NUMBER OF PRIME FACTORS WITH A GIVEN MULTIPLICITY

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Abstract. Let $k \geqslant 1$ be a natural number and $\omega_k(n)$ denote the number of distinct prime factors of a natural number n with multiplicity k. We estimate the first and the second moments of the functions ω_k with $k \geqslant 1$. Moreover, we prove that the function $\omega_1(n)$ has normal order $\log\log n$ and the function $(\omega_1(n) - \log\log n)/\sqrt{\log\log n}$ has a normal distribution. Finally, we prove that the functions $\omega_k(n)$ with $k \geqslant 2$ do not have normal order $F(n)$ for any nondecreasing nonnegative function F.

1. Introduction

Let $\omega(n)$ be the number of distinct prime factors of a natural number n. Since the probability that a prime number $p \leqslant n$ divides n is considered to be $1/p$, the expected value of $\omega(n)$ is

$$\sum_{p\leqslant n} \frac{1}{p}$$

which is asymptotic to $\log\log n$ by Mertens' Theorem [15, Theorem 2.7(d)]. The first moment of $\omega(n)$ can be considered to be a way of verifying this heuristic on average. Due to the studies of Sathe [18], Selberg [19], Delange [2], [5] and Saidak [17, Eq. (6)], the behaviour of the function $\omega(n)$ on average is known by the estimate

$$\sum_{n\leqslant x} \omega(n) = x\log\log x + bx + \sum_{j=1}^{m} \frac{(-1)^{j-1}G^{(j)}(1)}{j}\frac{x}{\log^j x} + O\left(\frac{x}{\log^{m+1} x}\right) \tag{1}$$

where

$$b := \gamma_0 - \sum_p \sum_{j=2}^{\infty} \frac{1}{jp^j}, \qquad G(s) := \frac{(s-1)\zeta(s)}{s}, \tag{2}$$

γ_0 denotes the Euler-Mascheroni constant, the sum $\sum_p$ runs over all prime numbers and m is a fixed natural number. Thus the behaviour of $\omega(n)$ on average is similar to $\log\log n$ and a natural question to ask is how large the deviation $|\omega(n) - \log\log n|$ on average can be. For this purpose, the concept of normal order is defined as follows [10]. Let $f, F : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be two functions such that F is nondecreasing. Then $f(n)$ is said to have *normal order* $F(n)$ if for any $\epsilon > 0$, the number of $n \leqslant x$ that do not satisfy the inequality

$$(1-\epsilon)F(n) < f(n) < (1+\epsilon)F(n)$$

is $o(x)$ as $x \to \infty$. The original definition in [10] is given for increasing F, here we extend this definition in order to include constant functions.

In [10] (see also [11, Section 22.11]), Hardy and Ramanujan proved that $\omega(n)$ has normal order $\log\log n$. In [22], Turán showed that

$$\sum_{n\leqslant x}(\omega(n)-\log\log x)^2 \ll x\log\log x$$

from which it follows that the number of $3\leqslant n\leqslant x$ satisfying the inequality

$$\frac{|\omega(n)-\log\log n|}{\sqrt{\log\log n}} > h(x)$$

is $o(x)$ as $x\to\infty$ for any increasing function $h(x)\to\infty$ as $x\to\infty$ which again implies that $\omega(n)$ has normal order $\log\log n$. Moreover, in [8], Erdős and Kac proved the remarkable result that the function $(\omega(n)-\log\log n)\,/\,(\log\log n)^{1/2}$, $n\geqslant 3$, has a normal distribution. More precisely, for $a,b\in\mathbb{R}$ with $a\leqslant b$, they proved that

$$\lim_{x\to\infty}\frac{1}{x}\left|\left\{3\leqslant n\leqslant x:\, a\leqslant\frac{\omega(n)-\log\log n}{\sqrt{\log\log n}}\leqslant b\right\}\right| = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-u^2/2}\,du.$$

The idea behind Erdős-Kac's proof was essentially probabilistic. One can find a probabilistic proof of their result in the paper of Billingsley [1]. One can also see the approaches using the method of moments in the work of Delange [3], [4], Vilkas [23], Misevičius [14], and Granville-Soundararajan [9]. Also, further developments of probabilistic ideas led Kubilius [13] and Shapiro [20] to prove a generalization of the Erdős-Kac Theorem independently. Their result is applicable to what are called strongly additive functions. An interested reader can find a comprehensive treatment of it in the monograph of Elliott [6], [7].

In this work, we consider some refined versions of the $\omega(.)$ function through the following set up. For a prime number p and a natural number $n\geqslant 1$, let $\nu_p(n)$ be the multiplicity of p in the unique factorization of n, that is, $\nu_p(n)$ is the unique integer such that $p^{\nu_p(n)}\mid n$ but $p^{\nu_p(n)+1}\nmid n$. For natural numbers $k,n\geqslant 1$, define

$$\omega_k(n) := \sum_{\substack{p\mid n\\ \nu_p(n)=k}} 1$$

which counts the number of prime factors of n with multiplicity k. Note that the usual $\omega(.)$ function can be partitioned into the functions $\omega_k(.)$ with $k\geqslant 1$ as

$$\omega(n)=\sum_{k\geqslant 1}\omega_k(n)$$

for all $n\in\mathbb{N}$. We first prove the following result about the summatory functions of $\omega_k(.)$ with $k\geqslant 1$.

Theorem 1. *Define*

$$P(k) := \sum_p \frac{1}{p^k}, \quad (k\geqslant 2) \tag{3}$$

where the sum runs over all prime numbers. Let the constant b and the function $G(s)$ be defined by (2). We have

$$\sum_{n\leqslant x}\omega_1(n) = x\log\log x + (b-P(2))x + \sum_{j=1}^{m}\frac{(-1)^{j-1}G^{(j)}(1)}{j}\frac{x}{\log^j x} + O\left(\frac{x}{\log^{m+1} x}\right)$$

for any fixed $m\in\mathbb{N}$. Moreover, for $k\geqslant 2$, we have

$$\sum_{n\leqslant x}\omega_k(n) = (P(k)-P(k+1))x + O\left(x^{\frac{k+1}{3k-1}}\log^2 x\right).$$

Next, we consider the second moments of the functions ω_k with $k\geqslant 1$ and prove the following theorem.

Theorem 2. *Let $P(k)$ be defined by (3) and define*

$$C_1 := (b-2P(2))(b+1) + \frac{\pi^2}{6} + P^2(2) + 2P(3) - P(4) \tag{4}$$

and

$$C_k := (P(k)-P(k+1))\,(P(k)-P(k+1)+1) - P(2k) + 2P(2k+1) - P(2k+2).$$

We have

$$\sum_{n\leqslant x}\omega_1^2(n) = x\,(\log\log x)^2 + (1+2b-2P(2))\,x\log\log x + C_1 x + O\left(\frac{x\log\log x}{\log x}\right)$$

and

$$\sum_{n\leqslant x}\omega_k^2(n) = C_k x + O\left(x^{\frac{k+1}{3k-1}}\log^2 x\right), \quad (k\geqslant 2).$$

Analogous to the usual $\omega(.)$ function, we have the following corollary about the function $\omega_1(.)$ and its normal order.

Corollary 2.1. *Let C_1 be defined by (4). We have*

$$\sum_{n\leqslant x}(\omega_1(n)-\log\log n)^2 = x\log\log x + C_1 x + O\left(\frac{x\log\log x}{\log x}\right).$$

Let $h(x)$ be an increasing function such that $h(x)\to\infty$ as $x\to\infty$. Then the number of natural numbers $3\leqslant n\leqslant x$ such that

$$\frac{|\omega_1(n)-\log\log n|}{\sqrt{\log\log n}} \geqslant h(x)$$

is $o(x)$ and thus $\omega_1(n)$ has normal order $\log\log n$.

Similar to the Erdős-Kac Theorem, we also prove that the function $(\omega_1(n)-\log\log n)\,/\,(\log\log n)^{1/2}$, $n\geqslant 3$, has a normal distribution.

Theorem 3. *Let $a,b\in\mathbb{R}$ with $a\leqslant b$. We have*

$$\lim_{x\to\infty}\frac{1}{x}\left|\left\{3\leqslant n\leqslant x:\ a\leqslant\frac{\omega_1(n)-\log\log n}{\sqrt{\log\log n}}\leqslant b\right\}\right| = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-u^2/2}\,du.$$

Here we would like to note that since the function $\omega_1(.)$ is not strongly additive, Theorem 3 is not covered by the aforementioned result of Kubilius and Shapiro.

Recall that the main terms for the summatory functions of ω_1 and ω_1^2 are $x\log\log x$ and $x(\log\log x)^2$, respectively. Since

$$\sum_{n\leqslant x}(\omega_1(n)-\log\log n)^2=\sum_{n\leqslant x}\omega_1(n)^2-2\sum_{n\leqslant x}\omega_1(n)\log\log n+\sum_{n\leqslant x}(\log\log n)^2, \tag{5}$$

the main terms of the three sums on the right-hand side of (5) cancel out by partial summation and we obtain the first assertion of Corollary 2.1. However, we do not have such a cancellation for ω_k with $k\geqslant 2$. The average value of ω_k with $k\geqslant 2$ is $(P(k)-P(k+1))$ by Theorem 1 and we have

$$\sum_{n\leqslant x}(\omega_k(n)-(P(k)-P(k+1)))^2=\left(C_k-(P(k)-P(k+1))^2\right)x$$
$$+O\left(x^{\frac{k+1}{3k-1}}\log^2 x\right)$$

by Theorems 1 and 2. Since

$$C_k-(P(k)-P(k+1))^2=\sum_p\left(\left(\frac{1}{p^k}-\frac{1}{p^{k+1}}\right)-\left(\frac{1}{p^k}-\frac{1}{p^{k+1}}\right)^2\right)\neq 0,$$

the analogous sum to (5) for ω_k with $k\geqslant 2$ is $\gg x$ which is of the same order of magnitude as the second moment of ω_k. This makes us wonder whether the functions $\omega_k(n)$ with $k\geqslant 2$ have normal order $F(n)$ for some nondecreasing function $F:\mathbb{N}\to\mathbb{R}_{\geqslant 0}$ which is the content of the following theorem.

Theorem 4. *Let $k\geqslant 2$ be a fixed integer. Then the function $\omega_k(n)$ does not have normal order $F(n)$ for any nondecreasing function $F:\mathbb{N}\to\mathbb{R}_{\geqslant 0}$.*

Let $\Omega(n)$ be the number of prime factors of n counted with multiplicity. Hardy and Ramanujan [10] and Turán [22] showed that the function $\Omega(n)$ has normal order $\log\log n$. Let $\Omega_k(n)$ be the number of prime factors of n with multiplicity k, counted with weight k. Then

$$\Omega(n)=\sum_{k\geqslant 1}\Omega_k(n).$$

Since $\Omega_k(n)=k\omega_k(n)$, similar deductions can be made for the function $\Omega_k(n)$ by our results above such as the function $\Omega_1(n)$ has normal order $\log\log n$ and the function $(\Omega_1(n)-\log\log n)/(\log\log n)^{1/2}$, $n\geqslant 3$, has a normal distribution. We can also show that the functions $\Omega_k(n)$ with $k\geqslant 2$ do not have normal order $F(n)$ for any nondecreasing nonnegative function F. Finally, we remark that one can consider analogous questions for the set of shifted prime numbers instead of the set of natural numbers. We intend to investigate such prime analogues of our results on a future occasion.

2. Proof of Theorem 1

The proof of Theorem 1 relies on the following general result.

Proposition 2.1. *Let $g : \mathbb{N} \to \mathbb{C}$ be a function such that $|g(p)| \leqslant 1$ for all prime numbers p. For a natural number $k \geqslant 1$, define*

$$a_{g,k}(n) := \sum_{\substack{p \mid n \\ \nu_p(n) \geqslant k+1}} (1 + g(p) + g(p)^2 + ... + g(p)^{\nu_p(n)-(k+1)}) \tag{6}$$

with the convention that empty sum is taken to be zero. Define

$$c_{g,k} := \sum_p \frac{1}{p^k(p - g(p))}.$$

Then we have

$$\sum_{n \leqslant x} a_{g,k}(n) = c_{g,k}x + O\left(x^{\frac{k+2}{3k+2}} \log^2 x\right) \tag{7}$$

where the implied constant is absolute.

Proof. Let $s = \sigma + it$, $\sigma, t \in \mathbb{R}$, and define

$$A_{g,k}(s) := \sum_{n=1}^{\infty} \frac{a_{g,k}(n)}{n^s}, \quad (\sigma > 1).$$

First we show that

$$A_{g,k}(s) = \zeta(s) \sum_p \frac{1}{p^{ks}(p^s - g(p))}, \qquad (\sigma > 1). \tag{8}$$

Note that

$$\sum_p \frac{1}{p^{ks}(p^s - g(p))} = \sum_p \left(\frac{1}{p^{(k+1)s}} + \frac{g(p)}{p^{(k+2)s}} + \frac{g(p)^2}{p^{(k+3)s}} + ... \right) = \sum_{n=1}^{\infty} \frac{b_{g,k}(n)}{n^s} \tag{9}$$

where

$$b_{g,k}(n) := \begin{cases} g(p)^{\alpha-(k+1)} & \text{if } n = p^\alpha, \ \alpha \geqslant k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\zeta(s) \sum_{n=1}^{\infty} \frac{b_{g,k}(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\sum_{d|n} b_{g,k}(d)}{n^s}$$

and

$$\sum_{d|n} b_{g,k}(d) = \sum_{p|n} \sum_{j=k+1}^{\nu_p(n)} g(p)^{j-(k+1)} = a_{g,k}(n),$$

the identity in (8) follows for $\sigma > 1$. Since the series in (9) is absolutely convergent for $\sigma > 1/(k+1)$, by (8) we obtain an analytic continuation of the Dirichlet series $A_{g,k}(s)$ for $\sigma > 1/(k+1)$.

Now, we apply Perron's formula, [21, Lemma 3.12]. Note that

$$|a_{g,k}(n)| \leqslant \sum_{\substack{p|n \\ \nu_p(n) \geqslant k+1}} (\nu_p(n) - k) \ll \log n \tag{10}$$

and

$$\sum_{n=1}^{\infty}\frac{|a_{g,k}(n)|}{n^\sigma}\leqslant\sum_{n=1}^{\infty}\frac{1}{n^\sigma}\sum_{\substack{p|n\\ \nu_p(n)\geqslant k+1}}(\nu_p(n)-k)=\zeta(\sigma)\sum_p\frac{1}{p^{k\sigma}(p^\sigma-1)}\ll\frac{1}{\sigma-1}$$

as $\sigma\to 1^+$. Let $x>2$ be half of an odd integer and let T be a real number with $2\leqslant T\leqslant x$. By Perron's formula, we have

$$\sum_{n<x}a_{g,k}(n)=\frac{1}{2\pi i}\int\limits_{1+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}+iT}A_{g,k}(s)\frac{x^s}{s}\,ds+O\left(\frac{x\log^2 x}{T}\right).$$

By moving the line of integration above to the left and applying the residue theorem, we have

$$\sum_{n<x}a_{g,k}(n)=c_{g,k}x-(I_1+I_2+I_3)+O\left(\frac{x\log^2 x}{T}\right)$$

where

$$c_{g,k}:=\sum_p\frac{1}{p^k(p-g(p))},$$

$$I_1:=\frac{1}{2\pi i}\int\limits_{1+\frac{1}{\log x}+iT}^{\frac{1}{k+1}+\frac{1}{\log x}+iT}A_{g,k}(s)\frac{x^s}{s}\,ds,$$

$$I_2:=\frac{1}{2\pi i}\int\limits_{\frac{1}{k+1}+\frac{1}{\log x}+iT}^{\frac{1}{k+1}+\frac{1}{\log x}-iT}A_{g,k}(s)\frac{x^s}{s}\,ds,$$

$$I_3:=\frac{1}{2\pi i}\int\limits_{\frac{1}{k+1}+\frac{1}{\log x}-iT}^{1+\frac{1}{\log x}-iT}A_{g,k}(s)\frac{x^s}{s}\,ds.$$

For $\sigma\geqslant\frac{1}{k+1}+\frac{1}{\log x}$, we have

$$\left|\sum_p\frac{1}{p^{ks}(p^s-g(p))}\right|\ll\sum_p\frac{1}{p^{(k+1)\sigma}}\leqslant\sum_p\frac{1}{p^{1+\frac{1}{\log x}}}\leqslant\zeta\left(1+\frac{1}{\log x}\right)\ll\log x.$$

For $|t|\geqslant 2$, we know, [12, p. 25], that

$$|\zeta(s)|\ll\begin{cases}1 & \text{if}\quad \sigma>2,\\ \log|t| & \text{if}\quad 1\leqslant\sigma\leqslant 2,\\ |t|^{\frac{1-\sigma}{2}}\log|t| & \text{if}\quad 0\leqslant\sigma\leqslant 1,\\ |t|^{\frac{1}{2}-\sigma}\log|t| & \text{if}\quad \sigma\leqslant 0.\end{cases}$$

Thus we have

$$\begin{aligned} I_1 &\ll \frac{T^{1/2}(\log x)\log T}{T} \int\limits_{\frac{1}{k+1}+\frac{1}{\log x}}^{1} \left(\frac{x}{T^{1/2}}\right)^{\sigma} d\sigma + \frac{\log T}{T} \int\limits_{1}^{1+\frac{1}{\log x}} x^{\sigma}\, d\sigma \\ &\ll \frac{x\log^2 x}{T}. \end{aligned}$$

Similarly, we have $I_3 \ll \frac{x\log^2 x}{T}$. For I_2, we have

$$I_2 \ll x^{\frac{1}{k+1}} \log x \int\limits_0^2 \left|\frac{1}{k+1}+\frac{1}{\log x}+it\right|^{-1} dt + x^{\frac{1}{k+1}}(\log x)(\log T) \int\limits_2^T t^{\frac{1-\frac{1}{k+1}-\frac{1}{\log x}}{2}} \frac{1}{t}\, dt.$$

Note that

$$\left|\frac{1}{k+1}+\frac{1}{\log x}+it\right|^{-1} \leqslant \left|\frac{1}{k+1}+\frac{1}{\log x}\right|^{-1} \leqslant \log x$$

and

$$\begin{aligned} \int\limits_2^T t^{\frac{1-\frac{1}{k+1}-\frac{1}{\log x}}{2}} \frac{1}{t}\, dt &= \frac{2}{1-\frac{1}{k+1}-\frac{1}{\log x}} \left(T^{\frac{1-\frac{1}{k+1}-\frac{1}{\log x}}{2}} - 2^{\frac{1-\frac{1}{k+1}-\frac{1}{\log x}}{2}}\right) \\ &\ll T^{\frac{1}{2}\left(1-\frac{1}{k+1}\right)} \end{aligned}$$

where the implied constant is absolute. Thus we have

$$I_2 \ll x^{\frac{1}{k+1}} \log^2 x + x^{\frac{1}{k+1}} T^{\frac{1}{2}\left(1-\frac{1}{k+1}\right)} \log^2 x \ll x^{\frac{1}{k+1}} T^{\frac{1}{2}\left(1-\frac{1}{k+1}\right)} \log^2 x.$$

By combining the bounds for I_1, I_2 and I_3, we have

$$\sum_{n<x} a_{g,k}(n) = c_{g,k}x + O\left(\frac{x\log^2 x}{T}\right) + O\left(x^{\frac{1}{k+1}} T^{\frac{1}{2}\left(1-\frac{1}{k+1}\right)} \log^2 x\right).$$

Taking $T = x^{\frac{2k}{3k+2}}$ equates the error terms above and we obtain

$$\sum_{n<x} a_{g,k}(n) = c_{g,k}x + O\left(x^{\frac{k+2}{3k+2}} \log^2 x\right)$$

where the implied constant is absolute. By (10), adding the single term $a_{g,k}(\lfloor x\rfloor+1)$ to the left-hand side of the estimate above has contribution $\ll \log x$ and thus Proposition 2.1 follows. □

Now, we deduce Theorem 1 from Proposition 2.1.

Proof of Theorem 1. Let $g(p) = -1$ for all prime numbers p. Then, with this choice of $g(.)$, we have

$$a_k(n) := a_{g,k}(n) = \sum_{\substack{p|n \\ \nu_p(n)\geqslant k+1 \\ \nu_p(n)-k \text{ odd}}} 1 \tag{11}$$

which counts the number of prime factors of n whose multiplicities are of the form $k+l$ for some odd natural number l. By Proposition 2.1, we have

$$\sum_{n\leqslant x} a_k(n) = c_k x + O\left(x^{\frac{k+2}{3k+2}}\log^2 x\right), \qquad (k\geqslant 1) \tag{12}$$

where

$$c_k := \sum_p \frac{1}{p^k(p+1)}.$$

Note that

$$\begin{aligned} \omega_1(n) &= \omega(n) - a_1(n) - a_2(n), \\ \omega_k(n) &= a_{k-1}(n) - a_{k+1}(n), \qquad (k\geqslant 2). \end{aligned} \tag{13}$$

Hence, the desired result in Theorem 1 follows from (1) and (12). $\square$

3. Proof of Theorem 2

We start with the second moment of ω_1. We have

$$\sum_{n\leqslant x}\omega_1^2(n) = \sum_{n\leqslant x}\left(\sum_{\substack{p|n\\ p^2\nmid n}} 1\right)^2 = \sum_{n\leqslant x}\sum_{\substack{p|n,\, q|n\\ p^2\nmid n,\, q^2\nmid n}} 1 = \sum_{n\leqslant x}\omega_1(n) + \sum_{\substack{p,q\leqslant x\\ p\neq q}}\sum_{\substack{n\leqslant x\\ p|n,\, q|n\\ p^2\nmid n,\, q^2\nmid n}} 1.$$

Let

$$S(x) := \sum_{\substack{p,q\leqslant x\\ p\neq q}}\sum_{\substack{n\leqslant x\\ p|n,\, q|n\\ p^2\nmid n,\, q^2\nmid n}} 1 = \sum_{\substack{p,q\leqslant x\\ p\neq q}}\left(\left\lfloor\frac{x}{pq}\right\rfloor - \left\lfloor\frac{x}{pq^2}\right\rfloor - \left(\left\lfloor\frac{x}{p^2q}\right\rfloor - \left\lfloor\frac{x}{p^2q^2}\right\rfloor\right)\right)$$

and

$$\begin{aligned} S_1(x) &:= \sum_{\substack{p,q\leqslant x\\ p\neq q}}\left\lfloor\frac{x}{pq}\right\rfloor, \\ S_2(x) &:= \sum_{\substack{p,q\leqslant x\\ p\neq q}}\left\lfloor\frac{x}{pq^2}\right\rfloor, \\ S_3(x) &:= \sum_{\substack{p,q\leqslant x\\ p\neq q}}\left\lfloor\frac{x}{p^2q^2}\right\rfloor. \end{aligned}$$

Then, by symmetry, we have $S(x) = S_1(x) - 2S_2(x) + S_3(x)$. Now we consider $S_1(x)$ by closely following the argument in [17] due to Saidak. We have

$$S_1(x) = \sum_{\substack{pq\leqslant x\\ p\neq q}}\left\lfloor\frac{x}{pq}\right\rfloor = \sum_{\substack{pq\leqslant x\\ p\neq q}}\left(\frac{x}{pq} + O(1)\right). \tag{14}$$

For $pq\leqslant x$ and $q\geqslant 2$, we have $p\leqslant x/2$ and

$$\sum_{p\leqslant x/2}\frac{1}{p\log(x/p)} \ll \frac{\log\log x}{\log x}$$

by [16, Exercise 9.4.4]. Thus the contribution of the $O(1)$ term in (14) is

$$\ll \sum_{p\leqslant x/2}\sum_{q\leqslant x/p} 1 \ll x \sum_{p\leqslant x/2} \frac{1}{p\log(x/p)} \ll \frac{x\log\log x}{\log x}. \tag{15}$$

We have

$$\sum_{\substack{pq\leqslant x\\ p\neq q}} \frac{1}{pq} = \sum_{pq\leqslant x}\frac{1}{pq} - \sum_{p\leqslant\sqrt{x}}\frac{1}{p^2} = \sum_{pq\leqslant x}\frac{1}{pq} - \sum_{p}\frac{1}{p^2} + O\left(\frac{1}{\sqrt{x}}\right)$$

and by a result of Saidak, [17, Lemma 3], we have

$$\sum_{pq\leqslant x}\frac{1}{pq} = (\log\log x)^2 + 2b\log\log x + \frac{\pi^2}{6} + b^2 + O\left(\frac{\log\log x}{\log x}\right).$$

Thus

$$S_1(x) = x(\log\log x)^2 + 2bx\log\log x + \left(\frac{\pi^2}{6} + b^2 - P(2)\right)x + O\left(\frac{x\log\log x}{\log x}\right).$$

For $S_2(x)$, we have

$$S_2(x) = \sum_{\substack{pq^2\leqslant x\\ p\neq q}} \left\lfloor \frac{x}{pq^2}\right\rfloor = \sum_{\substack{pq^2\leqslant x\\ p\neq q}} \left(\frac{x}{pq^2} + O(1)\right).$$

The contribution of the $O(1)$ term above is

$$\ll \sum_{\substack{pq^2\leqslant x\\ p\neq q}} 1 \leqslant \sum_{\substack{pq\leqslant x\\ p\neq q}} 1 \ll \frac{x\log\log x}{\log x}$$

by the estimate in (15). Thus

$$S_2(x) = x\left(\sum_{pq^2\leqslant x}\frac{1}{pq^2} - P(3)\right) + O\left(\frac{x\log\log x}{\log x}\right).$$

We have

$$\sum_{pq^2\leqslant x}\frac{1}{pq^2} = \sum_{p\leqslant x/4}\frac{1}{p}\left(P(2) - \sum_{q>\sqrt{\frac{x}{p}}}\frac{1}{q^2}\right).$$

Let

$$L(u) := \sum_{q\leqslant u}\frac{1}{q} = \log\log u + b + R(u)$$

where

$$R(u) \ll \frac{1}{\log u}$$

by [15, Theorem 2.7(d)]. For $p \leqslant x/4$, we have

$$\sum_{q \geqslant \sqrt{\frac{x}{p}}} \frac{1}{q^2} = \int_{\sqrt{\frac{x}{p}}}^{\infty} \frac{1}{u} dL(u) = \int_{\sqrt{\frac{x}{p}}}^{\infty} \frac{du}{u^2 \log u} + R\left(\sqrt{\frac{x}{p}}\right) \frac{1}{\sqrt{\frac{x}{p}}} + \int_{\sqrt{\frac{x}{p}}}^{\infty} \frac{R(u)}{u^2}\, du$$

$$\ll \frac{1}{\log\left(\frac{x}{p}\right)}.$$

Thus

$$\begin{aligned}\sum_{pq^2 \leqslant x} \frac{1}{pq^2} &= \sum_{p \leqslant x/4} \frac{1}{p} \left(P(2) + O\left(\frac{1}{\log\left(\frac{x}{p}\right)} \right) \right) \\ &= P(2)\left(\log\log\left(\frac{x}{4}\right) + b + O\left(\frac{1}{\log x}\right)\right) + O\left(\sum_{p \leqslant x/4} \frac{1}{p \log\left(\frac{x}{p}\right)} \right) \\ &= P(2) \log\log x + bP(2) + O\left(\frac{\log\log x}{\log x} \right).\end{aligned}$$

Thus we have

$$S_2(x) = x\left(P(2)\log\log x + bP(2) - P(3)\right) + O\left(\frac{x \log\log x}{\log x} \right).$$

For $S_3(x)$, we have

$$S_3(x) = x \sum_{\substack{p^2q^2 \leqslant x \\ p \neq q}} \frac{1}{p^2q^2} + O\left(\sqrt{x}\log\log x\right)$$

and

$$\begin{aligned}\sum_{\substack{p^2q^2 \leqslant x \\ p \neq q}} \frac{1}{p^2q^2} &= \sum_{p^2q^2 \leqslant x} \frac{1}{p^2q^2} - \sum_{p^4 \leqslant x} \frac{1}{p^4} \\ &= \sum_{p \leqslant \sqrt{x}} \frac{1}{p^2} \left(P(2) + O\left(\sum_{q > \frac{\sqrt{x}}{p}} \frac{1}{q^2} \right) \right) - P(4) + O\left(\frac{1}{x^{3/4}} \right) \\ &= P^2(2) - P(4) + O\left(\sum_{p \leqslant \sqrt{x}} \frac{1}{p^2} \frac{p}{\sqrt{x}} \right) + O\left(\frac{1}{x^{1/2}} \right) \\ &= P^2(2) - P(4) + O\left(\frac{\log\log x}{\sqrt{x}} \right).\end{aligned}$$

Thus

$$S_3(x) = \left(P^2(2) - P(4)\right) x + O\left(\sqrt{x}\log\log x\right).$$

By using the first moment of ω_1 and the estimates above for $S_1(x)$, $S_2(x)$ and $S_3(x)$, we obtain the desired result for the second moment of $\omega_1(.)$.

Let $k \geqslant 2$. We have

$$\sum_{n \leqslant x} \omega_k^2(n) = \sum_{n \leqslant x} \omega_k(n) + \sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n) = \nu_q(n) = k}} 1.$$

For a natural number $\ell \geqslant k$ and distinct prime numbers p and q with $pq \leqslant x^{1/k}$, define

$$h(\ell, p, q, x) := \left\lfloor \frac{x}{p^\ell q^k} \right\rfloor - \left\lfloor \frac{x}{p^\ell q^{k+1}} \right\rfloor$$

which counts the number of $n \leqslant x$ such that $p^\ell \mid n$ and $\nu_q(n) = k$. Then

$$\sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n) = \nu_q(n) = k}} 1 = \sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \left(h(k, p, q, x) - h(k+1, p, q, x)\right). \tag{16}$$

Since

$$h(\ell, p, q, x) = \frac{x}{p^\ell}\left(\frac{1}{q^k} - \frac{1}{q^{k+1}}\right) + O(1),$$

we have

$$\begin{aligned}
\sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \sum_{\substack{n \leqslant x \\ \nu_p(n) = \nu_q(n) = k}} 1 &= \sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \left(\frac{x}{p^k}\left(\frac{1}{q^k} - \frac{1}{q^{k+1}}\right) - \frac{x}{p^{k+1}}\left(\frac{1}{q^k} - \frac{1}{q^{k+1}}\right) + O(1)\right) \\
&= x \sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)\left(\frac{1}{q^k} - \frac{1}{q^{k+1}}\right) + O\left(x^{1/k} \log\log x\right). \qquad (17)
\end{aligned}$$

For a real number r and a statement S, define

$$\mathbb{1}_S(r) := \begin{cases} r & \text{if S is true,} \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned}
&\sum_{\substack{p,q \\ pq \leqslant x^{1/k} \\ p \neq q}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)\left(\frac{1}{q^k} - \frac{1}{q^{k+1}}\right) \\
&\quad = \sum_{p \leqslant x^{1/k}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)\left(P(k) - P(k+1) - \mathbb{1}_{p \leqslant x^{\frac{1}{2k}}}\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right) + O\left(\frac{p^{k-1}}{x^{(k-1)/k}}\right)\right).
\end{aligned}$$

For the contribution of the error term above, we have

$$\sum_{p \leqslant x^{1/k}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right) \frac{p^{k-1}}{x^{(k-1)/k}} \ll \frac{1}{x^{(k-1)/k}} \sum_{p \leqslant x^{1/k}} \frac{1}{p} \ll \frac{\log\log x}{x^{(k-1)/k}}.$$

For the remaining terms, we have

$$\begin{aligned}
&\sum_{p \leqslant x^{1/k}} \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)\left(P(k) - P(k+1) - \mathbb{1}_{p \leqslant x^{\frac{1}{2k}}}\left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)\right) \\
&\quad = \left(P(k) - P(k+1)\right)^2 + O\left(\frac{1}{x^{(k-1)/k}}\right) - \sum_p \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)^2 + O\left(\frac{1}{x^{(2k-1)/(2k)}}\right) \\
&\quad = \left(P(k) - P(k+1)\right)^2 - P(2k) + 2P(2k+1) - P(2k+2) + O\left(\frac{1}{x^{(k-1)/k}}\right).
\end{aligned}$$

By (17) and the estimate above, we have

$$\sum_{\substack{p,q\\ pq\leqslant x^{1/k}\\ p\neq q}}\ \sum_{\substack{n\leqslant x\\ \nu_p(n)=\nu_q(n)=k}} 1=\Big((P(k)-P(k+1))^2-P(2k)+2P(2k+1)-P(2k+2)\Big)\,x$$

$$+O\left(x^{1/k}\log\log x\right). \tag{18}$$

By (16), (18) and Theorem 1, we obtain the desired result. □

4. Proof of Corollary 2.1

The first assertion in Corollary 2.1 follows immediately by the first and the second moments of ω_1 and partial summation. For the second part, let $h(x)$ be an increasing function such that $h(x)\to\infty$ as $x\to\infty$ and let $\mathcal{E}$ be the set of natural numbers n with $\frac{x}{\log x}\leqslant n\leqslant x$ such that

$$\frac{|\omega_1(n)-\log\log n|}{\sqrt{\log\log n}}\geqslant h(x).$$

Let $|\mathcal{E}|$ be the cardinality of $\mathcal{E}$. Then

$$\begin{aligned}\sum_{3\leqslant n\leqslant x}(\omega_1(n)-\log\log n)^2 &\geqslant \sum_{n\in\mathcal{E}}(\omega_1(n)-\log\log n)^2\\ &\geqslant h^2(x/\log x)\sum_{n\in\mathcal{E}}\log\log n\\ &\geqslant h^2(x/\log x)\,|\mathcal{E}|\log\log\left(x/\log x\right).\end{aligned} \tag{19}$$

By (19) and the fact that the left-hand side of (19) is $\ll x\log\log x$, we have

$$\frac{|\mathcal{E}|}{x}\ll\frac{\log\log x}{h^2(x/\log x)\log\log(x/\log x)}\to 0$$

as $x\to\infty$ since $h(x)\to\infty$ as $x\to\infty$. This finishes the proof of the second assertion of Corollary 2.1 since the remaining set of natural numbers with $n<x/\log x$ is already of size $o(x)$. □

5. Proof of Theorem 3

For $f=\omega,\omega_1$, let $r_f(n)$ be the ratio

$$r_f(n):=\frac{f(n)-\log\log n}{\sqrt{\log\log n}},\quad (n\geqslant 3),$$

and for $b\in\mathbb{R}$, let

$$D(f,x,b):=\frac{1}{x}\left|\{3\leqslant n\leqslant x: r_f(n)\leqslant b\}\right|$$

be the corresponding density function for sufficiently large x. Since $\omega(n)\geqslant\omega_1(n)$, we have $r_{\omega_1}(n)\leqslant r_\omega(n)$ for all $n\geqslant 3$. Thus $\{3\leqslant n\leqslant x: r_\omega(n)\leqslant b\}\subset\{3\leqslant n\leqslant x: r_{\omega_1}(n)\leqslant b\}$ and

$$D(\omega,x,b)\leqslant D(\omega_1,x,b) \tag{20}$$

for all $x\geqslant 3$. Let

$$\Phi(b):=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{b}e^{-\frac{u^2}{2}}\,du.$$

Then, by (20) and the Erdős-Kac Theorem, we have

$$\Phi(b) \leqslant \liminf_{x\to\infty} D(\omega_1, x, b). \tag{21}$$

Let $\epsilon > 0$ and define the set

$$\mathcal{A}(x,\epsilon) := \left\{3 \leqslant n \leqslant x : \frac{\omega(n)-\omega_1(n)}{\sqrt{\log\log n}} \leqslant \epsilon\right\}.$$

Let $\mathcal{A}^c(x,\epsilon)$ denote the complement of $\mathcal{A}(x,\epsilon)$ inside natural numbers up to x. Since

$$r_{\omega_1}(n) = r_\omega(n) + \frac{\omega_1(n)-\omega(n)}{\sqrt{\log\log n}},$$

we have

$$\begin{aligned}
\{3 \leqslant n \leqslant x : r_{\omega_1}(n) \leqslant b\} &= \left\{3 \leqslant n \leqslant x : r_\omega(n) + \frac{\omega_1(n)-\omega(n)}{\sqrt{\log\log n}} \leqslant b\right\} \\
&= \left\{3 \leqslant n \leqslant x : n \in \mathcal{A}(x,\epsilon),\, r_\omega(n) + \frac{\omega_1(n)-\omega(n)}{\sqrt{\log\log n}} \leqslant b\right\} \\
&\quad \cup \left\{3 \leqslant n \leqslant x : n \in \mathcal{A}^c(x,\epsilon),\, r_\omega(n) + \frac{\omega_1(n)-\omega(n)}{\sqrt{\log\log n}} \leqslant b\right\} \\
&\subseteq \left\{3 \leqslant n \leqslant x : r_\omega(n) \leqslant b+\epsilon\right\} \cup \{3 \leqslant n \leqslant x : n \in \mathcal{A}^c(x,\epsilon)\}.
\end{aligned}$$

Thus

$$D(\omega_1, x, b) \leqslant D(\omega, x, b+\epsilon) + \frac{1}{x}\left|\{3 \leqslant n \leqslant x : n \in \mathcal{A}^c(x,\epsilon)\}\right|. \tag{22}$$

Now we show that

$$\lim_{x\to\infty} \frac{1}{x}\left|\{3 \leqslant n \leqslant x : n \in \mathcal{A}^c(x,\epsilon)\}\right| = 0. \tag{23}$$

By (1) and Theorem (1), we have

$$\sum_{n\leqslant x} (\omega(n)-\omega_1(n)) \ll x.$$

Since

$$\begin{aligned}
\sum_{n\leqslant x} (\omega(n)-\omega_1(n)) &\geqslant \sum_{\substack{\frac{x}{\log x} \leqslant n \leqslant x \\ n\in\mathcal{A}^c(x,\epsilon)}} (\omega(n)-\omega_1(n)) \\
&> \epsilon \sum_{\substack{\frac{x}{\log x} \leqslant n \leqslant x \\ n\in\mathcal{A}^c(x,\epsilon)}} \sqrt{\log\log n} \\
&\geqslant \epsilon\sqrt{\log\log(x/\log x)}\left|\{x/\log x \leqslant n \leqslant x : n \in \mathcal{A}^c(x,\epsilon)\}\right|,
\end{aligned}$$

we have

$$\left|\{x/\log x \leqslant n \leqslant x : n \in \mathcal{A}^c(x,\epsilon)\}\right| \ll \frac{1}{\epsilon}\frac{x}{\sqrt{\log\log(x/\log x)}}$$

which gives (23) since the size of the remaining set $\{n < x/\log x : n \in \mathcal{A}^c(x,\epsilon)\}$ is already $o(x)$ as $x \to \infty$. By (22), (23) and the Erdős-Kac Theorem, we have

$$\limsup_{x\to\infty} D(\omega_1, x, b) \leqslant \Phi(b+\epsilon). \tag{24}$$

Since ϵ is arbitrary, by (21) and (24), we have $\lim_{x\to\infty} D(\omega_1, x, b) = \Phi(b)$ and this finishes the proof of Theorem 3. □

6. Proof of Theorem 4

Now, we prove that the functions $\omega_k(n)$ with $k \geqslant 2$ do not have normal order $F(n)$ for any nondecreasing function $F : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$.

First we assume that there exists $n_0 \in \mathbb{N}$ such that $F(n_0) > 0$. Then $F(n) > 0$ for $n \geqslant n_0$ since F is nondecreasing. Thus

$$\lim_{N\to\infty} \frac{|\{n \leqslant N : F(n) > 0\}|}{N} = 1.$$

For a natural number N, define the set

$$\mathcal{N}_0(N) := \{n \leqslant N : \omega_k(n) = 0\}.$$

Since

$$\sum_{\substack{n\leqslant N\\ n\notin\mathcal{N}_0(N)}} 1 = \sum_p \sum_{\substack{n\leqslant N\\ p^k\mid n\\ p^{k+1}\nmid n}} 1 \leqslant \sum_p \sum_{\substack{n\leqslant N\\ p^k\mid n}} 1 \leqslant N \sum_p \frac{1}{p^k},$$

we have

$$\frac{|\mathcal{N}_0(N)|}{N} \geqslant \frac{N - N\sum_p \frac{1}{p^k}}{N} = 1 - \sum_p \frac{1}{p^k} \geqslant 1 - \sum_p \frac{1}{p^2} > 1 - (\zeta(2) - 1) = 2 - \frac{\pi^2}{6} > 0.$$

Thus

$$\liminf_{N\to\infty} \left(\frac{|\{n \leqslant N : F(n) > 0\}|}{N} + \frac{|\mathcal{N}_0(N)|}{N} \right) > 1$$

and the cardinality of the set of $n \leqslant N$ for which $F(n) > 0$ and $\omega_k(n) = 0$ is not $o(N)$. Since for such n, the inequality

$$|\omega_k(n) - F(n)| > \frac{F(n)}{2}$$

is satisfied, we deduce that $\omega_k(n)$ does not have normal order $F(n)$.

Now assume that $F(n) = 0$ for all $n \in \mathbb{N}$. Then

$$\lim_{N\to\infty} \frac{|\{n \leqslant N : F(n) = 0\}|}{N} = 1.$$

Define

$$\mathcal{N}_1(N) := \{n \leqslant N : \omega_k(n) = 1\}.$$

Since

$$|\mathcal{N}_1(N)| \geqslant \sum_{\substack{n\leqslant N\\ \nu_2(n)=k\\ \nu_p(n)<k \text{ for all } p\geqslant 3}} 1 = \sum_{\substack{n\leqslant N\\ \nu_2(n)=k}} 1 - \sum_{\substack{n\leqslant N\\ \nu_2(n)=k\\ \nu_p(n)\geqslant k \text{ for some } p\geqslant 3}} 1$$
$$= \left\lfloor \frac{N}{2^k} \right\rfloor - \left\lfloor \frac{N}{2^{k+1}} \right\rfloor - \sum_{p\geqslant 3} \sum_{\substack{n\leqslant N/2^k\\ p^k|n\\ n \text{ is odd}}} 1$$
$$\geqslant \frac{N}{2^k} - \frac{N}{2^{k+1}} - \frac{N}{2^k}\sum_{p\geqslant 3}\frac{1}{p^k} - 1,$$

we have

$$\liminf_{N\to\infty} \frac{|\mathcal{N}_1(N)|}{N} \geqslant \frac{1}{2^k}\left(\frac{1}{2} - \sum_{p\geqslant 3}\frac{1}{p^k}\right) \geqslant \frac{1}{2^k}\left(\frac{1}{2} - \sum_{p\geqslant 3}\frac{1}{p^2}\right)$$
$$> \frac{1}{2^k}\left(\frac{1}{2} - \left(\frac{\pi^2}{6} - 1 - \frac{1}{4}\right)\right)$$
$$> 0.$$

Thus

$$\liminf_{N\to\infty}\left(\frac{|\{n\leqslant N : F(n)=0\}|}{N} + \frac{|\mathcal{N}_1(N)|}{N}\right) > 1$$

and the cardinality of the set of $n \leqslant N$ for which $F(n) = 0$ and $\omega_k(n) = 1$ is not $o(N)$. Since for such n, the inequality $|\omega_k(n) - F(n)| > F(n)/2$ is satisfied, we deduce that $\omega_k(n)$ does not have normal order $F(n)$. □

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