Mind the GAP: Amenability Constants and Arens Regularity of Fourier Algebras

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapters 1 and 2 are entirely background material.

Chapter 3 is largely a survey of known results, with additional remarks and comments that are not necessarily stated in the literature but are likely known.

I am the sole author of Chapter 4, which is based on [69].

Chapter 5 is based on joint work with Brian Forrest and Aasaimani Thamizhazhagan from [30].

I am the sole author of Chapter 6.

Abstract

This thesis aims to investigate properties of algebras related to the Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$, where G is a locally compact group.

For a Banach algebra $\mathcal{A}$ there are two natural multiplication operations on the double dual $\mathcal{A}^{**}$ introduced by Arens in 1971, and if these operations agree then the algebra $\mathcal{A}$ is said to be Arens regular. We study Arens regularity of the closures of $A(G)$ in the multiplier and completely bounded multiplier norms, denoted $A_M(G)$ and $A_{cb}(G)$ respectively. We prove that if a nonzero closed ideal in $A_M(G)$ or $A_{cb}(G)$ is Arens regular then G must be a discrete group.

Amenable Banach algebras were first studied by B.E. Johnson in 1972. For an amenable Banach algebra $\mathcal{A}$ we can consider its amenability constant $AM(\mathcal{A}) \geq 1$. We are particularly interested in collections of amenable Banach algebras for which there exists a constant $\lambda > 1$ such that the values in the interval $(1, \lambda)$ cannot be attained as amenability constants. If G is a compact group, then the central Fourier algebra is defined as $ZA(G) = ZL^1(G) \cap A(G)$ and endowed with the $A(G)$ norm. We study the amenability constant theory of $ZA(G)$ when G is a finite group.

Acknowledgements

> So, what do you say? Why not help one another on this lonely journey?
>
> *Solaire*
> **Dark Souls**

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Finally, to my advisors Brian Forrest and Matthew Wiersma, thank you. I spent most of my graduate career in a state of doubt, uncertainty, and non-existent self-confidence, and your steady encouragement has been my lifeline. You've always been kind and patient while I frantically try to justify my lack of progress since our last meeting, and I feel so lucky to call you my advisors. I would be amiss to not also mention my Master's supervisor, Nico Spronk, who brought me into the abstract harmonic analysis fold in the first place.

Dedication

This thesis is dedicated to my grandparents: Shirley Wilson, Doris Sawatzky, and Dr. Anton Sawatzky.

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Chapter 1

Introduction

Raise thy sword by the light,
and head to the place where the
sword's light gathers.

Dormin
Shadow of the Colossus

Much of abstract harmonic analysis is concerned with Banach algebras associated with a locally compact group G, and studying the relationship between Banach algebra properties and group properties. Two of the most important examples of Banach algebras arising from locally compact groups are the Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$, which were introduced by Pierre Eymard in [23, 1964]. For an abelian group we have the identification $A(G) \cong L^1(\widehat{G})$ where $\widehat{G}$ is the Pontryagin dual of G, using a generalization of the notion of the Fourier transform. A famed theorem of Leptin states that $A(G)$ has a bounded approximate identity if and only if G is amenable, demonstrating a deep link between the properties of $A(G)$ as an algebra and G as a group; indeed, $A(G)$ and $A(H)$ are isometrically isomorphic if and only if G and H are isomorphic groups [74] - a property shared by the group algebra $L^1(G)$ [75].

This thesis has two major themes. The first is studying amenability constants and gaps results of a particular subalgebra of $A(G)$, and the second being properties of the second dual of the closure of $A(G)$ in multiplier and completely bounded norms.

The study and notion of Banach algebra amenability was initiated by B.E. Johnson in [43, 1972], and from the beginning algebras of functions on groups have been featured as key examples in the theory. In Johnson's seminal work he proved that $L^1(G)$ was amenable if and only if G was amenable, and in [44, 1992] he provided an example that showed that the same did not hold for $A(G)$: indeed, it's possible for $A(G)$ to be non-abelian even if G is compact. Amenable algebras have an associated amenability constant $AM(\cdot)$: in the case of $L^1(G)$ this constant is always 1, while for $A(G)$ Johnson proved that for finite groups $AM(A(G)) = 1$ if and only if G was abelian, and if G was non-amenable then $AM(A(G)) \geq \frac{3}{2}$. This result was later generalized in [60, 2006] and [14, 2022] to hold for all groups such that $A(G)$ is amenable. Zhong-Jin Ruan introduced an alternative characterization of amenability that takes into account the operator space structure of an algebra in [58, 1995] called operator amenability. The Fourier algebra behaves more "nicely" with respect to this notion: $A(G)$ is operator amenable if and only if G is amenable, in which case $A(G)$ has an operator amenability constant of 1.

In [56, 1973] Daniel Rider proved that if ψ was a central idempotent in $L^1(G)$ for a compact group G such that $\|\psi\|_1 > 1$, then $\|\psi\|_1 > 1 + \frac{1}{300}$. This bound is certainly not sharp; a footnote by Yemon Choi in [13, 2016] notes that it is possible to improve the bounded to $1 + \frac{1}{80}$. However, the question of what is the correct sharp bound is open. Numerical calculations in Chapter 6 suggest that $\frac{1+\sqrt{2}}{2}$ is a sharp bound, at least for the case of finite groups. This is reminiscent of a bound discovered by Saeki in [65, 1986], where it was shown for an idempotent measure μ on a compact abelian group with $\|\mu\| > 1$, then $\|\mu\| \geq \frac{1+\sqrt{2}}{2}$ - and in this case, the bound is sharp.

In the study of amenability constants of algebras defined on groups, there have been natural questions about the relationship between amenability constant values and hereditary properties of the underlying groups. A conjecture in the original version of [50] asked if $AM(A(G/N)) \leq AM(A(G))$ for a closed normal subgroup N of G, which was later noted to be false via a reference to the Atlas of Finite Groups [16]. Curiously, in the counterexample given $|G| = 2160$ and $|G/N| = 1080$, although as we consider in an example in Chapter 4, there is a much simpler counterexample with $|G| = 32$ and $|G/N| = 16$. This demonstrates the utility of being able to search for counterexample using computers, which is a major theme throughout the amenability constant content featured in this thesis.

In [8, 2008] Azimifard, Samei, and Spronk investigated amenability properties of the centre of the group algebra, $ZL^1(G)$. In particular, they demonstrated that in the case where G is finite then because $ZL^1(G)$ is finite dimensional calculating the amenability constant amounts to calculating the L^1 norm of a specific idempotent in $ZL^1(G)$, which can be done by applying a formula that only requires knowledge of the irreducible character theory of G. They apply Rider's rudimentary $1+\frac{1}{300}$ gap result to show that the amenability constant of $ZL^1(G)$ is equal to 1 if and only if G is abelian, and in the case that G is non-abelian then the amenability constant is greater than $1 + \frac{1}{300}$.

For a compact group G, $ZL^1(G)$ can be viewed as a hypergroup algebra of the conjugacy classes of G. The dual of this hypergroup is $\widehat{G}$, the irreducible representations (up to unitary equivalence) of G. The hypergroup algebra of $\widehat{G}$ is $ZA(G)$, the central Fourier algebra of G. For a finite group $ZA(G)$ is just the class functions with the Fourier algebra norm. In this context $ZA(G)$ coincides as a set with $ZL^1(G)$, albeit with differing norms and multiplication. In [5, 2016], Mahmood Alaghmandan and Nico Spronk produced a formula for $AM(ZA(G))$ that mirrors the formula for $AM(ZL^1(G))$ from [8], and furthermore proved that $AM(ZA(G))$ has a gap constant of $\frac{2}{\sqrt{3}}$ (although very importantly, this bound was not claimed to be sharp), which notably at the time this was an improvement over the $1 + \frac{1}{300}$ gap known for $ZL^1(G)$. Fifteen days after this result was uploaded to arXiv, Yemon Choi uploaded [13, 2016], which built on [5] to show that $\frac{7}{4}$ is the sharp bound for $ZL^1(G)$.

In Chapter 2 we introduce basic notation and background information from abstract harmonic analysis, representation theory, and operator algebras that will be used throughout this thesis. Material from first-courses in abstract algebra, topology, and functional analysis will typically be used without comment, although the interested reader can refer to [20], [53], and [17] for introductions to those subjects.

In Chapter 3 we compile an overview of what is known about the amenability constant theory of a variety of well-known Banach algebras associated with groups, semigroups, and hypergroups.

Chapter 4 features an investigation into amenability constants of $ZA(G)$. The connection between $AM(ZL^1(G))$ and $AM(ZA(G))$ allows for much of the theory from $ZL^1(G)$ to be carried over to $ZA(G)$, however there are some key differences. Choi's proof in [13] depended on the fact that if N is a normal subgroup of G then $AM(ZL^1(G/N)) \leq AM(ZL^1(G))$. At first glance it would be reasonable to guess that this property should also hold for $AM(ZA(G))$, although as shown in this chapter that actually isn't the case. While the failure of the quotient hereditery property is at the moment a barrier preventing a proof of a sharp bound for $AM(ZA(G))$, much other work be carried over from the $ZL^1(G)$ case. For example, the class of groups for which $AM(ZA(G))$ respects quotients will have the $\frac{7}{4}$ gap. Furthermore, in [5] a formula was developed that calculates $AM(ZL^1(G))$ for a group with only two character degrees, while we are able to show that if instead one takes a group with only two conjugacy class sizes then a dual formula can be developed for $AM(ZA(G))$. These two formulas agree if a finite group has both two conjugacy class sizes and two character degrees.

For a Banach algebra $\mathcal{A}$, Arens demonstrated in [6, 1951] that it is possible to define two natural multiplication operations on the double dual $\mathcal{A}^{**}$, denoted in this thesis by $\square$ and $\odot$. Arens proved that $(\mathcal{A}^{**}, \square)$ and $(\mathcal{A}^{**}, \odot)$ are both Banach algebras, and if $\square$ and $\odot$ agree then the algebra $\mathcal{A}$ is said to be Arens regular. It is known that C^*-algebras are always Arens regular but for many algebras considered in abstract harmonic analysis it turns out that Arens regularity is a very restrictive condition. The group algebra $L^1(G)$ is Arens regular if and only if G is finite, and as proven by Forrest in [26, 1991] for $1 < p < \infty$ if the Figà-Talamanca-Herz algebra $A_p(G)$ is Arens regular then G must be a discrete group. When $p = 2$ then $A_2(G) = A(G)$, in which case a closed non-zero ideal in $A(G)$ being Arens regular is sufficient to force G to be discrete.

Being the predual of a von Neumann algebra, $A(G)$ has a natural operator space structure. Taking the norm closure of $A(G)$ within the multiplier and completely bounded multipliers of $A(G)$ results in the algebras $A_M(G)$ and $A_{cb}(G)$ respectively. The algebra $A_{cb}(G)$ was first studied by Brian Forrest in [28, 2004], and $A_M(G)$ was later studied in [29, 2013]. These algebras are more manageable to work with compared to $MA(G)$ and $M_{cb}(G)$, and share a number of properties with $A(G)$. Indeed, in Chapter 5 we study the Arens regularity of ideals of $A_M(G)$ and $A_{cb}(G)$. By showing that there is a bijection between the sets of topologically invariant means of $A(G), A_M(G)$, and $A_{cb}(G)$, we prove that any of these algebras having a unique topologically invariant mean is sufficient for the group G being discrete. Indeed, this condition is satisfied even when the only assumption is that an ideal is Arens regular, extending Forrest's result for $A(G)$ in [26].

Chapter 6 expands on the use of computational tools applied in Chapter 3 to other problems in abstract harmonic analysis. We provide insight on an approach to improving Rider's $1 + \frac{1}{300}$ bound on idempotents in $ZL^1(G)$, include Sage code for calculating norms of arbitrary functions in $A(G)$, and consider the collection of groups with absolutely idempotent characters.

Chapter 2

Preliminaries

Link
The Legend of Zelda

2.1 Banach Algebras

Definition 2.1.1. Let $\mathcal{A}$ be a Banach space over $\mathbb{C}$ with norm $\|\cdot\|$. If $\mathcal{A}$ is also an associative algebra that satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$ then we say that $\mathcal{A}$ is a *Banach algebra*.

Example 2.1.2. Let X be a locally compact Haussdorff space and denote the continuous complex-valued functions on X by $C(X)$. Let

- $C_0(X) = \{f \in C(X) : f \text{ vanishes at infinity}\}$
- $C_b(X) = \{f \in C(X) : f \text{ is bounded}\}$
- $C_C(X) = \{f \in C(X) : \text{supp}(f) \text{ is compact}\}$.

Each of $C_0(X)$ and $C_b(X)$ are Banach algebras with respect to pointwise multiplication of functions when endowed with the norm

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

In the case that X is itself compact then $C_0(X) = C(X)$ and the constant function $1(x) = 1$ is an identity for each algebra.

Definition 2.1.3. We call a Banach algebra $\mathcal{A}$ a C*-algebra if it posses a map $\mathcal{A} \to \mathcal{A}, a \mapsto a^*$ satisfying for $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$:

1. $(a^*)^* = a$
2. $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$
3. $(ab)^* = b^* a^*$
4. $\|aa^*\| = \|a\|^2$.

Definition 2.1.4. For a Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ be the space of the bounded linear operators on $\mathcal{H}$. A C^*-algebra $\mathcal{M}$ is called a *von Neumann algebra* if there is a Hilbert space $\mathcal{H}$ such that $\mathcal{M}$ is $*$-isomorphic to a unital subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the weak operator topology (WOT).

Theorem 2.1.5. *If $\mathcal{M}$ is a von Neumann algebra then there exists a (up to isomorphism) unique predual $\mathcal{M}_*$ of $\mathcal{M}$.*

Definition 2.1.6. Let $\mathcal{A}$ be a commutative Banach algebra. We define the *spectrum* of $\mathcal{A}$ as

$$\Delta(\mathcal{A}) = \{\phi : \mathcal{A} \to \mathbb{C} : \phi \text{ a nonzero algebra homomorphism}\}.$$

The maps in $\Delta(\mathcal{A})$ are called *characters*.

Remark 2.1.7. If $\mathcal{A}$ is in fact a unital Banach algebra there is a correspondence between $\Delta(\mathcal{A})$ and the space of maximal closed ideals of $\mathcal{A}$ with each maximal ideal being associated with the kernel of a character.

Definition 2.1.8. Let $\mathcal{A}$ be a Banach algebra and let E be a Banach space that is a left (right) module over $\mathcal{A}$ with action

$$\mathcal{A} \times E \to E, \quad (a, x) \mapsto a \cdot x, \quad E \times \mathcal{A} \to E, \quad (x, e) \mapsto x \cdot a.$$

If this action is bounded then we call E a *left (right) Banach $\mathcal{A}$-module.* If E is both a left Banach $\mathcal{A}$-module and a right Banach $\mathcal{A}$-module then we say that E is a *Banach $\mathcal{A}$-bimodule*

Example 2.1.9. Naturally, $\mathcal{A}$ is itself a Banach $\mathcal{A}$-bimodule where the left and right actions are just left and right multiplications respectively.

Example 2.1.10. Let $\mathcal{A}$ be a Banach algebra and E a Banach $\mathcal{A}$-bimodule. Then E^* is itself a Banach $\mathcal{A}$-bimodule with actions

- $\langle a \cdot \phi, x\rangle = \langle \phi, x \cdot a\rangle$ where $a \in \mathcal{A}, x \in E, \phi \in E^*$.
- $\langle \phi \cdot a, x\rangle = \langle \phi, a \cdot x\rangle$ where $a \in \mathcal{A}, x \in E, \phi \in E^*$.

Given a Banach algebra $\mathcal{A}$, there is not necessarily any clear way of defining a Banach algebra structure on the dual $\mathcal{A}^*$. However, it turns out that there are two natural ways of considering the double dual $\mathcal{A}^{**}$ as a Banach algebra. Consider the following operations:

1a) $\langle u \cdot T, v\rangle = \langle T, vu\rangle$ for every $u, v \in \mathcal{A}$ and $T \in \mathcal{A}^*$.

1b) $\langle T \odot m, u\rangle = \langle m, u \cdot T\rangle$ for every $u \in \mathcal{A}$ and $T \in \mathcal{A}^*$ and $m \in \mathcal{A}^{**}$.

1c) $\langle m \odot n, T\rangle = \langle n, T \odot m\rangle$ for every $T \in \mathcal{A}^*$ and $m, n \in \mathcal{A}^{**}$.

2a) $\langle T \Box u, v\rangle = \langle T, uv\rangle$ for every $u, v \in \mathcal{A}$ and $T \in \mathcal{A}^*$.

2b) $\langle m \Box T, u\rangle = \langle m, T \Box u\rangle$ for every $u \in \mathcal{A}$ and $T \in \mathcal{A}^*$ and $m \in \mathcal{A}^{**}$.

2c) $\langle m \Box n, T\rangle = \langle m, n \Box T\rangle$ for every $T \in \mathcal{A}^*$ and $m, n \in \mathcal{A}^{**}$.

The operations $\odot$ and $\Box$ are both valid Banach algebra multiplication operations on $\mathcal{A}^{**}$, and if they agree we call $\mathcal{A}$ an *Arens regular algebra*. Unless specified otherwise, we will assume to use the $\odot$ operation.

Definition 2.1.11. Let $\mathcal{A}$ be a Banach algebra and let $(e_\alpha)_\alpha$ be a net in the $\mathcal{A}$. We say that $(e_\alpha)_\alpha$ is a *bounded approximate identity* if $\sup_\alpha \|e_\alpha\| < \infty$ and

$$\lim_\alpha \|e_\alpha a - a\| = \lim_\alpha \|a e_\alpha - a\| = 0 \text{ for all } a \in \mathcal{A}.$$

Theorem 2.1.12 (Cohen's Factorization Theorem)**.** *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity $(e_\alpha)_\alpha$ and E a Banach $\mathcal{A}$ bimodule. If $x \in E$ and $e_\alpha \cdot x \to x$ then there exists $a \in \mathcal{A}$ and $y \in E$ such that $x = a \cdot y$.*

Definition 2.1.13. We call the space

$$UCB(\mathcal{A}) = \overline{\text{span}\{v \cdot T : v \in \mathcal{A}, T \in \mathcal{A}^*\}}^{-\|\cdot\|_{\mathcal{A}^*}}$$

the *uniformly continuous functionals on* $\mathcal{A}$.

We call $T \in \mathcal{A}^*$ a *(weakly) almost periodic functional* on $\mathcal{A}$ if

$$\{u \cdot T : u \in \mathcal{A}, \|u\|_{\mathcal{A}} \leq 1\}$$

is relatively (weakly) compact in $\mathcal{A}^*$ and we denote the space of all (weakly) almost periodic functionals on $\mathcal{A}$ by $AP(\mathcal{A})$ ($WAP(\mathcal{A})$).

It is well-known that $T \in \mathcal{A}^*$ is weakly almost periodic if and only if given two nets $\{u_\alpha\}_{\alpha\in\Omega_1}$ and $\{v_\beta\}_{\beta\in\Omega_2}$ in $\mathcal{A}$ we have that

$$\lim_\alpha \lim_\beta \langle T, u_\alpha v_\beta\rangle = \lim_\beta \lim_\alpha \langle T, u_\alpha v_\beta\rangle$$

whenever both limits exist. It follows that $\mathcal{A}$ is Arens regular if and only if $WAP(\mathcal{A}) = \mathcal{A}^*$.

Definition 2.1.14. We say that a closed subspace $X \subseteq \mathcal{A}^*$ is *invariant* if $u \cdot T \in X$ for every $u \in \mathcal{A}$ and $T \in X$.

Given a closed invariant subspace X of $\mathcal{A}^*$ and an $m \in X^*$, we define the linear operator $m_L : X \to \mathcal{A}^*$ by

$$\langle m_L(T), u\rangle = \langle m, u \cdot T\rangle$$

for every $T \in X$ and $u \in \mathcal{A}$. We say that X is *topologically introverted* if $m_L(T) \in X$ for every $m \in X^*$ and $T \in X$.

If X is topologically introverted then X^* can be made into a Banach algebra with the $\odot$ operation as follows:

1) For each $T \in X$ and $m \in X^*$, we define $T \odot m = m_L(T)$.

2) For each $T \in X$ and $n, m \in X^*$, we define $\langle m \odot n, T\rangle = \langle n, T \odot m\rangle$.

It is well-known that $AP(\mathcal{A}), WAP(\mathcal{A})$, and $UCB(\mathcal{A})$ are closed introverted subspaces of $\mathcal{A}^*$.

Definition 2.1.15. Let $\mathcal{A}$ be a commutative Banach algebra with maximal ideal space $\Delta(\mathcal{A})$. Let X be a closed submodule of $\mathcal{A}^*$ containing $\phi \in \Delta(\mathcal{A})$. Then $m \in X^*$ is called a *topologically invariant mean* (TIM) on X at ϕ if

i) $\|m\|_{X^*} = \langle m, \phi\rangle = 1$,

ii) $\langle m, v \cdot T \rangle = \phi(v)\langle m, T \rangle$ for every $v \in \mathcal{A}$ and $T \in X$.

We denote the set of topologically invariant means on X at ϕ by $TIM_{\mathcal{A}}(X, \phi)$.

Definition 2.1.16. Let I be a closed ideal in a commutative Banach algebra $\mathcal{A}$. We let

$$Z(I) = \{x \in \Delta(\mathcal{A}) : \ u(x) = 0 \text{ for all } u \in I\}.$$

Given a closed set $E \subseteq \Delta(\mathcal{A})$, we let

$$I(E) = \{u \in \mathcal{A} : \ u(x) = 0 \text{ for all } x \in E\}$$

and

$$I_0(E) = \{u \in \mathcal{A} \cap C_C(\Delta(\mathcal{A})) : \ \text{supp}(u) \cap E = \emptyset\}.$$

We say that a closed set $E \subseteq \Delta(\mathcal{A})$ is a *set of spectral synthesis* for $\mathcal{A}$ if the only closed ideal I of $\mathcal{A}$ with $Z(I) = E$ is $I(E)$. This condition is well-known to be equivalent to the statement that $I_0(E)$ is dense in $I(E)$.

2.2 Operator Spaces

Let $\mathcal{H}$ be a Hilbert space. A closed subspace $V \subseteq \mathcal{B}(\mathcal{H})$ is called a (concrete) *operator space*. For $n \in \mathbb{N}$ there is an induced norm $\|\cdot\|_n$ on $M_n(V)$, the space of $n \times n$ matrices on X via the inclusion

$$M_n(X) \subseteq M_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^n)$$

that satisfies the following properties for $v \in M_m(V), w \in M_n(V), \alpha \in M_{n,m}, \beta \in M_{m,n}$:

$$\|v \oplus w\| = \max\{\|v\|_m, \|w\|_n\} \tag{2.1}$$

and

$$\|\alpha v \beta\|_n \leq \|\alpha\| \cdot \|v\|_m \cdot \|\beta\|. \tag{2.2}$$

Equivalently, an (abstract) operator space is a normed vector space V along with a sequence of matricial norms $\|\cdot\|_n$ on $M_n(V)$ that satisfies 2.1 and 2.2.

For two operator spaces V and W and a linear map $\phi : V \to W$ we define the amplification map

$$\phi : M_n(V) \to M_n(W)$$

by

$$\phi_n([v_{i,j}]) = [\phi(v_{i,j})]$$

for $[v_{i,j}] \in M_n(V)$. We say that ϕ is *completely bounded* if

$$\|\phi\|_{cb} := \sup\{\|\phi_n\| : n \in \mathbb{N}\} < \infty.$$

A completely bounded map ϕ is called *completely contractive* if

$$\|\phi\|_{cb} \leq 1.$$

We will denote the space of all completely bounded maps from V into W by $\mathcal{CB}(V, W)$. If $V = W$ then we will just write $\mathcal{CB}(V)$ instead. The space $\mathcal{CB}(V, W)$ is itself an operator space via the isometric identification

$$M_n(\mathcal{CB}(V, W)) \cong \mathcal{CB}(V, M_n(W)).$$

For a linear function f on an operator space V, it is known that f is completely bounded if and only if it is bounded and $\|f\| = \|f\|_{cb}$. This induces a canonical operator space structure on $V^* \cong \mathcal{CB}(V, \mathbb{C})$.

All C^*-algebras are automatically operator spaces, and furthermore if M is a von Neumann algebra and M_* is the unique predual of M then the canonical identification of M_* into its second dual M^* induces an operator structure on M_*.

2.3 Banach Algebras Amenability

Let X and Y be Banach spaces, and let $u \in X \otimes Y$. Then define the *projective tensor product norm* on $X \otimes Y$ by

$$\|u\|_\gamma = \inf\left\{\sum_{i=1}^{n} \|x_i\| \cdot \|y_i\| : u = \sum_{i=1}^{n} \otimes y_i\right\}.$$

We call the completion of $X \otimes Y$ with respect to this norm the *projective tensor product* of X and Y, which is denoted $X \otimes^\gamma Y$. There are natural module actions of $\mathcal{A}$ on $\mathcal{A} \otimes^\gamma \mathcal{A}$ that are given by

$$a \cdot (b \otimes c) = (ab) \otimes c, \text{ where } a, b, c \in \mathcal{A}$$

and

$$(b \otimes c) \cdot a = b \otimes (ca) \text{ where } a, b, c \in \mathcal{A}.$$

If V and W are taken to be operator spaces, a similar construction can be done that incorporates the operator space structure. For $u \in M_n(V \otimes W)$, we define the *operator projective tensor product norm* by

$$\|u\|_\wedge = \inf\{\|\alpha\| \cdot \|v\| \cdot \|w\| \cdot \|\beta\| : u = \alpha(v \otimes w)\beta\}$$

where $v \in M_p(V), w \in M_q(W), \alpha \in M_{n,pq}, \beta \in M_{pq,n}$. We will refer to the completion of $V \otimes W$ with respect to $\|\cdot\|_\wedge$ by *the operator projective tensor product* of V and W, and we will denote this space by $V \widehat{\otimes} W$. There is a canonical identification

$$(V \widehat{\otimes} W)^* \cong \mathcal{CB}(V, W^*),$$

which demonstrates that $V \widehat{\otimes} W$ is itself an operator space.

We define the multiplication map $m : A \otimes A \to A$ by $m(a \otimes b) = ab$. As possible, we can extend m to maps $m : A \otimes^\gamma A \to A$ and $m : A \widehat{\otimes} A \to A$ respectively.

Let $\mathcal{A}$ be an operator space. We say that $\mathcal{A}$ is a *completely contractive* (*quantized*) Banach algebra if $m : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{A}$ is completely contractive (completely bounded). Let $\mathcal{A}$ be completely contractive Banach algebra and let V be an $\mathcal{A}$-bimodule. Then V is called an operator $\mathcal{A}$-bimodule if V is an operator space and the $\mathcal{A}$-bimodule operations

$$\mathcal{A} \widehat{\otimes} V \to V, \quad a \otimes v \mapsto a \cdot v$$

and

$$V \widehat{\otimes} \mathcal{A} \to V, \quad v \otimes a \mapsto v \cdot a$$

are completely bounded.

The notion of amenability of a Banach algebra was first introduced by B.E. Johnson in [43], and has been a fruitful area of research ever since. A brief introduction is offered here, but we recommend [62] for a more detailed survey of amenability.

Definition 2.3.1. Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \to X$ is called a *derivation* if for all $a, b \in \mathcal{A}$ then

$$D(ab) = a \cdot D(b) + D(a) \cdot b.$$

A derivation D is called an *inner derivation* if there exists $x \in X$ such that

$$D(a) = a \cdot x - x \cdot a.$$

Remark 2.3.2. If X is a Banach $\mathcal{A}$-bimodule then X^* is also a Banach $\mathcal{A}$-bimodule via the actions

$$\langle f \cdot a, x \rangle = \langle f, a \cdot x \rangle$$

and

$$\langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle,$$

where $a \in \mathcal{A}, x \in X$, and $f \in X^*$.

Definition 2.3.3. A *bounded approximate diagonal* (b.a.d.) for $\mathcal{A}$ is a bounded net $(d_\alpha)_\alpha$ in $\mathcal{A} \otimes^\gamma \mathcal{A}$ such that for $a \in \mathcal{A}$ then

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0 \tag{2.3}$$

and

$$am(d_\alpha) \to a. \tag{2.4}$$

Similarly, a *virtual diagonal* for $\mathcal{A}$ is defined as an element $D \in (\mathcal{A} \otimes^\gamma \mathcal{A})^{**}$ such that for $a \in \mathcal{A}$ the following holds:

$$a \cdot D = D \cdot a \tag{2.5}$$

and

$$a \cdot m^{**}D = a. \tag{2.6}$$

Definition 2.3.4. We call a Banach algebra $\mathcal{A}$ *amenable* if one of the following equivalent conditions holds.

1. For every Banach $\mathcal{A}$-bimodule X, every derivation $D : \mathcal{A} \to X^*$ is inner.
2. $\mathcal{A}$ has a bounded approximate diagonal.
3. $\mathcal{A}$ has a virtual diagonal.

We define the *amenability constant* of $\mathcal{A}$ by

$$AM(\mathcal{A}) = \inf\{\sup_\alpha \|\omega_\alpha\| : (\omega_\alpha) \text{ is a b.a.d. for } \mathcal{A}\}.$$

If $\mathcal{A}$ is not amenable then we set $AM(\mathcal{A}) = \infty$.

Proposition 2.3.5. *Let A and B be Banach algebras and let $\phi : A \to B$ be a continuous homomorphism with dense range. Then*

$$AM(A) \leq \|\phi\|^2 AM(B).$$

Here are some useful heriditery properties of amenability with respect to closed ideals, see [18, Proposition 2.4] and [62, Proposition 2.3.3].

Proposition 2.3.6. *Let $\mathcal{A}$ be an amenable Banach algebra and I a closed ideal in $\mathcal{A}$.*

i) If $\mathcal{A}$ has an identity $e_{\mathcal{A}}$ then $AM(\mathcal{A}) \geq \|e_{\mathcal{A}}\|$.

ii) If I has an identity e_I then I is amenable and $AM(I) \leq \|e_I\| AM(\mathcal{A})$.

iii) If e_α is a bounded approximate identity for I with bound C then I is amenable and

$$AM(I) \leq C^2 AM(\mathcal{A}).$$

iv) $AM(\mathcal{A}/I) \leq AM(\mathcal{A})$.

Remark 2.3.7. A copy of $\mathcal{A} \otimes^\gamma \mathcal{A}$ can be identified as living inside of $(\mathcal{A} \otimes^\gamma \mathcal{A})^{**}$. If there is a virtual diagonal which is an element of $\mathcal{A} \otimes^\gamma \mathcal{A} \subseteq (\mathcal{A} \otimes^\gamma \mathcal{A})^{**}$ then we call it a *diagonal*. In the case that $\mathcal{A}$ is a finite-dimensional commutative amenable Banach algebra, then $\mathcal{A}$ possesses a unique diagonal [34].

Definition 2.3.8. A collection of Banach algebras $\mathcal{C}$ is said to have an *amenability constant gap* if there exists $\lambda > 1$ such that for $A \in \mathcal{C}$, $AM(A) = 1$ if and only if $AM(A) < \lambda$. Any λ with this property is called an *amenability constant bound* (for $\mathcal{C}$), and if λ is the supremum of all possible amenability constant bounds then it is called the *sharp bound* (for $\mathcal{C}$).

Definition 2.3.9. We call a Banach algebra $\mathcal{A}$ *weakly amenable* if every derivation $D : \mathcal{A} \to \mathcal{A}^*$ is inner, where $\mathcal{A}$ is being viewed as a Banach $\mathcal{A}$-bimodule over itself.

Ruan introduced in [58, 1995] the notion of *operator amenability*, which takes a completely contractive Banach algebra's operator space structure into account.

Definition 2.3.10. Let $\mathcal{A}$ be a completely contractive Banach algebra. An *operator bounded approximate diagonal* (o.b.a.d.) for $\mathcal{A}$ is a bounded net $(d_\alpha)_\alpha$ in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that for $a \in \mathcal{A}$ then

$$a \cdot d_\alpha - d_\alpha \cdot a \to 0 \tag{2.7}$$

and

$$am(d_\alpha) \to a. \tag{2.8}$$

Similarly, an *operator virtual diagonal* for $\mathcal{A}$ is defined as an element $D \in (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ such that for $a \in \mathcal{A}$ the following holds:

$$a \cdot D = D \cdot a \tag{2.9}$$

and

$$a \cdot m^{**}D = a. \tag{2.10}$$

Definition 2.3.11. Let $\mathcal{A}$ be a completely contractive Banach algebra. We call $\mathcal{A}$ *operator amenable* if any of the following equivalent conditions is satisfied:

1. For every operator $\mathcal{A}$-bimodule V, every completely bounded derivation $D : \mathcal{A} \to E^*$ is inner.
2. $\mathcal{A}$ has an operator bounded approximate diagonal.
3. $\mathcal{A}$ has an operator virtual diagonal.

We define the *operator amenability constant* of $\mathcal{A}$ by

$$AM_{op}(\mathcal{A}) = \inf\{\sup_\alpha \|\omega_\alpha\| : (\omega_\alpha) \text{ is an o.b.a.d. for } \mathcal{A}\}.$$

If $\mathcal{A}$ is not operator amenable then we set $AM_{op}(\mathcal{A}) = \infty$.

Many properties of amenability also hold for operator amenability after making suitable adjustments. For example, Proposition 2.3.5 also holds in the operator space context:

Proposition 2.3.12. *Let A and B be completely contractive Banach algebras and let $\phi \in \mathcal{CB}(A, B)$ be a homomorphism with dense range. Then*

$$AM_{cb}(A) \leq \|\phi\|^2 AM_{cb}(B).$$

2.4 Locally Compact Groups

Throughout this thesis we will investigate properties of Banach algebras associated with groups, which will typically take the form of algebras of complex-valued functions on a given group. We will first review some background material from abstract harmonic analysis. See [25] or [38] for a more complete introduction to the subject.

Definition 2.4.1. Let G be a group equipped with a locally compact and Hausdorff topology such that the map $(a, b) \mapsto ab^{-1}$ from $G \times G \to G$ is continuous. Then we call G a *locally compact group.*

The motivation for considering specifically locally compact groups comes from the following essential theorem.

Theorem 2.4.2 (Existence of Haar Measure)**.** *If G is a locally compact group, then there exists a positive Radon measure μ that satisfies the condition that if $x \in G$ and E is measurable, then $\mu(xE) = \mu(E)$. Furthermore, μ is unique up to scaling. We call μ a* Haar measure *on G.*

Example 2.4.3.

1. If G is discrete then counting measure is a Haar measure on G.
2. If $G = \mathbb{Z}$ is viewed as a group with respect to addition then G is a discrete abelian group.
3. If G is compact then we will always work with the normalized Haar measure μ such that $\mu(G) = 1$.
4. If $G = \mathbb{R}^n$ with the normal topology, then Lebesgue measure is a Haar measure on G.
5. For a given integer n the group $GL(n, \mathbb{C})$ of complex-valued invertible $n \times n$ matrices with respect to matrix multiplication is locally compact, and non-abelian if $n > 1$.
6. Let $\mathbb{F}_n$ be the free group on n generators. Then $\mathbb{F}_n$ is a discrete, non-abelian group.
7. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Then $\mathbb{T}$ is a compact abelian group with respect to multiplication and the normal topology on $\mathbb{C}$.
8. Let G be an abelian group. Then

$$\widehat{G} = \{\sigma : G \to \mathbb{T} : \sigma \text{ a continuous homomorphism}\}$$

is a locally compact group with respect to the topology of uniform convergence on compact sets called the *dual group* of G. It's known that $\widehat{\widehat{G}} \cong G$ and $\widehat{\mathbb{Z}} \cong \mathbb{T}$.

Let G be a locally compact group and $1 \leq p \leq \infty$. We can define the Banach spaces $L^p(G, \mu)$ in the usual way, although for convenience we will just write $L^p(G)$ instead.

Example 2.4.4. We can define a *convolution product* on $L^1(G)$ by

$$f * g(x) = \int_G f(xy)g(y^{-1})d\mu(y), \text{ where } f, g \in L^1(G)$$

which allows us to consider $L^1(G)$ as a Banach algebra.

For a locally compact group G, let $\{U_\alpha\}_\alpha$ be a neighborhood basis of e consisting of relatively compact sets. Then $\left(\frac{1}{\mu(U_\alpha)} 1_{U_\alpha}\right)_\alpha$ is an approximate identity for $L^1(G)$ bounded by 1.

Example 2.4.5. $L^\infty(G)$ is a commutative von Neumann algebra with respect to pointwise (almost everywhere) function multiplication.

It is known that $L^\infty(G)$ can be identified with $L^1(G)^*$ via the dual pairing

$$\langle f, \phi \rangle = \int_G \phi f d\mu, \quad \text{where} f \in L^1(G), \phi \in L^\infty(G).$$

As per Theorem 2.1.5 this implies that $L^1(G)$ is the unique predual of $L^\infty(G)$. If $p = 2$ then $L^2(G)$ forms a Hilbert space with inner product $\langle f, g \rangle = \int_G f \overline{g} d\mu$ for $f, g \in L^2(G)$.

Example 2.4.6. By the Riesz representation theorem we know that the space of complex Radon measures on G, denoted $M(G)$, can be identified with $C_0(G)^*$ via the pairing

$$\langle f, \nu \rangle = \int_G f d\nu, \quad \text{where } f \in C_0(G), \nu \in M(G).$$

For $\nu, \rho \in M(G)$ we can define a product $\nu * \rho$ by

$$\langle f, \nu * \rho \rangle = \int_G \int_G f(xy) d\nu(x) d\rho(y), \quad \text{where } f \in C_0(G).$$

We can actually view $L^1(G)$ isometrically isomorphically as a subalgebra of $M(G)$ by associating $g \in L^1(G)$ with a measure in the following way

$$\langle f, g \rangle = \int_G fg d\mu(x), \quad f \in C_0(G),$$

where μ is Haar measure on G. Under this identification $f * g$ is the same for $f, g \in L^1(G)$ whether the convolution is undertaken in $L^1(G)$ or $M(G)$.

Definition 2.4.7. Let G be a locally compact group. A *left-invariant mean* on G is a positive functional $M \in L^\infty(G)^*$ that satisfies the following conditions:

i) $M(1) = \|M\| = 1$

ii) $M(x \cdot f) = M(f)$, where $x \cdot f(y) = f(x^{-1}y)$ for $x, y \in G$ and $f \in L^\infty(G)$.

If a left-invariant mean for G exists, then we say that G is an *amenable* group.

There are many equivalent characterizations of amenability. One of particular importance proposed by Reiter is as follows:

Proposition 2.4.8. *G is amenable if and only if there exists a net $(f_\alpha)_\alpha$ in $L^1(G)$ with $f_\alpha \geq 0, \|f_\alpha\|_1 = 1$ and*

$$\|\delta_x * f_\alpha - f_\alpha\|_1 \to 0 \qquad \textit{uniformly on compact sets,}$$

where $\delta_x \in M(G)$ is the point-mass measure on x.

Proposition 2.4.9. *Let G be an amenable locally compact group.*

i) If H is a closed subgroup of G then H is amenable.

ii) If N is a normal subgroup of G then G/N is amenable.

iii) If $G_1, ..., G_n$ are all amenable then $\prod_{i=1}^{n} G_i$ is amenable.

iv) Every compact or abelian group is amenable.

Example 2.4.10. The free group $\mathbb{F}_2 = \langle a, b\rangle$ is not amenable. Supposing otherwise, let $M \in L^\infty(\mathbb{F}_2)^*$ be a left-invariant mean. For $x \in \{a, b, a^{-1}, b^{-1}\}$ let E_x denote the set of reduced words in $\mathbb{F}_2$ beginning with x. It is clear that

$$\mathbb{F}_2 = \{e\} \cup E_a \cup E_b \cup E_{a^{-1}} \cup E_{b^{-1}}.$$

Observe also two possible other ways of writing $\mathbb{F}_2$ as a disjoint union:

$$\mathbb{F}_2 = E_a \cup aE_{a^{-1}} = E_b \cup bE_{b^{-1}}.$$

Then

$$1 = M(1_{E_a}) + M(1_{aE_{a^{-1}}}) = M(1_{E_b}) + M(1_{bE_{b^{-1}}}).$$

Applying the left-invariance of M yields a contraction because

$$1 = M(1_{\{e\}}) + M(1_{E_a}) + M(1_{E_{a^{-1}}}) + M(1_{E_b}) + M(1_{E_{b^{-1}}}) = M(1_{\{e\}}) + 2.$$

2.5 Amenable Semigroups

The definition of amenability can be extended to apply for semigroups, although we will limit ourselves to only considering discrete semigroups.

Definition 2.5.1. Let S be a semigroup. Then every $f \in \ell^1(S)$ can be written as

$$f = \sum_{s\in S} \alpha_s \delta_s$$

such that

$$\|f\|_1 = \sum_{s\in S} |\alpha_s| < \infty$$

with convolution defined as

$$(f * g)(t) = \sum_{rs=t} f(r)g(s), \text{ where } f, g \in \ell^1(S), t \in S.$$

We can identify

Remark 2.5.2. The definition of amenability given for groups required taking an inverse, which is not necessarily possible for semigroups as inverses and identities are not assumed to exist. We can get around this issue by defining an action of S on $\ell^1(S)$ by identifying each

$s \in S$ with the function $1_s(t) = \begin{cases} 1 & s = t \\ 0 & \text{otherwise} \end{cases}$. Then convolution of 1_s with $f \in \ell^1(S)$ induces the left action

$$1_s * f(t) = \sum_{lr=t} 1_s(l) f(r) = \sum_{sr=t} f(r).$$

With this idea in hand we are now free to define amenability for semigroups:

Definition 2.5.3. Let S be a discrete semigroup. We say that a positive functional $M \in \ell^\infty(S)^*$ is a *left-invariant mean* if M satisfies the following conditions:

i) $M(1) = \|M\| = 1$

ii) $M(1_s * f) = M(f)$, where $s \in S$ and $f \in \ell^\infty(S)$.

If a left-invariant mean for S exists, then we say that S is a left-*amenable* semigroup. A right-*amenable* semigroup is defined in a similar fashion. A semigroup that is both right-amenable and left-amenable is called *amenable*.

Notably, unlike as the case with groups, left-amenability and right-amenability are not equivalent. However, this is the case given that S is commutative:

Theorem 2.5.4. *[7, Theorem 4] Let S be a commutative semigroup. Then S is amenable.*

2.6 Representation Theory

2.6.1 Representation Theory for Finite Groups

We will first specifically discuss representation theory of finite groups, and then extend to the locally compact case. See Isaac's book [41] for more on finite group character theory.

Definition 2.6.1. Let G be a finite group. A (complex) *representation* of G is a homomorphism $\phi : G \to GL(n, \mathbb{C})$ for some n. Two representations $\phi : G \to GL(n, \mathbb{C})$ and $\psi : G \to GL(n, \mathbb{C})$ are called *equivalent* if there is an isomorphism $\alpha : \mathbb{C}^n \to \mathbb{C}^n$ such that $\psi(g) = \alpha \circ \phi \circ \alpha^{-1}(g)$.

Definition 2.6.2. A function $\chi : G \to \mathbb{C}$ is called a *character* of G if there is a representation ϕ of G such that $\chi(g) = \text{Trace } \phi(g)$. The value $\chi(e)$ is called the degree of χ and is denoted d_χ. If $d_\chi = 1$ then χ is called a *linear* character, and we denote the set of linear characters by $\mathcal{L}(G)$.

Definition 2.6.3. A function $\phi : G \to \mathbb{C}$ is called a *class function* if ϕ is constant on the conjugacy classes of G. Denote the set of conjugacy classes of G by $\text{Conj}(G)$. The *inner product* of two class functions ϕ and ψ is defined by

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{C \in Conj} |C| \phi(C) \overline{\psi(C)}.$$

Due to the fact that $\text{Trace}(AB) = \text{Trace}(BA)$ it follows that characters are automatically class functions on G.

Definition 2.6.4. A character χ is called *irreducible* if $\langle \chi, \chi \rangle = 1$. Denote the set of irreducible characters by $\text{Irr}(G)$.

It is known that equivalent representations induce equal characters, which creates a correspondence between characters and equivalence classes of equivalent representations. The following useful properties of characters will all come in handy later:

Proposition 2.6.5. *i) Characters are algebraic-integer valued.*

ii) Finite sums and products of characters are characters.

iii) $Irr(G)$ is a basis for the set of class functions on G.

iv) $|Irr(G)| = |Conj(G)|$.

v) G is abelian if and only if every character in $Irr(G)$ is linear.

vi) $\sum_{\chi \in Irr(G)} d_\chi^2 = |G|$.

vii) If $\chi \in Irr(G)$ with $d_\chi > 1$ then $\chi(g) = 0$ for some $g \in G$.

Proposition 2.6.6 (Schur Orthogonality I)**.** *Let $\chi_i, \chi_j \in Irr(G)$. Then*

$$\langle \chi_i, \chi_j \rangle = \frac{1}{|G|} \sum_{C \in Conj(G)} |C| \chi_i(C) \overline{\chi_j(C)} = \delta_{ij}.$$

Proposition 2.6.7 (Schur Orthogonality II). *Let $g, h \in G$, and let g^G and h^G denote the conjugacy classes containing g and h respectively. Then*

$$\sum_{\chi \in Irr(G)} \chi(g)\overline{\chi(h)} = \begin{cases} 0 & g^G \neq h^G \\ \dfrac{|G|}{|g^G|} & g^G = h^G \end{cases}.$$

These properties highly constrict the structure of $\mathrm{Irr}(G)$, and in particular imply that $\mathrm{Irr}(G)$ can be represented in a table known as the *character table* of G. Consider the dihedral group of order 8, $D_4 = \langle a, b : a^4 = b^2 = abab = e \rangle$, which has a 5×5 character table:

Class	e	a^2	b, a^2b	a, a^3	ab, a^3b
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Figure 2.1: Character table of D_4

Definition 2.6.8. For $\chi \in \mathrm{Irr}(G)$ let

$$Z(\chi) = \{g \in G : |\chi(g)| = d_\chi\}$$

denote the *center* of χ and let

$$\ker(\chi) = \{g \in G : \chi(g) = d_\chi\}$$

denote the *kernel* of χ, both of which are subgroups of G.

Proposition 2.6.9.

i) $G' = \bigcap_{\chi \in Irr(G)} \{\ker(\chi) : d_\chi = 1\}$, *where G' is the derived subgroup of G.*

ii) $|\mathcal{L}(G)| = |G : G'|$

iii) $Z(G) = \bigcap_{\chi \in Irr(G)} \{Z(\chi)\}$

iv) If $\chi \in Irr(G)$ and $G/Z(\chi)$ is abelian then $|G : Z(\chi)| = d_\chi^2$.

v) If $\chi \in Irr(G)$ then d_χ divides $|G : Z(\chi)|$.

Let $N \trianglelefteq G$. If $\chi \in \mathrm{Irr}(G)$ and $N \subseteq \ker(\chi)$ then $\hat{\chi}(gN) = \chi(g)$ determines an irreducible character of the group quotient G/N, and every element of $\mathrm{Irr}(G/N)$ can be constructed this way.

Definition 2.6.10. Let $H \leq G$, and let χ be a character of H. Denote the *induced character* by

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \chi^\circ(xgx^{-1}).$$

$$\text{where } \chi^\circ(g) = \begin{cases} \chi(g) & g \in H \\ 0 & g \notin H \end{cases}$$

Definition 2.6.11. Let $H \leq G$, and let χ be a character of G. Then χ_H denotes the *restriction to* H, defined by $\chi_H(h) = \chi(h)$ for $h \in H$.

Theorem 2.6.12 (Frobenius Reciprocity)**.** *Let $H \subseteq G$ and let ϕ be a character on H and ψ a character on G. Then*

$$\langle \phi, \psi_H \rangle = \langle \phi^G, \psi \rangle$$

It's important to know that for $H \leq G \leq K$ and $\chi \in \mathrm{Irr}(G)$ it is not necessarily true that either χ^K or χ_H are irreducible on K and H respectively. Still, there are partial results, such as the famed theorem of Clifford:

Theorem 2.6.13 (Clifford's Theorem)**.** *Let $N \trianglelefteq G$ and $\chi \in Irr(G)$. Let ϕ be an irreducible constituent of χ_N. For each $g \in G$ let $\phi^g(h) = \phi(ghg^{-1})$ denote the* conjugate character *of ϕ with respect to g, and let $\phi_1, ..., \phi_n$ denote the distinct conjugate characters of ϕ. Then*

$$\chi_N = \langle \chi_N, \phi \rangle \sum_{i=1}^{n} \phi_i.$$

2.6.2 Representation Theory for Locally Compact Groups

Let G be a locally compact group. For a Hilbert space $\mathcal{H}$, let

$$\mathcal{U}(\mathcal{H}) = \{U \in \mathcal{B}(\mathcal{H}) : U^*U = I = UU^*\}$$

denote the unitary operators on $\mathcal{H}$. A *continuous unitary representation* on G is a homomorphism

$$\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$$

for some Hilbert space $\mathcal{H}_\pi$ such that for any $\zeta, \eta \in \mathcal{H}_\pi$ then the function $\pi_{\zeta,\eta}(x) = \langle \pi(x)\zeta, \eta \rangle$ is continuous (we often call these functions *matrix coefficients*). We will denote the collection of equivalence classes of unitary representations on G by $\Sigma(G)$, where two representations are equivalent if they are equal up to a conjugation by a unitary.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Then we can form the tensor product $\mathcal{H} \otimes \mathcal{K}$ with inner product

$$\langle \zeta \otimes \eta, \phi \otimes \psi \rangle_{\mathcal{H}\otimes\mathcal{K}} = \langle \zeta, \phi \rangle_{\mathcal{H}} \langle \eta, \psi \rangle_{\mathcal{K}}, \text{ where } \zeta, \phi \in \mathcal{H} \text{ and } \eta, \psi \in \mathcal{K}.$$

For $\pi, \rho \in \Sigma(G)$, we can form another (continuous unitary) representation called the *tensor product* of π and ρ denoted by $\pi \otimes \rho : G \to \mathcal{U}(\mathcal{H}_\pi \otimes \mathcal{H}_\rho)$ defined as

$$\pi \otimes \rho(x)(\zeta \otimes \eta) = \pi(x)\zeta \otimes \rho(x)\eta, \text{ where } \zeta \in \mathcal{H}_\pi, \eta \in \mathcal{H}_\rho, x \in G.$$

If we have a continuous unitary representation π on G, then we can determine a representation on $L^1(G)$ via the operator-valued integral

$$\pi(f) = \int_G f(x)\pi(x)dx,$$

where $\pi(f)$ is defined by

$$\langle \pi(f)\zeta, \eta \rangle = \int_G f(x)\langle \pi(x)\zeta, \eta \rangle dx, \text{ where } \zeta, \eta \in \mathcal{H}(\pi) \text{ and } f \in L^1(G).$$

A particularly important unitary representation for our purposes is the *left regular representation* and is defined by $\lambda : G \to \mathcal{U}(L^2(G))$, where if $x, y \in G$ and $f \in L^2(G)$ then

$$\lambda(y)f(x) = f(y^{-1}x).$$

For $f \in L^1(G)$ and $g \in L^2(G)$ it is known that $\lambda(f)g = f * g$.

If G is a compact group denote the equivalences classes up to unitary equivalence of continuous irreducible unitary representations of G by $\widehat{G}$. Every representation $\pi \in \widehat{G}$ is finite-dimensional, so we can define the *character* of π by

$$\chi_\pi : \text{Trace} \circ \pi : G \to \mathbb{C}.$$

2.7 Fourier and Fourier-Stieltjes Algebres

Let $f \in L^1(G)$. Then we define the norm

$$\|f\|_* = \sup\{\|\pi(f)\| : \pi \in \Sigma(G)\}$$

In general $L^1(G)$ is not complete under this new norm, we denote the completion of $L^1(G)$ with respect to this $\|\cdot\|_*$ norm by

$$C^*(G) = \overline{L^1(G)}^{\|\cdot\|_*}.$$

The space $C^*(G)$ is called the *group C^*-algebra* of G. Now define a space of continuous functions

$$B(G) = \{\pi_{\zeta,\eta} : \pi \in \Sigma(G) \text{ and } \zeta, \eta \in \mathcal{H}_\pi\}$$

$$\|u\|_{B(G)} = \sup\left\{\left|\int_G f(x)u(x)dx\right| : f \in L^1(G), \|f\|_* \leq 1\right\}$$

We call $B(G)$ the *Fourier-Stieltjes algebra* of G. We can also define the *Fourier algebra* $A(G)$ by

$$A(G) = \overline{\text{span}\{\lambda_{f,g} : f, g \in L^2(G)\}}^{B(G)}$$

which is an ideal in $B(G)$. We define the *von Neumann algebra* of G by

$$VN(G) = \overline{\lambda(L^1(G))}^{\text{WOT}}.$$

As the name implies, is a von Neumann algebra. We have the following important facts about these algebras, which can all be found in [47]:

Theorem 2.7.1.

i) $A(G)^* = VN(G)$

ii) If G is abelian then $A(G) = L^1(\widehat{G})$.

iii) G is compact if and only if $A(G) = B(G)$.

iv) (Leptin's Theorem) G is amenable if and only if $A(G)$ has a bounded approximate identity.

v) If H is a closed subgroup of G then $A(G)|_H = A(H)$.

The fact that $A(G)$ is a predual of $VN(G)$ is important, because as noted in Theorem 2.1.5, the predual of a von Neumann algebra is always necessarily unique.

The Fourier algebra enjoys a nice regularity property that will prove to be very useful for calculations in Chapter 5. See [47, Proposition 2.3.2] for the construction.

Proposition 2.7.2. *Let G be a locally compact group, $K \subseteq G$ a compact subset, and $U \subseteq G$ an open subset such that $K \subseteq U$. Then there exists $u \in C_C(G) \cap A(G)$ such that*

$$|u| \leq 1, \ \operatorname{supp}(u) \subseteq U, \ u|_K = 1.$$

2.8 $ZL^1(G)$ and $ZA(G)$

Let G be locally compact. We will consider the center of $L^1(G)$, which is described as

$$Z(L^1(G)) = \{f \in L^1(G) : f * g = g * f \text{ for all } g \in L^1(G)\}.$$

As noted in [52], it is equivalent to write this algebra as

$$Z(L^1(G)) = \{f \in L^1(G) : f(xyx^{-1}) = f(y) \text{ for almost every } x, y \in G\}.$$

We will typically denote $Z(L^1(G))$ by $ZL^1(G)$ for convenience. It is clear that $ZL^1(G)$, being a subalgebra of $L^1(G)$, is a commutative Banach algebra.

The central Fourier algebra $ZA(G)$ was first introduced by Alaghmandan and Spronk in [5]. For a compact group G, set

$$ZA(G) = ZL^1(G) \cap A(G) = \{u \in A(G) : u(xyx^{-1}) = u(y) \text{ for all } x, y \in G\}$$

with respect to the $A(G)$ norm and pointwise multiplication.

Remark 2.8.1. Naturally because $A(G)$ is already a commutative algebra, $ZA(G)$ is not equal to the center of $A(G)$ because the center of a commutative algebra is itself.

Our interest in these algebras will be largely in the case where G is finite, in which case

$$ZA(G) = ZL^1(G) = \operatorname{span}\{\chi : \chi \in \operatorname{Irr}(G)\}.$$

Note that the equality above is just in terms of sets, they in general will have distinct Banach algebra structures. $ZA(G)$ is a Banach algebra with respect to pointwise multiplication and $\| \cdot \|_{A(G)}$, while $ZL^1(G)$ is a Banach algebra with respect to $L^1(G)$ convolution

$*$ and $\|\cdot\|_{L^1(G)}$. These algebras are actually even more deeply connected than they might appear on the surface, but first we need the language of hypergroups.

We call a non-empty set H a *discrete hypergroup* if there exists an associative convolution product on $\ell^1(H)$ and an involution $x \mapsto \overline{x}$ on H such that the following properties are satisfied:

1. For $x, y \in H$ then $\delta_x * \delta_y \geq 0$ and $\|\delta_x * \delta_y\| = 1$.
2. There exists a (necessarily unique) element $e \in H$ such that

$$\delta_e * \delta_x = \delta_x = \delta_x * \delta_e$$

   for all $x \in H$.
3. $\overline{\overline{x}} = x$ and $\delta_{\overline{x}} * \delta_{\overline{y}} = \delta_y * \delta_x$ for all $x, y \in H$.
4. $e \in \mathrm{supp}(\delta_x * \delta_y)$ if and only if $y = \overline{x}$.

Remark 2.8.2. In this thesis we will only consider examples of discrete hypergroups, but the locally compact case can be found in [42] (note that the author refers to hypergroups by the term "convos" instead).

Theorem 2.8.3. *[11, Theorem 1.3.26] Let H be a discrete hypergroup. Then there exists a positive Radon measure μ on H such that (when viewed as a linear functional on $C_0(G)$ by the Riesz representation theorem) then*

$$\mu(\delta_x * f) = \mu(f)$$

for $x \in H$. We call μ a Haar measure *for H. Furthermore, μ is unique up to scaling.*

Both of the algebras considered above can be regarded as semigroup algebras. For a compact group G let $\mathrm{Conj}(G)$ be the conjugacy classes of G and $\widehat{G}$ the irreducible representations of G. Then $ZL^1(G) \cong \ell^1(\mathrm{Conj}, \mu_{\mathrm{Conj}})$ and $ZA(G) \cong \ell^1(\hat{G}, \mu_{\hat{G}})$ where $\mu_{\mathrm{Conj}}(C) = |C|$ and $\mu_{\hat{G}}(\pi) = d_\pi^2$. Indeed, Conj and $\widehat{G}$ are dual hypergroups as shown in [2].

2.9 Multipliers of $A(G)$

A function $v : G \to \mathbb{C}$ is called a *multiplier* of $A(G)$ if $vu \in A(G)$ for all $u \in A(G)$. We denote the space of all such functions by $MA(G)$, which we call the *multiplier algebra* of

G. Let M_v denote the linear operator $M_v(u) = vu$ from $A(G)$ to itself. By the closed graph theorem M_v is bounded, so $M_v \in \mathcal{B}(A(G))$ and is hence equipped with an operator norm. This induces the *multiplier norm* on v via

$$\|v\|_M = \|M_v\|.$$

If for $v \in MA(G)$ the map M_v is completely bounded then we call v a *completely bounded multiplier*. The space of completely bounded multipliers on $A(G)$ is denoted by $M_{cb}(G)$, with norm

$$\|v\|_{cb} = \|M_v\|_{cb}.$$

There is chain of subalgebras

$$A(G) \subseteq B(G) \subseteq M_{cb}(G) \subseteq MA(G)$$

with

$$\|v\|_{A(G)} \geq \|v\|_{cb} \geq \|v\|_M.$$

Theorem 2.9.1. *For a locally compact group G, the following are equivalent:*

1. *G is amenable.*
2. *$B(G) = MA(G)$.*
3. *$B(G) = M_{cb}(G)$.*

There is a characterization of $M_{cb}(G)$ due to Jolissaint [45]:

Theorem 2.9.2. *For locally compact G and $v : G \to \mathbb{C}$ the following are equivalent:*

1. *$v \in M_{cb}(G)$*
2. *There exists a Hilbert space $\mathcal{H}$ and functions $f, g : G \to \mathcal{H}$ such that*

$$v(t^{-1}s) = \langle f(s), g(t) \rangle \text{ for all } s, t \in G.$$

If these conditions are satisfied then $\|v\|_{cb} = \inf \|f\|_\infty \|g\|_\infty$, where the infimum is taken over pairs as (2).

While $MA(G)$ and $M_{cb}(G)$ are interesting algebras in their own right, in the non-amenable case they can be large enough to be unwieldy. In Chapter 5 we will focus instead on the closures of $A(G)$ in the respective multiplier and cb-multiplier norms. Let

$$A_M(G) = \overline{A(G)}^{\|\cdot\|_M} \subseteq M(A(G)).$$

and

$$A_{cb}(G) = \overline{A(G)}^{\|\cdot\|_{cb}} \subseteq M_{cb}(A(G)).$$

Remark 2.9.3. Let $\mathcal{A}(G)$ denote either $A_{cb}(G)$ or $A_M(G)$. Consider the following map and its adjoints:

$$i : A(G) \to \mathcal{A}(G)$$

$$i^* : \mathcal{A}(G)^* \to VN(G)$$

$$i^{**} : VN(G)^* \to \mathcal{A}(G)^{**},$$

where i denotes the inclusion map. Since i has dense range, i^* is injective and as such is invertible with inverse i^{*-1} on $Range(i^*)$. It is easy to see that i^* is simply the restriction map. That is

$$i^*(T) = T_{|_{A(G)}}.$$

It will also be useful to view all of the above maps as embeddings. That is, when G is non-amenable $\mathcal{A}(G)^*$ can be viewed as a proper subset of $VN(G)$ and $VN(G)^*$ as a proper subset of $\mathcal{A}(G)^{**}$.

It is well known due to [23] that $\Delta(A(G)) \cong G$ via the point evaluation map $x \mapsto \phi_x$, where for $x \in G$ then $\phi_x \in \mathcal{A}(G)^*$ is defined by $\phi_x(u) = u(x)$ for $u \in \mathcal{A}(G)$. It follows immediately that $\Delta(A_M(G)) \cong G$ and $\Delta(A_{cb}(G)) \cong G$ due to the fact that the map i above is a contractive and injective homomorphism from $A(G)$ with dense range.

Recall the definitions of topologically invariant means and sets of spectral synthesis involved the spectrum of the algebras. The fact that $\Delta(\mathcal{A}(G))$ can be associated with G for the above algebras makes investigating these properties, as we do in Chapter 5, more straightforward.

Chapter 3

Survey of Amenability Constant Gaps

This was a triumph
I'm making a note here
Huge success
It's hard to overstate my
satisfaction

GLaDOS
Portal

A major theme in Chapter 4 is an investigation of the amenability constants of the central Fourier algebra $ZA(G)$. To that end, in this chapter we provide a survey of amenability constant gap results in the literature which will help put the work in Chapter 4 into context.

3.1 $L^1(G)$

The term "amenable Banach algebra" has a pleasing relationship with amenable groups, as demonstrated by the following result due to Johnson [43, 1972]:

Theorem 3.1.1 (Johnson's Theorem)**.** *For a locally compact group G, the group algebra $L^1(G)$ is amenable as a Banach algebra if and only if G is amenable as a group, in which case $AM(L^1(G)) = 1$.*

The characterization of amenability via bounded approximate diagonals had not yet been developed when [43] was first published, so the original proof of Johnson's Theorem utilized the inner derivation definition. We provide a short sketch of one direction of a proof of this result which was done by Stokke in [72] that is more illuminating for our purposes because it uses the bounded approximate diagonal of amenability:

Proof. $\Longleftarrow$ Assume that G is amenable, and let $(f_\alpha)_\alpha$ be a net as from Proposition 2.4.8. Because G is amenable we can choose a bounded approximate identity $(e_\beta)_\beta$ for $L^1(G)$ with $e_\beta \geq 0$, $\|e_\beta\|_1 = 1$, and

$$\|\delta_x * e_\beta - e_\beta * \delta_x\|_1 \to 0 \text{ uniformly in } x \text{ on compact subsets of } G.$$

By letting $\lambda = (\alpha, \beta)$ we can define a net $(d_\lambda)_\lambda$ in $L^1(G \times G) \cong L^1(G) \otimes^\gamma L^1(G)$ by

$$d_\lambda(s,t) = f_\alpha(s) e_\beta(st), \quad s, t \in G.$$

Because $\|f_\alpha\|_1 = \|e_\beta\|_1 = 1$ then it follows that $\|d_\lambda\|_1 = 1$. All that remains is to show that $(d_\lambda)_\lambda$ satisfies Equation 2.3 and Equation 2.4. Let $x, s, t \in G$ and $\lambda = (\alpha, \beta)$. Then we can see that

- $(\delta_x \cdot d_\lambda)(s,t) = (\delta_x * f_\alpha)(s)(\delta_x * e_\beta)(st)$
- $(d_\lambda \cdot \delta_x)(s,t) = f_\alpha(s)(e_\beta * \delta_x)(st)$.

It follows that

$$\begin{aligned}
&\|\delta_x \cdot d_\lambda - d_\lambda \cdot \delta_x\|_1 \\
&= \int_G \int_G |(\delta_x * f_\alpha)(s)(\delta_x * e_\beta)(st) - f_\alpha(s)(e_\beta * \delta_x(st))| ds dt \\
&= \int_G \int_G |(\delta_x * f_\alpha)(s)(\delta_x * e_\beta)(t) - f_\alpha(s)(e_\beta * \delta_x(t))| ds dt \\
&\leq \int_G \int_G |(\delta_x * f_\alpha)(s) - f_\alpha(s)|(\delta_x * e_\beta)(t) dt ds + \int_G \int_G f_\alpha(s)|(\delta_x * e_\beta)(t) - (e_\beta * \delta_x)(t)| dt ds \\
&= \|\delta_x * f_\alpha - f_\alpha\|_1 + \|\delta_x * e_\beta - e_\beta * \delta_x\|_1 \\
&\to 0 \text{ uniformly in } x \text{ on compact subsets.}
\end{aligned}$$

Take $\mu \in M(G)$. Because μ is a Radon measure by definition, it is inner regular, hence it follows that

$$\mu \cdot d_\lambda - d_\lambda \cdot \mu \to 0.$$

In particular, for $f \in L^1(G) \subseteq M(G)$ then

$$f \cdot d_\lambda - d_\lambda \cdot f \to 0.$$

Let $\phi \in L^\infty(G)$. Note that because $f_\alpha \geq 0$ and $\|f_\alpha\|_1 = 1$ then $\int_G f_\alpha(x)dx = 1$. Then

$$\begin{aligned}
\langle m(d_\gamma), \phi\rangle &= \langle d_\gamma, m^*(\phi)\rangle \\
&= \int_G \int_G f_\alpha(x) e_\beta(xy)\phi(xy)dydx \\
&= \int_G \int_G f_\alpha(x) e_\beta(y)\phi(y)dydx \\
&= \int_G e_\beta(y)\phi(y) \int_G f_\alpha(x)dxdx \\
&= \int_G e_\beta(y)\phi(y)dy \\
&= \langle e_\beta, \phi\rangle
\end{aligned}$$

This shows that $m(d_\gamma) = e_\beta$, hence $fm(d_\gamma) \to f$ for all $f \in L^1(G)$.

We have that (d_λ) satisfies both Equation 2.3 and 2.4, and that $\|d_\lambda\|_1 = 1$ for all λ, so it follows that $L^1(G)$ is amenable and that $AM(L^1(G)) = 1$. □

A key element in this proof was the fact that $L^1(G \times G) \cong L^1(G) \otimes^\gamma L^1(G)$. The analog of this property does not hold for all Banach algebras of G, as we will see next.

3.2 $A(G)$

The amenability constant theory of $A(G)$ was originally studied in by Johnson in [44], where it was shown that if G is finite then

$$AM(A(G)) = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} d_\chi^3. \tag{3.1}$$

Example 3.2.1. Applying Johnson's formula to the character table of D_4 (Table 2.1) yields that

$$AM(A(D_4)) = \frac{1}{8}(1 + 1 + 1 + 1 + 8) = \frac{3}{2}$$

Remark 3.2.2. This formula combined with Proposition 2.6.5 shows that $AM(A(G)) = 1$ if and only if G is abelian. It was proven by Runde in [60] that this holds more generally for locally compact groups.

Using Equation 3.1, Johnson showed that $\frac{3}{2}$ was a sharp amenability bound over finite groups:

Theorem 3.2.3. *[44] Let G be a finite group. Then the following are equivalent:*

1. *G is abelian.*
2. *$AM(A(G)) < \frac{3}{2}$.*
3. *$AM(A(G)) = 1$.*

This gap is sharp, as $AM(A(D_4)) = \frac{3}{2}$.

Remark 3.2.4. Given results like Johnson's Theorem and Leptin's theorem, it would be natural to hope that there is a correspondence between amenability of $A(G)$ as an algebra and G as a group. Tragically, this is not the case: Johnson squashed all such hopes by showing in [44] that $A(SO(3, \mathbb{R}))$ is a non-amenable algebra, despite the fact that $SO(3, \mathbb{R})$ is compact, hence amenable.

The search for a characterization of groups G such that $A(G)$ is amenable was completed by Forrest and Runde in [31], but first, we need a definition.

Definition 3.2.5. We call a group G *almost abelian* if G contains an abelian subgroup of finite index (this is also known as virtually abelian in the literature).

Theorem 3.2.6. *[31] For a locally compact group G the following are equivalent:*

1. *G is almost abelian.*
2. *$A(G)$ is amenable.*

Definition 3.2.7. Let G be a locally compact group. We call the subset

$$G_\Gamma = \{(x, x^{-1}) : x \in G\}$$

of $G \times G$ the *anti-diagonal* of G.

Runde noted in [60] the following connection between G_Γ and the amenability of $A(G)$:

Lemma 3.2.8. *[60] Let G be a locally compact group and denote G endowed with the discrete topology by G_d. Then the following are equivalent:*

1. *$A(G)$ is amenable.*
2. *$1_{G_\Gamma} \in B(G_d \times G_d)$.*

If either (hence both) of the above hold then

$$\|1_{G_\Gamma}\|_{B(G_d \times G_d)} \leq AM(A(G)).$$

Choi observed in [14] that the inequality in Lemma 3.2.8 is actually an equality if G is finite:

Proposition 3.2.9. *[14] Let G be a finite group. Then*

$$\|1_{G_\Gamma}\|_{A(G_d \times G_d)} = AM(A(G)).$$

This observation of Choi's led to an improvement to Theorem 3.2.3 that settles the amenability constant gap of $A(G)$.

Theorem 3.2.10. *[14] Let G be a locally compact group. Then the following are equivalent:*

1. *G is abelian.*
2. *$AM(A(G)) < \frac{3}{2}$.*
3. *$AM(A(G)) = 1$.*

This gap is sharp, as $AM(A(S_3)) = \frac{3}{2}$.

Choi's proof of the $\frac{3}{2}$ bound relied heavily on the following subgroup hereditary property.

Lemma 3.2.11. *If G is an almost abelian group and H is a subgroup of G then*

$$AM(A(H)) \leq AM(A(G)).$$

Recall that because $A(G)$ is the predual of the von Neumann algebra $VN(G)$, it has a natural structure as a completely contractive Banach algebra. This structure works very well in terms of operator projective tensor products:

Theorem 3.2.12. *[22] Let G be a locally compact group. Then $A(G \times G) \cong A(G) \widehat{\otimes} A(G)$.*

In many ways, operator amenability is the "right" notion for $A(G)$ as opposed to amenability, as demonstrated by the following result due to Ruan:

Theorem 3.2.13. *[58] For a locally compact group G, $A(G)$ is operator amenable if and only if G is an amenable group, in which case $AM_{op}(A(G)) = 1$.*

3.3 $B(G)$

The amenability constant theory of the Fourier-Stieltjes algebra $B(G)$ largely just reduces to the case where G is compact, in which case $A(G) = B(G)$.

Theorem 3.3.1. *For a locally compact group G, the following are equivalent.*

1. *$B(G)$ is amenable.*
2. *G is compact and almost abelian.*

The above theorem is really just saying that there is nothing additional to be gained by looking at amenability constants of $B(G)$ as opposed to $A(G)$. The story is very different when considering operator amenability instead:

Theorem 3.3.2 ([63]). *For a locally compact group the following are equivalent.*

1. *$AM_{cb}(B(G)) < 5$.*
2. *$AM_{cb}(B(G)) = 1$.*
3. *G is compact.*

In this theorem compactness replaces commutativity of the group as compared with our previous examples so far. What's interesting is that this bound is actually sharp: Runde and Spronk gave an example in [64] of a non-compact group G such that $AM_{cb}(B(G)) = 5$. It turns out that such groups are necessarily disconnected, as demonstrated by this result of Spronk:

Theorem 3.3.3 ([71]). *Let G be a connected group. Then the following are equivalent.*

1. *$B(G)$ is operator amenable.*
2. *G is compact.*

3.4 $A_p(G)$

Let G be a locally compact group, and let $p, q \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1$. Recall for a function $f : G \to \mathbb{C}$ we use the notation $\check{f}(x)$ to denote the function $\check{f}(x) = f(x^{-1})$. We will now define the Figà-Talamanca-Herz algebra $A_p(G)$, which consists of functions $f : G \to \mathbb{C}$ such that

$$f = \sum_{i=1}^{\infty} \phi_i * \check{\psi}_i$$

where $(\phi_i)_{i=1}^{\infty} \in L^p(G)$ and $(\psi_i)_{i=1}^{\infty} \in L^q(G)$ such that

$$\sum_{i=1}^{\infty} \|\phi_i\|_p \cdot \|\psi_i\|_q < \infty.$$

Define a norm on this set $\|f\|_{A_p(G)}$ as the infimum of of all sums satisfying (1) and (2). Equipping $A_p(G)$ with this norm and pointwise multiplication results in a Banach algebra, as originally shown by Herz in [37]. Because of the key fact that $A_2(G) = A(G)$, this class of algebras can be viewed as a generalization of the Fourier algebra.

Lemma 3.4.1. *[37, Theorem C] Let G be amenable and $p \in (1,\infty)$. Then there is a contractive inclusion*

$$A(G) \subseteq A_p(G) \text{ and } A_p(G) = \overline{A(G)}^{\|\cdot\|_{A_p(G)}}.$$

The amenability constant theory of the $A_p(G)$ algebras was first developed by Runde in [61, Theorem 2.9]

Theorem 3.4.2. *[61] Let G be a locally compact group and fix $p \in (1,\infty)$. Then the following are equivalent:*

1. *$A_p(G)$ is amenable and $AM(A_p(G)) = 1$.*
2. *G is abelian.*

This result was improved by Roydor in [57, Corollary 4.10]:

Theorem 3.4.3. *[57] Let G be a locally compact group and fix $p \in (1,\infty)$. Then the following are equivalent:*

1. $A_p(G)$ is amenable.

2. $AM(A_p(G)) = 1$.

3. There is a constant $\gamma_p > 1$ such that $AM(A_p(G)) < \gamma_p$.

4. G is abelian.

In the case that $p = 2$ then $\gamma_p = \frac{3}{2}$ is the sharp bound, but sharp γ_p bounds are not currently known for $p \neq 2$. A naive guess may be that $\gamma_p = 2^{p-2} + \frac{1}{2}$ if $1 \leq p \leq 2$ or $\gamma_p = 2^{1-p}+1$ because of formulas for gaps of norms of $p-$completely bounded isomorphisms found in [57], but as of the writing of this thesis the question of sharp amenability constant gaps for $A_p(G)$ when $p \neq 2$ is open.

When we move to the question of operator amenability of $A_p(G)$, things get trickier, as it is not clear how to give $A_p(G)$ an operator space structure when $p \neq 2$. Multiple authors have approached this problem. In [59] Runde used interpolation techniques originally due to Pisier to define operator analogues of the $A_p(G)$ algebras, which Runde denoted as $OA_p(G)$. Each $OA_p(G)$ is a completely contractive Banach algebra with a contractive inclusion $A_p(G) \subseteq OA_p(G)$, but when $p \neq 2$ it is possible for $A_p(G)$ and $OA_p(G)$ to fail to be isomorphic even as Banach spaces. An approach due to Daws in [19] endows $A_p(G)$ with a $p-$operator space structure, but has the limitation of assuming that G is amenable. Of most interest for this thesis is the column operator space structure utilized by Lambert, Neufang, and Runde in [48].

Lemma 3.4.4. *[48, Corollary 6.6]. Let G be amenable and $p \in (1, \infty)$. Endow $A_p(G)$ with the operator space structure as in [48]. The inclusion map from $A(G)$ into $A_p(G)$ has completely bounded norm at most $K_{\mathbb{G}}^2$, where $K_{\mathbb{G}}$ is Grothendiuk's constant.*

With regards to this structure $A_p(G)$ is a quantized Banach algebra, but is not guarenteed to be a completely contractive Banach algebra. Our full notion of operator amenability as described in 2.3.11 does not necessarily apply in this case, but as noted in [48] it is still possible to work with a version of operator amenability for quantized Banach algebras that utilized the inner derivation characterization of operator amenability.

Theorem 3.4.5. *[48, Theorem 7.3]*

Let G be a locally compact group and let $A_p(G)$ be a quantized Banach algebra as in [48]. Then the following are equivalent:

1. G is amenable.

2. $A_p(G)$ *is operator amenable for each* $p \in (1, \infty)$.

3. *There exists* $p \in (1, \infty)$ *such that* $A_p(G)$ *is operator amenable.*

Remark 3.4.6. This generalization of Theorem 3.2.13 unfortunately has the limitation of not actually having an operator amenability constant gap, as operator amenability constants are not defined for quantized Banach algebras that are not necessarily completely contractive.

3.5 $A_{cb}(G)$ and $A_M(G)$

Due to Theorem 2.9.1, we know that the amenability constant theory of $A_{cb}(G)$ and $A_M(G)$ only has the potential to be distinguishable from $A(G)$ when G is non-amenable.

In [32, Theorem 3.4] it was shown that over locally compact groups $A_{cb}(G)$ has an amenability constant gap of $\frac{2}{\sqrt{3}}$. This gap was later improved in the Master's thesis of Juselius, but is still not known to be sharp:

Corollary 3.5.1. *[46, Corollary 4.7] Let* G *be a locally compact group. Then the following are equivalent:*

1. G *is abelian.*

2. $AM(A_{cb}(G)) < \frac{9}{7}$.

3. $AM(A_{cb}(G)) = 1$.

The amenability constant theory of $A_M(G)$ has not yet been explicitly studied in the literature. We note that the following corollary follows immediately from the above theorem.

Corollary 3.5.2. *Let* G *be a locally compact group. Then*

$$AM(A_M(G)) \leq AM(A_{cb}(G)).$$

Proof. It is known that $A_{cb}(G) \subseteq A_M(G)$ and $\|\cdot\|_{cb} \geq \|\cdot\|_M$, so the inclusion map $A_{cb}(G) \subseteq A_M(G)$ is a contractive algebra homomorphism. It follows from Proposition 2.3.5 that

$$AM(A_M(G)) \leq AM(A_{cb}(G)).$$

□

Question 3.5.3. Does $A_M(G)$ have an amenability constant gap? If G is abelian then $A(G) = A_M(G)$ so $AM(A_M(G)) = 1$, but it is not known if the converse holds (although this seems extremely likely).

3.6 $L^1_0(G)$

Not all Banach algebras that are naturally defined with respect to a locally compact group G necessarily follow the amenability constant gap patterns we have seen so far. Take the augmentation ideal of $L^1(G)$, which is defined by

$$L^1_0(G) = \left\{ f \in L^1(G) : \int_G f(x)dx = 0 \right\}.$$

Suppose that G is a finite non-trivial group, then $1_e(x) - \mu(x)$ is the identity for $L^1_0(G)$. To see this, take $f \in L^1_0(G)$, then

$$\begin{aligned}
(1_e - \mu) * f(x) &= \sum_{y \in G} (1_e - \mu)(y) f(xy^{-1}) \\
&= f(x) - \frac{1}{|G|} \sum_{y \in G} f(xy^{-1}) \\
&= f(x) - \frac{1}{|G|} \sum_{z = xy^{-1} \in G} f(z) \\
&= f(x).
\end{aligned}$$

Clearly the same argument shows that $f * (1_e - \mu)(x) = f(x)$.

Now observe that

$$\begin{aligned}
\|1_e - \mu\|_1 &= \sum_{x \in G} \left| 1_e(x) - \frac{1}{|G|} \right| \\
&= 1 - \frac{1}{|G|} + \sum_{x \neq e} \frac{1}{|G|} \\
&= 1 - \frac{2}{|G|} + 1
\end{aligned}$$

$$= 2 - \frac{2}{|G|}.$$

Applying Proposition 2.3.6 (i) with $\mathcal{A} = L_0^1(G)$ and $e_I = 1_e - \mu$ yields that $AM(L_0^1(G)) \geq 2 - \frac{2}{|G|}$. An appeal to (ii) of the same proposition with $I = L_0^1(G)$ and $\mathcal{A}(G) = L^1(G)$ gives that $AM(L_0^1(G)) \leq 2 - \frac{2}{|G|}$, hence $AM(L_0^1(G)) = 2 - \frac{2}{|G|}$. Note that for every finite group G with $|G| > 2$ we have that $AM(L_0^1(G)) > 1$, so there cannot be an amenability constant gap.

3.7 $ZL^1(G)$ and $ZA(G)$

In [8] Azimifard, Samei, and Spronk investigated amenability properties of $ZL^1(G)$. In case that G is finite they constructed a formula for the amenability constant of $ZL^1(G)$:

Theorem 3.7.1. *[8, Theorem 1.8] Let G be a finite group. Then*

$$\sum_{\chi \in Irr(G)} d_\chi^2 (\chi \otimes \chi)$$

is the unique diagonal element for $ZL^1(G)$. The amenability constant of $ZL^1(G)$ can be calculated by the formula

$$AM(ZL^1(G)) = \frac{1}{|G|^2} \sum_{C, C' \in Conj(G)} |C||C'| \left| \sum_{\chi \in Irr(G)} d_\chi^2 \chi(C) \overline{\chi_\pi(C')} \right|. \tag{3.2}$$

The authors of [8] applied Rider's $\frac{1}{300}$ gap result for central idempotents of $L^1(G)$ (see Theorem 6.2.1 in Chapter 6) to show that $AM(ZL^1(G))$ has an amenability constant gap of $1 + \frac{1}{300}$. This was improved by Choi in [13] to arrive at a sharp bound:

Theorem 3.7.2. *[13, Theorem 1.2] Let G be a finite group. Then $AM(ZL^1(G)) = 1$ if and only if G is abelian, otherwise $AM(ZL^1(G)) \geq \frac{7}{4}$. This bound is sharp.*

A key ingredient in this proof was the following hereditary property with respect to quotients:

Lemma 3.7.3. *[8, Corollary 1.3] Let G be a finite group and N a normal subgroup of G. Then*

$$AM(ZL^1(G/N)) \leq AM(ZL^1(G)).$$

In [5, 2016], Mahmood Alaghmandan and Nico Spronk produced a formula for $AM(ZA(G))$ that mirrors the formula for $AM(ZL^1(G))$ from [8]:

Theorem 3.7.4. *[5, Proposition 4.2] Let G be a finite group. Then the element*

$$1_{Conj(G)_D} = \sum_{C \in Conj(G)} 1_{C \times C}$$

is a diagonal for $ZA(G)$ via the identification $ZA(G) \otimes^\gamma ZA(G) \cong ZA(G \times G)$. The amenability constant of $ZA(G)$ can then be calculated by the formula

$$AM(ZA(G)) = \frac{1}{|G|^2} \sum_{\chi, \chi' \in Irr(G)} d_\chi d_{\chi'} \left| \sum_{C \in Conj(G)} |C|^2 \chi(C) \overline{\chi'(C)} \right|. \qquad (3.3)$$

Furthermore, they proved that $AM(ZA(G))$ also has an amenability constant gap:

Theorem 3.7.5. *Let G be a finite group. Then $AM(ZA(G)) = 1$ if and only if G is abelian, otherwise $AM(ZA(G)) \geq \frac{2}{\sqrt{3}}$.*

Remark 3.7.6. Notably this bound is not proven to be sharp. We will discuss investigations regarding improving this bound at length in Chapter 4 and Chapter 6.

3.8 Hypergroup Algebras

The similarity between the formulas for $AM(ZL^1(G))$ and $AM(ZA(G))$ is no coincidence, but actually follows from a more general formula for amenability constants of hypergroup algebras.

Theorem 3.8.1. *[2, Theorem 3.7] Let H be a finite commutative hypergroup with Haar measure λ and dual $\hat{H}$. For $\chi \in \hat{H}$ let k_χ denote the hyperdimension of χ. Then we have that*

$$AM(\ell^1(H, \lambda)) = \frac{1}{\lambda(H)^2} \sum_{x,y \in H} \lambda(x)\lambda(y) \left| \sum_{\chi \in \hat{H}} k_\chi^2 \chi(x) \overline{\chi(y)} \right|. \qquad (3.4)$$

It is not possible to state an amenability gap theorem for $AM(\ell^1(H))$ that holds for all finite commutative hypergroups H. If $1 < p < \infty$ is fixed then a hypergroup containing two elements $H_p = \{0, a\}$ can be defined by setting

$$\delta_a * \delta_a = \frac{1}{p}\delta_0 + (10\frac{1}{p})\delta_a.$$

Using the above formula, Alaghmandan and Amini showed in [2, Example 3.8] that

$$AM(\ell^1(H_p)) = \frac{5p^2 - 2p + 1}{(p+1)^2}.$$

By varying p every value in $(1, 5)$ can be achieved as an amenability constant of some $\ell^1(H_p)$.

3.9 C^*-algebras

In [15, Corollary 2] Connes proved that amenable C^*-algebras are necessarily nuclear (see Chapter 2 of [12] for an introduction to nuclear C^*-algebras), and in [36, Theorem 3.1] Haagerup proved that nuclear C^*-algebras are amenable with amenability constant equal to 1. Ruan [58, Theorem 5.1] proved that C^*-algebras are amenable if and only if they are operator amenable. We know that $1 \leq AM_{op}(\cdot) \leq AM(\cdot)$ hence it follows that the operator amenability constant of an operator amenable C^*-algebras is 1. We summarize all of this in the following theorem.

Theorem 3.9.1. *Let $\mathcal{C}$ be a C^*-algebra. Then the following are equivalent.*

1. *$\mathcal{C}$ is nuclear.*
2. *$\mathcal{C}$ is amenable as a Banach algebra.*
3. *$\mathcal{C}$ is operator amenable as a completely contractive Banach algebra.*
4. *$AM(\mathcal{C}) = 1$*
5. *$AM_{op}(\mathcal{C}) = 1$*

3.10 Semigroup Algebras

For a discrete semigroup S it is known that if $\ell^1(S)$ is amenable then S is amenable, although unlike the group case the converse need not hold.

Theorem 3.10.1. *[18, Corollary 10.28] Let S be a semigroup. Then the following are equivalent.*

1. *$AM(\ell^1(S)) < 5$.*
2. *$AM(\ell^1(S)) = 1$.*
3. *S is an amenable group.*

In the case that S is a commutative semigroup, the range of allowable amenability constant values of $\ell^1(S)$ is further reduced:

Theorem 3.10.2. *[34] There is no commuative semigroup S such that*

$$5 < AM(\ell^1(S)) < 9.$$

Proposition 3.10.3. *[34] If S is a commutative semigroup such that every element of S is an idempotent, then $AM(S) = 4n + 1$ for some $n \in \mathbb{N}$. All such values are achieved.*

Chapter 4

Amenability Constants of $ZA(G)$

Every puzzle has an answer.

Professor Layton
Professor Layton

4.1 $AMZA$ and AMZ

In this chapter we will be examining the amenability constant theory of the central Fourier algebra, $ZA(G)$. We will always assume that G is a finite group, in which case $ZL^1(G)$ and $ZA(G)$ are both equal to the class functions on G, albeit with differing norms and products. Recall the formulas for the amenability constants of these algebras:

$$AM(ZL^1(G)) = \frac{1}{|G|^2} \sum_{C,C' \in \mathrm{Conj}(G)} |C||C'| \left| \sum_{\chi \in \mathrm{Irr}(G)} d_\chi^2 \chi(C) \overline{\chi(C')} \right|$$

and

$$AM(ZA(G)) = \frac{1}{|G|^2} \sum_{\chi,\chi' \in \mathrm{Irr}(G)} d_\chi d_{\chi'} \left| \sum_{C \in \mathrm{Conj}(G)} |C|^2 \chi(C) \overline{\chi'(C)} \right|.$$

For convenience we will set $AMZA(G) = AM(ZA(G))$ and $AMZ(G) = AM(ZL^1(G))$. Note that if you choose to only sum up the elements where $\chi = \chi'$ in the above formula then you get the same value as you get from only summing up the elements where $C = C'$ in the formula for $AMZ(G)$. We will call this quantity the *auxiliary minorant* of $AMZA(G)$ (or equivalently, of $AMZ(G)$). Matching the notation from [13], we denote it as follows:

$$\operatorname{ass}(G) = \frac{1}{|G|^2} \sum_{\chi \in \operatorname{Irr}(G)} d_\chi^2 \sum_{C \in \operatorname{Conj}(G)} |C|^2 |\chi(C)|^2.$$

It will often make sense to split up a calculation between $\operatorname{ass}(G)$ and $AMZA(G) - \operatorname{ass}(G)$, which we denote as $AMZA_{\text{off}}(G)$, and can be written as

$$AMZA_{\text{off}}(G) = \frac{1}{|G|^2} \sum_{\chi \neq \chi' \in \operatorname{Irr}(G)} d_\chi d_{\chi'} \left| \sum_{C \in \operatorname{Conj}(G)} |C|^2 \chi(C) \overline{\chi'(C)} \right|.$$

Theorem 3.7.2 gives that $\frac{7}{4}$ is a sharp amenability constant bound for $ZL^1(G)$, and while by Theorem 3.7.5 $AMZA(G)$ also has a gap, a sharp bound has not yet been proven for $ZA(G)$.

4.2 Structure of Sum

We wish to learn more about the behavior of

$$AMZA(G) = \frac{1}{|G|^2} \sum_{\chi, \chi' \in \operatorname{Irr}(G)} d_\chi d_{\chi'} \left| \sum_{C \in \operatorname{Conj}(G)} |C|^2 \chi(C) \overline{\chi'(C)} \right|.$$

Of particular interest is the inside sum. For $\chi, \chi' \in \operatorname{Irr}(G)$ let

$$\Phi(\chi, \chi') = \sum_{C \in \operatorname{Conj}(G)} |C|^2 \chi(C) \overline{\chi'(C)}.$$

Note the connection with the inner product formula for characters

$$\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum_{C \in \operatorname{Conj}(G)} |C| \chi(C) \overline{\chi'(C)}$$

A more condensed way of writing $AMZA(G)$ is

$$AMZA(G) = \frac{1}{|G|^2} \sum_{\chi,\chi' \in \mathrm{Irr}(G)} d_\chi d_{\chi'} |\Phi(\chi,\chi')|.$$

Because irreducible characters have values in the algebraic integers, we know that $|\Phi(\chi,\chi')| \in \mathbb{Z}$, although it turns out that this is still true even without taking the complex magnitude. First we need a lemma that we will use several times.

Lemma 4.2.1. *Let $\phi_1, \phi_2, ..., \phi_{|Z(G)|}$ be the distinct irreducible characters of $Z(G)$ for a finite group G. Let $\chi \in Irr(G)$ and let $\chi_{Z(G)}$ denote the restriction of χ to $Z(G)$. Then there exists ϕ_i such that $\chi_{Z(G)} = d_\chi \phi_i$. This allows us to define the pairwise-disjoint sets $A_i = \{\chi \in Irr(G) : \chi_{Z(G)} = d_\chi \phi_i\}$. Furthermore, $\sum_{\chi \in A_i} d_\chi^2 = |G : Z(G)|$.*

Proof. This follows by Clifford's Theorem, Theorem 2.6.13. □

Proposition 4.2.2. *Let $\chi, \chi' \in Irr(G)$. Then $\Phi(\chi,\chi')$ is an integer divisible by $|Z(G)|$ and $\Phi(\chi,\chi') = 0$ if $d'_\chi \chi_{Z(G)} \neq d_\chi \chi'_{Z(G)}$.*

Proof. For convenience let $Z = Z(G)$. Let $\mathrm{Irr}(Z) = \{\phi_1, .., \phi_{|Z|}\}$ and let $\{A_1, ..., A_n\}$ be the decomposition from Lemma 4.2.1. Let $\chi, \chi' \in \mathrm{Irr}(G)$. By the lemma there exist i and j such that $\chi \in A_i$ and $\chi' \in A_j$. Let $|g^G|$ denote the size of the conjugacy class of g in G. We adapt an argument from [67, Proposition 1] to see the following:

$$\begin{aligned}
\sum_{C \in \mathrm{Conj}(G)} |C|^2 \chi(C)\overline{\chi'(C)} &= \sum_{gZ \in G/Z} \frac{1}{d_\chi d_{\chi'}} |g^G| \chi(g)\overline{\chi'(g)} \sum_{z \in Z} \chi(z)\overline{\chi'(z)} \\
&= \langle \chi_Z, \chi'_Z \rangle \cdot |Z| \sum_{gZ \in G/Z} |g^G| \chi(g)\overline{\chi'(g)} \\
&= \delta_{ij} \cdot |Z| \sum_{gZ \in G/Z} |g^G| \chi(g)\overline{\chi'(g)}.
\end{aligned}$$

As noted in the proof of [67, Proposition 1], $\sum\limits_{gZ \in G/Z} |g^G| \chi(g)\overline{\chi'(g)}$ must be a rational algebraic integer, hence an integer. □

Corollary 4.2.3. *$AMZA(G) \cdot |G|^2$ is divisible by $|Z(G)|$*

We also get the following rough estimate of $AMZA(G/Z(G))$ compared to $AMZA(G)$.

Corollary 4.2.4. $AMZA(G) \geq \dfrac{AMZA(G/Z(G))}{|Z(G)|}$.

Proof. Let $Z = Z(G)$. By the proof of Proposition 4.2.2 we have that

$$AMZA(G) = \frac{|Z|}{|G|^2} \sum_{i=1}^{|Z|} \sum_{\chi,\chi' \in A_i} d_\chi d_{\chi'} \left| \sum_{gZ \in G/Z} |g^G| \chi(g) \overline{\chi'(g)} \right|.$$

We can identify A_1 with $\mathrm{Irr}(G/Z)$, and then using the fact that $|g^G| \geq |gZ^{G/Z}|$ it follows that $AMZA(G) \geq \frac{1}{|Z|} \cdot AMZA(G/Z)$, as desired. $\square$

Example 4.2.5. We will use the ideas explored in this section to help simplify an explicit calculation of $AMZA(G)$ for the group $G = \texttt{SmallGroup}(24,3) = SL(2,\mathbb{F}_3)$. The character table of G is listed in Figure 4.1, where $\zeta_6 = e^{\frac{\pi i}{3}}$. We also provide the character table of $Z(G) \cong C_2 = \{e, a\}$ in Figure 4.2 We begin by calculating $\mathrm{ass}(G)$. If $\chi \in \mathfrak{L}(G) =$

Class Size	K_1 1	K_2 1	K_3 4	K_4 4	K_5 6	K_6 4	K_7 4
χ_1	1	1	1	1	1	1	1
χ_2	1	1	ζ_6^4	ζ_6^2	1	ζ_6^4	ζ_6^2
χ_3	1	1	ζ_6^2	ζ_6^4	1	ζ_6^2	ζ_6^4
χ_4	2	-2	ζ_6^5	ζ_6	0	ζ_6^2	ζ_6^4
χ_5	2	-2	ζ_6	ζ_6^5	0	ζ_6^4	ζ_6^2
χ_6	2	-2	-1	-1	0	1	1
χ_7	3	3	0	0	-1	0	0

Figure 4.1: Character table of $SL(2,\mathbb{F}_3)$

Class	e	a
ρ_1	1	1
ρ_2	1	-1

Figure 4.2: Character table of C_2

$\{\chi_1, \chi_2, \chi_3\}$ then $|\chi| = 1$, so it follows that

$$\Phi(\chi, \chi) = \sum_{C \in \mathrm{Conj}(G)} |C|^2 = 102.$$

For $\chi \in \{\chi_4, \chi_5, \chi_6\}$ we see that

$$\Phi(\chi, \chi) = 4(4 + 4 + 16 + 16 + 16 + 16) = 288.$$

Finally, we have that

$$\Phi(\chi_7, \chi_7) = 9(9 + 9 + 36) = 486.$$

Then it follows that

$$\mathrm{ass}(G) = \frac{(3 \cdot 102 + 3 \cdot 288 + 486)}{24^2} = \frac{23}{8}$$

We will now calculate $AMZA_{\mathrm{off}}(G)$. First, some observations to reduce the number of calculations needed.

For $\chi, \chi' \in \mathrm{Irr}(G)$ it is clear that $\Phi(\chi, \chi') = \Phi(\chi', \chi)$. If χ' is real-valued then it is easy to see that $\Phi(\chi, \chi') = \Phi(\overline{\chi}, \chi')$. In the context of this example this is relevant because $\chi_2 = \overline{\chi_3}$ and $\chi_4 = \overline{\chi_5}$. If we list $\mathrm{Irr}(Z(G)) = \{\rho_1, \rho_2\}$ and let A_1 and A_2 be as in Lemma 4.2.1 then we can see that $A_1 = \{\chi_1, \chi_2, \chi_3, \chi_7\}$ and $A_2 = \{\chi_4, \chi_5, \chi_6\}$. By the proof of Proposition 4.2.2 we know that if $\chi \in A_1$ and $\chi' \in A_2$ then $\Phi(\chi, \chi') = 0$, so it suffices to just calculate Φ for characters within the same A_i. Finally, recall that $\zeta_6 + \zeta_6^5 = 1$ and $\zeta_6^4 + \zeta_6^2 = -1$. Then we can see that

- $\Phi(\chi_1, \chi_2) = 1 + 1 + 16\zeta_6^2 + 16\zeta_6^4 + 16\zeta_6^2 + 36 + 16\zeta_6^4 + 16\zeta_6^2 = 38 - 32 = 6$
- $\Phi(\chi_1, \chi_7) = 3 + 3 - 36 = 30$
- $\Phi(\chi_2, \chi_3) = 1 + 1 + 16\zeta_6^2 + 16\zeta_6^4 + 16\zeta_6^2 + 36 + 16\zeta_6^4 + 16\zeta_6^2 = 38 - 32 = 6$
- $\Phi(\chi_2, \chi_7) = 3 + 3 - 36 = -30$
- $\Phi(\chi_4, \chi_5) = 8 + 16\zeta_6^4 + 16\zeta_6^2 + 16\zeta_6^4 + 16\zeta_6^2 = 8 - 16 - 16 = -24$
- $\Phi(\chi_4, \chi_6) = 8 - 16\zeta_6^4 - 16\zeta_6 + 16\zeta_6^5 + 16\zeta_6^2 = 8 - 16 - 16 = -24$

By combing our observations above we have that

$$AMZA_{\mathrm{off}}(G) = 2 \cdot \frac{(6 + 3 \cdot 30 + 4 \cdot 24) + (6 + 3 \cdot 30 + 4 \cdot 24) + (6 + 3 \cdot 30 + 4 \cdot 24)}{24^2} = 2$$

It then follows that

$$AMZA(SL(2,\mathbb{F}_3)) = \frac{23}{8} + 2 = \frac{39}{8} = 4.875.$$

While we omit the calculation, it is worth noting that $AMZ(SL(2,\mathbb{F}_3)) = 5$, which demonstrates that $G = SL(2,\mathbb{F}_3)$ is an example of a group for which the amenability constants of $ZL^1(G)$ and $ZA(G)$ do not agree.

4.3 Frobenius Groups with Abelian Factor and Kernel

It is of interest to try to determine when the condition $AMZA(G) = AMZ(G)$ is satisfied. While this is not necessarily the case - see Example 4.2.5 - it often does hold. In this section we will demonstrate that the amenability constants of $ZA(G)$ and $ZL^1(G)$ always agree for a particular class of semidirect products.

Definition 4.3.1. We say that a finite group G is a *Frobenius group* if it has a finite, proper, non-trivial subgroup H that satisfies $H \cap gHg^{-1} = \{e\}$ for all $g \in G \setminus H$. It can be shown that $K = \left(G \setminus \bigcup_{g \in G} gHg^{-1}\right) \cup \{e\}$ is a subgroup of G and that $G = K \rtimes H$. We call H the *Frobenius complement* of G and K the *Frobenius kernel*.

We will consider the case when K and H are both abelian, which is a class of groups that includes the dihedral groups D_{2n}, where n is odd, and the affine group of the finite field of order q, $\mathrm{Aff}(\mathbb{F}_q)$, where q is an odd prime power.

Theorem 4.3.2. *Let $G = K \rtimes H$ be Frobenius, where K and H are abelian and have orders k and h respectively. Then $AMZA(G) = AMZL(G) = 1 + \frac{2(h^2-1)}{h}\left(1 - \frac{h-1}{k}\right)\left(1 - \frac{1}{k}\right)$.*

Proof. Throughout this proof we will use a number of facts from [4], in particular from the appendix and Proposition 3.3 . It is known that $\mathrm{Irr}(G)$ is comprised of h linear characters (the set of which we will designate $\mathfrak{L}(G)$, or $\mathfrak{L}$ for convenience) that come from composition of characters from $\mathrm{Irr}(H)$ and the quotient map $G \to G/K \cong H$ and $\frac{k-1}{h}$ many characters of degree h induced from characters in $\mathrm{Irr}(K)$. Furthermore, G has trivial centre, $\frac{k-1}{h}$ conjugacy classes of size h (which are all contained in K) and $h-1$ conjugacy classes of size k. Let

- $B_1 = \{Z(G)\} = \{e\}$
- $B_2 = \{C \in \mathrm{Conj}(G) : |C| = h\}$
- $B_3 = \{C \in \mathrm{Conj}(G) : |C| = k\}$

The elements in B_2 partition $K \setminus \{e\}$ and the elements in B_3 partition $G \setminus K$. We know by calculations in [4, Theorem 2.4] that

$$\mathrm{ass}(G) = h^2 - \frac{(h^2-1)(1+h(k-1)+(h-1)k^2)}{hk^2}.$$

Therefore, it suffices to calculate $AMZA_{\mathrm{off}}(G)$. Let $\chi \neq \chi' \in \mathrm{Irr}(G)$. Then by Schur orthogonality we have that

$$\begin{aligned}
&\left|\sum_{C \in \mathrm{Conj}(G)} |C|^2 \chi(C)\overline{\chi'(C)}\right| \\
&= \left|d_\chi d_{\chi'} + h \sum_{C \in B_2} h\chi(C)\overline{\chi'(C)} + k^2 \sum_{C \in B_3} \chi(C)\overline{\chi'(C)}\right| \\
&= \left|(1-h)d_\chi d_{\chi'} + (k^2 - hk) \sum_{C \in B_3} \chi(C)\overline{\chi'(C)} + h \sum_{C \in \mathrm{Conj}(G)} |C|\chi(C)\overline{\chi'(C)}\right| \\
&= \left|(1-h)d_\chi d_{\chi'} + (k^2 - hk) \sum_{C \in B_3} \chi(C)\overline{\chi'(C)}\right|.
\end{aligned}$$

If $\chi \notin \mathfrak{L}$ then χ is induced from an irreducible character on K, it follows that χ vanishes on $G \setminus \bigcup_{x \in G} xKx^{-1}$, so χ vanishes on H, hence χ vanishes on each $C \in B_2$. This allows us to see that

$$\begin{aligned}
&\sum_{\chi \text{ or } \chi' \notin \mathfrak{L}(G), \chi \neq \chi'} d_\chi d_{\chi'} \left|\sum_{C \in \mathrm{Conj}(G)} |C|^2 \chi(C)\overline{\chi'(C)}\right| \\
&= (h-1) \sum_{\chi, \chi' \notin \mathfrak{L}, \chi \neq \chi'} d_\chi^2 d_{\chi'}^2 + 2(h-1) \sum_{\chi \notin \mathfrak{L}, \chi' \in \mathfrak{L}} d_\chi^2 d_{\chi'}^2 \\
&= (h-1)\left(\frac{k-1}{h}\right)\left(\frac{k-1}{h} - 1\right) h^4 + 2(h-1)h\left(\frac{k-1}{h}\right) h^2.
\end{aligned}$$

On the other hand, if $\chi, \chi' \in \mathfrak{L}$ with $\chi \neq \chi'$ then $\chi_K = \chi'_K = 1$, so $\chi(C) = \chi'(C) = 1$ for all $C \in B_2$. Then by Schur orthogonality it follows that

$$\begin{aligned}
0 &= \sum_{C \in \mathrm{Conj}(G)} |C| \chi(C) \overline{\chi'(C)} \\
&= 1 + \sum_{C \in B_2} h \chi(C) \overline{\chi'(C)} + \sum_{C \in B_3} k \chi(C) \overline{\chi'(C)} \\
&= 1 + \frac{k-1}{h} h + k \sum_{C \in B_3} \chi(C) \overline{\chi'(C)} \\
&= k + k \sum_{C \in B_3} \chi(C) \overline{\chi'(C)}.
\end{aligned}$$

Rearranging the equation yields that

$$\sum_{C \in B_3} \chi(C) \overline{\chi'(C)} = -1.$$

We can then calculate that

$$\begin{aligned}
&\sum_{\chi, \chi' \in \mathfrak{L}, \chi \neq \chi'} d_\chi d_{\chi'} \left| \sum_{C \in \mathrm{Conj}(G)} |C|^2 \chi(C) \overline{\chi'(C)} \right| \\
&= \sum_{\chi, \chi' \in \mathfrak{L}, \chi \neq \chi'} \left| 1 + \sum_{C \in B_2} h^2 1^2 + \sum_{C \in B_3} k^2 \chi(C) \overline{\chi'(C)} \right| \\
&= h(h-1) \left| 1 + \frac{k-1}{h} h^2 - k^2 \right| \\
&= h(h-1) |1 + (k-1)h - k^2| \\
&= h(h-1)(h + k^2 - hk - 1).
\end{aligned}$$

Combining everything together, we have that

$$\begin{aligned}
&AMZA_{\mathrm{off}}(G) \\
&= \frac{1}{h^2k^2} \left[(h-1) \left(\frac{k-1}{h} \right) \left(\frac{k-1}{h} - 1 \right) h^4 + 2(h-1)(k-1)h^2 + h(h-1)(h + k^2 - hk - 1) \right].
\end{aligned}$$

We can now see that

$$\begin{aligned}
&AMZA_{\text{off}}(G) + \text{ass}(G)\\
&= \frac{2h^2}{k^2} - \frac{2h^2}{k} - \frac{2h}{k^2} + \frac{2}{hk^2} + 2h - \frac{2}{h} - \frac{2}{k^2} + \frac{2}{k} + 1\\
&= 1 + 2\frac{h^2-1}{h}\left(1 - \frac{h-1}{k}\right)\left(1 - \frac{1}{k}\right).
\end{aligned}$$

By appealing again to [4, Theorem 2.4], we get that $AMZA(G) = AMZ(G)$. □

4.4 Groups with Two Conjugacy Class Sizes

Definition 4.4.1. Let $cd(G)$ denote the set of dimensions of irreducible characters of G, and let $cc(G)$ denote the set of sizes of conjugacy classes of G. We say that G has *two character degrees* (or two *conjugacy class sizes*) if $|cd(G)| = 2$ (or $|cc(G)| = 2$).

There is a nice formula for $AMZ(G)$ in the two character degree case:

Theorem 4.4.2. *[4, Theorem 2.4]. Let G be a finite group with $cd(G) = \{1, m\}$. Then*

$$AMZ(G) = 1 + 2(m^2-1)\left(1 - \frac{1}{|G| \cdot |G'|} \sum_{C \in Conj(G)} |C|^2\right).$$

We can prove a dual formula for $AMZA(G)$ if instead we assume that there are only two possible sizes of conjugacy classes. First, we show a lemma.

Lemma 4.4.3. *Let G be a finite group. Then*

$$\frac{1}{|G|^2} \sum_{\chi,\chi' \in Irr(G)} d_\chi d_{\chi'} \left| \sum_{x \in Z(G)} \chi(x)\overline{\chi'(x)} \right| = 1.$$

Proof. Using the notation from Lemma 4.2.1, let $\chi \in A_i$ and $\chi' \in A_j$. Then by Schur orthogonality for $Z(G)$

$$\left| \sum_{x \in Z(G)} \chi(x)\overline{\chi(x)} \right| = |Z(G)| \cdot \langle d_\chi \phi_i | d_{\chi'} \phi_j \rangle_{Z(G)} = |Z(G)| \cdot d_\chi d_{\chi'} \delta_{ij},$$

so it follows that

$$\frac{1}{|G|^2}\sum_{\chi,\chi'\in\mathrm{Irr}(G)} d_\chi d_{\chi'}\left|\sum_{x\in Z(G)}\chi(x)\overline{\chi'(x)}\right| = \frac{1}{|G|^2}\sum_{i=1}^{|Z(G)|}\sum_{\chi,\chi'\in A_i}|Z(G)|d_\chi^2 d_{\chi'}^2$$
$$= \frac{1}{|G|^2}\sum_{i=1}^{|Z(G)|}|Z(G)|\cdot|G:Z(G)|^2$$
$$= 1.$$

□

Theorem 4.4.4. *Let G be a finite group with $cc(G)=\{1,s\}$. Then*

$$AMZA(G) = 1 + 2(s-1)\left(1 - \frac{|Z(G)|}{|G|^2}\sum_{\chi\in Irr(G)} d_\chi^4\right)$$

Proof. If $s=1$ then G is abelian, so $AMZA(G) = AMZ(G) = 1$, hence we can assume that $s>1$. Recall that we have the decomposition $AMZA(G) = AMZA_{\mathrm{off}}(G) + \mathrm{ass}(G)$. We will work with each component separately.

$$|G|^2 AMZA_{\mathrm{off}}(G) = \sum_{\chi\neq\chi'\in\mathrm{Irr}(G)} d_\chi d_{\chi'}\left|\sum_{C\in\mathrm{Conj}(G)}|C|^2\chi(C)\overline{\chi'(C)}\right|$$
$$= \sum_{\chi\neq\chi'\in\mathrm{Irr}(G)} d_\chi d_{\chi'}\left|\sum_{x\in Z(G)}\chi(x)\overline{\chi'(x)} + s\sum_{C\in\mathrm{Conj}(G),|C|>1} s\chi(C)\overline{\chi'(C)}\right|$$
$$= \sum_{\chi\neq\chi'\in\mathrm{Irr}(G)} d_\chi d_{\chi'}\left|(1-s)\sum_{x\in Z(G)}\chi(x)\overline{\chi'(x)} + s\sum_{C\in\mathrm{Conj}(G)}|C|\chi(C)\overline{\chi'(C)}\right|$$
$$= \sum_{\chi\neq\chi'\in\mathrm{Irr}(G)} d_\chi d_{\chi'}\left|(1-s)\sum_{x\in Z(G)}\chi(x)\overline{\chi'(x)}\right|$$
$$= (s-1)\sum_{\chi,\chi'\in\mathrm{Irr}(G)} d_\chi d_{\chi'}\left|\sum_{x\in Z(G)}\chi(x)\overline{\chi'(x)}\right| - (s-1)\sum_{\chi\in\mathrm{Irr}(G)} d_\chi^2\sum_{x\in Z(G)}|\chi(x)|^2$$

$$= |G|^2(s-1) + (1-s) \cdot |Z(G)| \cdot \left(\sum_{\chi \in \mathrm{Irr}(G)} d_\chi^4 \right).$$

And then for ass(G) we have

$$\begin{aligned} |G|^2 \mathrm{ass}(G) &= \sum_{\chi \in \mathrm{Irr}(G)} d_\chi^2 \sum_{C \in \mathrm{Conj}(G)} |C|^2 |\chi(C)|^2 \\ &= \sum_{\chi \in \mathrm{Irr}(G)} d_\chi^2 \left(s|G| + (1-s) \sum_{x \in Z(G)} |\chi(x)|^2 \right) \\ &= s|G|^2 + (1-s) \cdot |Z(G)| \cdot \left(\sum_{\chi \in \mathrm{Irr}(G)} d_\chi^4 \right). \end{aligned}$$

Adding them together and rearranging terms, we achieve formula

$$AMZA(G) = 1 + 2(s-1) \left(1 - \frac{|Z(G)|}{|G|^2} \sum_{\chi \in \mathrm{Irr}(G)} d_\chi^4 \right).$$

$\square$

Example 4.4.5. Let p be a prime. A group G is called p-extraspecial if $|G| = p^{2n+1}$ for some integer n, $|Z(G)| = p$, and $G/Z(G)$ is a non-trivial elementary abelian p-group. As noted in [4], such groups have two conjugacy class sizes and two character degrees, so both of the above formulas apply and yield the same result, namely that

$$AMZ(G) = AMZA(G) = 1 + 2\left(1 - \frac{1}{p^{2n}}\right)\left(1 - \frac{1}{p}\right).$$

This leads to the question: will $AMZ(G) = AMZA(G)$ always hold in the case that G has two conjugacy class sizes and two character degrees? Based on our earlier formulas, Y. Choi has observed that the answer is positive.

Theorem 4.4.6. *Let G be a finite group where $cd(G) = \{1, m\}$ and $cc(G) = \{1, s\}$. Then*

$$AMZA(G) = AMZ(G)$$

The following proof is based on a personal communication from Choi.

Proof. If either $s = 1$ or $m = 1$ then they both equal 1, in which case G is abelian and the result is trivial. Instead, we assume that $m, s > 1$. For notational convenience, let $k = |\mathrm{Irr}(G)| = |\mathrm{Conj}(G)|$, $Z = Z(G)$, and $\mathfrak{L} = \mathfrak{L}(G)$. Because $|G| = \sum_{C \in \mathrm{Conj}(G)} |C| = |Z| + (k - |Z|)s$ we can see that

$$\begin{aligned}
\sum_{C \in \mathrm{Conj}(G)} |C|^2 &= |Z| + (k - |Z|)s^2 \\
&= |Z| + s(|Z| + (k - |Z|)s) - s|Z| \\
&= |Z| + s|G| - s|Z| \\
&= |Z| + (k - |Z|)s + s|G| - sk \\
&= (s + 1)|G| - sk
\end{aligned}$$

Similarly, from $|G| = \sum_{\chi \in \mathrm{Irr}(G)} {d_\chi}^2 = |\mathfrak{L}| + (k - |\mathfrak{L}|)m^2$ it follows that

$$\begin{aligned}
\sum_{\chi \in \mathrm{Irr}(G)} {d_\chi}^4 &= |\mathfrak{L}| + (k - |\mathfrak{L}|)m^4 \\
&= |\mathfrak{L}| + m^2(|\mathfrak{L}| + (k - |\mathfrak{L}|)m^2) - m^2|\mathfrak{L}| \\
&= |\mathfrak{L}| + m^2|G| - m^2|\mathfrak{L}| \\
&= |\mathfrak{L}| + (k - |\mathfrak{L}|)m^2 + m^2|G| - km^2 \\
&= (m^2 + 1)|G| - km^2
\end{aligned}$$

Define the function

$$f(x, y) = x - 1 - \frac{xk - |G|}{|G|}\Big((y + 1)|G| - yk\Big).$$

The above calculations combined with Theorem 4.4.2 and Theorem 4.4.4 yields that $AMZ(G) - 1 = 2f(m^2, s)$ and $AMZA(G) - 1 = 2f(s, m^2)$. However, we can see that

$$f(x, y) = x - 1 + y + 1 - \frac{xk(y + 1)}{|G|} + \frac{(xk - |G|)(yk)}{|G|^2}$$

$$= x + y - \frac{xy + x + y}{|G|} + \frac{xyk^2}{|G|^2}.$$

In particular note that $f(x, y) = f(y, x)$, thus $AMZA(G) = AMZ(G)$, as desired. $\square$

Remark 4.4.7. By results in [24], for any integer n there exists groups G and H such that $|cd(G)| = |cc(H)| = 2$ and $|cc(G)| = |cd(H)| = n$. This tells us that the two conjugacy class size and two character degree conditions are possibly independent of each other, so there is no reason to expect that the conclusion of Theorem 4.4.6 will hold if only one of the sets $cd(G)$ and $cc(G)$ has size 2. Indeed, $G =$ `SmallGroup(256,10070)` is an example of a group with $|cc(G)| = 2$, $|cd(G)| = 3$, and $AMZ(G) \neq AMZA(G)$.

4.5 AMZA of Quotient Groups

Recall that an essential ingredient in Choi's [13] proof that $\frac{7}{4}$ is the sharp amenability bound for $ZL^1(G)$ is the fact that $AMZ(G) \geq AMZ(G/N)$ for $N \trianglelefteq G$. If we look at the collection of groups such that $AMZA(G)$ respects all possible quotients, then by utilizing similar techniques as in [13] we can prove that $\frac{7}{4}$ is a sharp amenability bound.

Theorem 4.5.1. *$\frac{7}{4}$ is the sharp amenability bound over the collection of finite groups G with the property that $AMZA(G) \geq AMZA(G/N)$ for all $N \trianglelefteq G$.*

Proof. By taking sufficiently large enough quotients of G, we can assume without loss of generality that G is non-abelian but possesses no non-abelian proper quotients. As noted in [13, Lemma 4.4 and Theorem 4.5], there are three possibilities:

- G has a non-trivial centre
- G has a trivial centre and a conjugacy class of size 2
- G has a trivial centre and no conjugacy classes of size 2.

The first two options correspond with G either being a two conjugacy class size and two character degree group, or being isomorphic D_{2p} for some odd prime p. Theorem 4.4.6 and Theorem 4.3.2 apply respectively, which shows that $AMZ(G) = AMZA(G)$ in those cases. If G has a trivial centre and no conjugacy classes of size 2 then [13, Proposition 4.12] yields that $AMZA(G) \geq \mathrm{ass}(G) \geq \frac{7}{4}$. $\square$

It can be shown computationally that all groups of order less than 192 satisfy the conditions of Theorem 4.5.1. The above prompts the question: can this argument apply to every finite group? As demonstrated by the next example, this is not the case.

Example 4.5.2. Let G = `SmallGroup(192,1022)` and choose $N \cong C_2$ in G such that $G/N \cong$ `SmallGroup(96,204)`, then $AMZA(G) = 13.4921875$ and $AMZA(G/N) = 15.53125$. This example also demonstrates that auxiliary minorant of AMZ does not always respect quotients. For these choices of G and N we have that $\mathrm{ass}(G) = 7.2109375$ and $\mathrm{ass}(G/N) = 8.265625$.

Chapter 5

Arens Regularity of Ideals in $A(G), A_{cb}(G),$ and $A_M(G)$

Pardon me, I was absorbed in thought.

Siegward Of Catarina
Dark Souls 3

5.1 Invariant Subspaces

Definition 5.1.1. Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let $x \in G$ define the isometry $L_x : \mathcal{A}(G) \to \mathcal{A}(G)$ by

$$L_x(u)(y) = u(xy),$$

for each $y \in G$.

The following proposition will prove useful.

Proposition 5.1.2. *Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let $X \subseteq \mathcal{A}(G)^*$ be a closed submodule. Let $x \in G$.*

i) If $Y = L_x^*(X)$, then Y is a closed submodule of $\mathcal{A}(G)^*$ and

$$u \cdot L_x^*(T) = L_x^*(L_x(u) \cdot T)$$

for every $u \in \mathcal{A}(G)$ and $T \in X$.

ii) Let $x \in G$. Then $L_x^*(\phi_e) = \phi_x$ where e denotes the identity of G.

iii) Let $m \in TIM_{\mathcal{A}(G)}(X, \phi_e)$. If $x \in G$, then $\phi_x \in L_x^*(X)$ and $L_{x^{-1}}^{**}(m) \in TIM_{\mathcal{A}(G)}(L_x^*(X), \phi_x)$.

Proof. Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let $X \subset \mathcal{A}(G)^*$ be a closed submodule. Let $x \in G$.

i) Since L_x^* is an isometry, it is clear that Y is a closed subspace.

Let $T \in X$ and $u, v \in \mathcal{A}(G)$. Observe that

$$\begin{aligned} \langle u \cdot L_x^*(T), v \rangle &= \langle L_x^*(T), uv \rangle \\ &= \langle T, L_x(uv) \rangle \\ &= \langle T, L_x(u) L_x(v) \rangle \\ &= \langle L_x(u) \cdot T, L_x(v) \rangle \\ &= \langle L_x^*(L_x(u) \cdot T), v \rangle \end{aligned}$$

Hence $u \cdot L_x^*(T) = L_x^*(L_x(u) \cdot T) \in L_x^*(X) = Y$.

ii) Let $u \in \mathcal{A}(G)$. Then

$$\langle L_x^*(\phi_e), u \rangle = \langle \phi_e, L_x(u) \rangle = L_x(u)(e) = u(x) = \langle \phi_x, u \rangle.$$

This shows that $L_x^*(\phi_e) = \phi_x$.

iii) Let $T = L_x^*(T_1)$ with $T_1 \in X$. If $u \in \mathcal{A}(G)$, we have

$$\begin{aligned} \langle L_{x^{-1}}^{**}(m), u \cdot T \rangle &= \langle m, L_{x^{-1}}^*(u \cdot T) \rangle \\ &= \langle m, L_{x^{-1}}^*(u \cdot L_x^*(T_1)) \rangle \\ &= \langle m, , L_{x^{-1}}^*(L_x^*((L_x(u) \cdot T_1)) \rangle \\ &= \langle m, L_x(u) \cdot T_1 \rangle \end{aligned}$$

$$\begin{aligned}
&= L_x(u)(e)\langle m, T_1\rangle \\
&= u(e)\langle m, T_1\rangle \\
&= u(e)\langle m, L^*_{x^{-1}}(L^*_x(T_1))\rangle \\
&= u(e)\langle L^{**}_{x^{-1}}(m), T\rangle
\end{aligned}$$

Since $L^{**}_{x^{-1}}$ is an isometry we have that

$$\|L^{**}_{x^{-1}}(m)\|_{\mathcal{A}^*} = \|m\|_{\mathcal{A}^*} = 1.$$

Finally, we have that

$$\langle L^{**}_{x^{-1}}(m), \phi_x\rangle = \langle m, L^*_{x^{-1}}(\phi_x)\rangle = \langle m, \phi_e\rangle = 1.$$

□

The next result follows immediately from Proposition 5.1.2 iii).

Corollary 5.1.3. *Let $\mathcal{A}(G)$ be any of $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let $x \in G$. Then*

$$|TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)| = |TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_e)|,$$

where $|\cdot|$ represents the cardinality of the underlying set.

This next result was proven for the $TIM_{A_M}(A_M(G)^*, \phi_e)$ case in [29].

Proposition 5.1.4. *Let $\mathcal{A}(G)$ be either $A_{cb}(G)$ or $A_M(G)$ and let $i : A(G) \to \mathcal{A}(G)$ be the inclusion map as in Remark 2.9.3. Let*

$$\mathcal{A}(G) \cdot VN(G) = \{u \cdot T : u \in \mathcal{A}(G), T \in VN(G)\}.$$

Then

i) $\mathcal{A}(G) \cdot VN(G) \subseteq UCB(A(G))$

ii) $i^*(v \cdot T) = v \cdot i^*(T)$ *for each* $v \in \mathcal{A}(G), T \in \mathcal{A}(G)^*$.

iii) $i^*(UCB(\mathcal{A}(G))) \subseteq UCB(A(G))$.

iv) If $\mathcal{A}(G)$ has a bounded approximate identity, then $\mathcal{A}(G) \cdot VN(G) = UCB(A(G))$.

v) $u \cdot T \in i^*(\mathcal{A}(G)^*)$ *for each* $u \in A(G), T \in VN(G)$.

Proof. i) Since $UCB(A(G))$ is a closed subspace of $VN(G)$ and since $A(G)$ is dense in $\mathcal{A}(G)$ with respect to the norm $\|\cdot\|_{\mathcal{A}(G)}$, to establish $i)$ we need only show that for any sequence $\{v_n\} \subset A(G)$ and $v \in \mathcal{A}(G)$ with $\|v_n - v\|_{\mathcal{A}(G)} \to 0$ and any $T \in VN(G)$, we have $\|v_n \cdot T - v \cdot T\|_{VN(G)} \to 0$. However, this follows immediately since for any $u \in A(G)$

$$\begin{aligned} | \langle v_n \cdot T - v \cdot T, u \rangle | &= | \langle (v_n - v) \cdot T, u \rangle | \\ &= | \langle T, (v_n - v)u \rangle | \\ &\leq \|T\|_{VN(G)} \|v_n - v\|_{\mathcal{A}(G)} \|u\|_{A(G)}. \end{aligned}$$

ii) Let $v \in \mathcal{A}(G)$, $T \in \mathcal{A}(G)^*$ and $u \in A(G)$. We have

$$\begin{aligned} \langle i^*(v \cdot T), u \rangle &= \langle v \cdot T, i(u) \rangle \\ &= \langle T, vi(u) \rangle \\ &= \langle T, i(vu) \rangle \\ &= \langle i^*(T), vu \rangle \\ &= \langle v \cdot i^*(T), u \rangle \end{aligned}$$

Hence $i^*(v \cdot T) = v \cdot i^*(T)$.

iii) Because of i) above we need only show that $i^*(v \cdot T) \in \mathcal{A}(G) \cdot VN(G)$ for any $v \in \mathcal{A}(G)$ and $T \in \mathcal{A}(G)^*$. However, this follows immediately from ii).

iv) Let $(u_n \cdot T_n)_n \subseteq \mathcal{A}(G) \cdot VN(G) \subseteq VN(G)$ be a sequence converging to some $L \in VN(G)$. Because $\mathcal{A}(G)$ has a bounded approximate identity we can apply Theorem 2.1 by viewing $VN(G)$ as a Banach $\mathcal{A}(G)$-bimodule. This yields that $L = u \cdot T$ for $u \in \mathcal{A}(G), T \in VN(G)$, hence $L \in \mathcal{A}(G) \cdot VN(G)$. Thus $\mathcal{A}(G) \cdot VN(G)$ is closed, and in particular is a closed subspace of $UCB(G)$ due to $i)$. We have that $A(G) \cdot VN(G) \subseteq \mathcal{A}(G) \cdot VN(G)$, so it follows by the density of $A(G) \cdot VN(G)$ in $UCB(A(G))$ that $\mathcal{A}(G) \cdot VN(G)$ is dense in $UCB(A(G))$, hence $\mathcal{A} \cdot VN(G) = UCB(A(G))$.

v) Let $u \in A(G), T \in VN(G)$. Then we can define a linear functional on $\mathcal{A}(G)$ by

$$\varphi_{u,T}(v) = \langle T, uv \rangle$$

for each $v \in \mathcal{A}(G)$. Note that this functional is well-defined because $\mathcal{A}(G)$ consists of multipliers of $A(G)$, so $uv \in A(G)$. It is clear that $\varphi_{u,T}$ has norm at most $\|u\|_{A(G)}\|T\|_{VN(G)}$, and moreover, that this linear functional agrees with $u \cdot T$ on $A(G)$ and as such $u \cdot T = i^*(\varphi_{u,T})$.

□

Theorem 5.1.5. *Let $\mathcal{A}(G)$ denote either $A_{cb}(G)$ or $A_M(G)$. For any locally compact group, $i^{**}(TIM_{A(G)}(VN(G), \phi_x)) \subseteq TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$. Moreover,*

$$i^{**} : TIM_{A(G)}(VN(G), \phi_x) \to TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$$

is a bijection.

Proof. We will first show that $i^{**}(TIM_{A(G)}(VN(G), \phi_x)) \subseteq TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$.

Let $m \in TIM_{A(G)}(VN(G), \phi_x))$. Let $v \in \mathcal{A}(G)$ and $T \in \mathcal{A}(G)^*$. By the density of $A(G)$ in $\mathcal{A}(G)$ there exists $\{u_n\} \subset A(G)$ such that $\|u_n - v\|_{\mathcal{A}(G)} \to 0$. Since $\|u_n - v\|_\infty \leq \|u_n - v\|_{\mathcal{A}(G)}$ it follows that $u_n(x) \to v(x)$. Next, we note that in a similar manner to the proof of Proposition 5.1.4 i), we can show that $u_n \cdot T \to v \cdot T$ in the norm $\|\cdot\|_{\mathcal{A}(G)^*}$ for each $T \in \mathcal{A}(G)^*$. Appealing this time to Proposition 5.1.4 ii), it follows that

$$\begin{aligned}
\langle i^{**}(m), v \cdot T\rangle &= \lim_{n\to\infty} \langle i^{**}(m), u_n \cdot T\rangle \\
&= \lim_{n\to\infty} \langle m, i^*(u_n \cdot T)\rangle \\
&= \lim_{n\to\infty} \langle m, u_n \cdot i^*(T)\rangle \\
&= \lim_{n\to\infty} u_n(x)\langle m, i^*(T)\rangle \\
&= v(x)\langle m, i^*(T)\rangle \\
&= v(x)\langle i^{**}(m), T\rangle.
\end{aligned}$$

This shows that $i^{**}(TIM_{A(G)}(VN(G), \phi_x)) \subseteq TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$.

We next show that $i^{**} : TIM_{A(G)}(VN(G), \phi_x) \to TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$ is injective. To see this, we first note that if $m_1, m_2 \in TIM_{A(G)}(VN(G), \phi_x)$ with $m_1 \neq m_2$, then there exists an $T \in VN(G)$ for which

$$\langle m_1, T\rangle \neq \langle m_2, T\rangle.$$

Next choose $u_0 \in A(G)$ with $u_0(x) = 1$ (this is possible for example by an application of Proposition 2.7.2). Then

$$\langle m_1, u_0 \cdot T\rangle = \langle m_1, T\rangle \neq \langle m_2, T\rangle = \langle m_2, u_0 \cdot T\rangle.$$

Since $u_0 \cdot T \in \mathcal{A}(G)^*$, we have

$$\begin{aligned}
\langle i^{**}(m_1), u_0 \cdot T\rangle &= \langle m_1, i^*(u_0 \cdot T)\rangle \\
&= \langle m_1, u_0 \cdot T\rangle \\
&\neq \langle m_2, u_0 \cdot T\rangle \\
&= \langle m_2, i^*(u_0 \cdot T)\rangle \\
&= \langle i^{**}(m_2), u_0 \cdot T\rangle
\end{aligned}$$

so that $i^{**}(m_1) \neq i^{**}(m_2)$.

Finally, we show that $i^{**} : TIM_{A(G)}(VN(G), \phi_x) \to TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$ is surjective.

Let $M \in TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$. First note that if $u, v \in A(G)$, with $u(x) = 1 = v(x)$ and if $T \in VN(G)$, then $u \cdot T$ and $v \cdot T$ are in $\mathcal{A}(G)^*$ and

$$\langle M, u \cdot T\rangle = \langle M, v \cdot (u \cdot T)\rangle = \langle M, u \cdot (v \cdot T)\rangle = \langle M, v \cdot T\rangle.$$

Pick a $u_0 \in A(G)$ with $\|u_0\|_{A(G)} = 1$ and $u_0(x) = 1$. We can define $m_M \in A(G)^{**}$ by

$$\langle m_M, T\rangle = \langle M, u_0 \cdot T\rangle$$

for $T \in VN(G)$.

Note that $\|m_M\|_{A(G)^{**}} \leq 1$. It is clear from the observation above that if $v \in A(G)$ is such that $v(x) = 1$, then $\langle m_M, v \cdot T\rangle = \langle m_M, T\rangle$. We also have that

$$\langle m_M, \phi_x\rangle = \langle M, u_0 \cdot \phi_x\rangle = \langle M, u_0(x)\phi_x\rangle = \langle M, \phi_x\rangle = 1.$$

That is, $m_M \in TIM_{A(G)}(VN(G), \phi_x)$.

Finally, if $T \in \mathcal{A}(G)^*$, then

$$\langle i^{**}(m_M), T\rangle = \langle m_M, i^*(T)\rangle = \langle M, u_0 \cdot i^*(T)\rangle = \langle M, u_0.T\rangle = \langle M, T\rangle.$$

Therefore, $i^{**}(m_M) = M$.

□

Definition 5.1.6. Given a locally compact group G we let $b(G)$ denote the smallest cardinality of a neighbourhood basis at the identity e for G.

The next corollary follows immediately from the previous theorem and from Hu [39].

Corollary 5.1.7. *Let G be a non-discrete locally compact group. Let $\mathcal{A}(G)$ be $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Then*

$$\mid TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)) \mid = 2^{2^{b(G)}}.$$

In particular, $\mathcal{A}(G)^*$ admits a unique topologically invariant mean if and only if G is discrete.

We now turn our attention to ideals in the algebra $\mathcal{A}(G)$ where $\mathcal{A}(G)$ is any of $A(G)$, $A_{cb}(G)$ or $A_M(G)$.

Remark 5.1.8. In several of the proofs to come we will have a setup where I is a closed ideal in $\mathcal{A}(G)$ and $x \notin Z(I)$. By local compactness of G we can find a relatively compact open set V containing x. Because $Z(I)$ is closed then $U = V \cap (G \setminus Z(I))$ is an open neighborhood of x that is disjoint from $Z(I)$. By applying Proposition 2.7.2 to $\{x\} \subseteq U$ we know there exists a function $u_0 \in A(G) \cap C_C(G)$ such that $\text{supp}(u_0) \subseteq U$ and $u_0(x) = 1$. It follows that $u_0 \in I$.

Lemma 5.1.9. *Let $\mathcal{A}(G)$ be $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$. Assume that $x \notin Z(I)$. Let $M \in TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$. Then $M \in (I^\perp)^\perp$.*

Proof. Let $T \in I^\perp$ and take u_0 and U as in Remark 5.1.8.

Since $T \in I^\perp$, we have that for any $u \in \mathcal{A}(G)$ that

$$\langle u_0 \cdot T, u \rangle = \langle T, u_0 u \rangle = 0$$

so $u_0 \cdot T = 0$. However, since $M \in TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$, and since $u_0(x) = 1$,

$$\langle M, T \rangle = \langle M, u_0 \cdot T \rangle = 0.$$

□

Remark 5.1.10. The previous lemma shows that if I is a closed ideal in $\mathcal{A}(G)$ with $x \notin Z(I)$, and if $M \in TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$, then we can view M as an element $\hat{M}$ of I^{**} in a canonical way. Specifically, if $T \in I^{**}$ and T_1 is any extension of T we can define

$$\hat{M}(T) = M(T_1).$$

Note that $\hat{M}$ is well-defined since $M \in (I^{\perp})^{\perp}$. We claim that $\hat{M} \in TIM_I(I^*, \phi_{x_{|_I}})$. To see that this is the case we note that

$$\|\hat{M}\|_{I^*} = \|M\|_{\mathcal{A}(G)^{**}} = M(\phi_x) = \hat{M}(\phi_{x_{|_I}}).$$

If $u \in I$, then if $T_{1_{|_I}} = T$, then $u \cdot T_{1_{|_I}} = u \cdot T$ and as such

$$\hat{M}(u \cdot T) = M(u \cdot T_1) = u(x)M(T_1) = \phi_{x_{|_I}}(u)\hat{M}(T).$$

This gives us a map $\Gamma : TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x) \to TIM_I(I^*, \phi_{x_{|_I}})$, given by

$$\Gamma(M) = \hat{M}.$$

Theorem 5.1.11. *Let $\mathcal{A}(G)$ be $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$. Assume that $x \notin Z(I)$ and let Γ be as above. Then Γ is a bijection.*

Proof. Assume that $M_1, M_2 \in TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$ and that $T_0 \in \mathcal{A}(G)^*$ is such that $M_1(T_0) \neq M_2(T_0)$. Let $T \in I^* = T_{0_{|_I}}$. Then

$$\hat{M}_1(T) = M_1(T_0) \neq M_2(T_0) = \hat{M}_2(T)$$

so

$$\Gamma(M_1) = \hat{M}_1 \neq \hat{M}_2 = \Gamma(M_2)$$

and hence Γ is injective.

Next we let $m \in TIM_I(I^*, \phi_{x_{|_I}})$. We let $u_0 \in I$ be such that $u_0(x) = 1$ with $\|u_0\|_{\mathcal{A}(G)} = 1$. First, observe that $(u_0 \cdot T)_{|_I} = u_0 \cdot (T_{|_I})$ for each $T \in \mathcal{A}(G)^*$. Then we define $M \in \mathcal{A}(G)^{**}$ by

$$\langle M, T\rangle = \langle m, u_0 \cdot (T_{|_I})\rangle = \langle m, T_{|_I}\rangle$$

for each $T \in \mathcal{A}(G)^*$. We have that

$$\|M\|_{\mathcal{A}(G)^{**}} \leq \|m\|_{I^{**}}\|u_0\|_{\mathcal{A}(G)} = 1.$$

Moreover

$$\langle M, \phi_x \rangle = \langle m, u_0 \cdot \phi_{x_{|_I}} \rangle = u_0(x)\langle m, \phi_{x_{|_I}} \rangle = 1.$$

Next let $T \in \mathcal{A}(G)^*$ and let $u \in \mathcal{A}(G)$. Then

$$\begin{aligned}
\langle M, u \cdot T \rangle &= \langle m, u_0 \cdot (u \cdot T)_{|_I} \rangle \\
&= \langle m, (u_0 u) \cdot T_{|_I} \rangle \\
&= (u_0 u)(x)\langle m, T_{|_I} \rangle \\
&= u(x)(u_0(x)\langle m, T_{|_I} \rangle) \\
&= u(x)\langle m, u_0 \cdot (T_{|_I}) \rangle \\
&= u(x)\langle M, T \rangle
\end{aligned}$$

It follows that $M \in TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x)$. Finally, if $T \in I^*$ and if $T_1 \in \mathcal{A}(G)$ with $T_{1_{|_I}} = T$, then

$$\begin{aligned}
\langle \hat{M}, T \rangle &= \langle M, T_1 \rangle \\
&= \langle m, T_{1_{|_I}} \rangle \\
&= \langle m, T \rangle
\end{aligned}$$

Hence $\Gamma(M) = \hat{M} = m$ and Γ is surjective.

□

The following result follows immediately from Corollary 5.1.7.

Corollary 5.1.12. *Let $\mathcal{A}(G)$ be $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$. Assume that $x \notin Z(I)$. If G is a non-discrete group, then*

$$|TIM_I(I^*, \phi_{x_{|_I}})| = 2^{2^{b(G)}}.$$

In particular, I^* admits a unique topological invariant mean if and only if G is discrete.

Lemma 5.1.13. *Let $\mathcal{A}$ be a commutative Banach algebra with maximal ideal space $\Delta(\mathcal{A})$. Let X be a closed submodule of $\mathcal{A}^*$ containing $\phi \in \Delta(\mathcal{A})$. Let $M \in TIM_{\mathcal{A}}(\mathcal{A}, \phi)$. Let $m = M_{|_X}$ be the restriction of M to X. Then $m \in TIM_{\mathcal{A}}(X, \phi)$.*

In particular, if we let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$, I be a closed ideal in $\mathcal{A}(G)$ with $x \notin Z(I)$ and X is any of $UCB(I)$, $WAP(I)$ or $AP(I)$, we have that $\phi_{x_{|_I}} \in X$ and hence that for any $M \in TIM_I(I^*, \phi_{x_{|_I}})$, if $m = M_{|X}$ we get that $m \in TIM_I(X, \phi_{x_{|_I}})$.

Proof. Let $M \in TIM_{\mathcal{A}}(\mathcal{A}, \phi)$. Let $m = M_{|X}$ be the restriction of M to X. We have that

$$1 = \langle M, \phi \rangle = \langle m, \phi \rangle = \|m\|_{X^*}$$

and that if $u \in \mathcal{A}$ and $T \in X$, then

$$\langle m, u \cdot T \rangle = \langle M, u \cdot T \rangle = \langle \phi, u \rangle \langle M, T \rangle = \langle \phi, u \rangle \langle m, T \rangle.$$

Hence $m \in TIM_{\mathcal{A}}(X, \phi)$.

Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$ and I a closed ideal in $\mathcal{A}(G)$ with $x \notin Z(I)$. Let $X = UCB(I)$. Let u_0 and U be as in Remark 5.1.8. Then $u_0 \in I$. Moreover, if $v \in I$,

$$\langle u_0 \cdot \phi_{x_{|_I}}, v \rangle = \langle \phi_{x_{|_I}}, u_0 v \rangle = u_0(x) v(x) = \langle \phi_{x_{|_I}}, u_0 v \rangle.$$

Hence $\phi_{x_{|_I}} = u_0 \cdot \phi_{x_{|_I}} \in UCB(I)$.

To see that $\phi_{x_{|_I}} \in AP(I)$ note that $\{u(x) | \|u\|_I \leq 1\} = \{\lambda \in \mathbb{C} | \mid \lambda \mid \leq 1\}$ is compact and hence

$$\{u \cdot \phi_{x_{|_I}} | \|u\|_I \leq 1\} = \{\lambda \phi_{x_{|_I}} \in \mathbb{C} | \mid \lambda \mid \leq 1\}$$

is compact in I^* so $\phi_{x_{|_I}} \in AP(I)$. As $AP(I) \subseteq WAP(I)$ we also have that $\phi_{x_{|_I}} \in WAP(I)$. □

Theorem 5.1.14. *Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$. Assume that $x \notin Z(I)$. The restriction map $R : TIM_I(I^*, \phi_{x_{|_I}}) \to TIM_I(UCB(I), \phi_{x_{|_I}})$ is a bijection. In particular, if G is non-discrete, then*

$$\mid TIM_{\mathcal{A}(G)}(\mathcal{A}(G)^*, \phi_x) \mid = \mid TIM_{\mathcal{A}(G)}(UCB(\mathcal{A}(G), \phi_x) \mid = 2^{2^{b(G)}}.$$

Proof. Let u_0 and U be as in Remark 5.1.8. Then $u_0 \in I$ and

$$\phi_{x_{|_I}} = u_0 \cdot \phi_{x_{|_I}} \in UCB(I).$$

Next we let $M \in TIM_I(I^*, \phi_{x_{|_i}})$. Let $m = R(M)$. It follows from Lemma 5.1.13 that $m \in TIM_I(UCB(I), \phi_{x_{|_I}})$.

If $M_1, M_2 \in TIM_I(I^*, \phi_{x_{|_I}})$ with $M_1 \neq M_2$, then there exists a $T \in I^*$ for which

$$\langle M_1, T\rangle \neq \langle M_2, T\rangle.$$

Let u_0 and U be as in Remark 5.1.8, then $u_0 \cdot T \in UCB(I)$ with

$$\langle M_1, u_0 \cdot T\rangle = \langle M_1, T\rangle \neq \langle M_2, T\rangle = \langle M_2, u_0 \cdot T\rangle.$$

This shows that $R(M_1) \neq R(M_2)$ and hence R is injective.

Next, let $m \in TIM_I(UCB(I), \phi_{x_{|_I}})$. Pick a $u_0 \in I$ with $\|u_0\|_{A(G)} = 1 = u_0(x)$. Define $M \in I^{**}$ by

$$\langle M, T\rangle = \langle m, u_0 \cdot T\rangle, \qquad T \in I^*.$$

Since $u_0(x) = 1$, it follows that

$$\langle M, \phi_{x_{|_I}}\rangle = \langle m, u_0 \cdot \phi_{x_{|_I}}\rangle = \langle m, \phi_{x_{|_I}}\rangle = 1.$$

From this and the fact that $\|u_0\|_{A(G)} = 1$, we get that $\|M\| = 1$.

Next, if $v \in I, T \in I^*$, then

$$\langle M, v \cdot T\rangle = \langle m, u_0 \cdot (v \cdot T)\rangle = \langle m, v \cdot (u_0 \cdot T)\rangle = v(x)\langle m, u_0 \cdot T\rangle = v(x)\langle M, T\rangle.$$

This shows that $M \in TIM_I(I^*, \phi_{x_{|_I}})$.

Finally, if $T \in UCB(I))$, then

$$\langle M, T\rangle = \langle m, u_0 \cdot T\rangle = \langle m, T\rangle$$

since $m \in TIM_I(UCB(I), \phi_{x_{|_I}})$. Therefore, $R(M) = m$ and R is surjective.

□

Remark 5.1.15. In the proof of the previous theorem we were able to explicitly show how each $m \in TIM_I(UCB(I), \phi_{x_{|_I}})$ extends to an element $M \in TIM_I(I^*, \phi_{x_{|_I}})$. The next proposition shows that such extensions hold in greater generality.

We need the following lemma.

Lemma 5.1.16. *Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$ with $Z(I)$ being a set of spectral synthesis for $\mathcal{A}(G)$. Assume that $x \notin Z(I)$. Then $\{x\}$ is a set of spectral synthesis for I.*

Proof. Let $u \in I$ be such that $u(x) = 0$. Let $E = Z(I)$. Let $\epsilon > 0$. Since E is a set of spectral synthesis for $\mathcal{A}(G)$, we can find $w \in \mathcal{A}(G) \cap C_C(G)$ such that $\mathrm{supp}(w) \cap E = \emptyset$ and

$$\|u - w\|_{\mathcal{A}(G)} < \frac{\epsilon}{2}.$$

Since $u(x) = 0$, this means that $|w(x)| < \frac{\epsilon}{2}$.

We will go through a slightly adjusted procedure as done in Remark 5.1.8. By Proposition 2.7.2 we can find $v \in I \cap C_C(G)$ and an open neighborhood U of x such that $\mathrm{supp}(v) \cap E = \emptyset$, $v(x) = w(x)$ on U and $\|v\|_{\mathcal{A}(G)} \leq |w(x)|$. Then $w - v \in I \cap C_C(G)$ with $\mathrm{supp}(w - v) \cap (E \cup \{x\}) = \emptyset$ and

$$\|u - (w - v)\|_{\mathcal{A}(G)} \leq \|u - w\|_{\mathcal{A}(G)} + \|v\|_{\mathcal{A}(G)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Proposition 5.1.17. *Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$ with $Z(I)$ being a set of spectral synthesis for $\mathcal{A}(G)$. Assume that $x \notin Z(I)$. Assume also that $Y \subset X$ are two closed submodules of I^* each containing $\phi_{x_{|_I}}$. Let $m \in TIM_I(Y, \phi_{x_{|_I}})$. Then there exists some $M \in TIM_I(X, \phi_{x_{|_I}})$ such that $M_{|_Y} = m$.*

Proof. Let $m \in TIM_I(Y, \phi_{x_{|_I}})$. By the Hahn-Banach Theorem we can find a $\Psi \in X^*$ so that

$$\|\Psi\|_{X^*} = \|m\|_{Y^*} = 1.$$

Next we let

$$S = \{u \in I \mid \|u\|_{\mathcal{A}(G)} = u(x) = 1\}.$$

Then S is a commutative semigroup under pointwise multiplication and is hence amenable due to Theorem 2.5. Let $\Phi \in \ell_\infty(S)^*$ be an invariant mean. Then for each $T \in X$ we define $f_T : S \to \mathbb{C}$ by

$$f_T(u) = \langle \Psi, u \cdot T \rangle.$$

It follows that $f_T \in \ell_\infty(S)$ as $|f_T(u)| \leq \|T\|_X$ for each $u \in S$. Moreover, if $v \in S$, then

$$f_{v \cdot T}(u) = \langle \Psi, u \cdot (v \cdot T) \rangle = \langle \Psi, vu \cdot T \rangle = L_v(f_T)(u)$$

where L_v is the left translation operator on $\ell_\infty(S)$.

Next, we let
$$\langle M, T\rangle = \langle \Phi, f_T\rangle$$
for each $T \in X$. Note that $f_{\alpha T_1+\beta T_2} = \alpha f_{T_1} + \beta f_{T_2}$ and $|\ \langle M, T\rangle\ | \leq \|T\|_X$ so that in fact $M \in X^*$ with $\|M\|_{X^*} \leq 1$.

Now if $T = \phi_{x_{|_I}}$, then
$$f_T(u) = \langle \Psi, u \cdot \phi_{x_{|_I}}\rangle = 1$$
for every $u \in S$ and hence $\langle M, \phi_{x_{|_I}}\rangle = 1$ and $\|M\|_{X*} = 1$.

Finally, since Φ is a left invariant mean on $\ell_\infty(S)^*$ we have that if $u \in S$, then

$$\langle M, u \cdot T\rangle = \langle \Phi, f_{u \cdot T}\rangle = \langle \Phi, L_u(f_T)\rangle = \langle M, T\rangle.$$

Finally, we must show that $\langle M, u \cdot T\rangle = u(x)\langle M, T\rangle$ for all $u \in I$, and $T \in X$.

First we will show that if $v \in I$ and if there exists a neighborhood U of x such that $v(y) = 1$ on U, then $\langle M, v \cdot T\rangle = \langle M, T\rangle$ for all $T \in X$. To see why this is the case, we choose $u \in I$ such that $u(x) = 1 = \|u\|_{\mathcal{A}(G)}$ with $u(y) = 0$ if $y \notin U$. Then $uv = u$, hence

$$\langle M, v \cdot T\rangle = \langle M, u \cdot (v \cdot T)\rangle = \langle M, uv \cdot T\rangle = \langle M, u \cdot T\rangle = \langle M, T\rangle.$$

Next assume that $v \in I$ satisfies $v(y) = 0$ on some neighborhood U of x. Since $1 \in B(G)$, the function $1 - u$ is a multiplier of I, that is $(1 - v)w \in I$ for every $w \in I$ and hence $(1 - v) \cdot T \in X$. We note that $1 - v(y) = 1$ on U. Again choosing $u \in I$ such that $u(x) = 1 = \|u\|_{\mathcal{A}(G)}$ with $u(y) = 0$ if $y \notin U$. Once more we get that

$$(\langle M, T\rangle - \langle M, v \cdot T\rangle) = \langle M, (1 - v) \cdot T\rangle = \langle M, u \cdot ((1 - v) \cdot T)\rangle = \langle M, u \cdot T\rangle = \langle M, T\rangle.$$

Hence $\langle M, v \cdot T\rangle = 0$.

Now let $u \in I$ be such that $u(x) = 0$. By Lemma 5.1.16, $\{x\}$ is a set of spectral synthesis for I. It follows that we can find a sequence of functions $\{w_n\} \subset I$ and a sequence $\{U_n\}$ of open neighbourhoods of x such that each w_n has compact support, $w_n(y) = 0$ for all $y \in U_n$ and $\lim\limits_{n\to\infty} \|u - w_n\|_{\mathcal{A}(G)} = 0$. In particular, from what we have seen above, $\langle M, w_n \cdot T\rangle = 0$ and hence
$$\langle M, u \cdot T\rangle = \langle M, (u - w_n) \cdot T\rangle$$
Moreover,
$$\lim_{n\to\infty} \|u \cdot T - w_n \cdot T\|_{I^*} = 0,$$

as hence we have that $\langle M, u\rangle = 0$. Finally, choose $u \in I$ be such that $u(x) = 1$. Once more choose $v \in I$ so that $v = 1$ on a neighbourhood U of x. Then $u - v(x) = 0$ and

$$\langle M, (u - v) \cdot T\rangle = 0$$

which means that

$$\langle M, u \cdot T\rangle = \langle M, v \cdot T\rangle = v(x)\langle M, T\rangle = u(x)\langle M, T\rangle.$$

For here, if $u(x) \neq 0$, let $w = \dfrac{1}{u(x)}u$. Then

$$\langle M, u \cdot T\rangle = \left\langle M, u(x)\left(\frac{1}{u(x)}u \cdot T\right)\right\rangle = u(x)\left\langle M, \left(\frac{1}{u(x)}u \cdot T\right)\right\rangle = u(x)\langle M, T\rangle.$$

This shows that $M \in TIM_I(X, \phi_{x_{|_I}})$.

Finally, if $T \in Y$, then

$$f_T(u) = \langle \Psi, u \cdot T\rangle = \langle m, u \cdot T\rangle = \langle m, T\rangle$$

for all $u \in S$. In particular, $\langle M, T\rangle = \langle \Phi, f_T\rangle = \langle m, T\rangle$ so $M_{|_Y} = m$ as desired.

□

Theorem 5.1.18. *Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$. Assume that $x \notin Z(I)$. Then $WAP(I)$ has a unique topologically invariant mean at $\phi_{x_{|_I}}$.*

Proof. The fact that $TIM_I(WAP(I), \phi_{x_{|_i}}) \neq \emptyset$ follows immediately from the observation that every TIM on I^* restricts to a TIM on $WAP(I)$ and we know that $TIM_I(I^*, \phi_{x_{|_i}}) \neq \emptyset$.

Next, we let $n, m \in TIM_I(WAP(I), \phi_{x_{|_i}})$. Then there exits a net $\{u_\alpha\}_{\alpha\in\Omega} \subseteq I$ so that

$$m = \lim_{\alpha\in\Omega}\{u_\alpha\}$$

in the weak* topology on I^{**}. Hence, for each $T \in I^*$,

$$\langle n \odot m, T\rangle = \lim_\alpha \langle n \odot u_\alpha, T\rangle = \lim_\alpha \langle n, u_\alpha \cdot T\rangle = \lim_\alpha u_\alpha(x)\langle n, T\rangle.$$

But we also know that

$$1 = \langle n, \phi_{x_{|_I}} \rangle = \lim_{\alpha} \langle u_\alpha, \phi_{x_{|_I}} \rangle = \lim_{\alpha} u_\alpha(x).$$

Hence $n \odot m = n$. But we also know that $n \odot m = m \odot n = m$. Hence, $n = m$ and the TIM is unique. $\square$

Corollary 5.1.19. *Let $\mathcal{A}(G)$ be one of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a closed ideal in $\mathcal{A}(G)$. If $UCB(I) \subseteq WAP(I)$, then G is discrete.*

Proof. Assume that G is non discrete and that $x \notin Z(I)$. Let $M_1, M_2 \in TIM_I(I^*, \phi_{x_{|_i}})$ with $M_1 \neq M_2$, then $m_1 = M_{1_{|_{WAP(I)}}}, m_2 = M_{2_{|_{WAP(I)}}} \in TIM_I(WAP(I), \phi_{x_{|_i}})$. Let $T \in I^*$ be such that $\langle M_1, T \rangle \neq \langle M_2, T \rangle$ and choose $u \in I$ so that $u(x) = 1$. Then $u \cdot T \in UCB(I) \subseteq WAP(I)$, and

$$\langle m_1, u \cdot T \rangle = \langle M_1, u \cdot T \rangle = \langle M_1, T \rangle \neq \langle M_2, T \rangle = \langle M_2, u \cdot T \rangle = \langle m_2, u \cdot T \rangle$$

which contradicts the uniqueness of the TIM on $WAP(I)$. $\square$

5.2 Arens Regularity of Ideals in $A(G)$, $A_{cb}(G)$ and $A_M(G)$

In this section, we will apply what we know about topologically invariant means to questions concerning the possible Arens regularity of ideals in $A(G)$, $A_{cb}(G)$, and $A_M(G)$. The key observation is the following which improves on [26, Corollary 3.13]:

Theorem 5.2.1. *Let $\mathcal{A}(G)$ be any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let I be a non-zero closed ideal in $\mathcal{A}(G)$. If I is Arens regular, then G is discrete.*

Proof. If I is Arens regular, then $I^* = WAP(I)$. Hence I^* has a unique topologically invariant mean. However, by Corollary 5.1.12, this implies that G must be discrete.

$\square$

The following corollary is immediate. See also [26, Theorem 3.2] and [29, Corollary 3.9].

Corollary 5.2.2. *Let $\mathcal{A}(G)$ be any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. If $\mathcal{A}(G)$ is Arens regular, then G is discrete.*

Corollary 5.2.3. *Let G be non-discrete. If $\mathcal{A}(G)$ is one of $A(G)$, $A_{cb}(G)$ or $A_M(G)$, then $\mathcal{A}(G)$ has no non-zero reflexive closed ideal.*

Lemma 5.2.4. *Let $\mathcal{A}(G)$ be any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let H be a subgroup of G. If $\mathcal{A}(G)$ is Arens regular, then so is $\mathcal{A}(H)$. In particular, if H is amenable, then H is finite.*

Proof. As G is discrete, H is open in G. In this case, the restriction map $R : \mathcal{A}(G) \to \mathcal{A}(H)$ is a contractive homomorphism that is also surjective. As such $\mathcal{A}(H)$ is Arens regular.

The last statement is simply [51, Proposition 3.3].

□

Theorem 5.2.5. *Let I be a closed ideal of $A(G)$ with a bounded approximate identity that is Arens regular. Then I is finite-dimensional.*

Proof. Assume that $I \subseteq A(G)$ is Arens regular. By Theorem 5.2.1 G must be discrete. It is well-known that the predual of a von Neumann algebra is weakly sequentially complete, hence $A(G)$ is weakly sequentially complete. It follows that I is weakly sequentially complete due to the fact that I is a closed subalgebra of $A(G)$, so because I is Arens regular and also has a bounded approximate identity then [73, Theorem 3.3] provides that I is in fact unital. It follows that $1_{G\setminus Z(I)} \in I$. In particular, $G \setminus Z(I)$ must be compact and hence finite. This shows that I is finite-dimensional.

□

Remark 5.2.6. The fact that $A(G)$ is weakly sequentially complete was crucial in establishing the previous theorem. Unfortunately, we do not know whether or not either or both of $A_{cb}(G)$ or $A_M(G)$ would be weakly sequentially complete.

For the remainder of this section, we will assume that G is a discrete group.

Lemma 5.2.7. *Let $\mathcal{A}(G)$ be any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let H be a proper amenable subgroup of G. If $I_{\mathcal{A}(G)}(H)$ is Arens regular, then H is finite and $\mathcal{A}(G)$ is also Arens regular.*

Proof. Since H is proper, there exists an $x \in G \setminus H$. The ideal $I_{\mathcal{A}(G)}(xH)$ is isometrically isomorphic to $I_{\mathcal{A}(G)}(H)$ and hence is also Arens regular.

If $u \in \mathcal{A}(H)$, then the function u° defined by $u^\circ(y) = u(y)$ if $y \in H$ and $u^\circ(y) = 0$ if $y \in G \setminus H$ is in $\mathcal{A}(G)$. Now if $R : \mathcal{A}(G) \to \mathcal{A}(H)$ is the restriction map, then R is contractive homomorphism that maps $I_{\mathcal{A}(G)}(xH)$ onto $\mathcal{A}(H)$. In particular, $\mathcal{A}(H) = A(G)$ is also Arens regular. It follows that H is finite.

Let $\mathcal{B}$ be the algebra $1_H\mathcal{A}(G) \oplus_1 1_{G\setminus H}\mathcal{A}(G)$. Then B is a commutative Banach algebra and the mapping $\Gamma : \mathcal{A}(G) \to \mathcal{B}$ given by $\Gamma(u) = (1_H u, 1_{G\setminus H} u)$ is a continuous isomorphism that maps $I_{\mathcal{A}(G)}(H)$ isometrically onto the ideal $(I_{\mathcal{A}(G)}(H), 0)$ in $\mathcal{B}$. Since $1_{G\setminus H}\mathcal{A}(G)$ is finite-dimensional, it is Arens regular. We get that

$$(1_H\mathcal{A}(G) \oplus_1 1_{G\setminus H}\mathcal{A}(G))^{**} = (1_H\mathcal{A}(G))^{**} \oplus_1 (1_{G\setminus H}\mathcal{A}(G))^{**}$$

which is commutative since each of its components is commutative. Hence $1_H\mathcal{A}(G) \oplus_1 1_{G\setminus H}\mathcal{A}(G)$ is Arens regular, and so is $\mathcal{A}(G)$. $\square$

Definition 5.2.8. Let $\mathcal{R}(G)$, the *coset ring of G, denote the Boolean ring of sets generated by cosets of subgroups of G. A subset E of G is in $\mathcal{R}(G)$ if and only if*

$$E = \bigcup_{i=1}^{n} \left(x_i H_i \setminus \bigcup_{j=1}^{m_i} b_{i,j} K_{i,j} \right),$$

where H_i is a subgroup of G, $x_i \in G$, $K_{i,j}$ is a subgroup of H_i, and $b_{i,j} \in K_{i,j}$.

By $\mathcal{R}_a(G)$, the amenable coset ring of G, we will mean all sets of the form

$$E = \bigcup_{i=1}^{n} \left(x_i H_i \setminus \bigcup_{j=1}^{m_i} b_{i,j} K_{i,j} \right),$$

where H_i is an amenable subgroup of G, $x_i \in G$, $K_{i,j}$ is a subgroup of H_i, and $b_{i,j} \in K_{i,j}$.

Theorem 5.2.9. *Let $\mathcal{A}(G)$ be any of the algebras $A(G)$, $A_{cb}(G)$ or $A_M(G)$. Let $E \in \mathcal{R}_a(G)$ and $I_{\mathcal{A}(G)}(E)$ be non-zero and Arens regular. Then either E is finite and $\mathcal{A}(G)$ is also Arens regular, or G is amenable and $I_{\mathcal{A}(G)}(E)$ is finite-dimensional.*

Proof. We begin by first assuming that $E = \bigcup_{i=1}^{n} x_i H_i$. In this case, we will prove the conclusion by induction on n. That is we let $P(n)$ be the statement that if $E = \bigcup_{i=1}^{n} x_i H_i$ is a proper subset of G and if $I_{\mathcal{A}(G)}(E)$ is Arens regular, then E is finite and $\mathcal{A}(G)$ is Arens regular.

If $n = 1$, $E = xH$ where H is a proper amenable subgroup. Since $I_{\mathcal{A}(G)}(H)$ is isometrically isomorphic to $I_{\mathcal{A}(G)}(xH)$, Lemma 5.2.7 shows that E is finite and $\mathcal{A}(G)$ is Arens regular.

Assume that $P(n)$ is true for all $n \leq k$. Let $E = \bigcup\limits_{i=1}^{k+1} x_i H_i$ where each H_i is an amenable subgroup of G. By translating if necessary we can assume that $x_{k+1} = e$. If $H_{k+1} \subseteq \bigcup\limits_{i=1}^{k} x_i H_i$, then we have $E = \bigcup\limits_{i=1}^{k} x_i H_i$ and we are done. So we may assume that

$$F = H_{k+1} \setminus \left(\bigcup_{i=1}^{k} x_i H_i \right) \neq \emptyset.$$

Note that $H_{k+1} \setminus F \in \mathcal{R}(H_{k+1})$ and

$$I_{\mathcal{A}(H)}(H_{k+1} \setminus F) = I_{\mathcal{A}(G)}(E)_{|H_{k+1}}.$$

In particular, since the restriction map is a homomorphism, $I_{\mathcal{A}(H)}(H_{k+1} \setminus F)$ is Arens regular. But as $H_{k+1} \setminus F \in \mathcal{R}(H_{k+1})$ and H_{k+1} is amenable, we have that $\mathcal{A}(H) = A(H)$ and $I_{\mathcal{A}(H)}(H_{k+1} \setminus F)$ has a bounded approximate identity. It then follows from Theorem 5.2.5 that F is finite.

Next we observe that E is the disjoint union of $\bigcup\limits_{i=1}^{k} x_i H_i$ and the finite set F. But as F is finite we can proceed in a manner similar to that of the proof of Lemma 5.2.7 to conclude that $I_{\mathcal{A}(G)}(\bigcup\limits_{i=1}^{k} x_i H_i)$ is also Arens regular. From here the induction hypothesis tells us that $\bigcup\limits_{i=1}^{k} x_i H_i$ is finite. And as $F = H_{k+1} \setminus (\bigcup\limits_{i=1}^{k} x_i H_i)$ is also finite, H_{k+1} is finite. Hence E is finite as well.

If we assume that

$$E = \bigcup_{i=1}^{n} \left(x_i H_i \setminus \bigcup_{j=1}^{m_i} b_{i,j} K_{i,j} \right),$$

where H_i is an amenable subgroup of G, $x_i \in G$, $K_{i,j}$ is a subgroup of H_i. We have two cases. The first is that $\bigcup\limits_{i=1}^{n} x_i H_i \neq G$. If this is the case, then if $E_1 = \bigcup\limits_{i=1}^{n} x_i H_i$ then

$E \subseteq E_1$ and hence the non-zero closed ideal $I_{\mathcal{A}(G)}(E_1)$ is contained in the Arens regular ideal $I_{\mathcal{A}(G)}(E)$ and is therefore also Arens regular. But we have seen above that this means that E_1 is finite. It follows that E is also finite. As before, this would imply that $\mathcal{A}(G)$ would also be Arens regular.

Finally, if we assume that $\bigcup_{i=1}^{n} x_i H_i = G$. Then by [27, Corollary 3.3] one of the H_i's has finite index in G. Since each H_i is amenable, so is G. This means that we can express $G \backslash E$ as a disjoint union $\bigcup_{l=1}^{m} F_l$ where each F_l is a translate of an element of the coset ring of one of the open amenable subgroups $K_{i,j}$. Moreover, this means that $I_{\mathcal{A}(G)}(E) = I_{A(G)}(E)$ has a bounded approximate identity [27, Theorem 3.20]. It now follows from Theorem 5.2.5 that this ideal is finite-dimensional.

□

Chapter 6

A Computational Approach to Banach Algebras on Finite Groups

The ending isn't any more important than any of the moments leading to it

Dr. Rosalene
To the Moon

6.1 A Word on Numerical Computations

This chapter features investigations and analysis of Banach algebras on finite groups using computational methods, namely the computer algebra systems SageMath (shortened to Sage) and GAP [33]. The nature of these inquiries is simply of an attempt to get a sense of behavior of these objects on a larger scale than is possible by hand. The algorithms and implementations are by no means optimized or intended to be used in computer-aided proofs, and is not meant for a high-performance computing or algorithmic analysis context. Instead, our motivation is one of "experimental mathematics": the computational tools discussed in this chapter are primarily used to search for counter-examples and to help inspire conjectures that can then be proven explicitly.

The decision to write a function in GAP or Sage depends on what is required. Sage excels at general numerical calculations and has a variety of useful linear algebra methods implemented (in particular, being able to call a Sage method for finding the singular values of a matrix is quite useful for calculating $\|\cdot\|_{A(G)}$), while GAP is better-suited for computations involving group theory. Of special importance is GAP's SmallGroup library which allows for cycling through all groups of a given order less than 2000 (except for those of order 1024). Character tables for sufficiently small groups can be easily accessed through the GAP Character Table Library, which among other sources draws from the famed Atlas of Finite Groups [16]. It is possible to run GAP inside of a Sage environment, which allows us to enjoy the best of both worlds of GAP's group theory tools and Sage's superior handling of decimal calculations, however this process is somewhat cumbersome so we will only do so when necessary.

The standard interface from Sage into GAP is currently quite slow. As such, we have chosen to use the libGAP package, which allows for much more efficient computations. The `libgap.function_factory()` function will be our go-to method for creating GAP objects that are interactable within a Sage enviroment.

It should be noted that many of our numerical calculations involve floating point arithmetic, which when done with default settings often has small amounts of error based on the way that Sage stores numbers. The values calculated in this thesis have all been rounded to five digits to avoid such issues, but it is important to be aware that using a computer algebra system in this way can fail to be completely exact.

6.2 Rider's Theorem

We state the famed gap theorem of Rider:

Theorem 6.2.1. *[56, Lemma 5.2] Let G be a compact group, and ψ an idempotent in $ZL^1(G)$ such that $\|\psi\|_1 > 1$. Then $\|\psi\|_1 > \frac{301}{300}$.*

While Rider's result is powerful, it is known to not be sharp. However, an analogue by Saeki for abelian groups actually does have a sharp bound:

Theorem 6.2.2. *[65] Let G be a locally compact abelian group, and ψ an idempotent in $ZL^1(G)$ such that $\|\psi\|_1 > 1$. Then $\|\psi\|_1 \geq \frac{1+\sqrt{2}}{2}$. This bound is sharp.*

Recall that if G is finite then $ZL^1(G)$ is equal to the span of span $\mathrm{Irr}(G)$. By [25, Equation (5.20)] we can see that all idempotents in $ZL^1(G)$ are sums of $d_\chi \chi$ for $\chi \in \mathrm{Irr}(G)$, so for the purposes of investigating Rider's theorem it suffices to calculate the norms of each $d_\chi \chi$. Figure 6.1 has an implementation of this. While the algorithm is simple and straightforward, we note the choice to use Sage instead of GAP. GAP is not currently capable of calculating complex norms involving decimal approximations of non-rationals. The non-official FUtil package is supposedly equipped to do so, but we have observed that the `SqrtDecimalApproximation()` method often led to non-completing programs and would therefore not recommend its use for these purposes.

Figure 6.1 has been run for groups of order less than 256 and so far the minimum value achieved has been $\dfrac{1+\sqrt{2}}{2}$, which matches the bound from Saeki's bound on abelian groups. Based on these calculations we form the following conjecture:

Conjecture 6.2.3. Let G be a finite group, and ψ an idempotent in $ZL^1(G)$ such that $\|\psi\|_1 > 1$. Then $\|\psi\|_1 \geq \frac{1+\sqrt{2}}{2}$.

Remark 6.2.4. If the above conjecture does indeed hold, it may be possible to extend the result to the compact case.

6.3 Fourier Algebra Norm

Recall that the left-regular representation λ as viewed as a representation on $L^1(G)$ is defined as

$$\lambda : L^1(G) \to \mathcal{U}(L^2(G)),\ \lambda(f)(g) = f * g.$$

We implement an algorithm for calculating $\|\cdot\|_{A(G)}$ using the approach from [68]:

Theorem 6.3.1. *[68, Lemma 2] Let G be a finite group, $f \in A(G)$, and $s_1, ..., s_n$ the singular values of $\lambda(f)$. Then*

$$\|f\|_{A(G)} = \sum_{i=1}^{n} s_i.$$

Let $G = \{x_1, ..., x_n\}$ and $f \in A(G)$. Then $\lambda(f)$ can be written as the matrix

$$\lambda(f) = \frac{1}{n} \begin{bmatrix} f(x_1{x_1}^{-1}) & f(x_1{x_2}^{-1}) \cdots\cdots & f(x_1{x_n}^{-1}) \\ \vdots & \ddots & \vdots \\ f(x_n{x_1}^{-1}) & \cdots\cdots\cdots & f(x_n{x_n}^{-1}) \end{bmatrix}.$$

The matrix $\lambda(f)$ can be calculated by using `convMatrix` (Figure 6.2). Sage has a built-in method for calculating the singular value decomposition of a given matrix, so actually calculating the Fourier algebra norm is quite straightforward. This is done by `ANorm()` (Figure 6.3).

In the implementation of both `convMatrix()` and `ANorm` f is actually a list of length $|G|$ with the ith entry corresponding to the value of f on the ith element of the `Elements(G)` listing of G. Figure 6.4 and Figure 6.5 construct 1_Γ and $1_{C\times C}$ as functions on $G \times G$ that are compatible with `ANorm()`.

Recall from Equation 3.2.9 and Theorem 3.7.4 that $AM(A(G)) = \|1_{G_\Gamma}\|_{A(G\times G)}$ and $AMZA(G) = \|1_{\mathrm{Conj}(G)_D}\|_{A(G\times G)}$, so it is possible to calculate $AM(A(G))$ and $AMZA(G)$ using Figure 6.3. However, we note that Equation 3.1 and Equation 3.3 are much easier to use instead.

6.4 AIC Groups

Rider's gap theorem requires looking at $\chi \in \mathrm{Irr}(G)$ such that $\|d_\chi \chi\|_1 > 1$. In this section we consider the other case.

Definition 6.4.1. Let $\chi \in \mathrm{Irr}(G)$. We call χ an *absolutely idempotent character* if $\|d_\chi \chi\|_1 = 1$. If every character of $\mathrm{Irr}(G)$ is absolutely idempotent, then we call G an *AIC* group.

The following is well-known, and allows for a simple condition of checking whether or not a group is AIC:

Proposition 6.4.2. *Let G be a finite group. Then χ is an absolutely idempotent character of G if and only if $|\chi(x)| \in \{0, d_\chi\}$ for all $x \in G$.*

Proof. Because $|\chi(x)| \leq d_\chi$ for all $x \in G$ then it follows that

$$1 = \frac{1}{|G|}\sum_{x\in G} |\chi(x)|^2 \leq \frac{d_\chi}{|G|}\sum_{x\in G}|\chi(x)| = \|d_\chi \chi\|_1.$$

In particular, if χ is absolutely idempotent then the inequality above must be an equality, which certainly forces the condition that $|\chi(x)| \in \{0, d_\chi\}$. The converse is clear as well: if $|\chi(x)| \in \{0, d_\chi\}$ then $|\chi(x)|^2 = d_\chi |\chi(x)|$, which forces $\|d_\chi \chi\|_1 = 1$ by the calculation above. $\square$

An implementation of function in GAP that checks if a group is AIC or not using this condition can be found in Figure 6.6.

There is a relationship between AIC groups and amenability of the group algebra, such as in following result:

Theorem 6.4.3 ([3]). *Let $G = \oplus_{i \in I} G_i$, where each G_i is a finite group. Then every maximal ideal of $ZL^1(G)$ has a bounded approximate identity if and only if all but finitely many of the G_i are AIC.*

For finite groups the AIC condition implies nilpotency.

Proposition 6.4.4 ([10]). *Let G be a finite group. If G is AIC then G is nilpotent, and if G is nilpotent class 2 then G is AIC.*

6.5 $AMZA$ vs AMZ

The connection between $AMZA$ and AMZ motivates a question whose answer could suggest a solution to the sharp gap of $AMZA$:

Question 6.5.1. When are $AMZA$ and AMZ equal?

In Chapter 4 we addressed the question of when $AMZA$ and AMZ are equal in several specific cases. Here we look at a computational approach. These constants are straightforward to calculate using GAP as done in Figures 6.7 and 6.8. Using these programs we can calculate a table of groups of order less than 100 that satisfy $AMZA(G) \neq AMZ$

At least for small-order groups, the property $AMZA(G) \neq AMZ(G)$ seems relatively uncommon; Table 6.1 consists of 173 groups out of the 851 non-abelian groups of order less than 100.

One of the first things we can note just by looking at the first two entries of Table 6.1 is that neither of the two quantities will be strictly greater or lesser than the other: After all, $AMZA(S_4) \geq AMZ(S_4)$ and $AMZA(SL(2,3)) \leq AMZ(SL(2,3))$. Also, there seems to be a significant amount of restriction on what kind of orders of groups are possible. The only orders that show up in the above table are

$$24, 48, 60, 64, 72, 80, 96.$$

In particular, all of the examples have even order. While this may lead to asking whether or not $AMZA(G) \neq AMZ(G)$ for odd order G, it turns out this is not the case: we have inequality for the group `SmallGroup(567, 16)`, and indeed this is the smallest odd-order group for which $AMZA(G) \neq AMZ(G)$ holds. It appears that $AMZA(G) \neq AMZL(G)$ often holds when G is a non-monomial group (that is, G contains an irreducible character that is not induced from a linear character on a subgroup). There are 24 non-monomial groups of order less than 100 listed. The 21 that have the property that $AMZA(G) \neq AMZL(G)$ are listed in Table 6.2, and the three that satisfy $AMZA(G) = AMZL(G)$ are listed in Table 6.3. This suggests that being non-monomial and having $AMZA(G) = AMZL(G)$ is rare, more study could result in a condition that characterizes this condition.

It is also worthwhile to note that $AMZA(G) = AMZ(G)$ often seems to hold when G is AIC. $G =$ `SmallGroup(128,36)` is the first example of an AIC group for which equality does not hold. Still, even when $AMZ(G) \neq AMZA(G)$ it appears that the quantities are still close in value, which suggests that it may be possible to bound $|AMZA(G) - AMZ(G)|$. Further study of this approach using computational methods could be quite helpful in the pursuit of finding a sharp amenability constant bound for $AMZA(G)$, not only for AIC groups but all finite groups.

6.6 Code and Outputs

The group IdGroup Id column in the following tables refers to the ordering from GAP's `IdGroup()` function. The group description is from the `StructureDescription()` function, although it is important to note that `StructureDescription()` is not invariant under group isomorphism and should not be relied upon for comparisons between groups. As for the figures representing functions in GAP and Sage, the code as presented here is not necessarily capable of running as-is. In particular, in Sage whitespace (which is usually in the form of tabs but here is represented by spaces for purposes of code presentation) determines the logic of the program, copy and pasting the code may result in a program that is not able to run without making the whitespace match up with the intended structure.

```
def L1Norm():
 min = 2
 for k in [6..255]:
 K = libgap.function_factory('AllSmallGroups')(k)
 for grp in K:
  T = libgap.function_factory('CharacterTable')(grp)
  Irreps = libgap.function_factory('Irr')(T).sage()
  Cc = libgap.function_factory('SizesConjugacyClasses')(T).sage()
  g = libgap.function_factory('Order')(grp).sage()
  for i in [0..len(Irreps)-1]:
   phi = Irreps[i]
   d = phi[0]
   if d > 1:
    m = sum([abs((phi[j] * d * Cc[j])) for j in [0..len(phi)-1]])/g
    if ((m > 1.0001) and (m < min)):
     min = m
     print(libgap.function_factory('StructureDescription')(G))
     print(sum([abs((phi[j] * d * Cc[j])) for j in [0..len(phi)-1]])/g)
  return
```

Figure 6.1: Sage method for calculating minimum $L^1(G)$ norms of central idempotents

```
def convMatrix(grp, f):
 L = libgap.function_factory('Elements')(grp)
 #Provides an ordered list of elements of the group
 n = libgap.function_factory('Size')(grp).sage()
 M = Matrix(CDF, n)
 #Constructs a n x n matrix of zeros
 for i in [0..n-1]:
  for j in [0..n-1]:
   y = L[i]*libgap.function_factory('Inverse')(L[j])
   p = (libgap.function_factory('Position')(L,y)).sage() - 1
   #Need to account for Position() returning an index beginning at 1
   M[i,j]=f[p]*(1/n)
 return(M)
```

Figure 6.2: Sage code for calculating $\lambda(f)$

```
def ANorm(grp,f):
 A = convMatrix(grp,f)
 t = sum(A.singular_values())
 return(t)
```

Figure 6.3: Sage code for calculating $\|f\|_{A(G)}$

Group Id	Group Description	$AMZA(G)$	$AMZ(G)$
(24, 3)	SL(2,3)	4.875	5.
(24, 12)	S4	7.0972	7.0833
(48, 15)	(C3 x D8) : C2	5.0139	5.1875
(48, 16)	(C3 : Q8) : C2	5.0139	5.1875
(48, 17)	(C3 x Q8) : C2	5.0139	5.1875
(48, 18)	C3 : Q16	5.0139	5.1875
(48, 28)	C2 . S4 = SL(2,3) . C2	9.4583	10.438
(48, 29)	GL(2,3)	9.4583	10.438
(48, 30)	A4 : C4	7.0972	7.0833
(48, 32)	C2 x SL(2,3)	4.875	5.
(48, 33)	((C4 x C2) : C2) : C3	4.875	5.

(48, 48)	C2 x S4	7.0972	7.0833
(60, 5)	A5	21.969	22.653
(60, 7)	C15 : C4	6.44	6.0133
(64, 8)	(C2 x D8) : C4	4.2188	4.2812
(64, 9)	(C2 x Q8) : C4	4.2188	4.2812
(64, 10)	(C8 : C4) : C2	4.2188	4.2812
(64, 11)	(C2 x C2) . ((C4 x C2) : C2) = (C4 x C2) . (C4 x C2)	4.2188	4.2812
(64, 12)	(C4 : C8) : C2	4.2188	4.2812
(64, 13)	(C2 x C2) . ((C4 x C2) : C2) = (C4 x C2) . (C4 x C2)	4.2188	4.2812
(64, 14)	(C2 x C2) . ((C4 x C2) : C2) = (C4 x C2) . (C4 x C2)	4.2188	4.2812
(64, 32)	(C2 x C2 x C2 x C2) : C4	4.8516	4.6797
(64, 33)	(C4 x C2 x C2) : C4	4.8516	4.6797
(64, 34)	(C4 x C4) : C4	4.8516	4.6797
(64, 35)	(C4 x C4) : C4	4.8516	4.6797
(64, 36)	(C2 . ((C4 x C2) : C2) = (C2 x C2) . (C4 x C2)) : C2	4.8516	4.6797
(64, 37)	C2 . ((C2 x C2 x C2) : C4) = (C4 x C2) . (C4 x C2)	4.8516	4.6797
(64, 41)	D16 : C4	4.1172	3.9297
(64, 42)	(C4 . D8 = C4 . (C4 x C2)) : C2	4.1172	3.9297
(64, 43)	C2 . ((C8 x C2) : C2) = C8 . (C4 x C2)	4.1172	3.9297
(64, 46)	C16 : C4	4.1172	3.9297
(64, 128)	(C8 x C2) : (C2 x C2)	4.2188	4.2812
(64, 129)	(C2 x C2 x Q8) : C2	4.2188	4.2812
(64, 130)	(C2 x D16) : C2	4.2188	4.2812
(64, 131)	(C8 x C2) : (C2 x C2)	4.2188	4.2812
(64, 132)	(C2 x Q16) : C2	4.2188	4.2812
(64, 133)	(C2 x QD16) : C2	4.2188	4.2812
(64, 134)	(C4 x C4) : (C2 x C2)	5.125	4.9609
(64, 135)	(C8 : (C2 x C2)) : C2	5.125	4.9609
(64, 136)	(C4 x C4) : (C2 x C2)	5.125	4.9609
(64, 137)	((C2 x Q8) : C2) : C2	5.125	4.9609
(64, 138)	(C2 x C2 x C2 x C2) : (C2 x C2)	5.125	4.9609
(64, 139)	(C4 x C2 x C2) : (C2 x C2)	5.125	4.9609
(64, 140)	(C2 x D16) : C2	4.2188	4.2812
(64, 141)	(C2 x QD16) : C2	4.2188	4.2812
(64, 142)	(C4 : Q8) : C2	4.2188	4.2812

(64, 143)	C4 : Q16	4.2188	4.2812
(64, 144)	(C4 x D8) : C2	4.2188	4.2812
(64, 145)	(C4 x Q8) : C2	4.2188	4.2812
(64, 152)	(C2 x QD16) : C2	4.1172	3.9297
(64, 153)	(C2 x D16) : C2	4.1172	3.9297
(64, 154)	(C2 x Q16) : C2	4.1172	3.9297
(64, 155)	(C4 : Q8) : C2	4.2188	4.2812
(64, 156)	Q8 : Q8	4.2188	4.2812
(64, 157)	(C4 : Q8) : C2	4.2188	4.2812
(64, 158)	Q8 : Q8	4.2188	4.2812
(64, 159)	((C2 x C2) . (C2 x C2 x C2)) : C2	4.2188	4.2812
(64, 160)	(C2 x C2) . (C2 x D8) = (C4 x C2) . (C2 x C2 x C2)	4.2188	4.2812
(64, 161)	(C2 x (C4 : C4)) : C2	4.2188	4.2812
(64, 162)	(C2 x (C4 : C4)) : C2	4.2188	4.2812
(64, 163)	((C8 x C2) : C2) : C2	4.2188	4.2812
(64, 164)	(Q8 : C4) : C2	4.2188	4.2812
(64, 165)	(Q8 : C4) : C2	4.2188	4.2812
(64, 166)	(Q8 : C4) : C2	4.2188	4.2812
(64, 190)	C16 : (C2 x C2)	4.1172	3.9297
(64, 191)	(C2 x Q16) : C2	4.1172	3.9297
(64, 257)	(C8 x C2) : (C2 x C2)	3.3906	3.1094
(64, 258)	(C8 x C2) : (C2 x C2)	3.3906	3.1094
(64, 259)	(C2 x Q16) : C2	3.3906	3.1094
(72, 3)	Q8 : C9	4.875	5.
(72, 15)	((C2 x C2) : C9) : C2	10.231	11.009
(72, 22)	(C6 x S3) : C2	6.0926	6.4907
(72, 23)	(C6 x S3) : C2	6.0926	6.4907
(72, 24)	(C3 x C3) : Q8	6.0926	6.4907
(72, 25)	C3 x SL(2,3)	4.875	5.
(72, 40)	(S3 x S3) : C2	12.019	10.417
(72, 42)	C3 x S4	7.0972	7.0833
(72, 43)	(C3 x A4) : C2	10.231	11.009
(80, 15)	(C5 x D8) : C2	5.745	6.0475
(80, 16)	(C5 : Q8) : C2	5.745	6.0475
(80, 17)	(C5 x Q8) : C2	5.745	6.0475

(80, 18)	C5 : Q16	5.745	6.0475
(80, 29)	(C5 : C8) : C2	5.47	5.23
(80, 31)	C20 : C4	5.47	5.23
(80, 33)	(C5 : C8) : C2	5.47	5.23
(80, 34)	(C10 x C2) : C4	5.47	5.23
(96, 3)	((C4 x C2) : C4) : C3	8.0625	9.
(96, 13)	(C2 x C2 x S3) : C4	4.7604	4.4271
(96, 14)	(C3 : C8) : C4	5.0139	5.1875
(96, 15)	(C3 : C8) : C4	5.0139	5.1875
(96, 16)	(C3 x (C4 : C4)) : C2	5.0139	5.1875
(96, 17)	(C3 : Q8) : C4	5.0139	5.1875
(96, 29)	C3 : (C4 . D8 = C4 . (C4 x C2))	5.0139	5.1875
(96, 30)	(C3 x (C8 : C2)) : C2	4.7604	4.4271
(96, 31)	C3 : (C2 . ((C4 x C2) : C2) = (C2 x C2) . (C4 x C2))	4.7604	4.4271
(96, 32)	(C3 x (C8 : C2)) : C2	5.0139	5.1875
(96, 33)	(C3 x D16) : C2	6.7118	7.2552
(96, 34)	(C3 : Q16) : C2	6.7118	7.2552
(96, 35)	(C3 x Q16) : C2	6.7118	7.2552
(96, 36)	C3 : Q32	6.7118	7.2552
(96, 39)	(C2 x (C3 : C8)) : C2	5.0139	5.1875
(96, 40)	((C3 : C8) : C2) : C2	4.7604	4.4271
(96, 41)	(C6 x C2 x C2) : C4	4.7604	4.4271
(96, 42)	(C3 x Q8) : C4	5.0139	5.1875
(96, 43)	C3 : (C2 . ((C4 x C2) : C2) = (C2 x C2) . (C4 x C2))	4.7604	4.4271
(96, 44)	(C3 x D8) : C4	5.0139	5.1875
(96, 64)	((C4 x C4) : C3) : C2	12.118	13.115
(96, 65)	A4 : C8	7.0972	7.0833
(96, 66)	SL(2,3) : C4	9.4583	10.438
(96, 67)	SL(2,3) : C4	9.4583	10.438
(96, 69)	C4 x SL(2,3)	4.875	5.
(96, 70)	(C2 x C2 x C2 x C2) : C6	8.4531	8.0312
(96, 71)	(C4 x C4) : C6	8.4531	8.0312
(96, 72)	(C4 x C4) : C6	8.4531	8.0312
(96, 74)	((C8 x C2) : C2) : C3	4.875	5.
(96, 85)	(C2 x (C3 : Q8)) : C2	5.0139	5.1875

(96, 86)	(C4 x (C3 : C4)) : C2	5.0139	5.1875
(96, 89)	(C12 x C2) : (C2 x C2)	5.0139	5.1875
(96, 90)	(C2 x C4 x S3) : C2	5.0139	5.1875
(96, 91)	(C2 x C4 x S3) : C2	5.0139	5.1875
(96, 92)	(C4 x (C3 : C4)) : C2	5.0139	5.1875
(96, 93)	(C2 x C2 x (C3 : C4)) : C2	5.0139	5.1875
(96, 95)	C12 : Q8	5.0139	5.1875
(96, 96)	C3 : ((C2 x C2) . (C2 x C2 x C2))	5.0139	5.1875
(96, 97)	C3 : ((C2 x C2) . (C2 x C2 x C2))	5.0139	5.1875
(96, 101)	(C2 x C4 x S3) : C2	5.0139	5.1875
(96, 102)	(C2 x C4 x S3) : C2	5.0139	5.1875
(96, 103)	(C3 x (C4 : C4)) : C2	5.0139	5.1875
(96, 104)	(C2 x (C3 : Q8)) : C2	5.0139	5.1875
(96, 105)	(C4 x (C3 : C4)) : C2	5.0139	5.1875
(96, 115)	C24 : (C2 x C2)	4.7604	4.4271
(96, 116)	(C2 x (C3 : Q8)) : C2	4.7604	4.4271
(96, 118)	C24 : (C2 x C2)	6.1562	5.8854
(96, 121)	C24 : (C2 x C2)	6.1562	5.8854
(96, 122)	(Q8 x S3) : C2	6.1562	5.8854
(96, 125)	(Q8 x S3) : C2	6.1562	5.8854
(96, 138)	C2 x ((C3 x D8) : C2)	5.0139	5.1875
(96, 139)	(C6 x D8) : C2	4.7604	4.4271
(96, 140)	C2 x ((C3 : Q8) : C2)	5.0139	5.1875
(96, 142)	(C2 x C2 x (C3 : C4)) : C2	5.0139	5.1875
(96, 143)	(C4 x (C3 : C4)) : C2	5.0139	5.1875
(96, 144)	(C6 x C2 x C2) : (C2 x C2)	5.0139	5.1875
(96, 145)	(C2 x C4 x S3) : C2	5.0139	5.1875
(96, 146)	(C2 x C2 x (C3 : C4)) : C2	5.0139	5.1875
(96, 147)	(C6 x D8) : C2	5.0139	5.1875
(96, 148)	C2 x ((C3 x Q8) : C2)	5.0139	5.1875
(96, 149)	(C6 x Q8) : C2	4.7604	4.4271
(96, 150)	C2 x (C3 : Q16)	5.0139	5.1875
(96, 151)	C3 : (C4 : Q8)	5.0139	5.1875
(96, 153)	(C6 x Q8) : C2	5.0139	5.1875
(96, 154)	(C6 x Q8) : C2	5.0139	5.1875

(96, 156)	(C2 x D24) : C2	4.7604	4.4271
(96, 157)	(C2 x (C3 : C8)) : C2	5.0139	5.1875
(96, 158)	(C2 x (C3 : Q8)) : C2	4.7604	4.4271
(96, 185)	(C2 x C2) : (C3 : Q8)	9.0208	9.4583
(96, 186)	C4 x S4	7.0972	7.0833
(96, 187)	(C2 x S4) : C2	9.0208	9.4583
(96, 188)	C2 x (C2 . S4 = SL(2,3) . C2)	9.4583	10.438
(96, 189)	C2 x GL(2,3)	9.4583	10.438
(96, 192)	GL(2,3) : C2	9.4583	10.438
(96, 194)	C2 x (A4 : C4)	7.0972	7.0833
(96, 195)	(C2 x C2 x A4) : C2	9.0208	9.4583
(96, 198)	C2 x C2 x SL(2,3)	4.875	5.
(96, 200)	C2 x (((C4 x C2) : C2) : C3)	4.875	5.
(96, 201)	((C2 x C2 x C2) : (C2 x C2)) : C3	6.1562	5.6875
(96, 202)	(C2 x Q8) : C6	6.1562	5.6875
(96, 203)	(C2 x C2 x Q8) : C3	8.0625	9.
(96, 211)	(C6 x C2 x C2) : (C2 x C2)	4.2708	3.7708
(96, 214)	(C6 x Q8) : C2	4.2708	3.7708
(96, 216)	(C12 x C2) : (C2 x C2)	4.2708	3.7708
(96, 217)	(Q8 x S3) : C2	4.2708	3.7708
(96, 226)	C2 x C2 x S4	7.0972	7.0833
(96, 227)	((C2 x C2 x C2 x C2) : C3) : C2	12.118	13.115

Table 6.1: $|G| < 100$ and $AMZA(G) \neq AMZ(G)$

Group Id	Group Description	$AMZA(G)$	$AMZ(G)$
(24, 3)	SL(2,3)	4.875	5.
(48, 28)	C2 . S4 = SL(2,3) . C2	9.4583	10.438
(48, 29)	GL(2,3)	9.4583	10.438
(48, 32)	C2 x SL(2,3)	4.875	5.
(48, 33)	((C4 x C2) : C2) : C3	4.875	5.
(60, 5)	A5	21.969	22.653
(72, 3)	Q8 : C9	4.875	5.
(72, 25)	C3 x SL(2,3)	4.875	5.
(96, 3)	((C4 x C2) : C4) : C3	8.0625	9.

(96, 66)	SL(2,3) : C4	9.4583	10.438
(96, 67)	SL(2,3) : C4	9.4583	10.438
(96, 69)	C4 x SL(2,3)	4.875	5.
(96, 74)	((C8 x C2) : C2) : C3	4.875	5.
(96, 188)	C2 x (SL(2,3) . C2)	9.4583	10.438
(96, 189)	C2 x GL(2,3)	9.4583	10.438
(96, 192)	GL(2,3) : C2	9.4583	10.438
(96, 198)	C2 x C2 x SL(2,3)	4.875	5.
(96, 200)	C2 x (((C4 x C2) : C2) : C3)	4.875	5.
(96, 201)	((C2 x C2 x C2) : (C2 x C2)) : C3	6.1562	5.6875
(96, 202)	(C2 x Q8) : C6	6.1562	5.6875
(96, 203)	(C2 x C2 x Q8) : C3	8.0625	9.

Table 6.2: G non-monomial and $AMZA(G) \neq AMZ(G)$

Group Id	Group Description	$AMZA(G)$	$AMZ(G)$
(96, 190)	(C2 x SL(2,3)) : C2	11.927	11.927
(96, 191)	(SL(2,3) . C2) : C2	11.927	11.927
(96, 193)	SL(2,3) : (C2 x C2)	11.927	11.927

Table 6.3: G non-monomial and $AMZA(G) = AMZ(G)$

```
DiagConjIndicator := function(grp)
 local e1, e2, L, i, x, lst, DP;
 DP:= DirectProduct(grp, grp)
 lst:= Elements(DP);
 e1:=Projection(DP,1);
 e2:=Projection(DP,2);
 L := [];
 for i in [1 .. Size(DP)] do
  x:= lst[i];
  if IsConjugate(grp,Image(e1,x), (Image(e2,x))) then
   Add(L, 1);
  else
   Add(L,0);
  fi;
 od;
 return L;
end;
```

Figure 6.4: GAP method for calculating 1_{G_Γ}

```
antiDiagIndicator := function(grp)
 local e1, e2, L, i, x, lst, DP;
 DP:= DirectProduct(grp, grp)
 lst := Elements(DP);
 e1:=Projection(DP,1);
 e2:=Projection(DP,2);
 L := [];
 for i in [1 .. Size(DP)] do
  x:= lst[i];
  if Image(e1,x) = Inverse(Image(e2,x)) then
   Add(L, 1);
  else
   Add(L,0);
  fi;
 od;
return L;
end;
```

Figure 6.5: GAP method for calculating $1_{\mathrm{Conj}(G)_D}$

```
isAIC:= function(grp)
 local tbl, phi, x;
 tbl := CharacterTable(grp);
 for phi in Irr(tbl) do
  for x in [1..NrConjugacyClasses(tbl)] do
   if (phi[x]*ComplexConjugate(phi[x]) <> phi[1]^2) and (phi[x] <> 0) then
    return false;
   fi;
  od;
 od;
 return true;
end;
```

Figure 6.6: GAP code for calculating if a group is AIC

```
AMZ:= function(grp)
 local clssizes, n, kg, phi, i, j, tbl;
 tbl := CharacterTable(grp);
 clssizes:= SizesConjugacyClasses( tbl );
 n:= Size( tbl );      # group order
 kg:= NrConjugacyClasses( tbl );
 return Sum([1..kg], i-> Sum([1..kg], j-> clssizes[i] * clssizes[j] *
 AbsoluteValue(Sum(Irr(tbl), phi -> phi[1]^2 * phi[i] *
 ComplexConjugate(phi[j])))))/n^2;
end;
```

Figure 6.7: GAP code for calculating AMZ

```
AMZA:= function(grp)
 local clssizes, n, kg, phi, psi, i, tbl, irreps,;
 tbl := CharacterTable(grp);
 irreps := Irr(tbl);
 clssizes:= SizesConjugacyClasses( tbl );
 n:= Size( tbl );      # group order
 kg:= NrConjugacyClasses( tbl );
 return Sum(irreps, phi -> Sum(irreps, psi -> phi[1] * psi[1]
* AbsoluteValue(Sum([1..kg], i -> clssizes[i]^2 * phi[i] *
ComplexConjugate(psi[i])))))/n^2;
end;
```

Figure 6.8: GAP code for calculating AMZA

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