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A Generalization of Roth's Theorem in Function Fields

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1. Introduction

For $n \in \mathbb{N} = \{1, 2, \dots\}$, let $D_3([1, n])$ denote the maximal cardinality of an integer subset of $[1, n]$ containing no nontrivial 3-term arithmetic progression. In a fundamental paper [9], Roth proved that $D_3([1, n]) \ll n/\log\log n$. His result was later improved by Heath-Brown [4] and Szemerédi [11] to $D_3([1, n]) \ll n/(\log n)^{\alpha}$ for some small positive constant $\alpha > 0$ ($\alpha = 1/20$ in [11]). By introducing the notion of Bohr sets, Bourgain [2; 3] further improved this bound and showed that $D_3([1, n]) \ll n(\log\log n)^2/(\log n)^{2/3}$.

One can consider Roth's theorem in function fields. Let $\mathbb{F}_q[t]$ be the ring of polynomials over the finite field $\mathbb{F}_q$. For $N \in \mathbb{N}$, let $\mathcal{S}_N$ be the subset of $\mathbb{F}_q[t]$ containing all polynomials of degree strictly less than N and let $|\mathcal{S}_N|$ be the cardinality of $\mathcal{S}_N$. We denote by $D_3(\mathcal{S}_N)$ the maximal cardinality of a subset of $\mathcal{S}_N$ containing no nontrivial 3-term arithmetic progression. When q is not divisible by 2, the result of Meshulam in [8, Thm. 1.2] implies that $D_3(\mathcal{S}_N) \ll |\mathcal{S}_N|/\log|\mathcal{S}_N|$, which is a sharper estimate than its integer analogue. Meshulam's method is applicable to all finite abelian groups of odd order. The additional abelian group structure also allows him to provide a beautiful short proof of the 3-term arithmetic progression problem. In [6], Lev extended Meshulam's result to any finite abelian group G such that $2G = \{2g \mid g \in G\}$ is nontrivial.

An important point for function field arithmetic is that, because there are many signs (i.e., nonzero elements in $\mathbb{F}_q$), as the finite field gets larger it becomes a family of questions, each with respect to a fixed choice of signs for the terms. Although the appearance of abelian group structure on the underlying set makes the original 3-term arithmetic progression problem easier, the approach of Meshulam does not work for other choices of signs.

One can formulate a generalization of Meshulam's result in $\mathbb{F}_q[t]$ as follows. For $s \in \mathbb{N}$ with $s \geq 3$, let $\mathbf{g} = (g_1, \dots, g_s)$ be a vector of nonzero elements of $\mathbb{F}_q[t]$ satisfying $g_1 + \dots + g_s = 0$. Let $D_{\mathbf{g}}(\mathcal{S}_N)$ denote the maximal cardinality of a set $A \subseteq \mathcal{S}_N$ for which the equation $g_1x_1 + \dots + g_sx_s = 0$ is never satisfied by distinct elements $x_1, \dots, x_s \in A$. In the special case that $\mathbf{g} = (1, -2, 1)$, we have $D_{\mathbf{g}}(\mathcal{S}_N) = D_3(\mathcal{S}_N)$. In [7] it was proved that if $g_i \in \mathbb{F}_q \setminus \{0\}$ ($1 \leq i \leq s$) then

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$D_{\mathbf{g}}(\mathcal{S}_N) \ll_{\mathbf{g}} |\mathcal{S}_N|/(\log|\mathcal{S}_N|)^{s-2}$. Their proof is an application of the circle method for $\mathbb{F}_q[t]$. It also combines the observation that $\mathcal{S}_N$ is a finite vector space over $\mathbb{F}_q$ with the fact that, for an $\mathbb{F}_q$-linear transformation T, the map composition $g_i \circ T$ is equal to $T \circ g_i$ (see [7, Lemma 2]).

In this paper, we further generalize the setting in [7] by allowing $g_i \in \mathbb{F}_q[t] \setminus \{0\}$ $(1 \le i \le s)$. Since multiplication by a nonzero element of $\mathbb{F}_q[t]$ is no longer commutative with an $\mathbb{F}_q$-linear transformation, the approach of [7] fails to yield an effective result for this general setting. In order to bound $D_{\mathbf{g}}(\mathcal{S}_N)$ for general $\mathbf{g}$, we establish a version of the circle method for $\mathbb{F}_q[t]$ that is more flexible than that of [7]. We also employ a modification of the Bohr set technology developed by Bourgain in [2] and [3]. Then we can prove the following result.

THEOREM 1. *For $s \in \mathbb{N}$ with $s \ge 3$, let $\mathbf{g} = (g_1, \dots, g_s)$ with $g_i \in \mathbb{F}_q[t] \setminus \{0\}$ $(1 \le i \le s)$ and $g_1 + \cdots + g_s = 0$. Then there exists a constant $C = C(\mathbf{g}; q) > 0$ such that*

$$D_{\mathbf{g}}(\mathcal{S}_N) \le C|\mathcal{S}_N| \left(\frac{(\log\log|\mathcal{S}_N|)^2}{\log|\mathcal{S}_N|} \right)^{\frac{2(s-2)^2}{4s-9}}. \tag{1}$$

A major difference between a "Bohr set in integers" (for definition, see [3, 0.11]) and a "Bohr set in $\mathbb{F}_q[t]$" (see Definition 1) is that the latter is a vector space over $\mathbb{F}_q$. This additional structure allows us to compute explicitly the size of "major arcs" (see Lemma 2). Thus, our estimates of the major arc contribution (see the proof of Lemma 11(i)) differ significantly from their integer analogues in [3, (6.6)–(6.9)]. The appearance of vector space structure also allows the analogue of [3, integral (6.14)] in our analysis to be zero. Thus, the case considered in [3, Sec. 7] is not required in our argument. Finally, an important technique used in [3] is to replace a probability measure by a convolution of two probability measures. In the integer case, the resulting errors are well controlled (see [3, Lemmas 3.16 and 3.29]) but do lead to difficulties in constructing new Bohr sets. However, analogous errors in our setting are zero (see Lemma 7(ii)), since a Bohr set in $\mathbb{F}_q[t]$ inherits vector space structure. Because of these advantages, the density increment arguments in [3] are greatly simplified in this paper. Finally, we would like to remark that Sanders [10] has recently improved Bourgain's result to $D_3([1, n]) \ll n(\log\log n)^5/\log n$. In future work, we will show how his method can be implemented to improve the result in this paper.

2. Preliminaries

We begin this section by introducing the Fourier analysis for function fields. Let $\mathbb{A} = \mathbb{F}_q[t]$ and $\mathbb{K} = \mathbb{F}_q(t)$ be the field of fractions of $\mathbb{A}$. Let $\mathbb{K}_\infty = \mathbb{F}_q((1/t))$ be the completion of $\mathbb{K}$ at ∞. We may write each element $\alpha \in \mathbb{K}_\infty$ in the form $\alpha = \sum_{i \le v} a_i t^i$ for some $v \in \mathbb{Z}$ and $a_i = a_i(\alpha) \in \mathbb{F}_q$ $(i \le v)$. If $a_v \ne 0$, we define $\operatorname{ord} \alpha = v$. We adopt the convention that $\operatorname{ord} 0 = -\infty$. Also, it is often convenient to refer to a_{-1} as being the residue of α, denoted by $\operatorname{res} \alpha$. Let $\mathbb{T} =$

$\{\alpha \in \mathbb{K}_\infty \mid \operatorname{ord} \alpha < 0\}$. Given any Haar measure $d\alpha$ on $\mathbb{K}_\infty$, we normalize it in such a manner that $\int_{\mathbb{T}} 1\, d\alpha = 1$. We are now equipped to define the exponential function on $\mathbb{K}_\infty$. Suppose that the characteristic of $\mathbb{F}_q$ is p. Let $e(z)$ denote $e^{2\pi i z}$, and let $\operatorname{tr}\colon \mathbb{F}_q \to \mathbb{F}_p$ denote the familiar trace map. There is a nontrivial additive character $e_q\colon \mathbb{F}_q \to \mathbb{C}^\times$ defined for each $a \in \mathbb{F}_q$ by taking $e_q(a) = e(\operatorname{tr}(a)/p)$. This character induces a map $e\colon \mathbb{K}_\infty \to \mathbb{C}^\times$ by defining, for each element $\alpha \in \mathbb{K}_\infty$, the value of $e(\alpha)$ to be $e_q(\operatorname{res} \alpha)$. The orthogonality relation underlying the Fourier analysis of $\mathbb{F}_q[t]$, established in [5, Lemma 1], takes the form

$$\int_{\mathbb{T}} e(h\alpha)\, d\alpha = \begin{cases} 1 & \text{if } h = 0, \\ 0 & \text{if } h \in \mathbb{F}_q[t] \setminus \{0\}. \end{cases}$$

Finally, we denote by $\operatorname{rk} Z$ the rank of a matrix Z and by $\ker \Lambda$ the kernel of a function Λ.

LEMMA 2. *Let $W \subset \mathbb{A}$ be a finite vector space, and let $\hat{W}$ denote the character group of W. Let $\Lambda\colon \mathbb{T} \to \hat{W}$ be a function defined such that, for each $\alpha \in \mathbb{T}$ and $x \in W$, we have*

$$\Lambda(\alpha)(x) = e(\alpha x).$$

(i) *For any $\alpha, \beta \in \mathbb{T}$, $\Lambda(\alpha + \beta) = \Lambda(\alpha)\Lambda(\beta)$.*
(ii) $\ker \Lambda = \{\alpha \in \mathbb{T} \mid \operatorname{res}(\alpha x) = 0$ *for all* $x \in W\}$.
(iii) $\mathbb{T}/\ker \Lambda \cong \hat{W}$.
(iv) $\operatorname{meas}(\ker \Lambda) = |W|^{-1}$.
(v) *For $\alpha \in \mathbb{T}$ and $\alpha \notin \ker \Lambda$, we have $\sum_{x \in W} e(\alpha x) = 0$.*

Proof. (i) For any $\alpha, \beta \in \mathbb{T}$ and $x \in W$,

$$\Lambda(\alpha + \beta)(x) = e((\alpha + \beta)x) = e(\alpha x)e(\beta x) = (\Lambda(\alpha)\Lambda(\beta))(x).$$

(ii) Let $\mathcal{A} = \{\alpha \in \mathbb{T} \mid \operatorname{res}(\alpha x) = 0 \text{ for all } x \in W\}$, which is a subset of $\ker \Lambda$. Suppose there exists a $\beta \in \ker \Lambda \setminus \mathcal{A}$. Then there exists a $z \in W$ such that $\operatorname{res}(\beta z) \neq 0$. Thus,

$$\{\operatorname{res}(\beta x) \mid x \in W\} = \mathbb{F}_q.$$

Since $e = e_q \circ \operatorname{res} = e_p \circ \operatorname{tr} \circ \operatorname{res}$ and since tr is surjective, we deduce that

$$\{\Lambda(\beta)(x) \mid x \in W\} = \{e(n/p) \mid n = 0, \dots, p-1\}.$$

It follows that $\beta \notin \ker \Lambda$, a contradiction. Therefore, $\ker \Lambda = \mathcal{A}$.

(iii) Let r be the dimension of W, and let $\{z_1, \dots, z_r\}$ be a basis for W. Let m be the maximal degree of $z_i \in \mathbb{A}$ $(1 \leq i \leq r)$. Then we can write

$$z_i = a_{i,0} + a_{i,1}t + \cdots + a_{i,m}t^m \quad (1 \leq i \leq r)$$

with $a_{i,j} \in \mathbb{F}_q$ $(1 \leq i \leq r, 0 \leq j \leq m)$. Let $\beta = \sum_{l<0} b_l t^l \in \mathbb{T}$. Then

$$\operatorname{res}(\beta z_i) = a_{i,0}b_{-1} + \cdots + a_{i,m}b_{-m-1} \quad (1 \leq i \leq r).$$

Write

$$Z = (a_{i,j})_{1 \leq i \leq r,\, 0 \leq j \leq m} \quad \text{and} \quad \mathbf{b} = (b_{-1}, \dots, b_{-m-1}).$$

It follows from part (ii) of the lemma that $\beta \in \ker\Lambda$ if and only if $Z\mathbf{b} = \mathbf{0}$. Let $\mathcal{M} = t^{-m-1}\mathbb{T}$. Thus, $(\ker\Lambda)/\mathcal{M}$ is isomorphic to the null space of Z. Since $z_1, \dots, z_r$ are linearly independent, we see that $\operatorname{rk} Z = r$. Then we have

$$(\ker\Lambda)/\mathcal{M} \cong \mathbb{F}_q^{(m+1)-r}.$$

Since

$$\mathbb{T}/\mathcal{M} \cong \mathbb{F}_q^{m+1} \quad \text{and} \quad \mathbb{T}/\ker\Lambda \cong (\mathbb{T}/\mathcal{M})/(\ker\Lambda/\mathcal{M}),$$

we see that

$$|\mathbb{T}/\ker\Lambda| = q^r = |W|.$$

By part (i), Λ is a surjective homomorphism from $\mathbb{T}$ to $\hat{W}$. It then follows from the first isomorphism theorem that

$$\mathbb{T}/\ker\Lambda \cong \hat{W}.$$

(iv) Let r be the dimension of W. Because the dimension of $\hat{W}$ is also r, by part (iii) there exist $\beta_i \in \mathbb{T}$ $(1 \le i \le q^r)$ such that

$$\mathbb{T} = \bigsqcup_{i=1}^{q^r}(\beta_i + \ker\Lambda).$$

Thus,

$$1 = q^r \cdot \operatorname{meas}(\ker\Lambda),$$

from which it follows that

$$\operatorname{meas}(\ker\Lambda) = q^{-r} = |W|^{-1}.$$

(v) If $\alpha \notin \ker\Lambda$, then $\Lambda(\alpha)$ is a nontrivial character of W. Therefore,

$$\sum_{x \in W} e(\alpha x) = \sum_{x \in W} \Lambda(\alpha)(x) = 0. \qquad \square$$

DEFINITION 1. For $\alpha = \sum_{i \le v} a_i t^i \in \mathbb{K}_\infty$, let $\{\alpha\} = \sum_{i<0} a_i t^i$. For $N \in \mathbb{N}$, $d \in \mathbb{N} \cup \{0\}$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d) \in \mathbb{T}^d$, and $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$, we define

$$V(N; \boldsymbol{\theta}; \mathbf{n}) = \{x \in \mathbb{A} \mid \operatorname{ord} x < N \text{ and } \operatorname{ord}\{x\theta_j\} < -n_j \ (1 \le j \le d)\}.$$

We say that U is a *Bohr space* if there are $a \in \mathbb{A} \setminus \{0\}$ and $V = V(N; \boldsymbol{\theta}; \mathbf{n})$ such that $U = aV = \{ax \mid x \in V\}$. We also say that the *length* of $\mathbf{n}$ is d, denoted by $\operatorname{length}(\mathbf{n}) = d$.

REMARK 1. Since ord is a non-Archimedean valuation, a Bohr space is a finite vector space over $\mathbb{F}_q$. Also, by taking $a = 1$ and $d = 0$ in Definition 1, we see that the set $\mathcal{S}_N$ is also a Bohr space.

LEMMA 3. *For $N \in \mathbb{N}$, $d \in \mathbb{N} \cup \{0\}$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d) \in \mathbb{T}^d$, and $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$, let $|\mathbf{n}| = n_1 + \cdots + n_d$ and write*

$$\theta_j = \sum_{l \le -1} b_{j,l} t^l \quad (1 \le j \le d).$$

For each i, $1 \le i \le N$, *let*

$$C_i = (b_{1,-1-N+i}, \dots, b_{1,-n_1-N+i}, \dots, b_{d,-1-N+i}, \dots, b_{d,-n_d-N+i})^{\mathrm{T}}.$$

Let $K = K(N;\boldsymbol{\theta};\mathbf{n})$ *be a* $|\mathbf{n}| \times N$ *matrix defined by*

$$K = (C_1, \dots, C_N).$$

Then

$$|V(N;\boldsymbol{\theta};\mathbf{n})| = q^{N-\operatorname{rk} K}.$$

Proof. If $x \in \mathbb{A}$ satisfies $\operatorname{ord} x < N$ then we write $x = a_{N-1}t^{N-1} + \cdots + a_1 t + a_0$, where $a_i \in \mathbb{F}_q$ $(0 \le i \le N-1)$. We have

$$\{x\theta_j\} = \sum_{k \le -1} (a_{N-1}b_{j,k-(N-1)} + \cdots + a_0 b_{j,k})t^k \quad (1 \le j \le d).$$

Thus, $\operatorname{ord}\{x\theta_j\} < -n_j$ $(1 \le j \le d)$ if and only if

$$a_{N-1}b_{j,k-(N-1)} + \cdots + a_0 b_{j,k} = 0 \quad (-n_j \le k \le -1,\ 1 \le j \le d);$$

that is,

$$(b_{j,k-(N-1)}, \dots, b_{j,k})(a_{N-1}, \dots, a_0)^{\mathrm{T}} = 0 \quad (-n_j \le k \le -1,\ 1 \le j \le d).$$

It follows that $x \in V(N;\boldsymbol{\theta};\mathbf{n})$ if and only if $(a_{N-1}, \dots, a_1, a_0)$ is a solution of $K\mathbf{y} = 0$. Thus,

$$|V(N;\boldsymbol{\theta};\mathbf{n})| = q^{N-\operatorname{rk} K}.$$

This completes the proof of the lemma. □

In what follows, unless stated otherwise we have $a \in \mathbb{A} \setminus \{0\}$, $N \in \mathbb{N}$, $d \in \mathbb{N} \cup \{0\}$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d) \in \mathbb{T}^d$, and $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$.

LEMMA 4. (i) *Let* $\boldsymbol{\theta}' = (\boldsymbol{\theta}, \theta_{d+1}) \in \mathbb{T}^{d+1}$ *and* $\mathbf{n}' = (\mathbf{n}, 1) \in \mathbb{N}^{d+1}$. *Then*

$$V(N;\boldsymbol{\theta}';\mathbf{n}') \subseteq V(N;\boldsymbol{\theta};\mathbf{n}) \quad \textit{and} \quad |V(N;\boldsymbol{\theta}';\mathbf{n}')| \ge q^{-1}|V(N;\boldsymbol{\theta};\mathbf{n})|.$$

(ii) *For* $N \ge 2$, *let* $M = N-1$ *and* $\mathbf{m} = \mathbf{n} + (1, \dots, 1) \in \mathbb{N}^d$. *Then*

$$V(M;\boldsymbol{\theta};\mathbf{m}) \subseteq V(N;\boldsymbol{\theta};\mathbf{n}) \quad \textit{and} \quad |V(M;\boldsymbol{\theta};\mathbf{m})| \ge q^{-(d+1)}|V(N;\boldsymbol{\theta};\mathbf{n})|.$$

Proof. Let C_i $(1 \le i \le N)$ and K be defined as in Lemma 3. Write

$$\theta_j = \sum_{l \le -1} b_{j,l}t^l \quad (1 \le j \le d+1).$$

(i) By Definition 1 we have $V(N;\boldsymbol{\theta}';\mathbf{n}') \subseteq V(N;\boldsymbol{\theta};\mathbf{n})$. For each i with $1 \le i \le N$, write

$$C_i' = (b_{1,-1-N+i}, \dots, b_{1,-n_1-N+i}, \dots, b_{d,-1-N+i}, \dots, b_{d,-n_d-N+i}, b_{d+1,-1-N+i})^{\mathrm{T}}.$$

Then

$$K' = K(N;\boldsymbol{\theta}';\mathbf{n}') = (C_1', \dots, C_N') = \begin{pmatrix} K \\ \mathbf{b} \end{pmatrix},$$

where

$$\mathbf{b} = (b_{d+1,-N}, \dots, b_{d+1,-1}).$$

We see that $\operatorname{rk} K' \le 1 + \operatorname{rk} K$. By Lemma 3, we have

$$|V(N; \boldsymbol{\theta}'; \mathbf{n}')| = q^{N-\operatorname{rk} K'} \ge q^{-1} q^{N-\operatorname{rk} K} = q^{-1} |V(N; \boldsymbol{\theta}; \mathbf{n})|.$$

(ii) For each i with $1 \le i \le M$, write

$$\begin{aligned} C_i'' &= (b_{1,-1-M+i}, \dots, b_{1,-m_1-M+i}, \dots, b_{d,-1-M+i}, \dots, b_{d,-m_d-M+i})^{\mathrm{T}} \\ &= (b_{1,-N+i}, \dots, b_{1,-n_1-N+i}, \dots, b_{d,-N+i}, \dots, b_{d,-n_d-N+i})^{\mathrm{T}}. \end{aligned}$$

Then

$$K'' = K(M; \boldsymbol{\theta}; \mathbf{m}) = (C_1'', \dots, C_M'').$$

Thus, K'' is row equivalent to

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix},$$

where

$$K_1 = \begin{pmatrix} b_{1,-N+1} & b_{1,-N+2} & \cdots & b_{1,-1} \\ b_{2,-N+1} & b_{2,-N+2} & \cdots & b_{2,-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{d,-N+1} & b_{d,-N+2} & \cdots & b_{d,-1} \end{pmatrix}$$

and

$$K_2 = (C_1, \dots, C_{N-1}).$$

It follows that

$$\operatorname{rk} K'' \le d + \operatorname{rk} K_2 \le d + \operatorname{rk} K.$$

By Lemma 3 we have

$$|V(M; \boldsymbol{\theta}; \mathbf{m})| = q^{M-\operatorname{rk} K''} \ge q^{N-1-d-\operatorname{rk} K} = q^{-(d+1)} |V(N; \boldsymbol{\theta}; \mathbf{n})|,$$

which completes the proof of the lemma. □

LEMMA 5. *Let* $U = aV(N; \boldsymbol{\theta}; \mathbf{n})$ *with* $n_j > \operatorname{ord} a$ $(1 \le j \le d)$. *Let* $U' = V(N'; \boldsymbol{\theta}; \mathbf{n}')$, *where*

$$N' = N + \operatorname{ord} a \quad \textit{and} \quad \mathbf{n}' = \mathbf{n} - (\operatorname{ord} a, \dots, \operatorname{ord} a) \in \mathbb{N}^d.$$

Then we have

$$U \subseteq U'.$$

Proof. Let $y \in U$ and $\mathbf{n}' = (n_1', \dots, n_d') \in \mathbb{N}^d$. Then there exists an $x \in V(N; \boldsymbol{\theta}; \mathbf{n})$ with $y = ax$. Since

$$\operatorname{ord}(a\{x\theta_j\}) = \operatorname{ord} a + \operatorname{ord}(\{x\theta_j\}) \le \operatorname{ord} a - n_j = -n_j' < 0 \quad (1 \le j \le d),$$

we have $a\{x\theta_j\} = \{ax\theta_j\}$. Therefore, $y = ax \in V(N'; \boldsymbol{\theta}; \mathbf{n}')$ and the lemma follows. □

LEMMA 6. *For* $g_1, \dots, g_s \in \mathbb{A}$, *let* $g = g_1 g_2 \cdots g_s$ *and* $n = \operatorname{ord} g$. *For each* $j \in \{1, \dots, s\}$, *let* $h_j = g/g_j$. *Let* $U = aV(N; \boldsymbol{\theta}; \mathbf{n})$ *with* $N > n$ *and let* $\tilde{U} = aV(\tilde{N}; \boldsymbol{\theta}; \tilde{\mathbf{n}})$, *where*

$$\tilde{N} = N - n \quad \textit{and} \quad \tilde{\mathbf{n}} = \mathbf{n} + (n, \ldots, n) \in \mathbb{N}^d.$$

Then we have

$$h_j \tilde{U} \subseteq U \quad \textit{and} \quad |\tilde{U}| \geq q^{-n(d+1)}|U|.$$

Proof. By Lemma 5, for each h_j $(1 \leq j \leq s)$ we have

$$\begin{aligned} h_j \tilde{U} &= a h_j V(\tilde{N}; \boldsymbol{\theta}; \tilde{\mathbf{n}}) \\ &\subseteq aV(\tilde{N} + \operatorname{ord} h_j; \boldsymbol{\theta}; \tilde{\mathbf{n}} - (\operatorname{ord} h_j, \ldots, \operatorname{ord} h_j)) \\ &\subseteq aV(\tilde{N} + n; \boldsymbol{\theta}; \tilde{\mathbf{n}} - n(1, \ldots, 1)) \\ &= aV(N; \boldsymbol{\theta}; \mathbf{n}) = U. \end{aligned}$$

In addition, from Lemma 4(ii) it follows that

$$|\tilde{U}| \geq q^{-n(d+1)}|U|.$$

This completes the proof of the lemma. □

Definition 2. Let $V \subset \mathbb{A}$ be a finite vector subspace over $\mathbb{F}_q$. We say that $\nu : \mathbb{A} \to \mathbb{R}$ is a *probability measure on* $\mathbb{A}$ *defined by* V if it satisfies

$$\nu(x) = \begin{cases} 1/|V| & \text{if } x \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Also, for $S \subseteq \mathbb{A}$ we define

$$\nu(S) = \sum_{x \in S} \nu(x).$$

Lemma 7. *Let V_1, V_2 be finite vector subspaces of $\mathbb{A}$, and let ν_1, ν_2 be the probability measures on $\mathbb{A}$ defined by V_1, V_2 (respectively). For $x \in \mathbb{A}$, define*

$$\nu_1 * \nu_2(x) = \sum_{y \in \mathbb{A}} \nu_2(y)\nu_1(x - y).$$

(i) *We have*

$$\nu_1 * \nu_2 = \nu_2 * \nu_1. \tag{2}$$

(ii) *Suppose that $V_2 \subseteq V_1$. Then*

$$\nu_1 * \nu_2 = \nu_1. \tag{3}$$

(iii) *Suppose that $V_2 \subseteq V_1$, and let $\alpha \in \mathbb{K}_\infty$. Suppose that $e(\alpha z) = 1$ for each $z \in V_2$. Then, for any subset $S \subseteq \mathbb{A}$, we have*

$$\sum_{x \in S} \nu_1(x) e(\alpha x) = \sum_{y \in \mathbb{A}} \nu_1(y)\nu_2(S - y)e(\alpha y); \tag{4}$$

here $S - y = \{x - y \mid x \in S\}$. In particular, if $\alpha = 0$ then (4) implies that

$$\nu_1(S) = \sum_{y \in \mathbb{A}} \nu_1(y)\nu_2(S - y). \tag{5}$$

Proof. (i) For any $x \in \mathbb{A}$,

$$\nu_1 * \nu_2(x) = \sum_{y\in\mathbb{A}} \nu_2(y)\nu_1(x-y) = \sum_{z\in\mathbb{A}} \nu_2(x-z)\nu_1(z) = \nu_2 * \nu_1(x).$$

(ii) Let $z \in V_1$. Since $V_2 \subseteq V_1$, for $y \in V_2$ we have $z - y \in V_1$. Therefore,

$$\begin{aligned}\nu_1 * \nu_2(z) &= \sum_{y\in\mathbb{A}} \nu_2(y)\nu_1(z-y) = \sum_{y\in V_2} \nu_2(y)\nu_1(z-y)\\ &= |V_1|^{-1} \sum_{y\in V_2} \nu_2(y) = |V_1|^{-1} = \nu_1(z).\end{aligned}$$

Let $x \in V \setminus V_1$; then $\nu_1(x) = 0$. If $\nu_1 * \nu_2(x) \neq 0$ then there exists a $y \in V_2$ such that $x - y \in V_1$. Since $V_2 \subseteq V_1$, we have $x \in V_1$—a contradiction. Thus,

$$\nu_1 * \nu_2(x) = 0 = \nu_1(x).$$

In both cases, $\nu_1 * \nu_2 = \nu_1$.

(iii) Let $\alpha \in \mathbb{K}_\infty$, and suppose that $e(\alpha z) = 1$ for each $z \in V_2$. Then, by parts (i) and (ii),

$$\begin{aligned}\sum_{x\in S} \nu_1(x)e(\alpha x) &= \sum_{x\in S} \nu_1 * \nu_2(x)e(\alpha x)\\ &= \sum_{x\in S}\sum_{y\in\mathbb{A}} \nu_1(y)\nu_2(x-y)e(\alpha x)\\ &= \sum_{y\in\mathbb{A}} \nu_1(y)e(\alpha y)\sum_{x\in S} \nu_2(x-y)e(\alpha(x-y))\\ &= \sum_{y\in\mathbb{A}} \nu_1(y)e(\alpha y)\sum_{x\in S} \nu_2(x-y)\\ &= \sum_{y\in\mathbb{A}} \nu_1(y)\nu_2(S-y)e(\alpha y).\end{aligned}$$

This completes the proof of the lemma. □

3. Density Increments

For $s \in \mathbb{N}$ with $s \geq 3$, let $g_1, \dots, g_s \in \mathbb{F}_q[t] \setminus \{0\}$ with $g_1 + \cdots + g_s = 0$. Let $g = g_1 g_2 \cdots g_s$. Write $n = \operatorname{ord} g$. Let $U = aV(N; \boldsymbol{\theta}; \mathbf{n})$ with $N > n$. Let $\tilde{U}$ and h_j $(1 \leq j \leq s)$ be defined as in Lemma 6, and let

$$V_j = h_j\tilde{U} \quad (1 \leq j \leq s).$$

Then we have

$$g_1V_1 + \cdots + g_sV_s = g\tilde{U} + \cdots + g\tilde{U} = g\tilde{U}.$$

Also, it follows from Lemma 6 that

$$V_j \subseteq U \quad (1 \leq j \leq s).$$

Let μ be the probability measure on $\mathbb{A}$ defined by U. For each j $(1 \leq j \leq s)$, let ν_j be the probability measure on $\mathbb{A}$ defined by V_j. By Lemma 7(ii) we have

$$\mu * \nu_j = \mu \quad (1 \leq j \leq s).$$

Let $B' \subset \mathbb{A}$ with $\mu(B') = \delta_1$. Suppose that the equation $g_1 x_1 + \cdots + g_s x_s = 0$ is never satisfied by distinct elements $x_1, \ldots, x_s \in B'$. Write $d = \text{length}(\mathbf{n})$ and $C_0 = 2^{2s-1}\binom{s}{2}$. Suppose that

$$\log|U| - (\log q)n(d+1) \geq 2\log\delta_1^{-2} + 2\log C_0. \tag{6}$$

Recall that $\tilde{U} = aV(\tilde{N}, \boldsymbol{\theta}, \tilde{\mathbf{n}})$ for $\tilde{N} = N - n$ and that $\tilde{\mathbf{n}} = \mathbf{n} + (n, \ldots, n)$. By Lemma 6, we have

$$\begin{aligned} d &= \text{length}(\mathbf{n}) = \text{length}(\tilde{\mathbf{n}}), \\ |\tilde{U}| &\geq q^{-n(d+1)}|U|. \end{aligned} \tag{7}$$

Then from (6) we obtain

$$|\tilde{U}|^{1/2} \geq \max\{\delta_1^{-2}, C_0\}. \tag{8}$$

Hereafter, we fix ε as follows:

$$0 < \varepsilon \leq (4s)^{-1}\delta_1. \tag{9}$$

LEMMA 8 (Density Increment I). *Suppose that, for each $y \in \mathbb{A}$, there exists a $j = j(y) \in \{1, \ldots, s\}$ such that*

$$|\nu_j(B' + y) - \delta_1| \geq 2s\varepsilon.$$

Then there exist $i \in \{1, \ldots, s\}$ and $z \in \mathbb{A}$ such that

$$\nu_i(B' + z) \geq \delta_1 + \varepsilon.$$

Proof. For each $y \in \mathbb{A}$, we have

$$\sum_{j=1}^{s} |\nu_j(B' + y) - \delta_1| \geq 2s\varepsilon.$$

Hence

$$\sum_{j=1}^{s} \sum_{y \in \mathbb{A}} \mu(y)|\nu_j(B' + y) - \delta_1| = \sum_{y \in \mathbb{A}} \mu(y) \sum_{j=1}^{s} |\nu_j(B' + y) - \delta_1| \geq 2s\varepsilon.$$

It follows that there exist $i \in \{1, \ldots, s\}$ such that

$$\sum_{y \in \mathbb{A}} \mu(y)|\nu_i(B' + y) - \delta_1| \geq 2\varepsilon.$$

For each $x \in \mathbb{A}$, from Definition 2 it is clear that $\mu(x) = \mu(-x)$. By (5), we have

$$\delta_1 = \mu(B') = \sum_{x \in \mathbb{A}} \mu(x)\nu_i(B' - x) = \sum_{y \in \mathbb{A}} \mu(y)\nu_i(B' + y).$$

Therefore,

$$\sum_{y\in\mathbb{A}}\mu(y)(\nu_i(B'+y)-\delta_1)=0.$$

Combining these estimates establishes the existence of a $z\in\mathbb{A}$ such that

$$\nu_i(B'+z)\geq\delta_1+\varepsilon.$$

This completes the proof of the lemma. □

In the rest of this section we assume that there exists a $y\in\mathbb{A}$ such that

$$|\nu_j(B'+y)-\delta_1|<2s\varepsilon\quad(1\leq j\leq s),\tag{10}$$

since the complementary case has already been treated in Lemma 8. Write $B=B'+y$. From (9) it follows that

$$\tfrac{1}{2}\delta_1<\nu_j(B)<2\delta_1\quad(1\leq j\leq s).\tag{11}$$

For $\alpha\in\mathbb{T}$, define

$$f_j(\alpha;B)=\sum_{x\in B}\nu_j(x)e(\alpha g_j x)\quad(1\leq j\leq s)$$

and

$$F(\alpha;B)=\prod_{j=1}^{s}f_j(\alpha;B).$$

To estimate $F(\alpha;B)$, we apply the circle method for $\mathbb{F}_q[t]$. Let $W=g_1V_1+\cdots+g_sV_s=g\tilde{U}$ and let $\Lambda\colon\mathbb{T}\to\hat{W}$ be defined as in Lemma 2. The set of *major arcs* $\mathfrak{M}$ is defined to be

$$\mathfrak{M}=\ker\Lambda.$$

Also, we denote by $\mathfrak{m}=\mathbb{T}\setminus\mathfrak{M}$ the complementary set of *minor arcs*. By Lemma 2(iv),

$$\operatorname{meas}\mathfrak{M}=|W|^{-1}=|\tilde{U}|^{-1}.$$

Let $\alpha\in\mathfrak{m}$. Since $g_jV_j=W$, it follows from Lemma 2(v) that, for each $j\in\{1,\dots,s\}$,

$$\sum_{y\in V_j}e(\alpha g_j y)=\sum_{x\in W}e(\alpha x)=0.\tag{12}$$

LEMMA 9 (Density Increment II). *For each $l\in\{1,\dots,s\}$ and $\alpha\in\mathbb{T}$, define*

$$\tilde{V}_l=h_l aV(\tilde{N};\tilde{\boldsymbol{\theta}};\tilde{\mathbf{m}}),$$

where

$$\tilde{\boldsymbol{\theta}}=(\boldsymbol{\theta},\{\alpha ga\})\quad\textit{and}\quad\tilde{\mathbf{m}}=(\tilde{\mathbf{n}},1).$$

(i) *We have $\tilde{V}_l\subseteq V_l$ and, for any $y\in\tilde{V}_l$, $e(\alpha g_l y)=1$. Also,*

$$|\tilde{V}_l|\geq q^{-1-n(d+1)}|U|,$$

$$\operatorname{length}(\tilde{\mathbf{m}})-\operatorname{length}(\tilde{\mathbf{n}})=1.$$

(ii) *Suppose there exist $j\in\{1,\dots,s\}$ and $\alpha\in\mathfrak{m}$ such that*

$$|f_j(\alpha;B)|\geq2(2s+1)\varepsilon.$$

Let $\tilde{\nu}_j$ be the probability measure on $\mathbb{A}$ defined by $\tilde{V}_j$. Then there exists a $z \in \mathbb{A}$ such that

$$\tilde{\nu}_j(B+z) \geq \delta_1 + \varepsilon.$$

Proof. (i) In view of the definition of V_l, by Lemma 4 we have

$$\tilde{V}_l \subseteq h_l a V(\tilde{N}; \boldsymbol{\theta}; \tilde{\mathbf{n}}) = V_l.$$

Let $y \in \tilde{V}_l$. Then there exists an $x \in V(\tilde{N}; \tilde{\boldsymbol{\theta}}; \tilde{\mathbf{m}})$ such that $y = h_l a x$. Since $g = g_l h_l$, we have

$$\operatorname{ord}\{\alpha g_l y\} = \operatorname{ord}\{\alpha g_l h_l a x\} = \operatorname{ord}\{\alpha g a x\} = \operatorname{ord}\{\{\alpha g a\} x\} < -1.$$

Thus, $e(\alpha g_l y) = 1$. When we combine Lemma 4(i) with (7), the remaining statement follows.

(ii) By part (i) and Lemma 7(iii), we have

$$f_j(\alpha; B) = \sum_{y \in B} \nu_j(y) e(\alpha g_j y) = \sum_{y \in \mathbb{A}} \nu_j(y) \tilde{\nu}_j(B-y) e(\alpha g_j y).$$

Since $\alpha \in \mathfrak{m}$, we see from (12) that

$$\sum_{y \in \mathbb{A}} \nu_j(y) e(\alpha g_j y) = \frac{1}{|V_j|} \sum_{y \in V_j} e(\alpha g_j y) = 0.$$

Combining the previous two equalities yields

$$f_j(\alpha; B) = \sum_{y \in \mathbb{A}} \nu_j(y)(\tilde{\nu}_j(B-y) - \nu_j(B)) e(\alpha g_j y),$$

which implies that

$$\sum_{y \in \mathbb{A}} \nu_j(y) |\tilde{\nu}_j(B-y) - \nu_j(B)| \geq 2(2s+1)\varepsilon.$$

By part (i) and Lemma 7(iii) we see that

$$\sum_{y \in \mathbb{A}} \nu_j(y) \tilde{\nu}_j(B-y) = \nu_j(B);$$

therefore,

$$\sum_{y \in \mathbb{A}} \nu_j(y)(\tilde{\nu}_j(B-y) - \nu_j(B)) = 0.$$

Combining these estimates establishes the existence of an $x \in \mathbb{A}$ such that

$$\tilde{\nu}_j(B-x) - \nu_j(B) \geq (2s+1)\varepsilon.$$

Let $z = -x$. Now, by (10), we have

$$\tilde{\nu}_j(B+z) \geq \nu_j(B) + (2s+1)\varepsilon \geq \delta_1 + \varepsilon.$$

This completes the proof of the lemma. $\square$

For a subset $\mathcal{W} \subseteq \{1, \ldots, s\}$, let $\mathcal{W}^c = \{1, \ldots, s\} \setminus \mathcal{W}$ be the complement of $\mathcal{W}$. Also, define

$$F(\alpha; B; \mathcal{W}) = \prod_{i \in \mathcal{W}} f_i(\alpha; B).$$

LEMMA 10. *Let $\mathcal{T} = \{j_1, j_2\} \subseteq \{1, \ldots, s\}$. Then*

$$\int_{\mathbb{T}} |F(\alpha; B; \mathcal{T})| \, d\alpha \le 2\delta_1 |\tilde{U}|^{-1}.$$

Proof. It follows from the Cauchy–Schwarz inequality that

$$\begin{aligned} \int_{\mathbb{T}} |F(\alpha; B; \mathcal{T})| \, d\alpha &= \int_{\mathbb{T}} |f_{j_1}(\alpha; B) f_{j_2}(\alpha; B)| \, d\alpha \\ &\le \left(\int_{\mathbb{T}} |f_{j_1}(\alpha; B)|^2 \, d\alpha \right)^{1/2} \left(\int_{\mathbb{T}} |f_{j_2}(\alpha; B)|^2 \, d\alpha \right)^{1/2}. \end{aligned}$$

For each $i \in \{1, 2\}$, by the definition of $f_{j_i}(\alpha; B)$ we have

$$\int_{\mathbb{T}} |f_{j_i}(\alpha; B)|^2 \, d\alpha = \sum_{x, y \in B} \nu_{j_i}(x) \nu_{j_i}(y) \int_{\mathbb{T}} e(\alpha g_{j_i}(x - y)) \, d\alpha = \sum_{x \in B} \nu_{j_i}(x)^2.$$

From (11) it follows that

$$\sum_{x \in B} \nu_{j_i}(x)^2 = |V_{j_i}|^{-1} \sum_{x \in B} \nu_{j_i}(x) = |\tilde{U}|^{-1} \sum_{x \in B} \nu_{j_i}(x) \le 2\delta_1 |\tilde{U}|^{-1}.$$

Thus,

$$\int_{\mathbb{T}} |F(\alpha; B; \mathcal{T})| \, d\alpha \le 2\delta_1 |\tilde{U}|^{-1}.$$

This completes the proof of the lemma. □

LEMMA 11. (i) *We have*

$$\int_{\mathfrak{m}} |F(\alpha; B)| \, d\alpha \ge 2^{-s-1} \delta_1^s |\tilde{U}|^{-1}.$$

(ii) *If*

$$\varepsilon = 2^{-2-\frac{4}{s-2}} (2s+1)^{-1} \delta_1^{1+\frac{1}{s-2}},$$

then there exists a $j \in \{1, \ldots, s\}$ such that

$$\sup_{\alpha \in \mathfrak{m}} |f_j(\alpha; B)| \ge 2(2s+1)\varepsilon.$$

Proof. (i) Let $\mathbf{x} = (x_1, \ldots, x_s) \in B^s$. We have

$$\begin{aligned} \int_{\mathbb{T}} F(\alpha; B) \, d\alpha &= \sum_{\mathbf{x} \in B^s} \prod_{j=1}^{s} \nu_j(x_j) \int_{\mathbb{T}} e(\alpha(g_1 x_1 + \cdots + g_s x_s)) \, d\alpha \\ &= \sum_{\substack{\mathbf{x} \in B^s \\ g_1 x_1 + \cdots + g_s x_s = 0}} \prod_{j=1}^{s} \nu_j(x_j). \end{aligned}$$

Recall that the equation $g_1x_1+\cdots+g_sx_s=0$ is never satisfied by distinct elements $x_1,\ldots,x_s\in B'$. Since $B=B'+y$ and $g_1+\cdots+g_s=0$, it follows that $g_1x_1+\cdots+g_sx_s=0$ is never satisfied by distinct elements $x_1,\ldots,x_s\in B$. By (11) and the definition of ν_j $(1\le j\le s)$ we know that

$$\begin{aligned}\sum_{\substack{\mathbf{x}\in B^s\\ g_1x_1+\cdots+g_sx_s=0}}\prod_{j=1}^{s}\nu_j(x_j) &\le \sum_{1\le j_1<j_2\le s}\ \sum_{\substack{\mathbf{x}\in B^s\\ g_1x_1+\cdots+g_sx_s=0\\ x_{j_1}=x_{j_2}}}\prod_{j=1}^{s}\nu_j(x_j)\\ &\le \sum_{1\le j_1<j_2\le s}|\tilde{U}|^{-2}\sum_{\substack{x_j\in B\\ j\ne j_1,j_2}}\ \prod_{\substack{j=1\\ j\ne j_1,j_2}}^{s}\nu_j(x_j)\\ &\le C_1 2^{s-2}\delta_1^{s-2}|\tilde{U}|^{-2},\end{aligned}$$

where $C_1=\binom{s}{2}$. Then

$$\int_{\mathbb{T}}F(\alpha;B)\,d\alpha\le 2^{s-2}C_1\delta_1^{s-2}|\tilde{U}|^{-2}.$$

For $\alpha\in\mathfrak{M}$, we have

$$F(\alpha;B)=\sum_{\mathbf{x}\in B^s}\left(\prod_{j=1}^{s}\nu_j(x_j)\right)e(\alpha(g_1x_1+\cdots+g_sx_s))=\prod_{j=1}^{s}\nu_j(B).$$

It follows from (11) that

$$\int_{\mathfrak{M}}F(\alpha;B)\,d\alpha=\left(\prod_{j=1}^{s}\nu_j(B)\right)\operatorname{meas}(\mathfrak{M})\ge 2^{-s}\delta_1^{s}|\tilde{U}|^{-1}.$$

By (8), we have

$$\delta_1^{-2}\le|\tilde{U}|^{1/2}\quad\text{and}\quad 2^{2s-1}C_1=C_0\le|\tilde{U}|^{1/2}.$$

Therefore,

$$2^{2s-2}C_1\delta_1^{-2}|\tilde{U}|^{-1}\le 2^{-1}.$$

Combining these three inequalities now yields

$$\int_{\mathfrak{m}}|F(\alpha;B)|\,d\alpha\ge\left|\int_{\mathfrak{M}}F(\alpha;B)\,d\alpha\right|-\left|\int_{\mathbb{T}}F(\alpha;B)\,d\alpha\right|\ge 2^{-s-1}\delta_1^{s}|\tilde{U}|^{-1}.$$

(ii) Let $\mathcal{T}=\{1,2\}$. By Lemma 10, we have

$$\int_{\mathbb{T}}|F(\alpha;B;\mathcal{T})|\,d\alpha\le 2\delta_1|\tilde{U}|^{-1}.$$

Also, by part (i) we have

$$\left(\sup_{\alpha\in\mathfrak{m}}|F(\alpha;B;\mathcal{T}^c)|\right)\int_{\mathbb{T}}|F(\alpha;B;\mathcal{T})|\,d\alpha\ge\int_{\mathfrak{m}}|F(\alpha;B)|\,d\alpha\ge 2^{-s-1}\delta_1^{s}|\tilde{U}|^{-1}.$$

After combining these two estimates, we see that

$$\sup_{\alpha\in\mathfrak{m}}|F(\alpha;B;\mathcal{T}^c)|\ge 2^{-s-2}\delta_1^{s-1}.$$

Since $|\mathcal{T}^c| = s-2$, there exists a $j \in \{1,\dots,s\}$ such that

$$\sup_{\alpha\in\mathfrak{m}} |f_j(\alpha; B)| \geq 2^{-1-\frac{4}{s-2}}\delta_1^{1+\frac{1}{s-2}}.$$

Since $\varepsilon = 2^{-2-\frac{4}{s-2}}(2s+1)^{-1}\delta_1^{1+\frac{1}{s-2}}$, it follows that

$$\sup_{\alpha\in\mathfrak{m}} |f_j(\alpha; B)| \geq 2(2s+1)\varepsilon.$$

This completes the proof of the lemma. □

In the rest of this section we assume that, for each $i \in \{1,\dots,s\}$,

$$\sup_{\alpha\in\mathfrak{m}} |f_i(\alpha; B)| < 2(2s+1)\varepsilon,$$

since the complementary case was treated in Lemma 9.

LEMMA 12. (i) *For each* $i \in \{1,\dots,s\}$, *define*

$$\mathfrak{m}'_i = \left\{\alpha\in\mathfrak{m} \;\middle|\; |f_i(\alpha; B)| \geq 2^{\frac{-s-3}{s-2}}\delta_1^{1+\frac{1}{s-2}}\right\}.$$

Then there exists a $j \in \{1,\dots,s\}$ *such that*

$$\int_{\mathfrak{m}'_j} |F(\alpha; B)|\,d\alpha \geq s^{-1}2^{-s-2}\delta_1^s|\tilde{U}|^{-1}.$$

(ii) *Let* j *and* $\mathfrak{m}'_j$ *be defined as in part* (i). *For* $\tau\in\mathbb{R}$ *with* $\tau > 0$, *define*

$$\mathfrak{m}_\tau = \{\alpha\in\mathfrak{m}'_j \mid \tfrac{1}{2}\tau\delta_1 < |f_j(\alpha; B)| < 2\tau\delta_1\}.$$

Then there exists a τ *with* $2^{\frac{-s-3}{s-2}}\delta_1^{\frac{1}{s-2}} \leq \tau \leq 4(2s+1)\varepsilon\delta_1^{-1}$ *such that*

$$\int_{\mathfrak{m}_\tau} |F(\alpha; B; \{j\}^c)|\,d\alpha \geq C_2\frac{\delta_1^{s-1}}{\tau(1+\log\delta_1^{-1})}|\tilde{U}|^{-1},$$

where $C_2 = s^{-1}2^{-s-3}(3s-1)^{-1}(s-2)$.

(iii) *Let* j *be chosen as in part* (i), *and let* $\mathfrak{m}_\tau$ *be defined as in part* (ii). *Then there exists a* $k \neq j$ *such that*

$$\int_{\mathfrak{m}_\tau} |f_k(\alpha; B)|^2\,d\alpha \geq 2^{-s+3}C_2\frac{\delta_1^2}{\tau(1+\log\delta_1^{-1})}|\tilde{U}|^{-1}.$$

Proof. (i) Let $\alpha\in\mathfrak{m}\setminus\bigcup_{i=1}^s\mathfrak{m}'_i$. Then, for each $i\in\{1,\dots,s\}$,

$$|f_i(\alpha; B)| < 2^{\frac{-s-3}{s-2}}\delta_1^{1+\frac{1}{s-2}}.$$

Let $\mathcal{T} = \{1,2\}$. By Lemma 10, we have

$$\int_{\mathbb{T}} |F(\alpha; B; \mathcal{T})|\,d\alpha \leq 2\delta_1|\tilde{U}|^{-1}.$$

Thus,

$$\int_{\mathfrak{m}\setminus\bigcup_{i=1}^s\mathfrak{m}'_i} |F(\alpha; B)|\,d\alpha < 2^{-s-2}\delta_1^s|\tilde{U}|^{-1}.$$

Now combining this estimate with Lemma 11(i) yields

$$\int_{\bigcup_{i=1}^{s}\mathfrak{m}'_i} |F(\alpha;\, B)|\, d\alpha > 2^{-s-2}\delta_1^s|\tilde{U}|^{-1}.$$

Hence there exists a $j \in \{1, \ldots, s\}$ such that

$$\int_{\mathfrak{m}'_j} |F(\alpha;\, B)|\, d\alpha \geq s^{-1}2^{-s-2}\delta_1^s|\tilde{U}|^{-1}.$$

(ii) Let $\tau_0 = 2^{\frac{-s-3}{s-2}}\delta_1^{\frac{1}{s-2}}$. For each $n \in \mathbb{N}$, let $\tau_n = 2\tau_{n-1}$. Then

$$\bigcup_{i=0}^{n}\left(\frac{1}{2}\tau_i\delta_1, 2\tau_i\delta_1\right) = \left(\frac{1}{2}\tau_0\delta_1, 2\tau_n\delta_1\right).$$

Let $n = [\log_2 \tau_0^{-1}] + 1$, where $[x]$ denotes the greatest integer not exceeding a real number x. Since $\tau_n = 2^n\tau_0$, we see from (9) that

$$2\tau_n\delta_1 = 2^{n+1}\tau_0\delta_1 > 2\delta_1 > 2(2s+1)\varepsilon.$$

For each $\alpha \in \mathfrak{m}'_j$ we have $2^{\frac{-s-3}{s-2}}\delta_1^{1+\frac{1}{s-2}} \leq |f_j(\alpha;\, B)| < 2(2s+1)\varepsilon$, so it follows that

$$\left(2^{\frac{-s-3}{s-2}}\delta_1^{1+\frac{1}{s-2}}, 2(2s+1)\varepsilon\right) \subseteq \left(\frac{1}{2}\tau_0\delta_1, 2\tau_n\delta_1\right).$$

Therefore,

$$\mathfrak{m}'_j \subseteq \bigcup_{i=0}^{n}\mathfrak{m}_{\tau_i}.$$

Suppose that, for each τ_i $(0 \leq i \leq n)$,

$$\int_{\mathfrak{m}_{\tau_i}} |F(\alpha;\, B;\, \{j\}^c)|\, d\alpha < C_2\frac{\delta_1^{s-1}}{\tau_i(1+\log\delta_1^{-1})}|\tilde{U}|^{-1};$$

here $C_2 = s^{-1}2^{-s-3}(3s-1)^{-1}(s-2)$. Since

$$n+1 \leq -\log_2\tau_0 + 2 < (3s-1)(s-2)^{-1}(1+\log\delta_1^{-1}),$$

it follows that

$$\begin{aligned}
\int_{\mathfrak{m}'_j} |F(\alpha;\, B)|\, d\alpha &\leq 2\sum_{i=0}^{n}\tau_i\delta_1\int_{\mathfrak{m}_{\tau_i}}|F(\alpha;\, B;\, \{j\}^c)|\, d\alpha \\
&< 2\sum_{i=0}^{n}\tau_i\delta_1 C_2\frac{\delta_1^{s-1}}{\tau_i(1+\log\delta_1^{-1})}|\tilde{U}|^{-1} \\
&\leq 2C_2(3s-1)(s-2)^{-1}(1+\log\delta_1^{-1})\frac{\delta_1^s}{1+\log\delta_1^{-1}}|\tilde{U}|^{-1} \\
&= s^{-1}2^{-s-2}\delta_1^s|\tilde{U}|^{-1},
\end{aligned}$$

which contradicts part (i). Then there exists a $\tau = \tau_i$ for some $0 \leq i \leq n$ such that

$$\int_{\mathfrak{m}_\tau} |F(\alpha;\, B;\, \{j\}^c)|\, d\alpha \geq C_2\frac{\delta_1^{s-1}}{\tau(1+\log\delta_1^{-1})}|\tilde{U}|^{-1}.$$

By the definition of τ_i, we have $\tau \geq \tau_0 = 2^{\frac{-s-3}{s-2}} \delta_1^{\frac{1}{s-2}}$. In addition, $\emptyset \neq \mathfrak{m}_\tau \subseteq \mathfrak{m}'_j$. For $\alpha \in \mathfrak{m}_\tau$, since $\frac{1}{2}\tau\delta_1 < |f_j(\alpha; B)| < 2(2s+1)\varepsilon$ it follows that $\tau < 4(2s+1)\varepsilon\delta_1^{-1}$.

(iii) Let $\{j_1, j_2\} \subseteq \{j\}^c$. Then, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\int_{\mathfrak{m}_\tau} &|f_{j_1}(\alpha; B)||f_{j_2}(\alpha; B)|\, d\alpha \\ &\leq \left(\int_{\mathfrak{m}_\tau} |f_{j_1}(\alpha; B)|^2\, d\alpha\right)^{1/2} \left(\int_{\mathfrak{m}_\tau} |f_{j_2}(\alpha; B)|^2\, d\alpha\right)^{1/2}.\end{aligned}$$

Suppose that

$$\int_{\mathfrak{m}_\tau} |f_{j_1}(\alpha; B)|^2\, d\alpha \geq \int_{\mathfrak{m}_\tau} |f_{j_2}(\alpha; B)|^2\, d\alpha.$$

Let $k = j_1$. By (11), we have

$$\begin{aligned}\int_{\mathfrak{m}_\tau} |F(\alpha; B; \{j\}^c)|\, d\alpha &\leq (2\delta_1)^{s-3} \int_{\mathfrak{m}_\tau} |f_{j_1}(\alpha; B)||f_{j_2}(\alpha; B)|\, d\alpha \\ &\leq (2\delta_1)^{s-3} \int_{\mathfrak{m}_\tau} |f_k(\alpha; B)|^2\, d\alpha.\end{aligned}$$

We saw in part (ii) that

$$\int_{\mathfrak{m}_\tau} |F(\alpha; B; \{j\}^c)|\, d\alpha \geq C_2 \frac{\delta_1^{s-1}}{\tau(1+\log \delta_1^{-1})} |\tilde{U}|^{-1}.$$

Combining these two estimates results in

$$\int_{\mathfrak{m}_\tau} |f_k(\alpha; B)|^2\, d\alpha \geq 2^{-s+3} C_2 \frac{\delta_1^2}{\tau(1+\log \delta_1^{-1})} |\tilde{U}|^{-1}.$$

This completes the proof of the lemma. □

For the rest of this section, let $j, k \in \{1, \ldots, s\}$ and let $\mathfrak{m}_\tau$ satisfy Lemma 12. In view of the definitions of $\mathfrak{M}$ and $\mathfrak{m}_\tau$, we may assume that there exists a $Q_1 \in \mathbb{N}$ such that

$$\mathfrak{m}_\tau = \bigsqcup_{i=1}^{Q_1} (\beta_i + \mathfrak{M}).$$

Write $\bar{\beta}_i = \beta_i + \mathfrak{M}$ $(1 \leq i \leq Q_1)$. Since $\mathbb{T}/\mathfrak{M}$ is an $\mathbb{F}_q$-vector space, we assume without loss of generality that $\{\bar{\beta}_1, \ldots, \bar{\beta}_Q\}$ is a maximal linearly independent subset of $\{\bar{\beta}_1, \ldots, \bar{\beta}_{Q_1}\}$.

LEMMA 13 (Density Increment III). *For $l \in \{1, \ldots, s\}$, define*

$$V'_l = h_l a V(\tilde{N}; \boldsymbol{\theta}'; \mathbf{m}'),$$

where

$$\boldsymbol{\theta}' = \boldsymbol{\theta} \cup \{\{\beta_i g a\} \mid 1 \leq i \leq Q\} \quad \textit{and} \quad \mathbf{m}' = (\tilde{\mathbf{n}}, \underbrace{1, \ldots, 1}_{Q\ copies}).$$

(i) *We have* $V_l' \subseteq V_l$ *and, for each* $y \in V_l'$, $e(\beta_i g_l y) = 1$ $(1 \le i \le Q)$. *Furthermore,*

$$|V_l'| \ge q^{-Q-n(d+1)}|U|,$$

$$\text{length}(\mathbf{m}') - \text{length}(\tilde{\mathbf{n}}) = Q.$$

(ii) *Let* ν_k' *be the probability measure on* $\mathbb{A}$ *defined by* V_k'. *Then there exists a* $z \in \mathbb{A}$ *such that*

$$\nu_k'(B+z) \ge 2^{-s+2} C_2 \frac{\delta_1}{\tau(1+\log \delta_1^{-1})},$$

where C_2 *is defined as in Lemma 12(ii).*

Proof. (i) The result follows from a similar argument as in Lemma 9(i).

(ii) Fix $\alpha \in \mathfrak{m}_\tau$ and write $\bar{\alpha} = \alpha + \mathfrak{M}$. Then there exist $b_1, \ldots, b_Q \in \mathbb{F}_q$ such that

$$\bar{\alpha} = b_1 \bar{\beta}_1 + \cdots + b_Q \bar{\beta}_Q.$$

Let $y \in V_k'$. Since $g_k V_k' \subseteq g_k V_k$, it follows from the definition of $\mathfrak{M}$ that

$$e(\alpha g_k y) = e((b_1 \beta_1 + \cdots + b_Q \beta_Q) g_k y).$$

Note that $b_i y \in V_k'$ $(1 \le i \le Q)$. By part (i) of the lemma, we see that

$$e(\alpha g_k y) = \prod_{i=1}^{Q} e(\beta_i g_k b_i y) = 1.$$

By (4) we deduce that, for each $\alpha \in \mathfrak{m}_\tau$,

$$f_k(\alpha; B) = \sum_{y \in \mathbb{A}} \nu_k(y) \nu_k'(B-y) e(\alpha g_k y).$$

Hence

$$\begin{aligned} \int_{\mathbb{T}} \left| \sum_{y \in \mathbb{A}} \nu_k(y) \nu_k'(B-y) e(\alpha g_k y) \right|^2 d\alpha &\ge \int_{\mathfrak{m}_\tau} \left| \sum_{y \in \mathbb{A}} \nu_k(y) \nu_k'(B-y) e(\alpha g_k y) \right|^2 d\alpha \\ &= \int_{\mathfrak{m}_\tau} |f_k(\alpha; B)|^2 \, d\alpha. \end{aligned}$$

Since

$$\int_{\mathbb{T}} \left| \sum_{y \in \mathbb{A}} \nu_k(y) \nu_k'(B-y) e(\alpha g_k y) \right|^2 d\alpha = \sum_{y \in \mathbb{A}} \nu_k(y)^2 \nu_k'(B-y)^2,$$

it follows from the foregoing estimates and Lemma 12(iii) that

$$\sum_{y \in \mathbb{A}} \nu_k(y)^2 \nu_k'(B-y)^2 \ge \int_{\mathfrak{m}_\tau} |f_k(\alpha; B)|^2 \, d\alpha \ge 2^{-s+3} C_2 \frac{\delta_1^2}{\tau(1+\log \delta_1^{-1})} |\tilde{U}|^{-1}.$$

Therefore,

$$\left(\max_{y \in \mathbb{A}} \nu_k'(B-y) \right) \cdot |V_k|^{-1} \cdot \sum_{y \in \mathbb{A}} \nu_k(y) \nu_k'(B-y) \ge 2^{-s+3} C_2 \frac{\delta_1^2}{\tau(1+\log \delta_1^{-1})} |\tilde{U}|^{-1}.$$

Note that $|V_k| = |\tilde{U}|$. Combining (5) and (11) with part (i) now yields that

$$\sum_{y\in\mathbb{A}} \nu_k(y)\nu'_k(B-y) = \nu_k(B) < 2\delta_1.$$

Hence there exists a $z \in \mathbb{A}$ such that

$$\nu'_k(B+z) = \max_{y\in\mathbb{A}} \nu'_k(B-y) \geq 2^{-s+2} C_2 \frac{\delta_1}{\tau(1+\log\delta_1^{-1})}.$$

This completes the proof of the lemma. □

LEMMA 14. *Let* $\emptyset \neq \mathcal{I} \subseteq \{1,\dots,Q\}$. *For each* $i \in \mathcal{I}$, *let* $a_i \in \mathbb{C}$ *and define* $\varphi_i : \mathbb{A} \to \mathbb{R}$ *by*

$$\varphi_i(x) = \mathrm{Re}(a_i e(\beta_i g_j x)),$$

where $\mathrm{Re}(z)$ *denotes the real part of* $z \in \mathbb{C}$. *Define* $\Omega = \Omega(\mathcal{I}; \mathbf{a}) : \mathbb{A} \to \mathbb{R}$ *by*

$$\Omega(x) = \prod_{i\in\mathcal{I}} (1+\varphi_i(x)).$$

Then we have

$$\sum_{\mathbf{x}\in\mathbb{A}} \nu_j(x)\Omega(x) = 1.$$

Proof. For a complex number z, we denote by $\bar{z}$ its conjugate. Note that

$$\begin{aligned}\Omega(x) &= \prod_{i\in\mathcal{I}} (1+\varphi_i(x)) \\ &= \prod_{i\in\mathcal{I}} \left(1 + \frac{1}{2} a_i e(\beta_i g_j x) + \frac{1}{2}\overline{a_i e(\beta_i g_j x)}\right) \\ &= 1 + \sum_{1\leq h\leq|\mathcal{I}|} \left(\frac{1}{2}\right)^h \sum_{\substack{j_1,\dots,j_h\in\mathcal{I} \\ j_1<\dots<j_h \\ \eta_1,\dots,\eta_h=\pm1}} \left(\prod_{l=1}^h b_{j_l}\right) e((\eta_1\beta_{j_1}+\cdots+\eta_h\beta_{j_h})g_j x),\end{aligned}$$

where

$$b_{j_l} = \begin{cases} a_{j_l} & \text{if } \eta_l = 1, \\ \bar{a}_{j_l} & \text{if } \eta_l = -1. \end{cases}$$

Therefore,

$$\begin{aligned}&\sum_{x\in\mathbb{A}} \nu_j(x)\Omega(x) \\ &\quad= \sum_{x\in\mathbb{A}} \nu_j(x) \\ &\qquad + \sum_{x\in\mathbb{A}} \nu_j(x) \sum_{1\leq h\leq|\mathcal{I}|} \left(\frac{1}{2}\right)^h \sum_{\substack{j_1,\dots,j_h\in\mathcal{I} \\ j_1<\dots<j_h \\ \eta_1,\dots,\eta_h=\pm1}} \left(\prod_{l=1}^h b_{j_l}\right) e((\eta_1\beta_{j_1}+\cdots+\eta_h\beta_{j_h})g_j x) \\ &\quad= 1 + \sum_{1\leq h\leq|\mathcal{I}|} \left(\frac{1}{2}\right)^h \sum_{\substack{j_1,\dots,j_h\in\mathcal{I} \\ j_1<\dots<j_h \\ \eta_1,\dots,\eta_h=\pm1}} \left(\prod_{l=1}^h b_{j_l}\right) \sum_{x\in\mathbb{A}} \nu_j(x) e((\eta_1\beta_{j_1}+\cdots+\eta_h\beta_{j_h})g_j x).\end{aligned}$$

Since $\{j_1, \ldots, j_h\} \subseteq \{1, \ldots, Q\}$, it follows that $\bar{\beta}_{j_1}, \ldots, \bar{\beta}_{j_h}$ are $\mathbb{F}_q$-linearly independent. Thus, $\eta_1\beta_{j_1} + \cdots + \eta_h\beta_{j_h} \in \mathfrak{m}$. On recalling (12), we have

$$\sum_{x \in \mathbb{A}} \nu_j(x) e((\eta_1\beta_{j_1} + \cdots + \eta_h\beta_{j_h}) g_j x) = 0;$$

hence

$$\sum_{x \in \mathbb{A}} \nu_j(x)\Omega(x) = 1.$$

This completes the proof of the lemma. $\square$

LEMMA 15 (Density Increment IV). (i) *Let τ be defined as in Lemma 12(ii), and let $\gamma = \tau/20$. For each $i \in \{1, \ldots, Q\}$, there exists a $b_i \in \mathbb{C}$ with $|b_i| = \frac{1}{4}$ such that*

$$\mathrm{Re}(b_i f_j(\beta_i; B)) > \gamma\delta_1.$$

For $x \in \mathbb{R}$, define

$$\varphi_i(x) = \mathrm{Re}(b_i e(\beta_i g_j x)) \quad (1 \le i \le Q).$$

Then there exist absolute constants $C_3, C_4 > 0$ such that, whenever

$$Q > \frac{10 s\varepsilon \log \delta_1^{-1}}{C_3 \gamma^2 \delta_1},$$

there is a subset $\mathcal{I}_0 \subseteq \{1, \ldots, Q\}$ satisfying

$$|\mathcal{I}_0| \le \left(400 C_3^{-1} C_4 s 2^{\frac{s+3}{s-2}}\right)\varepsilon\delta^{-1-\frac{1}{s-2}} \quad \textit{and} \quad \sum_{x \in B} \nu_j(x) \prod_{i \in \mathcal{I}_0} (1 + \varphi_i(x)) > \delta_1 + \varepsilon.$$

(ii) *Suppose that Q and $\mathcal{I}_0$ satisfy the described properties, and let $\sigma = |\mathcal{I}_0|$. For each $l \in \{1, \ldots, s\}$, define*

$$\bar{V}_l = h_l a V(\tilde{N}; \bar{\boldsymbol{\theta}}; \bar{\mathbf{m}}),$$

where

$$\bar{\boldsymbol{\theta}} = \boldsymbol{\theta} \cup \{\{\beta_i g a\} \mid i \in \mathcal{I}_0\} \quad \textit{and} \quad \bar{\mathbf{m}} = (\tilde{\mathbf{n}}, \underbrace{1, \ldots, 1}_{\sigma \text{ copies}}).$$

Then $\bar{V}_l \subseteq V_l$ and, for any $y \in \bar{V}_l$, $e(\beta_i g_l y) = 1$ $(i \in \mathcal{I}_0)$. Furthermore,

$$|\bar{V}_l| \ge q^{-\sigma - n(d+1)}|U|,$$

$$\mathrm{length}(\bar{\mathbf{m}}) - \mathrm{length}(\tilde{\mathbf{n}}) = \sigma.$$

(iii) *Let $\bar{V}_j$ be defined as in part (ii), and let $\bar{\nu}_j$ be the probability measure on $\mathbb{A}$ defined by $\bar{V}_j$. Then there is a $z \in \mathbb{A}$ such that*

$$\bar{\nu}_j(B + z) > \delta_1 + \varepsilon.$$

Proof. (i) Fix i with $1 \le i \le Q$. Since $\beta_i \in \mathfrak{m}_\tau$, we have $\frac{1}{2}\tau\delta_1 < |f_j(\beta_i; B)| < 2\tau\delta_1$. Then either $|\mathrm{Re}(f_j(\beta_i; B))| > \frac{1}{4}\tau\delta_1$ or $|\mathrm{Im}(f_j(\beta_i; B))| > \frac{1}{4}\tau\delta_1$, where $\mathrm{Im}(z)$ is the imaginary part of $z \in \mathbb{C}$. Hence there exists a $b_i \in \mathbb{C}$ with $|b_i| = \frac{1}{4}$ such that $\mathrm{Re}(b_i f_j(\beta_i; B)) > \gamma\delta_1$ with $\gamma = \frac{1}{20}\tau$. Then $|\varphi_i(x)| < \frac{1}{2}$ and

$$\sum_{x\in B}\varphi_i(x)\nu_j(x)=\mathrm{Re}\left(b_i\left(\sum_{x\in B}\nu_j(x)e(\beta_i g_j x)\right)\right)=\mathrm{Re}(b_i f_j(\beta_i;B))>\gamma\delta_1.$$

For each $\emptyset\neq\mathcal{I}\subseteq\{1,\ldots,Q\}$, we deduce from Lemma 14 that

$$\sum_{x\in\mathbb{A}}\nu_j(x)\prod_{i\in\mathcal{I}}(1+\varphi_i(x))=1.$$

By [3, Prop. (∗)] there exist absolute constants $C_3, C_4>0$ such that, for each $\emptyset\neq\mathcal{I}\subseteq\{1,\ldots,Q\}$, we can identify a subset $\mathcal{I}_0\subseteq\mathcal{I}$ satisfying

$$\frac{C_3\gamma|\mathcal{I}|}{\log\delta_1^{-1}}\leq|\mathcal{I}_0|\leq\frac{C_4\gamma|\mathcal{I}|}{\log\delta_1^{-1}}$$

and

$$\sum_{x\in B}\nu_j(x)\prod_{i\in\mathcal{I}_0}(1+\varphi_i(x))>\left(1+\frac{\gamma|\mathcal{I}_0|}{2}\right)\nu_j(B).$$

Suppose that $Q>\frac{10s\varepsilon\log\delta_1^{-1}}{C_3\gamma^2\delta_1}$. We now take a subset $\mathcal{I}\subseteq\{1,\ldots,Q\}$ such that

$$\frac{10s\varepsilon\log\delta_1^{-1}}{C_3\gamma^2\delta_1}\leq|\mathcal{I}|\leq\frac{20s\varepsilon\log\delta_1^{-1}}{C_3\gamma^2\delta_1}.$$

Then there exists a subset $\mathcal{I}_0\subseteq\mathcal{I}$ satisfying

$$\frac{10s\varepsilon}{\gamma\delta_1}\leq|\mathcal{I}_0|\leq\frac{20C_4s\varepsilon}{C_3\gamma\delta_1}$$

and

$$\sum_{x\in B}\nu_j(x)\prod_{i\in\mathcal{I}_0}(1+\varphi_i(x))>\left(1+\frac{\gamma|\mathcal{I}_0|}{2}\right)\nu_j(B)\geq(1+5s\varepsilon\delta_1^{-1})\nu_j(B).$$

By (9) and (10) it follows that $\nu_j(B)\geq\delta_1-2s\varepsilon$ and $0<s\varepsilon\delta_1^{-1}<\frac{1}{4}$. Thus,

$$(1+5s\varepsilon\delta_1^{-1})\nu_j(B)\geq(1+5s\varepsilon\delta_1^{-1})(\delta_1-2s\varepsilon)>\delta_1+\varepsilon.$$

Also, since $\gamma=\frac{1}{20}\tau$ and $\tau\geq 2^{\frac{-s-3}{s-2}}\delta_1^{\frac{1}{s-2}}$ we have

$$|\mathcal{I}_0|\leq\frac{20C_4s\varepsilon}{C_3\gamma\delta_1}=\frac{400C_4s\varepsilon}{C_3\tau\delta_1}\leq(400C_3^{-1}C_4s2^{\frac{s+3}{s-2}})\varepsilon\delta_1^{-1-\frac{1}{s-2}}.$$

(ii) This part can be proved using a similar argument as for Lemma 9(i).

(iii) Write $\Omega(x)=\prod_{i\in\mathcal{I}_0}(1+\varphi_i(x))$. Since $e(\beta_i g_j y)=1$ ($i\in\mathcal{I}_0$) for any $y\in\bar{V}_j$, we find that, for any $x\in\mathbb{A}$,

$$\Omega(x)=\Omega(x-y).$$

Then, by part (ii) and Lemma 7(ii),

$$\begin{aligned}\sum_{x\in B}\nu_j(x)\Omega(x)&=\sum_{x\in B}\left(\sum_{y\in\mathbb{A}}\nu_j(x-y)\bar{\nu}_j(y)\right)\Omega(x-y)\\&=\sum_{w\in\mathbb{A}}\sum_{x\in B}\nu_j(w)\bar{\nu}_j(x-w)\Omega(w)\\&=\sum_{w\in\mathbb{A}}\nu_j(w)\Omega(w)\bar{\nu}_j(B-w).\end{aligned}$$

It now follows from Lemma 14 that

$$\sum_{w\in\mathbb{A}} \nu_j(w)\Omega(w) = 1.$$

Note that, for any $w \in \mathbb{A}$, we have $\nu_j(w) \geq 0$, $\Omega(w) \geq 0$, and $\bar{\nu}_j(B-w) \geq 0$. Thus, by the two previous equalities,

$$\sum_{x\in B} \nu_j(x)\Omega(x) \leq \max_{w\in\mathbb{A}} \bar{\nu}_j(B-w) = \max_{z\in\mathbb{A}} \bar{\nu}_j(B+z).$$

Using part (i), we obtain

$$\sum_{x\in B} \nu_j(x)\Omega(x) > \delta_1 + \varepsilon.$$

Combining these two inequalities reveals that there is a $z \in \mathbb{A}$ such that

$$\bar{\nu}_j(B+z) > \delta_1 + \varepsilon.$$

This completes the proof of the lemma. □

4. Summary

Let $U = aV(N;\boldsymbol{\theta};\mathbf{n})$, n, d, h_j, V_j, ν_j $(1 \leq j \leq s)$, B', and δ_1 be defined as in the beginning of Section 3. Recall that

$$\log|U| - (\log q)n(d+1) \geq 2\log\delta_1^{-2} + 2\log C_0,$$

where $C_0 = 2^{2s-1}\binom{s}{2}$. In the following steps, we summarize the density increments established in Section 3. For the balance of this section, all implicit constants depend only on $\mathbf{g} = (g_1,\ldots,g_s)$ and q.

Step 1. Suppose that, for each $y \in \mathbb{A}$, there exists a $i = i(y) \in \{1,\ldots,s\}$ such that

$$|\nu_i(B'+y) - \delta_1| \geq 2s\varepsilon.$$

By Lemmas 6 and 8, there exist $i_0 \in \{1,\ldots,s\}$, $V_{i_0} = h_{i_0}aV(\tilde{N};\boldsymbol{\theta};\tilde{\mathbf{n}})$, and $z_0 \in \mathbb{A}$ such that

$$\begin{gathered} \nu_{i_0}(B'+z_0) \geq \delta_1 + \varepsilon, \\ \tilde{d} = \text{length}(\tilde{\mathbf{n}}), \quad \tilde{d} - d = 0, \\ |V_{i_0}| \geq q^{-n(d+1)}|U|. \end{gathered} \tag{13}$$

Step 2. Suppose there exists a $y \in \mathbb{A}$ such that

$$|\nu_i(B'+y) - \delta_1| < 2s\varepsilon \quad (1 \leq i \leq s).$$

Write $B = B' + y$, and let $f_i(\alpha; B)$ $(1 \leq i \leq s)$ be defined as in Section 3.

Step 2.1. Suppose there exist an $i_1 \in \{1,\ldots,s\}$ and an $\alpha \in \mathfrak{m}$ such that

$$|f_{i_1}(\alpha; B)| \geq 2(2s+1)\varepsilon.$$

By Lemma 9, there exist $\tilde{V}_{i_1} = h_{i_1}aV(\tilde{N};\tilde{\boldsymbol{\theta}};\tilde{\mathbf{m}})$ and $z_1 \in \mathbb{A}$ such that

$$\begin{gathered}\tilde{\nu}_{i_1}(B+z_1) \ge \delta_1+\varepsilon,\\ \tilde{d} = \text{length}(\tilde{\mathbf{m}}), \quad \tilde{d}-d=1,\\ |\tilde{V}_{i_1}| \ge q^{-1-n(d+1)}|U|.\end{gathered} \tag{14}$$

Step 2.2. Suppose that

$$\sup_{\alpha\in\mathfrak{m}}|f_i(\alpha;B)| < 2(2s+1)\varepsilon \quad (1\le i\le s).$$

Step 2.2.1. By Lemmas 12 and 13, there exist $i_2 \in \{1,\dots,s\}$,

$$V'_{i_2} = h_{i_2}aV(\tilde{N};\boldsymbol{\theta}';\mathbf{n}'),$$

and $z_2 \in \mathbb{A}$ such that, for some τ with $\delta_1^{\frac{1}{s-2}} \ll \tau \ll \varepsilon\delta_1^{-1}$,

$$\begin{gathered}\nu'_{i_2}(B+z_2) \gg \frac{\delta_1}{\tau(1+\log\delta_1^{-1})},\\ d' = \text{length}(\mathbf{n}'), \quad d'-d=Q,\\ |V'_{i_2}| \ge q^{-Q-n(d+1)}|U|.\end{gathered} \tag{15}$$

Step 2.2.2. Let Q be defined as in the paragraph before Lemma 12 (the same Q as the one in (15)). Suppose that it satisfies the condition of Lemma 15(i)—that is,

$$Q \gg \frac{\varepsilon\log\delta_1^{-1}}{\tau^2\delta_1}.$$

By parts (ii) and (iii) of Lemma 15, there exist $i_3 \in \{1,\dots,s\}$,

$$\bar{V}_{i_3} = h_{i_3}aV(\tilde{N};\bar{\boldsymbol{\theta}};\bar{\mathbf{m}}),$$

and $z_3 \in \mathbb{A}$ such that

$$\begin{gathered}\bar{\nu}_{i_3}(B+z_3) > \delta_1+\varepsilon,\\ \bar{d} = \text{length}(\bar{\mathbf{m}}), \quad 0<\bar{d}-d \ll \varepsilon\delta_1^{-1-\frac{1}{s-2}},\\ |\bar{V}_{i_3}| \ge q^{-(\bar{d}-d)-n(d+1)}|U|.\end{gathered} \tag{16}$$

REMARK 2. When

$$\varepsilon = 2^{-2-\frac{4}{s-2}}(2s+1)^{-1}\delta_1^{1+\frac{1}{s-2}},$$

it follows from Lemma 11(ii) that there exists a $j \in \{1,\dots,s\}$ such that

$$\sup_{\alpha\in\mathfrak{m}}|f_j(\alpha;B)| \ge 2(2s+1)\varepsilon.$$

Then applying (13) and (14) suffices to increase the density.

5. Proof of Theorem 1

The goal of this section is to prove the following theorem, which is a generalization of Theorem 1.

THEOREM 1*. *For $s\in\mathbb{N}$ with $s\ge 3$, let $\mathbf{g}=(g_1,\dots,g_s)$ with $g_i\in\mathbb{F}_q[t]\setminus\{0\}$ $(1\le i\le s)$ and $g_1+\dots+g_s=0$. Let $U=aV(N;\boldsymbol{\theta};\mathbf{n})$ with* length$(\mathbf{n}) \le M$, *where $M = M(\mathbf{g}; q) > 0$. Define $D_{\mathbf{g}}(U)$ to be the maximal cardinality of a subset $A \subseteq U$ for which the equation*

$$g_1x_1 + \cdots + g_sx_s = 0 \tag{17}$$

is never satisfied by distinct elements $x_1, \ldots, x_s \in A$. Then there exists a constant $C = C(\mathbf{g}; q; M) > 0$ such that

$$D_{\mathbf{g}}(U) \leq C|U|\left(\frac{(\log\log|U|)^2}{\log|U|}\right)^{\frac{2(s-2)^2}{4s-9}}. \tag{18}$$

REMARK 3. According to Remark 1 (see Section 2), $\mathcal{S}_N$ is a Bohr space of length 0. Therefore, Theorem 1 follows directly from Theorem 1*.

LEMMA 16. *Let $u, v \in \mathbb{R}$ with $u, v > 0$. Suppose that a sequence $\{\Delta_i\}_{i\in\mathbb{N}\cup\{0\}}$ satisfies*

$$0 < \Delta_0 < 1 \quad \text{and} \quad \Delta_i \geq \Delta_{i-1} + v\Delta_{i-1}^{1+u} (i \in \mathbb{N}).$$

Let $m = \min\{i \in \mathbb{N} \mid \Delta_i > 1\}$. Then there is a constant $E = E(u, v) > 0$ such that

$$m \leq E\Delta_0^{-u}.$$

Proof. Let $L_1 = [(v\Delta_0^u)^{-1}] + 1$. For each $i \in \mathbb{N}$ let $L_{i+1} = L_i + l_i$, where $l_i = [(v\Delta_{L_i}^u)^{-1}] + 1$. Since for each $j \in \mathbb{N}$ we have

$$\Delta_j \geq \Delta_{j-1} + v\Delta_{j-1}^{1+u} \geq \Delta_{j-1} + v\Delta_0^{1+u},$$

it follows that

$$\Delta_{L_1} \geq \Delta_0 + L_1 \cdot (v\Delta_0^{1+u}) > 2\Delta_0.$$

Note that if

$$\Delta_{L_i} > 2^i\Delta_0$$

for $i \in \mathbb{N}$, then

$$\Delta_{L_{i+1}} \geq \Delta_{L_i} + l_i \cdot (v\Delta_{L_i}^{1+u}) > 2\Delta_{L_i} > 2^{i+1}\Delta_0.$$

Thus, by induction, for each $i \in \mathbb{N}$ we have

$$\Delta_{L_i} > 2^i\Delta_0.$$

Take $r = [\log_2 \Delta_0^{-1}] + 1$. Then $2^r\Delta_0 > 1$ and so $\Delta_{L_r} > 1$. Thus,

$$\begin{aligned}
m \leq L_r &= L_1 + l_1 + \cdots + l_{r-1} \\
&\leq (v\Delta_0^u)^{-1} + (v\Delta_{L_1}^u)^{-1} + \cdots + (v\Delta_{L_{r-1}}^u)^{-1} + r \\
&< (v\Delta_0^u)^{-1} + (v \cdot 2^u \cdot \Delta_0^u)^{-1} + \cdots + (v \cdot 2^{u(r-1)}\Delta_0^u)^{-1} + r \\
&< \Delta_0^{-u}v^{-1}\sum_{i=0}^{\infty}(2^{-u})^i + r.
\end{aligned}$$

Observe that $r \leq \log_2 \Delta_0^{-1} + 1 < u^{-1}\Delta_0^{-u} + 1$. Let $E = v^{-1}(1 - 2^{-u})^{-1} + u^{-1} + 1$. Then $m \leq E\Delta_0^{-u}$, which completes the proof of the lemma. □

Proof of Theorem 1.* Write $n = \operatorname{ord}(g_1 \cdots g_s)$. Let $A \subseteq U = aV(N; \boldsymbol{\theta}; \mathbf{n})$ with $N > n$, and let $\delta = |A|/|U|$. Suppose that (17) is never satisfied by distinct elements $x_1, \dots, x_s \in A$. Let $u = \frac{1}{s-2} - \frac{1}{2(s-2)^2}$, and let $v = 2^{-2-\frac{4}{s-2}}(2s+1)^{-1}$. Write $\delta_0 = \delta$, $N_0 = N$, $\boldsymbol{\theta}_0 = \boldsymbol{\theta}$, $\mathbf{n}_0 = \mathbf{n}$, and $d_0 = \operatorname{length}(\mathbf{n}_0)$. Suppose that

$$\log|U| - (\log q)n(d_0+1) < 2\log\delta^{-2} + 2\log C_0,$$

where $C_0 = 2^{2s-1}\binom{s}{2}$. Then

$$\delta < C_0^{1/2} q^{n(d_0+1)/4} |U|^{-1/4},$$

whence there exists an $E_0 = E_0(\mathbf{g}; q; M) > 0$ such that

$$\delta < E_0\left(\frac{(\log\log|U|)^2}{\log|U|}\right)^{\frac{2(s-2)^2}{4s-9}}. \tag{19}$$

It remains to consider the case where

$$\log|U| - (\log q)n(d_0+1) \geq 2\log\delta^{-2} + 2\log C_0.$$

We now increase the density of A in Bohr spaces by repeatedly applying the process described in Section 4. For each $i \in \mathbb{N}$, write δ_i for the density obtained at the ith step and ε_i for the density *increment* taken at the ith step. We divide all the steps into three stages.

Stage 1. We start from $\delta_0 = \delta$. At the ith step, we increase δ_{i-1} to δ_i by taking $\varepsilon_i = v\delta_{i-1}^{1+u}$ and applying procedure (13), (14), or (16). Thus we obtain

$$\begin{gathered} U_i = a_i V(N_i; \boldsymbol{\theta}_i; \mathbf{n}_i), \qquad d_i = \operatorname{length}(\mathbf{n}_i), \\ \delta_i \geq \delta_{i-1} + v\delta_{i-1}^{1+u}, \\ 0 \leq d_i - d_{i-1} \ll \delta^{u-\frac{1}{s-2}}, \\ |U_i| \geq q^{-(d_i-d_{i-1})-n(d_{i-1}+1)}|U_{i-1}|. \end{gathered} \tag{20}$$

Stage 2. If at some point (say, at the jth step) procedure (15) is required, then for some τ with $\delta_{j-1}^{\frac{1}{s-2}} \ll \tau \ll \delta_{j-1}^{u}$ we get

$$\begin{gathered} U_j = a_j V(N_j; \boldsymbol{\theta}_j; \mathbf{n}_j), \qquad d_j = \operatorname{length}(\mathbf{n}_j), \\ \delta_j \gg \delta_{j-1}\tau^{-1}(1+\log\delta_{j-1}^{-1})^{-1}, \\ 0 < d_j - d_{j-1} = Q \ll \delta_{j-1}^{u}(\log\delta_{j-1}^{-1})\tau^{-2}, \\ |U_j| \geq q^{-(d_j-d_{j-1})-n(d_{j-1}+1)}|U_{j-1}|. \end{gathered} \tag{21}$$

Stage 3. We continue the process and take $\varepsilon_i = v\delta_{i-1}^{1+\frac{1}{s-2}}$ at the ith step $(i > j)$. By applying procedures (13) and (14) (see Remark 2 in the previous section), we obtain

$$\begin{gathered} U_i = a_i V(N_i; \boldsymbol{\theta}_i; \mathbf{n}_i), \qquad d_i = \operatorname{length}(\mathbf{n}_i), \\ \delta_i \geq \delta_{i-1} + v\delta_{i-1}^{1+\frac{1}{s-2}}, \\ 0 \leq d_i - d_{i-1} \leq 1, \\ |U_i| \geq q^{-(d_i-d_{i-1})-n(d_{i-1}+1)}|U_{i-1}|. \end{gathered} \tag{22}$$

Thus, procedure (15) is applied at most once. The process terminates once we derive a δ_k for which

$$\delta_k + v\delta^{1+\frac{1}{s-2}} > 1 \tag{23}$$

or

$$\log|U_k| - (\log q)n(d_k + 1) < 2\log\delta_k^{-2} + 2\log C_0.$$

We now consider two cases.

Case 1: Procedure (15) *is never required.* In other words, only Stage 1 is needed. It then follows from Lemma 16 that

$$k \ll \delta^{-u}.$$

By (20) and this estimate, we have

$$d_k - d_0 = \sum_{i=1}^{k}(d_i - d_{i-1}) \ll k\delta^{u-\frac{1}{s-2}} \ll \delta^{-\frac{1}{s-2}}$$

and

$$n\sum_{i=0}^{k}(d_i + 1) \le n(k+1)(d_k+1) \ll k^2\delta^{u-\frac{1}{s-2}} \ll \delta^{-u-\frac{1}{s-2}}.$$

Hence there exists an $E_1 = E_1(\mathbf{g}; q; M) > 0$ such that

$$2\log C_0 + 2\log\delta^{-2} + (\log q)(d_k - d_0) + (\log q)n\sum_{i=0}^{k}(d_i+1) \le E_1\delta^{-u-\frac{1}{s-2}}.$$

Now suppose that

$$\delta > \left(\frac{E_1}{\log|U|}\right)^{\frac{2(s-2)^2}{4s-9}},$$

and note that $-u - \frac{1}{s-2} = \frac{9-4s}{2(s-2)^2}$. Then

$$\begin{aligned}\log|U| &\ge E_1\delta^{-u-\frac{1}{s-2}}\\ &\ge 2\log C_0 + 2\log\delta^{-2} + (\log q)(d_k - d_0) + (\log q)n\sum_{i=0}^{k}(d_i+1).\end{aligned}$$

It follows from (20) and this estimate that

$$\begin{aligned}&\log|U_k| - (\log q)n(d_k+1)\\ &\quad\ge -(\log q)(d_k - d_0) - (\log q)n\sum_{i=0}^{k}(d_i+1) + \log|U|\\ &\quad\ge 2\log\delta^{-2} + 2\log C_0\\ &\quad\ge 2\log\delta_k^{-2} + 2\log C_0.\end{aligned}$$

Thus, the iterating process terminates only because of (23). We then increase the density δ_k to $\delta_{k+1} \ge \delta_k + v\delta^{1+\frac{1}{s-2}} > 1$, a contradiction. Therefore,

$$\delta \le \left(\frac{E_1}{\log|U|}\right)^{\frac{2(s-2)^2}{4s-9}} \le (E_1)^{\frac{2(s-2)^2}{4s-9}}\left(\frac{(\log\log|U|)^2}{\log|U|}\right)^{\frac{2(s-2)^2}{4s-9}}. \tag{24}$$

Case 2: Procedure (15) *is applied at the* jth *step.* By Lemma 16, $j \ll \delta^{-u}$. By an argument similar to the one used in Case 1, we deduce that

$$d_{j-1} - d_0 \ll \delta^{-\frac{1}{s-2}} \quad \text{and} \quad n \sum_{i=0}^{j-1} (d_i + 1) \ll \delta^{-u-\frac{1}{s-2}}.$$

Continuing with the iteration (22) until the process ends, we conclude by Lemma 16 that

$$d_k - d_j \le k - j \ll \delta_j^{-\frac{1}{s-2}}.$$

Now, by (21) and (22), we have

$$\begin{gathered} d_i - d_j \le i - j \ (i > j), \\ k - j \ll \tau^{\frac{1}{s-2}} \delta_{j-1}^{-\frac{1}{s-2}} (1 + \log \delta_{j-1}^{-1})^{\frac{1}{s-2}}, \\ Q \ll \tau^{-2} \delta_{j-1}^{u} (\log \delta_{j-1}^{-1}), \\ \delta_{j-1}^{\frac{1}{s-2}} \ll \tau \ll \delta_{j-1}^{u}. \end{gathered}$$

It follows that

$$\begin{aligned} & n \sum_{i=j}^{k} (d_i + 1) \\ & \quad = n \sum_{i=j+1}^{k} (d_i - d_j) + n(k - j + 1)(d_j + 1) \\ & \quad \le n(k-j)^2 + n(k - j + 1)(Q + (d_{j-1} + 1)) \\ & \quad \ll (1 + \log \delta_{j-1}^{-1})^2 \Big(\tau^{\frac{2}{s-2}} \delta_{j-1}^{-\frac{2}{s-2}} + \tau^{\frac{1}{s-2} - 2} \delta_{j-1}^{-\frac{1}{s-2} + u} + \tau^{\frac{1}{s-2}} \delta_{j-1}^{-\frac{1}{s-2}} \delta^{-\frac{1}{s-2}} \Big) \\ & \quad \ll (1 + \log \delta_{j-1}^{-1})^2 \Big(\delta_{j-1}^{\frac{2u-2}{s-2}} + \delta_{j-1}^{\frac{1}{s-2}(\frac{1}{s-2} - 2)} \delta_{j-1}^{-\frac{1}{s-2} + u} + \delta_{j-1}^{\frac{u-1}{s-2}} \delta^{-\frac{1}{s-2}} \Big). \end{aligned}$$

Recall that $u = \frac{1}{s-2} - \frac{1}{2(s-2)^2}$. We have

$$\frac{-3}{s-2} + \frac{1}{(s-2)^2} + u = -u - \frac{1}{s-2} \le \frac{u-2}{s-2} < \frac{2u-2}{s-2}.$$

Since $\delta_{j-1} \ge \delta$, it follows that

$$\begin{aligned} \delta_{j-1}^{\frac{2u-2}{s-2}} + \delta_{j-1}^{\frac{1}{s-2}(\frac{1}{s-2} - 2)} \delta_{j-1}^{-\frac{1}{s-2} + u} + \delta_{j-1}^{\frac{u-1}{s-2}} \delta^{-\frac{1}{s-2}} & \ll \delta^{\frac{2u-2}{s-2}} + \delta^{\frac{-3}{s-2} + \frac{1}{(s-2)^2} + u} + \delta^{\frac{u-2}{s-2}} \\ & \ll \delta^{-u-\frac{1}{s-2}}. \end{aligned}$$

Thus,

$$n \sum_{i=j}^{k} (d_i + 1) \ll (1 + \log \delta_{j-1}^{-1})^2 \delta^{-u-\frac{1}{s-2}} \le (1 + \log \delta^{-1})^2 \delta^{-u-\frac{1}{s-2}}.$$

Similarly, we have

$$
\begin{aligned}
d_k - d_0 &= (d_k - d_j) + (d_j - d_{j-1}) + (d_{j-1} - d_0) \\
&\leq (k - j) + Q + (d_{j-1} - d_0) \\
&\ll (1 + \log \delta^{-1})^2 \delta^{-u - \frac{1}{s-2}}.
\end{aligned}
$$

Once we combine the preceding estimates, it is clear that

$$
\begin{aligned}
(d_k - d_0) + n \sum_{i=0}^{k} (d_i + 1) &= (d_k - d_0) + n \sum_{i=j}^{k} (d_i + 1) + n \sum_{i=0}^{j-1} (d_i + 1) \\
&\ll (1 + \log \delta^{-1})^2 \delta^{-u - \frac{1}{s-2}} \\
&= (1 + \log \delta^{-1})^2 \delta^{\frac{-4s+9}{2(s-2)^2}}.
\end{aligned}
$$

Hence there exists a constant $E_2 = E_2(\mathbf{g}; q; M) > 0$ such that

$$
\begin{aligned}
2 \log C_0 + 2 \log \delta^{-2} + (\log q)(d_k - d_0) + (\log q) n \sum_{i=0}^{k} (d_i + 1) \\
\leq E_2 \delta^{\frac{-4s+9}{2(s-2)^2}} (1 + \log \delta^{-1})^2.
\end{aligned}
$$

Combining this estimate with (20)–(22) yields

$$
\begin{aligned}
&\log|U_k| - (\log q) n (d_k + 1) \\
&\quad \geq -(\log q)(d_k - d_0) - (\log q) n \sum_{i=0}^{k} (d_i + 1) + \log|U| \\
&\quad \geq 2 \log C_0 + 2 \log \delta^{-2} - E_2 \delta^{\frac{-4s+9}{2(s-2)^2}} (1 + \log \delta^{-1})^2 + \log|U|.
\end{aligned}
$$

If $\delta < q/(\log|U|)^c$, where $c = \frac{2(s-2)^2}{4s-9}$, then the proof is complete. So it remains only to consider the case when $\delta \geq q/(\log|U|)^c$ (i.e., $1 + \log \delta^{-1} \leq c \log\log|U|$). Suppose

$$
\delta > \left(\frac{E_2 (c \log\log|U|)^2}{\log|U|} \right)^{\frac{2(s-2)^2}{4s-9}}.
$$

Then

$$
\log|U| \geq E_2 \delta^{\frac{-4s+9}{2(s-2)^2}} (c \log\log|U|)^2 \geq E_2 \delta^{\frac{-4s+9}{2(s-2)^2}} (1 + \log \delta^{-1})^2
$$

and so

$$
\log|U_k| - (\log q) n (d_k + 1) \geq 2 \log \delta^{-2} + 2 \log C_0 \geq 2 \log \delta_k^{-2} + 2 \log C_0.
$$

A contradiction results when we argue as in Case 1. Therefore,

$$
\delta \leq (c^2 E_2)^{\frac{2(s-2)^2}{4s-9}} \cdot \left(\frac{(\log\log|U|)^2}{\log|U|} \right)^{\frac{2(s-2)^2}{4s-9}}. \tag{25}
$$

If we let $C = \max\left\{E_0, E_1^{\frac{2(s-2)^2}{4s-9}}, (c^2 E_2)^{\frac{2(s-2)^2}{4s-9}}\right\}$, then the theorem follows from the combination of (19), (24), and (25). □

REMARK. After the paper was submitted, T. Bloom extended the recent improvement of Roth's theorem on 3-term arithmetic progression by Sanders to obtain an improvement of $D_{\mathbf{g}}(\mathcal{S}_N)$. For more details, see [1].

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References

[1] T. Bloom, *Translation invariant equations and the method of Sanders,* Bull. London Math. Soc. (to appear).
[2] J. Bourgain, *On triples in arithmetic progression,* Geom. Funct. Anal. 9 (1999), 968–984.
[3] ———, *Roth's theorem on progressions revisited,* J. Anal. Math. 104 (2008), 155–192.
[4] D. R. Heath-Brown, *Integer sets containing no arithmetic progressions,* J. London Math. Soc. (2) 35 (1987), 385–394.
[5] R. M. Kubota, *Waring's problem for* $\mathbb{F}_q[x]$, Dissertationes Math. (Rozprawy Mat.) 117 (1974).
[6] V. F. Lev, *Progression-free sets in finite abelian groups,* J. Number Theory 104 (2004), 162–169.
[7] Y.-R. Liu and C. V. Spencer, *A generalization of Roth's theorem in function fields,* Int. J. Number Theory 5 (2009), 1149–1154.
[8] R. Meshulam, *On subsets of finite abelian groups with no 3-term arithmetic progressions,* J. Combin. Theory Ser. A 71 (1995), 168–172.
[9] K. F. Roth, *On certain sets of integers,* J. London Math. Soc. 28 (1953), 104–109.
[10] T. Sanders, *On Roth's theorem on progressions,* Ann. of Math. (2) 174 (2011), 619–636.
[11] E. Szemerédi, *Integer sets containing no arithmetic progressions,* Acta Math. Hungar. 56 (1990), 155–158.

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