Analysis of an Optimal Stopping Problem Arising from Hedge Fund Investing *

Xinfu Chen[†], David Saunders[‡], John Chadam[§]

June 19, 2019

Abstract

We analyze the optimal withdrawal time for an investor in a hedge fund with a first-loss or shared-loss fee structure, given as the solution of an optimal stopping problem on the fund's assets with a piecewise linear payoff function. Assuming that the underlying follows a geometric Brownian motion, we present a complete solution of the problem in the infinite horizon case, showing that the continuation region is a finite interval, and that the smooth-fit condition may fail to hold at one of the endpoints. In the finite horizon case, we show the existence of a pair of optimal exercise boundaries and analyze their properties, including smoothness and convexity.

*The authors thank Ben Djerroud, Kurt Henry, Luis Seco, and Mohammad Shakourifar, for helpful comments and suggestions. Chen is supported by NSF-DMS-1516344. David Saunders gratefully acknowledges support from an NSERC Discovery Grant.

[†]Department of Mathematics, University of Pittsburgh, xinfu@pitt.edu.

[‡]Corresponding Author. Department of Statistics and Actuarial Science, University of Waterloo, dsaunders@uwaterloo.ca.

[§]Department of Mathematics, University of Pittsburgh, chadam@pitt.edu.

1 Introduction

Traditionally, a "two and twenty" fee structure has been very common in the hedge fund industry. Under such an arrangement, investors pay the fund manager a flat fee of 2% of assets under management, together with a performance fee of 20% of the profits. The performance fee is essentially a call option on the underlying hedge fund value. More recently, a number of variants on the traditional fee structure have been proposed. "High watermark" provisions stipulate that fees for a given period are only paid when the fund value exceeds the previous maximum since the initial investment. See Goetzmann et al. [2003], Panageas and Westerfield [2009], Guasoni and Obłój [2013] for mathematical treatments. "First loss" and "shared loss" structures require the hedge fund manager to contribute capital to insure investors against losses on their investment in the fund. The resulting fee structures resemble portfolios of options, with the investor's position being equivalent to a long position in the fund, a short position in a call option on the fund (the performance fee), and a long position in a put option bear spread[1] on the fund. In a first-loss fee structure, He and Kou [2018] consider the portfolio selection decision of the fund manager, and its impact on the utility of both the manager and the hedge fund investor. Djerroud et al. [2016] discuss the motivation for first-loss and shared-loss structures, and analyze the contracts using a risk-neutral valuation approach.

In this paper, we consider the liquidation timing decision of the investor in a hedge fund with a first-loss and/or shared-loss fee structure. We assume that the investor seeks to maximize the risk-neutral expected value of the payoff and that the value of the investment in the hedge fund, $\{X_t^x\}_{t\geq 0}$, follows a geometric Brownian motion:

$$dX_t^x = rX_t^x dt + \sigma X_t^x dW_t, \qquad X_0^x = x \tag{1.1}$$

where $r > 0$ is the risk-free interest rate, $\sigma > 0$ is the fund volatility, and $\{W_t\}_{t\geq 0}$ is a standard Browian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$, the standard augmentation of the filtration generated by W, satisfying the usual conditions. If we consider an infinite horizon, the investor's optimal stopping problem becomes the optimization

$$V(x) = \sup_{\tau\in\mathcal{T}} \ \mathbb{E}[e^{-r\tau} g(X_\tau^x)] \tag{1.2}$$

where $\mathcal{T}$ is the set of all $\mathbb{F}$ stopping times and τ is interpreted as the time at which the investor withdraws from the fund. If there is a finite investment horizon T, the optimal stopping problem becomes

$$v(x,T) = \sup_{\tau\in\mathcal{T}_{[0,T]}} \ \mathbb{E}[e^{-r\tau} g(X_\tau^x)] \tag{1.3}$$

where $\mathcal{T}_{[0,T]}$ is the set of all $\mathbb{F}$ stopping times such that $\tau \leqslant T$ almost surely. Under both the first-loss

[1] Buying a put option with a higher strike price and selling a put option with a lower strike price.

and the shared-loss fee structures, the payoff function g has the general form

$$g(x) = \begin{cases} A + Bx & \text{if } 0 \leqslant x \leqslant \kappa, \\ q + (1-q)x & \text{if } \kappa \leqslant x \leqslant 1, \\ p + (1-p)x & \text{if } 1 \leqslant x \end{cases} \tag{1.4}$$

where (A, B, p, q, κ) is a set of parameters satisfying:

$$B \geqslant 1 \geqslant q > 0, \quad p \in (0,1), \quad \kappa \in (0,1), \quad A := q + (1 - q - B)\kappa \geqslant 0.$$

Examples of such fee arrangements include the following. Suppose that the initial investment is $x = 1$, and the hedge fund manager sets aside an amount c to cover a fixed fraction θ of the investor's losses. $\theta = 1$ corresponds to first-loss protection, while $\theta < 1$ is a shared-loss structure. If, in return, the manager keeps the fraction α of the investor's profits as a performance fee, then the payoff to the investor becomes:

$$g(x) = \begin{cases} c + x & x \leqslant 1 - \frac{c}{\theta} \\ \theta + (1-\theta)x & 1 - \frac{c}{\theta} \leqslant x \leqslant 1 \\ \alpha + (1-\alpha)x & x \geqslant 1 \end{cases} \tag{1.5}$$

(when $\theta \leqslant c$, only the first two pieces of the above payoff are relevant). This equivalent to (1.4) with $A = c, B = 1, p = \alpha, q = \theta$, and $\kappa = 1 - \frac{c}{\theta}$.

Alternatively, rather than setting money aside, the manager may make an investment w in the fund. In this case, when losses occur, then investor is then entitled to be compensated from the manager's share of the total investment. Once again, assuming that the proportion $\theta \in (0,1]$ of the investor's losses is covered, and that the manager is entitled to the fraction $\alpha \in (0,1)$ of the investor's profits as a performance fee, the payoff to the investor becomes:

$$g(x) = \begin{cases} (1+w)x & x \leqslant \frac{\theta}{w+\theta} \\ \theta + (1-\theta)x & \frac{\theta}{w+\theta} \leqslant x \leqslant 1 \\ \alpha + (1-\alpha)x & x \geqslant 1 \end{cases} \tag{1.6}$$

This is equivalent to (1.4) with $A = 0, B = 1 + w, q = \theta, p = \alpha, \kappa = \frac{\theta}{w+\theta}$.

In (1.2) and (1.3), we have formulated the investor's optimal withdrawal time problem using risk-neutral pricing (i.e. by maximizing the expected present value of the payoff under the risk-neutral measure over all eligible stopping times). Were the underlying assets of the hedge fund to be tradable in the market, then, under the assumptions of the Black-Scholes model, this value would give the arbitrage-free price of the investor's payoff (see, e.g. Karatzas and Shreve [1998]). However, in this case the underlying asset is the portfolio managed by the hedge fund, and is typically not tradable.[2] An alternative approach would be to assume a utility function that models investor preferences and maximize (discounted) expected utility. This is the approach taken in the analysis of He and Kou [2018], who employ the S-shaped utility

[2]A similar issue arises in other applications of the Black-Scholes model outside of its original context, such as Merton's structural credit risk model Merton [1974].

function from cumulative prospect theory. The disadvantage of the utility approach is that it requires making strong assumptions regarding the nature of investor preferences.

Note that g is non-negative and Lipschitz continuous with Lipschitz constant B. In the above formulation, for normalization purposes, we have assumed that the initial fund investment (against which profits and losses are measured) was equal to 1. Figure 1 shows an example of the payoff with $B = q = 1$. In this case, $g(S) = 1 + (1-p)\max\{S-1, 0\} - \max\{\kappa - S, 0\}$, being a portfolio of unit cash (initial

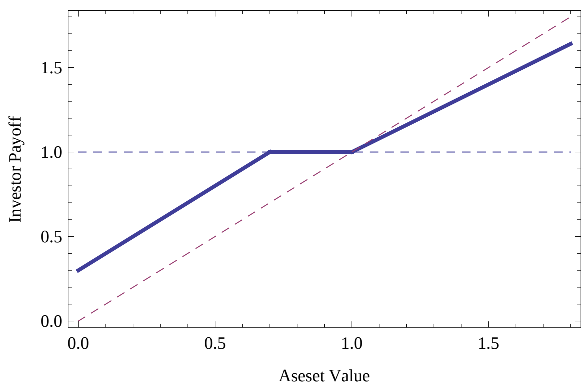

Figure 1: Payoff function g, with $B = 100\%$, $q = 100\%$, $p = 20\%$ and $\kappa = 70\%$.

investment), a long position of (1-p) units of a call option with strike price one, and a short position in a put option with strike price κ. We can view the investor as owning the fund, giving to the hedge fund manager 20% of all of the profits in return for insurance of the losses of the investor, up to a maximum of 30% of the initial investment. Other variations on this structure are possible, all leading to payoffs of the form (1.4). For example, in many cases the manager will make a direct investment in the fund, and the investor's losses will be partially covered by this investment. See Djerroud et al. [2016] for details.

The purpose of this paper is to provide a rigorous mathematical analysis for the value functions V and v, as well as the optimal stopping times that attain the suprema in (1.3) and (1.2), for the pay-off function given by (1.4).

The value function V for the optimal stopping problem (1.2) can be found explicitly. When $p \geqslant q$, g is concave and it is optimal to exercise immediately. Otherwise, g is concave on $[0, 1]$, and convex on $[\kappa, \infty)$; there are two stopping boundaries, satisfying $\kappa \leqslant S_1 < 1 < S_2$, and it is optimal to stop at the first time either of these boundaries is reached. The smooth-fit condition $V' = g'$ always holds at S_2, but it may fail at S_1 depending on the values of the parameters. The finite horizon problem (1.3) inherits from the infinite horizon case the property of having two exercise boundaries $s_1(T) < 1 < s_2(T)$. Furthermore, $\lim_{T\to\infty} s_i(T) = S_i$, and $\lim_{T\to 0} s_i(T) = 1$, and s_1 is decreasing and s_2 is increasing. In addition, we show that $\ln s_1$ is convex and $\ln s_2$ is concave.

The results are mainly derived by considering the value functions V, v as the unique solutions of variational inequalities, and then employing analytical techniques. Solution of the perpetual optimal stopping problem (to find V) requires relatively elementary methods. To analyze the finite horizon problem, we consider the time derivative $u = \partial v/\partial T$ of the value function. We note that u satisfies a Stefan problem (in our case, with a delta function for the initial condition), an observation dating back to Schatz [1969] (see also van Moerbeke [1976]). Our approach involves analyzing a regularized version of the Stefan problem directly, employing uniform estimates to derive properties of the limiting solution as the regularizing parameter ε tends to zero, and then verifying that an appropriate integral of this Stefan problem solution is indeed the value function of the optimal stopping problem. Based on this approximation, and a detailed and careful argument, we are able to derive smoothness and convexity properties of the stopping boundaries.

Convexity of free boundaries in general, and optimal stopping boundaries in particular, is a technical and challenging topic. Friedman and Jensen [1977] considered the convexity of the free boundary in a Stefan problem and the dam problem, by studying level curves of the solution. In optimal stopping applications, Chen et al. [2008] and Ekström [2004] independently studied the convexity of the free boundary for the American put option on a non-dividend paying asset. Chen et al. [2013] showed that the boundary can fail to be convex in certain circumstances when the asset pays dividends.

The approach to convexity of the boundaries in this paper is similar to that taken in Chen et al. [2008]. There are both conceptual and technical aspects of our work on the current problem that distinguish it from the earlier work on the American put. The conceptual contributions of the paper include the following:

1. The simplification in the method for approximating the singular initial value by Gaussians, compared to the method employed in Chen et al. [2008], is significant, and in our opinion naturally connects to the relationship between diffusion processes and Gaussian distributions. This construction may further lead to an effective numerical algorithm and its justification.

2. The existing literature on the convexity of free boundaries typically addresses problems with only one boundary.[3] Here, we provide a treatment of a problem with two boundaries.[4]

The technical issues include the following:

1. The payoff function is neither convex nor concave, which significantly complicates the analysis.

2. The problem possesses two free boundaries, rather than a single one. Additional difficulties arise from the fact that both of these boundaries emanate from the same point.

3. The value function may fail to satisfy the smooth fit condition, in both the infinite and finite horizon cases. In the finite horizon case, this is associated with the possibility of a discontinuity in the first

[3]The boundaries in the Wiener sequential testing problem appear to be concave/convex, see Gapeev and Peskir [2004], however we are not aware of a proof.

[4]In addition to the American capped call option studied in Broadie and Detemple [1995], another financial application in which a pair of stopping boundaries appears is in the analysis of continuous instalment options, in which fees for the option are paid for throughout its lifetime, rather than simply at initiation, and investors may stop either to realize the payoff, or to stop paying fees. See, for example, Ciurlia and Roko [2005], Kimura [2009], Yang and Yi [2009].

order derivative of the lower boundary.

4. Last but not least, the above considerations force additional requirements on the behavior at time zero of the approximating Stefan problem, which makes the problem of finding an appropriate approximating initial condition more challenging. Nonetheless, we have discovered a method to approximate the delta function by Gaussians, which actually greatly simplifies the analysis both in our case and for the American put. This simplification, compared to the method employed in Chen et al. (2008), is significant, and in our opinion naturally connects to the relationship between diffusion processes and Gaussian distributions. The construction may in addition lead to effective numerical algorithms and their justification.

We note that our formulation of the problem ignores some aspects of the fee agreement that may be present in practice. First of all, we do not consider the fee for assets under management. If this is taken to be a lump sum paid at the time of initial investment, it will have no impact on the analysis of the timing decision. Secondly, if there is a maintenance fee to be paid continuously, it may be modeled by modifying the geometric Brownian motion (1.1) to have a constant dividend rate; this may have an impact on the nature of the solution of the optimal stopping problem. Thirdly, there may be a penalty for withdrawing investments from the hedge fund, which would alter the structure of the payoff g. The impact of these potential modifications is the subject of ongoing research.

The remainder of the paper is structured as follows. Section Two analyzes the perpetual optimal stopping problem (1.2). Section Three considers the finite horizon problem (1.3), introduces the Stefan problem for $u := \partial v/\partial T$ and its regularization, and employs these to derive basic properties of the value function v. Section Four uses a more detailed analysis of the Stefan problem to derive convexity properties of the boundaries $s_1(T)$ and $s_2(T)$. Section Five discusses how to recover the value function v from the solution u of the underlying Stefan problem, and Section Six concludes.

2 Perpetual Problem

In this section, we analyze the infinite horizon optimal stopping problem (1.2), by finding an expression for the value function. The expression is explicit, except for the need (for certain combinations of the parameters) to solve a nonlinear algebraic equation.

The payoff function g is Lipschitz continuous, and nonnegative, but not smooth. We first dispense with the mathematically trivial case when $p \geqslant q$.

Lemma 2.1. *Consider the optimal stopping problem $V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}[e^{-r\tau} g(X_\tau^x)]$, where g is concave, nonnegative, and continuous on $[0,\infty)$. Then $V = g$ and it is optimal to stop immediately.*

Consequently, if g is given by (1.4) with $p \geqslant q$, then $V = g$ and it is optimal to stop immediately.

Proof. Set $\mathcal{A} = \{(a,b) | g(x) \leqslant a + bx \ \ \forall x \in [0,\infty)\}$. For each $(a,b) \in \mathcal{A}$ and $\tau \in \mathcal{T}$, $a \geqslant g(0) \geqslant 0$ and

$$\mathbb{E}[e^{-r\tau} g(X_\tau^x)] \leqslant \mathbb{E}[e^{-r\tau}(a + bX_\tau^x)] \leqslant a\mathbb{E}[e^{-r\tau}] + bx \leqslant a + bx$$

by applying Theorem II.77.5 from Rogers and Williams [1994] (pages 189-190). Taking the supremum over $\tau \in \mathcal{T}$ and infimum over $(a,b) \in \mathcal{A}$ we obtain $V(x) \leqslant \min_{(a,b)\in\mathcal{A}}\{a+bx\} = g(x)$. $\square$

Throughout the remainder of the paper, we always assume that $q > p$.

2.1 General Properties of the Value Function

In this subsection we first establish a few properties of the value function V and then convert the problem into a free boundary problem for an ordinary differential equation. We shall use the Dynamic Programming Principle (Pham [2009], page 97, El Karoui [1981], Theorem 1.17, pages 95–97): for any stopping time $\sigma \in \mathcal{T}$,

$$V(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[e^{-r\tau} g(X_\tau^x)\mathbf{1}_{\tau<\sigma} + e^{-r\sigma} V(X_\sigma^x)\mathbf{1}_{\sigma\leq\tau}\right]. \tag{2.1}$$

For future use, we introduce a constant β and the Black-Scholes operator L by

$$\beta := \frac{2r}{\sigma^2}, \qquad Lf(x) := \frac{\sigma^2 x^2}{2} f''(x) + rxf'(x) - rf(x). \tag{2.2}$$

It is easy to check that for given constants $x_0 > 0$ and $v_0 \in \mathbb{R}$, the solution to the initial value problem

$$LW = 0 \text{ in } (0,\infty), \qquad W(x_0) = g(x_0), \quad W'(x_0) = v_0 \tag{2.3}$$

is unique and is given by

$$W(x, x_0, v_0) = g(x_0)\left(\frac{\beta}{1+\beta}\frac{x}{x_0} + \frac{1}{1+\beta}\left(\frac{x}{x_0}\right)^{-\beta}\right) + \frac{v_0 x_0}{1+\beta}\left(\frac{x}{x_0} - \left(\frac{x}{x_0}\right)^{-\beta}\right). \tag{2.4}$$

Now we consider a few basic properties of the value function V. The proof of the following lemma uses standard techniques, and is consequently omitted.[5]

Lemma 2.2. *Let g be given by* (1.4) (with $q > p$) *and V be defined by* (1.2). *Then the following holds:*

(i) *V is increasing, Lipschitz continuous, and $g(x) \leqslant V(x) \leqslant \min\{A + Bx, g(x) + (A+B)x^{-\beta}\}$ for each $x \in (0,\infty)$. In particular,*

$$V = g \text{ on } [0,\kappa], \qquad \lim_{x\to\infty}\Big\{V(x) - g(x)\Big\} = 0. \tag{2.5}$$

(ii) *If $V(a) = g(a)$ for some $a \in [\kappa, 1)$, then $V(x) = g(x)$ for all $x \in [0, a]$.*

(iii) *If $V(b) = g(b)$ for some $b \in (1,\infty)$, then $V(x) = g(x)$ for all $x \in [b,\infty)$.*

(iv) *$V(1) > g(1)$.*

(v) *If $V > g$ on $(a,b) \subset [\kappa,\infty)$, then $LV = 0$ on (a,b) and $V|_{[a,b]} \in C^\infty([a,b])$.*

[5] The proof is available as an appendix upon request from the reader.

(vi) *Let*

$$S_1 := \inf\{x \in [\kappa, 1) | V(x) > g(x)\}, \qquad S_2 = \sup\{x > 1 \mid V(x) > g(x)\}.$$

Then $S_1 \in [\kappa, 1)$*,* $S_2 \in (1, \infty)$*, and*

$$V > g \ \text{on} \ (S_1, S_2), \qquad V = g \ \text{on} \ [0, S_1] \cup [S_2, \infty).$$

(vii) $V'(S_2) = g'(S_2)$.

(viii) *If* $S_1 > \kappa$*, then* $V'(S_1) = g'(S_1)$*; if* $S_1 = \kappa$*, then* $V'(S_1) \in [g'(\kappa+), g'(\kappa-)] = [1 - q, B]$.

(ix) *For each fixed* $x \in (0, \infty)$*, let*

$$\tau_x^* := \inf\{t \geqslant 0 \mid V(X_t^x) = g(X_t^x)\} = \inf\{t \geqslant 0 \mid X_t^x \in [0, S_1] \cup [S_2, \infty)\}. \tag{2.6}$$

Then τ_x^* *is an optimal stopping time, i.e.*

$$V(x) := \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau} g(X_\tau^x)] = \mathbb{E}[e^{-r\tau_x^*} g(X_{\tau_x^*}^x)].$$

We can translate part of Lemma 2.2 as follows:

Theorem 1. *Let* g *be given by* (1.4) ($q > p$) *and* V *be the value function defined in* (1.2)*. Then* V*, together with some unknown constants* $S_1 \in [\kappa, 1)$ *and* $S_2 \in (1, \infty)$*, solve the free boundary problem:*

$$\begin{cases} LV = 0, \quad V > g & \text{in } (S_1, S_2), \\ V = g & \text{in } [0, S_1] \cup [S_2, \infty), \\ V'(S_2) = g'(S_2), & \\ V'(S_1+) \in [g'(S_1+), g'(S_1-)]. & \end{cases} \tag{2.7}$$

Remark 2.1. *We call* S_1 *and* S_2 *the free boundaries since a priori, they are unknown. The optimal stopping time* τ_x^* *can be interpreted as follows: it is suboptimal to continue holding the option once the asset value* X_t^x *drops below* S_1*, as it would be better to receive payment immediately. Similarly, it would be suboptimal to hold the option when the asset value is higher than* S_2*, since it is optimal to lock in gains when* $X_t^x \geqslant S_2$*.*

Remark 2.2. *Typically the underlying problem is formulated as a viscosity solution (see Crandall et al. [1992] or Touzi [2013]) of the variational inequality*

$$\min\{-LV,\ V - g\} = 0 \ \text{ in } \ (0, \infty), \qquad V(0) = g(0), \quad \lim_{x \to \infty} \{V(x) - g(x)\} = 0.$$

Instead of using the viscosity approach, we can use an alternative approach by defining the inequality $-LV \geqslant 0$ *in* $(0, \infty)$ *as follows:* $V \geqslant \tilde{V}$ *on* $[a, b]$ *if* $\tilde{V}$ *is the solution of* $L\tilde{V} = 0$ *in* (a, b) *with boundary condition* $\tilde{V}(a) = V(a), \tilde{V}(b) = V(b)$*.*

Another interpretation of $-LV \geqslant 0$ at non-smooth points of V is that V'' has an upper bound. This implies that $V'(S+) \leqslant V'(S-)$ for any $S \in (0,\infty)$, since otherwise V'' would be a positive delta function at S. As $V > g$ in (S_1, S_2), this condition implies the **smooth fit condition** *$V'(S_2) = g'(S_2)$ and the general* **consistent fit condition** *$V'(S_1+) \in [g'(S_1+), g'(S_1-)]$. Economically, smooth-fit fails at S_1 when it is optimal to withdraw from the fund exactly when the funds provided by the manager to compensate the investor's losses have been exhausted.*

From a probabilistic point of view, the most intuitive interpretation of $-LV \geqslant 0$ is that $\{e^{-rt}V(X_t^x)\}_{t\geq 0}$ is a super-martingale (V is r-excessive for X).

2.2 Solution of the Perpetual Problem

Here we solve the free boundary problem (2.7), for unknown $(V, S_1, S_2) \in \mathrm{Lip}([0,\infty)) \times [\kappa, 1) \times (1,\infty)$.

For parameters $S_1 \in [\kappa, 1)$ and $v_1 \in [1-q, B]$, the solution of the initial value problem $LW = 0$ on $(0,\infty)$ with $W|_{x=S_1} = g(S_1)$, $W'|_{x=S_1} = v_1$ has the form $W(x, S_1, v_1) = C_1 x + C_2 x^{-\beta}$ with

$$C_1 = 1 - q + \frac{v_1 - (1-q)}{1+\beta} + \frac{\beta q}{(1+\beta)S_1}, \tag{2.8}$$

$$C_2 = \frac{S_1^\beta}{1+\beta}\Big(q - [v_1 - (1-q)]S_1\Big). \tag{2.9}$$

Note that $v_1 < g(S_1)/S_1$. One then can verify from (2.4) that $W''(\cdot, S_1, v_1) > 0$ on $(0,\infty)$.

If we look for $S_2 > 1$ such that $W = g$ and $W' = g'$ at S_2, we must solve

$$C_1 S_2 + C_2 S_2^{-\beta} = p + (1-p)S_2, \qquad C_1 - \beta C_2 S_2^{-\beta-1} = 1 - p,$$

which are equivalent to

$$C_1 = (1-p) + \frac{\beta p}{1+\beta}\frac{1}{S_2}, \qquad C_2 = \frac{p}{1+\beta}S_2^\beta.$$

Eliminating S_2, we see that it is necessary and sufficient for C_1 and C_2 to obey the relation

$$C_1 = (1-p) + \frac{\beta p}{1+\beta}\left(\frac{(1+\beta)C_2}{p}\right)^{-1/\beta}. \tag{2.10}$$

Substituting (2.8) and (2.9) into (2.10) and dividing both sides by q we obtain the condition for (S_1, v_1) :

$$0 = \frac{p}{q} - 1 + \frac{\beta}{1+\beta}\,\frac{1}{S_1}\Big\{1 - \Big(\frac{p}{q}\Big)^{1+\frac{1}{\beta}}\Big[1 - \frac{v_1 - (1-q)}{q}S_1\Big]^{-\frac{1}{\beta}}\Big\} + \frac{v_1 - (1-q)}{q(1+\beta)}. \tag{2.11}$$

Set $\eta = p/q$. If $S_1 \in (\kappa, 1)$, then $v_1 = g'(S_1) = 1 - q$, so (2.11) becomes $S_1 = H(\eta)$ where:

$$H(z) = \begin{cases} \frac{\beta}{1+\beta}\cdot\frac{1-z^{1+\frac{1}{\beta}}}{1-z} & \text{if } z \neq 1, \\ 1 & \text{if } z = 1. \end{cases} \tag{2.12}$$

The following result is elementary.

Lemma 2.3. *$H \in C^\infty([0,\infty)), H' > 0$ on $[0,\infty)$, $H(1) = 1$, and $\lim_{z\to\infty} H(z) = \infty$.*

The following result follows immediately from the above calculations together with Theorem 1 and the strict convexity of W on $[S_1, S_2]$ which ensures that $W > g$ on (S_1, S_2).

Lemma 2.4. *If $S_1 \in (\kappa, 1)$, then $S_1 = H(\eta) > \kappa$, $S_2 = H(\frac{1}{\eta})$, and*

$$V(x) = \begin{cases} W(x, S_1, 1-q) & \text{if } x \in [S_1, S_2], \\ g(x) & \text{if } x \in [0, S_1] \cup [S_2, \infty). \end{cases} \tag{2.13}$$

Finally, we give a complete characterization of the value function.

Theorem 2. *The solution of the free boundary problem (2.7) is uniquely given as follows:*
1. If $\kappa \leqslant H(\eta)$, then $S_1 = H(\eta), S_2 = H(\frac{1}{\eta})$, and V is given by (2.13).
2. If $\kappa > H(\eta)$, then

$$S_1 = \kappa, \quad S_2 = \left(\frac{1-\delta}{\eta}\right)^{\frac{1}{\beta}} \kappa,$$

and

$$V(x) = \begin{cases} W(x, \kappa, 1-q+\frac{\delta q}{\kappa}) & \text{if } x \in [\kappa, S_2], \\ g(x) & \text{if } x \in [0, \kappa] \cup [S_2, \infty) \end{cases}$$

where δ is the unique root of $F(\cdot) = 0$ on $[0, 1-\eta)$ and

$$F(z) := z - \beta\eta^{1+\frac{1}{\beta}}\left((1-z)^{-\frac{1}{\beta}} - 1\right) + (1+\beta)(1-\eta)\Big(H(\eta) - \kappa\Big).$$

Proof. If $S_1 = \kappa$, we set $v_1 = 1 - q + \delta q/\kappa$, where $\delta \in [0, 1 - A/q]$, and (2.11) becomes $F(\delta) = 0$. Note that $F(0) = (1+\beta)(1-\eta)(H(\eta)-\kappa)$, $F'' < 0$ on $[0,1)$, $F'(1-\eta) = 0, F(1-\eta) > 0$, and $F(1) = -\infty$.

Suppose $\kappa < H(\eta)$. There is no solution to $F(\cdot) = 0$ on $[0, 1-\eta]$ and the solution of $F(\cdot) = 0$ in $(1-\eta, 1)$ yields $S_2 < \kappa$ which cannot be used. Thus, $S_1 \in (\kappa, 1)$ and we have the assertion of Lemma 2.4.

Suppose $\kappa \geqslant H(\eta)$. We must have $S_1 = \kappa$, $\delta \in [0,1)$ and $F(\delta) = 0$. There are two solutions to $F(\delta) = 0$, $\delta_1 \in [0, 1-\eta)$ and $\delta_2 \in (1-\eta, 1)$. One can verify that setting $\delta = \delta_2$ yields $S_2 \in (0, \kappa)$, which thus cannot lead to a solution of (2.7). On the other hand, $\delta = \delta_1$ leads to $S_2 > 1$. It is then clear that V as specified above yields the unique solution of (2.7). □

Using the above result, we can examine the free boundaries in the infinite horizon case under different financial conditions. Figure 2 shows S_1 and S_2 as β varies for the case of the payoff (1.5). Here, we have set the performance fee $\alpha = p = 0.5$, so that the hedge fund manager keeps half of the profits, and $c = 0.1, \theta = q = 1.0$, so that the investor's losses are completely compensated, up to 10% of the initial investment. In the notation of our general problem here, we have $A = c = 0.1$, and $B = 1$. We see that when β is small (corresponding to relatively low interest rates and high volatility), the boundaries are far apart, and the lower boundary is at the bound $\kappa = 1 - \frac{c}{\theta} = 0.9$. In contrast, when β is large (corresponding to relatively high interest rates and low volatility), the continuation region is much smaller, and the investor will likely withdraw from the contract very early on. For smaller values of θ, implying less downside protection (and lower values of κ), the flat portion of the curve for S_1 as a function of β is shorter, as with less downside protection, it is more often optimal to exit the investment earlier.

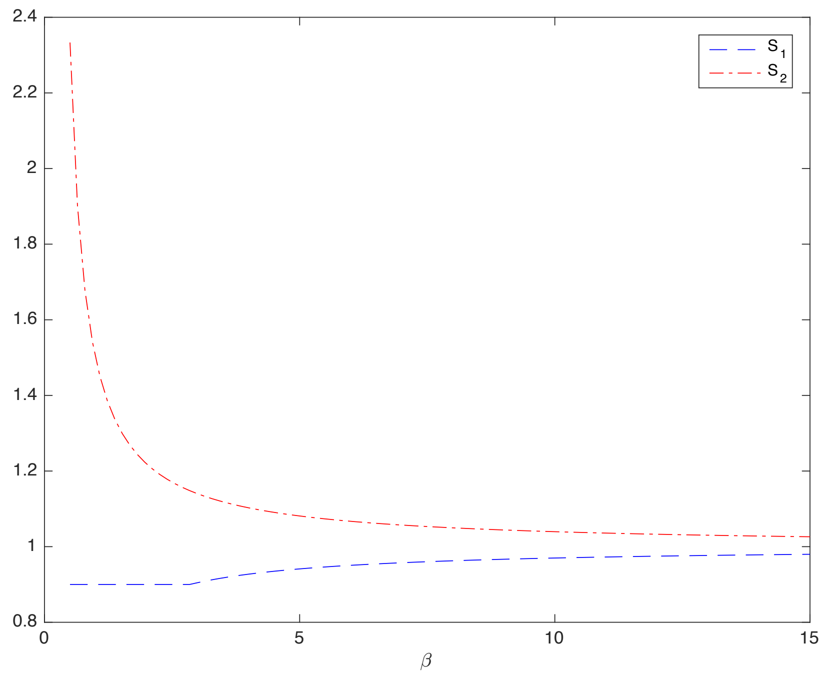

Figure 2: Upper and lower boundaries S_1 and S_2 for the optimal stopping problem with payoff (1.5) with $\alpha = 0.5$, $\theta = 1$, $c = 0.1$, for different values of β.

In Figure 3 we have fixed the value of β at 2.5 (this would arise, for example, if $r = 0.05$, and $\sigma = 0.2$). We still consider the payoff (1.5) with $\alpha = 0.5$, and $c = 0.1$, but vary the value of θ, giving the fraction of the investor's losses that are covered by the fund manager. We also plot the lower bound κ for S_1 (notice that this parameter varies with θ since $\kappa = 1 - \frac{c}{\theta}$ for this fee structure). The upper and lower boundaries behave intuitively. For low levels of protection, it is optimal to exercise early (the case $\theta = 0.5$ corresponds to $p = q$, whence $V = g$ by Lemma 2.1 and it is optimal to exercise immediately). For large levels of protection, the continuation region (S_1, S_2) is much larger (and, in particular, $S_1 = \kappa$), implying that it is optimal to invest in the fund for a longer period of time to accrue both the benefits of the upside performance and the downside protection.

A more detailed analysis of the sensitivity of the investment value to different parameter values in the 'European' case (i.e. with a fixed final date T, and no early withdrawal) is contained in Djerroud et al. [2016].

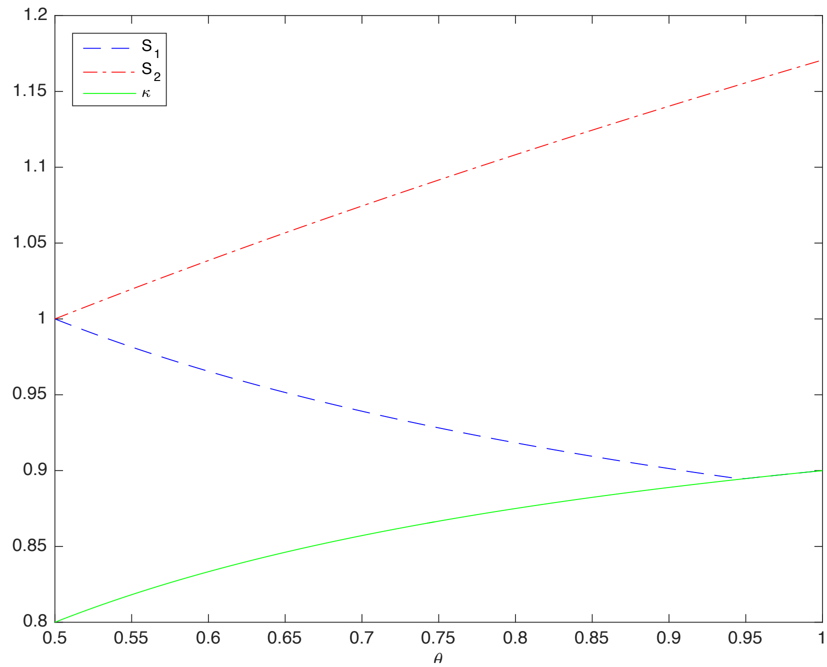

Figure 3: Upper and lower boundaries S_1 and S_2 for the optimal stopping problem with payoff (1.5) with $\alpha = 0.5$, $c = 0.1$, $\beta = 2.5$, for different values of θ.

3 Finite Horizon Case

In this section, we analyze the finite horizon optimal stopping problem; i.e, the value function given in (1.3), where $\mathcal{T}_{[0,T]}$ is the set of all stopping times τ such that $0 \leqslant \tau \leqslant T$ almost surely.

As in the infinite horizon case, if $p \geqslant q$, it is optimal to exercise immediately, and $v(x, T) = g(x)$ for all T. Thus, in the sequel, we always assume that $p < q$.

3.1 Basic Properties of the Finite Horizon Problem

We begin with the following stability result. While we believe the result to be well-known, we are unaware of a precise reference:

Lemma 3.1. *Let h be such that $\|h-g\|_{L^\infty([0,\infty))} < \infty$ and let $v_h(x,T) = \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}[e^{-r\tau} h(X_\tau^x)]$. Then $\|v - v_h\|_{L^\infty([0,\infty)\times[0,\infty))} \leqslant \|h - g\|_{L^\infty([0,\infty))}$.*

Proof. Let $\varepsilon > 0$ and τ_g be an ε-optimal stopping time for v. Then

$$v(x,T) - v_h(x,T) \leqslant \mathbb{E}[e^{-r\tau_g} g(X_{\tau_g}^x)] + \varepsilon - \mathbb{E}[e^{-r\tau_g} h(X_{\tau_g}^x)] \leqslant \|g - h\|_{L^\infty[0,\infty)} + \varepsilon. \tag{3.1}$$

Since ε is arbitrary, $v(x,T) - v_h(x,T) \leqslant \|g-h\|_{L^\infty[0,\infty)}$. The proof of $v_h(x,T) - v(x,T) \leqslant \|g-h\|_{L^\infty([0,\infty))}$ is similar. Thus, $|v - v_h| \leqslant \|g - h\|_{L^\infty([0,\infty))}$. $\square$

Since g is not smooth, in applications, this stability results allows us to replace g by its smooth regularization.

Next, we establish a relation between v and V. While we expect the following result be well-known in greater generality than presented here, we are unaware of a precise reference in the general case (for the American put, see Karatzas and Shreve [1998], Corollary 7.3, page 70, or Chen and Chadam [2007], Theorem 2.3, pages 1619-1620, for an analytic approach).

Lemma 3.2. *For each $x \in [0,\infty)$ and $0 \leqslant T_1 \leqslant T_2$,*

$$g(x) = v(x,0) \leqslant v(x,T_1) \leqslant v(x,T_2) \leqslant V(x), \qquad \lim_{T\to\infty} v(x,T) = V(x).$$

Proof. Since $\mathcal{T}_{[0,T_1]} \subseteq \mathcal{T}_{[0,T_2]} \subseteq \mathcal{T}$, it is immediate that $v(x,T)$ is increasing in T and $g(x) = v(x,0) \leqslant v(x,T_1) \leqslant v(x,T_2) \leqslant V(x)$. In particular, for a fixed x, the $\lim_{T\to\infty} v(x,T)$ is well-defined.

Since $v(x,T) \leqslant V(x)$ for all T, we have that $\lim_{T\to\infty} v(x,T) \leqslant V(x)$. Let τ_x^*, given by (2.6), be the optimal stopping time for the perpetual problem. Then by Fatou's Lemma:

$$\lim_{T\to\infty} v(x,T) \geqslant \varliminf_{T\to\infty} \mathbb{E}[e^{-r\ \tau_x^*\wedge T} g(X^x_{\tau_x^*\wedge T})] \geqslant \mathbb{E}[\varliminf_{T\to\infty} e^{-r\ \tau_x^*\wedge T} g(X^x_{\tau_x^*\wedge T})] = \mathbb{E}[e^{-r\tau_x^*} g(X^x_{\tau_x^*})] = V(x).$$

This completes the proof. $\square$

Since $g(x) \leqslant v(x,T) \leqslant V(x) = g(x)$ when $x \in [0,S_1] \cup [S_2,\infty)$, we have the following:

Proposition 1. *The value function v is the unique continuous viscosity solution of*

$$\min\left\{\frac{\partial v}{\partial T} - Lv, v - g\right\} = 0 \quad in \quad (0,\infty)\times(0,\infty), \tag{3.2}$$

$$v(\cdot,0) = g(\cdot), \qquad v(x,T) = g(x) \quad \forall\, x \in [0,S_1]\cup[S_2,\infty), T \geqslant 0. \tag{3.3}$$

Proof. That v is a viscosity solution is a standard derivation; see for example, Touzi [2013], pages 95–99. We omit the details. $\square$

In general, some boundary conditions on v are necessary in order to ensure uniqueness; for example, boundedness near the origin and linear growth at ∞ would also uniquely identify v. For simplicity, we have assigned conditions based on properties of the value function that had already been demonstrated.

Consider the stopping and continuation regions at time to expiry T,

$$\mathcal{S}_T = \{x \geqslant 0 | v(x,T) = g(x)\}, \qquad \mathcal{C}_T = \{x > 0 | v(x,T) > g(x)\}.$$

The following Lemma lists analogous properties in the finite horizon case to those given for the perpetual case in Lemma 2.2.

Lemma 3.3. *For all $T \in (0,\infty)$, the following holds:*

1. $1 \in \mathcal{C}_T$.

2. If $a \in (S_1,1)\cap\mathcal{S}_T$, then $[0,a] \subseteq \mathcal{S}_T$.

3. If $b \in (1,S_2)\cap\mathcal{S}_T$, then $[b,\infty) \subseteq \mathcal{S}_T$.

Proof. 1. Let $\underline{v}$ be the solution of the parabolic problem $\partial\underline{v}/\partial T = L\underline{v}$ in $(0,\infty)^2$, $\underline{v}(\cdot,0)=g$. Then $\underline{v}(x,T)=\mathbb{E}[e^{-rT}g(X_T^x)] \leqslant v(x,T)$. Since $g'(1-) < g'(1+)$, one can check that $\underline{v}(1,T) > g(1)$ for every $0 < T \ll 1$. Thus, $v(1,T) > g(1)$ for small T. Since v is increasing in T, we see that $v(1,T) > g(1)$ for all $T > 0$.

2. Suppose $a \in (S_1,1)\cap\mathcal{S}_T$. If the assertion $[0,a]\subseteq\mathcal{S}_T$ is not true, then $F(\cdot) := v(\cdot,T) - g(\cdot)$ on $[S_1,a]$ will attain a positive local maximum at some $\hat{x} \in (S_1,a)$. Also, v is a smooth solution of $\partial v/\partial T = Lv$ in a neighborhood of $(\hat{x},T)$. Thus, $F''(\hat{x}) \leqslant 0, F'(\hat{x}) = 0, F(\hat{x}) > 0$. This implies that

$$0 < -LF(\hat{x}) = -v_T + Lg \leqslant Lg = -rq,$$

which is impossible. Thus, $[0,a] \in \mathcal{S}_T$.

3. Using $Lg = -rp$ on $(1,\infty)$, the proof follows in a manner analogous to the previous step.

□

The above lemma immediately implies the existence of free boundaries:

Lemma 3.4. *For each $T > 0$, define*

$$s_1(T) := \inf\{x > 0 | v(x,T) > g(x)\}, \qquad s_2(T) = \sup\{x > 0 | v(x,T) > g(x)\}. \tag{3.4}$$

Then

$$v(\cdot,T) > g(\cdot) \ \text{in} \ (s_1(T), s_2(T)), \qquad v(\cdot,T) = g \ \text{on} \ [0, s_1(T)] \cup [s_2(T),\infty).$$

Furthermore, $s_2(\cdot)$ is an increasing function, $s_1(\cdot)$ is a decreasing function, and

$$\lim_{T\to\infty} s_1(T) = S_1, \quad \lim_{T\to\infty} s_2(T) = S_2.$$

In the remainder of this paper, we shall study the free boundaries $x = s_1(\cdot)$ and $x = s_2(\cdot)$. Besides smoothness, we shall show that $\ln s_1(\cdot)$ is a convex function and $\ln s_2(\cdot)$ is concave function.

Remark 3.1. *The fits $v_x(s_2(T),T) = g'(s_2(T))$ and $v_x(s_1(T)+,T) \in [g'(s_1(T)+), g'(s_1(T)-)]$ are typically hard to establish for viscosity solutions due to the lack of regularity. Here they can be proven by two facts: (i) the viscosity solution of (3.1) is unique and (ii) taking the limit of the regularization one can construct a viscosity solution satisfying $v_x \in C([\kappa,\infty)\times(0,\infty))$, so the smooth fit conditions are automatically satisfied.*

3.2 Formal Derivation of a Stefan Problem

As outlined earlier, our basic strategy is to analyze the Stefan problem that is solved (at first formally) by the time derivative of the value function, and then to derive (rigorously) properties of the value function from the properties of the Stefan problem solution. In this section, we present a formal derivation of the

Stefan problem, and outline the strategy to derive its properties. A rigorous verification will be given in Section 5.

We begin with the assumption that $\kappa \leqslant H(\eta)$, so for the infinite horizon problem we have the smooth-fit free boundary condition. It is more convenient to carry out this analysis after having performed a change of variables as follows. We write the functions in the previous section as $v(S,T)$ and $s_j(T)$. We introduce new variables

$$x = \ln S, \qquad t = \frac{\sigma^2}{2}T, \qquad \mathcal{L} = \frac{\partial^2}{\partial x^2} + (\beta-1)\frac{\partial}{\partial x} - \beta$$
$$x_j(t) = \ln s_j(T), \qquad u = \frac{2}{(q-p)\sigma^2}\frac{\partial v}{\partial T} = \frac{1}{q-p}\frac{\partial v}{\partial t}.$$

Assume that the boundaries s_j are smooth. In the image of the continuation region, we should have $u_t = \mathcal{L}u$, since v should be a classical solution of $v_T = Lv$ in $\mathcal{C}$. On the boundary of this region (i.e. on the images of the curves s_j) we should have $u = 0$ (by considering the left time derivative $\frac{\partial v}{\partial T}(s_j(T), T-)$ and using that $v = g$ in $\mathcal{S}$). To derive a second condition on the boundary, assume that the smooth fit condition $(v-g)_S = 0$ at $s_j(T)$ holds. Differentiating with respect to T at $s_j(T)$ gives $(v-g)_{SS}\dot{s}_j + v_{ST} = 0$, and thus

$$\frac{ds_j}{dT} = -\frac{v_{ST}(s_j(T),T)}{(v-g)_{SS}}, \quad j = 1,2. \tag{3.5}$$

Now, on the boundaries,

$$0 = v_T - Lv \quad = -\tfrac{\sigma^2 S^2}{2}(v_{SS} - g_{SS}) - Lg \Longrightarrow (v-g)_{SS} = -\tfrac{2Lg}{\sigma^2 S^2}.$$

Thus

$$\frac{dx_j}{dt} = \frac{2}{\sigma^2 s_j}\frac{ds_j}{dT} = -\frac{2}{\sigma^2 s_j}\frac{v_{ST}(s_j(T),T)}{(v-g)_{SS}} = \frac{s_j v_{ST}}{Lg(s_j)} = \frac{(q-p)\sigma^2}{2Lg(s_j)}u_x.$$

Note that $Lg = -rp$ in $(1,\infty)$ and $Lg = -rq$ in $(\kappa, 1)$, hence, we

$$\ell_j \frac{dx_j}{dt} = -u_x(x_j(t),t) \tag{3.6}$$

where:

$$\ell_1 = \frac{2qr}{(q-p)\sigma^2}, \quad \ell_2 = \frac{2pr}{(q-p)\sigma^2} < \ell_1\ . \tag{3.7}$$

Since it will turn out that $\dot{x}_2(t) > 0 > \dot{x}_1(t)$, from now on we use the notation $x_+(t) = x_2(t)$, $\ell_+ = \ell_2$, $x_-(t) = x_1(t)$, $\ell_- = \ell_1$.

We show that $s_j(0+) = 1$, i.e. $x_\pm(0+) = 0$. Thus, at time zero, $u = \frac{2}{\sigma^2}\cdot\frac{1}{q-p}\cdot\frac{\partial v}{\partial T} = \frac{2}{\sigma^2(q-p)}Lg = \delta(x)$. Thus, formally, u, together with free boundary $x_\pm$, is the solution of the free boundary problem:

$$\begin{cases} u_t - \mathcal{L}u = 0 & t>0, \quad x_-(t) < x < x_+(t), \\ u = 0 & t > 0, \quad x = x_\pm(t), \\ \ell_\pm \dot{x}_\pm(t) = -u_x(x_\pm(t),t) & t>0, \quad x = x_\pm(t), \\ x_+(0) = x_-(0) = 0, & \\ \lim_{t\to 0} u(\cdot,t) = \delta(\cdot), & \end{cases} \tag{3.8}$$

where $\ell_- = \ell_1$ and $\ell_+ = \ell_2$ is given by (3.7), and $\delta(\cdot)$ is the Dirac delta function.

Note that the problem (3.8) does not depend on κ. Hence, as $t \to \infty$ we have

$$\lim_{t\to\infty} x_-(t) = \ln H(\eta), \qquad \lim_{t\to\infty} x_+(t) = \ln H(\eta^{-1}).$$

Now consider two cases:

1. Suppose $\kappa \leqslant H(\eta)$. Then the function v can be recovered from the solution of (3.8) alone.

2. Suppose $\kappa > H(\eta)$. Then there exists a finite time $t^* > 0$ such that

$$x_-(t^*) = \ln \kappa, \qquad x'_-(t^*) < 0.$$

Then the continuation of the solution after $t > t^*$ should be replaced by the solution of

$$\begin{cases} \tilde{u}_t - \mathcal{L}\tilde{u} = 0 & t > t^*, \quad \ln\kappa < x < \tilde{x}_+(t), \\ \ell_+ \dot{\tilde{x}}_+(t) = -\tilde{u}_x(\tilde{x}_+(t), t), & t > t^*, \\ \tilde{u}(\ln(\kappa), t) = 0, \quad \tilde{u}(\tilde{x}_+(t), t) = 0, & t > t^*, \\ \tilde{x}_+(t^*) = x_+(t^*), \quad \tilde{u}(\cdot, t^*) = u(\cdot, t^*). \end{cases} \tag{3.9}$$

The analysis of this free boundary problem with one free boundary is not more difficult than that of the free boundary problem (3.8) which has two free boundaries to consider.

3.3 Regularization of The Stefan Problem

The singularity of the initial condition in (3.8) makes it somewhat difficult to analyze directly. Consequently, we study its regularization. The stability result Lemma 3.1 allows us to replace u_0 by its smooth regularization. Thus, for each small $\varepsilon > 0$, we study

$$\begin{cases} u^\varepsilon_t - \mathcal{L}u^\varepsilon = 0 & \text{for } \ t > 0, \quad x^\varepsilon_-(t) < x < x^\varepsilon_+(t) \\ u^\varepsilon = 0 & \text{for } \ t \geqslant 0, \quad x = x^\varepsilon_\pm(t) \\ \ell^\varepsilon_\pm \frac{dx^\varepsilon_\pm}{dt}(t) = -u^\varepsilon_x(x^\varepsilon_\pm(t), t), & \text{for } \ t \geqslant 0, \\ x^\varepsilon_\pm(0) = x^\varepsilon_{\pm,0}, & \\ u^\varepsilon(x, 0) = u^\varepsilon_0(x) & \text{for } \ x^\varepsilon_{-,0} \leqslant x \leqslant x^\varepsilon_{+,0} \end{cases} \tag{3.10}$$

where $x^\varepsilon_{\pm,0}$ and u^ε_0 are carefully selected initial data, and $\ell^\varepsilon_\pm$ are carefully selected parameters. As $\varepsilon \searrow 0$, we want

$$x^\varepsilon_{\pm,0} \to 0, \qquad \ell^\varepsilon_\pm \to \ell_\pm, \quad \int_{x^\varepsilon_{-,0}}^{x^\varepsilon_{+,0}} u_{0\varepsilon}(x)dx \to 1.$$

We extend u^ε over $\mathbb{R} \times (0, \infty)$ by $u^\varepsilon = 0$ for $t \geqslant 0$, $x \in (-\infty, x^\varepsilon_-(t)) \cup (x^\varepsilon_+(t), \infty)$. The function v^ε will be defined by $v^\varepsilon(x,t) = g^\varepsilon(x) + (q-p)\int_0^t u^\varepsilon(x,\tau)d\tau$ for some suitably chosen g^ε. It will be shown that v^ε is the value function with payoff function g^ε. Thus, by comparison, $\|v^\varepsilon - v\|_{L^\infty(\mathbb{R}\times[0,\infty))} \leqslant \|g^\varepsilon - g\|_{L^\infty(\mathbb{R})}$. After showing that $g^\varepsilon \to g$, we find that $v^\varepsilon \to v$ and $x^\varepsilon_\pm \to x_\pm$. The boundaries $x^\varepsilon_\pm$ are smooth and monotone, and it will be shown that x_+^ε is concave and x_-^ε is convex, and these properties carry over to their respective limits. The convexity properties are the most challenging. The strategy for their proof is as follows. First, it is shown that with a careful choice of the initial condition u_0^ε the signs of the derivatives of u^ε have the pattern given in Figure 4.

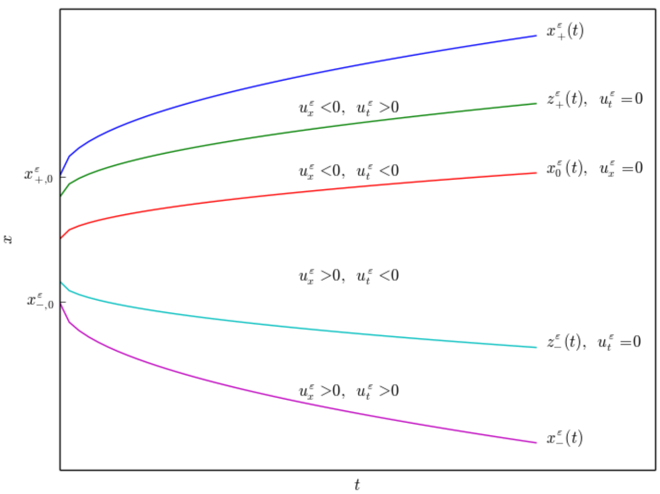

Figure 4: Signs of derivatives of u^ε in the approximating Stefan problem.

The next step is to differentiate (with respect to t) the two boundary conditions for u^ε (i.e. $u^\varepsilon = 0$ and $u_x^\varepsilon = -\ell_\pm^\varepsilon \dot{x}_\pm^\varepsilon$) on the boundaries $x_\pm^\varepsilon$ to obtain

$$u_x^\varepsilon(x_\pm^\varepsilon(t),t)\dot{x}_\pm^\varepsilon(t) + u_t^\varepsilon(x_\pm^\varepsilon(t),t) = 0, \tag{3.11}$$

$$u_{xt}^\varepsilon(x_\pm^\varepsilon(t),t) + u_{xx}^\varepsilon(x_\pm^\varepsilon(t),t)\dot{x}_\pm^\varepsilon(t) = -\ell_\pm^\varepsilon \ddot{x}_\pm^\varepsilon. \tag{3.12}$$

This gives that $\dot{x}_\pm^\varepsilon(t) = -\phi^\varepsilon(x_\pm^\varepsilon(t),t)$, where

$$\phi^\varepsilon(x,t) = \frac{u_t^\varepsilon(x,t)}{u_x^\varepsilon(x,t)}.$$

Differentiating ϕ^ε with respect to x at $x_\pm^\varepsilon(t)$ and using (3.11) and (3.12) yields

$$\phi_x^\varepsilon(x^\varepsilon(t),t) = \frac{u_{tx}^\varepsilon u_x^\varepsilon - u_t^\varepsilon u_{xx}^\varepsilon}{(u_x^\varepsilon)^2} = \frac{u_x^\varepsilon(u_{tx}^\varepsilon + u_{xx}^\varepsilon \dot{x}_\pm^\varepsilon) - u_{xx}^\varepsilon(u_t^\varepsilon + u_x^\varepsilon \dot{x}_\pm^\varepsilon)}{(u_x^\varepsilon)^2} = \frac{u_{tx}^\varepsilon + u_{xx}^\varepsilon \dot{x}_\pm^\varepsilon}{u_x^\varepsilon} = \frac{\ddot{x}_\pm^\varepsilon}{\dot{x}_\pm^\varepsilon}.$$

Therefore,

$$\ddot{x}_\pm^\varepsilon(t) = \dot{x}_\pm^\varepsilon(t)\phi_x(x_\pm^\varepsilon(t),t) = -\phi^\varepsilon(x_\pm^\varepsilon(t),t)\phi_x^\varepsilon(x_\pm^\varepsilon(t),t). \tag{3.13}$$

From the signs of the derivatives of u^ε in Figure 4, we see that $\phi^\varepsilon(x_-^\varepsilon(t),t) > 0$ and $\phi^\varepsilon(x_+^\varepsilon(t),t) < 0$, so that the asserted convexity properties of $x_\pm^\varepsilon$ will follow if it can be shown that $\phi_x^\varepsilon(x_\pm^\varepsilon(t),t) < 0$. For a careful choice of the initial condition u_0^ε, this can be proved using the PDEs satisfied by ϕ^ε and $\psi^\varepsilon := \phi_x^\varepsilon$ and a maximum principle argument.

4 The Stefan Problem

In this section, we study (3.10), establishing certain properties of the solution $(u^\varepsilon, x^\varepsilon_\pm)$. Since we directly connect v with u^ε, we omit most of the process of taking the limit as $\varepsilon \downarrow 0$ to obtain a classical solution of (3.8). In order to carry out the strategy outlined in Section 3.3, we require that the solutions u^ε have sufficient regularity, including on the boundaries $x^\varepsilon_\pm$. For this to hold, the conditions on the boundaries $x^\varepsilon_\pm$, and the initial condition u^ε_0 must satisfy consistency conditions (see, for example, Friedman [1964], Chapter 3). The zeroth order consistency condition comes from matching the values of the initial condition and the boundary conditions at time zero, and leads to:

$$u^\varepsilon_0 > 0 \text{ in } (x^\varepsilon_{-,0}, x^\varepsilon_{+,0}), \quad u^\varepsilon_0(x^\varepsilon_{\pm,0}) = 0. \tag{4.1}$$

Note that since $u^\varepsilon(x^\varepsilon_\pm(t), t) = 0$, differentiation gives $u^\varepsilon_t + u^\varepsilon_x \dot{x}^\varepsilon_\pm = 0$, so $u^\varepsilon_t = -\dot{x}^\varepsilon_\pm u^\varepsilon_x$ on $x = x^\varepsilon_\pm(t)$. Using the second boundary condition $\dot{x}^\varepsilon_\pm = -\frac{1}{\ell^\varepsilon_\pm} u^\varepsilon_x$ we obtain $u^\varepsilon_t = \frac{1}{\ell^\varepsilon_\pm}(u^\varepsilon_x)^2$, which leads to the first order consistency condition:

$$\ell^\varepsilon_\pm \mathcal{L} u^\varepsilon_0(x^\varepsilon_{\pm,0}) = (u^\varepsilon_{0x}(x^\varepsilon_{\pm,0}))^2. \tag{4.2}$$

The following result can be proved using well-known techniques from the analysis of the Stefan problem. A sketch of the proof is given in the Appendix.

Lemma 4.1. *Assume that $u^\varepsilon_0 \in C^4([x^\varepsilon_{-,0}, x^\varepsilon_{+,0}])$ and satisfies (4.1) and (4.2). Then (3.10) admits a unique classical solution $(x^\varepsilon_+, x^\varepsilon_-, u^\varepsilon)$ with the following properties:*

$$\begin{gathered} x^\varepsilon_\pm \in C^\infty((0,\infty)) \cap C^{2+\alpha/2}([0,\infty)) \qquad \forall \alpha \in (0,1), \\ u^\varepsilon \in C^\infty(\cup_{t>0}[x^\varepsilon_-(t), x^\varepsilon_+(t)] \times \{t\}) \cap C^{3+\alpha,(3+\alpha)/2}\left(\cup_{t\geq 0}[x^\varepsilon_-(t), x^\varepsilon_+(t)] \times \{t\}\right), \\ \dot{x}^\varepsilon_+(t) > 0 > \dot{x}^\varepsilon_-(t), \quad u^\varepsilon(x,t) > 0 \quad \forall\, x \in (x^\varepsilon_-(t), x^\varepsilon_+(t)), t \geqslant 0. \end{gathered}$$

The arguments in the appendix also yield the following estimate (used in the appendix to derive global existence for (3.10) from local existence). Recall that $x_+(t) = \ln s_2(T), x_-(t) = \ln s_1(T)$ with $T = 2t/\sigma^2$.

Theorem 3. *As $\varepsilon \searrow 0$, $x^\varepsilon_\pm \to x_\pm$. Also, $x_\pm \in C^\infty((0,\infty)) \cap C([0,\infty))$, $\pm x_\pm(t) > 0$ for $t > 0$, and*

$$x_\pm(0+) = 0, \qquad \left|\frac{dx_\pm(t)}{dt}\right| \leqslant \frac{|x_\pm(t)|}{\ell_\pm\sqrt{4\pi t^3}} \exp\left(-\tfrac{|x_\pm(t)|^2}{4t} - \tfrac{(\beta-1)x_\pm(t)}{2} - \tfrac{(\beta+1)^2}{4}t\right) \quad \forall\, t > 0. \tag{4.3}$$

The differential inequality in (4.3) implies that $x_\pm(t) = O(\sqrt{t|\ln t|})$ as $t \searrow 0$. We omit the details.

Next, we proceed to show that under additional conditions on u^ε_0, the signs of the derivatives of the solution u^ε to problem (3.10) have the pattern displayed in Figure 4.

Lemma 4.2. *Suppose, in addition to the conditions (4.1), (4.2), assumed in Lemma 4.1, that there exists $x^\varepsilon_0 \in (x^\varepsilon_{-,0}, x^\varepsilon_{+,0})$ such that $u^\varepsilon_{0x} > 0$ in $[x^\varepsilon_{-,0}, x^\varepsilon_0)$, $u^\varepsilon_{0x} < 0$ in $(x^\varepsilon_0, x^\varepsilon_{+,0}]$, and $u^\varepsilon_{0xx}(x^\varepsilon_0) < 0$. Then there exists a smooth curve $x^\varepsilon_0(t)$ with $x^\varepsilon_0(0) = x^\varepsilon_0$ such that $u^\varepsilon_x(x^\varepsilon_0(t), t) = 0$, $u^\varepsilon_x(x,t) > 0$ for $x \in [x^\varepsilon_-(t), x^\varepsilon_0(t))$, and $u^\varepsilon_x(x,t) < 0$ for $x \in (x^\varepsilon_0(t), x^\varepsilon_+(t)]$. Furthermore, $u^\varepsilon_t(x^\varepsilon_0(t), t) < 0$.*

Proof. Let $D^\varepsilon = \{(x,t)|x_-^\varepsilon(t) < x < x_+^\varepsilon(t), t > 0\}$. The Strong Maximum Principle (Friedman [1964], Theorem 3.1, pages 34–38) implies that $u^\varepsilon > 0$ in D^ε. Since $u^\varepsilon = 0$ on $(-\infty, x_-^\varepsilon(t)] \cup [x_+^\varepsilon(t), \infty)$ there exists a point $x_0^\varepsilon(t) \in (x_-^\varepsilon(t), x_+^\varepsilon(t))$ where $u^\varepsilon(\cdot,t)$ attains its (positive) maximum, and at which $u_x^\varepsilon(x_0^\varepsilon(t),t) = 0$. From Hopf's boundary point lemma (Friedman [1964], Theorem 3.14, page 49), we have that $u_x^\varepsilon(x_-^\varepsilon(t),t) > 0$ and $u_x^\varepsilon(x_+^\varepsilon(t),t) < 0$. By differentiating, we see that $w^\varepsilon = u_x^\varepsilon$ solves $\partial_t w^\varepsilon - \mathcal{L}w^\varepsilon = 0$. Uniqueness of the point $x_0^\varepsilon(t)$ in $(x_-^\varepsilon, x_+^\varepsilon)$ at which $u_x^\varepsilon(x_0^\varepsilon(t),t) = w^\varepsilon(x_0^\varepsilon(t),t) = 0$ then follows since w^ε has only one sign change on the parabolic boundary of D^ε (i.e. the single root at time 0; the number of roots of $w(\cdot,t) = 0$ for a fixed t of the solution of the equation $\partial_t w^\varepsilon - \mathcal{L}w^\varepsilon = 0$ must be decreasing in t, see Sattinger [1969], Theorem 4 and Corollary 5, page 84[6]).

To prove smoothness, it is enough to show that $u_{xx}^\varepsilon(x_0^\varepsilon(t),t) < 0$, and then apply the Implicit Function Theorem. Let $\tilde{u}^\varepsilon = e^{\beta t}u^\varepsilon$, $\tilde{w}^\varepsilon = \tilde{u}_x^\varepsilon$, and note that $\tilde{w}_t^\varepsilon = \tilde{w}_{xx}^\varepsilon + (\beta - 1)\tilde{w}_x^\varepsilon$ in D^ε. For $\delta > 0$, let $t_1(\delta) = \inf\{t > 0|\tilde{w}^\varepsilon(x_+^\varepsilon(t),t) = -\delta\}$, $t_2(\delta) = \inf\{t > 0|\tilde{w}^\varepsilon(x_-^\varepsilon(t),t) = \delta\}$, $t^*(\delta) = \min(t_1(\delta), t_2(\delta))$, and note that $\lim_{\delta\downarrow 0} t^*(\delta) = \infty$ (if t^* were bounded along some sequence δ_n tending to zero, passing to a convergent subsequence would yield a point on one of the lateral boundaries of D^ε at which $w^\varepsilon = 0$, contradicting what was shown above). Fix $c < 1$. For small enough $\delta > 0$, both $\tilde{w}^\varepsilon \pm \delta$ have only a single sign change on the parabolic boundary of $D^{\varepsilon,\delta} = \{(x,t)|x_-^\varepsilon(t) < x < x_+^\varepsilon(t), 0 < t < ct^*(\delta)\}$. Therefore, using the argument above, at each $0 \leqslant t < ct^*$ there exist unique $y_\pm^{\varepsilon,\delta}(t) \in D^{\varepsilon,\delta}$ such that $y_+^{\varepsilon,\delta}(t) < x_0^\varepsilon(t) < y_-^{\varepsilon,\delta}(t)$, and $w^\varepsilon(y_\pm^\varepsilon(t),t) = \pm\delta$. Furthermore, Sard's Theorem (see, e.g., Guillemin and Pollack [1974], pages 39–45), ensures the existence of a sequence $\delta_n \downarrow 0$ such that $y_\pm^{\varepsilon,\delta_n}$ are smooth curves. Consider the domains $B^{\varepsilon,\delta_n} \subseteq D^{\varepsilon,\delta_n}$, defined by $B^{\varepsilon,\delta_n} = \{(x,t)|y_+^{\varepsilon,\delta_n}(t) < x < y_-^{\varepsilon,\delta_n}(t), 0 < t < ct^*(\delta_n)\}$, and note that the assumption that $u_{0xx}^\varepsilon(x_0^\varepsilon) < 0$ ensures that for small enough δ_n, $\tilde{w}^\varepsilon$ attains its minimum value of $-\delta_n$ on $y_-^{\varepsilon,\delta_n}$, and its maximum value of δ_n on $y_+^{\varepsilon,\delta_n}$. Thus for small enough δ_n, $\tilde{w}_x^\varepsilon < 0$ on the entire parabolic boundary of $B^{\varepsilon,\delta}$ (by applying the Hopf Boundary point lemma on $y_\pm^{\varepsilon,\delta_n}$, and smoothness and the fact that $u_{0xx}^\varepsilon(x_0^\varepsilon) < 0$ for $t = 0$). Since $(\tilde{w}_x^\varepsilon)_t = (\tilde{w}_x^\varepsilon)_{xx} + (\beta-1)(\tilde{w}_x^\varepsilon)_x$, the Strong Maximum Principle implies that $\tilde{w}_x^\varepsilon < 0$ in all of B^{ε,δ_n}, and in particular $w_x^\varepsilon(x_0^\varepsilon(t),t) = u_{xx}^\varepsilon(x_0^\varepsilon(t),t) < 0$ for $0 \leqslant t < ct^*(\delta_n)$. Since $t^*(\delta_n) \to \infty$, smoothness of $x_0^\varepsilon(t)$ follows.

Finally, on $x_0^\varepsilon(t)$,

$$u_t^\varepsilon(x_0^\varepsilon(t),t) = u_{xx}^\varepsilon(x_0^\varepsilon(t),t) + (\beta-1)u_x^\varepsilon(x_0^\varepsilon(t),t) - \beta u^\varepsilon(x_0^\varepsilon(t),t) = u_{xx}^\varepsilon(x_0^\varepsilon(t),t) - \beta u^\varepsilon(x_0^\varepsilon(t),t) < 0.$$

□

Lemma 4.3. *Suppose, in addition to the conditions assumed in Proposition 4.2, that there exist* $z_{\pm,0}^\varepsilon$ *with* $x_{-,0}^\varepsilon < z_{-,0}^\varepsilon < x_0^\varepsilon < z_{+,0}^\varepsilon < x_{+,0}^\varepsilon$ *such that:* $\mathcal{L}u_0^\varepsilon > 0$ *for* $x \in [x_-^\varepsilon, z_{-,0}^\varepsilon) \cup (z_{+,0}^\varepsilon, x_+^\varepsilon]$, $\mathcal{L}u_0^\varepsilon < 0$ *for* $x \in (z_{-,0}^\varepsilon, z_{+,0}^\varepsilon)$, *and* $\mathcal{L}u_0^\varepsilon = \frac{1}{\ell_\pm^\varepsilon}(u_{0x}^\varepsilon)^2$ *at* $x = x_{\pm,0}^\varepsilon$. *Then there exist smooth functions* $z_\pm^\varepsilon(t)$ *satisfying:*

$$x_-^\varepsilon(t) < z_-^\varepsilon(t) < x_0^\varepsilon(t) < z_+^\varepsilon(t) < x_+^\varepsilon(t) \tag{4.4}$$

[6]The result in this reference is stated for a cylindrical domain; however the result immediately generalizes to our case with the same proof.

such that $u^\varepsilon_t > 0$ *if* $x \in [x^\varepsilon_-(t), z^\varepsilon_-(t)) \cup (z^\varepsilon_+(t), x^\varepsilon_+(t)]$, $u^\varepsilon_t < 0$ *if* $x \in (z^\varepsilon_-(t), z^\varepsilon_+(t))$, *and* $u^\varepsilon_t = 0$ *iff* $x = z^\varepsilon_\pm(t)$.

Proof. Since $u^\varepsilon \equiv 0$ on the boundaries $x^\varepsilon_\pm$, differentiating yields $u^\varepsilon_t + \dot{x}^\varepsilon_\pm u^\varepsilon_x = 0$, so $u^\varepsilon_t = -\dot{x}^\varepsilon_\pm u^\varepsilon_x = \frac{1}{\ell^\varepsilon_\pm}(u^\varepsilon_x)^2 > 0$ on $x = x^\varepsilon_\pm$. Furthermore, from Proposition 4.2 we have that $u^\varepsilon_t < 0$ on $x = x^\varepsilon_0(t)$. The result then follows by applying the same arguments as in Proposition 4.2 on the domains $D^\varepsilon_1 = \{(x,t) : x^\varepsilon_-(t) < x < x^\varepsilon_0(t), t > 0\}$ and $D^\varepsilon_2 = \{(x,t) : x^\varepsilon_0(t) < x < x^\varepsilon_+(t), t > 0\}$. □

The following result asserts that there is an indeed an initial condition u^ε_0 satisfying all of our requirements (the third requirement is used in the proof of convexity below). It turns out that the sum of a Gaussian function and a linear function suffices. The proof is given in the Appendix.

Lemma 4.4. *There exist functions* $u^\varepsilon_0 : [x^\varepsilon_{-,0}, x^\varepsilon_{+,0}] \to \mathbb{R}_+$ *satisfying:*

1. $u^\varepsilon_0(x^\varepsilon_{\pm,0}) = 0$, $u^\varepsilon_0 > 0$ *in* $(x^\varepsilon_{-,0}, x^\varepsilon_{+,0})$, *and there exists* $x^\varepsilon_0 \in (x^\varepsilon_{-,0}, x^\varepsilon_{+,0})$ *such that* $u^\varepsilon_{0x} > 0$ *in* $[x^\varepsilon_{-,0}, x^\varepsilon_0)$, $u^\varepsilon_{0x} < 0$ *in* $(x^\varepsilon_0, x^\varepsilon_{+,0}]$.

2. $\mathcal{L}u^\varepsilon_0 = \frac{1}{\ell^\varepsilon_\pm}(u^\varepsilon_{0x})^2$ *at* $x = x^\varepsilon_{\pm,0}$, *and there exist* $z^\varepsilon_{\pm,0} \in (x^\varepsilon_{-,0}, x^\varepsilon_{+,0})$, *with* $z^\varepsilon_{-,0} < x^\varepsilon_0 < z^\varepsilon_{+,0}$, *such that* $\mathcal{L}u^\varepsilon_0 > 0$ *in* $[x^\varepsilon_{-,0}, z^\varepsilon_{-,0}) \cup (z^\varepsilon_{+,0}, x^\varepsilon_{+,0}]$, $\mathcal{L}u^\varepsilon_0 < 0$ *in* $(z^\varepsilon_{-,0}, z^\varepsilon_{+,0})$, $\mathcal{L}u^\varepsilon_0 = 0$ *at* $z^\varepsilon_{\pm,0}$.

3. $$\frac{\partial}{\partial x}\left(\frac{\mathcal{L}u^\varepsilon_0}{u^\varepsilon_{0x}}\right) < 0 \quad \text{in } [x^\varepsilon_{-,0}, z^\varepsilon_{-,0}] \cup [z^\varepsilon_{+,0}, x^\varepsilon_{+,0}].$$

4. $u^\varepsilon_0 \in C^4([x^\varepsilon_{-,0}, x^\varepsilon_{+,0}])$. *Extending* u^ε_0 *by zero on* $(-\infty, x^\varepsilon_{-,0}] \cup [x^\varepsilon_{+,0}, \infty)$ *we have*
$$\int_{\mathbb{R}} u^\varepsilon_0(x)\, dx = 1.$$

Theorem 4. *Suppose that* u^ε_0 *satisfies all the properties enumerated in Lemma 4.4. Then the function* x^ε_+ *is concave, and* x^ε_- *is convex.*

Proof. As outlined in Section 3.2 we consider the function $\phi^\varepsilon = \frac{u^\varepsilon_t}{u^\varepsilon_x}$ in $\{(x,t) : t \geqslant 0, x \in [x^\varepsilon_-(t), z^\varepsilon_-(t)] \cup [z^\varepsilon_+(t), x^\varepsilon_+(t)]\}$. Differentiating gives

$$0 = \left(\frac{\partial}{\partial t} - \mathcal{L}\right) u^\varepsilon_t = \left(\frac{\partial}{\partial t} - \mathcal{L}\right)(\phi^\varepsilon \cdot u^\varepsilon_x) = \phi^\varepsilon \left(\frac{\partial}{\partial t} - \mathcal{L}\right) u^\varepsilon_x + u^\varepsilon_x(\phi^\varepsilon_t - \phi^\varepsilon_{xx} - (\beta - 1)\phi^\varepsilon_x) - 2u^\varepsilon_{xx}\phi^\varepsilon_x,$$

so that

$$\phi^\varepsilon_t - \phi^\varepsilon_{xx} - \left(\beta - 1 + \tfrac{2u^\varepsilon_{xx}}{u^\varepsilon_x}\right)\phi^\varepsilon_x = \phi^\varepsilon_t - \phi^\varepsilon_{xx} - b^\varepsilon \phi^\varepsilon_x = 0, \tag{4.5}$$

where

$$b^\varepsilon(x) = \beta - 1 + \tfrac{2u^\varepsilon_{xx}}{u^\varepsilon_x} \in C^{1+\alpha,(1+\alpha)/2}.$$

Differentiating again, and defining $\psi^\varepsilon = \phi^\varepsilon_x$ yields

$$\psi^\varepsilon_t - \psi^\varepsilon_{xx} - b^\varepsilon \psi^\varepsilon_x - b^\varepsilon_x \psi^\varepsilon = 0. \tag{4.6}$$

Next, we investigate the boundary behaviour of ψ^ε. Recalling that on $x^\varepsilon_\pm(t)$, $\phi^\varepsilon(x^\varepsilon_\pm(t),t) = -\dot{x}^\varepsilon_\pm(t)$, we have that $\ddot{x}^\varepsilon_\pm(t) = -\frac{d}{dt}\phi^\varepsilon = -(\phi^\varepsilon_t + \phi^\varepsilon_x \dot{x}^\varepsilon_\pm) = -(\phi^\varepsilon_{xx} + b^\varepsilon \phi^\varepsilon_x - \phi^\varepsilon \phi^\varepsilon_x)$ on $x^\varepsilon_\pm(t)$. Furthermore, by (3.13), $\ddot{x}^\varepsilon_\pm = -\phi^\varepsilon \phi^\varepsilon_x$, and equating these two expressions for $\ddot{x}^\varepsilon_\pm$ yields $\phi^\varepsilon_{xx}(x^\varepsilon_\pm(t),t) = \phi^\varepsilon_x(x^\varepsilon_\pm(t),t) \cdot \big(2\phi^\varepsilon(x^\varepsilon_\pm(t),t) - b^\varepsilon(x^\varepsilon_\pm(t),t)\big)$; thus,

$$\psi^\varepsilon_x(x^\varepsilon_\pm(t),t) = \psi^\varepsilon(x^\varepsilon_\pm(t),t)\left[2\phi^\varepsilon(x^\varepsilon_\pm(t),t) - b^\varepsilon(x^\varepsilon_\pm(t),t)\right]. \tag{4.7}$$

We proceed to show that $\psi^\varepsilon(x^\varepsilon_\pm(t),t) < 0$, which by (3.13) implies that x^ε_+ is concave and x^ε_- is convex. First, notice that $\phi^\varepsilon < 0$ on x^ε_+, $\phi^\varepsilon(x,0) < 0$ on $(z^\varepsilon_{+,0}, x^\varepsilon_{+,0}]$, and $\phi^\varepsilon(z^\varepsilon_+(t),t) = 0$ for $t \geqslant 0$. These observations, together with (4.5) and the Hopf Boundary Point Lemma imply that $\psi^\varepsilon < 0$ on z^ε_+. Furthermore, it is assumed (condition 3 in Lemma 4.4) that $\psi^\varepsilon(x,0) < 0$ on $[z^\varepsilon_{0+}, x^\varepsilon_{+,0}]$. Suppose that there exists $t > 0$ such that $\psi^\varepsilon(x^\varepsilon_+(t),t) \geqslant 0$, and let $t_0 = \inf\{t > 0 : \psi^\varepsilon(x^\varepsilon_+(t),t) \geqslant 0\} > 0$. Then, using (4.6), and noting that b^ε and b^ε_x are smooth and bounded on $\bar{D}^\varepsilon_3 = \{(x,t) : z^\varepsilon_+(t) \leqslant x \leqslant x^\varepsilon_+(t), 0 \leqslant t \leqslant t_0\}$, by the Strong Maximum Principle we have that $0 = \psi^\varepsilon(x^\varepsilon_+(t_0),t_0)$ is the maximum of ψ^ε in $\bar{D}^\varepsilon_3$. The Hopf Boundary Point Lemma then implies that $\psi^\varepsilon_x(x^\varepsilon_+(t_0),t_0) > 0$. But $\psi^\varepsilon(x^\varepsilon_+(t_0),t_0) = 0$ and $\psi^\varepsilon_x(x^\varepsilon_+(t_0),t_0) > 0$ contradict (4.7), and thus we must have $\psi^\varepsilon(x^\varepsilon_+(t),t) < 0$ for all t. The proof that $\psi^\varepsilon(x^\varepsilon_-(t),t) < 0$ follows from a similar argument. □

Proposition 2. *Assume that $\kappa > H(\eta)$. Then there there is a $t^* > 0$ such that $x^\varepsilon_-(t^*) = \ln\kappa$. Denote by $(u^\varepsilon, x^\varepsilon_+)$ for $t > t^*$ the solution $(\tilde{u}, \tilde{x}_+)$ of (3.9). Then the function x^ε_+ is concave.*

The proof follows along the same lines as above.

5 Recovering the Value Function

In this section, we discuss how to recover the value function v from the solution of the Stefan problem u. Also, for notational simplicity, we abuse the notation $g(x)$ and $v(x,t)$ for the original functions $g(S)$ and $v(S,T)$ with $S = e^x$ and $t = \sigma^2 T/2$. For simplicity, we assume that $\kappa \leqslant H(\eta)$. Then $x_+(\infty) = \ln H(1/\eta)$ and $x_-(\infty) = \ln H(\eta)$. Extend u^ε by zero outside of the region $\mathcal{D}^\varepsilon = \{(x,t) : t > 0, x^\varepsilon_-(t) < x < x^\varepsilon_+(t)\}$, and given a function g^ε (to be defined later), we define:

$$v^\varepsilon(x,t) = g^\varepsilon(x) + (q-p)\int_0^t u^\varepsilon(x,s)\,ds, \quad \forall x \in \mathbb{R}, t \geqslant 0. \tag{5.1}$$

For convenience, we define:

$$T^\varepsilon(x) = \begin{cases} \infty & x \in (-\infty, \ln H(\eta)] \cup [\ln H(1/\eta), \infty), \\ T^\varepsilon_\pm(x) & x^\varepsilon_+(\infty) > x \geqslant x^\varepsilon_+(0) \quad \text{or} \quad x^\varepsilon_-(\infty) < x \leqslant x^\varepsilon_-(0), \\ 0 & x^\varepsilon_-(0) < x < x^\varepsilon_+(0), \end{cases}$$

where $t = T^\varepsilon_\pm$ is the inverse function of $x = x^\varepsilon_\pm(t)$, $0 \leqslant t < \infty$. Note that: $u^\varepsilon > 0$ if $t > T^\varepsilon(x)$, $u^\varepsilon = 0$ if $t \leqslant T^\varepsilon(x)$ and u^ε is Lipschitz continuous in $\mathbb{R} \times [0,\infty)$. We obtain:

$$v^\varepsilon(x,t) = g^\varepsilon(x) + (q-p)\int_{T^\varepsilon(x)\wedge t}^t u^\varepsilon(x,s)\,ds.$$

Direct differentiation yields:

$$\frac{1}{q-p}\frac{\partial v^\varepsilon}{\partial t} = u^\varepsilon = u_0^\varepsilon + \int_0^t \frac{\partial u^\varepsilon}{\partial t}(x,s)\,ds$$
$$= u_0^\varepsilon + \int_{T^\varepsilon(x)\wedge t}^t \frac{\partial u^\varepsilon}{\partial t}(x,s)\,ds \in C(\mathbb{R}\times[0,\infty))$$

and:

$$\frac{1}{q-p}\Big(\frac{\partial v^\varepsilon}{\partial x} - g_x^\varepsilon\Big) = \int_0^t \frac{\partial u^\varepsilon}{\partial x}(x,s)\,ds = \int_{T^\varepsilon(x)\wedge t}^t \frac{\partial u^\varepsilon}{\partial x}(x,s)\,ds \in C(\mathbb{R}\times[0,\infty)).$$

When $x_+^\varepsilon(0) \leqslant x \leqslant x_+^\varepsilon(t)$,

$$\frac{1}{q-p}\Big(\frac{\partial^2 v^\varepsilon}{\partial x^2} - g_{xx}^\varepsilon\Big) = \int_{T^\varepsilon(x)}^t \frac{\partial^2 u^\varepsilon}{\partial x^2}(x,s)\,dx - \frac{\partial u^\varepsilon}{\partial x}(x,T_+^\varepsilon(x))\cdot\frac{dT_+^\varepsilon}{dx}$$
$$= \int_{T_+^\varepsilon(x)}^t \frac{\partial^2 u^\varepsilon}{\partial x^2}(x,s)\,dx + \ell_+^\varepsilon.$$

Similarly,

$$\frac{1}{q-p}\Big(\frac{\partial^2 v^\varepsilon}{\partial x^2} - g_{xx}^\varepsilon\Big) - \int_{T^\varepsilon(x)\wedge t}^t \frac{\partial^2 u^\varepsilon}{\partial x^2}\,dx = \begin{cases} \ell_+^\varepsilon & x_+^\varepsilon(0) < x < x_+^\varepsilon(t) \\ 0 & x_-^\varepsilon(0) \leqslant x \leqslant x_+^\varepsilon(0) \\ \ell_-^\varepsilon & x_-^\varepsilon(t) < x < x_-^\varepsilon(0) \\ 0 & x \in (-\infty, x_-^\varepsilon(t)] \cup [x_+^\varepsilon(t),\infty). \end{cases}$$

Thus, $v_t^\varepsilon, v_x^\varepsilon - g_x^\varepsilon \in C(\mathbb{R}\times[0,\infty))$, $v_{xx}^\varepsilon - g_{xx}^\varepsilon \in L^\infty(\mathbb{R}\times[0,\infty))$. Consequently, since $u_t^\varepsilon - \mathcal{L}u^\varepsilon = 0$ when $t > T^\varepsilon(x)$, we have

$$\frac{\partial v^\varepsilon}{\partial t} - \mathcal{L}v^\varepsilon = -\mathcal{L}g^\varepsilon + (q-p)[u_0^\varepsilon - \ell_+^\varepsilon \mathbf{1}_{(x_+^\varepsilon(0),x_+^\varepsilon(t))} - \ell_-^\varepsilon \mathbf{1}_{(x_-^\varepsilon(t),x_-^\varepsilon(0))}].$$

Now we define g^ε as the unique solution of

$$\mathcal{L}g^\varepsilon = (q-p)[u_0^\varepsilon - \ell_-^\varepsilon \mathbf{1}_{(-\infty,x_-^\varepsilon(0))} - \ell_+^\varepsilon \mathbf{1}_{(x_+^\varepsilon(0),\infty)}]. \tag{5.2}$$

Then we have:

$$\frac{\partial v^\varepsilon}{\partial t} - \mathcal{L}v^\varepsilon = (q-p)[\ell_-^\varepsilon \mathbf{1}_{(-\infty,x_-^\varepsilon(t))} + \ell_+^\varepsilon \mathbf{1}_{(x_+^\varepsilon(t),\infty)}] \geqslant 0 \quad \text{on } \mathbb{R}\times(0,\infty)$$

and furthermore $v^\varepsilon \geqslant g^\varepsilon$ on $\mathbb{R}\times[0,\infty)$ so v^ε is the solution of the variational inequality

$$\min\Big\{\frac{\partial v^\varepsilon}{\partial t} - \mathcal{L}v^\varepsilon\ ,\ v^\varepsilon - g^\varepsilon\Big\} = 0 \quad \text{on } \mathbb{R}\times(0,\infty), \qquad v^\varepsilon = g^\varepsilon \quad \text{on } \mathbb{R}\times\{0\}$$

and therefore by the Comparison Principle

$$\|v^\varepsilon - v\|_{L^\infty(\mathbb{R}\times[0,\infty))} \leqslant \|g^\varepsilon - g\|_{L^\infty(\mathbb{R})}. \tag{5.3}$$

Note that

$$\mathcal{L}g = (q-p)\Big\{\delta(x) - \ell_-\mathbf{1}_{(-\infty,0)} - \ell_+\mathbf{1}_{(0,\infty)}\Big\}.$$

Thus,

$$\frac{1}{q-p}\mathcal{L}\Big(g^\varepsilon - g\Big) \quad = \quad u_0^\varepsilon - \delta(x) + \ell_-^\varepsilon \mathbf{1}_{[x_-^\varepsilon(0),0)} + \ell_+^\varepsilon \mathbf{1}_{[0,x_+^\varepsilon(0))} + (\ell_- - \ell_-^\varepsilon)\mathbf{1}_{(-\infty,0)} + (\ell_+ - \ell_+^\varepsilon)\mathbf{1}_{(0,\infty)}.$$

The Green's function of the operator $\mathcal{L}$ is

$$G(x,y) = \frac{1}{1+\beta}\begin{cases} e^{x-y}, & x<y \\ e^{\beta(y-x)}, & y\leqslant x, \end{cases}$$

so

$$\frac{g^\varepsilon(x)-g(x)}{q-p} = \int_{-\infty}^{\infty} G(x,y)\left(u_0^\varepsilon(y) - \delta(y) + \ell_-^\varepsilon \mathbf{1}_{[x_-^\varepsilon(0),0)} + \ell_+^\varepsilon \mathbf{1}_{(0,x_+^\varepsilon(0)]}\right) dy + (\ell_+ - \ell_+^\varepsilon)\int_0^\infty G(x,y)dy + (\ell_- - \ell_-^\varepsilon)\int_{-\infty}^0 G(x,y)dy.$$

Since $\int_{-\infty}^{\infty} u_0^\varepsilon(x)\,dx = 1$ and $u_0^\varepsilon(x) = 0$ if $x \leqslant x_-^\varepsilon(0)$ or $x \geqslant x_+^\varepsilon(0)$, we derive that

$$\int_{\mathbb{R}} \Big(u_0^\varepsilon(y) - \delta(y)\Big)G(x,y)dy = \int_{x_{-,0}^\varepsilon}^{x_{+,0}^\varepsilon} \left(G(x,y) - G(x,0)\right) u_0^\varepsilon(y)\,dy.$$

Hence,

$$\frac{|g^\varepsilon(x)-g(x)|}{q-p} \leqslant \|G_y\|_\infty \int_{x_-^\varepsilon(0)}^{x_+^\varepsilon(0)} |y| u_0^\varepsilon(y)\,dy + \|G\|_\infty \left(\ell_+^\varepsilon |x_-^\varepsilon(0)| + \ell_-^\varepsilon |x_+^\varepsilon(0)|\right) + |\ell_+^\varepsilon - \ell_+| + |\ell_-^\varepsilon - \ell_-|$$
$$\leqslant \max(x_+^\varepsilon(0), |x_-^\varepsilon(0)|)\left(1 + \frac{\ell_+^\varepsilon + \ell_-^\varepsilon}{1+\beta}\right) + |\ell_+^\varepsilon - \ell_+| + |\ell_-^\varepsilon - \ell_-|.$$

Sending $\varepsilon \searrow 0$ we have

$$\lim_{\varepsilon\searrow 0} \|v - v^\varepsilon\|_{L^\infty(\mathbb{R}\times[0,\infty))} = 0.$$

Since the free boundary of v^ε is $x = x_\pm^\varepsilon(t)$, by its convexity and the uniform estimate of its derivative on $[\delta,\infty)$ ($\delta>0$), we see that

$$x_\pm(t) = \lim_{\varepsilon\searrow 0} x_\pm^\varepsilon(t).$$

Thus $x_+(\cdot)$ is concave and $x_-(\cdot)$ is convex. We summarize our result as follows:

Theorem 5. *Let v be solution of* (1.3) *with $q>p$ and let s_j be the function derived in Lemma 3.4. Then $\ln s_1(T)$ is a convex function and $\ln s_2(T)$ is a concave function. Also, $s_2 \in C^\infty((0,\infty))\cap C([0,\infty))$ with $s_2(0)=1$ and $s_2'(T)>0$ for all $T>0$.*

If $k \leqslant H(\eta)$, then $s_1 \in C^\infty((0,\infty))\cap C([0,\infty))$ with $s_1(0)=1$ and $s_1'(T)<0$ for all $T>0$.

If $k > H(\eta)$, then there exists $T^* \in (0,\infty)$ such that $s_1'(T^*-)<0$ and $s_1(T)=\kappa$ for all $T\geqslant T^*$.

6 Concluding Remarks

Hedge funds are an important and popular investment vehicle. Concerns regarding traditional fee structures have caused fund investors and managers to develop innovative new structures which seek to improve risk sharing and better align incentives. We analyze the optimal withdrawal time for an investor in a hedge fund with a first-loss or shared-loss fee structure. We consider the resulting optimal stopping problem in both the finite and infinite horizon cases. In the infinite horizon case, when the investment process is assumed to follow a geometric Brownian motion, a closed form solution for the value function is available, and the continuation region is a finite interval. In the finite horizon case, there exists a pair of optimal exercise boundaries, which we show to be monotone, with the upper boundary concave, and the lower boundary convex (in the log-stock price coordinates).

There are a number of interesting directions for future research on this, and related problems. The asymptotic behavior of the stopping boundaries near expiry could be studied. The inclusion of real-world aspects of hedge-fund contracts, such as a fee for assets under management, or a penalty for withdrawal of funds, could be added to the problem formulation. Rather than employing risk-neutral valuation, a utility maximization perspective could be taken (indeed, one could even consider a game in which the hedge fund manager controls the portfolio and the investor decides on the withdrawal time, both seeking to maximize their expected own expected utilities). Finally, extensions to alternative asset processes, such as jump diffusions, may be considered.

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A Appendix

A.1 Proof of Lemma 4.1

Fix $\alpha \in (0,1)$ and $0 < h \ll 1$. We first establish existence locally in time (i.e. for $t \in [0,h]$). We define by $\mathbf{X}$ the subset of all $(x_+, x_-) \in C^{(3+\alpha)/2}([0,h]) \times C^{(3+\alpha)/2}([0,h])$ satisfying:

$$\begin{cases} x_\pm(0) = x^\varepsilon_{\pm,0}, & \dot{x}_\pm(0) = -\frac{1}{\ell^\varepsilon_\pm} u^\varepsilon_{0,x}(x^\varepsilon_{\pm,0}), \\ \pm\dot{x}_\pm \geqslant 0 \quad \text{in} \quad [0,h], & \|\dot{x}_\pm - \dot{x}_\pm(0)\|_{C^{(1+\alpha)/2}([0,h])} \leqslant 1. \end{cases}$$

For $(x_+, x_-) \in \mathbf{X}$ we define $Q = \cup_{0<t\leq h}\{(x_-(t), x_+(t)) \times \{t\}\}$ and let u be the solution of the initial boundary value problem:

$$\begin{cases} u_t - \mathcal{L}u = 0 & \text{in} \quad Q \\ u(\cdot, 0) = u^\varepsilon_0 & \text{on} \quad [x^\varepsilon_{-,0}, x^\varepsilon_{+,0}], \\ u(x_\pm(t), t) = 0 & \forall\, t \in (0,h]. \end{cases}$$

We can transform the above into a problem on a cylindrical domain by considering:

$$u(x,t) = U(z,t), \quad z = \frac{2x - (x_+(t) + x_-(t))}{x_+(t) - x_-(t)}, \quad x = \frac{x_+(t) + x_-(t)}{2} + \frac{x_+(t) - x_-(t)}{2} z. \tag{A.1}$$

Then the above problem becomes:

$$\begin{cases} U_t - a^2(t)U_{zz} - b(z,t)U_z - \beta U = 0 & \text{in } (-1,1) \times (0,h], \\ U(\pm 1, t) = 0 & \text{for } \ t \in (0,h], \\ U(z,0) = U_0(z) := u^\varepsilon_0(\frac{x^\varepsilon_{+,0} + x^\varepsilon_{-,0}}{2} + \frac{x^\varepsilon_{+,0} - x^\varepsilon_{-,0}}{2} z) & \text{for } \ z \in [-1,1], \end{cases}$$

where

$$a(t) = \frac{2}{x_+(t) - x_-(t)}, \quad b(z,t) = \frac{\beta - 1 + (1+z)x'_+(t) + (1-z)x'_-(t)}{x_+(t) - x_-(t)}. \tag{A.2}$$

Note that the zeroth order and first order compatibility conditions are satisfied: $U_0(\pm 1) = 0$ and

$$a^2(0)U_0'' + b(z,0)U' + \beta U_0\Big|_{z=\pm 1} = \mathcal{L}u_0(x^\varepsilon_{\pm,0}) + x'_\pm(0)u'_0(x^\varepsilon_{\pm,0}) = \mathcal{L}u_0(x^\varepsilon_{\pm,0}) - \frac{1}{\ell^\varepsilon_\pm} {u^\varepsilon_{0x}}^2(x^\varepsilon_{\pm,0}) = 0.$$

Note that there exists a constant $C_{1\varepsilon}$ such that

$$\|a,b\|_{C^{(1+\alpha)/2}}([0,h]) \leqslant C_{1\varepsilon}, \qquad a(t) \geqslant \frac{2}{x^\varepsilon_{+,0} - x^\varepsilon_{-,0}} \quad \forall t \in [0,h].$$

Thus there exists another constant $C_{2\varepsilon}$ such that

$$\|U\|_{C^{3+\alpha,\frac{3+\alpha}{2}}([0,h]\times[-1,1])} + \|U_x\|_{C^{2+\alpha,\frac{2+\alpha}{2}}([0,h]\times[-1,1])} \leqslant C_{2\varepsilon}.$$

Notice that, by Hopf's Lemma,

$$\pm\frac{\partial u}{\partial x}(x_\pm(t),t) = \pm a(t)\frac{\partial U}{\partial z}(\pm 1,t) < 0 \quad \forall\, t \in [0,h].$$

We now define $\mathbf{T} : (x_+, x_-) \to (\tilde{x}_+, \tilde{x}_-)$ by:

$$\tilde{x}_\pm(t) = x^\varepsilon_{\pm,0} - \int_0^t \frac{a(t')}{\ell^\varepsilon_\pm}\frac{\partial U}{\partial z}(\pm 1, t')\,dt' \quad \forall\, t \in [0,h]. \tag{A.3}$$

Then $\tilde{x}_\pm(0) = x^\varepsilon_{\pm,0}$, $\pm\dot{\tilde{x}}_\pm > 0$ and $\dot{\tilde{x}}_\pm = -\frac{1}{\ell^\varepsilon_\pm}u_x(x_\pm(t),t) \in C^{\frac{2+\alpha}{2}}([0,h])$. In addition,

$$\|\dot{\tilde{x}}_\pm - \dot{x}_\pm(0)\|_{C^{(1+\alpha)/2}([0,h])} \leqslant Ch^{\frac{1-\alpha}{2}}\|\dot{\tilde{x}}_\pm - \dot{x}_\pm(t)\|_{C^1([0,h])} \leqslant C_{3\varepsilon}\cdot h^{\frac{1-\alpha}{2}} \leqslant 1$$

for sufficiently small h. Thus, $\mathbf{T}$ maps $\mathbf{X}$ to itself. Since $\|\dot{\tilde{x}}_\pm\|_{C^{2+\alpha/2}([0,h])} \leqslant C_\varepsilon$ we see that $\mathbf{T}$ is a compact mapping. Thus, by Schauder's Fixed Point Theorem, $\mathbf{T}$ has a fixed point, which gives a solution of (3.10) for $t \in [0,h]$. By taking h smaller if necessary, one can show that $\mathbf{T}$ is a contraction. Thus this solution is unique. By a bootstrap argument, one can show that

$$x^\varepsilon_\pm \in C^\infty((0,h]) \cap C^{2+\alpha/2}([0,h]), \qquad U_z \in C^\infty([-1,1]\times(0,h]) \cap C^{2+\alpha,1+\alpha/2}([-1,1]\times[0,h]).$$

In order to derive global existence from local existence, we require some a priori bounds.

Fix an arbitrary $t_0 > 0$ in the known time existence interval. Set $L = x^\varepsilon_+(t_0)$. We extend u^ε_0 to $\mathbb{R}$ by setting $u^\varepsilon_0(x) = 0$ for $x \in (-\infty, x^\varepsilon_{-,0}] \times [x^\varepsilon_{+,0}, \infty)$. Now let K be the solution of:

$$\begin{gathered}(\partial_t - \mathcal{L})K = 0 \ \text{ in } \ (-\infty, L)\times(0,t_0], \\ K(L,t) = 0 \quad \forall\, t \in [0,t_0], \quad K(x,0) = u^\varepsilon_0(x) \quad \forall x \in (-\infty, L].\end{gathered}$$

Since $u^\varepsilon_0 \geqslant 0$, we have $K > 0$ in $(-\infty, L)\times(0,t_0]$. Comparing K with u^ε we find that $u^\varepsilon \leqslant K$ on $(-\infty, L]\times[0,t_0]$. Since $u^\varepsilon(L,t_0) - K(L,t_0) = 0$ and $u^\varepsilon(x,t_0) - K(x,t_0) < 0$ for $x < L$, we have:

$$0 \leqslant |u^\varepsilon_x(L,t_0)| \leqslant |K_x(L,t_0)|.$$

Using the fundamental solution gives:

$$\begin{aligned} K(x,t) &= \frac{e^{-\frac{\beta-1}{2}x - \frac{(\beta+1)^2}{4}t}}{\sqrt{4\pi t}}\int_{-\infty}^L e^{\frac{\beta-1}{2}y}u^\varepsilon_0(y)\Big(e^{-(x-y)^2/(4t)} - e^{-(x+y-2L)^2/(4t)}\Big)dy. \\ K_x(L,t) &= \frac{e^{-\frac{\beta-1}{2}L - \frac{(\beta+1)^2}{4}t}}{\sqrt{4\pi t}}\int_{-\infty}^L e^{\frac{\beta-1}{2}y}u^\varepsilon_0(y)\frac{y-L}{t}e^{-(L-y)^2/(4t)}dy. \end{aligned}$$

Since u_0 is supported on $[x^\varepsilon_{-,0}, x^\varepsilon_{+,0}]$, we thus obtain:

$$\Big|K_x(L,t)\Big| \leqslant \frac{L - x^\varepsilon_{-,0}}{\sqrt{4\pi t^3}} e^{-|L-x^\varepsilon_{+,0}|^2/(4t)-(\beta-1)L/2-(\beta+1)^2t/4} \int_{\mathbb{R}} e^{\frac{\beta-1}{2}y} u^\varepsilon_0(y)dy.$$

Assume for simplicity that $\int_{\mathbb{R}} e^{\frac{\beta-1}{2}y} u^\varepsilon_0(y)dy \leqslant 1$. Then

$$0 \leqslant \ell^\varepsilon_+ \frac{dx^\varepsilon_+}{dt}(t_0) = -\frac{\partial u(L,t_0)}{\partial x} \leqslant \frac{(L - x^\varepsilon_{-,0}) e^{-(L-x^\varepsilon_{+,0})^2/(4t_0)+\frac{\beta-1}{2}L-\frac{(\beta+1)^2}{4}t_0}}{\sqrt{4\pi t_0^3}}$$

Replacing L by $x^\varepsilon_+(t_0)$, we then obtain the estimate for $|\dot{x}^\varepsilon_+(t_0)|$. After a similar analysis for $\dot{x}^\varepsilon_-(t_0)$ and replacing t_0 by arbitrary $t>0$ we hence obtain the following:

$$\Big|\frac{dx^\varepsilon_\pm(t)}{dt}\Big| \leqslant \frac{|x^\varepsilon_\pm(t) - x^\varepsilon_{\mp,0}|}{\ell^\varepsilon_\pm \sqrt{4\pi t^3}} \exp\Big(-\frac{[x^\varepsilon_\pm(t) - x^\varepsilon_{\pm,0}]^2}{4t} - \frac{[\beta-1]x^\varepsilon_\pm(t)}{2} - \frac{(\beta+1)^2 t}{4}\Big) \quad \forall\, t>0,$$

which implies global existence.

Finally we use conservation of energy (integrating $u^\varepsilon - \mathcal{L}u^\varepsilon = 0$ and $e^{\beta t}(u^\varepsilon_t - \mathcal{L}u^\varepsilon) = 0$) to derive

$$\begin{aligned}
\int_{x^\varepsilon_{-0}}^{x^\varepsilon_{+,0}} u^\varepsilon_0(x)dx &\geqslant \ell^\varepsilon_+[x^\varepsilon(t) - x^\varepsilon_{+,0}] + \ell^\varepsilon_-[x^\varepsilon_-(t) - x^\varepsilon_{-0}],\\
e^{-\beta t}\int_{x^\varepsilon_{-0}}^{x^\varepsilon_{+,0}} u^\varepsilon_0(x)dx &\leqslant \int_{x^\varepsilon_-(t)}^{x^\varepsilon_+(t)} u^\varepsilon(x,t)dt + \ell^\varepsilon_+[x^\varepsilon_+(t) - x^\varepsilon_{+,0}] + \ell^\varepsilon_-[x^\varepsilon_{-0} - x^\varepsilon_-(t)]\\
&\leqslant \Big[\|u^\varepsilon(\cdot,t)\|_{L^\infty(\mathbb{R})} + \ell^\varepsilon_- + \ell^\varepsilon_+\Big][x^\varepsilon_+(t) - x^\varepsilon_-(t)].
\end{aligned}$$

Note that $\int_{\mathbb{R}} u^\varepsilon_0(x)dx = 1$ and

$$0 \leqslant u^\varepsilon(x,t) \leqslant \sup_{z\in\mathbb{R}} \frac{e^{-z^2/(4t)-(\beta-1)z/2-(\beta+1)^2t/4}}{\sqrt{4\pi t}} \int_{\mathbb{R}} u^\varepsilon_0(y)dy.$$

Thus, there exists a positive constant C that does not depend on ε such that

$$[x^\varepsilon_{+,0} - x^\varepsilon_{-0}] + \frac{1}{\min\{\ell^\varepsilon_+, \ell^\varepsilon_-\}} \geqslant x^\varepsilon_+(t) - x^\varepsilon_-(t) \geqslant \frac{\sqrt{t}}{C[1+\sqrt{t}]} \quad \forall\, t>0.$$

We remark that for fixed $\delta > 0$, $|\dot{x}^\varepsilon_\pm|$ is uniformly (in ε) bounded on $[\delta,\infty)$. In view of (A.2), we find that $a, 1/a, b, a_t, b_t$ are bounded uniformly in ε in $[\delta,\infty)$. After a bootstrapping argument, we can establish ε-independent bounds for derivatives of arbitrary higher order on $[\delta,\infty)$. Thus, by compactness, along a sequence of $\varepsilon \searrow 0$ we have $x^\varepsilon_\pm \to x_\pm$ for some $x_\pm \in C^\infty((0,\infty))$. It is easy to see that $x_\pm \in C([0,\infty))$ and $x_\pm(0) = 0$. Finally, from the analysis in Section 5, one sees that $x_+(t) = \ln s_2(2t/\sigma^2)$ and $x_-(t) = \ln s_1(2t/\sigma^2)$ are indeed the free boundaries for the original problem for v, which are unique. Thus, as $\varepsilon \searrow 0$, the whole sequence $\{x^\varepsilon_\pm\}$ approaches $x_\pm$. This completes the proof of Lemma 4.1 and Theorem 3.

A.2 Proof of Lemma 4.4

Define:

$$Q(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \qquad y_\pm^\varepsilon = \pm\sqrt{-2\ln(\sqrt{2\pi}\varepsilon\ell_\pm)} \quad \Longrightarrow \quad Q(y_\pm^\varepsilon) = \varepsilon\ell_\pm. \tag{A.4}$$

Without loss of generality assume that $\ell_+ \leqslant \ell_-$, so $y_+^\varepsilon \geqslant |y_-^\varepsilon|$, and $Q(y_+^\varepsilon) \leqslant Q(y_-^\varepsilon)$. Set

$$A^\varepsilon(y) = \frac{1}{y_+^\varepsilon - y_-^\varepsilon}\Big\{(y - y_-^\varepsilon)Q(y_+^\varepsilon) + (y_+^\varepsilon - y)Q(y_-^\varepsilon)\Big\} = O(\varepsilon) \text{ on } [y_-^\varepsilon, y_+^\varepsilon]. \tag{A.5}$$

For some $m_\varepsilon \approx 1$ to be defined later, we define

$$x_{\pm,0}^\varepsilon = \varepsilon y_\pm^\varepsilon, \quad u_0^\varepsilon(x) = \frac{1}{\varepsilon m_\varepsilon}\left(Q\left(\frac{x}{\varepsilon}\right) - A^\varepsilon\left(\frac{x}{\varepsilon}\right)\right), \quad \ell_\pm^\varepsilon = \frac{({u_0^\varepsilon}')^2(x_{0,\pm}^\varepsilon)}{\mathcal{L}u_0^\varepsilon(x_\pm^\varepsilon)}. \tag{A.6}$$

We now verify that such defined $(x_{\pm,0}^\varepsilon, u_0^\varepsilon, \ell_\pm^\varepsilon)$ serves our need.

1. It is immediate from the definition of A^ε that $u_0^\varepsilon(x_{\pm,0}^\varepsilon) = 0$.

   Next we show that $u_0^\varepsilon > 0$ in $(x_{-,0}^\varepsilon, x_{+,0}^\varepsilon)$. Define G^ε by $G^\varepsilon(y) = Q(y) - A^\varepsilon(y)$. Then clearly $G^\varepsilon(y_-^\varepsilon) = G^\varepsilon(y_+^\varepsilon) = 0$, and

   $$G_y^\varepsilon = -yQ(y) - \frac{Q(y_+^\varepsilon) - Q(y_-^\varepsilon)}{y_+^\varepsilon - y_-^\varepsilon}, \qquad G_{yy}^\varepsilon(y) = (y^2 - 1)Q(y).$$

   Thus, G^ε is concave on $(-1, 1)$ and convex on $(-\infty, -1] \cup [1, \infty)$. Now using $Q(y_\pm^\varepsilon) = \varepsilon\ell_\pm$, we have

   $$G_y^\varepsilon(y_-^\varepsilon) = Q(y_-^\varepsilon)\left(-y_-^\varepsilon - \frac{1}{y_+^\varepsilon - y_-^\varepsilon}\left(\frac{\ell_+}{\ell_-} - 1\right)\right),$$

   $$G_y^\varepsilon(y_+^\varepsilon) = Q(y_+^\varepsilon)\left(-y_+^\varepsilon - \frac{1}{y_+^\varepsilon - y_-^\varepsilon}\left(1 - \frac{\ell_-}{\ell_+}\right)\right).$$

   since $\pm y_\pm^\varepsilon \to \infty$ as $\varepsilon \searrow 0$, we see that $G_y^\varepsilon(y_+^\varepsilon) < 0 < G_y^\varepsilon(y_-^\varepsilon)$. Since $G_{yy}^\varepsilon > 0$ on $(-\infty, -1) \cup (1, \infty)$ we see that $G_y^\varepsilon(y) > 0$ on $[y_-^\varepsilon, -1]$ and $G_y^\varepsilon(y) < 0$ on $[1, y_+^\varepsilon]$. As $G_{yy}^\varepsilon < 0$ in $(-1, 1)$, there exists a unique $y_0^\varepsilon = o(\varepsilon)$ such that $G_y(y_0^\varepsilon) = 0$. Setting $x_0^\varepsilon = \varepsilon y_0^\varepsilon = o(\varepsilon^2)$ we have $u_{0x}^\varepsilon < 0$ in $(x_0^\varepsilon, x_{+,0}^\varepsilon]$, $u_{0x}^\varepsilon > 0$ in $[x_{-,0}^\varepsilon, x_0^\varepsilon)$, and $u_{0xx}^\varepsilon(x_0^\varepsilon) < 0$. The first requirement for the assertion of Lemma 4.4 is proved.

2. Denoting $y = x/\varepsilon$. We now calculate

   $$\begin{aligned} u_{0x}^\varepsilon &= \frac{1}{m_\varepsilon\varepsilon^2}\Big(Q_y(y) - A_y^\varepsilon(y)\Big) = \frac{Q(y)}{m_\varepsilon\varepsilon^2}\left(-y - \frac{Q(y_+^\varepsilon)}{Q(y)}\frac{1 - \frac{\ell_-}{\ell_+}}{(y_+^\varepsilon - y_-^\varepsilon)}\right) \\ &= \frac{Q(y)}{m_\varepsilon\varepsilon^2}\left(-y + \frac{O(1)}{\sqrt{|\ln\varepsilon|}}\right). \end{aligned}$$

Using $Q' = -yQ$, $Q'' = (-1+y^2)Q$, and $Q''' = (3y - y^3)Q$, we derive that

$$\begin{aligned}
\mathcal{L}u_0^\varepsilon &= \frac{1}{m_\varepsilon \varepsilon^3}\Big(Q'' + (\beta-1)\varepsilon(Q' - A') - \beta\varepsilon^2(Q - A)\Big)\\
&= \frac{Q}{m_\varepsilon \varepsilon^3}\left(y^2 - 1 - (\beta-1)\varepsilon y - \beta\varepsilon^2 - \frac{(\beta-1)\varepsilon A' - \beta\varepsilon^2 A}{Q}\right)\\
&= \frac{Q}{m_\varepsilon \varepsilon^3}\Big(y^2 - 1 - (\beta-1)\varepsilon y + O(1)\varepsilon\Big).
\end{aligned}$$

Hence,

$$\ell_\pm^\varepsilon := \frac{({u_0^\varepsilon}')^2(x_{\pm,0}^\varepsilon)}{\mathcal{L}u_0^\varepsilon(x_{\pm,0}^\varepsilon)} = \frac{Q(y_\pm^\varepsilon)}{m_\varepsilon \varepsilon}\Big(1 + \frac{O(1)}{|y_\pm|^2}\Big) = \frac{\ell_\pm}{m_\varepsilon}\Big\{1 + \frac{O(1)}{|\ln\varepsilon|}\Big\}.$$

In addition,

$$\begin{cases} \mathcal{L}u_0^\varepsilon < 0 & \text{if } |y| \geqslant 1 + O(\varepsilon),\\ \mathcal{L}u_0^\varepsilon > 0 & \text{if } |y| < 1 - O(\varepsilon). \end{cases}$$

Finally,

$$\frac{d}{dx}\mathcal{L}u_0^\varepsilon = \frac{Q}{m_\varepsilon \varepsilon^4}\Big[y(3 - y^2) + O(\varepsilon + \varepsilon y)\Big].$$

Thus, there exist unique $z_\pm^\varepsilon = \varepsilon[\pm 1 + O(\varepsilon)]$ such that

$$\mathcal{L}u_0^\varepsilon > 0 \text{ in } [x_{-,0}^\varepsilon, z_-^\varepsilon) \cup (z_+^\varepsilon, x_{+,0}^\varepsilon], \quad \mathcal{L}u_0^\varepsilon(z_\pm^\varepsilon) = 0, \quad (\mathcal{L}u_0^\varepsilon)'(z_\pm^\varepsilon) \neq 0, \quad \mathcal{L}u_0^\varepsilon < 0 \text{ in } (z_-^\varepsilon, z_+^\varepsilon).$$

This establishes the second property.

3. For simplicity of notation, let $q = u_0^\varepsilon$ for this calculation:

$$\begin{aligned}
{u_{0x}^\varepsilon}^2 \frac{d}{dx}\left(\frac{\mathcal{L}u_0^\varepsilon}{u_{0x}^\varepsilon}\right) &= q'\mathcal{L}q' - q''\mathcal{L}q\\
&= q'\{q''' + (\beta-1)q'' - \beta q'\} - q''\{q'' + (\beta-1)q' - \beta q\}\\
&= q'q''' - q''^2 - \beta q'^2 + \beta q q''\\
&= \frac{Q^2}{\varepsilon^4 m_\varepsilon^2}\Big((-y - \tfrac{A'}{Q})(3y - y^3) - (1-y^2)^2 - \beta\varepsilon^2(-y + O(1))(y^2 - 1)\Big)\\
&= -\frac{Q^2}{\varepsilon^4 m_\varepsilon^2}\Big(1 + y^2 + \tfrac{yA^{\varepsilon\prime}}{Q}(3 - y^2) + O(\varepsilon^2|\ln\varepsilon|^3)\Big).
\end{aligned}$$

Without loss of generality $|y_+^\varepsilon| \geqslant |y_-^\varepsilon|$ so that $Q(y_+^\varepsilon) \leqslant Q(y_-^\varepsilon)$. To analyze $\Delta = \frac{yA^{\varepsilon\prime}}{Q}(3 - y^2)$, notice that

$$A^{\varepsilon\prime} = \frac{Q(y_+^\varepsilon) - Q(y_-^\varepsilon)}{y_+^\varepsilon - y_-^\varepsilon} < 0.$$

When $y \in [\sqrt{3}, y_+^\varepsilon]$, $\Delta \geqslant 0$. When $|y| \leqslant \sqrt{3}$, $\Delta = o(\varepsilon)$. When $y \in [y_-^\varepsilon, -\sqrt{3}]$, using $y_+^\varepsilon - y_-^\varepsilon \geqslant 2|y_-^\varepsilon|$, we can derive that

$$\Delta = \frac{y[Q(y_+^\varepsilon) - Q(y_-^\varepsilon)]}{Q(y)(y_+^\varepsilon - y_-^\varepsilon)}(3 - y^2) \geqslant \frac{(3 - y^2)|y|}{y_+^\varepsilon - y_-^\varepsilon}\,\frac{Q(y_-^\varepsilon)}{Q(y)}\Big(1 - \frac{\ell_+}{\ell_-}\Big) \geqslant \frac{3 - y^2}{2}.$$

Thus, when ε is small, we have

$$u_{0x}^{\varepsilon\,2}\frac{d}{dx}\left(\frac{\mathcal{L}u_0^\varepsilon}{u_{0x}^\varepsilon}\right) < 0 \text{ on } \ [x_{-,0}^\varepsilon, x_{+,0}^\varepsilon].$$

Consequently, u_0^ε satisfies the third requirement.

4. Finally, we define

$$m_\varepsilon = \int_{y_-^\varepsilon}^{y_+^\varepsilon} \Big\{Q(y) - A^\varepsilon(y)\Big\}dy.$$

Then $\int_{\mathbb{R}} u_0^\varepsilon(x)dx = 1$. Notice that $|A^\varepsilon| \leqslant \max\{Q(y_+^\varepsilon), Q(y_-^\varepsilon)\} = O(\varepsilon)$ and $y_\pm^\varepsilon = O(\sqrt{|\ln\varepsilon|})$. Hence, as $\varepsilon \searrow 0$,

$$m_\varepsilon = \int_{y_-^\varepsilon}^{y_+^\varepsilon} \Big(\frac{1}{\sqrt{2\pi}}e^{-y^2/2} + O(\varepsilon)\Big)dy = \frac{1}{\sqrt{2\pi}}\int_{y_-^\varepsilon}^{y_+^\varepsilon} e^{-y^2/2}dy + O(\varepsilon|\ln\varepsilon|) \to 1.$$

This completes the construction of approximating data, and the proof of Lemma 4.4.