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Journal of Number Theory 116 (2006) 1–20
JOURNAL OF Number Theory
www.elsevier.com/locate/jnt

A weighted Turán sieve method☆

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Received 19 August 2004; revised 6 January 2005
Available online 6 April 2005
Communicated by D. Goss

Abstract

We develop a weighted Turán sieve method and applied it to study the number of distinct prime divisors of $f(p)$ where p is a prime and $f(x)$ a polynomial with integer coefficients.

MSC: 11N37; 11N80

Keywords: Turan sieve; Prime divisors of polynomials; Normal order method

1. Introduction

For $n \in \mathbb{N}$, let $\omega(n)$ denote the number of distinct prime divisors of n. Hardy and Ramanujan [4] proved in 1917 that the normal order of $\omega(n)$ is $\log\log n$. In other words, given any $\varepsilon > 0$, as $x \to \infty$, we have

$$\#\left\{n \leqslant x \;\middle|\; n \text{ satisfies } |\omega(n) - \log\log n| > \varepsilon \log\log n\right\} = o(x).$$

☆ Research partially supported by an NSERC discovery grant.
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0022-314X/$ - see front matter
doi:10.1016/j.jnt.2005.02.003

The method they used was rather complicated and involving difficult sieve methods. In 1934, Turán [15] gave a greatly simplified proof of the Hardy–Ramanujan result by considering the second moment of $\omega(n)$. He proved that

$$\sum_{n \leqslant x} \big(\omega(n) - \log\log x\big)^2 \ll x \log\log x;$$

from which the normal order of $\omega(n)$ is easily deduced. Turán's original derivation of the Hardy–Ramanujan Theorem was essentially probabilistic and concealed in it an elementary sieve method. In [7], the authors introduced the Turán sieve method and applied it to probabilistic Galois theory problems. In [8], the authors extended this sieve to a combinatorial setting. More precisely, if X is a finite bipartite graph with partite sets A and B, then

$$\sum_{a \in A} \left(\deg a - \frac{1}{|A|} \sum_{b \in B} \deg b \right)^2 = \sum_{b_1, b_2 \in B} \deg(b_1, b_2) - \frac{1}{|A|} \left(\sum_{b \in B} \deg b \right)^2,$$

where $\deg x$ is the degree of the vertex x and $\deg(b_1, b_2)$ is the number of vertices of A incident with both b_1 and b_2. This equality was used as a starting point to investigate a variety of combinatorial questions in [8]. It is clear that a 'weighted' version of the above can be derived in a straightforward way. Indeed, if $\lambda : A \to \mathbb{C}$ is any function, one may set

$$\delta(A) = \sum_{a \in A} \lambda(a)$$

and show that

$$\sum_{a \in A} \lambda(a) \left(\deg a - \frac{1}{\delta(A)} \sum_{b \in B} \delta(b) \right)^2 = \sum_{b_1, b_2 \in B} \delta(b_1, b_2) - \frac{1}{\delta(A)} \left(\sum_{b \in B} \delta(b) \right)^2,$$

where

$$\delta(b) = \sum_{(a,b) \in X} \lambda(a)$$

and

$$\delta(b_1, b_2) = \sum_{\substack{(a,b_1) \in X \\ (a,b_2) \in X}} \lambda(a).$$

The notation $(a, b) \in X$ means that a and b are adjacent. We may also consider the special situation $\lambda(a) \geqslant 0$ and $\lambda(a) \geqslant 1$ for a in a subset A' of A. The sum

$$\sum_{a \in A'} \left(\deg a - \frac{1}{|A|} \sum_{b \in B} \deg b \right)^2$$

is dominated by

$$\sum_{a \in A} \lambda(a) \left(\deg a - \frac{1}{|A|} \sum_{b \in B} \deg b \right)^2 .$$

In this way, one can develop an 'enveloping sieve'. Even in the context of the 'classical' Turán sieve as discussed in [7], this is a new perspective. Thus, rather than developing this idea in full generality as indicated above, we will develop it in the classical setting using a specific example which we now describe.

Let p be a prime number. In 1935, Erdös [1] proved that the normal order of $\omega(p-1)$ is $\log\log p$. In 1951, Haselgrove [5] established that the normal order of $\omega(p+a)$ is also $\log\log p$ for any $a \in \mathbb{Z}$, $a \neq 0$. Let $f(x) \in \mathbb{Z}[x]$ be an irreducible polynomial and $f(x) \neq cx$ for some constant c. In 1953, Prachar [12] proved that

$$\sum_{p \leqslant x} \omega(f(p)) \gg \pi(x) \log\log x.$$

His result was improved in 1956 by Halberstam [3] where he showed that the normal order of $\omega(f(p))$ is $\log\log p$. More precisely, for any $\varepsilon > 0$, as $x \to \infty$, Halberstam proved that

$$\#\Big\{ p \leqslant x \;\Big|\; p \text{ satisfies } |\omega(f(p)) - \log\log p| > \varepsilon \log\log p \Big\} = o\big(\pi(x)\big),$$

where $\pi(x)$ is the number of primes $\leqslant x$. This provided a generalization of Haselgrove's theorem.

The proofs of the above 'prime analogues' of the Hardy–Ramanujan theorem were rather complicated as they followed the original approach of Hardy and Ramanujan stated at the outset of this paper. Moreover, they involved deep results on primes in arithmetic progressions. In this paper, by combining the second moment method of Turán and a technique of Selberg in [13], we develop a weighted Turán sieve method. The second moment approach allows us to eliminate complicated sieve methods while Selberg's technique helps us to transform the question from primes to integers. More precisely, we show that to consider the normal order of $\omega(f(p))$, it suffices to consider the second moment of $\omega(f(n))$ for a natural number n.

We prove the following theorem.

Theorem 1. *Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients and $f(x) \neq cx^e$ for some constants $c \in \mathbb{Z}$ and $e \in \mathbb{N}$. Write*

$$f(x) = f_1(x)^{e_1} f_2(x)^{e_2} \cdots f_r(x)^{e_r},$$

where $f_i(x) \in \mathbb{Z}[x]$ are distinct irreducible polynomials. We have

$$\sum_{p \leqslant x} \big(\omega(f(p)) - r \log\log x\big)^2 \ll \pi(x) \log\log x.$$

From Theorem 1, we can derive the following corollary, which is slightly general and stronger than Halberstam's result [3, Theorem 2].

Corollary 1. *Define $f(x)$ as in Theorem 1. For any $\varepsilon > 0$, we have*

$$\#\Big\{p \leqslant x \;\Big|\; p \text{ satisfies } \big|\omega(f(p)) - r \log\log p\big| > \big(\log\log p\big)^{1/2+\varepsilon}\Big\} \ll \frac{\pi(x)}{(\log\log x)^{2\varepsilon}}.$$

From this, we conclude that the normal order of $\omega(f(p))$ is $r \log\log p$.

2. The lemmas

Let $f(x) \in \mathbb{Z}[x]$ and $f(x) \neq cx^e$ for some $c \in \mathbb{Z}$, $e \in \mathbb{N}$. Write

$$f(x) = f_1(x)^{e_1} f_2(x)^{e_2} \cdots f_r(x)^{e_r},$$

where $f_i(x)$'s are distinct irreducible polynomials in $\mathbb{Z}[x]$. Since $\omega(f(p))$ is the number of distinct prime divisors of $f(p)$, without loss of generality, we can assume $e_1 = e_2 = \cdots = e_r = 1$.

Let $d(f)$ and $c(f)$ denote the discriminant and the leading coefficient of $f(x)$, respectively. For a prime p, define

$$\tilde{\omega}(f(p)) = \#\Big\{q \text{ is a prime} \;\Big|\; q \mid f(p) \text{ and } q \nmid c(f)d(f)\Big\}.$$

We have

Lemma 1. *Let $f(x) = f_1(x) f_2(x) \cdots f_r(x) \in \mathbb{Z}[x]$, where $f_i(x)$'s are distinct irreducible polynomials. We have*

$$\tilde{\omega}(f(p)) = \tilde{\omega}(f_1(p)) + \tilde{\omega}(f_2(p)) + \cdots + \tilde{\omega}(f_r(p)).$$

Proof. It suffices to prove that all prime divisors of $f_1(p), f_2(p), \ldots, f_r(p)$ are distinct except the ones dividing $c(f)d(f)$. Let q be a prime which satisfies

$$q \mid f_i(p) \quad \text{and} \quad q \mid f_j(p), \quad \text{for } i \neq j.$$

Since $q \mid f_i(p)$ and $q \mid f_j(p)$, $p \pmod q$ is a double root of the polynomial $\bar{f}(x)$, the reduction of $f(x) \pmod q$. In other words, $p \pmod q$ is a common root of $\bar{f}$ and $\bar{f}'$, the derivative of $\bar{f}$. It follows that the resultant $R(\bar{f}, \bar{f}')$ vanishes modulo q (see [6, V, Section 10] for more details). Since

$$R(f, f') = c(f)^{2d-1} d(f),$$

where d is the degree of $f(x)$, thus

$$R(\bar{f}, \bar{f}') = 0 \pmod q \quad \text{implies that} \quad q \mid c(f)d(f).$$

Hence, the prime divisors of $f_i(p)$ and $f_j(p)$ are distinct unless they divide $c(f)d(f)$. From the definition of $\tilde{\omega}$, we have

$$\tilde{\omega}(f(p)) = \tilde{\omega}(f_1(p)) + \tilde{\omega}(f_2(p)) + \cdots + \tilde{\omega}(f_r(p)).$$

This completes the proof of Lemma 1. $\square$

The following Lemma tells us that Theorem 1 can be reduced to the case of irreducible polynomials.

Lemma 2. *Let $f(x) \in \mathbb{Z}[x]$ and $f(x) \neq cx^e$. Suppose we have*

$$\sum_{p \leqslant x} \big(\omega(g(p)) - \log\log x\big)^2 \ll \pi(x) \log\log x,$$

whenever $g(x) \in \mathbb{Z}[x]$ is an irreducible polynomial and $g(x)$ is not a multiple of x. Then,

$$\sum_{p \leqslant x} \big(\omega(f(p)) - r\log\log x\big)^2 \ll \pi(x) \log\log x,$$

where r is the number of distinct irreducible polynomials dividing $f(x)$.

Proof. Define $\tilde{\omega}$ as before. Since there are only finitely many primes dividing $c(f)d(f)$, we have

$$\omega(f(p)) = \tilde{\omega}(f(p)) + O(1).$$

Thus,

$$\begin{aligned}\sum_{p\leqslant x}\big(\omega(f(p))-r\log\log x\big)^2 &= \sum_{p\leqslant x}\big(\tilde{\omega}(f(p))+O(1)-r\log\log x\big)^2\\ &\ll \sum_{p\leqslant x}\big(\tilde{\omega}(f(p))-r\log\log x\big)^2+O\big(\pi(x)\big).\end{aligned}$$

Similarly, for an irreducible polynomial $g(x)$, we have

$$\sum_{p\leqslant x}\big(\tilde{\omega}(g(p))-\log\log x\big)^2=\sum_{p\leqslant x}\big(\omega(g(p))-\log\log x\big)^2+O\big(\pi(x)\big).$$

Thus, from the assumption of the lemma, we have

$$\sum_{p\leqslant x}\big(\tilde{\omega}(g(p))-\log\log x\big)^2\ll\pi(x)\log\log x. \tag{1}$$

Also, to prove this lemma, it suffices to prove

$$\sum_{p\leqslant x}\big(\tilde{\omega}(f(p))-r\log\log x\big)^2\ll\pi(x)\log\log x.$$

We have seen in Lemma 1 that

$$\tilde{\omega}(f(p))=\tilde{\omega}(f_1(p))+\tilde{\omega}(f_2(p))+\cdots+\tilde{\omega}(f_r(p)).$$

Since each $f_i(x)$ is irreducible, from Eq. (1), we have

$$\begin{aligned}&\sum_{p\leqslant x}\big(\tilde{\omega}(f(p))-r\log\log x\big)^2\\ &\quad=\sum_{p\leqslant x}\Big(\big(\tilde{\omega}(f_1(p))-\log\log x\big)+\cdots+\big(\tilde{\omega}(f_r(p))-\log\log x\big)\Big)^2\\ &\quad\ll\sum_{p\leqslant x}\big(\tilde{\omega}(f_1(p))-\log\log x\big)^2+\cdots+\sum_{p\leqslant x}\big(\tilde{\omega}(f_r(p))-\log\log x\big)^2\\ &\quad\ll\pi(x)\log\log x.\end{aligned}$$

This completes the proof of Lemma 2. $\square$

From Lemma 2, we see that to prove Theorem 1, it suffices to prove

$$\sum_{p\leqslant x}\left(\omega(f(p))-\log\log x\right)^2\ll \pi(x)\log\log x,$$

where $f(x)\in\mathbb{Z}[x]$ is an irreducible polynomial and $f(x)\neq cx$. In the following discussion, we will assume $f(x)$ is irreducible.

Consider the constant term $f(0)$ of $f(x)$. Since $f(x)\neq cx$ and $f(x)$ is irreducible, we have $f(0)\neq 0$. Define

$$\omega_0(f(p))=\#\Big\{q \text{ is a prime}\,\Big|\, q\mid f(p) \text{ and } q\mid f(0)\Big\}.$$

Also, for $A\in\mathbb{N}$ and $\alpha\in\mathbb{R}$ with $0<\alpha<1$, define

$$\omega_1(f(p))=\#\Big\{q \text{ is a prime}\,\Big|\, q\mid f(p),\ q\leqslant(\log x)^A, \text{ and } q\nmid f(0)\Big\},$$

$$\omega_2(f(p))=\#\Big\{q \text{ is a prime}\,\Big|\, q\mid f(p),\ (\log x)^A<q\leqslant x^{\alpha}, \text{ and } q\nmid f(0)\Big\}$$

and

$$\omega_3(f(p))=\#\Big\{q \text{ is a prime}\,\Big|\, q\mid f(p),\ q>x^{\alpha}, \text{ and } q\nmid f(0)\Big\}.$$

Thus,

$$\omega(f(p))=\omega_0(f(p))+\omega_1(f(p))+\omega_2(f(p))+\omega_3(f(p)).$$

Choices of A and α will be made later.

Lemma 3. *Let $f(x)\in\mathbb{Z}[x]$ be irreducible and $f(x)\neq cx$. Suppose we have*

$$\sum_{p\leqslant x}\left(\omega_2(f(p))-\log\log x\right)^2\ll \pi(x)\log\log x.$$

Then,

$$\sum_{p\leqslant x}\left(\omega(f(p))-\log\log x\right)^2\ll \pi(x)\log\log x.$$

Proof. Notice that $f(0)$ have at most $\log_2 f(0)$ many divisors. Thus, we have

$$\omega_0(f(p))=O(1).$$

Also, for $p \leqslant x$, $f(p) \ll x^d$ where d is the degree of $f(x)$. For any integer $n \leqslant x^d$, there are at most d/α many primes q satisfying $q > x^\alpha$ and $q|n$. Thus, we have

$$\omega_3(f(p)) = O(1).$$

It follows that

$$\begin{aligned}
&\sum_{p \leqslant x} \big(\omega(f(p)) - \log\log x\big)^2 \\
&\quad = \sum_{p \leqslant x} \big(\omega_1(f(p)) + \omega_2(f(p)) + O(1) - \log\log x\big)^2 \\
&\quad \ll \sum_{p \leqslant x} \omega_1^2(f(p)) + \sum_{p \leqslant x} \big(\omega_2(f(p)) - \log\log x\big)^2 + O\big(\pi(x)\big).
\end{aligned}$$

Write

$$\begin{aligned}
\sum_{p \leqslant x} \omega_1^2(f(p)) &= \sum_{p \leqslant x} \sum_{\substack{q,l \leqslant (\log x)^A \\ q|f(p), l|f(p) \\ q \nmid f(0), l \nmid f(0)}} 1 \\
&\leqslant \sum_{\substack{q,l \leqslant (\log x)^A \\ q \neq l}} \sum_{\substack{p \leqslant x \\ ql|f(p)}} 1 + \sum_{q \leqslant (\log x)^A} \sum_{\substack{p \leqslant x \\ q|f(p)}} 1.
\end{aligned}$$

Let $\rho_f(q)$ be the number of solutions of

$$f(a) \equiv 0 \pmod{q}, \quad \text{where } 0 \leqslant a < q.$$

By Chinese remainder theorem, we have

$$\#\Big\{p \leqslant x \;\Big|\; p \text{ satisfies } ql|f(p)\Big\} = \sum_{i=1}^{\rho_f(q)\rho_f(l)} \sum_{\substack{p \leqslant x \\ p \equiv a_i \,(\mathrm{mod}\, ql)}} 1,$$

where a_i's are solutions of $f(a) \equiv 0 \pmod{ql}$ for $0 \leqslant a < ql$. Since the degree of $f(x)$ is d, thus $\rho_f(q) \leqslant d$. Applying results of Brun–Titchmarsh and Montgomery–Vaughan [10], [14, pp. 73–76], for primes q and l which are $\leqslant (\log x)^A$, we have

$$\#\big\{p \leqslant x \;\big|\; p \text{ satisfies } ql|f(p)\big\} \ll \frac{2\pi(x)\rho_f(q)\rho_f(l)}{ql} \ll \frac{\pi(x)}{ql}.$$

Similarly,

$$\#\{p \leqslant x \mid p \text{ satisfies } q \mid f(p)\} \ll \frac{2\pi(x)\rho_f(q)}{q} \ll \frac{\pi(x)}{q}.$$

Using the classical Mertens theorem [9], we get

$$\begin{aligned}\sum_{p \leqslant x} \omega_1^2(f(p)) &\ll \sum_{q,l \leqslant (\log x)^A} \frac{\pi(x)}{ql} + \sum_{q \leqslant (\log x)^A} \frac{\pi(x)}{q} \\ &\ll \pi(x)(\log\log\log x)^2.\end{aligned}$$

Combining all the above estimates, by the assumption of the lemma, we have

$$\begin{aligned}&\sum_{p \leqslant x} (\omega(f(p)) - \log\log x)^2 \\ &\quad \ll \sum_{p \leqslant x} (\omega_2(f(p)) - \log\log x)^2 + O\big(\pi(x)(\log\log\log x)^2\big) \\ &\quad \ll \pi(x)\log\log x.\end{aligned}$$

This completes the proof of Lemma 3. $\square$

We recall that $\rho_f(q)$ is the number of solutions of

$$f(a) \equiv 0 \pmod{q} \quad \text{where } 0 \leqslant a < q.$$

It is well-known that (see, for example [2, Lemma 7]).

Lemma 4. *For an irreducible polynomial $f(x) \in \mathbb{Z}[x]$, we have*

$$\sum_{q \leqslant x} \frac{\rho_f(q)}{q} = \log\log x + O(1).$$

Lemma 5. *Define*

$$E(x) = \sum_{\substack{(\log x)^A < q \leqslant x^\alpha \\ q \nmid f(0)}} \frac{\rho_f(q)}{q}.$$

Suppose we have

$$\sum_{p \leqslant x} \bigl(\omega_2(f(p)) - E(x)\bigr)^2 \ll \pi(x) \log\log x.$$

Then,

$$\sum_{p \leqslant x} \bigl(\omega_2(f(p)) - \log\log x\bigr)^2 \ll \pi(x) \log\log x.$$

Proof. From Lemma 4, we have

$$\log\log x = E(x) + \sum_{\substack{q \leqslant (\log x)^A \\ q \nmid f(0)}} \frac{\rho_f(q)}{q} + \sum_{\substack{x^\alpha < q \leqslant x \\ q \nmid f(0)}} \frac{\rho_f(q)}{q} + \sum_{q \mid f(0)} \frac{\rho_f(q)}{q} + O(1).$$

Thus,

$$\begin{aligned}
&\sum_{p \leqslant x} \bigl(\omega_2(f(p)) - \log\log x\bigr)^2 \\
&\quad \ll \sum_{p \leqslant x} \bigl(\omega_2(f(p)) - E(x)\bigr)^2 + \sum_{p \leqslant x} \Biggl(\sum_{q \leqslant (\log x)^A} \frac{\rho_f(q)}{q}\Biggr)^2 \\
&\qquad + \sum_{p \leqslant x} \Biggl(\sum_{x^\alpha < q \leqslant x} \frac{\rho_f(q)}{q}\Biggr)^2 + O\bigl(\pi(x)\bigr).
\end{aligned}$$

Applying Lemma 4, we have

$$\sum_{q \leqslant (\log x)^A} \frac{\rho_f(q)}{q} \ll \log\log\log x \quad \text{and} \quad \sum_{x^\alpha < q \leqslant x} \frac{\rho_f(q)}{q} = O(1).$$

Thus,

$$\sum_{p \leqslant x} \bigl(\omega_2(f(p)) - \log\log x\bigr)^2 \ll \sum_{p \leqslant x} \bigl(\omega_2(f(p)) - E(x)\bigr)^2 + O\bigl(\pi(x)(\log\log\log x)^2\bigr).$$

Applying the assumption of the lemma, we obtain

$$\sum_{p \leqslant x} \bigl(\omega_2(f(p)) - \log\log x\bigr)^2 \ll \pi(x) \log\log x.$$

This completes the proof of Lemma 5. □

Lemma 6. *Let $z = x^{\delta}$ with $0 < \delta < 1$. Suppose we have*

$$\sum_{z<p\leqslant x} \bigl(\omega_2(f(p)) - E(x)\bigr)^2 \ll \pi(x)\log\log x.$$

Then,

$$\sum_{p\leqslant x} \bigl(\omega_2(f(p)) - E(x)\bigr)^2 \ll \pi(x)\log\log x.$$

Proof. Notice that there are at most $O(\log z)$ prime divisors of $f(p)$ for $p\leqslant z$. Thus,

$$\begin{aligned}\sum_{p\leqslant x} \bigl(\omega_2(f(p)) - E(x)\bigr)^2 &= \sum_{z<p\leqslant x} \bigl(\omega_2(f(p)) - E(x)\bigr)^2 + \sum_{p\leqslant z} \bigl(\omega_2(f(p)) - E(x)\bigr)^2 \\ &= \sum_{z<p\leqslant x} \bigl(\omega_2(f(p)) - E(x)\bigr)^2 + O\bigl(\pi(z)(\log z)^2\bigr).\end{aligned}$$

Applying the assumption of the lemma, we obtain

$$\sum_{p\leqslant x} \bigl(\omega_2(f(p)) - E(x)\bigr)^2 \ll \pi(x)\log\log x. \qquad \square$$

3. Proof of Theorem 1

Let $f(x) \in \mathbb{Z}[x]$ be an irreducible polynomial and $f(x) \neq cx$. Define

$$\omega_2(f(p)) = \#\Bigl\{q \text{ is a prime } \Bigm| q \mid f(p),\ (\log x)^A < q \leqslant x^{\alpha}, \text{ and } q \nmid f(0)\Bigr\}$$

and

$$E(x) = \sum_{\substack{(\log x)^A < q \leqslant x^{\alpha} \\ q \nmid f(0)}} \frac{\rho_f(q)}{q}.$$

From Lemmas 2, 3, 5 and 6, we see that to prove Theorem 1, it suffices to prove

$$\sum_{z<p\leqslant x} \bigl(\omega_2(f(p)) - E(x)\bigr)^2 \ll \pi(x)\log\log x,$$

where $z = x^{\delta}$ with $0 < \delta < 1$. We will choose the constants A, α, and δ later.

As we stated in the Introduction, we will estimate the quantity

$$\sum_{z<p\leqslant x} \big(\omega_2(f(p)) - E(x)\big)^2$$

by first transforming it into

$$\sum_{n\leqslant x} \big(\omega_2(f(n)) - E(x)\big)^2,$$

where n is a natural number. Hence, we can omit the use of deeper theorems concerning primes in arithmetic progressions. Define

$$P(z) = \prod_{\substack{l\leqslant z\\ l \text{ is a prime}}} l.$$

For $d|P(z)$, let λ_d be real numbers which satisfy

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_d = 0 \quad \text{if } d > z.$$

Thus, we have

$$\begin{aligned}\sum_{z<p\leqslant x} \big(\omega_2(f(p)) - E(x)\big)^2 &\leqslant \sum_{n\leqslant x} \big(\omega_2(f(n)) - E(x)\big)^2 \cdot \left\{\sum_{d|(n,P(z))} \lambda_d\right\}^2 \\ &= \sum_{\substack{d_1,d_2\leqslant z\\ d_1,d_2|P(z)}} \lambda_{d_1}\lambda_{d_2} \cdot \left\{\sum_{\substack{n\leqslant x\\ [d_1,d_2]|n}} \big(\omega_2(f(n)) - E(x)\big)^2\right\},\end{aligned}$$

where $[d_1, d_2]$ is the least common multiple of d_1 and d_2. Let $\sum'$ denote the sum over all $d_1, d_2 \leqslant z$ with d_1, d_2 dividing $P(z)$. We have

$$\sum_{z<p\leqslant x} \big(\omega_2(f(p)) - E(x)\big)^2 \leqslant S_1 + S_2 + S_3,$$

where

$$S_1 = {\sum}' \lambda_{d_1}\lambda_{d_2} \sum_{\substack{n\leqslant x\\ [d_1,d_2]|n}} \omega_2^2(f(n)),$$

$$S_2 = -2E(x) \sum{}' \lambda_{d_1} \lambda_{d_2} \sum_{\substack{n \leqslant x \\ [d_1, d_2] | n}} \omega_2(f(n))$$

and

$$S_3 = \big(E(x)\big)^2 \sum{}' \lambda_{d_1} \lambda_{d_2} \sum_{\substack{n \leqslant x \\ [d_1, d_2] | n}} 1.$$

Consider S_3 first. We have

$$\begin{aligned} S_3 &= \big(E(x)\big)^2 \sum{}' \lambda_{d_1} \lambda_{d_2} \left[\frac{x}{[d_1, d_2]} \right] \\ &= x\big(E(x)\big)^2 \sum{}' \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} + O\left(\big(E(x)\big)^2 \sum{}' |\lambda_{d_1} \lambda_{d_2}| \right). \end{aligned} \tag{2}$$

Consider S_2 now. For a prime q, we have

$$\sum_{\substack{n \leqslant x \\ [d_1, d_2] | n}} \omega_2(f(n)) = \sum_{\substack{n \leqslant x \\ [d_1, d_2] | n}} \sum_{\substack{(\log x)^A < q \leqslant x^\alpha \\ q | f(n), q \nmid f(0)}} 1 = \sum_{\substack{(\log x)^A < q \leqslant x^\alpha \\ q \nmid f(0)}} \sum_{\substack{n \leqslant x \\ [d_1, d_2] | n \\ q | f(n)}} 1.$$

For $q \nmid f(0)$, let $a_1, a_2, \ldots, a_{\rho_f(q)}$ be solutions of

$$f(a) \equiv 0 \pmod{q} \quad \text{where } 0 \leqslant a < q.$$

For $q \nmid [d_1, d_2]$, if $[d_1, d_2] | n$ and $q | f(n)$, this implies that

$$n \equiv \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{\rho_f(q)} \pmod{q[d_1, d_2]},$$

where $\hat{a}_i \equiv a_i \pmod{q}$, $\hat{a}_i \equiv 0 \pmod{[d_1, d_2]}$, and $0 \leqslant \hat{a}_i < q[d_1, d_2]$.

For $q | [d_1, d_2]$, if n is an integer with $[d_1, d_2] | n$ and $q | f(n)$, then $q | n$ and $q | f(n)$. It follows that $q | f(0)$, which is impossible. Hence, such an n does not exist. To summarize, for $q \nmid f(0)$, we have

$$\#\Big\{ n \leqslant x \;\Big|\; n \text{ satisfies } [d_1, d_2] | n \text{ and } q \,|\, f(n) \Big\} = \begin{cases} 0 & \text{if } q | [d_1, d_2], \\ \dfrac{x \rho_f(q)}{q[d_1, d_2]} + O\big(\rho_f(q)\big) & \text{otherwise.} \end{cases}$$

Thus, we have

$$\sum_{\substack{n\leqslant x\\ [d_1,d_2]|n}} \omega_2(f(n)) = \sum_{\substack{(\log x)^A<q\leqslant x^\alpha\\ q\nmid[d_1,d_2],q\nmid f(0)}} \left\{\frac{x\rho_f(q)}{q[d_1,d_2]} + O\big(\rho_f(q)\big)\right\}$$
$$= \sum_{\substack{(\log x)^A<q\leqslant x^\alpha\\ q\nmid[d_1,d_2],q\nmid f(0)}} \frac{x\rho_f(q)}{q[d_1,d_2]} + O\big(\pi(x^\alpha)\big).$$

The last equality follows from the fact that $\rho_f(q)\leqslant d$, the degree of $f(x)$. Hence, we have

$$S_2 = -2E(x)\sum{}' \lambda_{d_1}\lambda_{d_2}\left\{\sum_{\substack{(\log x)^A<q\leqslant x^\alpha\\ q\nmid[d_1,d_2],q\nmid f(0)}} \frac{x\rho_f(q)}{q[d_1,d_2]} + O\big(\pi(x^\alpha)\big)\right\}$$
$$= -2xE(x)\sum{}' \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]} \sum_{\substack{(\log x)^A<q\leqslant x^\alpha\\ q\nmid[d_1,d_2],q\nmid f(0)}} \frac{\rho_f(q)}{q} + O\Big(\pi\big(x^\alpha\big)E(x)\sum{}' |\lambda_{d_1}\lambda_{d_2}|\Big). \tag{3}$$

Consider S_1. For primes q and l, we have

$$\sum_{\substack{n\leqslant x\\ [d_1,d_2]|n}} \omega_2^2(f(n)) = \sum_{\substack{n\leqslant x\\ [d_1,d_2]|n}} \left(\sum_{\substack{(\log x)^A<q\leqslant x^\alpha\\ q|f(n),q\nmid f(0)}} 1\right)^2$$
$$= \sum_{\substack{(\log x)^A<q,l\leqslant x^\alpha\\ q\neq l,ql\nmid f(0)}} \sum_{\substack{n\leqslant x\\ [d_1,d_2]|n\\ ql|f(n)}} 1 + \sum_{\substack{(\log x)^A<q\leqslant x^\alpha\\ q\nmid f(0)}} \sum_{\substack{n\leqslant x\\ [d_1,d_2]|n\\ q|f(n)}} 1.$$

Notice that for $ql\nmid f(0)$,

$$\#\Big\{n\leqslant x \,\Big|\, n \text{ satisfies } [d_1,d_2]|n \text{ and } ql\,|\,f(n)\Big\}$$
$$= \begin{cases} 0 & \text{if } q|[d_1,d_2] \text{ or } l|[d_1,d_2],\\ \dfrac{x\rho_f(q)\rho_f(l)}{ql[d_1,d_2]} + O\big(\rho_f(q)\rho_f(l)\big) & \text{otherwise.}\end{cases}$$

Consider

$$\sum_{\substack{(\log x)^A < q, l \leqslant x^\alpha \\ q \neq l, ql \nmid f(0)}} \sum_{\substack{n \leqslant x \\ [d_1, d_2] | n \\ ql | f(n)}} 1 = \sum_{\substack{(\log x)^A < q, l \leqslant x^\alpha \\ q \neq l \\ ql \nmid [d_1, d_2],\, ql \nmid f(0)}} \left\{ \frac{x\rho_f(q)\rho_f(l)}{ql[d_1, d_2]} + O\big(\rho_f(q)\rho_f(l)\big) \right\}$$

$$= \sum_{\substack{(\log x)^A < q, l \leqslant x^\alpha \\ q \neq l \\ ql \nmid [d_1, d_2],\, ql \nmid f(0)}} \frac{x\rho_f(q)\rho_f(l)}{ql[d_1, d_2]} + O\big((\pi(x^\alpha))^2\big).$$

Notice that

$$\sum_{\substack{(\log x)^A < q, l \leqslant x^\alpha \\ q \neq l \\ ql \nmid [d_1, d_2],\, ql \nmid f(0)}} \frac{x\rho_f(q)\rho_q(l)}{ql[d_1, d_2]}$$

$$= \sum_{\substack{(\log x)^A < q, l \leqslant x^\alpha \\ q \nmid [d_1, d_2],\, l \nmid [d_1, d_2] \\ q \nmid f(0),\, l \nmid f(0)}} \frac{x\rho_f(q)\rho_f(l)}{ql[d_1, d_2]} - \sum_{\substack{(\log x)^A < q \leqslant x^\alpha \\ q \nmid [d_1, d_2], q \nmid f(0)}} \frac{x(\rho_f(q))^2}{q^2[d_1, d_2]}$$

$$= \frac{x}{[d_1, d_2]} \left(\sum_{\substack{(\log x)^A < q \leqslant x^\alpha \\ q \nmid [d_1, d_2], q \nmid f(0)}} \frac{\rho_f(q)}{q} \right)^2 + O\left(\frac{x}{(\log x)^A [d_1, d_2]} \right).$$

Also, we have seen in the calculation of S_2 that

$$\sum_{\substack{(\log x)^A < q \leqslant x^\alpha \\ q \nmid f(0)}} \sum_{\substack{n \leqslant x \\ [d_1, d_2] | n \\ q | f(n)}} 1 = \sum_{\substack{(\log x)^A < q \leqslant x^\alpha \\ q \nmid [d_1, d_2], q \nmid f(0)}} \frac{x\rho_f(q)}{q[d_1, d_2]} + O\big(\pi(x^\alpha)\big).$$

Thus,

$$S_1 = x \sum{}' \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1, d_2]} \left(\sum_{\substack{(\log x)^A < q \leqslant x^\alpha \\ q \nmid [d_1, d_2], q \nmid f(0)}} \frac{\rho_f(q)}{q} \right)^2 + x \sum{}' \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1, d_2]} \sum_{\substack{(\log x)^A < q \leqslant x^\alpha, \\ q \nmid [d_1, d_2], q \nmid f(0)}} \frac{\rho_f(q)}{q}$$
$$+ O\left((\pi(x^\alpha))^2 \sum{}' |\lambda_{d_1}\lambda_{d_2}| \right) + O\left(\frac{x}{(\log x)^A} \sum{}' \frac{|\lambda_{d_1}\lambda_{d_2}|}{[d_1, d_2]} \right). \tag{4}$$

Combining Eqs. (2), (3), and (4), we obtain

$$\begin{aligned}
&\sum_{z<p\leqslant x} (\omega_2(f(p)) - E(x))^2 \\
&\quad \leqslant T_1 + T_2 + O\left(\left(\pi(x^\alpha)\right)^2 {\sum}' |\lambda_{d_1}\lambda_{d_2}|\right) + O\left(\frac{x}{(\log x)^A} {\sum}' \frac{|\lambda_{d_1}\lambda_{d_2}|}{[d_1, d_2]}\right), \qquad (5)
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= x {\sum}' \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1, d_2]} \left(\sum_{\substack{(\log x)^A < q \leqslant x^\alpha \\ q \nmid [d_1,d_2], q \nmid f(0)}} \frac{\rho_f(q)}{q} - E(x) \right)^2 \\
&= x {\sum}' \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1, d_2]} \left(\sum_{\substack{(\log x)^A < q \leqslant x^\alpha \\ q \mid [d_1,d_2], q \nmid f(0)}} \frac{\rho_f(q)}{q} \right)^2 \qquad (6)
\end{aligned}$$

and

$$T_2 = x {\sum}' \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1, d_2]} \sum_{\substack{(\log x)^A < q \leqslant x^\alpha, \\ q \nmid [d_1,d_2], q \nmid f(0)}} \frac{\rho_f(q)}{q}.$$

We write

$$\begin{aligned}
T_2 &= x {\sum}' \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1, d_2]} \left(E(x) - \sum_{\substack{(\log x)^A < q \leqslant x^\alpha, \\ q \mid [d_1,d_2], q \nmid f(0)}} \frac{\rho_f(q)}{q} \right) \\
&= x E(x) {\sum}' \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1, d_2]} - x {\sum}' \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1, d_2]} \sum_{\substack{(\log x)^A < q \leqslant x^\alpha, \\ q \mid [d_1,d_2], q \nmid f(0)}} \frac{\rho_f(q)}{q}. \qquad (7)
\end{aligned}$$

The term

$$x E(x) {\sum}' \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1, d_2]}$$

will be the dominant one. Our goal is to minimize this quantity, subject to the condition that $\lambda_1 = 1$. The analysis of this expression is identical to a similar expression that occurs in the Selberg upper bound sieve. As the details of this analysis are well-known (see for example, [11, pp. 140–143]), we will be very brief. Using a technique of Selberg's [13], we can choose λ_d so that

$$\sum{}' \frac{\lambda_{d_1}\lambda_{d_2}}{[d_1, d_2]} \ll \frac{1}{\log x} \tag{8}$$

and

$$|\lambda_d| \leqslant 1. \tag{9}$$

Thus, from Eqs. (6)–(9), we have

$$T_1 \ll x \sum{}' \frac{1}{[d_1, d_2]} \left(\sum_{\substack{(\log x)^A < q \leqslant x^\alpha \\ q \mid [d_1, d_2], q \nmid f(0)}} \frac{\rho_f(q)}{q} \right)^2$$

and

$$T_2 \ll \frac{x \log\log x}{\log x} + x \sum{}' \frac{1}{[d_1, d_2]} \sum_{\substack{(\log x)^A < q \leqslant x^\alpha \\ q \mid [d_1, d_2], q \nmid f(0)}} \frac{\rho_f(q)}{q}.$$

Notice that there are at most $\log_2[d_1, d_2]$ many prime factors of $[d_1, d_2]$. Also, for each $q \mid [d_1, d_2]$, $\rho_f(q) \leqslant d$, the degree of $f(x)$. Hence,

$$\sum_{\substack{(\log x)^A < q \leqslant x^\alpha, \\ q \mid [d_1, d_2], q \nmid f(0)}} \frac{\rho_f(q)}{q} \leqslant \frac{d \ \log_2[d_1, d_2]}{(\log x)^A} \ll \frac{\log z}{(\log x)^A}.$$

Thus,

$$x \sum{}' \frac{1}{[d_1, d_2]} \sum_{\substack{(\log x)^A < q \leqslant x^\alpha, \\ q \mid [d_1, d_2], q \nmid f(0)}} \frac{\rho_f(q)}{q} \ll \frac{x \log z}{(\log x)^A} \cdot \sum{}' \frac{1}{[d_1, d_2]}$$

and

$$x \sum' \frac{1}{[d_1, d_2]} \left(\sum_{\substack{(\log x)^A < q \leqslant x^\alpha \\ q \| [d_1, d_2]}} \frac{\rho_f(q)}{q} \right)^2 \ll \frac{x(\log z)^2}{(\log x)^{2A}} \cdot \sum' \frac{1}{[d_1, d_2]}.$$

The last sum is easily estimated as follows. Noting that

$$\frac{1}{[d_1, d_2]} = \frac{(d_1, d_2)}{d_1 d_2} \quad \text{and} \quad (d_1, d_2) = \sum_{e|(d_1, d_2)} \varphi(e),$$

where (d_1, d_2) is the greatest common divisor of d_1 and d_2 and φ the Euler function, we can write

$$\frac{1}{[d_1, d_2]} = \frac{1}{d_1 d_2} \sum_{e|d_1, d_2} \varphi(e).$$

Inserting this fact into the sum in question, interchanging summations, it is clear that

$$\sum' \frac{1}{[d_1, d_2]} \leqslant (\log z)^3$$

by standard estimates of elementary number theory. Thus,

$$T_1 \ll \frac{x(\log z)^5}{(\log x)^{2A}}$$

and

$$T_2 \ll \pi(x) \log\log x + \frac{x(\log z)^4}{(\log x)^A}.$$

Combining all the estimates together, from Eq. (5), we obtain

$$\sum_{z < p \leqslant x} \big(\omega_2(f(p)) - E(x)\big)^2$$
$$\ll \pi(x) \log\log x + \frac{x(\log z)^5}{(\log x)^A} + O\Big(\big(\pi(x^\alpha) z\big)^2\Big).$$

By choosing $A = 7$, $\alpha = \delta = 1/6$ (note: $z = x^{\delta}$), we obtain

$$\sum_{z<p\leqslant x} \bigl(\omega_2(f(p)) - E(x)\bigr)^2 \ll \pi(x) \log\log x.$$

From Lemmas 3, 5, and 6, it follows that for any irreducible polynomial $f(x) \in \mathbb{Z}[x]$, $f(x) \neq cx$, we have

$$\sum_{p\leqslant x} (\omega(f(p)) - \log\log x)^2 \ll \pi(x) \log\log x.$$

Now, let $f(x) \in \mathbb{Z}[x]$ be a general polynomial which is divisible by r distinct irreducible polynomials and $f(x) \neq cx^e$. From Lemma 2, we conclude that

$$\sum_{p\leqslant x} (\omega(f(p)) - r\log\log x)^2 \ll \pi(x) \log\log x.$$

This completes the proof of Theorem 1.

It will be interesting to see if the methods of this paper can be used to show the analogue of the Erdös–Kac theorem holds for $\omega(f(p))$. More precisely, can one prove that for $f(x)$ defined as in Theorem 1,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \leqslant x,\ \frac{\omega(f(p)) - r\log\log p}{\sqrt{r\log\log p}} \leqslant \gamma \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\gamma} e^{-t^2/2}\, dt.$$

The case $f(x)$ is irreducible ($r = 1$) was first proved by Halberstam [3] using more difficult methods.

We hope that the techniques here will find wider applications, especially, in the context where strong theorems, such as the Bombieri–Vinogradov theorem are not available. One may also consider analogues of Theorem 1 for polynomials of several variables. This investigation we relegate to future work.

Acknowledgments

We thank the referee for helpful remarks.

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