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Analysis of the Optimal Time to Withdraw Investments from Hedge Funds with Alternative Fee Structures

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We study the optimal stopping problem arising from an investor determining the optimal time to withdraw from a hedge fund investment with a shared loss fee structure and a positive fee for assets under management. The optimal solution is characterized as the first exit time of the fund value from a bounded region with upper and lower stopping boundaries. In the infinite horizon case, we present the complete solution to the optimal stopping problem, while in the finite horizon case we derive a pair of coupled integral equations for the stopping bounds, and present an asymptotic analysis of the stopping boundaries for small time.

Keywords: Optimal Stopping, Hedge Funds, Variational Inequalities, Asymptotic Analysis

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1. Introduction

Fee structures in the hedge fund industry have been the subject of much recent discussion and innovation. Traditional fee structures, in which the manager charges a 2% fee for assets under management and keeps 20% of profits as an incentive fee (the so-called 2 and 20 structures) have been criticized both for resulting in fees that are too high, and for incentivizing risk taking on the part of the fund manager. For further discussions, see Kouwenberg and Ziemba (2007), and Hodder and Jackwerth (2007).

Recent innovations include first-loss and shared loss structures, in which the insurer provides some downside protection to fund investors in return for a higher percentage of upside participation. As compensation for paying the losses first, the incentive fee is usually higher, typically about 40%. He and Kou (2018) compared the first-loss and traditional fee structures by solving utility maximizing problems for the hedge fund manager, and found that under certain parameter assumptions, the utilities of both the investor and the fund manager were improved. Djerroud et al. (2016) studied shared-loss fee structures from an option pricing perspective; Meng et al. (2019) considered the risk-return tradeoff for both investors and managers with shared-loss fee structures; Bhaduri et al. (2018) considered both pricing and risk-management for related hedge-fund fee structures, with payoffs to investors resembling those of asset-backed securities, or high-yield bonds.

In this paper, we study the optimal stopping problem arising from an investor determining the optimal time to withdraw from a hedge fund investment with a first-loss or shared-loss fee structure and a positive fee for assets under management. The optimal solution is characterized as the first exit time of the fund value from a bounded region with upper and lower stopping boundaries. In the infinite horizon case, we present the complete solution to the optimal stopping problem, while in the finite horizon case we derive a pair of coupled integral equations for the stopping bounds, and present an asymptotic analysis of the stopping boundaries for small time.

The paper closest to the current work is Chen et al. (2020). In that paper, the authors studied the investor's (optimal stopping) problem of when to withdraw the investment from the fund under the assumption that the fund assets follow a geometric Brownian motion, with no penalty for withdrawal, or fee for assets under management. The infinite horizon optimal stopping problem was solved explicitly, and various properties of the finite horizon problem, including the existence and convexity properties of optimal stopping boundaries were studied. This paper builds on that work in three main ways. First of all, it introduces a running fee for assets under management. Adding this realistic feature to the model adds significant complexity (it is analogous to introducing a dividend rate in the American option pricing problem), with even the solution of the infinite horizon problem being more complicated. Secondly, we derive an early exercise decomposition of the value of the investor's position, and use this to obtain a pair of coupled integral equations for the optimal stopping boundaries. Finally, we perform an asymptotic analysis of the small-time behaviour of the boundaries, employing a strategy of comparison with related "European" prices and boundaries due to Lamberton (1995). The analysis is significantly more complicated in our case however, owing to the existence of two stopping boundaries, and the lack of global convexity of the payoff function.

1.1 *Managerial Implications*

A critical aspect of the management of investments in hedge funds with first-loss fee structures is the ability of the investor to time the withdrawal of their money. Barr (2011) states that "[t]he downside for managers is that, if they suffer a big monthly loss, they lose their own capital quickly. And first-loss capital providers can pull their money fast to protect their investment". Weiss (2018) notes that billionaire hedge fund manager John Paulson resorted to employing first-loss fee structures in order to attract investors; again, the possibility of early withdrawal of investors' funds was crucial: "[w]hile Prelude and its two peers supply most of the capital in first-loss strategies, they almost never lose any of it. That's because they can shut down an account once most of the hedge fund manager's capital is gone."

The optimal stopping strategies determined in this paper can be directly applied by investors in hedge funds with the new fee structures (e.g. managers of funds of funds), governing their decisions of the time to shut down their accounts. Interestingly, although the articles in the financial press cited above mainly focus on the withdrawal of funds in the event of significant losses, we find that there are two stopping boundaries, one corresponding to large losses, and the other to large gains. The situation is similar to that which arises with American continuous installment options (see, e.g. Ciurlia and Roko (2005), Kimura (2009)). There is an obvious incentive to withdraw when there are substantial losses, as the insurance against investor losses has basically been exhausted (this is analogous to exercising the option when it is in the money). However, when gains are very large, the cost of the performance fee, which may be interpreted as an option that the fund investor has provided to the fund manager, becomes too large, and the probability of the investor's downside protection taking effect becomes miniscule. At this point, the investor would be better off under another fee structure (this is analogous to the situation when a continuous installment option is exercised/cancelled because the ongoing installment fees are too expensive when the option is very deep out of the money, and unlikely to expire with a positive payoff). Recall, as noted above that while a 20% performance fee is standard in traditional fee structures, performance fees of 40%, or even 50%, can be present in first-loss structures.

Depending on the technical sophistication of the investor, and the nature of the fee contract, the results of this paper could be used in two ways. The simplest would be to use the semi-analytical formulas for the early exercise boundaries in the infinite horizon case, outlined in Section Three. This requires only the solution of a system of two nonlinear equations. The more advanced approach, given that the investment contract has a finite horizon, would be to solve numerically the coupled integral equations presented in Section Four (perhaps employing the asymptotic results of Section Five in order to specify the small-time behaviour), and to use the resulting numerical boundaries to determine the best time to withdraw from the hedge fund investment.

1.2 *Structure of the Paper*

The remainder of the paper is structured as follows. The second section introduces notation, presents the investor's payoff in the shared and first-loss hedge fund fee structures, and defines the optimal stopping problem. The third section presents the solution of the infinite horizon optimal stopping problem. The fourth section gives an analysis of the finite horizon problem, including properties of the stopping boundaries, the early exercise decomposition, and the integral equations for the boundaries. The fifth section presents the small-time asymptotic analysis of the stopping boundaries.

2. Investor Payoffs and Optimal Stopping Model

In this paper, we assume the hedge fund assets follow a geometric Brownian motion,

$$\begin{aligned} dX_t^x &= (r-\delta)X_t^x dt + \sigma X_t^x dW_t, \quad X_0^x = x, \quad t \geqslant 0, \\ X_t^x &= x\exp\{(r-\delta-\frac{1}{2}\sigma^2)t + \sigma W_t\}. \end{aligned} \tag{2.1}$$

where $r > 0$ is the risk-free rate, $\delta > 0$ is the fee rate for assets under management, $\sigma > 0$ is the volatility and W is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$. We employ the standard right-continuous completion of the filtration generated by W; when we consider stopping times, they are with respect to this filtration, which we denote by $\mathbb{F}$. We consider the problem of finding the investor's optimal withdrawal time by optimizing the present value of the expected payoff in a risk-neutral world.[1]

For the infinite horizon problem, the value function becomes:

$$V(x) = \sup_{\tau \in \mathscr{T}} \mathbb{E}_{\mathbb{Q}}[e^{-r\tau} g(X_\tau^x)], \tag{2.2}$$

where $\mathscr{T}$ is the set of all $\mathbb{F}$-stopping times. On the other hand, if the hedge fund has a maturity date T, then the

[1] In general, the assets of a hedge fund may not be directly tradable, and so the assumptions required for risk-neutral valuation may be questioned. However, first-loss investors often provide capital to funds that invest in relatively liquid strategies. Weiss (2018) notes that "[f]irst-loss providers also generally prefer to back strategies whose assets can be easily sold should the trades sour". Exploration of the optimal stopping problem under the real-world measure and a model for fund investor preferences is a possible subject for future research.

value function at the current time (taken to be 0) is:

$$V(x,T) = \sup_{\tau \in \mathscr{T}_{[0,T]}} \mathbb{E}_{\mathbb{Q}}[e^{-r\tau} g(X_\tau^x)], \tag{2.3}$$

where $\tau \in \mathscr{T}_{[0,T]}$ is the set of all $\mathbb{F}$-stopping times such that $0 \leqslant \tau \leqslant T$, and $g(x)$ is the payoff function for the first-loss and shared-loss fee structures.

2.1 *Hedge Fund Fee Structures*

Chen et al. (2020) explain the various payoff functions $g(x)$ for first-loss and shared-loss fee structures in detail. In this section, we briefly summarize the payoffs for each fee structure. Roughly speaking, the manager has two ways to provide downside protection for the investor. First, she can set up an escrow account, which sets aside funds to compensate the investor's losses. Second, she can invest her own money in the fund and insure the investor's losses from her own share. In both cases, the investor's loss is paid in full unless the escrow account or the manager's share is wiped out.

2.1.1 *First-Loss*

First, suppose the manager sets up an escrow account and let $c,\ 0 < c < 1$ be the escrow amount, a percentage of the initial investment x. Then the payoff function to the investor is

$$g(X_T^x) = \begin{cases} \alpha X_T^x + (1-\alpha) X_T^x, & X_T^x \geqslant x, \\ x, & (1-c)x \leqslant X_T^x \leqslant x, \\ cx + X_T^x, & X_T^x \leqslant cx, \end{cases} \tag{2.4}$$

Next, the manager contributes her own capital into the fund. Let $\omega \in (0,1)$ be the proportion of the investor's initial capital contributed by the manager, so that the total initial investment is $(1+\omega)x$. Then the payoff to the investor is

$$g(X_T^x) = \begin{cases} \alpha X_T^x + (1-\alpha) X_T^x, & X_T^x \geqslant x, \\ x, & \frac{1}{1+\omega} x \leqslant X_T^x \leqslant x, \\ (1+\omega) X_T^x, & X_T^x \leqslant \frac{1}{1+\omega} x. \end{cases} \tag{2.5}$$

2.1.2 *Shared-Loss*

For the shared-loss fee structure, the manager covers the proportion θ of the investor's losses from an escrow account. If $c \geqslant \theta$, which implies that the escrow account cannot be exhausted, the payoff to the investor is

$$g(X_T^x) = \begin{cases} \alpha X_T^x + (1-\alpha) X_T^x, & X_T^x \geqslant x, \\ \theta + (1-\theta) X_T^x, & X_T^x \leqslant x. \end{cases} \tag{2.6}$$

If $c < \theta$, then the payoff to the investor is

$$g(X_T^x) = \begin{cases} \alpha X_T^x + (1-\alpha) X_T^x, & X_T^x \geqslant x, \\ \theta + (1-\theta) X_T^x, & (1-\frac{c}{\theta})x \leqslant X_T^x \leqslant x, \\ cx + X_T^x, & X_T^x \leqslant (1-\frac{c}{\theta})x. \end{cases} \tag{2.7}$$

The final case is when the manager invests ωx in the fund and covers the proportion θ of the investor's losses from her own share. The payoff is

$$g(X_T^x) = \begin{cases} \alpha X_T^x + (1-\alpha)X_T^x, & X_T^x \geqslant x, \\ \theta + (1-\theta)X_T^x, & \frac{\theta}{\omega+\theta}x \leqslant X_T^x \leqslant x, \\ (1+\omega)X_T^x, & X_T^x \leqslant \frac{\theta}{\omega+\theta}x. \end{cases} \tag{2.8}$$

Without loss of generality, we assume the investor's initial contribution is 1. Then, as in Chen et al. (2020), under both the first-loss and shared-loss fee structures, the payoff function $g(x)$ can be written in the following form

$$g(x) = \begin{cases} A + Bx, & 0 \leqslant x \leqslant \kappa \\ q + (1-q)x, & \kappa \leqslant x \leqslant 1, \\ p + (1-p)x, & 1 \leqslant x, \end{cases} \tag{2.9}$$

where $B \geqslant 1 \geqslant q > A \geqslant 0$, $p \in (0,1)$ and $\kappa = (B-(1-q))^{-1}(q-A)$.

3. Optimal Withdrawal Time: Infinite Horizon Case

In this section, we derive the value function $V(x)$ in (2.2) for the infinite horizon case. The results here generalize those of Chen et al. (2020, Section 2). We begin by establishing some properties of $V(x)$ and showing that $V(x)$ is the unique viscosity solution of a variational inequality. We then propose a solution and verify that it solves the variational inequality. By uniqueness, the proposed solution is our desired value function $V(x)$. Some numerical examples are also presented.

3.1 *Definitions and Properties*

It is well-known that under quite general assumptions, the value function V of the the infinite horizon problem (2.2) is a viscosity solution of the following variational inequality:

$$\min\Big(rV - LV, V - g\Big) = 0, \tag{3.1}$$

where L is the infinitesimal general of the process X. For convenience, we recall here the definition of a viscosity solution in this context (see, e.g. Reikvam (1998), Pham (2009, Definition 4.2.1, Page 63) or Touzi (2013, Definition 6.3, Page 68)). Let L be the infinitesimal generator of the process X, which operates on smooth functions W as

$$LW(x) = (r-\delta)x\frac{\partial W}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 W}{\partial x^2}.$$

Definition 3.1. *Let $W \in C([0,\infty),\mathbb{R})$. Then,*

1. *W is a viscosity super-solution of* (3.1) *if*

$$\min\Big(rW(x_0) - L\varphi(x_0), W(x_0) - g(x_0)\Big) \geqslant 0 \tag{3.2}$$

for all smooth functions φ and all $x_0 \in (0,\infty)$ such that $W-\varphi$ attains a local minimum at x_0.

2. *W is a viscosity sub-solution of* (3.1) *if*

$$\min\Big(rW(x_0) - L\psi(x_0), W(x_0) - g(x_0)\Big) \leqslant 0 \tag{3.3}$$

for all smooth functions ψ *and all* $x_0 \in (0,\infty)$ *such that* $W - \psi$ *attains a local maximum at* x_0.

W is called a viscosity solution of (3.1) *if it is both a viscosity super-solution and a viscosity sub-solution.*

Proposition 3.1. *Consider the value function* $V(x)$ *in* (2.2). *The following properties hold:*

a. *For* $x \in [0,\infty)$, $V(x)$ *is non-decreasing, Lipschitz continuous, and* $\lim_{x\to\infty} \frac{V(x)}{g(x)} = 1$.

b. *If* $p \geqslant q$, *then* $V(x) = g(x)$.

Proof. The monotonicity and Lipschitz continuity of V are standard, and follow from the fact that g is increasing and Lipschitz. Since $V \geqslant g$, $\liminf_{x\to\infty} V(x)/g(x) \geqslant 1$. Chen et al. (2020) show that $V_0 \sim g$ as $x \to \infty$, where V_0 is the value function when $\delta = 0$. Since $V \leqslant V_0$, $V \sim g$ follows. (b) follows from the fact that $e^{-rt}S_t$ is a $\mathbb{Q}$ supermartingale, and when $p \geqslant q$, g is increasing and concave. □

Theorem 3.1. *The value function* $V(x)$ *in* (2.2) *is the unique viscosity solution of*

$$\min\Big(rV - LV, V - g\Big) = 0. \tag{3.4}$$

satisfying $V(0) = A$ *and* $V \sim g$ *as* $x \to \infty$.

Proof. From part (a) Proposition 3.1, V satisfies a linear growth condition. Then, by Pham (2009, Theorem 5.2.1, Page 97-99), V is the unique viscosity solution of (3.4). □

By Theorem 3.1, we know that solving (2.2) is equivalent to finding a function satisfying (3.4). The remainder of the section is devoted to finding a more explicit form for $V(x)$. Given Proposition 3.1, we only consider the case $q > p$.

We divide the interval $[0,\infty)$ into the stopping region $\mathscr{S}$ and the continuation region $\mathscr{C}$:

$$\mathscr{S} = \{x | V(x) = g(x)\}, \quad \mathscr{C} = \{x | V(x) > g(x)\}.$$

By Theorem 2.1, we can easily deduce that V satisfies the Cauchy-Euler equation $rV - LV = 0$ (see Pham (2009, Lemma 5.2.2, Page 100)) in $\mathscr{C}$.

Now, consider the equation

$$LW - rW = 0, \tag{3.5}$$

with initial conditions $W(x_0) = z_0$ and $W'(x_0) = z_1$. Let $\beta = \frac{2r}{\sigma^2} > 0$ and $\gamma = \frac{2\delta}{\sigma^2} > 0$. The general solution of (3.5) is of the form,

$$W(x) = C_1 x^{m_1} + C_2 x^{m_2}, \tag{3.6}$$

where

$$m_1 = \frac{-(\beta-\gamma-1) + \sqrt{(\beta-\gamma-1)^2 + 4\beta}}{2}, \quad m_2 = \frac{-(\beta-\gamma-1) - \sqrt{(\beta-\gamma-1)^2 + 4\beta}}{2},$$

and

$$C_1 = x_0^{-m_1} \frac{x_0 z_1 - z_0 m_2}{m_1 - m_2}, \quad C_2 = x_0^{-m_2} \frac{z_0 m_1 - x_0 z_1}{(m_1 - m_2)}. \tag{3.7}$$

Remark 3.1. *From* (3.7)*, we obtain,*

$$W''(x) = C_1 m_1 (m_1 - 1) x^{m_1 - 2} + C_2 m_2 (m_2 - 1) x^{m_2 - 2}.$$

It is easy to show that $m_1 > 1$, $m_2 < 0$, and hence if either $C_1 > 0, C_2 \geqslant 0$ or $C_1 \geqslant 0, C_2 > 0$ holds, then $W(x)$ is strictly convex on $(0, \infty)$.

Proposition 3.2. *Suppose that $q > p$, then,*

a. $[0, \kappa] \subseteq \mathscr{S}$.

b. $1 \in \mathscr{C}$.

c. If $a < 1$ and $a \in \mathscr{S}$, then $[0, a] \subseteq \mathscr{S}$.

d. If $b > 1$ and $b \in \mathscr{S}$, then $[b, \infty] \subseteq \mathscr{S}$.

Proof. a. Note that $e^{-rt} X_t^x$ is a positive supermartingale, and $g(x) \leqslant A + Bx$. Thus, on $[0, \kappa]$ we have $V(x) \leqslant A + Bx = g(x)$, so $[0, \kappa] \subseteq \mathscr{S}$.

b. Suppose $1 \in \mathscr{S}$. Consider test functions of the form $\varphi(x) = 1 - M_n + M_n \exp(n(x-1))$, where $M_n = \frac{\xi}{n}$ and $\xi \in (1-q, 1-p)$. Clearly, $\varphi(1) = 1 = g(1)$, and $\varphi(x) < g(x)$ for x close to 1, $\varphi'(1) = \xi$, and $\varphi''(1) = n\xi$. By the super-solution property, we should have $r\varphi(1) - L\varphi(1) \geqslant 0$, but

$$r\varphi(1) - L\varphi(1) = r - (r - \delta)\xi - \frac{1}{2}\sigma^2 n \xi < 0$$

for n large enough. Hence $1 \in \mathscr{C}$.

c. If $a \leqslant \kappa$, $[0, \kappa] \subseteq \mathscr{S}$ implies $[0, a] \subseteq \mathscr{S}$. So we only consider the case when $\kappa < a < 1$. Suppose $\tilde{x} = \sup\{x \in \mathscr{C}, x \leqslant a\}$ exists. Then $V(\tilde{x}) = g(\tilde{x}) = q + (1-q)\tilde{x}$ and $V'(\tilde{x}) = g'(\tilde{x}) = 1 - q$ (Touzi, 2013, Theorem 4.9, page 48). Also, from (2.8) and (2.9), $V(x) = C_1 x^{m_1} + C_2 x^{m_2}$ in the component of $\mathscr{C}$ containing $\tilde{x} - \varepsilon$ for some $\varepsilon > 0$, where

$$C_2 = \frac{\tilde{x}^{-m_2}((q + (1-q)\tilde{x})m_1 - \tilde{x}(1-q))}{m_1 - m_2} = \frac{\tilde{x}^{-m_2}(q + (1-q)(m_1 - 1)\tilde{x})}{m_1 - m_2} > 0.$$

By Remark 2.1, $V(x)$ is strictly convex and will always be above its tangent line. Because $q + (1-q)x \geqslant g(x)$ for $x \leqslant \tilde{x}$, $x \in \mathscr{C}$ for $x < \tilde{x}$, contradicting $\kappa \in \mathscr{S}$.

d. Suppose $\tilde{x} = \inf\{x > b | x \in \mathscr{C}\}$ exists. Then $V(\tilde{x}) = p + (1-p)\tilde{x}$ and $V'(\tilde{x}) = (1-p)$. Again, $V(x) = C_1 x^{m_1} + C_2 x^{m_2}$ in the component of $\mathscr{C}$ containing $\tilde{x} + \varepsilon$ for some $\varepsilon > 0$. As above, it is easy to verify $C_1 > 0$, $C_2 > 0$ and hence $V(x)$ is strictly convex. Again, by the continuity and strict convexity of $V(x)$, it follows that $x \in \mathscr{C}$ for all $x > \tilde{x}$. But,

$$\lim_{x \to \infty} \frac{V(x)}{g(x)} = \lim_{x \to \infty} \frac{C_1 x^{m_1} + C_2 x^{m_2}}{p + (1-p)x} = \lim_{x \to \infty} \frac{C_1 m_1 x^{m_1 - 1} + C_2 m_2 x^{m_2 - 1}}{1 - p} = \infty.$$

contradicting $V \sim g$ as $x \to \infty$.

□

From Proposition 3.2, we can define the stopping boundaries as follows:

$$S_1 := \inf\{x \in [\kappa, 1) | V(x) > g(x)\}, \quad S_2 := \sup\{x > 1 | V(x) > g(x)\},$$

with $\kappa < S_1 < 1$, $S_2 > 1$, $\mathscr{C} = (S_1, S_2)$ and $\mathscr{S} = [0, S_1] \cup [S_2, \infty]$. In the next section, we will show $S_2 < \infty$ and derive the value function $V(x)$ when $q > p$.

3.2 *The Value Function*

For the case without dividends, Chen et al. (2020) proved that the continuation region starts either at $S_1, \kappa \leqslant S_1 < 1$ with the smooth-fit condition or at κ without satisfying the smooth-fit condition. The analogue of these results in our case is given by the following Proposition. $V'(x+)$ is the derivative from the right at x (which exists due to the continuity and monotonicity properties proved earlier).

Proposition 3.3. *(Smooth-Fit Condition) If $\kappa < S_1 < 1$, then $V'(S_1) = g'(S_1)$. If $S_2 < \infty$, $V'(S_2) = g'(S_2)$. If $S_1 = \kappa$, then $V'(S_1+) \in [1-q, B]$.*

Proof. The only part of the Proposition that does not following immediately from standard results (Touzi, 2013, Theorem 4.9, page 48) is the case when $S_1 = \kappa$. In this case, the fact that $V'(S_1+) \geqslant 1-q$ follows from $V \geqslant g$. The assumption that $V'(S_1+) > B$ leads to a contradiction by constructing a test function in a manner similar to the proof that $1 \in \mathscr{S}$ (part (b) of Proposition 3.2). □

Next, we introduce some notation. Let $W(x; x_0, v_0)$, $C_1(x_0, v_0)$ and $C_2(x_0, v_0)$ denote $W(x) = C_1 x^{m_1} + C_2 x^{m_2}$ with initial values $W(x_0) = q + (1-q)x_0$ and $W'(x_0) = v_0$. The proof of the following result requires only elementary calculus.

Lemma 3.1. a. *For $x_0 \in (0, \infty)$, $C_1(x_0, 1-q)$ is decreasing in x_0, $C_2(x_0, 1-q)$ is increasing in x_0 and $W(x; x_0, 1-q)$ is a strictly convex function on $(0, \infty)$. For $0 < x_1 < x_2$, $W(x; x_1, 1-q) > W(x; x_2, 1-q)$ for all $x \geqslant x_2$.*

b. *For $\frac{(q+(1-q)\kappa)m_2}{\kappa} \leqslant v_0 \leqslant \frac{(q+(1-q)\kappa)m_1}{\kappa}$, $W(x; \kappa, v_0)$ is a strictly convex function on $(0, \infty)$. In particular, $W(x; \kappa, \frac{qm_1 + (1-q)\kappa m_1}{\kappa}) = \kappa^{-(m_1-1)}(1 - q + \frac{q}{\kappa})x^{m_1} > g(x)$. Moreover, for $x > \kappa$, $W(x; \kappa, v_0)$ is increasing in v_0.*

According to Proposition 3.3, either $S_1 > \kappa$, and smooth fit holds, in which case V agrees with $W(x; S_1, 1-q)$ on (S_1, S_2) with $W'(S_2; S_1, 1-q) = 1-p$ (assuming $S_2 < \infty$), or $S_1 = \kappa$ and V agrees with $W(x; S_1, v_0)$ with $v_0 \in [1-q, B)$ on (S_1, S_2) again with $W'(S_2; S_1, v_0) = 1-p$. The following Proposition allows us to distinguish between these two cases (and show that $S_2 < \infty$), thereby providing an explicit characterization of the value function V. The proof is given in the appendix.

Proposition 3.4. *Suppose $p < q$, and let $h(x) = W(x; \kappa, 1-q) - (p + (1-p)x)$.*

a. *There exists a unique $x_* > \kappa$ such that $h'(x_*) = 0$.*

b. *If $h(x_*) \geqslant 0$, there exists a unique solution $(S_1, S_2) \in [\kappa, 1) \times (1, \infty)$ to $W(S_2; S_1, 1-q) = p + (1-p)S_2$, $W'(S_2; S_1, 1-q) = 1-p$.*

c. *If $h(x_*) < 0$, there exists a unique solution $(v_0, S_2) \in (1-q, 1-p) \times (1, \infty)$ to $W(S_2; \kappa, v_0) = p + (1-p)S_2$, $W'(S_2; \kappa, v_0) = 1-p$.*

Last, in the following theorem, we provide the explicit form of $V(x)$ and show that it is indeed the viscosity solution to (3.4).

Theorem 3.2. *Suppose $p < q$.*

a. *If $h(x_*) \geqslant 0$, then the value function $V(x)$ is*

$$V(x) = \begin{cases} g(x), & x \in [0, S_1], \\ W(x; S_1, 1-q), & x \in (S_1, S_2), \\ g(x), & x \in [S_2, \infty). \end{cases} \tag{3.8}$$

where $(S_1, S_2) \in [\kappa, 1) \times (1, \infty)$ is the unique solution of $W(S_2; S_1, 1-q) = p + (1-p)S_2$, $W'(S_2; S_1, 1-q) = 1-p$.

b. *If $h(x_*) < 0$, then the value function $V(x)$ is*

$$V(x) = \begin{cases} g(x), & x \in [0, \kappa], \\ W(x; \kappa, v_0), & x \in (\kappa, S_2), \\ g(x), & x \in [S_2, \infty), \end{cases} \tag{3.9}$$

where $(v_0, S_2) \in (1-q, \infty) \times (1, \infty)$ is the unique solution of the $W(S_2; \kappa, v_0) = p + (1-p)S_2$, $W'(S_2; \kappa, v_0) = 1-p$.

Proof. For V as defined above, $V \geqslant g$ by the convexity of W. The verification is standard except at $x = \kappa$. Since $V(\kappa) = g(\kappa)$, the sub-solution property holds immediately at κ. Suppose ψ is a smooth test function satisfying $\psi \geqslant V$ and $\psi(\kappa) = V(\kappa)$. Then:

$$B = V'(\kappa^-) = \lim_{x \to \kappa^-} \frac{g(x) - g(\kappa)}{x - \kappa} \leqslant \lim_{x \to \kappa^-} \frac{\varphi(x) - \varphi(\kappa)}{x - \kappa} = \varphi'(\kappa^-) = \varphi'(\kappa)$$
$$V'(\kappa^+) = \lim_{x \to \kappa^+} \frac{V(x) - V(\kappa)}{x - \kappa} \geqslant \lim_{x \to \kappa^+} \frac{\varphi(x) - \varphi(\kappa)}{x - \kappa} = \varphi'(\kappa^+) = \varphi'(\kappa),$$

which leads to the inequality $B \leqslant \varphi(\kappa) \leqslant v_0$. However $V'(\kappa+) \leqslant 1 - p < 1 \leqslant B$. Therefore, no such smooth function φ exists and the super-solution condition holds vacuously. □

3.3 *Numerical Examples*

In this section, we investigate how the parameters of the optimal stopping problem affect the stopping boundaries and the value function. By Theorem 3.2 we conclude with the following steps to solve for $V(x)$ when $q > p$ for the infinite horizon case.

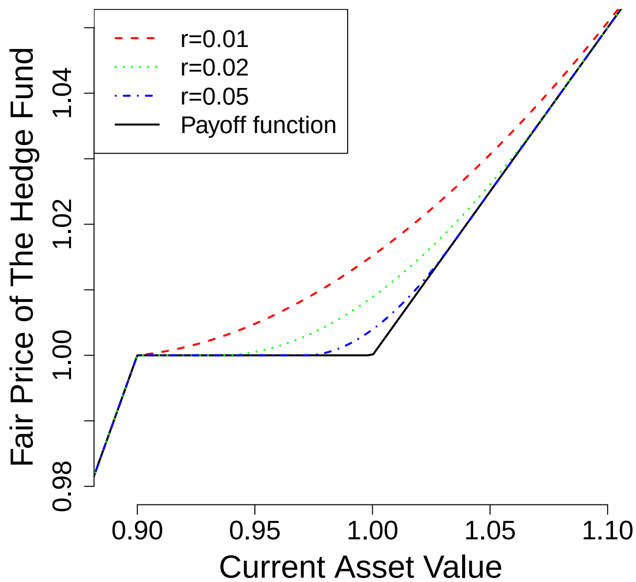
1. Solve $h'(x_*) = 0$ and calculate $h(x_*)$.

2. If $h(x_*) \geqslant 0$, then $V(x)$ is of the form (3.8) and (S_1, S_2) are solutions of $W(S_2; S_1, 1-q) = p + (1-p)S_2$, $W'(S_2; S_1, 1-q) = 1-p$.

3. If $h(x_*) < 0$, then $V(x)$ is of the form (3.9) and (v_0, S_2) are solutions of $W(S_2; \kappa, v_0) = p + (1-p)S_2$, $W'(S_2; \kappa, v_0) = 1-p$.

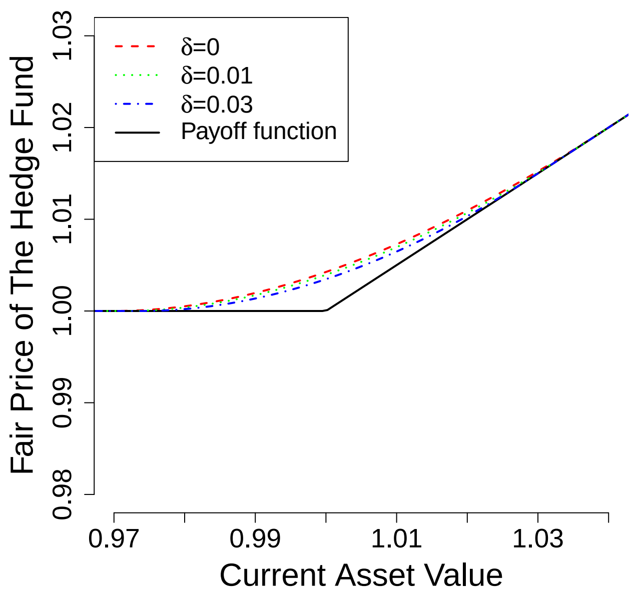
The following figures are obtained by fixing any two of the three parameters r, δ, σ and letting the remaining parameter change in its reasonable range. From the Figures, we can observe that the value function increases and the continuation region becomes wider as r decreases or σ increases in all the fee structures, while δ has a relatively small impact on the value function.

FIG. 1. Escrow First-loss



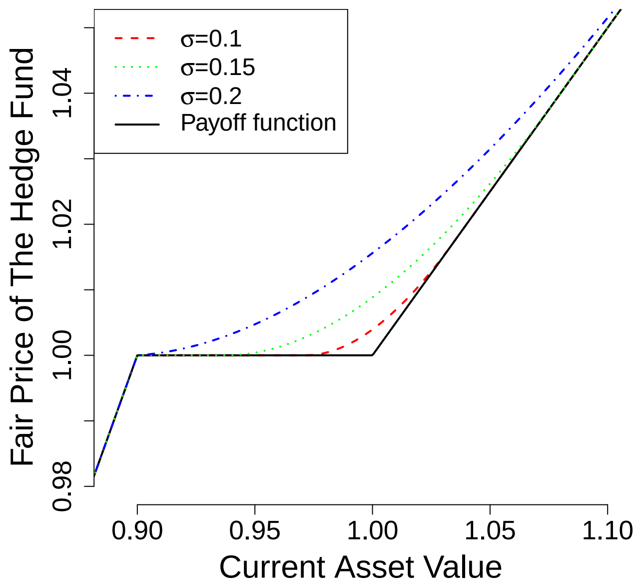
$x = 1, \delta = 0.01, \sigma = 0.1, A = 0.1, B = 1, p = 0.5, q = 1$

FIG. 2. Escrow First-loss



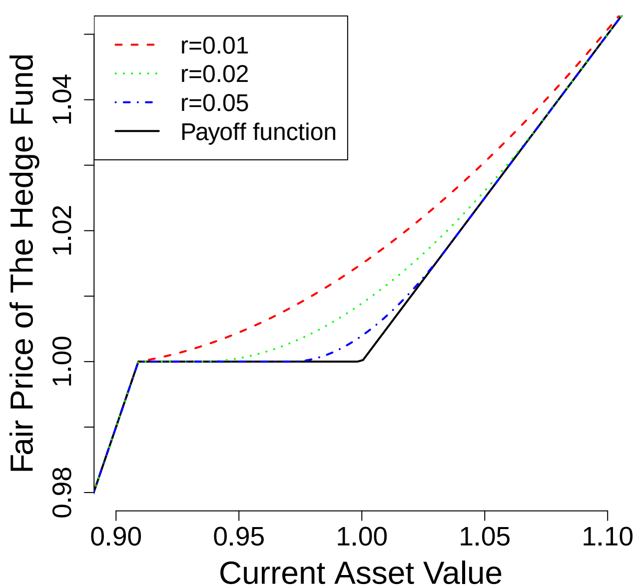
$x = 1, \delta = 0.01, \sigma = 0.1, A = 0.1, B = 1, p = 0.5, q = 1$

FIG. 3. Escrow First-loss



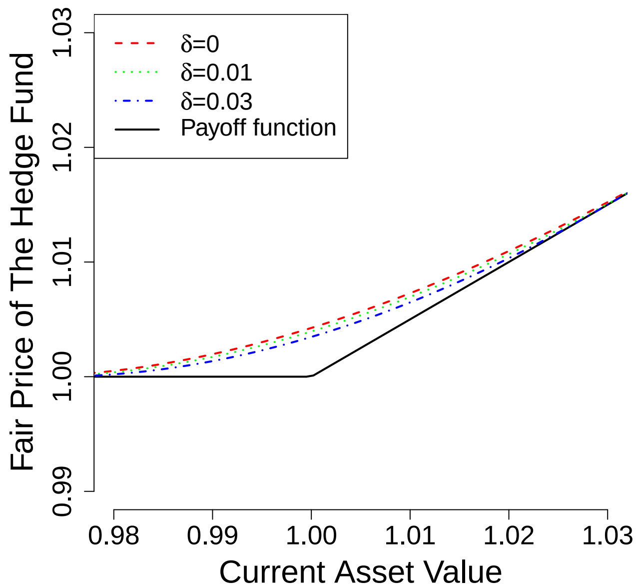
$x = 1, \delta = 0.01, \sigma = 0.1, A = 0.1, B = 1, p = 0.5, q = 1$

FIG. 4. Non-escrow First-loss



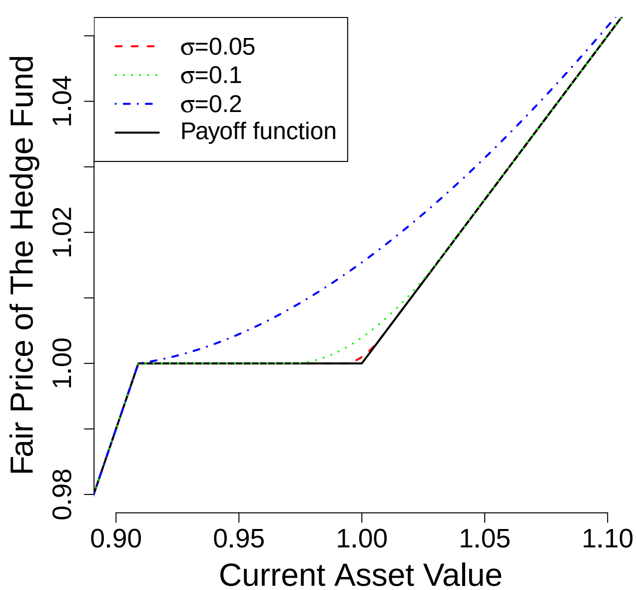
$x = 1, \delta = 0.01, \sigma = 0.1, A = 0, B = 1.1, p = 0.5, q = 1$

FIG. 5. Non-escrow First-loss



$x = 1, \delta = 0.01, \sigma = 0.1, A = 0, B = 1.1, p = 0.5, q = 1$

FIG. 6. Non-escrow First-loss



$x = 1, \delta = 0.01, \sigma = 0.1, A = 0, B = 1.1, p = 0.5, q = 1$

4. Optimal Withdrawal Time: Finite Horizon Case

4.1 *Introduction*

In this section, we study the optimal stopping problem in (2.3) for the finite horizon case. Similar to the previous section, we first derive several properties of the value function $V(x,T)$. Next, we prove monotonicity and continuity properties of the early exercise boundaries. Finally, motivated by Ciurlia and Roko (2005); Detemple (2005); Kimura (2009), we derive the early exercise premium representation.

4.2 *Definitions and Properties*

Similar to the infinite horizon case, $V(x,T)$ solves the variational inequality

$$\min\left(rV - LV + \frac{\partial V}{\partial T}, V - g\right) = 0. \tag{4.1}$$

in the viscosity sense, for which we use the following standard definition.

Definition 4.1. *Let $W \in C([0,\infty)\times[0,\infty),\mathbb{R})$. Then,*

1. *W is a viscosity super-solution of* (4.1) *if*

$$\min\left(rW(x_0,t_0) - L\varphi(x_0,t_0) + \frac{\partial \varphi}{\partial T}(x_0,t_0), W(x_0,t_0) - g(x_0)\right) \geqslant 0, \tag{4.2}$$

   for all smooth functions φ and all $(x_0,t_0) \in (0,\infty)\times(0,\infty)$ such that $W-\varphi$ attains a local minimum at (x_0,t_0).

2. *W is a viscosity sub-solution of* (4.1) *if*

$$\min\left(rW(x_0,t_0) - L\psi(x_0,t_0) + \frac{\partial \psi}{\partial T}(x_0,t_0), W(x_0,t_0) - g(x_0)\right) \leqslant 0, \tag{4.3}$$

   for all smooth functions ψ and all $(x_0,t_0) \in (0,\infty)\times(0,\infty)$ such that $W-\psi$ attains a local maximum at (x_0,t_0).

W is called a viscosity solution of (4.1) *if it is both a super-solution and sub-solution.*

The proof of the following is identical to that of the corresponding result in Chen et al. (2020).

Proposition 4.1. *For $x \in [0,\infty)$. The value function $V(x,T)$ is increasing in x, increasing in T and $\lim_{T\to\infty} V(x,T) = V(x)$.*

Recall that $V(x) = g(x)$ when $p \geqslant q$. Then, Proposition 4.1 implies $g(x) = V(x) \geqslant V(x,T) = g(x)$. Again, we only need to consider the case when $q > p$. The following Theorem connects (4.1) to our optimal stopping problem (2.3).

Theorem 4.1. *The value function is the unique viscosity solution of*

$$\min\left(rV - LV + \frac{\partial V}{\partial T}, V - g\right) = 0, \tag{4.4}$$

satisfying $V(x,0) = g(x)$, $V(0,T) = A$, $V(x,T) \sim g(x)$ as $x \to \infty$ for all $T > 0$ and $\lim_{T\to\infty} V(x,T) = V(x)$.

Proof. From Proposition 4.1, we can easily obtain that $V(x,T)$ is locally bounded for all $(t,x) \in [0,\infty) \times [0,\infty)$. By Touzi (2013, Theorem 7.7, Pages 96-99), $V(x,T)$ is the unique viscosity solution of (4.4). $\square$

Next, define the sections of the stopping region and continuation region at each time T as follows:

$$\mathscr{S}_T = \{x | V(x,T) = g(x)\}, \quad \mathscr{C}_T = \{x | V(x,T) > g(x)\}.$$

The following properties of these sets generalize the case $\delta = 0$ from Chen et al. (2020), and can be proved in the same way.

Proposition 4.2. *Suppose that $q > p$, then,*

1. $[0,\kappa] \subseteq \mathscr{S}_T$.
2. $1 \in \mathscr{C}_T$.
3. *If $a < 1$ and $a \in \mathscr{S}_T$, then $[0,a] \subseteq \mathscr{S}_T$.*
4. *If $b > 1$ and $b \in \mathscr{S}_T$, then $[b,\infty] \subseteq \mathscr{S}_T$.*

By Proposition 4.2, we can define the two stopping boundaries at maturity T as follows:

$$S_-(T) := \inf\{x | V(x,T) > g(x)\}, \quad S_+(T) := \sup\{x | V(x,T) > g(x)\}.$$

Next, we summarize some basic properties of $S_-(T)$ and $S_+(T)$.

Lemma 4.1.

1. *$S_-(T)$ and $S_+(T)$ are continuous functions.*
2. $\lim_{T \downarrow 0} S_-(T) = \lim_{T \downarrow 0} S_+(T) = 1$.
3. *The smooth-fit condition holds on the upper boundary, $\lim_{x \to S_+(T)} V(x,T) = g'(S_+(T)) = 1 - p$. Furthermore, if $S_1 > \kappa$, the smooth fit condition holds on the lower boundary as well, i.e. $\lim_{x \to S_-(T)} V_x(x,T) = g'(S_-(T)) = 1 - q$.*

Proof. The first two results can be proved using the argument in Theorem 3.1 in De Angelis (2015), while the third result can be proved following the same strategy as for the American put, see Peskir and Shiryayev (2006, pages 381–382). $\square$

4.3 *Early Exercise Representation and Integral Equations*

We now derive the early exercise representation for V, and a pair of coupled integral equations for $S_\pm(T)$. Throughout, we assume that $q > p$, and $S_1 > \kappa$. From the above, we have that V solves:

$$\begin{cases} V(x,0) = g(x), \\ \lim_{x \to S_-(T)} V(x,T) = q + (1-q) S_-(T), \\ \lim_{x \to S_-(T)} V_x(x,T) = 1 - q, \\ \lim_{x \to S_+(T)} V(x,T) = p + (1-p) S_+(T), \\ \lim_{x \to S_+(T)} V_x(x,T) = 1 - p. \end{cases} \tag{4.5}$$

Along with the information that $S_-(T)$ and $S_+(T)$ never intersect with each other and are locally bounded and continuous, we can find a representation for $V(x,T)$ by applying the change-of variable formula on curves (Peskir, 2005).

Theorem 4.2. *V defined by* (2.3) *satisfies:*

$$
\begin{aligned}
V(x,T) =& V_e(x,T) + \delta \int_0^T x e^{-\delta(T-\tau)} \Big(B + (1-q-B)\Phi(d_1(x,\kappa,T-\tau)) \\
&- (1-q)\Phi(d_1(x,S_1(\tau),T-\tau)) + (1-p)\Phi(d_1(x,S_2(\tau),T-\tau)) \Big) d\tau \\
&+ r\int_0^T e^{-r(T-\tau)} \Big(A(1-\Phi(d_2(x,\kappa,T-\tau))) \\
&+ q(\Phi(d_2(x,\kappa,T-\tau)) - \Phi(d_2(x,S_1(\tau),T-\tau)) + p\Phi(d_2(x,S_2(\tau),T-\tau))) \Big) d\tau,
\end{aligned} \tag{4.6}
$$

where $\Phi(\cdot)$ *is the standard normal cumulative distribution function,*

$$
d_1(x,y,t) = \frac{\log(x/y) + (r-\delta+\sigma^2/2)t}{\sigma\sqrt{t}}, \quad d_2(x,y,t) = d_1(x,y,t) - \sigma\sqrt{t}.
$$

and $V_e(x,T) := \mathbb{E}_{\mathbb{Q}}[e^{-rT} g(X_T^x)]$ *is the corresponding European-style value function.*

Proof. Note that the variable T is time to maturity, which implies that our time is running backward. Since it is convenient to apply the change-of-variable formula when time is running forward, we introduce the following notation. Define

$$
\begin{aligned}
&\tilde{V}(x,t;T) := V(x,T-t), \quad \tilde{Z}(x,t;T) := e^{-rt} V(x,t;T), \\
&\tilde{S}_-(t) := S_-(T-t), \quad \tilde{S}_+(t) := S_+(T-t),
\end{aligned}
$$

where $\tilde{V}(X_t,t;T)$ is the value process at the current time t, $0 \leqslant t \leqslant T$ and $\tilde{Z}(X_t,t;T)$ is the discounted value process at time t, $0 \leqslant t \leqslant T$. Applying Peskir's change-of-variable formula (Peskir, 2005, Theorem 2.1, Remark 2.3 and Remark 2.5) on $\tilde{Z}(X_t,t;T)$ leads to

$$
\begin{aligned}
\tilde{Z}(X_T,T;T) =& \tilde{Z}(X_0,0;T) + \int_0^T \frac{\partial \tilde{Z}(X_t,t;T)}{\partial t} \mathbf{1}_{\{X_t \notin \{\tilde{S}_-(t),\tilde{S}_+(t)\}\}} dt \\
&+ \int_0^T \frac{\partial \tilde{Z}(X_t,t;T)}{\partial x} \mathbf{1}_{\{X_t \notin \{\tilde{S}_-(t),\tilde{S}_+(t)\}\}} dX_t \\
&+ \frac{1}{2}\int_0^T \sigma^2 X_t^2 \frac{\partial^2 \tilde{Z}(X_t,t;T)}{\partial x^2} \mathbf{1}_{\{X_t \notin \{\tilde{S}_-(t),\tilde{S}_+(t)\}\}} dt \\
&+ \frac{1}{2}\int_0^T \Big(\frac{\partial Z(X_t+,t;T)}{\partial x} - \frac{\partial Z(X_t-,t;T)}{\partial x}\Big) \mathbf{1}_{\{X_t=\tilde{S}_-(t)\}} d\ell_t^{\tilde{S}_-} \\
&+ \frac{1}{2}\int_0^T \Big(\frac{\partial Z(X_t+,t;T)}{\partial x} - \frac{\partial Z(X_t-,t;T)}{\partial x}\Big) \mathbf{1}_{\{X_t=\tilde{S}_+(t)\}} d\ell_t^{\tilde{S}_+}
\end{aligned} \tag{4.7}
$$

where

$$
\ell_t^{\tilde{S}_\pm} := \mathbb{P}\left[\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{\tilde{S}_\pm(s)-\varepsilon < X_s < \tilde{S}_\pm(s)+\varepsilon\}} \sigma^2 X_s^2 ds \right], \quad i = 1,2 \tag{4.8}
$$

is the local time of X_s at the curve $\tilde{S}_i(s)$ for $s \in [0,t]$. Furthermore, note that

$$
\frac{\partial \tilde{Z}(X_t,t;T)}{\partial x} = e^{-rt} \frac{\partial \tilde{V}(X_t,t;T)}{\partial x}, \quad \frac{\partial^2 \tilde{Z}(X_t,t;T)}{\partial x^2} = e^{-rt} \frac{\partial^2 \tilde{V}(X_t,t;T)}{\partial x^2},
$$

$$\frac{\partial \tilde{Z}(X_t,t;T)}{\partial t} = -re^{-rt}\tilde{V}(X_t,t;T) + e^{-rt}\frac{\partial \tilde{V}(X_t,t;T)}{\partial t}.$$

Substituting the above into (4.7), we can verify that

$$\begin{aligned} e^{-rT}\tilde{V}(X_T,T;T) =& \tilde{V}(X_0,0;T) + \int_0^T e^{-rt}\frac{\partial \tilde{V}(X_t,t;T)}{\partial x}\mathbf{1}_{\{X_t\notin\{\tilde{S}_-(t),\tilde{S}_+(t)\}\}}dX_t \\ &+ \int_0^T e^{-rt}\Big(\frac{\sigma^2X_t^2}{2}\frac{\partial^2\tilde{V}(X_t,t;T)}{\partial x^2} - r\tilde{V}(X_t,t;T) + \frac{\partial \tilde{V}(X_t,t;T)}{\partial t}\Big)\mathbf{1}_{\{X_t\notin\{\tilde{S}_-(t),\tilde{S}_+(t)\}\}}dt. \end{aligned} \tag{4.9}$$

Next, knowing that $\tilde{V}(X_t,t;T) = g(X_t)$ on the stopping region,

$$\begin{aligned} \tilde{V}(X_t,t;T) =& \mathbf{1}_{\{X_t\leqslant\kappa\}}(A+BX_t) + \mathbf{1}_{\{\kappa<X_t\leqslant\tilde{S}_-(t)\}}(q+(1-q)X_t) \\ &+ \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}V(X_s,s) + \mathbf{1}_{\{X_t\geqslant\tilde{S}_+(t)\}}(p+(1-p)X_t). \end{aligned}$$

For simplicity, we define

$$\begin{aligned} f_1(x,t) &:= \mathbf{1}_{\{x<\kappa\}}(A+Bx) + \mathbf{1}_{\{\kappa<x<\tilde{S}_-(t)\}}(q+(1-q)x) + \mathbf{1}_{\{x>\tilde{S}_+(t)\}}(p+(1-p)x) \\ \frac{\partial f_1(x,t)}{\partial x} &:= \mathbf{1}_{\{x<\kappa\}}B + \mathbf{1}_{\{\kappa<x<\tilde{S}_-(t)\}}(1-q) + \mathbf{1}_{\{x>\tilde{S}_+(t)\}}(1-p). \end{aligned}$$

Then it is easy to obtain the following expressions,

$$\tilde{V}(X_t,t;T)\mathbf{1}_{\{X_t\notin\{\tilde{S}_-(t),\tilde{S}_+(t)\}\}} = f_1(X_t,t) + \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\tilde{V}(X_t,t;T), \tag{4.10}$$

$$\begin{aligned} \frac{\partial \tilde{V}(X_t,t;T)}{\partial x}\mathbf{1}_{\{X_t\notin\{\tilde{S}_-(t),\tilde{S}_+(t)\}\}} &= \frac{\partial f_1(X_t,t)}{\partial x} + \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\frac{\partial \tilde{V}(X_t,t;T)}{\partial x} \\ &= \mathbf{1}_{\{X_t<\kappa\}}B + \mathbf{1}_{\{\kappa<X_t<\tilde{S}_-(t)\}}(1-q) \\ &\quad + \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\frac{\partial \tilde{V}(X_t,t;T)}{\partial x} + \mathbf{1}_{\{X_t>\tilde{S}_+(t)\}}(1-p), \end{aligned} \tag{4.11}$$

$$\frac{\partial^2 \tilde{V}(X_t,t;T)}{\partial x^2}\mathbf{1}_{\{X_t\notin\{\tilde{S}_-(t),\tilde{S}_+(t)\}\}} = \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\frac{\partial^2 \tilde{V}(X_t,t;T)}{\partial x^2}, \tag{4.12}$$

$$\frac{\partial \tilde{V}(X_t,t;T)}{\partial t}\mathbf{1}_{\{X_t\notin\{\tilde{S}_-(t),\tilde{S}_+(t)\}\}} = \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\frac{\partial \tilde{V}(X_t,t;T)}{\partial t}. \tag{4.13}$$

Substituting (4.10), (4.11), (4.12) and (4.13) into (4.9), we have

$$\begin{aligned} e^{-rT}\tilde{V}(X_T,T;T) =& \tilde{V}(X_0,0;T) + \int_0^T e^{-rt}\Big(\frac{\partial f_1(X_t,t)}{\partial x} + \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\frac{\partial \tilde{V}(X_t,t;T)}{\partial x}\Big)dX_t \\ &+ \int_0^T e^{-rt}\Big(\frac{\sigma^2X_t^2}{2}\mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\frac{\partial^2\tilde{V}(X_t,t;T)}{\partial x^2} \\ &- r\big(f_1(X_t,t) + \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\tilde{V}(X_t,t;T)\big) + \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}\frac{\partial \tilde{V}(X_t,t;T)}{\partial t}\Big)dt \\ =& \tilde{V}(X_0,0;T) + \int_0^T e^{-rt}\frac{\partial f_1(X_t,t)}{\partial x}dX_t \\ &+ \int_0^T \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}e^{-rt}(r-\delta)X_t\frac{\partial \tilde{V}(X_t,t;T)}{\partial x}dt \\ &+ \int_0^T \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}}e^{-rt}\sigma X_t\frac{\partial \tilde{V}(X_t,t;T)}{\partial x}dW_t \end{aligned}$$

$$
\begin{aligned}
&+\int_0^T \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}} e^{-rt}\big(\frac{\sigma^2 X_t^2}{2}\frac{\partial^2 \tilde{V}(X_t,t;T)}{\partial x^2} - r\tilde{V}(X_t,t;T) + \frac{\partial \tilde{V}(X_t,t;T)}{\partial t}\big)dt \\
&-\int_0^T re^{-rt} f_1(X_t,t)dt \\
=&\tilde{V}(X_0,0;T) + \int_0^T e^{-rt}(r-\delta)X_t\frac{\partial f_1(X_t,t)}{\partial x}dt + \int_0^T e^{-rt}\sigma X_t \frac{\partial f_1(X_t,t)}{\partial x}dW_t \\
&+\int_0^T \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}} e^{-rt}\Big((r-\delta)X_t\frac{\partial \tilde{V}(X_t,t;T)}{\partial x} \\
&+\frac{\sigma^2 X_t^2}{2}\frac{\partial^2 \tilde{V}(X_t,t;T)}{\partial x^2} - r\tilde{V}(X_t,t;T) + \frac{\partial \tilde{V}(X_t,t;T)}{\partial t}\Big)dt \\
&+\int_0^T \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}} e^{-rt}\sigma X_t \frac{\partial \tilde{V}(X_t,t;T)}{\partial x}dW_t - \int_0^T re^{-rt} f_1(X_t,t)dt. \qquad (4.14)
\end{aligned}
$$

Note that on the continuation region, $V(x,T)$ satisfies the PDE $LV - rV - V_T = 0$, so $\tilde{V}_t + L\tilde{V} - r\tilde{V} = 0$. Thus, we can further simplify (4.14) as follows,

$$
\begin{aligned}
e^{-rT}\tilde{V}(X_T,T;T) =&\tilde{V}(X_0,0;T) + \int_0^T e^{-rt}(r-\delta)X_t\frac{\partial f_1(X_t,t)}{\partial x}dt + \int_0^T e^{-rt}\sigma X_t \frac{\partial f_1(X_t,t)}{\partial x}dW_t \\
&+\int_0^T \mathbf{1}_{\{\tilde{S}_-(t)<X_t<\tilde{S}_+(t)\}} e^{-rt}\sigma X_t \frac{\partial \tilde{V}(X_t,t;T)}{\partial x}dW_t - \int_0^T re^{-rt} f_1(X_t,t)dt. \qquad (4.15)
\end{aligned}
$$

Taking expectations and applying Fubini's Theorem on (4.15), we obtain,

$$
\begin{aligned}
E[e^{-rT}\tilde{V}(X_T,T;T)] =&\tilde{V}(X_0,0;T) + \int_0^T (r-\delta)e^{-rt}E\left[\frac{\partial f_1(X_t,t)}{\partial x}X_t\right]dt - \int_0^T re^{-rt}E[f_1(X_t,t)]dt \\
=&\tilde{V}(X_0,0;T) - \delta\int_0^T e^{-rt}E\left[\frac{\partial f_1(X_t,t)}{\partial x}X_t\right]dt \\
&+r\int_0^T e^{-rt}\left(E\left[\frac{\partial f_1(X_t,t)}{\partial x}X_t\right] - E[f_1(X_t,t)]\right)dt.
\end{aligned}
$$

Note that $\tilde{V}(X_T,T;T) = g(X_T)$, so $E[e^{-rT}\tilde{V}(X_T,T;T)] = E[e^{-rT}g(X_T)]$. After rearranging terms, the value function $V(x,0;T)$ has the early exercise premium integral representation:

$$
\begin{aligned}
\tilde{V}(x,0;T) =&V_e(x,T) + \delta\int_0^T e^{-rt}E\left[\frac{\partial f_1(X_t,t)}{\partial x}X_t\right]dt \\
&+r\int_0^T e^{-rt}\left(E[f_1(X_t,t)] - E\left[\frac{\partial f_1(X_t,t)}{\partial x}X_t\right]\right)dt. \qquad (4.16)
\end{aligned}
$$

Now, we can calculate the expectations in (4.16) separately. First, we write down each expectation explicitly

$$
\begin{aligned}
V_e(x,T) =&e^{-rT}(E[1_{\{X_T<\kappa\}}(A+BX_T)] + E[\mathbf{1}_{\{\kappa<X_T<1\}}(q+(1-q)X_T)] \\
&+E[\mathbf{1}_{\{X_T>1\}}(p+(1-p)X_T)]), \qquad (4.17)
\end{aligned}
$$

$$
E\left[\frac{\partial f_1(X_t,t)}{\partial x}X_t\right] = E[\mathbf{1}_{\{X_t<\kappa\}}BX_t + \mathbf{1}_{\{\kappa<X_t<\tilde{S}_-(t)\}}(1-q)X_t + \mathbf{1}_{\{X_t>\tilde{S}_+(t)\}}(1-p)X_t], \qquad (4.18)
$$

$$
\begin{aligned}
E[f_1(X_t,t)] =&E[\mathbf{1}_{\{X_t<\kappa\}}(A+BX_t) + \mathbf{1}_{\{\kappa<X_t<\tilde{S}_-(t)\}}(q+(1-q)X_t) + \mathbf{1}_{\{X_t>\tilde{S}_+(t)\}}(p+(1-p)X_t)] \\
=&E[\mathbf{1}_{\{X_t<\kappa\}}A + \mathbf{1}_{\{\kappa<X_t<\tilde{S}_-(t)\}}q + \mathbf{1}_{\{X_t>\tilde{S}_+(t)\}}p] + E\left[\frac{\partial f_1(X_t,t)}{\partial x}X_t\right]. \qquad (4.19)
\end{aligned}
$$

Note that since X_t follows a log-normal distribution, we can easily obtain:

$$E\left[\frac{\partial f_1(X_t,t)}{\partial x}X_t\right] = xe^{(r-\delta)t}\Big(B+(1-q-B)\Phi(d_1(x,\kappa,t)-(1-q)\Phi(d_1(x,\tilde{S}_-(t),t)) \\ +(1-p)\Phi(d_1(x,\tilde{S}_+(t),t))\Big), \quad (4.20)$$

$$E[f_1(X_t,t)]-E\left[\frac{\partial f_1(X_t,t)}{\partial x}X_t\right] = A(1-\Phi(d_2(x,\kappa,t))) \\ +q(\Phi(d_2(x,\kappa,t))-\Phi(d_2(x,\tilde{S}_-(t),t)) \\ +p\Phi(d_2(x,\tilde{S}_+(t),t))). \quad (4.21)$$

Next, substituting (4.20) and (4.21) into (4.16) yields the following integral representation for $\tilde{V}(X_0,0;T)$:

$$\tilde{V}(X_0,0;T) = V_e(x,T)+\delta\int_0^T xe^{-\delta t}\Big(B+(1-q-B)\Phi(d_1(x,\kappa,t) \\ -(1-q)\Phi(d_1(x,\tilde{S}_-(t),t))+(1-p)\Phi(d_1(x,\tilde{S}_+(t),t))\Big)dt \\ +r\int_0^T e^{-rt}\Big(A(1-\Phi(d_2(x,\kappa,t))) \\ +q(\Phi(d_2(x,\kappa,t))-\Phi(d_2(x,\tilde{S}_-(t),t))+p\Phi(d_2(x,\tilde{S}_+(t),t)))\Big)dt. \quad (4.22)$$

After reverting to our original notation, this is the desired result. □

Thus, by Theorem 4.2 and the boundary conditions (4.5), the optimal stopping boundaries $S_-(T)$ and $S_+(T)$ satisfy the following coupled pair of integral equations:

$$q+(1-q)S_-(T) = V_e(S_1(T),T)+\delta\int_0^T S_-(T)e^{-\delta(T-\tau)}\Big(B+(1-q-B)\Phi(d_1(S_-(T),\kappa,T-\tau) \\ -(1-q)\Phi(d_1(S_-(T),S_1(\tau),T-\tau))+(1-p)\Phi(d_1(S_-(T),S_2(\tau),T-\tau))\Big)d\tau \\ +r\int_0^T e^{-r(T-\tau)}\Big(A(1-\Phi(d_2(S_-(T),\kappa,T-\tau))) \\ +q(\Phi(d_2(S_-(T),\kappa,T-\tau))-\Phi(d_2(S_-(T),S_1(\tau),T-\tau)) \\ +p\Phi(d_2(S_-(T),S_2(\tau),T-\tau))\Big)d\tau \quad (4.23)$$

$$p+(1-p)S_+(T) = V_e(S_2(T),T)+\delta\int_0^T S_+(T)e^{-\delta(T-\tau)}\Big(B+(1-q-B)\Phi(d_1(S_+(T),\kappa,T-\tau) \\ -(1-q)\Phi(d_1(S_+(T),S_1(\tau),T-\tau))+(1-p)\Phi(d_1(S_+(T),S_2(\tau),T-\tau))\Big)d\tau \\ +r\int_0^T e^{-r(T-\tau)}\Big(A(1-\Phi(d_2(S_+(T),\kappa,T-\tau))) \\ +q(\Phi(d_2(S_+(T),\kappa,T-\tau))-\Phi(d_2(S_+(T),S_1(\tau),T-\tau)) \\ +p\Phi(d_2(S_+(T),S_2(\tau),T-\tau))\Big)d\tau. \quad (4.24)$$

5. Exercise Boundaries Near Maturity

In this section, we study the asymptotic behaviour of the stopping boundaries $S_{\pm}(T)$ for small T. In particular, we show that as $T \searrow 0$:

$$S_{\pm}(T) \sim 1 \pm \sigma\sqrt{T(-\log T)} \tag{5.1}$$

We follow the strategy employed by Lamberton (1995) in the case of the American put. Translated into our context, this consists of the following steps:

- Show that for the European option with payoff g, and price $V^e(x,T) = \mathbb{E}_{\mathbb{Q}}[e^{-rT}g(X_T^x)]$, and for $T > 0$ small enough, there exist two boundaries $S_-^e(T) < 1 < S_+^e(T)$ such that $V^e(S_-^e(T),T) = g(S_-^e(T))$, $V^e(S_+^e(T),T) = g(S_+^e(T))$.
- Derive the small-time behaviour of $S_{\pm}^e(T)$.
- Show that for small T, the boundaries $S_{\pm}(T)$ are close to $S_{\pm}^e(T)$. In particular, for T small enough, there exists a $C > 0$ such that:

$$0 \leqslant S_-^e(T) - S_-(T) \leqslant C\sqrt{T}, \quad 0 \leqslant S_+(T) - S_+^e(T) \leqslant C\sqrt{T}. \tag{5.2}$$

- Infer the asymptotic behaviour of $S_{\pm}(T)$ from that of $S_{\pm}^e(T)$.

Implementing the strategy in this case is significantly more complicated than in the case of the American put covered by Lamberton (1995), for two main reasons. First of all, we need to deal with two boundaries rather than a single one. Secondly, our payoff function is more complicated; in particular it lacks the convexity that aids in the analysis of the American put.

We need the following simple results, whose proofs are omitted (the first is a straightforward calculation, while the second and third follow from elementary calculus).

Lemma 5.1. *Let $X_t^x = x\exp\{(r-\delta+\frac{\sigma^2}{2})t + \sigma W_t\}$. Then, for $0 \leqslant a \leqslant b < \infty$,*

$$\mathbb{Q}[a < X_t^x < b] = \Phi(d_2(x,a,t)) - \Phi(d_2(x,b,t)), \tag{5.3}$$

$$\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\{a<X_t^x<b\}}X_t^x] = xe^{(r-\delta)t}(\Phi(d_1(x,a,t)) - \Phi(d_1(x,b,t)), \tag{5.4}$$

where Φ is the standard normal cumulative distribution function,

$$d_1(x,y,t) = \frac{\log(x/y) + (r-\delta+\sigma^2/2)t}{\sigma\sqrt{t}}, \ \textit{and} \ d_2(x,y,t) = d_1(x,y,t) - \sigma\sqrt{t}.$$

Lemma 5.2. *Let $C(x,K,T) = \mathbb{E}_{\mathbb{Q}}[e^{-rT}\max(X_T^x - K, 0)]$ denote the price of a European call option with strike price K, maturity date T, and current stock value x. Then as $T \searrow 0$:*

$$C(x,K,T) - \max(x-K,0) = \begin{cases} O(\sqrt{T}), & \textit{if } x = K \\ o(\sqrt{T}), & \textit{if } x \neq K. \end{cases} \tag{5.5}$$

Lemma 5.3. *a. Suppose the function $f_1(x)$ is smooth on the interval $[a_1,b_1]$. If the following conditions are satisfied*

1. *$f_1(a_1) < 0$, $f_1(b_1) > 0$, $f_1'(a_1) > 0$, $f_1'(b_1) > 0$.*
2. *$f_1(x)$ has a unique inflection point $x_2 \in (a_1,b_1)$ satisfying $f_1''(x_2) = 0$. Moreover, $f_1''(x) < 0$ for $x < x_2$ and $f_1''(x) > 0$ for $x > x_2$.*

3. *For any point $x_1 \in (a_1, b_1)$ such that $f_1'(x_1) = 0$, $f_1(x_1) < 0$.*

Then $f_1(x) = 0$ has a unique solution on the interval (a_1, b_1).

b. *Suppose the function $f_2(x)$ is smooth and strictly convex on the interval $[a_2, \infty)$ with $f(a_2) > 0$. If there exists a constant b_2 such that $f_2(x) \leqslant 0$ for all $x \geqslant b_2$, then the equation $f_2(x) = 0$ has a unique solution on the interval (a_2, ∞).*

Proposition 5.1. *There exists $T_e > 0$ such that for all $T \in (0, T_e]$, $V_e(1,T) > 1$, and there exists a unique $S_-^e(T) \in (\kappa, 1)$ and a unique $S_+^e(T) \in (1, \infty)$ satisfying $V_e(S_-^e(T), T) = g(S_-^e(T))$ and $V_e(S_+^e(T), T) = g(S_+^e(T))$ respectively.*

Proof. Noting that $g(x) = (A + Bx) - (B - 1 + q)(x - \kappa)_+ + (q - p)(x - 1)_+$, we have:

$$V_e(x,T) = Ae^{-rT} + xe^{-\delta T}B - (B-1+q)C(x,\kappa,T) + (q-p)C(x,1,T). \tag{5.6}$$

Let $u_T(x) = V_e(x,T) - q - (1-q)x$. Recalling that $\kappa = (q-A)(B-1+q)^{-1}$ and using (5.5), we have:

$$\begin{aligned} u_T(\kappa) &= Ae^{-rT} + \kappa B e^{-\delta T} - q - (1-q)\kappa - (B-1+q)C(\kappa,\kappa,T) + (q-p)C(\kappa,1,T) \\ &\leqslant Ae^{-rT} + \kappa B - q - (1-q)\kappa - (B-1+q)C(\kappa,\kappa,T) + (q-p)C(\kappa,1,T) \\ &= Ae^{-rT} - q + \kappa(B-1+q) - (B-1+q)C(\kappa,\kappa,T) + (q-p)C(\kappa,1,T) \\ &= A(e^{-rT} - 1) - (B-1+q)C(\kappa,\kappa,T) + (q-p)C(\kappa,1,T). \end{aligned} \tag{5.7}$$

By (5.5) and the fact that $e^{-rT} - 1 = O(T)$, it can be verified that $C(\kappa,\kappa,T) = O(\sqrt{T})$ converges slower than the other terms for T sufficiently small. Since $C(\kappa,\kappa,T)$ is always positive, we must have $u_T(\kappa) < 0$ for some T small enough. Similarly, we also have

$$u_T(1) = Ae^{-rT} + (Be^{-\delta T} - 1) - (B-1+q)C(1,\kappa,T) + (q-p)C(1,1,T) > 0 \tag{5.8}$$

for T sufficiently small (implying $V_e(1,T) > 1$). Therefore, we can conclude that there must exist a T_1 small enough such that for all $T \leqslant T_1$, $u_T(\kappa) < 0$ and $u_T(1) > 0$. Next, differentiating $u_T(x)$ with respect to x, we obtain,

$$\begin{aligned} u_T'(x) = B(e^{-\delta T} - 1) + (B-1+q)(1 - e^{-\delta T}\Phi(d_1(x,\kappa,T)) \\ + (q-p)e^{-\delta T}\Phi(d_1(x,1,T)) \end{aligned} \tag{5.9}$$

from which it immediately follows that $u_T'(\kappa) > 0$ and $u_T'(1) > 0$ for T small enough. Now, differentiating with respect to x again, we have,

$$u_T''(x) = -(B-1+q)\frac{e^{-\delta T}}{\sigma x\sqrt{T}}\varphi(d_1(x,\kappa,T)) + (q-p)\frac{e^{-\delta T}}{\sigma x\sqrt{T}}\varphi(d_1(x,1,T)). \tag{5.10}$$

Noting that $d_1(x,1,T) = d_1(x,\kappa,T) + C_1(T)$, where $C_1(T) = \frac{\log \kappa}{\sigma\sqrt{T}} < 0$, we have:

$$u_T''(x) = \frac{e^{-\delta T}\varphi(d_1(x,\kappa,T))}{\sigma x\sqrt{T}}\Big(-(B-1+q) + (q-p)e^{-C_1(T)d_1(x,\kappa,T) - \frac{1}{2}C_1(T)^2}\Big). \tag{5.11}$$

Now, let $h_T(x) = -(B-1+q) + (q-p)e^{-C_1 d_1(x,\kappa,T) - \frac{1}{2}C_1^2}$, so that roots of h_T are the same as those of u_T''. It can easily be verified that $\lim_{x\to 0+} h_T(x) = -(B-1+q) < 0$, $\lim_{x\to\infty} h_T(x) = \infty$ and $h_T(x)$ is strictly increasing in x. So, for each fixed T, we must have a unique root $x_2(T) \in (0,\infty)$ such that $h_T(x_2(T)) = 0$. Letting $C_2 = \frac{q-p}{B-1+q}$, a simple calculation yields

$$x_2(T) = \kappa^{\frac{1}{2}} \exp\left(-\left(r - \delta + \left(\frac{1}{2} - \frac{\log C_2}{\log \kappa}\right)\sigma^2\right) T\right). \tag{5.12}$$

Moreover, since $\kappa < \kappa^{\frac{1}{2}} < 1$ and $x_2(T)$ converges to $\kappa^{\frac{1}{2}}$ as $T \to 0$, there must exist a T_3 small enough such that $x_2(T) \in (\kappa, 1)$ for all $T \leqslant T_3$. Taking $T_e = \min(T_1, T_2, T_3)$, we have that $u_T(\kappa) < 0$, $u_T(1) > 0$, $u_T'(\kappa) > 0$, $u_T'(1) > 0$, and u_T'' is strictly negative on $(\kappa, x_2(T))$, and strictly positive on $(x_2(T), 1)$ for $T \leqslant T_e$. Finally, substituting $u_T'(x_1(T)) = 0$ into the definition of $u_T(x)$ and simplifying yields:

$$\begin{aligned} u_T(x_1(T)) = q(e^{-rT} - 1) - (q - A)(1 - \Phi(d_2(x_1(T), \kappa, T))) \\ - (q-p)e^{-rT}\Phi(d_2(x_1(T), 1, T)) < 0 \end{aligned} \tag{5.13}$$

Lemma 5.3 (a) implies that $u_T(x) = 0$ must attain a unique root on $(\kappa, 1)$ for every fixed $T \leqslant T_e$, i.e. there is a unique $S_-^e(T)$ such that $V_e(S_-^e(T), T) = g(S_-^e(T))$.

To prove $V_e(x,T) = g(x)$ attains a unique root on $(1,\infty)$, we let $v_T(x) = V_e(x,T) - p - (1-p)x$. Note that $u_T(1) = v_T(1)$. So, $v_T(1) > 0$ for $T \leqslant T_e$. Since $v_T''(x) = u_T''(x)$, $v_T''(x) > 0$ on $(1,\infty)$ and $v_T(x)$ is strictly convex on $(1,\infty)$ for all $T \leqslant T_e$. Moreover, since $V_e(x,T) \leqslant V(x,T) = g(x)$ for $x \geqslant S_+(T)$ and $1 < S_+(T) \leqslant S_2 < \infty$, it can be easily verified that $v_T(x) = V_e(x,T) - p - (1-p)x \leqslant 0$ for all $x \geqslant S_+(T)$. By Lemma 5.3 (b), we obtain that there exists a unique root $S_+^e(T) \in (1,\infty)$ satisfying $V_e(S_+^e(T), T) = g(S_+^e(T))$ for all $T \leqslant T_e$. □

The following depends only on the fact that $V(x,T) \geqslant V_e(x,T)$.

Lemma 5.4. *For $T \leqslant T_e$:*

$$S_-(T) \leqslant S_-^e(T) \leqslant S_+^e(T) \leqslant S_+(T) \tag{5.14}$$

Proof. Since $V(x,T) \geqslant V_e(x,T)$, we have $g(S_-(T)) = V(S_-(T),T) \geqslant V_e(S_-(T),T)$ and $g(S_+(T)) = V(S_+(T),T) \geqslant V_e(S_+(T),T)$. Since $V^e > g$ on $(S_-^e(T), S_+^e(T))$, we must have we must have $S_-^e(T) \geqslant S_-(T)$ and $S_+^e(T) \leqslant S_+(T)$. □

We next give a rough result on the rate of convergence of the boundaries $S_-^e(T)$ and $S_+^e(T)$ to one in small-time (Lamberton (1995, Lemma 2.2) proves an analogous property for the American put).

Lemma 5.5.

$$\lim_{T\to 0+} \frac{S_-^e(T) - 1}{\sqrt{T}} = -\infty, \qquad \lim_{T\to 0+} \frac{S_+^e(T) - 1}{\sqrt{T}} = \infty.$$

Proof. By the previous Lemma, we have

$$1 = \lim_{T\to 0+} S_-(T) \leqslant \lim_{T\to 0+} S_-^e(T) \leqslant \lim_{T\to 0+} S_+^e(T) \leqslant \lim_{T\to 0+} S_+(T) = 1. \tag{5.15}$$

Next, note that for T small enough, we have a unique $S_-^e(T)$ such that $q + (1-q)S_-^e(T) = V_e(S_-^e(T), T)$. Then, by (5.6), and a simple rearrangement, we obtain

$$\begin{aligned} A(e^{-rT} - 1) + S_-^e(T)B(e^{-\delta T} - 1) + (q-p)C(S_-^e(T), 1, T) \\ - (B - 1 + q)\Big(C(S_-^e(T), \kappa, T) - (S_-^e(T) - \kappa)\Big) = 0 \end{aligned} \tag{5.16}$$

Using put-call parity (with dividends), we obtain from (5.16):

$$\frac{1-S^e_-(T)}{\sqrt{T}}e^{-\delta T} = \frac{A(e^{-rT}-1)}{(q-p)\sqrt{T}} + \frac{BS^e_-(T)(e^{-\delta T}-1)}{\sqrt{T}} + \frac{\mathbb{E}_{\mathbb{Q}}\left[e^{-rT}\max\left(1-X_T^{S^e_-(T)},0\right)\right]}{\sqrt{T}} + \frac{e^{-\delta T}-e^{-rT}}{\sqrt{T}} - \frac{(B-1+q)\left(C(S^e_-,\kappa,T)-(S^e_-(T)-\kappa)\right)}{(q-p)\sqrt{T}} \quad (5.17)$$

By Lemma 5.2 and elementary calculus, all the terms on the right hand side of the above equation tend to zero, except for the "put-option" term. Thus:

$$\begin{aligned}
\eta_1 &:= \liminf_{T\to 0+}\frac{1-S^e_-(T)}{\sqrt{T}} = \liminf_{T\to 0+}\frac{e^{-rT}}{\sqrt{T}}\mathbb{E}_{\mathbb{Q}}\left[\max\left(1-X_T^{S^e_-(T)},0\right)\right] \\
&= \liminf_{T\to 0}\frac{1}{\sqrt{T}}\mathbb{E}_{\mathbb{Q}}\left[\max\left(1-S^e_-(T)\exp\left((r-\delta-\tfrac{\sigma^2}{2})T+\sigma\sqrt{T}\cdot Z\right),0\right)\right] \\
&\geqslant \mathbb{E}_{\mathbb{Q}}\left[\max\left(\liminf_{T\to 0+}\frac{1-S^e_-(T)}{\sqrt{T}} + \lim_{T\to 0+}\frac{S^e_-(T)\left(1-\exp\left((r-\delta-\frac{\sigma^2}{2})T+\sigma\sqrt{T}\cdot Z\right)\right)}{\sqrt{T}},0\right)\right]
\end{aligned}$$

where $Z\sim N(0,1)$, and we have used Fatou's Lemma. If $\eta_1\in[0,\infty)$ then we get $\eta_1 \geqslant \mathbb{E}_{\mathbb{Q}}[\max(\eta_1-\sigma Z,0)]$, which leads to a contradiction as $\mathbb{Q}(\{\eta_1-\sigma Z<0\})>0$ implies $\mathbb{E}_{\mathbb{Q}}[\max(\eta_1-\sigma Z,0)] > \eta_1-\sigma\mathbb{E}_{\mathbb{Q}}[Z]=\eta_1$. Thus $\eta_1=\infty$. The proof for $S^e_+(T)$ is similar. □

Theorem 5.1. *For $T\geqslant 0$, let*

$$\psi_1(T) = \frac{-\log S^e_-(T)}{\sigma\sqrt{T}}, \quad \psi_2(T) = \frac{\log S^e_+(T)}{\sigma\sqrt{T}}.$$

As $T\searrow 0$:

$$\psi_1(T)^2 e^{\frac{\psi_1(T)^2}{2}} \sim \frac{(q-p)\sigma}{(rq+(1-q)\delta)\sqrt{2\pi T}}, \quad \psi_2(T)^2 e^{\frac{\psi_2(T)^2}{2}} \sim \frac{(q-p)\sigma}{(rq+(1-q)\delta)\sqrt{2\pi T}}, \quad (5.18)$$

and furthermore,

$$1-S^e_-(T)\sim\sigma\sqrt{T(-\log T)}, \quad \textit{and} \quad S^e_+(T)-1\sim\sigma\sqrt{T(-\log T)}. \quad (5.19)$$

Proof. Let $y(T)=S^e_-(T)$ for convenience. A simple rearrangement of $V^e(y(T),T)=q+(1-q)y(T)$ yields:

$$\mathbb{E}_{\mathbb{Q}}[e^{-rT}g(X_T^{y(T)})] = qe^{-rT}+(1-q)y(T)e^{-\delta T} + \mathbb{E}_{\mathbb{Q}}[e^{-rT}\left(\mathbf{1}_{\{X_T^{y(T)}\leqslant\kappa\}}(A-q+(B-1+q)X_T^{y(T)})\right)] + \mathbb{E}_{\mathbb{Q}}[e^{-rT}\left(\mathbf{1}_{\{X_T^{y(T)}\geqslant 1\}}(p-q+(q-p)X_T^{y(T)})\right)]. \quad (5.20)$$

So:

$$q(1-e^{-rT})+(1-q)y(T)(1-e^{-\delta T}) = \mathbb{E}_{\mathbb{Q}}[e^{-rT}\left(\mathbf{1}_{\{X_T^{y(T)}\leqslant\kappa\}}(A-q+(B-1+q)X_T^{y(T)})\right)] + \mathbb{E}_{\mathbb{Q}}[e^{-rT}\left(\mathbf{1}_{\{X_T^{y(T)}\geqslant 1\}}(p-q+(q-p)X_T^{y(T)})\right)]. \quad (5.21)$$

Noting that $q(1-e^{-rT})+(1-q)y(T)(1-e^{-\delta T}) \sim (rq+(1-q)\delta)T$ and $\lim_{T\to 0^+} \mathbb{E}_{\mathbb{Q}}[e^{-rT}\big(\mathbf{1}_{\{X_T \leqslant \kappa\}}(A-q+(B-1+q)X_T^{y(T)})\big)]/T = 0$, (5.21) becomes

$$\frac{(rq+(1-q)\delta)T}{q-p} \sim \mathbb{E}_{\mathbb{Q}}[e^{-rT}(X_T^{y(T)}-1)^+]. \tag{5.22}$$

Now, let

$$\alpha_1(T) = \frac{-\log S_-^e(T) - (r-\delta-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}.$$

By Lemma 5.5, we have $\lim_{T\to 0^+} \alpha_1(T) = \infty$ and $\lim_{T\to 0^+} \sqrt{T}\alpha_1(T) = 0$, from which we obtain:

$$\frac{(rq+(1-q)\delta)T}{q-p} \sim \mathbb{E}_{\mathbb{Q}}[(e^{\sigma\sqrt{T}Z} - e^{\sigma\sqrt{T}\alpha_1(T)})^+]. \tag{5.23}$$

with $Z \sim N(0,1)$. Next, let

$$f(T) := \mathbb{E}_{\mathbb{Q}}[(e^{\sigma\sqrt{T}Z} - e^{\sigma\sqrt{T}\alpha_1(T)})^+] = \mathbb{E}_{\mathbb{Q}}[\big(e^{\sigma\sqrt{T}Z} - e^{\sigma\sqrt{T}\alpha_1(T)}\big)\mathbf{1}_{\{Z>\alpha_1(T)\}}]. \tag{5.24}$$

Using the inequality $|e^x - 1 - x| \leqslant \frac{x^2}{2}e^{|x|}$, we get:

$$\begin{aligned} &|f(T) - \mathbb{E}_{\mathbb{Q}}[\sigma\sqrt{T}(Z-\alpha_1(T))\mathbf{1}_{\{Z>\alpha_1(T)\}}]| \\ &\qquad \leqslant \frac{\sigma^2 T}{2}\mathbb{E}_{\mathbb{Q}}[Z^2 e^{\sigma\sqrt{T}|Z|}\mathbf{1}_{\{Z>\alpha_1(T)\}}] + \frac{\sigma^2 T\alpha_1(T)^2}{2}e^{\sigma\sqrt{T}|\alpha_1(T)|}\mathbb{Q}[Z>\alpha_1(T)]. \end{aligned} \tag{5.25}$$

Since $\lim_{T\to 0^+} \alpha_1(T) = \infty$ and $\lim_{T\to 0^+} \sqrt{T}\alpha_1(T) = 0$, both terms in (5.25) are $o(T)$ and we have $|f(T) - \mathbb{E}_{\mathbb{Q}}[\sigma\sqrt{T}(Z-\alpha_1(T))\mathbf{1}_{\{Z>\alpha_1(T)\}}]| = o(T)$. Then, from (5.23), we have:

$$\begin{aligned} \frac{(rq+(1-q)\delta)T}{(q-p)} &\sim \mathbb{E}_{\mathbb{Q}}[\sigma\sqrt{T}(Z-\alpha_1(T))\mathbf{1}_{\{Z>\alpha_1(T)\}}] \\ &= \frac{\sigma\sqrt{T}}{\sqrt{2\pi}\alpha_1(T)^2 e^{\alpha_1(T)^2/2}}\int_0^\infty x e^{-x-\frac{x^2}{2\alpha_1(T)^2}}\,dx. \end{aligned} \tag{5.26}$$

Since $\lim_{T\to 0^+} \int_0^\infty x e^{-x-\frac{x^2}{2\alpha_1(T)^2}}\,dx = \int_0^\infty x e^{-x}dx = 1$,

$$\frac{(rq+(1-q)\delta)T}{(q-p)} \sim \frac{\sigma\sqrt{T}}{\sqrt{2\pi}\alpha_1(T)^2 e^{\alpha_1(T)^2/2}}. \tag{5.27}$$

So:

$$\psi_1(T)^2 e^{\frac{\psi_1(T)^2}{2}} \sim \alpha_1(T)^2 e^{\frac{\alpha_1(T)^2}{2}} \sim \frac{(q-p)\sigma}{(rq+(1-q)\delta)\sqrt{2\pi T}}. \tag{5.28}$$

where we have used that $\psi_1(T) - \alpha_1(T) = O(\sqrt{T})$. and $\lim_{T\to 0^+} \alpha_1(T) \to \infty$. Since $\lim_{T\to 0^+} \frac{\psi_1(T)}{\log\psi_1(T)} = \infty$, it is then elementary that $1-y(T) \sim -\log(y(T)) \sim \sigma\sqrt{T(-\log T)}$. Again, the proof for $S_+^e(T)$ is similar. □

To conclude, we must show that $S_-(T), S_+(T)$ are sufficiently close (within $\sqrt{T}$) to the corresponding

"European" boundaries $S^e_-(T), S^e_+(T)$. To accomplish this, we need the following bounds on the derivative of V, whose proof is given in the appendix.

Lemma 5.6. *For small $T > 0$:*

$$\sup_{(S_-(T), S^e_-(T))} \frac{\partial}{\partial x} V(\cdot, T) \leqslant (1-q) + o(1) \tag{5.29}$$

$$\sup_{(S^e_+(T), S_+(T))} \frac{\partial}{\partial x} V(\cdot, T) \leqslant (1-p) + o(1) \tag{5.30}$$

Proposition 5.2. *There exist constants $C > 0$ and $T' > 0$ such that for all $0 < T \leqslant T'$,*

$$0 \leqslant S^e_-(T) - S_-(T) \leqslant C\sqrt{T}, \tag{5.31}$$

and

$$0 \leqslant S_+(T) - S^e_+(T) \leqslant C\sqrt{T}. \tag{5.32}$$

Proof. Note that $V(x,T)$ is a classical solution of $V_T = LV - rV$ on $(S_-(T), S_+(T))$ and $\frac{\partial V}{\partial x}(S_-(T), T) = 1 - q$. Taylor's formula yields:

$$\begin{aligned} V(S^e_-(T), T) &= q + (1-q)S^e_-(T) + \frac{(S^e_-(T) - S_-(T))^2}{2} \frac{\partial^2 V}{\partial x^2}(\xi_1(T), T), \\ &= V_e(S^e_-(T), T) + \frac{(S^e_-(T) - S_-(T))^2}{2} \frac{\partial^2 V}{\partial x^2}(\xi_1(T), T). \end{aligned}$$

where $\xi_1(T) \in (S_-(T), S^e_-(T))$. The early exercise premium representation (4.6), together with the facts that $0 \leqslant \Phi \leqslant 1$ and $\kappa < S^e_-(T) \leqslant 1$ yields:

$$\frac{(S^e_-(T) - S_-(T))^2}{2} \frac{\partial^2 V}{\partial x^2}(\xi_1(T), T) = V(S^e_-(T), T) - V_e(S^e_-(T), T) \leqslant KT \tag{5.33}$$

for some $K \geqslant 0$. Using $S_-(T) < \xi_1(T) < S^e_-(T)$ and $\frac{\partial V}{\partial T}(x,T) \geqslant 0$ gives:

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2}(\xi_1(T), T) &= \frac{2}{\sigma^2 \xi_1(T)^2} \left(\frac{\partial V}{\partial T}(\xi_1(T), T) - (r-\delta)\xi_1(T) \frac{\partial V}{\partial x}(\xi_1(T), T) + rV(\xi_1(T), T) \right) \\ &\geqslant \frac{2r}{\sigma^2 \xi_1(T)^2} ((V(\xi_1(T), T) - \xi_1(T) \frac{\partial V}{\partial x}(\xi_1(T), T))) \\ &\geqslant \frac{2r}{\sigma^2 \xi_1(T)^2} (q + (1-q)\xi_1(T) - \xi_1(T) \frac{\partial V}{\partial x}(\xi_1(T), T)) \geqslant c > 0. \end{aligned}$$

for T small enough, by Lemma 5.6. Thus

$$(S^e_-(T) - S_-(T))^2 \leqslant C_1 T \Rightarrow S^e_-(T) - S_-(T) \leqslant C_1 \sqrt{T}, \tag{5.34}$$

for some $C_1 > 0$ and $0 < T \leqslant T_1$. The proof of (5.32) is similar. □

Finally, by Theorem 5.1, we can easily obtain

$$\sigma\sqrt{T(-\log T)} \sim 1 - S_-(T), \quad \sigma\sqrt{T(-\log T)} \sim S_+(T) - 1. \tag{5.35}$$

6. Conclusion

This paper analyzes an optimal stopping problem for a investor with a piecewise linear payoff function, where the underlying follows a geometric Brownian motion, corresponding to a hedge fund with a continuous fee for assets under management deducted (or, equivalently, the price process for a stock paying a continuous dividend yield). We present a complete solution of the problem in the infinite horizon case. In the finite horizon case, we describe the shape of the continuation region, characterize the stopping boundaries using a coupled pair of integral equations, and present an asymptotic analysis of the boundaries in small time.

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A. Appendix

Proof of Proposition 3.4: Let $H(x,z) = W(x;z,1-q) - (p + (1-p)x)$. Then $h(x) = H(x,\kappa)$, $H_x(x,z) = W_x(x;z,1-q) - (1-p)$ and $H_{xx}(x,z) = W_{xx}(x;z,1-q)$, so that $H(\cdot,z)$ is a strictly convex function for each $z \in [\kappa,1]$, $H_x(\kappa,z) \leqslant H_x(z,z) = (p-q) < 0$, and $H_x(\cdot,z) \to \infty$ as $x \to \infty$. Thus, for each $z \in [\kappa,1]$ there is a unique minimizer $\tilde{x}(z)$ of $H(\cdot,z)$, at which $W_x(\tilde{x}(z);z,1-q) = 1-p$. Taking $x_* = \tilde{x}(\kappa)$ proves a). Furthermore, note that by the strict convexity of W, when $z \in [\kappa,1)$, $H(x,z) > 0$ for $x \in [\kappa,1]$.

Define $\tilde{u}(z) = H(\tilde{x}(z),z) = \inf_{x\in[\kappa,\infty)} H(x,z)$. Then $\tilde{u}$ is continuous, strictly decreasing, and $\tilde{u}(\kappa) = h(x_*)$, $\tilde{u}(1) < 0$. If $h(x_*) \geqslant 0$, then there exists a unique $\tilde{z} \in [\kappa,1)$ such that $\tilde{u}(\tilde{z}) = 0$. Taking $S_1 = \tilde{z}$, $S_2 = \tilde{x}(S_1)$ then yields b). (Since $\tilde{u}(\tilde{z}) = H(\tilde{x}(\tilde{z}),\tilde{z}) = 0$, and $\tilde{z} \in [\kappa,1)$, we must have $S_2 = \tilde{x}(\tilde{z}) > 1$.)

If $h(x_*) < 0$, define $G(x,v) = W(x;\kappa,v) - (p + (1-p)x)$. As before, for $v < 1-p$, G is strictly convex, with $G_x(\kappa,v) < 0$ and $G_x(x,v) \to \infty$ as $x \to \infty$, so that there exists a unique $\hat{x}(v)$ at which $G_x(\hat{x}(v),v) = 0$, and G is minimized. Letting, $\hat{u}(v) = G(\hat{x}(v),v)$, we have that $\hat{u}(1-q) < 0$, and $\hat{u}$ is continuous and strictly increasing. For v large enough, $\hat{u}(v)$ is positive, so there exists a unique $\hat{v}$ at which $\hat{u}(\hat{v}) = 0$. Taking $S_2 = \hat{x}(\hat{v})$ then yields c). Convexity again implies that $S_2 > 1$.

Proof of Lemma 5.6

Proof. Note that V is increasing and Lipschitz continuous in x (uniformly on any $[0,\bar{T}]$) by Touzi (2013, Proposition 4.7, pages 46-47). Let $x,y \in (S_-(T), S^e_-(T))$ with $x \geqslant y$. By the Dynamic Programming Principle (Touzi (2013, page 41)), for any stopping time $\theta \in \mathscr{T}_{[0,T]}$

$$V(x,T) = \sup_{\tau \in \mathscr{T}_{[0,T]}} \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\tau<\theta} e^{-r\tau} g(X^x_\tau) + \mathbf{1}_{\tau \geqslant \theta} e^{-r\theta} V(T-\theta, X^x_\theta)]$$

so that

$$V(x,T) - V(y,T) \leqslant \sup_{\tau \in \mathscr{T}_{[0,T]}} \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{\tau<\theta} e^{-r\tau}(g(X^x_\tau) - g(X^y_\tau)) \\ + \mathbf{1}_{\tau \geqslant \theta} e^{-r\theta}(V(T-\theta, X^x_\theta) - V(T-\theta, X^y_\theta))]. \quad \text{(A.1)}$$

Define:

$$\theta = \inf\{t > 0,\ X^x_t = 1 \text{ or } X^y_t = \kappa\}, \quad \text{(A.2)}$$

to obtain:

$$V(x,T) - V(y,T) \leqslant (x-y)(1-q) + C(x-y)\mathbb{Q}(\theta \leqslant \tau) \leqslant (x-y)(1-q) + C(x-y)\mathbb{Q}(\theta \leqslant T).$$

where C is the Lipschitz constant of V, and we have suppressed the dependence of θ on x and y. Take $\eta > \kappa$. Then since $S_-(T) \to 1$, for T small enough

$$\theta \geqslant \bar{\theta} = \inf\{t > 0,\ X^{S^e_-(T)}_t = 1 \text{ or } X^\eta_t = \kappa\} \quad \text{(A.3)}$$

and $\bar{\theta}$ does not depend on the choice of x,y. Thus:

$$\frac{V(x,T) - V(y,T)}{x-y} \leqslant (1-q) + C\mathbb{Q}(\bar{\theta} \leqslant T). \quad \text{(A.4)}$$

The final term is bounded by the constant C multiplied by the sum of the probabilities that the process X started at η hits κ before T, and that X started at $S^e_-(T)$ hits 1 before T. Both of these probabilities can be shown to be $o(1)$ using the explicit form of the hitting time distribution of a geometric Brownian motion (the first trivially, and the second using the estimate (5.19)). The proof for $x,y \in (S^e_+(T), S_+(T))$ is again similar. □