# Quantum superchannels on the space of quantum channels

by

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## Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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## Abstract

Quantum channels, defined as completely-positive and trace-preserving maps on matrix algebras, are an important object in quantum information theory. In this thesis we are concerned with the space of these channels. This is motivated by the study of quantum superchannels, which are maps whose input and output are quantum channels.

Rather than taking the domain to be the space of all linear maps, as has been done in the past, we motivate and define superchannels by considering them as transformations on the operator system spanned by quantum channels. Extension theorems for completely positive maps allow us to apply the characterisation theorem for superchannels, [3, Theorem 1], to this smaller set of maps. These extensions are non unique, showing two different superchannels act the same on all input quantum channels, and so this new definition on the smaller domain captures more precisely the action of superchannels as transformations between quantum channels. The non uniqueness can affect the auxilliary dimension needed for the characterisation as well as the tensor product of the superchannels.

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# Table of Contents

# Chapter 1

# Introduction

Quantum channels are a fundamental object studied in quantum information [23, 27]. Defined as completely positive trace-preserving (CPTP) maps between operators on Hilbert spaces, they map quantum states to quantum states. Since quantum states are positive operators with trace one, the natural domain and range of quantum channels is taken to be the ideal of trace class operators inside the space $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space. Quantum superchannels are one step up from this, transformations between quantum channels.

Quantum superchannels were introduced in [3] to describe the most general transformation of quantum channels, and have been used as a model of quantum circuit boards with the ability to replace quantum channels [6]. Recent work has used superchannels to define the entropy of quantum channels [16], and study dynamical resource theories such as entanglement [18], magic [38], and coherence [25]. Concepts from quantum channels, such as entanglement-breaking and dephasing, have been extended to the superchannel case [2, 30] to understand how these properties can be introduced as channels change.

In [3] and [16] the domain of these superchannels is taken to be the set of quantum operations which are completely positive trace non-increasing maps between matrix algebras. In finite dimensions the span of these maps gives all of the linear maps on these matrix algebras. A characterisation of all superchannels in these papers describes them as the action of two ordinary quantum channels, a "pre-processing" and "post-processing" channel.

This thesis takes a different approach, defining superchannels on the space spanned by quantum channels, as well as looking at the properties of this space. It is worth noting that although superchannels are not uniquely determined by their action on quantum channels, they are uniquely determined by their actions on channels in larger spaces. That is, what a superchannel $S$ does to a "non physical" map is determined by what $S \otimes \mathrm{id}_n$ does to physical maps i.e. channels, see Remark 3.2.10.

In Chapter 1 we discuss operator systems, and show how Arveson's extension theorem ultimately derives from the Hahn-Banach extension theorem. This will be the source of the non-uniqueness of extended superchannels, which we explore in Chapter 3. A key tool for studying channels are Choi-Jamiołkowski matrices, and they are introduced here as well, as are quantum channels.

In Chapter 2 we define quantum superchannels and prove their dilation theorem. The uniqueness condition for this dilation is given. We also briefly mention some recent uses superchannels have had, in defining the entropy of quantum channels, and defining resource theories for certain classes of quantum channels.

Chapter 3 is based on the author's paper, [12], where we explore properties of the operator system spanned by quantum channels. A simple definition of what a superchannel on this space should be is given. We call these objects "QSCs" to distinguish them from superchannels defined on the whole space of linear maps. Most importantly they are shown to be CP maps which allows their extension to the full space of linear maps, making them restrictions of standard superchannels. These extensions are shown to be non-unique, however, and this has implications for the superchannel dilation theorem. A generalisation of Choi's theorem on the extreme points of sets of CP maps is proved, giving necessary and sufficient conditions for extreme points of the set of CP maps which are equal on subspaces.

Chapter 4 looks at a few classes of superchannels which have appeared in recent papers. These are largely motivated by the study of resource theories. Since these resource theories are defined as the collection of superchannels preserving a certain set of quantum channels, the concept of QSC can be enlightening. In general these classes of superchannels are not uniquely defined by their action on quantum channels, thus as far as maps on channels are concerned, they can be implemented differently. It is interesting that certain definitions do give unique QSCs though, such as the Schur product superchannels. The class of coherence-breaking superchannels highlights the trouble caused by not being careful of the domain on which the superchannels properties are defined, since breaking the coherence of *all* linear maps is a much stronger condition than just doing so for quantum channels. As a result some theorems need to be reformulated.

Finally, in Chapter 5 we use many of the same techniques, and the same space of quantum channels, but switch from looking at superchannels to looking at symmetries. The purpose of this is to give a reformulation of Wigner's theorem, but defined on channels instead of states. This follows the paper [4] which gives symmetries on quantum operations, which are trace non-increasing completely-positive maps, as mentioned above. Using the results from this paper we give a different proof of their main theorem. We also discuss how the notion of a quantum channel symmetry differs.

In the remainder of this chapter we discuss the background theory of completely positive maps and operator systems. We then give an axiomatic derivation of quantum channels, to motivate the similar definition of superchannels.

## 1.1 Operator systems and completely positive maps

For notation let $M_n$ be the space of $n \times n$ matrices over the complex numbers and let $B(\mathcal{H})$ be the space of bounded operators a Hilbert space $\mathcal{H}$. $M_n(\mathcal{H}) = M_n \otimes \mathcal{H}$ is the space of $n \times n$ matrices with entries in $\mathcal{H}$. Similarly for a $C^*$-algebra $\mathcal{A}$ we have $M_n(\mathcal{A})$ as $n \times n$ matrices with entries in $\mathcal{A}$, and there is a unique norm making this space into a $C^*$-algebra.

Denote the positive elements of a $C^*$-algebra as $\mathcal{A}^+ = \{p \in \mathcal{A} : p \geq 0\}$. This is a *cone*; i.e., a convex set which is closed under scaling by non-negative real numbers. For every natural number $n$ the space $M_n(\mathcal{A})$ inherits a cone of positive elements $M_n(\mathcal{A})^+$. This sequence of cones is called the *matrix order* on $\mathcal{A}$.

**Definition 1.1.1.** An *operator system* $\mathcal{S}$ is a subspace of a unital $C^*$-algebra which contains the unit and is self-adjoint; i.e., $\mathcal{S} = \mathcal{S}^* = \{a^* : a \in \mathcal{S}\}$.

Each operator system $\mathcal{S}$ comes with an induced matrix order via

$$M_n(\mathcal{S})^+ = M_n(\mathcal{S}) \cap M_n(\mathcal{A})^+.$$

We will often require maps defined on operator systems to be extended to the $C^*$-algebra containing it. A key tool for this is Krein's theorem. First a lemma.

**Lemma 1.1.2.** *Let $\mathcal{S}$ be an operator system and $\mathcal{B}$ a $C^*$-algebra. Then any unital contraction $\phi : \mathcal{S} \to \mathcal{B}$ is a positive map.*

*Proof.* Let $a \in \mathcal{S}$ be positive. We first show its true for functionals. Take a unital contraction $f : \mathcal{S} \to \mathbb{C}$. The spectrum of $a$, $\sigma(a)$, is compact and so its closed convex hull is in the intersection of closed disks containing it. Let $\{z : |z - \lambda| \leq r\}$ be such a disk. Then $\sigma(a - \lambda 1) \subseteq \{z : |z| \leq r\}$, and as $a - \lambda 1$ is normal, the spectral radius is the norm, so $\|a - \lambda 1\| \leq r$. This implies $|f(a - \lambda 1)| \leq \|f\| r = r$ and hence $f(a) \in \{z : |z - \lambda| \leq r\}$. Since $\lambda$ and $r$ were arbitrary, $f(a)$ is in the closed convex hull of the spectrum of $a$ and must be positive.

For $\phi : \mathcal{S} \to \mathcal{B}$ we can let $\mathcal{B} = B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and for a unit norm $x$ define $f(a) = \langle x | \phi(a) x \rangle$. This is a unital contraction, so is positive, and as $x$ was arbitrary we have $\phi(a)$ is positive. □

**Theorem 1.1.3** (Krein)**.** *Let $\mathcal{S}$ be an operator system in the $C^*$-algebra $\mathcal{A}$, and $\phi : \mathcal{S} \to \mathbb{C}$ a positive linear functional. Then there is a positive linear functional $\widetilde{\phi} : \mathcal{A} \to \mathbb{C}$ extending $\phi$.*

*Proof.* First note that $\|\phi\| \leq \phi(1)$ where $1 \in \mathcal{S}$ is the unit. This is clear for self-adjoint elements $a \in \mathcal{S}$ since

$$-\|a\| \cdot 1 \leq a \leq \|a\| \cdot 1$$

so by positivity of $\phi$

$$-\|a\|\phi(1) \leq \phi(a) \leq \|a\|\phi(1);$$

i.e., $|\phi(a)| \leq \|a\|\phi(1)$. For an arbitrary $a \in \mathcal{S}$ choose $\lambda \in \mathbb{C}$, $|\lambda| = 1$ such that $|\phi(a)| = \lambda\phi(a) = \phi(\lambda a)$. Because $\phi(\lambda a)$ is a real number we get

$$\phi(\lambda a) = \phi(\lambda a)^* = \phi((\lambda a)^*)$$

where the last equality is because positive maps are self-adjoint. This allows us to write it as $\phi$ acting on a self-adjoint element $\phi(\lambda a) = \phi(\frac{\lambda a + (\lambda a)^*}{2})$ and thus

$$\phi\left(\frac{\lambda a + (\lambda a)^*}{2}\right) \leq \frac{\|\lambda a + (\lambda a)^*\|}{2}\phi(1) \leq \frac{\|\lambda a\| + \|(\lambda a)^*\|}{2}\phi(1) = \|a\|\phi(1).$$

Now if $\phi(1) = 0$ we have $\|\phi\| = 0$ so the zero map will extend $\phi$. Otherwise $\phi(1)$ is positive and so we can define a positive functional $\psi$ via

$$\psi(a) = \phi(a)\phi(1)^{-1}.$$

We have $\|\psi\| = \psi(1) = 1$, and using the Hahn-Banach theorem we get an extension $\widetilde{\psi} : \mathcal{A} \to \mathbb{C}$ with $\|\widetilde{\psi}\| = 1$. Since it is a unital contraction, it is a positive map by the previous lemma. Finally, define

$$\widetilde{\phi}(a) = \widetilde{\psi}(a)\phi(1), \quad \forall a \in \mathcal{A}.$$

This is a positive map extending $\phi$. □

For a linear map $\phi : \mathcal{S} \to \mathcal{T}$, define $\phi_n : M_n(\mathcal{S}) \to M_n(\mathcal{T})$ by $\phi_n((a_{i,j})) = (\phi(a_{i,j}))$. That is, $\phi_n = \phi \otimes \mathrm{id}_n$ where $\mathrm{id}_n$ is the identity map on $M_n$. Call $\phi$ *completely positive* (CP) if $\phi_n$ is positive for all $n$; i.e.,

$$\phi^{(n)}(M_n(\mathcal{S})^+) \subseteq M_n(\mathcal{T})^+.$$

To show two spaces define the same operator system it is necessary to show their matrix orders are the same and for this we use a complete order isomorphism.

**Definition 1.1.4.** Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. A linear map $\phi : \mathcal{S} \to \mathcal{T}$ is a *complete order isomorphism* provided it is bijective, and both $\phi$ and $\phi^{-1}$ are completely positive.

The following theorem fully characterises completely positive maps into $B(\mathcal{H})$.

**Theorem 1.1.5** (Stinespring)**.** *If $\mathcal{A}$ is a unital $C^*$-algebra and $\phi : \mathcal{A} \to B(\mathcal{H})$ is a completely positive map, then there exists a Hilbert space $\mathcal{K}$, a unital *-homomorphism $\pi : \mathcal{A} \to B(\mathcal{K})$, and a bounded operator $V : \mathcal{H} \to \mathcal{K}$ such that*

$$\phi(a) = V^*\pi(a)V.$$

**Remark 1.1.6.** When $\mathcal{H} = \mathbb{C}^n$ there is a unitary $U : \mathcal{K} \to A \otimes \mathbb{C}^E$ such that

$$\pi(a) = U^*(a \otimes I_E)U$$

for some dimension $E$. In particular, for a CP map $\phi : M_d \to M_r$, we can write Stinespring's theorem as

$$\phi(\rho) = V^*(\rho \otimes I_E)V \tag{1.1}$$

for $V : \mathbb{C}^r \to \mathbb{C}^d \otimes \mathbb{C}^E$.

**Corollary 1.1.7.** *$\phi : M_d \to M_r$ is completely positive if and only if it has a Kraus representation i.e. there exist operators $V_k : \mathbb{C}^r \to \mathbb{C}^d$ (called Kraus operators) with*

$$\phi(\rho) = \sum_{k=1}^{E} V_k^* \rho V_k \tag{1.2}$$

*Proof.* Use Stinespring's theorem to get the operator $V$ and space $\mathbb{C}^E$ and take an orthonormal basis $\{e_k\}$ for $E$. Then using $I_E = \sum_{k=1}^{E} |e_k\rangle\langle e_k|$ define $V_k = \langle e_k| \otimes V$ which acts as

$$\langle v|V_k|w\rangle = \langle v \otimes e_k|V|w\rangle, \quad v \in \mathbb{C}^d, w \in \mathbb{C}^r$$

and gives (1.2) via (1.1). □

While the Kraus/Stinespring representations are not unique, they are unique up to isometry when the dimension $E$ is minimal. In this case there is a linearly independent set of Kraus operators.

**Theorem 1.1.8** ([9]). *If $\{V_k\}_{k=1}^E$ and $\{W_k\}_{k=1}^N$ are Kraus operators describing the same map, with $N \geq E$, then there is an isometry $U = (u_{i,j}) : \mathbb{C}^E \to \mathbb{C}^N$ such that*

$$W_i = \sum_{j=1}^{E} u_{i,j} V_j$$

**Definition 1.1.9.** For a linear map $\phi : M_d \to M_r$ its *dual* map is a linear map $\phi^* : M_r \to M_d$ defined by

$$\mathrm{Tr}(Y\phi(X)) = \mathrm{Tr}(\phi^*(Y)X), \quad \forall X \in M_d,\ Y \in M_r.$$

If $\phi$ has a representation $\phi(X) = \sum_i V_i^* X V_i$, then $\phi^*(Y) = \sum_i V_i Y V_i^*$.

**Remark 1.1.10.** Let $\mathcal{C}_1(\mathcal{H})$ be trace class operators on a Hilbert space. Then, for $\phi : \mathcal{C}_1(\mathcal{H}) \to \mathcal{C}_1(\mathcal{K})$ we can also define a dual map $\phi^* : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ which satisfies $\mathrm{Tr}(Y\phi(X)) = \mathrm{Tr}(\phi^*(Y)X)$ for all $X \in \mathcal{C}_1(\mathcal{H})$, $Y \in \mathcal{B}(\mathcal{K})$. In physics, the dagger map is defined by treating $\mathrm{Tr}(Y^*X)$ as an inner product, despite the inputs being in different spaces. It can be given by $\phi^\dagger(Y) = \phi^*(Y^*)^*$. For CP maps and for maps on finite dimensional spaces we have $\phi^* = \phi^\dagger$.

### Extending CP maps

We will make use of the following extension theorem for CP maps, see [1] and [28, Theorem 7.5] for proofs.

**Theorem 1.1.11** (Arveson's extension theorem)**.** *Let $\mathcal{A}$ be a $C^*$-algebra, $\mathcal{S}$ an operator system contained in $\mathcal{A}$, and $\phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ a completely positive map. Then there exists a completely positive map, $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, extending $\phi$.*

For the case $\mathcal{B}(\mathcal{H}) = M_n$ the proof follows from Krein's theorem. Given a map $\phi : \mathcal{S} \to M_n$, define the functional $s_\phi : M_n(\mathcal{S}) \to \mathbb{C}$ via

$$s_\phi((a_{i,j})) = \frac{1}{n} \sum_{i,j} \phi(a_{i,j})_{(i,j)}.$$

We can also invert this, given a functional $s : M_n(\mathcal{S}) \to \mathbb{C}$ we define $\phi_s : \mathcal{S} \to M_n$ with the formula

$$(\phi_s(a))_{(i,j)} = n \cdot s(a \otimes E_{i,j}).$$

where $E_{i,j}$ are the matrix units in $M_n$. It can then be shown ([28, Theorem 6.1]) that $s_\phi$ being positive is equivalent to $\phi$ being completely positive. The positive extension of $s_\phi$ via Krein's theorem corresponds to a completely positive extension of $\phi$. Note that, by the proof of Krein's theorem, ultimately the extension is done using the Hahn-Banach theorem and it does not guarantee uniqueness. This fact will manifest when we extend maps defined on the space of quantum channels to the whole space of linear maps.

### Choi-Jamiołkowski Matrices

Let $E_{i,j}$, $1 \leq i, j \leq d$ denote the matrix units in $M_d$. Any linear map is determined by its action on basis elements so for a linear $L : M_d \to M_r$ we get a vector space isomorphism from $\mathrm{Lin}(M_d, M_r)$ onto $M_d(M_r)$ via $L \mapsto (L(E_{i,j}))$. This matrix defines the map and vice versa.

**Definition 1.1.12.** The matrix $C_L := (L(E_{i,j})) = \sum_{i,j} E_{i,j} \otimes L(E_{i,j})$ is called the *Choi matrix* or Choi-Jamiołkowski matrix of the map $L$.

$$C_L = \begin{pmatrix} L(E_{11}) & L(E_{12}) & \cdots & L(E_{1d}) \\ L(E_{21}) & L(E_{22}) & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ L(E_{d1}) & \cdots & \cdots & L(E_{dd}) \end{pmatrix}$$

We will frequently make use of an important theorem from Choi [9], which says that a map $\phi : M_d \to M_r$ is CP if and only if its Choi matrix $C_\phi \in M_{dr}$ is positive as a matrix. Since $\sum_{i,j} E_{i,j} \otimes E_{i,j} = \sum_{i,j} |e_i \otimes e_i\rangle\langle e_j \otimes e_j| \geq 0$ the complete positivity of $\phi$ ensures $C_\phi = \mathrm{id}_n \otimes \phi(\sum_{i,j} E_{i,j} \otimes E_{i,j})$ is positive, so one direction is clear. The other direction can be shown by taking the spectral decomposition of the positive matrix $C_\phi$ as $\sum_i v_i v_i^*$ for $v_i \in \mathbb{C}^{dr}$. Then write each vector as $d$ blocks of $r \times 1$ vectors,

$$v_i = \sum_{k=1}^{d} \sum_{l=1}^{r} (v_i)_{k,l} |e_k \otimes e_l\rangle$$

and define the $r \times d$ matrices $V_i = \sum_{k,l} (v_i)_{k,l} E_{l,k}$. Then we have

$$(I \otimes V_i) \left( \sum_j |e_j \otimes e_j\rangle \right) = \sum_{j,l} (v_i)_{j,l} |e_j \otimes e_l\rangle = v_i.$$

It follows that $\phi(\rho) = \sum_i V_i \rho V_i^*$ and so $\phi$ is completely positive. We have thus proved:

**Theorem 1.1.13** (Choi, [9])**.** *A linear map $\phi : M_d \to M_r$ is completely positive if and only if its Choi matrix $C_\phi \in M_d(M_r)$ is positive.*

### Partial Trace

For an operator $T \in M_d \otimes M_r$ the partial trace $\mathrm{Tr}_2 T \in M_d$ is the unique operator satisfying

$$\langle \phi | \mathrm{Tr}_2 T | \psi \rangle = \sum_k \langle \phi \otimes e_k | T | \psi \otimes e_k \rangle, \quad \forall \phi, \psi \in \mathbb{C}^d.$$

That is, $\mathrm{Tr}_2 = \mathrm{id}_1 \otimes \mathrm{Tr}$. We remark that $\mathrm{Tr}_2 : M_d \otimes M_r \to M_d$ is the unique linear map satisfying any of the following equivalent conditions:

1. $\mathrm{Tr}_2[(A \otimes I_r)B] = A \,\mathrm{Tr}_2 B$ for all $A \in M_d$, $B \in M_d \otimes M_r$

2. $\mathrm{Tr}_2[B(A \otimes I_r)] = \mathrm{Tr}_2(B)A$ for all $A \in M_d$, $B \in M_d \otimes M_r$

3. $\mathrm{Tr}((A \otimes I_r)B) = \mathrm{Tr}(A\,\mathrm{Tr}_2(B))$ for all $A \in M_d$, $B \in M_d \otimes M_r$.

We can, of course, define $\mathrm{Tr}_1$ when tracing out the first space, or more generally $\mathrm{Tr}_i$ on the $i$-th leg of a tensor product of $M_{n_1} \otimes \cdots \otimes M_{n_k}$.

**Remark 1.1.14.** As we will be considering maps on tensor products of matrix spaces, the notation for partial trace can be messy. When tracing out system $M_{d_i}$ we will use the notation $\mathrm{Tr}_{d_i}$, or when tracing out both $M_{d_i}$ and $M_{r_i}$ we use $\mathrm{Tr}_{d_i r_i}$. So, for example tracing out the second domain space and the first range space is a map

$$\mathrm{Tr}_{r_1 d_2} : M_{d_1} \otimes M_{r_1} \otimes M_{d_2} \otimes M_{r_2} \to M_{d_1} \otimes M_{r_2}$$
$$\mathrm{Tr}_{r_1 d_2}[A \otimes B \otimes C \otimes D] = A \otimes D \cdot \mathrm{Tr}(B)\,\mathrm{Tr}(C).$$

The following formula is useful way to retrieve the linear map associated with a Choi matrix.

**Theorem 1.1.15.** *For a linear map $\phi : M_d \to M_r$ with Choi matrix $C_\phi$ we have for any input $\rho \in M_d$*

$$\phi(\rho) = \mathrm{Tr}_d[C_\phi(\rho^T \otimes I_r)] \tag{1.3}$$

*Proof.* Choose an orthonormal basis $\{|e_k\rangle\}_k$ for $\mathbb{C}^d$. Then since $\langle e_j|\rho|e_i\rangle = \rho_{ji}$ we compute

$$\begin{aligned}
\mathrm{Tr}_d[C_\phi(\rho^T \otimes I_r)] &= \sum_k (\langle e_k| \otimes I_r)[\sum_{ij} E_{ij} \otimes \phi(E_{ij})(\rho^T \otimes I_r)](|e_k\rangle \otimes I_r) \\
&= \sum_{ijk} \langle e_k|E_{ij}\rho^T|e_k\rangle \otimes \phi(E_{ij}) \\
&= \sum_{ijk} \langle e_k||e_i\rangle\langle e_j|\rho^T|e_k\rangle \otimes \phi(E_{ij}) \\
&= \sum_{ij} \rho_{ij}\phi(E_{ij}) = \phi(\rho)
\end{aligned}$$

as required. □

## 1.2 Quantum channels

In this section we want to give an axiomatic description of quantum channels. We follow the notes in [29]. Important properties about their Choi matrices, and about the maximally entangled state are also discussed.

A quantum system is given by a complex Hilbert space. Pure states are given by unit vectors $|\psi\rangle$ in this Hilbert space. Every pure state has an associated positive, trace 1 operator, given by $|\psi\rangle\langle\psi|$, the rank one projection on $\mathbb{C}|\psi\rangle$, and any positive trace 1 operator can be interpreted as a statistical mixture of pure states, which is how we define the general notion of a quantum state. Here we focus on finite dimensions.

**Definition 1.2.1.** A *quantum state* is given by a density matrix $\rho \in M_d$, that is, a positive matrix satisfying

$$\mathrm{Tr}[\rho] = 1.$$

Note that density matrices span to give the whole space of matrices. This means that when we are defining quantum channels we can take the domain and range to simply be $M_d$ and $M_r$. To see what other properties channels must satisfy we have to understand the most basic operation applied to a quantum state, that of a measurement.

A $K$-outcome *measurement system* is given by $K$ operators $\{M_1, \ldots, M_K\} \subset \mathcal{L}(\mathbb{C}^d, \mathbb{C}^r)$, satisfying $\sum_k M_k^* M_k = I_d$. When the system is in state $\rho$ the probability of observing outcome $k$ is $\mathrm{Tr}[M_k^* M_k \rho]$. Afterwards the system is in the state with density matrix $\sum_k M_k \rho M_k^*$.

The combination of quantum systems is described via tensor products. Given two state spaces $M_{d_1}$ and $M_{d_2}$ the combined state space is $M_{d_1} \otimes M_{d_2}$. Given two measurement systems, $\{M_1, \ldots, M_K\} \subset \mathcal{L}(\mathbb{C}^{d_1}, \mathbb{C}^{r_1})$ and $\{N_1, \ldots, N_J\} \subset \mathcal{L}(\mathbb{C}^{d_2}, \mathbb{C}^{r_2})$ we get two maps,

$$\phi_1(\rho_1) = \sum_{k=1}^{K} M_k \rho_1 M_k^*$$
$$\phi_2(\rho_2) = \sum_{j=1}^{J} N_j \rho_2 N_j^*.$$

The measurement system of the combined system is the $KJ$-outcome measurement given by $\{M_k \otimes N_j\}$ which has map

$$\phi_{12}(\rho) = \sum_{k,j} (M_k \otimes N_j)\rho(M_k \otimes N_j)^*.$$

Note that $\phi_{12}(\rho_1 \otimes \rho_2) = \phi_1(\rho_1) \otimes \phi_2(\rho_2)$.

Based off these requirements of quantum mechanics we can describe some axioms that quantum channels should satisfy

**Axiom 1.** A quantum channel should be a linear map $\phi : M_d \to M_r$ which sends quantum states to quantum states.

**Axiom 2.** The identity map $\mathrm{id} : M_d \to M_d$ should be a valid quantum channel.

**Axiom 3.** If $\phi_1 : M_{d_1} \to M_{r_1}$ and $\phi_2 : M_{d_2} \to M_{r_2}$ are two quantum channels then there is a quantum channel

$$\phi_{12} : M_{d_1} \otimes M_{d_2} \to M_{r_1} \otimes M_{r_2}$$

which satisfies $\phi_{12}(\rho_1 \otimes \rho_2) = \phi_1(\rho_1) \otimes \phi_2(\rho_2)$ for all states $\rho_1 \in M_{d_1}$, $\rho_2 \in M_{d_2}$.

Since quantum states have trace 1 and span the whole space the first axiom implies that quantum channels should be *trace-preserving* (TP), that is,

$$\mathrm{Tr}[\phi(\rho)] = \mathrm{Tr}[\rho], \quad \forall \rho \in M_d.$$

The first axiom also implies that $\phi$ should be a positive map. But the second and third axioms imply that $\phi \otimes \mathrm{id}_n$ should be a valid channel for all $n \in \mathbb{N}$, and hence $\phi$ should be a completely positive map. If it has Kraus representation $\phi(\rho) = \sum_i V_i \rho V_i^*$, the trace-preserving condition requires $\sum_i V_i^* V_i = I_d$. By the previous discussion of measurement systems every map with such Kraus operators satisfies the axioms. Thus we define:

**Definition 1.2.2.** A *quantum channel* is a completely-positive trace-preserving (CPTP) linear map $\phi : M_d \to M_r$.

**Remark 1.2.3.** An alternative way to derive quantum channels as CPTP maps is to first describe the evolution of a closed quantum system, which is given by a unitary map. Evolution of an open quantum system is then described as a unitary evolution acting on a larger system (i.e. the original system coupled with the environment), and then restricting to the original system with a partial trace. Thus, it is a unitary dilation, and still a completely-positive map.

**Example 1.2.4.** Unitary conjugation and isometry conjugation are examples of channels. They are clearly CP maps and $\mathrm{Tr}[V\rho V^*] = \mathrm{Tr}[V^*V\rho] = \mathrm{Tr}[\rho]$ so they are trace-preserving. Call such maps unitary channels and isometric channels respectively.

**Remark 1.2.5.** For a trace-preserving map $\phi : M_d \to M_r$, the Choi matrix satisfies

$$\begin{aligned}\mathrm{Tr}_r\, C_\phi &= \sum_{i,j} E_{i,j} \otimes \mathrm{Tr}[\phi(E_{i,j})] \\ &= \sum_i E_{i,i} = I_d.\end{aligned}$$

Thus a matrix $C_\phi \in M_d \otimes M_r$ represents a CPTP map if and only if it is positive, and the partial trace satisfies $\mathrm{Tr}_r\, C_\phi = I_d$.

For example the matrix $\frac{1}{r} I_d \otimes I_r$ satisfies this, as does any matrix of the form $I_d \otimes \rho$ for a density matrix $\rho$.

**Example 1.2.6** (Maximally entangled state)**.** Given a basis $\{|e_i\rangle\}$ for $\mathbb{C}^d$ denote the unnormalized maximally entangled pure state as

$$|\phi_+^d\rangle = \sum_i |e_i\rangle \otimes |e_i\rangle$$

and the corresponding (unnormalized) density matrix $\phi_+^d = |\phi_+^d\rangle\langle\phi_+^d| = \sum_{i,j} E_{i,j} \otimes E_{i,j} \in M_d \otimes M_d$. Note that the partial trace on either system gives the identity matrix.

For a matrix $\rho = (\rho_{k,l}) \in M_d$ we have $(\rho \otimes I_d)\phi_+^d = \sum_{i,j,k} \rho_{k,i} E_{k,j} \otimes E_{i,j}$ and similarly $(I_d \otimes \rho^T)\phi_+^d = \sum_{i,j,k} \rho_{i,k} E_{i,j} \otimes E_{k,j}$. Thus we have the identity

$$(\rho \otimes I_d)\phi_+^d = (I_d \otimes \rho^T)\phi_+^d. \tag{1.4}$$

We can express the Choi matrix of a linear map $\phi : M_d \to M_r$ as

$$C_\phi = \mathrm{id}_d \otimes \phi(\phi_+^d)$$

**Remark 1.2.7.** Equation (1.4) can be generalised to a state written using different bases. We can assume $d_A = d_B$. If $\{e_j^A\}$ is an orthonormal basis for $\mathbb{C}^{d_A}$, and $\{e_j^B\}$ an orthonormal basis for $\mathbb{C}^{d_B}$, then for a matrix $\rho \in M_{d_A}$ we have

$$(\rho \otimes I_{d_B}) \left( \sum_j |e_j^A\rangle \otimes |e_j^B\rangle \right) = (I_{d_A} \otimes \rho') \left( \sum_j |e_j^A\rangle \otimes |e_j^B\rangle \right)$$

where $\rho' = U^{-1}\rho^T U$ where $U$ is the unitary transforming the bases.

# Chapter 2

# Superchannels

In this chapter we discuss the standard notion of quantum superchannels and prove the dilation theorem.

## 2.1 Definition of quantum superchannel

Quantum superchannels are linear maps preserving quantum channels in a complete sense.

**Definition 2.1.1.** A *quantum superchannel* is a linear map $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ which has the following three properties:

1. CP preserving: $S$ sends CP maps to CP maps.
2. Completely CP preserving: For any $d, r$, if $\mathrm{id}_{d,r}$ is the identity map acting on $\mathcal{L}(M_d, M_r)$ then $S \otimes \mathrm{id}_{d,r}$ is CP preserving.
3. TP preserving: $S$ sends TP maps to TP maps.

Every quantum superchannel has an induced map $\widetilde{S} : M_{d_1}(M_{r_1}) \to M_{d_2}(M_{r_2})$ which acts on the Choi matrices of linear maps. It is given by

$$\widetilde{S}(C_\phi) = C_{S(\phi)}, \quad \phi \in \mathcal{L}(M_{d_1}, M_{r_1}).$$

Given two superchannels $S_1$, $S_2$ the composition, when defined, satisfies

$$\widetilde{S_2 \circ S_1} = \widetilde{S}_2 \circ \widetilde{S}_1.$$

By Choi's theorem, property 1 of superchannels implies that $\widetilde{S}$ is a positive map. Given a map $\phi \in \mathcal{L}(M_{d_1} \otimes \mathbb{C}, M_{r_1} \otimes M_n)$ we may expand $\phi = \sum_k f_k \otimes g_k$, for $f_k \in \mathcal{L}(M_{d_1}, M_{r_1})$ and $g_k \in \mathcal{L}(\mathbb{C}, M_n)$. We then have

$$\begin{aligned}
\widetilde{S \otimes \mathrm{id}_{1,n}}(C_\phi) &= C_{S \otimes \mathrm{id}_{1,n}(\phi)} \\
&= \sum_{i,j} E_{i,j} \otimes 1 \otimes \left[ \sum_k S(f_k) \otimes g_k \right] (E_{i,j} \otimes 1) \\
&= \sum_k C_{S(f_k)} \otimes g_k(1).
\end{aligned}$$

We can use this to show $\widetilde{S}$ is completely-positive. Let $C_\phi$ be positive, then expanding $\phi$ as before,

$$\begin{aligned}
\widetilde{S} \otimes \mathrm{id}_n(C_\phi) &= \widetilde{S} \otimes \mathrm{id}_n \left( \sum_{i,j} E_{i,j} \otimes 1 \otimes \sum_k f_k(E_{i,j}) \otimes g_k(1) \right) \\
&= \sum_k \widetilde{S}(C_{f_k}) \otimes g_k(1) \\
&= \widetilde{S \otimes \mathrm{id}_{1,n}}(C_\phi).
\end{aligned}$$

Since $C_\phi$ is positive, $\phi$ is CP, and since $S \otimes \mathrm{id}_{1,n}$ is a superchannel, it is completely CP preserving. So this last term is positive. Thus $\widetilde{S} \otimes \mathrm{id}_n$ is a positive map, and $n$ was arbitrary. Thus we have shown:

**Theorem 2.1.2.** *If $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ is a quantum superchannel then the induced map*

$$\widetilde{S} : M_{d_1}(M_{r_1}) \to M_{d_2}(M_{r_2})$$

*is completely-positive.*

**Remark 2.1.3.** Given standard bases $\{e_i^d\}_i$ and $\{e_k^r\}_k$ for $\mathbb{C}^d$ and $\mathbb{C}^r$, then for any $i, j, k, l$ we can define a map $\mathcal{E}_{i,j,k,l} : M_d \to M_r$ as

$$\mathcal{E}_{i,j,k,l}(\rho) = \langle e_i^d | \rho | e_j^d \rangle |e_k^r\rangle\langle e_l^r| = \mathrm{Tr}[E_{i,j}^* \rho] E_{k,l}.$$

With the Hilbert-Schmidt inner product, the set $\{\mathcal{E}_{i,j,k,l}\}_{i,j,k,l}$ forms an orthonormal basis for $\mathcal{L}(M_d, M_r)$. We can express the Choi matrix of the induced map of a superchannel in terms of this basis.

**Lemma 2.1.4.** *For a quantum superchannel $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ we have*

$$C_{\widetilde{S}} = \sum_{i,j,k,l} C_{\mathcal{E}_{i,j,k,l}} \otimes C_{S(\mathcal{E}_{i,j,k,l})} \in M_{d_1}(M_{r_1}) \otimes M_{d_2}(M_{r_2}). \tag{2.1}$$

*Proof.* This follows from the fact that $C_{\mathcal{E}_{i,j,k,l}} = E_{i,j} \otimes E_{k,l}$ and thus,

$$\begin{aligned}\sum_{i,j,k,l} C_{\mathcal{E}_{i,j,k,l}} \otimes C_{S(\mathcal{E}_{i,j,k,l})} &= \sum_{i,j,k,l} (E_{i,j} \otimes E_{k,l}) \otimes \widetilde{S}(E_{i,j} \otimes E_{k,l}) \\ &= \mathrm{id}_{d_1 r_1} \otimes \widetilde{S}(\phi_+^{d_1 r_1}) \\ &= C_{\widetilde{S}},\end{aligned}$$

as required. □

Note that after commuting the tensor product factors, the linear map

$$(\mathrm{id}_{d_1 r_1} \otimes S)[\sum_{i,j,k,l} \mathcal{E}_{i,j,k,l} \otimes \mathcal{E}_{i,j,k,l}]$$

has Choi matrix given by Equation 2.1. Thus, we can view the map

$$\Phi_+ := \sum_{i,j,k,l} \mathcal{E}_{i,j,k,l} \otimes \mathcal{E}_{i,j,k,l} \in \mathcal{L}(M_d \otimes M_d, M_r \otimes M_r)$$

as the linear map equivalent of the maximally entangled state $\phi_+^d$. However, it is not a quantum channel since it is not trace-preserving.

**Remark 2.1.5.** If $d_1 = d_2 = 1$ then each quantum channel corresponds to a density matrix in $M_{r_i}$ and so a superchannel corresponds to a quantum channel. In terms of linear maps this is

$$(1 \mapsto \rho) \xrightarrow{S} (1 \mapsto \sigma)$$

and the induced map on Choi matrices is simply the channel $\phi(\rho) = \sigma$.

Using the reverse Choi formula, Equation (1.3), for a quantum superchannel $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ we have for any linear map $\phi : M_{d_1} \to M_{r_1}$

$$C_{S(\phi)} = \widetilde{S}(C_\phi) = \mathrm{Tr}_{d_1 r_1}[C_{\widetilde{S}}(C_\phi^T \otimes I_{d_2 r_2})] \tag{2.2}$$

## 2.2 Characterisation theorem for superchannels

In this section we present a theorem which characterises all quantum superchannels. This is analogous to the Stinespring dilation theorem for completely-positive maps.

First, note that given a CP map, the map $\phi \mapsto \phi \otimes \mathrm{id}_e$ is also CP, and satisfies the properties of a superchannel. Also, since composition of CP maps is CP, given any quantum channel $\psi$, the map $S(\phi) = \psi \circ \phi$ is a superchannel. It is completely CP preserving since $S \otimes \mathrm{id}_{d,r}(\phi) = (\psi \otimes \mathrm{id}_r) \circ \phi$ is a composition of CP maps.

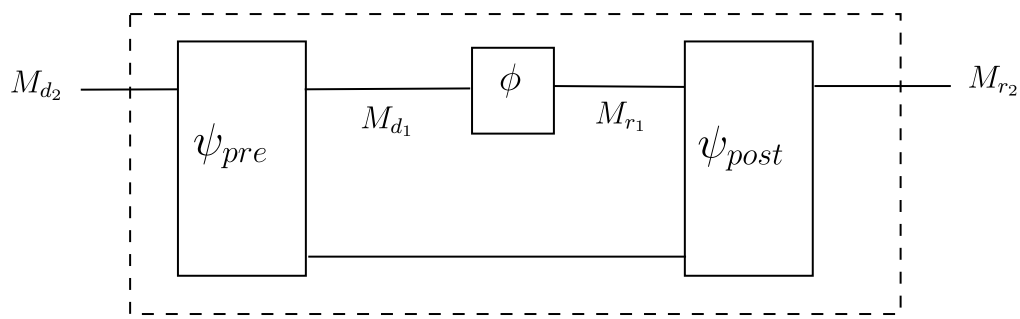
The characterisation theorem says that tensoring with the identity channel, and composition with other channels gives you every quantum superchannel.

**Theorem 2.2.1** ([3]). *If $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ is a quantum superchannel then there exists two quantum channels $\psi_{pre} : M_{d_2} \to M_{d_1} \otimes M_e$ and $\psi_{post} : M_{r_1} \otimes M_e \to M_{r_2}$ such that*

$$S(\phi) = \psi_{post} \circ (\phi \otimes id_e) \circ \psi_{pre}$$

*where $e$ is the dimension of an auxilliary space.*

**Remark 2.2.2.** We can represent this theorem with a diagram as follows:




Like a circuit, it is read from left to right. Any input state in $M_{d_2}$ is transformed to a state in $M_{r_2}$. Given an input channel $\phi$ the output channel is everything inside the dashed lines. From a physical perspective this makes sense, as quantum channels represent the most general description of the evolution of a quantum system, so channels themselves should transform by the use of other channels on different size spaces.

We will break the proof of this up into two parts, following [16]. First we show that every superchannel has a Choi matrix satisfying certain partial trace identities.

When we trace out the range spaces $r_2$ we have trivial dependence on the space $r_1$. That is, the domain part of the output quantum channel depends only on the domain part of the input channel. This is Theorem 2.2.3.

The second part, Theorem 2.2.8, shows that any linear map satisfying these partial trace relations can be written as the composition of pre and post processing channels. Thus it is a quantum superchannel and every superchannel must have this form.

### Proof of the first part

Let $X \in M_A \otimes M_B \otimes M_C$ be a self-adjoint matrix and suppose

$$\operatorname{Tr}(X(Y \otimes Z)) = 0$$

for all $Z \in M_C$ and all $Y \in M_A \otimes M_B$ with $\operatorname{Tr}_2 Y = 0$. So it is orthogonal in the Hilbert-Schmidt inner product to such matrices. Note that any matrix in $M_A \otimes \operatorname{span}\{I_B\} \otimes M_C$ will satisfy this. In fact every such $X$ will be in this space because it is of dimension $A \cdot 1 \cdot C$ which is the dimension of the orthogonal complement to all matrices $Y \otimes Z$ with $\operatorname{Tr}_2 Y = 0$. This also implies $X = \operatorname{Tr}_2(X) \otimes \frac{1}{B} I_B$ where we insert the tensor in the second position.

Recall that Choi matrices of channels $\phi : M_d \to M_r$ are exactly the positive matrices with $\operatorname{Tr}_r C_\phi = I_d$.

**Theorem 2.2.3** ([16])**.** *If $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ is a quantum superchannel, then $\operatorname{Tr}_{d_1 r_2} C_{\widetilde{S}} = I_{r_1} \otimes I_{d_2}$ and $\operatorname{Tr}_{r_2} C_{\widetilde{S}} = (\operatorname{Tr}_{r_1 r_2} C_{\widetilde{S}}) \otimes \frac{1}{r_1} I_{r_1}$.*

*Proof.* Take any $\rho \in M_{d_2}$ and multiply the reverse Choi formula (Equation (2.2)) by $\rho \otimes I_{r_2}$. Take the partial trace $\operatorname{Tr}_{r_2}$ to get

$$I_{d_2} \cdot \rho = \operatorname{Tr}_{d_1 r_1}[\operatorname{Tr}_{r_2}(C_{\widetilde{S}})(C_\phi^T \otimes I_{d_2})]\rho$$

where we used the partial trace formula from subsection 1.1, and the fact that $C_{S(\phi)}$ is the Choi matrix of a channel and so its partial trace is the identity. Now take the trace to get

$$\begin{aligned}
\operatorname{Tr}(\rho) &= \operatorname{Tr}(\operatorname{Tr}_{d_1 r_1}[\operatorname{Tr}_{r_2}(C_{\widetilde{S}})(C_\phi^T \otimes I_{d_2})]\rho) \\
&= \operatorname{Tr}(\operatorname{Tr}_{r_2}(C_{\widetilde{S}})(C_\phi^T \otimes I_{d_2}) I_{d_1 r_1} \otimes \rho) \\
&= \operatorname{Tr}(\operatorname{Tr}_{r_2}(C_{\widetilde{S}})(C_\phi^T \otimes \rho)).
\end{aligned} \tag{2.3}$$

This holds for any input $\rho$ and any $\phi$ which is trace-preserving (i.e. $\operatorname{Tr}_{r_1} C_\phi = I_{d_1}$) in particular it holds for $\frac{1}{r_1} I_{d_1} \otimes I_{r_1}$. So for any $Y = C_\phi - \frac{1}{r_1} I_{d_1} \otimes I_{r_1}$ with $\operatorname{Tr}_{r_1} Y = 0$ we have for all $\rho$

$$\operatorname{Tr}(\operatorname{Tr}_{r_2}(C_{\widetilde{S}})(Y^T \otimes \rho)) = 0.$$

Hence, by the comments before the theorem, we conclude

$$\operatorname{Tr}_{r_2} C_{\widetilde{S}} = (\operatorname{Tr}_{r_1 r_2} C_{\widetilde{S}}) \otimes \frac{1}{r_1} I_{r_1}.$$

Put this into Equation (2.3) and take $C_\phi = \frac{1}{r_1} I_{d_1} \otimes I_{r_1}$ to get

$$\begin{aligned} \operatorname{Tr}(\rho) &= \operatorname{Tr}[(\operatorname{Tr}_{r_1 r_2}(C_{\widetilde{S}}) \otimes \frac{1}{r_1} I_{r_1})(\frac{1}{r_1} I_{d_1} \otimes I_{r_1} \otimes \rho)] \\ &= \frac{1}{r_1^2} \operatorname{Tr}[\operatorname{Tr}_{d_1 r_2}(C_{\widetilde{S}}) \rho \otimes I_{r_1}] \\ &= \frac{1}{r_1} \operatorname{Tr}[\operatorname{Tr}_{d_1 r_1 r_2}(C_{\widetilde{S}}) \rho], \end{aligned}$$

which implies $\operatorname{Tr}_{d_1 r_1 r_2}(C_{\widetilde{S}}) = r_1 I_{d_2}$. Finally, using this we have

$$\operatorname{Tr}_{d_1 r_2}(C_{\widetilde{S}}) = \operatorname{Tr}_{d_1}(\operatorname{Tr}_{r_1 r_2}(C_{\widetilde{S}}) \otimes \frac{1}{r_1} I_{r_1}) = I_{r_1} \otimes I_{d_2}.$$

which completes the proof. □

### Schmidt decomposition and purification

**Definition 2.2.4.** For a positive matrix $\rho \in M_n$ a *purification* is matrix $|\psi\rangle\langle\psi| \in M_n \otimes M_m$ for some $m$ and some vector $|\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$ such that

$$\rho = \operatorname{Tr}_2 |\psi\rangle\langle\psi|$$

**Proposition 2.2.5** (Schmidt Decomposition)**.** *If $\rho_{AB} = |\psi\rangle\langle\psi| \in M_n \otimes M_m$ with $\operatorname{Tr}_2 \rho_{AB} = \rho_A$ and $\operatorname{Tr}_1 \rho_{AB} = \rho_B$ then $\rho_A$ and $\rho_B$ have the same non-zero eigenvalues $\lambda_j$. Furthermore*

$$|\psi\rangle = \sum_j \sqrt{\lambda_j} |e_j^A\rangle \otimes |e_j^B\rangle \tag{2.4}$$

*where $\{e_j^{A,B}\}$ are orthonormal eigenvectors of $\rho_A$, resp. of $\rho_B$.*

*Proof.* Take an orthonormal basis $\{e_j^A\}$ of eigenvectors for $\rho_A$ then write $|\psi\rangle = \sum_j |e_j^A\rangle \otimes |h_j^B\rangle$ for some vectors $|h_j^B\rangle \in \mathbb{C}^m$. Compute the partial trace

$$\sum_{j,k} \langle h_j^B | h_k^B \rangle |e_k^A\rangle\langle e_j^A| = \sum_j \lambda_j |e_j^A\rangle\langle e_j^A| = \rho_A$$

and therefore $\langle h_j^B | h_k^B\rangle = \lambda_j \delta_{jk}$. Finally, we can define the vectors $|e_j^B\rangle = \frac{1}{\sqrt{\lambda_j}} |h_j^B\rangle$ for $\lambda_j > 0$ and complete to an orthonormal basis. □

**Proposition 2.2.6.** *Any positive matrix $\rho \in M_n$ has a purification $|\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^E$ with the dimension $E \leq rank(\rho)$. For any other purification $|\phi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$ with $m \geq E$ there is an isometry $V : \mathbb{C}^E \to \mathbb{C}^m$ with $(I_n \otimes V)|\psi\rangle = |\phi\rangle$.*

*Proof.* Diagonalise $\rho = \sum_j \lambda_j |e_j\rangle\langle e_j|$ and take $|\psi\rangle = \sum_j \sqrt{\lambda_j}|e_j\rangle \otimes |h_j\rangle$ where $\{h_j\}$ is another orthonormal basis for $\mathbb{C}^n$. Any other purification has a Schmidt decomposition as in (2.4) with a different basis and we can relate them via an isometry. □

**Remark 2.2.7.** The uniqueness of purification is a special case of the uniqueness of Kraus operators/Stinespring representation. Consider the CP map $1 \mapsto \rho$. This has Choi matrix equal to $\rho$. Thus, any two purifications gives two sets of Kraus operators, which are unique up to isometry. Similarly, we can use the uniqueness of purification of a Choi matrix to derive the isometry relating Kraus operators. In general, for a CP map the minimal number of Kraus operators is equal to the rank of the Choi matrix, and this is called the Choi-Kraus rank of the map.

### Proof of the second part

For ease of notation we interpret the identity $\mathrm{Tr}_{r_2} C_{\widetilde{S}} = (\mathrm{Tr}_{r_1 r_2} C_{\widetilde{S}}) \otimes \frac{1}{r_1} I_{r_1}$ as being inserted in the second tensor position. Similarly, throughout this proof we may write linear maps acting on tensor products in the wrong order, so as to simplify the notation of the maximally entangled state $\phi_+^{r_1} \otimes \phi_+^{d_2}$. As long as we keep track of which component the linear maps are acting on it is okay.

**Theorem 2.2.8.** *If $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ is a linear map with $C_{\widetilde{S}} \geq 0$, $\mathrm{Tr}_{d_1 r_2} C_{\widetilde{S}} = I_{r_1} \otimes I_{d_2}$ and $\mathrm{Tr}_{r_2} C_{\widetilde{S}} = (\mathrm{Tr}_{r_1 r_2} C_{\widetilde{S}}) \otimes \frac{1}{r_1} I_{r_1}$ then there exists two CPTP maps $\psi_{pre} : M_{d_2} \to M_{d_1 e}$ and $\psi_{post} : M_{r_1 e} \to M_{r_2}$ such that for any linear $\phi : M_{d_1} \to M_{r_1}$ we have*

$$S(\phi) = \psi_{post} \circ (\phi \otimes id_e) \circ \psi_{pre}$$

*Proof.* Since the partial trace of the maximally entangled state gives the identity we can use the hypothesis to construct two purifications of $\mathrm{Tr}_{r_2} C_{\widetilde{S}}$ and of $I_{d_2}$. The isometries relating these purifications will give the "pre" and "post" maps. To that end, let $A_{d_3} \in M_{d_1} \otimes M_{r_1} \otimes M_{d_2} \otimes M_{r_2} \otimes M_{d_3}$ be a purification of $C_{\widetilde{S}}$ and $B_e \in M_{d_1} \otimes M_{d_2} \otimes M_e$ be a purification of $\frac{1}{r_1} \mathrm{Tr}_{r_1 r_2} C_{\widetilde{S}}$.

Note that $A_{d_3}$ is also a purification of $\mathrm{Tr}_{r_2} C_{\widetilde{S}}$ and so is $B_e \otimes \phi_+^{r_1}$ since we are assuming $\mathrm{Tr}_{r_2} C_{\widetilde{S}} = (\mathrm{Tr}_{r_1 r_2} C_{\widetilde{S}}) \otimes \frac{1}{r_1} I_{r_1}$. Using the isometry relating them we get an isometric channel $\mathcal{V} : M_e \otimes M_{r_1} \to M_{r_2} \otimes M_{d_3}$ such that

$$A_{d_3} = \mathrm{id}_{d_1 r_1 d_2} \otimes \mathcal{V} \left( B_e \otimes \phi_+^{r_1} \right)$$

and so

$$C_{\widetilde{S}} = \mathrm{Tr}_{d_3} A_{d_3} = \mathrm{id}_{d_1 r_1 d_2} \otimes \psi_{post} \left( B_e \otimes \phi_+^{r_1} \right) \tag{2.5}$$

where $\psi_{post} = \mathrm{Tr}_{d_3} \circ \mathcal{V}$.

For the other channel, note that

$$\mathrm{Tr}_{d_1 e} B_e = \frac{1}{r_1} \mathrm{Tr}_{d_1} \mathrm{Tr}_{r_1 r_2} C_{\widetilde{S}} = \frac{1}{r_1} \mathrm{Tr}_{r_1} (I_{r_1} \otimes I_{d_2}) = I_{d_2}.$$

Thus, as a purification of the identity, there is an isometric channel $\psi_{pre} : M_{d_2} \to M_{d_1 e}$ such that

$$B_e = \mathrm{id}_{d_2} \otimes \psi_{pre} \left( \phi_+^{d_2} \right).$$

Using Equation (2.5) we have

$$C_{\widetilde{S}} = (\mathrm{id}_{r_1 d_2} \otimes ((\mathrm{id}_{d_1} \otimes \psi_{post}) \circ (\mathrm{id}_{r_1} \otimes \psi_{pre}))) \left( \phi_+^{r_1} \otimes \phi_+^{d_2} \right). \tag{2.6}$$

The reverse Choi formula tells us that for any $\rho \in M_{d_2}$ and input map $\phi : M_{d_1} \to M_{r_2}$ the output channel acts as

$$\begin{aligned} S(\phi)(\rho) &= \mathrm{Tr}_{d_2}[S(C_\phi)(\rho^T \otimes I_{r_2})] \\ &= \mathrm{Tr}_{d_1 r_1 d_2}[C_{\widetilde{S}}(C_\phi^T \otimes \rho^T \otimes I_{r_2})] \\ &= \mathrm{Tr}_{d_1 r_1 d_2}[(\mathrm{id}_{r_1 d_2} \otimes ((\mathrm{id}_{d_1} \otimes \psi_{post}) \circ (\mathrm{id}_{r_1} \otimes \psi_{pre}))) \\ &\qquad \left( \phi_+^{r_1} \otimes \phi_+^{d_2} \right) (C_\phi^T \otimes \rho^T \otimes I_{r_2})] \end{aligned}$$

The terms being traced out can be cycled in the partial trace, so $C_\phi^T \otimes I_{r_2}$ can be cycled to the left side of the trace. The only terms affected by tracing out $d_2$ are the first component of $\mathrm{id}_{d_2} \otimes \psi_{pre}(\phi_+^{d_2})$ and $\rho^T$. By expanding out Stinespring representations we can bring these terms together. We can see the effect of this by letting $\psi_{pre}(\sigma) = V^*\sigma V$ for some matrix $V$ and calculate

$$\begin{aligned}
&\mathrm{Tr}_{d_2}[\mathrm{id}_{d_2} \otimes \psi_{pre}(\phi_+^{d_2})\rho^T \otimes I_{d_1} \otimes I_e] \\
&\qquad = \mathrm{Tr}_{d_2}[(I_{d_2} \otimes V^*)(\phi_+^{d_2})(I_{d_2} \otimes V)\rho^T \otimes I_{d_1} \otimes I_e] \\
&\qquad = \mathrm{Tr}_{d_2}[(I_{d_2} \otimes V^*)(\phi_+^{d_2})(I_{d_2} \otimes \rho)(I_{d_2} \otimes V)] \\
&\qquad = \mathrm{Tr}_{d_2}[(\phi_+^{d_2})]V^*\rho V = \psi_{pre}(\rho)
\end{aligned}$$

where in the second equation we used the property of the maximally entangled state on $\rho \otimes I_{d_2}$. Overall we get

$$S(\phi)(\rho) = \mathrm{Tr}_{d_1 r_1}[C_\phi^T \otimes I_{r_2} \left(\mathrm{id}_{d_1 r_1} \otimes \psi_{post} \left(\phi_+^{r_1} \otimes \psi_{pre}(\rho)\right)\right)].$$

To finish compute the partial trace over $r_1$

$$\begin{aligned}
\mathrm{Tr}_{r_1}[C_\phi^T \otimes I_{r_1}(I_{d_1} \otimes \phi_+^{r_1})] &= \mathrm{Tr}_{r_1}[\sum_{i,j} E_{j,i} \otimes I_{r_1} \otimes \phi(E_{i,j})(I_{d_1} \otimes \phi_+^{r_1})] \\
&= \sum_{i,j} E_{j,i} \otimes \phi(E_{i,j})
\end{aligned}$$

giving

$$\begin{aligned}
S(\phi)(\rho) &= \mathrm{Tr}_{d_1}\left[\mathrm{id}_{d_1} \otimes \psi_{post}\left((\sum_{i,j} E_{j,i} \otimes \phi(E_{i,j}) \otimes I_e)(\psi_{pre}(\rho) \otimes I_{r_1})\right)\right] \\
&= \psi_{post}\left(\sum_{i,j} \phi(E_{i,j}) \otimes \mathrm{Tr}_{d_1}\left[(E_{j,i} \otimes I_e)\psi_{pre}(\rho)\right]\right) \\
&= \psi_{post}\left(\sum_{i,j} \phi(E_{i,j}) \otimes (\langle e_i| \otimes I_e)(\psi_{pre}(\rho))(|e_j\rangle \otimes I_e)\right) \\
&= \psi_{post}\left((\phi \otimes \mathrm{id}_e)(\psi_{pre}(\rho))\right)
\end{aligned}$$

which completes the proof. □

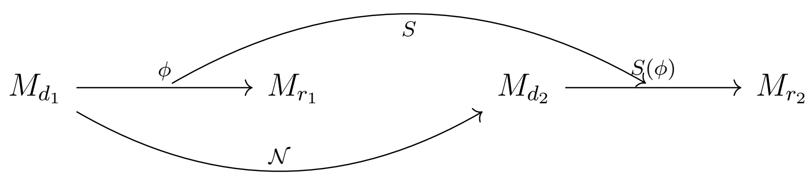
Theorems 2.2.3 and 2.2.8 give Theorem 2.2.1.

**Remark 2.2.9.** In the original proof of the superchannel characterisation theorem in [3], it is derived in a slightly different way, by using the uniqueness of Kraus operators to construct the isometries. This is ultimately equivalent to using purification of Choi matrices. The alternative set of Kraus operators come from the non-trivial fact that every for superchannel $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ there is a unital CP map (i.e. the dual of a channel) $\mathcal{N} : M_{d_1} \to M_{d_2}$ such that

$$\operatorname{Tr}_{r_2}(\widetilde{S}) = \mathcal{N} \circ \operatorname{Tr}_{r_1} . \tag{2.7}$$

By writing the CP maps $\widetilde{S}$ and $\mathcal{N}$ in terms of their Kraus operators this equation gives two different sets of Kraus operators for the same map.

Similar to the conditions $\operatorname{Tr}_{d_1 r_2} C_{\widetilde{S}} = I_{r_1} \otimes I_{d_2}$ and $\operatorname{Tr}_{r_2} C_{\widetilde{S}} = (\operatorname{Tr}_{r_1 r_2} C_{\widetilde{S}}) \otimes \frac{1}{r_1} I_{r_1}$, Equation (2.7) reflects the fact that range part of the input channel has no effect on the domain part of the output channel. Here is a diagram showing some of the various maps:




Indeed, in [16] these conditions are shown to be equivalent. If Equation (2.7) holds, then

$$\begin{aligned}
\operatorname{Tr}_{r_2} C_{\widetilde{S}} &= \operatorname{Tr}_{r_2}(\mathrm{id}_{d_1 r_1} \otimes \widetilde{S}(\sum_{i,j,k,l} E_{i,j} \otimes E_{k,l} \otimes E_{i,j} \otimes E_{k,l})) \\
&= \mathrm{id}_{d_1 r_1} \otimes \mathcal{N}(\sum_{i,j} E_{i,j} \otimes I_{r_1} \otimes E_{i,j}) \\
&= C_{\mathcal{N}} \otimes I_{r_1}.
\end{aligned}$$

where in the last line we swapped around the tensors to make it clear that this implies $\operatorname{Tr}_{r_1 r_2} C_{\widetilde{S}} = r_1 C_{\mathcal{N}}$. Similarly, $\operatorname{Tr}_{r_1 r_2 d_1} C_{\widetilde{S}} = r_1 I_{d_2}$ since $\mathcal{N}$ is unital, which implies $\operatorname{Tr}_{d_1 r_2} C_{\widetilde{S}} = I_{d_2} \otimes I_{r_1}$.

On the other hand, if $\operatorname{Tr}_{d_1 r_2} C_{\widetilde{S}} = I_{r_1} \otimes I_{d_2}$ and $\operatorname{Tr}_{r_2} C_{\widetilde{S}} = (\operatorname{Tr}_{r_1 r_2} C_{\widetilde{S}}) \otimes \frac{1}{r_1} I_{r_1}$ then by defining $\mathcal{N}$ via

$$C_{\mathcal{N}} = \frac{1}{r_1} \operatorname{Tr}_{r_1 r_2} C_{\widetilde{S}}$$

then the same equations imply it satisfies Equation (2.7).

**Remark 2.2.10.** Given a superchannel with characterisation as in Theorem 2.2.1, its Choi matrix is defined by the map given by composition of $\psi_{post}$ and $\psi_{pre}$ More specifically, if we define a map $Q : M_{r_1} \otimes M_{d_2} \to M_{d_1} \otimes M_{r_2}$ by

$$Q = (\mathrm{id}_{d_1} \otimes \psi_{post}) \circ (Swap^{r_1 d_1 \longrightarrow d_1 r_1} \otimes \mathrm{id}_e) \circ (\mathrm{id}_{r_1} \otimes \psi_{pre})$$

then, up to rearranging the tensor factors, this has the same Choi matrix as $\widetilde{S}$. Here $Swap^{r_1 d_1 \longrightarrow d_1 r_1}$ sends $A \otimes B \in M_{r_1} \otimes M_{d_1}$ to $B \otimes A \in M_{d_1} \otimes M_{r_1}$. The swap map is equal to $\sum_{i,j,k,l} \mathcal{E}^T_{i,j,k,l} \otimes \mathcal{E}_{i,j,k,l}$.

We can show this using Equation (2.1), let $E^d_{i,j}$ denote the matrix units in $M_d$, then

$$C_{\widetilde{S}} = \sum_{i,j,k,l} (E^{d_1}_{i,j} \otimes E^{r_1}_{k,l}) \otimes (\mathrm{id}_{d_2} \otimes \psi_{post} \circ (\mathcal{E}_{i,j,k,l} \otimes \mathrm{id}_e) \circ \psi_{pre})(\sum_{a,b} E^{d_2}_{a,b} \otimes E^{d_2}_{a,b}).$$

Now using $\mathcal{E}^T_{i,j,k,l}(E^{r_1}_{a,b}) = \delta_{(a,b),(k,l)} E^{d_1}_{i,j}$, the Choi matrix of $Q$ is

$$\begin{aligned} C_Q &= \sum_{a,b,c,d} (E^{r_1}_{a,b} \otimes E^{d_2}_{c,d}) \otimes (\mathrm{id}_{d_1} \otimes \psi_{post}) \circ (\sum_{i,j,k,l} \mathcal{E}^T_{i,j,k,l} \otimes \mathcal{E}_{i,j,k,l} \otimes \mathrm{id}_e)(E^{r_1}_{a,b} \otimes \psi_{pre}(E^{d_2}_{c,d})) \\ &= \sum_{k,l,c,d,} (E^{r_1}_{k,l} \otimes E^{d_2}_{c,d}) \otimes \sum_{i,j} E^{d_1}_{i,j} \otimes \psi_{post} \circ (\mathcal{E}_{i,j,k,l} \otimes \mathrm{id}_e) \circ \psi_{pre}(E^{d_2}_{c,d}). \end{aligned}$$

This is the same sum as $C_{\widetilde{S}}$, but with differently ordered tensors.

Since the Choi matrix is positive we see that $Q$ is a CPTP map. Thus, every superchannel has a Choi matrix defined by a quantum channel. This also gives a useful formula for comparing the pre and post-processing maps of a superchannel, which we use in the next theorems.

### 2.2.1 Uniqueness and minimal dimension

The choice of pre and post-processing channels in Theorem 2.2.1 are not unique. For example, given any density matrix $\sigma$, if we replace $\psi_{pre}$ with $\sigma \otimes \psi_{pre}$, and $\psi_{post}$ with $\mathrm{Tr}_1 \otimes \psi_{post}$, then it describes the same superchannel. However, in the proof of Theorem 2.2.8 we constructed the pre-processing channel as an isometry, and the dimension $e$ bounded by the rank of $\mathrm{Tr}_{r_1 r_2} C_{\widetilde{S}}$. Under these conditions the characterisation theorem is unique up to unitary.

**Definition 2.2.11.** Let $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ be a quantum superchannel with $e = \text{rank}(\text{Tr}_{r_1r_2} C_{\widetilde{S}})$. A *minimal dilation* for $S$ is given by a pair $(\psi_{pre}, \psi_{post})$, where $\psi_{post} : M_{r_1} \otimes M_e \to M_{r_2}$ is a channel, and $\psi_{pre} : M_{d_2} \to M_{d_1} \otimes M_e$ is an isometric channel, such that

$$S(\phi) = \psi_{\text{post}} \circ (\phi \otimes \text{id}_e) \circ \psi_{\text{pre}}.$$

**Lemma 2.2.12.** *If $\rho_{AB} = |\psi\rangle\langle\psi|$ is a purification of $\rho_A \in M_A$, then*

$$|\psi\rangle = (\sqrt{\rho_A} V \otimes I_B)|\phi_+^d\rangle$$

*for some unitary $V$.*

*Proof.* We can assume the purification is a square matrix i.e. $|\psi\rangle \in M_d \otimes M_d$. Take a Schmidt-decomposition of $|\psi\rangle$,

$$|\psi\rangle = \sum_j \sqrt{\lambda_j}|e_j^A\rangle \otimes |e_j^B\rangle,$$

and let $U_A$, $U_B$ be the unitaries sending $\{e_j^A\}$ and $\{e_j^B\}$ to the standard basis. Recall that each $\lambda_j$ is an eigenvalue of $\rho_A$. We then have

$$\begin{aligned}|\psi\rangle &= (\sqrt{\rho_A} \otimes I_B)(U_A \otimes U_B)|\phi_+^d\rangle \\ &= (\sqrt{\rho_A} U_A \otimes I_B)(I_A \otimes U_B)|\phi_+^d\rangle \\ &= (\sqrt{\rho_A} U_A U_B^T \otimes I_B)|\phi_+^d\rangle,\end{aligned}$$

where in the last equation we used the property of the maximally entangled state, Equation (1.4). □

**Theorem 2.2.13** ([17]). *Let $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ be a quantum superchannel with a minimal dilation $(\psi_{pre}, \psi_{post})$. If there are other channels $\Psi_{post}$ : $M_{r_1} \otimes M_{e'} \to M_{d_1} \otimes M_{r_2}$, and $\Psi_{pre} : M_{d_2} \to M_{d_1} \otimes M_{e'}$, satisfying*

$$S(\phi) = \Psi_{post} \circ (\phi \otimes id_{e'}) \circ \Psi_{pre}$$

*with $\Psi$ isometric and $e' \leq e$, then $e = e'$ and there is a unitary channel, $\mathcal{U} : M_e \to M_{e'}$, such that*

$$\Psi_{post} = \psi_{post} \circ (id_{r_1} \otimes \mathcal{U}^{-1})$$

*and*

$$\Psi_{pre} = (id_{d_1} \otimes \mathcal{U}) \circ \psi_{pre}$$

*Proof.* Recall the map from Remark 2.2.10, informally given by $Q = \psi_{post} \circ \psi_{post} = \Psi_{post} \circ \Psi_{pre}$. If we omit the $Swap^{r_1 d_1 \to d_1 r_1}$ map we get

$$\operatorname{Tr}_{r_1} C_Q = \mathrm{id}_{d_2 d_1} \otimes \psi_{post}\Big(\sum_{k,l} E^{d_2}_{k,l} \otimes I_{r_1} \otimes \psi_{pre}(E^{d_2}_{k,l})\Big).$$

Since channels are trace-preserving we have

$$r_1 \cdot \operatorname{Tr}_e \psi_{pre}(E^{d_2}_{k,l}) = \operatorname{Tr}_{r_2}(\mathrm{id}_{d_1} \otimes \psi_{post}[I_{r_1} \otimes \psi_{pre}(E^{d_2}_{k,l})])$$

which implies

$$\begin{aligned} \operatorname{Tr}_{r_1 r_2} C_Q &= r_1 \cdot \operatorname{Tr}_e[\mathrm{id}_{d_2} \otimes \psi_{pre}(\phi^{d_2}_+)] \\ &= r_1 \operatorname{Tr}_e C_{\psi_{pre}}. \end{aligned}$$

This also holds for $\Psi_{pre}$ with $\operatorname{Tr}_{e'}$. Since both channels are isometric, this equation implies that $\mathrm{id}_{d_2} \otimes \psi_{pre}(\phi^{d_2}_+)$ and $\mathrm{id}_{d_2} \otimes \Psi_{pre}(\phi^{d_2}_+)$ are purifications for $\operatorname{Tr}_{r_1 r_2} C_Q$. Since $e$ is defined as the minimal dimension achieving this purification, we have $e' \geq e$ and so they are equal. Furthermore, by the uniqueness of purification, there is a unitary $\mathcal{U} : M_e \to M_{e'}$ such that

$$(\mathrm{id}_{d_2 d_1} \otimes \mathcal{U})(\mathrm{id}_{d_2} \otimes \psi_{pre}(\phi^{d_2}_+)) = \mathrm{id}_{d_2} \otimes \Psi_{pre}(\phi^{d_2}_+)$$

and hence we conclude,

$$\Psi_{pre} = (\mathrm{id}_{d_1} \otimes \mathcal{U}) \circ \psi_{pre}.$$

We now show the same unitary works for the post-processing maps. What we want to show is

$$\mathrm{id}_e \otimes \mathrm{id}_{r_1} \otimes \psi_{post}(\phi^e_+ \otimes \phi^{r_1}_+) = \mathrm{id}_e \otimes \mathrm{id}_{r_1} \otimes [\Psi_{post} \circ (\mathrm{id}_{r_1} \otimes \mathcal{U}^{-1})](\phi^e_+ \otimes \phi^{r_1}_+) \tag{2.8}$$

where we omit the swap maps putting the tensors in the correct order. We can then conclude the maps have the same Choi matrix, and hence are the same map. Since they define $C_Q$, $\psi_{post}$ acts on the $M_e$ component of $C_{\psi_{pre}}$ in the same way $\Psi_{post}$ acts on the $M_{e'}$ component of $C_{\Psi_{pre}}$. Thus

$$\mathrm{id}_{\widetilde{e}} \otimes \mathrm{id}_{r_1} \otimes \psi_{post}(A_{\widetilde{e}e} \otimes \phi^{r_1}_+) = \mathrm{id}_{\widetilde{e}} \otimes \mathrm{id}_{r_1} \otimes \Psi_{post}((\mathrm{id}_{\widetilde{e}} \otimes \mathcal{U}) A_{\widetilde{e}e} \otimes \phi^{r_1}_+) \tag{2.9}$$

where $A_{\widetilde{e}e} \in M_{\widetilde{e}} \otimes M_e$ is $C_{\psi_{pre}}$ restricted so it only acts on the support of $\operatorname{Tr}_e C_{\psi_{pre}}$. This support has dimension $\widetilde{e} = e$. If $A_{\widetilde{e}} = \operatorname{Tr}_e A_{\widetilde{e}e}$, then by the previous lemma, there is a unitary $V$ such that

$$A_{\widetilde{e}e} = (\sqrt{A_{\widetilde{e}}} V \otimes I_e)\phi^e_+(V^* \sqrt{A_{\widetilde{e}}} \otimes I_e).$$

Equation (2.9) now gives Equation (2.8) if we conjugate by $V^* \sqrt{A_{\widetilde{e}}}^{-1} \otimes I_e \otimes I_{r_1} \otimes I_{r_1}$. Note that, by construction, $A_{\widetilde{e}}$ is invertible. □

## 2.3 Some uses of superchannels

In this section we look at two recent uses of superchannels: defining entropy functions for channels, and defining the set of free operations in a dynamical quantum resource theory.

### Entropy of a channel

One of the uses of quantum superchannels has been to formulate a theory of entropy functions on quantum channels. This is done in an analogous way to quantum states. The entropy of a state can be viewed as some measure of information or noise. Under a reversible process it should be unchanged. We can quickly show that the only reversible quantum channels are unitary maps.

**Theorem 2.3.1.** *If a quantum channel $\phi : M_d \to M_d$ has an inverse channel then it is a unitary channel.*

*Proof.* Since $\phi$ and $\phi^{-1}$ are channels we can expand in terms of Kraus operators, $\phi(\rho) = \sum_i V_i \rho V_i^*$, $\sum_i V_i^* V_i = I_d$ and $\phi^{-1}(\rho) = \sum_i W_i \rho W_i^*$, $\sum_i W_i^* W_i = I_d$. Then

$$\phi^{-1} \circ \phi(\rho) = \sum_{i,j} W_j V_i \rho V_i^* W_j^* = \rho.$$

for all inputs $\rho \in M_d$. Since each term in this sum is positive we have $W_j V_i = c_{j,i} I_d$, some $c_{i,j} \in \mathbb{C}$. Further,

$$V_j^* V_k = V_j^* (\sum_i W_i^* W_i) V_k = \sum_i \overline{c_{i,j}} c_{i,k} I_d = \lambda_{j,k} I_d.$$

We can use this to show that each $V_i$ is just a positive multiple of a unitary. Using polar decomposition we have for some unitaries $U_i$,

$$V_i = \sqrt{V_i^* V_i} U_i = \sqrt{\lambda_{ii}} U_i.$$

But then

$$V_j^* V_k = \sqrt{\lambda_{jj} \lambda_{kk}} U_j^* U_k = \lambda_{j,k} I_d$$

which gives

$$U_k = \frac{\lambda_{jk}}{\sqrt{\lambda_{kk} \lambda_{jj}}} U_j.$$

Thus we can replace each Kraus operator $V_i$ with a multiple of the same unitary. □

For a state $\rho \in M_d$, the *von Neumann entropy* is given by $H(\rho) = -\operatorname{Tr}[\rho \log_2 \rho]$. There are other entropy functions on states, and a general property they must satisfy is monotonicity under the action of a random unitary channel. The idea here is that, unlike unitary transformations, random unitaries represent loss of information about the transformation of a state, and so can increase the noise. A random unitary channel is given by

$$\phi(\rho) = \sum_i p_i U_i \rho U_i^*$$

where $U_i$ are unitaries and $\{p_i\}_i$ is a probability distribution. Let $\mathcal{D}(M_d)$ be the set of density matrices in $M_d$. A function $f : \mathcal{D}(M_d) \to \mathbb{R}$ is *monotonic* if for any random unitary channel $\phi$ it satisfies

$$f(\phi(\rho)) \geq f(\rho), \quad \forall \rho.$$

Similarly, when defining the entropy of a quantum channel, it should be unchanged by reversible evolution, meaning a superchannel where the pre and post processing channels are unitary channels. Define a random unitary superchannel as one of the form

$$S(\phi) = \sum_i p_i \mathcal{U}_i^{post} \circ \phi \circ \mathcal{U}_i^{pre}$$

where $\mathcal{U}_i^{post} : M_r \to M_r$, and $\mathcal{U}_i^{pre} : M_d \to M_d$ are unitary channels, and $\{p_i\}_i$ a probability distribution.

One entropy function on channels was given in [19]. First, on density matrices we can use the von Neumann entropy to define the *quantum relative entropy* between two states as

$$H(\rho \| \sigma) := \begin{cases} \operatorname{Tr}[\rho(\log_2 \rho - \log_2 \sigma)] & \text{if } \operatorname{supp}(\rho) \subseteq \operatorname{supp}(\sigma) \\ +\infty & \text{otherwise} \end{cases}$$

Note that this allows us to express the von Neumann entropy of a state $\phi \in M_d$ in a different way

$$H(\rho) = \log_2 d - H(\rho \| \frac{1}{d} I_d).$$

Using this, define the quantum relative entropy between two channels as

$$H(\phi_1 \| \phi_2) := \sup_{\rho \in M_n \otimes M_d} H(\operatorname{id}_n \otimes \phi_1(\rho) \| \operatorname{id}_n \otimes \phi_2(\rho))$$

where the supremum is taken over states $\rho \in M_n \otimes M_d$, with $n$ of arbitrary size. Then given a channel $\phi : M_d \to M_r$ we can define its entropy by

$$H(\phi) = \log_2 r - H(\phi \| \delta)$$

where $\delta : M_d \to M_r$ is the completely depolarizing channel, $\delta(\rho) = \frac{1}{r} \operatorname{Tr}(\rho) I_r$. Another entropy function on channels is given in [10] as

$$H_{QC}(\phi) = H(\frac{1}{d} C_\phi) - \log_2 d \tag{2.10}$$

#### Resource theories

Recent work on superchannels have used them to study dynamical resource theories such as entanglement [18], magic [38], and coherence [25].

The intuition behind resource theories is wanting to perform some operation on a quantum system but only having access to some subset of states and channels. These are referred to as the free states and free channels. States outside of this set may allow you to do further operations and are thus a valuable resource. The standard example of a resource theory is that of entanglement.

**Definition 2.3.2.** Let $\mathcal{F}$ be a map which assigns to any two spaces $M_d$, $M_r$, a subset of channels $\mathcal{F}(M_d, M_r) \subseteq \{\phi : M_d \to M_r \,|\, \phi \text{ CPTP}\}$. Then $\mathcal{F}$ defines a *quantum resource theory* if the following holds:

1. $\mathcal{F}(M_d, M_d)$ contains the identity map $\mathrm{id}_d$
2. If $\phi_1 \in \mathcal{F}(M_{d_1}, M_{d_2})$, and $\phi_2 \in \mathcal{F}(M_{d_2}, M_{d_3})$, then $\phi_2 \circ \phi_1 \in \mathcal{F}(M_{d_1}, M_{d_3})$.

In this case $\mathcal{F}(M_d, M_r)$ is called the set of *free operations*, and $\mathcal{F}(M_d) := \mathcal{F}(\mathbb{C}, M_d)$ is called the set of *free states*.

**Example 2.3.3.** In the resource theory of entanglement, the set of free operations are the local operations and classical communication (LOCC) channels, [8]. This is a subset of separable channels, which are maps whose Kraus operators can be written as a tensor product.

A dynamical quantum resource theory is one where the free states are replaced by quantum channels, and the free operations by a set of quantum superchannels preserving this set of channels. Note that one way of achieving this is to look at superchannels where the pre and post processing maps are the free channels in a resource theory.

**Remark 2.3.4.** It is interesting to note that the definition of free superchannels in a dynamical resource theory only requires the action of those superchannels on a set of quantum channels. Thus, although the definition of superchannel has it being defined on the larger space of all linear maps, being defined on the space spanned by quantum channels would have the same effect. This point motivates the next chapter.

# Chapter 3

# Axiomatic Approach to Quantum Superchannels

This chapter is based on the author's paper [12].

In this chapter we propose a different definition of superchannels. In particular, we take the domain to be the span of quantum channels which in general is not all linear maps between operators. Considering the Choi matrices [9] of such maps allows us to define an operator system and make use of Stinespring's theorem [35]. Arveson's extension theorem [1, Theorem 1.2.3, Theorem 1.1.11] allows the same characterisation of the more general superchannels to apply to this smaller class of maps.

We then show that these extensions are non unique, meaning different extensions give superchannels whose action on quantum channels is the same. This shows that the usual definition of superchannel results in different maps which have the same effect on quantum channels. This provides evidence that this new definition is more natural as a description of maps on channels.

Consequences of the non uniqueness of these extensions are then explored. It is shown there are no trace-preserving extensions, and that the tensor product can be affected. The extreme points of the set of extensions is examined in a generalisation of a theorem by Choi [9].

## 3.1 The space of quantum channels

We want to define linear maps on the the space spanned by quantum channels. First we define this space, in two different ways. Simply as a space of linear maps, and as a space of block matrices. Note that the Choi matrix of a quantum channel is a block matrix where the diagonal blocks each have trace one and the off diagonal blocks have trace zero.

**Definition 3.1.1.** Given positive integers $d, r \geq 1$ define

$$SCPTP(d, r) := \text{span}\{\phi \,|\, \phi : M_d \to M_r \text{ is a CPTP map}\} \subset \mathcal{L}(M_d, M_r).$$

Also define $S(d, r) \subset M_d(M_r)$ to be the set of block matrices $(P_{i,j})$ such that for all $1 \leq i, j \leq d$, $\text{Tr}(P_{i,i}) = \text{Tr}(P_{j,j})$ and for $i \neq j$ $\text{Tr}(P_{i,j}) = 0$.

There is a natural way to define a matrix order on the space $SCPTP(d, r)$: for an $n \times n$ matrix $(\phi_{i,j})$ of maps, with each $\phi_{i,j} \in SCPTP(d, r)$, define $\Phi : M_d \to M_n(M_r)$ by $\Phi(x) = (\phi_{i,j}(x))$. Then $(\phi_{i,j}) \geq 0$ if and only if $\Phi$ is completely positive.

Note in the case $d = 1$ we have $M_1 = \mathbb{C}$ and since any such linear map is defined by its value at 1 we have an isomorphism $\mathcal{L}(M_1, M_r) \cong M_r$ via $\phi \mapsto \phi(1)$. The positive matrices span the whole space so in fact $SCPTP(1, r) \cong M_r$. With the order on $SCPTP(d, r)$ that we just defined this is a complete order isomorphism. Similarly $S(1, r) \subset M_1(M_r) = M_r$ and since there is just one block $P_{1,1}$ with no restriction we get all the $r \times r$ matrices in $S(1, r)$. Thus $SCPTP(1, r)$ is order isomorphic to $S(1, r)$.

**Theorem 3.1.2.** *$S(d, r)$ is an operator system and is completely order isomorphic to $SCPTP(d, r)$ via the Choi map*

$$\begin{aligned} R : SCPTP(d, r) &\to S(d, r) \\ \phi &\mapsto C_\phi. \end{aligned}$$

*Proof.* It is clear that $S(d, r)$ contains the identity matrix and the linearity of the trace ensures it is a subspace. If $X = (P_{i,j})$ is a block matrix then the adjoint is $X^* = (P_{j,i}^*)$ and for any block $\text{Tr}(P) = \overline{\text{Tr}(P^*)}$. This implies that $S(d, r)$ is self-adjoint and hence an operator system.

Next we show $\phi \mapsto C_\phi$ is an isomorphism between $SCPTP(d, r)$ and $S(d, r)$. This is the correct range because the Choi matrix for a quantum channel is in $S(d, r)$. It is injective because any linear map is defined by its Choi matrix. To prove surjectivity we use the fact about operator systems that any $X \in S(d,r)$ can be written in terms of four positive matrices $P_i \in S(d,r)$, $1 \leq i \leq 4$, as

$$X = (P_1 - P_2) + i(P_3 - P_4).$$

As they are positive $\mathrm{Tr}(P_i) = 0$ only if $P_i = 0$. Thus we can scale each $P_i$ by a factor $1/\mathrm{Tr}(P_i)$ to make it into a Choi matrix associated with a CPTP map. This proves any $X \in S(d,r)$ is in the span of Choi matrices of quantum channels.

Finally we show it is a complete order isomorphism. For a matrix of maps in $SCPTP(d,r)$ the condition to be positive is

$$(\phi_{i,j})_{i,j} \geq 0 \iff \Phi \text{ CP} \iff (\Phi(E_{k,l}))_{k,l} \geq 0 \iff \left((\phi_{i,j}(E_{k,l}))_{i,j}\right)_{k,l} \geq 0.$$

The corresponding matrix of Choi matrices can be written

$$\left(C_{\phi_{i,j}}\right)_{i,j} = \left((\phi_{i,j}(E_{k,l}))_{k,l}\right)_{i,j}$$

To conclude note that the shuffle which maps $M_m(M_n(\mathcal{A}))$ to $M_n(M_m(\mathcal{A}))$, where $\mathcal{A}$ is a $C^*$-algebra, is a $*$-isomorphism and hence preserves positivity. $\square$

**Remark 3.1.3.** A tensor product of linear maps gives a map on the tensor product of the spaces so there is an inclusion

$$SCPTP(d_1, r_1) \otimes SCPTP(d_2, r_2) \subseteq SCPTP(d_1 d_2, r_1 r_2).$$

We can show this is generally a strict inclusion and the spaces are not equal. The description of $S(d,r)$ allows us to do a dimension count giving the dimension of $SCPTP(d,r)$ as $d^2 r^2 - d^2 + 1$. So for the tensor product space we have dimension $(d_1^2 r_1^2 - d_1^2 + 1)(d_2^2 r_2^2 - d_2^2 + 1)$ but for the space on the right we have dimension $d_1^2 d_2^2 r_1^2 r_2^2 - d_1^2 d_2^2 + 1$ which is generally larger. The difference is

$$d_1^2 d_2^2 (r_1^2 + r_2^2 - 2) - d_1^2 (r_1^2 - 1) - d_2^2 (r_2^2 - 1)$$

and this is non-negative. We endow this tensor space with an order by regarding its elements as members of the operator system $SCPTP(d_1 d_2, r_1 r_2)$.

**Remark 3.1.4.** The domain of quantum superchannels, $\mathcal{L}(M_{d_1}, M_{r_1})$ can be understood as the span of *quantum operations*, that is, completely-positive trace non-increasing maps. These can be interpreted as describing probabilistic evolution of a state, as opposed to deterministic evolution given by quantum channels. A quantum operation $\phi : M_d \to M_r$ has a positive Choi matrix satisfying $\operatorname{Tr}_r C_\phi \leq I_d$. We can see that see these span the whole space of matrices by considering any positive $E \in M_d \otimes M_r$. We have

$$(\|E\| I_d \otimes I_r - E) \geq 0,$$

and as the partial trace is a positive map

$$\operatorname{Tr}_r \left( \frac{E}{r\|E\|} \right) \leq \operatorname{Tr}_r \left( \frac{1}{r} I_d \otimes I_r \right) = I_d.$$

Thus $E$ is in the span of the Choi matrices of quantum operations.

### 3.1.1 Basis for $S(2,2)$

For qubit quantum channels, $\phi : M_2 \to M_2$, the space of Choi matrices, $S(2,2)$, is a 13 dimensional space of matrices. Here we describe a simple basis for this space.

For clarity, a generic element of $S(2,2)$ looks like

$$\begin{pmatrix} v & * & b & * \\ * & w & * & -b \\ a & * & x & * \\ * & -a & * & y \end{pmatrix}$$

where $v + w = x + y$.

To get a basis for $S(2,2)$, take the 16 standard matrix units $\{E_{ij}\}$ for $M_4$, keep the eight which are block off diagonal, and replace $\{E_{11}, E_{22}, E_{33}, E_{44}, E_{31}, E_{42}, E_{13}, E_{24}\}$ with $\{X_1, X_2, X_3, X_4, X_5\}$ where

$$\begin{aligned} X_1 &= E_{11} + E_{44} \\ X_2 &= E_{22} + E_{44} \\ X_3 &= E_{33} - E_{44} \\ X_4 &= E_{31} - E_{42} \\ X_5 &= E_{13} - E_{24} \end{aligned}$$

For example take a diagonal $A \in S(2,2)$

$$A = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \end{pmatrix} = v \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + w \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + x \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
$$= vX_1 + wX_2 + xX_3$$

because $v + w = x + y$. Similarly the requirements of the trace on the off diagonal blocks are taken care of by $X_4$ and $X_5$.

The matrices $X_1$ and $X_2$ correspond directly to quantum channels but $X_3$ doesn't, as it is not positive. For $X_3$ we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = X_3$$

so we can replace $X_3$ with $E_{11} + E_{33}$, which is positive and describes a quantum channel. Similarly, for the non-positive $X_4$ and $X_5$, we can replace them with the matrices

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = I_4 + X_4 + X_5$$
$$\begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & i \\ i & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \end{pmatrix} = I_4 + iX_4 - iX_5$$

which are positive.

## 3.2 QSCs

To motivate our new definition of quantum superchannels it is worth recalling from Chapter 1, some of the reasons behind the definition for ordinary quantum channels.

Two simple requirements were that channels be linear maps that take quantum states to quantum states. This gives the trace-preserving condition. The requirements that quantum systems combine using tensor products, and that the identity map is a valid channel is what implies the completely positive condition. For superchannels we similarly require they be linear maps which takes channels to channels, and that the tensor of any two superchannels is again a superchannel on the combined space.

To distinguish our new definition of quantum superchannel from the standard one we use the terminology "QSC". The term superchannel will still refer to the objects defined in Chapter 2.

**Definition 3.2.1.** Given two spaces of quantum channels $SCPTP(d_i, r_i)$, $i = 1, 2$, a $QSC$ is a linear map $\Gamma : SCPTP(d_1, r_1) \to SCPTP(d_2, r_2)$ which satisfies

1. if $\phi$ is CPTP then $\Gamma(\phi)$ is CPTP
2. given any other dimensions $d_3, r_3 \in \mathbb{N}$ and the identity map

$$\mathrm{id}_{d_3,r_3} : SCPTP(d_3, r_3) \to SCPTP(d_3, r_3)$$

then

$$\Gamma \otimes \mathrm{id}_{d_3,r_3} : SCPTP(d_1, r_1) \otimes SCPTP(d_3, r_3) \to SCPTP(d_2, r_2) \otimes SCPTP(d_3, r_3)$$

sends CPTP maps to CPTP maps.

**Remark 3.2.2.** Elements of $SCPTP(d, r)$ scale the trace of density matrices and this scaling factor is preserved by QSCs. Consider a QSC $\Gamma : SCPTP(d_1, r_1) \to SCPTP(d_2, r_2)$. Suppose $\phi \in SCPTP(d_1, r_1)$ satisfies $\mathrm{Tr}\, \phi(X) = c \,\mathrm{Tr}\, X$ some constant $c$ and that $\mathrm{Tr}\, \Gamma(\phi)(Y) = k \,\mathrm{Tr}\, Y$ some constant $k$. Decompose $\phi$ as a span of quantum channels

$$\phi = \sum_i c_i \phi_i$$

and use the trace condition on $\phi$ to see

$$\sum_i c_i = c.$$

Now since $\Gamma$ is linear and sends TP maps to TP maps we get

$$k \,\mathrm{Tr}\, Y = \mathrm{Tr}\, \Gamma(\phi)(Y) = \sum_i c_i \,\mathrm{Tr}\, \Gamma(\phi_i)(Y) = c \,\mathrm{Tr}\, Y$$

and so $k = c$.

Since any CP map in $SCPTP(d, r)$ scales the trace by a positive number, property 2 in the definition of QSC implies $\Gamma \otimes \mathrm{id}_{d_3,r_3}$ sends all CP maps to CP maps, not just trace-preserving ones. In fact, this property is equivalent to $\Gamma \otimes \mathrm{id}_{d_3,r_3}$ preserving CP maps, as both $\Gamma$ and $\mathrm{id}_{d_3,r_3}$ preserve the trace-scaling of their inputs, and so will send TP maps to TP maps. Thus the "completely CPTP-preserving" property of QSC's can be replaced by a "completely CP-preserving" one.

Let $R_i : SCPTP(d_i, r_i) \to S(d_i, r_i)$ be the complete order isomorphism sending $\phi$ to $C_\phi$. If $\Gamma$ is a QSC it induces a map $\widetilde{\Gamma} : S(d_1, r_1) \to S(d_2, r_2)$ via

$$\widetilde{\Gamma} = R_2 \circ \Gamma \circ R_1^{-1}.$$

Explicitly this acts as $\widetilde{\Gamma}(C_\phi) = C_{\Gamma(\phi)}$. It is useful to study this map because the properties of QSC's implies that $\widetilde{\Gamma}$ is completely positive. Note that by Choi's theorem $\widetilde{\Gamma}$ sends positive matrices to positive matrices.

**Theorem 3.2.3.** *If $\Gamma : SCPTP(d_1, r_1) \to SCPTP(d_2, r_2)$ preserves CPTP maps then it is a QSC if and only if $\widetilde{\Gamma}$ is completely positive.*

*Proof.* Recall that $S(1, n) = M_n$. Take the identity map on $SCPTP(1, n)$ which has as its induced map on $S(1, n)$ the identity on $M_n$. Let $C_\phi \in S(d_1, r_1) \otimes M_n$ be a positive matrix. If $R_i : SCPTP(d_i, r_i) \to S(d_i, r_i)$ are the Choi isomorphisms we can write

$$\widetilde{\Gamma} \otimes \mathrm{id}_n(C_\phi) = (R_2 \otimes R_3)(\Gamma \otimes \mathrm{id}_{1,n})(\phi).$$

Then the second property of QSC's implies that $\widetilde{\Gamma} \otimes \mathrm{id}_n$ sends positive matrices to positive matrices for all $n$. Thus $\widetilde{\Gamma}$ is a completely positive map.

For the converse, suppose $\widetilde{\Gamma}$ is completely positive and note that

$$\mathrm{id}_{d_3,r_3} : SCPTP(d_3, r_3) \to SCPTP(d_3, r_3)$$

is a QSC, and thus has a completely positive induced map. For any CP map $\phi$ we have

$$\Gamma \otimes \mathrm{id}_{d_3,r_3}(\phi) = (R_2^{-1} \otimes R_3^{-1})(\widetilde{\Gamma} \otimes \widetilde{\mathrm{id}_{d_3,r_3}})(C_\phi)$$

and this is a CP map since $\widetilde{\Gamma}$ and $\widetilde{\mathrm{id}_{d_3,r_3}}$ are both completely positive so their tensor is a positive map. □

Thus the second property in the definition of QSC can be replaced by the requirement that QSCs be completely positive.

**Remark 3.2.4.** A tensor product of two QSCs,

$$\Gamma_1 \otimes \Gamma_2 : SCPTP(d_1, r_1) \otimes SCPTP(d_3, r_3) \to SCPTP(d_2, r_2) \otimes SCPTP(d_4, r_4)$$

will send CPTP maps in the domain to CPTP maps in the range, but it is not a QSC as its domain is not $SCPTP(d_1 d_3, r_1 r_3)$.

**Remark 3.2.5** (QSC vs quantum superchannel)**.** Superchannels and QSCs are defined in similar ways, although on a different space of maps. In [3] and [16] the definition of superchannel used the space of all linear maps as its domain and range. Since superchannels use the whole vector space of linear maps, tools such as the Choi matrix can be applied to them. This is not the case for QSCs since the space spanned by Choi matrices of quantum channels doesn't contain the standard matrix units, so the Choi matrix cannot be defined. Similarly, Stinespring's theorem and Kraus representations of CP maps don't apply to maps on an operator system, but only to maps defined on the full $C^*$-algebra.

The next theorem allows us to extend QSCs and treat them as restrictions of superchannels.

**Theorem 3.2.6.** *Every QSC extends to a quantum superchannel.*

*Proof.* Let $\Gamma$ be a QSC. Recall Arveson's extension theorem, 1.1.11 about completely-positive maps on operator systems. Since $\widetilde{\Gamma}$ is CP it has a CP extension with domain all of $M_{d_1}(M_{r_1})$. Call this extension $\widetilde{S}$.

Define the matrix order on $\mathcal{L}(M_d, M_r)$ in the same way as for $SCPTP(d, r)$ and a similar proof to Theorem 3.1.2 shows the Choi isomorphism $\phi \mapsto C_\phi$ is also a complete order isomorphism between $\mathcal{L}(M_d, M_r)$ and $M_d(M_r)$. Thus $\widetilde{S}$ corresponds to a map $S$ which is an extension of $\Gamma$. We will show $S$ is a quantum superchannel.

Any TP map $f \in \mathcal{L}(M_{d_1}, M_{r_1})$ will have a Choi matrix that has trace one on the diagonal blocks and trace zero on the off diagonal blocks. Thus $C_f \in S(d_1, r_1)$ and so we can write $f$ as a linear combination of CPTP maps. Using the linearity of $S$ we can see that $S(f)$ is a TP map.

To see it is completely CP preserving, take a matrix of CP maps $(\phi_{i,j})$. Then $(C_{\phi_{i,j}})$ is a matrix of positive matrices and since $\widetilde{S}$ is completely positive, $\widetilde{S}^{(n)}$ maps it to another matrix of positive matrices. □

**Remark 3.2.7.** The extension of a QSC is not unique. For example, let $d_1 = 2$, let $r_1$ be arbitrary size, and let $d_2 = r_2 = 1$. Define $\widetilde{\Gamma}_1, \widetilde{\Gamma}_2 : M_2(M_{r_1}) \to M_1(M_1)$ via

$$\begin{aligned} \widetilde{\Gamma}_1 \left( \begin{pmatrix} \phi(E_{11}) & \phi(E_{12}) \\ \phi(E_{21}) & \phi(E_{22}) \end{pmatrix} \right) &= \mathrm{Tr}(\phi(E_{11})), \\ \widetilde{\Gamma}_2 \left( \begin{pmatrix} \phi(E_{11}) & \phi(E_{12}) \\ \phi(E_{21}) & \phi(E_{22}) \end{pmatrix} \right) &= \mathrm{Tr}(\phi(E_{22})). \end{aligned} \tag{3.1}$$

These are different maps in general but are identical when restricted to the space of quantum channels $S(2, r_1)$. They are easily seen to be linear maps which send CPTP maps to 1. To see that they are completely positive take $V_1 = \begin{pmatrix} I_{r_1} \\ 0 \end{pmatrix}$, $V_2 = \begin{pmatrix} 0 \\ I_{r_1} \end{pmatrix}$ then

$$\widetilde{\Gamma}_i(C_\phi) = \mathrm{Tr}(V_i^* C_\phi V_i).$$

**Remark 3.2.8.** Define the *depolarizing channel* $\delta_1 : M_{d_1} \to M_{r_1}$

$$\delta_1(\rho) = \frac{Tr(\rho)}{r_1} I_{r_1}$$

and similarly $\delta_2 : M_{d_2} \to M_{r_2}$. Then the Choi matrices are $C_{\delta_1} = \frac{1}{r_1} I_{d_1 r_1} = \frac{1}{r_1} C_{r_1 \delta_1}$. For $\widetilde{\Gamma}$ to be unital we require

$$\widetilde{\Gamma}(I_{d_1 r_1}) = \widetilde{\Gamma}(C_{r_1 \delta_1}) = I_{d_2 r_2} = C_{r_2 \delta_2}$$

but since $\widetilde{\Gamma}(C_{r_1 \delta_1}) = C_{\Gamma(r_1 \delta_1)}$ unital is equivalent to requiring $\Gamma(r_1 \delta_1) = r_2 \delta_2$. Thus the depolarizing channel is the *order unit* of the operator system $SCPTP(d, r)$, see [28].

### 3.2.1 QSC with no TP extension

A QSC is defined by a CP map $\widetilde{\Gamma} : S(d_1, r_1) \to S(d_2, r_2)$ which sends block matrices of trace $\lambda d_1$ to block matrices of trace $\lambda d_2$ where $\lambda$ is the trace scaling factor of the linear map associated with the block matrix (with $\lambda = 1$ for CPTP maps and their Choi matrix). Thus the map

$$\frac{d_1}{d_2} \widetilde{\Gamma}$$

is a CPTP map. If we extend $\widetilde{\Gamma}$ to a superchannel $\widetilde{S} : M_{d_1}(M_{r_1}) \to M_{d_2}(M_{r_2})$ then in general it is not the case that $\widetilde{S}$ is TP.

Consider a map $M_2(M_2) \to M_2(M_2)$ defined by

$$\begin{aligned} E_{11} &\mapsto \text{Diag}(a_1, a_2, a_3, a_4) = A \\ E_{22} &\mapsto \text{Diag}(b_1, b_2, b_3, b_4) = B \\ E_{33} &\mapsto \text{Diag}(c_1, c_2, c_3, c_4) = C \\ E_{44} &\mapsto \text{Diag}(d_1, d_2, d_3, d_4) = D \end{aligned}$$

and all other standard basis matrices get sent to 0.

Since $E_{11} + E_{33}$, $E_{11} + E_{44}$, $E_{22} + E_{33}$, and $E_{22} + E_{44}$ are in $S(2, 2)$ for this map to restrict to give a QSC we require $A + C$, $A + D$, $B + C$, and $B + D$ to be in $S(2, 2)$ and have the same trace (since $\frac{d_1}{d_2} = 1$) i.e. they must have trace 2 and both diagonal blocks each have trace 1. In other words,

$$\begin{aligned} a_1 + c_1 + a_2 + c_2 &= 1 \\ a_3 + c_3 + a_4 + c_4 &= 1 \\ a_1 + d_1 + a_2 + d_2 &= 1 \\ a_3 + d_3 + a_4 + d_4 &= 1 \end{aligned}$$

and similarly with $b_i$ replacing $a_i$. This implies $a_1 + a_2 = b_1 + b_2$ and $a_3 + a_4 = b_3 + b_4$.

For this to be a trace-preserving map we require $\sum_i a_i = \sum_i b_i = \sum_i c_i = \sum_i d_i = 1$.

So for a particular choice of $A, B, C, D$ which give a QSC with no TP extension consider

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This QSC is defined by the matrices $A+C$ and $B+C$. Any choice of $a_i$ and $c_i$ must satisfy $a_1 + c_1 = \frac{1}{2}$, $a_2 + c_2 = \frac{1}{2}$, $a_3 + c_3 = 1$, and $a_4 + c_4 = 0$ to be the same QSC.

However for an extension to be a positive map we require all $a_i, b_i, c_i, d_i$, $1 \leq i \leq 4$ to be non-negative. Thus $b_3 + c_3 = 0 \implies b_3 = c_3 = 0$ but since $a_3 + c_3 = 1$ we conclude $a_3 = 1$ in any extension. Similarly since $b_2 + c_2 = 0 \implies c_2 = 0$ but then $a_2 + c_2 = \frac{1}{2} \implies a_2 = \frac{1}{2}$ in any extension. Already we have $a_2 + a_3 > 1$ so it cannot be TP.

### 3.2.2 Tensoring QSCs

Take two QSCs,

$$\Gamma_1 : SCPTP(d_1, r_1) \to SCPTP(d_2, r_2)$$

and

$$\Gamma_2 : SCPTP(d_3, r_3) \to SCPTP(d_4, r_4).$$

Extend each to a superchannel $S_1$, $S_2$ respectively. Then $S_1 \otimes S_2$ is a superchannel on the combined spaces and it restricts to give a QSC on $SCPTP(d_1 d_2, r_1 r_2)$. This is not necessarily unique.

To see an example of this note that $SCPTP(1, 2)$ is a 4-dimensional space and it contains the following four maps between $\mathbb{C} \to M_2$

$$L_{1,1} : 1 \mapsto E_{1,1}$$
$$L_{1,2} : 1 \mapsto E_{1,2}$$
$$L_{2,1} : 1 \mapsto E_{2,1}$$
$$L_{2,2} : 1 \mapsto E_{2,2}$$

Now, $SCPTP(2, 1)$ is a 1-dimensional space containing $I_2 \mapsto 1$. Define other maps from $M_2$ to $\mathbb{C}$ by the following Choi matrices in $M_2(M_1)$

$$C_{\phi_{1,1}} = E_{1,1}$$
$$C_{\phi_{1,2}} = E_{1,2}$$
$$C_{\phi_{2,1}} = E_{2,1}$$
$$C_{\phi_{2,2}} = E_{2,2}$$

so for example $\phi_{1,1}$ sends $E_{1,1}$ to 1 and everything else to 0. None of these are in the space of quantum channels $S(2,1)$.

Note that $\phi_{i,i} \otimes L_{i,i}(E_{k,k} \otimes 1) = \delta_{i,k} E_{i,i}$ and thus $\phi_{i,i} \otimes L_{i,i}$ has Choi matrix $E_{i,i} \otimes E_{i,i}$. Thus the Choi matrix of $\sum_i \phi_{i,i} \otimes L_{i,i}$ is the maximally entangled state

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \in M_2(M_2)$$

which is in the space of quantum channels $S(2,2)$.

We can use this to give an example of how tensoring QSCs depends on their extension. Define the following two superchannels $S_1, S_2 : \mathcal{L}(M_2, M_1) \to \mathcal{L}(M_2, M_1)$ by

$$\begin{aligned} S_1(\phi_{1,1}) &= \phi_{1,1} \\ S_1(\phi_{2,2}) &= \phi_{2,2} \\ S_1(\phi_{1,2}) = S_1(\phi_{2,1}) &= 0 \end{aligned}$$

and let $S_2 = \mathrm{id}_{2,1}$ the identity superchannel. Then these are two different extensions of the same QSC. Taking also the identity superchannel $\mathrm{id}_{1,2}$ on $\mathcal{L}(M_1, M_2)$ we have that the maps $S_1 \otimes \mathrm{id}_{1,2}$ and $S_2 \otimes \mathrm{id}_{1,2}$ have a different effect on the quantum channel $\sum_i \phi_{i,i} \otimes L_{i,i}$.

This shows that tensoring QSCs by extending them to superchannels gives you a different QSC on the larger space depending on how you extend them. Thus we can tensor two QSCs together and not fully define how they should act on the full space of quantum channels.

**Remark 3.2.9.** We have to be careful when tensoring Choi matrices and interpreting the result as the Choi matrix of a channel. For the induced map on Choi matrices $\widetilde{S_1 \otimes S_2} \neq \widetilde{S}_1 \otimes \widetilde{S}_2$, since

$$C_{\phi_1} \otimes C_{\phi_2} = \sum_{ij} E_{i,j} \otimes \phi_1(E_{i,j}) \otimes \sum_{k,l} E_{k,l} \otimes \phi_2(E_{k,l})$$

is not the same as

$$C_{\phi_1 \otimes \phi_2} = \sum_{i,j,k,l} E_{i,j} \otimes E_{k,l} \otimes \phi_1(E_{i,j}) \otimes \phi_2(E_{k,l}).$$

It's possible to create the Choi matrix of a channel from tensoring the Choi matrices of non channels if you ignore this fact. But in general if a simple tensor of maps gives a quantum channel then the individual maps should be quantum channels (or scalar multiples of channels). More specifically, if $\phi_1 \otimes \phi_2$ is a trace-preserving map for some linear $\phi_1$ and $\phi_2$ then for any inputs $X$ and $Y$

$$\begin{aligned}\operatorname{Tr}(X) \cdot \operatorname{Tr}(Y) = \operatorname{Tr}(X \otimes Y) &= \operatorname{Tr}(\phi_1 \otimes \phi_2(X \otimes Y)) \\ &= \operatorname{Tr}(\phi_1(X)) \operatorname{Tr}(\phi_2(Y)).\end{aligned}$$

Taking basis matrix units $E_{i,i}$ we then have $1 = \operatorname{Tr}(\phi_1(E_{i,i})) \cdot \operatorname{Tr}(\phi_2(E_{j,j}))$. Letting $i$ vary, it's clear that $\phi_1$ must multiply the trace by a constant scaling factor, and similarly for $\phi_2$.

**Remark 3.2.10.** Superchannels are determined by their action on channels on larger dimensional spaces. If $S_1$ and $S_2$ define the same QSC and also define the define the same QSC when tensored with the identity superchannel, they are the same map. That is, if in addition to being equal on channels, if $S_1 \otimes \mathrm{id}_{n,n}(\phi) = S_2 \otimes \mathrm{id}_{n,n}(\phi)$ for all channels $\phi$, for all $n$, then $S_1 = S_2$.

To see this, note that we only need to show $S_1$ and $S_2$ are equal on CP trace non-increasing maps (quantum operations) as these span the space. So let $\phi$ be such a map. There is another operation $\phi'$ such that $\phi + \phi'$ is a channel, and thus $S_1(\phi + \phi') = S_2(\phi + \phi')$. Now take two distinct channels $\psi_1, \psi_2 \in \mathcal{L}(M_n, M_n)$ and we see that $\phi \otimes \psi_1 + \phi' \otimes \psi_2$ is a CPTP map. This implies $S_1(\phi) \otimes \psi_1 + S_1(\phi') \otimes \psi_2 = S_2(\phi) \otimes \psi_1 + S_2(\phi') \otimes \psi_2$. Combining these two equations we have

$$(S_1(\phi) - S_2(\phi)) \otimes \psi_1 = (S_1(\phi) - S_2(\phi)) \otimes \psi_2.$$

Hence, $S_1(\phi) = S_2(\phi)$.

### 3.2.3 Choi matrix of equivalent extensions

Suppose $S_1$ and $S_2$ are superchannels. Note that since $C_{\widetilde{S_1}} - C_{\widetilde{S_2}} = C_{\widetilde{S}_1 - \widetilde{S_2}}$ we have by Theorem 2.2.3

$$\operatorname{Tr}_{r_2} C_{\widetilde{S}_1 - \widetilde{S}_2} = \operatorname{Tr}_{r_1 r_2}[C_{\widetilde{S_2} - \widetilde{S_2}}] \otimes \frac{1}{r_1} I_{r_1}.$$

If we follow some of the same steps in the proof of this theorem, we can see

$$\begin{aligned}(S_1 - S_2)(\phi)(\rho) &= \operatorname{Tr}_{d_2}[C_{(\widetilde{S}_1 - \widetilde{S}_2)(\phi)}(\rho^T \otimes I_{r_2})] \\ &= \operatorname{Tr}_{d_1 r_1 d_2}[C_{\widetilde{S}_1 - \widetilde{S}_2}(C_\phi^T \otimes \rho^T \otimes I_{r_2}]\end{aligned}$$

for all $\rho \in M_d$. Now suppose $S_1$ and $S_2$ are extensions of the same QSC i.e. they agree on the space if channels. Then if $\phi$ is trace-preserving, this equation is 0. Thus,

$$\operatorname{Tr}[\operatorname{Tr}_{r_2}(C_{\widetilde{S}_1-\widetilde{S}_2})(C_\phi^T \otimes \rho)] = 0$$

As this holds for all matrices with $\operatorname{Tr}_{r_1} C_\phi = I_{d_1}$ and $\rho \in M_{d_2}$ we can conclude that the $M_{d_1}$ component of $C_{\widetilde{S}_1-\widetilde{S}_2}$ has trace 0. We have thus shown:

**Theorem 3.2.11.** *If $S_1$ and $S_2$ are superchannels extending the same QSC then*

$$\operatorname{Tr}_{d_1} C_{\widetilde{S}_1-\widetilde{S}_2} = 0$$

We can illustrate this with an example.

**Example 3.2.12.** Consider superchannels $S_1$, $S_2$, acting on $M_2(M_2) \to M_2(M_2)$, so $16 \times 16$ Choi matrices. For $i = 1, 2$ write their Choi matrices as

$$C_{\widetilde{S}_i} = \begin{bmatrix} A_i & * & & \\ * & B_i & & \\ & & C_i & * \\ & & * & D_i \end{bmatrix} \in M_2(M_2) \otimes M_2(M_2).$$

Here $A_i, B_i, C_i, D_i \in M_2(M_2)$. As in section 3.2.1, since $E_{1,1} + E_{3,3}$ and $E_{2,2} + E_{4,4}$ represent channels in $M_2(M_2)$, $S_1$ and $S_2$ must agree on these matrices. Thus $A_1 + C_1 = A_2 + C_2$, and $B_1 + D_1 = B_2 + D_2$. They must also agree on the $*$ entries, and we can ignore what the blank parts of their Choi matrices look like. Taking the partial trace:

$$\begin{aligned} \operatorname{Tr}_{d_1} C_{\widetilde{S}_1-\widetilde{S}_2} &= \operatorname{Tr}_{d_1} \begin{bmatrix} A_1 - A_2 & 0 & & \\ 0 & B_1 - B_2 & & \\ & & C_1 - C_2 & 0 \\ & & 0 & D_1 - D_2 \end{bmatrix} \\ &= \begin{bmatrix} A_1 - A_2 + C_1 - C_2 & 0 \\ 0 & B_1 - B_2 + D_1 - D_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

## 3.3 Superchannel dilation for different extensions

Recall the proof of the characterisation theorem for superchannels, Theorem 2.2.1. One can show that any superchannel $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ has an associated unital CP map $\mathcal{N} : M_{d_1} \to M_{d_2}$ such that

$$\operatorname{Tr}_{r_2}(\widetilde{S}) = \mathcal{N} \circ \operatorname{Tr}_{r_1} . \tag{3.2}$$

For a QSC $\Gamma_1 : SCPTP(d_1, r_1) \to SCPTP(d_2, r_2)$ we have for any unital map $\mathcal{N} : M_{d_1} \to M_{d_2}$

$$\operatorname{Tr}_{r_2} \widetilde{\Gamma}(C_\phi) = \mathcal{N}(\operatorname{Tr}_{r_1} C_\phi)$$

however since the Stinespring dilation theorem only applies to maps defined on a full $C^*$-algebra, not operator systems, we cannot use this to get two equivalent sets of Kraus operators. Similarly, the Choi matrix requires the map to act on the standard basis elements in $M_n$ which a QSC does not have access to. Hence, we cannot apply the same proof of the characterisation theorem for superchannels to QSCs.

The extension in Theorem 3.2.6 shows that, by extending a QSC to a superchannel we can still describe it as

$$\Gamma(\phi) = \psi_{post} \circ (\phi \otimes \mathrm{id}_e) \circ \psi_{pre}, \quad \forall \phi \in SCPTP(d, r).$$

Since extending to superchannels is non-unique, these pre and post processing channels will depend on which superchannel it is extended to, even when requiring a minimal dilation.

The minimal dimension $e$ from Theorem 2.2.13 will depend on the extension of the QSC. Consider the non unique extensions from Equation (3.1) and set $r_1 = 2$, the superchannel Choi matrices are

$$C_{\widetilde{\Gamma}_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = E_{11} + E_{22},$$

$$C_{\widetilde{\Gamma}_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{33} + E_{44}.$$

In $M_{d_1}(M_{r_1}) = M_2(M_2)$, we have $\operatorname{Tr}_{r_1} E_{11} = E_{11} \in M_2$, $\operatorname{Tr}_{r_1} E_{22} = E_{11} \in M_2$, etc. Thus

$$\operatorname{Tr}_{r_1} \operatorname{Tr}_{r_2} C_{\widetilde{\Gamma}_1} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},$$
$$\operatorname{Tr}_{r_1} \operatorname{Tr}_{r_2} C_{\widetilde{\Gamma}_2} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},$$

which have equal rank. However, another equivalent extension is given by any convex combination $\widetilde{\Gamma} = p_1\widetilde{\Gamma}_1 + p_2\widetilde{\Gamma}_2$ for $p_1, p_2 > 0$, $p_1 + p_2 = 1$. This has Choi matrix

$$C_{\widetilde{\Gamma}} = \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_2 \end{pmatrix},$$

which reduces to

$$\operatorname{Tr}_{r_1} \operatorname{Tr}_{r_2} C_{\widetilde{\Gamma}} = \begin{pmatrix} 2p_1 & 0 \\ 0 & 2p_2 \end{pmatrix},$$

and this has greater rank.

**Remark 3.3.1.** The set of possible superchannel extensions of a QSC is a convex set. In the example given, the extensions with minimal $e$ are extreme points. A natural question to ask is whether it is generally true that the extensions which give minimal dimensions $e$ are extreme points.

Define $CP[M_n, M_m; K]$ to be CP maps from $M_n$ to $M_m$ which send the identity to a fixed $K \geq 0$. This is a convex set. The following theorem from [9] characterises the extreme points in terms of the Kraus operators:

**Theorem 3.3.2.** *A map $\phi \in CP[M_n, M_m; K]$ is extreme if and only if it admits an expression $\phi(A) = \sum_i V_i^* A V_i$ for all $A \in M_n$ such that $\sum_i V_i^* V_i = K$ and $\{V_i^* V_j\}_{ij}$ is a linearly independent set.*

Using the same proof it was noted in [24] that for the set of unital, trace-preserving CP maps $\phi$ is an extreme point if and only if it has Kraus operators $\{V_i\}_i$ such that $\{V_i^* V_j \bigoplus V_j V_i^*\}_{ij}$ is linearly independent.

For a CP map $\phi : M_n \to M_m$ with Kraus representation $\phi(A) = \sum_i V_i^* A V_i$ the (Hilbert-Schmidt) dual map $\phi^* : M_m \to M_n$ is given by $\phi^*(B) = \sum_i V_i B V_i^*$.

**Definition 3.3.3.** Let $\mathcal{S}$ be a subspace of $M_n$, and let $\mathcal{T}$ be a subspace of $M_m$. For a CP map $\Phi : M_n \to M_m$ define the convex set $CP[M_n, M_m; \mathcal{S}, \mathcal{T}, \Phi]$ to be CP maps from $M_n$ to $M_m$ which are equal to $\Phi$ on $\mathcal{S}$ and whose duals are equal to the dual of $\Phi$ on $\mathcal{T}$.

The proof of the following makes use of the same approaches as the proof of Theorem 3.3.2 from [9]. Namely, it uses the fact that a CP map has a minimal set of Kraus operators such that they are linearly independent and any other set can be related to it via an isometry.

Here we assume $\mathcal{S}$ and $\mathcal{T}$ have self-adjoint spanning sets i.e. are self-adjoint spaces.

**Theorem 3.3.4.** *A map $\phi \in CP[M_n, M_m; \mathcal{S}, \mathcal{T}, \Phi]$ is extreme if and only if it admits an expression $\phi(A) = \sum_i V_i^* A V_i$ for all $A \in M_n$ such that for any self-adjoint spanning sets $\{A_k\}_k$ for $\mathcal{S}$ and $\{B_l\}_l$ for $\mathcal{T}$ the set*

$$\{\bigoplus_k V_i^* A_k V_j \bigoplus_l V_j B_l V_i^*\}_{ij}$$

*is linearly independent.*

*Proof.* For the forward direction, suppose $\phi \in CP[M_n, M_m; \mathcal{S}, \mathcal{T}, \Phi]$ is extreme and take a minimal set of Kraus operators i.e. a linearly independent set $\{V_i\}_i$ with $\phi(A) = \sum_i V_i^* A V_i$. Choose self-adjoint spanning sets $\{A_k\}_k$ for $\mathcal{S}$ and $\{B_l\}_l$ for $\mathcal{T}$. Suppose there exist constants $\{\lambda_{ij}\}_{ij}$ such that

$$\sum_{ij} \lambda_{ij} \bigoplus_k V_i^* A_k V_j \bigoplus_l V_j B_l V_i^* = 0.$$

By taking the adjoint of this sum we see that $\{\overline{\lambda}_{ji}\}_{ij}$ is another set satisfying this. This implies $\{\lambda_{ij} \pm \overline{\lambda}_{ji}\}_{ij}$ do as well and if we show both these sets are the zero set then it will imply $\{\lambda_{ij}\}_{ij} = \{0\}$. Thus we may assume $(\lambda_{ij})_{ij}$ is a self-adjoint matrix. Also scale so that $-I \leq (\lambda_{ij})_{ij} \leq I$.

Define maps $\Psi_\pm : M_n \to M_m$ via

$$\Psi_\pm(A) = \sum_i V_i^* A V_i \pm \sum_{ij} \lambda_{ij} V_i^* A V_j.$$

Let $I + (\lambda_{ij})_{ij} = (\alpha_{ij})^*(\alpha_{ij})$ so that $\sum_k \overline{\alpha}_{ki}\alpha_{kj} = \lambda_{ij} + \delta_{ij}1$. Then if $W_i = \sum_i \alpha_{ij}V_j$ we can compute to get $\Psi_+ = \sum_i W_i^* A W_i$ and we can do similar for $\Psi_-$. This also shows that

$$\Psi_\pm^*(B) = \sum_i V_i B V_i^* \pm \sum_{ij} \lambda_{ij} V_j B V_i^*.$$

Thus $\Psi_\pm$ are in $CP[M_n, M_m; \mathcal{S}, \mathcal{T}, \phi]$.

We now have $\phi = \frac{1}{2}(\Psi_+ + \Psi_-)$ and so since it is extreme $\phi = \Psi_+$, say. The minimality of the set $\{V_i\}$ implies that $(\alpha_{ij})_{ij}$ is an isometry which gives $(\lambda_{ij})_{ij} = 0$ and we are done.

Conversely, assume $\phi$ has form $\phi(A) = \sum_i V_i^* A V_i$ and

$$\{\bigoplus_k V_i^* A_k V_j \bigoplus_l V_j B_l V_i^*\}_{ij}$$

is linearly independent for any self-adjoint spanning sets $\{A_k\}_k$ for $\mathcal{S}$ and $\{B_l\}_l$ for $\mathcal{T}$. This implies $\{V_i\}_i$ is linearly independent since $\sum_i \lambda_i V_i = 0$ would imply for any arbitrary summand that $\sum_{ij} \lambda_i V_j^* C V_i = 0$.

If $\phi$ is not extreme, say $\phi = \frac{1}{2}(\Psi_1 + \Psi_2)$ for $\Psi_1(A) = \sum_p W_p^* A W_p$ and $\Psi_2(A) = \sum_q Z_q^* A Z_q$, then we can write $W_p$ and $Z_q$ in terms of $V_i$. But if $W_p = \sum_i \alpha_{pi} V_i$ we have

$$\sum_i V_i^* A_k V_i = \sum_p W_p^* A_k W_p = \sum_{pij} \overline{\alpha}_{pi}\alpha_{pj} V_i^* A_k V_j$$

for any $A_k$ in the spanning set. Similarly for the dual we have

$$\sum_i V_i B_k V_i^* = \sum_{pij} \alpha_{pj}\overline{\alpha}_{pi} V_j B_k V_i^*.$$

Therefore $\sum_p \overline{\alpha}_{pi}\alpha_{pj} = \delta_{ij}$ or else we would have a linear dependency. This implies $(\alpha_{pi})_{pi}$ is an isometry and so the Kraus operators it relates define the same map; i.e., $\phi = \Psi_1$ and so $\phi$ is extreme. □

**Remark 3.3.5.** This last theorem immediately applies to the set of superchannels $\widetilde{S} : M_{d_1}(M_{r_1}) \to M_{d_2}(M_{r_2})$ which are extensions of the same QSC; i.e., are equal on the space $\mathcal{S} = S(d_1, r_1)$. A spanning set consisting of Choi matrices of quantum channels may be chosen. In this case the space $\mathcal{T}$ may be chosen to be zero. For a trace-preserving superchannel we can consider the set of TP extensions of the underlying

QSC, these are the ones with $\mathcal{T} = \text{span}\{I_{d_2r_2}\}$ being sent to $\text{span}\{I_{d_1r_1}\}$ (since $\phi$ being trace-preserving is equivalent to $\phi^*$ being unital). As shown for some QSCs this set of extensions is empty.

**Example 3.3.6** (Unitary superchannels)**.** If $U_1 \in M_{d_1}$ and $U_2 \in M_{r_1}$ are unitaries then it's easy to see conjugation by $U_1 \otimes U_2$ is a superchannel since if $\phi$ is a TP map, then

$$\begin{aligned}\text{Tr}_{r_2}[U_1 \otimes U_2(C_\phi)(U_1 \otimes U_2)^*] &= \text{Tr}_{r_2}[U_1 \otimes U_2(\sum_{ij} E_{ij} \otimes \phi(E_{ij}))U_1^* \otimes U_2^*] \\ &= \sum_{ij}(U_1 E_{ij} U_1^*) \cdot \text{Tr}(U_2 \phi(E_{ij}) U_2^*) \\ &= U_1(\sum_i E_{ii})U_1^* = I_{d_2},\end{aligned}$$

so it satisfies the TP-preserving condition.

In fact every unitary superchannel is of this form. Let $\mathcal{U}(n)$ denote the unitaries in $M_n$.

**Theorem 3.3.7.** *If $U \in \mathcal{U}(dr)$ is a unitary such that the map $\widetilde{S} : M_d(M_r) \to M_d(M_r)$ with $\widetilde{S}(C) = UCU^*$ is a superchannel then there exists unitaries $U_1 \in \mathcal{U}(d)$ and $U_2 \in \mathcal{U}(r)$ such that $U = U_1 \otimes U_2$.*

We delay the proof of this until Chapter 5, where we also prove it in the case of anti-unitary maps preserving channels, see Theorem 5.1.4.

By Theorem 3.3.4 any unitary superchannel $\widetilde{S}$ is an extreme point of the set of extensions of the underlying QSC. They will also always have minimal dimension $e$ for the characterisation theorem since the rank of $\text{Tr}_{r_1} \text{Tr}_{r_2} C_{\widetilde{S}}$ will be 1. To see this write the matrix units of $M_d(M_r)$ as $E_{ij} \otimes F_{kl}$, $1 \le i, j \le d$, $1 \le k, l \le r$, where $E_{ij} \in M_d$ and $F_{kl} \in M_r$ are the standard matrix units in their spaces. Then since $S(C) = U_1 \otimes U_2 C U_1^* \otimes U_2^*$ for $U_1 \in \mathcal{U}(d)$, $U_2 \in \mathcal{U}(r)$ we have

$$C_{\widetilde{S}} = \sum_{i,j} \sum_{k,l} E_{i,j} \otimes F_{k,l} \otimes U_1 E_{i,j} U_1^* \otimes U_2 F_{k,l} U_2^*.$$

Now applying $\text{Tr}_{r_1} \text{Tr}_{r_2}$ traces out the 2nd and 4th term giving

$$\begin{aligned}\text{Tr}_{r_1} \text{Tr}_{r_2} C_{\widetilde{S}} &= r \cdot \sum_{i,j} E_{i,j} \otimes U_1 E_{i,j} U_1^* \\ &= \text{Diag}(U_1, \dots, U_1) \sum_{i,j} E_{i,j} \otimes E_{i,j} \text{Diag}(U_1^*, \dots, U_1^*).\end{aligned}$$

Since $\mathrm{Diag}(U_1, \dots, U_1)$ has full rank and $\sum_{i,j} E_{i,j} \otimes E_{i,j}$ has rank 1 the overall matrix has rank 1.

### 3.3.1 Extensions given by UCP maps

An interesting space of superchannels to study are those with $r_1 = r_2 = 1$. In this case the dimension of $S(d, 1)$ is 1, and in particular the only Choi matrix corresponding to a channel is the identity matrix. Thus any unital completely-positive (UCP) map $\widetilde{S} : M_{d_1} \to M_{d_2}$ is a superchannel and all these maps define the same QSC.

The minimal dimension of the superchannel characterisation of such maps is given by the rank of the Choi matrix. This allows us to give a nice example of a superchannel, which is an extreme point of the set of extensions of its QSC, but which does not have minimal dimension $e$.

Fix $d_1 = d_2 = 3$. The anti-symmetric *Werner-Holevo* channel is given by the map $\phi : M_3 \to M_3$

$$\phi(\rho) = \frac{\mathrm{Tr}[\rho] I_3 - \rho^T}{2}.$$

This has Kraus operators

$$K_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} (|e_1\rangle\langle e_2| - |e_2\rangle\langle e_1|)$$
$$K_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} (|e_1\rangle\langle e_3| - |e_3\rangle\langle e_1|)$$
$$K_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} (|e_3\rangle\langle e_2| - |e_2\rangle\langle e_3|)$$

Ignoring the scaling factor of 2, the pairs $K_i K_j^*$ are:

$$K_1 K_1^* = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|, \quad K_2 K_2^* = |e_1\rangle\langle e_1| + |e_3\rangle\langle e_3|,$$
$$K_3 K_3^* = |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3|, \quad K_1 K_2^* = |e_2\rangle\langle e_3|,$$
$$K_1 K_3^* = |e_1\rangle\langle e_3|, \quad K_2 K_3^* = -|e_1\rangle\langle e_2|, \quad K_2 K_1^* = |e_3\rangle\langle e_2|,$$
$$K_3 K_1^* = |e_3\rangle\langle e_1|, \quad K_3 K_2^* = -|e_2\rangle\langle e_1|.$$

The set $\{K_i K_j^*\}_{i,j}$ is thus seen to be linearly independent. By Theorem 3.3.4 the channel $\phi$ is an extreme point of the set of extensions. However the rank of its Choi matrix is the number of Kraus operators, 3. Since any unitary map has Choi rank 1 this cannot be minimal.

## 3.4 Conclusion

Our results show that defining superchannels to act on the space of quantum channels gives a different class of maps in comparison to the original definition of superchannels. The standard definition of superchannel can be recovered by extending to the full set of linear maps and this extension is not unique. Therefore many different quantum superchannels can restrict to the same QSC, which means they are effectively the same as maps on channels. Thus, if we are really only concerned with the action of a superchannel on quantum channels, then we are really only concerned with the corresponding QSC.

However, as shown in Remark 3.2.10, we can make sense of what a superchannel does outside the space of channels by defining its action on channels on larger spaces. Since the action of $S$ and $S \otimes \mathrm{id}_{n,n}$ on channels uniquely determines what $S$ does to all maps in $\mathcal{L}(M_d, M_r)$.

It would be interesting to see if there is a "best" choice of extension. For example it may be that the minimal dimension $e$ for superchannel characterisation occurs for extreme points of the set of extensions, but this was not proved here and is an open question. We were able to show that extreme points do not always give this minimal dimension. It was also shown that TP extensions are not always available. It is unclear what the restrictions are on the choice of extension.

Not much is known about the operator system of quantum channels. It might be worth considering how the action of a map on this space determines the form of its possible Stinespring representations, and whether this affects the characterisation of superchannels.

# Chapter 4

# Types of superchannels

In this chapter we discuss several types of superchannels: entanglement breaking superchannels, dephasing superchannels, coherence breaking superchannels, and stabilizer-preserving superchannels and how they relate to their underlying QSC.

Many of these types of superchannels are motivated by their use in describing resource theories. It is thus interesting to note that they generally are not uniquely defined by how they transform quantum channels but by their action on the whole space of linear maps. The first type, Schur product maps is an exception to this. The extension of a QSC to a Schur product map is unique.

## 4.1 Schur product superchannels

In this section we discuss superchannels given by the Schur product map, and show they are unique extensions of QSC's. We then use them to describe the form of some mixed unitary superchannels, inspired by work on factorizable channels.

**Definition 4.1.1.** Given two matrices $A = (a_{ij})$, $B = (b_{ij}) \in M_d$ the *Schur product* is the matrix

$$A \circ B = (a_{ij} b_{ij}).$$

For a matrix $C \in M_d$ the *Schur product map* is denoted

$$\begin{aligned} S_C : M_d &\to M_d \\ S_C(\rho) &= C \circ \rho \end{aligned}$$

**Theorem 4.1.2.** *$S_C$ is completely positive if and only if $C \geq 0$.*

*Proof.* If $S_C$ is completely positive then since the matrix with all entries equal to 1 is positive, we must have $C$ is positive. For the other direction, assume $C \geq 0$. The Choi matrix of $S_C$ is

$$C_{S_C} = \sum_{i,j} c_{i,j} E_{i,j} \otimes E_{i,j}$$

which is a self-adjoint matrix since $C$ is. It also has the same non-zero eigenvalues as $C$ since it sends all $|e_k\rangle \otimes |e_l\rangle$ to 0 for $k \neq l$, and on the remaining subspace it acts identically to $C$

$$C_{S_C}(|e_k\rangle \otimes |e_k\rangle) = \sum_i c_{i,k} |e_i\rangle \otimes |e_i\rangle$$

$$C|e_k\rangle = \sum_i c_{i,k} |e_i\rangle$$

Hence by Choi's theorem $S_C$ is completely positive. □

**Definition 4.1.3.** A *correlation matrix* is a positive matrix $C = (c_{i,j})$, with $c_{i,i} = 1$ for all $i$.

**Corollary 4.1.4.** *$S_C$ is a quantum channel if and only if $C$ is a correlation matrix.*

*Proof.* The trace-preserving condition requires $I = \operatorname{Tr}_2 C_{S_C} = \sum_i c_{i,i} E_{i,i}$. Thus $c_{i,i} = 1$. □

**Definition 4.1.5.** Fix a basis $\{|e_i\rangle\}_{i=1}^d$. A quantum channel $\phi : M_d \to M_d$ is a *dephasing* channel if for all $i$ we have

$$\langle e_i|\phi(\rho)|e_i\rangle = \langle e_i|\rho|e_i\rangle$$

for all states $\rho$.

Since they don't affect the diagonal elements of density matrices, which represents the classical information of a state, dephasing channels can be interpreted as a form of purely quantum noise.

**Example 4.1.6.** The phase flip channel $\phi : M_2 \to M_2$ is given by

$$\phi(\rho) = (1-p)\rho + pZ\rho Z$$

where $p \in (0,1)$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. For $p = \frac{1}{2}$ it diagonalizes the state.

**Theorem 4.1.7.** *A channel $\phi : M_d \to M_d$ is a dephasing channel if and only if it is a Schur product channel.*

*Proof.* Schur product channels multiply the diagonal by 1 so one direction is clear. If $\phi$ preserves the diagonals of states, then the $(i,i)$-th entry of the $(j,j)$-th block of the Choi matrix is

$$(C_\phi)_{(j,j),(i,i)} = \langle e_i|\phi(E_{j,j})|e_i\rangle = \delta_{i,j}.$$

Thus the diagonals of the Choi matrix match that of a Schur product Choi matrix. For a positive matrix $A = (a_{i,j})$ the entries satisfy

$$|a_{i,j}|^2 \leq a_{ii}a_{jj}.$$

Therefore, most of the off diagonals of $C_\phi$ will be zero except possibly the $(i,j)$-th entry of the $(i,j)$-th block. This corresponds to a Schur product map. □

We are interested in superchannels given by Schur product maps on Choi matrices. Setting all dimensions equal, $d_1 = r_1 = d_2 = r_2 = d$, these are maps from $M_d(M_d)$ to itself.

**Theorem 4.1.8** ([30])**.** *A Schur product map $S_C : M_d(M_d) \to M_d(M_d)$ is a superchannel if and only if the matrix $C \in M_d(M_d)$ is a correlation matrix where every block has constant diagonal i.e. $C = ([C_{ij}])_{i,j=1}^d$ where for each $i, j$ the matrix $[C_{ij}] \in M_d$ has constant diagonal.*

*Proof.* Suppose $C$ has that form. If $\phi$ is a channel, then since $\mathrm{Tr}(\phi(E_{i,j})) = \delta_{i,j}$, and each matrix $C_{i,j}$ has constant diagonal, we have

$$\mathrm{Tr}_2(C \circ C_\phi) = \mathrm{Tr}_2(\sum_{i,j} E_{i,j} \otimes (C_{i,j} \otimes \phi(E_{i,j}))) = \sum_i E_{i,i} = I_d.$$

So $S_C$ preserves trace-preserving maps, and since it is completely positive it is a superchannel.

Conversely, if $S_C$ is a superchannel and $\phi$ a channel, the trace of the $(i,j)$-th block of $S_C(C_\phi)$ must be $\delta_{i,j}$:

$$\delta_{i,j} = \sum_k \langle e_i \otimes e_k|C \circ C_\phi|e_j \otimes e_k\rangle = \sum_k c_{(i,j),(k,k)}(C_\phi)_{(i,j),(k,k)}. \tag{4.1}$$

For each $k$, the matrix $I_d \otimes E_{k,k}$ is the Choi matrix of a channel. Inserting it into Equation (4.1) gives $1 = c_{(i,i),(k,k)}$ for all $i, k$. Thus $C$ must be a correlation matrix.

To see the other blocks of $C$ have constant diagonal, we need a channel whose Choi matrix has off diagonal elements. Let

$$dC_\phi = I_d \otimes I_d + E_{i,j} \otimes E_{k,k} + E_{j,i} E_{k,k} - E_{i,j} \otimes E_{l,l} - E_{j,i} \otimes E_{l,l}.$$

This satisfies the trace-preserving condition. To see it is positive let $|\psi\rangle = \sum_{i,k} a_{i,k} |e_i \otimes e_k\rangle$ and compute

$$\langle \psi | C_\phi | \psi \rangle = \bar{a}_{i,k} a_{j,k} + \bar{a}_{j,k} a_{i,k} - \bar{a}_{i,l} a_{j,l} - \bar{a}_{j,l} a_{i,l} + \sum_{i,k} |a_{i,k}|^2.$$

Using the identity $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1 \bar{z}_2)$ for $z_1, z_2 \in \mathbb{C}$ we can see that this is positive. Hence $C_\phi$ is the Choi matrix of a channel. Tracing the $(i,j)$-th block of $C \circ C_\phi$ gives

$$c_{(i,j),(k,k)} - c_{(i,j),(l,l)} = \delta_{i,j}$$

and thus each block of $C$ has constant diagonal. □

**Definition 4.1.9.** A superchannel $S : \mathcal{L}(M_d, M_d) \to \mathcal{L}(M_d, M_d)$ is a *dephasing superchannel* if it preserves the diagonals of Choi matrices of channels i.e for all channels $\phi$ and all $i, j \in \{1, \ldots, d\}$ it satisfies

$$\langle e_i | S(\phi)(E_{j,j}) | e_i \rangle = \langle e_i | \phi(E_{j,j}) | e_i \rangle$$

As with dephasing channels on states, the dephasing superchannel is designed to be a form of quantum noise, leaving untouched the classical; i.e., diagonal effects of any input channel.

**Theorem 4.1.10** ([30])**.** *A superchannel $S$ is a dephasing superchannel if and only if its action on Choi matrices is a Schur product map.*

*Proof.* Since the diagonal entries of correlation matrices are all 1 it is clear that Schur product superchannels will be dephasing.

Note that the Choi matrix of a Schur product superchannel is

$$C_{S_C} = \sum_{i,j,k,l} c_{(i,j),(k,l)} E_{(i,j),(k,l)} \otimes E_{(i,j),(k,l)} \in M_d \otimes M_d \otimes M_d \otimes M_d.$$

So the only non zero elements are the $((i,j),(k,l))$-th entries in the $((i,j),(k,l))$-th blocks. We will show that a dephasing superchannel has the same form of Choi matrix.

If $S$ is a superchannel (on Choi matrices) then the matrix

$$C_S^{(i,k)} := (I_{d^2} \otimes \langle e_i \otimes e_k|)C_S(I_{d^2} \otimes |e_i \otimes e_k\rangle) \in M_{d^2}$$

has as its $((i',j'),(k',l'))$-th entry the $((i,i),(k,k))$-th element of block $((i',j'),(k',l'))$. For a Schur product superchannel this is matrix is $E_{(i,i),(k,k)}$.

Using the reverse Choi formula we have

$$C_{S(\phi)} = \mathrm{Tr}_{d_1 r_1}[C_S(C_\phi^T \otimes I_{r_2})]$$

and if $S$ is dephasing then this gives

$$\begin{aligned} \langle e_i \otimes e_k|C_\phi|e_i \otimes e_k\rangle &= \mathrm{Tr}[I_{d_1 r_1} \otimes \langle e_i \otimes e_k|(C_S)I_{d_1 r_1} \otimes |e_i \otimes e_k\rangle C_\phi^T] \\ &= \mathrm{Tr}[C_S^{(i,k)} C_\phi^T] \end{aligned} \tag{4.2}$$

We can use this to show that the only non-zero entry of $C_S^{(i,k)}$ is the $((i,i),(k,k))$-th element. First, $C_S^{(i,k)}$ is positive, so it's diagonals entries are non-negative. Note that Choi matrices of the form

$$E_{(1,1),(i_1,i_1)} + E_{(2,2),(i_2,i_2)} + \ldots + E_{(d,d),(i_d,i_d)}$$

are positive, with constant trace down the diagonal blocks, and partial trace to give the identity. Thus they correspond to channels and can be inserted into Equation (4.2). By choosing such matrices without $E_{(i,i),(k,k)}$ we get all the other diagonal entries of $C_S^{(i,k)}$ summing to 0. So they are zero and hence

$$C_S^{(i,k)} = E_{(i,i),(k,k)}$$

as with Schur product channels.

This shows the diagonal of the Choi matrix of a dephasing superchannel matches that of the Choi matrix of a Schur product map. Then since $C_S$ is positive, for the other entries we can, as in Theorem 4.1.7, use the fact that for a positive matrix $(a_{i,j})$, $|a_{i,j}|^2 \leq a_{ii}a_{jj}$. □

**Remark 4.1.11.** In the definition of dephasing superchannel, and in the proof of Theorem 4.1.10 we only required action on channels. Indeed, we explicitly used Choi matrices of channels. Thus it is a property of the underlying QSC of a superchannel.

We can show that if a superchannel is given by a Schur product then it is the only Schur product channel extending its corresponding QSC. Indeed, suppose $S_A$ and $S_B$ are Schur product superchannels which have the same action on the space of quantum channels. In $M_d(M_r)$ the standard basis elements which are block off-diagonal are in $S(d,r)$ and thus all the off diagonals in the blocks of $A$ and $B$ must match. Thus the only elements needed to be checked are the diagonals of the blocks of $A$ and $B$. However considering the action on elements such as

$$\sum_{k=1}^{\lfloor \frac{r}{2} \rfloor} E_{(i,i),(k,k)} - \sum_{l=\lfloor \frac{r}{2} \rfloor}^{2\lfloor \frac{r}{2} \rfloor} E_{(i,i),(l,l)}$$

we can equate these. For example, in $S(2,2)$ all we require is

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Hence we have shown dephasing QSCs have unique extensions to superchannels.

**Theorem 4.1.12.** *If $S$ is a dephasing superchannel which extends a QSC*

$$\Gamma : SCPTP(d_1, r_1) \to SCPTP(d_2, r_2),$$

*then for any other superchannel $S'$ extending $\Gamma$, we have $S = S'$.*

### 4.1.1 Mixed unitary Schur product

Recall the completely depolarising channel $\delta_d : M_d \to M_d$ is defined as $\delta_d(\rho) = \mathrm{tr}_d(\rho) I_d$ where $tr_d$ is normalised trace. Let $\mathcal{U}(d)$ be the $d \times d$ unitary group. A mixed unitary map is one which is a convex combination of unitary conjugations. The completely depolarising channel is a mixed unitary channel as

$$\delta_d(\rho) = \frac{1}{d^2} \sum_{a,b=0}^{d-1} W_{a,b} \rho W_{a,b}^*$$

where $W_{a,b}$ are the Weyl-Heisenberg unitaries, see [39].

In [20] factorizable completely-positive maps are studied to give a reformulation of the Connes embedding conjecture. A quantum channel $\phi$ was called factorizable of degree $d$ if and only if $\delta_d \otimes \phi$ is mixed unitary.

In [21] the following theorem is proved:

**Theorem 4.1.13.** *Let $C \in M_k$ be a correlation matrix and let $d \in \mathbb{N}$. The map $\delta_d \otimes S_C : M_d \otimes M_k \to M_d \otimes M_k$ is mixed unitary if and only if*

$$C \in conv(\mathcal{F}_k(d))$$

*where $\mathcal{F}_k(d) = \{(tr_d(U_i^* U_j))_{i,j=1}^k \in M_k : U_1, \ldots, U_k \in \mathcal{U}(d)\}$*

*Proof.* If it is mixed unitary then for some probability distribution $\{p_l\}_{l=1}^M$ and unitaries $V_l \in \mathcal{U}(dk)$ we have $\delta_d \otimes S_C(A) = \sum_{l=1}^M p_l V_l A V_l^*$. Take $A = I_d \otimes E_{i,i}$ to get

$$\begin{aligned}\delta_d \otimes S_C(I_d \otimes E_{i,j}) &= I_d \otimes E_{i,i} \\ &= \sum_{l=1}^M p_l V_l (I_d \otimes E_{i,i}) V_l^* \\ &= \sum_{l=1}^M p_l \sum_{s,t=1}^k V_{l,s,i} V_{l,t,i}^* \otimes E_{s,t},\end{aligned}$$

where for the last equation we expanded $V_l = \sum_{i,j=1}^k V_{l,i,j} \otimes E_{i,j} \in M_d \otimes M_k$. Each $V_l$ is block diagonal since for $s = t \neq i$ we have

$$0 = \sum_{l=1}^M p_l V_{l,s,i} V_{l,s,i}^*$$

which is a sum of positive terms, implying $V_{l,s,i} = 0$ for $s \neq i$. Hence $V_l = \bigoplus_{i=1}^k V_{l,i,i}$ and each $V_{l,i,i} \in \mathcal{U}(d)$. Now for $i \neq j$, let $A = I_d \otimes E_{i,j}$ to see

$$\begin{aligned}\delta_d \otimes S_C(I_d \otimes E_{i,j}) &= c_{i,j} I_d \otimes E_{i,j} \\ &= \sum_{l=1}^M p_l \sum_{s,t=1}^k (V_{l,s,i} V_{l,t,j}^* \otimes E_{s,t}) \\ &= \left( \sum_{l=1}^M p_l V_{l,i,i} V_{l,j,j}^* \right) \otimes E_{i,j}.\end{aligned}$$

Finally, since $c_{i,j} = tr_d(c_{i,j} I_d)$ we have

$$c_{i,j} = \sum_{l=1}^{M} p_l \ \mathrm{tr}_d(V_{l,i,i} V_{l,j,j}^*)$$

which completes the first direction of the proof.

For the converse, the set of mixed unitary maps is a convex set, so we can consider $C = (\mathrm{tr}_d(U_i U_j^*))_{i,j=1}^k \in \mathcal{F}_k(d)$. If $\{W_l\}_{l=1}^{d^2}$ are the Weyl-Heisenberg unitaries in $\mathcal{U}(d)$ then defining $\widetilde{W_{l,l'}} = \bigoplus_{i=1}^k W_{l'} U_i W_l \in \mathcal{U}(dk)$. A calculation then shows that for $A = (A_{i,j}) \in M_k(M_d)$ we have

$$\delta_d \otimes S_C(A) = \frac{1}{d^4} \sum_{l,l'=1}^{d^2} \widetilde{W_{l,l'}}(A_{i,j}) \widetilde{W_{l,l'}^*}$$

which completes the proof. □

This theorem can be modified to describe mixed unitary Schur product superchannels. Note that the unitaries are of different sizes. For example consider products of unitaries, if we follow the reverse direction of the proof of Theorem 4.1.13 then a matrix of the form

$$C_{(i,j),(k,l)} = (tr_d(U_i^* V_k^* V_l U_j)), \quad U_i, V_i \in \mathcal{U}(d), 1 \leq i \leq k$$

does give a mixed unitary map but the forward direction does not follow.

When tensoring two superchannels

$$\widetilde{S}_1 : M_{d_1}(M_{r_1}) \to M_{d_2}(M_{r_2}),$$

$$\widetilde{S}_2 : M_{d_3}(M_{r_3}) \to M_{d_4}(M_{r_4})$$

we view the tensor $\widetilde{S}_1 \otimes \widetilde{S}_2$ as sending $M_{d_1 d_3}(M_{r_1 r_3}) \to M_{d_2 d_4}(M_{r_2 r_4})$. That is, we are interested in the induced map $\widetilde{S_1 \otimes S_2}$.

**Remark 4.1.14.** If we write $d = d_1 d_2$ for $d_1, d_2 \in \mathbb{N}$ we can regard the completely depolarizing channel as a superchannel $\delta_{d_1 d_2} : M_{d_1}(M_{d_2}) \to M_{d_1}(M_{d_2})$. To see the TP-preserving condition suppose $\phi$ is a TP map so that $\mathrm{Tr}_2 C_\phi = I_{d_1}$. This implies $\mathrm{Tr}\, C_\phi = d_1$ so then $\delta_{d_1 d_2}(C_\phi) = \frac{1}{d_2} I_{d_1} \otimes I_{d_2}$ and the partial trace of this gives identity.

**Theorem 4.1.15.** *Let $C \in M_{k^2}$ be a superchannel correlation matrix and let $d = d_1 d_2$ for $d_1, d_2 \in \mathbb{N}$. The superchannel $\delta_d \otimes S_C : M_{d_1 k}(M_{d_2 k}) \to M_{d_1 k}(M_{d_2 k})$ is a convex combination of unitary superchannels if and only if*

$$C \in conv(\mathcal{F}_k^2(d_1, d_2))$$

*where*

$$\mathcal{F}_k^2(d_1, d_2) = \{(tr_{d_1}(U_i^* U_j) \cdot tr_{d_2}(V_m^* V_n))_{(i,j),(m,n)=(1,1)}^{(k,k)} \in M_{k^2} : U_i \in \mathcal{U}(d_1), V_i \in \mathcal{U}(d_2)\}.$$

*Proof.* Suppose $\delta_{d_1 d_2} \otimes S_C(A_{ij}) = \sum_l^M p_l (U_l \otimes V_l)(A_{ij})(U_l \otimes V_l)^*$ for unitaries $U_l \in \mathcal{U}(d_1 k)$ and $V_l \in \mathcal{U}(d_2 k)$.

Following the proof of Theorem 4.1.13, if $W_i = U_i \otimes V_i$ is written as a block matrix in $M_{k^2}(M_{d_1 d_2})$, then it can be shown that it is block diagonal

$$W_i = \bigoplus_{s=1}^{k} (\bigoplus_{a=1}^{k} W_{i,(s.s),(a,a)}) = \sum_{s,a=1}^{k} W_{i,(s.s),(a,a)} \otimes E_{s,s} \otimes E_{a,a}$$

where each $W_{i,(s,s),(a,a)} \in \mathcal{U}(d_1 d_2)$. But, if we write

$$U_i = \sum_{m,n=1}^{k} U_{i,(m,n)} \otimes E_{m,n} \in M_{d_1} \otimes M_k, \quad \text{some } U_{i,(m,n)} \in M_{d_1}$$
$$V_i = \sum_{p,q=1}^{k} V_{i,(p,q)} \otimes E_{p,q} \in M_{d_2} \otimes M_k, \quad \text{some } V_{i,(p,q)} \in M_{d_2}$$

then equating $W_i = U_i \otimes V_i$ shows that the off diagonal blocks of $U_i$ and $V_i$ are zero and so the diagonal blocks are unitaries, and that $W_{i,(s,s),(a,a)} = U_{i,(s,s)} \otimes V_{i,(a,a)}$. We then have

$$\begin{aligned} c_{(ik,jl)} &= \sum_{m=1}^{M} p_m tr_{d_1 d_2}(W_{m,(i,i),(k,k)} W_{m,(j,j),(l,l)}^*) \\ &= \sum_{m=1}^{M} p_m tr_{d_1 d_2}(U_{m,(i,i)} \otimes V_{m,(k,k)}(U_{m,(j,j)} \otimes V_{m,(l,l)})^*) \\ &= \sum_{m=1}^{M} p_m tr_{d_1}(U_{m,(i,i)} U_{m,(j,j)}^*) tr_{d_2}(V_{m,(k,k)} V_{m,(l,l)}^*) \end{aligned}$$

which proves the forward direction of the theorem. The proof of the converse is the same as that of Theorem 4.1.13. □

**Remark 4.1.16.** Different factorisations of $d = d_1 d_2$ will give different sets $\mathcal{F}_k^2(d_1, d_2)$ and they are all subsets of $\mathcal{F}_{k^2}(d)$.

## 4.2 Entanglement-breaking superchannels

Here we look at entanglement-breaking superchannels which are a generalisation of the well known entanglement-breaking maps. These were introduced in [2].

**Definition 4.2.1.** A CP map $\phi : M_{d_1} \otimes M_{d_2} \to M_{r_1} \otimes M_{r_2}$ is $d_1 r_1 : d_2 r_2$ *separable* if it can be written as $\phi = \sum_i f_i \otimes g_i$ for CP maps $f_i : M_{d_1} \to M_{r_1}$, $g_i : M_{d_2} \to M_{r_2}$. This is equivalent to the Choi matrix $C_\phi$ being $d_1 r_1 : d_2 r_2$ *separable* meaning it can be written, after commuting tensor factors, as $C_\phi = \sum_i A_i \otimes B_i$ for positive operators $A_i \in M_{d_1} \otimes M_{r_1}$ and $B_i \in M_{d_2} \otimes M_{r_2}$.

**Definition 4.2.2.** A CP map $\phi : M_d \to M_r$ is *entanglement-breaking* if for any $n \in \mathbb{N}$ and any positive $\rho \in M_n \otimes M_d$ the matrix $\mathrm{id}_n \otimes \phi(\rho)$ is separable. This is equivalent to the Choi matrix $C_\phi$ being $d : r$ separable.

**Definition 4.2.3.** A superchannel $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ is an *entanglement-breaking superchannel* if for every $d_3, r_3 \in \mathbb{N}$ and CP map $\phi \in \mathcal{L}(M_{d_3} \otimes M_{d_1}, M_{r_3} \otimes M_{r_1})$ the map $\mathrm{id}_{d_3, r_3} \otimes S(\phi)$ is separable.

We show this is equivalent to the induced map on Choi matrices $\widetilde{S} : M_{d_1}(M_{r_1}) \to M_{d_2}(M_{r_2})$ being entanglement breaking.

**Theorem 4.2.4** ([2]). *Let $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ be a quantum superchannel. Then the following are equivalent:*

*(i) $S$ is an entanglement-breaking superchannel*

*(ii) $C_{\widetilde{S}}$ is $d_1 r_1 : d_2 r_2$ separable*

*(iii) $\widetilde{S}$ is an entanglement-breaking map.*

*Proof.* The equivalence of $(ii)$ and $(iii)$ is a well-known property of entanglement breaking maps.

Recall the map $\Phi_+$ from Remark 2.1.3 given by

$$\Phi_+ = \sum_{i,j,k,l} \mathcal{E}_{i,j,k,l} \otimes \mathcal{E}_{i,j,k,l} \in \mathcal{L}(M_{d_1} \otimes M_{d_1}, M_{r_1} \otimes M_{r_1})$$

As noted, after commuting the tensor factors in the Choi matrix of $\mathrm{id}_{d_1r_1} \otimes S[\Phi_+]$ we get the Choi matrix of $C_{\widetilde{S}}$. Now if $(i)$ holds it implies $\mathrm{id}_{d_1r_1} \otimes S[\phi_+]$ is $d_1r_1 : d_2r_2$ separable, which from the definition, means we can write

$$C_{\widetilde{S}} = \sum_i A_i \otimes B_i$$

for positive $A_i \in M_{d_1} \otimes M_{r_1}$, and $B_i \in M_{d_2} \otimes M_{r_2}$. Hence, $\widetilde{S}$ is separable, and $(i)$ implies $(ii)$ and $(iii)$.

To see how $(ii)$ implies $(i)$, consider an arbitrary CP map $\phi \in \mathcal{L}(M_{d_3} \otimes M_{d_1}, M_{r_3} \otimes M_{r_1})$. Then using the reverse Choi formula, we have

$$\begin{aligned} C_{\mathrm{id}_{d_3r_3} \otimes S(\phi)} &= \mathrm{Tr}_{d_3r_3d_1r_1}[C_{\widetilde{\mathrm{id}_{d_3r_3} \otimes S}}(C_\phi^T \otimes I_{d_3r_3} \otimes I_{d_2r_2})] \\ &= \mathrm{Tr}_{d_3r_3d_1r_1}[(\phi_+^{d_3} \otimes \phi_+^{r_3} \otimes C_{\widetilde{S}})(C_\phi^T \otimes I_{d_3r_3} \otimes I_{d_2r_2})] \\ &= \mathrm{Tr}_{d_1r_1}[I_{d_3} \otimes I_{r_3} \otimes C_{\widetilde{S}}(C_\phi^{\Gamma_1} \otimes I_{d_2r_2})]. \end{aligned}$$

Where $\Gamma_1$ is the partial transpose on system $M_{d_1} \otimes M_{r_1}$. Here we equated $C_{\widetilde{\mathrm{id}_{d_3r_3} \otimes S}}$ and $(\phi_+^{d_3} \otimes \phi_+^{r_3} \otimes C_{\widetilde{S}})$ since we are tracing out the system $d_3$ and $r_3$. Now, writing

$$C_{\widetilde{S}} = \sum_i A_i \otimes B_i$$

for positive $A_i \in M_{d_1} \otimes M_{r_1}$, and $B_i \in M_{d_2} \otimes M_{r_2}$ we see that

$$\begin{aligned} C_{\mathrm{id}_{d_3r_3} \otimes S(\phi)} &= \sum_i \mathrm{Tr}_{d_1r_1}[(I_{d_3} \otimes I_{r_3} \otimes A_i)C_\phi^{\Gamma_1}] \otimes B_i \\ &= \sum_i \mathrm{Tr}_{d_1r_1}[(I_{d_3} \otimes I_{r_3} \otimes A_i^T)C_\phi] \otimes B_i \end{aligned}$$

which is a sum of tensors of positive operators, which implies $\mathrm{id}_{d_3r_3} \otimes S(\phi)$ is separable. Since $\phi$ was abritrary, by the definition of entanglement-breaking superchannel this completes the proof. □

Focusing on the entanglement-breaking of channels specifically, we can see how QSCs affect this. The concept of entanglement-breaking superchannel does not apply to QSCs unless they are extended to be superchannels. Whether or not the resulting superchannel is entanglement-breaking depends on the extension. For example, consider superchannels acting between spaces of Choi matrices $M_2(M_1) \to M_2(M_1)$. Written in term of their Choi matrices, the following two maps define the same QSC,

$$C_{\widetilde{S}_1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

$$C_{\widetilde{S}_2} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

however the first is separable and the second is entangled. Thus $S_1$ is entanglement-breaking and $S_2$ is not, though they are extensions of the same QSC.

This implies that a superchannel can be entanglement-breaking on all quantum channels without being entanglement-breaking in general.

**Remark 4.2.5.** Just as positive partial transpose (PPT) maps generalise entanglement-breaking channels, one can define a notion of PPT superchannel. One way to define these superchannels is to set the pre and post processing channels as PPT maps. A more general way is to require the Choi matrix to be PPT. Superchannels with separable Choi matrices give an example of this, see [18] and [37].

## 4.3 Coherence-breaking superchannels

In this section we discuss some of the coherence-breaking superchannels from [31] and [26]. These were introduced to study the resource theory of coherence, extended to channels. Although we use the definitions of coherence breaking superchannels from [26], the results need to be stated more carefully, as a property being defined on all linear maps vs just quantum channels can have large consequences. This is essentially the moral of this thesis.

Coherence is responsible for many quantum effects, such as the interference patterns found in the double-slit experiment, [33]. For a density matrix, decoherence is the process whereby the off-diagonal terms are reduced to 0, due to noise from the environment. This represents the quantum state becoming a classical probabilistic mixture.

**Definition 4.3.1.** Given a fixed basis $\{e_i\}$ for $\mathbb{C}^n$, the *completely dephasing channel* $\mathcal{D}_n : M_n \to M_n$ is defined by

$$\mathcal{D}_n(\rho) = \sum_{i=1}^{n} |e_i\rangle\langle e_i|\rho|e_i\rangle\langle e_i|.$$

A state $\rho \in M_n$ is *incoherent* if $\mathcal{D}_n(\rho) = \rho$.

A channel $\phi : M_d \to M_r$ is a *maximally incoherent operation* (MIO) if it maps incoherent states to incoherent states. This is equivalent to $\phi$ satisfying

$$\mathcal{D}_r \circ \phi \circ \mathcal{D}_d = \phi \circ \mathcal{D}_d.$$

A channel $\phi : M_d \to M_r$ is a *classical channel* if it satisfies

$$\phi = \mathcal{D}_r \circ \phi \circ \mathcal{D}_d.$$

**Example 4.3.2.** As they leave the diagonals untouched, a Schur product channel is an example of a maximally incoherent operation.

**Definition 4.3.3.** The *completely dephasing superchannel* is given by

$$\Delta_i : \mathcal{L}(M_{d_i}, M_{r_i}) \to \mathcal{L}(M_{d_i}, M_{r_i})$$
$$\Delta_i(\phi) = \mathcal{D}_{r_i} \circ \phi \circ \mathcal{D}_{d_i}.$$

A superchannel $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ is a *maximally incoherent superchannel* (MISC) if it maps classical channels to classical channels. This is equivalent to $S$ satisfying

$$\Delta_2 \circ S \circ \Delta_1 = S \circ \Delta_1.$$

**Theorem 4.3.4** ([31])**.** *If $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ is a superchannel then $\Delta_2 \circ S$ and $S \circ \Delta_1$ satisfy*

$$C_{\widetilde{\Delta_2} \circ \widetilde{S}} = id_{d_1 r_1} \otimes \mathcal{D}_{d_2} \otimes \mathcal{D}_{r_2}(C_{\widetilde{S}})$$

*and*

$$C_{\widetilde{S} \circ \widetilde{\Delta_1}} = \mathcal{D}_{d_1} \otimes \mathcal{D}_{r_1} \otimes id_{d_2 r_2}(C_{\widetilde{S}}).$$

*Proof.* For any input $C_\phi$ the action of $\widetilde{\Delta_2}$ is

$$\begin{aligned}\widetilde{\Delta_2}(C_\phi) &= C_{\Delta_2(\phi)}\\ &= C_{\mathcal{D}_{r_2}\circ\phi\circ\mathcal{D}_{d_2}}\\ &= (\mathrm{id}_{d_2}\otimes\mathcal{D}_{r_2})\circ(\mathrm{id}_{d_2}\otimes\phi)\circ(\mathrm{id}_{d_2}\otimes\mathcal{D}_{d_2})(\phi_+^{d_2})\\ &= (\mathcal{D}_{d_2}^T\otimes\mathcal{D}_{r_2})\circ(\mathrm{id}_{d_2}\otimes\phi)(\phi_+^{d_2})\\ &= (\mathcal{D}_{d_2}\otimes\mathcal{D}_{r_2})(C_\phi).\end{aligned}$$

Since $C_{\widetilde{\Delta_2}\circ\widetilde{S}} = \mathrm{id}_{d_1r_1}\otimes\widetilde{\Delta_2}(C_{\widetilde{S}})$ the first result follows. For the second equation, we have a similar description for the action of $\widetilde{\Delta_1}$, and then

$$\begin{aligned}C_{\widetilde{S}\circ\widetilde{\Delta_1}} &= (\mathrm{id}_{d_1r_1}\otimes\widetilde{S})\circ(\mathrm{id}_{d_1r_1}\otimes\mathcal{D}_{d_1}\otimes\mathcal{D}_{r_1})(\phi_+^{d_1r_1})\\ &= (\mathcal{D}_{d_1}^T\otimes\mathcal{D}_{r_1}^T\otimes\mathrm{id}_{d_2r_2})\circ(\mathrm{id}_{d_1r_1}\otimes\widetilde{S})(\phi_+^{d_1r_1})\\ &= \mathcal{D}_{d_1}\otimes\mathcal{D}_{r_1}\otimes\mathrm{id}_{d_2r_2}(C_{\widetilde{S}}).\end{aligned}$$

□

**Theorem 4.3.5** ([31])**.** *A superchannel $S$ is a MISC if and only if*

$$\mathcal{D}_{d_1}\otimes\mathcal{D}_{r_1}\otimes\mathcal{D}_{d_2}\otimes\mathcal{D}_{r_2}(C_{\widetilde{S}}) = \mathcal{D}_{d_1}\otimes\mathcal{D}_{r_1}\otimes id_{d_2r_2}(C_{\widetilde{S}}).$$

*Proof.* By definition of MISC we have

$$C_{\widetilde{\Delta_2}\circ\widetilde{S}\circ\widetilde{\Delta_1}} = C_{\widetilde{S}\circ\widetilde{\Delta_1}}.$$

Using the previous theorem the right side of this equation is

$$\mathcal{D}_{d_1}\otimes\mathcal{D}_{r_1}\otimes\mathrm{id}_{d_2r_2}(C_{\widetilde{S}})$$

while the left side can be expanded as

$$\begin{aligned}C_{\widetilde{\Delta_2}\circ\widetilde{S}\circ\widetilde{\Delta_1}} &= \mathrm{id}_{d_1r_1}\otimes\mathcal{D}_{d_2}\otimes\mathcal{D}_{r_2}(C_{\widetilde{S}\circ\widetilde{\Delta_1}})\\ &= \mathcal{D}_{d_1}\otimes\mathcal{D}_{r_1}\otimes\mathcal{D}_{d_2}\otimes\mathcal{D}_{r_2}(C_{\widetilde{S}})\end{aligned}$$

□

In [26] the notion of a coherence-breaking superchannel is introduced.

**Definition 4.3.6.** A channel $\phi : M_d \to M_r$ is a *coherence-breaking channel* (CBC) if it maps any state to an incoherent state.

A superchannel $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ is a *coherence-breaking superchannel* (CBSC) if it sends any linear map $\phi \in \mathcal{L}(M_{d_1}, M_{r_1})$ to a coherence-breaking channel.

**Remark 4.3.7.** If $\phi$ is a CBC with respect to the standard basis, then it satisfies

$$\mathrm{Tr}[E_{i,j} \otimes E_{k,l} C_\phi] = 0$$

for any $i, j$, and $k \neq l$. It has a block diagonal Choi matrix,

$$C_\phi = \left(\begin{array}{cc|cc} * & 0 & * & 0 \\ 0 & * & 0 & * \\ \hline * & 0 & * & 0 \\ 0 & * & 0 & * \end{array}\right)$$

CBSC's have a similar property, the reduced Choi matrix $C^2_{\widetilde{S}} := \mathrm{Tr}_{d_1 r_1} C_{\widetilde{S}}$ satisfies

$$\mathrm{Tr}[E_{i,j} \otimes E_{k,l} C^2_{\widetilde{S}}] = 0$$

for any $i, j$, and $k \neq l$. The Choi matrix of $\widetilde{S}$ will consist of $d_1 \cdot r_1$ block diagonal matrices in $M_{d_2}(M_{r_2})$.

This similarity between CBSC's and CBC's motivates the definition of CBSC involving action on *all* linear maps. If we only require the underlying QSC to be coherence-breaking we won't necessarily recover a CBSC. For example, the following superchannel sends every quantum channel to a CBC but not every linear map: define $\widetilde{S} : M_2(M_1) \to M_2(M_2)$ as

$$\begin{aligned} \widetilde{S}(E_{1,1}) &= I_4, \\ \widetilde{S}(E_{2,2}) &= I_4, \\ \widetilde{S}(E_{1,2}) &= E_{1,4}, \\ \widetilde{S}(E_{2,1}) &= E_{4,1}. \end{aligned} \tag{4.3}$$

Suitably scaled this is a superchannel. Since it outputs Choi matrices which are not block diagonal it cannot be a CBSC. Only $E_{1,1} + E_{2,2}$ is Choi matrices of channels in $M_2(M_1)$. Their outputs are diagonal Choi matrices which correspond to CBC's.

Thus, as a QSC, it only outputs coherence-breaking channels, despite not being a CBSC.

If we change the action of $S$ to give $\widetilde{S}(E_{1,2}) = \widetilde{S}(E_{2,1}) = 0$ then it defines the same QSC but it now only outputs diagonal Choi matrices. This alternative extension is a CBSC.

**Definition 4.3.8.** A superchannel $S$ is a *strong coherence-breaking superchannel* if, for every input channel $\phi$, $S(\phi)$ is a classical channel.

**Remark 4.3.9.** Being a strong coherence breaking superchannel is a property of the underlying QSC. For decoherence with respect to the standard basis, classical channels have diagonal Choi matrices. Thus, the previous discussion shows that not every strong coherence-breaking superchannel is a CBSC. The superchannel defined in Equation (4.3) is a strong coherence-breaking superchannel but not a CBSC.

This also shows that strong coherence-breaking superchannels may have non-diagonal Choi matrices.

The definition of strong coherence-breaking superchannel is equivalent to saying that for any channel $\phi$,

$$\Delta_2 \circ S(\phi) = S(\phi).$$

Note, however, that while $\Delta_2 \circ S = S$ is sufficient for a superchannel to be strong coherence-breaking, it is not necessary, as shown in the previous two remarks. Indeed, the superchannel defined in Equation (4.3) satisfies

$$\mathcal{D}_{d_2} \otimes \mathcal{D}_{r_2} \circ \widetilde{S}(E_{1,2}) = 0$$

and so $\Delta_2 \circ S \neq S$.

The following theorem shows that Choi matrix being block diagonal in the incoherent basis is enough for a superchannel to strong coherence-breaking.

**Theorem 4.3.10.** *If a superchannel's Choi matrix is of the form*

$$C_{\widetilde{S}} = \sum_{i,j,k} a_{i,j,k} |e_i^1\rangle\langle e_j^1| \otimes |e_k^2\rangle\langle e_k^2|$$

*where* $\{e_i^k\}_i \subset \mathbb{C}^{d_k} \otimes \mathbb{C}^{r_k}$ *is the incoherent basis for system* $k$*, then it is a strong coherence-breaking superchannel.*

*Proof.* Using Theorem 4.3.4 we have

$$C_{\widetilde{\Delta_2 \circ S}} = \mathrm{id}_{d_1 r_1} \otimes \mathcal{D}_{d_2} \otimes \mathcal{D}_{r_2}(C_{\widetilde{S}}) = C_{\widetilde{S}}$$

Therefore, $\Delta_2 \circ S = S$, which implies $S$ is strong coherence-breaking. □

## 4.4 Stabilizer-preserving channels

In [32] the action of superchannels on stablizer-preserving channels is considered. We will quickly introduce the stabilizer formalism. The *Pauli* operators are

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The 1-qubit Pauli group is $\mathcal{P}_1 = (\pm 1, \pm i)\{I_2, X, Y, Z\}$. The $n$-qubit Pauli group is

$$\mathcal{P}_n = (\pm 1, \pm i)\{A_1 \otimes A_2 \otimes \ldots \otimes A_n | A_i \in \mathcal{P}_1\}.$$

An $n$-qubit state $|\psi\rangle \in \mathbb{C}^{2^n}$ is a *stabilizer pure state* if there is an abelian subgroup $\mathcal{S} \subset \mathcal{P}_n$, $|\mathcal{S}| = 2^n$ such that $A|\psi\rangle = |\psi\rangle$, for all $A \in \mathcal{S}$.

A stabilizer circuit is one where the gates are operators from the normalizer of the Pauli groups. The Gottesman-Knill theorem says that stabilizer circuits can be perfectly simulated in polynomial time by a probabilistic classical computer. Non-stabilizer states are called magic states, and the use of magic states combined with stabilizer circuits can achieve universal quantum computation, [15, 27, 11].

With this in mind, magic states are seen to be a valuable resource. A resource theory of magic is one in which the free states are some or all of the stabilizer states. In this subsection we look at channels which preserve these stabilizer states, particularly completely stabilizer-preserving operations. It is worth nothing that this is a larger class than stabilizer circuits, see [22].

**Definition 4.4.1.** A density matrix $\rho \in M_d$ is a *stabilizer state* if it is in the convex hull of pure stabilizer states. Denote the set of stabilizer states in $M_d$ as $\mathrm{STAB}(d)$.

Completely stabilizer-preserving operations were defined in [34].

**Definition 4.4.2.** A channel $\phi : M_d \to M_r$ is a *completely stabilizer-preserving operation* (CSPO) if for any $n \in \mathbb{N}$ we have

$$\phi \otimes \mathrm{id}_n(\rho) \in \mathrm{STAB}(rn), \quad \forall \rho \in \mathrm{STAB}(dn).$$

It is also shown that CSPOs are exactly the maps whose Choi matrix is a stabilizer state.

**Theorem 4.4.3** ([34])**.** *The map $\phi : M_d \to M_r$ is a CSPO if and only if*

$$\frac{1}{d}C_\phi \in STAB(dr).$$

**Definition 4.4.4.** A superchannel $S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ is *completely CSPO-preserving* if for any $d_3, r_3 \in \mathbb{N}$, and CSPO $\phi : M_{d_1} \otimes M_{d_3} \to M_{r_1} \otimes M_{r_3}$ the map $S \otimes \mathrm{id}_{d_3,r_3}(\phi)$ is a CSPO.

**Theorem 4.4.5** ([32])**.** *$S : \mathcal{L}(M_{d_1}, M_{r_1}) \to \mathcal{L}(M_{d_2}, M_{r_2})$ is a completely CSPO-preserving superchannel if and only if*

$$\frac{1}{r_1 d_2}C_{\widetilde{S}} \in STAB(d_1 r_1 d_2 r_2).$$

Although they have the same action on channels (and hence CSPO's), extensions of a QSC can be completely CSPO-preserving or not depending on how the extension is done. In [34] the following example of a non-stabilizer state is given to show that a map which preserves stabilizer states does not necessarily do so completely:

$$\begin{aligned}\frac{1}{2}\begin{pmatrix} 1 & e^{i\frac{\pi}{4}} \\ e^{-i\frac{\pi}{4}} & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -e^{i\frac{\pi}{4}} \\ -e^{-i\frac{\pi}{4}} & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ = \frac{1}{2}\begin{pmatrix} 1 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & 1 & 0 & -e^{i\frac{\pi}{4}} \\ e^{-i\frac{\pi}{4}} & 0 & 1 & 0 \\ 0 & -e^{-i\frac{\pi}{4}} & 0 & 1 \end{pmatrix} \notin \mathrm{STAB}(4)\end{aligned}$$

As the Choi matrix of a map $\widetilde{S} : M_2(M_1) \to M_2(M_1)$ this defines a superchannel since it is a unital map, and $S(2,1) = \mathrm{span}\{I_2\}$. An alternative extension of the same QSC is given by the maximally entangled state

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

which is a stabilizer state, and thus gives a completely CSPO-preserving superchannel. Similarly, $\frac{1}{4}I_4$ gives a stabilizer state with the same QSC. So we can conclude that

- Action on CSPOs does not uniquely define a superchannel.
- Completely preserving CSPO's does not mean every other equivalent superchannel preserves them completely.
- Different completely CSPO-preserving superchannels can have the same QSC.

**Remark 4.4.6.** As with PPT maps extending the notion of entanglement-breaking maps, another way to study the resource theory of magic is to use the discrete Wigner function. States with positive discrete Wigner function generalise stabilizer states, and completely positive-Wigner-preserving operations (CPWP) generalise completely stabilizer-preserving operations. The natural way to apply superchannels to this is to look at superchannels which preserve CPWP channels in a complete sense. These turn out to be those whose Choi matrix has positive discrete Wigner function, see [38].

# Chapter 5

# Channel Symmetries

In this chapter we discuss the paper [4] giving a slightly altered proof of the main theorem. We also discuss how it differs for the space of quantum channels.

This chapter is a bit of a change from the previous ones since, while it uses many of the same techniques and some of the same results, it is no longer about superchannels. Rather than linear maps on states, Wigner's theorem considers a broader set of maps on quantum states which drops any requirement of linearity. All that is required is that certain probabilities are preserved. As a result, both unitary and anti-unitary maps are allowed. Similarly, here we will look at maps on quantum channels which preserve some of the structure of the space, without the same requirements as superchannels. Nevertheless, a similar theorem as the characterisation of superchannels, Theorem 2.2.1, will be proved, but with the pre and post processing channels replaced with unitary or anti-unitary maps.

## 5.1 Splitting unitaries

In this section we prove that unitary maps which preserve quantum channels split into the tensor of unitaries on the domain and range spaces. We use results from the original paper on quantum superchannels, [6], and give the proofs.

**Theorem 5.1.1** ([6]). *If $A \in M_d \otimes M_r$ satisfies $\mathrm{Tr}[AC] = 1$ for all Choi matrices of channels $C \in M_d \otimes M_r$ then $A = \rho \otimes I_r$ for a trace one matrix $\rho \in M_d$. If $A \geq 0$ then $\rho \geq 0$.*

*Proof.* Let $E \in M_d \otimes M_r$ be a positive matrix such that $\operatorname{Tr}_r E = P \leq I_d$ and $\sigma$ some density matrix in $M_r$. Then for $B = (I_d - P) \otimes \sigma$ we have $\operatorname{Tr}_r[B+E] = I_d - P + P = I_d$, so $B + E$ is the Choi matrix of a channel. Then since $I_d \otimes \sigma$ is the Choi matrix of a channel

$$\begin{aligned}\operatorname{Tr}[AE] &= 1 - \operatorname{Tr}[AB] \\ &= 1 - \operatorname{Tr}[A(I_d \otimes \sigma)] + \operatorname{Tr}[A(P \otimes \sigma)] \\ &= \operatorname{Tr}[A(P \otimes \sigma)].\end{aligned}$$

Define $\rho = \operatorname{Tr}_r[A(I_d \otimes \sigma)]$. Using the last equation we compute,

$$\begin{aligned}\operatorname{Tr}[AE] &= \operatorname{Tr}[A(P \otimes \sigma)] \\ &= \operatorname{Tr}[A(I_d \otimes \sigma)(P \otimes I_r)] \\ &= \operatorname{Tr}[\operatorname{Tr}_r[A(I_d \otimes \sigma)]P] \\ &= \operatorname{Tr}[\rho P] \\ &= \operatorname{Tr}[(\rho \otimes I_r)E].\end{aligned}$$

This holds for all positive $E$ which implies $A = \rho \otimes I_r$. The matrix $\rho$ must have trace 1 since for any Choi matrix of a channel $C$ we have $1 = \operatorname{Tr}[(\rho \otimes I_r)C] = \operatorname{Tr}[\rho \operatorname{Tr}_r[C]] = \operatorname{Tr}[\rho]$. Positivity of $\rho$ from $A$ follows from the formula defining $\rho$. $\square$

**Theorem 5.1.2** ([6]). *If $S : M_{d_1}(M_{r_1}) \to M_{d_2}(M_{r_2})$ is a linear map which preserves Choi matrices of channels then there is a trace-preserving linear map $N : M_{d_2} \to M_{d_1}$ such that*

$$S^*(\rho \otimes I_{r_2}) = N(\rho) \otimes I_{r_1}, \quad \rho \in M_{d_2}.$$

*If $S$ is completely positive then $N$ is a quantum channel.*

*Proof.* For any density matrix $\rho \in M_{d_2}$, and any Choi matrix of a channel $C \in M_{d_1} \otimes M_{r_1}$, we have

$$\begin{aligned}\operatorname{Tr}[S^*(\rho \otimes I_{r_2})C] &= \operatorname{Tr}[\rho \otimes I_{r_2} S(C)] \\ &= 1\end{aligned}$$

Hence by the previous theorem $S^*(\rho \otimes I_{r_2}) = \rho' \otimes I_{r_1}$ for some trace one matrix $\rho' \in M_{d_1}$. For any density matrix $\sigma \in M_{r_1}$ we can take $\rho' = \operatorname{Tr}_{r_1}[S^*(\rho \otimes I_{r_2})(I_{d_1} \otimes \sigma)]$. In particular let $\sigma = \frac{1}{r_1} I_{r_1}$ to define

$$N(\rho) := \frac{1}{r_1} \operatorname{Tr}_{r_1}[S^*(\rho \otimes I_{r_2})]$$

If $S$ is completely positive, then $N$ is the composition of three completely positive maps, hence is CP. $\square$

We need the following result from [13, Theorem 2.3].

**Theorem 5.1.3.** *If $U \in M_d \otimes M_r$ is a unitary such that for all $X, Y \in M_d$, and $Z \in M_r$ with $\operatorname{Tr} Z = 0$ it satisfies*

$$\operatorname{Tr}[U(X \otimes Z)U^*(Y \otimes I_r)] = 0,$$

*then there exists unitaries $U_1 \in M_d$, and $U_2 \in M_r$, such that $U = U_1 \otimes U_2$.*

**Theorem 5.1.4.** *If $U \in \mathcal{U}(dr)$ is a unitary such that the map $\widetilde{S} : M_d(M_r) \to M_d(M_r)$ with $\widetilde{S}(C) = UCU^*$ preserves the Choi matrices of channels, then there exists unitaries $U_1 \in \mathcal{U}(d)$ and $U_2 \in \mathcal{U}(r)$ such that $U = U_1 \otimes U_2$. The same is true if $\widetilde{S}(C) = UC^TU^*$.*

*Proof.* First the unitary conjugation. Unitary conjugations are completely positive so by Theorem 5.1.2 we have

$$\widetilde{S}^*(\rho \otimes I_{r_2}) = N(\rho) \otimes I_{r_1}, \quad \rho \in M_{d_2}$$

for some quantum channel $N : M_{d_2} \to M_{d_1}$. Now for any $X, Y \in M_d$ and $Z \in M_r$ with $\operatorname{Tr} Z = 0$ we have

$$\operatorname{Tr}[U(X \otimes Z)U^*(Y \otimes I_r)] = \operatorname{Tr}[(X \otimes Z)(N(Y) \otimes I_r)] = 0.$$

By the previous theorem $U = U_1 \otimes U_2$.

If $\widetilde{S}(C) = UC^TU^*$ preserves channels, we again get a linear map $N$ with $\widetilde{S}^*(\rho \otimes I_{r_2}) = N(\rho) \otimes I_{r_1}$, and similarly,

$$\operatorname{Tr}[U(X^T \otimes Z^T)U^*(Y \otimes I_r)] = \operatorname{Tr}[S(X \otimes Z)(Y \otimes I_r)] = 0$$

for any matrices $X, Y$ and $Z$ with $\operatorname{Tr} Z = 0$. So the unitary $U = U_1 \otimes U_2$. □

## 5.2 Wigner's Theorem

Wigner's theorem says that any bijective map, $\mathcal{H} \to \mathcal{H}$, on a Hilbert space that preserves the *transition probability* $|\langle\phi|\psi\rangle|$ between any two vectors is given by a unitary or anti-unitary map. An equivalent version of Wigner's theorem can be given in terms of density matrices. Let $\mathcal{D}(\mathcal{H})$ be the set of density matrices acting on $\mathcal{H}$.

**Definition 5.2.1.** A *state space symmetry* is a bijective map $S : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H})$ which satisfies

$$S(p\rho + (1-p)\sigma) = pS(\rho) + (1-p)S(\sigma), \quad \forall \rho, \sigma \in \mathcal{D}(\mathcal{H}), p \in [0,1]$$

Wigner's theorem then says that every state space symmetry is given by either a unitary map $\rho \mapsto U\rho U^*$ or anti-unitary map $\rho \mapsto U\rho^T U^*$, where $U \in \mathcal{B}(\mathcal{H})$ is a unitary.

**Remark 5.2.2.** As shown in Theorem 2.3.1, the only quantum channels which are reversible are the unitary channels. So as far as physically realistic operations are concerned, the only state symmetries are unitary maps.

**Remark 5.2.3.** There are many different ways of formulating Wigner's theorem, depending on how you define the term symmetry. As mentioned, it was originally stated in terms of maps which preserve the transition probability, and can be reformulated in terms of map which preserve the structure of the space of quantum states. Another type of symmetry is a map which preserves the algebraic structure of the space of observables, that is, self-adjoint operators in $\mathcal{B}(\mathcal{H})$. See [14] for a discussion on linking these concepts together in the more general setting of von Neumann algebras.

Recall that a *quantum operation* is a completely positive trace non-increasing map $\phi : M_d \to M_r$. In terms of Choi matrices, the set of operations is just the positive matrices $E \in M_d(M_r)$ with $\operatorname{Tr} E \leq I_d$. Define $OP(d,r)$ to be the set of quantum operations from $M_d$ to $M_r$, and note that its span is the whole space of linear maps.

**Definition 5.2.4.** An *operation space symmetry* is a bijective map $S : OP(d_1, r_1) \to OP(d_2, r_2)$ which sends the null operation to the null operation and satisfies

$$S(p\phi + (1-p)\psi) = pS(\phi) + (1-p)S(\psi), \quad \forall \phi, \psi \in OP(d_1, r_1), p \in [0,1].$$

In [4] it is shown that every operation space symmetry and state space symmetry extends to give a bijective linear map between the spaces spanned by the sets they act on.

They also preserve the set of quantum channels:

**Theorem 5.2.5.** *If $S$ is an operation space symmetry then it sends quantum channels to quantum channels.*

*Proof.* Let $\phi$ be a quantum channel. Channels are operations and so $S(\phi)$ is an operation. We can complete $S(\phi)$ to a channel i.e. choose a quantum operation $\psi$ such that for some channel $\phi'$ we have

$$\phi' = S(\phi) + \psi.$$

Applying the inverse:

$$S^{-1}(\phi') = \phi + S^{-1}(\psi).$$

This implies $S^{-1}(\psi) = 0$ since otherwise $S^{-1}(\phi')$ would be a trace-increasing map, and thus not an operation. Therefore

$$S(\phi) = \phi',$$

which completes the proof. □

Also in [4] the following theorem is shown

**Theorem 5.2.6.** *If $S : \mathcal{L}(M_d, M_r) \to \mathcal{L}(M_d, M_r)$ is an operation space symmetry then the corresponding map on Choi matrices $\widetilde{S} : M_d(M_r) \to M_d(M_r)$ is state space symmetry.*

We can use this to give an alternative proof of their main theorem, that every operation space symmetry is given by composing the input operation with two state symmetries on either side.

**Theorem 5.2.7.** *If $S : M_d(M_r) \to M_d(M_r)$ is an operation space symmetry, then there are state space symmetries $S_{post} : M_r \to M_r$ and $S_{pre} : M_d \to M_d$, both either unitary or anti-unitary, such that*

$$S(\phi) = S_{post} \circ \phi \circ S_{pre}.$$

*Proof.* By Theorem 5.2.6 the induced map on Choi matrices is a state space symmetry. Consider first the case where $\widetilde{S}$ is a unitary map. Then for any map $\phi$ we have $\widetilde{S}(C_\phi) = UC_\phi U^*$. This is a completely positive map which sends channels to channels and thus is a superchannel. Hence by Theorem 5.1.4 the unitary splits as $U = U_1 \otimes U_2$. Now, using the fact that for the maximally entangled state $(A \otimes I)|\phi_+\rangle = (I \otimes A^T)|\phi_+\rangle$, we have

$$\begin{aligned}
\widetilde{S}(C_\phi) &= U_1 \otimes U_2 C_\phi U_1^* \otimes U_2^* \\
&= \sum_{i,j} U_1 E_{i,j} U_1^* \otimes U_2 \phi(E_{i,j}) U_2^* \\
&= (\mathrm{id}_d \otimes \mathcal{U}_2) \circ (\mathrm{id}_d \otimes \phi) \circ (\mathcal{U}_1 \otimes \mathrm{id}_d)(\phi_+) \\
&= (\mathrm{id}_d \otimes \mathcal{U}_2) \circ (\mathrm{id}_d \otimes \phi) \circ (\mathrm{id}_d \otimes \mathcal{U}_1^T)(\phi_+) \\
&= C_{\mathcal{U}_2 \circ \phi \circ \mathcal{U}_1^T}.
\end{aligned}$$

Thus $S(\phi) = \mathcal{U}_2 \circ \phi \circ \mathcal{U}_1^T$.

If $\widetilde{S}$ is an anti-unitary map, $\widetilde{S}(C_\phi) = \mathcal{U} \circ (\mathcal{T}_d \otimes \mathcal{T}_r)(C_\phi) = U C_\phi^T U^*$, then since it preserves channels Theorem 5.1.4 again applies. That is, $U = U_1 \otimes U_2$. Now we can go through the same calculation to get

$$S(\phi) = (\mathcal{U}_2 \circ \mathcal{T}_r) \circ \phi \circ (\mathcal{U}_1 \circ \mathcal{T}_d)^T$$

as required. □

Similar to what we have done for QSCs and superchannels, we can try to define channel symmetries using the operator system of quantum channels. Following the approach of operation symmetries we may require it to respect convex combinations, and preserve the null operation, which then implies it extends to give a unique linear map.

**Definition 5.2.8.** A *channel symmetry* is a bijective linear map

$$S : SCPTP(d_1, r_1) \to SCPTP(d_2, r_2)$$

which preserves the set of quantum channels.

Note that we cannot even in principle apply Theorem 5.2.6 to channel symmetries. While the induced map on Choi matrices will preserve states, it cannot be a state symmetry as the set of Choi matrices of channels is a strict subset of the set of states. An extension of such a map to the full space of linear maps need not be bijective.

Indeed, consider the identity QSC on $S(2,1) = \mathrm{span}\{I_2\}$. This is clearly a channel symmetry. It can be extended to the identity superchannel on $M_2(M_1)$ which will give an operation symmetry. But it can also be extended to the superchannel which sends the off diagonal basis elements, $E_{1,2}, E_{2,1} \in M_2$ to zero. This demonstrates two facts:

1. The superchannel extension of a QSC which is a channel symmetry does not have to be an operation symmetry or even bijective.

2. Unitary superchannels are not unique extensions of their QSC i.e. there might be a non-unitary superchannel with the same action on quantum channels.

If we impose the extra condition of being positive in both directions then an extension of a channel symmetry gives an operation symmetry:

**Theorem 5.2.9.** *If $S$ is a channel symmetry with a positive extension to the whole space of linear maps then it preserves quantum operations. If, in addition, the extension has a positive inverse, then it is an operation symmetry.*

*Proof.* Let $E$ be the Choi matrix of a quantum operation. For some other operation $B$ we have $E+B$ is the Choi matrix of a channel. Since $\widetilde{S}$ preserves channels we have

$$\widetilde{S}(E+B) = \widetilde{S}(E) + \widetilde{S}(B) = C,$$

where $C$ is the Choi matrix of a channel. $\widetilde{S}(E)$ and $\widetilde{S}(B)$ are positive operators summing to give a channel so they must be operations.

If $S$ has a positive inverse the same applies and so it's a bijective linear map preserving quantum operations. □

Any unitary or anti-unitary map will obviously give both a channel and operation symmetry. If a channel symmetry is a QSC then it will extend to give a map which preseves operations but may not have a positive inverse.

**Remark 5.2.10.** While completely positive maps on operator systems have a completely positive extension to the whole $C^*$-algebra, it is not true of positive maps defined on operator systems. The counterexample Arveson gave used the fact that positive maps with norm 2 could not be extended. A recent paper, [5], gave some other counterexamples, showing that positive maps with norm 1 cannot necessarily be extended.

They also mention the possibility that the fractional dimension of the operator system compared with its $C^*$-algebra may be a limiting factor to constructing these non-extendable positive maps. For the space of quantum channels, $SCPTP(d,r)$, its fractional dimension is

$$\frac{d^2r^2 - d^2 + 1}{d^2r^2} = 1 - \frac{1}{r^2}(1 - \frac{1}{d^2}).$$

Generally this fraction is large compared to the examples they gave of $1/2$ and $1/4$. For $r = 1$ and large $d$ this can be made arbitrarily small. Thus there may be a large class of extendable positive maps in this case.

# Chapter 6

# Conclusion

Given the importance and ubiquity of quantum channels, the space $SCPTP(d, r)$ is an interesting and worthwhile space to study. We have shown that, as an operator system, we can use the theory of completely-positive maps on this space to give a sensible definition of superchannels, and that this definition is distinct from the standard definition. Nevertheless, these QSC's can be extended to give quantum superchannels, and the non-uniqueness of the extension leads to different characterisations, implying a different physical implementation.

Since quantum superchannels are mainly motivated to study the evolution of quantum channels, understanding the behaviour and structure of QSC's may be the better way to describe this evolution. We have seen that for many of the classes of superchannels which have been studied, the effect they have on quantum channels is not enough to define them. On the other hand, for Schur product superchannels the concept of QSC and superchannel coincide. It would be interesting to better understand what restrictions the space $SCPTP(d, r)$ puts on superchannels to get this uniqueness of extension. For example, in Theorem 3.2.11 we saw two of the same extensions of a QSC satisfy:

$$\mathrm{Tr}_{d_1} C_{\widetilde{S}_1 - \widetilde{S}_2} = 0$$

For which classes of maps does this always hold? And can we say any more?

The set of extensions of a QSC is a convex set. The extreme points of this set were characterised by Theorem 3.3.4 in terms of their Kraus operators. At the same time, the minimal dimension in the superchannel characterisation theorem is given by Choi-Kraus rank of the reduced Choi matrix of a superchannel. So it would seem that the extensions which have minimal dimension $e$ tend to be given by the extreme points, and some of the examples given confirm this. However we were not able to prove it. We were able to give a counterexample to the converse.

It would also be nice to better understand $SCPTP(d,r)$. Perhaps there is a nice basis for this space, which could give a theorem like Choi's theorem for CP maps on it. For large $r$ this space is increasingly large compared to $\mathcal{L}(M_d, M_r)$, which could affect the possibility of non-unique extensions of QSC's on this space for higher dimensions.

In general the theory of quantum superchannels has many avenues to explore, due to the wide body of knowledge on quantum channels. One topic we touched on was factorizable superchannels, which gave a generalisation of the concept for channels. It is always tempting to think of superchannels whenever CP maps on tensor spaces are considered. For example, in [36] a *quantum no-signalling (QNS) correlation* is a quantum channel $\Phi : M_X \otimes M_Y \to M_A \otimes M_B$ which satisfies

$$\operatorname{Tr}_A \Phi(\rho) = 0 \text{ whenever } \operatorname{Tr}_X(\rho) = 0$$

and

$$\operatorname{Tr}_B \Phi(\rho') = 0 \text{ whenever } \operatorname{Tr}_Y(\rho') = 0.$$

If such a map is unital then it is a superchannel (acting on the space of Choi matrices), since for a trace-preserving map $\phi$ we have

$$\operatorname{Tr}_Y(C_\phi - \frac{1}{|Y|} I_X \otimes I_Y) = 0$$

and thus by the second condition

$$\operatorname{Tr}_B(\Phi(C_\phi - \frac{1}{|Y|} I_X \otimes I_Y)) = 0$$

and so if $\Phi(I_X \otimes I_Y) = I_A \otimes I_B$ it implies $\operatorname{Tr}_B \Phi(C_\phi) = \frac{|B|}{|Y|} I_A$, which means up to scaling it is TP preserving.

Another area where the space $SCPTP(d,r)$ is involved is in the study of channel symmetries. We were able to show that they are distinct from operation symmetries, which leads to issues in trying to give a Wigner theorem for these objects. Here the problem of extending positive maps comes up. Perhaps there is a better definition of a channel symmetry which more accurately takes into account the structure of the space. For example, as an operator system, complete order isomorphisms are the natural maps. Forcing complete order isomoprhisms to preserve channels would allow us to extend them to operation symmetries.

Here we only considered finite dimensions but there has been work done on infinite dimensional quantum superchannels. In [7] they take the domain to be $\mathrm{CB}(\mathcal{M}, \mathcal{N})$ which is the space of weak*-continuous, completely-bounded maps between separable von Neumann algebras. Inside this is the set of normal, unital CP maps which are the quantum channels. Then a quantum superchannel is a normal CP map $S : \mathrm{CB}(\mathcal{M}_1, \mathcal{N}_1) \to \mathrm{CB}(\mathcal{M}_1, \mathcal{N}_1)$ which preserves the set of quantum channels. The characterisation theorem for superchannels is proved for these maps. Of course we can ask many of the same questions about these objects as we did for the finite dimensional ones. In particular, what happens if we just consider the space of quantum channels?

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