^3$, so $H^*(\mathrm{SO}_3;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^4)$. Also, $H^*(\mathrm{SU}_3;\mathbb{Z}/2) \cong \mathbb{Z}/2[c_2, c_3]/(c_2^2, c_3^2)$, with $|c_2| = 3$ and $|c_3| = 5$ [29, §8].

Lemma 3.4.45. *$H^2(W;\mathbb{Z}/2) \cong \mathbb{Z}/2$.*

Proof. The class $x \in E_2^{0,1} = H^1(\mathrm{SO}_3;\mathbb{Z}/2)$ supports a differential because $H^1(\mathrm{SU}_3;\mathbb{Z}/2) = 0$. Since the Serre spectral sequence is first-quadrant, the only option is a transgressing $d_2\colon E_2^{0,1} \to E_2^{2,0}$. Therefore $\dim H^2(W;\mathbb{Z}/2) \geq 1$. One can also see that this is an upper bound. Since $H^2(\mathrm{SU}_3;\mathbb{Z}/2) = 0$ as well, any additional classes in $E_2^{2,0} = H^2(W;\mathbb{Z}/2)$ have to be killed by a differential. But the only differential that could kill those classes is the transgressing d_2 we just mentioned, and x is the only nonzero element of $H^1(\mathrm{SO}_3;\mathbb{Z}/2)$, so there cannot be anything else in $H^2(W;\mathbb{Z}/2)$. □

This is enough to get the cohomology ring: we already know H^0, H^1, and H^2 for the Wu manifold; Poincaré duality tells us $H^3(W;\mathbb{Z}/2) \cong \mathbb{Z}/2$, H^4 vanishes, and $H^5 \cong \mathbb{Z}/2$. Therefore there must be generators z_2 and z_3 for the cohomology ring in degrees 2 and 3, respectively, and their squares vanish for degree reasons. And by Poincare duality $z_2z_3 \neq 0$, so it is the generator of H^5. Therefore the cohomology ring is as we claimed.

Next we must determine the Steenrod squares. The fibration (3.4.43) pulls back from the universal SO_3-bundle $\mathrm{SO}_3 \to E\mathrm{SO}_3 \to B\mathrm{SO}_3$ via the classifying map f_P for P, inducing a map of Serre spectral sequences that commutes with the differentials. We draw this map in Figure 3.4. This map is an isomorphism on the line $E_2^{0,*}$, so $x \in E_2^{0,1}(\mathrm{SU}_3)$ pulls back from the generator $x \in E_2^{0,1}(E\mathrm{SO}_3)$ — and therefore $d_2(x) = z_2$ pulls back from a class in $E_2^{2,0} = H^2(B\mathrm{SO}_3;\mathbb{Z}/2)$. The only nonzero class in that degree is w_2, so $f_P^*(w_2) = z_2$, i.e. $w_2(P) = z_2$.

The Leibniz rule that in the Serre spectral sequence for SU_3, $d_2(x^2) = 2xd_2(x) = 0$. But because $H^2(\mathrm{SU}_2;\mathbb{Z}/2) = 0$, some differential must kill x^2. Apart from d_2, the only option is the transgressing $d_3\colon E_3^{0,2} \to E_3^{3,0}$, forcing $d_3(x^2) = z_3$. A similar argument in the Serre spectral sequence for $E\mathrm{SO}_3$ shows that in that spectral sequence, $d_3(x^2) = w_3$; therefore $f_P^*(w_3) = z_3$ and $w_3(P) = z_3$. Pullback commutes with Steenrod squares and $\mathrm{Sq}^1(w_2) = w_3$, so $\mathrm{Sq}^1(z_2) = z_3$. Finally, $f_P^*(w_2w_3) = z_2z_3$, and the Wu formula implies $\mathrm{Sq}^2(w_3) = w_2w_3$, so $\mathrm{Sq}^2(z_3) = z_2z_3$. We have computed all the Steenrod squares that could be nonzero for degree reasons, and along the way shown $w(P) = 1 + z_2 + z_3$: the higher-degree Stiefel-Whitney classes of a principal SO_3-bundle vanish. □

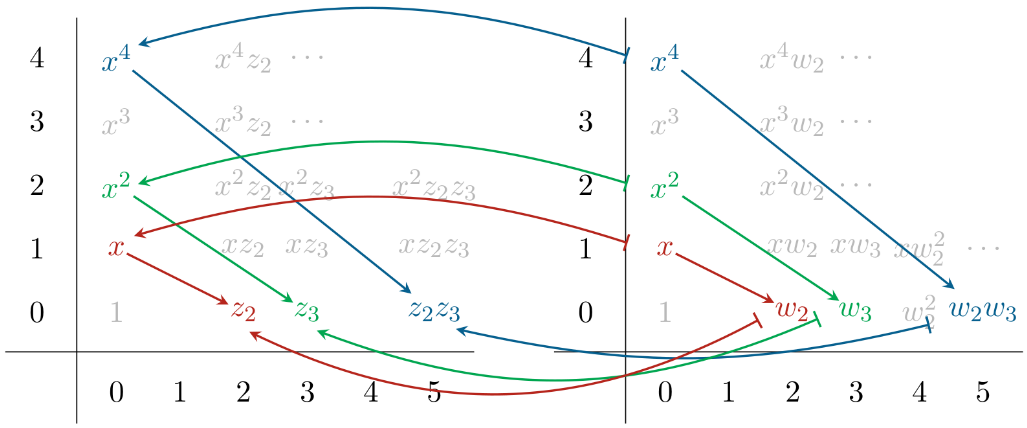


Figure 3.4: The fiber bundle $\mathrm{SO}_3 \to \mathrm{SU}_3 \to W$ pulls back from the universal SO_3-bundle $\mathrm{SO}_3 \to E\mathrm{SO}_3 \to B\mathrm{SO}_3$, inducing a map of Serre spectral sequences. This map commutes with differentials and is the identity on $E_2^{0,\bullet} = H^*(\mathrm{SO}_3; \mathbb{Z}/2)$, allowing us to conclude that w_2 pulls back to z_2, w_3 pulls back to z_3, and $w_2 w_3$ pulls back to $z_2 z_3$. This is a picture proof of part of Proposition 3.4.40.

3.5. Evaluating on the Anomaly

With the knowledge of the generating manifold for the $\mathbb{Z}/2$ in degree 5 as the Wu manifold, we can consider evaluating the anomaly of the theory with the field content given in §3.2. Since G_8 acts trivially on the scalars and the graviton only the remaining three fields could have anomalies.

Definition 3.5.1. The global anomaly for a fermion on a Riemannian manifold M in a a representation R coupled to background G gauge field is given by an invertible field theory with partition function the exponential of an η-invariant of the Dirac operator, $\eta_{M,R}(\mathcal{D})$[246, Section 4.3].

- For gauginos it is given by $\mathcal{A}_{1/2} = \exp(\pi i \eta_{M,R}(\mathcal{D})/2)$ [245, 99].
- For gravitinos it is given by $\mathcal{A}_{3/2} = \exp(\pi i \eta_{\text{gravitino}}/2)$ where

$$\eta_{\text{gravitino}} = \eta_{M,R}(\mathcal{D}_{\text{Dirac}\times TW}) - 2\eta_{M,R}(\mathcal{D}), \tag{3.5.2}$$

and $\eta_{M,R}(\mathcal{D}_{\mathrm{Dirac}\times TW})$ is the Dirac operator acting on the spinor bundle tensored with the tangent bundle [136].

For the remainder of the paper we will drop the M subscript label.

Lemma 3.5.3. *If $R = \sum_i R_i$ then $\eta_{M,\sum_i R_i}(\mathcal{D}) = \sum_i \eta_{M,R_i}(\mathcal{D})$.*

The anomaly for the vector boson is not given in terms of an η-invariant, but we assume that it is also an invertible theory, and we show that it also vanishes. The next section is dedicated to showing:

Theorem 3.5.4. *The total anomaly (global and perturbative) of 4d $\mathcal{N} = 8$ supergravity arising from the gaugino, vector boson, and gravitino, vanishes on the Wu manifold.*

3.5.1. Evaluating on the Wu manifold

The full anomaly denoted by $\mathcal{A}$ can be written schematically as

$$\text{``}\mathcal{A} = \mathcal{A}^{\mathrm{pert}}_{1/2} \otimes \mathcal{A}^{\mathrm{pert}}_{1} \otimes \mathcal{A}^{\mathrm{pert}}_{3/2} \otimes \mathcal{A}^{\mathrm{np}}_{1/2} \otimes \mathcal{A}^{\mathrm{np}}_{1} \otimes \mathcal{A}^{\mathrm{np}}_{3/2}\text{''} \tag{3.5.5}$$

where we have split up each part of the perturbative and nonperturbative anomaly coming from the gaugino, vector boson, and gravitino. Technically speaking, separating the anomaly in this way is not something that can be done canonically. By (3.3.4) the nontopological part arises as a quotient of the invertible theory by the topological theories. We write the anomaly in such a way in order to make it organizationally more clear. The Adams computation shows that the free part of $\Omega_6^{\mathrm{Spin}\text{-}\mathrm{SU}_8}$ is nontrivial but it was shown in [193, 30] that in fact the entire perturbative component of the anomaly vanishes.

The vector bosons can be defined without choosing a spin structure, and therefore the partition function of their anomaly field theory factors through the quotient by fermion parity. That is, the tangential structure is

$$\mathrm{SO} \times G_8 = (\mathrm{Spin}\text{-}\mathrm{SU}_8)/\{\pm 1\}\,. \tag{3.5.6}$$

We will proceed in understanding the perturbative anomalies by isolating $\mathcal{A}^{\mathrm{pert}}_1$.

Lemma 3.5.7. *The perturbative anomaly for the vector bosons independently vanishes.*

Proof. With the knowledge that the manifold generator for the anomaly is the Wu manifold, we will further restrict to the SO_3 inside of G_8; we are left to computing $\Omega_6^{\mathrm{SO}}(B\mathrm{SO}_3) \otimes \mathbb{Q}$, which isolates the free summand. For the degree we are after, we can compute the bordism group via the AHSS. We take the E^2 page of

$$E^2_{p,q} = H_p(B\mathrm{SO}_3, \Omega_q^{\mathrm{SO}}(\mathrm{pt})) \Longrightarrow \Omega_6^{\mathrm{SO}}(B\mathrm{SO}_3)\,, \tag{3.5.8}$$

where

$$\Omega_*^{\mathrm{SO}}(\mathrm{pt}) = \{\,\mathbb{Z}\,,\,0\,,\,0\,,\,0\,,\mathbb{Z}\,,\,\mathbb{Z}/2\,,\,0\,,\dots\}\,, \tag{3.5.9}$$

and tensor with $\mathbb{Q}$. This is equivalent to the E_∞ page, as all differentials vanish, and is given by

$$\begin{array}{c|ccccccc}
6 & 0 & & & & & & \\
5 & 0 & 0 & & & & & \\
4 & \mathbb{Q} & 0 & 0 & 0 & \mathbb{Q} & 0 & \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & \mathbb{Q} & 0 & 0 & 0 & \mathbb{Q} & 0 & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array} \tag{3.5.10}$$

We see that the perturbative anomaly of the vector boson vanishes. □

Corollary 3.5.11. *The perturbative anomalies from the fractional spin particles vanish on their own.*

Having established this corollary, we may now pullback the anomaly in (3.5.5) to the nonperturbative part, and the equation becomes literally true.

The η-invariant for the contributions in $\mathcal{A}^{\mathrm{np}}_{1/2} \otimes \mathcal{A}^{\mathrm{np}}_{3/2}$ is therefore a bordism invariant, and in particular the η-invariant is computed as two times some other representation and is twice another bordism invariant. In order to see this, we consider how **56**, **28**, and **8** split via our fourfold embedding of SU_2 into G_8 for the Wu manifold. We see that **56** gives the dimension of the alternating three forms in 8-dimensions, **28** the dimension of alternating two forms, and **8** is the defining representation. The branchings are given by

$$\mathbf{56} \to 2(10 \times \mathbf{2} + 2 \times \mathbf{4}), \tag{3.5.12}$$
$$\mathbf{28} \to 2(3 \times \mathbf{3} + 5 \times \mathbf{1}), \tag{3.5.13}$$
$$\mathbf{8} \to 4 \times \mathbf{2}\,, \tag{3.5.14}$$

where the right hand side is in terms of $\mathfrak{su}_2$ representations. In increasing numerical order, they are the trivial, defining, adjoint, and Sym^4 representation. To show this, notice that $\mathbf{8}$ splits as $V^{\oplus 4}$ when we pull back to SU_2, where $V = \mathbf{2}$. We then consider the ways of splitting the alternating three forms. This can be done as

$$\wedge^2 V \otimes \wedge^1 V \otimes \wedge^0 V \otimes \wedge^0 V = \mathbb{C} \otimes V \otimes \mathbb{C} \otimes \mathbb{C} \tag{3.5.15}$$

in 12 ways, essentially partitioning 3 into a sum of length 2. The $\mathbb{C}$ for both $\wedge^2 V$ and $\wedge^0 V$ show that they are isomorphic as representations. It can also split into

$$\wedge^1 V \otimes \wedge^1 V \otimes \wedge^1 V \otimes \wedge^0 V = V \otimes V \otimes V \otimes \mathbb{C} \tag{3.5.16}$$

in 4 ways. The fact that the third tensor product of the defining representation decomposes as $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{2} + \mathbf{2} + \mathbf{4}$, gives us (3.5.12). Similarly, the two forms can be split into

$$\wedge^2 V \otimes \wedge^0 V \otimes \wedge^0 V \otimes \wedge^0 V \quad \text{and} \quad \wedge^1 V \otimes \wedge^1 V \otimes \wedge^0 V \otimes \wedge^0 V \tag{3.5.17}$$

in 4 ways and 6 ways, respectively. The fact that $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} + \mathbf{3}$, establishes (3.5.13).

To argue that the anomaly vanishes, we also want to show that $\eta_{\mathbf{R}}(\mathcal{D}_{\mathrm{Dirac}})$ is an integer. But since the local anomaly for the fermion vanished, the η-invariant is a bordism invariant. This can be seen from the Atiyah-Patodi-Singer (APS) index theorem, and the index for a Dirac operator makes sense on a 6-manifold. Due to the special features of the Wu-manifold, we can instead just work with representations when evaluating the anomaly. The gaugino was in the representation $\mathbf{56}$, and via the branching in (3.5.12), this is 4 times the η-invariant of some other representation; this implies $\mathcal{A}^{\mathrm{np}}_{1/2}$ is zero.

As a spin 3/2 particle, the gravitino contains a spinor index as well as a Lorentz index, therefore in order to use (3.5.2) for the anomaly, we need to use the fact that the tangent bundle of the Wu manifold is an associated bundle.

Lemma 3.5.18. *The tangent bundle of the Wu manifold W is given by*

$$TW = \mathrm{SU}(3) \times_{\mathrm{SO}(3)} \frac{\mathrm{su}_3}{\mathfrak{so}_3}\,.$$

Proof. The fact that the Wu manifold is a homogeneous space allows us to use the following general procedure to construct its tangent bundle. For $H \subset G$ is a closed subgroup of a Lie group G, we have the following exact sequence of adjoint representations of H:

$$1 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 1. \tag{3.5.19}$$

The canonical principal H-bundle $H \to G/H$ gives an exact functor from representations of H to vector bundles over G/H. This gives a corresponding sequence of vector bundles:

$$1 \longrightarrow G \times_H \mathfrak{h} \longrightarrow G \times_H \mathfrak{g} \longrightarrow G \times_H \mathfrak{g}/\mathfrak{h} \longrightarrow 1. \tag{3.5.20}$$

There is an isomorphism $G \times_H \mathfrak{g}/\mathfrak{h} \to T(G/H)$ shown in [33]. Let $p : G \to G/H$ and L_X be the left invariant vector field generated by $X \in \mathfrak{h}$. Then the mapping of $(g, X+\mathfrak{h}) \in G\times(\mathfrak{g}/\mathfrak{h})$ to $T_g p \cdot L_X(g) \in T_{gH}(G/H)$ gives the isomorphism. Specifically for our problem, we have the SO_3-bundle $\mathrm{SU}_3 \to W$, which by the present construction gives the desired result. □

Remark 3.5.21. This is an example of the "mixing construction": for a principal G-bundle $P \to M$ and a G-representation V, the space $P \times_G V$ is a vector bundle over M with rank equal to the dimension of V.

We are now left to understand $\frac{\mathrm{su}_3}{\mathfrak{so}_3}$ as a representation of SO_3. The Lie algebra su_3 is an SU_3-representation, and restricting, it is also an SO_3 representation of dimension 8. But the $\mathbf{8}$ of su_3 branches as $\mathbf{8} \to \mathbf{1}+\mathbf{1}+\mathbf{3}+\mathbf{3}$ in $\mathfrak{so}_3$, so quotienting by $\mathfrak{so}_3$ then eliminates one of the $\mathbf{3}$ summands. Then $\eta(\mathcal{D}_{\mathrm{Dirac}\times TW}) = (\mathbf{1}+\mathbf{1}+\mathbf{3})\,\eta(\mathcal{D}_{\mathrm{Dirac}})$, which means the gravitino contributes $3\eta(\mathcal{D}_{\mathrm{Dirac}})$. By the branching in (3.5.14), $\eta(\mathcal{D}_{\mathrm{Dirac}})$ of $\mathbf{8}$ in su_8 is determined by $\mathbf{2}$ of su_2, and using Lemma 3.5.3, we have a multiple of 4 worth of $\eta_{\mathbf{2}}(\mathcal{D}_{\mathrm{Dirac}})$ and that determines $\eta_{\mathrm{gravitino}}$. Then the anomaly $\mathcal{A}^{\mathrm{np}}_{3/2}$ associated to $\eta_{\mathrm{gravitino}}$ vanishes per the above discussion for the gauginos.

We now move onto the nonperturbative anomaly from the vector bosons, which is accessible from $\Omega_5^{\mathrm{SO}}(B\mathrm{SO}_3)$. By applying (3.5.9) to the AHSS, we only need to consider $H_5(B\mathrm{SO}_3, \mathbb{Z})$ as well as the $\mathbb{Z}/2$ element in bidegree $(0, 5)$. One can evaluate the torsion part of $H_5(B\mathrm{SO}_3; \mathbb{Z})$ by the universal coefficient theorem, and looking at $H^6(B\mathrm{SO}_3; \mathbb{Z})$. We find that this is given by $w_2 w_3$ of the SO_3 bundle and is nontrivial on the Wu manifold. Then the AHSS says $\Omega_5^{\mathrm{SO}}(B\mathrm{SO}_3) = \mathbb{Z}/2 \times \mathbb{Z}/2$ detected by the bordism invariants $\int w_2(TM)w_3(TM)$ and $\int w_2(P)w_3(P)$; these are generated by W with trivial bundle, and W with the principal SO_3-bundle. We see that while the bosonic anomaly is in principle $\mathbb{Z}/2 \times \mathbb{Z}/2$ valued, and coupling to spin structure eliminates one of the $\mathbb{Z}/2$. Using (3.5.13), for the representation of the vector boson, the anomaly is also twice of something as a bordism invariant. This is reasonable since the anomaly of multiple particles is the tensor product of their anomalies [10]. The anomaly for the vector bosons is 2 times something as a bordism invariant, since the perturbative part vanished, and considering that we have argued that everything else

[10] For the gaugino and gravitino we could employ the decomposition of representations directly to the η-invariant. In the case of the vector boson, we use the fact that direct sums of representations goes to tensor products of anomalies.

in (3.5.5) vanishes aside from $\mathcal{A}_1^{\mathrm{np}}$, we have that $\mathcal{A} = \mathcal{A}_1^{\mathrm{np}}$. But $\mathcal{A}$ is $\mathbb{Z}/2$ valued, and with $\mathcal{A}_1^{\mathrm{np}}$ equating to 0 mod 2, the full anomaly vanishes, thus establishing proposition 3.5.4.

Chapter 4

Classification of 5d Topological Orders

The first part of this chapter is based off [147] proves a theorem about a special type of

4.1. Introduction

Just as multifusion 1-categories describe the fusion of quasiparticle excitations — 1-spacetime-dimensional objects, aka line operators — in topological phases of matter, multifusion 2-categories (first introduced in [85]) describe the fusion of 2-spacetime-dimensional "quasistring" excitations, aka surface operators. Except in very low dimensions, a typical topological phase can have quasistring excitations which are not determined by the quasiparticle excitations, and multifusion 2-categories are vital for the construction and classification of topological phases in medium dimension [179, 178, 149].

Recall that a multifusion 1-category $\mathcal{C}$ is *fusion* if the endomorphism algebra $\Omega\mathcal{C} = \mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}})$ is trivial, i.e. isomorphic to $\mathbb{C}$, where $1_{\mathcal{C}} \in \mathcal{C}$ denotes the monoidal unit [90]. There are two reasonable categorifications of this notion when $\mathcal{C}$ is a multifusion 2-category. The stronger generalization, which we will call *strongly fusion* [1], is to ask that the endomorphism 1-category $\Omega\mathcal{C} = \mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}})$ be trivial, i.e. equivalent to $\mathbf{Vec}_{\mathbb{C}}$. The weaker notion, which we will call merely *fusion*, is to ask only that $\Omega^2\mathcal{C} = \mathrm{End}_{\Omega\mathcal{C}}(1_{1_{\mathcal{C}}})$ be trivial, where $1_{1_{\mathcal{C}}} \in \Omega\mathcal{C}$ is the identity object. A *fusion 2-category* is a finite semisimple monoidal 2-category that has left and right duals for objects and a simple monoidal unit. Physically, if $\mathcal{C}$ describes the surface operators in a topological phase, then $\Omega\mathcal{C}$ describes the line operators and $\Omega^2\mathcal{C}$ describes the vertex (0-spacetime-dimensional) operators.

[1] In [85] the notion of strongly fusion is referred to as *endotrivial*

The classification of fusion 1-categories is extremely rich [91, 154, 205]. The simplest examples are the *grouplike*, aka pointed, fusion 1-categories, whose isomorphism classes of simple objects form a group G under the fusion product. These are famously classified by ordinary group cohomology $\mathrm{H}^3_{\mathrm{gp}}(G; \mathrm{U}(1))$. But there are many nongrouplike examples. The classification of (merely) fusion 2-categories is similarly rich, since it includes the classification of braided fusion 1-categories [85, Construction 2.1.19]. The main result of this note shows a dramatic difference with the strongly fusion case:

Theorem A. If $\mathcal{C}$ is a strongly fusion 2-category, then the equivalence classes of indecomposable objects of $\mathcal{C}$ form a finite group under the fusion product.

We also address the "fermionic" case where $\Omega\mathcal{C} \cong \mathbf{SVec}$:

Theorem B. If $\mathcal{C}$ is a fusion 2-category with $\Omega\mathcal{C} \cong \mathbf{SVec}$, then the equivalence classes of indecomposable objects of $\mathcal{C}$ form a finite group, which is a central double cover of the group $\pi_0\mathcal{C}$ of components of $\mathcal{C}$ (see Definition 4.2.11).

In particular, Theorem B asserts that the components of $\mathcal{C}$ do form a group.

Remark 4.1.1. Just as grouplike fusion 1-categories in which the simple objects form a group G are classified by $\mathrm{H}^3_{\mathrm{gp}}(G; \mathrm{U}(1))$, the strongly fusion 2-categories with simple objects G are classified by $\mathrm{H}^4_{\mathrm{gp}}(G; \mathrm{U}(1))$ [85, Remark 2.1.17]. In the fermionic case, if one additionally assumes that the actions of $\mathrm{End}(1_\mathcal{C}) \cong \mathbf{SVec}$ on $\mathrm{End}(X)$ given by tensoring on the left and on the right agree, then one can show that the options with $\pi_0\mathcal{C} = G$ are classified by "extended group supercohomology" $\mathrm{SH}^4_{\mathrm{gp}}(G)$ defined in [239]. There is a canonical map $\mathrm{SH}^4_{\mathrm{gp}}(G) \to \mathrm{H}^2_{\mathrm{gp}}(G; \mathbb{Z}_2)$ which takes an extended supercohomology class to its *Majorana layer*; the group of simple objects in $\mathcal{C}$ is the corresponding central extension $\mathbb{Z}_2.G$. Although in general Majorana layers of supercohomology classes have no reason to be trivial, we were unable to find an example where the extension $\mathbb{Z}_2.G$ did not split.

The outline of our paper is as follows. Section 4.2 reviews the definition of multifusion 2-category from [85]. In particular, we recall their notion of "component" of a semisimple 2-category in §4.2. The proofs of Theorems A and B occupy §4.3 and §4.4, respectively.

In future work, we will use these theorems to give a complete classification of 5-spacetime-dimensional topological orders.

4.2. Semisimple and multifusion 2-categories

The definition and basic theory of semisimple and multifusion 2-categories were first introduced in [85]. Since this theory is new, we take this section to review the main features.

Recall that a 2-category $\mathcal{C}$ is $\mathbb{C}$-*linear* if all hom-sets of 2-morphisms are vector spaces over $\mathbb{C}$, and both 1- and 2-categorical compositions of 2-morphisms are bilinear.

Definition 4.2.1. An object in a linear 2-category is *decomposible* if it is equivalent to a direct sum of nonzero objects, and *indecomposable* if it is nonzero and not decomposable.

Remark 4.2.1. We will slightly abuse the language and use the terms "simple" and "indecomposable" interchangeably. A *simple* object X in a 2-category is one such that any faithful 1-morphism $A \hookrightarrow X$ is either 0 or an equivalence. In finite semisimple 2-categories all indecomposable objects are simple [85].

In particular the objects which we consider in the 2-category will only be sums of finitely many simple objects, and decompositions are unique up to permutations. In our goal to define a semisimple 2-category, we present some definitions for the higher categorical generalization of the notion of idempotent splitting and idempotent completeness for 1-categories, also discussed in [109].

Definition 4.2.2. A 2-category $\mathcal{C}$ is *locally idempotent complete* if for all objects $A, B \in \mathcal{C}$, the 1-category $\hom_{\mathcal{C}}(A, B)$ is idempotent complete. It is *locally finite semisimple* if $\hom_{\mathcal{C}}(A, B)$ is furthermore a finite semisimple $\mathbb{C}$-linear category (i.e. an abelian $\mathbb{C}$-linear category with finitely many isomorphism classes of simple object and in which every object decomposes as a finite direct sum of simple objects).

In what follows, we will assume $\mathcal{C}$ is a locally idempotent complete 2-category.

Definition 4.2.3. A *separable monad* is a unital algebra object $p \in \hom_{\mathcal{C}}(A, A)$, for an object A, whose multiplication $m : p \circ p \to p$ admits a section as a p-p bimodule.

Definition 4.2.4. A *(unital) condensation* in a 2-category $\mathcal{C}$ is an adjunction $f \dashv g \equiv (f : A \leftrightarrows B : g, \eta : 1_A \to g \circ f, \epsilon : f \circ g \to 1_B)$ which is *separable* in the sense that its counit ϵ admits a section, i.e. if there exists a 2-morphsism $\phi : 1_B \to f \circ g$ which is the right inverse of ϵ. When there is such a separable adjuction, we will write "$A \twoheadrightarrow B$," and say that A *condenses onto* B.

Definition 4.2.5. A separable monad p is *separably split* if there exists a separable adjunction $f \dashv g$ and $g \circ f \cong p$. A *separable splitting* is a choice of this isomorphism.

Proposition 4.2.6 ([85, Proposition 1.3.4]). A separable monad in $\mathcal{C}$ which admits a separable splitting, admits a unique up-to-equivalence separable splitting.

Admitting a separable splitting implies that the adjunction $f \dashv g$ admits A as an Eilenberg-Moore object. In 1-categories, this is can be seen as module decomposition by forming a projector from an idempotent. The subtlety in 2-categories is that now there is no "orthogonal complement" to the projector, as in 1-categories.

Definition 4.2.7. A 2-category $\mathcal{C}$ is *2-idempotent complete* if it is locally idempotent complete and every separable monad splits.

Remark 4.2.2. Requiring the unitality of p and the existence of a unit for adjunction in the 2-category case differs slightly from the situation in 1-categories. In 1-categories there is an equality of $p^2 = p$ but there is no equality of 1 and p. [109] developed a nonunital version of separable monad for 2-categories and showed that if $\mathcal{C}$ has adjoints for 1-morphisms, then the notion of 2-idempotent completion in Definition 4.2.7 and in [109] agree.

Definition 4.2.8. A $\mathbb{C}$-linear 2-category $\mathcal{C}$ is *finite semisimple* if: it has finitely many isomorphism classes of simple objects; it is locally finite semisimple; has adjoints for 1-morphisms; has direct sums of objects; and is 2-idempotent complete.

Definition 4.2.9. A *multifusion 2-category* is a monoidal finite semisimple 2-category in which all objects have duals.

Remark 4.2.3. As noted in [85, Definition 2.1.6], in a fusion 2-category, left and right duals are the same.

The 1-categorical Schur's Lemma says that in a semisimple 1-category, if two indecomposable objects are related by a nonzero morphism, then they are isomorphic. This result fails in 2-categories, but [85, Proposition 1.2.19] provides the following replacement.

Proposition 4.2.10 (Categorical Schur's Lemma). If $f : X \to Y$ and $g : Y \to Z$ are nonzero 1-morphisms between indecomposable objects in a semisimple 2-category $\mathcal{C}$, then $gf : X \to Z$ is nonzero.

Proof. This follows from Proposition 4.2.12 below, since the composition of condensations is a condensation and since condensations with nonzero target are nonzero. □

In particular, "related by a nonzero morphism" defines an equivalence relation on the indecomposable objects of $\mathcal{C}$. (Note that, since every 1-morphism is required to have an adjoint, if there is a nonzero morphism $f : X \to Y$, then there is a nonzero morphism $f^* : Y \to X$.)

Definition 4.2.11. The set of *components* of $\mathcal{C}$, denoted $\pi_0\mathcal{C}$, is the set of equivalence classes of indecomposable objects for the equivalence relation "related by a nonzero morphism."

The structure of each component is fully determined by (the endomorphism category of) any representative object. Indeed:

Proposition 4.2.12. Suppose $X, Y \in \mathcal{C}$ are simple objects connected by a nonzero 1-morphism $f : X \to Y$. Then there is a condensation $X \twoheadrightarrow Y$. In particular, Y is the image of a simple algebra object in the fusion 1-category $\mathrm{End}_{\mathcal{C}}(X)$.

Proof. Choose g to be the right adjoint to f; it exists because all morphisms in a semisimple 2-category are required to have adjoints. The counit $\epsilon : f \circ g \to 1_Y$ is a nonzero 1-morphism in the semisimple 1-category $\mathrm{End}_{\mathcal{C}}(Y)$ with simple target, and so has a section. $\square$

4.3. Proof of Theorem A

We begin this section by developing the necessary graphical calculus in order to prove the main results. One important feature we will discuss is the state-operator map for fusion 2-categories and its interplay with duality. For an object $X \in \mathcal{C}$, we denote $\int_{S^1_b} X$ as the wrapping of X around a boundary-framed S^1, see Figure 4.1. This integral is a map $\int_{S^1_b} : \mathcal{C} \to \Omega\mathcal{C}$. This integral is an example of the general calculus of dualizability as in [106], and also arising from the cobordism hypothesis [11]. The boundary framing of the cylinder in Figure 4.1 is attained from the framing of the annulus, where the annulus framing is given by the restriction of the two-dimensional blackboard framing, see Figure 4.2. One could then take the framed annulus and pull the annulus into a cylinder. This results in a cylinder appropriately framed to be compatible with the state-operator map.

We now describe this operation $\int_{S^1_b}$ algebraically. Because we are working with a fusion 2-category each object has a dual and we have a unit $\eta_X : 1_{\mathcal{C}} \to X \otimes X^*$. It corresponds to the half-circle with framing as in Figure 4.3 part (a). Also, since all 1-morphisms have adjoints, there is a right adjoint $\eta_X^* : X \otimes X^* \to 1_{\mathcal{C}}$, corresponding to the framed half-circle in Figure 4.3 part (b). These two half-circles compose to an annulus whose framing can be continuously deformed to framing in Figure 4.2. All together, we find the algebraic definition:

$$\int_{S^1_b} X := \eta_X^* \circ \eta_X.$$

The vertex operators of X are by definition the 2-morphisms $1_X \Rightarrow 1_X$. They are precisely the operators that can be inserted in the interior hole in Figure 4.2; the blackboard framing is arranged so that this can happen. Such an insertion may be pulled down and thought of as a map $1_{1_{\mathcal{C}}} \to \int_{S^b_1} X$ as in Figure 4.4. In other words, $\hom(1_{1_{\mathcal{C}}}, \int_{S^1_b} X)$ is the vector space of ways for the vacuum line to end on $\int_{S^1_b} X$. This is the physical/geometric proof of the *state-operator correspondence*. Algebraically, we have:

Lemma 4.3.1 (State-Operator Correspondence). In a multifusion 2-category there is an isomorphism $\mathrm{End}_{\mathrm{End}_{\mathcal{C}}(X)}(1_X) \cong \hom_{\Omega\mathcal{C}}(1_{1_{\mathcal{C}}}, \int_{S^1_b} X)$.

Proof. The duality of X with X^* provides an equivalence of $\mathrm{End}_{\mathcal{C}}(X) \cong \hom(1_{\mathcal{C}}, X \otimes X^*)$.

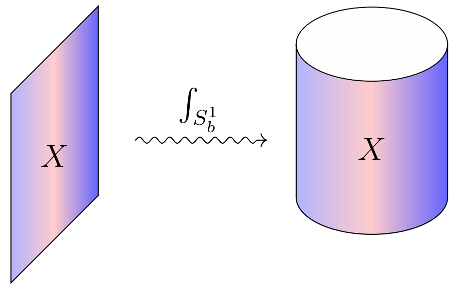


Figure 4.1: Wrapping the operator around a circle.

This equivalence identifies 1_X with η_X, and so in particular $\mathrm{End}(1_X) \cong \mathrm{End}(\eta_X)$, where the left-hand side is computed in $\mathrm{End}_{\mathcal{C}}(X)$ and the right-hand side is computed in $\hom(1_{\mathcal{C}}, X \otimes X^*)$. For any adjunctible 1-morphism $f : A \to B$ in a 2-category, $\mathrm{End}_{\hom(A,B)}(f) \cong \hom_{\mathrm{End}\,A}(1_{1_A}, f^* \circ f)$. Taking $f = \eta_X$, with $A = 1_{\mathcal{C}}$ and $B = X \otimes X^*$, completes the proof. $\square$

In particular, X is simple if and only if $\hom_{\Omega\mathcal{C}}(1_{1_{\mathcal{C}}}, \int_{S^1_b} X)$ is one-dimensional. In the strongly fusion case lemma 4.3.1 implies:

Proposition 4.3.2. Suppose $\mathcal{C}$ is strongly fusion. Then $X \in \mathcal{C}$ is indecomposable if and only if $\int_{S^1_b} X = \mathbb{C}$. $\square$

We now consider the tensor product of two indecomposable objects $X \otimes Y$ mapped by the integral $\int_{S^1_b}$. This represents a cylinder within a cylinder as on the left of Figure 4.5.

In general, we see that $\int_{S^1_b}$ is not monoidal: a cylinder within a cylinder is not the same as two adjacent cylinders. However, in the strongly fusion case, if X and Y are simple then we may collapse down the inner cylinder via the state operator map into the vacuum line. We may then collapse the outer cylinder. All together we find:

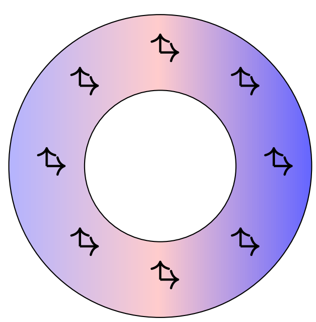


Figure 4.2: The framing on the annulus.

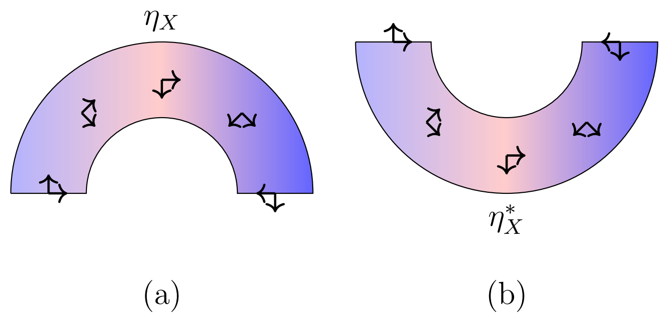


Figure 4.3: η_X^* is by definition the universal map such that the composition with η_X can be filled. The resulting framing of $\eta_X^* \circ \eta_X$ is homotopic to the blackboard framing of Figure 4.2.

Corollary 4.3.3. In a strongly fusion 2-category, the tensor product of indecomposable objects is indecomposable. □

This allows us to complete the proof of Theorem A:

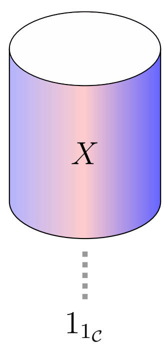


Figure 4.4: State operator map.

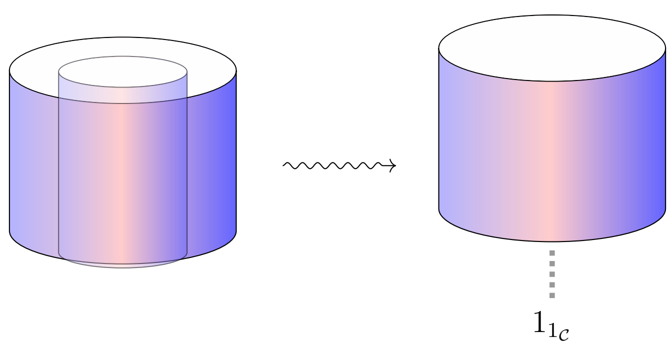


Figure 4.5: Removing the inner operator by collapsing it down to the vacuum.

Proof of Theorem A. If $X \in \mathcal{C}$ is a simple object, then X^* is as well (since $\mathrm{End}(X) \cong \mathrm{End}(X^*)$), and hence so is $X \otimes X^*$ (by Corollary 4.3.3). Since $\eta_X : 1_{\mathcal{C}} \to X \otimes X^*$ is nonzero, the simple objects $1_{\mathcal{C}}$ and $X \otimes X^*$ are in the same component. However, the fact that there are no lines in the strongly fusion case means that $1_{\mathcal{C}}$ is the only simple object in its component. ☐

Remark 4.3.1. We can consider working over the real numbers, which is the same as having an anti-linear involution (time-reversal). In this case, an indecomposable object is not absolutely simple, and Theorem A is no longer true. We can see this already at the level of fusion 1-categories. Consider a $\mathbb{Z}_3$ fusion 1-category with three objects $\{1, x, x^{-1}\}$ over $\mathbb{C}$. Over the real numbers, we can exchange x and x^{-1} by the involution. There will be two objects 1 and X over the real numbers, where $X \cong x + x^{-1}$, so that it is invariant under the involution. Schur's lemma states that over the complex numbers, indecomposable means that the endomorphisms of the object is just $\mathbb{C}$. But over the real numbers, the endomorphisms are a division ring, and we have the fusion $X^2 = X + 2$.

4.4. Proof of Theorem B

If we try to repeat the proof from §4.3 when $\Omega\mathcal{C} = \mathrm{End}(1_{\mathcal{C}}) \cong \mathbf{SVec}$, the first snag arises in Proposition 4.3.2. Indecomposability of X implies that the ordinary vector space $\hom(1_{1_{\mathcal{C}}}, \int_{S^1_b} X)$ is one-dimensional. This measures the even part of the super vector space $\int_{S^1_b} X$, but says nothing about the odd part. On the other hand, the super vector space $\int_{S^1_b} X$ is the superalgebra of vertex operators on X, and so it is supercommutative because we have the freedom to move operators around each other on the surface of the cylinder. Furthermore, since we are working in a semisimple 2-category, this supercommutative algebra is finite dimensional and semisimple.

Lemma 4.4.1. The only finite-dimensional semisimple supercommutative superalgebra A with one-dimensional bosonic part is $\mathbb{C}$.

Proof. If $x \in A$ is a nonzero odd element, then it is nilpotent (since $x^2 = xx = -xx$ and so $x^2 = 0$). Thus the principle ideal generated by x is proper. On the other hand, it is not a direct summand of A as an A-module because projection operators are bosonic, and so the only projection operators in A are 0 and 1. This contradicts the semisimplicty of A. ☐

This implies the fermionic versions of Proposition 4.3.2 and Corollary 4.3.3.

To complete the proof of Theorem B, it suffices to observe that if $X \in \mathcal{C}$ is indecomposable, then, since the unit map $\eta_X : 1_{\mathcal{C}} \to X \otimes X^*$ is nonzero, $X \otimes X^*$ is an indecomposable object in the identity component of $\mathcal{C}$, and so invertible, and thus X is invertible. Indeed, since $\Omega\mathcal{C} \cong \mathbf{SVec}$, by Proposition 4.2.12 there are precisely two simple objects in the identity component of $\mathcal{C}$, corresponding to the two simple superalgebras $\mathbb{C}$ and $\mathrm{Cliff}(1)$, and $\mathrm{Cliff}(1)$ is famously Morita-invertible.

Remark 4.4.1. In fact, $X \otimes X^*$ is always trivial, and never the nontrivial simple object $\mathrm{Cliff}(1)$. Indeed, it is a general fact of monoidal higher categories that if an object is invertible, then its inverse is its dual. One can also see this directly by running the proof of Proposition 4.2.12 for the nonzero 1-morphism $f = \eta_X$. Then $g = \eta_X^*$, and the simple algebra in question is the composition $p = gf = \eta_X^* \circ \eta_X = \int_{S^1_b} X = \mathbb{C}$.

4.5. Introduction

A physical system described by a Hamiltonian is *gapped* when the spectrum of eigenvalues for the Hamiltonian has a gap between the lowest energy state and the vacuum. Such systems prevent the existence of particles that are arbitrarily light. A *gapped phase* is an equivalence class of gapped systems. Systems that can be continuously deformed into each other without closing the energy gap are considered to be in the same phase. The low-energy limit of gapped phases may exhibit topological behaviour. Such is true for some quantum field theories, which flow in the infrared to topological theories [121]. All of the dynamical degrees of freedom can be integrated out, leaving only the topological excitations. The study of gapped phases in various dimensions has led to interest regarding the topological nature of extended objects, or operators, in these phases. In nontrivial cases, the content of operators and defects, as well as the algebraic structure of how they interact, compile into a topological order [241].

The classification of topological orders has been an interesting problem that combines the mathematics of higher category theory with the physics of gapped topological phases. By now the classification in lower dimensions is understood. In (1+1)-dimensions, topological orders are classified by their spectrum of point operators together with anomaly information that manifests as a class in ordinary or supercohomology. Some other well studied situation are in (2+1)d where topological orders with nondegenerate local ground states are classified by modular tensor categories [242], and in (3+1)d where topological order with nondegenerate local ground states are (modulo a few subtleties) always described by finite group gauge theories [179, 181, 149, 145].

This paper addresses the classification in (4+1)d. We focus on the case of super topological orders, i.e. topological orders defined over the category **SVec** of super vector spaces, because the existence of super fibre functors makes this case technically easier. Following the strategy of [179, 181], the first step is to condense out all of the line operators in the topological order. The resulting topological order has no line operators, and our first result is a classification of these:

$$\{\text{super (4+1)d topological orders with no lines}\} = \{\text{symplectic finite Abelian groups}^{2}\}. \tag{4.5.1}$$

By reducing along a Lagrangian subgroup, we furthermore show that every super (4+1)d topological order can be condensed all the way to the vacuum via a gapped topological boundary:

$$\{\text{super (4+1)d topological orders}\}/\text{Morita equivalence}^{3} = \{1\}. \tag{4.5.2}$$

This is to be expected, as it agrees with the cobordism classification proposed by [158]: a Morita-nontrivial super (4+1)d topological order should have a nontrivial gravitational anomaly detectable on (5+1)d spin manifolds, but every (5+1)d spin manifold is spin-nullcobordant.

By studying a spectral sequence introduced in [149, 145], the classification (4.5.2) allows us to compute the analogous group for bosonic topological orders. We find that there is an isomorphism:

$$\{\text{bosonic (4+1)d topological orders}\}/\text{Morita equivalence} \cong \mathbb{Z}_2^\infty. \tag{4.5.3}$$

In other words, there are infinitely many pairwise-Morita-inequivalent bosonic (4+1)d topological orders (and each has a gapped boundary to its time-reversal). This disagrees with the cobordism prediction: the cobordism group of (5+1)d oriented manifolds is trivial. The origin of the disagreement, and indeed of the answer (4.5.3), is in (2+1)d: the Witt groups $\mathcal{W}$ and $S\mathcal{W}$ of Morita equivalence classes of bosonic and super modular tensor categories, studied in [61], are very large, whereas the cobordism classification would have predicted a classification in terms of the central charge alone.

The outline of our paper is as follows. Section 4.6 starts off by explaining how to reduce the set of operators in a (4+1)d topological order to only the surface operators, and how to see that their monoidality is given by a finite Abelian group. In principle this procedure

[2] We give the definition of symplectic finite Abelian group at the start of section 4.7.

[3] By definition, two topological orders are Morita equivalent if they be separated by a gapped topological interface.

works for the bosonic and fermionic case, up to a small caveat that is remarked upon. In that section though, we give the explanation specifically for super topological orders. We then review some aspects of fusion and sylleptic 2-categories to understand the nature of how surface operators pair up given three ambient dimensions. The build up is to see by way of a cohomology calculation that (4+1)d topological orders are parametrized by a symplectic form carried by the finite group of surface operators, establishing (4.5.1).

Section 4.7 outlines the method of symplectic reduction and its relation to Morita equivalence. This allows us to prove (4.5.2) that (4+1)d super topological orders all admit a gapped boundary. We furthermore give relationships between the bulk and boundary theories, where we interpret the bulk (4+1)d theory to be a higher form of "centre" for the boundary theory. To juxtapose with Section 4.6, we present a *bosonic* example of how the centre construction goes through. Lastly, we address the question of lifting boundary theories into the bulk, and obstructions in doing so.

Section 4.8 explains how we recover a bosonic theory from a fermionic theory plus extra information in "descent data", and computes the group of Morita equivalence of (4+1)d bosonic topological orders.

In many parts of the paper we will also draw analogies to lower dimensional theories when instructive.

4.6. 5-dimensional Super Topological Orders

4.6.1. Condensing out the lines

An $(n+1)$-dimensional super topological order is defined in [171, 172, 173, 149] to be a multifusion n-supercategory $\mathcal{A}$ with trivial centre.[4] Triviality of the center is an axiomatization of the principle of remote detectability. For our purposes we will be considering only the *fusion* case. By this, we mean that there are no nontrivial 0-dimensional operators. This is to say that the ground state of our topological order is nondegenerate [253]. The principle of remote detectability, along with the fusion condition, implies that all codimension-1 operators arise as condensation descendants [149, Theorem 4]. In an arbitrary 5d[5] topological order given by the fusion 4-category $\mathcal{A}$, we therefore only need to consider operators of

[4] All of our "n-categories" are "weak." For example, a "2-category" is a bicategory. Multifusion 2-categories were first introduced by [85], and the n-category generalization was developed in [149, 174].

[5] For the remainder of this paper whenever the dimension of an extended object or phase is given without the time component specified, we will take that dimension to represent the full spacetime dimension.

codimension-2 and higher. We will focus on the *super* case in which $\mathcal{A}$ is enriched over $\mathbf{SVec}$.

We will deem two 5d topological orders as being *Morita equivalent* if they can be separated by a gapped 4d topological interface; this is also known as *Witt equivalence*. One way to produce a Morita equivalence is to perform a categorical condensation [109], where the condensation wall that separates the two phases is gapped and described by its own higher category of operators.

The first main step in our classification of 5d topological orders is to use the method outlined in [181, 179] to condense out all the lines in any super 5d topological order. Here is a streamlined version of their construction, written in the language of [109, 149]:

Within the super fusion 4-category $\mathcal{A}$ describing the topological order, there is a symmetric super fusion 1-category $\Omega^3\mathcal{A}$ of line operators. Suppose that we choose a functor $F : \Omega^3\mathcal{A} \to \mathbf{SVec}$ of symmetric super fusion 1-categories. Such F is called a *fibre functor*, and in the super case always exists [79]; since $\mathcal{A}$ is assumed to be fusion, F is unique up to isomorphism, although not up to unique isomorphism.

This F can be "suspended" to a functor $\Sigma^3 F : \Sigma^3\Omega^3\mathcal{A} \to \Sigma^3\mathbf{SVec}$, where $\Sigma^3\Omega^3\mathcal{A} \subset \mathcal{A}$ is the sub 4-category of operators which arise as condensation descendants from line operators, and $\Sigma^3\mathbf{SVec}$ is the 4-category of operators in the vacuum 5d super topological order. This $\Sigma^3 F$ makes the 4-category $\Sigma^3\mathbf{SVec}$ into a module for the fusion 4-category $\Sigma^3\Omega^3\mathcal{A}$. We may induce (aka base change) this module along the inclusion $\Sigma^3\Omega^3\mathcal{A} \subset \mathcal{A}$ to produce an $\mathcal{A}$-module

$$\mathcal{M} := \mathcal{A} \otimes_{\Sigma^3\Omega^3\mathcal{A}} \Sigma^3\mathbf{SVec}.$$

We set $\mathcal{B} := \mathrm{End}_{\mathcal{A}}(\mathcal{M})$ to be the super fusion 4-category of $\mathcal{A}$-linear endomorphisms of $\mathcal{M}$; then $\mathcal{M}$ is a Morita equivalence $\mathcal{A} \simeq \mathcal{B}$[6]. Because we started with a fibre functor on the full category of line operators in $\mathcal{A}$, there are no nontrivial line operators in $\mathcal{B}$, i.e. $\Omega^3\mathcal{B} = \mathbf{SVec}$.

Remark 4.6.1. In the case of bosonic topological orders, to condense all the lines would require choosing a bosonic fibre functor $\Omega^3\mathcal{A} \to \mathbf{Vec}$. Such a functor exists if and only if there are no emergent fermions [79].

Since $\Omega^3\mathcal{B} = \mathbf{SVec}$, the result of [149, Theorem 5] implies that $\mathcal{B} = \Sigma^2\mathcal{C}$, where $\mathcal{C} := \Omega^2\mathcal{B}$ is the (sylleptic) fusion 2-category of surface operators (and junctions between them); the statement $\mathcal{B} = \Sigma^2\Omega^2\mathcal{B}$ means that all three- and four-dimensional "membrane" objects can be built as condensation descendants of surface operators. But $\Omega\mathcal{C} = \Omega^3\mathcal{B} = \mathbf{SVec}$, i.e. it is *strongly super fusion*:

[6]This construction presently outlined also goes by the name *deequivariantization*.

Definition 4.6.1 ([152]). A super fusion 2-category $\mathcal{C}$ is *strongly super fusion* if $\Omega\mathcal{C} := \mathrm{End}_{\mathcal{C}}(1) \cong \mathbf{SVec}$.

An object in a (super) fusion 2-category is *indecomposable* if it is nonzero and cannot be written as a direct sum of nonzero objects; recall from [85] that in a (super) fusion 2-category, an object is indecomposible iff it is simple. Two indecomposable objects are in the same *component* if they are related by a nonzero morphism; the set of components of a (super) fusion 2-category $\mathcal{C}$ is denoted $\pi_0\mathcal{C}$. The second main step in our classification of 5d topological orders is a classification of strongly fusion 2-categories that we established in [152]:

Theorem 4.6.2 ([152, Theorem B]). If $\mathcal{C}$ is a (super) fusion 2-category with $\Omega\mathcal{C} \cong \mathbf{SVec}$, then every indecomposable object of $\mathcal{C}$ is invertible. The equivalence classes of indecomposable objects in $\mathcal{C}$ form a finite group, which is a central double cover of the group $\pi_0\mathcal{C}$ of components of $\mathcal{C}$ (in particular, $\pi_0\mathcal{C}$ is a group).

Since an invertible object always has the same endomorphisms as the identity, Theorem 4.6.2 implies in particular that the endomorphisms of any indecomposable object in $\mathcal{C}$ is equivalent to $\mathbf{SVec}$, a super version of the condition called "endotriviality" in [85].

4.6.2. Sylleptic and symplectic groups: bosonic case

In any 5d topological order, the surface operators have three ambient dimensions in which they can compose. Thus the fusion 2-category $\mathcal{C}$ is 3-monoidal, aka *sylleptic*. The definition of sylleptic monoidal 2-category, which can be found in full in the appendix of [213], simplifies dramatically in the strongly fusion case.

To warm up, in this section we discuss the case of bosonic strongly fusion 2-categories, where sylleptic structures are classified by the Eilenberg–MacLane cohomology introduced in [87]. Indeed, suppose that $\mathcal{C}$ is bosonic strongly fusion, meaning that it is a fusion 2-category with $\Omega\mathcal{C} = \mathbf{Vec}$. The bosonic case of Theorem 4.6.2 is [152, Theorem A], which says that the indecomposable objects in $\mathcal{C}$ form a finite group M, equal to the group of components since $\mathcal{C}$ is forced to be endotrivial.

The full data of the monoidal structure on $\mathcal{C}$ consists of: a tensor functor $\otimes$, given by the group law on M; an associator $\alpha_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)$; and a pentagonator

$\pi_{x,y,z,w}$

$$\begin{tikzcd}
 & (w \otimes x) \otimes (y \otimes z) \arrow[dr, "\alpha"] & \\
((w \otimes x) \otimes y) \otimes z \arrow[ur, "\alpha"] \arrow[d, "\alpha \otimes I"'] & \Uparrow \pi & w \otimes (x \otimes (y \otimes z)) \\
(w \otimes (x \otimes y)) \otimes z \arrow[rr, "\alpha"'] & & w \otimes ((x \otimes y) \otimes z) \arrow[u, "I \otimes \alpha"']
\end{tikzcd} \tag{4.6.2}$$

which must satisfy a certain equation that we will not reproduce in full. But by endotriviality, α is no data: there is up to isomorphism a unique equivalence $(x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z)$ for every triple of indecomposable object (x, y, z). After trivializing α, the equation for π says simply that it is a 4-cocycle in ordinary group cohomology with coefficients in $\mathbb{C}^\times$. We will henceforth adopt the following notation. Given a group M (Abelian if $n \geq 2$), we will write $M[n]$ for the Eilenberg–Mac Lane space more typically written $K(M, n)$, and $\mathrm{H}^k(-)$ without coefficients always means ordinary cohomology with $\mathbb{C}^\times$ coefficients $\mathrm{H}^k(-; \mathbb{C}^\times)$. To summarize the above discussion, we find that bosonic strongly fusion 2-categories with $\mathcal{C}$ with $\pi_0 \mathcal{C} = M$ are classified by

$$[\pi] \in \mathrm{H}^4_{\mathrm{gp}}(M) := \mathrm{H}^4(M[1]; \mathbb{C}^\times). \tag{4.6.3}$$

Suppose $\mathcal{C}$ is a monoidal 2-category with tensor bifunctor $\otimes$, associator α, and pentagonator π. A *braiding* on $\mathcal{C}$ consists of a natural (in both variables) equivalence $b_{x|y} : x \otimes y \to y \otimes x$,[7] together with *hexagonators* $R_{(x|-,-)}$ and $S_{(-,-|x)}$ that provide the monoidality of b:

$$\begin{tikzcd}
 & (y \otimes x) \otimes z \arrow[r, "\alpha"] & y \otimes (x \otimes z) \arrow[dr, "b_{x|z}"] & \\
(x \otimes y) \otimes z \arrow[ur, "b_{x|y}"] \arrow[dr, "\alpha"'] & \Downarrow \; R_{(x|y,z)} & & y \otimes (z \otimes x) \\
 & x \otimes (y \otimes z) \arrow[r, "b_{x|yz}"] & (y \otimes z) \otimes x \arrow[ur, "\alpha"'] &
\end{tikzcd} , \quad
\begin{tikzcd}
 & (z \otimes y) \otimes x \arrow[r, "b_{zy|x}"] & x \otimes (z \otimes y) \arrow[dr, "\alpha"] & \\
z \otimes (y \otimes x) \arrow[ur, "\alpha"] \arrow[dr, "b_{y|x}"'] & \Downarrow \; S_{(z,y|x)} & & (x \otimes z) \otimes y \\
 & z \otimes (x \otimes y) \arrow[r, "\alpha"] & (z \otimes x) \otimes y \arrow[ur, "b_{z|x}"'] &
\end{tikzcd} . \tag{4.6.4}$$

R and S must solve various equations. When α and π are trivial, these equations say first

[7] We write the braiding as $b_{x|y}$ rather than $b_{x,y}$ to be consistent with later notation for Eilenberg–Mac Lane cocycles. Higher Eilenberg–Mac Lane cocycles are like AT&T sales pitch: "More bars in more places."

that for each x, $R_{(x|-,-)}$ and $S_{(-,-|x)}$ are 2-cocycles[8], and they furthermore assert:

$$
\begin{tikzcd}[column sep=small]
& & z\otimes y\otimes x \arrow[drr, "(b_{x|y})^{-1}"] & & \\
y\otimes z\otimes x \arrow[urr, "b_{y|z}"] & & \cong & & z\otimes x\otimes y \arrow[ll, phantom, "R_{(x|z,y)}\Leftarrow"] \\
y\otimes x\otimes z \arrow[u, "b_{x|z}"] \arrow[drr, "(b_{x|y})^{-1}"'] \arrow[rr, phantom, "R^{-1}_{(x|y,z)}\Rightarrow"] & & & & x\otimes z\otimes y \arrow[u, "b_{x|z}"'] \arrow[uull, bend right, "b_{x|zy}"'] \\
& & x\otimes y\otimes z \arrow[urr, "b_{y|z}"'] \arrow[uull, bend left, "b_{x|yz}"'] & &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& & z\otimes y\otimes x \arrow[drr, "(b_{x|y})^{-1}"] & & \\
y\otimes z\otimes x \arrow[urr, "b_{y|z}"] \arrow[rr, phantom, "S^{-1}_{(y,x|z)}\Leftarrow"] & & \cong & & z\otimes x\otimes y \\
y\otimes x\otimes z \arrow[u, "b_{x|z}"] \arrow[drr, "b_{y|x}"'] \arrow[uurr, bend right, "b_{yx|z}"'] & & & & x\otimes z\otimes y \arrow[u, "b_{x|z}"'] \arrow[ll, phantom, "S_{(x,y|z)}\Rightarrow"] \\
& & x\otimes y\otimes z \arrow[urr, "b_{y|z}"'] \arrow[uurr, bend left, "b_{xy|z}"] & &
\end{tikzcd}
\tag{4.6.5}
$$

$$
\begin{tikzcd}[column sep=small]
& w\otimes y\otimes z\otimes x \arrow[dr, "b_{w|yz}"] \arrow[d, phantom, "\Uparrow S"] & \\
w\otimes x\otimes y\otimes z \arrow[ur, "b_{x|yz}"] \arrow[rr, "b_{wx|yz}"] \arrow[dr, "b_{wx|y}"'] & {} & y\otimes z\otimes w\otimes x \\
& y\otimes w\otimes x\otimes z \arrow[ur, "b_{wx|z}"'] \arrow[u, phantom, "R^{-1}\Uparrow"] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=tiny]
& & w\otimes y\otimes z\otimes x \arrow[dr, "b_{y|w}"'] \arrow[drr, "b_{w|yz}"] & & \\
w\otimes x\otimes y\otimes z \arrow[urr, "b_{x|yz}"] \arrow[r, "b_{x|y}"] \arrow[drr, "b_{wx|y}"'] & w\otimes y\otimes x\otimes z \arrow[ur, "b_{y|x}"] \arrow[dr, "b_{w|y}"] \arrow[ul, phantom, "\Leftarrow R^{-1}"] \arrow[dl, phantom, "\Rightarrow S"] & \cong & y\otimes w\otimes z\otimes x \arrow[r, "b_{w|z}"] \arrow[ur, phantom, "\Rightarrow R^{-1}"] \arrow[dr, phantom, "\Leftarrow S"] & y\otimes z\otimes w\otimes x \\
& & y\otimes w\otimes x\otimes z\,. \arrow[ur, "b_{x|z}"] \arrow[urr, "b_{wx|z}"'] & &
\end{tikzcd}
\tag{4.6.6}
$$

The unlabeled isomorphisms are the naturality of b. If b is also trivial, then (4.6.5) and (4.6.6) simply say:

$$R^{-1}_{(x|y,z)}\, R_{(x|z,y)} = S^{-1}_{(y,x|z)}\, S_{(x,y|z)}, \tag{4.6.7}$$

$$R^{-1}_{(wx|y,z)}\, S_{(w,z|yz)} = R^{-1}_{(x|y,z)}\, R^{-1}_{(w|y,z)}\, S_{(w,x|y)}\, S_{(w,x|z)}\,. \tag{4.6.8}$$

Suppose that we are in the bosonic strongly fusion case. Then α and b are automatically trivial, but π may not be. In this case, the equivalent equations (4.6.6) and (4.6.8), as well as the requirements that $R_{x|-,-}$ and $S_{-,-|x}$ be 2-cocycles, receive corrections by π. (The equivalent equations (4.6.5) and (4.6.7) do not require corrections, because π only appears when we need to coherently tensor four or more objects.) The full result is that (π, R, S) are together the data of what is sometimes called an "Abelian cocycle," and what we will call a *braided cocycle*: they define a class in the Eilenberg–Mac Lane cohomology

$$[\pi, R, S] \in \mathrm{H}^4_{\mathrm{br}}(M; \mathbb{C}^\times) := \mathrm{H}^5(M[2]; \mathbb{C}^\times). \tag{4.6.9}$$

Finally, suppose that $\mathcal{C}$ is a braided monoidal 2-category. A *syllepsis* v for $\mathcal{C}$ is an

[8] These are 3-cochains if we include the x variable, but not ordinary 3-cocycles.

isomorphism $v_{x||y} : b_{x|y} \overset{\sim}{\Rightarrow} b_{y|x}^{-1}$ for each x, y such that the diagram

$$\begin{tikzcd} b_{\ell|xy} \arrow[d, Rightarrow, "v_{\ell,xy}"] & & b_{\ell|x}\, b_{\ell|y} \arrow[ll, Leftarrow, "R_{(\ell|x,y)}"'] \arrow[d, Rightarrow, "v_{\ell||x}\, v_{\ell||y}"] \\ b_{xy|\ell}^{-1} & & b_{x|\ell}^{-1}\, b_{y|\ell}^{-1}\,. \arrow[ll, Leftarrow, "S_{(x,y|\ell)}"] \end{tikzcd} \tag{4.6.10}$$

commutes. In the bosonic strongly fusion case where α and b are trivial, v enhances (π, R, S) to a *sylleptic cocycle*, defining a class

$$[\pi, R, S, v] \in \mathrm{H}^4_{\mathrm{syl}}(M; \mathbb{C}^\times) := \mathrm{H}^6(M[3]; \mathbb{C}^\times). \tag{4.6.11}$$

In general, a theory with (only) grouplike p-spacetime dimensional objects with q-ambient dimensions (hence $p+q$ total spacetime dimensions) should be classified by degree $(p+q+1)$ cohomology of $M[q]$[9]. The original paper [87] calculates the values of $\mathrm{H}^p(A[q]; B)$ for small values of p, q and arbitrary Abelian groups A, B. In particular, writing $\widehat{A} := \hom(A, \mathbb{C}^\times)$ and $M_2 := \hom(\mathbb{Z}_2, M)$, sylleptic strongly fusion 2-categories $\mathcal{C}$ with $\pi_0\mathcal{C} = M$ are classified by

$$\mathrm{H}^6(M[3]; \mathrm{U}(1)) \cong \widehat{M_2} \oplus \widehat{\textstyle\bigwedge^2 M}\,,$$

where $\bigwedge^2 M := \frac{M \otimes M}{(m \otimes m)}$ denotes the alternating 2-forms on M. We will now explain the meaning of these two summands $\widehat{M_2}$ and $\widehat{\bigwedge^2 M}$. Further discussion can be found in [60, §2.1].

The summand $\widehat{M_2}$ measures the following [150]: given an invertible surface operator $m \in M$, consider wrapping the surface operator around a Klein bottle. This requires choosing an equivalence $m \cong m^{-1}$, since the Klein bottle is not orientable. We have such an equivalence exactly when $m \in M_2$, in which case, by endotriviality, the equivalence is unique up to isomorphism. It also requires choosing a Pin structure on the Klein bottle; let's choose the nonbounding Pin structure. Then this Klein bottle wrapped with $m \in M_2$ will evaluate to some element of $\mathbb{C}^\times$. This gives the map $M_2 \to \mathbb{C}^\times$, or in other words the element of $\widehat{M_2}$. Since the Klein bottle embeds into $\mathbb{R}^4 \subset \mathbb{R}^5$, this class in $\widehat{M_2}$ depends only

[9]When p is large, the required cohomology theory is not ordinary cohomology. Indeed, any theory will have k-dimensional operators built by inserting decoupled k-dimensional topological theories, and for large enough k there are nontrivial invertible k-dimensional topological field theories. For most purposes the presence of these decoupled operators does not affect the physics. However, these operators can arise as "higher fusion coefficients" for fusion of lower-dimensional operators. The result of this is that classifications by ordinary cohomology must be corrected in high dimensions.

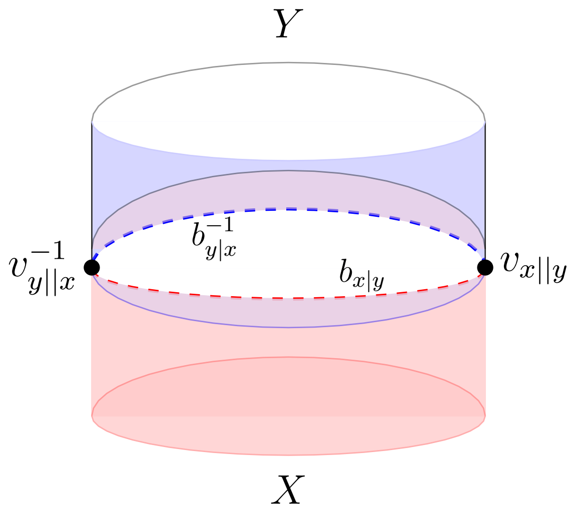


Figure 4.6: The two domed cylinders in red and blue represent two objects $X, Y \in \mathcal{C}$ respectively, living in four dimensions. The purple coloured regions show the domes of the objects. Initially, we can think of one object being above the other. The dashed lines indicate places where the two sheets pass over each other in the fourth dimension, with the colour indicating which is above. The two marked points show where one of the surfaces crosses over the other in the fifth dimension, changing the order of which surface is above and below. The change in color of the dotted circle represents the fact that after the syllepsis, the object which was initially on top, is now on the bottom.

on the braiding data and not the sylleptic form.

The summand $\widehat{\bigwedge^2 M}$ measures the following. Given surfaces with three ambient dimensions, then to "braid" them means passing them around each other in a two-parameter family, topologically a two-sphere. This procedure results in a phase factor that depends antisymmetrically on the inputs. In terms of the data of a sylleptic 2-category, this antisymmetric pairing is given by $\omega(x, y) = v_{x||y} - v_{y||x}$, where v is a 2-cocycle and represents the sylleptic data. This is because v tells how the surfaces go from above to below one another in the four dimension when we consider the double braiding of two surfaces. At two locations, the surfaces switch places by going into the fifth dimension. This process is depicted in Figure 4.6.

4.6.3. Sylleptic and symplectic groups: fermionic case

We turn now to the fermionic case, which is the main focus of this paper. As explained in §2.1, we are specifically interested in sylleptic strongly fusion super 2-categories $\mathcal{C}$. By definition, the line operators in such a 2-category are $\Omega\mathcal{C} = \mathbf{SVec}$. The simple lines consist of the identity line 1 and a fermion line f, corresponding to the super vector spaces $\mathbb{C}^{1|0}$ and $\mathbb{C}^{0|1}$ respectively. By Theorem 4.6.2, the components $\pi_0\mathcal{C}$ form a group M. The identity component, and hence every component, contains two simple objects. This identity component is a copy of $\Sigma\mathbf{SVec}$, equivalent to the 2-category of superalgebras and their super bimodules. The identity object $\mathbf{1}$ corresponds to the superalgebra $\mathbb{C}$, and the other simple object, which following [88] we will call the *Cheshire object* c, corresponds to the superalgebra Cliff(1). It is a fun exercise that the self-braiding $c \otimes c \to c \otimes c$ is given by the fermion f [110]. The invertible operators in the identity component form the symmetric monoidal higher group $(\Sigma\mathbf{SVec})^\times = \mathbb{C}^\times[2].\{1, f\}[1].\{\mathbf{1}, c\}[0]$ where the Postnikov extension data are given by $\mathrm{Sq}^2 : \{\mathbf{1}, c\} \to \{1, f\}$ and $(-1)^{\mathrm{Sq}^2} : \{1, f\} \to \mathbb{C}^\times$.

The collection of invertible operators in $\mathcal{C}$ is an extension of shape $(\Sigma\mathbf{SVec})^\times.\pi_0\mathcal{C}$. As in §4.6.2, we will encode that $\mathcal{C}$ is sylleptic by placing the (invertible) objects $\{\mathbf{1}, c\}.\pi_0\mathcal{C}$ in degree 3. In other words, setting $M := \pi_0\mathcal{C}$, we are interested in extensions of shape:

$$(\Sigma\mathbf{SVec})^\times[3].M[3]. \tag{4.6.12}$$

The classification of arbitrary extensions of this shape is somewhat complicated. But we know one thing more: the fermion f, and hence also its condensate c, are invisible. This is sometimes referred to as a local fermion, and any theory with this feature couples to spin structure and is equipped with a $\mathbb{Z}_2$ fermion parity symmetry that induces a grading on the Hilbert space. In the language of group theory, one can think of this as saying that the extension (4.6.12) is a "central extension," and so classified by untwisted cohomology (of $M[3]$) with coefficients in $(\Sigma\mathbf{SVec})^\times$.

Cohomology with coefficients in $(\Sigma\mathbf{SVec})^\times$ is called *(extended) supercohomology* $\mathrm{SH}^\bullet$. The name is due to [239] given in the context of condensed matter and lattice constructions, but had appeared in the mathematics literature beforehand as a generalized cohomology theory. See [110] for a more topological treatment. By the Atiyah-Hirzebruch spectral sequence, $\mathrm{SH}^\bullet$ is built out of three "layers" corresponding to the three homotopy groups of $(\Sigma\mathbf{SVec})^\times$. The bottom (Majorana) layer records whether the group of simple objects $\{\mathbf{1}, c\}.M$ is or is not a split extension. The second (Gu-Wen) layer records whether the isomorphism given by the braiding on two objects is even or odd; the fermion in particular braids with itself up to a sign rather than braiding trivially. The top layer records the associator data, i.e. a bosonic anomaly, of a suitable bosonic shadow to the fermionic theory [22]. There is a map $\mathrm{H}^\bullet(M[3]; \mathbb{C}^\times) \to \mathrm{SH}^\bullet(M[3])$ corresponding to viewing a bosonic theory as a fermionic one.[10]

[10] It was predicted in [161] that the classification of fermionic theories with symmetries in d dimensions

Proposition 4.6.3. For M any arbitrary Abelian group, $\mathrm{SH}^6(M[3]) \cong \widehat{\bigwedge^2 M} = \hom\left(\bigwedge^2 M\,, \mathbb{C}^\times\right)$, the space of alternating 2-forms.

Proof. We converge to the supercohomology by way of the Atiyah-Hirzebruch spectral sequence $\mathrm{SH}^6(M[3]) \Leftarrow \mathrm{H}^\bullet(M[3]; \mathrm{SH}^\bullet(\mathrm{pt}))$. The entries on the E_2 page can be filled in from the formulas in [60, 87]. This data assembles as:

$$E_2^{i,j} = \begin{array}{c|ccccc} j & & & & & \\ & & & & & \\ \mathbb{Z}_2 & \hom(M,\mathbb{Z}_2) & \mathrm{Ext}(M,\mathbb{Z}_2) & \hom(M,\mathbb{Z}_2) & \dots & \\ \mathbb{Z}_2 & \hom(M,\mathbb{Z}_2) & \mathrm{Ext}(M,\mathbb{Z}_2) & \hom(M,\mathbb{Z}_2) & \widehat{M_2} \oplus \hom\left(\bigwedge^2 M\,, \mathbb{Z}_2\right) & \\ \mathbb{C}^\times & \widehat{M} & 0 & \hom(M,\mathbb{Z}_2) & \widehat{M_2} \oplus \widehat{\bigwedge^2 M} & \\ \hline & 3 & 4 & 5 & 6 & i\,. \end{array} \tag{4.6.13}$$

The entries which include hom and Ext in degree three through five are all isomorphic to $\widehat{M_2}$, where M_2 denotes the 2-torsion of M, and the hat denotes Pontryagin duality. Specifically, $\hom(M, \mathbb{Z}_2) = (\widehat{M})_2$, and $\mathrm{Ext}(M, \mathbb{Z}_2) = \widehat{M_2}$, which can be seen from the short exact sequence $\mathbb{Z}_2 \xrightarrow{(-1)^x} \mathbb{C}^\times \xrightarrow{x^2} \mathbb{C}^\times$. The d_2 differential are given by:

$$d_2 : E_2^{i,2} = \mathrm{H}^i(M[3]\,; \mathbb{Z}_2) \to E_2^{i+2,1} = \mathrm{H}^{i+2}(M[3]\,; \mathbb{Z}_2) \qquad X \mapsto \mathrm{Sq}^2 X \tag{4.6.14}$$

$$d_2 : E_2^{i,1} = \mathrm{H}^i(M[3]\,; \mathbb{Z}_2) \to E_2^{i+2,0} = \mathrm{H}^{i+2}(M[3]\,; \mathbb{C}^\times) \qquad X \mapsto (-1)^{\mathrm{Sq}^2 X}\,. \tag{4.6.15}$$

Notice that because we are really looking at Eilenberg-MacLane spaces in degree three, we do not need to consider the entries in degree lower than three due to the Hurewicz's theorem:

$$\mathrm{H}^\bullet(M[3]\,; A) = 0 \text{ for } \bullet < 3\,. \tag{4.6.16}$$

We claim that $\mathrm{Sq}^2 : \mathrm{H}^3(M[3]; \mathbb{Z}_2) \to \mathrm{H}^5(M[3]; \mathbb{Z}_2)$ is an isomorphism. To see this, note first that $\mathrm{H}^3(M[3]; \mathbb{Z}_2) \cong \hom(M; \mathbb{Z}_2)$ by Hurewicz. Now given $\mu \in \mathrm{H}^3(M[3]; \mathbb{Z}_2)$, we can construct the pullback $\mu^* : \mathrm{H}^\bullet(\mathbb{Z}_2[3]; \mathbb{Z}_2) \to \mathrm{H}^\bullet(M[3]; \mathbb{Z}_2)$. The ring $\mathrm{H}^\bullet(\mathbb{Z}_2[3]; \mathbb{Z}_2)$ is a polynomial ring in the generators $T, \mathrm{Sq}^1 T, \mathrm{Sq}^2 T, \dots$ where T has degree 3. In particular,

is given by (twisted) spin cobordism $\Omega^{d+1}_{\mathrm{Spin}}(M)$. The Atiyah-Hirzebruch spectral sequence then allows us to compile the information in the first three layers to compute an approximation of spin cobordism, this recovers supercohomology. In low dimensions supercohomology well approximates spin cobordism, but as the dimensions get higher, the approximation is more crude and more information coming from the deeper layers may be necessary. In our case however, the supercohomology approximation is exact: while spin cobordism has layers below the Majorana layer, these layers do not contribute to cohomology of $M[3]$ because of the Hurewicz theorem.

$\mathrm{H}^3(\mathbb{Z}_2[3];\,\mathbb{Z}_2) = \{0, T\}$, with $\mu^*(T) = \mu$, and $\mathrm{H}^5(\mathbb{Z}_2[3];\,\mathbb{Z}_2) = \{0, \mathrm{Sq}^2 T\}$. Since Sq^2 is natural, we have $\mathrm{Sq}^2(\mu^* T) = \mu^*(\mathrm{Sq}^2 T)$, confirming the claim. Thus the d_2 differentials $E_2^{3,1} \to E_2^{5,0}$ and $E_2^{3,2} \to E_2^{5,1}$ are isomorphisms. The d_2 differentials supported in bidegrees $(4,1)$ and $(4,2)$ are injections by essentially the same argument. Namely, for each $m \in M_2 = \hom(\mathbb{Z}_2, M)$,we can restrict $\widehat{M_2} = \mathrm{H}^4(M[3];\mathbb{Z}_2)$ along the map $m^* : \mathrm{H}^4(\mathbb{Z}_2[3];\mathbb{Z}_2) \to \mathrm{H}^4(M[3];\mathbb{Z}_2)$. The only element in $\mathrm{H}^4(\mathbb{Z}_2[3];\mathbb{Z}_2)$ is $\mathrm{Sq}^1 T$, which is not annihilated by Sq^2. Again by naturality, the d_2 from $E_2^{4,1} \to E_2^{6,0}$ and $E_2^{4,2} \to E_2^{6,1}$ are injections.

All together, the E_3 page reads:

$$E_3^{i,j} = \begin{array}{c|cccc c} j & & & & & \\ & & & & & \\ \mathbb{Z}_2 & 0 & 0 & * & * & \\ \mathbb{Z}_2 & 0 & 0 & 0 & * & \\ \mathbb{C}^\times & \widehat{M} & 0 & 0 & \widehat{\bigwedge^2 M} & \\ \hline & 3 & 4 & 5 & 6 & i\,. \end{array} \tag{4.6.17}$$

In particular in total degree 6 the spectral sequence stabilizes on page 3, with the only nonzero entry being $\widehat{\bigwedge^2 M}$ in bidegree $(6,0)$. $\square$

Remark 4.6.18. We note that $\mathrm{H}^6(M[3];\mathbb{C}^\times) \simeq \widehat{M_2} \oplus \widehat{\bigwedge^2 M}$ classifies 5d bosonic topological phases, but the $\widehat{M_2}$ is killed by a differential in the spectral sequence for supercohomology. Thus a bosonic sylleptic form contains more information than its superization.

Thus we find:

Theorem 4.6.4. The set of fermionic (4+1)d topological orders with no lines is equal to the set of symplectic Abelian groups.

For the definition of *symplectic Abelian group* we refer the reader to §4.7.1.

Proof. The principle of remote detectability for topological orders ensures that there are no invisible operators (trivial centre). In detail, the "trivial centre" requirement for a sylleptic fusion 2-category is that its symmetric centre — its full subcategory on those objects x for which $v_{x||-} = v_{-||x}^{-1}$ — should be trivial. As explained at the end of §2.2, the class in $\widehat{\bigwedge^2 M}$ precisely records the antisymmetric pairing $\langle x, y\rangle = v_{x||y} v_{y||x}$. When applied to the group of surfaces, it means that the symplectic pairing is nondegenerate. $\square$

Remark 4.6.19. To make contact with lower dimensions, consider the familiar case of bosonic 3d topological orders. These are given by modular tensor categories (MTC). In the Abelian case, with M a group, the braiding data of the MTC data is determined by a class in $\mathrm{H}^4(M[2])$. This is isomorphic to the group of quadratic functions on M. The full braiding of lines is given by the symmetric pairing

$$\begin{aligned} M \otimes M &\to \mathbb{C}^\times \\ a \otimes b &\mapsto \frac{q(a+b)}{q(a)\, q(b)}\,, \end{aligned} \tag{4.6.20}$$

where q is a quadratic function. In 3d this is a one-parameter family in which the lines pass around each other in a circle and we get a phase factor because it is a one-dimensional motion, and one dimension lower than line operator is a phase. Furthermore, this phase depends symmetrically on the two inputs, and here "nondegenerate" means that the symmetric pairing is nondegenerate.

If M_2 is trivial, then $\mathrm{H}^4(M[2]; \mathrm{U}(1)) \cong \mathrm{Sym}^2 \widehat{M}$ by completing the square. In general, the map $\mathrm{H}^4(M[2]) \to \mathrm{Sym}^2 \widehat{M}$ has kernel. This is analogous to the kernel $\widehat{M_2}$ of the map $\mathrm{H}^6(M[2]; \mathrm{U}(1)) \to \mathrm{SH}^6(M[3]) = \widehat{\bigwedge^2 M}$. And indeed a similar analysis as in Proposition 4.6.3 shows that this kernel dies when going to fermionic theories and $\mathrm{SH}^4(M[2]) \cong \mathrm{Sym}^2 \widehat{M}$.

4.7. 5d Topological order from the boundary

4.7.1. Symplectic Reduction to Isotropic Subspaces

The symplectic form ω on M gives M the structure of a symplectic Abelian group.

Definition 4.7.1. A *symplectic Abelian group* is an Abelian group G together with an isomorphism $\omega : M \to \widehat{M}$, with $\widehat{M} = \hom(M, \mathbb{C}^\times)$ such that $\omega(g, g) = 1$ for every $g \in M$.

This definition implies an alternating feature, $\omega = \widehat{\omega}^{-1}$. Recall that by definition $\bigwedge^2 M = \frac{M \otimes M}{m \otimes m}$, so a map $\omega : M \to \widehat{M}$ is the same data as a map $\omega : M \otimes M \to \mathrm{U}(1)$. This map solves $\omega(g, g) = 1$ for all g iff it factors through $\bigwedge^2 M$.

Example. An example of a symplectic Abelian group is when M is a product of groups $B \times \widehat{B}$ with $\omega((b_1, f_1), (b_2, f_2)) = f_1(b_2) \cdot f_2(b_1)^{-1}$.

If M is a cyclic group then M does not admit a symplectic form. Call the generator for the cyclic group t, then $\omega(t, t) = 1$ but $\omega(t^a, t^b) = 1^{ab} = 1$, and ω is not an isomorphism.

Suppose M is a symplectic Abelian group and $N \subset M$ is a subgroup. The *symplectic orthogonal* $N^\perp$ is the subgroup $\{m \in M \text{ s.t. } \omega(m, n) = 1 \text{ for all } n \in N\}$. It is the subgroup corresponding to $\widehat{M/N} \subset \widehat{M}$ under the isomorphism $\omega : M \cong \widehat{M}$. From this description, we see that $|M| = |N| \times |N|^{\perp}$. A subgroup $L \subset M$ is *Lagrangian* if $L = L^{\perp}$ as subgroups of M. Thus L is Lagrangian exactly when $\omega|_L$ is trivial and $|L| = \sqrt{|M|}$. A *Lagrangian splitting* of M is a direct sum decomposition $M = L \oplus L'$ where both L and L' are Lagrangian. The symplectic form on M then identifies $L' \cong \widehat{L}$.

Proposition 4.7.2 (Darboux theorem for finite groups). Every symplectic finite Abelian group admits a Lagrangian splitting.[11]

The following proof is essentially given in [57, Lemma 5.2].

Proof. Every finite Abelian group canonically factors as a direct sum of subgroups for different p, and the symplectic form cannot mix different primes. We thus reduce to the case where the group in consideration M has order p^k for some prime p. We give the $p = 2$ case for clarity, and the proof generalizes for other primes. Pick an element $x \in M$ of maximal order, say 2^a. Then $x^{2^{a-1}}$ is nontrivial and we choose an element y such that the pair $\omega(x^{2^{a-1}}, y) \neq 1$. We use the fact that $x^{2^{a-1}}$ is order 2 and so by inspecting $\omega(x^{2^{a-1}}, y) = \omega(x, y)^{2^{a-1}}$, which is itself also order 2, we see that y has order at least 2^a. But a was maximal, so we have found two subgroups, generated by x and y, both of order 2^a. We note that these two groups are transverse because an alternating form vanishes on a cyclic subgroup. Let N denote the subgroup generated by x and y. It is a product of cyclic groups $\mathbb{Z}_{2^a} \times \mathbb{Z}_{2^a}$, which are themselves each Lagrangians in N. The restriction of ω to N is the canonical split pairing $\omega(x, y)$, of x pairing with y. By construction, $\omega|_N$ is nondegenerate. Thus N and $N^{\perp}$ are transverse ($N \cap N^{\perp} = 0$), so $M = N \oplus N^{\perp}$. By induction of the previous procedure, $N^{\perp}$ can be further split into something Lagrangian, therefore M has a Lagrangian splitting.

□

4.7.2. Lagrangian subgroups as boundary theories

We now turn to investigate the boundary (3+1)d theory of a 5d theory which also has only surfaces, and make some relations with the bulk. The boundary is a braided strongly fusion 2-supercategory $\mathcal{L}$ with objects being surfaces that have an L group fusion rule. The braiding β is a class in $\mathrm{SH}^5(L[2])$, which in this case, antisymmetrically pairs objects.[12]

[11] While it is true that all such M admit a Lagrangian subgroup, it is not the case that any Lagrangian at all fits into the sequence $\widehat{L} \hookrightarrow M \to L$, see [57, Example 5.4]

[12] The analogue of this class for a (1+1)d boundary to a (2+1)d bulk would be a class $\alpha \in \mathrm{SH}^3(L[1])$ that provides associator information regarding the lines in the (1+1)d theory.

We can think of the bulk (4+1)d theory as the *sylleptic centre* of $\mathcal{L}$ denoted by $\mathcal{Z}_{(2)}(\mathcal{L})$, where the objects have fusion rule M with a sylleptic structure ω. Since all 1-morphisms are trivial in the strongly fusion case, all the data is encoded in R, S and of particular importance is the class in $\mathrm{SH}^5(L[2])$ encoding the braiding.

Definition 4.7.3. Let $\mathcal{L}$ be a braided monoidal 2-category. An object in the *sylleptic centre* $\mathcal{Z}_{(2)}(\mathcal{L})$ is a pair $(x, v_{x||-})$. A 1-morphism from $(x, v_{x||-})$ to $(x', v_{x'||-})$ is a one morphism $f : x \to x'$ in $\mathcal{L}$ such that the following diagram commutes for all $y \in \mathcal{L}$:

$$
\begin{tikzcd}
x \otimes y \arrow[rr, Rightarrow] \arrow[dr, "b_{x|y}"'] \arrow[dd] & {} \arrow[d, phantom, "\Downarrow v_{x||y}"] & x \otimes y \arrow[dd] \\
{} \arrow[r, phantom, "\Rightarrow b_{f|y}"] & y \otimes x \arrow[ur, "b_{y|x}"'] \arrow[dd] \arrow[r, phantom, "\Rightarrow b_{y|f}"] & {} \\
x' \otimes y \arrow[rr, Rightarrow] \arrow[dr, "b_{x'|y}"'] & {} \arrow[d, phantom, "\Downarrow v_{x'||y}"] & x' \otimes y \\
& y \otimes x' \arrow[ur, "b_{y|x'}"'] &
\end{tikzcd}
$$

where the 2-morphism on the back face is the identity. The two morphisms are defined in the same manner as in $\mathcal{L}$.

Lemma 4.7.4. If $\mathcal{L}$ is a strongly fusion braided fusion 2-category, then $\mathcal{Z}_{(2)}(\mathcal{L})$ contains no lines.

Proof. Consider the identity object $(\mathbf{1}, v_{\mathbf{1}||-})$ of $\mathcal{Z}_{(2)}(\mathcal{L})$, where $v_{\mathbf{1}||x}$ is the following isomorphism:

$$
\begin{tikzcd}
\mathbf{1} \otimes x \arrow[rr, bend left, "b_{\mathbf{1}|x}"] \arrow[rr, bend right, "b^{-1}_{x|\mathbf{1}}"'] \arrow[rr, phantom, "\Downarrow v_{\mathbf{1}||x}"] & & x \otimes \mathbf{1}
\end{tikzcd} .
$$

A priori, v could be any x-dependent $\mathbb{C}^\times$ number satisfying a 1-cocycle relation i.e. $v \in \widehat{L}$. But since $\mathbf{1}$ is the identity object, $b_{\mathbf{1}|x}$ and $b_{x|\mathbf{1}}$ are both trivialized, and the identity object of the centre is the one such that v is also trivialized, so we take $v_{\mathbf{1}||-} = 1$. Now consider morphisms of the identity object, which is a morphism from $\nVdash \to \mathbf{1}$, or $\mathrm{id}_{\mathbf{1}}$. Then, we have the following 3-cell filling:

$$\begin{tikzcd}
\mathbf{1} \otimes x \arrow[rr, bend left, "b_{\mathbf{1}|x}"] \arrow[rr, bend right, "b^{-1}_{x|\mathbf{1}}"'] \arrow[rr, phantom, "\Downarrow v_{\mathbf{1}||x}"] \arrow[d] & & x \otimes \mathbf{1} \arrow[d] \\
\mathbf{1} \otimes x \arrow[rr, bend left, "b_{\mathbf{1}|x}"] \arrow[rr, bend right, "b^{-1}_{x|\mathbf{1}}"'] \arrow[rr, phantom, "\Downarrow v_{\mathbf{1}||x}"] & & x \otimes \mathbf{1}\,,
\end{tikzcd}$$

where the vertical maps are just identity maps. But, because we are in the 2-category, the only 3-cell is the identity. □

Remark 4.7.1. More generally, if $\mathcal{B}$ is any braided monoidal 2-category, then $\Omega\mathcal{Z}_{(2)}(\mathcal{B})$ is a full sub-1-category of $\Omega\mathcal{B}$. However, the analogous statements for $\mathcal{Z}_{(1)}$ and for 3-categories fail. For definiteness, Lemma 4.7.4 is to spell out the details of the case we care about.

Proposition 4.7.5. The sylleptic center of a trivially braided fusion 2-category $\mathcal{L}$ is $\widehat{\mathcal{L}} \times \mathcal{L}$ and the sylleptic form is the canonical one.

Proof. The trivial braiding indicates that $[\pi, R, S]$ in (4.6.9) are trivial. As a consequence, the diagram in (4.6.10) reduces down so that v satisfies the equation $v_{\ell||xy} = v_{\ell||x} v_{\ell||y}$, thus $v_{\ell||-}$ is a homomorphism $\mathcal{L} \to \mathbb{C}^\times$. The object $(x\,, v_{x||-})$ in the sylleptic centre is therefore an element of $(\mathcal{L}\,, \widehat{\mathcal{L}})$. □

Example. For clarity let us work bosonically in this example instead of using supercategories. Suppose M admits L as a lagrangian, and take $L = \mathbb{Z}_3$. A particularly simple class of braided fusion 2-category is $\mathcal{L} = 2\mathbf{Vec}^\beta[\mathbb{Z}_3]$, where $\beta \in \mathrm{H}^5(\mathbb{Z}_3[2])$. A computation shows that $\mathrm{H}^5(\mathbb{Z}_3[2]) = \widehat{\bigwedge^2 \mathbb{Z}_3} = 0$, which means the only category is $2\mathbf{Vec}[\mathbb{Z}_3]$ with the sylleptic centre

$$\mathcal{Z}_{(2)}(2\mathbf{Vec}[\mathbb{Z}_3]) = 2\widehat{\mathbf{Vec}[\mathbb{Z}_3]} \times 2\mathbf{Vec}[\mathbb{Z}_3]. \tag{4.7.2}$$

In general, if L was a group such that $\beta \neq 0$, then $\mathcal{Z}_{(2)}(\mathcal{L}) = \widehat{\mathcal{L}}\,.\,\mathcal{L}$, a nontrivial extension of the boundary category. In terms of the groups, (4.7.2) implies that $M = \widehat{L} \times L$, where $\widehat{L} = M/L^{\perp} = M/L$. Therefore, M fits into the short exact sequence $\widehat{L} \hookrightarrow M \twoheadrightarrow L$.

Remark 4.7.3. The centre gives the corresponding Djikgraaf-Witten (DW) theory for the boundary, with anomaly given by a class in $\mathrm{SH}^6(M[3])$. The act of going from the sylleptic centre to the boundary can be done by first "forgetting" the sylleptic structure, and then applying a Dirichlet boundary condition aka a braided map from a braided monoidal centre to the boundary. The objects in the kernel of this map are precisely the "Wilson lines" of the DW theory. The boundary condition contains not only a condensation $\widehat{L}$ but furthermore a trivialization of $\omega|_{\widehat{L}}$, which is given by a class in $\mathrm{SH}^5(\widehat{L}[3])$.

We can also ask which boundary theories can be lifted to the bulk; this is the equivalent of finding a splitting of the bulk to boundary map. The objects in the image of the splitting map are the " 't Hooft lines" of the DW theory. A priori there can be an obstruction to the lifting [60], which means that the lines in the bulk do not split neatly as a direct sum of "electric" and "magnetic" lines. There exists an obstruction for a braided 4-cocycle $\{\pi(-,-,-,-), R_{(-|-,-)}, S_{(-,-|-)}\}$ to have sylleptic structure given by

$$\theta : \mathrm{H}^5(L[2]) \to \mathrm{Ext}(L, \widehat{L})\,, \tag{4.7.4}$$

with the kernal of this map precisely given by $\widehat{L}_2$. The map $\mathrm{H}^6(L[3]) \to \mathrm{H}^5(L[2])$ maps between the two $\widehat{L}_2$ subgroups, with $\mathrm{H}^6(L[3])$ attained from $\mathrm{H}^6(M[3])$ via a restriction map. The subgroup of $\widehat{L}_2$ in $\mathrm{H}^5(L[2])$ contains information regarding the data of π, R, S. The remainder of the group is braiding information that cannot be lifted to being sylleptic. There is furthermore a map from $\mathrm{H}^6(L[3]) \to \mathrm{SH}^6(L[3])$ that surjects onto $\widehat{\bigwedge^2 L}$. This is summarized in the following diagram:

$$\begin{tikzcd}
\mathrm{H}^6(L[3]) \simeq \underbrace{\widehat{L}_2} \oplus \widehat{\bigwedge^2 L} \arrow[r, two heads] \arrow[d, hook] & \mathrm{SH}^6(L[3]) \simeq \widehat{\bigwedge^2 L} \arrow[d] \\
\mathrm{H}^5(L[2]) \arrow[r] & \mathrm{SH}^5(L[2]).
\end{tikzcd}$$

The map from $\mathrm{SH}^6(L[3]) \to \mathrm{SH}^5(L[2])$ is therefore the zero map as composition of the left vertical map and the horizontal map gives zero; we then have:

Proposition 4.7.6. Only the fermionic boundary theories with trivial braiding can be extended, in a way such that multiplication data is consistent with the lift to the bulk, to a sylleptic form.

Remark 4.7.5. It is possible that all surfaces on the boundary can be lifted, but not necessarily canonically. In the case of a 3d $\mathbb{Z}_3$ DW theory with nontrivial anomaly, this has a Dirichlet boundary condition where the lines obey $\mathbb{Z}_3$ fusion rule. The bulk however has lines that obey $\mathbb{Z}_9$ fusion rule. Any line on the boundary can be lifted, but there is no way to do this in a way that is compatible with the tensor product. The lines on the boundary cube to the trivial line, but lifting it to the bulk means that the cube is nontrivial.

4.7.3. Morita trivial 5d phases

If M admits L as a Lagrangian subspace then the corresponding 5d topological order upon symplectic reduction is Morita equivalent to the trivial theory. More succinctly this is know as being *Morita trivial*. This reduction procedure is depicted physically in Figure 4.7.

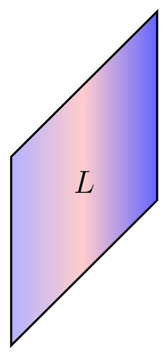


Figure 4.7: The wall is braided fusion 2-category with objects in $L^{\perp}$, separating the original theory $\mathcal{A}$ from the vacuum. Similar to the case of quantum Hamiltonian reductions, the wall is a bimodule for the two categories on either side.

Example. Consider in (1+1)d the category $\mathcal{I}$ given by $\mathbf{Vec}^{\alpha=0}[\mathbb{Z}_2]$, where $\alpha \in \mathrm{H}^3(\mathbb{Z}_2[1])$ is the trivial associator. Then the (2+1)d bulk theory is $\mathcal{T} = \mathcal{Z}(\mathbf{Vec}[\mathbb{Z}_2]) = \mathbf{Vec}[\mathbb{Z}_2] \times \mathbf{Vec}[\mathbb{Z}_2]$. Condensing out a $\mathbb{Z}_2$ subgroup from $\mathcal{T}$ amounts to the reduction $(\mathbb{Z}_2 \times \mathbb{Z}_2) \mathbin{/\!\!/} \mathbb{Z}_2 = \mathbb{Z}_2^{\perp}/\mathbb{Z}_2 = \{*\}$. Physically, this is equivalent to taking the (2+1)d Toric code and condensing out the m or e particle. The lines left "unscreened" are in $\mathbb{Z}_2^{\perp}$, and another identification by $\mathbb{Z}_2$ gives the trivial theory.

Theorem 4.6.4 relates 5d theories to symplectic Abelian groups and by Proposition 4.7.2, we see that:

Proposition 4.7.7. All 5d super topological orders are Morita trivial.

A 5d phase which is not Morita trivial has boundary conditions that are necessarily gapless, an immediate consequence is:

Corollary 4.7.8. All 4d fermionic boundaries can be gapped.

While there are 4d fermionic gapless theories, by introducing the appropriate interactions we can introduce a gap and hence there is no robust gapless phase. [13] We now present the reverse story and the way of reconstructing a theory from the vacuum. We will show that every fermionic 5d topological order can be built non-canonically by gauging a one-form symmetry $\widehat{L}[1]$, and a zero-form symmetry G, both acting on the vacuum. If the set of lines, $\Omega^3\mathcal{A}$, is super Tannakian, then $G = \mathrm{Aut}\,(F)$ and $\Omega^3\mathcal{A} \cong \mathbf{SRep}(G)$. The first step of condensing out the lines can be "undone" by gauging the group G which acts on the group M of surfaces. The symplectic form associated with M is now a G-equivariant class $\widetilde{\omega} \in \mathrm{H}^6(M[3]/G)$.

After condensing the lines, there is a similar map $C : \Omega^2\mathcal{A} \to 2\mathbf{SVec}$ which tells how to condense surfaces by choosing Lagrangian subspaces. Gauging by the dual group $\widehat{L}$ which acts on the vacuum then undoes this procedure. For Abelian group, gauging by a group or the dual group is always possible by the notion of "electromagnetic-duality". The important point to stress is that the choice of Lagrangian subgroup L from M was not canonical, and so doing the gauging by $\widehat{L}$ is also not canonical. In contrast, the zero-form group G is canonically determinded based on the lines of $\mathcal{A}$.

Remark 4.7.6. This two step procedure can not necessarily be combined to a one step condensation by a "2-group" symmetry $\widehat{L}[1].G$. We take for example the 5d toric code, with a $G = \mathbb{Z}_3$ action that permutes the three strings. A nontrivial extension of $\widehat{L}[1]$ by G will spoil the duality between switching the electic and magnetic lines.

Example. We give an analogous story by considering the (2+1)d Toric code. This is only analogous because the theory is not a symplectic Abelian group, rather the pairing is symmetric. We choose the set $\mathcal{R} = \{1, e\}$ to be Tannakian from the set $\{1, e, m, f\}$ of all the lines. As an $\mathcal{R}$ module, the Toric code is $\mathcal{R} \oplus m\mathcal{R}$. The map F takes R and condenses it to the vacuum. This forms a gapped (1+1)d boundary where, as an $\mathcal{R}$ module, the lines $\{1, m\}$ live. The group G is the group generated by $\{1, m\}$, as can be seen when we consider the fact that a zero-form symmetry in (1+1)d is sourced by lines.

Remark 4.7.7. Since M is a group of surface operators which are codimension-3 it defines a two-form symmetry. This group has an anomaly that is precisely the symplectic form. For a general isotropic $N \subset M$ we can consider gauging the N-symmetry. The importance of

[13] Our definition of topological order is such that an invertible phase is considered as the trivial topological order. In concluding Corollary 4.7.8 we are using the fact that "topological order" means "topological phase up to invertible phases", and thus for a topological order to have a gapped boundary, this means that the corresponding phase has a gapped interface to an invertible phase.

being isotropic is to ensure that that the symmetry is non-anomalous and can be gauged. By gauging the symmetry we build a gapped domain wall between the original theory M and a new theory given by $M /\!\!/ N$. In the gauging procedure, N screens out those operators in M which do not commute with N, and so the unscreened operators are $N^\perp$. But also the gauging procedure identifies the operators in N. The result is that the new procedure is described by the *symplectic reduction* $N^\perp/N$.[14] We note that $M /\!\!/ N$ itself is naturally symplectic by defining $\omega([a],[a']) = \omega(a,a')$ where $[a],[a']$ are classes in $N^\perp/N$, i.e. $a, a' \in N^\perp$, and they are defined up to shifting by $b, b' \in N$. If $N = L$ is Lagrangian, $L^\perp/L = \{*\}$, and so we do not have to assume that L participated in a Lagrangian splitting to show Morita triviality in proposition 4.7.7.

4.8. Bosonic 5-dimensional Topological Orders

The passage from bosonic to super topological orders is much like going from $\mathbb{R}$ and extending to $\mathbb{C}$, its algebraic closure. Consider a time reversal symmetry $\mathbb{Z}_2^T$ that acts $\mathbb{C}$-antilinearly and squares to the identity. Working with an algebra A of operators over the complex numbers with a $\mathbb{Z}_2^T$ symmetry is the same as working over the real numbers. The $\mathbb{Z}_2^T$ *descends* A into $A_\mathbb{R}$, an $\mathbb{R}$ algebra, so that $A = A_\mathbb{R} \otimes \mathbb{C}$. In the same spirit as the 0-categorical case, there is a way to 1-categorically extend **Vec** to **SVec**, where the latter is "algebraically closed"[144].

A bosonic topological order $\mathcal{A}$ is equipped with an action of the categorified Galois group $\mathrm{Gal}(\mathbf{SVec}/\mathbf{Vec}) = \mathbb{Z}_2^F[1]$, and *Galois descent* says that the algebra of a bosonic higher category can be considered as the algebra of a $\mathbb{Z}_2^F[1]$-equivariant higher supercategory. As remarked in Section 4.6, the fibre functor F may not allow for complete condensation of the lines if we are working bosonically. If the lines are Tannakian i.e. $\mathbf{Rep}(G)$ then we can condense out all the lines, the problem then reduces to the analogous problem discussed in the previous sections, with the symplectic form a class in $\mathrm{H}^6(M[3])$. If the lines are $\mathbf{Rep}(G, z)$, where z is a central element of order two, then it is always possible to condense to only $\{1, f\}$, i.e. **SVec**. Furthermore, in a 5d bosonic theory not only are there surface operators, but there are nontrivial 3d "membrane" operators. The surfaces operators still form a group under fusion by [152, Theorem A]; in this dimension the surfaces and lines can always unlink. But now either of the lines $\{1, f\}$ can wrap membranes, each detecting the other. This data compiles into a bosonic 3-category $\mathcal{A}$, with $\pi_0\mathcal{A} = \mathbb{Z}_2^F$. The "magnetic membrane" is the unique invertible object in the nontrivial component that enacts the $\mathbb{Z}_2^F$

[14]For an associative algebra A, gauging by the action of a connected and simply connected Lie group G is also called *quantum Hamiltonian reduction*.

one-form symmetry and will square to something in the identity component. The whole 3-category is describable by an extension

$$\mathbb{C}^\times[5].\underbrace{\mathbb{Z}_2}_{\{1,f\}}[4].\underbrace{(\mathbb{Z}_2.M)}_{\text{surfaces}}[3].\mathbb{Z}_2^F[2]\,; \tag{4.8.1}$$

$\mathbb{C}^\times[5]$ means "four-form $\mathbb{C}^\times$ symmetry" the $\mathbb{Z}_2$ in surfaces is given by $\{\mathbf{1}, c\}$, which are the two simple objects in $\Sigma\mathbf{SVec}$ as stated before in §4.6.3, with the caveat that now $c^2 \cong c \oplus c$. The fibre

$$\mathbb{C}^\times[5].\mathbb{Z}_2[4].(\mathbb{Z}_2.M)[3] = (\mathbb{C}^\times[5].\mathbb{Z}_2[4].\mathbb{Z}_2[3]).M[3] \tag{4.8.2}$$

is the 2-category of surfaces, and the base $\mathbb{Z}_2^F[2]$ are the two components of the 3-category. We can make a simplification of the fibre as follows. Any surface in $s \in M$ actually corresponds to two surfaces s_1 or s_2, being off from each other by the c. But because we have the magnetic brane, $\mathcal{M}$, we can act with this brane on either of the surfaces. The intersection of $\mathcal{M}$ with s_1 or s_2 is either the line 1 or f, however we know that $\mathcal{M}$ acting on c gives f. Therefore, it is possible to identify which $s_{1,2}$ is the one that is also "charged" with c. This gives us the freedom to always choose the "neutral" line, and so the term $\mathbb{Z}_2[3]$ can be ignored. Left with only the surfaces in M, we may condense them all out via the procedure in §4.7.1. We are left with only having to understand the $\mathbb{Z}_2^F[2]$ objects.

The fermionic Witt group inherits an action by $\mathbb{Z}_2^F[1]$ due to the fact that the spectrum [15]

$\mathrm{S}\mathcal{W}^\bullet = (\Sigma^{n-1}\mathbf{SVec})^\times$. $\mathcal{W}^\bullet$ is then the fixed-point spectrum of $\mathbb{Z}_2^F[1]$ via categorified Galois descent [149]. Therefore the cohomology of $\mathcal{W}^\bullet(\mathrm{pt})$ is given by the twisted $\mathrm{S}\mathcal{W}^\bullet$-cohomology, $\mathrm{S}\mathcal{W}^\bullet(\mathbb{Z}_2^F[2])$, of the space $\mathbb{Z}_2^F[2] = B(\mathbb{Z}_2^F[2])$. We compute this twisted cohomology by the following Atiyah-Hirzebruch spectral sequence:

$$\mathrm{H}^i(\mathbb{Z}_2^F[2]; \mathrm{S}\mathcal{W}^j(\mathrm{pt})) \Rightarrow \mathrm{S}\mathcal{W}^{i+j}(\mathbb{Z}_2^F[2]) = \mathcal{W}^{i+j}(\mathrm{pt})\,. \tag{4.8.3}$$

The homotopy groups of $\mathrm{S}\mathcal{W}^\bullet(\mathrm{pt})$ in low degrees are given by

$$\begin{aligned} &\mathrm{S}\mathcal{W}^0(\mathrm{pt}) = \mathbb{C}^\times\,, \quad \mathrm{S}\mathcal{W}^1(\mathrm{pt}) = \mathbb{Z}_2\,, \quad \mathrm{S}\mathcal{W}^2(\mathrm{pt}) = \mathbb{Z}_2 \\ &\mathrm{S}\mathcal{W}^3(\mathrm{pt}) = 0\,, \quad \mathrm{S}\mathcal{W}^4(\mathrm{pt}) = \mathrm{S}\mathcal{W}\,, \quad \mathrm{S}\mathcal{W}^5(\mathrm{pt}) = 0\,, \quad \mathrm{S}\mathcal{W}^6(\mathrm{pt}) = 0\,. \end{aligned} \tag{4.8.4}$$

In degree four, $\mathrm{S}\mathcal{W}$ known as the *fermionic Witt group* gives the set of (2+1)d super topological orders modulo gapped interfaces. Another way to think about this group is that

[15] Spectrum can be substituted interchangeably with the term "generalized cohomology theory", which was used in the introduction.

it gives the anomalies for 3d super MTCs [16], and two theories related by a gapped interface have the same anomaly.

The E_2 page is therefore:

$$
E_2^{ij} = \begin{array}{c|cccccccccc}
j & & & & & & & & & & \\
& & & & & & & & & & \\
0 & 0 & 0 & \dots & & & & & & & \\
0 & 0 & 0 & \dots & & & & & & & \\
\mathrm{S}\mathcal{W} & \mathrm{S}\mathcal{W} & 0 & \hom(\mathbb{Z}_2, \mathrm{S}\mathcal{W}) & \dots & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
\mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2^2 & \mathbb{Z}_2^2 \dots & & & \\
\mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2^2 & \mathbb{Z}_2^2 \dots & & & \\
\mathbb{C}^\times & \mathbb{C}^\times & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_4 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \\
\hline
& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & i\,.
\end{array}
\tag{4.8.5}
$$

The *twisted* d_2 differentials are:

$$
\begin{aligned}
d_2 : & E_2^{i,2} = \mathrm{H}^i(\mathbb{Z}_2^F[2]\,;\mathbb{Z}_2) \to E_2^{i+2,1} = \mathrm{H}^{i+2}(\mathbb{Z}_2^F[2]\,;\mathbb{Z}_2) && X \mapsto \mathrm{Sq}^2 X + TX \\
d_2 : & E_2^{i,1} = \mathrm{H}^i(\mathbb{Z}_2^F[2]\,;\mathbb{Z}_2) \to E_2^{i+2,0} = \mathrm{H}^{i+2}(\mathbb{Z}_2^F[2]\,;\mathbb{C}^\times) && X \mapsto (-1)^{\mathrm{Sq}^2 X + TX}\,,
\end{aligned}
\tag{4.8.6}
$$

where T is the generator of $\mathrm{H}^\bullet(\mathbb{Z}_2^F[2]\,;\mathbb{Z}_2)$ in degree two. The E_3 page is

$$
E_3^{ij} = \begin{array}{c|cccccccccc}
j & & & & & & & & & & \\
& & & & & & & & & & \\
0 & 0 & 0 & \dots & & & & & & & \\
0 & 0 & 0 & \dots & & & & & & & \\
\mathrm{S}\mathcal{W} & \mathrm{S}\mathcal{W} & 0 & \hom(\mathbb{Z}_2, \mathrm{S}\mathcal{W}) & \dots & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
\mathbb{Z}_2 & 0 & 0 & \mathbb{Z}_2 & 0 & 0 & \mathbb{Z}_2 & \dots & & & \\
\mathbb{Z}_2 & 0 & 0 & 0 & \mathbb{Z}_2 & 0 & 0 & 0 & 0 & \dots & \\
\mathbb{C}^\times & \mathbb{C}^\times & 0 & 0 & 0 & \mathbb{Z}_4 & \mathbb{Z}_2 & 0 & 0 & 0 & \\
\hline
& 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & i\,.
\end{array}
\tag{4.8.7}
$$

Remark 4.8.8. The generators of $\mathrm{H}^5(\mathbb{Z}_2^F[2]\,;\mathbb{Z}_2)$ are $\mathrm{Sq}^2\mathrm{Sq}^1 T$ and $T\mathrm{Sq}^1 T$. The d_2 differential

[16]This is a braided fusion category with trivial centre which is equipped with a "ribbon structure," which allows the corresponding (2+1)-dimensional TQFT to be placed on any oriented manifold. The TQFT is said to be *isotropic*.

annihilates $\left(\mathrm{Sq}^2\mathrm{Sq}^1 T + T\mathrm{Sq}^1 T\right)$ leaving a $\mathbb{Z}_2$ in bidegree (5,2). The $\mathbb{Z}_2$'s and $\mathbb{Z}_4$ in total degree four survive on E_∞ [149, Remark V.2]. The main result in [150] implies that the $\mathbb{Z}_2$ in bidegree $(5,0)$ survives on E_∞.

There is potentially a d_3 differential that maps $\hom(\mathbb{Z}_2, \mathrm{S}\mathcal{W}) \to E_3^{5,2} = \mathbb{Z}_2$, after which the spectral sequence stabilizes in total degree 6. Thus $\mathcal{W}^6(\mathrm{pt})$ is the kernel of this d_3. By [61, Proposition 5.18] we have

$$\mathrm{S}\mathcal{W} = \mathrm{S}\mathcal{W}_{\mathrm{pt}} \oplus \mathrm{S}\mathcal{W}_2 \oplus \mathrm{S}\mathcal{W}_\infty\,, \tag{4.8.9}$$

where $\mathrm{S}\mathcal{W}_{\mathrm{pt}}$ is generated by the Witt classes of Abelian super MTC, $\mathrm{S}\mathcal{W}_2$ is an elementary Abelian 2-group, and $\mathrm{S}\mathcal{W}_\infty$ is a free group of countable rank. It was proved in [206, Theorem, 7.2] that $\mathrm{S}\mathcal{W}_2$ is a group of infinite rank [17], which means that on E_∞ the entry in (2,4) will also have infinite rank even after the d_3 differential. As a result, $\mathcal{W}^6(\mathrm{pt})$ is also a group of infinite rank. By construction, $\mathcal{W}^6(\mathrm{pt})$ is the group of Morita equivalence classes of 5d topological orders, and so we have verified equation (4.5.3):

Theorem 4.8.1. There are infinitely many 5d bosonic topological orders which are "chiral" in the sense that they only admit gapless boundary. □

This starkly contrasts our conclusion in section 4.7.1 for the fermionic case, where $\mathrm{S}\mathcal{W}^6(\mathrm{pt})$ was trivial. The source of the difference lies in the fact that the magnetic membrane lives in the bosonic world. If we were to "fermionize" all of the bosonic theories, i.e. couple to spin structure, then the infinite rank group would trivialize.

To gain a more physical intuition for these ungappable and chiral bosonic objects we comment on their construction in a manner similar to [94], used for SPTs. The main takeaway for SPTs is that when constructing an SPT, we can place lower-dimensional invertible phases along homology cycle representatives dual to Stiefel-Whitney classes. This is what was done for the dual of the generalized double semion model in 5d to show that it is equivalent to a twisted Dijkgraaf-Witten dual stacked with lower dimensional SPT phases.

This takeaway leads to a construction of the chiral 5d phases gauranteed by Theorem 4.8.1. Pick a spin-MTC $\mathcal{C}$ representing an order-2 class in $\mathrm{S}\mathcal{W}_2$ that is in the kernel of the d_3 differential. We place the 3d topological order built from $\mathcal{C}$ along a representative of w_2, by this we mean we place $\mathcal{C}$ along the homology cycle that is dual to w_2 (and away from w_2 we can just flood the phase with the vacuum). The choice of representative for w_2 should not change the theory, the reason for this culminates from the fact that $\mathcal{C}^2$ is super-Witt trivial

[17] In particular the spin MTC $\mathrm{SO}(2n+1)_{2n+1}, n \geq 1$, are pairwise Morita inequivalent.

and furthermore $\mathcal{C}$ is in the kernel of d_3. The fact $\mathcal{C}$ is order 2 has to do with protecting our theory under changes of representatives by a $\mathbb{Z}_2^F[1]$-symmetry. Being in the kernel of d_3 is telling us that changes of triangulation that might lead to higher order anomalies do not show up.

To see why any 4d boundary theory can not be gapped, note that a representative of w_2 in the bulk will end along a representative of w_2 on the boundary. But $\mathcal{C}$ is nontrivial, so that representative of w_2 on the 4d boundary will necessarily carry a 2d chiral theory, namely a chiral edge mode for $\mathcal{C}$. For instance, suppose $\mathcal{C}$ is $\mathrm{SO}(2n+1)_{2n+1}$, or some product thereof that is within the kernel of d_3. Then the 4d boundary condition will see chiral WZW modes supported on a representative of w_2.

Chapter 5

Noninvertible Symmetries and Fusion 2-Categories

The first part of this chapter is based off [251], which works with noninvertible symmetries at the level of 1-categories, in $(2{+}1)d$.

The second part of this chapter is based off [78] which works with noninvertible symmetries in 2-categories.

5.1. Introduction

The study of topological operators in quantum field theories has given many insights into the nature of what a full quantum field theory consists of. The topological operators provide a vast simplification from the space of all possible operators that a theory may possess, and the formalism to understand them is through topological quantum field theories (TQFTs). A particularly useful feature of TQFTs is their ability to describe, and in some cases classify, the infrared phases of gauge theories and gapped phases of matter. Among the classification of topological phases are those phases which are nontrivially ordered, also known as "long range entangled" phases or *topological orders*. The topological properties of the phase are independent of spacetime or internal symmetries, and only depend on the global structure of the manifold that the phase lives on. In such long range entangled phases in $(n{+}1)$-dimensions there exists extended topological operators, with the structure of an n-category, the classifications for low values of n have been given in [242, 179, 181, 177, 149].

In this work we restrict to topological theories in three spacetime dimensions, with a focus on the line operators that are the *anyons*. The classification of topological orders in three spacetime dimensions is given by modular tensor categories (MTCs) [1]; for the purposes of this paper, we will represent these MTCs by 3d Chern-Simons theories, where the details about the framing of our underlying three-manifold is unimportant. Given the spectrum of line operators, one can perform *anyon condensation*, which is an action in three dimensions that also goes by the name of "gauging a one-form symmetry", or more generally "gauging a categorical symmetry". When an anyon generates a one-form symmetry, it has abelian fusion rules, as higher-form symmetries are always abelian groups [113]. The anyon is deemed an *abelian anyon* and the action of condensing abelian anyons is well studied in the literature [138, 13, 31, 179, 190].

When the anyon has nonabelian fusion rules, i.e. a *nonabelian anyon*, we must shift to a categorical point of view to understand condensation [109, 48]. In the categorical framework we see the anyon, or set of anyons that condense, as being part of an algebra object. More specifically, a *special Frobenius algebra* in the category $\mathcal{C}$. From here on out, $\mathcal{C}$ denotes the uncondensed theory we start with, or in condensed matter parlance the "parent theory". Condensing the algebra leads to the "child" theory $\mathcal{D}$, where some of the lines in the parent have been projected out, or confined on an interface that arises in the process of going from parent to child. In the case where the child theory is the vacuum, the interface that separates $\mathcal{C}$ and the vacuum is deemed to be a *gapped boundary* of $\mathcal{C}$. In order to go from $\mathcal{C}$ to the vacuum one condenses a *Lagrangian* algebra object $\mathcal{A}_\ell$, where $(\dim \mathcal{A}_\ell)^2 = \dim \mathcal{C} = \sum_{\lambda\in\mathcal{C}}(\dim\lambda)^2$, where the sum ranges over all lines in $\mathcal{C}$ and we use dimension to mean quantum dimension. In the literature, the use of the phrase "anyon condensation" is at times used to apply solely to those integer spin, i.e. bosonic anyons, which give a Lagrangian algebra, and condense $\mathcal{C}$ to the vacuum [169]. For Lagrangian algebras, it is a theorem that

Theorem 5.1.1. [61] For $\mathcal{F}$ a fusion category and $\mathcal{C} = \mathcal{Z}(\mathcal{F})$. There is a bijection between the sets of Lagrangian algebras in $\mathcal{C}$ and indecomposable $\mathcal{F}$-module categories.

The role of the fusion category in the above theorem is played by the lines on the interface, that we denote as $\mathcal{F}$, separating $\mathcal{C}$ and $\mathcal{D}$. While the procedure for determining the lines of the child theory when gauging a one-form symmetry is clear, there are few examples in the literature that perform nonabelian condensation at the level of the spectrum of lines for an MTC. We set out to outline an algorithm for performing nonabelian condensation, i.e. determining the modules of the condensation algebra in an efficient way, and perform many nontrivial examples of determining not only the spectrum of lines in the child theory but also their quantum dimensions.

[1] Modular tensor categories technically only classify topological orders up to an invertible phase

For our purposes, we will weaken the notion of condensation only being applicable for Lagrangian algebras and apply the condensation procedure, which involves finding modules of algebra objects, to a variety of algebras. The reason for doing this is because the condensation procedure has uses that go beyond just looking for gapped boundaries, and one of our goals is to provide examples that emphasize the other merits. It is natural to expect that not all anyons in $\mathcal{C}$ can be condensed because some do not correspond to an algebra. Using our algorithm we will give examples of how to decide if an algebra is condensable. With the tools for nonabelian condensation developed, we can apply them to verify conformal embeddings given in [61], and also to other cases where one might ask if two MTCs are Morita equivalent. This gives us a way to construct the interface, i.e. bimodules, between the two theories. Moreover we can use nonabelian condensation to understand the decomposition of characters in 2d topological cosets, which have been useful in describing the IR phases in [82].

In many instances taking all the bosons anyons and condensing them out may cause lines to split, but in such a way that preserves the quantum dimension. As a first step in generalizing beyond bosonic condensation, we look at fermion condensation where by fermion we mean a line with half integer spin. On the other hand, we will use the term *local fermion* to describe the condensed line. As we will see, a nonabelian fermion may be split into one that is abelian, and we can furthermore sequentially condense out the abelian part. We will investigate how this relates to the (super)modular invariants of the parent theory, and see what further insights the condensation algebra can give regarding modular invariants.

Along the way we will enlarge the notion of which anyons can be condensed, beyond bosons and fermions to a general spin $1/n$ object, if we also couple to an appropriate background n-structure [37]. We also observe that not all modular invariants correspond to gapped interfaces, like those that arise from Lagrangian algebras, as noted in [163, 58]. One way this fails to be true is that there are "charge conjugation" modular invariants that reflect some symmetry of the parent theory. Furthermore, an algebra that is at least symmetric Frobenius will result in a modular invariant, however, these need not be Lagrangian and therefore the modular invariant is not a truly gapped interface. We further supplement the analysis given in the references with more explicit examples of exotic idempotent modular invariants, and relationships between the modular invariants and condensation algebras. In the same manner as for supermodular invariants, we look to the higher modular invariants corresponding to condensing a $1/n$ line to support our claim that these lines can be condensed.

With a comprehensive understanding of gauging, we next aim to understand how to construct the center of the fusion category on the wall that separates $\mathcal{C}$ and $\mathcal{D}$, i.e. reconstructing $\mathcal{C}$ to some degree by ungauging the algebra used to reach the child theory. In particular we want to start off with information about the "wall category", this consists of the lines that can be confined on the wall. These are the lines that are projected out in going from $\mathcal{C} \to \mathcal{D}$, as well as the lines of the child theory $\mathcal{D}$. We will slightly abuse notation and call this fusion category $\mathcal{F}$ (this is a surface defect, but contains two kinds of lines); note that the lines which are totally confined cannot lift to the child $\mathcal{D}$, so there is no braided structure on the 2d surface that separates the two phases. The lines in $\mathcal{D}$ however can be moved to the surface $\mathcal{F}$ via a functor, and is the reason for our abuse of notation. We use the consistency relations mentioned in [107], and others which we elaborate on, to show in some nontrivial cases that the data of the S-matrix elements of $\mathcal{C}$ can be constructed. The data we start out with involves the S-matrix elements of the lines in $\mathcal{D}$, as well as fusion information of the wall category. Constructing $\mathcal{C}$ is not a very methodical process and there is no known procedure that exists in general. We gain an intuition from the examples in this work on how much information we can reasonably extract, given our initial data.

The layout of the paper is as follows, in §5.2 we give a mathematical formalism associated to gauging a categorical symmetry in terms of condensation algebras. We follow up by giving explicit examples of how to compute using this formalism by applying it to 3d Chern-Simons theories, and finding the lines of the child theory. More nontrivial examples of gauging are given in appendix A.2. In §5.3 we look at modular invariants and see how in some cases we can identify which algebra objects of the parent theory can lead to a modular invariant. We also introduce supermodular invariants and remark on their feature, as well as discuss generalizations to higher modular invariants that are motivated by the spin of the anyon one can condense. In §5.4 we give the consistency relations involving the lines on the wall category and see how to determine S-matrix elements of the parent theory. We also explain the information that we will provide regarding the fusion category, to be able to determine its center. We will put the consistency relations to use in a couple of examples namely in reconstructing the Toric code from the vacuum and $\mathrm{SU}(3)_3$ from $\mathrm{Spin}(8)_1$. In appendix A.3 we do a nontrivial example with reconstructing the S-matrix of $\mathrm{SU}(2)_{10}$ from $\mathrm{Spin}(5)_1$.

5.2. Overview of Gauging

We will perform condensation via a method of introducing *idempotents*. The formalism developed using idempotents and condensation monads in precisely what is needed to do nonabelian condensation, and it furthermore generalizes to higher categories [109] [2]. With this rigorous framework in place, the well known notions of anyon condensation in 3d, or simple current extensions in 2d VOAs, can be encapsulated in a common language that generalizes to higher dimensions. To better interpret the mathematical formalism we restrict out attention from general n-categories to modular tensor categories, and in particular 3d Chern-Simons. Already here, many of the properties that generalize to n-categories are manifest, and computationally tractable. We will give some examples of performing a familiar task of condensing abelian anyons by this method, while also shedding light on some of the subtleties that traditional methods miss. Having some familiarity with the steps involved in the procedure will be crucial when we generalize to the nonabelian story.

We first review the properties of idempotents, working just with a linear monoidal 1-category $\mathcal{C}$. For an object $X \in \mathcal{C}$ (we will later use $\mathcal{C}$ as our parent MTC, and X as our anyons) an idempotent is an endomorphism $\varphi : X \to X$ such that $\varphi \circ \varphi = \varphi$. For the purpose of this paper, the categories we will consider are all *idempotent complete*. This means that we can write φ using a pair of morphisms $f : X \to Y$ and $g : Y \to X$ as $\varphi = g \circ f$ so that Y is a direct summand of X and is the image of φ. We will also work in a finite setting, so that any decomposition into direct sums, is a finite decomposition into simple objects. Such finiteness conditions are a key feature associated with "topological settings" and generalize to higher categories where the finiteness properties are captured by the axioms of a multifusion category [3].

The idempotent φ will also be referred to as a condensation algebra in $\mathcal{C}$, and to perform a condensation, we first must select a finite set of lines to build this semisimple object. The condensation algebra consists of the data $\varphi \in \mathcal{C}$ as well as a multiplication map $\varphi \times \varphi \to \varphi$ and a co-multiplication map $\varphi \to \varphi \times \varphi$, and a set of axioms given in figure 5.1 and figure 5.2 where the line with an arrow denotes φ.

It is known that these axioms for φ make it into a nonunital special Frobenius algebra. Condensing this algebra means to flood spacetime with a fine network of lines corresponding to the algebra, and satisfying the axioms of associative (co)multiplication, and composition of comultiplication and multiplication. The importance of the axioms is to insure that the choice of which network to flood spacetime with is immaterial, i.e. works for any cellulation of spacetime. With this we can assemble a topological interface, which is two-dimensional interface that is populated by the one dimensional algebra. Since we were able to build this higher dimensional object from the lower dimensional lines in $\mathcal{C}$, this interface will be called

[2] For a discussion specified to 2-categories see [85], where the notion of condensation is referred to as "separable adjunction".

[3] As an example in lower categories, if one is working in representation theory, the finiteness conditions we consider boil down to the axioms when working with a semisimple finite dimensional algebra.

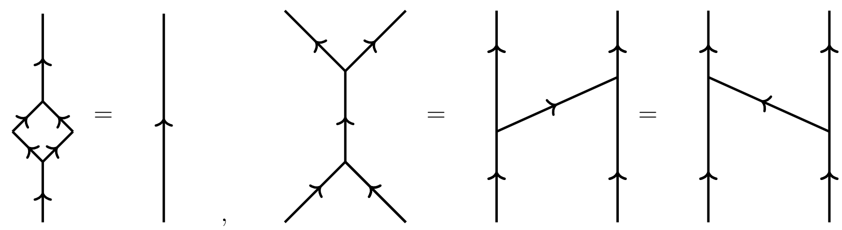


Figure 5.1: The diagram on the left is the axiom that multiplication and comultiplication can be composed into φ, i.e. all bubbles can be closed. The diagram on the right shows that the composition of comultiplcation and multiplication can be decomposed as a composition of ($\mathrm{id}_\varphi\times$multiplcation) and (comultiplcation$\times\mathrm{id}_\varphi$) or (multiplication$\times\mathrm{id}_\varphi$) and ($\mathrm{id}_\varphi\times$comultiplication)

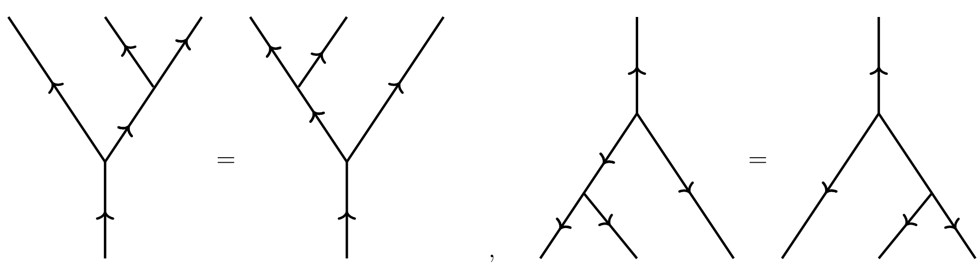


Figure 5.2: The left diagram shows that multiplication is associatve. The right diagram shows that comultiplication is coassociative

a *condensation descendant* to reflect its fundamental structure, and this notion can be used to classify topological orders as in [149, 145]. It is a fact that in 3d Chern-Simons theory, all the surfaces are built out of lines and thus are descendants of the condensation algebra [36]. Two interfaces are isomorphic if the two condensation algebras are Morita equivalent. This fact about interfaces will play a role later in our discussion of modular invariants for $\mathcal{C}$. The term anyon condensation is sometimes used in the literature to refer to the case when we condense an φ with the lines that comprise φ actually forming a Lagrangian algebra in $\mathcal{C}$ [169, 141].

Physically speaking, condensing out a Lagrangian algebra creates a gapped boundary for $\mathcal{C}$ [155, 180]. We will refer to anyon condensation in a looser manner that can be done for any "reasonable" condensation object, and not necessarily a Lagrangian algebra. In addition, we will consider condensing out anyons that are not only bosons, but have spin $1/n$ for $n \geq 2$. By enlarging the definition we will be able to employ our algorithm for anyon condensation to $\mathcal{C}$ that do not have Lagrangian algebras and gain insight into modular invariants that do not correspond to gapped boundaries, as well as constructing the lines of the child theory. It will also highlight how our computational methods naturally generalize.

While the physical interpretation of condensation corresponding to filling a submanifold with a network of lines is inspiring, we still have yet to fill in the technical details of computing the new spectrum using the condensation algebra and the data of the lines in $\mathcal{C}$. Let us explain by considering the spectrum of a G_k Chern-Simons theory which is given by all the integral representations at level k. Such representations are labeled by their highest weight λ. which can be expanded in a basis of fundamental weights as

$$\lambda = \sum_{i=0}^{r} \lambda_i \, \omega_i \,, \tag{5.2.1}$$

where $[\lambda_0, \lambda_1, \ldots, \lambda_r]$ are the Dynkin labels of λ, and $r \equiv \operatorname{rank} \mathfrak{g}$. The spectrum of G_k consists of all non-negative integer solutions to the equation

$$\lambda_0 + (\lambda, \theta) \equiv k \,, \tag{5.2.2}$$

where $(\cdot, \cdot)$ is the scalar product of $\mathfrak{g}$, and θ is the highest root vector. A line given by a representation λ has topological spin and quantum dimension given by

$$h_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2(k + h^\vee)} \,, \quad \text{q-dim}_\lambda = \prod_{\alpha \in \Delta_+} \frac{\sin\left(\frac{\pi(\lambda+\rho,\alpha)}{k+h^\vee}\right)}{\sin\left(\frac{\pi(\rho,\alpha)}{k+h^\vee}\right)} \,, \tag{5.2.3}$$

where Δ_+ denotes the positive roots of $\mathfrak{g}$, ρ is the Weyl vector, and $h^\vee$ is the dual Coxeter number. We will first focus on the case where the lines in the algebra have a grouplike fusion structure. This is known as gauging a one-form symmetry group. A well known method of gauging a one-form symmetry is to select the anyon generator a for the cyclic group, and compute the monodromy charge (induced by a) defined by

$$Q(\lambda) = h_\lambda + h_a - h_{\lambda \times a} \tag{5.2.4}$$

for all the other anyons λ. When the charge between the generator and a line is nontrivial mod 1, then that line is projected out of the spectrum and does not survive the gauging. Of the lines that remain, we break them up into orbits of the symmetry. While this procedure works for $\mathcal{C}$ with one-form symmetries, it does not generalize well to nonabelian anyons. Our understanding of why lines split is also obscured by computing monodromy charges, and in certain cases that we will see later on, projecting out lines based on their monodromy charge hides some of the subtleties of finding orbits when gauging a one-form symmetry, especially if the generator of the one-form symmetry is not bosonic. Furthermore, if we wanted to condense a general set of abelian anyons, this method becomes inefficient.

In order to formalize gauging one-form symmetries we consider a group homomorphism $\mu : G \to \mathcal{C}^\times$ from a finite group G to the set of invertible topological lines, denoted $\mathcal{C}^\times$, one can produce the norm element

$$N = \bigoplus_{g \in G} \mu(g) \ \in \mathcal{C}^\times, \tag{5.2.5}$$

which has the structure of a categorified idempotent. To see this structure, we first introduce the notion of a *fiber functor* $F : \mathbf{Vec}[G] \to \mathbf{Vec}$. The objects in the domain of F are G-graded vector spaces, and written as formal sums $\bigoplus_{g \in G} \mathrm{V}_g \cdot g$ where $\mathrm{V}_g \in \mathbf{Vec}$. The homomorphism F from the group algebra to the one-dimensional vector space is a choice of one dimensional representation for the group. There is also an adjoint of the fiber functor

$$F^* : \mathbf{Vec} \to \mathbf{Vec}[G], \text{ with } F^*(\mathbb{1}) = \bigoplus_{g \in G} g\,, \tag{5.2.6}$$

which is sensible since the one-dimensional vector in $\mathbf{Vec}$ is an algebra, the map F^* takes it to another algebra. The element $\bigoplus_{g \in G} g \in \mathbf{Vec}[G]$ is an idempotent whose image is $\mathbf{Vec}$, and the homomorphism μ is equivalent to giving a monoidal functor from $\mathbf{Vec}[G]$ to $\mathcal{C}$ and preserves idempotents, therefore $\mu\left(\bigoplus_{g \in G} g\right)$ is also idempotent.

What this fiber functor does at the level of lines is that it takes the algebra built out of lines and sends it to the vacuum. This makes manifesting the idempotent nature of the condensation algebra as products of the vacuum with itself again gives the vacuum. Furthermore, a physical way to view F^* in the realm of topological phases described by 3d Chern-Simons is that it builds a phase by starting from the vacuum $\nVdash$ and inserting the algebra of lines, similar in spirit to the construction of phases via the methods in [188].

As an example, take the object $\varphi = 0 + 1$ where 1 is a $\mathbb{Z}_2$ object, and we know an isomorphism $1 \times 1 \overset{F}{\simeq} 0$. In an attempt to make φ into an algebra, we need a map from

$$(0+1) \times (0+1) \xrightarrow{m} (0+1) \tag{5.2.7}$$

the only interesting data is the map from $1 \times 1 \to (0+1)$, as the other values from distribution take a canonical value. One can use some multiple of the isomorphism for $1 \times 1 \simeq 0$ to write

$$1 \times 1 \xrightarrow{(\lambda F, 0)} (0+1)\,, \quad \lambda \in \mathbb{C}\,, \tag{5.2.8}$$

for each of the components of φ. It appears that there are infinitely many unital multiplicative maps m one can use, but up to isomorphism there is only a single map.

The result of condensing the norm in (5.2.5), as per the prescription of flooding spacetime by the algebra, is the familiar notion of summing over G-bundles on spacetime, or the ways to insert G-flux to each wall of the cellulation of spacetime. After the condensation, we get a new phase which we denote as the child theory $\mathcal{D}$. In a 3d theory, the surface operators serve as interfaces between the vacuum $\nVdash$ and itself, and thus given by $\mathrm{End}_{\mathcal{C}}(\nVdash)$. Applying this same intuition to the line operators that are the actual objects of the 1-category $\mathcal{C}$ tells us that they also exist as endomorphisms. To take into account also the G-group action, we note that by the map μ, the lines of $\mathcal{C}$ are a G-module by right multiplication. We can form $\mathcal{C} \underset{\mathbf{Vec}[\mathrm{G}]}{\otimes} \nVdash$ i.e. by tensoring with the one-dimensional module, which identifies operators that have the same image under the fiber functor; the result is still a $\mathcal{C}$ module by left multiplication. We therefore see that the objects of $\mathcal{D}$ are given by

$$\mathrm{End}_{\mathcal{C}}\left(\mathcal{C} \underset{\mathbf{Vec}[\mathrm{G}]}{\otimes} \nVdash\right)\,, \tag{5.2.9}$$

where we are taking $\mathcal{C}$-linear endomorphisms. Formula (5.2.9) is equivalent to

$$\left(\mathcal{C} \underset{\mathbf{Vec}[\mathrm{G}]}{\otimes} \nVdash\right)^G\,, \tag{5.2.10}$$

which are the G-invariant operators in $\mathcal{C} \underset{\mathbf{Vec}[\mathrm{G}]}{\otimes} \nVdash$. The G-invariant operators are reasonable to consider because $\mathcal{C} \underset{\mathbf{Vec}[\mathrm{G}]}{\otimes} \nVdash$ itself still had a residual G-action.

We now tell an analogous story for condensing nonabelian anyons, which is sometimes known as gauging a categorical symmetry, as the fusion rules of nonabelian anyons do not exhibit a grouplike structure. We therefore replace G by a fusion category $\mathcal{G}$, which has an action by the topological lines of $\mathcal{C}$, and a monoidal fiber functor $\mathcal{F} : \mathcal{G} \to \mathbf{Vec}$. An idempotent in $\mathcal{G}$ takes the form of a sum of nonabelian anyons, and the fiber functor again identifies it with the vacuum. The operators after the condensation is formally given by

$$\left(\mathcal{C} \underset{\mathcal{G}}{\otimes} \mathbf{Vec}\right)^{\mathcal{G}} . \tag{5.2.11}$$

Suppose we had another fusion category $\mathcal{G}'$, with a map $\mathcal{G}' \to \mathbf{Vec}$, that is Morita equivalent to $\mathcal{G}$ given by the $\mathcal{G}$-linear endomorphisms of $\mathbf{Vec}$, i.e. $\mathcal{G}' = \mathrm{End}_{\mathcal{G}}(\mathbf{Vec})$. We can then consider $\mathcal{C}//\mathcal{G}//\mathcal{G}' = \mathcal{C}$, this gives the notion of "ungauging" the categorical symmetry and reconstructing $\mathcal{C}$. Ungauging is in practice difficult to do at the level of MTCs, and amounts to being as difficult as constructing the Drinfeld center of another fusion category [4]. We will study ungauging in more depth in a later section when we attempt to reconstruct the S-matrix of a parent theory, starting with a collection of data from the child theory.

5.2.1. Condensing Abelian Anyons

We now put the formalism into practice by consider some examples of condensating an abelian anyon, or equivalently gauging a one-form symmetry. We start with two elementary examples $\mathrm{SU}(3)_3$ and $\mathrm{SU}(4)_4$, where the generator of the one-form symmetry in the former is a boson, and the latter is a fermion [231, 1]. In the latter case, the child theory will contain a local fermion and we must couple to spin structure. The data of the spectrum for

[4]Note that in the case of gauging a regular symmetry, ungauging amounts to gauging the "dual symmetry". Since the notion of a dual symmetry does not exist for categorical symmetries, then the analogue of ungauging becomes a hard problem of reconstructing the parent theory in the bulk.

$SU(3)_3$ is given by the integer solutions to $\lambda_0 + \lambda_1 + \lambda_2 \equiv 3$. Thus we have the lines

$SU(3)_3$	λ	h	q-dim
0	$[0,0,3]$	0	1
1	$[0,3,0]$	1	1
2	$[3,0,0]$	1	1
3	$[0,1,2]$	$2/9$	2
4	$[1,2,0]$	$8/9$	2
5	$[2,0,1]$	$5/9$	2
6	$[0,2,1]$	$5/9$	2
7	$[2,1,0]$	$8/9$	2
8	$[1,0,2]$	$2/9$	2
9	$[1,1,1]$	$1/2$	3,

where the first column assigns a number to label each of the Dynkin labels, the third column gives the spins, and the final column gives the quantum dimension. The notation we adopt for naming the lines is the same as that used in the KAC program [20]. As directed by (5.2.5) we form the idempotent $\varphi = 0 + 1 + 2$; by applying the fiber functor we identify this as the new vacuum. Now we use (5.2.10) to compute the operator content of the gauged theory. There is a monoidal functor that moves a line $\ell \in \mathcal{C}$ to the surface formed out of a network φ by multiplying ℓ with the newly condensed vacuum, i.e. $\varphi \times \ell$. Physically, what this functor does is to take a line in the bulk and zoom out so that the line is very close to the surface. Everything in this setting is topological except for the distance from the line to the surface. This is the same as finding the modules of φ, given by:

$$\begin{aligned}
\varphi \times 0 &= \varphi\,,\\
\varphi \times 1 &= 1 + 2 + 0\,,\\
\varphi \times 2 &= 2 + 0 + 1\,,\\
\varphi \times 3 &= 3 + 4 + 5\,,\\
\varphi \times 4 &= 4 + 5 + 3\,,\\
\varphi \times 5 &= 5 + 3 + 4\,,\\
\varphi \times 6 &= 6 + 7 + 8\,,\\
\varphi \times 7 &= 7 + 8 + 6\,,\\
\varphi \times 8 &= 8 + 6 + 7\,,\\
\varphi \times 9 &= 9_1 + 9_2 + 9_3\,,
\end{aligned} \tag{5.2.12}$$

where we have used the fusion rules for the lines in $\mathrm{SU}(3)_3$. Since we do not write down all the elements $m \in \mathcal{C}$ such that there is a map $\varphi \times m \to m$, what we mean here and for the rest of the paper by the "modules of φ" is actually the free modules $m = \varphi \times \ell$ for some $\ell \in \mathcal{C}$. By "free", we mean that the map $\varphi \times m \to m$ is multiplication in φ. The free modules generate the category of all modules, in particular if φ is separable, then the category of φ-modules is semisimple, and every module is a direct sum of simple summands of free modules. Therefore, writing down (5.2.12) is sufficient information to be able to tell what are all the simple summands of $\varphi \times \ell$.

Not all of the lines define a different representation of $\mathrm{SU}(3)_3/\mathbb{Z}_3$. When we mod out by the group $\mathbb{Z}_3$, two lines in SU(3) may be indistinguishable in the child theory because any set of lines which differ by a gauge transformation, should be identified. For this example where the lines that condense are bosons, lines which differ by a gauge transformation, but have different spins (mod 1) should not be identified because one could still tell them apart via the individual spins. Said more precisely, the lines are grouped into orbits, and all the lines in a given orbit have the same spin and quantum dimension, as is expected from lines that are indistinguishable. Note that the fusion of $\varphi \times 9$ involves three copies of 9. In this case 9 is said to fit into a *short orbit* because it is fixed by some elements of φ. We therefore "split" the line 9 giving a degeneracy index, up to the order of the stabilizer of 9 in φ, with the constraint that the sum of the quantum dimensions or the split lines is conserved.

In terms of the free modules, the set of module maps

$$\hom_\varphi(\varphi \times \ell,\, \varphi \times k) = \hom(\ell,\, \varphi \times k)\,, \quad k \in \mathcal{C}. \tag{5.2.13}$$

This allows us to answer the question of which simple summands of ℓ in $\varphi \times \ell$ match which simple summands of $\varphi \times k$. In the case of $\ell = 9$ and $k = 9$, then we have

$$\hom_\varphi(\varphi \times 9, \varphi \times 9) = \hom(9, \varphi \times 9)\,, \tag{5.2.14}$$

where the copies of 9 in $\varphi \times 9$ index the simple summands of $\varphi \times 9$.

We have that the semisimple objects (or orbits) are

$$\{\varphi,\, (3+4+5),\, (6+7+8),\, 9_1,\, 9_2,\, 9_3\}\,, \tag{5.2.15}$$

but there is still the task to take the G-invariant operators. Thus, $(3+4+5)$ and $(6+7+8)$ are projected out, and lines that are degenerate are never grouped into the same semisimple object. Thus lines of the gauged theory are

$$\{\varphi,\, 9_1,\, 9_2,\, 9_3\}\,, \tag{5.2.16}$$

and they correspond to the lines of $\mathrm{Spin}(8)_1$.

We end this example by noting that there exists a conformal embedding $\mathrm{SU}(3)_3 \subset \mathrm{Spin}(8)_1$ at the level of affine Lie algebras. At the level of 3d Chern Simons, the subalgebra plays the role of the parent theory, and the lines of the child $\mathrm{Spin}(8)_1$ are direct sums of parent theory lines. The natural way to see this is to treat the 3d MTC as $\mathbf{Rep}(V)$ and $\mathbf{Rep}(W)$, for $W \subset V$ as 2d VOAs and V a W-module. One might also want to make an analogy to the 2d GKO coset picture for $\frac{\mathrm{Spin}(8)_1}{\mathrm{SU}(3)_3}$, where the characters of $\mathrm{Spin}(8)_1$ decompose as sums of characters of $\mathrm{SU}(3)_3$ by the formula

$$\chi_\lambda(q) = \sum_\Lambda b_\lambda^\Lambda(q)\chi_\Lambda(q)\,, \quad \lambda = s, v, c \in \mathrm{Spin}(8)_1, \quad \Lambda \in \mathrm{SU}(3)_3\,. \tag{5.2.17}$$

Since the coset is topological, the q-expansion of the branching function $b_\lambda^\Lambda(q)$ is finite, and in particular

$$\chi_s = \chi_v = \chi_c = \chi_{[1,1,1]}\,, \tag{5.2.18}$$

which gives us a check that the three characters χ_s, χ_v, χ_c corresponding to the three spinors of Spin(8) correspond to the line 9 which split into three copies. The triality symmetry also shows up in the fact that anyon condensation cannot tell apart which of the 9_i should be the two spinors or the vector.

We now move onto $\mathrm{SU}(4)_4$ with the main goal to point out some of the subtleties when the generator is a fermion. We also use this opportunity to introduce the notion of *sequential condensation*, which will be important when we move onto nonabelian condensation. The data of the spectrum for $\mathrm{SU}(4)_4$ is given by the integer solutions to $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 \equiv 4$.

Thus we have 35 lines [5]:

$SU(4)_4$	λ	h	q-dim
0	$[0,0,0,4]$	0	1
1	$[0,0,4,0]$	$3/2$	1
2	$[0,4,0,0]$	2	1
3	$[4,0,0,0]$	$3/2$	1
4	$[0,0,1,3]$	$15/64$	2.613125929753
5	$[0,1,3,0]$	$95/64$	2.613125929753
6	$[1,3,0,0]$	$111/64$	2.613125929753
$\vdots$			
34	$[1,1,1,1]$	$15/16$	9.656854249492 .

To gauge the one-form $\mathbb{Z}_4$ symmetry generated by line 3, we proceed with a two step process. We first condense out the abelian boson which is line 2 by forming $\varphi = 0+2$ and performing the procedure in (5.2.12). The unconfined lines in the following table are listed in the first column, with their constituent $SU(4)_4$ lines in the second column:

$SU(4)_4 \xrightarrow{\varphi}$	$SU(4)_4$	h	q-dim
0	0	0	1
1	$1+3$	$3/2$	1
2	$8+10$	$9/16$	3.414213562373
3	$9+11$	$9/16$	3.414213562373
4	$16+18$	$5/16$	3.414213562373
5	$17+19$	$21/16$	3.414213562373
6	$24+26$	1	5.828427124746
7	$25+27$	$1/2$	5.828427124746
8	28_1	$3/4$	2.414213562373
9	28_2	$3/4$	2.414213562373
10	29_1	$5/4$	2.414213562373
11	29_2	$5/4$	2.414213562373
12	34_1	$15/16$	4.828427124746
13	34_2	$15/16$	4.828427124746 .

We are left with an abelian spin 1/2 line, which is also condensible. The caveat to the use of the fiber functor F, is that now F passes onto a super fiber functor $F : \mathbf{Vec}[G] \to \mathbf{SVec}$

[5] See KAC for the full spectrum. The program also has the ability to produce the spectrum after condensing an abelian boson.

[79]. Physically, this makes the line into a local fermion and also requires the child theory to couple to spin structure. By forming the condensation algebra $\widetilde{\varphi} = 0 + 1$ in the table for $\mathrm{SU}(4)_4 \xrightarrow{\varphi}$ we find that

$$\begin{aligned}
&\widetilde{\varphi} \times 0 = \widetilde{\varphi}\,, && \widetilde{\varphi} \times 7 = 7 + 6\,,\\
&\widetilde{\varphi} \times 1 = \widetilde{\varphi}\,, && \widetilde{\varphi} \times 8 = 8 + 10\,,\\
&\widetilde{\varphi} \times 2 = 2 + 3\,, && \widetilde{\varphi} \times 9 = 9 + 11\,,\\
&\widetilde{\varphi} \times 3 = 3 + 2\,, && \widetilde{\varphi} \times 10 = 10 + 8\,,\\
&\widetilde{\varphi} \times 4 = 4 + 5\,, && \widetilde{\varphi} \times 11 = 11 + 9\,,\\
&\widetilde{\varphi} \times 5 = 5 + 4\,, && \widetilde{\varphi} \times 12 = 12_1 + 12_2\,,\\
&\widetilde{\varphi} \times 6 = 6 + 7\,, && \widetilde{\varphi} \times 13 = 13_1 + 13_2\,,
\end{aligned} \tag{5.2.19}$$

with the lines that are unconfined

$$\begin{array}{c|c}
\ell & \text{q-dim} \\
\hline
\varphi & 1 \\
(6+7) & 5.828427124746 \\
(8+10) & 2.414213562373 \\
(9+11) & 2.414213562373\,.
\end{array} \tag{5.2.20}$$

In terms of the Dynkin indices of $\mathrm{SU}(4)_4$ the lines above read

$$\begin{array}{c|c}
\ell & h \\
\hline
\varphi & 1 \\
6 = ([1,0,1,2] + [1,2,1,0]) & 1 \\
7 = ([0,1,2,1] + [2,1,0,1]) & 1/2 \\
8 = 9 = [0,2,0,2] & 3/4 \\
10 = 11 = [2,0,2,0] & 1/4\,.
\end{array} \tag{5.2.21}$$

After condensing the fermion, the algebra gives a natural grouping where lines with spins that differ by 1/2 are identified. The semisimple objects now have simple components which differ by 1/2, i.e. equivalence up to a fermion.

When we were only focused on bosonic condensation, then the lines of any child theory must have constituent objects that are all of equivalent spin mod 1 in the parent, in order to be in the unconfined sector. A subtlety to mention here is that in doing identifications up to spin 1/2 lines, the lines now do not have a definite spin. One way to understand this is that the algebra which includes a fermion is only associative and not commutative, and thus loses the braided structure that condensation algebras with bosons would have. This forces the algebra to only be able to fill in two-dimensions as shown in figure 5.3.

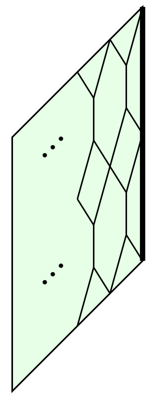


Figure 5.3: The physical picture of condensation looks like inserting a fine mesh of the algebra that takes the form of a surface when zoomed out. The dark line at the boundary represents a module for the algebra.

More precisely, an associative multiplication that takes place in one space dimension, when given to a one dimensional particle worldline in the time direction, grants a way for the line to fill in two-dimensions. Taking φ with its associative multiplication is a two-dimensional surface and the modules for the algebra look like a boundary condition whereas a bimodule is an interface on the surface. It is therefore also natural to view gauging an associative algebra as gauging a 2d surface operator that implements a zero-form global symmetry. We elaborate more explicitly on this point in §5.3.2. If in addition the algebra also had a braiding, then there are two directions for multiplication, and the algebra can fill in three-dimensions. In the new phase given by flooding with the commutative algebra, one can reasonably ask about the spins of the lines. But without the knowledge of how to flood 3d space, then it is not sensible to talk about spins of modules or bimodules.

One could also perform the two step condensation in one step, by choosing the algebra $\varphi = 0+1+2+3$ in $\mathrm{SU}(4)_4$, which generates the full $\mathbb{Z}_4$ symmetry. This algebra consists of two lines that are spin 0 and two that are spin $\frac{1}{2}$ mod 1, so this is regarded as a fermion condensation. One can check that the unconfined lines for this algebra are

ℓ	q-dim
$\varphi = 0+1+2+3$	1
$(24+25+26+27)$	5.828427124746
(28_1+29_1)	2.414213562373
(28_2+29_2)	2.414213562373

(5.2.22)

which matches the data in equation (5.2.20), upon matching the labels for lines. It is important to note that while fusing φ with line 28 (and 29) technically gives four lines $(28_1 + 29_1 + 28_2 + 29_2)$, the largest grouping we could have is $(28_1 + 29_1)$ and $(28_2 + 29_2)$ because the same line can not be grouped with itself. We end this example by noting that there are nonabelian bosons in the spectrum. By condensing those boson out, using the details in the next section, we find the embedding $\mathrm{SU}(4)_4 \subset \mathrm{Spin}(15)_1$. The lines of $\mathrm{Spin}(15)_1$ in terms of the dynkin labels of $\mathrm{SU}(4)_4$ are given by

$$\begin{aligned}
0 &= [0,0,0,4] + [0,4,0,0] + [0,1,2,1] + [2,1,0,1]\,, \\
1 &= [0,0,4,0] + [4,0,0,0] + [1,2,1,0] + [1,0,1,2]\,, \\
2 &= 2[1,1,1,1]\,.
\end{aligned} \tag{5.2.23}$$

Since the spectrum is large, another way to arrive at the same result is from the coset perspective. This is by considering $\frac{\mathrm{Spin}(15)_1}{\mathrm{SU}(4)_4}$, which is topological in the sense that the central charge of the numerator matches that of the denominator. The three characters of $\mathrm{Spin}(15)_1$ exactly decompose into the characters of $\mathrm{SU}(4)_4$ with the labels on the right hand side of the equality in the above equations.

The one-form generators need not be bosonic nor fermionic, as was the case in the last two examples. The one-form generator could have a more general rational value for its spin. Just like how we moved from integer spin lines to half integer spin lines we introduced a $\mathbb{Z}_2$ grading by enlarging the fiber functor to map to supervector spaces, a general $\frac{1}{n}$ anyon when condensed would lead to a $\mathbb{Z}_n$ graded vector space. This might be at odds physically with what is natural, due to the fact that one demands a *Hilbert pairing* in a physical Hilbert space. This is a pairing with no null vectors i.e. $\langle x|x\rangle > 0$ for $x \neq 0$ in the Hilbert space. Applying the Hilbert pairing to a vector purely in the i-th graded piece of the Hilbert space pairs it with another vector in the i-th graded piece and returns a real number. However, tensoring two purely i-th graded vectors should give a vector in the $2i$-th graded piece. Therefore, introducing a Hilbert pairing would be an unnatural morphism in our category of $\mathbb{Z}_n$-graded vector spaces. Nevertheless, one can still make use of (5.2.10) for a condensation algebra that includes the one-form generator, and perform condensation as purely an algebraic manipulation. Sequential condensation can also be generalized this way, to include a boson and a spin $1/n$ anyon with the resulting object having simple components with spins that differ by $1/n$. We will show an example with

$SU(2)_4$ here; the spectrum for this theory consists of 5 lines given by

$SU(2)_4$	λ	h	q-dim
0	$[0,4]$	0	1
1	$[4,0]$	1	1
2	$[1,3]$	1/8	1.732050807569
3	$[3,1]$	5/8	1.732050807569
4	$[2,2]$	1/3	2 .

Condensing the abelian boson splits the spin 1/3 line into two copies. Similar to how we can pass to a super fiber functor, we now let $F : \mathbf{Vec}[G] \to \mathbf{r\text{-}Vec}$ which sends $\varphi = 0 + 4_1 + 4_2$ to the new vacuum, while coupling to a r-spin structure. Two other examples where a similar effect takes place is $Sp(8)_1$ and $Spin(7)_2$.

5.2.2. Condensing Nonabelian Anyons

The formalism for finding the operators after gauging a categorical symmetry "generated" by a nonabelian anyon bears resemblance to the case of a regular symmetry, however due to the potentially complicated fusion structure of the MTCs, the nonabelian condensation can have complicated modules to work out. We will present an algorithm that is useful in practice to find the lines of the child theory. While this algorithm in principle works for any number of lines, the process quickly becomes complicated when the number of lines is large, the condensation algebra involves multiple lines, or when the fusion of nonabelian lines decomposes into many simple objects. The difficulty in performing the computation comes from assigning the proper quantum dimensions to each of the child lines, and grouping the lines from the parent that are equivalent under the fiber functor as in (5.2.11). We believe the best way to proceed is through examples. We begin with a well known and considerably elementary example of condensing the nonabelian boson in $SU(2)_{10}$. In Appendix A.2 we give more nontrivial examples of performing nonabelian condensation by using this algorithm.

We align with the notation commonly used in the anyon condensation literature for this example instead of using KAC's notation. The data of the spectrum of $SU(2)_{10}$ consists of

11 lines given by

$SU(2)_{10}$	λ	h	q-dim
0	$[0,10]$	0	1
1	$[1,9]$	$1/16$	1.931851652578
2	$[2,8]$	$1/6$	2.732050807569
3	$[3,7]$	$5/16$	3.346065214951
4	$[4,6]$	$1/2$	3.732050807569
5	$[5,5]$	$35/48$	3.863703305156
6	$[6,4]$	1	3.732050807569
7	$[7,3]$	$21/16$	3.346065214951
8	$[8,2]$	$5/3$	2.732050807569
9	$[9,1]$	$33/16$	1.931851652578
10	$[10,0]$	$5/2$	1 .

the condensation algebra we take is $\varphi = 0 + 6$. Interestingly, the lowest-energy eigenspace of this anyon is the **7**-dimensional representation of SU(2). There is a well known "cross product" map $\mathbf{7} \otimes \mathbf{7} \to \mathbf{7}$, and correspondingly we get a multiplication map $6 \times 6 \to 6$. The condensation algebra above is therefore a version of the octonions. The modules are

$$\begin{aligned}
&\varphi \times 0 = \varphi && \varphi \times 6 = 6 + (0+2+4+6+8) \\
&\varphi \times 1 = 1 + (5+7)\,, && \varphi \times 7 = 7 + (1+3+5+7) \\
&\varphi \times 2 = 2 + (4+6+8)\,, && \varphi \times 8 = 8 + (2+4+6) \\
&\varphi \times 3 = 3 + (3+5+7+9)\,, && \varphi \times 9 = 9 + (3+5) \\
&\varphi \times 4 = 4 + (2+4+6+8+10)\,, && \varphi \times 10 = 10 + (4)\,. \\
&\varphi \times 5 = 5 + (1+3+5+7+9)\,, &&
\end{aligned} \tag{5.2.24}$$

We use parenthesis to denote the lines which came from fusing with 6 in φ. The lines that split in $SU(2)_{10}$ are the lines that appear multiple times when fused with the vacuum φ. The multiplicity dictates the number of copies the line splits up into, just as in the abelian case. Therefore we have

$$\begin{aligned}
&3 \to 3_1 + 3_2\,, && 6 \to 6_1 + 6_2 \\
&4 \to 4_1 + 4_2\,, && 7 \to 7_1 + 7_2\,. \\
&5 \to 5_1 + 5_2\,. &&
\end{aligned} \tag{5.2.25}$$

By using our knowledge that the quantum dimension should be conserved in the condensed phase, we work our way down the list of lines assigning a subscript label to the lines which split. Without loss of generality, we are free to assign the subscript so that the larger subscript values appear first in the list of lines, when reading right to left, in (5.2.24). As an example, we write the subscripts in (5.2.24) as

$$\begin{aligned}
&\varphi\times 0=\varphi && \varphi\times 6=6_1+(0+2+4_2+6_2+8_2)\\
&\varphi\times 1=1+(5_2+7_2)\,, && \varphi\times 7=7_1+(1+3_2+5_2+7_2)\\
&\varphi\times 2=2+(4_2+6_2+8)\,, && \varphi\times 8=8+(2+4_2+6_2)\\
&\varphi\times 3=3_1+(3_2+5_2+7_2+9)\,, && \varphi\times 9=9+(3_2+5_2)\\
&\varphi\times 4=4_1+(2+4_2+6_2+8+10)\,, && \varphi\times 10=10+(4_2)\,.\\
&\varphi\times 5=5_1+(1+3_2+5_2+7_2+9)\,, &&
\end{aligned} \tag{5.2.26}$$

Notice that while lines 5 and 7 both split, in our convention we only take 5_2 and 7_2 to be group, which is indicated by the parenthesis. A similar story goes for 4 and 6. Now we need to assign quantum dimensions to the lines the split and group together the lines that have the same quantum dimension. Since the line 1 does not split and itself has quantum dimension 1.93..., let us greedily assign this value to 5_2 and 7_2 because 1 appears with 5_2 and 7_2 frequently when we find the modules of φ. Then we form a grouping of lines $(1+5_2+7_2)$. Next, suppose we greedily assign the quantum dimension 2.73..., which is that of line 2 and 8, to both 4_2 and 6_2. Then we form the group $(2+4_2+6_2+8)$ of lines. We now consider $\varphi\times 3$, where we have the group (5_2+7_2) from earlier, and we can form the group (3_2+9) by assigning quantum dimension 1.93... to 3_2, which is the quantum dimension of 9. This leaves 1.41... for the quantum dimension of 3_1, by conservation. From $\varphi\times 1$ we learned that $(1+5_2+7_2)$ are condensed to the same line in the child theory, and we just learned that line 9 and 3_2 should also be condensed to the same group. We will keep these two lines separate, even though they share the same quantum dimension. We will subsequently see why we do not join them when we look at $\varphi\times 5$. For now, consider $\varphi\times 4$ which again contains $(2+4_2+6_2+8)$, something we already determined from $\varphi\times 2$ should be grouped, due to quantum dimension. This leaves 4_1 with q-dim 1, which is exactly the same quantum dimension as 10, so we condense them into the same line and have (4_1+10). From $\varphi\times 5$ we see that since 5_2 was assigned q-dim 1.93... then 5_1 also has q-dim 1.93... by the conservation of quantum dimension. However, since lines that spit should not be condensed into the same line, 5_1 gets condensed into (3_2+5_1+9) while 5_2 gets condensed into $(1+5_2+7_2)$. Because 5_1 and 5_2 have the same quantum dimension, we can exchange the two lines, so it is irrelevant whether we take 5_1 or 5_2 to be grouped with the former or the latter. We proceed to $\varphi\times 6$ and $\varphi\times 7$ and from here we learn that 6_1 should have q-dim 1, and 7_1 should have q-dim $1.41\ldots$. We will slightly abuse notation and denote the actual vacuum of the child theory as $\varphi = 0 + 6_1$, which makes sense as an abelian object coming from grouping 0 and 6_1, and can be given the properties of an idempotent. After the condensation we have the lines

$$\begin{array}{c|c} \ell & \text{q-dim} \\ \hline \varphi = 0 + 6_1 & 1 \\ (4_1 + 10) & 1 \\ (3_1 + 7_1) & 1.41421356237 \\ (1 + 5_2 + 7_2) & 1.931851652578 \\ (3_2 + 5_1 + 9) & 1.931851652578 \\ (2 + 4_2 + 6_2 + 8) & 2.732050807569\,. \end{array} \tag{5.2.27}$$

The final step is to project out the lines in which the spins from the parent theory do not agree. Therefore we only have

$$\{(0 + 6_1),\ (4_1 + 10),\ (3_1 + 7_1)\}$$

at the end of bosonic condensation, which correspond to the three lines in $\mathrm{Spin}(5)_1$. The nonabelian spin 1/2 line labeled 4 in $\mathrm{SU}(2)_{10}$ is now abelian after condensing the nonabelian boson, so we can further sequentially condense out $(4_1 + 10)$ and only be left with the vacuum line. It can be checked that the full algebra $\mathcal{A}_\ell = (0 + 6 + 4 + 10)$ in $\mathcal{C} = \mathrm{SU}(2)_{10}$ is a Lagrangian algebra object, and therefore condensing the algebra leads to a gapped interface [140]. Furthermore, since a fermion was condensed out the last step, the resulting theory couples in spin structure.

We will run through another example of using the algorithm with $(G_2)_3$. The spectrum consists of 6 lines given by

$(G_2)_3$	λ	h	q-dim
0	$[0,0,3]$	0	1
1	$[0,1,2]$	2/7	3.791287847478
2	$[0,2,1]$	2/3	5.791287847478
3	$[0,3,0]$	8/7	3.791287847478
4	$[1,0,1]$	4/7	3.791287847478
5	$[1,1,0]$	1	4.791287847478

We condense the algebra $\varphi = 0 + 5$ and see that in the modules the lines that repeat are 2 and 5, and splits into

$$2 \to 2_1 + 2_2 + 2_3\,, \quad 5 \to 5_1 + 5_2\,.$$

The lines with subscripts written using our previous prescription is listed on the right:

$$\begin{aligned}
&\varphi \times 0 = \varphi\,, && \varphi \times 0 = \varphi\,,\\
&\varphi \times 1 = 1 + (2+3+4+5)\,, && \varphi \times 1 = 1 + (2_3+3+4+5_2)\,,\\
&\varphi \times 2 = 2 + (1+2+2+3+4+5)\,, && \varphi \times 2 = 2_1 + (1+2_2+2_3+3+4+5_2)\,,\\
&\varphi \times 3 = 3 + (1+2+4+5)\,, && \varphi \times 3 = 3 + (1+2_3+4+5_2)\,,\\
&\varphi \times 4 = 4 + (1+2+3+5)\,, && \varphi \times 4 = 4 + (1+2_3+3+5_2)\,,\\
&\varphi \times 5 = 5 + (0+1+2+3+4+5)\,, && \varphi \times 5 = 5_1 + (0+1+2_3+3+4+5_2)\,.
\end{aligned} \tag{5.2.28}$$

We start with $\varphi \times 1$ and greedily assigning the quantum dimension of lines 1, 3, and 4 to 2_3 and 5_2; this gives us the group $(1 + 2_3 + 3 + 4 + 5_2)$. When we look at $\varphi \times 2$ we notice that some of the lines in parenthesis already appeared in $\varphi \times 1$, where we decided to group them together. We leave 2_1 and 2_2 separated and not grouped, due to the fact stated earlier that we do not group lines together which split from the same parent line. When we consider $\varphi \times 5$ there is 5_1 which we group with 0, since the q-dim is 1, and again we have $(1 + 2_3 + 3 + 4 + 5_2)$ reappearing. At the end of the condensation we have the lines

ℓ	q-dim
$\varphi = 0 + 5_1$	1
$(4_1 + 10)$	1
2_1	1
2_2	1
$(1 + 2_3 + 3 + 4 + 5_2)$	3.791287847478 ,

(5.2.29)

but we project out $(1 + 2_3 + 3 + 4 + 5_2)$ because the lines do not all have the same spin. We see that condensing the line 5 in the parent theory results in 5_1 being identified with the vacuum. Furthermore, the lines $2_1, 2_2$ have the right q-dim to both be abelian lines, which they must be or else one of them will have a quantum dimension that is less than 1.

One may wonder how to determine if our choice of condensation algebra is valid, in the sense that it will lead to a consistent child phase? In order for the child phase to be consistent, it must be true that the lines within the modules can be consistently assigned quantum dimension, while obeying the conservation requirement. In the process of constructing the modules of an algebra, if the quantum dimension for a line that has been split is reduced to a value that is smaller than the smallest number on the list of q-dim from the original spectrum, yet still not abelian, then our algorithm can rule out the condensation algebra. We stress that to generalize the notion of "condensability", a canonical way of being able to assign quantum dimensions is key.

As a tractable example consider $(G_2)_2$ which has a simple spectrum given by

$(G_2)_2$	λ	h	q-dim
0	$[0,0,2]$	0	1
1	$[0,1,1]$	1/3	2.879385241572
2	$[0,2,0]$	7/9	2.532088886238
3	$[1,0,0]$	2/3	1.879385241572

We can consider three algebras $\varphi_1 = 0+1$, $\varphi_2 = 0+2$, and $\varphi_3 = 0+3$. The three modules are given by

$$\begin{aligned}
&\varphi_1 \times 0 = \varphi_1 && \varphi_2 \times 0 = \varphi_2 && \varphi_3 \times 0 = \varphi_3 \\
&\varphi_1 \times 1 = 1_1 + (0 + 1_2 + 2_2 + 3) && \varphi_2 \times 1 = 1_1 + (1_2 + 2_2 + 3_2) && \varphi_3 \times 1 = 1_1 + (1_2 + 2) \\
&\varphi_1 \times 2 = 2_1 + (1_2 + 2_2 + 3) && \varphi_2 \times 2 = 2_1 + (0 + 1_2 + 2_2) && \varphi_3 \times 2 = 2 + (1_2 + 3) \\
&\varphi_1 \times 3 = 3 + (1_1 + 2_2) && \varphi_2 \times 3 = 3_1 + (1_2 + 3_2) && \varphi_3 \times 3 = 3 + (0 + 2)\,,
\end{aligned}$$

each one having issues that we now point out. In the module for φ_1, the grouping (1_2+2_2+3) that we give the q-dim $1.87\ldots$ means that the quantum dimension of 2_1 is less than 1. In the module for φ_2 the grouping $(1_2+2_2+3_2)$ that we assign q-dim $1.53\ldots$ means that the quantum dimension of 3_1 is less than 1. The module for φ_3 does not make 3 into an abelian line to join with the vacuum 0.

Another useful application of this notion of condensibility based on quantum dimenions is that we can see that the proper way to condense out nonabelian spin $\frac{1}{n}$ lines is to do so sequentially. In some cases, trying to pick an algebra that only includes a fermion, alike how we did for a nonabelian boson, will lead to quantum dimensions not being able to split properly. However if we condense the boson first resulting in an abelian fermion, then the quantum dimensions will be able to split properly[6] As an example consider $(F_4)_3$, the data of which is presented in appendix A.2. If we wanted to just naively condense the nonabelian fermion, the condensation algebra one can choose is $\varphi = 0+1$, which leads to the modules

$$\begin{aligned}
&\varphi \times 0 = \varphi\,, && \varphi \times 5 = 5_1 + (2_2 + 3_2 + 4_2 + 5_2 \\
&\varphi \times 1 = 1_1 + (0 + 1_2 + 2_2 + 4_2 + 7)\,, && \qquad + 5_3 + 6_2 + 8_2)\,,
\end{aligned}$$

[6]There are examples where condensing out a nonabelian fractional spin anyon is possible, namely in $\mathrm{Sp}(16)_1$.

$$
\begin{aligned}
\varphi \times 2 &= 2_1 + (1_2 + 2_2 + 3_2 + 4_2 + 5_3 + 8_2)\,, & \varphi \times 6 &= 6_1 + (4_2 + 5_3 + 6_2)\,,\\
\varphi \times 3 &= 3_1 + (2_2 + 3_2 + 5_3)\,, & \varphi \times 7 &= 7 + (1_2 + 4_2 + 8_2)\,,\\
\varphi \times 4 &= 4_1 + (1_2 + 2_2 + 4_2 + 5_3 + 6_2 + 7 + 8_2)\,, & \varphi \times 8 &= 8_1 + (2_2 + 4_2 + 5_3 + 6_2\\
& & &\qquad + 7 + 8_2)\,. \qquad (5.2.30)
\end{aligned}
$$

Greedily assigning the q-dim $4.49\ldots$ of 7 to the group $(1_2 + 2_2 + 3_2 + 4_2 + 5_3 + 6_2 + 7 + 8_2)$ leaves 3_1 with zero quantum dimension which contradicts the fact that the line 3 splits. To distribute $4.49\ldots$ among 3_1 and 3_2 would result in both of the lines being simple objects in the gauged theory, yet at least one would be nonabelian carrying q-dim less than $4.49\ldots$. In appendix A.2 we will show that by condensing the nonabelian boson first, that the spin $1/2$ line becomes abelian, and we can seqentially condense it.

5.3. Modular Invariants and Condensation

Having done a couple of examples where we find the lines of the child theory in the previous section, we now present some of the modular invariants of those theories, and others. It is well known that the modular invariants should correspond to the Frobenius algebra objects up to Morita equivalence. So in particular, there are modular invariants that correspond to nonabelian bosonic condensation; we will refer to them as "extension" modular invariants. This is not the end of the story as there also exists "permutation" modular invariants that pair up the lines with the same spin and in certain cases displays some symmetry of the theory. This is also referred to in the literature as the "charge conjugation" modular invariant. One might expect that these modular invariants arise from an algebra that includes a boson, but we can also find these permutation invariants in theories with no bosons at all! In this case, finding the condensation algebra for these invariants can be complicated. When the fusion rules are grouplike, it is more likely that we are able to determine what is the algebra that gives the permutation invariant. For abelian Chern-Simons theories, their unitary symmetries, documented in [81], is reflected by the modular invariants. Furthermore for $\mathrm{SU}(2)_k$ theories where there is an ADE classification of modular invariants [34, 166], it can be checked that the modular data as well as the F- and R-symbols reflect the symmetries given by the permutation modular invariants. Motivated by this, one could study the modular invariants that are not of the extension type, to reveal a subset of the symmetries of the nonabelian Chern-Simons, even though we are unable to check these symmetries entirely since we do not have knowledge of the F- and R-symbols for a general theory.

As an example of an algebra associated to a permutation, consider the toric code (=

$\mathrm{Spin}(16)_1$). There are two bosons and a fermion and there is a global $\mathbb{Z}_2$ symmetry which is usually called "electromagnetic duality" but which might as well be called charge conjugation. It is implemented by (the Morita equivalence class of) an algebra whose underlying object is $1 + \text{fermion}$. As another example one can consider is $\mathrm{Spin}(4)_1 = \mathrm{SU}(2)_1^2 = \text{semion}^2$. Its particles are the vacuum, a fermion, and two semions, and again 1+fermion is an algebra who implements a $\mathbb{Z}_2$ global symmetry. In this case that global symmetry switches the two semions. We will give more nontrivial examples such as $\mathrm{SU}(N)_1, (E_6)_1$, where we can explicitly see the association of a permutation modular invariant to an algebra.

While the modular invariants for the Lagrangian algebra correspond to gapped boundaries, the permutation types do not give gapped boundaries. This fact is manifest when we consider the embedding $\mathrm{SU}(3)_1 \times (E_6)_1 \subset (E_8)_1$. The product theory is abelian and contains 9 lines given by the following table, where spins of the $\mathrm{SU}(3)_1$ lines are on the horizontal axis, and the spins of the $(E_6)_1$ lines are on the vertical axis:

$$\begin{array}{c|ccc} \mathrm{SU}(3)_1 \times (E_6)_1 & 0 & 1/3 & 1/3 \\ \hline 0 & 0 & 1/3 & 1/3 \\ 2/3 & 2/3 & 1 & 1 \\ 2/3 & 2/3 & 1 & 1\,. \end{array} \tag{5.3.1}$$

The two Lagrangian algebras are given by the three lines on the diagonal, and the line 0 with the two off diagonal bosons. The nondiagonal modular invariants however are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{5.3.2}$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}, \tag{5.3.3}
$$

with the rows labeled by {{0,0}, {0,1},{0,2},{1,0},{1,1},{1,2},{2,0},{2,1},{2,2}} with the first entry a line in $\mathrm{SU}(3)_1$ and the second entry a line in $(E_6)_1$. Each matrix squares to the identity, and none corresponds to either of the Lagrangian algebras. In particular, the last two modular invariants correspond to the algebra $(0+2/3+2/3)$ and $(0+1/3+1/3)$ from the two separate theories. Therefore, they do not give gapped boundaries. In cases when the Lagrangian algebra contains a fermion we have to couple to spin structure in order to get the gapped boundary; this is because Lagrangian algebras require not only associativity but also commutativity. Therefore the gapped boundary will have to be seen through the super modular invariant. It is a natural generalization that coupling to higher spin structures can also make an algebra composed of $1/n$-spin anyons commutative.

5.3.1. Modular invariants for spin $1/n$ anyons

The first of these new modular invariants arising when $n=2$ is recognized as supermodular invariants. These are matrices M such that

$$
\begin{cases}
[M,S]=[M,T^2]=0\,, \\
T\,MT^{-1} \quad \text{is integral,} \\
(ST)M(ST)^{-1} \quad \text{has positive integral values.}
\end{cases} \tag{5.3.4}
$$

These exist when there are extension modular invariants coming from condensing a fermion. There are also supermodular invariants which are permutation matrices, but permute the lines with spins differing by 1/2.

Given the fact that some super modular invariants correspond to condensing out a fermion, let us consider $(E_7)_1$, which has an abelian line but is spin 3/4. When we tensor this theory with itself we get a fermion which generates a center $\mathbb{Z}_2$ one-form symmetry in the overall $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, and also an extension type super modular invariant

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{5.3.5}$$

This indicates that the fermion composed of the two 3/4 lines should be condensable. But then to allow the constituent lines of the fermion to also be condensable, we should allow the original abelian 3/4 lines to be "condensable", at least when we couple to proper background r-structure. Thus, the super modular invariant motivates us to naturally enlarge the notion of the fiber functor beyond **SVec**, as was discussed at the end of §5.2.1.

We can generalize the conditions for a supermodular invariant further to matrices $\mathcal{M}$, which pair up lines that differ by spin $1/n$, such that

$$[\mathcal{M}, \mathcal{T}^n] = [\mathcal{M}, T^n] = 0. \tag{5.3.6}$$

Such nontrivial $\mathcal{M}$ of extension type would fit in conjointly with the discussion in §5.2.1 about the possibility to condense a spin $1/n$ anyon. We denote $\mathcal{T} = T.S.T$ as the operation what replaces S in the search for (super)modular invariants. This is motivated by the fact that we can take our three dimensional theory and compactify the two spatial dimensions on a torus. The Hilbert space for the 3d theory restricted to the torus, has a basis given by conformal blocks i.e. the spectrum of lines, and comes with an action of a mapping class group of the torus.

We insert a defect along the time direction, as in figure 5.4, which intertwines the representation of the modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acting on the torus on each side of the defect. In particular, the matrices

$$\mathcal{T} : \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a \\ a+b \end{pmatrix}, \quad T : \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a+b \\ b \end{pmatrix} \tag{5.3.7}$$

give the Dehn twists on the torus. The matrices $\mathcal{T}^n$ and T^n also belong to the group

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \, a \equiv d \equiv 1 \mod n, \quad b \equiv c \equiv 0 \mod n \right\}, \tag{5.3.8}$$

which is a congruence subgroup of Γ. In the set of matrices $\mathcal{M}$ that satisfy (5.3.6), some might not correspond to interfaces that are built via true commutative algebra objects and

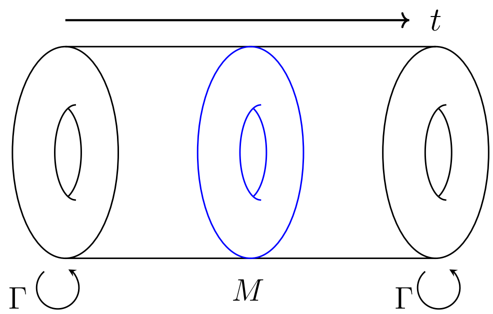


Figure 5.4: Each of the black tori represents the spatial dimensions of the 3d theory, with time running horizontally. The blue torus indicates a defect that can be placed in this quantum mechanics model at an instant in time. The black tori are both acted on by the modular group, so the defect M intertwines the two actions. The 2d theory on the black tori can in particular be the chiral or anti-chiral half of a WZW model.

thus do not contain the same physical interpretation as a modular invariant that came from condensing a Lagrangian algebra. These $\mathcal{M}$ only take the interpretation of intertwiners for $\Gamma(n)$ representations, in the same spirit as how there can exist modular invariants M that are intertwiners for Γ, i.e. matrices that commute with the modular actions, but do not come from Lagrangian algebras.

Nevertheless, to put these $\mathcal{M}$ into context, let us change perspectives from asking the categorical questions one can pose regarding the data of MTCs. If we look solely from a representation theory point of view, it is surprising that matrices in the representation of $\Gamma(n)$ can appear when we study MTCs. Given a representation of Γ, there can be endomorphisms of this representation as well as endomorphisms when we restrict to a subgroup $\Gamma(n)$. A reasonable question to ask is how one can construct the endomorphisms of $\Gamma(n)$, and where did they come from. It appears the condensation procedure we use can be useful to answering this question. Moreover, even the motivation for restricting to $\Gamma(n)$ representations is also clear as it came from observing the spins in the spectrum of anyons.

To make the discussion of using anyon condensation to find $\Gamma(n)$ representations more concrete, we give the explicit form of $\mathcal{M}$ in the examples $\mathrm{SU}(4)_2/\mathbb{Z}_2$, $(E_6)_1 \times (E_7)_1$ with the boson condensed out, and $\mathrm{Spin}(5)_1$. The spectrum of $\mathrm{SU}(4)_2$ is given by the following table on the left, and we can condense the boson:

$\mathrm{SU}(4)_2$	λ	h	q-dim
0	$[0,0,0,2]$	0	1
1	$[0,0,2,0]$	$3/4$	1
2	$[0,2,0,0]$	1	1
3	$[2,0,0,0]$	$3/4$	1
4	$[0,0,1,1]$	$5/16$	1.732050807569
5	$[0,1,1,0]$	$13/16$	1.732050807569
6	$[1,1,0,0]$	$13/16$	1.732050807569
7	$[1,0,0,1]$	$5/16$	1.732050807569
8	$[1,0,1,0]$	$2/3$	2
9	$[0,1,0,1]$	$5/12$	2

$\xrightarrow{\varphi=(0+2)}$

$\mathrm{SU}(4)_2/\mathbb{Z}_2$	ℓ	q-dim
0	$\varphi=(0+2)$	1
1	$(1+3)$	1
2	8_1	1
3	8_2	1
4	9_1	1
5	9_2	1 .

We notice that $8_{1,2}$ and $9_{1,2}$ differ by $3/4 \equiv -1/4 \mod 1$, so we consider the following matrices for $\mathcal{M}$ that pair up lines with spins that differ by $-1/4$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \tag{5.3.9}$$

and one can check that both commute with $\mathcal{T}^4 = (T^{-1}.S.T^{-1})^4$ and $(T^{-1})^4$. Here, T and S are those of the theory after condensing the boson i.e. $\mathrm{SU}(4)_2/\mathbb{Z}_2$. If we proceed in our usual manner of finding modules for an algebra object, we can consider the modules of $\varphi = 0+1$ in the table for $\mathrm{SU}(4)_2/\mathbb{Z}_2$ and we get

$$\begin{aligned} &\varphi \times 0 = \varphi && \varphi \times 3 = 3+4 \\ &\varphi \times 1 = \varphi && \varphi \times 4 = 4+3 \\ &\varphi \times 2 = 2+5 && \varphi \times 5 = 2+5\,. \end{aligned} \tag{5.3.10}$$

Therefore, the first of the two matrices in (5.3.9) corresponds to this φ.

The spectrum of $(E_6)_1 \times (E_7)_1$ contains 6 lines given by

$(E_6)_1 \times (E_7)_1$	ℓ	h	q-dim
0	$\{0,0\}$	0	1
1	$\{1,1\}$	17/12	1
2	$\{2,0\}$	2/3	1
3	$\{0,1\}$	3/4	1
4	$\{1,0\}$	2/3	1
5	$\{2,1\}$	17/12	1

and we see that by condensing $\varphi = 0 + 3$ the other lines are grouped as $(1+4)$ and $(2+5)$. The explicit matrix that corresponds to this condensation is

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \tag{5.3.11}$$

which can be checked commutes with $\mathcal{T}^4$ and $(T^{-1})^4$. Just as with $\mathrm{SU}(4)_2/\mathbb{Z}_2$, we can construct another $\mathcal{M}$ by grouping the lines by $(1+2)$ and $(4+5)$, but this is not what φ produces, so is unphysical.

The spectrum of $\mathrm{Spin}(5)_2$ is given by the following table on the left, where the boson can be condensed

$\mathrm{Spin}(5)_2$	λ	h	q-dim
0	$[0,0,2]$	0	1
1	$[2,0,0]$	1	1
2	$[0,1,1]$	1/4	2.236067977500
3	$[1,1,0]$	3/4	2.236067977500
4	$[0,2,0]$	3/5	2
5	$[1,0,1]$	2/5	2

$\xrightarrow{\varphi=(0+1)}$

ℓ	q-dim
$\varphi = (0+1)$	1
4_1	1
4_2	1
5_1	1
5_2	1 .

In this case, the spins of the child theory are all fifth roots of unity, and thus $T^5 = \mathrm{id}$. We also find that $\mathcal{T}^5$ is proportional to the identity, and thus all matrices satisfy (5.3.6), indicating there is a plethora of possible condensable algebras if we couple to background

r-structure [7].

We end the discussion on generalizing modular invariants with the case of $(G_2)_2$, which does not have such an $\mathcal{M}$ as in (5.3.6). Even though the spectrum contains two lines that differ by 1/3, the spin 1/3 line here is nonabelian. It was shown earlier that this spin $\frac{1}{3}$ was also not condensable, by the criterion we gave for a condensation in §5.2.2. If one were to consider the matrices that paired up the lines differing by spin 1/3 such as

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{5.3.12}$$

one would find that none commute with $\mathcal{T}^3$. This further supports our claim that there are no condensations possible, and that if one were to condense a spin $1/n$ line, then it must be abelian.

5.3.2. Modular invariants of tensored theories

We now consider in more depth what modular invariants one finds when we tensor theories. In this case, some of the lines may become bosons when combined with other lines, but are still not condensable algebras. This reinforces the fact that it is not the anyon necessarily that is crucial, but the algebra object. Just because some anyons might be nonabelian bosons, does not mean they belong to a condensation algebra, e.g. the Fibonacci category has no gapped boundary for any tensor product of the theory with itself [59]. When one considers a tensored theory such as $(G_k)^n$, there is an inherent symmetry group with order $n!$ that permutes the theories among themselves and is also reflected in the modular invariants of the tensored theory. From a physical point of view, recall that automorphisms of the theory are zero-form symmetries and therefore enacted by surface operators for our purposes. We will illustrate this explicitly in the example $(E_7)_1^3$. In a Reshetikhin- Turaev type theory, all of the surfaces arise as condensation descendants of lines by means described in §5.2. In this way we can think of the permutation modular invariants as being built from algebras.

To make contact with the previous section, we first look at the nondiagonal modular

[7] In addition to the matrices that correspond to algebras, we also get matrices that do not correspond to algebras since any general 6×6 matrix satisfies (5.3.6).

invariants of $\mathrm{SU}(3)_3$ given by

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&1&1&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&3 \end{pmatrix}, \tag{5.3.13}$$

which is a permutation invariant and the extension invariant, from gauging the one-form symmetry. There is a new nondiagonal super modular invariant given by

$$\begin{pmatrix} 1&1&1&0&0&0&0&0&0&1\\ 1&1&1&0&0&0&0&0&0&1\\ 1&1&1&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0&1 \end{pmatrix}, \tag{5.3.14}$$

which is the result of sequentially condensing out either of the three fermions in (5.2.16).

Moving onto $\mathrm{SU}(2)_{10}$, the nondiagonal modular invariants are

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1&0 \end{pmatrix}. \tag{5.3.15}$$

There also exist super modular invariants for this theory, given by

$$\begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&1&0&1&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&1&0&1&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&1&0&1&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&1&0&1&0&0&0&1 \end{pmatrix}, \tag{5.3.16}$$

the first is the permutation type that corresponds to condensing out the algebra $\varphi = 0 + 10$. By computing the modules of φ one can see that indeed the lines are paired as given by the left matrix:

$$\begin{aligned} &\varphi \times 0 = \varphi && \varphi \times 6 = 6 + 4 \\ &\varphi \times 1 = 1 + 9\,, && \varphi \times 7 = 7 + 3 \\ &\varphi \times 2 = 2 + 8\,, && \varphi \times 8 = 8 + 2 \\ &\varphi \times 3 = 3 + 7\,, && \varphi \times 9 = 9 + 1 \\ &\varphi \times 4 = 4 + 6\,, && \varphi \times 10 = \varphi\,. \end{aligned}$$

$$\varphi \times 5 = 5_1 + 5_2\,, \tag{5.3.17}$$

The latter modular invariant corresponds to condensing out the Lagrangian algebra, which included a fermion.

We give another example of finding the algebra that gives a permutation invariant by considering $\mathrm{SU}(N)_1$ with $N = 2n+1$. This is an abelian theory with $\mathbb{Z}_{2n+1}$ fusion rules, and associator $\kappa \in \mathrm{H}^3(\mathbb{Z}_{2n+1};\mathrm{U}(1))$ that is trivial. The algebras up to Morita equivalence i.e. the modules of the fusion category with fusion rules G and associator κ, are in bijection with subgroups $H \subset G$ and $\beta \in \mathrm{C}^2(H;\mathrm{U}(1))$ with $d\beta = \kappa|_H$. There is always the trivial subgroup, and the whole group itself. These give the diagonal modular invariant, and the permutation modular invariant – with the condensation algebra built by all of the lines $\varphi = 0 + 1 + \ldots + 2n$. For $\mathrm{SU}(N)_1$ with $N = 2n$, the associator is nontrivial and given by n mod $2n$. This is an obstruction to creating an algebra out of all the anyons, but we can form an associative algebra from the even anyons $\varphi = 0 + 2 + \ldots + 2n - 2$ which corresponds to the charge conjugation modular invariant.

We now present a theory that is formed as a tensor product of three copies of $(E_7)_1$. The spectrum is given by

$(E_7)_1^3$	$\{\ell_1, \ell_2, \ell_3\}$	h	q-dim
0	$\{0,0,0\}$	0	1
1	$\{0,0,1\}$	3/4	1
2	$\{0,1,0\}$	3/4	1
3	$\{0,1,1\}$	3/2	1
4	$\{1,0,0\}$	3/4	1
5	$\{1,0,1\}$	3/2	1
6	$\{1,1,0\}$	3/2	1
7	$\{1,1,1\}$	9/4	1 .

There are indeed five nondiagonal modular invariants of $(E_7)_1^3$, three of which give $\mathbb{Z}_2$

symmetries

$$\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&1&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \tag{5.3.18}$$

$$\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \tag{5.3.19}$$

and two which give a $\mathbb{Z}_3$ symmetry

$$\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}. \tag{5.3.20}$$

The three $\mathbb{Z}_2$'s are interfaces between any two of the three $(E_7)_1$ theories, and the $\mathbb{Z}_3$ symmetry allows us to cyclically go between the three $(E_7)_1$'s. For more discussion on these surface defects see [36, 175]. As for the super modular invariants of this product theory, we find 15 in total: 6 that were already mentioned and 9 new ones. Of the new matrices are idempotents:

$$\begin{pmatrix} 1&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&1 \end{pmatrix}, \tag{5.3.21}$$

formed from $\overline{(0+3)}(0+6)+\overline{(4+7)}(1+7)$ and its transpose,

$$\begin{pmatrix} 1&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&1 \end{pmatrix}, \tag{5.3.22}$$

formed from $\overline{(0+3)}(0+5)+\overline{(4+7)}(2+7)$ and its conjugate, and

$$\begin{pmatrix} 1&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&1 \end{pmatrix}, \tag{5.3.23}$$

formed from $\overline{(0+5)}(0+6)+\overline{(2+7)}(1+7)$ and its conjugate. There are furthermore matrices that are not idempotent, but whose elements grow as 2^{n-1} where n is the power in which the matrix is raised

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.3.24}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \tag{5.3.25}$$

The three total matrices in (5.3.24) and (5.3.25) are formed from

$$|0+6|^2 + |1+7|^2\,, \tag{5.3.26a}$$

$$|0+5|^2 + |2+7|^2\,, \tag{5.3.26b}$$

$$|0+3|^2 + |4+7|^2\,. \tag{5.3.26c}$$

It is natural to expect that the lines in the super modular invariant are grouped such that they differ by $\frac{1}{2}$ in spin. One can check that by condensing out $(0+3)$, $(0+5)$, $(0+6)$, that the lines which remain are $(4+7)$, $(2+7)$, and $(4+7)$ respectively. Thus the equations in (5.3.21) (5.3.22) (5.3.23) are the ones that "mix" two choices of condensation, and the expressions in (5.3.26) take each condensate individually.

5.4. Ungauging Anyons

We now consider starting off with some child theory $\mathcal{D}$ which is obtained from condensing some algebra in a parent $\mathcal{C}$, and present a method for studying the S-matrix elements of $\mathcal{C}$. The two MTCs $\mathcal{C}$ and $\mathcal{D}$ separated by an interface $\mathcal{F}$, and both acting on $\mathcal{F}$ by a braided monoidal map $\mathcal{C} \boxtimes \overline{\mathcal{D}} \to \mathcal{Z}(\mathcal{F})$. Here, $\overline{\mathcal{D}}$ means the category with opposite braiding, and $\mathcal{Z}$ means Drinfeld center. This has the structure of a braided monoidal category [162, 92] with braiding given by

$$\begin{aligned} \mathcal{Z}(\mathcal{F}) := \{(w, \beta_x) \,|\, w, x \in \mathcal{F} \text{ and } \beta_x : w \otimes x \to x \otimes w \\ \text{is natural in } x, \text{ such that } \beta_{x \otimes y} = \beta_y \otimes \beta_x\}. \end{aligned} \tag{5.4.1}$$

Thus there are two actions $\mathcal{C} \to \mathcal{Z}(\mathcal{F})$ and $\overline{\mathcal{D}} \to \mathcal{Z}(\mathcal{F})$, which commute [8]. This implies that $\mathcal{C}$ and $\overline{\mathcal{D}}$ are each other's commutants in $\mathcal{Z}(\mathcal{F})$ i.e. if we know $\mathcal{D}$ and $\mathcal{F}$ and $\overline{\mathcal{D}} \to \mathcal{Z}(\mathcal{F})$, then we can compute $\mathcal{C}$. It is precisely the subcategory of $\mathcal{Z}(\mathcal{F})$ of all objects that braid trivially with everything in $\mathcal{D}$, and similarly in the other order. In this way it is possible to reconstruct $\mathcal{C}$ from its "boundary" $\mathcal{F}$ [9]. The composition $\mathcal{C} \to \mathcal{Z}(\mathcal{F}) \to \mathcal{D}$ is dominant, in that every object is a direct summand of objects in the image. On the other hand $\mathcal{D} \to \mathcal{Z}(\mathcal{F}) \to \mathcal{C}$ is not dominant, as we have already learned from gauging condensation algebras.

We now review the details of the consistency relations that we will be using to reconstruct S of the parent. Consider a boundary line ℓ that is confined to the interface $\mathcal{F}$, and another line α on the wall brought in by moving it from the bulk $\mathcal{D}$. There is strictly speaking more information that α carries in the bulk, which might have been forgotten by moving to the boundary, but we can still uplift α from the wall back in to the bulk $\mathcal{D}$. Since α exists as a child line, it can be restricted back to the parent, where it can pass around ℓ. In particular, if on the wall we have the configuration α then ℓ, we can commute the two lines by lifting α into either of the bulks, which gives it a dimension to move around ℓ. So we have a configuration of ℓ then α, as summarized in figure 5.5. On the $\mathcal{D}$ side, α is passing an invisible line since ℓ does not lift off the wall. On the parent side, both α and ℓ can be restricted to their respective lines belonging to the theory $\mathcal{C}$. In general, both α and ℓ are semisimple with respect to the lines of $\mathcal{C}$, thus there can be multiple choices for restrictions. Since the two ways of α passing ℓ are equivalent, then $S_{\alpha,\ell} = 0$ in the parent, where the 0 denotes the fact that the braiding in the child theory is trivial among these two lines.

This is just stressing that the functor from $\mathcal{D} \to \mathcal{F}$ is also central. The compatibility for the lines in $\mathcal{C}$ with the lifting procedure is if

$$\mathcal{C} = \{\text{relative center of}\, \mathcal{Z}(\mathcal{F}; \mathcal{D})\}\,. \tag{5.4.2}$$

[8] Actually, the map $\mathcal{C} \boxtimes \overline{\mathcal{D}} \to \mathcal{Z}(\mathcal{F})$ is an equivalence.

[9] The term boundary is used a bit loosely because I don't mean a true boundary condition, but an interface to some other TFT. A boundary condition is a special case where it is an interface to the vacuum.

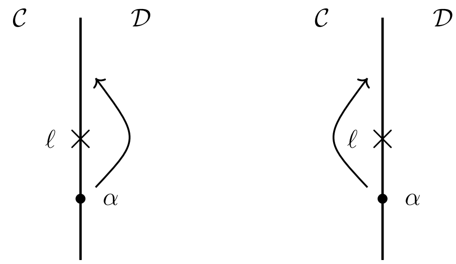


Figure 5.5: We give a top down view of the interface, which is represented by the solid line, that separates theories $\mathcal{C}$ and $\mathcal{D}$. Suppose that α is a line that exists in the parent theory, but lifts off to the child theory. Then it can pass by the totally confined object in two equivalent ways.

By definition, an object $X \in \mathcal{Z}(\mathcal{F};\mathcal{D})$ is an underlying object $\underline{X} \in \mathcal{F}$ together with half-braidings $\underline{X} \otimes Y \sim Y \otimes \underline{X}$ for all $Y \in \mathcal{F}$, monoidality, and commutativity with $\mathcal{D} \subset \mathcal{F}$. Furthermore, as can be seen in figure 5.6 given a, b lines on the wall where $a \in \mathcal{C}$ and $b \in \mathcal{D}$ originally, if we move a around b, then we move b around a, the two actions commute. In this case the S-matrix of the child can directly give the S-matrix elements of the parent, and we just need to "pull-back" the data.

Already in the case where $\mathcal{D}$ is the child theory as a result of condensing an abelian line from $\mathcal{C}$, it is nontrivial to use the consistency relations explained above to construct the S-matrix of $\mathcal{C}$. One could ask the obvious question which is "what is the minimum data of $\mathcal{F}$ and $\mathcal{D}$ that needs to be given to determind $\mathcal{C}$ uniquely?" This question goes beyond the scope and this paper, and perhaps does not even have a general answer for any MTC $\mathcal{C}$. For our purposes we will provide the content of the line spectrum and fusion rules on the interface $\mathcal{F}$, as well as the S-matrix of the child theory which can be calculated as in [83], all in terms of the simple objects of $\mathcal{C}$.

We consider an example where the fusion information of the category $\mathcal{F}$ is not enough to construct the exact parent theory (even though we might be able to attain the S-matrix), and we also need to give extra data in form of the associator. Let us suppose that $\mathcal{D}$ is trivial, and let $\mathcal{F} = \mathbf{Vec}^{\omega}[\mathbb{Z}_p]$ for p an odd prime. The fusion rules are independent of the cocycle $\omega \in \mathrm{H}^3(\mathbb{Z}_p;\mathrm{U}(1))$ known as the associator. By the Bockstein homomorphism for the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathrm{U}(1) \to 0\,, \tag{5.4.3}$$

ω is mapped to $\mathrm{H}^4(\mathbb{Z}_p;\mathbb{Z})$, so $\beta(\omega) \in \mathrm{H}^4(\mathbb{Z}_p;\mathbb{Z})$. There also exists a "squaring" map that goes from

$$\mathrm{H}^2(\mathbb{Z}_p;\mathbb{Z}) \to \mathrm{H}^4(\mathbb{Z}_p;\mathbb{Z})\,. \tag{5.4.4}$$

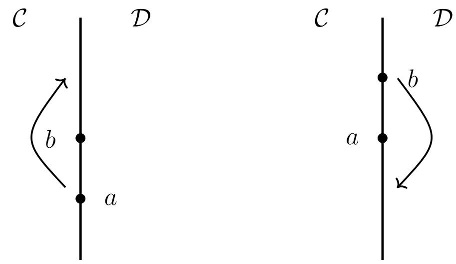


Figure 5.6: Since either a or b may lift off the wall, the configuration obtained from passing either one around the other by going into the respective bulk is equivalent.

Furthermore, the automorphisms of $\mathbb{Z}_p$ permute the entries in $\mathrm{H}^2(\mathbb{Z}_p;\mathbb{Z})$ and so permute the ω such that $\beta(\omega) = \text{Square}$. The three possibilities that ω can take are,

$$\omega = 0, \quad \beta(\omega) = \text{Square}, \quad \beta(\omega) = \text{non-Square}. \tag{5.4.5}$$

The parent is just the Drinfeld center of $\mathcal{F}$, so when $\omega = 0$, we denote $\mathcal{C}_0 = \mathbb{Z}_p \times \mathbb{Z}_p$, and for both of the other values of ω, the parent is $\mathcal{C}_1 = \mathbb{Z}_{p^2}$. At the level of groups, the map $\mathcal{C}_0 \to \mathcal{F}$ takes $(a,b) \to [b]$, where a and b are valued mod p. In other words, the line labeled $[j] \in \mathcal{F}$ is

$$[j] = \{(0,j),(1,j),\ldots,(p-1,j)\}\,, \tag{5.4.6}$$

i.e. comes from p many lines in the parent. In the case of $\mathcal{C}_1$ the map takes $(ap+b) \to [b]$. Given this, one could not tell the case of $\mathcal{C}_0$ and $\mathcal{C}_1$ apart because in either of the ways that we label lines in the two parents, the label shows up as $[b]$ when you move to the wall [10]. Thus without giving the associator for the wall category $\mathcal{F}$, the fusion of the lines on $\mathcal{F}$ is not sufficient to give a unique parent in this example.

5.4.1. Analysis of Ising $\boxtimes$ $\overline{\text{Ising}}$

Before we explicitly reconstruct S-matrix elements, it is useful to use the consistency relations and apply them to evaluate B elements where by $B(a,b)$ we mean the result $\frac{S_{ab}}{S_{1b}}$, where S_{ab} is the trace of the full braiding of lines a,b. Let $\mathcal{C}$ be Ising $\boxtimes$ $\overline{\text{Ising}}$, and by condensing $\varphi = \mathbb{1}\overline{\mathbb{1}} + \epsilon\bar{\epsilon}$, the child theory is the Toric code with

$$\begin{aligned} &(\mathbb{1}\overline{\mathbb{1}} + \epsilon\bar{\epsilon}) = 1\,, \quad (\mathbb{1}\bar{\epsilon} + \epsilon\overline{\mathbb{1}}) = f, \\ &\sigma\bar{\sigma}_1 + \sigma\bar{\sigma}_2 = e + m\,. \end{aligned} \tag{5.4.7}$$

[10] If we are also given some fusion information about the parent, then we could at least distinguish $\mathcal{C}_0$ from $\mathcal{C}_1$.

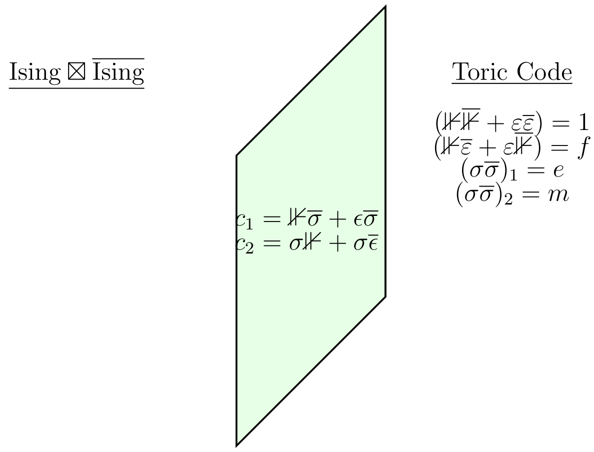


Figure 5.7: All lines of the fusion category on the wall are written in terms of the data of the parent theory. The lines that can not lift off the wall are c_1 and c_2. The data of the Toric Code is drawn in the bulk but can be brought to the wall.

The lines that are totally confined are given by

$$c_1 = \nVdash\overline{\sigma} + \epsilon\overline{\sigma}\,, \quad c_2 = \sigma\overline{\nVdash} + \sigma\overline{\epsilon}\,. \tag{5.4.8}$$

The picture one should have in mind is given by figure 5.7.

- **Braiding of lines that exist in the child theory**

Suppose we wanted to determine $S_{\sigma\overline{\sigma},\sigma\overline{\sigma}}$ in the parent. There is a relationship between the S-matrix of Toric code and Ising ⊠ $\overline{\text{Ising}}$. This is like a "restriction" map onto the parent theory from the child theory, and is a less expensive way of recovering the some of the S-matrix elements of the parent, without needing the full machinery of the fusion category on the wall. This says that if we can build a line in the parent theory, as some data that comes from the child theory, then we can restrict the S-matrix from the child MTC to get the S-matrix of the parent. We know that the line $e+m$ in the child restricts to $\sigma\overline{\sigma}$ in the parent. Since we know $S_{e+m,e+m} = 0$, the restriction of this across the boundary is zero. Indeed with knowledge of $\mathcal{C}$ we find [11]

$$S_{\sigma\overline{\sigma},\sigma\overline{\sigma}} = (R^{\sigma\overline{\sigma},\sigma\overline{\sigma}}_{\nVdash\overline{\nVdash}})^2\, d_{\nVdash\overline{\nVdash}} + (R^{\sigma\overline{\sigma},\sigma\overline{\sigma}}_{\epsilon\overline{\epsilon}})^2\, d_{\epsilon\overline{\epsilon}} + (R^{\sigma\overline{\sigma},\sigma\overline{\sigma}}_{\nVdash\overline{\epsilon}})^2\, d_{\nVdash\overline{\epsilon}} + (R^{\sigma\overline{\sigma},\sigma\overline{\sigma}}_{\epsilon\overline{\nVdash}})^2\, d_{\epsilon\overline{\nVdash}} = 0\,. \tag{5.4.9}$$

While some elements can be restricted, in general we will need to have more knowledge of the fusion rules of the totally confined lines to understand the braiding in the parent

[11]The S-matrix elements $S_{a,b}$ are given by $R^{a,b}_i R^{b,a}_i$.

theory. Thus we need to know $c_1 \times c_2 = e + m$. From the values of $B(1, f)$, $B(e+m, 1)$ and $B(e+m, f)$ in the Toric code, by restriction we get $B(1, f)$ restricts to

$$\begin{aligned} B(\nVdash\overline{\nVdash}, \nVdash\bar{\epsilon}) = 1\,, \quad B(\epsilon\bar{\epsilon}, \nVdash\bar{\epsilon}) = 1 \\ B(\nVdash\overline{\nVdash}, \epsilon\overline{\nVdash}) = 1\,, \quad B(\epsilon\bar{\epsilon}, \epsilon\overline{\nVdash}) = 1\,. \end{aligned} \tag{5.4.10}$$

Furthermore $B(e+m, 1)$ and $B(e+m, f)$ restrict to

$$B(\sigma\bar{\sigma}, \nVdash\bar{\epsilon}) = -2\,, \quad B(\sigma\bar{\sigma}, \epsilon\overline{\nVdash}) - 2\,,$$

and

$$B(\sigma\bar{\sigma}, \nVdash\overline{\nVdash}) = 2\,, \quad B(\sigma\bar{\sigma}, \epsilon\bar{\epsilon}) = 2\,.$$

- **Braiding of totally confined lines in the parent**

The next task to understand is how the confined lines on the wall, $\nVdash\bar{\sigma} + \epsilon\bar{\sigma}$ and $\sigma\overline{\nVdash} + \sigma\bar{\epsilon}$, braid in the parent theory. These two lines do not lift to the Toric code side, so we can not simply restrict the S-matrix from the Toric code to get the braiding. To answer this, suppose the line $\nVdash\bar{\epsilon} + \epsilon\overline{\nVdash}$ is brought in from the child theory to the wall. On the wall, $(\nVdash\bar{\epsilon} + \epsilon\overline{\nVdash}) \times (\nVdash\bar{\sigma} + \epsilon\bar{\sigma}) \xrightarrow{\cong} (\nVdash\bar{\sigma} + \epsilon\bar{\sigma}) \times (\nVdash\bar{\epsilon} + \epsilon\overline{\nVdash})$ because $(\nVdash\bar{\epsilon} + \epsilon\overline{\nVdash})$ lifts off to the Toric code side as f, and so we can bring it around $\nVdash\bar{\sigma} + \epsilon\bar{\sigma}$. Furthermore $\nVdash\bar{\sigma} + \epsilon\bar{\sigma}$ restricted to the parent becomes $\nVdash\bar{\sigma}$ or $\epsilon\bar{\sigma}$ and similarly $\nVdash\bar{\epsilon} + \epsilon\overline{\nVdash}$ becomes $\nVdash\bar{\epsilon}$ or $\epsilon\overline{\nVdash}$. Thus, we consider the braidings $B(\nVdash\bar{\sigma}, \nVdash\bar{\epsilon})$, $B(\epsilon\bar{\sigma}, \nVdash\bar{\epsilon})$. An important fact to notice is that the lines $\{1, f, c_1\}$, as a subcategory of the wall fusion category, have the same fusion rules as the Ising category. Here, c_1 has the fusion rules as the σ line. Therefore, $B(\nVdash\bar{\sigma}, 1\bar{\epsilon}) = -\sqrt{2}$ in the parent theory to reflect the fact that $B(\sigma, f) = -\sqrt{2}$ in Ising. We notice that the spin of $\epsilon\overline{\nVdash}$ is the negative of the spin of $\nVdash\bar{\epsilon}$ in the parent, so the braiding should have a relative negative i.e. $B(\nVdash\bar{\sigma}, \epsilon\overline{\nVdash}) = \sqrt{2}$. Due to the restriction of $\nVdash\bar{\sigma} + \sigma\bar{\epsilon}$ from the wall to the parent, then

$$B(\epsilon\bar{\sigma}, \nVdash\bar{\epsilon}) = -\sqrt{2}, \qquad B(\epsilon\bar{\sigma}, \epsilon\overline{\nVdash}) = \sqrt{2}\,. \tag{5.4.11}$$

The next object to consider is $B(\sigma\overline{\nVdash}, \nVdash\bar{\epsilon})$, which is natural to consider after lifting c_2 to the parent. Similar to before, we notice that $\{1, f, c_2\}$ also can be used to create a Ising subcategory. Therefore

$$\begin{aligned} B(\sigma\overline{\nVdash}, \epsilon\overline{\nVdash}) = B(\sigma, f) = -\sqrt{2}\,, \quad & B(\sigma\overline{\nVdash}, \nVdash\bar{\epsilon}) = \sqrt{2}\,, \\ B(\sigma\bar{\epsilon}, \epsilon\overline{\nVdash}) = -\sqrt{2}\,, \qquad\qquad & B(\sigma\bar{\epsilon}, \nVdash\bar{\epsilon}) = \sqrt{2}\,. \end{aligned}$$

We now consider the braiding of $\nVdash\overline{\sigma}$ and $\sigma\overline{\nVdash}$, or in general the braiding of two lines both comprising of σ in the parent theory. The braiding of $B(\sigma\overline{\nVdash}, \sigma\overline{\epsilon})$ in the parent is the restriction of c_1 and c_2 from the wall. This is analogous to asking about the braiding of two particles that behave like σ in the Ising category, but we know $B(\sigma, \sigma) = 0$, so $B(\sigma\overline{\nVdash}, \sigma\overline{\epsilon}) = 0$. The next braidings to consider is $B(\sigma\overline{\sigma}, \sigma\overline{\nVdash})$ and $B(\sigma\overline{\sigma}, \sigma\overline{\epsilon})$. First examine the fusion of $\sigma\overline{\sigma}$ with c_1 and c_2 on the wall fusion category, and notice that $c_1 \times c_2 = e + m$ and so can be moved off the wall to the Toric code side. If we consider on the wall $B(\sigma\overline{\sigma}, c_1 \times c_2)$, which after moving to the Toric code is $B(e + m, e + m) = 0$, this implies that one of $B(\sigma\overline{\sigma},\, c_1)$, $B(\sigma\overline{\sigma}, c_2)$ is equal to zero. But c_1 and c_2 should be symmetric as particles because they play the same role in the subIsing category, and so both braidings in the parent theory should be zero. Thus we have

$$B(\sigma\overline{\sigma}, \nVdash\overline{\sigma}) = B(\sigma\overline{\sigma}, \epsilon\overline{\sigma}) = 0\,, \tag{5.4.12}$$

$$B(\sigma\overline{\sigma}, \sigma\overline{\nVdash}) = B(\sigma\overline{\sigma}, \sigma\overline{\epsilon}) = 0\,. \tag{5.4.13}$$

5.4.2. Reconstructing the Toric Code

We will now apply the consistency relations to a simple example of the Toric code to solve for actual S-matrix elements. This MTC consists of four simple objects $\{1, e, m, f\}$. It has following fusion and braiding rules

$$e \times e = 1, \quad m \times m = 1, \quad e \times m = f\,,$$
$$B(e, e) = B(m, m) = 1, \quad B(e, m) = -1\,.$$

To help with computing the matrix elements, we give some S-matrix identities involving products and linearity; for a, b, c, d simple lines we have

$$S_{a,b\times c} = \sum_{\ell} S_{a,\ell} N^{\ell}_{b,c} = \frac{S_{a,b}S_{a,c}}{S_{a,0}}\,, \tag{5.4.14a}$$

$$S_{a,b+c} = S_{a,b} + S_{b,c}\,, \quad S_{a+b,c} = S_{a,c} + S_{b,c}\,. \tag{5.4.14b}$$

The Toric code has two kinds of bosonic anyon condensation given by $\varphi = 1+e$ or $\varphi = 1+m$. If we condense with $\varphi = 1 + m$, the remaining aynons $\{e, f\}$ will be confined on the wall, unable to lift to the child theory. Hence the child phase $\mathcal{D}$ is just the vacuum φ. On the other hand, the wall category which is just a fusion category consists of wall vacuum $1 + m$ (which in this case is identical to the condensed vacuum) and the remaining confining anyons are grouped into a single module, $e + f$. Now let us try to reconstruct the Toric code from the above condensed phase $\mathcal{D}$ and the wall category; the confined lines on the wall have a natural embedding in the Toric code. We assume the fusion rules of the confined line with φ are known:

$$m \times f = e, \quad m \times e = f\,. \tag{5.4.15}$$

From the lifting property of $1+m$ to be able to go to the $\mathcal{D}$ side of the wall, we start off with the fact that

$$\frac{S_{1+m,e+f}}{S_{1,e+f}} = 1 + \frac{S_{m,e}+S_{m,f}}{S_{1,e}+S_{1,f}} = 0\,. \tag{5.4.16}$$

using (5.4.14a) we see that

$$S_{m,e\times f} = \frac{S_{m,e}S_{m,f}}{S_{m,1}} = S_{m,m}\,, \tag{5.4.17a}$$

$$S_{m,e} = S_{m,m\times f} = \frac{S_{m,m}S_{m,f}}{S_{1,m}}\,, \quad S_{m,f} = S_{m,m\times e} = \frac{S_{m,m}S_{m,e}}{S_{1,m}}\,. \tag{5.4.17b}$$

From (5.4.17b) we have the two equations

$$S_{m,f}S_{1,m} = S_{m,m}S_{m,e}\,, \tag{5.4.18}$$

$$S_{m,e}S_{1,m} = S_{m,m}S_{m,f}\,, \tag{5.4.19}$$

and combining the two equations we have

$$S_{m,m} - S_{1,m} = 0, \quad \text{or} \quad S_{m,e,} + S_{m,f} = 0\,.$$

But by (5.4.16), the latter can not be zero, thus we have $S_{m,m} = S_{1,m}$. Another important relationship is

$$S_{1,e\times m} = \frac{S_{1,e}S_{1,m}}{S_{1,f}} \to S_{1,f}^2 = S_{1,e}S_{1,m}.$$

but $S_{1,e}$ and $S_{1,m}$ are equivalent, and $S_{1,e} \neq -S_{1,f}$ by (5.4.16), so the only consistent choice is

$$S_{1,f} = S_{1,e} = S_{1,m}. \tag{5.4.20}$$

We now use a fact from the S-matrix of the child theory, which is the value of

$$S_{1+m,1+m} = S_{1,1} + 2S_{1,m} + S_{m,m} = 1\,. \tag{5.4.21}$$

To get the value of $S_{1,1}$ we use

$$S_{1,m\times m} = \frac{S_{1,m}^2}{S_{1,1}} \to S_{1,1}^2 = S_{1,m}^2\,. \tag{5.4.22}$$

But there are two choices to be made for the value in (5.4.22). Suppose we take

$$S_{1,1} = S_{1,m}. \tag{5.4.23}$$

We see immediately from (5.4.21) that $S_{11} = \frac{1}{2}$. Then by using (5.4.16) and (5.4.17b) we see

$$\begin{aligned} S_{m,e} + S_{m,f} &= -1\,, \\ S_{m,e} &= S_{m,f}\,, \end{aligned}$$

thus $S_{m,e} = S_{m,f} = -\frac{1}{2}$. Finally, to get $S_{f,f}$ notice that

$$S_{f,f} = S_{f,e\times m} = \frac{S_{f,e}S_{f,m}}{S_{1,f}}\,,$$

so $S_{f,f} = \frac{1}{2}$. With this and the other equations relating different S-matrix elements, as well as the symmetry between e and m, we can fully determine S of the Toric code parent theory. One could wonder what happens if we had made the other choice in (5.4.23) by taking $S_{1,1} = -S_{1,m}$. If we consider

$$S_{1,m} = S_{1,e\times f} = \frac{S_{1,e}S_{1,f}}{S_{1,1}} \tag{5.4.24}$$

we get that $S_{1,1} = S_{1,f}$, coupled with the earlier fact that $S_{1,f} = S_{1,m}$, leads to a contradiction.

It is important to remark that in our reconstruction of the parent S-matrix we assumed the associator with respect to the fusion ring of $(1+m)$ and $(e+m)$ was trivial. However, because the lines are the group ring for the group $\mathbb{Z}_2$ and $\mathrm{H}^3(\mathbb{Z}_2;\mathrm{U}(1)) = \mathbb{Z}_2$, there also exists a nontrivial associator. Had we chosen the nontrivial associator, the parent theory would be $\mathrm{SU}(2)_1 \boxtimes \overline{\mathrm{SU}(2)_1}$ aka the semion anti-semion theory. Let x denote the nontrivial element confined on the wall such that $(xx) = 1$. Giving x a central structure amounts to defining $\beta_{x,-} : x\times - \to -\times x$, in which the only data is $\beta_{x,x} \in \mathbb{C}$. We require that braiding with the trivial element is trivial

$$\widetilde{x}(xx) \xrightarrow{\beta_{x,1}=1} (xx)\widetilde{x}\,,$$

and also the hexagon identity applies

$$\begin{tikzcd}
& \widetilde{x}(xx) \arrow[r, "\alpha=-1"] \arrow[dl, "\beta_{\widetilde{x},1}"'] & (\widetilde{x}x)x \arrow[dr, "\beta_{x,\widetilde{x}}"] & \\
(xx)\widetilde{x} \arrow[dr, "\alpha=-1"'] & & & (x\widetilde{x})x \arrow[dl, "\alpha=-1"] \\
& x(x\widetilde{x}) \arrow[r, "\beta_{x,\widetilde{x}}"'] & x(\widetilde{x}x) & .
\end{tikzcd}$$

This implies that $\beta_{x,x}^2 = -1$ so $\beta_{x,x} = \pm\sqrt{-1}$. If the associator was trivial, then $\beta_{x,x} = \pm\sqrt{1}$ and that's why x would have lifted to either a boson or a fermion in the toric code. This implies that when we choose different associators that the S-matrix in the parent theory will be different.

If the fusion rules on the wall are not a group, then there is a set of associators, which are solutions to some polynomial equation. In general none of the solutions have to be trivial. In contrast, for grouplike fusion rules, one of the solutions is just a constant and deserves to be called trivial. In the examples that we will consider the fusion category of the wall as well as the child theory will be bosonic, and H_k the parent theory conformally embeds into G_1 of the child. Therefore, the natural algebra object of the parent is a sum of bosonic anyons. We will use this fact to reconstruct the S-matrix elements of the parent, without the need to solve for the possible associators of the wall fusion category; it is surprising that it suffices to only utilize facts about relative centers and the fusion rules on the wall. In general, given a theory with finitely many anyons, there can be infinitely many fusion rings, but there are only a finite number of categorifications. The fact that in our examples we are reconstructing a parent that comes from a conformal embedding may contribute to the fact that we did not have to give the associator, yet still landed on equations that consistently produced an S-matrix.

5.4.3. Reconstructing $\mathrm{SU}(3)_3$

For the case of reconstructing $\mathrm{SU}(3)_3$ from $\mathrm{Spin}(8)_1$ we will use the consistency relations to show the relationships among S-matrix elements, we will then comment on how to obtain the explicit values. As we did for the Toric code, we will split up finding the S-matrix into different cases.

- **S-matrix element with only the vacuum line**

From §5.2.1 the lines $\varphi = 0+1+2$ was the condensation algebra, so it can lift off the wall to the parent or child theory. On the wall, there are three ways for the line $(0+1+2)$ to lift into the parent side, and go around the $(0+1+2)$ on the wall. This is like saying we have three equations from restricting $S_{\varphi,\varphi}$ to the parent (restricting $(0+1+2)$ back to parent), namely

$$S_{0,(0+1+2)} = \tfrac{1}{2},\ S_{1,(0+1+2)} = \tfrac{1}{2},\ S_{2,(0+1+2)} = \tfrac{1}{2}. \tag{5.4.25}$$

In more colloquial terms, for each one of the lift to the parent side $\{0,1,2\}$, we could have taken that "lift element", moved it to the child where it becomes φ, and then gone around φ in the child theory where $S_{\varphi,\varphi} = \frac{1}{2}$. Since each element of $\{0,1,2\}$ is treated on "equal footing" in terms of being in φ, then each element S_{ij} in (5.4.25) should be equal to $\frac{1}{6}$, by distribution.

- **S-matrix elements containing the line 9**

From the wall to the child side, 9 has three lifts as $(9_1+9_2+9_3)$, resulting in the other three nontrivial lines of $\mathrm{Spin}(8)_1$. Each of the lifts has an S-matrix element $S_{\varphi,9_j} = \frac{1}{2}$ in the child, thus $S_{\varphi,9_1}+S_{\varphi,9_2}+S_{\varphi,9_3} = \frac{3}{2}$. When we restrict back to the parent side $(9_1+9_2+9_3)$ restricts to 9, and $(0+1+2)$ has three ways to restrict to the parent; figure 5.8 therefore gives the equations

$$S_{\varphi,9_1}+S_{\varphi,9_2}+S_{\varphi,9_3} = S_{\varphi,(9_1+9_2+9_3)} \xrightarrow{\text{parent}} S_{\varphi,9} = S_{0,9}+S_{1,9}+S_{2,9} = \tfrac{3}{2}\,, \tag{5.4.26}$$

and so

$$S_{0,9} = S_{1,9} = S_{2,9} = \tfrac{1}{2}\,.$$

The confined lines are $(3+4+5)$ and $(6+7+8)$, since neither of these two lines lift to the child theory, the line 9 can be braided around them by going to the child side. Restricting this to the parent means

$$S_{k,9} = 0\,, \quad k \in \{3,4,5,6,7,8\}.$$

To determine $S_{9,9}$ in the parent consider taking both of the 9's and bringing them to the wall, then we get $(9_1+9_2+9_3)$ next to each other. We can lift them to the child side in

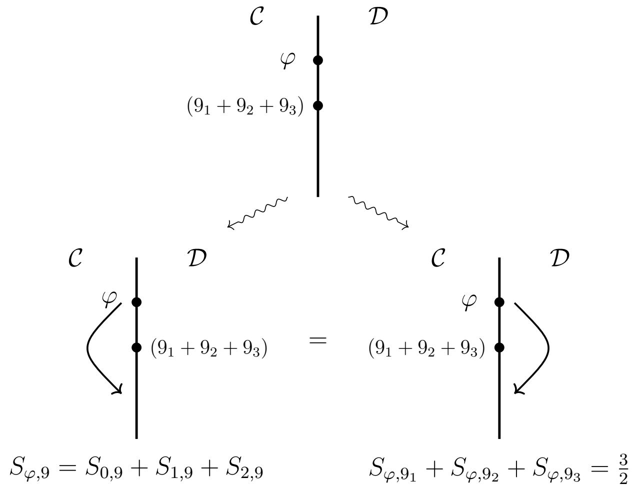


Figure 5.8: The two ways of passing φ around the totally confined line on the wall are equivalent, and this relates the S-matrix elements.

three ways, and go around each other. The sum $\sum_{i,j} S_{9_i,9_j} = -\frac{1}{2}$ in the child, and therefore in the parent we have

$$S_{9,9} = S_{(9_1+9_2+9_3),(9_1+9_2+9_3)} = -\frac{1}{2}\,.$$

- **S-matrix of totally confined lines and the vacuum**

We now determine the braiding of the totally confined lines with φ, and with themselves in the parent. This is the most complicated case. We first recognize that since φ can go around either $(3+4+5)$ or $(6+7+8)$ by moving to the child side, then as per figure 5.5 we get the equations

$$\begin{aligned} S_{3,0} + S_{3,1} + S_{3,2} &= 0\,, \\ S_{4,0} + S_{4,1} + S_{4,2} &= 0\,, \\ S_{5,0} + S_{5,1} + S_{5,2} &= 0\,, \end{aligned} \tag{5.4.27}$$

as well as

$$\begin{aligned} S_{6,0} + S_{6,1} + S_{6,2} &= 0\,, \\ S_{7,0} + S_{7,1} + S_{7,2} &= 0\,, \\ S_{8,0} + S_{8,1} + S_{8,2} &= 0\,. \end{aligned} \tag{5.4.28}$$

Our method of using the relative center properties is not quite enough to solve for the matrix elements. We now employ our knowledge of the fusion of the lines on the wall, which we assume were given to us in the beginning. For simplicity of writing, let $a = S_{3,0}$, $b = S_{3,1}$, $c = S_{3,2}$. Motivated by taking 3 and encircling it around $(3+4+5)$ we consider the following fusions:

$$S_{3,3} = S_{3,4\times 2} = \frac{S_{3,4}S_{3,2}}{S_{3,0}}\,, \tag{5.4.29a}$$

$$S_{3,4} = S_{3,3\times 1} = \frac{S_{3,3}S_{3,1}}{S_{3,0}}\,, \tag{5.4.29b}$$

$$S_{3,5} = S_{3,3\times 2} = \frac{S_{3,3}S_{3,2}}{S_{3,0}}\,. \tag{5.4.29c}$$

Furthermore by inspecting other fusion relations we have

$$S_{3,4} = S_{3,5\times 2} = \frac{S_{3,5}S_{3,2}}{S_{3,0}}\,, \tag{5.4.30a}$$

$$S_{3,5} = S_{3,4\times 1} = \frac{S_{3,4}S_{3,1}}{S_{3,0}}\,. \tag{5.4.30b}$$

We can plug (5.4.29b) into (5.4.29a) to get $a^2 = bc$. Also, by setting (5.4.29b) equal to (5.4.30a) and (5.4.29c) equal to (5.4.30b) we get $c^2 = ab$ and $b^2 = ac$. All together we have the system

$$a+b+c=0, \quad a^2 = bc, \quad b^2 = ac, \quad c^2 = ab, \tag{5.4.31}$$

which has the solution $\{a,b,c\} = \{a, a\omega, a\omega^2\}$ and $\{a, a\omega^2, a\omega\}$, where ω is a cube root of unity. We notice that if a is real, which it is because $a = S_{3,0}$ is just the quantum dimension of 3, divided by $D = \sqrt{\sum_i \text{q-dim}_i^2}$, then the two solutions are complex conjugates. The next piece of information which we can draw from the fusion rules on the wall is from using the Verlinde formula. Consider the fact that $3 \times 3 = 6 + 8$, then we have

$$1 = N_{3,3}^{8} = \sum_a \frac{S_{3,a}S_{3,a}S^*_{8,a}}{S_{0,a}}\,. \tag{5.4.32}$$

But $S^*_{8,a} = S_{3,a}$ because $3 \times 3 = 0 + 9$, so we can write the above formula as

$$1 = \sum_a \frac{S^3_{3,a}}{S_{0,a}}\,. \tag{5.4.33}$$

We know that given $S_{3,0}$, then $S_{3,1} = S_{3,0}\,\omega$ and $S_{3,2} = S_{3,0}\,\omega^2$. Note that this also satisfies the first equation in (5.4.27). The same holds true for $S_{3,3}$ and $S_{3,6}$ and can be easily seen from the fusion rules, i.e.

$$S_{3,4} = S_{3,3}\,\omega\,, \quad S_{3,5} = S_{3,3}\,\omega^2\,, \tag{5.4.34a}$$
$$S_{3,7} = S_{3,6}\,\omega\,, \quad S_{3,8} = S_{3,6}\,\omega^2\,. \tag{5.4.34b}$$

To use (5.4.33), we need to relate both $S_{3,3}$ and $S_{3,6}$ to $S_{3,0}$, so then the sum can be written with only a single unknown variable. In order to make the relations manifest we use the following fusion rules

$$S_{3,3\times 3} \to S_{3,0}(S_{3,6}+S_{3,8}) = S_{3,3}^2\,, \tag{5.4.35a}$$
$$S_{3,3\times 4} \to S_{3,0}(S_{3,6}+S_{3,7}) = S_{3,3}S_{3,4}\,, \tag{5.4.35b}$$
$$S_{3,3\times 5} \to S_{3,0}(S_{3,7}+S_{3,8}) = S_{3,3}S_{3,5}\,, \tag{5.4.35c}$$
$$S_{3,3\times 6} \to S_{3,0}(S_{3,1}+S_{3,9}) = S_{3,3}S_{3,6}\,, \tag{5.4.35d}$$
$$S_{3,3\times 7} \to S_{3,0}(S_{3,2}+S_{3,9}) = S_{3,3}S_{3,7}\,, \tag{5.4.35e}$$
$$S_{3,3\times 8} \to S_{3,0}(S_{3,0}+S_{3,9}) = S_{3,3}S_{3,8}\,, \tag{5.4.35f}$$
$$S_{3,3\times 9} \to S_{3,0}(S_{3,1}+S_{3,4}+S_{3,5}) = S_{3,3}S_{3,9}\,. \tag{5.4.35g}$$

By using the relations in (5.4.34) and the fact that $S_{3,9}=0$ we can simplify the equations in (5.4.35) into

$$S_{3,0}S_{3,6}(1+\omega^2) = S_{3,3}^2\,, \tag{5.4.36a}$$
$$S_{3,0}S_{3,6}(1+\omega) = S_{3,3}^2\,\omega\,, \tag{5.4.36b}$$
$$S_{3,0}S_{3,6}(\omega+\omega^2) = S_{3,3}^2\,\omega^2\,, \tag{5.4.36c}$$
$$S_{3,0}^2\,\omega = S_{3,3}S_{3,6}\,, \tag{5.4.36d}$$
$$S_{3,0}^2\,\omega^2 = S_{3,3}S_{3,6}\,\omega\,, \tag{5.4.36e}$$
$$S_{3,0}^2 = S_{3,3}S_{3,6}\,\omega^2\,. \tag{5.4.36f}$$

The sum of equations (5.4.36a) and (5.4.36c) along with (5.4.36f) gives

$$S_{3,0}^3\,(1+\omega^2)^{-1} = S_{3,3}^3\,; \tag{5.4.37}$$

by cubing (5.4.36f) and using (5.4.37) we find

$$S_{3,0}^3(1+\omega^2) = S_{3,6}^3\,. \tag{5.4.38}$$

By the fact that (3+4+5) are grouped together, then $S_{3,0} = S_{4,0} = S_{5,0}$ and $S_{0,6} = S_{0,7} = S_{0,8}$

by duality of $\{6,7,8\}$ with $\{5,4,3\}$. The fusion $S_{0,3\times 6} = S_{0,0}(S_{0,1}+S_{0,9}) = S_{0,3}S_{0,6}$ gives

$$(S_{3,0}-S_{0,0})(S_{3,0}+S_{0,0}) = \frac{1}{2}S_{0,0}\,, \tag{5.4.39}$$

where all the quantities are positive. Assuming that the two factors on the left of the equality correspond to either $\frac{1}{2}$ or $S_{0,0}$ on the right, it must therefore be that $S_{3,0}+S_{0,0} = \frac{1}{2}$ and $S_{3,0}-S_{0,0} = S_{0,0}$. We can therefore boil down (5.4.33) to

$$\begin{aligned}
1 =& \frac{3S_{3,0}^3}{\frac{1}{2}S_{3,0}} + \frac{3S_{3,3}^3}{S_{3,0}} + \frac{3S_{3,6}^3}{S_{3,0}} \\
=& \frac{3S_{3,0}^3}{\frac{1}{2}S_{3,0}} + \frac{3S_{3,0}^3\,(1+\omega^2)^{-1}}{S_{3,0}} + \frac{3S_{3,0}^3(1+\omega^2)}{S_{3,0}}
\end{aligned} \tag{5.4.40}$$

which gives $S_{3,0} = \frac{1}{3}$. We summarize the relationships as follows,

$$\begin{tikzcd}
S_{3,3} \arrow[r, "\omega"] & S_{3,4} \arrow[r, "\omega"] & S_{3,5} \arrow[r, "*"] & S_{3,6} \arrow[r, "\omega"] & S_{3,7} \arrow[r, "\omega"] & S_{3,8} \\
S_{3,0} \arrow[u] \arrow[urrr] \arrow[d, "\frac{1}{2}"] & & & & & \\
S_{0,0} = S_{1,0} = S_{2,0} & & & & &
\end{tikzcd}$$

where the arrow from $S_{3,5}$ to $S_{3,6}$ reflects the fact that the S-matrix elements are conjugates of each other. The arrows from $S_{3,0}$ to $S_{3,3}$ and $S_{3,6}$ reflect equations (5.4.37) and (5.4.38).

We can construct the analogues of (5.4.29) and (5.4.30), by encircling 4 and 5 around $(3+4+5)$. We have

$$S_{4,3} = S_{4,4\times 2} = \frac{S_{4,4}S_{4,2}}{S_{4,0}}\,, \qquad S_{5,3} = S_{5,4\times 2} = \frac{S_{5,4}S_{5,2}}{S_{5,0}}\,, \tag{5.4.41a}$$

$$S_{4,4} = S_{4,3\times 1} = \frac{S_{4,3}S_{4,1}}{S_{4,0}}\,, \qquad S_{5,4} = S_{5,3\times 1} = \frac{S_{5,3}S_{5,1}}{S_{5,0}}\,, \tag{5.4.41b}$$

$$S_{4,5} = S_{4,3\times 2} = \frac{S_{4,3}S_{4,2}}{S_{4,0}}\,. \qquad S_{5,5} = S_{5,3\times 2} = \frac{S_{5,3}S_{5,2}}{S_{5,0}}\,, \tag{5.4.41c}$$

as well as

$$S_{4,4} = S_{4,5\times 2} = \frac{S_{4,5}S_{4,2}}{S_{4,0}}\,, \qquad S_{5,4} = S_{5,5\times 2} = \frac{S_{5,5}S_{5,2}}{S_{5,0}}\,, \tag{5.4.42a}$$

$$S_{4,5} = S_{4,4\times 1} = \frac{S_{4,4}S_{4,1}}{S_{3,0}}\,. \qquad\qquad S_{5,5} = S_{5,4\times 1} = \frac{S_{5,4}S_{5,1}}{S_{5,0}}\,. \tag{5.4.42b}$$

Just like the case with $S_{3,0}$ we find

$$S_{4,1} = S_{4,0}\,\omega\,, \quad S_{4,2} = S_{4,0}\,\omega^2\,, \tag{5.4.42c}$$

$$S_{5,1} = S_{5,0}\,\omega\,, \quad S_{5,2} = S_{5,0}\,\omega^2\,, \tag{5.4.42d}$$

where $S_{4,0} = S_{5,0} = S_{3,0}$ due to their quantum dimensions. The relations among $S_{4,-}$ and $S_{5,-}$ are summarized by:

$$S_{4,3} \xrightarrow{\omega} S_{4,4} \xrightarrow{\omega} S_{4,5} \xrightarrow{*} S_{4,6} \xrightarrow{\omega} S_{4,7} \xrightarrow{\omega} S_{4,8}\,,$$

$$S_{5,3} \xrightarrow{\omega} S_{5,4} \xrightarrow{\omega} S_{5,5} \xrightarrow{*} S_{5,6} \xrightarrow{\omega} S_{5,7} \xrightarrow{\omega} S_{5,8}\,.$$

Lastly, recall that $S_{4,3}$ and $S_{4,5}$ can be related to $S_{3,0}$ by our previous analysis, so all the nontrivial S-matrix elements that we could not obtain from restricting the child theory, we can relate to $S_{3,0}$.

We now make a concluding remark about reconstructing the parent S-matrix. When we were considering the totally confined lines, as well as the child theory, all of the lines were direct sums of simple lines in the parent theory. In this sense, we already knew about the spectrum and fusion of the parent theory, though still, it can be nontrivial to construct the S-matrix elements as we have seen. But one tool we gain is the Verlinde formula, which is fundamentally important and also will be used in appendix A.3. One can wonder if it is possible to completely construct the parent lines through only the fusion information of the wall category.

5.5. Gauging Noninvertible symmetries: from a 2-category perspective

One of the most exciting prospects of generalized symmetries is the study of noninvertible symmetry operators. These are topological but instead of having a grouplike composition, their interactions are described by a general higher category. For grouplike global symmetries, the anomaly determines whether or not the symmetry can be gauged. The classification of such anomalies is well known to be captured by an invertible theory one dimension higher. Further, they can be classified using spectral sequences for group cohomology, and, more generally, for cobordisms as formalized in [100, 158]. Provided the anomaly vanishes, the gauging procedure will in general reshuffle the topological content, and in some cases add new richness into the theory in form of noninvertible operators [156, 45]. When gauging discrete abelian groups, what manifests is a dual group, which upon gauging takes us back to the original theory. The notion of *condensation* was introduced in [109] as a generalization of gauging, which applies to noninvertible symmetries. One particularly useful perspective of condensing a symmetry involves starting from the vacuum theory and proliferating in space (or perhaps in some subspace) a network of operators for that symmetry which fill out a new phase [209, 44]. Since this procedure is fully topological one can imagine running this procedure backwards and constructing a topological boundary between some phase and the vacuum. If one can go back and forth with no obstruction, then the symmetry is nonanomalous.

The purpose of this article is to generalize the notion of an anomaly for a symmetry, to an anomaly for a noninvertible symmetry. We will focus on noninvertible surface operators, for which the natural mathematical setting is a 2-category. For other applications of 2-categories in the physics literature, we refer the reader to [21, 16, 24, 203]. In general, the 2-category $\mathfrak{C}$ can have more structure such as a braiding, where the braiding takes place along the morphisms of $\mathfrak{C}$, or a syllepsis, and we will consider both cases. If one is in a setting were the surfaces are fully symmetric, we will show that a higher analogue of Deligne's theorem in [79] holds. More precisely, it was first announced in [148], that for any symmetric fusion 2-category $\mathfrak{S}$, there exists a fibre 2-functor $\mathrm{Fib} : \mathfrak{S} \to 2\mathbf{SVec}$ to the 2-category of super-2-vector spaces. In this sense, in the fully symmetric case, there is no obstruction to condensing all the operators, if we allow for emergent fermions. In this work we will consider both the cases of condensing to $2\mathbf{Vec}$, the 2-category of 2-vector spaces, and to $2\mathbf{SVec}$, where the latter involves working fermionically by condensing a fermionic algebra. This is the noninvertible analogue of being able to gauge a symmetry. In this article, we are mainly concerned with theories that have surface operators belonging to a fusion 2-category $\mathfrak{C}$ that can at least braid with each other, but are not fully symmetric. Since $\mathfrak{C}$ is not fully symmetric, there is no universal target that all the operators can condense to. We instead consider a related question which involves finding a subcategory of surface operators that enjoy more levels of monoidality than the general surface operators in the ambient category. One such example is given by the extra data of the aforementioned syllepsis, which can be thought of as anomaly cancellation data associated to the braiding [12]. It is then a meaningful question to ask what happens to the ambient category upon

[12]The additional level of monoidality means that the surface can secretly ascend to a higher dimension. For example, surfaces can braid in four total dimensions, but the data of being sylleptic means that some set of surfaces can lift to five dimensions.

condensing the subcategory. While working in a 2-category, if it so happens that there exists a procedure to go to the vacuum theory, then there will be no anomalies for any noninvertible symmetry, as all of them will have been "gauged". This idea will be useful in theories of gravity where it is expected that, not only there are no global symmetries, but also no noninvertible symmetries. For more on global symmetries arising in gravitational settings see [9, 134, 118, 7, 230].

Building on the work of the first author [77, 74], the main results of this article are proven in §5.7. More precisely, we present the result of condensing noninvertible surfaces in an ambient 2-category, with subsequent corollaries involving changing the properties of the condensation monad, also called separable algebra.

Theorem C. For $\mathfrak{B}$ a braided multifusion 2-category, condensing a braided separable algebra B in $\mathfrak{B}$ results in a multifusion 2-category.

Theorem D. For $\mathfrak{S}$ a sylleptic multifusion 2-category, condensing a symmetric separable algebra B in $\mathfrak{S}$, results in a braided multifusion 2-category.

Theorem E. For $\mathfrak{S}$ a sylleptic multifusion 2-category, condensing a symmetric separable algebra B in the symmetric center of $\mathfrak{S}$, results in a sylleptic multifusion 2-category. Further, if $\mathfrak{S}$ is symmetric, then condensing B yields a symmetric multifusion 2-category.

The auxiliary results of this article build off the main theorems by exploring particularly nice cases where the resulting category after condensation is "grouplike", in addition to being braided, or sylleptic. We call these categories *strongly fusion*, and the operator content is essentially captured by the surfaces [152]. The reader interested in applications of the main theorems can go to §5.8 for explicit examples of condensations within 2-categories, which in the right setting, yield strongly fusion categories. In particular, we show that every symmetric fusion 2-category can be condensed to a symmetric strongly fusion 2-category. For theories described by strongly fusion 2-categories, the obstruction to condensing to the vacuum is given by a cohomology class, which we compute when the 2-category is braided. In addition, we show that the obstruction to condense a symmetric strongly fusion 2-category to the 2-category of super-2-vector spaces vanishes. Thereby establishing the following result:

Theorem F. Every symmetric fusion 2-category admits a fibre 2-functor to $2\mathbf{SVec}$.

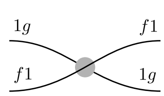
The above theorem categorifies [79].

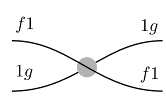
We now outline the contents of this article: In §5.6 we explain the graphical calculus used for braided, sylleptic, and symmetric monoidal 2-categories. We also discuss algebras, and the relationship between modules and condensation. In §5.7 we prove the main theorems about braided or sylleptic monoidal 2-category, and the result of condensing separable algebras that are respectively braided or symmetric. We examine specific examples of condensing separable algebras in connected and disconnected 2-categories that are interesting for physical applications in §5.8; we find that in some cases, the 2-category becomes *strongly fusion*. In §5.9 we perform cohomology computations for theories described by the braided and symmetric strongly fusion 2-categories, and report on the obstruction to condensing the theory to the vacuum.

5.6. Preliminaries on 2-Categories

5.6.1. Graphical Calculus

We begin by setting up the fundamental definitions and explaining the computational language of string diagrams. We work within a monoidal 2-category $\mathfrak{C}$ with monoidal unit I and monoidal product $\Box$ in the sense of definition 2.3 of [212]. Thanks to the coherence theorem of [129], we may assume without loss of generality that $\mathfrak{C}$ is strict cubical (in the sense that it satisfiesthe conditions of definition 2.26 of [212]). In this setting, we use the graphical calculus of [120], as described in [77] (see also [73]). In particular, we will often omit the monoidal product $\Box$ from our notation. In addition, identity 1-morphisms are denoted using the symbol 1. Further, the interchanger is depicted using by the string diagram below on the left, and its inverse by that on the right:



,


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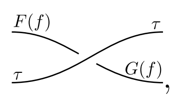
The lines represent 1-morphisms and their composition is read from top to bottom. The string diagrams are then read from left to right, and the coupons represent 2-morphisms. The regions between the lines represent objects of the 2-category, which are specified uniquely by the 1-morphisms.

We also need to recall the graphical conventions related to 2-natural transformations from [120]. In the present article, these will exclusively be used for the braiding, which will be introduced below. Let $F, G : \mathfrak{A} \to \mathfrak{B}$ be two (weak) 2-functors, and let $\tau : F \Rightarrow G$ be 2-natural transformation. This means that, for every object A in $\mathfrak{A}$, we have a 1-morphism

$\tau_A : F(A) \to G(A)$, and for every 1-morphism $f : A \to B$ in $\mathfrak{A}$, we have a 2-isomorphism

$$\begin{tikzcd}
F(A) \arrow[r, "\tau_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\overset{\tau_f}{\nearrow}"] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\tau_B"'] & G(B),
\end{tikzcd}$$

The collection of these 2-isomorphisms has to satisfy the obvious coherence relations. In our graphical language, we will depict the 2-isomorphism τ_f using the following diagram on the left, and its inverse using the diagram on the right:



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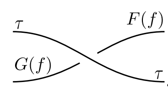

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Braided Monoidal 2-Categories

Let $\mathfrak{B}$ a braided monoidal 2-category in the sense of definition 2.3 of [212]. In particular, $\mathfrak{C}$ is a monoidal 2-category, so that we use I to denote its monoidal unit, and $\Box$ to denote its monoidal product. The coherence theorem of [128] allows us to assume that $\mathfrak{B}$ is a semi-strict braided monoidal 2-category. In particular, the underlying monoidal 2-category is strict cubical. Further, $\mathfrak{B}$ comes equipped with a braiding b, which is an adjoint 2-natural equivalence given on objects A, B in $\mathfrak{B}$ by

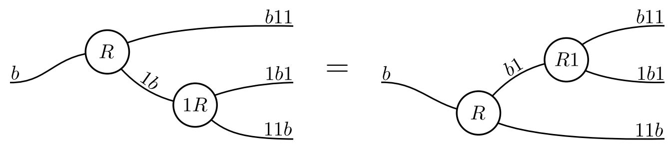
$$b_{A,B} : A\Box B \to B\Box A.$$

Further, there are two invertible modifications R and S, which are given on the objects A, B, C of $\mathfrak{B}$ by

$$\begin{tikzcd}
ABC \arrow[rr, "b"] \arrow[dr, "b1"'] & \arrow[d, phantom, "\Downarrow R"] & BCA, \\
& BAC \arrow[ur, "1b"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
ABC \arrow[rr, "b_2"] \arrow[dr, "1b"'] & \arrow[d, phantom, "\Downarrow S"] & CAB \\
& ACB \arrow[ur, "b1"'] &
\end{tikzcd}$$

where the subscript in b_2 records that the braiding occurs between the first two objects on the left and the next ones. On the other hand, b means that the braiding occurs between the first object on the left and the next ones. These two modifications are subject to the following relations:

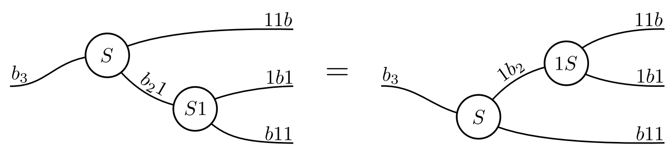
a. For all objects A, B, C, D in $\mathfrak{B}$, we have



(5.6.1)

in $Hom_{\mathfrak{B}}(ABCD, BCDA)$.

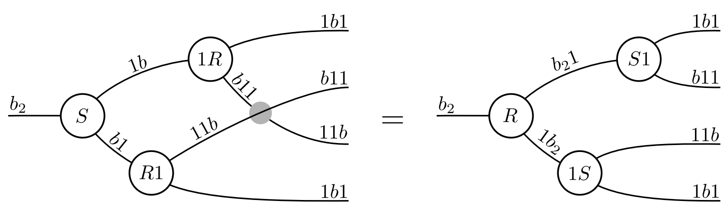
b. For all objects A, B, C, D in $\mathfrak{B}$, we have



(5.6.2)

in $Hom_{\mathfrak{B}}(ABCD, DABC)$,

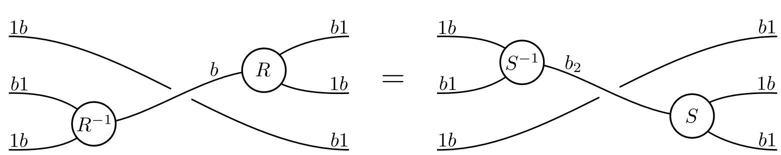
c. For all objects A, B, C, D in $\mathfrak{B}$, we have



(5.6.3)

in $Hom_{\mathfrak{B}}(ABCD, CDAB)$,

d. For all objects A, B, C in $\mathfrak{B}$, we have



(5.6.4)

in $Hom_{\mathfrak{B}}(ABC, CBA)$,

e. For all objects A in $\mathfrak{B}$, the adjoint 2-natural equivalences

$$b_{A,I} : A\Box I \to I\Box A \text{ and } b_{I,A} : I\Box A \to A\Box I$$

are the identity adjoint 2-natural equivalences,

f. For all objects A, B, C in $\mathfrak{B}$, the 2-isomorphisms $R_{A,B,C}$ and $S_{A,B,C}$ are equal to the identity 2-isomorphism whenever either A, B, or C is equal to I.

In each of the $Hom_{\mathfrak{B}}$ above, the first set of objects is given by the top most region bound by 1-morphism, and the second set of objects is given by the bottom most region.

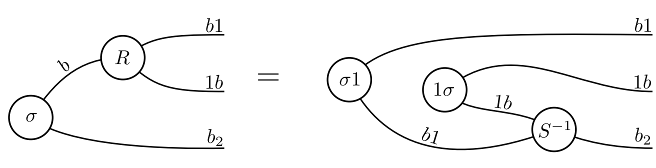
Sylleptic and Symmetric Monoidal 2-Categories

Our work will also involve sylleptic monoidal 2-categories (see definition 2.3 of [212]), these are braided monoidal 2-categories equipped with an additional structure called a *syllepsis*. Without loss of generality, we may assume that every sylleptic monoidal 2-category $\mathfrak{S}$ semi-strict. (This follows from a slight generalization of [131].) This means that $\mathfrak{S}$ is a semi-strict braided monoidal 2-category equipped with an invertible modification σ given on the objects A, B of $\mathfrak{S}$ by

$$\begin{tikzcd}
AB \arrow[rr, equal] \arrow[dr, "b"'] & \arrow[d, phantom, "\Downarrow\sigma"] & AB. \\
& BA \arrow[ur, "b"'] &
\end{tikzcd}$$

Furthermore, the invertible modification σ satisfies the following relations:

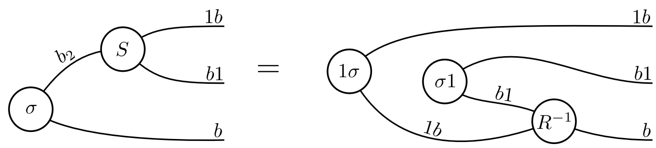
a. For all objects A, B, C of $\mathfrak{S}$, we have



(5.6.5)

in $Hom_{\mathfrak{B}}(ABC, ABC)$,

b. For all objects A, B, C of $\mathfrak{S}$, we have



(5.6.6)

in $Hom_{\mathfrak{B}}(ABC, ABC)$,

c. For all objects A, B of $\mathfrak{S}$, the 2-isomorphisms $\sigma_{A,B}$ is the identity 2-morphism whenever either A or B is equal to I.

We give a physical interpretation of syllepsis for surfaces. Namely, two surfaces existing in 5d braid by passing one around around each other in a 2 parameter family. The surfaces can exchange the order of which one is on top by going into the fifth dimension and using the syllepsis.

Finally, we will also consider symmetric monoidal 2-categories. Thanks to the main result of [131], every symmetric monoidal 2-category is equivalent to a semi-strict symmetric monoidal 2-category that is to a semi-strict sylleptic monoidal 2-category $\mathfrak{S}$ as defined above, whose syllepsis satisfies

$$\sigma_{B,A} \circ b_{A,B} = b_{A,B} \circ \sigma_{A,B}\,, \tag{5.6.7}$$

for every object A, B in $\mathfrak{S}$. Physically speaking, if the surface operators have enough freedom to move around each other, such as in six ambient spacetime dimensions, then this is automatic.

5.6.2. Algebras and Modules

Let $\mathfrak{C}$ be a strict cubical monoidal 2-category. We recall the definition of an algebra in $\mathfrak{C}$ expressed using our graphical calculus from [77]. These objects were introduced under the name pseudo-monoidal in [62]. The definition of an algebra in an arbitrary monoidal 2-category using our graphical conventions may be found in [73].

Definition 5.6.1. An algebra in $\mathfrak{C}$ consists of:

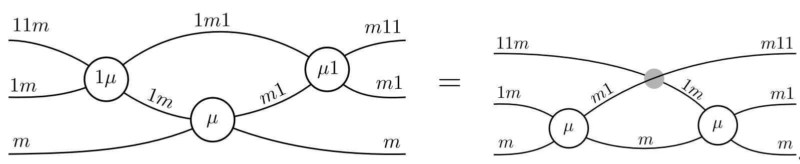
1. An object A of $\mathfrak{C}$;
2. Two 1-morphisms $m : A\Box A \to A$ and $i : I \to A$;

3. Three 2-isomorphisms

$$\begin{tikzcd} A \arrow[rr, equal] \arrow[dr, "i1"'] & \arrow[d, phantom, "\lambda\Downarrow"] & A \\ & AA, \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} AAA \arrow[r, "m1"] \arrow[d, "1m"'] \arrow[dr, phantom, "\overset{\mu}{\nearrow}"] & AA \arrow[d, "m"] \\ AA \arrow[r, "m"'] & A, \end{tikzcd} \qquad \begin{tikzcd} & AA \arrow[dr, "m"] \arrow[d, phantom, "\Downarrow\rho"] & \\ A \arrow[ur, "1i"] \arrow[rr, equal] & & A, \end{tikzcd}$$

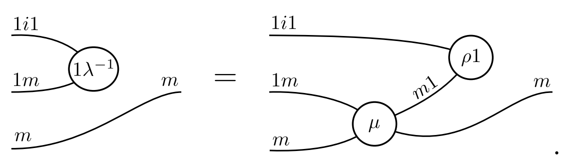
satisfying:

a.



(5.6.8)

b.



(5.6.9)

Let us now recall the definition of a right A-module in $\mathfrak{C}$ from [77]. Once, again the definition in a general monoidal 2-category may be found in [73].

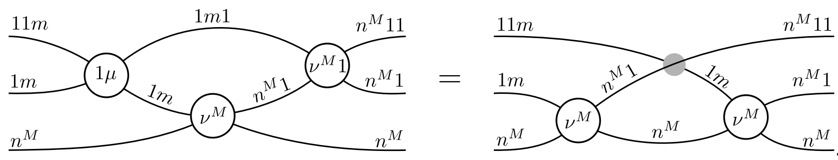
Definition 5.6.2. A right A-module in $\mathfrak{C}$ consists of:

1. An object M of $\mathfrak{C}$;
2. A 1-morphism $n^M : M\Box A \to M$;
3. Two 2-isomorphisms

$$\begin{tikzcd} MAA \arrow[r, "n^M1"] \arrow[d, "1m"'] \arrow[dr, phantom, "\overset{\nu^M}{\nearrow}"] & MA \arrow[d, "n^M"] \\ MA \arrow[r, "n^M"'] & M, \end{tikzcd} \qquad \begin{tikzcd} & MA \arrow[dr, "n^M"] \arrow[d, phantom, "\Downarrow\rho^M"] & \\ M \arrow[ur, "1i"] \arrow[rr, equal] & & M, \end{tikzcd}$$

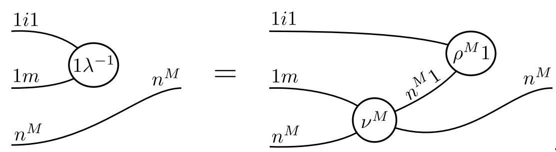
satisfying:

a.



(5.6.10)

b.



(5.6.11)

The definitions of a right A-module 1-morphism and that of a right A-module 2-morphism in $\mathfrak{C}$ may be found in [77]. These objects assemble into a 2-category as was proven in lemma 3.2.10 of [73].

Lemma 5.6.3. Let A be an algebra in a monoidal 2-category $\mathfrak{C}$. Right A-modules, right A-module 1-morphisms, and right A-module 2-morphisms form a 2-category, which we denote by $\mathbf{Mod}_{\mathfrak{C}}(A)$.

5.6.3. Higher Condensations and Separable Algebras

We now briefly review the notions of 2-condensations and 2-condensation monads. These notions were introduced in [109] as the categorifications of the notions of split surjection and idempotent.

Definition 5.6.4. A 2-*condensation* in a 2-category $\mathfrak{C}$ consists of two objects A and B, together with two 1-morphisms $f : A \leftrightarrows B : g$, and two 2-morphisms $\phi : f \circ g \Rightarrow Id_B$ and $\gamma : Id_B \Rightarrow f \circ g$, such that $\phi \cdot \gamma = Id_{Id_B}$.

The data of 2-condensation as in the above definition induces a 2-condensation monad on the object A.

Definition 5.6.5. A 2-condensation monad in $\mathfrak{C}$ is an object A together with a 1-morphism $e : A \to A$ and 2-morphisms $\mu : e \circ e \to e$ and $\delta : e \to e \circ e$, such that μ is associative, δ is coassociative, the Frobenius relations holds, and $\mu \cdot \delta = Id_e$.

We say that a 2-condensation monad can be split, if it can be extended to a 2-condensation. There is also a categorification of the concept of idempotent complete 1-category. Before we review this definition, let us recall that a 2-category $\mathfrak{C}$ is locally idempotent complete if for all objects $A, B \in \mathfrak{C}$, the 1-category $\hom_{\mathfrak{C}}(A, B)$ is idempotent complete.

Definition 5.6.6. We say that a 2-category is *Karoubi complete* if it is locally idempotent complete, and every 2-condensation monad can be split.

Physically, this means that any surface that arises as a condensation defect, i.e. a network of lower dimensional objects, is included in the 2-category.

The 2-category $\mathfrak{C}$ is locally finite semisimple if $\hom_{\mathfrak{C}}(A, B)$ is a finite semisimple $\mathbb{C}$-linear 1-category (i.e. an abelian $\mathbb{C}$-linear 1-category with finitely many isomorphism classes of simple object and in which every object decomposes as a finite direct sum of simple objects). We say that an object A of $\mathfrak{C}$ is simple if the identity 1-morphism Id_A is a simple object of the 1-category $\mathrm{End}_{\mathfrak{C}}(A)$.

Definition 5.6.7. A *finite semisimple* 2-category is a locally finite semisimple 2-category, that has adjoints for 1-morphisms, is Karoubi complete, has direct sums for objects, and has finitely many equivalence classes of objects.

Finite semisimple 2-categories were introduced in [86]. We have recalled an equivalent version of their definition (see theorem 3.1.7 [109]). Through proposition 1.4.5 of [86], any object in a finite semisimple 2-category is the direct sum of finitely many simple objects, i.e. surfaces.

Let us recall the following definition from [76]. Thanks to section 2.2 of [76], this is equivalent to the original definition given in [86].

Definition 5.6.8. A multifusion 2-category is a finite semisimple rigid monoidal 2-category. A fusion 2-category is a multifusion 2-category whose monoidal unit is a simple object.

Further, in a finite semisimple 2-category, two simple objects that have a nonzero 1-morphism between them are organized into the same component of $\mathfrak{C}$, denoted by $\pi_0(\mathfrak{C})$, due to the categorical Schur's lemma (see proposition 1.2.19 of [86]). In other words, $\pi_0(\mathfrak{C})$ only remembers objects up to condensation. We review the following definition from [149], due to its prevalence in section 5.8:

Definition 5.6.9. A multifusion 2-category $\mathfrak{C}$ is *bosonic strongly fusion* if the braided fusion 1-category $\Omega\mathfrak{C} = \mathrm{End}_{\mathfrak{C}}(\mathbb{1}_{\mathfrak{C}})$ is equivalent to **Vec**. It is *fermionic strongly fusion* if $\Omega\mathfrak{C} \simeq \mathbf{SVec}$.

In such a 2-category $\mathfrak{C}$, the main result of [152] shows that $\pi_0(\mathfrak{C})$ has grouplike fusion rules.

Definition 5.6.5 has been categorified further in [109] where the authors define an n-condensation monad for any n. Examples of 3-condensation monads are given by separable algebras in a monoidal 2-category as defined below. It is also convenient to introduce the notion of a rigid algebra, which can be traced back to [116]. Rigid algebras are a weakening of separable algebras, and were first considered in the setting of fusion 2-categories in [150]. We also point out that both of these definitions are thoroughly unpacked in section 2.1 of [77].

Definition 5.6.10. An algebra A in a monoidal 2-category $\mathfrak{C}$ is called rigid if the multiplication map $m : A \Box A \to A$ has a right adjoint m^* as an A-A-bimodule 1-morphism. A rigid algebra A in $\mathfrak{C}$ is called separable if the counit $\epsilon^m : m \circ m^* \Rightarrow Id_A$ witnessing that m^* is right adjoint to m as an A-A-bimodule 1-morphism has a section as an A-A-bimodule 2-morphism.

We will see the separability property appear in the theorems in section 5.7. In fact, these results holds more generally for any 3-condensation monad. For later use, we also record the following result, which is given by combining together proposition 3.1.2 of [77] and corollary 2.2.3 of [72].

Proposition 5.6.11. Let A be a separable algebra in a fusion 2-category $\mathfrak{C}$. Then, the 2-category $\mathbf{Mod}_{\mathfrak{C}}(A)$ is a finite semisimple 2-category.

The physical picture for condensing surfaces in a 2-category involves finding some gapped boundary of the initial 2-category $\mathfrak{C}$, and then possibly triggering another condensation in order to map to 2**SVec**, see figure 5.9. This bulk boundary point of view has been given the name of a "quiche", in [104]. The tensor unit of the boundary can be identified with a separable algebra A in $\mathfrak{C}$, and we denote it as $\mathbf{Mod}_{\mathfrak{C}}(A)$, the 2-category of A-modules in $\mathfrak{C}$. From this point of view, condensation along a specific direction of spacetime builds modules which usually causes the resulting 2-category to lose a level of monoidality, this is reflected in Theorems C and D. Theorem E, however, maintains the sylleptic property due to the extra condition of being in the symmetric center. For a description of condensation in 1-categories where modules are explicitly built, see [169, 251].

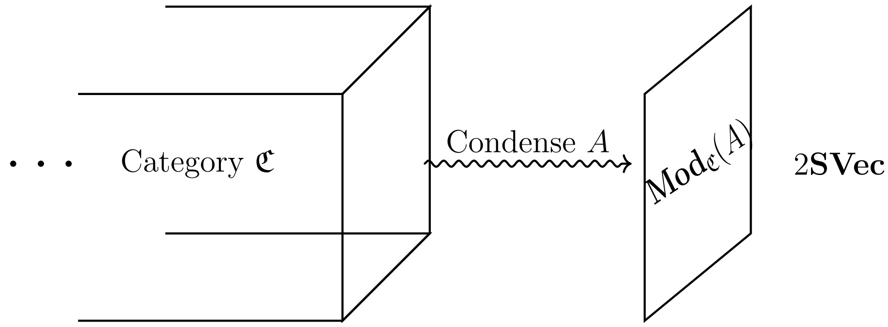


Figure 5.9: This gives a three dimensional view of condensing the algebra A, taking place in a 2-category $\mathfrak{C}$. The resulting boundary is the category $\mathbf{Mod}_{\mathfrak{C}}(A)$, and $2\mathbf{SVec}$ represents the "fermionic vacuum".

5.6.4. Relative Tensor Product

We now recall the definition of the relative tensor product over an algebra in a monoidal 2-category given in section 3 of [74]. These definitions will be important for the proofs of the main theorems in §5.7.2 and §5.7.3. We also give sufficient criterion for the 2-category of bimodules over an algebra to carry a monoidal structure.

Let us now fix an algebra A in a fusion 2-category $\mathfrak{C}$, together with M a right A-module in $\mathfrak{C}$, and N a left A-module in $\mathfrak{C}$ (for which we use the notations of appendix A of [77]). We begin by defining A-balanced 1-morphisms and 2-morphisms out of the pair (M, N).

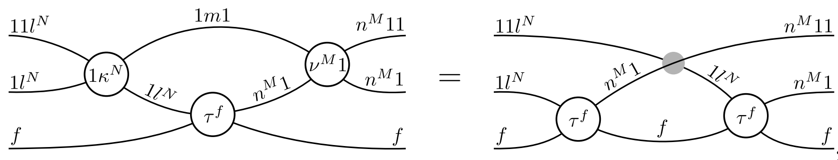
Definition 5.6.12. Let C be an object of $\mathfrak{C}$. An A-balanced 1-morphism $(M, N) \to C$ consists of:

1. A 1-morphism $f : M\Box N \to C$ in $\mathfrak{C}$;
2. A 2-isomorphism

$$\begin{tikzcd} MAN \arrow[r, "n^M 1"] \arrow[d, "1l^N"'] \arrow[dr, phantom, "\overset{\tau^f}{\Rightarrow}"] & MN \arrow[d, "f"] \\ MN \arrow[r, "f"'] & C, \end{tikzcd}$$

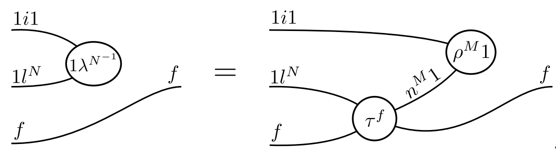
satisfying:

a.



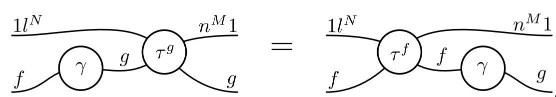
(5.6.12)

b.



(5.6.13)

Definition 5.6.13. Let C be an object of $\mathfrak{C}$, and $f, g : (M, N) \to C$ be two A-balanced 1-morphisms. An A-balanced 2-morphism $f \Rightarrow g$ is a 2-morphism $\gamma : f \Rightarrow g$ in $\mathfrak{C}$ such that



(5.6.14)

Definition 5.6.14. The relative tensor product of M and N over A, if it exists, is an object $M\Box_A N$ of $\mathfrak{C}$ together with an A-balanced 1-morphism $t_A : (M, N) \to M\Box_A N$ satisfying the following 2-universal property:

1. For every A-balanced 1-morphism $f : (M, N) \to C$, there exists a 1-morphism $\widetilde{f} : M\Box_A N \to C$ in $\mathfrak{C}$ and an A-balanced 2-isomorphism $\xi : \widetilde{f} \circ t_A \cong f$.

2. For any 1-morphisms $g, h : M\Box_A N \to C$ in $\mathfrak{C}$, and any A-balanced 2-morphism $\gamma : g \circ t_A \Rightarrow h \circ t_A$, there exists a unique 2-morphism $\zeta : g \Rightarrow h$ such that $\zeta \circ t_A = \gamma$.

The following result was established in theorem 3.1.6 of [74].

Theorem 5.6.15. Let A be a separable algebra in a Karoubi complete monoidal 2-category $\mathfrak{C}$. Then, the relative tensor product of any right A-module M and any left A-module N exists.

Using this result, it was shown in theorem 3.2.8 of [74] that the relative tensor product over A endows the 2-category $\mathbf{Bimod}_{\mathfrak{C}}(A)$ of A-A-bimodules in the Karoubi complete 2-category $\mathfrak{C}$ with a *weak* monoidal structure. In particular, all the relevant structures were exhibited using the 2-universal property of the relative tensor product over multiple separable algebras.

5.7. Braided and Symmetric Algebras

5.7.1. Definitions

Let $\mathfrak{B}$ be a semi-strict braided monoidal 2-category. The definition of a braided algebra in a braided monoidal 2-category, also called braided pseudo-monoid, can be traced back to [62]. Below we review this definition using the graphical calculus that we have previously introduced. We refer the reader to [194] for a version of this definition, resp. the next one, in a completely general braided, resp. sylleptic, monoidal 2-category.

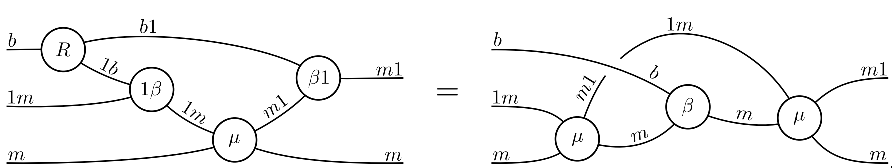
Definition 5.7.1. A braided algebra in $\mathfrak{B}$ consists of:

1. An algebra $(B, m, i, \lambda, \mu, \rho)$ in $\mathfrak{B}$;
2. A 2-isomorphisms

$$\begin{tikzcd}
 & AA \arrow[dr, "m"] \arrow[d, phantom, "\Downarrow\beta"] & \\
AA \arrow[ur, "b"] \arrow[rr, "m"'] & {} & A,
\end{tikzcd}$$

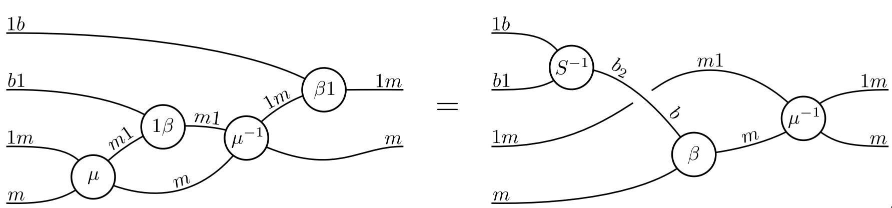
satisfying:

a.



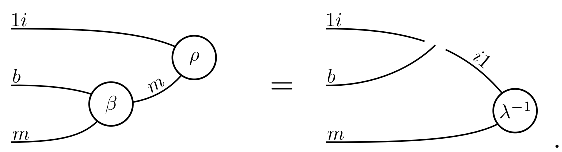
(5.7.1)

b.



(5.7.2)

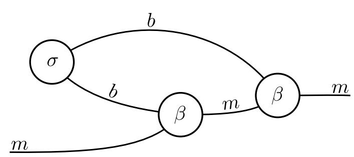
c.



(5.7.3)

Let $\mathfrak{S}$ be a semi-strict sylleptic monoidal 2-category. The definition of a symmetric algebra in $\mathfrak{S}$, also called symmetric pseudo-monoid, first appeared in [62]. We review this definition using our graphical calculus.

Definition 5.7.2. A symmetric algebra in $\mathfrak{S}$ is a braided algebra $(B, m, i, \lambda, \mu, \rho, \beta)$ such that



$$= Id_m. \tag{5.7.4}$$

Example. Braided algebras in the symmetric fusion 2-category 2**Vec** are exactly braided monoidal finite semisimple 1-categories. Symmetric algebras in the symmetric fusion 2-category 2**Vec** are exactly symmetric monoidal finite semisimple 1-categories.

5.7.2. The 2-Category of Modules over a Braided Algebra.

As before, we take $\mathfrak{B}$ to be a semi-strict braided monoidal 2-category. Furthermore, we will assume throughout that $\mathfrak{B}$ is a Karoubi complete 2-category.

Lemma 5.7.3. Let B a braided algebra in $\mathfrak{B}$. There is a 2-functor

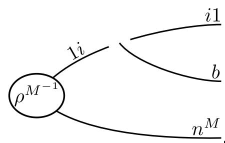
$$Ind^+ : \mathbf{Mod}_{\mathfrak{B}}(B) \to \mathbf{Bimod}_{\mathfrak{B}}(B),$$

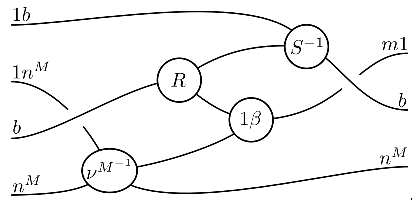
which is fully faithful on 2-morphisms.

Proof. Let M be a right B-module. The underlying right B-module of $Ind^+(B)$ is given by B. In the notations of [77], the left B-module structure on $Ind^+(B)$ is given by the 1-morphism

$$l^M : B\Box M \xrightarrow{b} M\Box B \xrightarrow{n^M} M$$

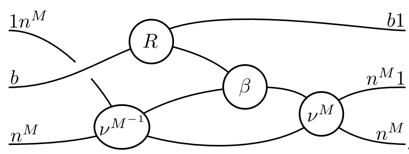
together with the 2-isomorphisms

$\lambda^M :=$ 

, $\kappa^M :=$ 

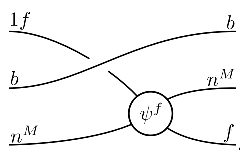
.

Further, the compatibility between the left and the right actions is given by the 2-isomorphism

$\beta^M :=$ 

.

Given a right B-module 1-morphism $f : M \to N$, the underlying right B-module 1-morphism of the B-B-module 1-morphism $Ind^+(f)$ is f. Its left B-module structure is given by

$\chi^f :=$ 

.

Given a right B-module 2-morphism $\gamma : f \Rightarrow g$, it is easy to check that γ is a B-B-bimodule 2-morphism $Ind^+(f) \Rightarrow Ind^+(g)$, so that we can set $Ind^+(\gamma) = \gamma$. It follows readily from the definitions that Ind^+ defines a strict 2-functor. Moreover, note that Ind^+ is fully faithful on 2-morphisms by construction. □

Remark 5.7.5. When constructing the 2-functor Ind^+, we have used the braiding b of $\mathfrak{B}$. Instead, we could have used its adjoint equivalence $b^\bullet$, and so doing obtained a 2-functor $Ind^- : \mathbf{Mod}_{\mathfrak{B}}(B) \to \mathbf{Bimod}_{\mathfrak{B}}(B)$.

Proposition 5.7.4. Let B a braided separable algebra in $\mathfrak{B}$. Then, $\mathbf{Mod}_{\mathfrak{B}}(B)$ is a monoidal 2-category with monoidal unit B.

Proof. Thanks to lemma 5.7.3, we can view $\mathbf{Mod}_{\mathfrak{B}}(B)$ as a sub-2-category of $\mathbf{Bimod}_{\mathfrak{B}}(B)$. For convenience, we will assume that this sub-2-category is replete. Now, as was recalled in section 5.6.4 above, the monoidal structure of $\mathbf{Bimod}_{\mathfrak{B}}(B)$ is given by the relative tensor product $\Box_B$, which is defined using the 2-universal property reviewed in definition 5.6.14.

Given M and N two right B-modules, we want to show that the B-B-bimodule $M\Box_B N$ is actually an object of the sub-2-category $\mathbf{Mod}_{\mathfrak{B}}(B)$. In order to prove this, we need to unfold the definition of the left B-module structure on $M\Box_B N$. Let us write $t : M\Box N \to M\Box_B N$, together with $\tau^t : t\circ(M\Box l^N) \cong t\circ(n^M\Box N)$, for the 2-universal B-balanced 1-morphism as in definition 5.6.14. Furthermore, note that for any C in $\mathfrak{B}$, $C\Box t$ equipped with $C\Box\tau^t$ is a 2-universal B-balanced 1-morphism. By remark 3.2.3 of [74], the 1-morphism $l^{M\Box_B N} : B\Box(M\Box_B N) \to M\Box_B N$ is induced by the 2-universal property of $B\Box t$ applied to the solid arrow diagram

$$\begin{tikzcd}
B\Box M\Box N \arrow[r, "1t"] \arrow[d, "(n^M\circ b)1"'] \arrow[dr, phantom, "\nearrow v^l"] & B\Box(M\Box_B N) \arrow[d, dotted, "l^{M\Box_B N}"] \\
M\Box N \arrow[r, "t"'] & M\Box_B N,
\end{tikzcd}$$

where the left bottom composite 1-morphism is equipped with the obvious B-balanced structure. The 1-morphism $n^{N\Box_B M} : (M\Box_B N)\Box B \to M\Box_B N$ is defined similarly. But, the 2-isomorphism

$$(t\circ 1n^N\circ R^{-1})\cdot(\tau^{t^{-1}}\circ b1) : t\circ(n^M\Box N)\circ(b\Box N) \cong t\circ(M\Box n^N)\circ b$$

is B-balanced. Thanks to the 2-universal property of the relative tensor product, this means that there exists a 2-isomorphism $\theta : l^{M\Box_B N} \cong n^{M\Box_B N}\circ b$. Furthermore, it also follows from the 2-universal property that θ promotes the identity right B-module 1-morphism on $M\Box_B N$ to a B-B-bimodule 1-equivalence from $M\Box_B N$ to $Ind^+(M\Box_B N)$. This proves that the objects of $\mathbf{Mod}_{\mathfrak{B}}(B)$ are closed under $\Box_B$. A similar argument shows that the 1-morphisms of $\mathbf{Mod}_{\mathfrak{B}}(B)$ are closed under $\Box_B$, which concludes the proof. $\Box$

Remark 5.7.6. We emphasize that $\mathbf{Mod}_{\mathfrak{B}}(B)$ is not a braided 2-category in general, as can be seen from example 5.7.2 below. Further, we also note that our proof of proposition 5.7.4 only used the existence of the relative tensor product over B for any B-B-bimodules in $\mathfrak{B}$. We refer the reader to remark 3.2.11 of [74] for a more thorough discussion. An analogous comment can be made with regards to lemma and 5.7.3 above and lemma 5.7.5 below.

In order to prove our next theorem, we need the following technical lemma.

Lemma 5.7.5. The 2-functor $F : \mathfrak{B} \to \mathbf{Mod}_{\mathfrak{B}}(B)$ given by sending the object C in $\mathfrak{B}$ to $C\Box B$ with its canonical right B-module structure is monoidal.

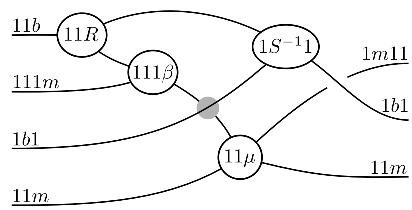
Proof. Let C and D be two objects of $\mathfrak{B}$. Firstly, note that $C\Box D\Box B$ satisfies the 2-universal property of $(C\Box B)\Box_B(D\Box B)$ in $\mathbf{Bimod}_{\mathfrak{B}}(B)$. More precisely, the B-B-bimodule

1-morphism

$$u_{C,D} : C\Box B\Box D\Box B \xrightarrow{1b1} C\Box D\Box B\Box B \xrightarrow{11m} C\Box D\Box B$$

admits a canonical B-balanced structure given by

$\tau^{u_{C,D}} :=$

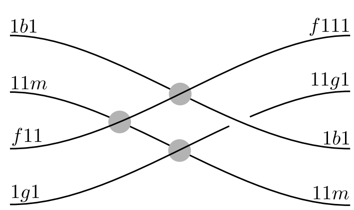


and satisfies the conditions of definition 5.6.14. In particular, this yields B-B-bimodule 1-equivalences $e_{C,D} : (C\Box B)\Box_B(D\Box B) \simeq (C\Box D)\Box B$ for every C and D in $\mathfrak{B}$ together with a B-balanced B-B-bimodule 2-ismorphism $\zeta_{C,D}$ as in the following diagram:

$$\begin{tikzcd}
C\Box B\Box D\Box B \arrow[rr, "u_{C,D}"] \arrow[dr, "t_{CB,DB}"'] & \arrow[d, phantom, "\Downarrow\zeta_{C,D}"] & C\Box D\Box B \\
& (C\Box B)\Box_B(D\Box B). \arrow[ur, "e_{C,D}"'] &
\end{tikzcd}$$

Secondly, observe that for any two 1-morphisms $f : C \to E$ and $g : D \to F$ in $\mathfrak{B}$, the B-B-bimodule 2-isomorphism

$\upsilon_{f,g} :=$



is B-balanced. Thus, thanks to the 2-universal property of the relative tensor product, we can use the 2-isomorphisms $\upsilon_{f,g}$ to promote the collection of the B-B-bimodule 1-equivalences $e_{C,D}$ for varying C and D to a 2-natural equivalence e.

Using the 2-universal property of the relative tensor product repeatedly (together with the variants over multiple algebras considered in section 3.2 of [74]), one constructs the remaining data necessary to endow F with a monoidal structure, and prove that they satisfy the relevant axioms from definition 2.5 of [212]. $\Box$

Theorem 5.7.6. Let $\mathfrak{B}$ be a braided multifusion 2-category, and B a braided separable algebra in $\mathfrak{B}$. Then, $\mathbf{Mod}_{\mathfrak{B}}(B)$ is a multifusion 2-category.

Proof. The 2-category $\mathbf{Mod}_{\mathfrak{B}}(B)$ is finite semisimple thanks to proposition 5.6.11. Further, we have shown in proposition 5.7.4 that it admits a monoidal structure. It therefore only remains to prove that it has duals. But, as $\mathfrak{B}$ is a multifusion 2-category, it has right and left duals. In particular, every object in the image of $F : \mathfrak{B} \to \mathbf{Mod}_{\mathfrak{B}}(B)$ has a right and a left dual. But, it was shown in lemma 3.1.1 of [77] that every right B-module M is the splitting of a 2-condensation monad (in $\mathbf{Mod}_{\mathfrak{B}}(B)$) supported on $M \Box B = F(M)$. Thence, it follows from lemma 5.5 of [71] that M has a right and a left dual, and thereby concludes the proof. $\square$

Following section 5.2 of [74], we say that a separable algebra B is connected if its unit 1-morphism $i : I \to B$ is simple. Under the equivalence

$$Hom_B(B, B) \simeq Hom_{\mathfrak{B}}(I, B)$$

of lemma 3.2.13 of [73], we have $Id_B \mapsto i$. Thus, B is a simple right B-module if and only if B is a connected algebra. Combined with the above theorem, this yields the following corollary.

Corollary 5.7.7. Let $\mathfrak{B}$ be a braided multifusion 2-category, and B a connected braided separable algebra in $\mathfrak{B}$. Then, $\mathbf{Mod}_{\mathfrak{B}}(B)$ is a fusion 2-category.

Example. Let $\mathcal{B}$ be a braided multifusion 1-category, that is a braided separable algebra in $2\mathbf{Vec}$. Then, $\mathbf{Mod}_{\mathbf{2Vec}}(\mathcal{B}) = \mathbf{Mod}(\mathcal{B})$ is the multifusion 2-category of finite semisimple right $\mathcal{B}$-module 1-categories with monoidal structure given by $\boxtimes_{\mathcal{B}}$ the relative Deligne tensor over $\mathcal{B}$. The braided separable algebra $\mathcal{B}$ is braided if and only if $\mathcal{B}$ is a fusion 1-category, in which case $\mathbf{Mod}(\mathcal{B})$ is a fusion 2-category. Finally, we note that it follows from a slight variant of proposition 2.4.7 of [76] that $\mathbf{Mod}(\mathcal{B})$ is braided if and only if $\mathcal{B}$ is symmetric.

5.7.3. The 2-Category of Modules over a Symmetric Algebra

In this section we give sufficient conditions for the 2-category of modules over a braided algebra to be itself braided. We also explain when the 2-category of modules is sylleptic or symmetric.

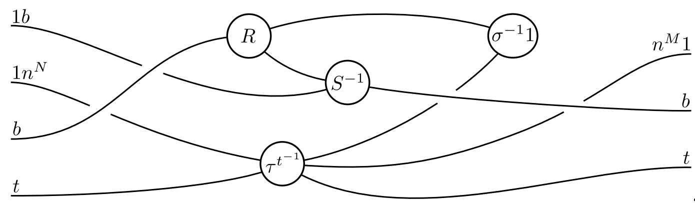
Theorem 5.7.8. Let $\mathfrak{S}$ be a Karoubi complete sylleptic monoidal 2-category, and B a symmetric separable algebra in $\mathfrak{S}$. Then, $\mathbf{Mod}_{\mathfrak{S}}(B)$ is a braided monoidal 2-category.

Proof. Without loss of generality, we may assume that $\mathfrak{S}$ is semi-strict. Our first task is to endow the monoidal 2-category $\mathbf{Mod}_{\mathfrak{S}}(B)$ with a braiding $\widetilde{b}$. To this end, let M and N be two right B-modules, and write

$$t_{M,N} : M\Box N \to M\Box_B N \text{ and } t_{N,M} : N\Box M \to N\Box_B M$$

for the 2-universal B-balanced right B-module 1-morphisms with structure 2-isomorphisms τ^t. We claim that the 1-morphism $t_{N,M} \circ b_{M,N} : M\Box N \to N\Box_B M$ in $\mathfrak{S}$ can be upgraded to a B-balanced right B-module 1-morphism. Namely, the B-balanced structure is given by the 2-isomorphism

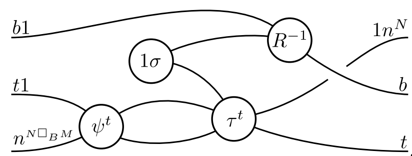
$\tau^{t\circ b} :=$



.

In order to check that $\tau^{t\circ b}$ satisfies axiom a of definition 5.6.14, we use the diagrams depicted in appendix A.4.1. Figure A.1 depicts the right hand-side of equation (5.6.12). By moving the indicated coupons to the top along the blue arrows, we arrive at figure A.2. Then, using equation (5.6.12) for τ^t on the blue coupons brings us to figure A.3. At this point, we use the definition of κ^M given in the proof of lemma 5.7.3 on the blue coupon, which leads us to contemplate figure A.4. Moving the coupon labeled $11\beta^{-1}$ to the left along the blue arrow, and then using equation (5.6.4) on the green coupons brings us to figure A.5. We arrive at figure A.6 by moving the blue coupons to the left along the blue arrows. Moving the coupon labeled $1R^{-1}$ to the right along the blue arrow and then applying equation (5.6.6) on the green coupons bring us to figure A.7. By moving the coupon labeled $1S$ to the right along the blue arrow and then make use of equation (5.6.5) on the green coupons bring us to figure A.8. Using equation (5.6.3) on the blue coupons, we arrive at figure A.9. We obtain figure A.10 by applying equation (5.6.1) on the blue coupons, using equation (5.6.2) on the green coupons, and moving the coupon labeled $1\beta^{-1}1$ to the right along the red arrow. Then, using equation (5.6.4) on the blue coupons and equation (5.7.4) on the green coupons, we arrive at figure A.11. Finally, we get to figure A.12, which depicts the left hand-side of equation (5.6.12), by moving the coupon labeled R to the right along the blue arrow and the coupon labeled $\beta^{-1}11$ to the left along the green arrow. Furthermore, equation (5.6.13) for $\tau^{t\circ b}$ follows from equation (5.6.13) for τ^t together with the fact that R, S, σ are modifications, combined with axiom f of definition 5.6.1 and axiom c of definition 5.6.1.

Moreover, the right B-module structure on $t_{N,M} \circ b_{M,N}$ is given by the 2-isomorphism

$\psi^{t \circ b} :=$ 

.

Thus, by the 2-universal property of $t_{M,N}$, the solid arrow diagram below can be filled by a B-balanced right B-module 2-isomorphism $\xi_{M,N}$:

$$\begin{tikzcd}
M\Box N \arrow[r, "t_{M,N}"] \arrow[d, "b_{M,N}"'] \arrow[dr, phantom, "\xi_{M,N} \Nearrow"] & M\Box_B N \arrow[d, dotted, "\widetilde{b}_{M,N}"] \\
N\Box M \arrow[r, "t_{N,M}"'] & N\Box_B M.
\end{tikzcd}$$

Furthermore, as $b_{M,N}$ is a 1-equivalence, the 2-universal property implies that the 1-morphism $\widetilde{b}_{M,N}$ is also an equivalence. Using the 2-universal property of the relative tensor product over B again, we find that the collection of the 1-equivalences $\widetilde{b}_{M,N}$ assembles to form a 2-natural equivalence $\widetilde{b}$. We upgrade $\widetilde{b}$ to an adjoint 2-natural equivalence by appealing to the 2-universal property.

We also have to construct invertible modifications $\widetilde{R}$ and $\widetilde{S}$ witnessing the coherence of the braiding $\widetilde{b}$ on $\mathbf{Mod}_{\mathfrak{S}}(B)$. As the monoidal structure on $\mathbf{Mod}_{\mathfrak{S}}(B)$ is not strict cubical, we need to use the fully weak definition of these modifications given in figure 2.3 of [212]. Let M, N, and P be three right B-modules, in order to construct the right B-module 2-isomorphism $\widetilde{R}_{M,N,P}$ we use the 2-universal property of the relative tensor product over two algebras following definition 3.2.6 of [74]. More precisely, let us consider the 3-dimensional commutative diagram whose back and front are depicted below:

$$\begin{tikzcd}
 & N\Box M\Box P \arrow[rr, equal] \arrow[dd] & & N\Box M\Box P \arrow[dr, "1b"] \arrow[dd] & \\
M\Box N\Box P \arrow[ur, "b1"] \arrow[dd] & & & & N\Box P\Box M \arrow[dd] \\
 & (N\Box_B M)\Box_B P \arrow[rr, "\alpha"'] & & N\Box_B(M\Box_B P) \arrow[dr, "1\widetilde{b}"'] & \\
(M\Box_B N)\Box_B P \arrow[ur, "\widetilde{b}1"'] \arrow[dr, "\alpha"'] & & \Downarrow \exists! \widetilde{R} & & N\Box_B(P\Box_B M), \\
 & M\Box_B(N\Box_B P) \arrow[rr, "\widetilde{b}"'] & & (N\Box_B P)\Box_B M \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

$$
\begin{tikzcd}
 & N\Box M\Box P \arrow[r, equal] \arrow[dr, phantom, "\Downarrow R"] & N\Box M\Box P \arrow[dr, "1b"] & \\
M\Box N\Box P \arrow[ur, "b1"] \arrow[dr, equal] \arrow[dd] & & & N\Box P\Box M \arrow[dd] \\
 & M\Box N\Box P \arrow[r, "b"'] \arrow[dd] & N\Box P\Box M \arrow[ur, equal] \arrow[dd] & \\
(M\Box_B N)\Box_B P \arrow[dr, "\alpha"'] & & & N\Box_B(P\Box_B M). \\
 & M\Box_B(N\Box_B P) \arrow[r, "\widetilde{b}"'] & (N\Box_B P)\Box_B M \arrow[ur, "\alpha"'] &
\end{tikzcd}
$$

All the vertical 1-morphisms are 2-universal (B, B)-balanced right B-module 1-morphisms, and all the square faces are filled by (B, B)-balanced right B-module 2-isomorphisms thanks to either the proof of lemma 3.2.7 of [74] or the construction of $\widetilde{b}$ given above. Thus, thanks to the 2-universal property of the relative tensor product, there exists a unique right B-module 2-isomorphism $\widetilde{R}$ such that the whole 3-dimensional prism is commutative. Furthermore, the collection of these assignments assemble into an invertible modification as can been seen using the 2-universal property of the relative tensor product over two algebras. The invertible modification $\widetilde{S}$ is constructed similarly.

Finally, one has to check that $\widetilde{R}$ and $\widetilde{S}$ together with the modifications supplied by the monoidal structure of $\mathbf{Mod}_{\mathfrak{S}}(B)$ satisfy the equations given in figures C.7 through C.14 of [212] hold. This follows from the 2-universal property of the relative tensor product over three and four algebras explained in the proof of theorem 3.2.8 of [74]. □

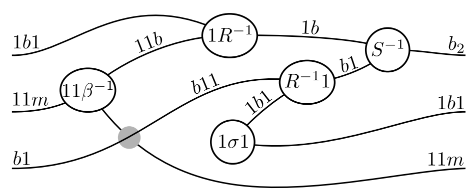
Proposition 5.7.9. Let $\mathfrak{S}$ be a Karoubi complete sylleptic monoidal 2-category, and B a symmetric separable algebra in $\mathfrak{S}$. Then, the monoidal 2-functor $F : \mathfrak{S} \to \mathbf{Mod}_{\mathfrak{S}}(B)$ of lemma 5.7.5 is braided.

Proof. Let C and D be any objects of $\mathfrak{S}$. Using the notations of lemma 5.7.5 and theorem

5.7.8, we can consider the following diagram:

$$\begin{tikzcd}
CBDB \arrow[r, "t"] \arrow[d, "b_2"'] \arrow[rr, bend left=40, "u"] \arrow[rr, phantom, bend left=20, "\Downarrow\zeta"] & (CB)\Box_B(DB) \arrow[r, "e"] \arrow[d, "\widetilde{b}"] \arrow[dl, phantom, "\overset{\xi}{\Rightarrow}"] & CDB \arrow[d, "b1"] \arrow[dl, phantom, "\overset{\exists!\epsilon}{\Rightarrow}"] \\
DBCB \arrow[r, "t"'] \arrow[rr, bend right=40, "u"'] \arrow[rr, phantom, bend right=20, "\Downarrow\zeta^{-1}"] & (DB)\Box_B(CB) \arrow[r, "e"'] & DCB.
\end{tikzcd} \tag{5.7.7}$$

Further, the outer square can be filled using the B-balanced right B-module 2-isomorphism ς given by:

$\varsigma :=$ 

.

Thus, thanks to the 2-universal property of the relative tensor product over B, the right hand-side square of the commutative diagram (5.7.7) can be filled by a right B-module 2-isomorphism ϵ such that its full composite is equal to ς. Further, it follows from the same 2-universal property that the collection of these 2-isomorphism defines an invertible modification. Finally, one checks that the axioms of definition 2.5 of [212] hold for ϵ using the 2-universal property of the relative tensor product over one and two algebras. $\square$

Note that $\mathbf{Mod}_{\mathfrak{S}}(B)$ is not sylleptic in general. Nonetheless, under favourable circumstances, this is in fact the case. We begin by recalling the following definition from section 5.3 of [46].

Definition 5.7.10. Let $\mathfrak{S}$ be a sylleptic fusion 2-category. The symmetric center of $\mathfrak{S}$, denoted by $\mathcal{Z}_{(3)}(\mathfrak{S})$ is the full sub-2-category of $\mathfrak{S}$ on those objects C such that

$$\sigma_{D,C} \circ b_{C,D} = b_{C,D} \circ \sigma_{C,D}$$

for every D in $\mathfrak{S}$.

Remark 5.7.8. It follows immediately from the definitions that $\mathcal{Z}_{(3)}(\mathfrak{S})$ is a (semi-strict) symmetric monoidal 2-category (see also theorem 5.2 of [46]).

Proposition 5.7.11. Let $\mathfrak{S}$ be a Karoubi complete sylleptic monoidal 2-category, and B a symmetric separable algebra in $\mathcal{Z}_{(3)}(\mathfrak{S})$. Then, $\mathbf{Mod}_{\mathfrak{S}}(B)$ is a sylleptic monoidal 2-category.

Proof. Without loss of generality, we may assume that $\mathfrak{S}$ is semi-strict. We have already endowed the 2-category $\mathbf{Mod}_{\mathfrak{S}}(B)$ with a braided monoidal structure. Moreover, using the notation of the proof of theorem 5.7.8, for every right B-modules M and N in $\mathfrak{S}$, we can consider the following right B-module 2-isomorphism

$$\begin{tikzcd}
M\Box N \arrow[rr, "t"] \arrow[drr, "b"] \arrow[dd, equal] & & M\Box_B N \arrow[dr, "\widetilde{b}"] & \\
& \arrow[r, phantom, "\overset{\sigma}{\Longrightarrow}"] & N\Box M \arrow[u, phantom, "\xi\Uparrow"] \arrow[r, "t"] \arrow[dll, "b"] \arrow[d, phantom, "\overset{\xi}{\Longrightarrow}"] & N\Box_B M \arrow[dl, "\widetilde{b}"] \\
M\Box N \arrow[rr, "t"'] & & M\Box_B N. &
\end{tikzcd}$$

The above right B-module 2-isomorphism is B-balanced. In order to see this, we use the diagrams depicted in appendix A.4.2. Figure A.13 depicts the left hand-side of equation (5.6.14) of definition 5.6.13 for the above 2-isomorphism. Applying equation (5.6.14) for ξ on the blue coupons brings us to figure A.14. By inserting the definition of $\tau^{t\circ b}$ given in the proof of theorem 5.7.8, we arrive at figure A.15. Then, using equation (5.6.14) for ξ on the blue coupons leads us to figure A.16. Inserting the definition of $\tau^{t\circ b}$ once again, we get to figure A.17. In order to get to figure A.18, we first use the equation given in definition 5.7.10 on the blue coupons and the strand immediately on top of it, and then move the left most coupon labeled σ along the green arrow. Then, applying equation (5.6.5) on the blue coupons brings to figure A.19. Using equation (5.6.6) on the blue coupons, followed by moving the freshly created coupon labeled σ down along the green arrow, and cancelling the pair of red coupons brings us to figure A.20. But, figure A.20 depicts the right hand-side of equation (5.6.14), so the proof of the claim is finished.

Then, thanks to the 2-universal property of the relative tensor product, this yields a 2-isomorphism $\widetilde{\sigma}_{M,N}$ as in the diagram below

$$\begin{tikzcd}
M\Box_B N \arrow[rr, equal] \arrow[dr, "\widetilde{b}"'] & \arrow[d, phantom, "\Downarrow\widetilde{\sigma}"] & M\Box_B N. \\
& N\Box_B M \arrow[ur, "\widetilde{b}"'] &
\end{tikzcd}$$

Further, it follows from the 2-universal property of the realtive tensor product that the collection of the 2-isomorphisms $\widetilde{\sigma}_{M,N}$ for varying M and N defines an invertible modification. Finally, one has to check that $\widetilde{\sigma}$ defines a syllepsis on the braided monoidal 2-category $\mathbf{Mod}_{\mathfrak{S}}(B)$, i.e. that the equations given in figure C.15 and C.16 of [212] hold. This follows from the 2-universal property of the relative tensor product over one and two algebras explained in section 3 of [74]. □

We now consider the case when $\mathfrak{S}$ is symmetric monoidal.

Corollary 5.7.12. Let $\mathfrak{S}$ be a Karoubi complete symmetric monoidal 2-category, and B a symmetric separable algebra in $\mathfrak{S}$. Then, $\mathbf{Mod}_{\mathfrak{S}}(B)$ is a symmetric monoidal 2-category.

Proof. If $\mathfrak{S}$ is symmetric, then $\mathcal{Z}_{(3)}(\mathfrak{S}) = \mathfrak{S}$, which implies that $\mathbf{Mod}_{\mathfrak{S}}(B)$ is sylleptic. Further, it follows from the definition of the syllepsis, that $\mathbf{Mod}_{\mathfrak{S}}(B)$ is in fact symmetric if $\mathfrak{S}$ is symmetric. □

Lemma 5.7.13. Let $\mathfrak{S}$ be a Karoubi complete sylleptic monoidal 2-category, and B a symmetric separable algebra in $\mathcal{Z}_{(3)}(\mathfrak{S})$. Then, the braided monoidal functor $F : \mathfrak{S} \to \mathbf{Mod}_{\mathfrak{S}}(B)$ of proposition 5.7.5 is sylleptic. In particular, if $\mathfrak{S}$ is symmetric, then F is symmetric.

Proof. The first part follows from the construction and the 2-universal property of the relative tensor product over B. The last part is immediate as a symmetric monoidal 2-functor is nothing but a sylleptic monodial 2-functor between symmetric monoidal 2-categories (see definition 2.5 of [212]). □

Remark 5.7.9. Analogously to what was noted in remark 5.7.6, the proofs of all the above results in this section only used the existence of the relative tensor product over B for any B-modules in $\mathfrak{S}$.

Finally, if $\mathfrak{S}$ is a sylleptic multifusion 2-category, proposition 5.7.11 can be strengthened. We begin by the following lemma.

Lemma 5.7.14. Let $\mathfrak{S}$ be a sylleptic fusion 2-category. Then, its symmetric center $\mathcal{Z}_{(3)}(\mathfrak{S})$ is generated under direct sums by the union of some of the connected components of $\mathfrak{S}$. In particular, it is a symmetric fusion 2-category, and it contains the connected components of the identity of $\mathfrak{S}$.

Proof. Observe that, by definition, $\mathcal{Z}_{(3)}(\mathfrak{S})$ is a full sub-2-category of $\mathfrak{S}$. Further, note that $\mathcal{Z}_{(3)}(\mathfrak{S})$ is closed under taking direct sums. Now, let S be an object of $\mathcal{Z}_{(3)}(\mathfrak{S})$. We wish to prove that if T is a simple object of $\mathfrak{S}$ given by the splitting of a 2-condensation monad on S, then T is in $\mathcal{Z}_{(3)}(\mathfrak{S})$. Given an arbitrary object C in $\mathfrak{S}$, it follows from the 2-universal property of the splitting of a 2-condensation monad that the syllepsis $\sigma_{T,C}$ and $\sigma_{C,T}$ are completely determined by $\sigma_{S,C}$ and $\sigma_{C,S}$. But, by hypothesis, we have $\sigma_{C,S} \circ b_{S,C} = b_{S,C} \circ \sigma_{C,S}$, so that $\sigma_{C,T} \circ b_{T,C} = b_{T,C} \circ \sigma_{C,T}$, which proves the claim. The second part follows from the observation that a connected component of a finite semisimple 2-category is necessarily a finite semisimple 2-category. □

Proposition 5.7.15. Let $\mathfrak{S}$ be a sylleptic multifusion 2-category, and B a symmetric separable algebra in $\mathcal{Z}_{(3)}(\mathfrak{S})$. Then, we have

$$\mathcal{Z}_{(3)}(\mathbf{Mod}_{\mathfrak{S}}(B)) \simeq \mathbf{Mod}_{\mathcal{Z}_{(3)}(\mathfrak{S})}(B).$$

Proof. It follows from the construction that the syllepsis on $\mathbf{Mod}_{\mathfrak{S}}(B)$ is constructed from the syllepsis on $\mathfrak{S}$. In particular there is a symmetric monoidal inclusion $\mathbf{Mod}_{\mathcal{Z}_{(3)}(\mathfrak{S})}(B) \subseteq \mathcal{Z}_{(3)}(\mathbf{Mod}_{\mathfrak{S}}(B))$. On the other hand, the free 2-functor $F : \mathfrak{S} \to \mathbf{Mod}_{\mathfrak{S}}(B)$ is sylleptic monoidal. In particular, for any object C of $\mathfrak{S}$, $F(C)$ is in $\mathcal{Z}_{(3)}(\mathbf{Mod}_{\mathfrak{S}}(B))$ if and only if C is in $\mathcal{Z}_{(3)}(\mathfrak{S})$. But, every object of $\mathbf{Mod}_{\mathfrak{S}}(B)$ is the splitting of a 2-condensation monad supported on an object of the form $F(C)$ for some C in $\mathfrak{S}$ by lemma 3.1.1 of [77]. Further, $\mathcal{Z}_{(3)}(\mathbf{Mod}_{\mathfrak{S}}(B))$ is a union of connected components of $\mathbf{Mod}_{\mathfrak{S}}(B)$ by lemma5.7.14 above, so that $\mathcal{Z}_{(3)}(\mathbf{Mod}_{\mathfrak{S}}(B)) \simeq \mathbf{Mod}_{\mathcal{Z}_{(3)}(\mathfrak{S})}(B)$ as desired. □

5.8. Specific 2-Category of Modules

In this section, we will examine the 2-categories of right modules associated to specific algebras. This can be thought of as condensing a 3-condensation monad. In order to be applicable to physical theories, we will consider the cases when the ambient 2-category is either totally disconnected or connected. In both cases, we will work bosonically and fermionically, where the later means that we work with super 2-categories. A subset of surface operators can be assembled to form a separable algebra as in the previous section; we may thus apply the theorems above to understand the effect of the condensation. Throughout, we work over the complex numbers (or any algebraically closed field of characteristic zero), we use G to denote a finite group, and E to denote a finite abelian group.

5.8.1. Totally Disconnected 2-Category

Bosonic case

Starting with the simplest case, suppose that the fusion 2-category of surface operators and their interactions is given by $2\mathbf{Vec}[G]$, the 2-category of G-graded 2-vector spaces. In particular, the (equivalence classes of) simple objects are given by $\mathbf{Vec}_g$ with $g \in G$. We can consider the algebra $\mathbf{Vec}[G]$ in $2\mathbf{Vec}[G]$ given by $\boxplus_{g\in G}\mathbf{Vec}_g$, the sum of the equivalence classes of simple objects.

Lemma 5.8.1. The left $2\mathbf{Vec}[G]$-module 2-category $2\mathbf{Vec}$ is equivalent to $\mathbf{Mod}_{2\mathbf{Vec}[G]}(\mathbf{Vec}[G])$, where $\mathbf{Vec}[G]$ is the fusion 1-category of G-graded vector spaces viewed as an algebra in $2\mathbf{Vec}[G]$ with the canonical grading. Further, $\mathbf{Vec}[G]$ is a separable algebra.

Proof. It is easy to check directly that $\mathbf{Mod}_{2\mathbf{Vec}[G]}(\mathbf{Vec}[G]) \simeq 2\mathbf{Vec}$ as left $2\mathbf{Vec}[G]$-module 2-categories. Further, one can check directly that $\mathbf{Vec}[G]$ is a separable algebra in $2\mathbf{Vec}[G]$. Alternatively, this follows from theorem 3.2.4 and corollary 3.3.7 of [77]. $\square$

Before moving on to the general case, we establish the following technical result. Recall from [73] that a left $2\mathbf{Vec}[G]$-module 2-category is 2-category equipped with a left action by $2\mathbf{Vec}[G]$. Note that this is equivalent to the data of an action of the group G. In particular, given a left $2\mathbf{Vec}[G]$-module 2-category $\mathfrak{M}$, we can consider the 2-category $\mathbf{LMod}_{\mathfrak{M}}(\mathbf{Vec}[G])$ of left $\mathbf{Vec}[G]$-modules in $\mathfrak{M}$, given by gauging the G-action on $\mathfrak{M}$. If $\mathfrak{M}$ is a finite semisimple 2-category, the G-action permutes the set of connected components of $\mathfrak{M}$.

Proposition 5.8.2. Let $\mathfrak{M}$ be a finite semisimple left $2\mathbf{Vec}[G]$-module 2-category. Then, we have

$$\pi_0(\mathbf{LMod}_{\mathfrak{M}}(\mathbf{Vec}[G])) \cong \pi_0(\mathfrak{M})/G.$$

Proof. We claim that it suffices to prove this result for $\mathfrak{M}$ an indecomposable finite semisimple left $2\mathbf{Vec}[G]$-module 2-category. Namely, it follows from lemma 5.2.3 of [74] that every finite semisimple left $2\mathbf{Vec}[G]$-module 2-category $\mathfrak{M}$ can be decomposed into a finite direct sum $\mathfrak{M} \simeq \boxplus_{i=1}^n \mathfrak{M}_i$ of indecomposable ones. From this, it follows that there is a bijection $\pi_0(\mathfrak{M}) \cong \coprod_{i=1}^n \pi_0(\mathfrak{M}_i)$ of sets compatible with the G-actions. This establishes the claim of sufficiency.

Now, note that it follows from the definition that a finite semisimple left $2\mathbf{Vec}[G]$-module 2-category is indecomposable if and only if the action of G on $\pi_0(\mathfrak{M})$ is transitive. Thus, it only remains to prove that if $\mathfrak{M}$ is an indecomposable finite semisimple left $2\mathbf{Vec}[G]$-module 2-category, then $\pi_0(\mathbf{Mod}_{\mathfrak{M}}(\mathbf{Vec}[G])) = *$.

To see this, note that thanks to theorem 5.1.2 of [73], there exists an algebra A in $2\mathbf{Vec}[G]$ such that $\mathfrak{M} \simeq \mathbf{Mod}_{2\mathbf{Vec}[G]}(A)$. Furthermore, by theorem 5.4.7 of [73], the algebra A is in fact rigid. But rigid algebras in $2\mathbf{Vec}[G]$ are precisely G-graded multifusion 1-categories, so that A is an G-graded multifusion 1-category. Moreover, as $\mathfrak{M}$ is indecomposable, A is indecomposable as a G-graded multifusion 1-category (see corollary 5.2.7 of [74]).

By inspection, there are equivalences of 2-categories

$$\mathbf{LMod}_{\mathfrak{M}}(\mathbf{Vec}[G]) \simeq \mathbf{Bimod}_{\mathbf{2Vec}[G]}(\mathbf{Vec}[G], A) \simeq \mathbf{Mod}_{\mathbf{2Vec}}(A),$$

where, on the right hand-side, we view A as a multifusion 1-category. Thus, by proposition 2.3.5 of [75], it is enough to prove that A is indecomposable as a multifusion 1-category. (A multifusion 1-category is "connected" in the sense of definition 2.3.1 of [75] if and only if it is indecomposable.) Finally, observe that a decomposition of A into a direct sum of two non-zero multifusion 1-categories would automatically be compatible with the G-grading. This is impossible by construction so we are done. $\square$

If $G = E$ is a finite abelian group, then $2\mathbf{Vec}[E]$ is braided fusion 2-category. Further, the algebra $\mathbf{Vec}[E]$ is actually braided. It is therefore sensible to consider the case when the 2-category of all surfaces is a braided fusion 2-category $\mathfrak{B}$, equipped with a braided monoidal inclusion $2\mathbf{Vec}[E] \subseteq \mathfrak{B}$. This allows us to view the separable algebra $\mathbf{Vec}[E]$ in $2\mathbf{Vec}[E]$ as living in $\mathfrak{B}$, and we can investigate the properties of 2-category obtained by the condensation of $\mathbf{Vec}[E]$ in $\mathfrak{B}$. The following result follows from theorem 5.7.6, the above proposition, and lemma 5.7.5.

Corollary 5.8.3. Given $\mathfrak{B}$ a braided fusion 2-category and $2\mathbf{Vec}[E] \subseteq \mathfrak{B}$ a braided monoidal inclusion, the 2-category $\mathbf{Mod}_{\mathfrak{B}}(\mathbf{Vec}[E])$ obtained by condensing $\mathbf{Vec}[E]$ is a fusion 2-category with $\pi_0(\mathbf{Mod}_{\mathfrak{B}}(\mathbf{Vec}[E])) \cong \pi_0(\mathfrak{B})/E$. Moreover, the canonical 2-functor $\mathfrak{B} \to \mathbf{Mod}_{\mathfrak{B}}(\mathbf{Vec}[E])$ is monoidal.

In particular, the condensation reorganizes the 2-category $\mathfrak{B}$ by identifying the connected components of surfaces which are related by the action of E. This is effectively gauging the E action on the components. The resulting fusion 2-category is in general not braided.

Example. Consider $2\mathbf{Vec}[\mathbb{Z}_4]$, with simple objects labeled by $\{\mathbf{Vec}_0, \mathbf{Vec}_1, \mathbf{Vec}_2, \mathbf{Vec}_3\}$ and fusion given by addition mod 4. Suppose we condense the algebra $\mathbf{Vec}_0 \boxplus \mathbf{Vec}_2$, which is $\mathbf{Vec}[\mathbb{Z}_2]$, the simple modules are then given by $\mathbf{Vec}_0 \boxplus \mathbf{Vec}_2$, and $\mathbf{Vec}_1 \boxplus \mathbf{Vec}_3$. As there is no 1-morphism between them, $\mathbf{Mod}_{\mathbf{2Vec}[\mathbb{Z}_4]}(\mathbf{Vec}[\mathbb{Z}_2])$ has two connected components. On the other hand, one sees that $\pi_0(2\mathbf{Vec}[\mathbb{Z}_4])/\mathbb{Z}_2$ has the same two connected components.

Remark 5.8.1. We give an example for which the 2-functor in corollary 5.8.3 is not necessarily braided, take $\mathfrak{B} = \mathcal{Z}(2\mathbf{Vec}[\mathbb{Z}_2])$, the Drinfeld center of $2\mathbf{Vec}[\mathbb{Z}_2]$, equipped with the canonical inclusion $2\mathbf{Vec}[\mathbb{Z}_2] \subseteq \mathcal{Z}(\mathbf{2Rep}(\mathbb{Z}_2))$. We can then condense the algebra $\mathbf{Vec}[\mathbb{Z}_2]$, and get

$$\mathbf{Mod}_{\mathcal{Z}(2\mathbf{Vec}[\mathbb{Z}_2])}(\mathbf{Vec}[\mathbb{Z}_2]) \simeq \mathbf{2Rep}(\mathbb{Z}_2).$$

Further, the monoidal 2-functor $\mathcal{Z}(2\mathbf{Vec}[\mathbb{Z}_2]) \to \mathbf{Mod}_{\mathcal{Z}(\mathbf{2Rep}(\mathbb{Z}_2))}(\mathbf{Vec}[\mathbb{Z}_2])$ of lemma 5.7.5 is identified with the monoidal forgetful 2-functor $\mathcal{Z}(\mathbf{2Rep}(\mathbb{Z}_2)) \to \mathbf{2Rep}(\mathbb{Z}_2)$, which is not braided.

The next result follows from proposition 5.7.11, lemma 5.7.13, and proposition 5.8.2.

Corollary 5.8.4. Let $\mathfrak{S}$ be a sylleptic fusion 2-category, with an inclusion $2\mathbf{Vec}[E] \subseteq \mathcal{Z}_{(3)}(\mathfrak{S})$, then $\mathbf{Mod}_{\mathfrak{S}}(\mathbf{Vec}[E])$ is a sylleptic fusion 2-category such that $\pi_0(\mathbf{Mod}_{\mathfrak{S}}(\mathbf{Vec}[E])) \cong \pi_0(\mathfrak{S})/E$. Furthermore, the canonical monoidal 2-functor $\mathfrak{S} \to \mathbf{Mod}_{\mathfrak{S}}(\mathbf{Vec}[E])$ is sylleptic.

Fermionic Case

We mirror the bosonic case and first consider the fusion 2-category $2\mathbf{SVec}[G]$ of G-graded super 2-vector spaces. In order to condense $2\mathbf{SVec}[G]$ to $2\mathbf{SVec}$, it is enough to consider the bosonic algebra $\mathbf{Vec}[G]$ given by the canonical monoidal inclusion $2\mathbf{Vec}[G] \subseteq 2\mathbf{SVec}[G]$. By direct inspection, we find that $\mathbf{Mod}_{2\mathbf{SVec}[G]}(\mathbf{Vec}[G]) \simeq 2\mathbf{SVec}$.

Let us now comment on the braided case. Namely, if $G = E$ is a finite abelian group, then $2\mathbf{SVec}[E]$ is a braided fusion 2-category. We can therefore consider $\mathfrak{B}$ a braided fusion 2-category containing $2\mathbf{SVec}[E]$. But, the inclusion $2\mathbf{Vec}[E] \subseteq 2\mathbf{SVec}[E]$ is braided, so this is exactly in the setup of corollary 5.8.3. Similar remarks holds for the sylleptic and symmetric cases.

5.8.2. Connected Category

Let $\mathfrak{B}$ be a braided fusion 2-category, then $\mathrm{End}_{\mathfrak{B}}(\mathbb{1})$, the endomorphisms of the identity surface, is a symmetric fusion 1-category, so that $\mathfrak{B}^0 = \mathbf{Mod}(\mathrm{End}_{\mathfrak{B}}(\mathbb{1}))$ is a symmetric fusion 2-category (see [76]). Here, $\mathfrak{B}^0$ denotes the identity component and is a prime candidate for a condensation.

Bosonic Case

Suppose that $\mathfrak{B}^0 = 2\mathbf{Rep}(G)$, i.e. the surfaces in the identity component of $\mathfrak{B}$ form the fusion 2-category $2\mathbf{Rep}(G)$. Here we think of $2\mathbf{Rep}(G)$ as the 2-category of finite semisimple 1-categories equipped with a G-action. One such object is given by $\mathbf{Vec}[G]$ with the canonical G-action. In this description, the monoidal product of two finite semisimple 1-categories $\mathcal{C}$ and $\mathcal{D}$ equipped with G-actions is given by their Deligne tensor product $\mathcal{C} \boxtimes \mathcal{D}$ equipped with the diagonal G-action. The fusion 2-category $2\mathbf{Rep}(G)$ is connected, which means that all the surfaces arise as networks of lines. We write φ for the symmetric algebra $\mathbf{Fun}(G, \mathbf{Vec})$ in $2\mathbf{Rep}(G)$. We note that the underlying object of φ is $\mathbf{Vec}[G]$. In the setting of fusion 1-categories, this corresponds to considering the symmetric algebra $\mathbb{C}[G]^*$ inside $\mathbf{Rep}(G)$. A module for φ is thus a way for the lines to end at the boundary.

We also point out that there is another model for $2\mathbf{Rep}(G)$, given by $\mathbf{Mod}(\mathbf{Rep}(G))$ (see lemma 1.3.8 of [74]). In the fermionic case, only this second model is available. It is therefore necessary to give an alternative description of φ in this model. The symmetric fusion 1-category $\mathbf{Vec}$ equipped with the canonical symmetric monoidal functor $\mathbf{Rep}(G) \to \mathbf{Vec}$ defines a symmetric algebra in $\mathbf{Mod}(\mathbf{Rep}(G))$. This algebra is separable thanks to theorem 3.2.4 and proposition 3.3.3 of [77] and theorem 2.3 of [92]. Moreover, under the equivalence of lemma 1.3.8 of [74], the algebra $\mathbf{Vec}$ in $\mathbf{Mod}(\mathbf{Rep}(G))$ corresponds to the algebra φ in the first model. It follows that $\mathbf{Mod}_{2\mathbf{Rep}(G)}(\varphi) \simeq \mathbf{Mod}_{\mathbf{Mod}(\mathbf{Rep}(G))}(\mathbf{Vec}) \simeq 2\mathbf{Vec}$.

Proposition 5.8.5. Let $\mathfrak{B}$ be a braided fusion 2-category with $2\mathbf{Rep}(G) \simeq \mathfrak{B}^0$ as braided fusion 2-categories. Then, condensing the braided separable algebra φ in $\mathfrak{B}$ yields a strongly fusion 2-category $\mathbf{Mod}_{\mathfrak{B}}(\varphi)$ equipped with a monoidal 2-functor $\mathfrak{B} \to \mathbf{Mod}_{\mathfrak{B}}(\varphi)$.

Proof. All but the strongly fusion part follow from theorem 5.7.6. We claim that $\mathbf{Mod}_{\mathfrak{B}}(\varphi)^0 = 2\mathbf{Vec}$, so that $\mathbf{Mod}_{\mathfrak{B}}(\varphi)$ is strongly fusion. Note that φ is an algebra in $2\mathbf{Rep}(G) \simeq \mathfrak{B}^0$. By corollary 2.3.6 of [76], this implies that the underlying object in $\mathfrak{B}$ of any simple right φ-module is supported in a single connected component of $\mathfrak{B}$. This shows that $\mathbf{Mod}_{\mathfrak{B}}(\varphi)^0 \simeq \mathbf{Mod}_{2\mathbf{Rep}(G)}(\varphi)^0 \simeq 2\mathbf{Vec}$. This finishes the proof of the claim. □

Remark 5.8.2. In the fusion 2-category $2\mathbf{Rep}(G)$ the algebra φ is actually symmetric, but we can not view φ as a symmetric algebra in $\mathfrak{B}$; this requires extra data in the ambient braided fusion 2-category $\mathfrak{B}$. Therefore φ is treated as a braided algebra when considered in $\mathfrak{B}$.

We give a physical explanation as to why condensing in the identity component in proposition 5.8.5 was sufficient to make $\mathfrak{B}$ strongly fusion: the objects in the identity component of $\mathfrak{B}$ are related to the identity surface by 2-condensations but if the identity component was condensed to just $2\mathbf{Vec}$ via φ, then all the 1-morphisms are trivial, hence the 2-category is strongly fusion.

Remark 5.8.3. Categorifying the main result of [**?**] and [204], we expect that if $\mathfrak{B}$ is braided fusion 2-category with $2\mathbf{Rep}(G) \simeq \mathfrak{B}^0$, then the fusion 2-category $\mathbf{Mod}_{\mathfrak{B}}(\varphi)$ admits a G-crossed braided structure.

Let $\mathfrak{S}$ be a sylleptic multifusion 2-cateogry. As a consequence of lemma 5.7.14, we find that any inclusion $2\mathbf{Rep}(G) \subseteq \mathfrak{S}$ of sylleptic fusion 2-categories automatically includes in the symmetric center of $\mathfrak{S}$. Namely, $2\mathbf{Rep}(G)$ is necessarily contained in the component of the identity of $\mathfrak{S}$. Combing this observation with proposition 5.7.11 and lemma 5.7.13 yields the following result.

Corollary 5.8.6. Let $\mathfrak{S}$ be a sylleptic fusion 2-category. Suppose that there is an inclusion $\mathfrak{S} \simeq 2\mathbf{Rep}(G)$, then $\mathbf{Mod}_{\mathfrak{S}}(\varphi)$ is a sylleptic strongly fusion 2-category. Furthermore, the canonical monoidal 2-functor $\mathfrak{S} \to \mathbf{Mod}_{\mathfrak{S}}(\varphi)$ is sylleptic.

Remark 5.8.4. We make a small physical point regarding the above corollary. Consider a setting in (3+1)d but *not* limited to considering only topological theories. The surface operators can be nontrivial even if the line operators have been condensed, as in the situation of corollary 5.8.6. If we are in a purely topological (3+1)d setting, then the there are actually no surface operators either because surfaces detect lines in this dimension. This means we are just in a situation of bosonic Dijkgraaf-Witten theory.

Fermionic Case

We consider the fusion 2-category $2\mathbf{Rep}(G, z) := \mathbf{Mod}(\mathbf{Rep}(G, z))$, where z is an emergent fermion in G, that is a central element of order 2. We are viewing $2\mathbf{Rep}(G, z)$ as so because there is no fermionic analogue of the model for $2\mathbf{Rep}(G)$ that was used in §5.8.2. We define the symmetric separable algebra $\varphi := \mathbf{SVec}$ in $2\mathbf{Rep}(G, z)$. More precisely, φ denotes $\mathbf{SVec}$ equipped with the canonical forgetful symmetric monoidal functor $\mathbf{Rep}(G, z) \to \mathbf{SVec}$. Let us examine the result of condensing φ. In this case, there is no obstruction to condensing to the vacuum.

Lemma 5.8.7. As left $2\mathbf{Rep}(G, z)$-module 2-categories, we have $2\mathbf{SVec} \simeq \mathbf{Mod}_{2\mathbf{Rep}(G,z)}(\varphi)$, where $\mathbf{SVec}$ is viewed as an algebra in $2\mathbf{Rep}(G, z)$ via $\mathbf{Rep}(G, z) \to \mathbf{SVec}$.

Proof. This follows from example 3.2.5 of [73]. □

Physically, we find that condensing φ gives a local fermion. The next proposition follows using a variant of the proof of proposition 5.8.5, with a slight change to φ.

Proposition 5.8.8. Let $\mathfrak{B}$ be a braided fusion 2-category, and assume $\mathfrak{B}^0 \simeq 2\mathbf{Rep}(G, z)$ as braided fusion 2-categories. Then, condensing the algebra $\varphi = \mathbf{SVec}$ in $\mathfrak{B}$ yields a fermionic strongly fusion 2-category $\mathbf{Mod}_{\mathfrak{B}}(\varphi)$ equipped with a monoidal 2-functor $\mathfrak{B} \to \mathbf{Mod}_{\mathfrak{B}}(\varphi)$.

5.9. Strongly Fusion Computations

5.9.1. Braided Strongly Fusion 2-Categories

In the previous section, we have seen examples of strongly fusion 2-categories arising from condensations. It was shown in [152] that such 2-categories have grouplike fusion rules. Said differently, a strongly fusion 2-category is a "grouplike" extension of operators in different dimensions [147, 153]. In particular, their classification essentially boils down to a cohomology computation problem. We now consider the case where our fusion 2-category is strongly fusion and only braided. For instance, this is what happens to a sylleptic fusion 2-category when we condense the algebra φ in the subcategory $\mathfrak{B}^0 = 2\mathbf{Rep}(G)$. Fermionic braided strongly fusion 2-categories are classified by supercohomology [239] and we expect that the cases we discuss here cover all the examples of braided strongly fusion 2-categories, but we do not prove this fact. Namely, in general, one ought to consider supercohomology with twisted coefficients, but we expect that this is not necessary for braided fermionic strongly fusion 2-categories. On the other hand, braided bosonic strongly fusion 2-categories with finite abelian group of surfaces given by E are completely classified by $\mathrm{H}^5(E[2];\mathbb{C}^\times)$. This holds because this cohomology theory has no twisted variant.

The classification of the physical theories described by braided strongly fusion 2-categories proceeds by identifying those fusion 2-categories that are related by a topological boundary. More precisely, fixing a finite abelian group of surfaces E, the associated physical theories are classified by generalized cohomology. In the fermionic case, the relevant spectrum of coefficients is $\mathcal{SW}^\bullet(\mathrm{pt})$, the super–Witt spectrum [149]. Its homotopy groups in low degrees are recalled below in (5.9.4). In the bosonic case, the classification requires twisted equivariant cohomology. We now discuss these computations in more detail.

Fermionic Case

Let $\mathfrak{B}$ be a braided fermionic strongly fusion 2-category, and write E for the finite abelian group of connected components. Physically, E is the group of "fundamental" surfaces in $\mathfrak{B}$ that do not arise as condensations. Further, such braided fusion 2-categories can be constructed by deforming the coherence structure of $2\mathbf{SVec}[E]$ using a class in the super-cohomology group $\mathrm{SH}^5(E[2])$. Here and in what follows, $E[2]$ denotes the second Eilenberg-MacLane space of E, and we note that the number in brackets denotes the codimension associated to the objects with fusion rules given by the group E. In fact, all braided fermionic strongly fusion 2-categories arise via this construction, but we do not prove this fact. On the other hand, the (3+1)d theory associated to $\mathfrak{B}$ has no codimension one operators that do not arise through a condensation. By remote detectability [149], which says that every object must link topologically with another object of the appropriate dimension, this is the same as assuming that there are no nontrivial point operators in the theory. Then, the obstruction to condensing the theory associated to $\mathfrak{B}$ to the vacuum is given by a class in $\mathcal{SW}^\bullet(E[2])$. Now, if the group $\mathcal{SW}^\bullet(E[2])$ vanishes the theory associated to $\mathfrak{B}$ is automatically Morita equivalent to the vacuum. Our goal is to understand for which abelian groups E the cohomology group $\mathcal{SW}^5(E[2])$ does not vanish. More precisely, there is a canonical map $\mathrm{SH}^5(E[2]) \to \mathcal{SW}^\bullet(E[2])$, which corresponds to taking the theory associated to a braided fermionic strongly fusion 2-category. We argue that the image of this map is non-trivial in general.

Since the fusion 2-category $\mathfrak{B}$ is strongly fusion, there are no nontrivial lines, but we still have $\{1, f\}$ in **SVec** from the fermionic nature of the 2-category. We denote the condensation surface arising from f as c which has fusion rule $c^2 = \mathbb{1}$. The content of the fusion 2-category forms a higher group extension

$$(\mathbb{C}^\times[4].\underbrace{\mathbb{Z}_2}_{\{1,f\}}[3].\underbrace{\mathbb{Z}_2}_{\{\mathbb{1},c\}}[2]).E[2]\,, \tag{5.9.1}$$

where the component $\mathbb{C}^\times[4]$ means "three form $\mathbb{C}^\times$ symmetry". Such extensions are classified by $\mathrm{SH}^5(E[2])$, which can be computed with the knowledge that the supercohomology of a point is built out of three layers:

$$\mathrm{SH}^0(\mathrm{pt}) = \mathbb{C}^\times\,, \quad \mathrm{SH}^1(\mathrm{pt}) = \mathbb{Z}_2\,, \quad \mathrm{SH}^2(\mathrm{pt}) = \mathbb{Z}_2\,, \tag{5.9.2}$$

We note in passing that these groups agree with the first three layers of spin cobordism. Then, there is a canonical map $\mathrm{SH}^\bullet \to \mathcal{SW}^\bullet$. Assuming that $\mathfrak{B}$ is classified by a class in $\mathrm{SH}^5(E[2])$, the associated fermionic theory can be condensed to the vacuum exactly if the image of this class in $\mathcal{SW}^\bullet(E[2])$ is trivial. In order to understand for which groups E this can happen, we use the following Atiyah-Hirzebruch spectral sequence

$$\mathrm{H}^i(E[2]; \mathcal{SW}^j(\mathrm{pt})) \Rightarrow \mathcal{SW}^{i+j}(E[2])\,. \tag{5.9.3}$$

The homotopy groups of $\mathrm{S}\mathcal{W}^\bullet(\mathrm{pt})$ in low degrees are given by

$$\begin{gathered}\mathcal{SW}^0(\mathrm{pt}) = \mathbb{C}^\times\,, \quad \mathcal{SW}^1(\mathrm{pt}) = \mathbb{Z}_2\,, \quad \mathrm{S}\mathcal{W}^2(\mathrm{pt}) = \mathbb{Z}_2\,, \\ \mathcal{SW}^3(\mathrm{pt}) = 0\,, \quad \mathcal{SW}^4(\mathrm{pt}) = \mathcal{SW}\,, \quad \mathcal{SW}^5(\mathrm{pt}) = 0\,, \quad \mathcal{SW}^6(\mathrm{pt}) = 0\,.\end{gathered} \tag{5.9.4}$$

In degree 4, $\mathcal{SW}$ gives the Witt group of slightly degenerate braided fusion 1-categories.

If E has no 2-torsion, then we find that $\mathcal{SW}^5(E[2]) = \mathrm{SH}^5(E[2]) = \mathrm{H}^5(E[2];\mathbb{C}^\times)$. But, it follows from [87] that the right most group is trivial, so that there are non-trivial theories in this case. On the other hand, we can assume that E is 2-torsion. Then, we see that in total degree 5, there are interesting non-zero contributions to the E_2-page of the spectral sequence. We first consider $E = \mathbb{Z}_{2^k}$. The E_2 page for (5.9.3) is then given by

$$E_2^{ij} = \begin{array}{c|ccccccc} j & & & & & & & \\ & & & & & & & \\ 0 & 0 & 0 & \dots & & & & \\ 0 & 0 & 0 & \dots & & & & \\ \mathcal{SW} & \mathcal{SW} & 0 & \mathrm{hom}(\mathcal{SW},\mathbb{Z}_2) & \dots & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & & \\ \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \mathrm{hom}(E,\mathbb{Z}_2) & \mathrm{Ext}(E,\mathbb{Z}_2) & \mathrm{Quad}(E,\mathbb{Z}_2) & \dots & \\ \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \mathrm{hom}(E,\mathbb{Z}_2) & \mathrm{Ext}(E,\mathbb{Z}_2) & \mathrm{Quad}(E,\mathbb{Z}_2) & \mathrm{Ext}(E,\mathrm{hom}(E,\mathbb{Z}_2)) & \\ \mathbb{C}^\times & \mathbb{C}^\times & 0 & \widehat{E} & 0 & \mathrm{Quad}(E,\mathbb{C}^\times) & \mathrm{Ext}(E,\mathrm{hom}(E,\mathbb{C}^\times)) & \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 & i\,, \end{array} \tag{5.9.5}$$

where Quad denotes the group of quadratic forms. In addition, the d_2 differentials are given by

$$\begin{aligned} d_2 :& E_2^{i,2} = \mathrm{H}^i(\mathbb{Z}_2[2]\,;\mathbb{Z}_2) \to E_2^{i+2,1} = \mathrm{H}^{i+2}(\mathbb{Z}_2[2]\,;\mathbb{Z}_2) && X \mapsto \mathrm{Sq}^2 X \\ d_2 :& E_2^{i,1} = \mathrm{H}^i(\mathbb{Z}_2[2]\,;\mathbb{Z}_2) \to E_2^{i+2,0} = \mathrm{H}^{i+2}(\mathbb{Z}_2[2]\,;\mathbb{C}^\times) && X \mapsto (-1)^{\mathrm{Sq}^2 X}\,. \end{aligned} \tag{5.9.6}$$

This implies that the E_3 page is given by

$$E_3^{ij} = \begin{array}{c|cccccccc} j & & & & & & & & \\ & & & & & & & & \\ 0 & 0 & 0 & \dots & & & & & \\ 0 & 0 & 0 & \dots & & & & & \\ \mathcal{SW} & \mathcal{SW} & 0 & \mathcal{SW}_2 & \dots & & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & & & \\ \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & 0 & 0 & \dots & & & \\ \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & 0 & 0 & 0 & \dots & & \\ \mathbb{C}^\times & \mathbb{C}^\times & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & i\,. \end{array} \tag{5.9.7}$$

Therefore, $\mathcal{SW}^5(\mathbb{Z}_{2^k}[2]) = 0$, so that the theory associated to $\mathfrak{B}$ with $E = \mathbb{Z}_{2^k}$ can be condensed to the vacuum.

If E is a product of groups, we use the fact that for any generalized cohomology theory $h^\bullet$ computed on pointed spaces X and Y, we have

$$h^\bullet(X \times Y) = h^\bullet(\mathrm{pt}) \oplus \widetilde{h}^\bullet(X) \oplus \widetilde{h}^\bullet(Y) \oplus \widetilde{h}^\bullet(X \wedge Y)\,, \tag{5.9.8}$$

where $\widetilde{h}$ represents reduced cohomology. We can see that the contribution of $\widetilde{\mathcal{SW}}^5(\mathbb{Z}_{2^k}[2] \wedge \mathbb{Z}_{2^k}[2])$ to $\mathcal{SW}^5(\mathbb{Z}_{2^k}[2] \times \mathbb{Z}_{2^k}[2])$, is nontrivial by comparing it with $\widetilde{\Omega}^5_{\mathrm{Spin}}(\mathbb{Z}_{2^k}[2] \wedge \mathbb{Z}_{2^k}[2])$. Spin cobordism gives the group of maps from the spin bordism groups into $\mathbb{C}^\times$, and when evaluated on a point gives $\Omega^\bullet_{\mathrm{Spin}}(\mathrm{pt}) = \{\mathbb{C}^\times, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{C}^\times, 0, 0, 0, \ldots\}$ in low degrees. We claim that it is sufficient to show that $\widetilde{\Omega}^5_{\mathrm{Spin}}(\mathbb{Z}_{2^k}[2] \wedge \mathbb{Z}_{2^k}[2])$ does not vanish. Namely, the bottom three layers of $\Omega^\bullet_{\mathrm{Spin}}(\mathrm{pt})$ agree with those of $\mathcal{SW}^\bullet(\mathrm{pt})$, and a fortiori with those of $\mathrm{SH}^\bullet(\mathrm{pt})$, as was shown in [149, 110]. Furthermore, these layers are the only ones that we need to consider in order to compute $\widetilde{\mathcal{SW}}^5(\mathbb{Z}_{2^k}[2] \wedge \mathbb{Z}_{2^k}[2]) = \mathrm{SH}^5(\mathbb{Z}_{2^k}[2] \wedge \mathbb{Z}_{2^k}[2])$. Here, it is crucial that we are using reduced cohomology. We have a quick-and-dirty way to check that $\widetilde{\Omega}^5_{\mathrm{Spin}}(\mathbb{Z}_{2^k}[2] \wedge \mathbb{Z}_{2^k}[2])$ is nonzero, via the Adams spectral sequence. There are two classes in degree 5, each giving free $\mathcal{A}(1)$-summands in $\mathrm{H}^\bullet(B(\mathbb{Z}_{2^k}[2] \times \mathbb{Z}_{2^k}[2]); \mathbb{Z}_2)$, so by Margolis' theorem, the corresponding two $\mathbb{Z}_2$ summands on the E_2-page of the Adams spectral sequence do not admit or receive any differentials. Thus $\widetilde{\Omega}^5_{\mathrm{Spin}}(\mathbb{Z}_{2^k}[2] \wedge \mathbb{Z}_{2^k}[2])$ is nontrivial. More generally, this also implies that if E is any group which contains a product of two 2-torsion groups, then the map $\mathrm{SH}^5(E[2]) \to \widetilde{\mathcal{SW}}^5(E[2])$ has non-zero image.

Bosonic Case

In order to classify the bosonic theories associated to braided bosonic strongly fusion 2-categories, it is convenient to work with the associated fermionic theories. This is analogous to how working with an algebra over the real numbers is equivalent to working with the complexified algebra together with the Galois action of $\mathbb{Z}_2^T$, given by complex conjugation. In this sense, the action of $\mathbb{Z}_2^T$ provides the necessary data to descend a complex algebra into a real one. The categorification of this classical setup was introduced in [144]. Namely, for symmetric fusion 1-categories, the algebraic closure of **Vec** is given by **SVec** and the Galois higher group Gal(**SVec**/**Vec**) is given by $\mathbb{Z}_2^F[1]$. This higher group agrees with the physical phenomenon of spin statistics, which says that fermions reverse sign under 360 degree rotation. Then, *Galois descent* asserts that the theory associated to a braided bosonic strongly fusion 2-category $\mathfrak{B}$ is completely described by the $\mathbb{Z}_2^F[1]$-equivariant theory associated to $\mathfrak{B} \boxtimes 2\mathbf{SVec}$. We can study the later using the equivariant Atiyah-Hirzebruch spectral sequence.

In general, the group of surfaces of $\mathfrak{B}$ is given by a finite abelian group E. We begin by showing that $\mathcal{W}^5(\mathrm{pt})$ does not vanish. That is, we wish to understand the twisted $\mathcal{SW}^\bullet$-cohomology with E_2 page given by:

$$\mathrm{H}^i(\mathbb{Z}_2^F[2]; \mathcal{SW}^j(\mathrm{pt})) \Rightarrow \mathcal{SW}^{i+j}(\mathbb{Z}_2^F[2]) = \mathcal{W}^{i+j}(\mathrm{pt}). \tag{5.9.9}$$

To arrive at the last equality, we use the fact that $\mathcal{W}^\bullet$ is the fixed point spectrum of $\mathcal{SW}^\bullet$ under the action of $\mathbb{Z}_2^F[1]$. The E_2 page is then given by:

$$E_2^{ij} = \begin{array}{c|cccccccccc}
j & & & & & & & & & & \\
 & & & & & & & & & & \\
0 & 0 & 0 & \dots & & & & & & & \\
\mathcal{SW} & \mathcal{SW} & 0 & \mathcal{SW}_2 & \dots & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
\mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2^2 & \mathbb{Z}_2^2 & \dots & & \\
\mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2^2 & \mathbb{Z}_2^2 & \dots & & \\
\mathbb{C}^\times & \mathbb{C}^\times & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_4 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & i\,.
\end{array} \tag{5.9.10}$$

The d_2 differentials are the twisted analogue of (5.9.16)

$$\begin{aligned}
d_2 :& E_2^{i,2} = \mathrm{H}^i(\mathbb{Z}_2[2]\,;\mathbb{Z}_2) \to E_2^{i+2,1} = \mathrm{H}^{i+2}(\mathbb{Z}_2[2]\,;\mathbb{Z}_2) && X \mapsto \mathrm{Sq}^2 X + \iota_2 X \\
d_2 :& E_2^{i,1} = \mathrm{H}^i(\mathbb{Z}_2[2]\,;\mathbb{Z}_2) \to E_2^{i+2,0} = \mathrm{H}^{i+2}(\mathbb{Z}_2[2]\,;\mathbb{C}^\times) && X \mapsto (-1)^{\mathrm{Sq}^2 X + \iota_2 X}\,,
\end{aligned} \tag{5.9.11}$$

and we find the E_3 page is given by

$$E_3^{ij} = \begin{array}{c|cccccccccc}
j & & & & & & & & & & \\
 & & & & & & & & & & \\
0 & 0 & 0 & \dots & & & & & & & \\
\mathcal{SW} & \mathcal{SW} & 0 & \mathcal{SW}_2 & \dots & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
\mathbb{Z}_2 & 0 & 0 & \mathbb{Z}_2 & 0 & 0 & \mathbb{Z}_2 & \dots & & & \\
\mathbb{Z}_2 & 0 & 0 & 0 & \mathbb{Z}_2 & 0 & 0 & \dots & & & \\
\mathbb{C}^\times & \mathbb{C}^\times & 0 & 0 & 0 & \mathbb{Z}_4 & \mathbb{Z}_2 & 0 & 0 & 0 & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & i\,.
\end{array} \tag{5.9.12}$$

The d_5 differential from $(0,4)$ records the obstruction to minimal modular extensions. It sends a class in $\mathcal{SW}$ to 0 if the minimal modular extension exists, and to 1 if it does not exist [149]. The main result of [150] shows that the possible d_5 vanishes. We therefore find that $\mathcal{W}^5(\mathrm{pt}) \cong \mathbb{Z}_2$.

Now, let us consider any finite abelian group E. It follows from (5.9.8), that

$$\mathcal{W}^5(E[2]) \cong \mathcal{W}^5(\mathrm{pt}) \oplus \widetilde{\mathcal{SW}}^5(E[2]) \oplus \widetilde{\mathcal{SW}^5}(\mathbb{Z}_2^F[2] \wedge E[2]).$$

It follows from what we have argued above in the fermionic case that the canonical map $\mathrm{H}^5(E[2];\mathbb{C}^\times) \to \widetilde{\mathcal{SW}}^5(E[2])$ is non-zero for a general finite abelian group E. As a consequence, the theory associated to a braided bosonic strongly fusion 2-category can not be condensed to the vacuum in general.

Before moving on the case of symmetric strongly fusion 2-categories, let us briefly remark that, in section 5.8, we have also considered examples when the condensation yields a sylleptic strongly fusion 2-category. The computations for the theories associated to these 2-categories were performed in [153], where the object of study was topological (4+1)d theories.

5.9.2. Symmetric Strongly Fusion 2-Categories

We now analyze the structure of symmetric strongly fusion 2-categories. More precisely, we will show below that every symmetric strongly fusion 2-category admits a fibre 2-functor to 2**SVec**. In the process, we will also show that every symmetric fermionic strongly fusion 2-category is completely determined by its groups of connected components. These computations establish the 2-Deligne theorem for symmetric fusion 2-categories. Namely, it follows from corollary 5.8.6 together with the obvious fermionic analogue, that every symmetric fusion 2-category admits a fibre 2-functor to a strongly fusion 2-category. Putting the above discussion together, we obtain the following theorem, which is a categorification of [79].

Theorem 5.9.1. Every symmetric fusion 2-category admits a fibre 2-functor to 2**SVec**.

We point out that this result was first announced in [148]. In addition, we expect that the above theorem can be used to classify symmetric fusion 2-categories. More precisely, every symmetric fusion 2-category should be equivalent to the symmetric monoidal 2-category of finite semisimple 2-representation of a “super 2-group”.

Fermionic Case

Let $\mathfrak{S}$ be a symmetric fermionic strongly fusion 2-category, and let us denote by E its abelian group of connected components. We now wish to understand what additional data besides E, if any, needs to be supplied to recover $\mathfrak{S}$. We begin by describing $\mathfrak{S}^\times$ the Picard sub-2-category of $\mathfrak{S}$, that is the maximal sub-2-category on the invertible objects and morphisms.

It has been established in [130] that the homotopy theory of symmetric monoidal 2-categories for which all objects and morphisms are invertible is equivalent to that of spectra with homotopy groups concentrated in degrees $0, 1$, and 2. In particular, the Picard 2-category $\mathfrak{S}^\times$ fits into the following fibre sequence of spectra

$$2\mathbf{SVec}^\times \to \mathfrak{S}^\times \to \mathrm{H}E \to \Sigma 2\mathbf{SVec}^\times,$$

where $\mathrm{H}E$ denotes the Eilenberg-MacLane spectrum associated to E. In particular, $\mathfrak{S}^\times$ is completely determined by the map of spectra $\mathrm{H}E \to \Sigma 2\mathbf{SVec}^\times$. Up to homotopy, such maps are classified by the group $\mathrm{SH}^7(E[4])$.

In order to compute the group $\mathrm{SH}^7(E[4])$, we invoke the Atiyah-Hirzebruch spectral sequence with the E_2-page:

$$\mathrm{H}^i(E[4]; \mathrm{SH}^j(\mathrm{pt})) \Longrightarrow \mathrm{SH}^{i+j}(E[4])\,. \tag{5.9.13}$$

We will show that the degree seven supercohomology group $\mathrm{SH}^7(E[4])$ vanishes for any finite abelian group E. Firstly, it follows from [87] that $\mathrm{H}^7(E[4]; \mathbb{C}^\times) = 0$ if E has no 2-torsion. In addition, the Hurewicz theorem shows that $\mathrm{SH}^7(E[4])$ can only be non-trivial if E has 2-torsion. We start with the case $E = \mathbb{Z}_{2^k}$, as explained in [217], the cohomology $\mathrm{H}^\bullet(\mathbb{Z}_{2^k}[n], \mathbb{Z}_2)$ is a polynomial ring $\mathbb{Z}_2[\mathrm{Sq}^I(\iota_n)]$ where the generator $\iota_n \in \mathrm{H}^n(\mathbb{Z}_{2^k}[n]; \mathbb{Z}_2)$ is in degree n, and $I = (i_1, i_2, \ldots, i_m)$ runs over all sequences such that $i_j \geq 2i_{j+1}$ of excess $e(I) < n$. This quantity is defined as $e(I) = i_2 - \sum_{j\geq 3} i_j$ and $\mathrm{Sq}^I x = \mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2} \ldots \mathrm{Sq}^{i_m} x$. If $i_m = 1$ then $\mathrm{Sq}^I x = \mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2} \ldots \mathrm{Sq}^{i_{m-1}} \beta_k x$ where β_k denotes the k-th power Bockstein for the short exact sequence

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_{2^{k+1}} \to \mathbb{Z}_{2^k} \to 0\,. \tag{5.9.14}$$

The E_2 page for (5.9.13) in terms of the generators then takes the form

$$E_2^{ij} = \begin{array}{c|cccccccccc} j & & & & & & & & & & \\ \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & 0 & 0 & \iota_4 & \beta_k\iota_4 & \mathrm{Sq}^2\iota_4 & \ldots & & \\ \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & 0 & 0 & \iota_4 & \beta_k\iota_4 & \mathrm{Sq}^2\iota_4 & (\mathrm{Sq}^3\iota_4,\, \mathrm{Sq}^2\beta_k\iota_4) & & \\ \mathbb{C}^\times & \mathbb{C}^\times & 0 & 0 & 0 & (-1)^{\iota_4} & 0 & (-1)^{\mathrm{Sq}^2\iota_4} & (-1)^{\mathrm{Sq}^2\beta_k\iota_4} & & \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & i\,. \end{array} \tag{5.9.15}$$

The d_2 differentials are given by:

$$\begin{aligned} d_2 : E_2^{i,2} &= \mathrm{H}^i(\mathbb{Z}_{2^k}[4]\,;\mathbb{Z}_2) \to E_2^{i+2,1} = \mathrm{H}^{i+2}(\mathbb{Z}_{2^k}[4]\,;\mathbb{Z}_2) && X \mapsto \mathrm{Sq}^2 X \\ d_2 : E_2^{i,1} &= \mathrm{H}^i(\mathbb{Z}_{2^k}[4]\,;\mathbb{Z}_2) \to E_2^{i+2,0} = \mathrm{H}^{i+2}(\mathbb{Z}_{2^k}[4]\,;\mathbb{C}^\times) && X \mapsto (-1)^{\mathrm{Sq}^2 X}\,, \end{aligned} \tag{5.9.16}$$

and there are d_2's leaving the entries in bidegrees $(4,2)$ and $(4,1)$ that carry the generator ι_4 to $\mathrm{Sq}^2\iota_4$ and are isomorphisms. Additionally, there are d_2 differentials leaving the entries in bidegrees $(5,1)$ and $(5,2)$ which are isomorphisms. In total degree seven, the E_3 page converges to the E_∞ page and we see that $\mathrm{SH}^7(\mathbb{Z}_{2^k}[4]) = 0$. If E is a product of groups, we can use (5.9.8), where the spaces are fourth Eilenberg-MacLane spaces of groups that are 2-torsion. Then the term corresponding to $\widetilde{h}^\bullet(X \wedge Y)$ for supercohomology will only begin to contribute in degree 8, and everything else vanishes. In summary, we have shown that $\mathrm{SH}^7(E[4]) = 0$ for any group E.

This implies that $\mathfrak{S}^\times \cong 2\mathbf{SVec}^\times \times E$ as symmetric monoidal 2-categories. In particular, $\mathrm{B}\mathbf{SVec} \times E$ is a full symmetric monoidal sub-2-category of $\mathfrak{S}$. But $\mathrm{B}\mathbf{SVec} \times E$ contains an object in every connected component of $\mathfrak{S}$, so that its Cauchy completion $Cau(\mathrm{B}\mathbf{SVec} \times E) \simeq 2\mathbf{SVec}[E]$ is equivalent to $\mathfrak{S}$ as a symmetric monoidal 2-category. Thus, we obtain the following result.

Proposition 5.9.2. Every symmetric fermionic strongly fusion 2-category is of the form $2\mathbf{SVec}[E]$ for some finite abelian group E.

In particular, every symmetric strongly fusion 2-category admits a fibre 2-functor to $2\mathbf{SVec}$.

Bosonic Case

For a symmetric bosonic strongly fusion 2-category, the obstruction to condensing to the symmetric fusion 2-category to $2\mathbf{Vec}$ is given by a class in $\mathrm{H}^7(E[4];\mathbb{C}^\times)$. The group $\mathrm{H}^{n+m+1}(E[n];\mathbb{C}^\times)$ may be thought of as parameterizing the ways for m spacetime dimensional objects to fuse in n ambient dimensions with fusion rule E. A computation of this cohomology group can be found in [87] where the authors obtained $\mathrm{H}^7(E[4];\mathbb{C}^\times) = \widehat{(E_2)}$, with E_2 the 2-torsion subgroup of E, and for a group A we denote $\widehat{A} = \hom(A, \mathbb{C}^\times)$. Even though this cohomology group is not necessarily trivial, the computation in 5.9.2 shows that if we work in a fermionic setting, then there is no obstruction to the existence of a fibre 2-functor to $2\mathbf{SVec}$. In particular, any symmetric bosonic strongly fusion 2-category admits a fibre 2-functor to $2\mathbf{SVec}$, which concludes the proof of theorem 5.9.1.

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APPENDIX

Appendix A

Topological Details

A.1. Spectrum of a Ring

Let us start off with a commutative K-algebra denoted S, the simplest being the algebra over $\mathbb{C}$. Define $\mathrm{Spec}(S) : R \to \hom(S, R)$. Where R is some set of test objects. If we let $R = \mathbb{C}$ then $\mathrm{Spec}(S)(\mathbb{C}) = \hom(S, \mathbb{C})$ taking values in sets, and hom is as $\mathbb{C}$-linear algebras. As an example let $S = \mathbb{C}[x]/\left(p(x) = 0\right)$ for some arbitrary polynomial $p(x)$ with complex roots. Then $\mathrm{Spec}(S) = \{\lambda \in \mathbb{C} \,|\, (x-\lambda)^{-1} \notin S\}$ i.e. the noninvertible $(x - \lambda)$. Therefore, given x there exists a homomorphism $\rho : S \to \mathbb{C}$ with $x \mapsto \lambda$. Conversely, given a map ρ, then the point $\rho(x) \in \mathrm{Spec}(S)$. In general, the map from S to the R -valued functions on $\mathrm{Spec}(S)(R)$ is neither injective nor surjective. However, if S has finite dimension, is separable, and $R = \mathbb{C}$ then this map is an isomorphism. This says that $\mathrm{Spec}(S)(\mathbb{C})$ in sets determines S, which makes $\mathrm{Spec}(S)$ into a sheaf as a space.

A.2. Further Examples of Nonabelian Condensation

$\mathbf{SU(3)_2 \times (\boldsymbol{G_2})_1}$: We continue with an example of a product theory; we review this example because this type of theory arises frequently when one considers using the folding trick. We first give the two constituent spectra

$\mathrm{SU}(3)_2$	λ	h	q-dim
0	$[0,0,2]$	0	1
1	$[0,2,0]$	$2/3$	1
2	$[2,0,0]$	$2/3$	1
3	$[1,1,0]$	$3/5$	1.618033988750
4	$[1,0,1]$	$4/15$	1.618033988750
5	$[0,1,1]$	$4/15$	1.618033988750

$(G_2)_1$	λ	h	q-dim
0	$[0,0,1]$	0	1
1	$[0,1,0]$	$2/5$	1.618033988750 .

The spectrum of the product theory consists of 12 lines given by

$\mathrm{SU}(3)_2 \times (G_2)_1$	$\{\ell_1, \ell_2\}$	h	q-dim
0	$\{0,0\}$	0	1
1	$\{1,0\}$	$2/3$	1
2	$\{2,0\}$	$2/3$	1
3	$\{0,1\}$	$2/5$	1.618033988750
4	$\{1,1\}$	$16/15$	1.618033988750
5	$\{2,1\}$	$16/15$	1.618033988750
6	$\{3,0\}$	$3/5$	1.618033988750
7	$\{4,0\}$	$4/15$	1.618033988750
8	$\{5,0\}$	$4/15$	1.618033988750
9	$\{3,1\}$	1	2.618033988750
10	$\{4,1\}$	$2/3$	2.618033988750
11	$\{5,1\}$	$2/3$	2.618033988750 ,

from which we can form the algebra $\varphi = 0 + 9$. The modules constructed from this algebra are given by

$$\begin{aligned}
&\varphi \times 0 = \varphi\,, && \varphi \times 6 = 6 + (3 + 9_2)\,,\\
&\varphi \times 1 = \varphi\,, && \varphi \times 7 = 7 + (4 + 10_2)\,,\\
&\varphi \times 2 = 2 + 11_1\,, && \varphi \times 8 = 8 + (5 + 11_2)\,,\\
&\varphi \times 3 = 3 + (6 + 9_2)\,, && \varphi \times 9 = 9_1 + (0 + 9_2 + 3 + 6)\,,\\
&\varphi \times 4 = 4 + (7 + 10_2)\,, && \varphi \times 10 = 10_1 + (1 + 4 + 7 + 10_2)\,,\\
&\varphi \times 5 = 5 + (8 + 11_2)\,, && \varphi \times 11 = 11_1 + (2 + 5 + 8 + 11_2)\,.
\end{aligned} \tag{A.2.1}$$

The quantum dimension of the last three lines on the table, are exactly off from the quantum dimensions of lines 3 through 8 by 1, hinting at the fact that those three lines will split. Indeed, if we greedily assign the quantum dimension of line 3 and 6 to 9_2, then the vacuum $0+9_1$ has quantum dimension 1. We similarly assign the quantum dimension for 10_2 and 11_2. By grouping based on quantum dimensions we get the lines

$$\begin{array}{c|c} \ell & \text{q-dim} \\ \hline \varphi=(0+9_1) & 1 \\ (1+10_1) & 1 \\ (2+11_1) & 1 \\ (3+6+9_2) & 1.618033988750 \\ (4+7+10_2) & 1.618033988750 \\ (5+8+11_2) & 1.618033988750\,, \end{array} \tag{A.2.2}$$

the last three are projected out because of the simple objects have different spins. The three remaining lines

$$\{\varphi=(0+9_1),\,(1+10_1),\,(2+11_1)\} \tag{A.2.3}$$

are the ones in $(E_6)_1$.

$\boldsymbol{(F_4)_3}$**:** The spectrum for this theory consists of 9 lines given by

$(F_4)_3$	λ	h	q-dim
0	$[0,0,0,0,3]$	0	1
1	$[0,0,0,1,2]$	1/2	5.449489742783
2	$[0,0,0,2,1]$	13/12	8.898979485566
3	$[0,0,0,3,0]$	7/4	4.449489742783
4	$[0,0,1,0,1]$	1	9.898979485566
5	$[0,0,1,1,0]$	13/8	10.898979485566
6	$[0,1,0,0,0]$	3/2	5.449489742783
7	$[1,0,0,0,1]$	3/4	4.449489742783
8	$[1,0,0,1,0]$	4/3	8.898979485566

from which we form the algebra $\varphi=0+4$. By inspecting the quantum dimension, we see that $4.449\ldots$ is the lowest that is not 1, and the other higher quantum dimensions can be partitioned into $4.449\ldots$ and 1. The modules constructed from this algebra are given by

$$\varphi\times 0=\varphi\,,$$

$$
\begin{aligned}
\varphi \times 1 &= 1_1 + (1_2 + 2_2 + 4_3 + 5_4 + 6_2 + 7 + 8_2)\,, \\
\varphi \times 2 &= 2 + (1_2 + 2_2 + 3 + 4_3 + 4_3 + 5_3 + 5_4 + 6_2 + 7 + 8_1 + 8_2)\,, \\
\varphi \times 3 &= 3 + (2_2 + 4_3 + 5_4 + 6_2 + 8_2)\,, \\
\varphi \times 4 &= 4_1 + (0 + 1 + 2_1 + 2_2 + 3 + 4_2 + 4_3 + 5_3 + 5_4 + 6 + 7 + 8_1 + 8_2)\,, \\
\varphi \times 5 &= 5_1 + (1 + 2_1 + 2_2 + 3 + 4_2 + 4_3 + 5_2 + 5_3 + 5_4 + 6 + 7 + 8_1 + 8_2)\,, \\
\varphi \times 6 &= 6_1 + (1_2 + 2_2 + 3 + 4_3 + 5_4 + 6_2 + 8_2)\,, \\
\varphi \times 7 &= 7 + (1_2 + 2_2 + 4_3 + 5_4 + 8_2)\,, \\
\varphi \times 8 &= 8_1 + (1_2 + 2_1 + 2_2 + 3 + 4_2 + 4_3 + 5_3 + 5_4 + 6_2 + 7 + 8_2)\,. \qquad \text{(A.2.4)}
\end{aligned}
$$

While the fusion structure is more complicated, one does notice the following grouping of lines to appear

$$(1_2 + 4_2 + 5_3 + 7 + 8_1)\,, \quad (2_2 + 3 + 4_3 + 5_4 + 6_2 + 8_2),$$

both of which we greedy assign q-dim 4.49…, which is that of line 3 and 7. Together with the remaining lines we form the groupings given by

ℓ	q-dim
$\varphi = (0 + 4_1)$	1
$(1_1 + 6_1)$	1
5_1	1
5_2	1
$(1_2 + 4_2 + 5_3 + 7 + 8_1)$	4.449489742783
$(2_2 + 3 + 4_3 + 5_4 + 6_2 + 8_2)$	4.449489742783 ,

(A.2.5)

the first four

$$\{\varphi = (0 + 4_1),\, (1_1 + 6_1),\, 5_1,\, 5_2\} \qquad \text{(A.2.6)}$$

are the lines of $\mathrm{Spin}(26)_1$, while the last two are projected out.

$(G_2)_4$: The spectrum for this theory consists of 9 lines given by

$(G_2)_4$	λ	h	q-dim
0	$[0,0,4]$	0	1
1	$[0,1,3]$	1/4	4.449489742783
2	$[0,2,2]$	7/12	8.898979485566
3	$[0,3,1]$	1	9.898979485566
4	$[0,4,0]$	3/2	5.449489742783
5	$[1,0,2]$	1/2	5.449489742783
6	$[1,1,1]$	7/8	10.898979485566
7	$[1,2,0]$	4/3	8.898979485566
8	$[2,0,0]$	5/4	4.449489742783

from which we form the algebra $\varphi = 0 + 3$. The modules constructed from this algebra are given by

$$\begin{aligned}
&\varphi \times 0 = \varphi \\
&\varphi \times 1 = 1 + (2_2 + 3_3 + 4_2 + 6_4 + 7_2) \\
&\varphi \times 2 = 2_1 + (1 + 2_2 + 3_2 + 3_3 + 4_2 + 5 + 6_3 + 6_4 + 7_1 + 7_2 + 8) \\
&\varphi \times 3 = 3_1 + (0 + 1 + 2_1 + 2_2 + 3_2 + 3_3 + 4_2 + 5 + 6_3 + 6_4 + 7_1 + 7_2 + 8) \\
&\varphi \times 4 = 4_1 + (1 + 2_2 + 3_3 + 4_2 + 5_2 + 6_4 + 7_2) \\
&\varphi \times 5 = 5_1 + (2_2 + 3_3 + 4_2 + 5_2 + 6_4 + 7_2 + 8) \\
&\varphi \times 6 = 6_1 + (1 + 2_1 + 2_2 + 3_1 + 3_2 + 4 + 5 + 6_2 + 6_3 + 6_4) \\
&\varphi \times 7 = 7 + (1 + 2_1 + 2_2 + 3_1 + 3_2 + 4 + 5 + 6_3 + 6_4 + 7_2 + 8) \\
&\varphi \times 8 = 8_1 + (2_2 + 3_3 + 5 + 6_4 + 7_2)\,.
\end{aligned} \tag{A.2.7}$$

By grouping based on quantum dimensions we greedily assign the dimension of line 8 and line 1, which is the lowest quantum dimension that is not 1, to the lines

$$(1 + 2_2 + 3_3 + 6_4 + 7_2)\,, \quad (2_1 + 6_3 + 7_1 + 8) \tag{A.2.8}$$

which appear repeatedly in the equations above. In summary the groupings are

$$
\begin{array}{c|c}
\ell & \text{q-dim} \\
\hline
\varphi = (0 + 3_1) & 1 \\
(4_1 + 5_1) & 1 \\
6_1 & 1 \\
6_2 & 1 \\
(1 + 2_2 + 3_3 + 6_4 + 7_2) & 4.4494897427830 \\
(2_1 + 6_3 + 7_1 + 8) & 4.4494897427830 \\
(3_2 + 5_2) & 5.4494897427830\,.
\end{array}
\tag{A.2.9}
$$

The last three lines are projected out due to the fact that the simple objects have different spins. The first four lines give those of $\mathrm{Spin}(14)_1$.

$\mathbf{SU(3)_5}$: It will be clear after this example that as the number of lines becomes even larger, finding the modules for a condensation algebra becomes a tedious task. The spectrum of this theory contains 21 lines given by

$SU(3)_5$	λ	h	q-dim
0	$[0,0,5]$	0	1
1	$[0,5,0]$	5/3	1
2	$[5,0,0]$	5/3	1
3	$[1,4,0]$	3/2	2.414213562373
4	$[4,0,1]$	7/6	2.414213562373
5	$[0,1,4]$	1/6	2.414213562373
6	$[3,0,2]$	3/4	3.414213562373
7	$[0,2,3]$	5/12	3.414213562373
8	$[2,3,0]$	17/12	3.414213562373
9	$[0,3,2]$	3/4	3.414213562373
10	$[3,2,0]$	17/12	3.414213562373

$SU(3)_5$	λ	h	q-dim
11	$[2,0,3]$	5/12	3.414213562373
12	$[4,1,0]$	3/2	2.414213562373
13	$[1,0,4]$	1/6	2.414213562373
14	$[0,4,1]$	7/6	2.414213562373
15	$[1,1,3]$	3/8	4.828427124746
16	$[1,3,1]$	25/24	4.828427124746
17	$[3,1,1]$	25/24	4.828427124746
18	$[2,2,1]$	1	5.828427124746
19	$[2,1,2]$	2/3	5.828427124746
20	$[1,2,2]$	2/3	5.828427124746

The modules for the algebra $\varphi = 0 + 18$, created by the nonabelian boson is

$$\varphi \times 0 = \varphi\,,$$
$$\varphi \times 1 = 1 + 19_1\,,$$
$$\varphi \times 2 = 2 + 20_1\,,$$
$$\varphi \times 11 = 11_1 + (8_2 + 11_2 + 14 + 17_2 + 20_3)\,,$$
$$\varphi \times 12 = 12 + (9_2 + 15_2 + 18_3)\,,$$
$$\varphi \times 13 = 13 + (10_2 + 16_2 + 19_3)\,,$$

$$
\begin{aligned}
\varphi \times 3 &= 3 + (6_2 + 15_2 + 18_3)\,, \\
\varphi \times 4 &= 4 + (7_2 + 16_2 + 19_3)\,, \\
\varphi \times 5 &= 5 + (8_2 + 17_2 + 20_3)\,, \\
\varphi \times 6 &= 6_1 + (3 + 6_2 + 9_2 + 15_2 + 18_3)\,, \\
\varphi \times 7 &= 7_1 + (4 + 7_2 + 10_2 + 16_2 + 19_3)\,, \\
\varphi \times 8 &= 8_1 + (5 + 8_2 + 11_2 + 17_2 + 20_3)\,, \\
\varphi \times 9 &= 9_1 + (6_2 + 9_2 + 12 + 15_2 + 18_3)\,, \\
\varphi \times 10 &= 10_1 + (7_2 + 10_2 + 13 + 16_2 + 19_3)\,.
\end{aligned}
\qquad
\begin{aligned}
\varphi \times 14 &= 14 + (11_2 + 17_2 + 20_3)\,, \\
\varphi \times 15 &= 15_1 + (3 + 6_2 + 9_2 + 12 \\
&\qquad + 15_2 + 18_2 + 18_3)\,, \\
\varphi \times 16 &= 16_1 + (4 + 7_2 + 10_2 + 13 \\
&\qquad + 16_2 + 19_2 + 19_3)\,, \\
\varphi \times 17 &= 17_1 + (5 + 8_2 + 11_2 + 14 \\
&\qquad + 17_2 + 20_2 + 20_3)\,, \\
\varphi \times 18 &= 18_1 + (0 + 3 + 6_2 + 9_2 + 12 \\
&\qquad + 15_1 + 15_2 + 18_2 + 18_3)\,, \\
\varphi \times 19 &= 19_1 + (1 + 4 + 7_2 + 10_2 + 13 \\
&\qquad + 16_1 + 16_2 + 19_2 + 19_3)\,,
\end{aligned}
\tag{A.2.10}
$$

$$
\varphi \times 20 = 20_1 + (2 + 5 + 8_2 + 11_2 + 14 + 17_1 + 17_2 + 20_2 + 20_3)\,, \tag{A.2.11}
$$

By closely examining the repeating structures within the modules, we can see the following grouping of lines

ℓ	q-dim	ℓ	q-dim
$\varphi = (0 + 18_1)$	1	$(3 + 6_2 + 9_2 + 15_2 + 18_3)$	2.414213562373
$(1 + 19_1)$	1	$(15_1 + 18_2)$	2.414213562373
$(2 + 20_1)$	1	$(4 + 7_2 + 10_2 + 13 + 16_2 + 19_3)$	2.414213562373
$(6_1 + 9_1)$	1	$(16_1 + 19_2)$	2.414213562373
$(7_1 + 8_1)$	1	$(5 + 8_2 + 11_2 + 14 + 17_2 + 20_3)$	2.414213562373
$(10_1 + 11_1)$	1	$(17_1 + 20_2)$	2.414213562373 .

(A.2.12)

$\mathbf{Sp(16)_1}$: We present this theory to give a nontrivial example of when nonabelian condensation for a line with non-integer spin can be performed after abelian condensation, in a consistent way. In the bulk of the paper, it was shown that for $(G_2)_2$ that there was no canonical way to group lines and assign quantum dimensions in any consistent way. But we will see in this simple example that the grouping of lines is canonical. The spectrum consists of 9 lines given by

$\mathrm{Sp}(16)_1$	λ	h	q-dim
0	$[0,0,0,0,0,0,0,0,1]$	0	1
1	$[0,0,0,0,0,0,0,1,0]$	2	1
2	$[0,0,0,0,0,0,1,0,0]$	77/40	1.902113032590
3	$[1,0,0,0,0,0,0,0,0]$	17/40	1.902113032590
4	$[0,0,0,0,0,1,0,0,0]$	9/5	2.618033988750
5	$[0,1,0,0,0,0,0,0,0]$	4/5	2.618033988750
6	$[0,0,0,0,1,0,0,0,0]$	13/8	3.077683537175
7	$[0,0,1,0,0,0,0,0,0]$	9/8	3.077683537175
8	$[0,0,0,1,0,0,0,0,0]$	7/5	3.236067977500 .

Upon condensing out the abelian boson we are left with the lines

$\mathrm{Sp}(16)_1/\mathbb{Z}_2$	ℓ	h	q-dim
0	$\varphi=(0+1)$	0	1
1	$(4+5)$	4/5	2.618033988750
2	8_1	7/5	1.618033988750
3	8_2	7/5	1.618033988750 ,

from which we sequentially condense $\widetilde{\varphi}=0+1$, noticing that this is a nonabelian spin $\frac{4}{5}$ line that usually would have been abelian after the boson condensation. Nevertheless, the modules are

$$\begin{aligned}
\widetilde{\varphi}\times 0 &= \widetilde{\varphi}\,,\\
\widetilde{\varphi}\times 1 &= 1_1+(0+1_2+2+3)\,,\\
\widetilde{\varphi}\times 2 &= 2+(1_2+3)\,,\\
\widetilde{\varphi}\times 3 &= 3+(1_2+2)\,,
\end{aligned}\tag{A.2.13}$$

from which we can see that the remaining lines are $\widetilde{\varphi}$ and (1_2+2+3) with quantum dimension equal to the golden ratio. As a remark, the modular invariants of $\mathrm{Sp}(16)_1$ only captures the abelian condensation, and not the second step. The spectrum of lines in $\mathrm{Sp}(16)_1/\mathbb{Z}_2$ have spins that are all of a common denominator, so the set of $\mathcal{M}$ contain more than just those which can be built from algebras.

$\mathbf{SU(4)_4/\mathbb{Z}_4}$: We consider an example of a nonsimply connected group to prime ourselves for the next example in this appendix. We will condense out an abelian line in $\mathrm{SU}(4)_4$, and follow up with a nonabelian condensation. After the abelian condensation the spectrum consists of 14 lines already given in §5.2.1. The algebra formed by the nonabelian boson, $\varphi = 0 + 6$ has as its modules

$$\begin{aligned}
&\varphi \times 0 = \varphi\,, && \varphi \times 8 = 8 + (6_3 + 7_2 + 11)\,,\\
&\varphi \times 1 = 1 + 7_3\,, && \varphi \times 9 = 9 + (6_3 + 7_2 + 10)\,,\\
&\varphi \times 2 = 2 + (3 + 4 + 5 + 12_2 + 13_2)\,, && \varphi \times 10 = 10 + (6_3 + 7_2 + 9)\,,\\
&\varphi \times 3 = 3 + (2 + 4 + 5 + 12_2 + 13_2)\,, && \varphi \times 11 = 11 + (6_3 + 7_2 + 8)\,,\\
&\varphi \times 4 = 4 + (2 + 3 + 5 + 12_2 + 13_2)\,, && \varphi \times 12 = 12_1 + (2 + 3 + 4 + 5 + 12_2 + 13_1 + 13_2)\,,\\
&\varphi \times 5 = 5 + (2 + 3 + 4 + 12_2 + 13_2)\,, && \varphi \times 13 = 13_1 + (2 + 3 + 4 + 5 + 12_1 + 12_2 + 13_2)\,.
\end{aligned}$$

$$\begin{aligned}
&\varphi \times 6 = 6_1 + (0 + 6_2 + 6_3 + 7_1 + 7_2 + 8 + 9 + 10 + 11)\,,\\
&\varphi \times 7 = 7_1 + (1 + 6_2 + 6_3 + 7_2 + 7_3 + 8 + 9 + 10 + 11)\,,
\end{aligned} \tag{A.2.14}$$

The natural grouping of the lines from the modules is

ℓ	q-dim
$\varphi = (0 + 6_1)$	1
$(1 + 7_3)$	1
$(12_1 + 13_1)$	1.414213562373
$(6_2 + 7_1)$	2.414213562373
$(6_3 + 7_2 + 8 + 9 + 10 + 11)$	2.414213562373
$(2 + 3 + 4 + 5 + 12_2 + 13_2)$	3.414213562373 ,

(A.2.15)

The last two lines are confined due to the differing spins, so we find the remaining lines are

$$\{\varphi = (0 + 6_1),\, (1 + 7_3),\, (12_1 + 13_1)\}\,.$$

$\mathbf{SU(2)_4^{\boxtimes 3}}$: In this example we construct a theory where we show how anyon condensation can give insights into the symmetries of the theory that we may not have expected at first sight. Consider $\mathrm{SU}(2)_4^3$, its abelian anyons form a $(\mathbb{Z}_2)^3$ group. All of these are condensable, but we choose only to condense the $(\mathbb{Z}_2)^2$ subgroup given by the lines $\{000, 110, 101, 011\}$.

Here, the numbers denote the lines coming from each of the $\mathrm{SU}(2)_4$ factors, the spectrum was given in §5.2.1 The result we will call $\mathrm{SU}(2)_4^{\circ 3}$ where the 'o' stands for "central product". The data of the spectrum consists of 17 lines and is given by

$\mathrm{SU}(2)_4^{\circ 3}$	$\{\ell_1,\ell_2,\ell_3\}$	h	q-dim
0	$\{0,0,0\}$	0	1
1	$\{0,0,1\}$	1	1
2	$\{0,0,4\}$	1/3	2
3	$\{0,4,0\}$	1/3	2
4	$\{0,4,4\}$	2/3	2
5	$\{0,4,4\}$	2/3	2
6	$\{2,2,2\}$	3/8	5.196152422706
7	$\{2,2,3\}$	7/8	5.196152422706
8	$\{4,0,0\}$	1/3	2

$\mathrm{SU}(2)_4^{\circ 3}$	$\{\ell_1,\ell_2,\ell_3\}$	h	q-dim
9	$\{4,0,4\}$	2/3	2
10	$\{4,0,4\}$	2/3	2
11	$\{4,4,0\}$	2/3	2
12	$\{4,4,0\}$	2/3	2
13	$\{4,4,4\}$	1	2
14	$\{4,4,4\}$	1	2
15	$\{4,4,4\}$	1	2
16	$\{4,4,4\}$	1	2 .

The 8-dimensional representation $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}$ of $\mathrm{SU}(2)^3$ gives a map $\mathrm{SU}(2)_4^{\circ 3}\to\mathrm{Sp}(8)_1$ which is conformal. The condensable anyons are any one of $\{13,14,15,16\}$, and one could wonder which algebra gives the conformal embedding. We will see that all four anyons can condense to give $\mathrm{Sp}(8)_1$. The problem inherently has a triality due to the three SU(2) factors, but given the spectrum data and the fact actually four lines can condense prompts us to believe that as an MTC, $\mathrm{SU}(2)_4^{\circ 3}$ has an extra symmetry that is S_4. Since the theory has 17 lines, there are 17! permutations that are potentially a symmetry of the theory. A permutation will be a symmetry if it preserves the full modular data. One can see that there are $3!\cdot 4!\cdot 6!$ permutations that preserve the spins and quantum dimensions. Out of these, a brute force check shows that there are exactly 24 permutations that also preserve the fusion rules. Finally, by looking at how these permutations compose, it is straightforward to show that they correspond to the group S_4 [1].

Instead of doing the complete analysis given above, we can see hints of an enlarged symmetry when we consider the theory after condensing each of the four nonabelian bosons. We present only the modules of $\varphi_1 = 0+13$, as the same procedure works for the other choices of condensate:

$$\begin{aligned}
&\varphi\times 0=\varphi\,, && \varphi\times 9 = 9+(3+14)\,,\\
&\varphi\times 1 = 1+13_2\,, && \varphi\times 10 = 10+(5+12)\,,\\
&\varphi\times 2 = 2+(11+15)\,, && \varphi\times 11 = 11+(2+15)\,,
\end{aligned}$$

[1]The S_4 preserves the S and T matrices of $\mathrm{SU}(2)_4^{\circ 3}$, but in principle one should also check the F- and R-symbols. We believe it should be possible to compute these symbols in terms of those of $\mathrm{SU}(2)_4$

$$
\begin{aligned}
&\varphi \times 3 = 3 + (9 + 14)\,, && \varphi \times 12 = 12 + (5 + 10)\,,\\
&\varphi \times 4 = 4 + (8 + 16)\,, && \varphi \times 13 = 13_1 + (0 + 1 + 13_2)\,,\\
&\varphi \times 5 = 5 + (10 + 12)\,, && \varphi \times 14 = 14 + (3 + 9)\,,\\
&\varphi \times 6 = 6_1 + (6_2 + 7_2)\,, && \varphi \times 15 = 15 + (2 + 11)\,,\\
&\varphi \times 7 = 7_1 + (6_2 + 7_2)\,, && \varphi \times 16 = 16 + (4 + 8)\,.\\
&\varphi \times 8 = 8 + (4 + 16)\,, &&
\end{aligned}
\tag{A.2.16}
$$

In total, the modules for $\varphi_1 = 0 + 13$, $\varphi_2 = 0 + 14$, $\varphi_3 = 0 + 15$, and $\varphi_4 = 0 + 16$ give the organization of lines as follows [2]:

φ_1	ℓ	q-dim
0	$(0 + 13_1)$	1
1	$(1 + 13_2)$	1
2	$(2 + 11 + 15)$	1
3	$(3 + 9 + 14)$	2
4	$(4 + 8 + 16)$	2
5	$(5 + 10 + 12)$	2
6	6_1	1.732050807568
7	7_1	1.732050807568
8	$6_2 + 7_2$	3.464101615137

φ_2	ℓ	q-dim
0	$(0 + 14_1)$	1
1	$(1 + 14_2)$	1
2	$(2 + 12 + 16)$	1
3	$(3 + 9 + 13)$	2
4	$(4 + 10 + 11)$	2
5	$(5 + 8 + 15)$	2
6	6_1	1.732050807568
7	7_1	1.732050807568
8	$6_2 + 7_2$	3.464101615137

φ_3	ℓ	q-dim
0	$(0 + 15_1)$	1
1	$(1 + 15_2)$	1
2	$(2 + 11 + 13)$	1
3	$(3 + 10 + 16)$	2
4	$(4 + 9 + 12)$	2
5	$(5 + 8 + 14)$	2
6	6_1	1.732050807568
7	7_1	1.732050807568
8	$6_2 + 7_2$	3.464101615137

φ_4	ℓ	q-dim
0	$(0 + 16_1)$	1
1	$(1 + 16_2)$	1
2	$(2 + 12 + 14)$	1
3	$(3 + 10 + 15)$	2
4	$(4 + 8 + 13)$	2
5	$(5 + 9 + 11)$	2
6	6_1	1.732050807568
7	7_1	1.732050807568
8	$6_2 + 7_2$	3.464101615137 .

[2] A priori there is an ambiguity in splitting the quantum dimension of 6 and 7 into its constituents. The way the dimensions were assigned is guided by the fact that there exists a conformal embedding.

From the tables above the unconfined lines are

ℓ	q-dim	ℓ	q-dim
$\varphi_1 = (0 + 13_1)$	1	$\varphi_1 = (0 + 14_1)$	1
$(1 + 13_2)$	1	$(1 + 14_2)$	1
6_1	1.732050807568	6_1	1.732050807568
7_1	1.732050807568	7_1	1.732050807568
$(5 + 10 + 12)$	2	$(4 + 10 + 11)$	2

(A.2.17)

ℓ	q-dim	ℓ	q-dim
$\varphi_1 = (0 + 15_1)$	1	$\varphi_1 = (0 + 16_1)$	1
$(1 + 15_2)$	1	$(1 + 16_2)$	1
6_1	1.732050807568	6_1	1.732050807568
7_1	1.732050807568	7_1	1.732050807568
$(4 + 9 + 12)$	2	$(5 + 9 + 11)$	2

(A.2.18)

where each choice of condensation gives a copy of $\mathrm{Sp}(4)_1$, hence the triality symmetry we were expecting should be enlarged to a group that can permute four objects.

A.3. Reconstruction of $\mathrm{SU}(2)_{10}$

One of the new features of this example is that when a line splits such that one part is confined and one part moves to the child, we have some different condition on the S-matrix element. To see this explicitly, consider from the following table

ℓ	confined/unconfined
$\varphi = 0 + 6_1$	unconfined
$(4_1 + 10)$	unconfined
$(3_1 + 7_1)$	unconfined
$(1 + 5_2 + 7_2)$	confined
$(3_2 + 5_1 + 9)$	confined
$(2 + 4_2 + 6_2 + 8)$	confined

(A.3.1)

the element $S_{(1+5_2+7_2),\varphi}$. Since φ can move past a totally confined line by going to the child theory, we would expect that

$$S_{1,\varphi} = S_{5,\varphi} = S_{7,\varphi} = 0 \tag{A.3.2}$$

in the parent theory. However, the last equality does not hold due to the fact that there is an unconfined line with 7_1 as a constituent object. When it is not the case that $S_{a,b}$ is between lines where a single line splits on the wall and into the child, then the consistency relations in §5.4 still hold. We will now run through the cases for the S-matrix.

- $S_{\text{confined,unconfined}}$

More explicitly, from φ, $(4_1 + 10)$, $(3_1 + 7_1)$ going around $(1 + 5_2 + 7_2)$ we see that

$$S_{1,0} + S_{1,6} = S_{5,0} + S_{5,6} = 0\,, \tag{A.3.3a}$$

$$S_{1,4} + S_{1,10} = S_{5,4} + S_{5,10} = 0\,, \tag{A.3.3b}$$

$$S_{1,3} + S_{1,7} = S_{5,3} + S_{5,7} = 0\,. \tag{A.3.3c}$$

From the unconfined lines brought around $(3_2 + 5_1 + 9)$ we have

$$S_{5,0} + S_{5,6} = S_{9,0} + S_{9,6} = 0\,, \tag{A.3.4}$$

$$S_{5,4} + S_{5,10} = S_{9,4} + S_{9,10} = 0\,, \tag{A.3.5}$$

$$S_{5,3} + S_{5,7} = S_{9,3} + S_{9,7} = 0\,. \tag{A.3.6}$$

Next consider the unconfined lines brought around $(2 + 4_2 + 6_2 + 8)$

$$S_{2,0} + S_{2,6} = S_{8,0} + S_{8,6} = 0\,, \tag{A.3.7}$$

$$S_{2,4} + S_{2,10} = S_{8,4} + S_{8,10} = 0\,, \tag{A.3.8}$$

$$S_{2,3} + S_{2,7} = S_{8,3} + S_{8,7} = 0\,. \tag{A.3.9}$$

- $S_{\text{confined,confined}}$

Here we apply the same logic as above for the S-matrix between two confined lines, using the intuition that one of confined line can be lifted to the parent theory making trivial braiding with other confined line in the wall. We list all of the relations for one confined line encircling another, in which the "moving" line does not involve a simple object that splits into a component on the wall and a component in the child. First consider $S_{(1+5_2+7_2),(1+5_2+7_2)}$, we expect three relations

$$S_{1,1} + S_{1,5} + S_{1,7} = 0\,, \tag{A.3.10}$$

$$S_{5,1} + S_{5,5} + S_{5,7} = 0. \tag{A.3.11}$$

The next term $S_{(3_2+5_1+9),(3_2+5_1+9)}$ gives equations

$$S_{5,3} + S_{5,5} + S_{5,9} = 0\,, \tag{A.3.12a}$$
$$S_{9,3} + S_{9,5} + S_{9,9} = 0. \tag{A.3.12b}$$

The last diagonal term is $S_{(2+4_2+6_2+8),(2+4_2+6_2+8)}$ and gives equations

$$S_{2,2} + S_{2,4} + S_{2,6} + S_{2,8} = 0\,, \tag{A.3.13a}$$
$$S_{8,2} + S_{8,4} + S_{8,6} + S_{8,8} = 0. \tag{A.3.13b}$$

We now look at the off diagonal terms of the S-matrix, starting off with $S_{(1+5_2+7_2),(3_2+5_1+9)}$, which gives the equations

$$S_{1,3} + S_{1,5} + S_{1,9} = 0\,, \tag{A.3.14a}$$
$$S_{5,3} + S_{5,5} + S_{5,9} = 0\,, \tag{A.3.14b}$$
$$S_{1,5} + S_{5,5} + S_{7,5} = 0\,, \tag{A.3.14c}$$
$$S_{1,9} + S_{5,9} + S_{7,9} = 0\,, \tag{A.3.14d}$$

where the first two equations arise from $(1+5_2+7_2)$ encircling (3_2+5_1+9) by moving into the parent, and the last two equations arise from (3_2+5_1+9) encircling $(1+5_2+7_2)$ by moving into the parent. For the next off diagonal component we consider $S_{(3_2+5_1+9),(2+4_2+6_2+8)}$, which gives equations

$$S_{5,2} + S_{5,4} + S_{5,6} + S_{5,8} = 0\,, \tag{A.3.15a}$$
$$S_{9,2} + S_{9,4} + S_{9,6} + S_{9,8} = 0\,, \tag{A.3.15b}$$
$$S_{3,2} + S_{5,2} + S_{9,2} = 0\,, \tag{A.3.15c}$$
$$S_{3,8} + S_{5,8} + S_{9,8} = 0\,. \tag{A.3.15d}$$

The final off-diagonal element $S_{(2+4_2+6_2+8),(1+5_2+7_2)}$ gives the equations

$$S_{1,2} + S_{1,4} + S_{1,6} + S_{1,8} = 0\,, \tag{A.3.16a}$$
$$S_{5,2} + S_{5,4} + S_{5,6} + S_{5,8} = 0\,, \tag{A.3.16b}$$
$$S_{2,1} + S_{2,5} + S_{2,7} = 0\,, \tag{A.3.16c}$$
$$S_{8,1} + S_{8,5} + S_{8,7} = 0\,. \tag{A.3.16d}$$

- $S_{\text{unconfined,unconfined}}$

We first consider using $S_{\varphi,\varphi} = \frac{1}{2}$ from the child an obtaining relations for the parent theory. The equations we get are

$$S_{0,0} + S_{0,9} = \frac{1}{2}\,, \tag{A.3.17a}$$

$$S_{0,9} + S_{9,9} = \frac{1}{2}\,. \tag{A.3.17b}$$

We next consider

$$\begin{aligned}\{S_{\varphi,(4_1+10)} = \frac{1}{2},\ S_{\varphi,(3_1+7_1)} = \frac{1}{\sqrt{2}},\ S_{(4_1+10),(3_1+7_1)} = -\frac{1}{\sqrt{2}},\\ S_{(4_1+10),(4_1+10)} = \frac{1}{2},\ S_{(3_1+7_1),(3_1+7_1)} = 0\}\,,\end{aligned} \tag{A.3.18}$$

which give the following relationships in the parent theory:

$$S_{0,4} + S_{0,10} = \frac{1}{2}, \quad S_{6,4} + S_{6,10} = \frac{1}{2}, \quad S_{0,4} + S_{6,4} = \frac{1}{2}, \quad S_{0,10} + S_{6,10} = \frac{1}{2}, \tag{A.3.19a}$$

$$S_{0,3} + S_{0,7} = \frac{1}{\sqrt{2}}\,, \quad S_{6,3} + S_{6,7} = \frac{1}{\sqrt{2}}\,, \quad S_{0,3} + S_{6,3} = \frac{1}{\sqrt{2}}\,, \quad S_{0,7} + S_{6,7} = \frac{1}{\sqrt{2}}, \tag{A.3.19b}$$

$$\begin{aligned}&S_{4,3} + S_{4,7} = -\frac{1}{\sqrt{2}}\,, \quad S_{10,3} + S_{10,7} = -\frac{1}{\sqrt{2}}\,, \quad S_{4,3} + S_{10,3} = -\frac{1}{\sqrt{2}}\,,\\ &S_{4,7} + S_{10,7} = -\frac{1}{\sqrt{2}}\,,\end{aligned} \tag{A.3.19c}$$

$$S_{4,4} + S_{4,10} = \frac{1}{2}\,, \quad S_{10,4} + S_{10,10} = \frac{1}{2}\,, \tag{A.3.19d}$$

$$S_{3,3} + S_{3,7} = 0\,, \quad S_{7,3} + S_{7,7} = 0. \tag{A.3.19e}$$

A.3.1. Implementing the fusion rules

- Matrix elements of the form $S_{1,-}$

Having used found all the relations we can by utilizing the relative center, we now employ the fusion rules of the wall category. We first determining the S-matrix elements of form $S_{1,-}$. Since $1 \times 1 = 0 + 2$ we can use the Verlinde formula for $N_{1,1}^0 = 1 = \sum_d \frac{S_{1,d}^2 S_{0,d}^*}{S_{0,d}}$; we also have $N_{1,1}^2$ but for now we will set that aside. By using the fact that $S_{0,d}^*$ is real, then the Verlinde formula gives

$$S_{1,0}^2 + S_{1,1}^2 + \ldots + S_{1,10}^2 = 1\,. \tag{A.3.20}$$

Another relation we will have to use frequently is (5.4.14a), in particular we need

$$S_{1,10\times 2} = \frac{S_{1,10}S_{1,2}}{S_{1,0}} = S_{1,8}\,, \tag{A.3.21}$$

$$S_{1,10\times 8} = \frac{S_{10,1}S_{1,8}}{S_{1,0}} = S_{1,2}\,,$$

these two equations imply that $S_{1,2}^2 = S_{1,8}^2$ so $S_{1,2} = \pm S_{1,8}$. We use this, along with the relations in (A.3.3a), (A.3.3b), and (A.3.3c) to simplify (A.3.20) to

$$2\left(S_{1,0}^2 + S_{1,2}^2 + S_{1,3}^2 + S_{1,4}^2\right) + S_{1,1}^2 + S_{1,9}^2 + S_{1,5}^2 = 1\,. \tag{A.3.22}$$

To proceed we first solve for $S_{1,5}$, From fusion we have the two equations

$$S_{1,0}S_{1,1\times 5} = S_{1,1}S_{1,5} = (S_{1,4} + S_{1,6})S_{1,0}\,, \tag{A.3.23}$$

$$S_{1,0}S_{1,9\times 5} = S_{1,9}S_{1,5} = (S_{1,4} + S_{1,6})S_{1,0}\,, \tag{A.3.24}$$

which can be combined to give

$$S_{1,5}(S_{1,9} - S_{1,1}) = 0\,, \tag{A.3.25}$$

so either $S_{1,5} = 0$ or $S_{1,1} = S_{1,9}$. If $S_{1,9} = S_{1,1}$, and we assume that $S_{1,1} \neq 0$, then from (A.3.3c), (A.3.10), and (A.3.14a) we find $S_{1,1} = -S_{1,9} - 2S_{1,5}$ so $S_{1,1} = -S_{1,5}$. But then by (A.3.10) we get $S_{1,7} = 0$, so $S_{1,3} = 0$. Furthermore, from

$$S_{1,1\times 2} = \frac{S_{1,1}S_{1,2}}{S_{1,0}} \tag{A.3.26}$$

then $S_{1,2} = 0 = S_{1,8}$, and it is then easy to derive that $S_{1,1} = S_{1,9} = 0$, which contradicts our initial assumption. Therefore we take $S_{1,5} = 0$, so that $S_{1,1} = -S_{1,9}$. With this (A.3.22) can be simplified to

$$2\left(S_{1,0}^2 + S_{1,1}^2 + S_{1,2}^2 + S_{1,3}^2 + S_{1,4}^2\right) = 1\,. \tag{A.3.27}$$

A natural next step to consider is replacing the different squares with as many of the same quantities as possible. To do this consider the fusion having to do with $S_{1,-}$:

$$S_{1,1\times 1} = S_{1,0} + S_{1,2} = \frac{S_{1,1}^2}{S_{1,0}}\,, \tag{A.3.28}$$

$$S_{1,9\times 9} = S_{1,0} + S_{1,2} = \frac{S_{1,9}^2}{S_{1,0}}\,,$$

$$
\begin{aligned}
S_{1,2\times 2} &= S_{1,0} + S_{1,2} + S_{1,4} = \frac{S_{1,2}^2}{S_{1,0}}\,, \\
S_{1,10\times 10} &= S_{1,0} + S_{1,2} + S_{1,4} = \frac{S_{1,10}^2}{S_{1,0}}\,, \\
S_{1,3\times 3} &= S_{1,0} + S_{1,2} + S_{1,4} + S_{1,6} = \frac{S_{1,3}^2}{S_{1,0}}\,, \\
S_{1,7\times 7} &= S_{1,0} + S_{1,2} + S_{1,4} + S_{1,6} = \frac{S_{1,7}^2}{S_{1,0}}\,, \\
S_{1,4\times 4} &= S_{1,0} + S_{1,2} + S_{1,8} + S_{1,4} + S_{1,6} = \frac{S_{1,4}^2}{S_{1,0}}\,, \\
S_{1,6\times 6} &= S_{1,0} + S_{1,2} + S_{1,8} + S_{1,4} + S_{1,6} = \frac{S_{1,6}^2}{S_{1,0}}\,, \\
S_{1,5\times 5} &= S_{1,0} + S_{1,10} + S_{1,2} + S_{1,8} + S_{1,4} + S_{1,6} = \frac{S_{1,5}^2}{S_{1,0}}\,,
\end{aligned}
$$

and recall that $S_{1,2} + S_{1,4} + S_{1,6} + S_{1,8} = 0$ by (A.3.16a). Then we can write, $S_{1,3}^2 = S_{1,0}^2 - S_{1,0}S_{1,8}$. We may simplify (A.3.22) even further to be

$$
2\Big[S_{1,0}^2 + \left(S_{1,0}^2 + S_{1,0}S_{1,2}\right) + \left(S_{1,0}^2 + S_{1,0}S_{1,2} + S_{1,0}S_{1,4}\right) + \left(S_{1,0}^2 - S_{1,0}S_{1,8}\right) + S_{1,0}^2\Big] = 1\,, \tag{A.3.29}
$$

$$
10S_{1,0}^2 + S_{1,0}\left(4S_{1,2} - 2S_{1,8} + 2S_{1,4}\right) = 1\,. \tag{A.3.30}
$$

We desire some relations between $S_{1,0}, S_{1,4}, S_{1,8}$, we can consider

$$
S_{1,0}S_{1,2\times 4} = -S_{1,0}S_{1,8} = -S_{1,10}S_{1,2}\,, \tag{A.3.31a}
$$

$$
S_{1,0}S_{1,8\times 4} = -S_{1,0}S_{1,2} = -S_{1,10}S_{1,8}\,, \tag{A.3.31b}
$$

$$
S_{1,0}S_{1,2\times 8} = S_{1,0}S_{1,10} + S_{1,10}S_{1,2} - S_{1,10}S_{1,4}\,. \tag{A.3.31c}
$$

By adding the first two equations we get

$$
\left(S_{1,0} - S_{1,10}\right)\left(S_{1,2} + S_{1,8}\right) = 0\,, \tag{A.3.32}
$$

from which we have either $S_{1,0} = S_{1,10}$ or $S_{1,2} = -S_{1,8}$. but the last of (A.3.28) would cause the former choice to run into a contradiction. We have thus determined $S_{1,2} = -S_{1,8}$ and so $S_{1,4} = -S_{1,6} = S_{1,0}$. We now try to relate $S_{1,2}$ with $S_{1,0}$, to do this consider the fact that

$$S_{1,2}^2 = S_{1,0}^2 + S_{1,0}S_{1,2} + S_{1,0}S_{1,4} \tag{A.3.33}$$

and can be simplified to

$$S_{1,2}(S_{1,2} - S_{1,0}) = 2S_{1,0}^2 \tag{A.3.34}$$

which is satisfied if $S_{1,2} = 2S_{1,0}$. We summarize how all of $S_{1,-}$ is related to $S_{1,0}$ by the following equations

$$\begin{aligned} &S_{1,1}^2 = 3S_{1,0}^2\,, && S_{1,2}^2 = 4S_{1,0}^2\,, && S_{1,3}^2 = 3S_{1,0}^2\,, && S_{1,4}^2 = S_{1,0}^2\,,\\ &S_{1,5}^2 = 0\,, && S_{1,6}^2 = S_{1,0}^2\,, && S_{1,7}^2 = 3S_{1,0}^2\,, && S_{1,8}^2 = 4S_{1,0}^2\,,\\ &S_{1,9}^2 = 3S_{1,0}^2\,, && S_{1,10}^2 = S_{1,0}^2\,, \end{aligned} \tag{A.3.35}$$

and therefore (A.3.27) becomes $24S_{1,0}^2 = 1$, and thus $S_{1,0} = \frac{1}{\sqrt{24}}$.

We now repeat a similar process to find the elements of $S_{2,-}$. We start off systematically by giving the fusion rules:

$$S_{2,0}\, S_{2,2\times 10} = S_{2,0}(S_{2,8})\,, \tag{A.3.36a}$$
$$S_{2,0}\, S_{2,2\times 1} = S_{2,0}(S_{2,1} + S_{2,3})\,, \tag{A.3.36b}$$
$$S_{2,0}\, S_{2,2\times 9} = S_{2,0}(S_{2,9} + S_{2,7})\,, \tag{A.3.36c}$$
$$S_{2,0}\, S_{2,2\times 2} = S_{2,0}(S_{2,0} + S_{2,2} + S_{2,4})\,, \tag{A.3.36d}$$
$$S_{2,0}\, S_{2,2\times 8} = S_{2,0}(S_{2,10} + S_{2,8} + S_{2,6})\,, \tag{A.3.36e}$$
$$S_{2,0}\, S_{2,2\times 3} = S_{2,0}(S_{2,1} + S_{2,3} + S_{2,5})\,, \tag{A.3.36f}$$
$$S_{2,0}\, S_{2,2\times 7} = S_{2,0}(S_{2,9} + S_{2,7} + S_{2,5})\,, \tag{A.3.36g}$$
$$S_{2,0}\, S_{2,2\times 4} = S_{2,0}(S_{2,2} + S_{2,4} + S_{2,6})\,, \tag{A.3.36h}$$
$$S_{2,0}\, S_{2,2\times 6} = S_{2,0}(S_{2,8} + S_{2,4} + S_{2,6})\,, \tag{A.3.36i}$$
$$S_{2,0}\, S_{2,2\times 5} = S_{2,0}(S_{2,3} + S_{2,7} + S_{2,5})\,. \tag{A.3.36j}$$

From (A.3.9) applied to (A.3.36j) then $S_{2,2}S_{2,5} = S_{2,0}S_{2,5}$ which gives us two conditions: either $S_{2,5} = 0$ or $S_{2,2} - S_{2,0} = 0$, or both. Let us consider first $S_{2,5} = 0$ without putting conditions on $S_{2,2} - S_{2,0}$ just yet. A remarkable fact is that we can show that this leads to a contradiction down the line, and thus was the incorrect choice. We go to (A.3.36) and massage the equations based off the assumption $S_{2,5} = 0$.

$$\begin{aligned}
(\text{A.3.36a}) &\to S_{2,2}S_{2,10} = S_{2,0}S_{2,8}\,, &\text{(A.3.37)}\\
(\text{A.3.36b}) + (\text{A.3.36c}) &\to (S_{2,2} - S_{2,0})(S_{2,1} + S_{2,9}) = 0\,, &\text{(A.3.38)}\\
(\text{A.3.36d}) + (\text{A.3.36e}) &\to S_{2,2}(S_{2,2} + S_{2,8}) = S_{2,0}(S_{2,0} + S_{2,10})\,, &\text{(A.3.39)}\\
(\text{A.3.36f}) &\to S_{2,2}S_{2,3} = (S_{2,1} + S_{2,3})S_{2,0}\,, &\text{(A.3.40)}\\
(\text{A.3.36g}) &\to S_{2,2}S_{2,7} = (S_{2,9} + S_{2,7})S_{2,0}\,, &\text{(A.3.41)}\\
(\text{A.3.36h}) - (\text{A.3.36i}) &\to S_{2,2}(S_{2,4} - S_{2,6}) = (S_{2,2} - S_{2,8})S_{2,0}\,, &\text{(A.3.42)}\\
(\text{A.3.36j}) &\to 0\,. &\text{(A.3.43)}
\end{aligned}$$

Equations (A.3.36h) and (A.3.36i) can be added to get $S_{2,2}(S_{2,4} + S_{2,6}) = S_{2,0}(S_{2,4} + S_{2,6})$, and therefore

$$(S_{2,2} - S_{2,0})(S_{2,4} + S_{2,6}) = 0\,. \qquad \text{(A.3.44)}$$

There are multiple possibilities to consider, either

1. $S_{2,2} - S_{2,0} = 0\,, \quad S_{2,4} + S_{2,6} = 0\,,$
2. $S_{2,2} - S_{2,0} = 0\,, \quad S_{2,4} + S_{2,6} \neq 0\,,$
3. $S_{2,4} + S_{2,6} = 0\,, \quad S_{2,2} - S_{2,0} \neq 0\,.$

Suppose we consider the first of the above cases. But then (A.3.40) would imply that $S_{2,1} = 0$, but it was solved already in (A.3.35) that $S_{2,1} \neq 0$, so we have a contradiction. The second case also leads to a contradiction by the same reason as the first condition. One can also check that the third case is invalid as well. Thus our assumption that $S_{2,5} = 0$ was incorrect. We amend this choice and instead let $S_{2,5} \neq 0$ but let $S_{2,2} - S_{2,0} = 0$. This does not run into the problem of earlier because if $S_{2,5} \neq 0$, then (A.3.36f) is not simply $S_{2,3} = S_{2,1} + S_{2,3}$, but rather $S_{2,3} = S_{2,1} + S_{2,3} + S_{2,5}$. We use this to simplify the equations in (A.3.36)

$$\begin{aligned}
(\text{A.3.36a}) &\to S_{2,2}S_{2,10} = S_{2,0}S_{2,8}\,, &\text{(A.3.45)}\\
(\text{A.3.36b}) &\to S_{2,3} = 0\,,\\
(\text{A.3.36c}) &\to S_{2,7} = 0\,,\\
(\text{A.3.36d}) &\to S_{2,0} + S_{2,4} = 0\,,\\
(\text{A.3.36e}) &\to S_{2,10} + S_{2,6} = 0\,,\\
(\text{A.3.36f}) &\to S_{2,1} + S_{2,5} = 0\,,
\end{aligned}$$

$$
\begin{aligned}
(\text{A.3.36g}) &\rightarrow S_{2,9} + S_{2,5} = 0\,,\\
(\text{A.3.36h}) &\rightarrow S_{2,2} + S_{2,6} = 0\,,\\
(\text{A.3.36i}) &\rightarrow S_{2,8} + S_{2,4} = 0\,,\\
(\text{A.3.36j}) &\rightarrow S_{2,3} + S_{2,7} = 0\,.
\end{aligned}
$$

The important part now is to relate everything back to $S_{2,0}$ and $S_{2,1}$, the latter which we already obtained. In total we have

$$
\begin{aligned}
&S_{2,0} = S_{2,2} = -S_{2,4} = -S_{2,6} = S_{2,8} = S_{2,10}\\
&S_{2,1} = -S_{2,5} = S_{2,9}.
\end{aligned}
\tag{A.3.46}
$$

Now using the Verlinde formula in the form $N_{2,2}^{0} = 1 = \sum_d \frac{S_{2,d}^2 S_{0,d}^*}{S_{0,d}}$ we have

$$
\begin{aligned}
1 &= 6S_{2,0}^2 + 3S_{2,1}^2\\
1 &= \frac{1}{2} + 6S_{2,0}^2\,,
\end{aligned}
\tag{A.3.47}
$$

thus $S_{2,0} = \frac{1}{\sqrt{12}}$.

We now skip to finding the matrix elements of $S_{5,-}$, this is because 5 behaves differently from the other lines. The fusion rules give

$$S_{5,0}\,S_{5,5\times 10} = S_{5,0}S_{5,5}\,, \tag{A.3.48a}$$
$$S_{5,0}\,S_{5,5\times 1} = S_{5,0}(S_{5,4} + S_{5,6})\,, \tag{A.3.48b}$$
$$S_{5,0}\,S_{5,5\times 9} = S_{5,0}(S_{5,4} + S_{5,6})\,, \tag{A.3.48c}$$
$$S_{5,0}\,S_{5,5\times 2} = S_{5,0}(S_{5,3} + S_{5,7} + S_{5,5})\,, \tag{A.3.48d}$$
$$S_{5,0}\,S_{5,5\times 8} = S_{5,0}(S_{5,3} + S_{5,7} + S_{5,5})\,, \tag{A.3.48e}$$
$$S_{5,0}\,S_{5,5\times 3} = S_{5,0}(S_{5,2} + S_{5,8} + S_{5,4} + S_{5,6})\,, \tag{A.3.48f}$$
$$S_{5,0}\,S_{5,5\times 7} = S_{5,0}(S_{5,2} + S_{5,8} + S_{5,4} + S_{5,6})\,, \tag{A.3.48g}$$
$$S_{5,0}\,S_{5,5\times 4} = S_{5,0}(S_{5,1} + S_{5,9} + S_{5,3} + S_{5,7} + S_{5,5})\,, \tag{A.3.48h}$$
$$S_{5,0}\,S_{5,5\times 6} = S_{5,0}(S_{5,1} + S_{5,9} + S_{5,3} + S_{5,7} + S_{5,5})\,, \tag{A.3.48i}$$
$$S_{5,0}\,S_{5,5\times 5} = S_{5,0}(S_{5,0} + S_{5,10} + S_{5,2} + S_{5,8} + S_{5,4} + S_{5,6})\,. \tag{A.3.48j}$$

manipulating the equations gives

$$
\begin{aligned}
(\text{A.3.48a}) &\to S_{5,5}(S_{5,10} - S_{5,0}) = 0\,, \\
(\text{A.3.48b}) - (\text{A.3.48c}) &\to S_{5,5}(S_{5,1} - S_{5,9}) = 0\,, \\
(\text{A.3.48d}) &\to S_{5,5}(S_{5,0} - S_{5,2}) = 0\,, \\
(\text{A.3.48e}) &\to S_{5,5}(S_{5,0} - S_{5,8}) = 0\,, \\
(\text{A.3.48f}) &\to S_{5,0}S_{5,3} = 0\,, \\
(\text{A.3.48g}) &\to S_{5,0}S_{5,7} = 0\,, \\
(\text{A.3.48h}) &\to S_{5,5}S_{5,4} = -S_{5,0}S_{5,5}\,, \\
(\text{A.3.48i}) &\to S_{5,5}S_{5,6} = -S_{5,0}S_{5,5}\,, \\
(\text{A.3.48j}) &\to S_{5,5}^2 = S_{5,0}(S_{5,0} + S_{5,10})\,,
\end{aligned}
\tag{A.3.49}
$$

We have some choices, from the first of the equations we could have $S_{5,5} = 0$ and also $S_{10,1} - S_{10,0} = 0$. But then that contradicts the last equation of the above. Now suppose that $S_{5,10} = S_{5,0}$ with $S_{5,5} \neq 0$. Then we get $S_{1,5} + S_{5,5} = 0$ from one of our previous equations. However, we said before that $S_{1,5}$ around equation (A.3.27) this was already zero, so then $S_{5,5}$ would also have to be zero which is a contradiction. So we need to have $S_{5,10} \neq S_{5,0}$ and $S_{5,5} = 0$. Because from earlier $S_{3,5} + S_{5,9} = 0$, then $S_{5,9} = 0$, and furthermore from (A.3.13a) and (A.3.3b) we have $S_{5,4} = -S_{5,10}$ with $S_{5,10} = -S_{5,0}$ in (A.3.48j). The relationships are summarized as

$$S_{5,1} = 0\,, \qquad S_{5,2} = -S_{2,1} = -\frac{2}{\sqrt{24}}\,, \qquad S_{5,3} = 0\,, \qquad S_{5,4} = S_{5,0}\,, \tag{A.3.50}$$

$$S_{5,5} = 0\,, \qquad S_{5,6} = -S_{5,0}\,, \qquad S_{5,7} = 0\,, \qquad S_{5,8} = S_{2,1}\,, \tag{A.3.51}$$

$$S_{5,9} = 0\,, \qquad S_{5,10} = S_{5,0}\,. \tag{A.3.52}$$

Then by the Verlinde formula we have

$$1 = \sum_a S_{5,a}^2 = 1/3 + 4S_{5,0}^2 \tag{A.3.53}$$

so $S_{5,0} = \frac{1}{\sqrt{6}}$.

A.4. Diagrams

A.4.1. Proof of Theorem 5.7.8

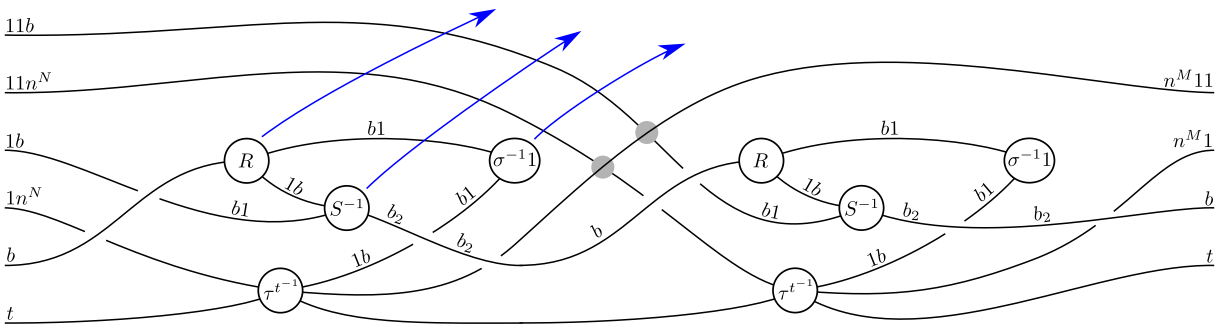

Figure A.1: Axiom a (Part 1)

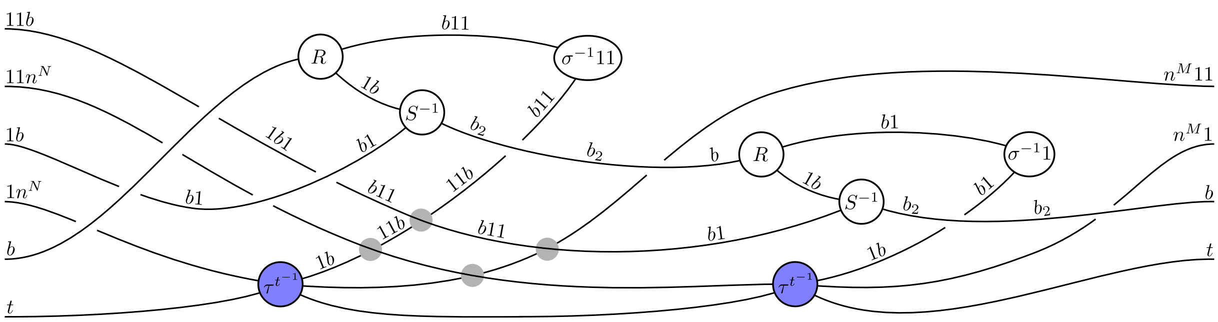

Figure A.2: Axiom a (Part 2)

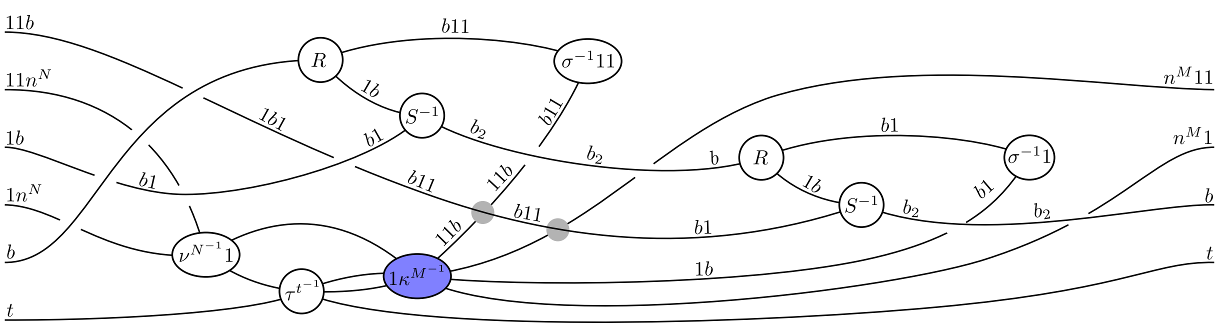

Figure A.3: Axiom a (Part 3)

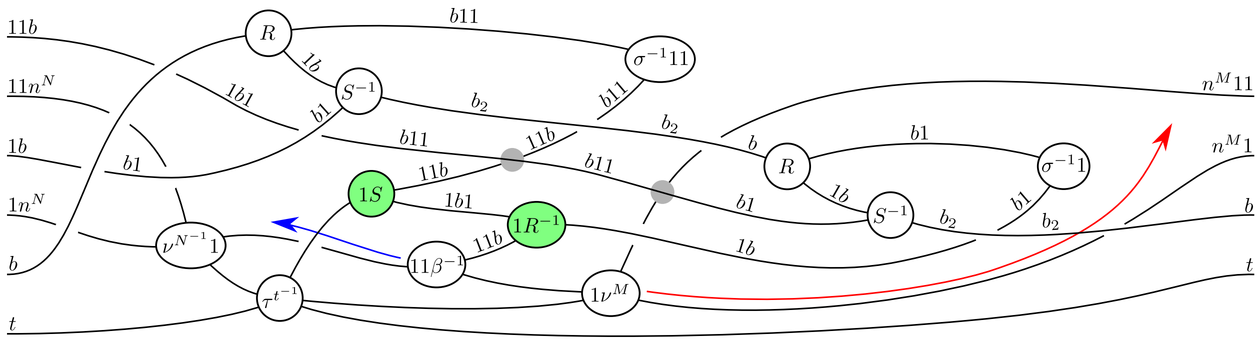

Figure A.4: Axiom a (Part 4)

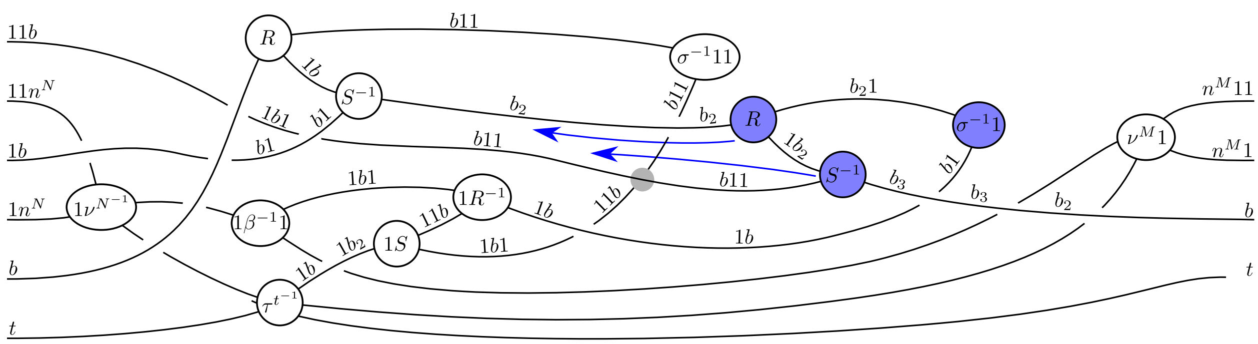

Figure A.5: Axiom a (Part 5)

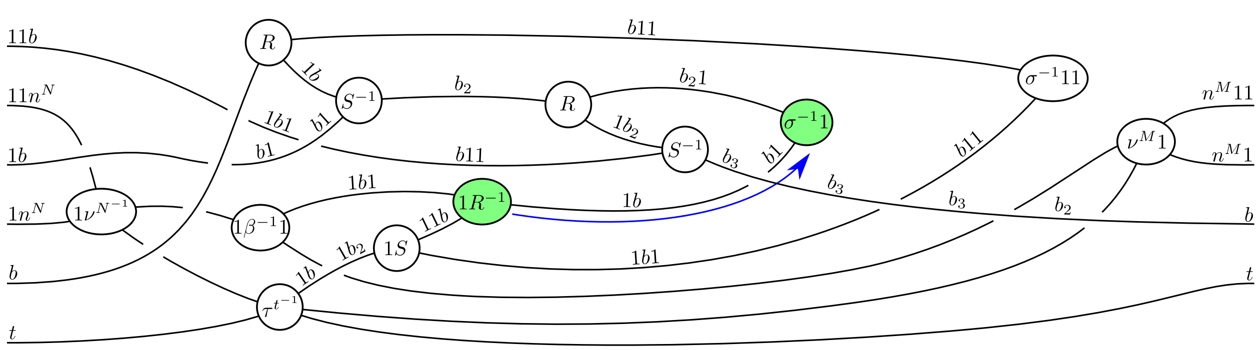

Figure A.6: Axiom a (Part 6)

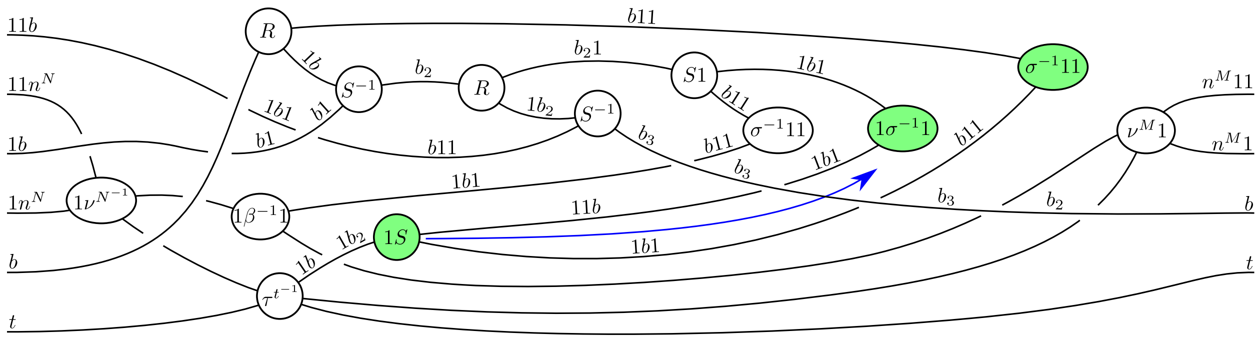

Figure A.7: Axiom a (Part 7)

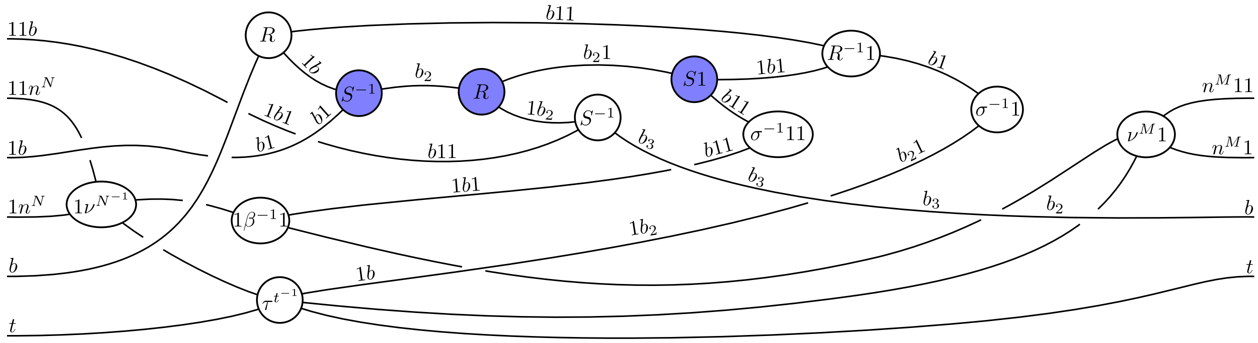

Figure A.8: Axiom a (Part 8)

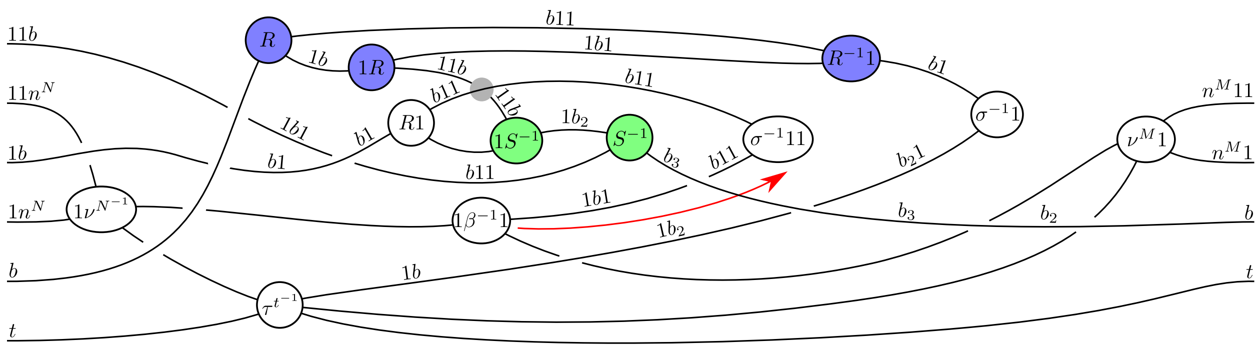

Figure A.9: Axiom a (Part 9)

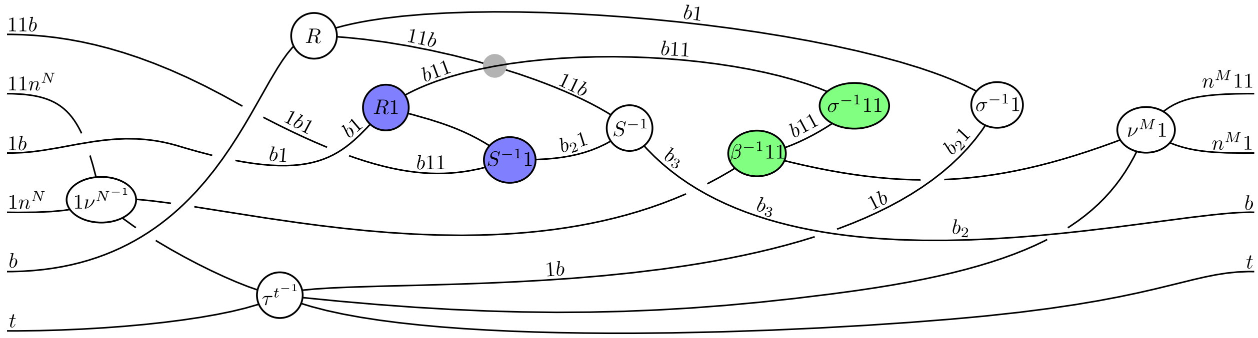

Figure A.10: Axiom a (Part 10)

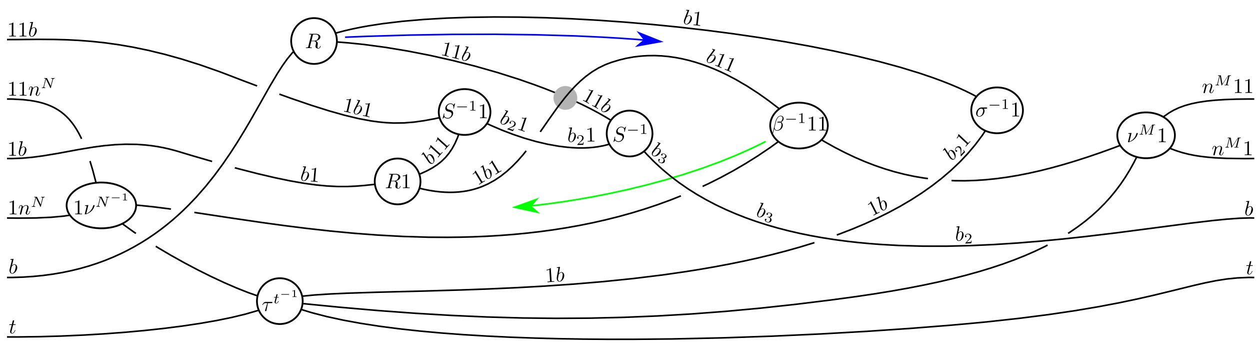

Figure A.11: Axiom a (Part 11)

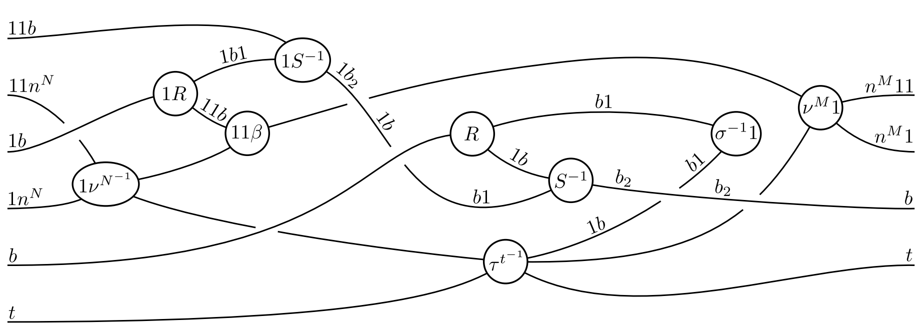

Figure A.12: Axiom a (Part 12)

A.4.2. Proof of Proposition 5.7.11

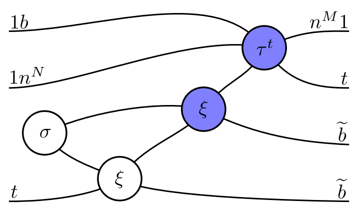

Figure A.13: Balanced structure (Part 1)

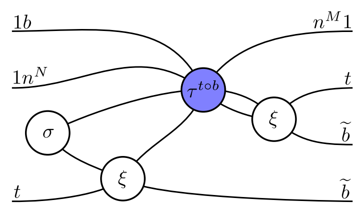

Figure A.14: Balanced structure (Part 2)

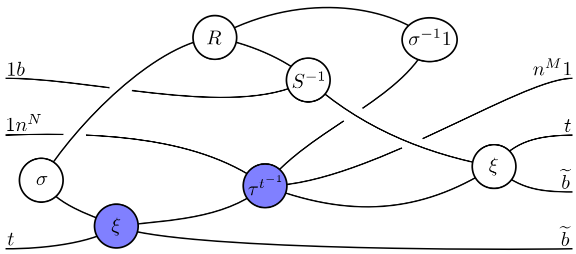

Figure A.15: Balanced structure (Part 3)

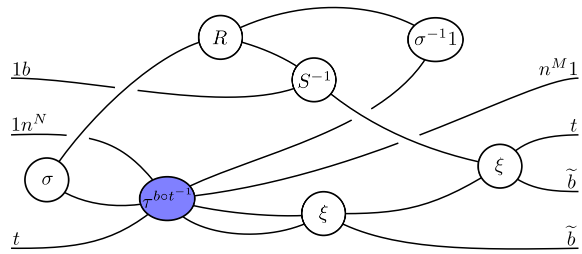

Figure A.16: Balanced structure (Part 4)

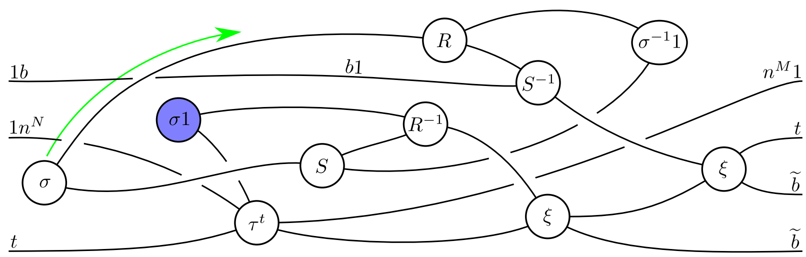

Figure A.17: Balanced structure (Part 5)

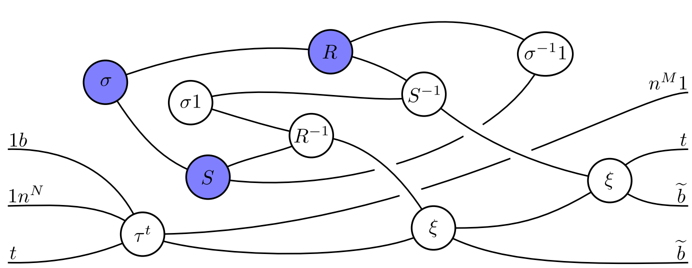

Figure A.18: Balanced structure (Part 6)

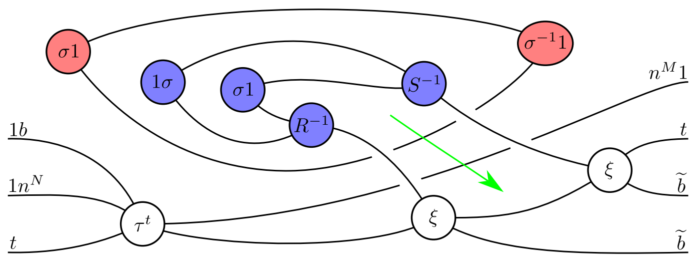

Figure A.19: Balanced structure (Part 7)

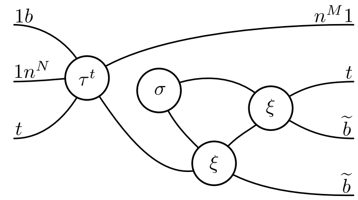

Figure A.20: Balanced structure (Part 8)