Properties of difference inclusions with computable reachable set

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Dynamical systems have important applications in science and engineering. For example, if a dynamical system describes the motion of a drone, it is important to know if the drone can reach a desired location; the dual problem of safety is also important: if an area is unsafe, it is important to know that the drone cannot reach the unsafe area. These types of problems fall under the area of reachability analysis and are important problems to solve whenever something is moving.

Computer algorithms have been used to solve these reachability problems. These algorithms are primarily viewed from a numerical simulation perspective, where guarantees about the dynamical system are made only in the short-term (i.e. on a finite time horizon). Yet, the important properties of dynamical systems often arise from their long-term (asymptotic) behaviours. Furthermore, in sensitive applications it may be important to determine if the system is provably safe or unsafe instead of approximately safe or unsafe. The theory of computation (or computability theory) can be used to investigate whether computer algorithms can determine weather a dynamical system is provably safe. Computability theory, broadly speaking, is a field of computer science that studies what kind of problems a computer can solve (or cannot solve).

For difference inclusions, a characterization of when the reachable set is computable was found by Pieter Collins. Difference inclusions, are one way of modelling discrete time dynamical systems with control. This thesis is an investigation into this characterization. Broadly, it is argued that this characterization is far to restrictive on the dynamical system to be of general practical use. For example, a continuous function f which maps the real line to itself, has a computable reachable set if and only if there is a metric d on the real line (which is equivalent to the standard metric) for which f is a contraction map with respect to d.

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Chapter 1

Introduction

Computational problems for dynamical systems involving a finite time horizon are broadly speaking tractable. For example, given a Lipschitz function f and a compact interval $[0, T]$ one can compute the solution to $\dot{x} = \mathrm{f}(x), x(0) = x_0$ to any given error tolerance using common integration methods. In contrast, problems involving an infinite time horizon such as "Does the solution reach a point y eventually?" or "Is y a recurrence point of the solution?" are more difficult to answer definitively—especially using computational methods.

Of particular interest to us, are computational methods which are equivalent to preforming a proof, sometimes called rigorous numerical methods or simply rigorous methods. Traditional numerical methods typically provide approximations as output but do not provide rigorous error bounds (rigorous, as in one could use the error output as a part of valid mathematical proof). In some sensitive applications it is important to be provably correct rather than approximately correct; in such situations rigorous methods must be used.

Rigorous methods inevitably fall under the purview of computability theory. Computability theory, broadly speaking, is a field of computer science that studies what kind of problems a computer can solve (or cannot solve). Although computability theory is well studied by computer scientists, it is still in its infancy when it comes to studying dynamical systems.

A central problem in computability theory are decision problems. Consider a question/problem that can be restated as a logical proposition P, such as $\exists t \geq 0 \;\; x(t) \in A$. The question/problem is said to be semidecidable (informally we may say, verifiable) if there is an algorithm A which can determinate that P is true, provided that P is true. For semi-decidability we do not require A will do anything useful if P is false, we only require that if P is true then A verifies that P is true. We say a question/problem is co-semidecidable if NOT(P) is semidecidable. That is, if P is false there is an algorithm A that will determine P is false. The question/problem is said to be decidable if it is both semidecidable and co-semidecidable.

Remark 1.0.1. *Many mundane mathematical propositions, such as $x = 0$, are traditionally viewed as undecidable (not decidable) over the real numbers. To see why consider a generic real number x, in the worst case x has no finite decimal representation say $x = x_0.x_1x_2x_3\ldots$, where $x_0 \in \mathbb{Z}$ and $x_n \in \{0, 1, 2, \ldots, 9\}$ $n \in \mathbb{N}$. Then, we can falsify (co-semidecide) the statement $x = 0$ for if $x \neq 0$ then we can check every $n \in \mathbb{N} \cup \{0\}$ sequentially for $x_n = 0$ or $x_n \neq 0$ since $x \neq 0$ for some n, $x_n \neq 0$ so the algorithm "check the digits and return false if a digit is non-zero" co-semidecideds, $x = 0$. However, if we are only capable of checking the digits one by one then it is impossible to verify that $x = 0$, since such a verification would require checking every digit and thus any algorithm would never stop. Hence, checking $x = 0$ is undecidable.*

Notice the above argument requires us to assume that the only way to check equality of real numbers is to check the digits. Fundamentally, computability theory requires us to pick a "representation" of our mathematical objects, different representations will (in general) yield different computable functions and elements. For example, above we represented real numbers by decimal expansion we could instead represent a real number by its unique canonical continued fraction or a sequence of open intervals $(a_n, b_n) \ni x$ of diameter 2^{-n}. These three representations yield different computability theories over the real numbers. According to [17] all three of this representations induce a computability theory in which the statement $x = 0$ is undecidable.

I generally regard the undecidability of mundane mathematical propositions as a defect of computability theory. When faced with a problem that is "undecidable" I take that as a sign that the problem in question is either too broad and more assumptions are needed or, more convenient representation is needed.

The details of computability theory in $\mathbb{R}^d$ or in separable metric spaces are quite complicated and frankly often not relevant to the problem at hand. I recommend [17] for the reader who is interested; For others I outline some relevant context of computability theory in Section 2.4.

The specific question this thesis seeks to answer is: What properties of discrete time dynamical systems with computable reachable sets have? To understand this question and why it is important we need some definitions.

Suppose (X, d) is a metric space and let U be a set. Consider a function $\mathrm{f} : X \times U \to X$ and a point $x_0 \in X$ then a sequence $\{x_n\}_{n=0}^{\infty}$ satisfying

$$x_{n+1} = \mathrm{f}(x_n, u_n) \tag{1.1}$$

is a controlled orbit of f or a trajectory of f for some controller $\{u_n\}_{n=0}^{\infty}$.

Questions 1.0.1. *Suppose that (X, d) is a metric space, U is some set, $\mathrm{f} : X \times U \to X$ is continuous in X, $x, y \in X$, and $A \subseteq X$ is non-empty.*

Name	*Description*	*Formal Statement*
Reachability	*Can a trajectory of f reach the point y with initial point x?*	*$\exists \{x_n\}_{n=0}^{\infty}$ a trajectory of f with $x_0 = x$ and $\exists n \in \mathbb{N} \cup \{0\}$ with $y = x_n$*
Invariance	*Do all trajectories of f starting in A remain in A?*	*$\forall \{x_n\}_{n=0}^{\infty}$ trajectories of f with $x_0 \in A$ have $x_n \in A$ for all $n \in \mathbb{N}$*
Recurrence	*Do some trajectories with initial point x recur to y?*	*$\exists \{x_n\}_{n=0}^{\infty}$ a trajectory of f with $x_0 = x$ and $\exists \{x_{n_k}\}_{k=0}^{\infty}$ with $\lim_{k\to\infty} x_{n_k} = y$.*
Asymptotic Attraction	*Do all trajectories of f, with initial point x, eventually lie in A?*	*$\forall \{x_n\}_{n=0}^{\infty}$ trajectories of f with $x_0 = x$ and the limit points of any subsequence of $\{x_n\}_{n=0}^{\infty}$ are in A ($\bigcap_{N\in\mathbb{N}} \overline{\bigcup_{n\geq N}\{x_n\}} \subseteq A$).*

There are several relevant interesting questions of similar form to the above which are not included in the table. We have just identified these four as the most important.

The literature surrounding these problems are primarily concerned with reachability and invariance, see [2, 3, 5, 6, 11, 16, 4, 10]. A central theme among these works (and indeed in the field) is the idea of solving a robust problem. It is necessary to solve a "robust problem" because the original problems are undecidable.

For example, let's assume that U is a singleton (so we are simply iterating a function), that $X = \mathbb{Q}$ and given any $x \in X$ the value $\mathrm{f}(x)$ can be determined exactly in finite time. The reachability problem becomes: given $x, y \in X$ does there exist an $n \in \mathbb{N}$ with $\mathrm{f}^{\circ n}(x) = y$? Notice, equality of two rational numbers is decidable in finite time, just write the numbers in lowest terms then the numbers are equal iff the numerators and denominators equal. Thus by assumption, if $\mathrm{f}^{\circ N}(x) = y$ for some N then we can compute the LHS for each n and check to see it equals y, stopping when see $\mathrm{f}^{\circ N}(x) = y$. So far this shows that reachability is semidecidable, in this case. For reachability to be decidable it must be co-semidecidable; however, if $\mathrm{f}^{\circ n}(x) \neq y$ for all n then, it seems like we would have to verify an infinite amount of points in are in $X \setminus \{y\}$ in order to falsify reachability. It turns out that in general the reachability problem is not co-semidecidable [4, 5].

In order to get around undecidability of the problem we instead assume that the dynamics satisfy a strong condition, which we can decide. First note that for points $x, y \in X$ exactly one of the following holds:

1. $\exists n \in \mathbb{N}$ with $\mathrm{f}^{\circ n}(x) = y$
2. $\exists I \subseteq X$ with I being f-invariant, $\mathrm{f}(x) \in I$, and $I \subseteq X \setminus \{y\}$.

The issue is we can't detect the second case (for technical reasons see Theorem 2.4.1); so we assume whenever 2 holds it is also true that

$2'$. $\exists I \subseteq X$ with I being compact, robustly f-invariant meaning that

$$\mathrm{f}(I) \subseteq \mathrm{int}(I),$$

$\mathrm{f}(x) \in I$, and $I \subseteq X \setminus \{y\}$.

We call $2'$ the robust alternative to 2. In Section 3.3.1 we provide an algorithm that semidecides $2'$. From here we can (somewhat arbitrarily) define a "robust" system as functions that satisfy exactly one of 1 or $2'$ for all $x, y \in X$.

Such systems were analyzed in [4, 5], where the authors found these "robust" systems are precisely the set of systems with computable reachable set.

The reachable set at $x \in X$ of the system in Equation (1.1) is

$$\mathrm{R}[x] = \{y \in X : \exists \{x_n\}_{n=0}^{\infty} \text{ a trajectory of f with } x_0 = x, \exists n \in \mathbb{N} \cup \{0\} \text{ and } y = x_n\}.$$

With some assumptions on U and f, such as compactness/finiteness of U and continuity of f in its variables, one can see that "inner" approximations of R are possible. That is, given $x \in X$ and $N \in \mathbb{N}$ we find all truncated trajectories, a sequence satisfying $\{x_n\}_{n=0}^{N}$ with $x_{n+1} = \mathrm{f}(x_n, u_n), u_n \in U$ for $n = 0, \dots, N-1$ and $x_0 = x$. The union of these truncated trajectories for some fixed $N \in \mathbb{N}$, is a inner/lower approximation of $\mathrm{R}[x]$. An over/upper approximation of $\mathrm{R}[x]$ would be a set O with $O \supseteq \mathrm{R}[x]$. For technical computability theory reasons, finding such an O requires that O is open and $\overline{\mathrm{R}[x]}$ is compact. Therefore, we say that O is an over/upper approximation of $\mathrm{R}[x]$ if $O \supseteq \overline{\mathrm{R}[x]}$, where $\overline{B}$ denotes the closure of the set $B \subseteq X$.

Inner approximations allows us to semidecide the reachability problem and over approximations allow us co-semidecide the (closed) reachability problem. Finding the over approximations is the hard part. However, finding a good over approximation is useful for certain problems/applications. For instance, suppose that $U \subseteq X$ is an unsafe set. Thus, we would like to avoid U. I see two main way to formulate this:

1. Strong safety at $x \in X$: $\overline{\mathrm{R}[x]} \subseteq X \setminus U$. That is every trajectory starting at $x \in X$ *must* avoid U.

2. Weak safety at $x \in X$: There is a trajectory of f, say $\{x_n\}_{n=0}^{\infty}$ with $x_n \notin U$ for all $n \in \mathbb{N} \cup \{0\}$. That is a trajectory starting at $x \in X$ *can* avoid U.

Strong safety implies weak safety. Strong safety at $x \in X$ can be verified by finding an over approximation of $\mathrm{R}[x]$; i.e find an open set O with $\overline{\mathrm{R}[x]} \subseteq O \subseteq X \setminus U$. Therefore, over approximations of $\mathrm{R}[x]$ are useful for verifying the safety of a system.

Over approximations of $\mathrm{R}[x]$ are also necessarily involved in finding approximations to the long term behaviour of all trajectories.

Recall that in [4, 5], the authors characterized the set of dynamical systems with computable reachable set and called such systems robust. This raises the question of: How can I tell when my system is robust? A plurality of my time as a PhD student has been spent trying to answer this question[1]. Unfortunately, I found it rather difficult to find a test or sufficient condition to determine if a system is robust. Indeed for a number of years, the only sufficient condition I came up with was: "the finite contractive case", that is, for all $u \in U$ the function $\mathrm{f}(\cdot, u)$ is a contraction map and U is finite—a very limited class of systems.

Instead I discovered a number of necessary conditions on robust systems. But these conditions were not what I wanted to see. These necessary conditions ensure that robust systems are remarkably stable; so stable that they seemed suspiciously like the finite contractive case. Because of this I spent years trying to prove that the finite contractive case was essentially the only case where a system would be robust, see Conjecture 4.0.1. Alas, I found a counterexample to Conjecture 4.0.1, Example 4.0.1 on August 21st, 2023 (two months and six days before this thesis is due! Yikes!).

Despite my poor conjecturing skills, this thesis does have some novel and interesting results. For example, I found some characterizations of global asymptotic stability for difference inclusions in Theorem 4.2.4. But more importantly I found that if the system from Equation (1.1) has $X = \mathbb{R}$ and U being a singleton (so $\mathrm{f} : \mathbb{R} \to \mathbb{R}$) then, the system is robust if and only if f has a globally asymptotic stable fixed point (Corollary 4.2.5.3). In the more general case where X is a connected metric space and U being compact, there are similar results involving globally asymptotic stable minimal invariant set (a minimal invariant set is a generalization of fixed points and periodic orbits), see Corollaries 4.2.5.1 and 4.2.5.2.

[1] In some ways this thesis is documentation of my struggles to make heads or tails out these robust systems.

Broadly speaking, we can interpret these results as saying: If a system is robust then, the space X is the basin of attraction for some attractor.

The structure of this thesis is as follows: In Chapter 2 some essential background knowledge is provided. In particular, Section 2.1 goes over the basics of point set topology, Section 2.2 comprehensively details how point set topology can be used on the subsets of a topological space, Section 2.3 explains the very basics of multifunction and notions of continuity for them, Section 2.4 gives a brief summery of computability theory as it pertains to mathematical analysis. Chapter 3 presents the necessary theory of dynamical systems applied to multifunctions. Section 3.3 introduces (and corrects when necessary) the theory of robust systems presented in [4, 5]. Finally, Chapter 4 provides an in depth analysis of robust systems and characterizes when a sufficiently continuous multifunction has a local/global asymptotic stable set.

The reader is assumed to be comfortable with typical mathematical notation as seen in elementary set theory and typical calculus, linear algebra, or real analysis courses. Such as, $\mathbb{R}$ denoting the set of real numbers, $\mathbb{N}$ denoting the set of natural numbers numbers (starting at 1), or $\overline{B}$ denoting the closure of the set B. It is also assumed the reader is comfortable with real analysis or point set topology, and in particular continuous functions.

Chapter 2

Background

2.1 Point set topology basics

This section is primarily for establishing notation, terminology and some important results. The proofs of the theorems in this section and chapter often not given. But almost any undergraduate level book on point set topology will have these proofs. For example see [18, 13].

Definition 2.1.1. *Let X be set, let $\mathcal{P}(X)$ be the power set of X and let $\tau \subseteq \mathcal{P}(X)$. Then, (X, τ) is said to be a topological space and τ a topology on X if τ satisfies:*

i) $\emptyset, X \in \tau$.

ii) for any $\mathcal{A} \subseteq \tau$ *we have* $\bigcup_{A \in \mathcal{A}} A \in \tau$.

iii) for any $A_1, A_2, \ldots, A_N \in \tau$ *we have* $\bigcap_{n=1}^{N} A_n \in \tau$.

When (X, τ) is a topological space the elements of τ are said to be open sets.

A topological space is a space in which we can define things like convergence and continuity of functions. These things are the underpinnings of mathematical analysis. The most common example of a topological space are metric spaces, and in particular the real line endowed with the standard notion of convergence. However, not all topological spaces are metric spaces, even in applications. For example, the topology of pointwise convergence is usually not a metric space. Similarly, the topology of uniform convergence on compacts is not a metric space, when the domain and range space of the functions are not compact.

We now provide some basic notations and definitions for topological spaces.

Definition 2.1.2. *Suppose that* (X, τ) *is a topological space. Then define:*

1. *The set* $C \subseteq X$ *is called a closed set of* (X, τ) *when* $C = X \setminus V$ *for some* $V \in \tau$. *(recall that* $A \setminus B = \{a \in A : a \notin B\}$ *for sets* A *and* B*)*

2. $\tau^c \subseteq \mathcal{P}(X)$ *to be the collection of closed sets of* (X, τ).

3. *The closure of a set, say* $A \subseteq X$, *denoted* $\overline{A}$ *or* $\mathrm{cl}(A)$ *to be the smallest closed set containing* A. *i.e.* $\overline{A} = \bigcap\{C \in \tau^c : C \supseteq A\}$.

4. *The interior of a set, say* $A \subseteq X$, *denoted* $\mathrm{int}(A)$ *to be the largest open subset of* A. *i.e* $\mathrm{int}(A) = \bigcup\{U \in \tau : U \subseteq A\}$.

5. *A neighborhood of a point* $x \in X$ *is a set* $N \subseteq X$ *with* $x \in \mathrm{int}(N)$. *A neighborhood,* N, *of* x *is said to be open (or closed) if* N *is open (or closed).*

6. $\mathcal{N}(x)$, *for* $x \in X$, *to be the set of all neighborhoods of* x.

7. τ_x, *for* $x \in X$, *to be the set of all open neighborhoods of* x.

8. $\overline{\mathcal{N}}(x)$, *for* $x \in X$, *to be the set of all closed neighborhoods of* x.

9. *A collection* $\mathcal{B}(x) \subseteq \mathcal{P}(X)$, *for* $x \in X$, *is called a local base for* x, *if for any* $V \in \tau_x$ *there is* $B \in \mathcal{B}(x)$ *with* $V \subseteq B$.

10. $\mathcal{O} \subseteq \tau$ *to be an open cover of a set* $A \subseteq X$ *when* $A \subseteq \bigcup_{O \in \mathcal{O}} O$. *A subcover of* $\mathcal{O}$ *is a subset of* $\mathcal{O}$ *which is an open cover of* A.

11. $K \subseteq X$, *is said to be a compact set if every open cover of* K *has a finite subcover (a subcover with finitely many elements).*

A topology on a set X does not appear out of thin air. Typically, a topology is generated by some distinguished open sets which do not satisfy the appropriate axioms. The quintessential example of this is are the open ϵ balls of a metric space, the union or intersection of two balls is not guaranteed to be a ϵ ball of some size. Despite this the ϵ balls are all we really need to consider in metric spaces, for the purposes of convergence and continuity. This is because the open ϵ balls form a base for the metric topology.

Definition 2.1.3. *Suppose that* X *is a set and* $\mathcal{B} \subseteq \mathcal{P}(X)$. *The family of sets* $\mathcal{B}$ *is called a base for* X *if it satisfies both:*

1. $\mathcal{B}$ *covers* X, *that is if* $x \in X$ *then there is a* $B \in \mathcal{B}$ *with* $x \in B$.

2. *for every* $A, B \in \mathcal{B}$ *and every* $x \in A \cap B$ *there is* $C \in \mathcal{B}$ *with* $x \in C \subseteq A \cap B$.

The family $\mathcal{B}$ *is called a sub-base (for* X*) if the family* $\mathcal{B}_2$ *is a base, where* $\mathcal{B}_2$ *is defined to be set of all finite intersections of elements of* $\mathcal{B}$.

Note that a base is also a sub-base. Given a family of sets $\mathcal{B}$ and for $\mathcal{B}_2$ being the the set of all finite intersections of $\mathcal{B}$ then, $\mathcal{B}_2$ always satisfies item 2 in the definition of a base. So for $\mathcal{B}_2$ to be base (and for $\mathcal{B}$ to be a sub-base) we need only check the covering condition, which is usually evident in practice.

Proposition 2.1.1. *Let* X *be a set and* $\mathcal{B}$ *be a base then,*

$$\tau := \left\{ \bigcup_{B \in \mathcal{B}'} B : \mathcal{B}' \subseteq \mathcal{B} \right\}$$

is a topology for X. *This topology is called the topology generated by* $\mathcal{B}$ *and* $\mathcal{B}$ *is said to be a base for* τ. *Further,* τ *is the smallest topology containing* $\mathcal{B}$ *i.e if* τ' *is a topology with* $\tau' \supseteq \mathcal{B}$ *then* $\tau \subseteq \tau'$. *The elements of* $\mathcal{B}$ *are called the basic open sets.*

If $\mathcal{S}$ *is a sub-base then*

$$\tau_2 := \left\{ \bigcup_{B \in \mathcal{B}'} B : \mathcal{B}' \subseteq \left\{ \bigcap_{S \in \mathcal{S}'} S : \mathcal{S} \supseteq \mathcal{S}' \text{ is finite} \right\} \right\}$$

is a topology for X. *This topology is called the topology generated by* $\mathcal{S}$ *and* $\mathcal{S}$ *is said to be a sub-base for* τ_2. *It is the smallest topology containing* $\mathcal{S}$. *The finite intersections of* $\mathcal{S}$ *are called the basic open sets.*

It is common to define a topology via a base or sub-base using Proposition 2.1.1. Notice that a given topology, τ is both a base and a sub-base. So every topology has a base (or sub-base) which generates it.

As expected from the ϵ balls of a metric space example, the basic open sets are all that's necessary to discuss convergence or the continuity of functions. Before we start this discussion in earnest we have some more definitions to recall.

Definition 2.1.4. *Let* (X, τ) *be a topological space. Then* (X, τ) *is said to be*

1. T_0 *or Kolmogorov if for all* $x, y \in X$, $x \neq y$ *there is an open set* U *with at least one of: i)* $x \in U \not\ni y$ *or ii)* $y \in U \not\ni x$.

2. T_1 *or Fréchet if for* $x, y \in X$, $x \neq y$ *there is an open set* U *with* $x \in U$ *and* $y \notin U$.

3. T_2 *or Hausdorff if for every* $x, y \in X$, $x \neq y$ *there are open sets* U, V *with* $x \in U$, $y \in V$ *and,* $U \cap V = \emptyset$.

4. *Regular, if for every closed set* C *and every point* $x \notin C$ *there are open sets* U, V *with* $U \ni x$, $V \supseteq C$ *and* $U \cap V = \emptyset$.

5. *Normal, if for any closed sets* C, K *with* $C \cap K = \emptyset$ *there are open sets* U, V *with* $U \supseteq C$, $V \supseteq K$ *and* $U \cap V = \emptyset$.

6. *Locally compact, if for every* $x \in X$ *there is a compact neighborhood of* x.

7. *First countable, if for every* $x \in X$ *there a countable local base of* x.

8. *Second countable, if there a countable base of* X.

This is a lot to take in at once. But a reader shouldn't worry to much about these definitions. They will be assumed when convenient and the interaction between them will not be important. For now one should notice that metric spaces are first countable normal Hausdorff topological spaces and that a set equipped with a pseudo-metric[1] may fail to be T_0 but is normal.

We (eventually) wish to discuss connected sets, to do so we should define the relative topology.

Proposition 2.1.2. *Let* (X, τ) *be a topological space and let* $Y \subseteq X$ *then* $(Y, \tau|_Y)$ *is a topological space, called the subspace or relative topology, where*

$$\tau|_Y = \{Y \cap U : U \in \tau\}.$$

If (X, τ) *is* T_0, T_1 *or Hausdorff then* $(Y, \tau|_Y)$ *is* T_0, T_1 *or Hausdorff, respectively.*

If Y *is closed in* X *and* (X, τ) *is regular or normal then* $(Y, \tau|_Y)$ *is regular or normal, respectively.*

If Y *is compact in* (X, τ) *then* Y *is compact in* $(Y, \tau|_Y)$.

Definition 2.1.5. *Let* (X, τ) *be a topological space and* $Y \subseteq X$. *The set* Y *is said to be connected if for any* $B \subseteq Y$ *the following holds:*

$$B \text{ is both open and closed in } (Y, \tau|_Y) \implies B = \emptyset \text{ or } B = Y.$$

[1] A pseudo-metric on a set X, say $\rho : X \times X \to \mathbb{R}$, is effectively a metric except it may have $\rho(x, y) = 0$ even if $x \neq y$

If X is connected in (X, τ) then we say (X, τ) is connected.

A set $C \subseteq X$ is called a connected component of X (or simply a component of X) if C is connected in X and for every connected $D \subseteq X$ with $C \subseteq D$ we have $C = D$. It can be shown that the components of X are disjoint and closed.

(X, τ) is said to be locally connected if for all $x \in X$ the set of all connected neighborhoods of X is a local base for x.

Connectedness is not of much importance in this section, but will be essential later. Connectedness of the space X will allow us to make certain basins of attraction to be as large as possible.

We should also define how to take products of topological spaces.

Proposition 2.1.3. *Let Λ be a set and for all $\lambda \in \Lambda$ let $(X_\lambda, \tau_\lambda)$ be a topological space. The set $\Pi_{\lambda\in\Lambda} X_\lambda$ has a base*

$$\mathcal{B}_\Lambda := \{\Pi_{\lambda\in\Lambda} U_\lambda : U_\lambda \in \tau_\lambda \text{ and } U_\lambda \neq X \text{ for only finitely many } \lambda\}.$$

The topology generated by this base, say τ_Λ, is called the product topology.

The following hold for the product topology:

1. *If every $(X_\lambda, \tau_\lambda)$ is T_0 then $(\Pi_{\lambda\in\Lambda} X_\lambda, \tau_\Lambda)$ is T_0.*
2. *If every $(X_\lambda, \tau_\lambda)$ is T_1 then $(\Pi_{\lambda\in\Lambda} X_\lambda, \tau_\Lambda)$ is T_1.*
3. *If every $(X_\lambda, \tau_\lambda)$ is T_2 then $(\Pi_{\lambda\in\Lambda} X_\lambda, \tau_\Lambda)$ is T_2.*
4. *If every $(X_\lambda, \tau_\lambda)$ is regular then $(\Pi_{\lambda\in\Lambda} X_\lambda, \tau_\Lambda)$ is regular.*
5. *If every $(X_\lambda, \tau_\lambda)$ is compact then $(\Pi_{\lambda\in\Lambda} X_\lambda, \tau_\Lambda)$ is compact.*
6. *If every $(X_\lambda, \tau_\lambda)$ is connected then $(\Pi_{\lambda\in\Lambda} X_\lambda, \tau_\Lambda)$ is connected.*

2.1.1 Convergence of points and nets in topological spaces

A very natural and practical object in analysis and metric spaces are sequences. A sequence in the set X is function $x : \mathbb{N} \to X$ we typically just write "let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence" where $x(n) = x_n$ is understood. Unfortunately, sequences alone cannot (in general) describe a topology, unlike in metric spaces. For example if X, Y are two metric spaces then a given function f : $X \to Y$ is continuous iff for every convergent sequence $\{x_n\}$ of X we have that $\{f(x_n)\}_{n\in\mathbb{N}}$ converges in Y. Another example, if X is a compact metric space then every sequence has a convergent sequence. Both of these examples may fail to be true if we replace "metric space" with "topological space".

If we want to recover these results for topological spaces we must use nets.

Definition 2.1.6. *Let $\mathcal{D}$ be a set and $\succeq$ be a relation on $\mathcal{D}$. We say that $(\mathcal{D}, \succeq)$ is a directed set if*

1. *$\succeq$ is reflexive, i.e. for all $x \in \mathcal{D}$ we have $x \succeq x$*
2. *$\succeq$ is transitive, i.e for all $x, y, z \in \mathcal{D}$ we have that if $x \succeq y$ and $y \succeq z$ then $x \succeq z$.*
3. *For any $x, y \in \mathcal{D}$ there is $z \in \mathcal{D}$ with $z \succeq x$ and $z \succeq y$.*

Often there is no confusion with the relation $\succeq$, so we will simply refer to $\mathcal{D}$ as a directed set.

Let X be a set. Then, the function $x : \mathcal{D} \to X$ is a net in/of X if $\mathcal{D}$ is directed set. Often we will simply say "let $\{x_d\}_{d\in\mathcal{D}}$ be a net" were $\mathcal{D}$ is understood to be a directed set.

Let $\{x_d\}_{d\in\mathcal{D}}$ be a net then, a subnet of $\{x_d\}_{d\in\mathcal{D}}$ is a net, say $\{y_a\}_{a\in A}$, with a function $m : A \to \mathcal{D}$ which satisfies

i. *$x \circ m = y$ or equivalently $x_{m(a)} = y_a$ for all $a \in A$.*

ii. *for each $D \in \mathcal{D}$ there is $a \in A$ such that if $b \succeq_A a$ then $m(b) = m_b \succeq_{\mathcal{D}} D$.*

From what I've seen, some authors use different definitions for what a net is, either they require either more or less stringent conditions on the domain $\mathcal{D}$. There is even less consensuses on what a subnet is. Notice that, subnet may be indexed by a set different set than original net.

The most notable example of nets are sequences. Another example of a net is when we take the directed set to be certain subsets of the open sets like the open neighborhoods of a point directed by inclusion. Anyhow, the reason we need to nets is so we can talk about their convergence in a topological space.

Definition 2.1.7. *Let (X, τ) be topological space and let $\{x_d\}_{d\in\mathcal{D}}$ be a net in X.*

1. *A point $x \in X$ is said to be a limit point of $\{x_d\}_{d\in\mathcal{D}}$ if for every open set $V \ni x$ there is a $D \in \mathcal{D}$ such that for all $d \succeq D$ $x_d \in V$.*

2. *If* $\{x_d\}_{d\in\mathcal{D}}$ *has a limit point then,* $\{x_d\}_{d\in\mathcal{D}}$ *is said to be convergent.*

3. *Let* $\lim_{d\in\mathcal{D}} x_d$ *or* $\lim_d x_d$ *be the set of all limit points of the net. If there exactly one limit point of* $\{x_d\}_{d\in\mathcal{D}}$ *then we consider* $\lim_{d\in\mathcal{D}} x_d$ *to the point which the net converges to.*

4. *A point* $x \in X$ *is said to be an accumulation point of the net* $\{x_d\}_{d\in\mathcal{D}}$ *if for every open set* $V \ni x$ *and every* $D \in \mathcal{D}$ *there is* $d \succeq D$ *with* $x_d \in V$*.*

5. *Let* $\mathrm{Acc}_{d\in\mathcal{D}} x_d$ *or* $\mathrm{Acc}_d x_d$ *denote the set of all accumulation points of* $\{x_d\}_{d\in\mathcal{D}}$*.*

Sometimes, we will write $x_d \to x$ *to denote* $\{x_d\}_{d\in\mathcal{D}}$ *converging to* x*, without explicit mention of* $\mathcal{D}$*.*

We can see the relevant convergence definitions for sequences and nets are very similar. One immediate wrinkle is that fact the limits of nets (and sequences) in topological spaces may not be unique. This is regrettable, but unavoidable. Even in pseudo-metric spaces we can have a convergence sequence having multiple limit points. Fortunately for us we are largely interested is topological space where the limits of nets are unique.

Proposition 2.1.4. *Let* (X, τ) *be a topological space.*

The following are equivalent:

1. (X, τ) *is Hausdorff.*

2. *Every net in* X *which converges, converges to exactly one point.*

Furthermore, the following are also equivalent:

a. *The net* $\{x_d\}_{d\in\mathcal{D}}$ *converges to the point* $x \in X$*.*

b. *Every subnet of* $\{x_d\}_{d\in\mathcal{D}}$ *converges to the point* $x \in X$*.*

Additionally, the following are also equivalent, for a net $\{x_d\}_{d\in\mathcal{D}}$*:*

i. $x \in X$ *is an accumulation point of* $\{x_d\}_{d\in\mathcal{D}}$*.*

ii. *There is a subnet of* $\{x_d\}_{d\in\mathcal{D}}$ *which converges to the point* $x \in X$*.*

It can be convenient to express convergence of nets with respect to a base or sub-base.

Proposition 2.1.5. *Let (X, τ) be a topological space, $x \in X$, $\mathcal{S}$ be a sub-base for τ and let $\mathcal{S}_x = \{S \in \mathcal{S} : x \in S\}$. Then, a net $\{x_d\}_{d \in \mathcal{D}}$ converges to x if and only if for all $S \in \mathcal{S}_x$ there is $D \in \mathcal{D}$ such that for all $d \succeq D$ we have $x_d \in S$.*

Proof. $\implies$ follows quickly from definitions, since the elements of $\mathcal{S}_x$ are all open sets of x.

$\impliedby$

Let $O \in \tau_x$ then since τ is generated by $\mathcal{S}$ and $x \in O$ we know that there are $S_1, \ldots, S_K \in \mathcal{S}$ with $x \in \bigcap_{k=1}^K S_k \subseteq O$. By definition, each $S_k \in \mathcal{S}_x$ and by assumption for each $k = 1, \ldots, K$ there is a $D_k \in \mathcal{D}$ with for all $d \succeq D_k$ we have $x_d \in S_k$. From the definition of a directed set we can infer that there is a $D \in \mathcal{D}$ with $D \succeq D_1, \ldots, D_K$ and thus for all $d \succeq D$ we have that $x_d \in \bigcap_{k=1}^K S_k \subseteq O$. Therefore, $\{x_d\}_{d \in \mathcal{D}}$ converges to x, as required. □

Nets also have the expected results for closed and compact sets.

Proposition 2.1.6. *Let (X, τ) be a topological space. Let $A \subseteq X$, the the following are equivalent:*

1. *A is closed.*
2. *$A = \overline{A}$.*
3. *for all $x \in X$ with $V \cap A \neq \emptyset$ for every $V \in \tau_x$ we have $x \in A$.*
4. *If $\{a_d\}_{d \in \mathcal{D}}$ is a net in A which converges then $\lim a_d \subseteq A$.*

Additionally, the following are also equivalent:

a. *A is compact.*

b. *If $\mathcal{C} \subseteq \mathcal{P}(A)$ is a family of closed sets in (A, τ_A) with the finite intersection property (for all $N \in \mathbb{N}$ and any $C_1, \ldots, C_N \in \mathcal{C}$ we have $\bigcap_{n=1}^N C_n \neq \emptyset$) we have that $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.*

c. *Every net $\{a_d\}_{d \in \mathcal{D}}$ in A has a subnet which converges to a point in A.*

Furthermore, (X, τ) is Hausdorff then every compact set is closed.

A compact set may fail to be closed in non-Hausdorff spaces. Notice that a singleton set is always compact by the open cover definition but a singleton set may not be closed, for example when the topology is $\tau = \{\emptyset, X\}$ and X contains at least two points.

Due to some strange topological spaces, the statement: every sequence has a convergent subsequence. Could be false, even for compact spaces. This is does not contradict Proposition 2.1.6 because every sequence would still have a convergent subnet. This convergent subnet may fail to be a subsequence because subnets are allowed to be indexed by a directed set of greater carnality than the original net.

2.1.2 Functions and convergence of functions in topological spaces

The principle object of study in mathematics is function. Perhaps, the most celebrated class of functions are the continuous functions.

Definition 2.1.8. *Let (X, τ), (Y, σ) be topological spaces and $\mathrm{f} : X \to Y$ then, f is said to be continuous at the point $x \in X$ if for every open set $V \ni f(x)$ we have that $\mathrm{f}^{-1}(V)$ is a neighborhood of x.*

The function f is said to be continuous on X or simply continuous if f is continuous at every point of X.

There are many useful and well known characterizations of continuous functions.

Theorem 2.1.1. *Let (X, τ), (Y, σ) be topological spaces and $\mathrm{f} : X \to Y$. The following are equivalent:*

1. *f is continuous on X.*
2. *For all $V \in \sigma$ we have that $\mathrm{f}^{-1}(V) \in \tau$.*
3. *For all $B \subseteq Y$ we have $\mathrm{int}(\mathrm{f}^{-1}(B)) \supseteq \mathrm{f}^{-1}(\mathrm{int}(B))$.*
4. *For all $C \in \sigma^c$ we have that $\mathrm{f}^{-1}(C) \in \tau^c$.*
5. *For all $B \subseteq Y$ we have $\overline{\mathrm{f}^{-1}(B)} \subseteq \mathrm{f}^{-1}\left(\overline{B}\right)$.*
6. *If $\{x_d\}_{d \in \mathcal{D}}$ is a net in X which converges to x then $\{\mathrm{f}(x_d)\}_{d \in \mathcal{D}}$ converges in Y to $\mathrm{f}(x)$.*
7. *For all $A \subseteq X$ we have that $\mathrm{f}\left(\overline{A}\right) \subseteq \overline{\mathrm{f}(A)}$.*

Furthermore, if Y is Hausdorff and compact then f *is continuous iff the set* $\mathrm{Gr}(\mathrm{f}) = \{(x,y) \in X \times Y : y = f(x)\}$ *is closed in the product topology of* $X \times Y$.

Continuous functions are most conveniently defined through their inverse images. The forward image of a continuous function also has some important properties.

Proposition 2.1.7. *Let (X,τ), (Y,σ) be topological spaces,* $\mathrm{f} : X \to Y$ *and let* $A \subseteq X$. *The following hold:*

1. *If A is compact in X then,* $\mathrm{f}(A)$ *is compact in Y.*
2. *If A is connected in X then,* $\mathrm{f}(A)$ *is connected in Y.*

Item 1 is essentially the extreme value theorem and Item 2 is essentially the intermediate value theorem for topological spaces.

2.1.3 Convergence of functions in topological spaces

We begin with some more notation and definitions.

Definition 2.1.9. *Let (X,τ), (Y,σ) be topological spaces. Define, the set of all functions from X to Y,*

$$Y^X = \{\mathrm{f} \,|\, \mathrm{f} : X \to Y\}.$$

We may also take $Y^X = \Pi_{x\in X} Y$ and define the topology of pointwise convergence on Y^X to be $\mathfrak{P}$, where $\mathfrak{P}$ is the product topology.

Define the set of all continuous functions from X to Y to be

$$C(X,Y) = \left\{\mathrm{f} \in Y^X \middle|\, \mathrm{f} \ \textit{ is continuous}\right\}.$$

We should justify the name for the topology of pointwise convergence.

Proposition 2.1.8. *Let (X,τ), (Y,σ) be topological spaces. A net of functions from X to Y,* $\{\mathrm{f}_d\}_{d\in\mathcal{D}}$ *converges to a function* f *in the pointwise topology if and only if for every $x \in X$ the net* $\{\mathrm{f}_d(x)\}_{d\in\mathcal{D}}$ *converges to* $\mathrm{f}(x)$.

Furthermore, if Y is Hausdorff (or regular) then $(Y^X, \mathfrak{P})$ is Hausdorff (or regular).

We note that the topology on X is irrelevant for pointwise convergence. The topology pointwise convergence has the unfortunate property that $C(X,Y)$ may not be closed, the classic example of this is $X = Y = [0,1]$ with the usual topology on $\mathbb{R}$ and $\mathrm{f}_n(x) = x^n, n \in \mathbb{N}$ then each f_n is continuous but the pointwise limit of f_n is discontinuous.

I only care for continuous functions, so I have a great desire for $C(X,Y)$ to be closed. In metric spaces the way to get continuous limits of sequences of continuous functions is to employ the uniform metric. The uniform metric is a metric on all of $C(X,Y)$ only when one of X, Y is compact, so sometimes we would consider convergence in the uniform metric on all the compact subsets of X. Assumedly, this train of thought can produce the following definition.

Definition 2.1.10. *Let (X,τ), (Y,σ) be topological spaces. Define the sub-base*

$$\mathfrak{W} = \big\{ \big\{ \mathrm{f} \in Y^X : K \subseteq \mathrm{f}^{-1}(U) \big\} \big| K \text{ is compact in } X \text{ and } U \in \sigma \big\}$$

for Y^X. Let the topology generated from $\mathfrak{W}$ be called the compact open topology and be denoted $\mathfrak{C}$.

The sub-basic open sets $\big\{ \mathrm{f} \in Y^X : K \subseteq \mathrm{f}^{-1}(U) \big\}$, for K compact and U open, bear some ideas of uniform convergence on compact sets. Indeed when, Y is a metric space then $(C(X,Y), \mathfrak{C})$ is the topology of uniform convergence on compact sets. Do note that they only coincide on $C(X,Y)$ not Y^X. Embarrassingly, in $(Y^X, \mathfrak{C})$ the set $C(X,Y)$ is not even closed.

Example 2.1.1. *Let $X = Y = [0,1]$ with the usual topology and define*

$$\mathrm{f}(x) = \begin{cases} 0 & x = 0 \\ \sin(\frac{1}{x}) & x \neq 0 \end{cases}$$

and for $n \in \mathbb{N}$

$$\mathrm{f}_n(x) = \begin{cases} 0 & x \leq \frac{1}{2\pi n} \\ \sin(\frac{1}{x}) & x > \frac{1}{2\pi n} \end{cases}$$

then, the sequence of continuous functions f_n converges to f, a discontinuous function, in the compact open topology.

To see this consider any K compact and U open in X with $K \subseteq \mathrm{f}^{-1}(U)$ then if $0 \notin K$ there is a $N \in \mathbb{N}$ with for all $n \geq N$ we have $\mathrm{f}(x) = \mathrm{f}_n(x)$ for all $x \in K$ so no matter what U is we have $K \subseteq \mathrm{f}_n^{-1}(U)$. On the other hand if $0 \in K$ then $\mathrm{f}(0) = 0 \in U$. For any $n \in \mathbb{N}$ and for $x \in K$ with $x \leq \frac{1}{2\pi n}$ we have $\mathrm{f}_n(x) = 0 \in U$, and for $x > \frac{1}{2\pi n}$ we have $\mathrm{f}(x) = \mathrm{f}_n(x) \in U$. Hence, for all $n \in \mathbb{N}$, $K \subseteq \mathrm{f}_n^{-1}(U)$. So f_n converges to f in the compact open topology.

As far as I can tell, people only consider the compact open topology on $C(X,Y)$ rather than Y^X. So from a functional standpoint $C(X,Y)$ not being closed in Y^X is irrelevant as $C(X,Y)$ must be closed in $(C(X,Y),\mathfrak{C})$. I find this deeply unsatisfying, but it is what people do.

The reason we consider the compact open topology is to get a topological version of the Arzelà–Ascoli theorem and more importantly a topological version of equicontinuity. The latter will be an interesting regularity assumption later on.

Definition 2.1.11. *Let $\mathcal{F} \subseteq X^Y$ where (X,τ), (Y,σ) are topological spaces. Then the set $\mathcal{F}$ is said to be evenly continuous at $(x,y) \in X \times Y$ if for any $O \in \sigma_y$ there are $U \in \tau_x$ and $V \in \sigma_y$ with for all $\mathrm{f} \in \mathcal{F}$ we have*

$$\mathrm{f}(x) \in V \implies \mathrm{f}(U) \subseteq O.$$

$\mathcal{F}$ is said to be evenly continuous at $x \in X$ if $\mathcal{F}$ is evenly continuous at (x,y) for all $y \in Y$. Similarly, $\mathcal{F}$ is said to be evenly continuous if it is evenly continuous for all $x \in X$.

The set $\mathcal{F}$ is said to be topologically equicontinuous at $(x,y) \in X \times Y$ if for any $O \in \sigma_y$ there are $U \in \tau_x$ and $V \in \sigma_y$ with with for all $\mathrm{f} \in \mathcal{F}$ we have

$$\mathrm{f}(U) \cap V \neq \emptyset \implies \mathrm{f}(U) \subseteq O.$$

$\mathcal{F}$ is said to be topologically equicontinuous at $x \in X$, if $\mathcal{F}$ is topologically equicontinuous at (x,y) for all $y \in Y$. Similarly, $\mathcal{F}$ is said to be topologically equicontinuous if it is topologically equicontinuous for all $x \in X$.

Unfortunately, some definitions are largely incomprehensible. Personally, I find sequences and nets easier to grasp, at a glance.

Proposition 2.1.9. *Let $(X,\tau),(Y,\sigma)$ be topological spaces and $\mathcal{F} \subseteq Y^X$ then:*

1. *$\mathcal{F}$ is evenly continuous at $(x,y) \in X \times Y$ if and only if for every net $\{(\mathrm{f}_n, x_n)\}_{n\in N} \subseteq \mathcal{F} \times X$ with $x_n \to x$ we have that: if $\mathrm{f}_n(x) \to y$ then $\mathrm{f}_n(x_n) \to y$.*
2. *$\mathcal{F}$ is topologically equicontinuous at $(x,y) \in X \times Y$ if and only if for every net $\{(\mathrm{f}_n, x_n)\}_{n\in N} \subseteq \mathcal{F} \times X$ with $x_n \to x$ we have that: $\mathrm{f}_n(x) \to y$ if and only if $\mathrm{f}_n(x_n) \to y$.*

Proof. To prove 1, assume that $\mathcal{F}$ is evenly continuous at $(x,y) \in X \times Y$, $\{(\mathrm{f}_n, x_n)\}_{n\in N} \subseteq \mathcal{F} \times X$ with $x_n \to x$ and $\mathrm{f}_n(x) \to y$. Let $O \in \sigma_y$, by even continuity we get $U \in \tau_x$, $V \in \sigma_y$ with for all $n \in N$ the implication

$$\mathrm{f}_n(x) \in V \implies \mathrm{f}_n(U) \subseteq O$$

holds. We have that $\mathrm{f}_n(x) \to y$ and $x_n \to x$, so there is a $M \in N$ with for all $n \geq M$ both $\mathrm{f}_n(x) \in V$ and $x_n \in U$. Therefore, by the above implication, we have $O \supseteq \mathrm{f}_n(U) \ni \mathrm{f}_n(x_n)$ for all $n \geq M$; as $O \in \sigma_y$ is arbitrary, $\mathrm{f}_n(x_n) \to y$, as needed.

Conversely, suppose that $\mathcal{F}$ is not evenly continuous at $(x, y) \in X \times Y$ then there is a $O \in \sigma_y$ for all $U \in \tau_x$ and for all $V \in \sigma_y$ there is a $\mathrm{f} = \mathrm{f}_{(U,V)} \in \mathcal{F}$ with

$$\mathrm{f}_{(U,V)}(x) \in V \text{ and } \mathrm{f}_{(U,V)}(U) \nsubseteq O.$$

Thus, we can choose $x_{(U,V)}$ with $\mathrm{f}_{(U,V)}(x_{(U,V)}) \in \mathrm{f}_{(U,V)}(U) \backslash O$. Note that $\big\{(\mathrm{f}_{(U,V)}, x_{(U,V)})\big\}_{(U,V) \in \tau_x \times \sigma_y}$ is a net of $\mathcal{F} \times X$ where $(U, V) \geq (P, J)$ if $P \subseteq U$ and $J \subseteq V$. With this in mind, one can show that $x_{(U,V)} \to x$ (since $x_{(U,V)} \in U \in \tau_x$) and $\mathrm{f}_{(U,V)}(x) \to y$ (since $\mathrm{f}_{(U,V)}(x) \in V \in \sigma_y$). But, $\mathrm{f}_{(U,V)}(x_{(U,V)}) \notin O \in \sigma_y$. Therefore, $\mathrm{f}_{(U,V)}(x) \to y \not\Longrightarrow \mathrm{f}_{(U,V)}(x_{(U,V)}) \to y$ and 1 holds.

For 2, we first assume that $\mathcal{F}$ is topologically equicontinuous at $(x, y) \in X \times Y$ and $\{(\mathrm{f}_n, x_n)\}_{n \in N} \subseteq \mathcal{F} \times X$ with $x_n \to x$. We must show that $\mathrm{f}_n(x) \to y \iff \mathrm{f}_n(x_n) \to y$. The proof of the $\Longrightarrow$ direction is very similar to the first half of the prove of Item 1. So we prove the $\Longleftarrow$ direction. Assume $\mathrm{f}_n(x_n) \to y$ and let $O \in \sigma_y$, by topologically equicontinuity there are sets $U \in \tau_x$, $V \in \sigma_y$ with

$$\mathrm{f}_n(U) \cap V \neq \emptyset \implies \mathrm{f}_n(U) \subseteq O.$$

for all $n \in N$. Since, $\mathrm{f}_n(x_n) \to y$ and $x_n \to x$ there is a $M \in N$ with for all $n \geq M$, we have both $\mathrm{f}_n(x_n) \in V$ and $x_n \in U$. Hence, for all $n \geq M$, $\mathrm{f}_n(U) \cap V \neq \emptyset$ and $O \supseteq \mathrm{f}_n(U) \ni \mathrm{f}_n(x)$; as $O \in \sigma_y$ is arbitrary, $\mathrm{f}_n(x) \to y$, as needed.

Conversely, again assume that $\mathcal{F}$ is not topologically equicontinuous at $(x, y) \in X \times Y$ then there is a $O \in \sigma_y$ for all $U \in \tau_x$ and for all $V \in \sigma_y$ there is a $\mathrm{f} = \mathrm{f}_{(U,V)} \in \mathcal{F}$ with

$$\mathrm{f}_{(U,V)}(U) \cap V \neq \emptyset \text{ and } \mathrm{f}_{(U,V)}(U) \nsubseteq O.$$

Consider, two cases $\mathrm{f}_{(U,V)}(x) \to y$ and $\mathrm{f}_{(U,V)}(x) \not\to y$. When, $\mathrm{f}_{(U,V)}(x) \to y$ we take $\mathrm{f}_{(U,V)}(x_{(U,V)}) \in \mathrm{f}_{(U,V)}(U) \setminus O$ as before and can see that $\mathrm{f}_{(U,V)}(x_{(U,V)}) \not\to y$. Therefore, $\mathrm{f}_{(U,V)}(x) \to y \not\Longrightarrow \mathrm{f}_{(U,V)}(x_{(U,V)}) \to y$, as needed. In the other case $\mathrm{f}_{(U,V)}(x) \not\to y$ we can instead take $\mathrm{f}_{(U,V)}(z_{(U,V)}) \in \mathrm{f}_{(U,V)}(U) \cap V$. Again we can see that $z_{(U,V)} \to x$ and $\mathrm{f}_{(U,V)}(z_{(U,V)}) \to y$ but $\mathrm{f}_{(U,V)}(x) \not\to y$. Therefore, $\mathrm{f}_{(U,V)}(z_{(U,V)}) \to y \not\Longrightarrow \mathrm{f}_{(U,V)}(x) \to y$ and 2 holds.

□

It is fairly clear that evenly continuous and/or topologically equicontinuous sets of functions must be contained in $C(X, Y)$. So we can safely restrict our thoughts to continuous functions while considering evenly continuous and/or topologically equicontinuous sets.

Note that we can see that topologically equicontinuous sets are evenly continuous. Under certain conditions even continuity is equivalent to topologically equicontinuity.

Proposition 2.1.10. *Let $(X,\tau),(Y,\sigma)$ be topological spaces with Y Hausdorff, $\mathcal{F} \subseteq Y^X$, and $x \in X$. If the set $\{\mathrm{f}(x) | \mathrm{f} \in \mathcal{F}\}$ is compact in Y then, the following are equivalent:*

1. *$\mathcal{F}$ is evenly continuous at x.*
2. *$\mathcal{F}$ is topologically equicontinuous at x.*

Proof. $2 \implies 1$ always holds.

For $1 \implies 2$, we assume 1 holds and recalling Proposition 2.1.9 we consider a net $\{(\mathrm{f}_n, x_n)\}_{n\in D}$ of $\mathcal{F} \times X$ with $x_n \to x$. We need only show that if $\mathrm{f}_n(x_n) \to y$ then, $\mathrm{f}_n(x) \to y$. So assume that $\mathrm{f}_n(x_n) \to y$ and we know that $\{\mathrm{f}_n(x)\}_{n\in D}$ has a convergent subnet by compactness. So let $\mathrm{f}_{n_k}(x) \to z \in Y$ and by even continuity we have that $\mathrm{f}_{n_k}(x_{n_k}) \to z$ but note that $\mathrm{f}_{n_k}(x_{n_k})$ is a subnet of $\mathrm{f}_n(x_n) \to y$ so $\mathrm{f}_{n_k}(x_{n_k}) \to z, y$.

Since Y is Hausdorff, the limits of nets are unique. Thus, $y = z$. And can see that $\mathrm{f}_{n_k}(x) \to y$. Furthermore, we conclude that every convergent subnet of $\{\mathrm{f}_n(x)\}_{n\in D}$ converges to y. By compactness, one can show that $\mathrm{f}_n(x) \to y$ which completes the proof. □

This brings us to the Arzelà–Ascoli theorem for topological spaces.

Theorem 2.1.2 (Arzelà–Ascoli theorem for topological spaces)**.** *Let (X,τ) be a locally compact regular Hausdorff space and (Y,σ) be a regular Hausdorff space. Let $\mathcal{F} \subseteq C(X,Y)$. Then, the following are equivalent:*

i) *$\mathcal{F}$ is compact in $(C(X,Y),\mathfrak{C})$.*

ii) *The following hold:*

- *1)* *$\mathcal{F}$ is closed in $(C(X,Y),\mathfrak{C})$.*
- *2)* *for all $x \in X$ the set $\{\mathrm{f}(x) | \mathrm{f} \in \mathcal{F}\}$ is compact.*
- *3)* *$\mathcal{F}$ is evenly continuous.*

iii) *The following hold:*

- *1)* *$\mathcal{F}$ is closed in $(C(X,Y),\mathfrak{C})$.*
- *2)* *for all $x \in X$ the set $\{\mathrm{f}(x) | \mathrm{f} \in \mathcal{F}\}$ is compact.*

3) $\mathcal{F}$ is topologically equicontinuous.

Proof. *ii* $\iff$ *iii* by Proposition 2.1.10.
To prove *iii* $\implies$ *i*, we will have to argue that every net $\{\mathrm{f}_n\}_{n\in D}$ of $\mathcal{F}$ has a convergent subnet. In particular we must construct a function f as the limit of the subnet and somehow argue that f is both continuous and in $\mathcal{F}$. The general nature of X, Y make it difficult to actually construct f. So instead we use the fact that the Cartesian product of compact sets in compact in the product topology, this will allows to find a subnet of $\{\mathrm{f}_n\}_{n\in D}$ with a pointwise limit.

Specifically,

$$\mathcal{F} \subseteq \prod_{x\in X} \{\mathrm{f}(x)|\, \mathrm{f} \in \mathcal{F}\}$$

the RHS is compact in the product topology, by Item 5 of Proposition 2.1.3. In this case the product topology is the topology of point wise convergence in Y^X. Also by Proposition 2.1.3, Item 3, $(Y^X, \mathfrak{P})$ is Hausdorff. Therefore, the pointwise closure of $\mathcal{F}$, say $\mathrm{cl}_{\mathfrak{P}}(\mathcal{F})$ is compact, thus the net $\{\mathrm{f}_n\}_{n\in D}$ of $\mathcal{F}$ has a subnet which converges pointwise to some function $\mathrm{f} \in Y^X$.

WLOG, assume that $\mathrm{f}_n \to \mathrm{f}$ pointwise and $\mathrm{f}_n \in \mathcal{F}$ for all $n \in D$. For the sake of contradiction, assume that $\{\mathrm{f}_n\}_{n\in D}$ does not converge to f in the compact open topology. So there is a compact set $K \subseteq X$ and open $V \subseteq Y$ such that for all $N \in D$ there is a $n_N \geq N$ with

$$\mathrm{f}_{n_N} \notin \left\{\mathrm{g} \in Y^X |\, \mathrm{g}(K) \subseteq V\right\} \ni \mathrm{f}$$

this means for all $N \in D$ there is a $x_{n_N} \in K$ with $\mathrm{f}_{n_N}(x_{n_N}) \notin V$. By compactness we may assume WLOG, that $x_{n_N} \to x \in K$. Note $\mathrm{f}(x) \in V$. We may further assume that $\mathrm{f}_{n_N}(x)$ converges to $y \in Y$ by compactness of $\{\mathrm{g}(x)|\, \mathrm{g} \in \mathcal{F}\}$. Since Y is Hausdorff and $\mathrm{f}_n \to \mathrm{f}$ converges pointwise $y = \mathrm{f}(x)$.

This means, by topological equicontinuity, that

$$\mathrm{f}_{n_N}(x) \to \mathrm{f}(x) \implies \mathrm{f}_{n_N}(x_{n_N}) \to \mathrm{f}(x).$$

But then $\mathrm{f}(x) \in V$ so for all large $N \in D$ we have that $\mathrm{f}_{n_N}(x_{n_N}) \in V$, a contradiction. Therefore, $\{\mathrm{f}_n\}_{n\in D}$ converges to f in $(Y^X, \mathfrak{C})$.

We know argue that $\mathrm{f} \in \mathcal{F} \subseteq C(X, Y)$. Since $\mathcal{F}$ is closed in $(C(X, Y), \mathfrak{C})$ and $\mathrm{f}_n \to \mathrm{f}$ in $(Y^X, \mathfrak{C})$ we need only show that f is continuous. Suppose that f is not continuous then, there is a net $\{x_\beta\}_{\beta\in B}$ converging to $x \in X$ such that $\mathrm{f}(x_\beta) \not\to \mathrm{f}(x)$ then by regularity of Y there is set $V \in \sigma$ with $\mathrm{f}(x) \in V$ but for all $b \in B$ there is $\beta_b \geq b$ with $\mathrm{f}(x_{\beta_b}) \notin \overline{V}$.

Moreover, since X is locally compact, we may assume that $x_\beta \in K$ for all β where U_x is a neighborhood of x with $\overline{U_x}$ compact.

Define, for all $n \in D$ and $\beta \in B$, $\mathrm{f}_{(n,\beta)} = \mathrm{f}_n$ and $x_{(n,\beta)} = x_\beta$ then, $\{\mathrm{f}_{(n,\beta)}(x_{(n,\beta)})\}_{(n,\beta)\in D\times B}$ is a net of Y, with $\mathrm{f}_{(n,\beta)} \to \mathrm{f}$ in $(Y^X, \mathfrak{C})$ and $x_{(n,\beta)} \to x$. Since $\mathrm{f}_{(n,\beta)} \to \mathrm{f}$ we have that $\mathrm{f}_{(n,\beta)}(x) \to \mathrm{f}(x)$ so by topological equicontinuity we have that $\mathrm{f}_{(n,\beta)}(x_{(n,\beta)}) \to \mathrm{f}(x)$ as well.

Also notice that for all $b \in B$ and $N \in D$ there is a $n_{(N,b)} \geq N$ with

$$\mathrm{f}_{n_{(N,b)}}(x_{\beta_b}) \notin \overline{V} \text{ and } \mathrm{f}_{n_{(N,b)}}, \mathrm{f} \in \left\{\mathrm{g} \in Y^X \middle| \, \mathrm{g}\big(\overline{U_x}\big) \subseteq V\right\}$$

since $\mathrm{f}_n \to \mathrm{f}$ in $(Y^X, \mathfrak{C})$. However, $\mathrm{f}_{n_{(N,b)}}(x_{\beta_b}) = \mathrm{f}_{(n_{(N,b)},\beta_b)}(x_{(n_{(N,b)},\beta_b)})$ is a subnet of $\mathrm{f}_{(n,\beta)}(x_{(n,\beta)})$ and so must converge to $\mathrm{f}(x)$ but this contradicts that $\mathrm{f}_{n_{(N,b)}}(x_{\beta_b}) \in Y \setminus \overline{V}$. Therefore, f is continuous and a holds.

Lastly we prove $i \implies ii$. Since $\mathcal{F}$ is compact and $(C(X,Y), \mathfrak{C})$ is Hausdorff, we know that $\mathcal{F}$ is closed in $(C(X,Y), \mathfrak{C})$, which is 1. For 2, let $x \in X$ and $\mathcal{V}$ be an open cover of $\{\mathrm{f}(x) | \, \mathrm{f} \in \mathcal{F}\}$ and define for $K \subseteq X$ compact and $V \in \sigma$,

$$T(K, V) = \{\mathrm{g} \in C(X,Y) | \, \mathrm{g}(K) \subseteq V\}.$$

By definition, the sets $T(K,V)$ are sub-basic sets for $(C(X,Y), \mathfrak{C})$ and so are open, in particular the set $\mathcal{W} = \{T(\{x\}, V) | V \in \mathcal{V}\}$ is an open cover of $\mathcal{F}$ (this follows quickly from $\mathcal{V}$ being an open cover of $\{\mathrm{f}(x) | \, \mathrm{f} \in \mathcal{F}\}$). Since $\mathcal{F}$ is compact, $\mathcal{W}$ has a finite sub-cover, so there are $V_k \in \mathcal{V}$, $k = 1, \ldots, N$ with

$$\mathcal{F} \subseteq \bigcup_{k=1}^{N} T(\{x\}, V_k)$$

and we see that for any $\mathrm{f} \in \mathcal{F}$ there is a $k = 1, \ldots, N$ with $\mathrm{f}(x) \in V_k$. Thus,

$$\{\mathrm{f}(x) | \, \mathrm{f} \in \mathcal{F}\} \subseteq \bigcup_{k=1}^{N} V_k$$

which is a finite sub-cover of $\mathcal{V}$ and $\{\mathrm{f}(x) | \, \mathrm{f} \in \mathcal{F}\}$ is compact.

For 3 we consider a net $\{(\mathrm{f}_n, x_n)\}_{n\in D}$ of $\mathcal{F} \times X$ with $x_n \to x$ and any $y \in Y$. We will show that $\mathrm{f}_n(x_n) \not\to y$ then, $\mathrm{f}_n(x) \not\to y$. So suppose $\mathrm{f}_n(x_n) \not\to y$ then, there is a open set $V \ni y$, a subnet $\{\mathrm{f}_{n_k}(x_{n_k})\}_{k\in D_2}$ with $\mathrm{f}_{n_k}(x_{n_k}) \in Y \setminus \overline{V}$ for all $k \in D_2$ and, by compactness, $\mathrm{f}_{n_k} \to \mathrm{f} \in \mathcal{F}$ in $(C(X,Y), \mathfrak{C})$. By local compactness of X and since x_n converges, we can assume that for all $J \in D_2$ that $L_K = \overline{\{x_{n_k} | k \geq J\}}$ is compact. Since f is continuous, for any open $O \ni \mathrm{f}(x)$ there is a $K \in D_2$ with $\mathrm{f}(L_J) \subseteq O$ and

$$\mathrm{f}_{n_k} \in T(L_J, O)$$

for all $k \geq J$ (as $T(L_J, O)$ is an open set of f). In particular $x \in L_J$, so $\mathrm{f}_{n_k}(x) \in O$ for all $k \geq K$. Thus $\mathrm{f}_{n_k}(x) \to \mathrm{f}(x)$. We now claim that $\mathrm{f}_{n_k}(x) \not\to y$, for if $\mathrm{f}_{n_k}(x) \to y$ then, in the above equation we may take $O = V$ and since $x_k \in L_J$ for $k \geq J$ we would have $\mathrm{f}_{n_k}(x_{n_k}) \in V$ a contradiction. By Proposition 2.1.9 $\mathcal{F}$ is evenly continuous. □

2.2 Topologies and convergence of subsets of topological spaces

Approximation of sets is essential for computing the reachable set of a dynamical system. To do approximations of sets it makes sense to investigate possible topologies on the subsets of a topological space—the so called hyperspace topologies. The most famous topology on subsets is the topology induced by the Hausdorff metric.

Proposition 2.2.1. *Let* (X, d) *be a metric space. Let* $\mathfrak{B}_X$ *be the nonempty bounded subsets of* X*. Define,*

$$\mathrm{d}_H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\mathrm{d}(a, b), \sup_{b\in B}\inf_{a\in A}\mathrm{d}(b, a)\right\}$$

for $A, B \in \mathfrak{B}_X$ *then,*

1. $(\mathfrak{B}_X, \mathrm{d}_H)$ *is a pseudo-metric space.*
2. $(\mathrm{cl}\,\mathfrak{B}_X, \mathrm{d}_H)$ *and* $(\mathcal{K}_X, \mathrm{d}_H)$ *are metric spaces and are complete if* X *is complete. Where* $\mathrm{cl}\,\mathcal{B}_X$ *is the set of all closed nonempty bounded subsets of* X *and* $\mathcal{K}_X$ *is the set of all nonempty compact subsets of* X*.*
3. $(\mathrm{cl}\,\mathfrak{B}_X, \mathrm{d}_H)$ *and* $(\mathcal{K}_X, \mathrm{d}_H)$ *are compact spaces if* X *is compact.*

Proof. The proofs can be found throughout Chapter 1 of [12]. □

While, $(\mathfrak{B}_X, \mathrm{d}_H)$ is a pseudo-metric space it often fails to be a metric space. One can see this from recognizing that $\mathrm{d}_H(A, B) = \mathrm{d}_H(\overline{A}, \overline{B})$, so $\mathrm{d}_H(A, \overline{A}) = 0$.

The Hausdorff metric is often the first and last topology on sets a mathematician is exposed to. This is for good reason too, if our interest is in approximation sets then, ideally we are working in metric space. Realistically, we are often working in $\mathbb{R}^n$ where $\mathrm{cl}\,\mathfrak{B}_{\mathbb{R}^n} = \mathcal{K}_{\mathbb{R}^n}$ and we even get completeness! Of course, there is more to life then bounded sets; for instance I think it's intuitive that the intervals $A_n = [0, n]$ converges to $[0, \infty)$ but the Hausdorff metric disagrees. To deal with the unbounded sets we turn to the Vietoris topologies.

Definition 2.2.1 (Vietoris topologies)**.** *Let* (X, τ) *be a topological space. Define, for all* $A \in \mathcal{P}(X)$

$$A^- = \{B \in \mathcal{P}(X) \setminus \{\emptyset\} : B \cap A \neq \emptyset\}$$

and

$$A^+ = \{B \in \mathcal{P}(X) \setminus \{\emptyset\} : B \subseteq A\}.$$

We define the following topologies on $\mathcal{P}(X) \setminus \{\emptyset\}$*:*

- *The lower Vietoris topology (l.v.t), denoted* τ_{LV} *is the topology generated by the sub-base* $\{U^- : U \in \tau\}$*.*
- *The upper Vietoris topology (u.v.t), denoted* τ_{UV} *is the topology generated by the base* $\{U^+ : U \in \tau\}$*.*
- *The Vietoris topology (v.t), denoted* τ_V *is the topology generated by the sub-base* $\{U^- : U \in \tau\} \cup \{U^+ : U \in \tau\}$*.*

Let $\mathcal{S} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$*, then we let* $(\mathcal{S}, \tau_{LV})$*,* $(\mathcal{S}, \tau_{UV})$ *and* $(\mathcal{S}, \tau_V)$ *to denote the relative topology on* $\mathcal{S}$ *of the l.v.t, u.v.t and v.t respectively. Similarly when we are contextually working in* $\mathcal{S}$ *with one of these topologies, we will take* A^- *to mean* $A^- \cap \mathcal{S}$ *and* A^+ *to mean* $A^+ \cap \mathcal{S}$ *without note.*

We will now develop some properties of the Vietoris topologies and later we will establish that the Vietoris topology on the compact sets is the topology induced by the Hausdorff metric. That is to say, when the Hausdorff metric is most useful the Vietoris topology does the same thing and more, since it can also deal with unbounded sets effectively.

You may be wondering why we have defined three topologies, instead of just one. This is mostly because often times we will only have convergence in say the u.v.t but not in l.v.t. Or more commonly for control theory, we can guarantee certain point to set functions are continuous with respect to l.v.t but we cannot easily guarantee continuity in u.v.t.

Remark 2.2.1 (the empty set in the Vietoris topologies)**.** *We could modify the definitions of* A^+ *and* A^- *to include the* $\emptyset$ *as an element. However, there is no real advantage to this from the perceptive of approximation of sets. Indeed doing these modification yields,*

$$\begin{aligned} \emptyset^- &= \{B \in \mathcal{P}(X) : B \cap \emptyset \neq \emptyset\} &&= \emptyset \\ \emptyset^+ &= \{B \in \mathcal{P}(X) : B \subseteq \emptyset\} &&= \{\emptyset\} \end{aligned}$$

In the case of the u.v.t and v.t, we would have $\emptyset$ *being an isolated point of* $\mathcal{P}(X)$*, an isolated point* p *of a topological space is a point with if* p_n *converges to* p *then* p_n *is eventually constant. Therefore, I claim that an isolated point (and the empty set) can only be approximated by itself. Moreover, the v.t is often Hausdorff on the closed sets, this means that*

$\{\emptyset\}$ would be a non-trivial closed and open set and the hyperspace would (almost) never be connected.

The case is more severe for the l.v.t, where the sub-base in Definition 2.2.1 fails to be a sub-base for $\mathcal{P}(X)$. We could just add in a set to the base with $\emptyset$ as a member, but any reasonable one I can come up with will yield the same issue as with the other Vietoris topologies.

The first order of business is to find the basic sets.

Proposition 2.2.2. *Let (X, τ) be a topological space.*

1. *The basic open sets of $(\mathcal{P}(X) \setminus \{\emptyset\}, \tau_{LV})$ are*

$$\left\{ \bigcap_{n=1}^{N} V_n^- : N \in \mathbb{N}, V_1, \ldots, V_N \in \tau \right\}.$$

2. *The collection of sets $\{V^+ : V \in \tau\}$ do in fact, form a base for $(\mathcal{P}(X) \setminus \{\emptyset\}, \tau_{UV})$.*

3. *We can take the basic open sets of $(\mathcal{P}(X) \setminus \{\emptyset\}, \tau_V)$ to be*

$$\{\mathfrak{V}(V_1, \ldots, V_N) : N \in \mathbb{N}, V_1, \ldots, V_N \in \tau\}$$

where

$$\mathfrak{V}(V_1, \ldots, V_N) = \left\{ A \in \mathcal{P}(X) \setminus \{\emptyset\} : A \subseteq \bigcup_{n=1}^{N} V_n, A \cap V_n \neq \emptyset \text{ for } n = 1, \ldots, N \right\}$$

for all $N \in \mathbb{N}$ and $V_1, \ldots, V_N \subseteq X$.

Proof. Item 1 is just the finite intersections of the defining sub-base, $\{V^- : V \in \tau\}$. Thus, by definition we need only check that the sub-base covers $\mathcal{P}(X) \setminus \{\emptyset\}$, which follows immediately from τ covering X.

For 2, one can show the collection of sets $\{V^+ : V \in \tau\}$ cover $\mathcal{P}(X) \setminus \{\emptyset\}$. For the other condition for a base, let $A \in W^+ \cap U^+$ for $U, W \in \tau$ then $A \subseteq U$ and $A \subseteq W$. So $A \subseteq U \cap W \in \tau$ and $A \in (U \cap W)^+$. Since, $(U \cap W)^+$ is in $\{V^+ : V \in \tau\}$ and $(U \cap W)^+ \subseteq W^+ \cap U^+$, it follows that $\{V^+ : V \in \tau\}$ is a base.

For 3, again one can show that the $\mathfrak{V}$'s cover $\mathcal{P}(X) \setminus \{\emptyset\}$. To see the other condition for a base holds let $A \in \mathfrak{V}(V_1, \dots, V_N) \cap \mathfrak{V}(U_1, \dots, U_M)$ where $N, M \in \mathbb{N}$, $V_1, \dots, V_N \in \tau$ and $U_1, \dots, U_M \in \tau$.

Firstly, it follows from definition that

$$A \subseteq \left(\bigcup_{n=1}^{N} V_n \right) \cap \left(\bigcup_{m=1}^{M} U_m \right) = \bigcup_{n=1}^{N} \bigcup_{m=1}^{M} V_n \cap U_m.$$

Now let, $W_k \subseteq X$,$k = 1, \dots, K$ be an enumeration of the set

$$\{U_m \cap V_n : U_m \cap V_n \cap A \neq \emptyset, n = 1, \dots N \text{ and } m = 1, \dots, M\}.$$

We claim that $A \in \mathfrak{V}(W_1, \dots, W_K)$, it is clear that the W_k cover A since $\bigcup_{k=1}^{K} W_k = \bigcup_{n=1}^{N} \bigcup_{m=1}^{M} V_n \cap U_m$. So we only need to show that $A \cap W_k \neq \emptyset$ for all $k = 1, \dots, K$, but this holds by the definition of the W_k's. Now we need to show that $\mathfrak{V}(W_1, \dots, W_K) \subseteq \mathfrak{V}(V_1, \dots, V_N) \cap \mathfrak{V}(U_1, \dots, U_M)$. But first, we claim that for all $n = 1, \dots, N$ there is an $m = 1, \dots, M$ with $U_m \cap V_n \cap A \neq \emptyset$. For if this was not true there would be an $n = 1, \dots, N$ with

$$\begin{aligned} \bigcup_{m=1}^{M} A \cap V_n \cap U_m &= \emptyset \\ A \cap V_n \cap \left(\bigcup_{m=1}^{M} U_m \right) &= \emptyset \\ A \cap V_n \cap A &\subseteq \emptyset \end{aligned}$$

but $\emptyset \neq A \cap V_n = A \cap V_n \cap A = \emptyset$, a contradiction. By symmetry we can conclude that for all $n = 1, \dots, N$ and any $m = 1, \dots, M$ there are $n_m = 1, \dots, N$ $m_n = 1, \dots, M$ with $U_{m_n} \cap V_n \cap A \neq \emptyset$ and $U_m \cap V_{n_m} \cap A \neq \emptyset$. Now let $B \in \mathfrak{V}(W_1, \dots, W_K)$, then $B \subseteq \bigcup_{k=1}^{K} W_k$ and as we argued before $\bigcup_{k=1}^{K} W_k = \bigcup_{n=1}^{N} \bigcup_{m=1}^{M} V_n \cap U_m = \left(\bigcup_{n=1}^{N} V_n \right) \cap \left(\bigcup_{m=1}^{M} U_m \right)$. So B is covered by both the V_n's and the U_m's. Since $B \cap W_k \neq \emptyset$ for $k = 1, \dots, K$ and by the previous claim we that for any $n = 1, \dots, N$ and any $m = 1, \dots, M$ there are $n_m = 1, \dots, N$ $m_n = 1, \dots, M$ with $U_{m_n} \cap V_n \cap A \neq \emptyset$ and $U_m \cap V_{n_m} \cap A \neq \emptyset$. And by definition $U_{m_n} \cap V_n = W_{k_n}$, $U_m \cap V_{n_m} = W_{k_m}$ for some $k_n, k_m = 1, \dots, K$ and in particular $\emptyset \neq B \cap W_{k_n} \subseteq B \cap V_n$ and $\emptyset \neq B \cap W_{k_m} \subseteq B \cap U_m$. Therefore, $B \in \mathfrak{V}(V_1, \dots, V_N) \cap \mathfrak{V}(U_1, \dots, U_M)$ and we can conclude that the collection in 3 is a base.

It remains to show that this base generates the Vietoris topology. It can be seen that given $\mathfrak{V}(V_1, \ldots, V_N)$ with $N \in \mathbb{N}, V_1, \ldots, V_N \in \tau$ we have that

$$\mathfrak{V}(V_1, \ldots, V_N) = \left(\bigcup_{n=1}^{N} V_n\right)^+ \cap \bigcap_{m=1}^{N} V_n^-$$

which is finite intersection of elements of $\{U^- : U \in \tau\} \cup \{U^+ : U \in \tau\}$. Thus, we need only show that given a $A \subseteq X$, $A \neq \emptyset$ and a finite intersection of elements of $\{U^- : U \in \tau\} \cup \{U^+ : U \in \tau\}$, say $\bigcap_{k=1}^{K} W_k^+ \cap \bigcap_{m=1}^{M} U_m^-$ where $W_1, \ldots, W_K \in \tau$ and $U_1, \ldots, U_M \in \tau$, $K, M \in \mathbb{N}$, with $A \in \bigcap_{k=1}^{K} W_k^+ \cap \bigcap_{m=1}^{M} U_m^-$ then, there is a $\mathfrak{V}(V_1, \ldots, V_N)$, $N \in \mathbb{N}$, $V_1, \ldots, V_N \in \tau$ with $A \in \mathfrak{V}(V_1, \ldots, V_N) \subseteq \bigcap_{k=1}^{K} W_k^+ \cap \bigcap_{m=1}^{M} U_m^-$.

So suppose that $A \in \bigcap_{k=1}^{K} W_k^+ \cap \bigcap_{m=1}^{M} U_m^-$ then, $A \in \bigcap_{k=1}^{K} W_k^+$ and it follows from the definition of W_k^+ that $\bigcap_{k=1}^{K} W_k^+ = \left(\bigcap_{k=1}^{K} W_k\right)^+$. So let $W = \bigcap_{k=1}^{K} W_k$ and we have that $W^+ \cap \bigcap_{m=1}^{M} U_m^- = \bigcap_{k=1}^{K} W_k^+ \cap \bigcap_{m=1}^{M} U_m^-$. Now define $V_m = U_m \cap W$ for $m = 1, \ldots, M$ and $V_{M+1} = W$. Then $A \cap V_m \neq \emptyset$ for $m = 1, \ldots, M+1$ and $A \subseteq W = \bigcup_{m=1}^{M+1} V_m$, so $A \in \mathfrak{V}(V_1, \ldots, V_{M+1})$. Now suppose that $B \in \mathfrak{V}(V_1, \ldots, V_{M+1})$ then as before $B \subseteq \bigcup_{m=1}^{M+1} V_m = W$ (so $B \in W^+$) and for $m = 1, \ldots, M$ we have $\emptyset \neq B \cap V_m = B \cap U_m \cap W \subseteq B \cap U_m$, so $B \in \bigcap_{m=1}^{M} U_m^-$. Thus, $\mathfrak{V}(V_1, \ldots, V_N) \subseteq \bigcap_{k=1}^{K} W_k^+ \cap \bigcap_{m=1}^{M} U_m^-$ and the result follows. □

Remark 2.2.2. *It is possible (and sometimes convenient) to define the Vietoris topologies in terms of basic/sub-basic open sets of X. For example, we take the sub-base for l.v.t to be*

$$\left\{B^- : B \in \mathcal{B}\right\}$$

where $\mathcal{B}$ is base for (X, τ) and a similar modification for a sub-base $\mathcal{S}$ of (X, τ).

The induced topologies for the Vietoris topologies of using basic/sub-basic open sets of X results in the same topologies presented so far. This is useful, because the open balls of a metric space are sub-basic (and not basic) and the open balls are easy to work with. If you like pain it's also convenient for definitions hyper hyperspace topologies; Topologies on $\mathcal{P}(\mathcal{P}(X) \setminus \{\emptyset\}) \setminus \{\emptyset\}$, by using sub-bases $\{(V^-)^- : V \in \tau\}$ for instance.

We now establish some properties of the Vietoris topologies given the underlining point space has certain properties. Restricting the topology to closed sets will yield the most properties.

Proposition 2.2.3. *Let (X, τ) be a topological space, $\mathcal{A}_X = \tau^c \setminus \{\emptyset\}$ be the set of all nonempty closed subsets of X and $\mathcal{K}_X$ be the nonempty compact subsets of X.*

1. *$(\mathcal{A}_X, \tau_{LV})$ is T_0 and connected.*

2. $(\mathcal{A}_X, \tau_{UV})$ *is connected and compact.*

3. *If* (X, τ) *is* T_1 *then,*

   3a) $(\mathcal{A}_X, \tau_{UV})$ *is* T_0.

   3b) $(\mathcal{A}_X, \tau_V)$ *is* T_1.

   3c) $(\mathcal{A}_X, \tau_V)$ *is regular, whenever* (X, τ) *is normal.*

   3d) $(\mathcal{A}_X, \tau_V)$ *and* $(\mathcal{K}_X, \tau_V)$ *is connected, whenever* (X, τ) *is connected.*

4. *If* (X, τ) *is regular then,*

   4a) $(\mathcal{A}_X, \tau_{UV})$ *is* T_0.

   4b) $(\mathcal{A}_X, \tau_V)$ *is Hausdorff* (T_2).

   4c) $(\mathcal{K}_X, \tau_V)$ *is Hausdorff* (T_2) *and regular.*

5. *If* (X, τ) *is not the trivial topology* ($\tau \neq \{\emptyset, X\}$) *then, both* $(\mathcal{A}_X, \tau_{LV})$ *and* $(\mathcal{A}_X, \tau_{UV})$ *are not* T_1.

6. (X, τ) *is Hausdorff and compact if and only if* $(\mathcal{A}_X, \tau_V)$ *is Hausdorff and compact.*

7. (X, τ) *is Hausdorff and locally compact if and only if* $(\mathcal{K}_X, \tau_V)$ *is Hausdorff and locally compact.*

8. *Suppose that* (X, d) *is a metric space, which induces the topology* (X, τ). *Then the Hausdorff metric,* d_H, *induces the Vietoris topology on* $\mathcal{K}_X$.

Proof. Some of the more difficult Items are proved in [15]. Specifically, Item 3c (Theorem 4.9.5 of [15]), Item 3d (Theorem 4.10 of [15]), Item 4c (Theorem 4.9.10 of [15]), Item 6 (Theorem 4.9.6 of [15]) and Item 7 (Theorem 4.9.12 of [15]). The proof of Item 8 can be found in [12] Theorem 1.30.

To prove Item 1, let $A, B \in \mathcal{A}_X$ with $A \neq B$ then, WLOG there is $a \in A \setminus B$ and $a \in X \setminus B \in \tau$ so $A \in (X \setminus B)^-$ but $B \notin (X \setminus B)^-$. Hence $(\mathcal{A}_x, \tau_{LV})$ is T_0. To show this space is connected, notice that any sub-basic open set U^- with $U \in \tau$ has $X \in U^-$. Thus every nonempty open set in τ_{LV} contains X. Therefore, it is impossible for two disjoint nonempty open sets to exist in $\mathcal{A}_X$ (sometimes this property is called hyper-connectedness). So $(\mathcal{A}_x, \tau_{LV})$ is connected.

To prove Item 2, let $\mathcal{W}, \mathcal{U} \in \tau_{UV}$ be disjoint open sets with $\mathcal{A}_X = \mathcal{W} \cup \mathcal{U}$. Then, WLOG $X \in \mathcal{W}$ and since $\mathcal{W}, \mathcal{U}$ are disjoint, we have that: for any basic open sets $W^+ \subseteq \mathcal{W}$,

$U^+ \subseteq \mathcal{U}$, $U, W \in \tau$ we have $W^+ \cap U^+ = \emptyset$. As $X \in \mathcal{W}$ we must have $X^+ \subseteq \mathcal{W}$ and $X^+ \cap U^+ = \emptyset$ for all $U^+ \subseteq \mathcal{U}$. Thus, $\mathcal{U}$ can only contain the empty set, which is not basic open set and $\mathcal{U} = \emptyset$. Therefore, $\mathcal{W} = \mathcal{A}_X$ and $(\mathcal{A}_X, \tau_{UV})$ is connected.

To see why $\mathcal{A}_X$ is compact in $(\mathcal{A}_X, \tau_{UV})$, let $\mathfrak{U}$ be an open cover of $\mathcal{A}_X$. Since X is closed in (X, τ), $X \in \mathcal{A}_X$. So there is a $U \in \tau$ with $U^+ \subseteq \mathcal{U} \in \mathfrak{U}$ for some $\mathcal{U}$ and $X \in U^+$. As $X \in U^+$ we have that $X \subseteq U$ but U is an open set in X. Thus, $X = U$. Notably, every $A \in \mathcal{A}_X$ also has $A \subseteq X$. It follows that $A \in U^+ \subseteq \mathcal{U} \in \mathfrak{U}$. Therefore, the subcover $\{\mathcal{U}\}$ is an open cover of $\mathcal{A}_X$, which is finite.

For Item 3a, let $A, B \in \mathcal{A}_X$ with $A \neq B$ then, WLOG there is $a \in A \setminus B$. Since X is T_1 the set $\{a\}$ is closed and so $B \subseteq X \setminus \{a\} \in \tau$. Thus $B \in (X \setminus \{a\})^+ \not\ni A$, so $(\mathcal{A}_X, \tau_{UV})$ is T_0.

The proof of Item 3b follows from examining the proofs of Item 1 and Item 3a. For when $A, B \in \mathcal{A}_X$ with $A \neq B$ and $a \in A \setminus B$, we found both $B \in (X \setminus \{a\})^+ \not\ni A$ and $A \in (X \setminus B)^- \not\ni B$.

For Item 4a, let $A, B \in \mathcal{A}_X$ with $A \neq B$ then, WLOG there is $a \in A \setminus B$. Since X is regular there are $V \in \tau_a$ and $U \supseteq B$, $U \in \tau$ with $V \cap U = \emptyset$. We can see that $U^+ \ni B$ but $U^+ \not\ni A$. So $(\mathcal{A}_X, \tau_{UV})$ is T_0.

To prove Item 4b, we continue from the proof of Item 4a. We can see that $V^- \ni A$ and $V^- \not\ni B$. Furthermore, $V^- \cap U^+ = \emptyset$, which proves $(\mathcal{A}_X, \tau_V)$ is T_2.

Lastly, for Item 5 we consider $A, B \in \mathcal{A}_X$ and since X is not the trivial topology we can take $B \subsetneq A$. Notice that any open $U^+ \ni A$ also has $B \in U^+$, so $(\mathcal{A}_X, \tau_{UV})$ is not T_1. Similarly, any open $V^- \ni B$ has $V^- \ni A$ as well, so $(\mathcal{A}_X, \tau_{LV})$ is not T_1. □

We can see that the u.v.t and l.v.t do not inherit the nice properties of X, unlike the Vietoris topology. This is why I bothered to define the concepts of T_0 and T_1; We will be using u.v.t and l.v.t, so the "non-niceness" of u.v.t and l.v.t are unavoidable. Of particular note is Item 5, where the u.v.t and l.v.t are almost never T_1, and so almost never have unique limits. We can also notice that the u.v.t and l.v.t are always connected, even when X is not[2]; So these topologies get extra properties for free. This gives more credence to u.v.t and l.v.t being degenerate topologies.

Item 8 establishes that the Vietoris topology does everything the Hausdorff metric does and more. So whenever possible we shall use the Vietoris topology. The most important situation for our purposes is when X is a regular/normal Hausdorff space (like when X is

[2]It is typical of any hyperspace topology to have stronger connectivity properties than the underlying space.

a metric space), which is convenient because that is when we have the most topological properties on the Vietoris topologies. In particular, we have that the full Vietoris topology is Hausdorff and so we have uniqueness of limits. Speaking of limits, they are basically required for approximations, so we should develop nets in the hyperspace as well.

Proposition 2.2.4. *Let (X, τ) be a topological space. Let $\{A_d\}_{d\in\mathcal{D}}$,$\{B_d\}_{d\in\mathcal{D}}$ be nets in $\mathcal{P}(X)\setminus\{\emptyset\}$ and let $A, B, C \in \mathcal{P}(X)\setminus\{\emptyset\}$. The following hold:*

1. *$A_d \to A$ in $(\mathcal{P}(X)\setminus\{\emptyset\}, \tau_{LV})$ if and only if for every $V \in \tau$ with $A \cap V \neq \emptyset$ there is a $D \in \mathcal{D}$ such that for all $d \geq D$ we have $A_d \cap V \neq \emptyset$.*
2. *$A_d \to A$ in $(\mathcal{P}(X)\setminus\{\emptyset\}, \tau_{UV})$ if and only if for every $U \in \tau$ with $A \subseteq U$ there is a $D \in \mathcal{D}$ such that for all $d \geq D$ we have $A_d \subseteq U$.*
3. *The following are equivalent:*
   - *3a) $A_d \to A$ in $(\mathcal{P}(X)\setminus\{\emptyset\}, \tau_V)$.*
   - *3b) $A_d \to A$ in l.v.t and $A_d \to A$ in u.v.t.*
   - *3c) For every $V \in \tau$ with $A \cap V \neq \emptyset$ and every $U \in \tau$ with $A \subseteq U$ there is a $D \in \mathcal{D}$ such that for all $d \geq D$ we have $A_d \cap V \neq \emptyset$ and $A_d \subseteq U$.*
4. *If $A_d \to A$ in the l.v.t then, $A_d \to \overline{A}$ and $\overline{A_d} \to \overline{A}$ in the l.v.t. Suppose that either X is normal or, X is regular and $\overline{A}$ is compact. If $A_d \to A$ in the u.v.t or v.t then, $A_d \to \overline{A}$ and $\overline{A_d} \to \overline{A}$ in the u.v.t or v.t respectively.*
5. *If $A_d \to A$ in the l.v.t and $C \subseteq A$ then, $A_d \to C$ in the l.v.t.*
6. *If $A_d \to A$ in the u.v.t and $C \supseteq A$ then, $A_d \to C$ in the u.v.t. Indeed, every net converges to X.*
7. *If $\{x_d\}_{d\in\mathcal{D}}$ is a net of X which converges to $x \in X$ then, the set net $A_d = \{x_d\}$ $d \in \mathcal{D}$ converges to $\{x\}$ in l.v.t, u.v.t and v.t.*
8. *If $A_d \to A$, $B_d \to B$ in the l.v.t, u.v.t or v.t then, the set net $A_d \cup B_d \to A \cup B$ in the l.v.t, u.v.t or v.t respectively.*
9. *If A, B are closed sets with $A \cap B \neq \emptyset$, $A_d \to A$, $B_d \to B$ in the u.v.t, $A_d \cap B_d \neq \emptyset, \forall d \in \mathcal{D}$, and either X is normal or, X is regular and both $A\setminus(A\cap B), B\setminus(A\cap B)$ have compact closure then, the set net $A_d \cap B_d \to A \cap B$ in the u.v.t.*
10. *Even if the hypothesis of Item 9 holds for the l.v.t or v.t then, it is possible that $\{A_d \cap B_d\}_{d\in\mathcal{D}}$ does not converge to $A \cap B$ in l.v.t or v.t, respectively.*

Proof. The proofs of Items 1 to 3 are a direct application of definitions and Proposition 2.1.5.

To prove Item 4 we have three cases to consider. Suppose that $A_d \to A$ in the l.v.t and V is an open set with $V \cap \overline{A} \neq \emptyset$ then, $A \cap V \neq \emptyset$ too and by Item 1 we have $A_d \cap V \neq \emptyset$ for all $d \geq D$ (by Item 1 this shows $A_d \to \overline{A}$ in l.v.t). Since $A_d \subseteq \overline{A_d}$, we have $\overline{A_d} \cap V \neq \emptyset$ for all $d \geq D$. Thus, $\overline{A_d} \to \overline{A}$ by Item 1.

Suppose $A_d \to A$ in the u.v.t. If X is normal or X is regular Hausdorff and $\overline{A}$ is compact then, for any open set $O \supseteq \overline{A}$ there is a $V \in \tau$ with $\overline{A} \subseteq V$ and $\overline{V} \subseteq O$. To see this consider $O \in \tau$ with $\overline{A} \subseteq O$ then, $X \setminus O$ and $\overline{A}$ are disjoint closed sets, when X is normal there are sets $V, W \in \tau$ with $V \cap W = \emptyset$ and $V \supseteq \overline{A}$, $W \supseteq X \setminus O$. We see that $V \subseteq X \setminus W$ by disjointness, since $X \setminus W$ is closed we have $\overline{V} \subseteq X \setminus W$, and $X \setminus W \subseteq O$ since, $W \supseteq X \setminus O$. Hence, $\overline{A} \subseteq V \subseteq \overline{V} \subseteq O$.

When X is instead regular and $\overline{A}$ is compact, let $O \supseteq \overline{A}$ with $O \in \tau$. Then, for each $a \in \overline{A}$ we have $a \notin X \setminus O$ and by regularity there are sets $V_a, W \in \tau$ with $V_a \cap W = \emptyset$ and $V_a \in \tau_a$, $W \supseteq X \setminus O$. Like in the case where X is normal we can argue that for all $a \in \overline{A}$ we have $\overline{V_a} \subseteq O$. Since $\bigcup_{a \in \overline{A}} V_a$ is an open cover of $\overline{A}$ there is a finite subcover, $V = \bigcup_{k=1}^{K} V_{a_k}$. As this is a finite union we have $\overline{V} = \bigcup_{k=1}^{K} \overline{V_{a_k}} \subseteq O$ and $\overline{A} \subseteq V$.

In either case $A \subseteq V$ and by Item 2 we know there is $D \in \mathcal{D}$ such that for all $d \geq D$ we have $A_d \subseteq V$ (by Item 2 this shows $A_d \to \overline{A}$ in u.v.t). Taking closures yields, $\overline{A_d} \subseteq \overline{V} \subseteq O$. Since O is an arbitrary open set with $\overline{A} \subseteq O$ we conclude $\overline{A_d} \to \overline{A}$ by Item 2.

The case where $A_d \to A$ in the v.t is an immediate consequence of the preceding arguments and Item 3b.

For Item 5, let $A_d \to A$ in the l.v.t and $C \subseteq A$. If $V \in \tau$ has $V \cap C \neq \emptyset$ then, $V \cap C \subseteq V \cap A \neq \emptyset$ too. Hence, there is a $D \in \mathcal{D}$ with for all $d \geq D$ such that $A_d \cap V \neq \emptyset$. Since $V \in \tau$ was an arbitrary open set with $V \cap C \neq \emptyset$, $A_d \to C$ by Item 1.

To prove Item 6, let $A_d \to A$ in the u.v.t and $C \supseteq A$. If $U \in \tau$ has $U \supseteq C$ then $U \supseteq C \supseteq A$ too. And there is a $D \in \mathcal{D}$ with for all $d \geq D$ such that $A_d \subseteq U$. Since U was an arbitrary open set with $U \supseteq C$ we have $A_d \to C$ in u.v.t, by Item 2. Notably, $X \supseteq A$ so $A_d \to X$, in this case.

But even if we do not assume a net $\{B_d\}_{d \in \mathcal{D}}$ converges we still have that $B_d \subseteq X$ for all $d \in \mathcal{D}$ and since the only open set containing X in (X, τ) is X we have that $B_d \to X$. So every net in $(\mathcal{P}(X) \setminus \{\emptyset\}, \tau_{UV})$ converges (and thus has a convergent subset). Therefore, $(\mathcal{P}(X) \setminus \{\emptyset\}, \tau_{UV})$ is compact.

For Item 7, suppose that $x_d \to x$ and consider any open set V with $x \in V$. It follows that $\{x\} \subseteq V$ and $V \cap \{x\} \neq \emptyset$. Furthermore, any open set U with either $\{x\} \subseteq U$ or

$U \cap \{x\} \neq \emptyset$ has $x \in U$. From here the result follows quickly from applying Items 1 to 3.

Item 8 has three cases. Firstly, assume $A_d \to A$, $B_d \to B$ in the l.v.t and let $V \in \tau$ have $V \cap (A \cup B) \neq \emptyset$. WLOG, we can assume that $V \cap A \neq \emptyset$ and for all large $d \in \mathcal{D}$ we have $\emptyset \neq A_d \cap V \subseteq (A_d \cup B_d) \cap V$. Thus, $A_d \cup B_d \to A \cup B$ in the l.v.t.

Secondly, assume that $A_d \to A$, $B_d \to B$ in the u.v.t and let $U \in \tau$ have $A \cup B \subseteq U$. Then, $A, B \subseteq U$ too. So for all large $d \in \mathcal{D}$ we have $A_d, B_d \subseteq U$ and hence $A_d \cup B_d \subseteq U$. Therefore, $A_d \cup B_d \to A \cup B$ in the u.v.t.

The last case where $A_d \to A$, $B_d \to B$ in the v.t follows from the preceding arguments and Item 3b.

For Item 9 we assume that $A \cap B \neq \emptyset$, $A_d \cap B_d \neq \emptyset, \forall d \in \mathcal{D}$, and $A_d \to A$, $B_d \to B$ in the u.v.t. Let $W \in \tau$ have $A \cap B \subseteq W$, then $A \setminus W, B \setminus W$ are closed disjoint sets.

One can show that when X is normal or, X is realgar and $A \setminus (A \cap B), B \setminus (A \cap B)$ have compact closure, that there are open sets disjoint open sets V, U with $V \supseteq A \setminus W$ and $U \supseteq B \setminus W$.

We have that $V \cup W \supseteq A$ and $V \cup W \supseteq B$, furthermore one can show $A \cap B \subseteq (V \cup W) \cap (U \cup W) \subseteq W$, since U, V are disjoint. Since, $A_d \to A, B_d \to B$ in u.v.t then, there is a $D \in \mathcal{D}$ for all $d \geq D$ with $A_d \subseteq V \cup W$ and $B_d \subseteq U \cup W$. Thus, $A_d \cap B_d \subseteq (V \cup W) \cap (U \cup W) \subseteq W$ for all $d \geq D$. Therefore, $A_d \cap B_d \to A \cap B$ in u.v.t.

Finally, for Item 10 let $\mathcal{D} = \mathbb{N}$, $X = \mathbb{R}$ with the usual topology and define

$$A_n = \left\{\frac{1}{n}, 1\right\}$$
$$B_n = \left\{\frac{-1}{n}, 1\right\}$$
$$A = B = \{0, 1\}.$$

Note, A_n, B_n, A, B are compact for all $n \in \mathbb{N}$ and that X is normal. Then, $A_n \to A$ and $B_n \to B$ but $A_n \cap B_n \to \{1\}$ in the l.v.t, u.v.t and v.t. Furthermore, in l.v.t, v.t $A_n \cap B_n \not\to A \cap B$; Note, $A_n \cap B_n \to A \cap B$ in the u.v.t by Item 6.

□

One who followed the proof of Proposition 2.2.4 would have found that Items 1 to 3 are the practical ways we show convergence in the Vietoris topologies. Onwards, we may not refer back to this theorem when showing convergence in the Vietoris topologies. Item 4 shows that the Vietoris topologies have a hard time distinguishing sets and their closure, much like the Hausdorff metric. Item 7 is a sanity check, if it did not hold then, the Vietoris topologies would have questionable usefulness for approximations. Items 8 to 10 are explorations of set convergence with some typical set operations, unions do not cause any trouble but intersections cause issues—even when we make some unwarranted assumptions to prevent the net from being nonempty.

Items 5 and 6 are interesting. Firstly, they more explicitly describe how the l.v.t and u.v.t do not have unique limits. The u.v.t in particular is shown to be highly degenerate, what with every net/sequence converging. However, both Items 5 and 6 suggest an order of the possible limits of a net. In particular for the l.v.t Item 5 suggests there is a largest limit of a net. Similarly, Item 6 suggests there is a smallest limit of a net in the u.v.t. This leads into the idea of Kuratowski convergence of sets.

Definition 2.2.2. *Let (X, τ) be a topological space. Let $\{A_d\}_{d\in\mathcal{D}}$ be a set net of X.*

The lower Kuratowski limit, lower limit, inner limit or liminf of the set net $\{A_d\}_{d\in\mathcal{D}}$ is defined to be

$$\operatorname*{Li}_{d\in\mathcal{D}} A_d := \{x \in X : \forall V \in \tau_x \, \exists D \in \mathcal{D} \, \forall d \geq D \text{ such that } V \cap A_d \neq \emptyset\}.$$

The upper Kuratowski limit, upper limit, outer limit or limsup of the net $\{A_d\}_{d\in\mathcal{D}}$ is defined to be

$$\operatorname*{Ls}_{d\in\mathcal{D}} A_d := \{x \in X : \forall V \in \tau_x \, \forall D \in \mathcal{D} \, \exists d \geq D \text{ such that } V \cap A_d \neq \emptyset\}.$$

The Kuratowski limit or K-limit of the net $\{A_d\}_{d\in\mathcal{D}}$ is defined to be

$$\operatorname*{KLim}_{d\in\mathcal{D}} A_d := \operatorname*{Ls}_{d\in\mathcal{D}} A_d = \operatorname*{Li}_{d\in\mathcal{D}} A_d$$

whenever, the above equality holds.

The Kuratowski limits are more immediately comprehensible than the Vietoris topologies. It's nice that each Kuratowski limit is a uniquely defined set, unlike limits in the Vietoris topologies. It is worthwhile to establish a number of basic results about Kuratowski limits.

Proposition 2.2.5. *Let (X, τ) be a topological space and $\{A_d\}_{d\in\mathcal{D}}$,$\{B_d\}_{d\in\mathcal{D}}$ be set nets of X. Let $\mathfrak{U}$ be the set of all open covers of X. For a set $A \subseteq X$ and $\mathcal{U} \in \mathfrak{U}$ define $\mathcal{U}_A = \{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. Assume that whenever the* KLim *of a net is mentioned in an equation below that the* KLim *of the net exists, unless otherwise noted. The following hold:*

1. *Suppose that X is regular then,*
   $\operatorname*{Li}_{d\in\mathcal{D}} A_d = \bigcap_{\mathcal{U}\in\mathfrak{U}} \bigcup_{D\in\mathcal{D}} \bigcap_{d\geq D} \bigcup_{U\in\mathcal{U}_{A_d}} U$ *and*
   $\operatorname*{Ls}_{d\in\mathcal{D}} A_d = \bigcap_{\mathcal{U}\in\mathfrak{U}} \bigcap_{D\in\mathcal{D}} \bigcup_{d\geq D} \bigcup_{U\in\mathcal{U}_{A_d}} U.$

2. $\operatorname*{Ls}_{d\in\mathcal{D}} A_d = \bigcap_{D\in\mathcal{D}} \overline{\bigcup_{d\geq D} A_d}.$

3. $\operatorname*{Li}_{d\in\mathcal{D}} A_d = \operatorname*{Li}_{d\in\mathcal{D}} \overline{A_d}$ *and* $\operatorname*{Ls}_{d\in\mathcal{D}} A_d = \operatorname*{Ls}_{d\in\mathcal{D}} \overline{A_d}.$

4. $\operatorname*{Li}_{d\in\mathcal{D}} A_d \subseteq \operatorname*{Ls}_{d\in\mathcal{D}} A_d.$

5. $\operatorname*{KLim}_{d\in\mathcal{D}} A_d$ *exists if and only if* $\operatorname*{Li}_{d\in\mathcal{D}} A_d \supseteq \operatorname*{Ls}_{d\in\mathcal{D}} A_d.$

6. $\operatorname*{KLim}_{d\in\mathcal{D}} A_d = A$ *if and only if* $A \subseteq \operatorname*{Li}_{d\in\mathcal{D}} A_d$ *and* $\operatorname*{Ls}_{d\in\mathcal{D}} A_d \subseteq A.$

7. *If there is a $D \in \mathcal{D}$ such that $B_d \subseteq A_d$ for all $d \geq D$ then,* $\operatorname*{Li}_{d\in\mathcal{D}} B_d \subseteq \operatorname*{Li}_{d\in\mathcal{D}} A_d$, $\operatorname*{Ls}_{d\in\mathcal{D}} B_d \subseteq \operatorname*{Ls}_{d\in\mathcal{D}} A_d$ *and* $\operatorname*{KLim}_{d\in\mathcal{D}} B_d \subseteq \operatorname*{KLim}_{d\in\mathcal{D}} A_d.$

8. *Let I be a set an index set and for each $i \in I$ let $\{A_d^i\}_{d\in\mathcal{D}}$ be a set net of X. Then,*

   8a) $\bigcup_{i\in I} \operatorname*{Li}_{d\in\mathcal{D}} A_d^i \subseteq \operatorname*{Li}_{d\in\mathcal{D}} \bigcup_{i\in I} A_d^i$, $\bigcup_{i\in I} \operatorname*{Ls}_{d\in\mathcal{D}} A_d^i \subseteq \operatorname*{Ls}_{d\in\mathcal{D}} \bigcup_{i\in I} A_d^i$ *and* $\bigcup_{i\in I} \operatorname*{KLim}_{d\in\mathcal{D}} A_d^i \subseteq \operatorname*{KLim}_{d\in\mathcal{D}} \bigcup_{i\in I} A_d^i.$
   If I is finite then, $\bigcup_{i\in I} \operatorname*{Ls}_{d\in\mathcal{D}} A_d^i = \operatorname*{Ls}_{d\in\mathcal{D}} \bigcup_{i\in I} A_d^i$ *and* $\bigcup_{i\in I} \operatorname*{KLim}_{d\in\mathcal{D}} A_d^i = \operatorname*{KLim}_{d\in\mathcal{D}} \bigcup_{i\in I} A_d^i.$

   8b) $\operatorname*{Li}_{d\in\mathcal{D}} \bigcap_{i\in I} A_d^i \subseteq \bigcap_{i\in I} \operatorname*{Li}_{d\in\mathcal{D}} A_d^i$, $\operatorname*{Ls}_{d\in\mathcal{D}} \bigcap_{i\in I} A_d^i \subseteq \bigcap_{i\in I} \operatorname*{Ls}_{d\in\mathcal{D}} A_d^i$ *and* $\operatorname*{KLim}_{d\in\mathcal{D}} \bigcap_{i\in I} A_d^i \subseteq \bigcap_{i\in I} \operatorname*{KLim}_{d\in\mathcal{D}} A_d^i.$

9. *If for all $d, k \in \mathcal{D}$ with $d \geq k$ we have $A_d \supseteq A_k$ then,* $\operatorname*{KLim}_{d\in\mathcal{D}} A_d$ *exists and is equal to* $\overline{\bigcup_{d\in\mathcal{D}} A_d}.$

10. *If for all $d, k \in \mathcal{D}$ with $d \geq k$ we have $A_d \subseteq A_k$ then,* $\operatorname*{KLim}_{d\in\mathcal{D}} A_d$ *exists and is equal to* $\bigcap_{d\in\mathcal{D}} \overline{A_d}.$

11. *If* $\{A_{d_j}\}_{j\in\mathcal{J}}$ *is a subnet of* $\{A_d\}_{d\in\mathcal{D}}$ *then,* $\operatorname*{Li}_{d\in\mathcal{D}} A_d \subseteq \operatorname*{Li}_{j\in\mathcal{J}} A_{d_j}$ *and* $\operatorname*{Ls}_{d\in\mathcal{D}} A_d \supseteq \operatorname*{Ls}_{j\in\mathcal{J}} A_{d_j}$. *Additionally, if* $\operatorname*{KLim}_{d\in\mathcal{D}} A_d$ *exists then,* $\operatorname*{KLim}_{d\in\mathcal{D}} A_d = \operatorname*{KLim}_{j\in\mathcal{J}} A_{d_j}$.

Proof. To prove Item 1 let $P_{\text{Li}} = \bigcap_{\mathcal{U}\in\mathfrak{U}} \bigcup_{D\in\mathcal{D}} \bigcap_{d\geq D} \bigcup_{U\in\mathcal{U}_{A_d}} U$ and $x \in \text{Li}_{d\in\mathcal{D}} A_d$ then, given a $\mathcal{U} \in \mathfrak{U}$ we have that there is a $U \in \mathcal{U}$ with $x \in U$. Since $x \in \text{Li}_{d\in\mathcal{D}} A_d$ there is a $D \in \mathcal{D}$ for all $d \geq D$ such that $U \cap A_d \neq \emptyset$ and $U \in \mathcal{U}_{A_d}$ for all $d \geq D$. Or in other symbols, $x \in \bigcup_{D\in\mathcal{D}} \bigcap_{d\geq D} \bigcup_{U\in\mathcal{U}_{A_d}} U$ for any $\mathcal{U} \in \mathfrak{U}$. Therefore, $x \in P_{\text{Li}}$.

Conversely, if $x \in P_{\text{Li}}$, let $V \in \tau_x$ then, since X is regular one can show that there is an open $W \in \tau_x$ with $\overline{W} \subseteq V$. Define,

$$\mathcal{U} = \left\{X \setminus \overline{W}, V\right\}.$$

One can see that $\mathcal{U}$ is an open cover of X, by noting that $X = \overline{W} \cup X \setminus \overline{W} \subseteq V \cup X \setminus \overline{W}$. Since $x \in P_{\text{Li}}$ we have there is $D \in \mathcal{D}$ for all $d \geq D$ there is a $U \in \mathcal{U}_{A_d}$ with $x \in U$. By definition of $\mathcal{U}$ we have $U = V$. Further, since $V \in \mathcal{U}_{A_d}$ we have $\emptyset \neq V \cap A_d$. Hence, $x \in \text{Li}_{d\in\mathcal{D}} A_d$.

Now let $P_{\text{Ls}} = \bigcap_{\mathcal{U}\in\mathfrak{U}} \bigcap_{D\in\mathcal{D}} \bigcup_{d\geq D} \bigcup_{U\in\mathcal{U}_{A_d}} U$. If $x \in \text{Ls}_{d\in\mathcal{D}} A_d$ then, given a $\mathcal{U} \in \mathfrak{U}$ we have that there is a $U \in \mathcal{U}$ with $x \in U$. Since $x \in \text{Ls}_{d\in\mathcal{D}} A_d$ for all $D \in \mathcal{D}$ there is a $d \geq D$ such that $U \cap A_d \neq \emptyset$ and so $U \in \mathcal{U}_{A_d}$. Or in other symbols, $x \in \bigcap_{D\in\mathcal{D}} \bigcup_{d\geq D} \bigcup_{U\in\mathcal{U}_{A_d}} U$ for any $\mathcal{U} \in \mathfrak{U}$. Therefore, $x \in P_{\text{Ls}}$.

Conversely, if $x \in P_{\text{Ls}}$, let $V \in \tau_x$ then, like before, there is a $\mathcal{U} \in \mathfrak{U}$ with if $U \in \mathcal{U}$ with $x \in U$ then, $U = V$. Since $x \in P_{\text{Ls}}$ we have for all $D \in \mathcal{D}$ there is a $d \geq D$ and a $U \in \mathcal{U}_{A_d}$ with $x \in U$. So, $x \in V \cap U$ and by assumed property of $\mathcal{U}$ we have $U = V$. Further, since $V \in \mathcal{U}_{A_d}$ we have $\emptyset \neq V \cap A_d$. Hence, $x \in \text{Ls}_{d\in\mathcal{D}} A_d$.

For Item 2, we seek to prove

$$\operatorname*{Ls}_{d\in\mathcal{D}} A_d = \bigcap_{D\in\mathcal{D}} \overline{\bigcup_{d\geq D} A_d}.$$

Let $x \in \text{Ls}_{d\in\mathcal{D}} A_d$, then for all $V \in \tau_x$ and every $D \in \mathcal{D}$ there is a $d \geq D$ with $V \cap A_d \neq \emptyset$. Thus, $V \cap \bigcup_{d\geq D} A_d \neq \emptyset$ for every $D \in \mathcal{D}$ and $V \in \tau_x$. So, $x \in \overline{\bigcup_{d\geq D} A_d}$ for all $D \in \mathcal{D}$ and $x \in \bigcap_{D\in\mathcal{D}} \overline{\bigcup_{d\geq D} A_d}$.

Conversely, when $x \in \bigcap_{D\in\mathcal{D}} \overline{\bigcup_{d\geq D} A_d}$ then for all $D \in \mathcal{D}$ we have $x \in \overline{\bigcup_{d\geq D} A_d}$. By a characterization of the closure, for all $V \in \tau_x$ we have $V \cap \bigcup_{d\geq D} A_d \neq \emptyset$ for all $D \in \mathcal{D}$. Hence, for all $D \in \mathcal{D}$ there is a $d \geq D$ with $V \cap A_d \neq \emptyset$. So $x \in \text{Ls}_{d\in\mathcal{D}} A_d$.

The proof of Item 3 follows immediately from the fact: for any $V \in \tau$ and $A \subseteq X$ we have $V \cap A \neq \emptyset \iff V \cap \overline{A} \neq \emptyset$.

To prove Item 4, let $x \in \mathrm{Li}_{d\in\mathcal{D}} A_d$, $V \in \tau_x$ and let $D \in \mathcal{D}$ be arbitrary. As $x \in \mathrm{Li}_{d\in\mathcal{D}} A_d$ we have that there is an $E \in \mathcal{D}$ for all $e \geq E$ with $A_e \cap V \neq \emptyset$. Since $\mathcal{D}$ is directed there is a $d \in \mathcal{D}$ with $d \geq E$ and $d \geq D$. As $d \geq E$ we have $A_d \cap V \neq \emptyset$. In summary, for any $V \in \tau_x$ and any $D \in \mathcal{D}$ there is a $d \geq D$ with $A_d \cap V \neq \emptyset$, so $x \in \mathrm{Ls}_{d\in\mathcal{D}} A_d$.

Item 5 follows immediately from the definition and Item 4.

Similarly, Item 6 follows from definitions and Item 5.

All inclusions of Item 7 follow quickly from elementary properties of directed sets, Items 1 and 5, and the fact that if $B \subseteq A$ then if $V \in \tau$ has $V \cap A \neq \emptyset$ then, $V \cap B \neq \emptyset$ too.

The "$\subseteq$" inclusions of Item 8a follow from if $V \cap A^j \neq \emptyset$ for some $j \in I$ then $V \cap (\bigcup_{i\in I} A^i) \neq \emptyset$. All quantification on $\mathcal{D}$ is goes through unhindered. The case where I is finite is a little more interesting, let $x \in \mathrm{Ls}_{d\in\mathcal{D}} \bigcup_{i\in I} A^i_d$ then for all $V \in \tau_x$, every $D \in \mathcal{D}$ there is a $d \geq D$ and an $i_d \in I$ with $A^{i_d}_d \cap V \neq \emptyset$. Note that if $\mathcal{D}$ in finite the result holds (also, who cares about this case?), so assume $\mathcal{D}$ is infinite. Then, there is a $j \in I$ with $i_d = j$ for infinitely many $d \in \mathcal{D}$, $d \geq D$. Ergo, $A^j_d \cap V \neq \emptyset$ for some $d \geq D$ and so $x \in \mathrm{Ls}_{d\in\mathcal{D}} A^j_d \subseteq \bigcup_{i\in I} \mathrm{Ls}_{d\in\mathcal{D}} A^i_d$, as required. Therefore, $\bigcup_{i\in I} \mathrm{Ls}_{d\in\mathcal{D}} A^i_d = \mathrm{Ls}_{d\in\mathcal{D}} \bigcup_{i\in I} A^i_d$. The same equality with KLim holds since if the KLim exists it is equal to the Ls.

For Item 8b, let $x \in \mathrm{Li}_{d\in\mathcal{D}} \bigcap_{i\in I} A^i_d$ then for any $V \in \tau_x$ there is a $D \in \mathcal{D}$ such that for all $d \geq D$ we have $V \cap (\bigcap_{i\in I} A^i_d) \neq \emptyset$. Thus for every $i \in I$ and all $d \geq D$ we have $V \cap A^i_d \neq \emptyset$ too. So $x \in \mathrm{Li}_{d\in\mathcal{D}} A^i_d$ for all $i \in I$, which is the result. The case for the limsup and KLim are very similar.

To prove Item 9, we will use Item 6 to show convergence. To show $\mathrm{Li}_{d\in\mathcal{D}} A_d \supseteq \overline{\bigcup_{d\in\mathcal{D}} A_d}$, suppose that $x \in \overline{\bigcup_{d\in\mathcal{D}} A_d}$ then, for all $V \in \tau_x$ we have $V \cap (\bigcup_{d\in\mathcal{D}} A_d) \neq \emptyset$. So, there is a $D \in \mathcal{D}$ with $V \cap A_D \neq \emptyset$ and since for all $d \geq D$ we have $A_d \supseteq A_D$ then, $V \cap A_d \neq \emptyset$ for all $d \geq D$. Therefore, $x \in \mathrm{Li}_{d\in\mathcal{D}} A_d = \mathrm{KLim}_{d\in\mathcal{D}} A_d$ and $\mathrm{Li}_{d\in\mathcal{D}} A_d \supseteq \overline{\bigcup_{d\in\mathcal{D}} A_d}$. The other inclusion follows from

$$\operatorname*{Ls}_{d\in\mathcal{D}} A_d = \bigcap_{D\in\mathcal{D}} \overline{\bigcup_{d\geq D} A_d} \subseteq \overline{\bigcup_{d\in\mathcal{D}} A_d}$$

recalling Item 2. Therefore, $\mathrm{KLim}_{d\in\mathcal{D}} A_d = \overline{\bigcup_{d\in\mathcal{D}} A_d}$.

Now we consider Item 10. Again we use Item 6. Let $x \in \bigcap_{d\in\mathcal{D}} \overline{A_d}$ then, for all $V \in \tau_x$ we have $V \cap (\bigcap_{d\in\mathcal{D}} \overline{A_d}) \neq \emptyset$. So, for all $d \in \mathcal{D}$ we have $V \cap \overline{A_d} \neq \emptyset \iff V \cap A_d \neq \emptyset$. Picking a fixed (but arbitrary) $D \in \mathcal{D}$ and considering any $d \geq D$ we still have $V \cap A_d \neq \emptyset$.

Thus $x \in \operatorname{Li}_{d\in\mathcal{D}} A_d$ and $\bigcap_{d\in\mathcal{D}} \overline{A_d} \subseteq \operatorname{Li}_{d\in\mathcal{D}} A_d$. For the other inclusion, notice that since $A_D \supseteq A_d$ for all $d \geq D$ and $D \in \mathcal{D}$ we have $\bigcup_{d\geq D} A_d = A_D$. So we can see that

$$\operatorname*{Ls}_{d\in\mathcal{D}} A_d = \bigcap_{D\in\mathcal{D}} \overline{\bigcup_{d\geq D} A_d} = \bigcap_{D\in\mathcal{D}} \overline{A_D}.$$

By Item 6 $\operatorname{KLim}_{d\in\mathcal{D}} A_d = \overline{\bigcap_{d\in\mathcal{D}} A_d}$.

Lastly, we prove Item 11. Let $x \in \operatorname{Li}_{d\in\mathcal{D}} A_d$ then for all $V \in \tau_x$ there is a $D \in \mathcal{D}$ such that for all $d \geq D$ we have $V \cap A_d \neq \emptyset$. Since $\{A_{d_j}\}_{j\in\mathcal{J}}$ is a subnet of $\{A_d\}_{d\in\mathcal{D}}$ there is a $J \in \mathcal{J}$ such that for all $j \geq J$ we have $d_j \geq D$ (by definition of a subnet, see Definition 2.1.6), so $V \cap A_{d_j} \neq \emptyset$ for all $j \geq J$ and $x \in \operatorname{Li}_{j\in\mathcal{J}} A_{d_j}$. Therefore, $\operatorname{Li}_{d\in\mathcal{D}} A_d \subseteq \operatorname{Li}_{j\in\mathcal{J}} A_{d_j}$.

For the inclusion $\operatorname{Ls}_{d\in\mathcal{D}} A_d \supseteq \operatorname{Ls}_{j\in\mathcal{J}} A_{d_j}$, consider a $x \in \operatorname{Ls}_{j\in\mathcal{J}} A_{d_j}$ then, pick any $V \in \tau_x$ and $D \in \mathcal{D}$ let $J \in \mathcal{J}$ have the property if $j \geq J$ then $d_j \geq D$. Since $x \in \operatorname{Ls}_{j\in\mathcal{J}} A_{d_j}$ there is $j \geq J$ with $A_{d_j} \cap V \neq \emptyset$ and $d_j \geq D$ by choice of J. Since $D \in \mathcal{D}$ and $V \in \tau_x$ are arbitrary we have that $x \in \operatorname{Ls}_{d\in\mathcal{D}} A_d$. Therefore, $\operatorname{Ls}_{d\in\mathcal{D}} A_d \supseteq \operatorname{Ls}_{j\in\mathcal{J}} A_{d_j}$.

Finally, suppose that $\operatorname{KLim}_{d\in\mathcal{D}} A_d$ exists then

$$\operatorname*{Ls}_{j\in\mathcal{J}} A_{d_j} \subseteq \operatorname*{Ls}_{d\in\mathcal{D}} A_d = \operatorname*{KLim}_{d\in\mathcal{D}} A_d = \operatorname*{Li}_{d\in\mathcal{D}} A_d \subseteq \operatorname*{Li}_{j\in\mathcal{J}} A_{d_j}$$

by definition of KLim and the other inclusion proved in for this item. Therefore, by Item 6 $\operatorname{KLim}_{d\in\mathcal{D}} A_d = \operatorname{KLim}_{j\in\mathcal{J}} A_{d_j}$. □

The nice thing about Kuratowski limits is that they are uniquely defined, unlike with limits in the Vietoris topologies. It is also a great convenience that the sets in the nets in Proposition 2.2.5 can be empty, we can even have convergence to the empty set. For example the set sequence $A_n = [n, \infty)$ for $n \in \mathbb{N}$ is decreasing so by Item 10 converges to $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$.

We can also see that the provocative names of the limsup and liminf are not entirely for show. The limsup/liminf of sets bear a number of similarities to the limsup/liminf of real numbers, especially on the extended real numbers. For example Item 4 of Proposition 2.2.5 says the limsup of sets is larger than the liminf of set, the same is true for real numbers. Similarly, Item 5 of Proposition 2.2.5 says that the only time a net converges is when the liminf is larger than the limsup. Even, Items 8 to 10 have an analogous result for real numbers, one can interpret the union of sets as the sup/max of numbers and the intersection of sets as the inf/min of numbers. Interestingly, one can show that if $\{r_d\}_{d\in\mathcal{D}}$ is a net of extended real numbers then, $\liminf_{d\in\mathcal{D}} r_d = \inf\{x : x \in \operatorname{Ls}_{d\in\mathcal{D}}\{r_d\}\}$ and $\limsup_{d\in\mathcal{D}} r_d = \sup\{x : x \in \operatorname{Ls}_{d\in\mathcal{D}}\{r_d\}\}$.

It turns out that limits in the Vietoris topologies and Kuratowski limits are closely related and often times interchangeable; Especially, with sufficient compactness of some sets.

Proposition 2.2.6. *Let (X,τ) be a topological space and $\{A_d\}_{d\in\mathcal{D}}$,$\{B_d\}_{d\in\mathcal{D}}$ be nonempty set nets of X. Let $\lim^{LV}_{d\in\mathcal{D}} A_d, \lim^{UV}_{d\in\mathcal{D}} A_d$ and $\lim^{V}_{d\in\mathcal{D}} A_d$ denote the set of limits of the net $\{A_d\}_{d\in\mathcal{D}}$ with respect to the l.v.t, u.v.t and Vietoris topology respectively in $\mathcal{P}(X)\setminus\{\emptyset\}$. Then, the following hold:*

1. *If $\{A_d\}_{d\in\mathcal{D}}$ converges in l.v.t then, $\operatorname*{Li}_{d\in\mathcal{D}} A_d \in \lim_{d\in\mathcal{D}}{}^{LV} A_d$ and $\operatorname*{Li}_{d\in\mathcal{D}} A_d$ is the largest set in $\lim_{d\in\mathcal{D}}{}^{LV} A_d$. Moreover, if $\operatorname*{Li}_{d\in\mathcal{D}} A_d \neq \emptyset$ then, $\lim_{d\in\mathcal{D}}{}^{LV} A_d \neq \emptyset$.*

2. *Suppose that X is regular then, $\operatorname*{Ls}_{d\in\mathcal{D}} A_d = \bigcap\left\{\overline{C} : C \in \lim_{d\in\mathcal{D}}{}^{UV} A_d\right\}$. Consequently, if $\operatorname*{Ls}_{d\in\mathcal{D}} A_d \in \lim_{d\in\mathcal{D}}{}^{UV} A_d$ then $\operatorname*{Ls}_{d\in\mathcal{D}} A_d$ is the smallest closed set in $\lim_{d\in\mathcal{D}}{}^{UV} A_d$. It is possible for $\operatorname*{Ls}_{d\in\mathcal{D}} A_d \notin \lim_{d\in\mathcal{D}}{}^{UV} A_d$.*

3. *If K is a compact set and $A_d \to K$ in the u.v.t then, $A_d \to \operatorname*{Ls}_{d\in\mathcal{D}} A_d$ in the u.v.t (so, $\operatorname*{Ls}_{d\in\mathcal{D}} A_d \in \lim_{d\in\mathcal{D}}{}^{UV} A_d$).*

4. *If K is a compact set and $A_d \to K$ in the u.v.t then, $\emptyset \neq K \cap \operatorname*{Ls}_{d\in\mathcal{D}} A_d$. When X is also regular we have $\operatorname*{Ls}_{d\in\mathcal{D}} A_d \subseteq \overline{K}$.*

5. *If $A_d \to \operatorname*{Li}_{d\in\mathcal{D}} A_d$ in the u.v.t then, $A_d \to \operatorname*{Li}_{d\in\mathcal{D}} A_d$ in the v.t.*

6. *If X is regular and $A_d \to A$ in v.t then, $\operatorname*{KLim}_{d\in\mathcal{D}} A_d = \overline{A}$.*

Proof. To prove 1 suppose that $A_d \to A$ for $A \subseteq X$, $A \neq \emptyset$. Let $a \in A$ and $V \in \tau_a$ then, $V \cap A \neq \emptyset$ and by Item 1 of Proposition 2.2.4 we have that there is a $D \in \mathcal{D}$ such that for all $d \geq D$ we have $V \cap A_d \neq \emptyset$. But this holds for all $V \in \tau_a$, so by definition $a \in \operatorname{Li}_{d\in\mathcal{D}} A_d$. Therefore, $\emptyset \neq A \subseteq \operatorname{Li}_{d\in\mathcal{D}} A_d$ and if $A_d \to \operatorname{Li}_{d\in\mathcal{D}} A_d$ in the l.v.t then, $\operatorname{Li}_{d\in\mathcal{D}} A_d$ is the largest such set.

To show $A_d \to \operatorname{Li}_{d\in\mathcal{D}} A_d$ in the l.v.t, suppose that $V \in \tau$ with $V \cap \operatorname{Li}_{d\in\mathcal{D}} A_d \neq \emptyset$. Then, there is $x \in \operatorname{Li}_{d\in\mathcal{D}} A_d \cap V$ and so $V \in \tau_x$. Since $x \in \operatorname{Li}_{d\in\mathcal{D}} A_d$ we have there is a $D \in \mathcal{D}$ such that for all $d \geq D$ we have $V \cap A_d \neq \emptyset$. And again by Item 1 of Proposition 2.2.4 this means that $A_d \to \mathrm{Li}_{d\in\mathcal{D}} A_d$ in the l.v.t. Note this also shows that if $\mathrm{Li}_{d\in\mathcal{D}} A_d \neq \emptyset$ then, ${\lim_{d\in\mathcal{D}}}^{LV} A_d \neq \emptyset$.

For 2, let $C \in \lim^{UV}_{d\in\mathcal{D}} A_d$ then by Item 2 of Proposition 2.2.4 for all $O \in \tau$ with $O \supseteq C$ there is a $D \in \mathcal{D}$ such that for all $d \geq D$ we have $A_d \subseteq O$. In particular we have that $\bigcup_{d\geq D} A_d \subseteq O$ and we see

$$\mathop{\mathrm{Ls}}_{d\in\mathcal{D}} A_d = \bigcap_{D\in\mathcal{D}} \overline{\bigcup_{d\geq D} A_d} \subseteq \overline{O}$$

by Item 2 of Proposition 2.2.5. This holds for all open $O \supseteq C$ and since

$$\overline{C} = \bigcap_{O\in\tau, O\supseteq C} \overline{O}$$

by regularity[3], we see that $\mathrm{Ls}_{d\in\mathcal{D}} A_d \subseteq \overline{C}$. This shows that $\mathrm{Ls}_{d\in\mathcal{D}} A_d \subseteq \bigcap\{\overline{C} : C \in \lim^{UV}_{d\in\mathcal{D}} A_d\}$. Now suppose that $x \notin \mathrm{Ls}_{d\in\mathcal{D}} A_d$ then by definition we know there is $V \in \tau_x$ and a $D \in \mathcal{D}$ with for all $d \geq D$ we have $A_d \cap V = \emptyset$. Thus, we have $A_d \subseteq X \setminus V$ for $d \geq D$. But from this we see that $A_d \to X \setminus V$ in the u.v.t, as any open set $O \supseteq X \setminus V$ will have $A_d \subseteq X \setminus V \subseteq O$ for all $d \geq D$. Thus $\{A_d\}_{d\in\mathcal{D}}$ convergences to the closed set $X \setminus V \not\ni x$ and $x \notin \bigcap\{\overline{C} : C \in \lim^{UV}_{d\in\mathcal{D}} A_d\}$.

It is possible for $\mathrm{Ls}_{d\in\mathcal{D}} A_d \notin \lim^{UV}_{d\in\mathcal{D}} A_d$. Consider the set sequence of $\mathbb{R}$ with the usual topology: $A_n = \{0\} \cup \{n\}$ for $n \in \mathbb{N}$. Then $\{0\} = \mathrm{Ls}_{n\in\mathbb{N}} A_n$. And any set A with $A_n \to A$ in the u.v.t has $[N, \infty) \cap \mathbb{N}$ as a subset for some $N \in \mathbb{N}$ (As $\bigcup_{n\geq N} A_n$ contains that set). Thus A_n does not converge to $\{0\}$ in the u.v.t.

To prove 3, let $O \in \tau$ with $O \supseteq \mathrm{Ls}_{d\in\mathcal{D}} A_d$. For all $x \in X \setminus \mathrm{Ls}_{d\in\mathcal{D}} A_d$ there is a $V_x \in \tau_x$ and a $D_x \in \mathcal{D}$ such that for all $d \geq D_x$ we have $A_d \cap V_x = \emptyset$. Define

$$\mathcal{V} = \left\{ V_x : x \in X \setminus \mathop{\mathrm{Ls}}_{d\in\mathcal{D}} A_d \right\} \cup \{O\},$$

it can be seen that $\mathcal{V}$ is an open cover of X; by noting that $X = \mathrm{Ls}_{d\in\mathcal{D}} A_d \cup X \setminus \mathrm{Ls}_{d\in\mathcal{D}} A_d$. Thus, $\mathcal{V}$ is an open cover of K and so there is finite subcover, $K \subseteq O \cup \bigcup_{n=1}^N V_{x_n}$ for some $N \in \mathbb{N}$.

Since, $A_d \to K$ in the u.v.t there is a $D_{\mathcal{V}} \in \mathcal{D}$ with for all $d \geq D_{\mathcal{V}}$ we have $A_d \subseteq O \cup \bigcup_{n=1}^N V_{x_n}$. Now let $D \in \mathcal{D}$ have $D \geq D_{\mathcal{V}}, D_{x_1}, D_{x_2}, \ldots, D_{x_N}$ then, for all $d \geq D$ we

[3] To show this equality, we need only prove $x \notin \overline{C} \implies x$ not in the intersection. When the space is regular and $x \notin \overline{C}$ there are open sets W, V with $W \supseteq \overline{C}$ and $V \ni x$ with $W \cap V = \emptyset$. But this means $x \notin \overline{W}$ and also note that $W \supseteq C$. So x is not in the intersection.

have

$$A_d \subseteq O \cup \bigcup_{n=1}^{N} V_{x_n} \text{ and } A_d \cap \left(\bigcup_{n=1}^{N} V_{x_n} \right) = \emptyset.$$

It follows that for all $d \geq D$ we have $A_d \subseteq O$. Therefore, $A_d \to \mathrm{Ls}_{d\in\mathcal{D}} A_d$ in the u.v.t.

Now consider 4. Proceed by contraposition, so suppose that $\mathrm{Ls}_{d\in\mathcal{D}} A_d \cap K = \emptyset$. It follows that, $K \subseteq X \setminus \mathrm{Ls}_{d\in\mathcal{D}} A_d$ and for all $y \in K$ there is $V_y \in \tau_y$, a $D_y \in \mathcal{D}$ such that for all $d \geq D_y$ we have $A_d \cap V_y = \emptyset$.

Hence, the sets $\{V_y : y \in K\}$ cover K, and by compactness there is a finite subcover. Let $y_1, y_2, \ldots, y_N \in K$ with $K \subseteq \bigcup_{n=1}^{N} V_{y_n}$ and let $D \geq D_{y_1}, \ldots, D_{y_N}$. Then, for all $d \geq D$ we have that $A_d \cap \bigcup_{n=1}^{N} V_{y_n} = \emptyset$. Meaning that $A_d \not\subseteq \bigcup_{n=1}^{N} V_{y_n}$, for all $d \geq D$. It can be shown that this precludes $A_d \to K$ in the u.v.t, since $\bigcup_{n=1}^{N} V_{y_n} \supseteq K$ and Item 2 of Proposition 2.2.4.

When X is regular and $A_d \to K$ in the u.v.t, we can apply Item 2 of this theorem to conclude that $\mathrm{Ls}_{d\in\mathcal{D}} A_d \subseteq \overline{K}$.

Now we show 5. Assume that $A_d \to \mathrm{Li}_{d\in\mathcal{D}} A_d$ in the u.v.t. It follows that $\mathrm{Li}_{d\in\mathcal{D}} A_d \neq \emptyset$. By 1 we have that $A_d \to \mathrm{Li}_{d\in\mathcal{D}} A_d$ in the l.v.t. Therefore, $A_d \to \mathrm{Li}_{d\in\mathcal{D}} A_d$ in both the l.v.t and the u.v.t. Hence, $A_d \to \mathrm{Li}_{d\in\mathcal{D}} A_d$ in v.t.

Lastly, we prove 6. So assume that $A_d \to A$ in v.t and let $a \in A$. Then, we know that $A \in \lim^{LV}_{d\in\mathcal{D}} A_d \cap \lim^{UV}_{d\in\mathcal{D}} A_d$ i.e $A_d \to A$ in both the l.v.t and u.v.t. And by Item 1 we know $A \subseteq \mathrm{Li}_{d\in\mathcal{D}} A_d$ since $\mathrm{Li}_{d\in\mathcal{D}} A_d$ is the largest l.v.t limit (note this also means that $\mathrm{Li}_{d\in\mathcal{D}} A_d$ must be closed from Item 4 of Proposition 2.2.4) and by Item 2 $\mathrm{Ls}_{d\in\mathcal{D}} A_d \subseteq \overline{A}$. Since, $\overline{A} \subseteq \mathrm{Li}_{d\in\mathcal{D}} A_d$ we can apply Item 5 of Proposition 2.2.5 and we have $\overline{A} = \mathrm{KLim}_{d\in\mathcal{D}} A_d$. □

It is safe to say that Item 1 of Proposition 2.2.6 tells us that there is no real reason to separate the concepts of convergence in the lower Kuratowski sense and in the l.v.t sense. This is because by Item 5 of Proposition 2.2.4 the collection of all sets which a given (set) net converges to in the l.v.t is given by the power set of the lower Kuratowski limit; in the notation of Proposition 2.2.6, $\lim^{LV}_{d\in\mathcal{D}} A_d = \mathcal{P}(\mathrm{Li}_{d\in\mathcal{D}} A_d) \setminus \{\emptyset\}$. In contrast the relationship between the upper Kuratowski limit and the u.v.t limits is more complex. In general they are not equivalent in the way the lower Kuratowski limit and the l.v.t limits are. However, under some compactness assumptions the upper Kuratowski limit and the u.v.t limits are the same; in the notation of Proposition 2.2.6 we have

$$\tau^c \cap \lim_{d\in\mathcal{D}}{}^{UV} A_d = \left\{ \overline{C} : \mathop{\mathrm{Ls}}_{d\in\mathcal{D}} A_d \subseteq \overline{C} \right\}$$

whenever $\operatorname{Ls}_{d\in\mathcal{D}} A_d \in \lim^{UV}_{d\in\mathcal{D}} A_d$ and X is regular. It makes sense to consider $\{\overline{C} : \operatorname{Ls}_{d\in\mathcal{D}} A_d \subseteq \overline{C}\}$ as an interval of closed sets i.e $[\operatorname{Ls}_{d\in\mathcal{D}} A_d, X] \cap \tau^c$. In this case it would be a closed interval since it contains its infimum, $\operatorname{Ls}_{d\in\mathcal{D}} A_d$ and its supremum, X. This interpretation may not make sense when $\operatorname{Ls}_{d\in\mathcal{D}} A_d \notin \lim^{UV}_{d\in\mathcal{D}} A_d$; like in the counterexample given in the proof of Item 2 of Proposition 2.2.6 where $\operatorname{Ls}_{n\in\mathbb{N}}\{0,n\} = \{0\}$ and $\{0,1\}$ has $\operatorname{Ls}_{n\in\mathbb{N}}\{0,n\} \subsetneq \{0,1\}$ but $\{0,1\} \notin \lim^{UV}_{n\in\mathbb{N}}\{0,n\}$. This subtle difference between upper Kuratowski convergence and u.v.t convergence has caused me great pains in the past and we will see later that it causes some more trouble when we describe the continuity of multifunctions.

There is one more big idea concerning Kuratowski convergence, ideas concerning how point selections from a net effect (or characterize!) Kuratowski convergence.

Definition 2.2.3. *Let (X,τ) be a topological space. Let $\{A_d\}_{d\in\mathcal{D}}$ be a set net of X.*

A point selection of the net $\{A_d\}_{d\in\mathcal{D}}$ is point net $\{a_d\}_{d\in\mathcal{D}}$ which satisfies $a_d \in A_d$ for all $d \in D$. Often we will simply write " $\{a_d \in A_d\}_{d\in\mathcal{D}}$" to mean that $\{a_d\}_{d\in\mathcal{D}}$ is a point selection of $\{A_d\}_{d\in\mathcal{D}}$.

Proposition 2.2.7. *Let (X,τ) be a topological space. Let $\{A_d\}_{d\in\mathcal{D}}$ be a set net of X. Define,*

$$P_{\mathrm{Li},A_d} = \left\{x \in X : \exists\{a_d \in A_d\}_{d\in\mathcal{D}} \text{ with } a_d \to x\right\}$$

and

$$P_{\mathrm{Ls},A_d} = \left\{x \in X : \exists\{a_d \in A_d\}_{d\in\mathcal{D}} \text{ and } x \in \operatorname{Acc}_{d\in\mathcal{D}} a_d\right\}.$$

Recall that $\operatorname{Acc}_{d\in\mathcal{D}} a_d$ is the set of all accumulation points of of the point net a_d, see Definition 2.1.7.

The following hold:

1. $P_{\mathrm{Ls},A_d} = \operatorname*{Ls}_{d\in\mathcal{D}} A_d$.
2. $P_{\mathrm{Li},A_d} \subseteq \operatorname*{Li}_{d\in\mathcal{D}} A_d$.
3. *If for all $x \in X$ there is $\mathcal{V} : \mathcal{D} \to \tau_x$ with: for all $V \in \tau_x$ there is a $D \in \mathcal{D}$ such that for all $d \geq D$ we have $\mathcal{V}(d) \subseteq V$. Then, $P_{\mathrm{Li},A_d} = \operatorname*{Li}_{d\in\mathcal{D}} A_d$.*
4. *If X is first countable (this means every point in X has a countable base) and $\mathcal{D} = \mathbb{N}$ with the usual ordering then, $P_{\mathrm{Li},A_n} = \operatorname*{Li}_{n\in\mathbb{N}} A_n$. This equality can fail when the aforementioned conditions are not satisfied.*

Proof. To prove 1, let $x \in \mathrm{Ls}_{d\in\mathcal{D}} A_d$ then, for all $V \in \tau_x$ and for all $D \in \mathcal{D}$ there is a $d = d(D,V) \geq D$ with $V \cap A_{d(D,V)} \neq \emptyset$. For each $D \in \mathcal{D}$ and $V \in \tau_x$ we can pick $x_{d(D,V)} \in A_d \cap V$. Let $\{a_d \in A_d\}_{d\in\mathcal{D}}$ and let

$$\hat{a}_d = \begin{cases} x_{d(D,V)} & d = d(D,V) \text{ for some } V \in \tau_x, D \in \mathcal{D} \\ a_d & \text{otherwise} \end{cases}$$

then, $\{\hat{a}_d\}_{d\in\mathcal{D}}$ is a point selection of $\{A_d\}_{d\in\mathcal{D}}$ which has a sub net which converges to x. So $x \in P_{\mathrm{Ls},A_d}$.

On the other hand, if $x \in P_{\mathrm{Ls},A_d}$ then let $\{a_d \in A_d\}$ with $x \in \mathrm{Acc}_{d\in\mathcal{D}}\, a_d$ and for all $V \in \tau_x$, for all $D \in \mathcal{D}$ there is a $d \geq D$ with $a_d \in V$. Since $a_d \in A_d$ we have $A_d \cap V \neq \emptyset$ under the same quantitation and by definition $x \in \mathrm{Ls}_{d\in\mathcal{D}} A_d$. This proves $P_{\mathrm{Ls},A_d} = \mathrm{Ls}_{d\in\mathcal{D}} A_d$ and 1 holds.

For 2 suppose that $x \in P_{\mathrm{Li},A_d}$ and let $\{a_d \in A_d\}$ have $a_d \to x$ then, for all $V \in \tau_x$ there is a $D \in \mathcal{D}$ with for all $d \geq D$ we have $a_d \in V$; again, this means $A_d \cap V \neq \emptyset$ under the same quantitation and so $x \in \mathrm{Li}_{d\in\mathcal{D}} A_d$.

To prove 3, we consider $x \in \mathrm{Li}_{d\in\mathcal{D}} A_d$. By assumption there is a $\mathcal{V} : \mathcal{D} \to \tau_x$ with the stated properties. Pick any $V \in \tau_x$ then, there are $D_{\mathcal{V}}, D_{\mathrm{Li},\mathcal{V}} \in \mathcal{D}$ with:

$$\forall d \geq D_{\mathcal{V}}\, \mathcal{V}(d) \subseteq V \text{ and } \mathcal{V}(d) \in \tau_x$$
$$\forall d \geq D_{\mathrm{Li},\mathcal{V}}\, \exists x_d \in A_d \cap \mathcal{V}(d) \subseteq A_d \cap V.$$

Let $\{a_d \in A_d\}_{d\in\mathcal{D}}$. Like before it is possible to make a net $\{\hat{a}_d\}_{d\in\mathcal{D}}$ with

$$\hat{a}_d = \begin{cases} x_d & x_d \in A_d \cap \mathcal{V}(d) \text{ for some } d \in \mathcal{D} \\ a_d & \text{otherwise} \end{cases}$$

and such a selection will necessarily converge to x. Hence $x \in P_{\mathrm{Li},A_d}$.

Lastly, 4 is an application of 3. Since every point $x \in X$ has a countable base say $\mathcal{B}(x) = \{B_n\}_{n\in\mathbb{N}}$ then, let $\mathcal{V}(n) = \bigcap_{k=1}^{n} B_n$ it is clear that for any $n \geq N$, $N \in \mathbb{N}$ we have $\mathcal{V}(n) \supseteq \mathcal{V}(N)$. Moreover since $\mathcal{B}(x)$ is a base for x we know that for any $V \in \tau_x$ there is a $N \in \mathbb{N}$ with $B_N \subseteq V$ and also

$$V \supseteq B_N \supseteq \mathcal{V}(N) \supseteq \mathcal{V}(n)$$

for all $n \geq N$. Thus $\mathbb{N}$ satisfies the required property in 3 and 4 follows.

The counterexample is: endow $\mathbb{R}$ with the co-countable topology that is:

$$\tau = \{V \subseteq \mathbb{R} : V = \mathbb{R} \setminus E \text{ where } E \subseteq \mathbb{R} \text{ is countable}\} \cup \{\emptyset\}$$

now define

$$A_n = (-\infty, -\tfrac{1}{n}] \cup [\tfrac{1}{n}, \infty)$$

We claim that $0 \in \mathrm{Li}_{n\in\mathbb{N}} A_n$ but $0 \notin P_{\mathrm{Li},A_n}$. Every open set $V \ni 0$ has $V = \mathbb{R} \setminus E$ where E is countable, we see $0 \notin E$ and that V is uncountable. Thus, there is a $x \in V$ with $x \neq 0$ and for $N \in \mathbb{N}$ with $|x| \geq \frac{1}{N}$ we have that all $n \geq N$ we must have $V \cap A_n \neq \emptyset$. Therefore, $0 \in \mathrm{Li}_{n\in\mathbb{N}} A_n$.

On the other hand, every convergent sequence in $(\mathbb{R}, \tau)$ is eventually constant. To see this consider: $V = \mathbb{R} \setminus E$ is open if and only if E is closed. By definition of the co-countable topology every countable set is closed. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence converging to x then $\{x_n : n \in \mathbb{N}\} \setminus \{x\}$ is countable and thus closed. If $\{x_n : n \in \mathbb{N}\} \setminus \{x\}$ were infinite then we can select a subsequence of x_n which lies in the closed set $\{x_n : n \in \mathbb{N}\} \setminus \{x\}$, such a subsequence cannot converge to x. But this contradicts that $x_n \to x$. Thus, $\{x_n : n \in \mathbb{N}\} \setminus \{x\}$ is finite which in turn means $\{x_n\}_{n\in\mathbb{N}}$ is eventually constant. One can see that any sequence $\{a_n \in A_n\}_{n\in\mathbb{N}}$ always has $a_n \neq 0$, so any such a_n may never converge to 0. Therefore, $0 \notin P_{\mathrm{Li},A_n}$ and $P_{\mathrm{Li},A_n} \neq \mathrm{Li}_{d\in\mathbb{N}} A_n$. □

The most important applications occur in metric spaces. And I've never seen a real world approximation which needed more than a sequence to be approximated. Therefore, Proposition 2.2.7 tells us, in these cases, that Kuratowski convergence can be thought of purely in terms of point selections from the sequence of sets.

2.3 Multifunctions and continuity thereof

Multifunctions will play a central role in this thesis.

Definition 2.3.1. *Let X and Y be sets. We say that* F *is a multifunction of X into Y, denoted* $\mathrm{F} : X \rightsquigarrow Y$ *when* F *is a function* $\mathrm{F} : X \to \mathcal{P}(Y)$*. For all $x \in X$ we adopt the notation*

$$\mathrm{F}[x] = \mathrm{F}(x)$$

and for all $B \subseteq X$ we define the image of B under F *to be*

$$\mathrm{F}[B] = \bigcup_{b \in B} \mathrm{F}[b].$$

To avoid degeneracy of F *mapping to the empty set we define*

$$\mathrm{Dom}\,(F) = \{x \in X : \mathrm{F}[x] \neq \emptyset\}$$

to be the natural domain (or domain) of F*. When* $\mathrm{Dom}\,(\mathrm{F}) = X$ *we say that* F *is a total multifunction, otherwise it is called a partial multifunction.*

Finally, suppose that P is property that sets in Y can have (finite, open, closed, etc.) then, F *is said to be P valued if for all* $x \in \mathrm{Dom}\,(\mathrm{F})$ *the set* $\mathrm{F}[x]$ *has P.*

One may question why we even bother with the notion of a multifunction. After all, they are simply ordinary functions which map points to sets. I would argue that in many applications where we want to map into sets, we don't actually want to treat a multifunction $\mathrm{F} : X \rightsquigarrow Y$ as a function $\mathrm{F} : X \to \mathcal{P}(Y)$. For example, let $B \subseteq X$. It is natural to consider the forward image of B under F, $\mathrm{F}(B)$. However, $\mathrm{F}(B) = \{\mathrm{F}(b) : b \in B\} \subseteq \mathcal{P}(Y)$ whereas $\mathrm{F}[B] \subseteq Y$. Oftentimes, it's the latter we are really interested in and the former just makes things either notationally inconvenient or overly complicated. There are other similar reasons to work with multifunctions rather than just working with functions which map to sets.

Example 2.3.1. *Let X, Y be sets and* $\mathrm{f} : X \to Y$*. Then the inverse image of* f *is a multifunction defined by*

$$\mathrm{f}^{-1}[y] = \{x \in X : y = f(x)\}$$

where $\mathrm{f}^{-1} : Y \rightsquigarrow X$ *and* $\mathrm{Dom}\left(\mathrm{f}^{-1}\right) = f(X)$*.*

We may also consider the function f *to be a multifunction, define*

$$\mathrm{f}[x] = \{f(x)\}.$$

More concretely, define $\mathrm{F} : \mathbb{R}^3 \rightsquigarrow \mathbb{R}$ *by*

$$\mathrm{F}[(a,b,c)] = \left\{x \in \mathbb{R} : ax^2 + bx + c = 0\right\} = \begin{cases} \left\{\frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a}\right\} & a \neq 0 \\ \left\{\frac{-c}{b}\right\} & a = 0, b \neq 0 \\ \mathbb{R} & a = b = c = 0 \\ \emptyset & a = b = 0, c \neq 0. \end{cases}$$

Sometimes, multifunctions are called point to set functions, set valued functions or multivalued functions. It is also typical to define multifunctions to take nonempty values, that is $\mathrm{Dom}\,(\mathrm{F}) = X$ when $\mathrm{F} : X \rightsquigarrow Y$.

Remark 2.3.1 (multifunctions and set to set maps)**.** *It is the case that every multifunction naturally induces a set to set function (A set to set function is a mapping from* $\mathcal{P}(X)$ *to* $\mathcal{P}(Y)$ *for some sets* X, Y*). Given* $\mathrm{F} : X \rightsquigarrow Y$ *we can define a map* $\hat{\mathrm{F}} : \mathcal{P}(X) \to \mathcal{P}(Y)$ *by for all* $B \subseteq X$*,* $\hat{\mathrm{F}}(B) = \mathrm{F}[B]$ *(note when* $B = \emptyset$ *we have* $\mathrm{F}[B] = \emptyset$*).*

Note that functions which map sets to sets may fail to be induced by a multifunctions. For example let $\mathrm{G} : \mathcal{P}(X) \to \mathcal{P}(X)$ *where* $X = \{a, b\}$*,* $a \neq b$*, defined by*

$$\mathrm{G}(B) = \begin{cases} \emptyset & B = \emptyset \\ \{a\} & B = \{a\}, \{b\} \\ \{b\} & B = \{a, b\}. \end{cases}$$

G *is not induced by a multifunction because* $\mathrm{G}(\{a,b\}) \neq \mathrm{G}(\{a\}) \cup \mathrm{G}(\{b\})$*.*

In contrast when, a set to set map is union preserving and sends the empty set to the empty set then, it is induced by a multifunction. Here a function $\hat{\mathrm{H}} : \mathcal{P}(X) \to \mathcal{P}(Y)$ *is union preserving if for all* $A, B \subseteq X$ *we have* $\hat{\mathrm{H}}(A \cup B) = \hat{\mathrm{H}}(A) \cup \hat{\mathrm{H}}(B)$*.*

In conclusion, the reader should always be mindful of definitions of mappings which are defined to be set to set functions. They often fail to be multifunctions and the results from this section may not apply.

We start our rigorous exploration of multifunctions by generalizing some basic concepts/relations of single valued functions.

Definition 2.3.2. *Let* X, Y *be sets with* $C \subseteq Y$ *and* $\mathrm{F} : X \rightsquigarrow Y$ *be a multifunction. Define,*

$$\mathrm{F}^{-}[C] = \{x \in X : \mathrm{F}[x] \cap C \neq \emptyset\}$$

to be the lower pre-image, the weak pre-image, or inverse, of C by F. Note, F^- can be considered a multifunction in its own right.

Also define,

$$\mathrm{F}^+[C] = \{x \in \mathrm{Dom}\,(\mathrm{F}) : \mathrm{F}[x] \subseteq C\}$$

to be the upper pre-image, strong pre-image, or core, of C by F.

Both the lower and upper pre-image are direct generalizations of the normal function pre-image. Let $\mathrm{f} : X \to Y$ then, $\mathrm{f}^{-1}(C) = \{x \in X : f(x) \in C\}$. The statement $\mathrm{f}(x) \in C$ has two equivalent meanings when we consider $\mathrm{f}(x)$ to be a set: 1) $\{\mathrm{f}(x)\} \cap C \neq \emptyset$ (the set $\{\mathrm{f}(x)\}$ touches C) and 2) $\{\mathrm{f}(x)\} \subseteq C$ (the set $\{\mathrm{f}(x)\}$ is contained in C). These are only equivalent since $\{f(x)\}$ is a singleton. This idea of "$\in$" generalizing to two ways when we move to sets also explains why there where two Vietoris topologies. Indeed, we will find later that these pre-images are closely related to continuity for the Vietoris topologies. For now we establish some basic facts/identities about multifunctions and their images/pre-images.

Proposition 2.3.1. *Let X, Y be sets with $A, B \subseteq X$ and $C, D \subseteq Y$. Let $\mathcal{S}_X \subseteq \mathcal{P}(X)$, $\mathcal{S}_Y \subseteq \mathcal{P}(Y)$ and $\mathrm{F} : X \rightsquigarrow Y$. Then, the following hold:*

1. *If $A \subseteq B$ then, $\mathrm{F}[A] \subseteq \mathrm{F}[B]$.*
2. *If $C \subseteq D$ then, $\mathrm{F}^-[C] \subseteq \mathrm{F}^-[D]$ and $\mathrm{F}^+[C] \subseteq \mathrm{F}^+[D]$.*
3. $\mathrm{F}\left[\bigcup_{S \in \mathcal{S}_X} S\right] = \bigcup_{S \in \mathcal{S}_X} \mathrm{F}[S]$ *and* $\mathrm{F}\left[\bigcap_{S \in \mathcal{S}_X} S\right] \subseteq \bigcap_{S \in \mathcal{S}_X} \mathrm{F}[S]$.
4. $\mathrm{F}^-\left[\bigcup_{S \in \mathcal{S}_Y} S\right] = \bigcup_{S \in \mathcal{S}_Y} \mathrm{F}^-[S]$ *and* $\mathrm{F}^-\left[\bigcap_{S \in \mathcal{S}_Y} S\right] \subseteq \bigcap_{S \in \mathcal{S}_Y} \mathrm{F}^-[S]$.
5. $\mathrm{F}^+\left[\bigcup_{S \in \mathcal{S}_Y} S\right] \supseteq \bigcup_{S \in \mathcal{S}_Y} \mathrm{F}^+[S]$ *and* $\mathrm{F}^+\left[\bigcap_{S \in \mathcal{S}_Y} S\right] = \bigcap_{S \in \mathcal{S}_Y} \mathrm{F}^+[S]$.
6. $X \setminus \mathrm{F}^-[C] = \mathrm{F}^+[Y \setminus C]$ *and* $X \setminus \mathrm{F}^+[C] = \mathrm{F}^-[Y \setminus C]$.
7. $\mathrm{F}^+[C] \subseteq \mathrm{F}^-[C]$.
8. $B \subseteq \mathrm{F}^+[\mathrm{F}[B]]$ *and* $B \subseteq \mathrm{F}^-[\mathrm{F}[B]]$.
9. $\mathrm{F}\left[\mathrm{F}^+[C]\right] \subseteq C$.

Proof. Item 1: When $y \in \mathrm{F}[A]$ then, by definition there is $a \in A$ with $y \in \mathrm{F}[a]$. By assumption $A \subseteq B$, so $a \in B$. Therefore, by definition $y \in \mathrm{F}[B]$.

Item 2: When $x \in \mathrm{F}^-[C]$ we have, by definition, $\emptyset \neq \mathrm{F}[x] \cap C$. As $C \subseteq D$ we have also have $\emptyset \neq \mathrm{F}[x] \cap D$ and $x \in \mathrm{F}^-[D]$.

When $x \in \mathrm{F}^+[C]$ we have, by definition, $\mathrm{F}[x] \subseteq C$. As $C \subseteq D$ we get $\mathrm{F}[x] \subseteq D$ and so $x \in \mathrm{F}^+[D]$.

Item 3: $y \in \mathrm{F}\left[\bigcup_{S \in \mathcal{S}_X} S\right]$ iff there is a $x \in \bigcup_{S \in \mathcal{S}_X} S$ with $y \in \mathrm{F}[x]$ iff there is a $S' \in \mathcal{S}_X$ and a $x \in S'$ with $y \in \mathrm{F}[x]$ iff there is a $S' \in \mathcal{S}_X$ with $y \in \mathrm{F}[S']$ iff $y \in \bigcup_{S \in \mathcal{S}_X} \mathrm{F}[S]$.

For the intersection: $y \in \mathrm{F}\left[\bigcap_{S \in \mathcal{S}_X} S\right]$ iff there is a $x \in \bigcap_{S \in \mathcal{S}_X} S$ with $y \in \mathrm{F}[x]$ iff there is a $x \in X$ such that for all $S \in \mathcal{S}_X$ $x \in S$ and $y \in \mathrm{F}[x]$. From here, we see that $\mathrm{F}[\{x\}] = \mathrm{F}[x]$ and by Item 1 we have $y \in \mathrm{F}[x] \subseteq \mathrm{F}[S]$ for all $S \in \mathcal{S}_X$. Hence $y \in \bigcap_{S \in \mathcal{S}_X} \mathrm{F}[S]$.

Item 4: This follows from Item 3, recalling that F^- is a multifunction in its own right.

Item 5: When $x \in \bigcup_{S \in \mathcal{S}_Y} \mathrm{F}^+[S]$ we have, $\mathrm{F}[x] \subseteq S'$ for some $S' \in \mathcal{S}_Y$. In any case, $\mathrm{F}[x] \subseteq S' \subseteq \bigcup_{S \in \mathcal{S}_Y} S$ by definition of unions. So $x \in \mathrm{F}^+\left[\bigcup_{S \in \mathcal{S}_Y} S\right]$.

Consider: $x \in \mathrm{F}^+\left[\bigcap_{S \in \mathcal{S}_Y} S\right]$ iff $\mathrm{F}[x] \subseteq \bigcap_{S \in \mathcal{S}_Y} S$ iff for all $S \in \mathcal{S}_Y$ we have $\mathrm{F}[x] \subseteq S$ iff for all $S \in \mathcal{S}_Y$ we have $x \in \mathrm{F}^+[S]$ iff $x \in \bigcap_{S \in \mathcal{S}_Y} \mathrm{F}^+[S]$. Therefore, $\mathrm{F}^+\left[\bigcap_{S \in \mathcal{S}_Y} S\right] = \bigcap_{S \in \mathcal{S}_Y} \mathrm{F}^+[S]$.

Item 6: $x \in X \setminus \mathrm{F}^-[C]$ iff $x \notin \mathrm{F}^-[C]$ iff $\mathrm{F}[x] \cap C = \emptyset$ iff $\mathrm{F}[x] \subseteq Y \setminus C$ iff $x \in \mathrm{F}^+[Y \setminus C]$.

Similarly, $x \in X \setminus \mathrm{F}^+[C]$ iff $x \notin \mathrm{F}^+[C]$ iff $\mathrm{F}[x] \not\subseteq C$ iff $\mathrm{F}[x] \cap Y \setminus C \neq \emptyset$ iff $x \in \mathrm{F}^-[Y \setminus C]$.

Item 7: Note when $C = \emptyset$ the statement holds. Otherwise, When $x \in \mathrm{F}^+[C]$ we have that $\mathrm{F}[x] \neq \emptyset$ (since $x \in \mathrm{Dom}(\mathrm{F})$) and $\mathrm{F}[x] \subseteq C$. So $\mathrm{F}[x] \cap C \neq \emptyset$ and $x \in \mathrm{F}^-[C]$.

Item 8: Let $b \in B$ then it follows that $\mathrm{F}[b] \subseteq \mathrm{F}[B]$. But by definition this means $b \in \mathrm{F}^+[\mathrm{F}[B]]$, which is the result.

The other identity follows from the above and Item 7.

Item 9: Suppose that $y \in \mathrm{F}\left[\mathrm{F}^+[C]\right]$ then, there is a $x \in \mathrm{F}^+[C]$ with $y \in \mathrm{F}[x] \subseteq C$. Therefore, $y \in C$. $\square$

Broadly we see that the forward, lower and upper, pre-images behave in ways we might expect from the single valued case. The big difference here is that the lower and upper pre-images are complementary (Item 6 of Proposition 2.3.1), and have are not as well behaved as the pre-image of a single valued function (Items 4 and 5 of Proposition 2.3.1). Notably, the pre-image of a single valued function distributes over both intersections and unions but the lower pre-image of a multifunction is only guaranteed to distribute over union but not over intersections. This will cause interesting problems later.

Note that, some of the set inclusions in Proposition 2.3.1 can be made into equalities, under certain conditions. For the sake of generality, we will not be assuming the conditions in future theorems. So there is no need to list such conditions in this work.

We also consider applying some basic set operations to multifunctions.

Proposition 2.3.2. *Let X, Y be sets with and let $\mathcal{F}$ be a collection of multifunctions from X to Y. Then, the following hold:*

1. *If $\mathrm{F}, \mathrm{G} \in \mathcal{F}$ $\mathrm{F} \subseteq \mathrm{G}$ (that is for all $x \in \mathrm{Dom}(\mathrm{F})$ we have $\mathrm{F}[x] \subseteq \mathrm{G}[x]$) then, for all $C \subseteq Y$ we have $\mathrm{F}^-[C] \subseteq \mathrm{G}^-[C]$ and $\mathrm{F}^+[C] \supseteq \mathrm{G}^+[C] \cap \mathrm{Dom}(\mathrm{F})$.*

2. *Define*
$$\mathrm{F}_\cup[x] = \bigcup_{\mathrm{F} \in \mathcal{F}} \mathrm{F}[x]$$
for $x \in X$ then, for all $C \subseteq Y$ we have $\mathrm{F}_\cup^-[C] = \bigcup_{\mathrm{F}\in\mathcal{F}} \mathrm{F}^-[C]$ and when the elements of $\mathcal{F}$ are total we have $\mathrm{F}_\cup^+[C] = \bigcap_{\mathrm{F}\in\mathcal{F}} \mathrm{F}^+[C]$.

Proof. To prove 1 let $x \in \mathrm{F}^-[C]$ then $\emptyset \neq \mathrm{F}[x] \cap C \subseteq \mathrm{G}[x] \cap C$ and $x \in \mathrm{G}^-[C]$. If $x \in \mathrm{G}^+[C] \cap \mathrm{Dom}(\mathrm{F})$ then $C \supseteq \mathrm{G}[x] \supseteq \mathrm{F}[x]$ and since $x \in \mathrm{Dom}(\mathrm{F})$ we have $x \in \mathrm{F}^+[C]$.

For 2 we see $x \in \mathrm{F}_\cup^-[C]$ iff $\left(\bigcup_{\mathrm{F}\in\mathcal{F}} \mathrm{F}[x]\right) \cap C \neq \emptyset$ iff $\exists\, \mathrm{F} \in \mathcal{F}$ with $\mathrm{F}[x] \cap C \neq \emptyset$ iff $\exists\, \mathrm{F} \in \mathcal{F}$ with $x \in \mathrm{F}^-[C]$ iff $x \in \bigcup_{\mathrm{F}\in\mathcal{F}} \mathrm{F}^-[C]$. Similarly, when the elements of F are total: $x \in \mathrm{F}_\cup^+[C]$ iff $\bigcup_{\mathrm{F}\in\mathcal{F}} \mathrm{F}[x] \subseteq C$ iff $\forall\, \mathrm{F} \in \mathcal{F}$ we have $\mathrm{F}[x] \subseteq C$ and $x \in \mathrm{Dom}(\mathrm{F}) = X$ iff $\forall\, \mathrm{F} \in \mathcal{F}$ we have $x \in \mathrm{F}^+[C]$ iff $x \in \bigcap_{\mathrm{F}\in\mathcal{F}} \mathrm{F}^+[C]$. □

The union of multifunctions is rather important case for us. When $\mathrm{f} : X \times U \to X$ is a function we can define $\mathrm{F}[x] = \mathrm{f}(\{x\} \times U) = \bigcup_{u \in U} \{\mathrm{f}(x, u)\}$. Then as defined in Chapter 1, Equation (1.1), a controlled orbit of f, $\{x_n\}_{n=0}^{\infty}$ satisfies $x_{n+1} \in \mathrm{F}[x_n]$ for $n \in \mathbb{N} \cup \{0\}$.

There are other basic set operations we could apply to multifunctions. However, we won't really need them and more pertinently the statements of such results would be rather technical and unenlightening.

We should also consider composition of multifunctions.

Definition 2.3.3. *Let X, Y, Z be sets, $\mathrm{F} : X \rightsquigarrow Y$ and $\mathrm{G} : Y \rightsquigarrow Z$ be multifunctions. Define the composition of G at F, $\mathrm{G} \circ \mathrm{F} : X \rightsquigarrow Z$ to be*

$$\mathrm{G} \circ \mathrm{F}[x] = \mathrm{G}[\mathrm{F}[x]] = \bigcup_{y \in \mathrm{F}[x]} \mathrm{G}[y]$$

for all $x \in X$*. We also define the square product of* G *at* F, $\mathrm{G} \,\square\, \mathrm{F} : X \rightsquigarrow Z$ *to be*

$$\mathrm{G} \,\square\, \mathrm{F}[x] = \bigcap_{y \in \mathrm{F}[x]} \mathrm{G}[y]$$

for all $x \in X$*.*

Both the square product and $\circ$ can be thought of as generalizations of function composition. By far, the more useful of the two is $\circ$. In fact, the square product will largely be ignored in this work. It is included only so to emphasize there are usually two ways concepts generalize to multifunctions from single valued functions.

Proposition 2.3.3. *Let* X, Y, Z *be sets,* $\mathrm{F} : X \rightsquigarrow Y$ *and* $\mathrm{G} : Y \rightsquigarrow Z$ *be multifunctions. The following hold:*

1. $(\mathrm{G} \circ \mathrm{F})^- = \mathrm{F}^- \circ \mathrm{G}^-$.
2. *For all* $B \subseteq Z$ *we have* $(\mathrm{G} \circ \mathrm{F})^+[B] = \mathrm{F}^+\big[\mathrm{G}^+[B]\big]$*. For this reason, we (in an abuse of notation) define* $(\mathrm{G} \circ \mathrm{F})^+ = \mathrm{F}^+ \circ \mathrm{G}^+$.
3. *Composition of multifunctions is associative. That is, let* A *be a set and* $\mathrm{H} : Z \rightsquigarrow A$*. Then,* $(\mathrm{H} \circ \mathrm{G}) \circ \mathrm{F} = \mathrm{H} \circ (\mathrm{G} \circ \mathrm{F})$.

Proof. To prove Item 1, suppose $B \subseteq Z$ is arbitrary. We must show that $(\mathrm{G} \circ \mathrm{F})^-[B] = \mathrm{F}^- \circ \mathrm{G}^-[B]$. Consider: $x \in (\mathrm{G} \circ \mathrm{F})^-[B]$ iff $\mathrm{G} \circ \mathrm{F}[x] \cap B \neq \emptyset$ iff $\exists y \in \mathrm{F}[x]$ with $\mathrm{G}[y] \cap B \neq \emptyset$ iff $\exists y \in \mathrm{F}[x] \cap \mathrm{G}^-[B] \neq \emptyset$ iff $x \in \mathrm{F}^-\big[\mathrm{G}^-[B]\big]$ iff $x \in \mathrm{F}^- \circ \mathrm{G}^-[B]$.

Now we prove Item 2, Let $B \subseteq Z$, keep Item 6 of Proposition 2.3.1 in mind and we see

$$\begin{aligned}
(\mathrm{G} \circ \mathrm{F})^+[B] &= X \setminus (\mathrm{G} \circ \mathrm{F})^-[Z \setminus B] \\
&= X \setminus \mathrm{F}^-\big[\mathrm{G}^-[Z \setminus B]\big] \quad \text{by Item 1} \\
&= \mathrm{F}^+\big[Y \setminus \mathrm{G}^-[Z \setminus B]\big] \\
&= \mathrm{F}^+\big[\mathrm{G}^+[Z \setminus (Z \setminus B)]\big] \\
&= \mathrm{F}^+\big[\mathrm{G}^+[B]\big].
\end{aligned}$$

Lastly, for Item 3 let $x \in X$. Consider

$$(\mathrm{H} \circ \mathrm{G}) \circ \mathrm{F}[x] = \bigcup_{y \in \mathrm{F}[x]} \mathrm{H} \circ \mathrm{G}[y] = \bigcup_{y \in \mathrm{F}[x]} \bigcup_{z \in \mathrm{G}[y]} \mathrm{H}[z] = \bigcup \{\mathrm{H}[z] : y \in \mathrm{F}[x], z \in \mathrm{G}[y]\},$$

also

$$\mathrm{H}\circ(\mathrm{G}\circ\mathrm{F})[x]=\bigcup_{z\in\mathrm{G}\circ\mathrm{F}[x]}\mathrm{H}[z]=\bigcup\left\{\mathrm{H}[z]:z\in\bigcup_{y\in\mathrm{F}[x]}\mathrm{G}[y]\right\}=\bigcup\{\mathrm{H}[z]:y\in\mathrm{F}[x],z\in\mathrm{G}[y]\}.$$

□

The last major non-topological concept we explore is the graph of a multifunction.

Definition 2.3.4. *Let X, Y be sets and $\mathrm{F}: X \rightsquigarrow Y$. We define the graph of F to be*

$$\mathrm{Graph}\,(\mathrm{F})=\{(x,y)\in X\times Y: y\in\mathrm{F}[x]\}.$$

The graph of a multifunction will be useful for defining continuity later. Note that every multifunction is characterized by its graph, moreover any subset of $X \times Y \supseteq L$ defines a multifunction by $\mathrm{F}[x] = \{y \in Y : (x, y) \in L\}$.

Proposition 2.3.4 (trivial selection theorem)**.** *Let X, Y be sets and $\mathrm{F}: X \rightsquigarrow Y$ be a total multifunction. Then, there is a $\mathcal{F} \subseteq Y^X$ with*

$$\mathrm{F}[x]=\bigcup_{\mathrm{f}\in\mathcal{F}}\{\mathrm{f}(x)\}$$

for all $x \in X$.

Outline of proof.

1. Define the set

$$\mathfrak{F}=\left\{\mathcal{F}\subseteq Y^X:\mathcal{F}\neq\emptyset,\bigcup_{\mathrm{f}\in\mathcal{F}}\{\mathrm{f}(x)\}\subseteq\mathrm{F}[x]\quad\forall x\in X\right\}$$

   and an equivalence relation on $\mathfrak{F}$ where $\mathcal{F}, \mathcal{G} \in \mathfrak{F}$ are equivalent, if $\bigcup_{\mathrm{f}\in\mathcal{F}}\{\mathrm{f}(x)\} = \bigcup_{\mathrm{f}\in\mathcal{G}}\{\mathrm{f}(x)\}$ for all $x \in X$. Let $\langle\mathcal{F}\rangle$ be the equivalence class of $\mathcal{F} \in \mathfrak{F}$. Note, $\mathfrak{F}$ is nonempty, since $\Pi_{x\in X}\,\mathrm{F}[x] \neq \emptyset$ by the axiom of choice.

2. Define a partial order, $\preceq$, on $\langle\mathfrak{F}\rangle := \{\langle\mathcal{F}\rangle : \mathcal{F} \in \mathfrak{F}\}$ where $\langle\mathcal{F}\rangle \preceq \langle\mathcal{G}\rangle$ if $\bigcup_{\mathrm{f}\in\mathcal{F}}\{\mathrm{f}(x)\} \subseteq \bigcup_{\mathrm{f}\in\mathcal{G}}\{\mathrm{f}(x)\}$ for all $x \in X$.

3. Apply Zorn's lemma to $(\langle \mathfrak{F} \rangle, \preceq)$ Simply, take unions of all representatives in a chain of $\langle \mathfrak{F} \rangle$ to find an upper bound.

4. Given a maximal element of $(\langle \mathfrak{F} \rangle, \preceq)$, say $\langle \mathcal{M} \rangle$, show $\mathrm{F}[x] = \bigcup_{\mathrm{f} \in \mathcal{M}} \{\mathrm{f}(x)\}$ by a contradiction argument. i.e if not we can define a total multifunction,

$$\mathrm{G}[x] = \begin{cases} \mathrm{F}[x] & \mathrm{F}[x] = \bigcup_{\mathrm{f} \in \mathcal{M}} \{\mathrm{f}(x)\} \\ \mathrm{F}[x] \setminus \bigcup_{\mathrm{f} \in \mathcal{M}} \{\mathrm{f}(x)\} & \mathrm{F}[x] \supsetneq \bigcup_{\mathrm{f} \in \mathcal{M}} \{\mathrm{f}(x)\} \end{cases}$$

and by the axiom of choice, there is a $\mathrm{g} \in \prod_{x \in X} \mathrm{G}[x]$. However, $\mathcal{M} \cup \{\mathrm{g}\}$ is equivalent $\mathcal{M}$ (since $\mathcal{M}$ is maximal) which contradicts the definition of G.

□

2.3.1 Continuity of multifunctions

It is essential to most theorems involving functions for those functions to be continuous. It is no different in the case of multifunctions. What is different is that there are many different kinds of continuity for multifunctions, all with their own subtleties and uses.

Recall, Theorem 2.1.1 which characterizes continuity of single valued functions. Many of these characterization involve the pre-image. So it is natural to generalize these statements using the lower/upper pre-image of a multifunction.

Definition 2.3.5. *Let (X, τ) and (Y, σ) be topological spaces. Let $\mathrm{F} : X \rightsquigarrow Y$ be a multifunction.*

1. F *is said to be lower semicontinuous (l.s.c) at the point $x \in X$ if, for all $V \in \sigma$ with $\mathrm{F}[x] \cap V \neq \emptyset$ we have that $\mathrm{F}^-[V]$ is a neighborhood of x.* F *is said to be l.s.c on a subset $B \subseteq X$ if, for all $b \in B$ we have that* F *is l.s.c at b.* F *is said to be l.s.c if,* F *is l.s.c on X.*

2. F *is said to be upper semicontinuous (u.s.c) at the point $x \in X$ if, $\emptyset \neq \mathrm{F}[x]$ and for all $V \in \sigma$ with $\mathrm{F}[x] \subseteq V$ we have that $\mathrm{F}^+[V]$ is a neighborhood of x.* F *is said to be u.s.c on a subset $B \subseteq X$ if, for all $b \in B$ we have that* F *is u.s.c at b.* F *is said to be u.s.c if,* F *is u.s.c on X.*

3. F *is said to be continuous at $x \in X$ if it is both l.s.c and u.s.c at x.* F *is said to be continuous on a subset $B \subseteq X$ if, for all $b \in B$ we have that* F *is continuous at b.* F *is said to be continuous if,* F *is continuous on X.*

Be aware that the terminology for continuity of multifunctions do not have consistent meaning across different works. Many people would call a lower semicontinuous multifunction (as defined above), a lower *hemi*-continuous multifunction. And would reserve the name lower semicontinuous for a different type of continuity. For example, (at time of writing) Wikipedia adopts this alternative terminology, however [12, 1] adopt the terminology of Definition 2.3.5.

Example 2.3.2. *In this example we consider multifunctions from* $\mathbb{R}$ *to* $\mathbb{R}$ *with the usual topology. Define,*

$$\mathrm{F}_1[x] = \begin{cases} \{0\} & x \leq 0 \\ (-1,1) & x > 0 \end{cases}$$

then, F_1 *is l.s.c at* $x = 0$ *but not u.s.c at* $x = 0$*. When* $x \neq 0$, F_1 *is continuous at* x*.*

To see why, suppose that $V \cap \mathrm{F}_1[0] \neq \emptyset$ *where* V *is open then,* $\mathrm{F}_1^-[V] = \mathbb{R}$ *which is an open set containing* 0*. Therefore,* F_1 *is l.s.c at* $x = 0$*. However,* $(-0.5, 0.5) \supseteq \mathrm{F}_1[0]$ *is open but for every* $\delta > 0$ *we have that* $\mathrm{F}_1[0+\delta] \not\subseteq (-0.5, 0.5)$*, hence* $\mathrm{F}_1^+[(-0.5, 0.5)] \not\ni x + \delta$ *and* F_1 *is not u.s.c at* $x = 0$*. When* $x \neq 0$, F_1 *is locally constant i.e there is an open* $U \ni x$ *with* $\mathrm{F}_1[x'] = \mathrm{F}_1[x]$ *for all* $x' \in U$*. Continuity follows quickly from this fact.*

Now consider,

$$\mathrm{F}_2[x] = \begin{cases} \{0\} & x < 0 \\ (-1,1) & x \geq 0 \end{cases}$$

then, F_2 *is not l.s.c at* $x = 0$ *but is u.s.c at* $x = 0$*. When* $x \neq 0$, F_2 *is continuous at* x*. We see that* $0.5 \in \mathrm{F}[0]$ *and that* $(0.25, 0.75) \ni 0.5$ *is open with* $\mathrm{F}[0] \cap (0.25, 0.75) \neq \emptyset$*. However, for all* $\delta > 0$ *we have* $0 - \delta \notin \mathrm{F}^-[(0.25, 0.75)] = [0, \infty)$*, so* F_2 *is not l.s.c at* $x = 0$*. On the other hand* F_2 *is u.s.c since,* $\mathrm{F}^+[V] = \mathbb{R}$ *is an open set of* 0*, for every open set* $V \supseteq \mathrm{F}_2[0] = (-1, 1)$*. When* $x \neq 0$, F_2 *is locally constant i.e there is an open* $U \ni x$ *with* $\mathrm{F}_2[x'] = \mathrm{F}_2[x]$ *for all* $x' \in U$*. Continuity follows quickly from this fact.*

Intuitively, I would say that both F_1 *and* F_2 *are discontinuous. Since both "suddenly" change at* $x = 0$*. This highlights that l.s.c and u.s.c are different but also inadequate in their own way.*

We now go through the painstaking process of trying to recover Theorem 2.1.1 for the different flavors of continuity for multifunctions.

Theorem 2.3.1 (lower semicontinuity at a point)**.** *Let* (X, τ) *and* (Y, σ) *be topological spaces. Let* $\mathrm{F} : X \rightsquigarrow Y$ *be a multifunction and* $x \in \mathrm{Dom}\,(\mathrm{F})$*. Let* $\hat{\mathrm{F}} : \mathrm{Dom}\,(\mathrm{F}) \to \mathcal{P}(Y) \backslash \{\emptyset\}$ *be the function,* $\hat{\mathrm{F}}(x') = \mathrm{F}[x']$ *for all* $x' \in \mathrm{Dom}\,(\mathrm{F})$*. The following are equivalent:*

1. F *is lower semicontinuous at* x.

2. *For every open set* $V \in \sigma$ *with* $V \cap \mathrm{F}[x] \neq \emptyset$ *there is a* $U \in \tau_x$ *with* $U \subseteq \mathrm{F}^-[V]$.

3. *For every* $B \subseteq Y$ *we have that if* $x \in \mathrm{F}^-[\mathrm{int}(B)]$ *then,* $x \in \mathrm{int}(\mathrm{F}^-[B])$.

4. *For every* $B \subseteq Y$ *we have that if* $x \in \overline{\mathrm{F}^+[B]}$ *then,* $x \in \mathrm{F}^+\left[\overline{B}\right]$.

5. *For every* $A \subseteq X$ *with* $x \in \overline{A}$ *we also have* $\mathrm{F}[x] \subseteq \overline{\mathrm{F}[A]}$.

6. $x \in \mathrm{int}(\mathrm{Dom}\,(\mathrm{F}))$ *and* $\hat{\mathrm{F}}$ *is continuous at* x *when we endow* $\mathcal{P}(Y) \setminus \{\emptyset\}$ *with the lower Vietoris topology (see Definition 2.2.1) and* $\mathrm{Dom}\,(\mathrm{F})$ *with the relative topology (see Proposition 2.1.2).*

7. $x \in \mathrm{int}(\mathrm{Dom}\,(\mathrm{F}))$ *and for every net* $\{x_d\}_{d\in\mathcal{D}}$ *in* $\mathrm{Dom}\,(\mathrm{F})$ *which converges to* x *we have that the set net* $\{\mathrm{F}[x_d]\}_{d\in D}$ *converges to* $\mathrm{F}[x]$ *in the lower Vietoris topology.*

8. $x \in \mathrm{int}(\mathrm{Dom}\,(\mathrm{F}))$ *and for every net* $\{x_d\}_{d\in\mathcal{D}}$ *in* $\mathrm{Dom}\,(\mathrm{F})$ *which converges to* x*, we have* $\mathrm{F}[x] \subseteq \mathrm{Li}_{d\in\mathcal{D}}\,\mathrm{F}[x_d]$ *(see Definition 2.2.2 for the definition of* Li*).*

9. $x \in \mathrm{int}(\mathrm{Dom}\,(\mathrm{F}))$ *and for every net* $\{x_d\}_{d\in\mathcal{D}}$ *in* $\mathrm{Dom}\,(\mathrm{F})$ *which converges to* x*, we have* $\mathrm{F}[x] \subseteq \mathrm{Ls}_{d\in\mathcal{D}}\,\mathrm{F}[x_d]$ *(See Definition 2.2.2 for the definition of* Ls*).*

Proof. 1 $\implies$ 2: When F is l.s.c we know that for any $V \in \sigma$ with $V \cap \mathrm{F}[x] \neq \emptyset$ the set $\mathrm{F}^-[V]$ is a neighborhood of x. Thus, there is an open set $U \in \tau_x$ with $U \subseteq \mathrm{F}^-[V]$, which is 2.

2 $\implies$ 3: Assume 2 holds and that $B \subseteq Y$ with $x \in \mathrm{F}^-[\mathrm{int}(B)]$. By definition of the lower pre-image we have that $\mathrm{F}[x] \cap \mathrm{int}(B) \neq \emptyset$. By 2 there is a $U \in \tau_x$ with $U \subseteq \mathrm{F}^-[\mathrm{int}(B)] \subseteq \mathrm{F}^-[B]$, taking the interior of this inclusion yields $x \in U \subseteq \mathrm{int}(\mathrm{F}^-[B])$. So 3 holds.

3 $\implies$ 5: Suppose that $x \in \overline{A}$, let $y \in \mathrm{F}[x]$ and $V \in \sigma_y$ be arbitrary. We have that $x \in \mathrm{F}^-[V]$ and by 3 this implies $x \in \mathrm{int}(\mathrm{F}^-[V])$. So, there is a $x' \in \mathrm{int}(\mathrm{F}^-[V]) \cap A$ (since $x \in \overline{A}$) moreover we also have $\mathrm{F}[x'] \cap V \neq \emptyset$ (since $\mathrm{int}(\mathrm{F}^-[V]) \subseteq \mathrm{F}^-[V]$). Therefore, $\mathrm{F}[A] \cap V \neq \emptyset$ for all $V \in \sigma_y$ and so $y \in \overline{\mathrm{F}[A]}$. It follows that $\mathrm{F}[x] \subseteq \overline{\mathrm{F}[A]}$ as required.

5 $\implies$ 1: We proceed by contraposition. If F is not l.s.c at x then $\exists V \in \sigma$ such that $\mathrm{F}^-[V]$ is not a neighborhood of x. Let $A = X \setminus \mathrm{F}^-[V]$, we see $x \in \overline{A}$ (since for all $U \in \tau_x$ we have $U \nsubseteq \mathrm{F}^-[V]$ which means $U \cap A \neq \emptyset$) and by Item 6 of Proposition 2.3.1 we have $A = \mathrm{F}^+[Y \setminus V]$. Thus, we see for all $z \in A$ we have $\mathrm{F}[z] \subseteq Y \setminus V$, which means $\mathrm{F}[A] \subseteq Y \setminus V$ and $\overline{\mathrm{F}[A]} \subseteq Y \setminus V$ as the latter set is closed. But $\mathrm{F}[x] \cap V \neq \emptyset$, so $\mathrm{F}[x] \nsubseteq \overline{\mathrm{F}[A]}$ as required.

2 $\implies$ 4: Let $B \subseteq Y$, we show the contrapositive of 4 holds. So suppose that $x \notin \mathrm{F}^+\big[\overline{B}\big]$ then by Item 6 of Proposition 2.3.1 we have $x \in \mathrm{F}^-\big[Y \setminus \overline{B}\big]$. As $Y \setminus \overline{B}$ is open, by 2 we know there is a $U \in \tau_x$ with $U \subseteq \mathrm{F}^-\big[Y \setminus \overline{B}\big] \subseteq \mathrm{F}^-[Y \setminus B]$. But this means that $U \cap \big(X \setminus \mathrm{F}^-[Y \setminus B]\big) = \emptyset$ and again by Item 6 of Proposition 2.3.1 we have $U \cap \mathrm{F}^+[B] = \emptyset$. Therefore, $x \notin \overline{\mathrm{F}^+[B]}$ as required.

4 $\implies$ 2: Let $V \in \sigma$ have $V \cap \mathrm{F}[x] \neq \emptyset$, so $x \in \mathrm{F}^-[V]$ and $x \notin \mathrm{F}^+[Y \setminus V]$. Let $B = Y \setminus V$ and we see $\overline{B} = Y \setminus V$ as $V \in \sigma$. But by the contrapositive of 4 of we know that $x \notin \overline{\mathrm{F}^+[Y \setminus V]}$. Hence, there is an open set $U \ni x$ with $U \cap \mathrm{F}^+[Y \setminus V] = \emptyset$ and taking complements we see $U \subseteq \mathrm{F}^-[V]$ as required.

2 $\implies$ 6: We claim that $\hat{\mathrm{F}}^{-1}(V^-) = \mathrm{F}^-[V]$ for any set $V \subseteq Y$, recall that $V^- = \{B \in \mathcal{P}(Y) \setminus \{\emptyset\} : B \cap V \neq \emptyset\}$. We see

$$\begin{aligned}
\hat{\mathrm{F}}^{-1}(V^-) &= \Big\{x \in \mathrm{Dom}\,(\mathrm{F}) : \hat{\mathrm{F}}(x) \in V^-\Big\} \\
&= \Big\{x \in \mathrm{Dom}\,(\mathrm{F}) : \hat{\mathrm{F}}(x) \cap V \neq \emptyset\Big\} \\
&= \{x \in \mathrm{Dom}\,(\mathrm{F}) : \mathrm{F}[x] \cap V \neq \emptyset\} \\
&= \{x \in X : \mathrm{F}[x] \cap V \neq \emptyset\} \\
&= \mathrm{F}^-[V].
\end{aligned}$$

Given an open set in l.v.t of $\hat{\mathrm{F}}(x)$ say $\mathcal{V}$ we know there is an open basic set of $\hat{\mathrm{F}}(x)$ contained in $\mathcal{V}$, by Item 1 of Proposition 2.2.2 we can take this basic open set to be $\bigcap_{n=1}^N V_n^-$ where are $V_1, \ldots, V_N \in \sigma$. By properties of the pre-image we have that $\hat{\mathrm{F}}^{-1}(\bigcap_{n=1}^N V_n^-) = \bigcap_{n=1}^N \hat{\mathrm{F}}^{-1}(V_n^-)$ and by a previous argument we must have $\hat{\mathrm{F}}^{-1}(\bigcap_{n=1}^N V_n^-) = \bigcap_{n=1}^N \mathrm{F}^-(V_n)$. By 2 for each $n = 1, \ldots, N$ there is $U_n \in \tau_x$ with $U_n \subseteq \mathrm{F}^-(V_n)$; hence $\bigcap_{n=1}^N U_n \subseteq \hat{\mathrm{F}}^{-1}(\bigcap_{n=1}^N V_n^-) \subseteq \hat{\mathrm{F}}^{-1}(\mathcal{V})$. Noting that $\bigcap_{n=1}^N U_n \subseteq \mathrm{Dom}\,(\mathrm{F})$ (since $U_n \subseteq \mathrm{F}^-(V_n) \subseteq \mathrm{Dom}\,(\mathrm{F})$ for each n) we also have $x \in \mathrm{int}(\mathrm{Dom}\,(\mathrm{F}))$. This shows 6.

6 $\iff$ 7: This equivalence follows quickly from the standard equivalences of functions continuous at a point preserving convergent nets which converge to that point.

7 $\implies$ 8: Let $\{x_d\}_{d \in \mathcal{D}}$ be a net in $\mathrm{Dom}\,(\mathrm{F})$ converging to x. By 7, when $\{\mathrm{F}[x_d]\}_{d \in \mathcal{D}}$ converges to $\mathrm{F}[x]$ in the l.v.t we apply Item 1 of Proposition 2.2.6 to get $\{\mathrm{F}[x_d]\}_{d \in \mathcal{D}}$ converging to $\mathrm{Li}_{d \in \mathcal{D}}\,\mathrm{F}[x_d]$ and by Item 5 of Proposition 2.2.4 we have $\mathrm{F}[x] \subseteq \mathrm{Li}_{d \in \mathcal{D}}\,\mathrm{F}[x_d]$.

8 $\implies$ 9: By Item 4 of Proposition 2.2.5 we have $\mathrm{Li}_{d \in \mathcal{D}}\,\mathrm{F}[x_d] \subseteq \mathrm{Ls}_{d \in \mathcal{D}}\,\mathrm{F}[x_d]$ for any net $\{x_d\}_{d \in \mathcal{D}}$ in $\mathrm{Dom}\,(\mathrm{F})$ converging to x. And in when 8 holds, we see $\mathrm{F}[x] \subseteq \mathrm{Li}_{d \in \mathcal{D}}\,\mathrm{F}[x_d] \subseteq \mathrm{Ls}_{d \in \mathcal{D}}\,\mathrm{F}[x_d]$. So $\mathrm{F}[x] \subseteq \mathrm{Ls}_{d \in \mathcal{D}}\,\mathrm{F}[x_d]$ and 9 holds.

9 $\implies$ 5: Let $x \in \overline{A}$ for some $A \subseteq X$. Since $x \in \operatorname{int}(\operatorname{Dom}(\mathrm{F}))$ there is a net, $\{x_d\}_{d\in\mathcal{D}}$ in $\operatorname{Dom}(\mathrm{F}) \cap A$, converging to x. By 9 we see

$$\mathrm{F}[x] \subseteq \operatorname*{Ls}_{d\in\mathcal{D}} \mathrm{F}[x_d] \subseteq \operatorname*{Ls}_{d\in\mathcal{D}} \mathrm{F}[A] = \overline{\mathrm{F}[A]}.$$

Note, $\operatorname{Ls}_{d\in\mathcal{D}} \mathrm{F}[x_d] \subseteq \operatorname{Ls}_{d\in\mathcal{D}} \mathrm{F}[A]$ follows from Item 7 of Proposition 2.2.5 and $\operatorname{Ls}_{d\in\mathcal{D}} \mathrm{F}[A] = \overline{\mathrm{F}[A]}$ follows quickly from definitions. Therefore, $\mathrm{F}[x] \subseteq \overline{\mathrm{F}[A]}$ and 5 holds. □

Note that by definition a multifunction can only be l.s.c, u.s.c or continuous at a point x if x is in the interior of the domain of the multifunction. This is mostly a stylistic decision, one could easily take the definitions for l.s.c, u.s.c to be relative to the domain of the multifunction. In which case, Theorem 2.3.1 would hold but the requirement for x to be in the interior of the domain would need to be dropped. When the multifunction is total, every point is trivially in the interior of the domain.

I find the most useful way to show a multifunction is l.s.c at x is to show that Item 2 holds. The implications of Theorem 2.3.1 are better discussed after we extend the result to lower semicontinuity on all of X.

Theorem 2.3.2 (lower semicontinuity on X)**.** *Let (X,τ) and (Y,σ) be topological spaces. Let $\mathrm{F} : X \rightsquigarrow Y$ be a total multifunction. Let $\hat{\mathrm{F}} : X \to \mathcal{P}(Y) \setminus \{\emptyset\}$ be the function, $\hat{\mathrm{F}}(x') = \mathrm{F}[x']$ for all $x' \in X$. The following are equivalent:*

1. *F is lower semicontinuous on X.*
2. *For every open set $V \in \sigma$ we have $\mathrm{F}^-[V] \in \tau$.*
3. *For every $B \subseteq Y$ we have that $\mathrm{F}^-[\operatorname{int}(B)] \subseteq \operatorname{int}(\mathrm{F}^-[B])$.*
4. *For every closed $C \in \sigma^c$ we have $\mathrm{F}^+[C] \in \tau^c$.*
5. *For every $B \subseteq Y$ we have that $\overline{\mathrm{F}^+[B]} \subseteq \mathrm{F}^+\left[\overline{B}\right]$.*
6. *For every $A \subseteq X$ we have $\mathrm{F}\left[\overline{A}\right] \subseteq \overline{\mathrm{F}[A]}$.*
7. *$\hat{\mathrm{F}}$ is continuous on X when we endow $\mathcal{P}(Y) \setminus \{\emptyset\}$ with the lower Vietoris topology (see Definition 2.2.1).*
8. *For all $x \in X$ and for every net $\{x_d\}_{d\in\mathcal{D}}$ in X which converges to x we have that the set net $\{\mathrm{F}[x_d]\}_{d\in D}$ converges to $\mathrm{F}[x]$ in the lower Vietoris topology. That is, for all $V \in \sigma$ with $\mathrm{F}[x] \cap V \neq \emptyset$ there is a $D \in \mathcal{D}$ such that for all $d \geq D$ we have $V \cap \mathrm{F}[x_d] \neq \emptyset$.*

9. *For all $x \in X$ and for every net $\{x_d\}_{d\in\mathcal{D}}$ in X which converges to x, we have $\mathrm{F}[x] \subseteq \mathrm{Li}_{d\in\mathcal{D}} \mathrm{F}[x_d]$ (see Definition 2.2.2 for the definition of Li).*

10. *For all $x \in X$ and for every net $\{x_d\}_{d\in\mathcal{D}}$ in X which converges to x, we have $\mathrm{F}[x] \subseteq \mathrm{Ls}_{d\in\mathcal{D}} \mathrm{F}[x_d]$ (see Definition 2.2.2 for the definition of Ls).*

11. *$\tilde{\mathrm{F}} : \mathcal{P}(X) \setminus \{\emptyset\} \to \mathcal{P}(Y) \setminus \{\emptyset\}$ defined by $\tilde{\mathrm{F}}(A) = \mathrm{F}[A]$ for $A \subseteq X$ is continuous when we endow $\mathcal{P}(X) \setminus \{\emptyset\}$ and $\mathcal{P}(Y) \setminus \{\emptyset\}$ with the lower Vietoris topology.*

12. *For every set net $\{A_d\}_{d\in\mathcal{D}}$ of $\mathcal{P}(X) \setminus \{\emptyset\}$ we have $\mathrm{F}[\mathrm{Li}_{d\in\mathcal{D}} A_d] \subseteq \mathrm{Li}_{d\in\mathcal{D}} \mathrm{F}[A_d]$.*

13. *For every set net $\{A_d\}_{d\in\mathcal{D}}$ of $\mathcal{P}(X) \setminus \{\emptyset\}$ we have $\mathrm{F}[\mathrm{Ls}_{d\in\mathcal{D}} A_d] \subseteq \mathrm{Ls}_{d\in\mathcal{D}} \mathrm{F}[A_d]$.*

Proof. The first ten equivalences follow swiftly from definitions and Theorem 2.3.1.

2 $\implies$ 11: Suppose that a nonempty set net $\{A_d\}_{d\in\mathcal{D}}$ of X converges to a nonempty set $A \subseteq X$ in the l.v.t. To show continuity of $\tilde{\mathrm{F}}$ we will show that $\tilde{\mathrm{F}}(A_d) \to \tilde{\mathrm{F}}(A)$ in the l.v.t. Note that by definition of $\tilde{\mathrm{F}}$ we can just show $\mathrm{F}[A_d] \to \mathrm{F}[A]$ in the l.v.t. Let $V \in \sigma$ have $V \cap \mathrm{F}[A] \neq \emptyset$ or equivalently we have $\mathrm{F}^{-}[V] \cap A \neq \emptyset$. By 2 $\mathrm{F}^{-}[V]$ is open, and since $A_d \to A$ in the l.v.t we apply Item 1 of Proposition 2.2.4 to get: there is a $D \in \mathcal{D}$ with for all $d \geq D$ we have $A_d \cap \mathrm{F}^{-}[V] \neq \emptyset$ or equivalently $\mathrm{F}[A_d] \cap V \neq \emptyset$. This holds for all open V with $V \cap \mathrm{F}[A]$. Hence, $\mathrm{F}[A_d] \to \mathrm{F}[A]$ in the l.v.t.

11 $\implies$ 12: Let $\{A_d\}_{d\in\mathcal{D}}$ be a nonempty set net of X. If $\mathrm{Li}_{d\in\mathcal{D}} A_d = \emptyset$ then $\mathrm{F}[\mathrm{Li}_{d\in\mathcal{D}} A_d] = \emptyset$ and 12 holds. So suppose that $\mathrm{Li}_{d\in\mathcal{D}} A_d \neq \emptyset$ an we know that by Item 1 of Proposition 2.2.6 that $A_d \to \mathrm{Li}_{d\in\mathcal{D}} A_d$ in the l.v.t. By definition of $\tilde{\mathrm{F}}$ and 11 we have that $\mathrm{F}[A_d] \to \mathrm{F}[\mathrm{Li}_{d\in\mathcal{D}} A_d]$ in the l.v.t. Again, by Item 1 of Proposition 2.2.6 we also have $\mathrm{F}[A_d] \to \mathrm{Li}_{d\in\mathcal{D}} \mathrm{F}[A_d]$ in the l.v.t and $\mathrm{Li}_{d\in\mathcal{D}} \mathrm{F}[A_d]$ is the largest set which $\{\mathrm{F}[A_d]\}_{d\in\mathcal{D}}$ converges to. Hence, $\mathrm{F}[\mathrm{Li}_{d\in\mathcal{D}} A_d] \subseteq \mathrm{Li}_{d\in\mathcal{D}} \mathrm{F}[A_d]$ and 12 holds.

12 $\implies$ 9: Suppose that a point net $\{x_d\}_{d\in\mathcal{D}}$ is a net of X which converges to some arbitrary $x \in X$. Then, $\{x_d\} \to \{x\}$ in the l.v.t and by 12 we have (noting $\mathrm{F}[z] = \mathrm{F}[\{z\}] = \tilde{\mathrm{F}}(\{z\})$ for all $z \in X$) $\mathrm{F}[x] \subseteq \mathrm{Li}_{d\in\mathcal{D}} \mathrm{F}[x_d]$ which is 9.

So far we have proven that the first twelve items are equivalent.

12 and 6 $\implies$ 13: Given $\{A_d\}_{d\in\mathcal{D}}$, define for all $D \in \mathcal{D}$ the set net $A_{\geq D} = \overline{\bigcup_{d\geq D} A_d}$ then, $\{A_{\geq D}\}_{D\in\mathcal{D}}$ is decreasing and by Item 10 of Proposition 2.2.5 we have $\mathrm{KLim}_{D\in\mathcal{D}} A_{\geq D} = \bigcap_{D\in\mathcal{D}} A_{\geq D}$. And in this case we have that $\mathrm{Li}_{D\in\mathcal{D}} A_{\geq D} = \mathrm{KLim}_{D\in\mathcal{D}} A_{\geq D} = \mathrm{Ls}_{d\in\mathcal{D}} A_d$ where that right most equality is from Item 2 of Proposition 2.2.5.

Now by 12 and 6 we have

$$\mathrm{F}\left[\operatorname*{Ls}_{d\in\mathcal{D}} A_d\right] = \mathrm{F}\left[\operatorname*{Li}_{D\in\mathcal{D}} A_{\geq D}\right] \subseteq \operatorname*{Li}_{D\in\mathcal{D}} \mathrm{F}[A_{\geq D}] = \operatorname*{Li}_{D\in\mathcal{D}} \mathrm{F}\left[\overline{\bigcup_{d\geq D} A_d}\right] \subseteq \operatorname*{Li}_{D\in\mathcal{D}} \overline{\bigcup_{d\geq D} \mathrm{F}[A_d]} = \operatorname*{Ls}_{d\in\mathcal{D}} \mathrm{F}[A_d]$$

where the rightmost equality follows from $\left\{\overline{\bigcup_{d\geq D} \mathrm{F}[A_d]}\right\}_{D\in\mathcal{D}}$ being decreasing and Items 2 and 10 of Proposition 2.2.5. Therefore, $\mathrm{F}[\mathrm{Ls}_{d\in\mathcal{D}} A_d] \subseteq \mathrm{Ls}_{d\in\mathcal{D}} \mathrm{F}[A_d]$ is true which is 13.

13 $\implies$ 10: Suppose that a point net $\{x_d\}_{d\in\mathcal{D}}$ is a net of X which converges to some arbitrary $x \in X$. Then, $\{x\} \subseteq \mathrm{Ls}_{d\in\mathcal{D}}$ and we see

$$\mathrm{F}[x] = \mathrm{F}[\{x\}] \subseteq \mathrm{F}\left[\operatorname*{Ls}_{d\in\mathcal{D}} \{x_d\}\right] \subseteq \operatorname*{Ls}_{d\in\mathcal{D}} \mathrm{F}[\{x_d\}] = \operatorname*{Ls}_{d\in\mathcal{D}} \mathrm{F}[x_d]$$

and 10 holds. □

Some of the Items of Theorem 2.3.2 are direct parallels of items in Theorem 2.1.1. The big difference being that there are now two pre-images to consider. If we believe the default definition of a function f being continuous on X means $\mathrm{f}^{-1}(V)$ is open in Y whenever V is open in X then, Item 2 of Theorem 2.3.2 is directly generalizes this idea replacing f^{-1} with F^{-}. Oddly, the complementary relation between the lower and upper pre-image causes Item 4 of Theorem 2.3.2 to use the upper pre-image rather than the lower. Therefore, one cannot naively replace the f^{-1}'s of Theorem 2.1.1 with F^{-}'s and have a correct theorem about lower semicontinuity.

Form a use in proofs standpoint, I find that Items 2, 6 and 9 are the most useful.

Interestingly, Items 7 and 11 of Theorem 2.3.2 tell us that lower semicontinuity and continuity with respect to the lower Vietoris topology are effectively the same thing. Since the lower Vietoris topology plays nicely with lower Kuratowski limits we get also get Items 9, 10, 12 and 13 for free. Item 9 is often convenient and it is rather unfortunate the upper semicontinuous multifunctions do not have an analogous property to this.

Theorem 2.3.3 (upper semicontinuity at a point)**.** *Let (X,τ) and (Y,σ) be topological spaces. Let $\mathrm{F} : X \rightsquigarrow Y$ be a multifunction and $x \in \mathrm{Dom}\,(\mathrm{F})$. Let $\hat{\mathrm{F}} : \mathrm{Dom}\,(\mathrm{F}) \to \mathcal{P}(Y)\setminus\{\emptyset\}$ be the function, $\hat{\mathrm{F}}(x') = \mathrm{F}[x']$ for all $x' \in \mathrm{Dom}\,(\mathrm{F})$. The following are equivalent:*

1. *F is upper semicontinuous at x.*
2. *For every open set $V \in \sigma$ with $\mathrm{F}[x] \subseteq V$ there is a $U \in \tau_x$ with $U \subseteq \mathrm{F}^{+}[V]$ or equivalently $\mathrm{F}[U] \subseteq V$.*

3. *For every* $B \subseteq Y$ *we have that if* $x \in \mathrm{F}^{+}[\mathrm{int}(B)]$ *then,* $x \in \mathrm{int}(\mathrm{F}^{+}[B])$*.*

4. *For every* $B \subseteq Y$ *we have that if* $x \in \overline{\mathrm{F}^{-}[B]}$ *then,* $x \in \mathrm{F}^{-}\left[\overline{B}\right]$*.*

5. $x \in \mathrm{int}(\mathrm{Dom}\,(\mathrm{F}))$ *and* $\hat{\mathrm{F}}$ *is continuous at* x *when we endow* $\mathcal{P}(Y)\setminus\{\emptyset\}$ *with the upper Vietoris topology (see Definition 2.2.1) and* $\mathrm{Dom}\,(\mathrm{F})$ *with the relative topology (see Proposition 2.1.2).*

6. $x \in \mathrm{int}(\mathrm{Dom}\,(\mathrm{F}))$ *and for every net* $\{x_d\}_{d\in\mathcal{D}}$ *in* $\mathrm{Dom}\,(\mathrm{F})$ *which converges to* x *we have that the set net* $\{\mathrm{F}[x_d]\}_{d\in D}$ *converges to* $\mathrm{F}[x]$ *in the upper Vietoris topology. That is, for every* $O \in \sigma$ *with* $\mathrm{F}[x] \subseteq O$ *there is a* $D \in \mathcal{D}$ *such that for all* $d \geq D$ *we have* $\mathrm{F}[x_d] \subseteq O$*.*

Proof. 1 $\Longrightarrow$ 2: When $V \in \sigma$ with $\mathrm{F}[x] \subseteq V$, we know by 1 that $\mathrm{F}^{+}[V]$ is a neighborhood of x. This means that there is a $U \in \tau_x$ with $U \subseteq \mathrm{F}^{+}[V]$, which is 2. Note that it follows immediately from the definition of the upper pre-image that $U \subseteq \mathrm{F}^{+}[V] \iff \mathrm{F}[U] \subseteq V$ when $U \subseteq \mathrm{Dom}\,(\mathrm{F})$.

2 $\Longrightarrow$ 3: Suppose that $x \in \mathrm{F}^{+}[\mathrm{int}(B)]$ where $B \subseteq Y$. Then, by 2 there is a $U \in \tau_x$ with $U \subseteq \mathrm{F}^{+}[\mathrm{int}(B)] \subseteq \mathrm{F}^{+}[B]$. Simply take the interior of this inclusion, to yield $U = \mathrm{int}(U) \subseteq \mathrm{int}(\mathrm{F}^{+}[B])$. As $x \in U$ we have that $x \in \mathrm{int}(\mathrm{F}^{+}[B])$ and 3 is affirmed.

3 $\Longrightarrow$ 4: Assuming that 3 holds we prove the contrapositive of 4. Suppose that $x \notin \mathrm{F}^{-}\left[\overline{B}\right]$ for $B \subseteq Y$. Then, $x \in \mathrm{F}^{+}\left[Y \setminus \overline{B}\right]$ by Item 6 of Proposition 2.3.1. Since $\mathrm{int}(Y \setminus \overline{B}) = Y \setminus \overline{B}$ we apply 3 and see $x \in U := \mathrm{int}(\mathrm{F}^{+}\left[Y \setminus \overline{B}\right]) \subseteq \mathrm{F}^{+}\left[Y \setminus \overline{B}\right]$. So,

$$\emptyset = U \cap X \setminus \mathrm{F}^{+}\left[Y \setminus \overline{B}\right] = U \cap \mathrm{F}^{-}\left[\overline{B}\right] \supseteq U \cap \mathrm{F}^{-}[B] = \emptyset$$

and we conclude that $x \notin \overline{\mathrm{F}^{-}[B]}$. As required.

4 $\Longrightarrow$ 1: Suppose that $V \in \sigma$ and $\mathrm{F}^{+}[V] \ni x$. Then, $x \notin \mathrm{F}^{-}[Y \setminus V] = \mathrm{F}^{-}\left[\overline{Y \setminus V}\right]$. By the contrapositive of 4 we have that $x \notin \overline{\mathrm{F}^{-}[Y \setminus V]}$, so there is $U \in \tau_x$ with $U \cap \mathrm{F}^{-}[Y \setminus V] = \emptyset$. Therefore, $U \subseteq X \setminus \mathrm{F}^{-}[Y \setminus V] = \mathrm{F}^{+}[V]$ and so $\mathrm{F}^{+}[V]$ is a neighborhood of x.

2 $\Longrightarrow$ 5: Let $V \in \sigma$ and then by Item 2 of Proposition 2.2.2 the collection $V^{+} = \{B \in \mathcal{P}(Y) \setminus \{\emptyset\} : B \subseteq V\}$ is an open basic set of the upper Vietoris topology. We claim

that $\hat{\mathrm{F}}^{-1}[V^+] = \mathrm{F}^+[V]$, indeed

$$\begin{aligned}\hat{\mathrm{F}}^{-1}(V^+) &= \left\{x \in \mathrm{Dom}\,(\mathrm{F}) : \hat{\mathrm{F}}(x) \in V^+\right\} \\ &= \left\{x \in \mathrm{Dom}\,(\mathrm{F}) : \hat{\mathrm{F}}(x) \subseteq V\right\} \\ &= \{x \in \mathrm{Dom}\,(\mathrm{F}) : \mathrm{F}[x] \subseteq V\} \\ &= \mathrm{F}^+[V].\end{aligned}$$

By 2 there is a $U \in \tau_x$ with $U \subseteq \mathrm{F}^+[V] = \hat{\mathrm{F}}^{-1}(V^+)$. Therefore, $\hat{\mathrm{F}}^{-1}(V^+)$ is a neighborhood of x and so $\hat{\mathrm{F}}$ is continuous at x.

5 $\implies$ 2: Suppose that $V \in \sigma$ has $\mathrm{F}[x] \subseteq V$ then, $\hat{\mathrm{F}}(x) \subseteq V$ which means $\hat{\mathrm{F}}(x) \in V^+$. Therefore, $x \in \hat{\mathrm{F}}^{-1}(V^+)$ and since $\hat{\mathrm{F}}$ is continuous at x and $x \in \mathrm{int}(\mathrm{Dom}\,(\mathrm{F}))$, $\hat{\mathrm{F}}(V^-)$ contains a set $U \in \tau_x$. By an argument in the 2 $\implies$ 5 part of this proof, we also have $\mathrm{F}^+[V] = \hat{\mathrm{F}}(V^+) \supseteq U$. Which proves 2.

5 $\iff$ 6: This equivalence follows quickly from the standard equivalences of functions continuous at a point preserving convergent nets which converge to that point. □

Theorem 2.3.4 (upper semicontinuity on X)**.** *Let (X, τ) and (Y, σ) be topological spaces. Let $\mathrm{F} : X \rightsquigarrow Y$ be a total multifunction. Let $\hat{\mathrm{F}} : \mathrm{Dom}\,(\mathrm{F}) \to \mathcal{P}(Y) \setminus \{\emptyset\}$ be the function, $\hat{\mathrm{F}}(x') = \mathrm{F}[x']$ for all $x' \in \mathrm{Dom}\,(\mathrm{F})$. The following are equivalent:*

1. *F is upper semicontinuous on X.*
2. *For every open set $V \in \sigma$ we have $\mathrm{F}^+[V] \in \tau$.*
3. *For every $B \subseteq Y$ we have that $\mathrm{F}^+[\mathrm{int}(B)] \subseteq \mathrm{int}(\mathrm{F}^+[B])$.*
4. *For every closed set $C \in \sigma^c$ we have $\mathrm{F}^-[C] \in \tau^c$.*
5. *For every $B \subseteq Y$ we have that $\overline{\mathrm{F}^+[B]} \subseteq \mathrm{F}^+\left[\overline{B}\right]$.*
6. *$\hat{\mathrm{F}}$ is continuous on X when we endow $\mathcal{P}(Y) \setminus \{\emptyset\}$ with the upper Vietoris topology (see Definition 2.2.1).*
7. *For every $x \in X$ and for every net $\{x_d\}_{d\in\mathcal{D}}$ in X which converges to x we have that the set net $\{\mathrm{F}[x_d]\}_{d\in D}$ converges to $\mathrm{F}[x]$ in the upper Vietoris topology. That is, for every $O \in \sigma$ with $\mathrm{F}[x] \subseteq O$ there is a $D \in \mathcal{D}$ such that for all $d \geq D$ we have $\mathrm{F}[x_d] \subseteq O$.*

8. $\tilde{\mathrm{F}} : \mathcal{P}(X) \setminus \{\emptyset\} \to \mathcal{P}(Y) \setminus \{\emptyset\}$ *defined by* $\tilde{\mathrm{F}}(A) = \mathrm{F}[A]$ *for* $A \subseteq X$ *is continuous when we endow* $\mathcal{P}(X) \setminus \{\emptyset\}$ *and* $\mathcal{P}(Y) \setminus \{\emptyset\}$ *with the upper Vietoris topology.*

Proof. The first seven equivalences of this theorem follows quickly from definitions and Theorem 2.3.3.

2 $\implies$ 8: Suppose that a nonempty set net $\{A_d\}_{d\in\mathcal{D}}$ of X converges to a nonempty set $A \subseteq X$ in the u.v.t. To show continuity of $\tilde{\mathrm{F}}$ we will show that $\tilde{\mathrm{F}}(A_d) \to \tilde{\mathrm{F}}(A)$ in the u.v.t. Note that by definition of $\tilde{\mathrm{F}}$ we can just show $\mathrm{F}[A_d] \to \mathrm{F}[A]$ in the u.v.t. Let $V \in \sigma$ have $V \supseteq \mathrm{F}[A]$ or equivalently we have $\mathrm{F}^+[V] \supseteq A$. By 2 $\mathrm{F}^+[V]$ is open, and since $A_d \to A$ in the u.v.t we apply Item 2 of Proposition 2.2.4 to get: there is a $D \in \mathcal{D}$ with for all $d \geq D$ we have $A_d \subseteq \mathrm{F}^+[V]$ or equivalently $\mathrm{F}[A_d] \subseteq V$. This holds for all open V with $V \supseteq \mathrm{F}[A]$, hence $\mathrm{F}[A_d] \to \mathrm{F}[A]$ in the u.v.t.

8 $\implies$ 7: Given any $x \in X$ and any net $\{x_d\}_{d\in\mathcal{D}}$ which converges to x we have that $\{x_d\} \to \{x\}$ in the u.v.t. By 8 we have that $\mathrm{F}[\{x_d\}] \to \mathrm{F}[\{x\}]$ in the u.v.t as well. Since, $\mathrm{F}[\{z\}] = \mathrm{F}[z]$ for all $z \in X$, we have that $\mathrm{F}[x_d] \to \mathrm{F}[x]$ in the u.v.t too. Thus, 7 is true. $\square$

The equivalences of Theorem 2.3.4 are unsurprising when we are already familiar with Theorem 2.3.2. What is surprising is what's not included in Theorem 2.3.4, I know of no analogue to Items 6, 9 and 10 of Theorem 2.3.2 which apply to u.s.c multifunctions. That being said under some additional assumptions we can recover some analogues of Items 9 and 10 of Theorem 2.3.2 for u.s.c multifunctions; however this ultimately is a consequence of a yet another (not yet mentioned, see Definition 2.3.6) type of continuity for multifunctions. Before we get explore this, we should first state the obvious theorems which combines l.s.c and u.s.c to give continuity.

Theorem 2.3.5 (Continuous multifunction (possibly without closed/compact values) at a point)**.** *Let* (X, τ) *and* (Y, σ) *be topological spaces. Let* $\mathrm{F} : X \rightsquigarrow Y$ *be a multifunction. Let* $\hat{\mathrm{F}} : X \to \mathcal{P}(Y) \setminus \{\emptyset\}$ *be the function,* $\hat{\mathrm{F}}(x') = \mathrm{F}[x']$ *for all* $x' \in X$*.*

1. F *is continuous at* x*.*

2. *For all* $V, O \in \sigma$ *with* $\mathrm{F}[x] \cap V \neq \emptyset$ *and* $\mathrm{F}[x] \subseteq O$ *there is a* $U \in \tau_x$ *with* $U \subseteq \mathrm{F}^-[V]$ *and* $U \subseteq \mathrm{F}^+[O]$*.*

3. *For every* $B \subseteq Y$ *we have that* $x \in \mathrm{F}^-[\mathrm{int}(B)] \implies x \in \mathrm{int}(\mathrm{F}^-[B])$ *and* $x \in \mathrm{F}^+[\mathrm{int}(B)] \implies x \in \mathrm{int}(\mathrm{F}^+[B])$*.*

4. *For every* $B \subseteq Y$ *we have that* $x \in \overline{\mathrm{F}^+[B]} \implies x \in \mathrm{F}^+\left[\overline{B}\right]$ *and* $x \in \overline{\mathrm{F}^-[B]} \implies x \in \mathrm{F}^-\left[\overline{B}\right]$*.*

5. $x \in \operatorname{int}(\operatorname{Dom}(\mathrm{F}))$ *and* $\hat{\mathrm{F}}$ *is continuous at* x *when we endow* $\mathcal{P}(Y) \setminus \{\emptyset\}$ *with the Vietoris topology (see Definition 2.2.1) and* $\operatorname{Dom}(\mathrm{F})$ *with the relative topology (see Proposition 2.1.2).*

6. $x \in \operatorname{int}(\operatorname{Dom}(\mathrm{F}))$ *and for every net* $\{x_d\}_{d \in \mathcal{D}}$ *in* $\operatorname{Dom}(\mathrm{F})$ *which converges to* x *we have that the set net* $\{\mathrm{F}[x_d]\}_{d \in D}$ *converges to* $\mathrm{F}[x]$ *in the Vietoris topology. That is, for every* $V, O \in \sigma$ *with* $\mathrm{F}[x] \cap V \neq \emptyset$ *and* $\mathrm{F}[x] \subseteq O$ *there is a* $D \in \mathcal{D}$ *such that for all* $d \geq D$ *we have* $\mathrm{F}[x_d] \cap V \neq \emptyset$ *and* $\mathrm{F}[x_d] \subseteq O$*.*

Proof. The proof follows quickly from Theorems 2.3.1 and 2.3.3 and definitions. □

Theorem 2.3.6 (Continuous multifunction (possibly without closed/compact values) on X)**.** *Let* (X, τ) *and* (Y, σ) *be topological spaces. Let* $\mathrm{F} : X \rightsquigarrow Y$ *be a total multifunction. Let* $\hat{\mathrm{F}} : X \to \mathcal{P}(Y) \setminus \{\emptyset\}$ *be the function,* $\hat{\mathrm{F}}(x') = \mathrm{F}[x']$ *for all* $x' \in X$*.*

1. F *is continuous on* X*.*

2. *For all* $V \in \sigma$ *we have* $\mathrm{F}^-[V] \in \tau$ *and* $\mathrm{F}^+[V] \in \tau$*.*

3. *For every* $B \subseteq Y$ *we have that* $\mathrm{F}^-[\operatorname{int}(B)] \subseteq \operatorname{int}(\mathrm{F}^-[B])$ *and* $\mathrm{F}^+[\operatorname{int}(B)] \subseteq \operatorname{int}(\mathrm{F}^+[B])$*.*

4. *For all* $C \in \sigma^c$ *we have* $\mathrm{F}^+[C] \in \tau^c$ *and* $\mathrm{F}^-[C] \in \tau^c$*.*

5. *For every* $B \subseteq Y$ *we have that* $\overline{\mathrm{F}^+[B]} \subseteq \mathrm{F}^+\left[\overline{B}\right]$ *and* $\overline{\mathrm{F}^-[B]} \subseteq \mathrm{F}^-\left[\overline{B}\right]$*.*

6. $\hat{\mathrm{F}}$ *is continuous at* x *when we endow* $\mathcal{P}(Y) \setminus \{\emptyset\}$ *with the Vietoris topology (see Definition 2.2.1).*

7. *For every net* $\{x_d\}_{d \in \mathcal{D}}$ *in* X *which converges to* $x \in X$ *we have that the set net* $\{\mathrm{F}[x_d]\}_{d \in D}$ *converges to* $\mathrm{F}[x]$ *in the Vietoris topology. That is, for every* $V, O \in \sigma$ *with* $\mathrm{F}[x] \cap V \neq \emptyset$ *and* $\mathrm{F}[x] \subseteq O$ *there is a* $D \in \mathcal{D}$ *such that for all* $d \geq D$ *we have* $\mathrm{F}[x_d] \cap V \neq \emptyset$ *and* $\mathrm{F}[x_d] \subseteq O$*.*

8. $\tilde{\mathrm{F}} : \mathcal{P}(X) \setminus \{\emptyset\} \to \mathcal{P}(Y) \setminus \{\emptyset\}$ *defined by* $\tilde{\mathrm{F}}(A) = \mathrm{F}[A]$ *for* $A \subseteq X$ *is continuous when we endow* $\mathcal{P}(X) \setminus \{\emptyset\}$ *and* $\mathcal{P}(Y) \setminus \{\emptyset\}$ *with the Vietoris topology.*

Proof. The proof follows quickly from Theorems 2.3.2, 2.3.4 and 2.3.5 and definitions. □

Often times, in practice only one of l.s.c or u.s.c is all that is needed in a proof. Which makes the use case for continuous multifunctions somewhat narrow. However, it will be seen later that in dynamical systems we often get a continuous multifunction for free. So it doesn't matter. The most notable property exclusive to a continuous multifunction is that the lower/upper pre-image of a clopen set (a closed and open set) is clopen; this fact can be useful in connected spaces.

We now discuss (approximately) three new types of continuity for multifunctions ones, which are closely related to Kuratowski limits.

Definition 2.3.6. *Let (X, τ) and (Y, σ) be topological spaces. Let $\mathrm{F} : X \rightsquigarrow Y$ be multifunction.*

1. F *is said to be inner semicontinuous at the point $x \in X$ if $\mathrm{F}[x] \neq \emptyset$ and for every net, say $\{x_d\}_{d\in\mathcal{D}}$, converging to x we have $\mathrm{F}[x] \subseteq \mathrm{Li}_{d\in\mathcal{D}} \mathrm{F}[x_d]$.* F *is said to be inner semicontinuous on a subset $B \subseteq X$ if, for all $b \in B$ we have that* F *is inner semicontinuous at x.* F *is said to be inner semicontinuous if,* F *is inner semicontinuous on X.*

2. F *is said to be outer semicontinuous (o.s.c) at the point $x \in X$ if $\mathrm{F}[x] \neq \emptyset$ and for every net, say $\{x_d\}_{d\in\mathcal{D}}$, converging to x we have $\mathrm{Ls}_{d\in\mathcal{D}} \mathrm{F}[x_d] \subseteq \mathrm{F}[x]$.* F *is said to be o.s.c on a subset $B \subseteq X$ if, for all $b \in B$ we have that* F *is o.s.c at x.* F *is said to be o.s.c if,* F *is o.s.c on X.*

3. F *is said to be Kuratowski continuous at the point $x \in X$ if $\mathrm{F}[x] \neq \emptyset$ and for every net, say $\{x_d\}_{d\in\mathcal{D}}$, converging to x we have $\mathrm{F}[x] = \mathrm{KLim}_{d\in\mathcal{D}} \mathrm{F}[x_d]$.* F *is said to be Kuratowski continuous on a subset $B \subseteq X$ if, for all $b \in B$ we have that* F *is Kuratowski continuous at x.* F *is said to be Kuratowski continuous if,* F *is Kuratowski continuous on X.*

Firstly, note that inner semicontinuity is effectively lower semicontinuity by Item 8 of Theorem 2.3.1, specifically a total multifunction is l.s.c at a point iff it is inner semicontinuous there. So there is not much to more to say about inner semicontinuity. On the other hand, outer semicontinuity and Kuratowski continuity are truly distinct concepts. However, it is more common in the literature to ignore the concept of (full) Kuratowski continuity in favor of simplifying speaking of l.s.c and o.s.c multifunctions. This reduces the number of definitions to consider.

We also note that o.s.c multifunctions are often called closed multifunctions (not to be confused with closed valued multifunctions) because the graph (see Definition 2.3.4) of a o.s.c multifunction is closed in the product topology. Moreover, it is not hard to see that if a multifunction is o.s.c at a point then the image at that point is a closed set. To see this recall that constant nets converge and constant set nets converge, in the upper Kuratowski sense, to the closure of the constant set. So only multifunctions with closed values can be o.s.c.

Example 2.3.3. *Consider $X = Y = [0, 1]$ in the usual topology on $\mathbb{R}$. Define*

$$\mathrm{F}_1[x] = \begin{cases} \{0, x^2, x^4, \dots, x^{2k}, \dots, \} & x < 1 \\ \{1\} & x = 1 \end{cases}$$

Then, F_1 is l.s.c and compact valued on X (therefore, inner semicontinuous) but is not o.s.c. To see why it is not o.s.c consider the sequence $\left\{2^{-\frac{1}{2k}}\right\}_{k\in\mathbb{N}}$ (which converges to 1) we see for all $k \in \mathbb{N}$ that $\frac{1}{2} = \left(2^{-\frac{1}{2k}}\right)^{2k} \in \mathrm{F}_1[2^{-\frac{1}{2k}}]$ but $\frac{1}{2} \notin \mathrm{F}_1[1]$.

Now, define

$$\mathrm{F}_2[x] = \begin{cases} \{0, x^2, x^4, \dots, x^{2k}, \dots, \} & x < 1 \\ [0, 1] & x = 1 \end{cases}$$

Then, F_2 is o.s.c (and u.s.c) and compact valued on X but is not l.s.c (therefore, not inner semicontinuous).

Now for $X = Y = \mathbb{R}$ in the usual topology on $\mathbb{R}$, we again consider,

$$\mathrm{F}_3[x] = \begin{cases} \{0\} & x < 0 \\ (-1, 1) & x \geq 0 \end{cases}$$

then, F_3 is u.s.c but is not l.s.c or o.s.c. It is not o.s.c since $\mathrm{Ls}_{n\in\mathbb{N}}\,\mathrm{F}_3[0] = \overline{(-1, 1)} \nsubseteq \mathrm{F}_3[0] = (-1, 1)$.

Lastly,

$$\mathrm{F}_4[x] = \begin{cases} \{\frac{1}{x}\} & x \neq 0 \\ \{0\} & x = 0 \end{cases}$$

is o.s.c and compact valued but not u.s.c or l.s.c. Note that F_4 is o.s.c at 0 since $\mathrm{Ls}_{n\in\mathbb{N}}\,\mathrm{F}_4[x_n] = \emptyset$, for a sequence $\{x_n \neq 0\}_{n\in\mathbb{N}}$ with $x_n \to 0$.

We now present some characterizations of o.s.c multifunctions.

Theorem 2.3.7. *Let (X, τ) and (Y, σ) be topological spaces. Let $\mathrm{F} : X \rightsquigarrow Y$ be multifunction. Suppose that $x \in \mathrm{Dom}\,(\mathrm{F})$. The following are equivalent:*

1. F *is o.s.c at* x. *That is, for every net* $\{x_d\}_{d\in\mathcal{D}}$ *of* $\mathrm{Dom}(\mathrm{F})$ *converging to* x *we have* $\mathrm{Ls}_{d\in\mathcal{D}}\,\mathrm{F}[x_d] \subseteq \mathrm{F}[x]$.

2. *For every net* $\{x_d\}_{d\in\mathcal{D}}$ *of* $\mathrm{Dom}(\mathrm{F})$ *converging to* x *we have* $\mathrm{Li}_{d\in\mathcal{D}}\,\mathrm{F}[x_d] \subseteq \mathrm{F}[x]$.

3. *For every net* $\{(x_d, y_d)\}_{d\in\mathcal{D}}$ *of* $\mathrm{Graph}(\mathrm{F})$ *which converges to* (x, y) *in the product topology, for some* $y \in Y$, *we have that* $y \in \mathrm{F}[x]$.

4. *For every net* $\{x_d\}_{d\in\mathcal{D}}$ *of* $\mathrm{Dom}(\mathrm{F})$ *converging to* x *and every point selection of* $\{\mathrm{F}[x_d]\}_{d\in\mathcal{D}}$, *say* $\{y_d \in \mathrm{F}[x_d]\}_{d\in\mathcal{D}}$, *which converges to* $y \in Y$ *we have that* $y \in \mathrm{F}[x]$.

5. *For every* $y \notin \mathrm{F}[x]$ *there is a* $V \in \sigma_y$ *and a* $U \in \tau_x$ *such that* $\mathrm{F}[U] \cap V = \emptyset$.

6. $\mathrm{F}[x] = \bigcap_{U\in\tau_x} \overline{\mathrm{F}[U]}$.

Proof. 1 $\implies$ 2: This implication follows from $\mathrm{Li}_{d\in\mathcal{D}}\,\mathrm{F}[x_d] \subseteq \mathrm{Ls}_{d\in\mathcal{D}}\,\mathrm{F}[x_d]$ for any net $\{x_d\}_{d\in\mathcal{D}}$, see Item 4 of Proposition 2.2.5.

2 $\implies$ 4: This implication follows from Item 2 of Proposition 2.2.7.

4 $\implies$ 3: Given any net $\{(x_d, y_d)\}_{d\in\mathcal{D}}$ of $\mathrm{Graph}(\mathrm{F})$ which converges to (x, y) for some $y \in Y$. We notice that since $(x_d, y_d) \in \mathrm{Graph}(\mathrm{F})$ we have $y_d \in \mathrm{F}[x_d]$ which means $\{y_d \in \mathrm{F}[x_d]\}_{d\in\mathcal{D}}$ is a convergent point selection of $\{\mathrm{F}[x_d]\}_{d\in\mathcal{D}}$. By 4 we have that $y \in \mathrm{F}[x]$ and so 3 holds.

3 $\implies$ 5: We proceed by contraposition. Suppose that 5 is false, then there is $y \in Y \setminus \mathrm{F}[x]$ such that for all $(U, V) \in \tau_x \times \sigma_y$ we have $\mathrm{F}[U] \cap V \neq \emptyset$. Hence, for all $(U, V) \in \tau_x \times \sigma_y$ there are $x_{(U,V)} \in U$ and $y_{(U,V)} \in \mathrm{F}\big[x_{(U,V)}\big] \cap V$. This forms a net where $\mathcal{D} = \tau_x \times \sigma_y$ where $(U, V) \preceq (W, O)$ iff $U \supseteq W$ and $V \supseteq O$. It is not hard to see that $\big\{(x_{(U,V)}, y_{(U,V)})\big\}_{(U,V)\in\tau_x\times\sigma_y}$ converges to (x, y) in the product topology; recalling that $y \notin \mathrm{F}[x]$ we find that 3 is false.

5 $\implies$ 6: The inclusion $\mathrm{F}[x] \subseteq \bigcap_{U\in\tau_x} \overline{\mathrm{F}[U]}$ always holds since $x \in U$ for all $U \in \tau_x$. We prove the other inclusion by contraposition. So suppose that $y \notin \mathrm{F}[x]$ then by 5 there are $V \in \sigma_y$ and $U' \in \tau_x$ with $\mathrm{F}[U'] \cap V = \emptyset$. From this we can deduce that $\bigcap_{U\in\tau_x} \overline{\mathrm{F}[U]} \cap V = \emptyset$ and so $y \notin \bigcap_{U\in\tau_x} \overline{\mathrm{F}[U]}$. This shows $\mathrm{F}[x] \supseteq \bigcap_{U\in\tau_x} \overline{\mathrm{F}[U]}$ and thus 6 holds.

6 $\implies$ 1: Suppose that $\{x_d\}_{d\in\mathcal{D}}$ of $\mathrm{Dom}(\mathrm{F})$ converging to x. Then for all $U \in \tau_x$ there is a $D_U \in \mathcal{D}$ with for all $d \geq D_U$ we have $x_d \in U$. Hence,

$$\overline{\mathrm{F}[U]} \supseteq \overline{\bigcup_{d\geq D_U} \mathrm{F}[x_d]}$$

for all $U \in \tau_x$ and by 6 we see $\mathrm{F}[x] = \bigcap_{U\in\tau_x} \overline{\mathrm{F}[U]} \supseteq \bigcap_{U\in\tau_x} \overline{\bigcup_{d\geq D_U} \mathrm{F}[x_d]}$. Also note that $\mathcal{D} \supseteq \{D_U : U \in \tau_x\}$ hence we can see

$$\mathrm{F}[x] \supseteq \bigcap_{U\in\tau_x} \overline{\bigcup_{d\geq D_U} \mathrm{F}[x_d]} \supseteq \bigcap_{D\in\mathcal{D}} \overline{\bigcup_{d\geq D} \mathrm{F}[x_d]} = \operatorname*{Ls}_{d\in\mathcal{D}} \mathrm{F}[x_d]$$

where the rightmost equality is Item 2 of Proposition 2.2.5. Therefore, 1 holds. □

Theorem 2.3.8. *Let (X,τ) and (Y,σ) be topological spaces. Let $\mathrm{F} : X \rightsquigarrow Y$ be a total multifunction. The following are equivalent:*

1. *F is o.s.c. That is, for all $x \in X$ and for every net $\{x_d\}_{d\in\mathcal{D}}$ converging to x we have $\mathrm{Ls}_{d\in\mathcal{D}} \mathrm{F}[x_d] \subseteq \mathrm{F}[x]$.*
2. *For all $x \in X$ and every net $\{x_d\}_{d\in\mathcal{D}}$ converging to x we have $\mathrm{Li}_{d\in\mathcal{D}} \mathrm{F}[x_d] \subseteq \mathrm{F}[x]$.*
3. *$\mathrm{Graph}\,(\mathrm{F})$ is closed in the product topology.*
4. *For all $x \in X$ and every net $\{x_d\}_{d\in\mathcal{D}}$ converging to x and every point selection of $\{\mathrm{F}[x_d]\}_{d\in\mathcal{D}}$, say $\{y_d \in \mathrm{F}[x_d]\}_{d\in\mathcal{D}}$, which converges to $y \in Y$ we have that $y \in \mathrm{F}[x]$.*
5. *For all $x \in X$ and every $y \notin \mathrm{F}[x]$ there is a $V \in \sigma_y$ and a $U \in \tau_x$ such that $\mathrm{F}[U] \cap V = \emptyset$.*
6. *For all $x \in X$ we have $\mathrm{F}[x] = \bigcap_{U\in\tau_x} \overline{\mathrm{F}[U]}$.*
7. *$\mathrm{Graph}\left(\mathrm{F}^-\right)$ is closed in the product topology.*

Proof. The proof of equivalences of Items 1, 2 and 4 to 6 follow quickly from Theorem 2.3.7 and definitions. The equivalence of Item 3 follows from recalling that a set is closed iff if it contains all the limits of its convergent nets.

This only leaves Item 7. Consider the transpose map $t : X \times Y \to Y \times X$ by $t(x,y) = (y,x)$ for all $(x,y) \in X \times Y$. It can be shown that this map is homeomorphism, that is to say t is continuous, the inverse map t^{-1} exists and is continuous when we endow the product topology on both $X \times Y$ and $Y \times X$.

When $\mathrm{Graph}\,(\mathrm{F})$ is closed (that is 3 holds)

$$\mathrm{Graph}\left(\mathrm{F}^-\right) = t(\mathrm{Graph}\,(\mathrm{F})) = (t^{-1})^{-1}(\mathrm{Graph}\,(\mathrm{F}))$$

is closed by continuity of t^{-1} see, Item 4 of Theorem 2.1.1. Similarly when $\mathrm{Graph}\left(\mathrm{F}^-\right)$ is closed (that is 7 holds) $t^{-1}(\mathrm{Graph}\left(\mathrm{F}^-\right)) = \mathrm{Graph}\,(\mathrm{F})$ is closed. This shows that Items 3 and 7 are equivalent. □

As mentioned before, o.s.c multifunctions are often called closed multifunctions. This is due to Item 3 of Theorem 2.3.8. I find all the items of Theorem 2.3.8 to be useful in proofs, but Items 1, 5 and 6 are my go to ones.

Recall, that for single valued functions having a closed graph is not equivalent to being continuous, even on functions $\mathbb{R}$ to $\mathbb{R}$. However, Theorem 2.1.1 says that when the range space is Hausdorff and compact then, having a closed graph is equivalent to being continuous. This suggests that by making compactness assumptions on the range space we can recover similar results connecting continuous multifunctions and o.s.c multifunctions. It turns out that we can get even sharper results, since limits in the upper Vietoris topology have connections to Kuratowski limits, see Proposition 2.2.6.

Theorem 2.3.9 (When are u.s.c multifunctions o.s.c?)**.** *Let (X, τ) and (Y, σ) be topological spaces. Let $\mathrm{F} : X \rightsquigarrow Y$ be u.s.c at $x \in X$. Then, F is o.s.c at x, if any of the following hold:*

1. *Y is regular and F is closed valued.*
2. *Y is Hausdorff and $\mathrm{F}[x]$ is compact.*
3. *X, Y are first countable, Y is Hausdorff and $\mathrm{F}[x]$ is closed.*

Proof. To prove 1 we apply Item 2 of Proposition 2.2.6. To elaborate, let $\{x_d\}_{d\in\mathcal{D}}$ be a net converging to x. By Item 6 of Theorem 2.3.3 we know the set net $\{\mathrm{F}[x_d]\}_{d\in\mathcal{D}}$ converges in the upper Vietoris topology to $\mathrm{F}[x]$; by Item 2 of Proposition 2.2.6 we have that $\mathrm{Ls}_{d\in\mathcal{D}}\, \mathrm{F}[x_d] \subseteq \overline{\mathrm{F}[x]} = \mathrm{F}[x]$. So F is o.s.c at x.

Now consider 2. Since Y is Hausdorff and $\mathrm{F}[x]$ is compact there are sets $V \in \sigma_y$ and $W \in \sigma$ with $W \supseteq \mathrm{F}[x]$ such that $W \cap V = \emptyset$. Indeed, for all $z \in \mathrm{F}[x]$ there are $V_z \in \sigma_y$ and $W_z \in \sigma_z$ such that $V_z \cap W_z = \emptyset$ by Hausdorffness. By compactness there are $z_1, \ldots, z_N \in \mathrm{F}[x]$ such that $\mathrm{F}[x] \subseteq \bigcup_{n=1}^N W_{z_n}$, now pick $V = \bigcap_{n=1}^N V_{z_n} \in \sigma_y$ and $W = \bigcup_{n=1}^N W_{z_n}$ to get $V \cap W = \emptyset$. Since F is u.s.c at x there is a $U \in \tau_x$ with $\mathrm{F}[U] \subseteq W$ (by Item 2 of Theorem 2.3.3) and we must have $\mathrm{F}[U] \cap V = \emptyset$. This shows Item 5 of Theorem 2.3.7 holds and F is o.s.c at x.

Lastly, we prove 3. Proceed by contradiction, suppose that F is not o.s.c at x then by Item 5 of Theorem 2.3.7 and first countablity there is a $y \notin \mathrm{F}[x]$ and countable collections of neighborhoods, $\{U_n\}_{n\in\mathbb{N}}$ of τ_x and $\{V_n\}_{n\in\mathbb{N}}$ of σ_y, with $U_{n+1} \subseteq U_n$, $V_{n+1} \subseteq V_n$, $V_n \subseteq Y \setminus \mathrm{F}[x]$ (since $\mathrm{F}[x]$ is closed), and $\mathrm{F}[U_n] \cap V_n \neq \emptyset$ for all $n \in \mathbb{N}$. Let $(x_n, y_n) \in X \times Y$ have $y_n \in \mathrm{F}[x_n] \cap V_n$ and $x_n \in U_n$ then $\mathrm{F}[x] \subseteq Y \setminus \overline{\{y_n : n \in \mathbb{N}\}}$, since $\{y_n\}_{n\in\mathbb{N}}$ is a sequence converging to y and Y is Hausdorff. Since F is u.s.c at x there is a $k \in \mathbb{N}$ with

$\mathrm{F}[U_k] \subseteq Y \setminus \overline{\{y_n : n \in \mathbb{N}\}}$ (by Item 2 of Theorem 2.3.3). But this is a contradiction since $y_k \in \mathrm{F}[x_k] \subseteq \mathrm{F}[U_k]$ and $y_k \notin Y \setminus \overline{\{y_n : n \in \mathbb{N}\}}$. Therefore, F is o.s.c at x. □

We see from Theorem 2.3.9 that in all the most important spaces, a closed valued u.s.c multifunction is also o.s.c. One may also wonder when an o.s.c is u.s.c, much like in the single valued case, when we have sufficient compactness everything works out.

Theorem 2.3.10 (When are o.s.c multifunctions u.s.c?)**.** *Let (X, τ) and (Y, σ) be topological spaces. Let $\mathrm{F} : X \rightsquigarrow Y$ be o.s.c at $x \in \mathrm{int}(\mathrm{Dom}\,(\mathrm{F}))$. Then,* F *is u.s.c at x, if there is a $U \in \tau_x$ and a compact $K \subseteq Y$ with $\mathrm{F}[U] \subseteq K$.*

Proof. Suppose that $\{x_d\}_{d \in \mathcal{D}}$ converges to x in X then there is a $D \in \mathcal{D}$ where $x_d \in U$ and so $\mathrm{F}[x_d] \subseteq \mathrm{F}[U] \subseteq K$. By Item 3 of Proposition 2.2.6 we know that $\mathrm{F}[x_d] \to \mathrm{Ls}_{d \in \mathcal{D}} \mathrm{F}[x_d]$ in the upper Vietoris topology; By o.s.c at x we have $\mathrm{Ls}_{d \in \mathcal{D}} \mathrm{F}[x_d] \subseteq \mathrm{F}[x]$ and Item 6 of Proposition 2.2.4 we know that $\mathrm{F}[x_d] \to \mathrm{F}[x]$ in the upper Vietoris topology. Hence, by Item 6 of Theorem 2.3.3 we know F is u.s.c at x. □

An immediate result of Theorem 2.3.9 is the following, which generalizes the "furthermore" of Theorem 2.1.1.

Corollary 2.3.10.1. *Let (X, τ) and (Y, σ) be topological spaces, with Y compact and Hausdorff. Let $\mathrm{F} : X \rightsquigarrow Y$ be a total multifunction. Then,* F *is u.s.c and closed valued if and only if* F *is o.s.c.*

Proof. When F is u.s.c and closed valued we know that it is also compact valued, as Y is compact. Since Y is also Hausdorff we apply Item 2 of Theorem 2.3.9 to get that F is o.s.c.

Conversely, when F is o.s.c then F is closed valued by Item 6 of Theorem 2.3.8. And noting that $\mathrm{F}[X] \subseteq Y$ and Y is compact, by Theorem 2.3.10 we have that F is u.s.c. □

We also consider a miscellaneous result, which attempts to recover Item 12 of Theorem 2.3.2 for o.s.c multifunctions.

Proposition 2.3.5. *Let (X, τ) and (Y, σ) be topological spaces. Let $\mathrm{F} : X \rightsquigarrow Y$ be o.s.c on X. If $K \subseteq X$ is compact and $\{A_d\}_{d \in \mathcal{D}}$ nonempty set net of X with $A_d \to K$ in the u.v.t then, we have that $\mathrm{Ls}_{d \in \mathcal{D}} \mathrm{F}[A_d] \subseteq \mathrm{F}[\mathrm{Ls}_{d \in \mathcal{D}} A_d]$.*

Proof. Let $y \in \mathrm{Ls}_{d \in \mathcal{D}} \mathrm{F}[A_d]$ then for all $V \in \sigma_y$ and for all $D \in \mathcal{D}$ there is a $d \geq D$ and a $x_d \in A_d$ with $V \cap \mathrm{F}[x_d] \neq \emptyset$. Define $(x_{(d,V)}, y_{(d,V)}) \in X \times Y$ to have $y_{(d,V)} \in V \cap \mathrm{F}\big[x_{(d,V)}\big]$

for all $d \in \mathcal{D}$ and $V \in \sigma_y$. This defines a net with ordering $(d, V) \succeq (e, W)$ iff $d \geq e$ and $V \subseteq W$. One can see that $y_{(d,V)} \to y$.

As $A_d \to K$, $x_{D,V} \in A_D$ for all $D \in \mathcal{D}$ and $V \in \sigma_y$, we have that $\{x_{D,V}\} \to K$ by Item 7 of Proposition 2.2.5. And by Item 4 of Proposition 2.2.6 we have that $\mathrm{Ls}_{(d,V)\in\mathcal{D}\times\sigma_y}\{x_{D,V}\} \cap K \neq \emptyset$. It follows that, $\left\{x_{(d,V)}\right\}_{(d,V)\in\mathcal{D}\times\sigma_y}$ has a convergent subnet whose limit is $x \in K$. Let $\{(x_\lambda, y_\lambda)\}_{\lambda\in\Lambda}$ be a subnet of $\left\{(x_{(d,V)}, y_{(d,V)})\right\}_{(d,V)\in\mathcal{D}\times\sigma_y}$ such that $x_\lambda \to x$ then, by Item 1 of Proposition 2.2.7 we know $x \in \mathrm{Ls}_{d\in\mathcal{D}} A_d$. Moreover, $y_\lambda \to y$ and by o.s.c (Item 4 of Theorem 2.3.8) we have $y \in \mathrm{F}[x] \subseteq \mathrm{F}[\mathrm{Ls}_{d\in\mathcal{D}} A_d]$. □

There are some remarkable theorems of continuous (single valued) functions we have not yet generalized to continuous multifunctions. Perhaps most notable theorems are: the extreme value theorem and intermediate value theorem.

Theorem 2.3.11 (Extreme value theorem for u.s.c multifunctions)**.** *Let (X, τ) and (Y, σ) be topological spaces. Let $\mathrm{F} : X \rightsquigarrow Y$ be u.s.c and compact valued. Then, for any compact $K \subseteq X$ we have that $\mathrm{F}[K]$ is compact.*

When F is merely l.s.c or o.s.c (but not u.s.c) and compact valued, $\mathrm{F}[K]$ can be non compact when $K \subseteq X$ is compact.

Proof. Suppose that $\mathcal{U}$ is an open cover of $\mathrm{F}[K]$ that is

$$\mathrm{F}[K] \subseteq \bigcup_{U\in\mathcal{U}} U$$

then for all $x \in K$ we have $\mathrm{F}[x] \subseteq \mathrm{F}[K]$ and so $\mathcal{U}$ is an open cover of $\mathrm{F}[x]$. Since F is compact valued then, $\mathrm{F}[x]$ is compact and let $\mathcal{U}_x$ be a finite sub-cover of $\mathrm{F}[x]$. We have that for all $x \in K$ that $\mathrm{F}[x] \subseteq \bigcup_{U\in\mathcal{U}_x} U$ and so $x \in \mathrm{F}^+\left[\bigcup_{U\in\mathcal{U}_x} U\right]$. It follows that

$$K \subseteq \bigcup_{x\in K} \mathrm{F}^+\left[\bigcup_{U\in\mathcal{U}_x} U\right],$$

and the RHS is an open cover of K since F is u.s.c. Since K is compact, there is a finite sub-cover and so there are $x_1, \ldots, x_N \in K$ with

$$K \subseteq \bigcup_{n=1}^{N} \mathrm{F}^+\left[\bigcup_{U\in\mathcal{U}_{x_n}} U\right] \subseteq \mathrm{F}^+\left[\bigcup_{n=1}^{N} \bigcup_{U\in\mathcal{U}_{x_n}} U\right]$$

(the rightmost $\subseteq$ follows from Item 5 of Proposition 2.3.1) and

$$\mathrm{F}[K] \subseteq \bigcup_{n=1}^{N} \bigcup_{U \in \mathcal{U}_{x_n}} U.$$

Since each $\mathcal{U}_{x_n}$ is finite, the RHS is a finite sub-cover of $\mathcal{U}$. Therefore, $\mathrm{F}[K]$ is compact.

To prove the second claim of the theorem, let $X = \mathbb{R}$ and $Y = \mathbb{R}$ with the usual topology. Define

$$\mathrm{F}[x] = \begin{cases} [0, 2^x] & x \in (0,1) \\ \{0\} & x \notin (0,1) \end{cases} \qquad \mathrm{G}[x] = \begin{cases} \left\{\frac{1}{x}\right\} & x > 0 \\ \{0\} & x \leq 0 \end{cases}$$

then $\mathrm{F}[[0,1]] = [0,2)$ and $\mathrm{G}[[0,1]] = [1,\infty) \cup \{0\}$ which are not compact. Moreover, F is l.s.c since for $x_0 = 0, 1$ $\mathrm{F}[x_0] \subseteq \mathrm{F}[x]$ and thus for any sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x_0 we have $\mathrm{F}[x_0] \subseteq \mathrm{Li}_{n\in\mathbb{N}} \mathrm{F}[x_n]$. So F is l.s.c at $x_0 = 0, 1$, other arguments can show that F is l.s.c everywhere else.

Similarly, one can show that G is o.s.c. $\square$

We do need compact valuedness of F in Theorem 2.3.11, since $\{x\}$ is always a compact set. Also note that a single valued function can always be thought of as a compact valued multifunction.

We now present a related result of when the images of multifunctions are closed sets.

Proposition 2.3.6. *Let (X, τ) and (Y, σ) be topological spaces. Let $\mathrm{F} : X \rightsquigarrow Y$ be o.s.c. Then, for every compact $K \subseteq X$ we have that $\mathrm{F}[K]$ is closed.*

It is possible for $\mathrm{F}[C]$ to not be closed even when $C \subseteq X$ is closed.

When F is not o.s.c but is l.s.c this result may fail, even when F is closed valued.

Proof. Let $y \in \overline{\mathrm{F}[K]}$ then for all $V \in \sigma_y$ there is a $x_V \in K$ and a $y_V \in V \cap \mathrm{F}[x_V]$. By compactness of K there is a subnet $\{x_{V_d}\}_{d\in\mathcal{D}}$ converging to $x \in K$, also we have that $\{y_{V_d} \in \mathrm{F}[x_{V_d}]\}_{d\in\mathcal{D}}$ converging to y. By o.s.c we have $\mathrm{F}[x] \supseteq \mathrm{Li}_{d\in\mathcal{D}} \mathrm{F}[x_{V_d}]$, it follows from Item 2 of Proposition 2.2.7 that $y \in \mathrm{Li}_{d\in\mathcal{D}} \mathrm{F}[x_{V_d}] \subseteq \mathrm{F}[x]$ and $y \in \mathrm{F}[x] \subseteq \mathrm{F}[K]$.

For the counter examples we can use F and G as defined in the proof of Theorem 2.3.11, as $\mathrm{F}[[0,1]] = [0,2)$ is not closed and $\mathrm{G}[[1,\infty)] = (0,1]$ is not closed. $\square$

We conclude this section with the intermediate value theorem.

Theorem 2.3.12. *Let (X, τ) and (Y, σ) be topological spaces and let $\mathrm{F} : X \rightsquigarrow Y$ be a l.s.c (or u.s.c) multifunction with connected values. Then, if $C \subseteq X$ is connected then $\mathrm{F}[C]$ is connected in Y.*

Proof. Assume that $C \subseteq X$ is connected and suppose that a nonempty set $B \subseteq \mathrm{F}[C]$ is a closed and open. For this proof we will assume $\mathrm{F} : C \to \mathrm{F}[C]$, as we must work with relative topology, see Definition 2.1.5.

Thus, we have that $\mathrm{F}[C] = B \cup \mathrm{F}[C] \setminus B$ and we claim for all $c \in C$ that $\mathrm{F}[c] \subseteq B$ or $\mathrm{F}[c] \subseteq \mathrm{F}[C] \setminus B$. For if $\mathrm{F}[c] \cap B \neq \emptyset$ and $\mathrm{F}[c] \cap \mathrm{F}[C] \setminus B \neq \emptyset$ then, $\mathrm{F}[c] = (B \cap \mathrm{F}[c]) \cup ((\mathrm{F}[C] \setminus B) \cap \mathrm{F}[c])$ and in $(\mathrm{F}[c], \sigma|_{\mathrm{F}[c]})$ both $B \cap \mathrm{F}[c]$ and $(\mathrm{F}[C] \setminus B) \cap \mathrm{F}[c]$ are closed and open nonempty sets, so $\mathrm{F}[c]$ is not connected. A contradiction.

However, that implies $\mathrm{F}^{-}[B] = \mathrm{F}^{+}[B]$. Indeed, $\mathrm{F}^{-}[B] \supseteq \mathrm{F}^{+}[B]$ follows from Item 7 of Proposition 2.3.1. If $c \in \mathrm{F}^{-}[B]$ then, $\mathrm{F}[c] \cap B \neq \emptyset$. From the above argument $\mathrm{F}[c] \subseteq B$ or $\mathrm{F}[c] \subseteq \mathrm{F}[C] \setminus B$, so $\mathrm{F}[c] \subseteq B$ must be the case. Thus, $c \in \mathrm{F}^{+}[B]$ and $\mathrm{F}^{-}[B] = \mathrm{F}^{+}[B]$. But B is closed and open, so $\mathrm{F}^{-}[B]$ is open (closed) since F is l.s.c (u.s.c) and $\mathrm{F}^{+}[B]$ is closed (open) since F is l.s.c (u.s.c). Therefore, we have a closed and open set $\mathrm{F}^{-}[B] \subseteq C$ so either $\mathrm{F}^{-}[B] = C$ or $\mathrm{F}^{-}[B] = \emptyset$. Since $B \neq \emptyset$ and $\mathrm{F}[C] \supseteq B$ there must be a $c \in C$ with $\mathrm{F}[c] \cap B \neq \emptyset$. Hence, $\mathrm{F}^{-}[B] \neq \emptyset$ and so $\mathrm{F}^{-}[B] = \mathrm{F}^{+}[B] = C$. In particular $\mathrm{F}^{+}[B] \supseteq C$. From this we see $\mathrm{F}[C] \subseteq \mathrm{F}\big[\mathrm{F}^{+}[B]\big] \subseteq B$ by Item 9 of Proposition 2.3.1 and so $\mathrm{F}[C] = B$. Therefore, $\mathrm{F}[C]$ is connected.

□

Note being connected valued is necessary, since $\{x\}$ is always connected.

2.3.2 Operations which preserve continuity of multifunctions

In practice one might not have much control over the multifunction under consideration. Nevertheless, it is natural to want the multifunction to be closed, compact or open valued in certain situations. Thus, it is practical to ask if $\mathrm{F} : X \rightsquigarrow Y$ is a "continuous" multifunction then, is the pointwise closure $\overline{\mathrm{F}}[x] = \overline{\mathrm{F}[x]}$ also "continuous"? Similar questions can be asked for interiors and even convex hulls. In this subsection we explore how (and when) continuity is preserved when we perform natural set operations on multifunctions.

First, we should approach an even more basic question. If $\mathrm{f} : X \to Y$ is continuous then, is $\{\mathrm{f}(x)\}$ continuous? To simplify notation, we identify the multifunction $\{\mathrm{f}(x)\}$ with $\mathrm{f}[x]$.

Proposition 2.3.7. *Let (X,τ) and (Y,σ) be topological spaces. Suppose that $\mathrm{f} : X \to Y$ is a function. The following hold:*

1. *If f is continuous at x (on X) then, $\{\mathrm{f}(\cdot)\}$ is continuous at x (on X).*

2. *If f has closed graph then, $\{\mathrm{f}(\cdot)\}$ has closed graph and so is o.s.c.*

Proof. Item 1 follows from the fact that $\mathrm{f}^{-1}(B) = \mathrm{f}^{-}[B] = \mathrm{f}^{+}[B]$ for all $B \subseteq Y$ and Item 2 Theorem 2.3.6.

Item 2 follows from $\mathrm{Graph}\,(\mathrm{f}) = \mathrm{Graph}\,(\{\mathrm{f}(\cdot)\})$. □

Another, basic question is that of composition of multifunctions, is the composition of two continuous multifunctions continuous?

Proposition 2.3.8. *Let (X,τ), (Y,σ) and (Z,ρ) be topological spaces. Let $\mathrm{F} : X \rightsquigarrow Y$, $\mathrm{G} : Y \rightsquigarrow Z$. The following hold:*

1. *If F,G are l.s.c or u.s.c then, $\mathrm{G} \circ \mathrm{F}$ is l.s.c or u.s.c respectively.*

2. *If F,G are u.s.c and compact valued then, $\mathrm{G} \circ \mathrm{F}$ is u.s.c and compact valued. Moreover, if Y and Z are also Hausdorff then, F,G and $\mathrm{G} \circ \mathrm{F}$ are o.s.c too.*

3. *Assume Z is regular, G is u.s.c, closed valued, F is u.s.c and compact valued. Then, $\mathrm{G} \circ \mathrm{F}$ and G are u.s.c and o.s.c.*

Proof. Firstly, consider 1. By Proposition 2.3.3 we know that $(\mathrm{G} \circ \mathrm{F})^{-} = \mathrm{F}^{-} \circ \mathrm{G}^{-}$ and $(\mathrm{G} \circ \mathrm{F})^{-} = \mathrm{F}^{+} \circ \mathrm{G}^{+}$ it follows that when $V \in \rho$ that $\mathrm{F}^{-} \circ \mathrm{G}^{-}[V] = \mathrm{F}^{-}\big[\mathrm{G}^{-}[V]\big]$ is open when F,G are l.s.c (by Item 2 of Theorem 2.3.6). Hence, $\mathrm{G} \circ \mathrm{F}$ is l.s.c. Similarly, when $V \in \rho$ that $\mathrm{F}^{+} \circ \mathrm{G}^{+}[V] = \mathrm{F}^{+}\big[\mathrm{G}^{+}[V]\big]$ is open when F,G are u.s.c (by Item 2 of Theorem 2.3.6). So, $\mathrm{G} \circ \mathrm{F}$ is u.s.c.

To prove 2, first note that $\mathrm{G} \circ \mathrm{F}$ is u.s.c by 1. And for all $x \in X$, $\mathrm{G} \circ \mathrm{F}[x] = \mathrm{G}[\mathrm{F}[x]]$ and $\mathrm{F}[x]$ is compact. By Theorem 2.3.11 $\mathrm{G}[\mathrm{F}[x]]$ is compact and $\mathrm{G} \circ \mathrm{F}$ is compact valued.

When Y is Hausdorff, we apply Item 2 of Theorem 2.3.9 to conclude that F,G and $\mathrm{G} \circ \mathrm{F}$ are o.s.c.

Finally, we prove 3. By assumptions and Item 1 of Theorem 2.3.9 we know that G is o.s.c. Since F is compact valued, for all $x \in X$ we know that $\mathrm{G} \circ \mathrm{F}[x] = \mathrm{G}[\mathrm{F}[x]]$ is closed by Proposition 2.3.6 we know that $\mathrm{G} \circ \mathrm{F}[x]$ is closed valued. In turn, we again apply Item 1 of Theorem 2.3.9 to conclude that $\mathrm{G} \circ \mathrm{F}[x]$ is o.s.c. The composition $\mathrm{G} \circ \mathrm{F}[x]$ is u.s.c by 1 of this Proposition. □

One should note that even in the case of single valued functions which map $\mathbb{R}$ to $\mathbb{R}$ the composition of two functions with closed graph can fail to be closed. Thus, we cannot expect the composition of two o.s.c multifunctions to be o.s.c.

We now consider applying some set operations to a multifunction, and we see how these operations preserve continuity. For notational convenience, given $\mathrm{F} : X \rightsquigarrow Y$ we define $\overline{\mathrm{F}} : X \rightsquigarrow Y$ and $\mathrm{clF} : X \rightsquigarrow Y$ to be $\overline{\mathrm{F}[x]} = \overline{\mathrm{F}}[x] = \mathrm{clF}[x]$ for all $x \in X$. Similarly, define $\mathrm{intF} : X \rightsquigarrow Y$ to be $\mathrm{intF}[x] = \mathrm{int}(\mathrm{F}[x])$ for all $x \in X$.

Proposition 2.3.9. *Let (X, τ) and (Y, σ) be topological spaces. Let $\mathrm{F} : X \rightsquigarrow Y$. The following holds:*

1. *If F is l.s.c at $x \in X$ then, $\overline{\mathrm{F}}$ is l.s.c at x.*
2. *If F is u.s.c at $x \in X$ and Y be normal then, $\overline{\mathrm{F}}$ is u.s.c at x.*
3. *If F is u.s.c at $x \in X$ and Y be regular then, $\overline{\mathrm{F}}$ is o.s.c at x.*
4. *If F is u.s.c at $x \in X$, $\overline{\mathrm{F}}[x]$ is compact and Y be regular then, $\overline{\mathrm{F}}$ is u.s.c and o.s.c at x.*
5. *Let $\mathcal{U}$ be an open cover of X. If F is l.s.c then, $\mathrm{F}_{\mathcal{U}} : X \rightsquigarrow X$ defined by $\mathrm{F}_{\mathcal{U}}[x] = \bigcup\{U \in \mathcal{U} : U \cap \mathrm{F}[x] \neq \emptyset\}$ for all $x \in X$ is l.s.c.*
6. *If F is is l.s.c at $x \in X$ and for all $z \in Z$ we have $\overline{\mathrm{int}(\overline{\mathrm{F}}[z])} = \overline{\mathrm{F}}[z]$ then, $\mathrm{int}\,\overline{\mathrm{F}}$ is l.s.c at x.*

Proof. For 1 we use Item 5 of Theorem 2.3.1. Let $A \subseteq X$ have $x \in \overline{A}$ then, by l.s.c of F we have $\mathrm{F}[x] \subseteq \overline{\mathrm{F}[A]}$. Taking closures of both sides yields

$$\overline{\mathrm{F}}[x] = \overline{\mathrm{F}[x]} \subseteq \overline{\mathrm{F}[A]} = \overline{\bigcup_{a \in A} \mathrm{F}[a]} = \overline{\bigcup_{a \in A} \overline{\mathrm{F}[a]}} = \overline{\overline{\mathrm{F}}[A]}$$

which shows that $\overline{\mathrm{F}}$ is l.s.c at x.

To see 2, recall Item 6 of Theorem 2.3.3 and suppose that $\{x_d\}_{d \in \mathcal{D}}$ is a net of $\mathrm{Dom}\,(\mathrm{F}) = \mathrm{Dom}\left(\overline{\mathrm{F}}\right)$ converging to x. Then, by u.s.c of F at x we know that $\mathrm{F}[x_d] \to \mathrm{F}[x]$ in the u.v.t. Since Y is normal we can apply Item 4 of Proposition 2.2.4 and so $\overline{\mathrm{F}[x_d]} \to \overline{\mathrm{F}[x]}$ in the u.v.t. It follows that $\overline{\mathrm{F}}$ is u.s.c at x.

We now consider 3. Again, suppose that $\{x_d\}_{d \in \mathcal{D}}$ is a net of $\mathrm{Dom}\,(\mathrm{F}) = \mathrm{Dom}\left(\overline{\mathrm{F}}\right)$ converging to x. Then, by u.s.c (Item 6 of Theorem 2.3.3) we know $\mathrm{F}[x_d] \to \mathrm{F}[x]$ in the u.v.t.

Now, by Item 2 of Proposition 2.2.6 we know $\operatorname{Ls}_{d\in\mathcal{D}} \mathrm{F}[x_d] = \bigcap\{\overline{C} : \mathrm{F}[x_d] \to C \text{ in the u.v.t}\}$. Hence, $\operatorname{Ls}_{d\in\mathcal{D}} \mathrm{F}[x_d] \subseteq \overline{\mathrm{F}[x]} = \overline{\mathrm{F}}[x]$ and by Item 3 of Proposition 2.2.5 we have $\operatorname{Ls}_{d\in\mathcal{D}} \overline{\mathrm{F}}[x_d] = \operatorname{Ls}_{d\in\mathcal{D}} \overline{\mathrm{F}[x_d]} = \operatorname{Ls}_{d\in\mathcal{D}} \mathrm{F}[x_d] \subseteq \overline{\mathrm{F}}[x]$. By definition, $\overline{\mathrm{F}}$ is o.s.c.

For 4, we can apply 3 to get that $\overline{\mathrm{F}}$ is o.s.c. Suppose that $\{x_d\}_{d\in\mathcal{D}}$ is a net of $\operatorname{Dom}(\mathrm{F}) = \operatorname{Dom}\left(\overline{\mathrm{F}}\right)$ converging to x. Then, by u.s.c of F at x we know that $\mathrm{F}[x_d] \to \mathrm{F}[x]$ in the u.v.t. Since Y is regular and $\overline{\mathrm{F}[x]}$ is compact, we can apply Item 4 of Proposition 2.2.4 and so $\overline{\mathrm{F}[x_d]} \to \overline{\mathrm{F}[x]}$ in the u.v.t. It follows that $\overline{\mathrm{F}}$ is u.s.c at x.

We now prove 5. Let $V \in \tau$ and $x \in \mathrm{F}^-_{\mathcal{U}}[V]$. Then, there is a $U \in \mathcal{U}$ with both $U \cap V \neq \emptyset$ and $U \cap \mathrm{F}[x] \neq \emptyset$. Since F is l.s.c we know that $\mathrm{F}^-[U]$ is an open set containing x. If $z \in \mathrm{F}^-[U]$ then, $\mathrm{F}[z] \cap U \neq \emptyset$ and by definition $U \subseteq \mathrm{F}_{\mathcal{U}}[z]$. But then,

$$\emptyset \neq U \cap V \subseteq \mathrm{F}_{\mathcal{U}}[z] \cap V,$$

so $z \in \mathrm{F}^-_{\mathcal{U}}[V]$. This shows that $x \in \mathrm{F}^-[U] \subseteq \mathrm{F}^-_{\mathcal{U}}[V]$, since $\mathrm{F}^-[U]$ and x was an arbitrary member of $\mathrm{F}^-_{\mathcal{U}}[V]$ we conclude that $\mathrm{F}^-_{\mathcal{U}}[V]$ is open. So $\mathrm{F}_{\mathcal{U}}$ is l.s.c.

Lastly, for 6 we can see that for any $V \in \sigma$ we have

$$\emptyset \neq V \cap \overline{\mathrm{F}}[z] = V \cap \overline{\operatorname{int}(\overline{\mathrm{F}[z]})} \iff \emptyset \neq V \cap \operatorname{int}(\overline{\mathrm{F}[z]}).$$

for all $z \in X$. It follows that $\operatorname{cl}\mathrm{F}^-[V] = \operatorname{int}\operatorname{cl}\mathrm{F}^-[V]$ for all $V \in \sigma$. By Item 1 of this Proposition, we know that $\operatorname{cl}\mathrm{F}$ is l.s.c at x. Thus, if $V \in \sigma$ has $V \cap \overline{\operatorname{int}(\overline{\mathrm{F}[z]})} \neq \emptyset$ we know that $\operatorname{cl}\mathrm{F}^-[V] = \operatorname{int}\operatorname{cl}\mathrm{F}^-[V]$ is a neighborhood of x. Therefore, $\operatorname{int}\overline{\mathrm{F}}$ is l.s.c at x. □

Proposition 2.3.9 broadly tells us that taking the pointwise closure of a multifunction preserves or enhances the continuity properties of the multifunction. The more delicate case is taking the pointwise interior. The Vietoris topologies thinks that taking interiors of sets is destructive, which means continuity with respect to these topologies may not be preserved. See Example 2.3.4. Also note that o.s.c multifunctions are always pointwise closed, therefore the interior of a o.s.c multifunction is seldom o.s.c.

Example 2.3.4. *Let $X = Y = \mathbb{R}$ with the usual topology. Let $\mathbb{Q}$ be the rational numbers. Define,*

$$\mathrm{F}[x] = \begin{cases} (-1,0) \cup (\mathbb{Q} \cap (0,1)) & x \neq 0 \\ (-1,1) & x = 0 \end{cases}$$

then F *is l.s.c at* 0 *but*

$$\operatorname{int}\mathrm{F}[x] = \begin{cases} (-1,0) & x \neq 0 \\ (-1,1) & x = 0 \end{cases}$$

is not l.s.c at 0.

Define,

$$G[x] = [-1, x]$$

for $x \geq 0$ *then,* G *is u.s.c (on* $[0, \infty)$*) but*

$$\text{int}G[x] = (-1, x)$$

is nowhere u.s.c (on $[0, \infty)$*). To see this, let* $x \in [0, \infty)$ *and note that* $(-1, x) = \text{int}G[x]$ *is an open set containing* $\text{int}G[x]$*. However, the sequence* $\left\{x + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ *converges to* x *from above and* $\text{int}G[x] \subsetneq \text{int}G\left[x + \frac{1}{n}\right]$*. This means that* $\left\{\text{int}G\left[x + \frac{1}{n}\right]\right\}_{n\in\mathbb{N}}$ *does not converge to* $\text{int}G[x]$ *in the u.v.t (see Item 2 of Proposition 2.2.4) and so is not u.s.c at* x *(by Item 6 of Theorem 2.3.3).*

We now consider when the union and/or intersection of multifunctions are continuous.

Proposition 2.3.10. *Let* (X, τ) *and* (Y, σ) *be topological spaces. Let* $x \in X$ *and* $\mathcal{F}$ *be a set of multifunctions which map* X *to* Y*. The following hold:*

1. *Let*

$$F_\cup[x] = \bigcup_{F\in\mathcal{F}} F[x]$$

for all $x \in X$.

1a) If every $F \in \mathcal{F}$ *is l.s.c at* x *then,* $F_\cup$ *is l.s.c at* x.

1b) If $\mathcal{F}$ *is finite and every* $F \in \mathcal{F}$ *is u.s.c at* x *then,* $F_\cup$ *is u.s.c at* x.

1c) If $\mathcal{F}$ *is finite and every* $F \in \mathcal{F}$ *is o.s.c at* x *then,* $F_\cup$ *is o.s.c at* x.

1d) Item 1a can fail when we replace "l.s.c" with continuous, u.s.c, or o.s.c.

2. *Let*

$$F_\cap[x] = \bigcap_{F\in\mathcal{F}} F[x]$$

for all $x \in X$.

2a) If every $F \in \mathcal{F}$ *is o.s.c at* x *and* $F_\cap[x] \neq \emptyset$ *then,* $F_\cap$ *is o.s.c at* x.

2b) If every $F \in \mathcal{F}$ *is o.s.c at* x*,* $x \in \text{int}(\text{Dom}\,(F_\cap))$*, there is a* $F \in \mathcal{F}$*,* $U \in \tau_x$ *and a compact set* $K \subseteq Y$ *with* $F[U] \subseteq K$ *then,* $F_\cap$ *is u.s.c and o.s.c at* x.

2c) *Even if every* $\mathrm{F} \in \mathcal{F}$ *is l.s.c on* X, $\mathrm{F}_\cap$ *is total and* $\mathcal{F}$ *is finite then,* $\mathrm{F}_\cap$ *can fail to be l.s.c.*

Proof. Firstly, we consider 1a. We will prove a slightly stronger result: Let $x \in X$, if for every $\mathrm{F} \in \mathcal{F}$ with $x \in \mathrm{Dom}(\mathrm{F})$ we have that F is l.s.c at x then, $\mathcal{F}_\cup$ is l.s.c at x.

Let $V \in \sigma$ have $V \cap \mathrm{F}_\cup[x] \neq \emptyset$ then, by Item 2 of Proposition 2.3.2 we have that $x \in \mathrm{F}_\cup^-[V] = \bigcup_{\mathrm{F}\in\mathcal{F}} \mathrm{F}^-[V]$. Then, there is a $\mathrm{F} \in \mathcal{F}$ such that $x \in \mathrm{F}^-[V]$, it follows that $x \in \mathrm{Dom}(\mathrm{F})$ and by assumption F is l.s.c at x. Thus, there is a $U \in \tau_x$ with $U \subseteq \mathrm{F}^-[V]$ and $U \subseteq \bigcup_{\mathrm{F}\in\mathcal{F}} \mathrm{F}^-[V] = \mathrm{F}_\cup^-[V]$. This shows that $\mathrm{F}_\cup$ is l.s.c at x.

The proof of 1b is similar. Let $O \in \sigma$ have $O \supseteq \mathrm{F}_\cup[x]$ then for every $\mathrm{F} \in \mathcal{F}$ we know $\mathrm{F}[x] \subseteq O$. Since every $\mathrm{F} \in \mathcal{F}$ is by assumption u.s.c, we know there is a $U_\mathrm{F} \in \tau_x$ with $U_\mathrm{F} \subseteq \mathrm{F}^+[O]$. Therefore, $U = \bigcap_{\mathrm{F}\in\mathcal{F}} U_\mathrm{F} \subseteq \bigcap_{\mathrm{F}\in\mathcal{F}} \mathrm{F}^+[O]$; since $\mathcal{F}$ is finite, $U \in \tau_x$. Again, by Item 2 of Proposition 2.3.2 we have that $\bigcap_{\mathrm{F}\in\mathcal{F}} \mathrm{F}^+[O] = \mathrm{F}_\cup^+[O]$. So we have shown that $\mathrm{F}_\cup^+[O] \supseteq U$, which shows that $\mathrm{F}_\cup$ is u.s.c at x, by Item 2 of Theorem 2.3.3.

Next, we consider 1c. Suppose that $\{x_d\}_{d\in\mathcal{D}}$ is a net of $\mathrm{Dom}(\mathrm{F}_\cup)$ converging to x. Then, by Item 8a of Proposition 2.2.5, finiteness of $\mathcal{F}$ and o.s.c the elements of $\mathcal{F}$ at x we have,

$$\operatorname*{Ls}_{d\in\mathcal{D}} \mathrm{F}_\cup[x_d] = \operatorname*{Ls}_{d\in\mathcal{D}} \bigcup_{\mathrm{F}\in\mathcal{F}} \mathrm{F}[x_d] = \bigcup_{\mathrm{F}\in\mathcal{F}} \operatorname*{Ls}_{d\in\mathcal{D}} \mathrm{F}[x_d] \subseteq \bigcup_{\mathrm{F}\in\mathcal{F}} \mathrm{F}[x] = \mathrm{F}_\cup[x].$$

By definition, this means $\mathrm{F}_\cup$ is o.s.c at x.

For 1d we provide a counterexample. Let $X = Y = [0,1] \subseteq \mathbb{R}$ with the usual topology. Define, $\mathrm{F}_n[x] = \{x^n\}$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Let $\mathcal{F} = \{\mathrm{F}_n : n \in \mathbb{N}\}$ then, every element of $\mathcal{F}$ is continuous and o.s.c but $\mathrm{F}_\cup$ is not u.s.c at 1 and is only o.s.c at 0 (as $\mathrm{F}_\cup[x]$ is closed iff $x = 0$). One can see the former by considering the sequence $\left\{\sqrt[n]{0.5}\right\}_{n\in\mathbb{N}}$ then $0.5 \in \mathrm{F}_\cup\left[\sqrt[n]{0.5}\right]$ for all $n \in \mathbb{N}$ but $\mathrm{F}_\cup[1] = \{1\}$. So $\left\{\mathrm{F}_\cup\left[\sqrt[n]{0.5}\right]\right\}_{n\in\mathbb{N}}$ cannot converge to $\{1\}$ in the u.v.t.

Next, we consider 2a. Suppose that $\{x_d\}_{d\in\mathcal{D}}$ is a net of X converging to x. Then,

$$\operatorname*{Ls}_{d\in\mathcal{D}} \mathrm{F}_\cap[x_d] = \operatorname*{Ls}_{d\in\mathcal{D}} \bigcap_{\mathrm{F}\in\mathcal{F}} \mathrm{F}[x_d] \subseteq \bigcap_{\mathrm{F}\in\mathcal{F}} \operatorname*{Ls}_{d\in\mathcal{D}} \mathrm{F}[x_d] \subseteq \bigcap_{\mathrm{F}\in\mathcal{F}} \mathrm{F}[x] = \mathrm{F}_\cap[x]$$

where the leftmost $\subseteq$ is by Item 8b of Proposition 2.2.5 and the rightmost $\subseteq$ is by o.s.c at x of every $\mathrm{F} \in \mathcal{F}$. This shows that $\mathrm{F}_\cap$ is o.s.c by definition.

Now for Item 2b, we recall Theorem 2.3.10. Note that when $\mathrm{F}_\cap$ is o.s.c (by 2a) and one map $\mathrm{F} \in \mathcal{F}$ has $\mathrm{F}[U] \subseteq K$ for K compact and $U \in \tau_x$ then, $\mathrm{F}_\cap[U] \subseteq K$ as well. Since $x \in \mathrm{int}(\mathrm{Dom}(\mathrm{F}_\cap))$ too, we can apply Theorem 2.3.10 to get that $\mathrm{F}_\cap$ is u.s.c at x too.

Lastly, for 2c we provide a counterexample. Let $X = Y = \mathbb{R}$ with the usual topology. Define, $\mathrm{F}_1[x] = \{x^2\} \cup \{1\}$ and $\mathrm{F}_2[x] = \{-x^2\} \cup \{1\}$ (which is the union of l.s.c multifunctions and so are l.s.c) then,

$$\mathrm{F}_\cap[x] = \begin{cases} \{1\} & x \neq 0 \\ \{0\} \cup \{1\} & x = 0 \end{cases}$$

which is not l.s.c at 0. □

2.4 A brief glimpse of computability

The genesis of this thesis involves a condition for a particular multifunction to be computable. In the section we give a mostly informal explanation of what it means for a function or multifunction to be computable.

Computability theory explores what kind of problems can be solved by a computer. This leads to an axiom/definition called the Church Turing Thesis:

Axiom 2.4.1 (Church Turing Thesis)**.** *A problem is computable (that is the problem can be solved by a computer) if and only if the problem can be solved by a Turing machine.*

A problem that can be solved by a Turing machine is called: decidable, computable or recursive. A problem which cannot be solved by a Turing machine is called: undecidable or incomputable.

I will not be defining a Turing machine, this would take a long time and ultimately there is no payoff for doing so in the context of this work. All I would like the reader to know of Turing machines is the following:

1. A (type I) Turing machine is essentially a partial function from $\mathbb{N}$ to $\mathbb{N}$.
2. The set of all Turing machines, $\mathfrak{C}_I$ sometimes called the computable functions, are closed under composition, multiplication and addition.
3. $\mathfrak{C}_I$ is a countable set.

In the context of analysis/topology we are often not working with $\mathbb{N}$. How do we use computability theory in such contexts? One way to bridge these to fields is to use type II computability theory, I refer the reader to [17]. The reader will not need to understand, anything substantial about computability theory to understand the results outside this section. For completeness we will give the reader a taste of type II computability theory bellow.

In type II computability theory, type II Turing machines are used. Type II Turing machines are allowed to run forever unlike regular Turing machines; this is useful if we want to describe typical algorithms used in numerical analysis. Since type II Turing machine are allowed to run forever, we can identify them as functions from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$ (we can also identify them as functions from $\mathbb{N}$ to $\mathbb{N}$ when convenient) and we let the set of all type II Turing machines to be $\mathfrak{C}_{II}$, this set is still countable. To work with a metric/topological space, one provides a *representation* of points in the space. The standard representation, for a separable metric space of interest (X, d), is a surjective partial function $\delta : \mathbb{N}^{\mathbb{N}} \to X$ which has: For some known enumeration of $\mathbb{Q}\times\{x_n :\in \mathbb{N}\}$, say $(q_1, q_2) : \mathbb{N} \to \mathbb{Q}\times\{x_n :\in \mathbb{N}\}$ where $\{x_n :\in \mathbb{N}\}$, is a known countable dense set of X, δ satisfies

$$\delta(c) = x \iff \{x\} = \bigcap_{n\in\mathbb{N}} \{y \in X : \mathrm{d}(q_1 \circ c(n), y) < q_2 \circ c(n)\}$$

for all $x \in X$ and $c \in \mathbb{N}^{\mathbb{N}}$.

For example the standard representation of $(\mathbb{R}, |\cdot|)$, can be achieved by letting the dense set of $\mathbb{R}$ be the rationals. Then, for $c \in \mathbb{N}^{\mathbb{N}}$, the sets $\{y \in X : \mathrm{d}(q_1 \circ c(n), y) < q_2 \circ c(n)\}$ are open intervals with rational endpoints and when $\delta(c) = x \in \mathbb{R}$ these intervals uniquely determine x. Note that not every $c \in \mathbb{N}^{\mathbb{N}}$ will correspond to a real number, each c with $x = \delta(c)$ for some x is called a name for x. The set of computable real numbers (under this representation) is $\delta(\mathfrak{C}_{II})$. A typical algorithm in numerical analysis, which outputs a number, does not output an exact solution. It outputs an approximation, usually with an error bound. This is what the standard representation tries to formalize, for each $n \in \mathbb{N}$, $q_1 \circ c(n)$ is an approximation and $q_2 \circ c(n)$ is an error bound.

A function $f : \mathbb{R} \to \mathbb{R}$ is computable, if there exists a type II machine $p_f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ with $f \circ \delta = \delta \circ p_f$ (whenever the LHS is defined). Suppose that we wish, to find the value of $f(x)$. First, we find a name for x, say c (so $\delta(c) = x$). Next, we allow the type II machine p_f to compute on that name, resulting in $p_f(c) \in \mathbb{N}^{\mathbb{N}}$. Finally, we interpret $p_f(c)$ to be a name of $f(x)$, so to get $f(x)$ we use δ and $f(x) = f \circ \delta(c) = \delta \circ p_f(c)$. Intuitively, the type II Turing machine p_f is doing the computation on a name in the domain to produce a name in the range.

The reader may notice that the above definitions of a computable number/function depends highly on the representation, δ. Indeed, different representations do yield different computable numbers/functions. Therefore, when we apply computability theory to analysis the words "computable" and "incomputable" is not absolute, and a reason I personally do not take incomputability results too seriously. On the other hand, there is a way to compare two representations and it can be shown that the standard representation above is the "best representation which is compatible with the topology" in some sense. Which is a strong argument for the relevance of computability theory to analysis.

The unfortunate reality of computability theory is that not every elementary relation/problem on $\mathbb{R}$ is computable. We need more descriptive language to know what kind of problems are at least somewhat computable.

Definition 2.4.1. *Let $P(x)$ be a logical proposition (it has a truth value of true or false, but not both) dependent on x, eg $P(x) : x = 0$.*

P is said to be semi-decidable or semi-computable when there is a Turing machine T such that for all x, if $P(x)$ is true then, T that can verify that $P(x)$ is true. When $P(x)$ is false we do not require anything of T.

P is said to be co-semi-decidable or co-semi-computable when there is a Turing machine T such that for all x, if $P(x)$ is false then, T that can verify that $P(x)$ is false. When $P(x)$ is true we do not require anything of T.

It can be shown that if P is both semi-decidable and co-semi-decidable then it is decidable.

Proposition 2.4.1. *Let $x, y \in \mathbb{R}$, the following hold:*

1. *The problem, determine if $x < y$ is semi-computable but not computable.*
2. *The problem, determine if $x \geq y$ is co-semi-computable but not computable.*
3. *The problem, determine if $x = y$ is co-semi-computable but not computable.*

Sketch of proof. Note that problems 1 and 2 are negations of each other, thus problem 1 is semi-computable iff problem 2 is co-semi-computable. Similarly, problem 1 is co-semi-computable iff problem 2 is semi-computable. Which means that if problem 1 is not computable then neither is problem 2.

Suppose that c_x is a name of x and c_y is a name of y. When $x < y$ there is an $N \in \mathbb{N}$ with $q_1 \circ c_x(N) + q_2 \circ c_x(N) < q_1 \circ c_y(N) - q_2 \circ c_y(N)$, note that this comparison can be made in finite time since we are comparing two rational numbers. This N can be found by searching exhaustively from $1, 2, \ldots$ until such an N is found, when it is found we know that the interval $(q_1 \circ c_x(N) - q_2 \circ c_x(N), q_1 \circ c_x(N) + q_2 \circ c_x(N)) \ni x$ lies to the left of the interval $(q_1 \circ c_y(N) - q_2 \circ c_y(N), q_1 \circ c_y(N) + q_2 \circ c_y(N)) \ni y$, so we also know $x < y$.

In contrast, when $x \geq y$ and in particular when $x = y$, merely checking the intervals we find $(q_1 \circ c_y(n) - q_2 \circ c_y(n), q_1 \circ c_y(n) + q_2 \circ c_y(n)) \ni x, y$ and $(q_1 \circ c_x(n) - q_2 \circ c_x(n), q_1 \circ c_x(n) + q_2 \circ c_x(n)) \ni x, y$ for all $n \in \mathbb{N}$. This does not tell us that x and y are equal; only that they are nearby to each-other. Therefore, I assert it is not possible to find if $x = y$ by only checking a finite number of intervals given by the names of x, y and conclude that problem 1 is not co-semi-computable. □

Proposition 2.4.1 is a specific example of a more general phenomenon. Mainly, it is semi-computable to determine if a point (or a compact set) is in an open set. A computable topological space (X, τ) is a Hausdorff, second countable, locally compact space, with some extra structure which allows one to define a standard representation similar to that given for metric spaces above, see [7, Chapter 3] or [17].

Theorem 2.4.1. *Let (X, τ) be a computable topological space. Let $x \in X$, $V \in \tau$, $C \in \tau^c$ and $K \subseteq X$ be compact. The following hold:*

1. *The problem, determine if $x \in V$ is semi-computable but not computable.*
2. *The problem, determine if $x \in C$ is co-semi-computable but not computable.*
3. *The problem, determine if $K \subseteq V$ is semi-computable but not computable.*
4. *The problem, determine if $K \cap V \neq \emptyset$ is semi-computable but not computable.*

To the best of my knowledge the above theorem is the most general it in terms of the type of sets considered and when we fix the predicate being considered. That is, suppose that $x \in X$ and $A \subseteq X$, the problem/predicate, $x \in A$, is only semi-computable when A is open.

We should also consider the most interesting result about computability theory (as it relates to analysis), namely that all computable functions are continuous.

Theorem 2.4.2. *Let $(X, \tau), (Y, \sigma)$ be computable topological spaces and $f : X \to Y$. If f is computable then, f is continuous.*

Theorem 2.4.2 discounts the simplest of discontinuous functions being computable, eg the sign function,

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

for $x \in \mathbb{R}$ is not continuous and so not computable. This is because the problem of: determine if $x = 0$ is not computable. So a computer cannot say $f(x) = 0$ with certainty, as it cannot tell if $x = 0$ is true.

Chapter 3

Basics of difference inclusions

The broad topic of this thesis is: How do we approximate the long term behaviour of dynamical systems? And, is it even possible to approximate the long term behaviour? To make this problem a little smaller we focus our efforts on discrete time dynamical systems and even more specifically difference inclusions.

Suppose that X is a set and $\mathrm{f} : X \to X$. The formula

$$x_{n+1} = \mathrm{f}(x_n)$$

for $n \in \mathbb{N} \cup \{0\}$ and $x_0 \in X$, is called an autonomous difference equation. The set of points $\{x_n\}_{n=0}^{\infty}$ which satisfy the difference equation is called a trajectory starting at x_0. Difference equations are some of the simplest dynamical systems. Most mathematical modelling doesn't use difference equations, it is more typical to use continuous time models like differential equations. However, most methods of approximating continuous time models, use difference equations. Moreover, some fields seem to prefer difference equations for their mathematical models, for example many economic models use difference equations.

Intuitively, one can consider the function f to be a "law of motion" or an "location update rule", f takes a point $x \in X$ and moves it to some other point $\mathrm{f}(x) \in X$. Note that the updated location $\mathrm{f}(x)$ depends only on f (the update rule) and x (the starting point); there is no concept of an intelligent agent who is at x and has multiple options to move. In some applications, these options are important and can be modelled by equations of the form:

$$x_{n+1} = \mathrm{f}(x_n, u_n)$$

where, $\mathrm{f} : X \times U \to X$, U is a set, $n \in \mathbb{N} \cup \{0\}$, $u_n \in U$ and $x_0 \in X$. Such equations we call difference equations with control. A sequence $\{x_n\}_{n=0}^{\infty}$ which satisfies the equation for some sequence $\{u_n\}_{n=0}^{\infty}$, is called a trajectory starting at x_0 with controller $\{u_n\}_{n=0}^{\infty}$. The controller can potentially be chosen to achieve certain goals, things like: Make sure the trajectory never enters an unsafe region, or make sure the trajectory is equal to a point y eventually.

A further generalization of this, are difference inclusions. Let $\mathrm{F} : X \rightsquigarrow X$, the formula

$$x_{n+1} \in \mathrm{F}[x_n]$$

for $n \in \mathbb{N} \cup \{0\}$ and $x_0 \in X$, is called an autonomous difference inclusion. A set of points $\{x_n\}_{n=0}^{\infty}$ which satisfy the difference inclusion is called a trajectory starting at x_0. A difference equation with control can always be written as a difference inclusion, consider $\mathrm{f} : X \times U \to X$ as before then, define $\mathrm{F}[x] = \mathrm{f}(\{x\} \times U)$ for all $x \in X$. Now any trajectory of F is also trajectory of f with some controller and vice versa.

As a broad remark, the nice continuity properties of $\mathrm{F}[x] = \mathrm{f}(\{x\} \times U)$ are not free by any means. Even when all the functions f are continuous, F is only guaranteed to be l.s.c (see Proposition 2.3.10). F may fail to be u.s.c, o.s.c, or closed/compact valued, even in relativity simple cases.

Henceforth, we will only be considering difference inclusions and difference equations.

3.1 Trajectories, invariant and viable sets of difference inclusions

In order to examine the long term behavior of difference inclusions we fist must understand the short term behavior, at least a little. To simplify notation we define $\mathbb{N}_z = \{z, z+1, z+2, \ldots, \}$ for $z \in \mathbb{Z}$.

Definition 3.1.1 (Trajectories of difference inclusions)**.** *Let X be a set, $\mathrm{F} : X \rightsquigarrow X$ be a multifunction and $\{x_n\}_{n\in\mathbb{N}_0}$ be a sequence of X. Then, $\{x_n\}_{n\in\mathbb{N}_0}$ is said to be a trajectory of* F *starting at x_0 if*

$$x_{n+1} \in \mathrm{F}[x_n]$$

for $n \in \mathbb{N} \cup \{0\}$. The point x_0 is called initial point/condition of the trajectory. Often we call a trajectory of F *starting at x_0, a trajectory of* F*. Sometimes we simply call a trajectory of* F*, a trajectory, when there is no confusion of the multifunction being considered.*

A nice thing about difference inclusions is that trajectories actually exist, in most circumstances.

Proposition 3.1.1. *Let X be a set, $\mathrm{F} : X \rightsquigarrow X$ be a total multifunction. Then, for any $x_0 \in X$ there is a trajectory of* F *starting at x_0.*

Proof. Since F is total and the axiom of choice, there is a $\mathrm{f} \in \Pi_{x\in X} \mathrm{F}[x] \subseteq Y^X$. Hence, the sequence defined by $x_{n+1} = \mathrm{f}(x_n)$ for $n \in \mathbb{N}_0$, also has $x_{n+1} = \mathrm{f}(x_n) \in \mathrm{F}[x_n]$. Which means that $\{x_n\}_{n\in\mathbb{N}_0}$ is a trajectory of F starting at x_0. □

To simplify our analysis, we will assume that a multifunction involved in a difference inclusions is total, unless otherwise stated.

A large difference between difference equations and difference inclusions is that: difference inclusions can have multiple distinct trajectories originating from the same point, whereas a difference equations can only have one. This makes it possible to ask all sorts of questions which are trivial in the difference equations case.

Proposition 3.1.2 (Limit of Trajectories is a Trajectory)**.** *Let (X, τ) be a topological space, $\mathrm{F} : X \rightsquigarrow X$ be a o.s.c multifunction and $\{x_{\lambda n}\}_{(\lambda,n)\in\Lambda\times\mathbb{N}_0}$ be a net with both:*

1. *For all $\lambda \in \Lambda$ and all $n \in \mathbb{N}_0$ we have $x_{\lambda(n+1)} \in \mathrm{F}[x_{\lambda n}]$. i.e for fixed λ the sequence $\{x_{\lambda n}\}_{n\in\mathbb{N}}$ is a trajectory of* F *with initial point $x_{\lambda 0}$.*

2. *For all $n \in \mathbb{N}_0$ the net $\{x_{\lambda n}\}_{\lambda\in\Lambda}$ converges to x_n.*

Then, $\{x_n\}_{n\in\mathbb{N}}$ is a trajectory of F *with initial point x_0.*

Proof. Recall (see Theorem 2.3.8) that since F is o.s.c we have that for any nets $\{x_d\}_{d\in\mathcal{D}}$, $\{y_d\}_{d\in\mathcal{D}}$ with $\lim_{d\in\mathcal{D}} x_d = x$, $\lim_{d\in\mathcal{D}} y_d = y$ and $y_d \in \mathrm{F}[x_d]$ then, $y \in \mathrm{F}[x]$.

Fix any $n \in \mathbb{N}_0$ then by 1 $x_{\lambda(n+1)} \in \mathrm{F}[x_{\lambda n}]$ for all $\lambda \in \Lambda$. By 2 and o.s.c of F take the limit of $x_{\lambda(n+1)}$ and $x_{\lambda n}$ over Λ to get $x_{n+1} \in \mathrm{F}[x_n]$. The conclusion follows. □

The assumption of F being a o.s.c multifunction is necessary for the conclusion of Proposition 3.1.2 to hold. For if F where not o.s.c at x_0 then there would be nets $\{x_{\lambda 1}\}_{\lambda\in\Lambda}$ converging to x_1 and $\{x_{\lambda 0}\}_{\lambda\in\Lambda}$ converging to x_0 with $x_{\lambda 1} \in \mathrm{F}[x_{\lambda 0}]$ for all $\lambda \in \Lambda$ but $x_1 \notin \mathrm{F}[x_0]$. Hence, this limit of trajectories cannot even "start".

Item 2 of Proposition 3.1.2 can be difficult to arrange for in practice, it is often easier so appeal to compactness arguments to get there.

Proposition 3.1.3. *Let (X, τ) be a topological space, $\mathrm{F} : X \rightsquigarrow X$ be a total o.s.c multifunction, and $\{x_{\lambda n}\}_{(\lambda,n)\in\Lambda\times\mathbb{N}_0}$ be a net satisfying $x_{\lambda(n+1)} \in \mathrm{F}[x_{\lambda n}]$ for all $\lambda \in \Lambda$ and all $n \in \mathbb{N}_0$. If one of the following holds,*

1. *There is a set sequence $\{K_n\}_{n\in\mathbb{N}_0}$ of compact sets of X with for all $\lambda \in \Lambda$ and all $n \in \mathbb{N}_0$ we have $x_{\lambda n} \in K_n$.*

2. F *is also compact valued and u.s.c. There is a compact set K_0 with $x_{\lambda 0} \in K_0$ for all $\lambda \in \Lambda$.*

Then, there a subnet of $\{x_{\lambda n}\}_{(\lambda,n)\in\Lambda\times\mathbb{N}_0}$ of the form $\{x_{\lambda_d n}\}_{(d,n)\in\mathcal{D}\times\mathbb{N}_0}$ with: For all $n \in \mathbb{N}_0$ the net $\{x_{\lambda_d n}\}_{d\in\mathcal{D}}$ converges to x_n and $\{x_n\}_{n\in\mathbb{N}_0}$ is a trajectory of F *with initial point x_0.*

Proof. Suppose that 1 holds then, we can regard the net $\{x_{\lambda n}\}_{(\lambda,n)\in\Lambda\times\mathbb{N}_0}$ of X to be a net of $\Pi_{n\in\mathbb{N}_0} K_n$. To elaborate let $x_{\lambda *} = \{x_{\lambda n}\}_{n\in\mathbb{N}_0}$ for $\lambda \in \Lambda$ then, $\{x_{\lambda *}\}_{\lambda\in\Lambda}$ is a net of $\Pi_{n\in\mathbb{N}_0} K_n$. By Item 5 of Proposition 2.1.6 we know that $\Pi_{n\in\mathbb{N}_0} K_n$ is compact with respect to the product topology; which means that $\{x_{\lambda *}\}_{\lambda\in\Lambda}$ has a convergent subnet, say $\{x_{\lambda_d *}\}_{d\in\mathcal{D}}$ converging to $\{x_n\}_{n\in\mathbb{N}_0} \in \Pi_{n\in\mathbb{N}_0} K_n$. It can be shown from the definition of the product topology (again see Proposition 2.1.3) that for all $n \in \mathbb{N}_0$ we have $x_{\lambda_d n} \rightarrow_{d\in\mathcal{D}} x_n$ and by assumption we have $x_{\lambda_d(n+1)} \in \mathrm{F}[x_{\lambda_d n}]$ for all $d \in \mathcal{D}$. Hence, we can apply Proposition 3.1.2 to get that $\{x_n\}_{n\in\mathbb{N}_0}$ is a trajectory.

When 2 holds, we can observe that F composed with itself $n \in \mathbb{N}$ times remains compact valued and u.s.c by Item 2 of Proposition 2.3.8. For $n = 0$ let $\mathrm{F}^{\circ 0}[x] = \{x\}$, for $n = 1$ let $\mathrm{F}^{\circ 1}[x] = \mathrm{F}[x]$ and $n \in \mathbb{N}_1$ define $\mathrm{F}^{\circ(n+1)}[x] = \mathrm{F} \circ \mathrm{F}^{\circ n}[x]$ for all $x \in X$. Then, $\mathrm{F}^{\circ 0}[K_0] = K_0$ is compact and for all $n \in \mathbb{N}$ we have $\mathrm{F}^{\circ n}[K_0]$ is compact by Theorem 2.3.11. Defining, $K_n = \mathrm{F}^{\circ n}[K_0]$ for $n \in \mathbb{N}$ allows us to apply 1 of this proposition for the result. □

The proof of Proposition 3.1.3 touches a concept more general than a trajectory, that of iterating a multifunction (composing a multifunction with itself). For many problems and in the most important scenarios, analysis of the iterates of a multifunction is equivalent to the analysis of trajectories.

Definition 3.1.2. *Let X be a set, $\mathrm{F} : X \rightsquigarrow X$ be a multifunction and $\{x_n\}_{n \in \mathbb{N}_0}$ be a sequence of X. Then, $\{x_n\}_{n \in \mathbb{N}_0}$ is said to be an iterative* F *selection starting at x_0 if*

$$x_n \in \mathrm{F}^{\circ n}[x_0]$$

for $n \in \mathbb{N} \cup \{0\}$. Where we define $\mathrm{F}^{\circ n}[x] = \mathrm{F} \circ \mathrm{F}^{\circ(n-1)}[x]$ with $\mathrm{F}^{\circ 1}[x] = \mathrm{F}[x]$ and $\mathrm{F}^{\circ 0}[x] = \{x\}$ for all $x \in X$ and $n \in \mathbb{N}$. Also define $(\mathrm{F}^{\circ n})^- = \mathrm{F}^{\circ n-} = \mathrm{F}^{-\circ n}$ and $(\mathrm{F}^{\circ n})^+ = \mathrm{F}^{\circ n+} = \mathrm{F}^{+\circ n}$, these notations are sensible by Proposition 2.3.3.

The point x_0 is called initial point/condition of the iterative F *selection. Often we call an iterative* F *selection starting at x_0, a* F *selection.*

Given a multifunction F, it may be natural to also consider set valued difference equations alongside F selections and trajectories of F. That is considering a sequence of sets which satisfy $X_{n+1} = \mathrm{F}[X_n]$ for $n \in \mathbb{N}_0$. I hope that it is clear that finite time behaviour of the set valued difference equation can be described by the finite time behaviour of (many) F selections with initial point in X_0. Infinite time behaviour of the set valued difference equation can also be adequately described by the infinite time behaviour of (many) F selections, when the space is first countable see Item 4 of Proposition 2.2.7. Thus, we will not concern ourselves with set valued difference equations and instead briefly consider F selections.

Proposition 3.1.4. *Let X be a set, $\mathrm{F} : X \rightsquigarrow X$ be a multifunction and $\{x_n\}_{n \in \mathbb{N}_0}$ be a sequence of X. If $\{x_n\}_{n \in \mathbb{N}_0}$ is a trajectory of* F *then, $\{x_n\}_{n \in \mathbb{N}_0}$ an* F *selection. The converse is false.*

Proof. Suppose that $\{x_n\}_{n \in \mathbb{N}_0}$ is a trajectory of F. We proceed by induction to prove that $x_n \in \mathrm{F}^{\circ n}[x_0]$ for all $n \in \mathbb{N}_0$. By definition we have that $\{x_0\} = \mathrm{F}^{\circ 0}[x_0]$, so the base case holds. So assume for $n \in \mathbb{N}$ that $x_n \in \mathrm{F}^{\circ n}[x_0]$ then, $\{x_n\} \subseteq \mathrm{F}^{\circ n}[x_0]$ and applying F to both sides of this inclusion yields,

$$x_{n+1} \in \mathrm{F}[x_n] \subseteq \mathrm{F}[\mathrm{F}^{\circ n}[x_0]] = \mathrm{F} \circ \mathrm{F}^{\circ n}[x_0] = \mathrm{F}^{\circ(n+1)}[x_0].$$

Since $\{x_n\}_{n\in\mathbb{N}_0}$ is a trajectory of F we have $x_{n+1} \in \mathrm{F}[x_n]$, which means $x_{n+1} \in \mathrm{F}^{\circ(n+1)}[x_0]$. Therefore, $\{x_n\}_{n\in\mathbb{N}_0}$ an F selection.

To see why the converse is false we provide a counterexample. Let $X = \mathbb{R}$ and define

$$\mathrm{F}[x] = \{0, x\}.$$

Then, the sequence defined by

$$x_n = \begin{cases} 1 & \text{n even} \\ 0 & \text{n odd} \end{cases}$$

for $n \in \mathbb{N}_0$, is an F selection with initial point 1 but not a trajectory of F. This is because any trajectory of F, say $\{y_n\}_{n\in\mathbb{N}_0}$ which has $y_1 = 0$ must have $y_2 \in \mathrm{F}[y_1] = \mathrm{F}[0] = \{0\}$, so $y_2 = 0$. By induction one can see that for all $n \in \mathbb{N}$ we must also have $y_n = 0$. Hence, $\{x_n\}_{n\in\mathbb{N}_0}$ is not a trajectory of F ($x_2 \neq 0$ but $x_1 = 0$). But it is a F selection with initial point 1, which can be inferred from the fact that $\mathrm{F}^{\circ n}[1] = \{0, 1\}$ for all $n \in \mathbb{N}$. □

A F selection is a poor model for a dynamical system, when $\{x_n\}_{n\in\mathbb{N}_0}$ is a F selection the value of x_N depends only on $N \in \mathbb{N}_0$, F and x_0; It does not depend on x_{N-1} at all. Despite this, F selections are sometimes more convenient to work with than trajectories. Indeed, the fundamental object object of study in this thesis, the reachable set, is more clearly described by F selections.

Definition 3.1.3 (Reachability)**.** *Let (X, τ) be a topological space,* $\mathrm{F} : X \rightsquigarrow X$ *and* $x, y \in X$*. Define a multifunction* $\mathrm{R} : X \rightsquigarrow X$ *by,*

$$\mathrm{R}\,[\mathrm{F}, x] = \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]$$

to be the reachable set of F *with initial point $x \in X$. For $N \in \mathbb{N}_0$, also define*

$$\mathrm{R}_N[\mathrm{F}, x] = \bigcup_{n\in\mathbb{N}_N} \mathrm{F}^{\circ n}[x].$$

When there is no confusion of the multifunction, F*, being considered we may simply write* $\mathrm{R}\,[x]$ *or* $\mathrm{R}_N[x]$*. Recall, that* $\overline{\mathrm{R}}$ *and* clR *denote the pointwise closure of the multifunction* R*. That is,* $\overline{\mathrm{R}}[x] = \mathrm{clR}\,[x] = \overline{\mathrm{R}[x]}$*.*

A point y is said to be reachable from x in finite time if $y \in \mathrm{R}\,[x]$. The point y is said to be eventually reachable from x if $y \in \overline{\mathrm{R}\,[x]}$.

Analysis of the closed reachable set is the focus of this thesis. So it is useful to see it from many different angles.

At first glance the reachable set and reachability is talking the points which a F selection can "reach". It turns out that that we can replace "F selection" with "trajectory of F".

Proposition 3.1.5. *Let X be a set, $\mathrm{F} : X \rightsquigarrow X$ be a total multifunction and $x, y \in X$. The following are equivalent:*

1. *$y \in \mathrm{R}[x]$. That is, y is reachable from x in finite time.*
2. *There is a* F *selection, $\{x_n\}_{n\in\mathbb{N}_0}$ with $x_0 = x$ and $x_N = y$ for some $N \in \mathbb{N}$.*
3. *There is a trajectory of* F*, $\{x_n\}_{n\in\mathbb{N}_0}$ with $x_0 = x$ and $x_N = y$ for some $N \in \mathbb{N}$.*

Proof. $3 \implies 2$: Follows from Proposition 3.1.4.

$2 \implies 1$: Since, there is a F selection, $\{x_n\}_{n\in\mathbb{N}_0}$ with $x_0 = x$ and $x_N = y$ for some $N \in \mathbb{N}$. We know that $y \in \mathrm{F}^{\circ N}[x]$ and by definition of the reachable set $\mathrm{F}^{\circ N}[x] \subseteq \mathrm{R}[x]$. So $y \in \mathrm{R}[x]$.

$1 \implies 3$: Given $y \in \mathrm{R}[x]$, we can see, by definition, that there is a $N \in \mathbb{N}$ with $y \in \mathrm{F}^{\circ N}[x]$. Let $x_N = y$, by definition of composition of multifunctions ($\mathrm{F}^{\circ N}[x] = \mathrm{F}[\mathrm{F}^{\circ(N-1)}[x]]$) there is a $x_{N-1} \in \mathrm{F}^{\circ(N-1)}[x]$ with $x_N \in \mathrm{F}[x_{N-1}]$. If $N = 1$ we now that $x_0 \in \mathrm{F}^{\circ(0)}[x] = \{x\}$. If $N > 1$ we can then pick a $x_{N-1} \in \mathrm{F}^{\circ(N-2)}[x]$ with $x_{N-1} \in \mathrm{F}[x_{N-2}]$. We can continue this argument to get a finite sequence $\{x_n\}_{n=1}^{N}$ with $x_N = y$, $x_0 = x$ and $x_{n+1} \in \mathrm{F}[x_n]$ for $n = 0, 1, \ldots, N-1$. Since F is total by Proposition 3.1.1 there is a trajectory $\{y_n\}_{n\in\mathbb{N}_0}$ with $y_0 = y = x_N$. Now define $x_n = y_{n-N}$ for $n \in \mathbb{N}_N$, one can see that $\{x_n\}_{n\in\mathbb{N}_0}$ is a trajectory starting at x with $x_N = y$ for some $N \in \mathbb{N}$. □

Proposition 3.1.5 justifies why the reachable set is relevant to dynamical systems. The reachable set is the set of all points a trajectory can reach/touch in finite time. Therefore, F selections and trajectories of F are dynamically equivalent for finite time. This is pleasing, after all when F is single valued there is no distinction between them. It is unfortunate that the same does not hold for "infinite time".

Proposition 3.1.6. *Let (X, τ) be a regular topological space, $\mathrm{F} : X \rightsquigarrow X$ be a compact valued u.s.c, o.s.c multifunction and $x, y \in X$. The following are equivalent:*

1. *$y \in \overline{\mathrm{R}[x]}$. That is, y is eventually reachable from x.*
2. *There is a* F *selection, $\{x_n\}_{n\in\mathbb{N}_0}$ with $x_0 = x$ and $y \in \overline{\{x_n : n \in \mathbb{N}\}}$.*

It is possible for $y \in \overline{\mathrm{R}[x]}$ but every trajectory of F, $\{x_n\}_{n\in\mathbb{N}_0}$ *with* $x_0 = x$ *has* $y \notin \overline{\{x_n : n \in \mathbb{N}\}}$.

Proof. 2 $\implies$ 1: Let $\{x_n\}_{n\in\mathbb{N}_0}$ be a F selection with $x_0 = x$ and $y \in \overline{\{x_n : n \in \mathbb{N}\}}$. It follows from Proposition 3.1.6 that $\{x_n : n \in \mathbb{N}\} \subseteq \mathrm{R}[x]$, taking closures of both sides yields 1.

1 $\implies$ 2: Suppose that $y \in \overline{\mathrm{R}[x]}$ then one can see that $y \in \overline{\bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[x]} \cup \overline{\mathrm{R}_{N+1}[x]}$ for all $N \in \mathbb{N}$. Since F is a compact valued u.s.c, o.s.c multifunction and X is regular, we can apply Items 2 and 3 of Proposition 2.3.8 to conclude that $\mathrm{F}^{\circ n}$ is compact valued u.s.c, o.s.c multifunction for all $n \in \mathbb{N}$. Also, by Item 1c of Proposition 2.3.10 we can see that $\bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[x]$ is an o.s.c multifunction for all $N \in \mathbb{N}$. In particular, for all $N \in \mathbb{N}$, $\bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[x]$ is a closed (and compact) set, meaning $y \in \bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[x] \cup \overline{\mathrm{R}_{N+1}[x]}$. If $y \in \bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[x]$ for some $N \in \mathbb{N}$ then, $y \in \mathrm{F}^{\circ n}[x]$, for some $n \in \mathbb{N}$. From here one can prove that $y \in \overline{\{x_n : n \in \mathbb{N}\}}$ for some F selection $\{x_n\}_{n\in\mathbb{N}_0}$ with $x_0 = x$. Otherwise, $y \in \bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[x] \cup \overline{\mathrm{R}_{N+1}[x]}$ and $y \notin \bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[x]$ for all $N \in \mathbb{N}$. Therefore, for all $N \in \mathbb{N}$ we have $y \in \overline{\mathrm{R}_{N+1}[x]}$ and $y \in \overline{\mathrm{R}[x]}$. So $y \in \bigcap_{N\in\mathbb{N}} \overline{\mathrm{R}_N[x]}$. By applying Item 2 of Proposition 2.2.5, we see $y \in \mathrm{Ls}_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x] = \bigcap_{N\in\mathbb{N}} \overline{\mathrm{R}_N[x]}$ and applying Item 1 of Proposition 2.2.7 gives us an F selection $\{x_n\}_{n\in\mathbb{N}_0}$ with $x_0 = x$ and $y \in \overline{\{x_n : n \in \mathbb{N}\}}$.

For the last statement of the theorem we provide a counterexample. Let $X = \mathbb{R}$ with the usual topology and consider

$$\mathrm{F}[x] = \{2x, e^x\}$$

for $x \in X$. Then, $0 \in \mathrm{clR}[-1]$ as $\{-2^n : n \in \mathbb{N}\} \subseteq \mathrm{R}[-1]$ and so $\left\{e^{-2^n} : n \in \mathbb{N}\right\} \subseteq \mathrm{R}[-1]$. The sequence $\left\{e^{-2^n}\right\}_{n\in\mathbb{N}}$ converges to 0 thus $0 \in \mathrm{clR}[-1]$.

However, every trajectory $\{x_n\}_{n\in\mathbb{N}_0}$ with $x_0 = -1$ has $0 \notin \overline{\{x_n : n \in \mathbb{N}_0\}}$. To see why, note that the set $\mathrm{F}[x]$ for $x > 0$ contains only positive numbers. Moreover, $x < 2x, e^x$ for $x > 0$. This this means if a trajectory $\{x_n\}_{n\in\mathbb{N}_0}$ with $x_0 = -1$ has $x_n > 0$ where $n \in \mathbb{N}$ is the smallest $k \in \mathbb{N}$ with $x_k > 0$ then $x_n < x_m$ for all $m \in \mathbb{N}_{n+1}$. Hence 0 is bounded away from any trajectory which a positive number. On the other hand, if $\{x_n\}_{n\in\mathbb{N}_0}$ with $x_0 = -1$ is always negative then it is $\{-2^n\}_{n\in\mathbb{N}_0}$ which also has $0 \notin \overline{\{-2^n : n \in \mathbb{N}_0\}}$. □

It is true that the closure of a trajectory starting at x must lie in $\mathrm{clR}[x]$. This means that the closed reachable set is at least somewhat useful for analyzing the long term dynamics of trajectories. However, Proposition 3.1.6 tells us that just because y is eventually reachable from x does not mean that y is "eventually reachable from x *via one trajectory*". On the other hand, Proposition 3.1.5 tells us that the smallest closed set containing every trajectory starting at x is $\mathrm{clR}\,[x]$. Considering that computability theory tells us that the problem: determine if $K \subseteq V$ is true, is semi-computable only when K is compact and V is open, see Theorem 2.4.1. Practically, this means if we want to over approximate all the trajectories starting at x then, we need the closure of the union of all such trajectories be a compact set at the very least. However, the closure of this union is $\mathrm{clR}\,[x]$, which may contain points we don't want.

Regardless, this leaves us with little option other than considering closed reachable set, to analyze the long term dynamics of trajectories. To this end, we explore some more concepts related to trajectories and the reachable set.

Definition 3.1.4. *Let (X, τ) be a topological space, $\mathrm{F} : X \rightsquigarrow X$ and $S \subseteq X$. Define the following:*

1. *The set S is called an invariant set of F, a forward invariant set of F, a deflationary set of F, or a sub-invariant set of F if*

$$\mathrm{F}[S] \subseteq S \text{ or equivalently } S \subseteq \mathrm{F}^{+}[S].$$

   When there is no confusion over the multifunction being considered we will simply say that S is invariant.

2. *Denote the set $\mathcal{I}_{\mathrm{F}}$ to be the set of all nonempty invariant sets of F. When there is no confusion over the multifunction being considered we will simply write $\mathcal{I}$.*

   Let $\mathrm{cl}\,\mathcal{I}$ denote the set of all closed invariant sets of F.

3. *The set S is called super-invariant set of F or an inflationary set of F if*

$$S \subseteq \mathrm{F}[S].$$

4. *The set S is called fixed set of F if*

$$\mathrm{F}[S] = S.$$

5. *The set S is called an viable set of F if*

$$S \subseteq \mathrm{F}^{-}[S].$$

   When there is no confusion over the multifunction being considered we will simply say that S is viable.

6. *Denote the set* $\mathcal{V}_{\mathrm{F}}$ *to be the set of all nonempty viable sets of* F*. When there is no confusion over the multifunction being considered we will simply write* $\mathcal{V}$*.*

   Let $\mathrm{cl}\,\mathcal{V}$ *denote the set of all closed viable sets of* F*.*

It is possible to also consider invariant sets of F^- or fixed sets of F^-. We have no real need to do so, in this work.

For trajectories and the reachable set, the important concepts are that of invariant and viable sets.

Proposition 3.1.7 (Invariant sets)**.** *Let* X *be a set,* $\mathrm{F} : X \rightsquigarrow X$ *be a total multifunction and* $S \subseteq X$*. The following are equivalent:*

1. $\mathrm{F}[S] \subseteq S$*. That is,* S *is an invariant set of* F*.*
2. $S \subseteq \mathrm{F}^+[S]$*.*
3. *For every* $x_0 \in S$ *and every trajectory of* F *starting at* x_0*, say* $\{x_n\}_{n\in\mathbb{N}_0}$*, which has* $x_n \in S$ *for all* $n \in \mathbb{N}$*.*
4. S *is an invariant set of* R*.*

Proof. $1 \implies 2$: Let $x \in S$ then by 1 we have that $\mathrm{F}[x] \subseteq S$ by definition of F^+ we have $x \in \mathrm{F}^+[S]$. Thus, 2 holds.

$2 \implies 3$: Let $x_0 \in S$ and let $\{x_n\}_{n\in\mathbb{N}_0}$ be a trajectory of F. We proceed by induction. Base case $n = 1$: As $x_0 \in S$ by 2 we have $x_0 \in \mathrm{F}^+[S]$. By definition of F^+ this means that $\mathrm{F}[x_0] \subseteq S$, so $x_1 \in \mathrm{F}[x_0] \subseteq S$.

Inductive step: Assume that $x_n \in S$ then similar to the base case $\mathrm{F}[x_n] \subseteq S$ and thus $x_{n+1} \in \mathrm{F}[x_n] \subseteq S$.

Therefore, $x_n \in S$ for all $n \in \mathbb{N}$ and 3 holds.

$3 \implies 4$: This implication follows very quickly from Proposition 3.1.5.

$4 \implies 1$: We know that $\mathrm{R}\,[S] \subseteq S$. By definition of the reachable set we also know $\mathrm{F}[x] \subseteq \mathrm{R}\,[x]$ for all $x \in X$, it follows that $\mathrm{F}[S] \subseteq \mathrm{R}\,[S] \subseteq S$. Which proves 1. □

From a general dynamical systems perspective, Item 3 of Proposition 3.1.7 is the best definition of an invariant set. However, for discrete time dynamics Item 1 is the easiest thing to check for. We know present the analogous result for viable sets.

Proposition 3.1.8 (Viable sets)**.** *Let X be a set, $\mathrm{F} : X \rightsquigarrow X$ and $S \subseteq X$. The following are equivalent:*

1. *$S \subseteq \mathrm{F}^{-}[S]$. That is, S is an viable set of* F*.*
2. *For every $x_0 \in S$ and there is a trajectory of* F *starting at x_0, say $\{x_n\}_{n\in\mathbb{N}_0}$, which has $x_n \in S$ for all $n \in \mathbb{N}$.*

Proof. 1 $\implies$ 2: Let $x_0 \in S$ then by 1 we know there is a trajectory $\{x_n \in \mathrm{F}[x_{n-1}]\}_{n\in\mathbb{N}}$ with $x_n \in S$ for all $n \in S$. In particular $x_1 \in S$ and $x_1 \in \mathrm{F}[x]$ thus $x_1 \in \mathrm{F}[x_0] \cap S \neq \emptyset$, by definition of F^{-} this means that $x_0 \in \mathrm{F}^{-}[S]$. Therefore, 2 holds.

2 $\implies$ 1: Let $x_0 \in S$. We construct an appropriate trajectory by by induction. Base case $n = 1$: As $x_0 \in S$ by 2 we have $x_0 \in \mathrm{F}^{-}[S]$. By definition of F^{-} this means that $\mathrm{F}[x_0] \cap S \neq \emptyset$, so pick $x_1 \in \mathrm{F}[x_0] \cap S$.

Inductive step: Assume that $x_m \in \mathrm{F}[x_{m-1}] \cap S$ for $m \leq n$ then similar to the base case $\mathrm{F}[x_n] \cap S \neq \emptyset$ and thus $x_{n+1} \in \mathrm{F}[x_n] \cap S$.

Therefore, $x_n \in S$ and $x_n \in \mathrm{F}[x_{n-1}]$ for all $n \in \mathbb{N}$ and 1 holds. □

Again, Item 2 of Proposition 3.1.8 is likely the best definition for a viable set from a general dynamical systems perspective. There are fewer equivalences for viable sets compared to invariant sets. I think this is just because viability is a much weaker condition than invariance.

We can see that the duality of F^{-} and F^{+} lives on in viable sets and invariant sets. That is to say, the difference between viable sets and invariant set is a "$\exists$" quantifier to a "$\forall$" quantifier; you can stay in a viable set but you must stay in an invariant set. As one might expect, when F is single valued, $\mathrm{F}^{-} = \mathrm{F}^{+}$, so there is no difference between a viable set and an invariant set.

Example 3.1.1. *Let $X = \mathbb{R}$ with the usual topology and define the maps*

$$\mathrm{f}_0(x) = \frac{1}{2}x$$
$$\mathrm{f}_1(x) = \frac{1}{2}x + \frac{1}{2}$$

and the multifunction

$$\mathrm{F}[x] = \{\mathrm{f}_0(x)\} \cup \{\mathrm{f}_1(x)\}.$$

Then, F *is compact valued, continuous and o.s.c.*

Sets if the form $[a,b]$, $(a,b]$, $[a,b)$ and (a,b) for $a<0$ and $b>1$ are invariant.

The fixed points of the functions, $\{0\},\{1\}$ are viable sets of F *but not invariant sets of* F.

The set $[0,1]$ is a fixed set of F, *and is both viable and invariant.*

We now present some basic observations about invariant and viable sets.

Proposition 3.1.9. *Let (X,τ) be a topological space and let $\mathrm{F}: X \rightsquigarrow X$ be a total multifunction. The following hold:*

1. $\mathcal{I}_{\mathrm{F}} \subseteq \mathcal{V}_{\mathrm{F}}$.
2. $X \in \mathcal{I}_{\mathrm{F}}$. $\emptyset \notin \mathcal{I}_{\mathrm{F}} \cup \mathcal{V}_{\mathrm{F}}$, *by definition, but* $\emptyset$ *is invariant and viable.*
3. *If* $I \in \mathcal{I}_{\mathrm{F}}$ *then,* $\mathrm{F}[I], \mathrm{F}^{+}[I] \in \mathcal{I}_{\mathrm{F}}$.
4. *If* $V \in \mathcal{V}_{\mathrm{F}}$ *then,* $\mathrm{F}^{-}[V] \in \mathcal{V}_{\mathrm{F}}$.
5. $I \in \mathcal{I}_{\mathrm{F}} \iff \mathrm{R}_0[I] = I \neq \emptyset$.
6. $\mathcal{I}_{\mathrm{F}} = \mathcal{I}_{\mathrm{R}}$.
7. *For all* $N \in \mathbb{N}_0$ *and* $x \in X$, $\mathrm{R}_N[x] \in \mathcal{I}_{\mathrm{F}}$.
8. *Suppose that* $\{x_n\}_{n\in\mathbb{N}_0}$ *is a trajectory of* F *then, for all* $N \in \mathbb{N}_0$ *we have* $\{x_n : n \in \mathbb{N}_N\} \in \mathcal{V}_{\mathrm{F}}$.
9. *When* $\emptyset \neq \mathcal{S} \subseteq \mathcal{I}_{\mathrm{F}}$ *we have that,* $\bigcup_{S\in\mathcal{S}} S \in \mathcal{I}_{\mathrm{F}}$ *and* $\bigcap_{S\in\mathcal{S}} S$ *is invariant. Furthermore, if* $\bigcap_{S\in\mathcal{S}} S \neq \emptyset$ *then,* $\bigcap_{S\in\mathcal{S}} S \in \mathcal{I}_{\mathrm{F}}$.
10. *When* $\emptyset \neq \mathcal{S} \subseteq \mathcal{V}_{\mathrm{F}}$ *we have that,* $\bigcup_{S\in\mathcal{S}} S \in \mathcal{V}_{\mathrm{F}}$.
11. *Given* $A \subseteq X$, *the set* $\mathrm{R}_0^{+}[A]$ *is the largest invariant set in* A.

Proof. To prove 1 we first note that by Item 7 of Proposition 2.3.1, for all sets B, $\mathrm{F}^{+}[B] \subseteq \mathrm{F}^{-}[B]$. Thus by definition we have if $I \in \mathcal{I}$, then

$$I \subseteq \mathrm{F}^{+}[I] \subseteq \mathrm{F}^{-}[I]$$

and so $I \in \mathcal{V}$.

Item 2 vacuously follows from definitions.

For 3, suppose that $I \in \mathcal{I}_{\mathrm{F}}$ then by definition $\mathrm{F}[I] \subseteq I$ by applying F to both sides of this inclusion we see that

$$\mathrm{F}[\mathrm{F}[I]] \subseteq \mathrm{F}[I].$$

So $\mathrm{F}[I] \in I \in \mathcal{I}_{\mathrm{F}}$. Also we have $I \subseteq \mathrm{F}^{+}[I]$, and apply F^{+} to both side this inclusion yields

$$\mathrm{F}^{+}[V] \subseteq \mathrm{F}^{+}\big[\mathrm{F}^{+}[V]\big].$$

It follows, $\mathrm{F}^{+}[I] \in F^{+}[I]$.

Similarly, for 4 suppose that $V \in \mathcal{V}_{\mathrm{F}}$. Then by definition $V \subseteq \mathrm{F}^{-}[V]$ and applying F^{-} to both sides of this inclusion we see

$$\mathrm{F}^{-}[V] \subseteq \mathrm{F}^{-}\big[\mathrm{F}^{-}[V]\big].$$

Hence, $\mathrm{F}^{-}[V] \in \mathcal{V}_{\mathrm{F}}$.

For 5, suppose that $I \in \mathcal{I}$,

$$\mathrm{F}^{\circ 2}[I] \subseteq \mathrm{F}[I] \subseteq I$$

by an induction proof we can see that $\mathrm{F}^{\circ n}[I]] \subseteq I$ for all $n \in \mathbb{N}$. Recalling that $\mathrm{F}^{\circ 0}[I] = I$, we see that $\mathrm{R}_0[I] = I$.

Conversely when a set $I \subseteq X$ has $\mathrm{R}_0[I] = I$, it follows from definition of R that $I = \mathrm{R}_0[I] \supseteq \mathrm{F}[I]$, so I is invariant.

Item 6, follows from Proposition 3.1.7.

Item 7 follows from the facts that $\mathrm{F} \circ \mathrm{R}_N = \mathrm{R}_{N+1}$ for all $N \in \mathbb{N}_0$ and $\mathrm{R}_{N+1} \subseteq \mathrm{R}_N$. These facts follow from definitions and Item 3 of Proposition 2.3.1. Given these facts we see, $\mathrm{F} \circ \mathrm{R}_N[x] \subseteq \mathrm{R}_N[x]$ for all $N \in \mathbb{N}_0$ and $x \in X$.

For 8, let $N \in \mathbb{N}$ and $k \geq N$ then since $\{x_n\}_{n \in \mathbb{N}_0}$ is a trajectory we have $x_{k+1} \in \mathrm{F}[x_k]$ but $x_{k+1} \in \{x_n : n \in \mathbb{N}_N\}$ so $\mathrm{F}[x_k] \cap \{x_n : n \in \mathbb{N}_N\} \neq \emptyset$ for all $N \in \mathbb{N}$ and $k \geq N$ this shows that $\{x_n : n \in \mathbb{N}_N\} \subseteq \mathrm{F}^{-}[\{x_n : n \in \mathbb{N}_N\}]$.

To see 9, let $\mathcal{S} \subseteq \mathcal{I}_{\mathrm{F}}$ and consider $\bigcup_{S \in \mathcal{S}} S$, we see that

$$\mathrm{F}\left[\bigcup_{S \in \mathcal{S}} S\right] = \bigcup_{S \in \mathcal{S}} \mathrm{F}[S] \subseteq \bigcup_{S \in \mathcal{S}} S$$

by Item 3 of Proposition 2.3.1. So $\bigcup_{S \in \mathcal{S}} S$ is invariant. Since $\mathcal{S} \neq \emptyset$ the union is nonempty as well, so $\bigcup_{S \in \mathcal{S}} S \in \mathcal{I}_{\mathrm{F}}$.

Now consider, $\bigcap_{S\in\mathcal{S}} S$, Similar to the last case, we see

$$\mathrm{F}\left[\bigcap_{S\in\mathcal{S}} S\right] \subseteq \bigcap_{S\in\mathcal{S}} \mathrm{F}[S] \subseteq \bigcap_{S\in\mathcal{S}} S$$

by Item 3 of Proposition 2.3.1. So $\bigcap_{S\in\mathcal{S}} S$ is invariant. By definition if $\bigcap_{S\in\mathcal{S}} S \neq \emptyset$ then, $\bigcap_{S\in\mathcal{S}} S \in \mathcal{I}_{\mathrm{F}}$.

Next, we prove 10. Let $\mathcal{S} \subseteq \mathcal{V}_{\mathrm{F}}$ and consider $\bigcup_{S\in\mathcal{S}} S$, we see that $S \subseteq \mathrm{F}^{-}[S]$ for all $S \in \mathcal{S}$. So

$$\bigcup_{S\in\mathcal{S}} S \subseteq \bigcup_{S\in\mathcal{S}} \mathrm{F}^{-}[S] = \mathrm{F}^{-}\left[\bigcup_{S\in\mathcal{S}} S\right]$$

by Item 4 of Proposition 2.3.1.

Lastly, we prove Item 11. Let $A \subseteq X$ then, recalling Item 2 of Proposition 2.3.2

$$\mathrm{R}_0^{+}[A] = \bigcap_{n\in\mathbb{N}_0} \mathrm{F}^{+\circ n}[A] = A \cap \bigcap_{n\in\mathbb{N}} \mathrm{F}^{+\circ n}[A] = A \cap \bigcap_{n\in\mathbb{N}_0} \mathrm{F}^{+\circ(n+1)}[A] = A \cap \mathrm{F}^{+}\left[\bigcap_{n\in\mathbb{N}_0} \mathrm{F}^{+\circ n}[A]\right]$$

where the right most equality comes from Item 5 of Proposition 2.3.1. Thus we see that $\mathrm{R}_0^{+}[A] = A \cap \mathrm{F}^{+}\left[\mathrm{R}_0^{+}[A]\right]$ and it follows that $\mathrm{R}_0^{+}[A] \subseteq A, \mathrm{F}^{+}\left[\mathrm{R}_0^{+}[A]\right]$. Therefore, we have that $\mathrm{R}_0^{+}[A]$ is a invariant set inside A.

To see why $\mathrm{R}_0^{+}[A]$ is the largest invariant set inside A, let $I \in \mathcal{I}_{\mathrm{F}}$ with $I \subseteq A$. Then, by Item 5 of this proposition we have that $\mathrm{R}_0[I] = I \subseteq A$ which means $I \subseteq \mathrm{R}_0^{+}[A]$. So every invariant set in A is contained in $\mathrm{R}_0^{+}[A]$.

□

We see that from Item 1 of Proposition 3.1.9 that invariant sets are viable. Reinforcing the idea that invariance is strong (and characterized by the strong/upper pre-image) and viability is weak (and characterized by the weak/lower pre-image). Feeding into this, are Items 5 and 8. Where Item 5 of Proposition 3.1.9 together with Proposition 3.1.5 says that invariant sets are just unions of *all* trajectories starting in the set and Item 8 of Proposition 3.1.9 suggests viable sets are just unions of *some* trajectories starting in the set.

Items 9 and 10 shows us that invariant/viable sets are closed under unions. However, only invariant sets behave sanely under intersection, see Example 3.1.2.

Example 3.1.2. *Let $X = \mathbb{R}$ with the usual topology and define the maps*

$$\mathrm{f}_0(x) = \frac{1}{2}x$$
$$\mathrm{f}_1(x) = \frac{1}{2}x + \frac{1}{2}$$

and the multifunction

$$\mathrm{F}[x] = \{\mathrm{f}_0(x)\} \cup \{\mathrm{f}_1(x)\}.$$

Then, F *is compact valued, continuous and o.s.c.*

The trajectories, $\left\{\mathrm{f}_0^{\circ n}\left(\frac{1}{2}\right)\right\}_{n\in\mathbb{N}_0}, \left\{\mathrm{f}_1^{\circ n}\left(\frac{1}{2}\right)\right\}_{n\in\mathbb{N}_0}$ *define viable sets,* $\left\{\mathrm{f}_0^{\circ n}\left(\frac{1}{2}\right) : n \in \mathbb{N}_0\right\}, \left\{\mathrm{f}_1^{\circ n}\left(\frac{1}{2}\right) : n \in \mathbb{N}_0\right\}$*, their intersection is* $\left\{\frac{1}{2}\right\}$*, which is not viable.*

With some extra assumptions, we can take closures of invariant (and viable) sets in a convenient way.

Proposition 3.1.10. *Let (X, τ) be a topological space and let $\mathrm{F} : X \rightsquigarrow X$. The following hold:*

1. *Assume* F *is l.s.c. If $I \in \mathcal{I}$ then $\overline{I} \in \mathcal{I}$. In particular, this means that* $\mathrm{cl}\,\mathcal{I} = \left\{\overline{I} : I \in \mathcal{I}\right\}$.
2. *Assume* F *is u.s.c. If $V \in \mathcal{V}$ then $\overline{V} \in \mathcal{V}$. In particular, this means that* $\mathrm{cl}\,\mathcal{V} = \left\{\overline{V} : V \in \mathcal{V}\right\}$.

Proof. For Item 1, we recall by Item 4 of Theorem 2.3.2 that, if F is l.s.c and C is closed then $\mathrm{F}^+[C]$ is closed. Also recall that F^+ is monotone so if $A \subseteq B$ then $\mathrm{F}^+[A] \subseteq \mathrm{F}^+[B]$.

Now let $I \in \mathcal{I}$ and we see that

$$\mathrm{F}^+\left[\overline{I}\right] \supseteq \mathrm{F}^+[I] \supseteq I$$

the left most set is closed, so we can take closures and find that

$$\mathrm{F}^+\left[\overline{I}\right] \supseteq \overline{I}.$$

So $\overline{I} \in \mathcal{I}$.

Lastly for Item 2, we recall that by Item 4 of Theorem 2.3.4 that, if F is u.s.c and C is closed then $\mathrm{F}^-[C]$ is closed. Also recall that F^- is monotone so if $A \subseteq B$ then $\mathrm{F}^-[A] \subseteq \mathrm{F}^-[B]$.

Now let $V \in \mathcal{V}$ and we see that

$$\mathrm{F}^{-}\left[\overline{V}\right] \supseteq \mathrm{F}^{-}[V] \supseteq V$$

the left most set is closed, so we can take closures and find that

$$\mathrm{F}^{-}\left[\overline{V}\right] \supseteq \overline{V}.$$

So $\overline{V} \in \mathcal{V}$. ☐

Proposition 3.1.10 allows us to work more or less exclusively with closed invariant/viable sets, provided we have sufficient continuity. This will be convenient later.

We now take the time to highlight an interesting connection between invariant sets and the reachable set. And also discuss some continuity proprieties of the reachable set.

Proposition 3.1.11. *Let (X, τ) be a topological space and let $\mathrm{F} : X \rightsquigarrow X$ be a total multifunction. The following hold:*

1. *For all $k, n \in \mathbb{N}_0$ we have $\mathrm{R}_n \circ \mathrm{F}^{\circ k} = \mathrm{F}^{\circ k} \circ \mathrm{R}_n = \mathrm{R}_{n+k} = \mathrm{R}_k \circ \mathrm{R}_n$.*
2. *For all $N \in \mathbb{N}_0$ and $x \in X$, $\mathrm{R}_N[x] = \bigcap\left\{I \in \mathcal{I}_{\mathrm{F}} : \mathrm{F}^{\circ N}[x] \subseteq I\right\}$.*
3. *For F l.s.c we have:*
   - *3a) For all $N \in \mathbb{N}_0$, both R_N and $\mathrm{cl}\mathrm{R}_N$ are l.s.c.*
   - *3b) For all $N \in \mathbb{N}_0$ and $x \in X$, $\mathrm{cl}\mathrm{R}_N[x] = \bigcap\left\{I \in \mathrm{cl}\,\mathcal{I}_{\mathrm{F}} : \mathrm{F}^{\circ N}[x] \subseteq I\right\}$.*
4. *When X is regular, F is compact valued, continuous, o.s.c and $\mathrm{cl}\mathrm{R}$ is compact valued then, for all $k, n \in \mathbb{N}$ we have $\mathrm{F}^{\circ k} \circ \mathrm{cl}\mathrm{R}_n = \mathrm{cl}\mathrm{R}_{n+k}$ is closed and compact.*

Proof. To prove 1, let $k, n \in \mathbb{N}_0$. We first show $\mathrm{R}_n \circ \mathrm{F}^{\circ k} = \mathrm{R}_{n+k}$. Let $x \in X$ and keeping definitions and Proposition 2.3.1 in mind we see (reading left to right)

$$\mathrm{R}_n \circ \mathrm{F}^{\circ k}[x] = \mathrm{R}_n\left[\mathrm{F}^{\circ k}[x]\right] = \bigcup_{j \geq n} \mathrm{F}^{\circ j}\left[\mathrm{F}^{\circ k}[x]\right] = \bigcup_{j \geq n} \mathrm{F}^{\circ j+k}[x] = \bigcup_{j \geq n+k} \mathrm{F}^{\circ j}[x] = \mathrm{R}_{n+k}[x].$$

Similarly, we can see

$$\mathrm{R}_{n+k}[x] = \bigcup_{j \geq n} \mathrm{F}^{\circ j+k}[x] = \bigcup_{j \geq n} \mathrm{F}^{\circ k}\left[\mathrm{F}^{\circ j}[x]\right] = \mathrm{F}^{\circ k}\left[\bigcup_{j \geq n} \mathrm{F}^{\circ j}[x]\right] = \mathrm{F}^{\circ k} \circ \mathrm{R}_n[x].$$

And for $\mathrm{R}_{n+k} = \mathrm{R}_k \circ \mathrm{R}_n$, we note that by definition the sets $\mathrm{R}_j[x]$ are decreasing in $j \in \mathbb{N}_0$. Now consider

$$\mathrm{R}_k \circ \mathrm{R}_n[x] = \mathrm{R}_k[\mathrm{R}_n[x]] = \bigcup_{j \geq k} \mathrm{F}^{\circ j}[\mathrm{R}_n[x]] = \bigcup_{j \geq k} \mathrm{R}_{n+j}[x] = \mathrm{R}_{n+k}[x].$$

Now we prove 2. For any $N \in \mathbb{N}_0$ consider the intersection, $\bigcap\{I \in \mathcal{I}_{\mathrm{F}} : \mathrm{F}^{\circ N}[x] \subseteq I\}$, by Item 7 of Proposition 3.1.9 we know that $\mathrm{R}_N[x] \in \mathcal{I}_{\mathrm{F}}$. Also by definition $\mathrm{R}_N[x] \supseteq \mathrm{F}^{\circ N}[x]$. Therefore, the intersection in question contains, $\mathrm{R}_N[x]$. Conversely, if $I \in \mathcal{I}_{\mathrm{F}}$ has $\mathrm{F}^{\circ N}[x] \subseteq I$ then, by applying R_0 to both sides of this inclusion yields,

$$\mathrm{R}_N[x] = \mathrm{R}_0\big[\mathrm{F}^{\circ N}[x]\big] \subseteq \mathrm{R}_0[I] = I.$$

The desired conclusion follows.

Item 3a is an application of the following facts: l.s.c is preserved under composition (Item 1 of Proposition 2.3.3), unions (Item 1a of Proposition 2.3.10), and taking closures (Item 1 of Proposition 2.3.9).

Next we consider 3b, which follows quickly from 2 and Item 1 of Proposition 3.1.10.

Finally, we prove 4. By assumptions and Items 2 and 3 of Proposition 2.3.8 we know that $\mathrm{F}^{\circ k}$ is compact valued, continuous, o.s.c for all $k \in \mathbb{N}$. From, Proposition 2.3.6 and Theorem 2.3.11 we see that $\mathrm{F}^{\circ k} \circ \mathrm{clR}\,[x]$ is closed and compact for all $x \in X$ and $k \in \mathbb{N}$. By 1 of this proposition, l.s.c of F, we see

$$\mathrm{R}_{k+1}[x] = \mathrm{F}^{\circ k} \circ \mathrm{R}\,[x] \subseteq \mathrm{F}^{\circ k} \circ \mathrm{clR}\,[x] = \mathrm{F}^{\circ k}\left[\overline{\mathrm{R}\,[x]}\right] \subseteq \overline{\mathrm{F}^{\circ k} \circ \mathrm{R}\,[x]} = \overline{\mathrm{R}_{k+1}[x]}.$$

Since, $\mathrm{F}^{\circ k} \circ \mathrm{clR}\,[x]$ is closed we have that $\mathrm{F}^{\circ k} \circ \mathrm{clR}\,[x] = \overline{\mathrm{R}_{k+1}[x]}$; moreover, it follows for all $n \in \mathbb{N}_2$ that $\mathrm{clR}_n[x]$ is compact and closed. Hence, $\mathrm{clR}_n[x]$ is compact and closed for all $n \in \mathbb{N}$ (the case for $n = 1$ is assumed). To achieve $\mathrm{F}^{\circ k} \circ \mathrm{clR}_n = \mathrm{clR}_{n+k}$ for all $k, n \in \mathbb{N}$, we can apply the same argument above replacing R, clR with $\mathrm{R}_n, \mathrm{clR}_n$.

□

Item 1 of Proposition 3.1.11 is effectively a semi-group property of a dynamical system, it comes up a lot. It is unfortunate that the semi-group property of Item 1 does not readily extend to the closed reachable set. But Item 4 gives at least some of these properties, under some regularity assumptions.

Items 2 and 3a are more notable. They give a characterization fo the (tails of) the reachable set and closed reachable set. These facts seem trivial once you see them. However, we will see later that we can phrase a number of important concepts purely from intersecting various invariant sets together.

3.2 Stability, stabilizability and long term behaviour

In this section we explore some basic concepts related to the long term behaviour of difference inclusions. Much like for the reachability problem, there are two main ways of viewing long term behaviour; one for selections and one for trajectories.

Definition 3.2.1. *Let (X, τ) be a topological space and $\mathrm{F} : X \rightsquigarrow X$. Define the following:*

1. *The omega limit set of* F *from x,*

$$\omega[x] := \operatorname*{Ls}_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x] = \bigcap_{N\in\mathbb{N}} \mathrm{cl}\mathrm{R}_N[x].$$

   Note, ω is a multifunction from X to X.

2. *The omega limit of a trajectory, $\{x_n\}_{n\in\mathbb{N}_0}$ of* F *is the set of accumulation points of the trajectory. That is $\mathrm{Acc}_{n\in\mathbb{N}}\, x_n$ or $\mathrm{Ls}_{n\in\mathbb{N}}\{x_n\}$ is the omega limit of the trajectory.*

As one might guess from Propositions 3.1.5 and 3.1.6, the omega limit set and the omega limits of trajectories are related but generally not equivalent. It is true that for a point $x \in X$ (using the notation in the definition) we have $\omega[x] \supseteq \mathrm{Ls}_{n\in\mathbb{N}}\{x_n\}$ when $\{x_n\}_{n\in\mathbb{N}_0}$ is a trajectory of F with $x_0 = x$; however there may be a point in $\omega[x]$ which is not in the omega limit of a single trajectory. Again see, Proposition 3.1.6 and the discussion thereafter.

Proposition 3.2.1. *Let (X, τ) be a topological space and $\mathrm{F} : X \rightsquigarrow X$. The following hold:*

1. *For all $N \in \mathbb{N}_0$ and $x \in X$ we have that, $\mathrm{cl}\mathrm{R}_N[x] = \bigcup_{n\geq N} \overline{\mathrm{F}^{\circ n}[x]} \cup \omega[x]$. In particular when,* F *is a compact valued u.s.c, o.s.c multifunction, and X is regular or Hausdorff we have $\mathrm{cl}\mathrm{R}_N[x] = \mathrm{R}_N[x] \cup \omega[x]$.*

2. *For a trajectory of* F, *$\{x_n\}_{n\in\mathbb{N}_0}$ with initial point x_0 we have $\omega[x_0] \supseteq \mathrm{Ls}_{n\in\mathbb{N}}\{x_n\}$.*

3. *For* F *l.s.c and $x \in \mathrm{Dom}\,(\omega)$ we have $\omega[x] \in \mathrm{cl}\,\mathcal{I}$.*

4. F *is u.s.c and $\{x_n\}_{n\in\mathbb{N}_0}$ be a trajectory then, $\mathrm{Ls}_{n\to\infty}\{x_n\} \in \mathrm{cl}\,\mathcal{V}$, provided any of the following hold:*

   4a) F *is closed valued and $\overline{\{x_n\}_{n\in\mathbb{N}_0}}$ is compact.*

   4b) F *is closed valued, compact valued and $\mathrm{Ls}_{n\to\infty}\{x_n\}$ is nonempty.*

Proof. First we prove 1. Let $N \in \mathbb{N}_0$ and $x \in X$, the inclusion $\mathrm{clR}_N[x] \supseteq \bigcup_{n \geq N} \overline{\mathrm{F}^{\circ n}[x]} \cup \omega[x]$ follows from definitions and properties of the closure. For the other inclusion, suppose that $y \in \overline{\mathrm{R}_N[x]}$ then one can see that $y \in \overline{\bigcup_{n=1}^{M} \mathrm{F}^{\circ n}[x]} \cup \overline{\mathrm{R}_{M+1}[x]}$ for all $M \in \mathbb{N}_N$.

In particular, for all $M \in \mathbb{N}_N$, $\overline{\bigcup_{n=N}^{M} \mathrm{F}^{\circ n}[x]} = \bigcup_{n=N}^{M} \overline{\mathrm{F}^{\circ n}[x]}$. So either $y \in \bigcup_{n=1}^{M} \overline{\mathrm{F}^{\circ n}[x]}$ for some $M \in \mathbb{N}$ or $y \in \mathrm{clR}_{M+1}[x]$ for all $M \in \mathbb{N}$. In the first case $y \in \bigcup_{n \geq N} \overline{\mathrm{F}^{\circ n}[x]}$ so we are done. In the other case, one can show that $y \in \bigcap_{M \in \mathbb{N}} \mathrm{clR}_M[x] = \omega[x]$, by definitions and noting the clR_k $k \in \mathbb{N}_0$ are decreasing. This shows that $\mathrm{clR}_N[x] = \bigcup_{n \geq N} \overline{\mathrm{F}^{\circ n}[x]} \cup \omega[x]$

To prove the "furthermore", when F is a compact valued u.s.c, o.s.c multifunction and X is regular or Hausdorff, we can apply Items 2 and 3 of Proposition 2.3.8 to conclude that $\mathrm{F}^{\circ n}$ is compact valued u.s.c, o.s.c multifunction for all $n \in \mathbb{N}$. In particular, each $\mathrm{F}^{\circ n}$ is closed valued, so $\bigcup_{n \geq N} \overline{\mathrm{F}^{\circ n}[x]} = \bigcup_{n \geq N} \mathrm{F}^{\circ n}[x]$ for all $N \in \mathbb{N}_0$. The conclusion follows.

To see why 2 holds recall that a trajectory is an F selection, by Proposition 3.1.4. So $\{x_n\}_{n \in \mathbb{N}_0}$ has $x_n \in \mathrm{F}^{\circ n}[x_0]$ for $n \in \mathbb{N}_0$. It follows from Item 1 of Proposition 2.2.7 that $\omega[x_0] = \mathrm{Ls}_{n \to \infty} \mathrm{F}^{\circ n}[x_0] \supseteq \mathrm{Ls}_{n \in \mathbb{N}}\{x_n\}$.

Item 3 follows quickly from the fact that when $x \in \mathrm{Dom}\,(\omega)$ we have for all $N \in \mathbb{N}_0$ that $\mathrm{clR}_N[x] \in \mathrm{cl}\,\mathcal{I}$, by Item 1 of Proposition 3.1.10 and Item 7 of Proposition 3.1.9. Then, by definition of ω, ω is the intersection of invariant sets and so is invariant by Item 9 of Proposition 3.1.9.

We can prove 4a and 4b in tandem. Let $y \in \mathrm{Ls}_{n \to \infty}\{x_n\} = \bigcap_{N \in \mathbb{N}} \overline{\bigcup_{n \geq N}\{x_n\}}$, by Item 2 of Proposition 3.1.10 and Item 8 of Proposition 3.1.9 we have that

$$\overline{\bigcup_{n \geq N} \{x_n\}} \subseteq \mathrm{F}^{-}\left[\overline{\bigcup_{n \geq N} \{x_n\}}\right]$$

for all $N \in \mathbb{N}$. Taking intersections on both sides yields,

$$\operatorname*{Ls}_{n \to \infty} \{x_n\} \subseteq \bigcap_{N \in \mathbb{N}} \mathrm{F}^{-}\left[\overline{\bigcup_{n \geq N} \{x_n\}}\right].$$

Since $y \in \mathrm{Ls}_{n \to \infty}\{x_n\}$ we see that the sets

$$C_N = \mathrm{F}[y] \cap \overline{\bigcup_{n \geq N} \{x_n\}}$$

are nonempty for all $N \in \mathbb{N}$. If either 4a or 4b is true then the C_N's are also closed and compact, moreover the C_N's are nested. Thus, we can apply the finite intersection theorem and so

$$\emptyset \neq \bigcap_{N\in\mathbb{N}} C_N = \bigcap_{N\in\mathbb{N}} \mathrm{F}[y] \cap \overline{\bigcup_{n\geq N}\{x_n\}} = \mathrm{F}[y] \cap \bigcap_{N\in\mathbb{N}} \overline{\bigcup_{n\geq N}\{x_n\}} = \mathrm{F}[y] \cap \operatorname*{Ls}_{n\to\infty}\{x_n\}.$$

and $y \in \mathrm{F}^{-}[\mathrm{Ls}_{n\to\infty}\{x_n\}]$ so $\mathrm{Ls}_{n\to\infty}\{x_n\}$ is viable as required.

□

Conventionally, in dynamical systems we are working with iterating a single valued function. And in this context, often we are trying to determine if the dynamics are approaching a fixed point or periodic cycle of the function. That is: Is the omega limit set equal to the set containing the points in the periodic cycle? In our case where we are considering difference inclusions, fixed points and periodic cycles aren't a very good general case to consider. So we introduce a generalization of fixed points and periodic cycles.

Definition 3.2.2. *Let X be a set and let P be a property subsets of X can have. Then, $S \subseteq X$ is said to the smallest (or the minimum) P set if every $A \subseteq X$ with property P also has $S \subseteq A$. That is S is contained in* ***every*** *set with property P.*

A set $L \subseteq X$ is said to the largest (or the maximum) P set if every $A \subseteq X$ with property P also has $L \supseteq A$. That is S is contains every set with property P.

We define $M \subseteq X$ to be a minimal P set when M has property P and the following implication holds: If $A \subseteq M$ has property P then, $A = M$. That is, M contains no proper subset with property P.

Example 3.2.1. *Let $X = \{1, 2, 3\}$ then, the smallest subset of X is $\emptyset$. There is no smallest nonempty subset of X. However, $\{1\}.\{2\}, \{3\}$ are minimal nonempty sets of X.*

Broadly, speaking A being the smallest set means everything is bigger than A. While A being minimal means nothing is smaller than A.

Definition 3.2.3. *Let (X, τ) be a topological space and $\mathrm{F} : X \rightsquigarrow X$. A set A is called a minimal invariant set (m.i.s) of F if it is a minimal set of $\mathrm{cl}\,\mathcal{I}$. That is, A is closed nonempty invariant and has no proper subsets which are closed nonempty and invariant.*

A set S is called the small set of F if it is the smallest set of $\mathrm{cl}\,\mathcal{I}$.

A set Q is called a minimal viable set (m.v.s) of F if it is a minimal set of $\mathrm{cl}\,\mathcal{V}$. That is, A is closed nonempty viable and has no proper subsets which are closed nonempty and viable.

Example 3.2.2. *Let $X = \mathbb{R}$ with the usual topology and define the maps*

$$\mathrm{f}_0(x) = \frac{1}{2}x$$
$$\mathrm{f}_1(x) = \frac{1}{2}x + \frac{1}{2}$$

and the multifunction

$$\mathrm{F}[x] = \{\mathrm{f}_0(x)\} \cup \{\mathrm{f}_1(x)\}.$$

The point $x = 0$ is a fixed point of the function f_0. This means $\{0\}$ is a m.i.s of f_0. Since f_0 is single valued $\{0\}$ is also a m.v.s. Note that $\{0\}$ is the unique m.i.s of f_0, moreover it is not hard to see that $\{0\}$ is also the small set of f_0.

The set,

$$A = [0, 1]$$

is the small set of F and is the unique m.i.s of F in X. The set A is viable but it is not a m.v.s of F. The sets $\{0\}, \{1\}$ are m.v.s of F.

Let $b_1, \ldots, b_k \in \{0, 1\}$ and let $\bar{x}$ be the fixed point of $\mathrm{f}_{b_1} \circ \cdots \circ \mathrm{f}_{b_k}$ then, $\{\bar{x}, \mathrm{f}_{b_k}(\bar{x}), \mathrm{f}_{b_{k-1}} \circ \mathrm{f}_{b_k}(\bar{x}), \ldots, \mathrm{f}_{b_2} \circ \cdots \circ \mathrm{f}_{b_k}(\bar{x})\}$ is a m.v.s of F.

Example 3.2.3 (M.i.s can be infinite even for single valued functions)**.** *Let X be the unit circle in the complex plane. Define*

$$\mathrm{f}\big(e^{i\theta}\big) = e^{i(\theta + 2\pi\alpha)}$$

where $i^2 = -1$, $\theta \in [0, 2\pi)$ and $\alpha \in [0, 1]$ is a constant. If $\alpha = \frac{p}{q}$ is rational and in lowest terms then, for every $z \in X$ the set $\{\mathrm{f}^{\circ n}(z)\}_{n=1}^{q}$ is a m.i.s (and a periodic orbit).

If α is irrational then, the unique minimal set is the entire space and thus uncountable.

For now the reader should think of a m.i.s as a generalization of a fixed point or periodic cycle, which can contain infinitely many points. We will justify this thought later. For now we explore when, m.i.s and m.v.s exist.

Theorem 3.2.1. *Let (X, τ) be a topological space and $\mathrm{F} : X \rightsquigarrow X$ is l.s.c. Suppose that for some $x \in X$, $\mathrm{clR}[x]$ is compact. Then, $\mathrm{clR}[x]$ contains a compact m.i.s.*

Proof. Define the set

$$\mathcal{J} = \{I \in \mathrm{cl}\,\mathcal{I} : I \subseteq \mathrm{clR}[x]\}.$$

This set is nonempty since $\mathrm{cl}\mathrm{R}[x] \in \mathrm{cl}\,\mathcal{I}$ is a nonempty closed invariant set for all $x \in X$, this follows from F being l.s.c. Also note that the elements of $\mathcal{J}$ are compact.

We seek to apply Zorn's Lemma to $\mathcal{J}$, so we wish to show that any totally ordered subset of $\mathcal{J}$ has a lower bound in $\mathcal{J}$. Let $\mathcal{C} \subseteq \mathcal{J}$ be totally ordered; i.e. given any $C_1, C_2 \in \mathcal{C}$ either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$. It follows that $\mathcal{C}$ has the finite intersection property. Since the elements of $\mathcal{C}$ are compact we know that $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$ by the finite intersection theorem. By Item 9 of Proposition 3.1.9, $\bigcap_{C\in\mathcal{C}} C \in \mathcal{I}$ and since the sets $C \in \mathcal{C}$ are closed we have $\bigcap_{C\in\mathcal{C}} C \in \mathrm{cl}\,\mathcal{I}$. Moreover, we still have that $\bigcap_{C\in\mathcal{C}} C \subseteq \mathrm{cl}\mathrm{R}\,[x]$. So, $\bigcap_{C\in\mathcal{C}} C \in \mathcal{J}$ and so $\mathcal{C}$ has a lower bound in $\mathcal{J}$. By Zorn's Lemma, $\mathcal{J}$ has a minimal element, say A, it is evident that A is m.i.s. For if $I \subseteq A$ and $I \in \mathrm{cl}\,\mathcal{I}$ then $I \in \mathcal{J}$ and since A is minimal in $\mathcal{J}$ we have $I = A$. □

Theorem 3.2.2. *Let (X, τ) be a topological space and $\mathrm{F} : X \rightsquigarrow X$ is closed valued u.s.c multifunction. Suppose that a trajectory of the* F*, say $\{x_n \in \mathrm{F}[x_{n-1}]\}_{n\in\mathbb{N}}$, has compact closure. Then, $\overline{\{x_n\}_{n\in\mathbb{N}}}$ contains a compact m.v.s.*

Proof. Define the set

$$\mathcal{J} = \left\{ V \in \mathrm{cl}\,\mathcal{V} : V \subseteq \overline{\{x_n\}_{n\in\mathbb{N}}} \right\}.$$

This set is nonempty since $\overline{\{x_n\}_{n\in\mathbb{N}}}$ is a nonempty closed viable set (i.e $\overline{\{x_n\}_{n\in\mathbb{N}}} \in \mathrm{cl}\,\mathcal{V}$), this follows from Item 2 of Proposition 3.1.10. Also note that the elements of $\mathcal{J}$ are compact.

We seek to apply Zorn's Lemma to $\mathcal{J}$, so we wish to show that any totally ordered subset of $\mathcal{J}$ has a lower bound in $\mathcal{J}$. Let $\mathcal{C} \subseteq \mathcal{J}$ be totally ordered; i.e. given any $C_1, C_2 \in \mathcal{C}$ either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$. It follows that $\mathcal{C}$ has the finite intersection property. Since the elements of $\mathcal{C}$ are compact we know that $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$ by the finite intersection theorem. We claim that $\bigcap_{C\in\mathcal{C}} C \in \mathcal{J}$; $\bigcap_{C\in\mathcal{C}} C$ is nonempty and closed so we need only show that it is viable. Let $x \in \bigcap_{C\in\mathcal{C}} C$ then, for all $C \in \mathcal{C}$, $x \in C$ and since C is viable

$$C_{\mathrm{F}} := \mathrm{F}[x] \cap C \neq \emptyset.$$

The sets C_{F} are nonempty compact and totally ordered, it is compact since F is closed valued and totally ordered since $\mathcal{C}$ is. Thus, we can apply the finite intersection theorem again

$$\emptyset \neq \bigcap_{C\in\mathcal{C}} C_{\mathrm{F}} = \bigcap_{C\in\mathcal{C}} \mathrm{F}[x] \cap C = \mathrm{F}[x] \cap \bigcap_{C\in\mathcal{C}} C$$

and by definition $x \in \mathrm{F}^{-}\left[\bigcap_{C\in\mathcal{C}} C\right]$. Therefore, $\bigcap_{C\in\mathcal{C}} C$ is viable, $\bigcap_{C\in\mathcal{C}} C \in \mathcal{J}$ and so $\mathcal{C}$ has a lower bound in $\mathcal{J}$. By Zorn's Lemma, $\mathcal{J}$ has a minimal element, say Q, it is evident that Q is m.v.s. For if $V \subseteq Q$ and $V \in \mathrm{cl}\,\mathcal{V}$ then $V \in \mathcal{J}$ and since Q is minimal in $\mathcal{J}$ we have $V = Q$. □

Theorems 3.2.1 and 3.2.2 tell us that with just a dash of compactness (and continuity) we know there are some minimal invariant/viable sets. There may be many such minimal invariant/viable sets. Without the compactness, there can be none. For example the function $\mathrm{f} : \mathbb{R} \to \mathbb{R}$, $\mathrm{f}(x) = x + 1$ has no compact invariant/viable sets and also no minimal invariant/viable sets.

Also note that instead of assuming that the closed reachable set is compact valued in Theorem 3.2.1 (or the closure of a trajectory is compact in Theorem 3.2.2), we can assume there is a closed compact invariant set, say I, (or closed compact viable set V) to conclude that there is a m.i.s in I (or a m.v.s in V).

We now characterize minimal invariant sets.

Theorem 3.2.3. *Let (X, τ) be a topological space, $\mathrm{F} : X \rightsquigarrow X$ be a l.s.c multifunction and A be a nonempty subset of X. Then, the following are equivalent:*

1. *A is a m.i.s.*
2. *For all $a \in A$ and for all $N \in \mathbb{N}_0$ we have $A = \mathrm{clR}_N[a]$.*
3. *For all $a \in A$ we have $A = \omega[a]$.*
4. *For all $a \in A$ and some $N \in \mathbb{N}_0$ we have $A = \mathrm{clR}_N[a]$.*

Proof. From Item 9 of Proposition 3.1.9 and Proposition 3.1.10, we have that for all $a \in A$ and every $N \in \mathbb{N}$ that the sets $\omega[a]$, $\overline{\mathrm{R}_N[a]}$ are nonempty closed and invariant.

Also note that if $A \in \mathrm{cl}\,\mathcal{I}$ and $a \in A$ then, for all $n \in \mathbb{N}_0$ we have that $\mathrm{F}^{\circ n}[a] \subseteq A$. It follows that for all $N \in \mathbb{N}_0$ we also have $\mathrm{clR}_N[a] \subseteq A$.

1 $\implies$ 2: Assume that 1 holds. Then, since $\mathrm{clR}_N[a] \in \mathrm{cl}\,\mathcal{I}$ for all $a \in A$ and $N \in \mathbb{N}_0$ and $A \in \mathrm{cl}\,\mathcal{I}$, we have $\mathrm{clR}_N[a] \subseteq A$. But A is a m.i.s, so $\mathrm{clR}_N[a]$ cannot be a proper subset. Therefore, $A = \mathrm{clR}_N[a]$.

2 $\implies$ 3: When 2 holds, we can simply take the intersections of the $\mathrm{clR}_N[a]$ over all $N \in \mathbb{N}_0$ to get $A = \bigcap_{N \in \mathbb{N}_0} \mathrm{clR}_N[a] = \omega[a]$ for all $a \in A$.

3 $\implies$ 4: Assume that 3 holds. We show that $A = \mathrm{clR}\,[a]$ for all $a \in A$ to prove the implication. Let $a \in A$. By definition of ω and 3 we have that

$$\mathrm{clR}\,[a] \supseteq \bigcap_{N \in \mathbb{N}_0} \mathrm{clR}_N[a] = \omega[a] = A.$$

So $A \subseteq \mathrm{clR}\,[a]$. But $a \in A = \omega[a] \in \mathrm{cl}\,\mathcal{I}$, this means that $\mathrm{clR}\,[a] \subseteq \omega[a]$. Therefore, $A = \mathrm{clR}\,[a]$ for all $a \in A$.

$4 \implies 1$: Suppose that 4 holds. Let B be nonempty closed invariant set of A then, for all $b \in B \subseteq A$, $\mathrm{clR}_N[b] \subseteq B \subseteq A$ for all $N \in \mathbb{N}_0$. But $b \in A$, hence by 4, $\mathrm{clR}_N[b] = A$ for some $N \in \mathbb{N}_0$. It follows that $A = B = \mathrm{clR}_N[b]$ and that A is a m.i.s. □

For those who have a very clear understanding of invariant sets and minimality, Theorem 3.2.3 is rather trivial. Nevertheless it is worth writing down. Of particular note are the "for all" quantifiers on the points $a \in A$. These quantifiers cannot be weakened. Indeed one way to interpret Item 2, setting $N = 1$, is: for all $a, b \in A$, a eventually reaches b. So the point to point long term reachability problem is symmetric within a m.i.s.

We now present some useful facts about m.i.s.

Proposition 3.2.2. *Let (X, τ) be a topological space, $\mathrm{F} : X \rightsquigarrow X$ be a l.s.c multifunction. Suppose that $A, B \subseteq X$ are m.i.s.*

1. *If $A \cap B \neq \emptyset$ or $\mathrm{R}_0[A] \cap \mathrm{R}_0[B] \neq \emptyset$ then, $A = B$.*
2. *$\overline{\mathrm{F}[A]} = A$. If A is compact and F is o.s.c then, $\mathrm{F}[A] = A$.*
3. *Assume X is compact. A is the small set (see Definition 3.2.3) if and only if A is the unique minimal invariant set.*

Proof. To prove 1, note that since A, B are m.i.s we see that $A = \mathrm{R}_0[A]$ and $B = \mathrm{R}_0[B]$ by Item 5 of Proposition 3.1.9. So we need only consider the case $A \cap B \neq \emptyset$. When $A \cap B \neq \emptyset$ then, $A \cap B \in \mathrm{cl}\,\mathcal{I}$ (by Item 9 of Proposition 3.1.9). Moreover, $A \cap B \subseteq A, B$. But A, B are minimal in $\mathrm{cl}\,\mathcal{I}$, so $A \cap B$ is not a proper subset of either A or B. Therefore, $A = A \cap B = B$ is the only possibility.

Now we prove 2. Since A is a m.i.s we have $\mathrm{F}[A] \subseteq A$ since, A is closed we also have $\overline{\mathrm{F}[A]} \subseteq A$. By Item 3 of Proposition 3.1.9 we know $\mathrm{F}[A]$ is invariant and by Proposition 3.1.10 we see that $\overline{\mathrm{F}[A]} \in \mathrm{cl}\,\mathcal{I}$. But A is a m.i.s, so $\overline{\mathrm{F}[A]}$ is not a proper subset of A. Hence, $A = \overline{\mathrm{F}[A]}$.

When A is compact and F is o.s.c we can apply Proposition 2.3.6 to get that $\mathrm{F}[A]$ is closed. So $A = \overline{\mathrm{F}[A]} = \mathrm{F}[A]$.

Lastly, to prove 3 we assume X is compact. And suppose that A is the small set of F. If B is an arbitrary m.i.s then $B \in \mathrm{cl}\,\mathcal{I}$. But A is the smallest set in $\mathrm{cl}\,\mathcal{I}$, so $A \subseteq B$. However, B is minimal in $\mathrm{cl}\,\mathcal{I}$ so A is not a proper subset of B. It follows $A = B$ and noting that a small set is always a m.i.s we can conclude that A is the unique m.i.s in X.

Conversely, suppose that A is the unique m.i.s in X. Let $I \in \mathrm{cl}\,\mathcal{I}$, we seek to show that $A \subseteq I$. Let $x \in I$ then, $\mathrm{clR}\,[x] \subseteq I \subseteq X$. As X is compact, we see that $\mathrm{clR}\,[x]$ is too.

So we can apply Theorem 3.2.1, and there is a m.i.s, say B, with $B \subseteq \mathrm{clR}[x]$. However by assumption A is the unique m.i.s in X, so $A = B$. And we see that $A \subseteq \mathrm{clR}[x] \subseteq I$. Therefore, $A \in \mathrm{cl}\,\mathcal{I}$ is contained in every set in $\mathrm{cl}\,\mathcal{I}$ and so is the smallest set in $\mathrm{cl}\,\mathcal{I}$. □

We see from Item 1 that distinct m.i.s cannot touch, nor can they reach each other. Thus they are dynamically isolated from each other. Note that this is a property that periodic cycles and fixed points enjoy. Item 2 tells us that a m.i.s is a fixed set of the multifunction, in many circumstances. This is also a a property that periodic cycles and fixed points enjoy. To see this suppose that $\mathrm{f} : X \to X$ is a function and $P = \{x_0, x_1 \ldots, x_N\}$ is a periodic cycle, so $x_{n+1} = \mathrm{f}(x_n)$ for $n = 0, \ldots, N-1$ and $x_0 = \mathrm{f}(x_N)$. Then, we see that $\mathrm{f}(\{x_0, x_1 \ldots, x_{N-1}, x_N\}) = \{x_1, x_2, \ldots, x_N, x_0\} = P$. Item 3 relates the smallest set in $\mathrm{cl}\,\mathcal{I}$ to the minimal sets of $\mathrm{cl}\,\mathcal{I}$. Note that when the space is not compact it is possible to have a unique minimal set which is not the small set; for example, in $\mathbb{R}$ consider $\mathrm{f}(x) = 2x$ then $\{0\}$ is the unique minimal in $\mathbb{R}$ but $[1, \infty)$ is a nonempty closed invariant which does not contain $\{0\}$.

We now give an an analogous result to Theorem 3.2.3 for minimal viable sets.

Theorem 3.2.4. *Let (X, τ) be a topological space, $\mathrm{F} : X \rightsquigarrow X$ be a u.s.c multifunction and Q be a nonempty subset of X. Then, the following are equivalent:*

1. *Q is a m.v.s.*
2. *If $\{x_n\}_{n\in\mathbb{N}} \subseteq Q$ is a trajectory of F then, $\overline{\{x_n\}_{n\in\mathbb{N}}} = Q$.*
3. *If $\{x_n\}_{n\in\mathbb{N}} \subseteq Q$ is a trajectory of F then, for all $N \in \mathbb{N}$ $\overline{\{x_n\}_{n=N}^{\infty}} = Q$.*
4. *If $\{x_n\}_{n\in\mathbb{N}} \subseteq Q$ is a trajectory of F then, $\mathrm{Ls}_{n\to\infty}\{x_n\} = Q$.*

Proof. The implication $3 \implies 2$ is immediate.

To show $2 \implies 1$ let, $V \in \mathrm{cl}\,\mathcal{V}$ with $V \subseteq Q$. Since V is viable there is a trajectory with $\{x_n\}_{n\in\mathbb{N}} \subseteq V$, since V is closed we also have $\overline{\{x_n\}_{n\in\mathbb{N}}} \subseteq V$ but by 2 we have $\overline{\{x_n\}_{n\in\mathbb{N}}} = Q$. Hence, $V = Q$ and 1 holds.

For $1 \implies 3$, let $\{x_n\}_{n\in\mathbb{N}} \subseteq Q$ be a trajectory of F and fix $N \in \mathbb{N}$. By Item 2 of Proposition 3.1.10 we know that $\overline{\{x_n\}_{n=N}^{\infty}} \in \mathrm{cl}\,\mathcal{V}$, as Q is closed and contains the trajectory we have that $\overline{\{x_n\}_{n=N}^{\infty}} \subseteq Q$. Since Q m.v.s we have that $Q = \overline{\{x_n\}_{n=N}^{\infty}}$.

Thus the first three items are equivalent.

For $4 \implies 2$, if $\{x_n\}_{n\in\mathbb{N}} \subseteq Q$ then $Q = \mathrm{Ls}_{n\to\infty}\{x_n\}$ is closed since $\mathrm{Ls}_{n\to\infty}\{x_n\}$ is. Thus $\overline{\{x_n\}_{n\in\mathbb{N}}} \subseteq Q$ and

$$\overline{\{x_n\}_{n\in\mathbb{N}}} = \{x_n\}_{n\in\mathbb{N}} \cup \operatorname*{Ls}_{n\to\infty}\{x_n\} = \{x_n\}_{n\in\mathbb{N}} \cup Q \supseteq Q.$$

Therefore, $\overline{\{x_n\}_{n\in\mathbb{N}}} = Q$ and 2 holds.

Lastly we prove $3 \implies 4$, let $\{x_n\}_{n\in\mathbb{N}} \subseteq Q$ and applying the definition of $\mathrm{Ls}_{n\to\infty}\{x_n\}$ with 3 we see

$$\operatorname*{Ls}_{n\to\infty}\{x_n\} = \bigcap_{N\in\mathbb{N}} \overline{\{x_n\}_{n=N}^{\infty}} = \bigcap_{N\in\mathbb{N}} Q = Q.$$

So 4 holds. □

Currently, I have no great insights into minimal viable set. They are notable in my mind, because when F is continuous with a compact m.i.s, A, then, A contains at least one m.v.s (this can be seen from Theorem 3.2.2). In some situations a m.i.s is "made up of" m.v.ss, that is, for some m.i.s , say$A$, we can have $\overline{\bigcup\{Q \subseteq A : Q \text{ is m.v.s}\}} = A$.

Example 3.2.4. *Let $X = \mathbb{R}$ with the usual topology and define the maps*

$$\mathrm{f}_0(x) = \frac{1}{2}x$$
$$\mathrm{f}_1(x) = \frac{1}{2}x + \frac{1}{2}$$

and the multifunction

$$\mathrm{F}[x] = \{\mathrm{f}_0(x)\} \cup \{\mathrm{f}_1(x)\}.$$

Then,

$$A = [0, 1]$$

is the unique m.i.s in X. The sets $\{0\}, \{1\}$ are m.v.s. Let $b_1, \ldots, b_k \in \{0, 1\}$ and let $\bar{x}$ be the fixed point of $\mathrm{f}_{b_1} \circ \cdots \circ \mathrm{f}_{b_k}$ then, $\left\{\bar{x}, \mathrm{f}_{b_k}(\bar{x}), \mathrm{f}_{b_{k-1}} \circ \mathrm{f}_{b_k}(\bar{x}), \ldots, \mathrm{f}_{b_2} \circ \cdots \circ \mathrm{f}_{b_k}(\bar{x})\right\}$ is a m.v.s. From this you can see that the union of all m.v.s is dense on A.

Example 3.2.5. *Let $X = \mathbb{R}$ with the usual topology and define the maps*

$$\mathrm{f}_1(x) = \frac{1}{2}$$
$$\mathrm{f}_2(x) = x^2$$

and the multifunction

$$\mathrm{F}[x] = \{\mathrm{f}_1(x)\} \cup \{\mathrm{f}_2(x)\}.$$

Then, the set

$$A = \left\{ \frac{1}{2^{2n}} : n \in \mathbb{N} \right\} \cup \left\{ \frac{1}{2}, 0 \right\}$$

is the unique m.i.s in X*. The sets* $\{0\}, \left\{\frac{1}{2}\right\}, \{-1\}, \{1\}$ *are the only m.v.s. So not all m.v.s are contained in a unique m.i.s. Moreover, the point* $\frac{1}{4}$ *is isolated in* A *and does not belong to a m.v.s. So the union of all m.v.s is not dense on* A*.*

We have not satisfactorily established why these minimal invariant/viable sets are related to the long term behaviour of difference inclusions. From a weak perspective, we already know that given sufficient compactness and continuity that m.i.ss show up in long term behaviour. To be explicit suppose that F is continuous and maps a topological space X to X, and there is a closed compact invariant set I. Then, for $x \in I$ the set $\mathrm{clR}\,[x]\,, \omega[x]$ are compact invariant sets and by Theorem 3.2.1 there is a m.i.s, say A, with $A \subseteq \mathrm{clR}\,[x] \subseteq I$. But $\omega[x]$ is also a closed compact invariant set and by a similar argument there is a m.i.s, say B, with $B \subseteq \omega[x]$. There could be many m.i.s in $\omega[x]$. From this we see that m.i.s have something to do with omega limits. When we make additional assumptions on the m.i.s this relationship becomes stronger.

Definition 3.2.4. *Let* (X, τ) *be a topological space and* $\mathrm{F} : X \rightsquigarrow X$ *be a multifunction.*

A set A *is said to be* F*-stable, invariantly stable,* $\mathcal{I}$*-stable or stable, if for all open* $O \supseteq A$ *there is a* $I \in \mathcal{I}$ *with both* $I \subseteq O$ *and* $\mathrm{int}(I) \supseteq A$*. i.e. the set of all invariant neighborhoods of* A *form a local base for* A*.*

A set which is not stable is said to be unstable.

A set A *is said to be locally asymptotically stable if it is* $\mathcal{I}$*-stable and there is an open set* $U \supseteq A$ *such that for all* $x \in U$ *we have* $A = \omega[x]$*.*

Similarly, a set B *is said to be stabilizable, viably stable, or* $\mathcal{V}$*-stable if for all if for all open* $O \supseteq B$ *there is a* $V \in \mathcal{V}$ *with both* $V \subseteq O$ *and* $\mathrm{int}(V) \supseteq B$*. i.e. the set of all viable neighborhoods of* B *form a local base for* A*.*

Stable and stabilizable sets are generalizations of Lyapunov stability; which is essentially "start close stay close" stability. Indeed, a stable set can be thought of as "start close *must* stay close", while a stabilizable set is "start close *can* stay close".

Usually, stability is stated in terms of fixed/equilibrium points rather than sets. But it is reasonable to consider a stable periodic cycle, for instance. And a periodic cycle is naturally thought of as a set. So it reasonable, even in more conventional settings, to consider the stability of sets.

Proposition 3.2.3. *Let (X, τ) be a topological space, $\mathrm{F} : X \rightsquigarrow X$ be a total multifunction and $A \subseteq X$ be a set. Then the following are equivalent:*

1. *A is stable.*
2. *For every open set $O \supseteq A$ there is an open set $U \subseteq O$ with $A \subseteq U$ and $\mathrm{F}^{\circ n}[x] \subseteq O$ for all $x \in U$ and $n \in \mathbb{N}$ (F stability).*
3. *For every open set $O \supseteq A$ the set $\mathrm{R}^{+}[O]$ is a neighborhood of A (R is u.s.c at A).*

Proof. 1 $\implies$ 2: Suppose O is an open neighborhood of A and by 1 there is an invariant neighborhood of A, $U \subseteq O$. For all $x \in \mathrm{int}(U)$ we have $\mathrm{F}^{\circ n}[x] \subseteq U \subseteq O$ for all $n \in \mathbb{N}$, as U is invariant. Therefore, $\mathrm{int}(U)$ is the required set for F stability.

2 $\implies$ 3: Given an open neighborhood of A, O, by 2 we have the open set $U \supseteq A$ with $\mathrm{F}^{\circ n}[U] \subseteq O$ for all $n \in \mathbb{N}$. Hence, $\mathrm{R}[U] = \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[U] \subseteq O$ and by definition of the upper pre-image we have $A \subseteq U \subseteq \mathrm{R}^{+}[O]$. Therefore, $\mathrm{R}^{+}[O]$ is a neighborhood of A.

3 $\implies$ 1: By 3 the set $\mathrm{R}^{+}[O]$ is a neighborhood of A, when $O \supseteq A$ is open. One can see from Item 11 of Proposition 3.1.9 that $\mathrm{R}^{+}[O] \cap O = \mathrm{R}_0^{+}[O]$ is an invariant neighborhood of A in O. Item 1 follows. □

Normally, Item 2 is taken as the definition for stability. Usually, instead of open sets O and U we use ϵ's and δ's; such a definition (using ϵ's and δ's) is compatible with Proposition 3.2.3 when the set A is compact.

Proposition 3.2.4. *Let (X, τ) be a topological space, $\mathrm{F} : X \rightsquigarrow X$ be a total multifunction and $A \subseteq X$ be a set. The following hold:*

1. *If A is stable and for some trajectory $\{x_n\}_{n\in\mathbb{N}_0}$ we have $A \cap \mathrm{Ls}_{n\in\mathbb{N}}\{x_n\} \neq \emptyset$ then $\{\{x_n\}\}_{n\in\mathbb{N}}$ converges to A in the upper Vietoris topology. Note that when X is also regular then, $\mathrm{Ls}_{n\to\infty}\{x_n\} \subseteq \overline{A}$.*
2. *If A is stable, F is l.s.c and X is regular then, $\overline{A}$ is invariant.*
3. *If A is invariant and R is u.s.c then, A is stable.*
4. *If A is a closed invariant set and $\overline{\mathrm{R}}$ is u.s.c then, A is stable.*

Proof. To prove 1, suppose that $a \in A \cap \mathrm{Ls}_{n\in\mathbb{N}}\{x_n\}$ for some trajectory $\{x_n\}_{n\in\mathbb{N}}$. Let $O \supseteq A$ be open and since A is stable there is an invariant I with $A \subseteq \mathrm{int}(I) \subseteq I \subseteq O$. So $\mathrm{int}(I) \ni a$ and $\mathrm{int}(I) \cap \mathrm{Ls}_{n\in\mathbb{N}}\{x_n\} \neq \emptyset$. It follows that, $\mathrm{int}(I) \cap \{x_n : n \in \mathbb{N}_0\} \neq \emptyset$ (as $\mathrm{Ls}_{n\in\mathbb{N}}\{x_n\} = \bigcap_{N\in\mathbb{N}_0} \{x_n : n \in \mathbb{N}_N\}$) but I is invariant. Thus, for some $N \in \mathbb{N}$ we have $\{x_n : n \in \mathbb{N}_N\} \subseteq I \subseteq O$. Since O is an arbitrary open set containing A, we have that $\{\{x_n\}\}_{n\in\mathbb{N}_0}$ converges to A in the u.v.t (by Item 2 of Proposition 2.2.4).

When X is regular then, we can see from Item 2 of Proposition 2.2.6 that $\mathrm{Ls}_{n\to\infty}\{x_n\} \subseteq \overline{A}$.

For 2, note that since X is regular,

$$\overline{A} = \bigcap_{O\in\tau, O\subseteq A} \overline{O}.$$

Since A is stable, one can see that

$$\bigcap_{O\in\tau, O\subseteq A} \overline{O} \supseteq \bigcap\left\{\overline{I} : \forall O \in \tau, \exists I \in \mathcal{I}, O \supseteq I \supseteq \mathrm{int}(I) \supseteq A\right\} \supseteq \overline{A}.$$

Hence, $\overline{A} = \bigcap\left\{\overline{I} : \forall O \in \tau, \exists I \in \mathcal{I}, O \supseteq I \supseteq \mathrm{int}(I) \supseteq A\right\}$ and since F is l.s.c, $\overline{A}$ is the intersection of closed invariant sets. By Item 9 of Proposition 3.1.9 we know that $\overline{A}$ is invariant.

To prove 3 note that if, R is u.s.c, A is invariant and $O \supseteq A$ is open then, $\mathrm{R}^+[O]$ is an open neighborhood of A ($\mathrm{R}[A] \subseteq A \subseteq O \implies A \subseteq \mathrm{R}^+[O]$ and $\mathrm{R}^+[O]$ is open by u.s.c). So by Item 3 of Proposition 3.2.3 we have that A is stable.

Similarly for 4, if clR is u.s.c and $O \supseteq A$ is open then, $A \subseteq \mathrm{clR}^+[O] \subseteq \mathrm{R}^+[O]$ (if $\mathrm{clR}[x] \subseteq O$ then $\mathrm{R}[x] \subseteq O$ as well). So $\mathrm{R}^+[O]$ is a neighborhood of A and by Item 3 of Proposition 3.2.3 we have that A is stable.

□

Stable sets are of interest due to Item 1; which reads as: if a trajectory touches a stable set (in finite or infinite time) then, the stable set contains the omega limit of the trajectory. So stable sets provide us with a "set upper bound" on the omega limits of trajectories.

Item 2 tells us that closed stable sets are invariant, in most circumstances. For this reason it makes sense to mostly consider closed stable sets. Note that if A is stable then, $\overline{A}$ may be unstable. Consider $\mathrm{f} : \mathbb{R} \to \mathbb{R}$, $\mathrm{f}(x) = 2x$ where $\mathbb{R}$ has the usual topology. Then, $A = (0, \infty)$ is stable since A is open and invariant but $\overline{A}$ is unstable.

Items 3 and 4 gives a condition where every invariant (or closed invariant) set is stable. This is rather strong conclusion and it comes from the continuity proprieties of the reachable set. We will see later that the upper semicontinuity of R or clR is even stronger than Proposition 3.2.4 might indicate.

3.3 The chain reachable set, robust invariance and computability

In [4], Collins showed that the closed reachable set is computable if and only if the reachable set is robust. This result sounds fantastic! If we can effectively approximate the closed reachable set we can answer many natural questions about the dynamics. Tragically, the reachable set being robust is a shocking strong condition of the dynamics of the system. In this section we present the some parts of Collins work, with my own thoughts and results interspersed.

First we must know what the chain reachable set is.

Definition 3.3.1 (chain reachable set, for metric spaces, given in [4])**.** *Let (X, d) be a metric space, $C \subseteq X$ and $\mathrm{F} : X \rightsquigarrow X$ be a multifunction. For $\epsilon > 0$ we denote the open ball of radius ϵ centered at $x \in X$ to be $\mathcal{B}_\epsilon(x) = \{y \in X : \mathrm{d}(x, y) < \epsilon\}$. For $\epsilon > 0$ we define $\mathcal{B}_\epsilon(C) = \bigcup_{c\in C} \mathcal{B}_\epsilon(c)$. Also, define $\mathrm{F}_\epsilon[x] = \mathcal{B}_\epsilon(\mathrm{F}[x])$ for all $x \in X$.*

Let $\epsilon > 0$, we define an ϵ-chain of F starting in C to be $\{y_n\}_{n=0}^N$ where $N \in \mathbb{N}$ and we have that: $y_{n+1} \in \mathrm{F}_\epsilon[y_n]$ and $y_0 \in C$ for $n = 0, \ldots, N-1$.

Define the chain reachable set

$$\mathrm{CR}(\mathrm{F}, C) = \Big\{x \in X : \forall\epsilon > 0\ \exists\{y_n\}_{n=0}^N \text{ an } \epsilon\text{-chain of } \mathrm{F} \text{ starting in } C,\ x = y_N\Big\}.$$

If the multifunction is understood we may instead write $\mathrm{CR}(C)$ to be the chain reachable set. The reachable set $\mathrm{R}[\mathrm{F}, C]$ is said to be robust if $\overline{\mathrm{R}[\mathrm{F}, C]} = \mathrm{CR}(\mathrm{F}, C)$.

One can see that

$$\mathrm{CR}(\mathrm{F}, C) = \bigcap_{\epsilon>0} \bigcup_{n=1}^{\infty} \mathrm{F}_\epsilon^{\circ n}[C] = \bigcap_{\epsilon>0} \mathrm{R}[\mathrm{F}_\epsilon, C].$$

Intuitively, an ϵ-chain is a trajectory of F, which is perturbed by some noise of size ϵ, at each time step. So the chain reachable set can be thought of as the set of points reachable by trajectories perturbed by "infinitesimally small" perturbations.

Collins credits Conley, the author of [8], with concept of chains. In [8, Chapter 2 Section 6], Conley speaks of a concept of chain recurrence. However, Conley defined chains for topological spaces. Something Collins also did, while redefining the chain reachable set, in [5].

Definition 3.3.2 (chain reachable set, for topological spaces, given in [5]). *Let (X, τ) be a topological space, $C \subseteq X$ and $\mathrm{F} : X \rightsquigarrow X$ be a multifunction. Let $\mathcal{U}$ be an open cover of X. Define $\mathrm{F}_{\mathcal{U}} : X \rightsquigarrow X$ to be $\mathrm{F}_{\mathcal{U}}[x] = \bigcup\{U \in \mathcal{U} : U \cap \mathrm{F}[x] \neq \emptyset\}$ for all $x \in X$.*

We define an $\mathcal{U}$-chain of F starting in C to be $\{y_n\}_{n=0}^{N}$ where $N \in \mathbb{N}$ and we have that: $y_{n+1} \in \mathrm{F}_{\mathcal{U}}[y_n]$ and $y_0 \in C$ for $n = 0, \ldots, N-1$.

Define the chain reachable set

$$\mathrm{CR}(\mathrm{F}, C) = \bigcap\{\mathrm{R}\left[\mathrm{F}_{\mathcal{O}}, C\right] : \mathcal{O} \text{ is an open cover of } X\}.$$

If the multifunction is understood we may instead write $\mathrm{CR}\left[C\right]$ to be the chain reachable set. The reachable set $\mathrm{R}\left[\mathrm{F}, C\right]$ is said to be robust if $\overline{\mathrm{R}\left[\mathrm{F}, C\right]} = \mathrm{CR}(\mathrm{F}, C)$.

I do not think that the chain reachable set is a multifunction as defined in Definitions 3.3.1 and 3.3.2. This is mostly a minor inconvenience as we are mostly interested in when the reachable set is robust. Speaking of which, the definition of a "robust" reachable set isn't technically precise, since the set C is not quantified. I assume this was just an oversight by Collins. So I will adopt my own terminology here: (1) We say that reachable set is robust at the set C if $\overline{\mathrm{R}\left[\mathrm{F}, C\right]} = \mathrm{CR}(\mathrm{F}, C)$. (2) We say that the reachable set is pointwise robust (or simply robust) if $\overline{\mathrm{R}\left[\mathrm{F}, x\right]} = \mathrm{CR}(\mathrm{F}, \{x\})$ for all $x \in X$. (3) We say that the reachable set is compactly robust if $\overline{\mathrm{R}\left[\mathrm{F}, K\right]} = \mathrm{CR}(\mathrm{F}, K)$ for all compact sets $K \subseteq X$.

If I had to guess, Collins meant either (1) or (3) for his definitions of robustness in his papers. For our purposes (2) is the most important, note that (3) implies (2).

It is not so clear that Definitions 3.3.1 and 3.3.2 are equivalent. Indeed I don't think they are. However, I believe that when F is u.s.c compact valued and $\mathrm{CR}(\mathrm{F}, C)$ (from either definition) is compact then, these definitions coincide.

More pertinently, I think that Collins, by mistake, does not use the chain reachable set as given in Definition 3.3.2. Rather, in key places in [5] he instead uses the following set.

Definition 3.3.3 (strong chain reachable set, accidentally used in [5]). *Let (X, τ) be a topological space, $C \subseteq X$ and $\mathrm{F} : X \rightsquigarrow X$ be a multifunction. Let $\mathcal{U}$ be an open cover of X. Define $\mathrm{F}_{\mathcal{U}} : X \rightsquigarrow X$ to be $\mathrm{F}_{\mathcal{U}}[x] = \bigcup\{U \in \mathcal{U} : U \cap \mathrm{F}[x] \neq \emptyset\}$ for all $x \in X$.*

Define the strong chain reachable set

$$\mathrm{sCR}(\mathrm{F}, C) = \bigcap\left\{\overline{\mathrm{R}\left[\mathrm{F}_{\mathcal{O}}, C\right]} : \mathcal{O} \text{ is an open cover of } X\right\}.$$

If the multifunction is understood we may instead write $\mathrm{sCR}\left[C\right]$ to be the strong chain reachable set. The reachable set $\mathrm{R}\left[\mathrm{F}, C\right]$ is said to be strongly robust at C if $\overline{\mathrm{R}\left[\mathrm{F}, C\right]} =$

$\mathrm{sCR}(\mathrm{F}, C)$. *The reachable set* R *or* clR *is said to be pointwise strongly robust (or simply strongly robust) if if* $\overline{\mathrm{R}\,[\mathrm{F}, x]} = \mathrm{sCR}(\mathrm{F}, \{x\})$ *for all* $x \in X$. *We say that the reachable set is compactly strongly robust if* $\overline{\mathrm{R}\,[\mathrm{F}, K]} = \mathrm{sCR}(\mathrm{F}, K)$ *for all compact sets* $K \subseteq X$.

In the proof of Lemma 4.4 in [5], the penultimate sentence, Collins writes:

> "Since $[\overline{\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, C]}]$ decreases on taking refinements [of the open cover $\mathcal{U}$], and converges to $[\mathrm{CR}(\mathrm{F}, C)]$, we must have $[\overline{\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, C]} \cap B = \emptyset]$ for some $\mathcal{U}$."[1]

Contextually, I believe the "converges to" means "intersects to". If this is true then, the above quote is false (or at least needs further clarification). For while the $\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, C]$ intersect to $\mathrm{CR}(\mathrm{F}, C)$, the closures, $\mathrm{cl}(\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, C])$, intersect to $\mathrm{sCR}(\mathrm{F}, C)$. Of course, it can be the case that CR = sCR, in which case this "error" is nothing more than an oversight. In Example 3.3.1 we present an example of a case where CR $\neq$ sCR, however in [5] computable Hausdorff spaces are considered, which are (among other things) first countable, regular and Hausdorff. The space considered in Example 3.3.1 is none of first countable, regular or Hausdorff; but it does illustrate that a proof of CR = sCR is at least a little non-trivial.

Example 3.3.1 (An (arguably irrelevant) example of CR $\neq$ sCR)**.** *Endow* $X = \mathbb{R}$ *with the co-countable topology that is:*

$$\tau = \{V \subseteq X : V = X \setminus E \textit{ where } E \subseteq X \textit{ is countable}\} \cup \{\emptyset\}$$

and consider the multifunction $\mathrm{F}[x] = \{0\}$ *for all* $x \in X$. *One can argue that* F *is continuous and compact valued, with respect to the co-countable topology.*

Note that given an uncountable set $A \subseteq X$, *we have* $\overline{A} = X$. *This is because the closed sets of* X *are the countable sets and* X.

We claim that $\mathrm{CR}(\mathrm{F}, C) = \{0\}$ *but* $\mathrm{sCR}(\mathrm{F}, C) = X$ *for all* $C \subseteq X$. *To see why* $\mathrm{sCR}(\mathrm{F}, C) = X$, *let* $\mathcal{U}$ *be an open cover of* X *then,* $\mathrm{F}_{\mathcal{U}}[C] = \bigcup\{U \in \mathcal{U} : 0 \in U\}$ *is open. Since every open set is uncountable, the closure of every open set is* X. *Hence,* $\overline{\mathrm{F}_{\mathcal{U}}[C]} = X$ *for every open cover* $\mathcal{U}$, *so by definition* $\mathrm{sCR}(\mathrm{F}, C) = X$.

On the other hand, we can see that $\mathrm{CR}(\mathrm{F}, C) = \{0\}$, *since if* $y \in X$ *and* $y \neq 0$ *then and we can pick an open cover* $\mathcal{U} = \{X \setminus \{y\}, X \setminus \{0\}\}$. *But then,*

$$\mathrm{F}_{\mathcal{U}}[B] = \bigcup\{U \in \mathcal{U} : 0 \in U\} = X \setminus \{y\}$$

[1] The equations within this quote were modified or corrected to have consistent notation with Definitions 3.3.1 to 3.3.3.

for all $B \subseteq X$. It follows that $\mathrm{R}[\mathrm{F}_{\mathcal{U}}, C] = X \setminus \{y\} \not\ni y$ *and by definition* $y \notin \mathrm{CR}(\mathrm{F}, C)$. *This holds for all* $y \in X$ *with* $y \neq 0$, *so* $\mathrm{CR}(\mathrm{F}, C) \subseteq \{0\}$. *The other inclusion, follows from* $\{0\} = \mathrm{R}[\mathrm{F}, C] \subseteq \mathrm{CR}(\mathrm{F}, C)$. *Therefore,* $\mathrm{CR}(\mathrm{F}, C) = \{0\}$.

We now find some conditions on when $\mathrm{CR} = \mathrm{sCR}$.

Theorem 3.3.1. *Let* (X, τ) *be a normal topological space,* $C \subseteq X$ *and* $\mathrm{F} : X \rightsquigarrow X$ *be a total multifunction.*

1. *For every open cover of* X, $\mathcal{U}$ *there is an open cover* $\mathcal{V}$ *with* $\overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]} \subseteq \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]$.
2. $\mathrm{CR}(C) = \mathrm{sCR}(C)$.

Proof. Item 1: Let $\mathcal{U}$ be on open cover of X. Note that $\mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]$ is open, since it is the union of sets in $\mathcal{U}$. So $X \setminus \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]$ is closed and by normality there is an open set W with

$$X \setminus \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C] \subseteq W \subseteq \overline{W} \subseteq \bigcup\{U \in \mathcal{U} : U \cap X \setminus \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C] \neq \emptyset\}.$$ [2]

Define,

$$\mathcal{V} = \{U \setminus \overline{W} : U \in \mathcal{U}\} \cup \{U \in \mathcal{U} : U \cap X \setminus \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C] \neq \emptyset\}.$$

It can be seen that $\mathcal{V}$ is an open cover of X. By construction the elements of $\mathcal{V}$ are open. To see why it covers X, let $y \in X$ then $y \in \overline{W}$ or $y \in X \setminus \overline{W}$. If $y \in \overline{W}$ then, $y \in \bigcup\{U \in \mathcal{U} : U \cap X \setminus \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C] \neq \emptyset\}$, and the U's involved in this union are in $\mathcal{V}$. If $y \in X \setminus \overline{W}$ then y is in an element of $\{U \setminus \overline{W} : U \in \mathcal{U}\}$, since $\mathcal{U}$ covers X.

Since every element of $\mathcal{V}$ is a subset of an element of $\mathcal{U}$ we have that for all $A \subseteq X$

$$\bigcup\{V \in \mathcal{V} : V \cap A \neq \emptyset\} \subseteq \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}.$$

It follows that $\mathrm{F}_{\mathcal{V}} \subseteq \mathrm{F}_{\mathcal{U}}$ and consequently

$$\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C] \subseteq \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C].$$

So $\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C] \cap X \setminus \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C] = \emptyset$, it follows that $\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]$ is a union of elements of $\{U \setminus \overline{W} : U \in \mathcal{U}\}$. Therefore, $\overline{W} \cap \mathrm{R}[\mathrm{F}_{\mathcal{V}}, C] = \emptyset$ and so $\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C] \subseteq X \setminus \overline{W} \subseteq X \setminus W$. Thus,

$$\overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]} \subseteq X \setminus W \subseteq X \setminus (X \setminus \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]) = \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]$$

[2] To see this note that if A is closed and O is open with $A \subseteq O$ then, $A \cap X \setminus O = \emptyset$. As $A, X \setminus O$ are closed and by normality there are open sets W, V with $W \supseteq A$, $V \supseteq X \setminus O$ and $W \cap V = \emptyset$. In particular, $W \subseteq X \setminus V$, so $\overline{W} \subseteq X \setminus V \subseteq X \setminus (X \setminus O) = O$.

and the claim holds.

Item 2: To prove $\mathrm{CR}(C) = \mathrm{sCR}(C)$, first note that $\mathrm{CR}(C) \subseteq \mathrm{sCR}(C)$ follows immediately from definitions. On the other hand, if $y \notin \mathrm{CR}(C)$ then, there is an open cover $\mathcal{U}$ with $y \notin \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]$. By Item 1 there is an open cover $\mathcal{V}$ with $\overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]} \subseteq \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]$, and $y \notin \overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]}$. Therefore, $y \notin \mathrm{sCR}(C)$ either. This shows that, $\mathrm{CR}(C) \supseteq \mathrm{sCR}(C)$, which concludes the proof. □

Theorem 3.3.2. *Let (X, τ) be a regular topological space, $C \subseteq X$ and $\mathrm{F} : X \rightsquigarrow X$ be a total multifunction. The following hold:*

1. *$y \notin \mathrm{CR}(C)$ if and only if there is a $W \in \tau_y$ and an open cover $\mathcal{U}$ of X with $\mathrm{R}[\mathrm{F}_{\mathcal{U}}, C] \cap \overline{W} = \emptyset$.*

2. *$\mathrm{CR}(C) = \mathrm{sCR}(C)$.*

Proof. Item 1: Suppose that $y \notin \mathrm{CR}(C)$ then, for some open cover $\mathcal{V}$, we have that $y \notin \mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]$. Since $\mathcal{V}$ is a cover of X, for some $V_y \in \mathcal{V}$ we have $y \in V_y$. Since X is regular there is a open $W \ni y$ with $\overline{W} \subseteq V_y$.

Define,

$$\mathcal{U} = \left\{ V \setminus \overline{W} : V \in \mathcal{V} \right\} \cup \{V_y\}.$$

The collection $\mathcal{U}$ consists of only open sets and $\mathcal{U}$ also covers X. To see this, let $x \in X$ then, either $x \in \overline{W}$ or $x \in X \setminus \overline{W}$. When $x \in \overline{W} \subseteq V_y$ then $x \in V_y \in \mathcal{U}$. When $x \in X \setminus \overline{W}$, we use the fact that $\mathcal{V}$ covers X, so there is a $V \in \mathcal{V}$ with $x \in V \cap X \setminus \overline{W} = V \setminus \overline{W} \in \mathcal{U}$. Moreover, since every element of $\mathcal{U}$ is a subset of an element of $\mathcal{V}$ we have that for all $A \subseteq X$

$$\bigcup \{U \in \mathcal{U} : U \cap A \neq \emptyset\} \subseteq \bigcup \{V \in \mathcal{V} : V \cap A \neq \emptyset\}.$$

It follows that $\mathrm{F}_{\mathcal{U}} \subseteq \mathrm{F}_{\mathcal{V}}$ and consequently

$$\mathrm{R}[\mathrm{F}_{\mathcal{U}}, C] \subseteq \mathrm{R}[\mathrm{F}_{\mathcal{V}}, C].$$

Since $y \notin \mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]$ we know that $y \notin \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]$. Thus, $\mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]$ is the union of sets in $\left\{ V \setminus \overline{W} : V \in \mathcal{V} \right\}$ and so $\mathrm{R}[\mathrm{F}_{\mathcal{U}}, C] \cap \overline{W} = \emptyset$.

Conversely, if there is a $W \in \tau_y$ and an open cover $\mathcal{U}$ of X with $\mathrm{R}[\mathrm{F}_{\mathcal{U}}, C] \cap \overline{W} = \emptyset$ then, $y \notin \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]$ it follows that $y \notin \mathrm{CR}(\mathrm{F}, C)$.

Item 2: The inclusion $\mathrm{CR}(C) \subseteq \mathrm{sCR}(C)$ follows immediately from definitions. On the other hand, if $y \notin \mathrm{CR}(C)$ then, by Item 1 we know that there is a $W \in \tau_y$ and an open cover $\mathcal{U}$ of X with $\mathrm{R}[\mathrm{F}_{\mathcal{U}}, C] \cap \overline{W} = \emptyset$. Hence, we see that $\overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]} \cap W = \emptyset$ and it follows that $y \notin \overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]}$. This shows that, $\mathrm{CR}(C) \supseteq \mathrm{sCR}(C)$, which concludes the proof. □

Recall that, metric spaces, pseudo metric spaces, and compact regular Hausdorff spaces are normal. So, Theorem 3.3.1 covers all the most important cases. The broader case of computable Hausdorff spaces is covered Theorem 3.3.2, since locally compact Hausdorff spaces are regular. This rectifies Lemma 4.4 of [5], which we effectively prove now.

Proposition 3.3.1. *Let (X,τ) be a topological space, $C \subseteq X$ be nonempty and $\mathrm{F} : X \rightsquigarrow X$ be a total multifunction.*

1. *Let $K \subseteq X$ be a closed compact set. If $K \cap \mathrm{sCR}(C) = \emptyset$ then, there is an open cover $\mathcal{U}$ of X with $K \cap \overline{\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, C\right]} = \emptyset$.*
2. *For any open cover $\mathcal{U}$ of X we have that $\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, C\right], \mathrm{CR}\left[C\right] \in \mathcal{I}_{\mathrm{F}} \cap \mathcal{I}_{\mathrm{F}_{\mathcal{U}}}$.*
3. *Assume X is a Hausdorff locally compact space, $\mathrm{F} : X \rightsquigarrow X$ is a compact valued u.s.c multifunction and the set $\mathrm{CR}(C)$ is compact. Then, for any open set $O \supseteq \mathrm{CR}(C)$ there is an open cover $\mathcal{U}$ of X with $\overline{\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, C\right]}$ compact and $\overline{\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, C\right]} \subseteq O$. (This is Lemma 4.4 of [5])*

Proof. Item 1: We proceed by contraposition, suppose that for all open covers $\mathcal{U}$ of X with $K \cap \overline{\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, C\right]} \neq \emptyset$, where $K \subseteq X$ is a closed compact set. Let $A^{\mathcal{U}} = K \cap \overline{\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, C\right]}$ for an open cover $\mathcal{U}$. We seek to apply the finite intersection theorem to the $A^{\mathcal{U}}$; the $A^{\mathcal{U}}$ are closed sets of K, moreover the collection of the $A^{\mathcal{U}}$ satisfy the finite intersection property. To see this, note that if $\mathcal{U}_1, \ldots, \mathcal{U}_N$ are open covers of X we can define an open cover $\mathcal{V} = \{\bigcap_{n=1}^{N} U_n : U_n \in \mathcal{U}_n \text{ for } n = 1, \ldots, N\}$, one can see that $\mathrm{R}\left[\mathrm{F}_{\mathcal{V}}, C\right] \subseteq \mathrm{R}\left[\mathrm{F}_{\mathcal{U}_n}, C\right]$ for $n = 1, \ldots, N$. Thus, $A^{\mathcal{V}} \subseteq \bigcap_{n=1}^{N} A^{\mathcal{U}_n}$ and since $A^{\mathcal{V}} \neq \emptyset$ we conclude that the collection of $A^{\mathcal{U}}$ has the finite intersection property.

Let $\mathfrak{U}$ be the set of all open covers of X, then by the finite intersection theorem we have that

$$\emptyset \neq \bigcap_{\mathcal{U} \in \mathfrak{U}} A^{\mathcal{U}} = \bigcap_{\mathcal{U} \in \mathfrak{U}} K \cap \overline{\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, C\right]} = K \cap \bigcap_{\mathcal{U} \in \mathfrak{U}} \overline{\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, C\right]} = K \cap \mathrm{sCR}(C).$$

Which concludes the proof.

Item 2: Let $\mathcal{U}$ be an open cover of X. Note that $\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, C\right] = \bigcup_{c \in C} \mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, c\right]$ and the union/intersection of invariant sets is invariant, by Item 9 of Proposition 3.1.9. Hence, we need only show that $\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, x\right] \in \mathcal{I}_{\mathrm{F}} \cap \mathcal{I}_{\mathrm{F}_{\mathcal{U}}}$ for all $x \in X$. Let $x \in X$ and by Item 7 of Proposition 3.1.9 we know $\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, x\right] \in \mathcal{I}_{\mathrm{F}_{\mathcal{U}}}$. Since $\mathrm{F}[z] \subseteq \mathrm{F}_{\mathcal{U}}[z]$ for all $z \in X$, we have that

$$\mathrm{F}[\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, x\right]] \subseteq \mathrm{F}_{\mathcal{U}}[\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, x\right]] \subseteq \mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, x\right]$$

so by definition $\mathrm{R}[\mathrm{F}_{\mathcal{U}}, x] \in \mathcal{I}_{\mathrm{F}}$.

Item 3: Let $O \supseteq \mathrm{CR}(C)$ then, since X is locally compact and Hausdorff, X is regular. From this one can make a covering argument to show that there is a open V, with $\mathrm{CR}(C) \subseteq V \subseteq \overline{V} \subseteq O$ with $\overline{V}$ compact. By Item 2 we have that, $\mathrm{F}[\mathrm{CR}[C]] \subseteq \mathrm{CR}[C] \subseteq V$ and we see that $\mathrm{CR}(C) \subseteq \mathrm{F}^+[V]$. Since F is u.s.c the set, $\mathrm{F}^+[V]$ is open and so $\mathrm{CR}(C) \subseteq V \cap \mathrm{F}^+[V]$. Again, via a covering argument one can show that there is open W with

$$\mathrm{CR}(C) \subseteq W \subseteq \overline{W} \subseteq V \cap \mathrm{F}^+[V].$$

Since X is regular $\mathrm{CR}(C) = \mathrm{sCR}(C)$ by Theorem 3.3.2. Since $\overline{V} \setminus W$ is compact and closed with $\mathrm{sCR}(C) \cap \overline{V} \setminus W = \emptyset$ we can can apply Item 1. So there is an open cover $\mathcal{U}$ of X with $\overline{V} \setminus W \cap \overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]} = \emptyset$.

Define the open cover

$$\mathcal{V} = \left\{U \setminus \left(\mathrm{F}\left[\overline{W}\right] \cup \overline{W}\right) : U \in \mathcal{U}\right\} \cup \{U \cap V : U \in \mathcal{U}\}.$$

Note that $\mathrm{F}\left[\overline{W}\right]$ is a compact and therefore closed set, this follows from X being Hausdorff, $\overline{W}$ being compact, F being an u.s.c compact valued multifunction and Theorem 2.3.11. Then, it can be shown that $\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C] \subseteq \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]$. From this, it follows that $\overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]} \cap \overline{V} \setminus W = \emptyset$, and

$$\overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]} \subseteq X \setminus (\overline{V} \setminus W) = W \cup X \setminus \overline{V}.$$

For the sake of contradiction, suppose that $\overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]} \cap X \setminus \overline{V} \neq \emptyset$. Then, since $X \setminus \overline{V}$ is open we also have $\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C] \cap X \setminus \overline{V} \neq \emptyset$. Let $n \in \mathbb{N}$ be the smallest $k \in \mathbb{N}$ with $\mathrm{F}_{\mathcal{V}}^{\circ k}[C] \cap X \setminus \overline{V}$. Note $n > 1$ as

$$\mathrm{F}_{\mathcal{V}}[C] \subseteq V \supseteq \mathrm{F}_{\mathcal{V}}\left[\overline{W}\right].$$

The inclusion, $V \supseteq \mathrm{F}_{\mathcal{V}}\left[\overline{W}\right]$ follows quickly from the construction of W (namely, $\mathrm{F}\left[\overline{W}\right] \subseteq V$) and $\mathcal{V}$. The inclusion $\mathrm{F}_{\mathcal{V}}[C] \subseteq V$, holds for similar reasons since, $\mathrm{F}[C] \subseteq \mathrm{R}[\mathrm{F}, C] \subseteq \mathrm{CR}(C) \subseteq W$.

By definition of n, there is $x \in \mathrm{F}_{\mathcal{V}}^{\circ(n-1)}[C]$ with $x \in \overline{V}$ and $\mathrm{F}_{\mathcal{V}}[x] \cap X \setminus \overline{V} \neq \emptyset$. Either $x \in \overline{W}$ or $x \notin \overline{W}$; in the case of $x \in \overline{W}$ we see that $\mathrm{F}_{\mathcal{V}}[x] \subseteq \mathrm{F}_{\mathcal{V}}\left[\overline{W}\right] \subseteq V$, a contradiction to $\mathrm{F}_{\mathcal{V}}[x] \cap X \setminus \overline{V} \neq \emptyset$. In the case of $x \notin \overline{W}$, then $x \in \overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]} \cap \overline{V} \setminus W$, a contradiction to $\overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]} \cap \overline{V} \setminus W = \emptyset$. Therefore, $\overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]} \cap X \setminus \overline{V} = \emptyset$; as $\overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]} \subseteq W \cup X \setminus \overline{V}$ we see that $\overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]} \subseteq W \subseteq O$. Since $\overline{W}$ is compact then, $\overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, C]}$ is compact. This concludes the proof.

□

Frankly, I think that both Definitions 3.3.1 and 3.3.2 aren't the best definitions for what Collins wanted to prove. Since he immediately characterizes the chain reachable set in terms of robust invariant sets.

Definition 3.3.4 (robust invariant set)**.** *Let (X, τ) be a topological space, $I \subseteq X$ and $\mathrm{F} : X \rightsquigarrow X$ be a multifunction.*

The set I is called a robust invariant set of F *if*

$$\mathrm{F}\left[\overline{I}\right] \subseteq \operatorname{int}(I).$$

When there is no confusion over the multifunction F *being considered we will simply refer to a robust invariant set of* F *as a robust invariant set.*

Let $\rho\mathcal{I}_{\mathcal{F}}$ be the set of of all nonempty robust invariant sets of F*. Let $\rho\,\mathrm{cl}\,\mathcal{I}_{\mathcal{F}}$ be the set of all closed nonempty robust invariant sets of* F*. Let $\rho o\mathcal{I}_{\mathcal{F}}$ be the set of all open nonempty robust invariant sets of* F*. Again we will omit the subscript* F *when there is no over the multifunction being considered.*

A set $V \subseteq X$ is called a robust viable set of F *if*

$$\overline{V} \subseteq \mathrm{F}^{-}[\operatorname{int}(V)].$$

When there is no confusion over the multifunction F *being considered we will simply refer to a robust viable set of* F *as a robust viable set.*

The robust invariant sets, are rather convenient computationally and they provide a way of actually rigorously over-approximating the reachable set. From a computability theory perspective, the robust invariant sets are intuitively the only invariant sets which can be found. Recall Item 3 of Theorem 2.4.1, let K be compact and V be open, the problem:

Detect if $K \subseteq V$ is true, given that $K \subseteq V$ is in fact true.

Can be solved by a computer. When K is not compact or V is not open this same problem cannot by solved by a computer[3]. If we want to detect an invariant set, I, then, we want to search for sets I with,

$$\mathrm{F}[I] \subseteq I.$$

[3]Well, its more complicated than this. A computer could not solve these problems using only the topological information described in Section 2.4. If more or other information is used then certain instances of the problem could be solvable.

But this inclusion is not computable, in general. To make this inclusion computable, we need $\mathrm{F}[I]$ to be compact and I to be open. Naively, this leads to us searching for sets I with

$$\overline{\mathrm{F}[I]} \subseteq \operatorname{int}(I).$$

When X is Hausdorff, $\overline{I}$ is compact and F is a compact valued continuous multifunction then,

$$\mathrm{F}\left[\overline{I}\right] = \overline{\mathrm{F}\left[\overline{I}\right]} = \overline{\mathrm{F}[I]}$$

and we arrive at

$$\mathrm{F}\left[\overline{I}\right] \subseteq \operatorname{int}(I);$$

The definition of a robust invariant set, with $\mathrm{F}\left[\overline{I}\right]$ compact.

We now present some grounding facts about robust invariant sets.

Proposition 3.3.2. *Let (X, τ) be a topological space and $\mathrm{F} : X \rightsquigarrow X$ be a total multifunction.*

1. *If $I \in \rho\mathcal{I}$ then, $\overline{I}, \operatorname{int}(I), \overline{\operatorname{int}(I)} \in \rho\mathcal{I}$. Note that $\overline{\operatorname{int}(I)}$ is a closed regular set; A set B is a closed regular set if $B = \overline{\operatorname{int}(B)}$.*
2. *If $I, J \in \rho\mathcal{I}$ then, $I \cup J \in \rho\mathcal{I}$. Furthermore, If $I, J \in \rho\mathcal{I}$ with $I \cap J \neq \emptyset$ then, $I \cap J \in \rho\mathcal{I}$.*
3. *If F is o.s.c, $K \subseteq X$ is a compact set and $I \in \rho\,\mathrm{cl}\mathcal{I}$ is compact with $\mathrm{F}[K] \subseteq \operatorname{int}(I)$ then, there is an open cover $\mathcal{U}$ of X with $\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, K\right] \subseteq \operatorname{int}(I)$.*
4. *Suppose F is l.s.c. Then, for every open cover $\mathcal{U}$ of X and $x \in X$ we have that $\mathrm{clR}\left[\mathrm{F}_{\mathcal{U}}, x\right] \in \rho\mathcal{I}_{\mathrm{F}} \cap \rho\mathcal{I}_{\mathrm{F}_{\mathcal{U}}}$.*

Proof. Item 1: Let $I \in \rho\mathcal{I}$, $A = \overline{I}$ then,

$$\mathrm{F}[A] = \mathrm{F}\left[\overline{I}\right] \subseteq \operatorname{int}(I) \subseteq \operatorname{int}(A)$$

so $A \in \rho\mathcal{I}$. Let $B = \operatorname{int}(I)$ then,

$$\mathrm{F}\left[\overline{B}\right] \subseteq \mathrm{F}\left[\overline{I}\right] \subseteq \operatorname{int}(I) = B$$

as $B \subseteq I$ and $B \in \rho\mathcal{I}$. The fact that $\overline{\operatorname{int}(I)} \in \rho\mathcal{I}$ follows directly from what we just proved.

Item 2: Suppose that $I, J \in \rho\mathcal{I}$ we have $\mathrm{F}\left[\overline{I}\right] \subseteq \mathrm{int}(I)$ or equivalently $\overline{I} \subseteq \mathrm{F}^+[\mathrm{int}(I)]$. Recalling Item 5 of Proposition 2.3.1 we see,

$$\overline{I \cup J} = \overline{I} \cup \overline{J} \subseteq \mathrm{F}^+[\mathrm{int}(I)] \cup \mathrm{F}^+[\mathrm{int}(J)] \subseteq \mathrm{F}^+[\mathrm{int}(I) \cup \mathrm{int}(J)] \subseteq \mathrm{F}^+[\mathrm{int}(I \cup J)]$$

and since $I \cup J \neq \emptyset$ we have that $I \cup J \in \rho\mathcal{I}$. Similarly,

$$\overline{I \cap J} \subseteq \overline{I} \cap \overline{J} \subseteq \mathrm{F}^+[\mathrm{int}(I)] \cap \mathrm{F}^+[\mathrm{int}(J)] = \mathrm{F}^+[\mathrm{int}(I) \cap \mathrm{int}(J)] = \mathrm{F}^+[\mathrm{int}(I \cap J)].$$

So when $I \cap J \neq \emptyset$ we have that $I \cap J \in \rho\mathcal{I}$.

Item 3: Since I and K are compact and F is o.s.c we apply Proposition 2.3.6, to conclude that $\mathrm{F}[I \cup K]$ is closed. Note that $\mathrm{F}[I] \subseteq \mathrm{int}(I)$ since I is a robust invariant set. We claim there is open cover $\mathcal{U}$ of X with

$$\mathrm{F}_{\mathcal{U}}[I \cup K] \subseteq \mathrm{int}(I).$$

For example, let $\mathcal{W}$ be any open cover of X and define

$$\mathcal{U} = \{W \setminus \mathrm{F}[I \cup K] : W \in \mathcal{W}\} \cup \{\mathrm{int}(I)\}.$$

The elements of $\mathcal{U}$ are open (since $\mathrm{F}[I \cup K]$ is closed). The collection $\mathcal{U}$ covers X, since $X \setminus \mathrm{F}[I \cup K]$ is covered by $\{W \setminus \mathrm{F}[I \cup K] : W \in \mathcal{W}\}$ and $\mathrm{F}[I \cup K]$ is covered by $\mathrm{int}(I)$. By construction, if $x \in I \cup K$ then $\mathrm{F}_{\mathcal{U}}[x] = \mathrm{int}(I) \subseteq \mathrm{int}(I)$. It follows that $\mathrm{F}_{\mathcal{U}}[I \cup K] \subseteq \mathrm{int}(I)$, as claimed.

Define, $X_0 = K$ and for $n \in \mathbb{N}$ define $X_{n+1} = \mathrm{F}_{\mathcal{U}}[X_n]$. We claim that $X_n \subseteq \mathrm{int}(I)$ for all $n \in \mathbb{N}$. Proceed by induction, for $n = 1$ we know that $\mathrm{F}_{\mathcal{U}}[K] \subseteq \mathrm{int}(I)$ by construction of $\mathcal{U}$. Assume that $X_n \subseteq \mathrm{int}(I)$ then, $\mathrm{F}[X_n] \subseteq \mathrm{F}[I]$ and by construction of $\mathcal{U}$ we have that $X_{n+1} = \mathrm{F}_{\mathcal{U}}[X_n] \subseteq \mathrm{F}_{\mathcal{U}}[I] \subseteq \mathrm{int}(I)$. Therefore, $X_n \subseteq \mathrm{int}(I)$ for all $n \in \mathbb{N}$ and it follows that $\mathrm{R}\left[\mathrm{F}_{\mathcal{U}}, K\right] \subseteq \mathrm{int}(I)$.

Item 4: Let $\mathcal{U}$ be an open cover of X and $x \in X$. By Item 5 of Proposition 2.3.9 we know that $\mathrm{F}_{\mathcal{U}}$ is l.s.c. It follows that $\mathrm{clR}\left[\mathrm{F}_{\mathcal{U}}, x\right]$ is a closed invariant set of $\mathrm{F}_{\mathcal{U}}$ (by Item 1 of Proposition 3.1.10). By invariance we have

$$\mathrm{F}_{\mathcal{U}}[\mathrm{clR}\left[\mathrm{F}_{\mathcal{U}}, x\right]] \subseteq \mathrm{clR}\left[\mathrm{F}_{\mathcal{U}}, x\right].$$

Since $\mathrm{F}_{\mathcal{U}}$ is open valued, the set $\mathrm{F}_{\mathcal{U}}[\mathrm{clR}\left[\mathrm{F}_{\mathcal{U}}, x\right]]$ is the union of open sets (by definition of the forward image of a multifunction). So the set $\mathrm{F}_{\mathcal{U}}[\mathrm{clR}\left[\mathrm{F}_{\mathcal{U}}, x\right]]$ is open and in particular we have that

$$\mathrm{F}_{\mathcal{U}}[\mathrm{clR}\left[\mathrm{F}_{\mathcal{U}}, x\right]] \subseteq \mathrm{int}(\mathrm{clR}\left[\mathrm{F}_{\mathcal{U}}, x\right]).$$

So $\mathrm{clR}\left[\mathrm{F}_{\mathcal{U}}, x\right] \in \rho\mathcal{I}_{\mathrm{F}_{\mathcal{U}}}$. Since $\mathrm{F}[y] \subseteq \mathrm{F}_{\mathcal{U}}[y]$ for all $y \in X$ we have that $\mathrm{F}[\mathrm{clR}\left[\mathrm{F}_{\mathcal{U}}, x\right]] \subseteq \mathrm{F}_{\mathcal{U}}[\mathrm{clR}\left[\mathrm{F}_{\mathcal{U}}, x\right]] \subseteq \mathrm{int}(\mathrm{clR}\left[\mathrm{F}_{\mathcal{U}}, x\right])$ and so $\mathrm{clR}\left[\mathrm{F}_{\mathcal{U}}, x\right] \in \rho\mathcal{I}_{\mathrm{F}}$ too.

□

Note that the infinite union and intersection of robust invariant sets may fail to be a robust invariant set, see Example 3.3.2. This is in contrast with ordinary invariant sets. A consequence of this is that while there is a largest invariant in a set A, there may not be a largest robust invariant in A.

Example 3.3.2 (There is no such thing as largest/smallest robust invariant sets)**.** *Let $X = \mathbb{R}$ with the usual topology and*

$$\mathrm{f}(x) = \begin{cases} 1.5x & x \leq 1 \\ 0.5x + 1 & x \geq 1 \end{cases}$$

then $[0,3]$ is invariant but not a robust invariant set since $\mathrm{f}(0) = 0 \notin (0,3)$. This means there is no largest compact robust invariant invariant set containing $[0,3]$, since any invariant set containing a negative number is unbounded.

Note that the sets (a,b) where $0 < a < 2 < b \leq \infty$ are all robust invariant sets, containing the fixed point 2. It follows that there is no largest robust invariant set contained in $[0,3]$; Simply take the union $\bigcup\{(a,b) : 0 \leq a < 2 < b \leq 3\} = (0,3)$, which is not robust. Hence the infinite union of robust invariant sets may not be a robust invariant set.

It also follows that the intersection of robust invariant sets containing 2, is $\{2\}$. Which is not a robust invariant set. So there is no smallest robust invariant set containing 2. And the infinite intersection of robust invariant sets may not be a robust invariant set.

We can now establish the relationship between the chain reachable set and robust invariant sets.

Theorem 3.3.3. *Let (X,τ) be a topological space, $C \subseteq X$, $K \subseteq X$ be a nonempty compact set and $\mathrm{F} : X \rightsquigarrow X$ be a total multifunction. Define the sets*

$$\rho\,\mathrm{R}(C) = \bigcap\{O : O \in \rho o\mathcal{I}_{\mathcal{F}}, \mathrm{F}[C] \subseteq O\}$$
$$s\rho\,\mathrm{R}(C) = \bigcap\{\overline{O} : O \in \rho o\mathcal{I}_{\mathcal{F}}, \mathrm{F}[C] \subseteq O\}.$$

1. *If F is continuous and X is regular then, $\mathrm{CR}(C) = \mathrm{sCR}(C) = s\rho\,\mathrm{R}(C)$.*

2. *Assume F is u.s.c. If $y \notin s\rho\,\mathrm{R}(C)$ then, there are open sets V, U with $V \ni y$, $U \supseteq C$ and $V \cap s\rho\,\mathrm{R}(U) = \emptyset$.*

3. *Assume X is a Hausdorff locally compact space and $\mathrm{F} : X \rightsquigarrow X$ is a compact valued continuous multifunction. If either, $\mathrm{CR}(K)$ is compact or there is set $O \in \rho o\mathcal{I}_{\mathcal{F}}$, $\mathrm{F}[K] \subseteq O$ with $\overline{O}$ compact then, $\mathrm{sCR}(K) = \mathrm{CR}(K) = \rho\,\mathrm{R}(K) = s\rho\,\mathrm{R}(K)$.*

Proof. Item 1: Since X is regular we can use Theorem 3.3.2 to get $\mathrm{CR}(C) = \mathrm{sCR}(C)$. So wee need only show that $\mathrm{sCR}(C) = s\rho\,\mathrm{R}(C)$. Let $y \in s\rho\,\mathrm{R}(C)$, and let $\mathcal{U}$ be an open cover of X. By Item 4 of Proposition 3.3.2 we know that $\overline{\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, C]}$ is a robust invariant set of F. And by Item 1 of Proposition 3.3.2 we have that $\mathrm{int}(\overline{\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, C]})$ is an open robust invariant set of F. As $y \in s\rho\,\mathrm{R}(C)$, we have that $y \in \overline{\mathrm{int}(\overline{\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, C]})} \subseteq \overline{\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, C]}$ and so $y \in \mathrm{sCR}(C)$.

Conversely, suppose that $y \notin s\rho\,\mathrm{R}(C)$ then there is a $O \in \rho o\mathcal{I}_{\mathcal{F}}$ with $\mathrm{F}[C] \subseteq O$ and $y \notin \overline{O}$. Since X is regular there are open sets V, U with $y \in V$, $\overline{O} \subseteq U$ and $V \cap U = \emptyset$. Define the open cover

$$\mathcal{U} = \left\{U \cap \mathrm{F}^{+}[O], X \setminus \overline{O}\right\}.$$

Note $\mathrm{F}^{+}[O]$ is open since F is u.s.c. Since O is a robust invariant set of F, we see that $\mathrm{F}[C] \subseteq O \subseteq \overline{O} \subseteq \mathrm{F}^{+}[O]$. And by definition of U, we have $\overline{O} \subseteq U$, as well. So

$$\mathrm{F}[C] \subseteq O \subseteq \overline{O} \subseteq U \cap \mathrm{F}^{+}[O].$$

As $\mathrm{F}[C] \subseteq O$, we have that if $W \in \mathcal{U}$ has $\mathrm{F}[C] \cap W \neq \emptyset$ then, $W = U \cap \mathrm{F}^{+}[O]$. Thus, $\mathrm{F}_{\mathcal{U}}[C] = U \cap \mathrm{F}^{+}[O]$. We claim that $U \cap \mathrm{F}^{+}[O]$ is an invariant set of $\mathrm{F}_{\mathcal{U}}$. Indeed, if $x \in U \cap \mathrm{F}^{+}[O]$ then $\mathrm{F}[x] \subseteq O$ and $\mathrm{F}[x] \cap X \setminus \overline{O} = \emptyset$. Therefore, $\mathrm{F}_{\mathcal{U}}[x] = U \cap \mathrm{F}^{+}[O] \subseteq U \cap \mathrm{F}^{+}[O]$ and $U \cap \mathrm{F}^{+}[O]$ is a invariant of $\mathrm{F}_{\mathcal{U}}$.

It follows that $\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, C] \subseteq \overline{U \cap \mathrm{F}^{+}[O]}$ and since $V \cap U = \emptyset$ we have that $V \cap \mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, C] = \emptyset$. This also means that $y \notin \overline{\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, C]}$ and so by definition $y \notin \mathrm{sCR}(C)$.

Item 2: Suppose that $y \notin s\rho\,\mathrm{R}(C)$ then there is $O \in \rho o\mathcal{I}$ and $\mathrm{F}[C] \subseteq O$ with $y \notin \overline{O}$. Since, $\overline{O}$ is closed there is an open set $V \ni y$ with $V \cap \overline{O} = \emptyset$. Let $U = \mathrm{F}^{+}[O]$, which is open since F is u.s.c. Further, $C \subseteq \mathrm{F}^{+}[O]$ since $\mathrm{F}[C] \subseteq O$. Also $\mathrm{F}[U] \subseteq O$, by Item 9 of Proposition 2.3.1. It follows from the definition of $s\rho\,\mathrm{R}$ that $s\rho\,\mathrm{R}(U) \subseteq \overline{O}$ and $V \cap \overline{O} = \emptyset$. So $s\rho\,\mathrm{R}(U) \cap V = \emptyset$, as required.

Item 3: Since X is a Hausdorff locally compact space we know that X regular. By Item 1 we have $\mathrm{CR}(K) = \mathrm{sCR}(K) = s\rho\,\mathrm{R}(K)$. One can see that $\rho\,\mathrm{R}(K) \subseteq s\rho\,\mathrm{R}(K)$. So we need only show that $\mathrm{sCR}(K) \subseteq \rho\,\mathrm{R}(K)$.

Let $O \in \rho o\mathcal{I}_{\mathcal{F}}$, $\mathrm{F}[K] \subseteq O$. If we are in the case where $\overline{O}$ is compact then, we can apply Item 3 of Proposition 3.3.2 (F is o.s.c by Item 2 of Theorem 2.3.9). So there an open cover $\mathcal{U}$ with $\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, K] \subseteq O$. It follows that $\mathrm{CR}(K) \subseteq O$. Noting that if $O_2 \in \rho o\mathcal{I}_{\mathcal{F}}$, $\mathrm{F}[K] \subseteq O_2$ then $O \cap O_2 \in \rho o\mathcal{I}_{\mathcal{F}}$ (by Item 2 of Proposition 3.3.2), $\mathrm{F}[K] \subseteq O \cap O_2$ and $\overline{O \cap O_2}$ is compact. We can make a the same argument again to show that $\mathrm{CR}(K) \subseteq O \cap O_2$. It follows that $\mathrm{CR}(K) \subseteq \rho\,\mathrm{R}(K)$.

In the case where $\mathrm{CR}(K)$ is compact, we begin by arguing that $\mathrm{CR}(K) \subseteq \mathrm{F}^{+}[O]$. Define the open cover

$$\mathcal{U} = \left\{\mathrm{F}^{+}[O], X \setminus \overline{O}\right\}.$$

much like in the proof of Item 1, one can show that $\mathrm{R}[\mathrm{F}_{\mathcal{U}}, K] \subseteq \mathrm{F}^{+}[O]$. By definition $\mathrm{CR}(K) \subseteq \mathrm{F}^{+}[O]$.

Now we can apply, Item 3 of Proposition 3.3.1. So there is an open cover $\mathcal{V}$ of X with $\overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, K]}$ compact and $\overline{\mathrm{R}[\mathrm{F}_{\mathcal{V}}, K]} \subseteq \mathrm{F}^{+}[O]$. Much like in the proof of Item 1 we know that $\mathrm{int}(\overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, K]})$ is an open robust invariant set of F. Also we have that $\mathrm{F}[K] \subseteq \mathrm{R}[\mathrm{F}_{\mathcal{U}}, K] \subseteq \mathrm{int}(\overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, K]})$. Again since the finite intersection of robust invariant sets is a robust invariant set (see Item 2 of Proposition 3.3.2), we have that $O \cap \mathrm{int}(\overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, K]}) \in \rho o \mathcal{I}_{\mathrm{F}}$ and $O \cap \mathrm{int}(\overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, K]}) \subseteq \overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, K]}$ is compact. However, this means we are in the first case. So $\mathrm{CR}(K) \subseteq \rho\,\mathrm{R}(K)$. □

Remark 3.3.1. *I believe that there is another error in [5]. Specifically in Theorem 4.5, in where a proof of $\mathrm{CR}(C) = \rho\,\mathrm{R}(C)$ is attempted. The first issue I see is when he writes (again I have converted the notion):*

> *"For any open cover $\mathcal{U}$, we have $[C \subseteq \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]]$ and $[\overline{\mathrm{F}[\mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]]} \subseteq \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]]$."*

I see no reason for $\overline{\mathrm{F}[\mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]]} \subseteq \mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]$ to be the case. Although, I have been unable to prove otherwise.

Later, he writes:

> *"To complete the proof, we let U be such that $C \subseteq U$ and $\mathrm{F}[\overline{U}] \subseteq U$, and need to show that $\mathrm{CR}(\mathrm{F}, C) \subseteq U$. We have $[\bigcup\{V \in \mathcal{U} : V \cap \mathrm{F}[\overline{U}] \neq \emptyset\} \subseteq U]$ for some open cover $\mathcal{U}$."*

The second sentence is simply false. For a counter example let $X = \mathbb{R}$ with the usual topology,

$$\mathrm{F}[x] = \{e^{-x}\} \quad U = (0, \infty)$$

then, $\mathrm{F}[\overline{U}] = (0, 1] \subseteq U$. But every open cover of $\mathbb{R}$, say $\mathcal{U}$ will have an open $W \in \mathcal{U}$ with $0 \in W$. So, $\mathrm{F}[\overline{U}] \cap W \neq \emptyset$ and $W \cap X \setminus U \neq \emptyset$; since 0 is a boundary point of both $\mathrm{F}[\overline{U}]$ and U. This means that $\bigcup\{V \in \mathcal{U} : V \cap \mathrm{F}[\overline{U}]\} \supseteq W$ and so $\bigcup\{V \in \mathcal{U} : V \cap \mathrm{F}[\overline{U}]\} \not\subseteq U$.

An open cover $\mathcal{U}$, with $\bigcup\{V \in \mathcal{U} : V \cap \mathrm{F}[\overline{U}] \neq \emptyset\} \subseteq U$ can be chosen (in the context of the quoted theorem), provided that $\overline{U}$ is compact. However, I see no reason for there to be such a U with $\overline{U}$ being compact in the context of the theorem.

All of that being said, Item 3 of Theorem 3.3.3 provides and adequate substitution for Theorem 4.5 of [5]. The extra assumption that F is also l.s.c is necessary for the computability of clR *later. Thus, even if Theorem 4.5 of [5] is incorrect then, the main results of [5] are not affected.*

The reason why Item 3 of Theorem 3.3.3 is important is because the compact robust invariant sets can be found by a computer (we argue this in Section 3.3.1). Thus the problem of verify $y \in \mathrm{CR}(C)$ is co-semi-computable (that is if $y \notin \mathrm{CR}(C)$ is true there is an algorithm that can prove that $y \notin \mathrm{CR}(C)$ is true). Via forward iteration of F, we can compute $\bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[C]$ for all $N \in \mathbb{N}$. This means that the problem of verify $y \in \overline{\mathrm{R}\,[C]}$ is semi-computable (if $y \in \overline{\mathrm{R}\,[C]}$ is true there is an algorithm that can prove that $y \in \overline{\mathrm{R}\,[C]}$ is true).

Therefore, when $\overline{\mathrm{R}\,[C]} = \mathrm{CR}(C)$ is true (that is when the closed reachable set is robust at C), the problem of verify $y \in \overline{\mathrm{R}\,[C]}$ is computable. The main result of [5, 4] is that $\overline{\mathrm{R}\,[C]} = \mathrm{CR}(C)$ is equivalent to $\overline{\mathrm{R}\,[C]}$ being computable.

Theorem 3.3.4 ((corrected) Theorem 4.10 of [5])**.** *Let (X, τ) be a computable Hausdorff space which is a topological manifold, $\mathrm{F} : X \rightsquigarrow X$ be a continuous compact valued multi-function and $C \subseteq X$ be compact. Then, the following are equivalent:*

1. clR *is computable (as a compact set) at* (F, C).

2. $\overline{\mathrm{R}\,[C]} = \mathrm{CR}(C)$ *is compact.*

Proof. We have not developed enough computability theory to prove this Theorem. So I point the reader to Theorem 4.10 of [5]. □

Note that in [5], Collins misstates Theorem 4.10. Collins forgot to write the "is compact" in Item 2; Careful analysis of the proof of Theorem 4.10 of [5] reveals he did in fact prove Theorem 3.3.4 as stated above.

Example 3.3.3 (Theorem 4.10 of [5] is misstated)**.** *We consider* $\mathrm{f} : [1, \infty) \to [1, \infty)$,

$$\mathrm{f}(x) = 2x$$

Then,

$$\overline{\mathrm{R}\,[x]} = \mathrm{R}\,[x] = \bigcup_{n \in \mathbb{N}} \{2^n x\}$$

is not compact, so it cannot be "computable as a compact set" but is robust. To prove this, note that

$$\mathrm{CR}\,[x] = \bigcap_{\epsilon > 0} \Big\{ y \in \mathbb{R} : \exists \{y_n\}_{n=1}^{\infty} \exists \{d_n\}_{n=1}^{\infty} \text{ with } |d_n| < \epsilon \\ s.t\ y_n = 2y_{n-1} + d_{n-1}, y_0 = x, d_0 = 0 \text{ and } \exists N_\epsilon \in \mathbb{N}, y = y_{N_\epsilon} \Big\}.$$

When $z \in \mathrm{CR}[x] \setminus \overline{\mathrm{R}[x]}$ *then, for* $\epsilon > 0$ *there is* $y_{N_\epsilon} = z$*, it can be shown that* $N_\epsilon \to \infty$ *as* $\epsilon \to 0$*.*

We are trying to show that the reachable set is robust at x*. For* $x \in \mathbb{R}$ $x \geq 1$*,* $y \in \mathrm{CR}[x] \setminus \overline{\mathrm{R}[x]}$ *and* $\epsilon > 0$ *with* $|x| > \epsilon$*. One can see that for any* $\{y_n\}_{n=1}^{\infty}, \{d_n\}_{n=1}^{\infty}$ *with* $|d_n| < \epsilon$ *and* $y_n = 2y_{n-1} + d_{n-1}, y_0 = x, d_0 = 0$ *we have*

$$|y_n| = |2y_{n-1} + d_1| \geq 2|y_{n-1}| - \epsilon$$

and by induction we can argue that

$$|y_n| \geq 2^n|x| - \sum_{k=0}^{n-1} 2^k \epsilon = 2^n|x| - \epsilon\frac{1-2^n}{1-2} = 2^n(|x| - \epsilon) + \epsilon.$$

So we can conclude that, since $y = y_{N_\epsilon}$ *and* $N_\epsilon \to \infty$ *as* $\epsilon \to 0$ *that* $|y| \to \infty$*; which is impossible. So* $\mathrm{CR}[x] = \overline{\mathrm{R}[x]}$ *for all* $x \in [1, \infty)$ *and the reachable set is robust but not compact.*

Remark 3.3.2. *One may wonder why Item 1 of Theorem 3.3.4, states that "*clR *is computable (as a compact set) at* (F, C)*". Before, in this work we spoke about being robust at* C*. The multifunction* F*, was always seen as an ambient constant, rather than as a variable/input. In [6, 4, 5] the computability reachable set is considered using both* (F, C) *as its variables.*

For simplicity I suppressed the variable F *in this thesis, however it is still essential for Theorem 3.3.4 to hold. Indeed, it makes sense to envision* R *to be computationally dependent on* F*. Since the values of* F *may be directly inaccessible and instead must be approximated.*

For example, let $\mathrm{f} : \mathbb{R} \to \mathbb{R}$ *be* $\mathrm{f}(x) = e^{-x}$*,* $\mathrm{F}[x] = \{\mathrm{f}(x)\}$ *for all* $x \in \mathbb{R}$ *and* $\mathbb{R}$ *have the usual topology. Then, if we wanted to compute* $\mathrm{f}(1), \mathrm{f}^{\circ 2}[1], \ldots,$ *(and thus* $\mathrm{R}[\mathrm{F}, 1]$*), how do we do this? Perhaps, the obvious answer is use* $\mathrm{f}_N[x] = \sum_{n=0}^{N} \frac{(-1)^n x^n}{n!}$ *for some* $N \in \mathbb{N}$ *as an approximation to* f *then iterate* f_N*. But now we computing* $\mathrm{R}[\mathrm{F}_N, 1]$ *where* $\mathrm{F}_N[x] = \{\mathrm{f}_N(x)\}$ *for* $x \in \mathbb{R}$*. And then we ask ourselves, does* $\mathrm{R}[\mathrm{F}_N, 1]$ *tend toward* $\mathrm{R}[\mathrm{F}, 1]$ *in some sense? Therefore, I assert that envisioning* R *to be computationally dependent on* F*, is at least reasonable.*

3.3.1 Robust invariant sets are detectable

In this subsection we present an algorithm, Algorithm 1, that can find a robust invariant set. It can be modified to find a robust invariant set with some additional desired properties; for instance, contains a prescribed point x.

The algorithm is based on the following observation about robust invariant sets.

Proposition 3.3.3. *Let (X, τ) be a topological space, $\emptyset \neq I \subseteq X$, and $\mathrm{F} : X \rightsquigarrow X$ be u.s.c. Assume that one of the following holds: (i) $\overline{I}$ is compact and X is regular. (ii) X is normal.*

Then, $I \in \rho\mathcal{I}$ if and only if there are open sets O, V with $\mathrm{F}\big[\overline{O}\big] \subseteq V$, $\overline{V} \subseteq O$, and $V \subseteq I \subseteq O$.

Note that for nonempty open sets O, V with $\mathrm{F}\big[\overline{O}\big] \subseteq V$ and $\overline{V} \subseteq O$ we know that $O, V \in \rho o\mathcal{I}$.

Proof. First, suppose that $I \in \rho\mathcal{I}$ then,

$$\mathrm{F}\big[\overline{I}\big] \subseteq \mathrm{int}(I) \iff \overline{I} \subseteq \mathrm{F}^{+}[\mathrm{int}(I)].$$

Let $V = \mathrm{int}(I)$ and note $V \subseteq I$. We claim that in both cases (i) and (ii) there is an open set O with $\overline{I} \subseteq O$ and $\overline{O} \subseteq \mathrm{F}^{+}[V]$.

In case (i), first note that $\mathrm{F}^{+}[V] \supseteq \overline{I}$ is an open set since F is u.s.c (Item 2 of Theorem 2.3.4) and since X is regular, for each $x \in \overline{I}$ there is a $U_x \in \tau_x$ with $\overline{U_x} \subseteq \mathrm{F}^{+}[V]$. These U_x, $x \in \overline{I}$ form an open cover of $\overline{I}$. By compactness there is finite subcover, $O = \bigcup_{n=1}^{N} U_{x_n} \supseteq \overline{I}$ with $\overline{O} \subseteq \mathrm{F}^{+}[V]$.

In case (ii), we see that $\overline{I} \cap X \setminus \mathrm{F}^{+}[V] = \emptyset$, and both $\overline{I}$ and $X \setminus \mathrm{F}^{+}[V]$ are closed. By normality, there are open sets $O \supseteq I$ and $W \supseteq X \setminus \mathrm{F}^{+}[V]$ with $O \cap W = \emptyset$. In particular, $O \subseteq X \setminus W$, so $\overline{O} \subseteq X \setminus W \subseteq X \setminus (X \setminus \mathrm{F}^{+}[V]) = \mathrm{F}^{+}[V]$.

In either case, we see that $O \supseteq \overline{I} \supseteq \overline{V}$, $O \supseteq \overline{I} \supseteq I$ and

$$\overline{O} \subseteq \mathrm{F}^{+}[V] \iff \mathrm{F}\big[\overline{O}\big] \subseteq V.$$

This shows that there are open sets V, O with $\mathrm{F}\big[\overline{O}\big] \subseteq V$, $\overline{V} \subseteq O$, and $V \subseteq I \subseteq O$.

Conversely, assume there are open sets V, O with $\mathrm{F}\big[\overline{O}\big] \subseteq V$, $\overline{V} \subseteq O$, and $V \subseteq I \subseteq O$. Then, since V is open with $V \subseteq I$ we must have $V \subseteq \mathrm{int}(I)$. Hence,

$$\mathrm{F}\big[\overline{I}\big] \subseteq \mathrm{F}\big[\overline{O}\big] \subseteq V \subseteq \mathrm{int}(I)$$

so $I \in \rho\mathcal{I}$.

Lastly, note that when O, V are nonempty open sets with $\mathrm{F}\big[\overline{O}\big] \subseteq V$ and $\overline{V} \subseteq O$ then,

$$\mathrm{F}\big[\overline{V}\big] \subseteq \mathrm{F}\big[\overline{O}\big] \subseteq V$$

so $V \in \rho o\mathcal{I}$, noting that $\operatorname{int}(V) = V$ since V is open. Similarly

$$\mathrm{F}\big[\overline{O}\big] \subseteq V \subseteq \overline{V} \subseteq O$$

so $O \in \rho o\mathcal{I}$. □

If we wish to find an robust invariant set I then, are strategy is to find open sets O, V with $\mathrm{F}\big[\overline{O}\big] \subseteq V$, $\overline{V} \subseteq O$, and $V \subseteq I \subseteq O$. The set V is an under-approximation and the set O is an over-approximation.

Lemma 3.3.1. *Suppose that (X, d) is a compact metric space, $\mathrm{F} : X \rightsquigarrow X$ is compact valued and continuous. Let $\mathcal{B}_r(x) = \{y \in X : \mathrm{d}(x, y) < r\}$ for $x \in X$ and $r > 0$. If $A \subseteq X$ and $r > 0$ then, let $A_r = \bigcup_{a\in A} \mathcal{B}_r(a)$*

For each $k \in \mathbb{N}$ let $\epsilon_k = 2^{-k}$ and let $\delta_k > 0$ satisfy for all $x \in X$ $\mathrm{F}[\mathcal{B}_{\delta_k}(x)] \subseteq \mathrm{F}[x]_{\epsilon_k}$. Further, suppose that for each $k \in \mathbb{N}$ there are finite sequences $\big\{x^k_n\big\}_{n=1}^{N_k}$, $\big\{y^k_m\big\}_{m=1}^{M_k}$ with $X \subseteq \bigcup_{n=1}^{N_k} \mathcal{B}_{\frac{\delta_k}{2}}\big(x^k_n\big)$ and $X \subseteq \bigcup_{m=1}^{M_k} \mathcal{B}_{\epsilon_k}\big(y^k_m\big)$.

1. *For $I^k_n = \big\{m \in [1, M_k] \cap \mathbb{N} | \, \mathrm{F}\big[x^k_n\big] \cap \mathcal{B}_{\epsilon_k}\big(y^k_m\big) \neq \emptyset\big\}$ we have*
$$\mathrm{F}\big[x^k_n\big]_{\epsilon_k} \subseteq \bigcup_{m\in I^k_n} \mathcal{B}_{\epsilon_{k-1}}\big(y^k_m\big) \subseteq \mathrm{F}\big[x^k_n\big]_{\epsilon_{k-2}}.$$
2. *For $J \subseteq [1, N_k] \cap \mathbb{N}$, $V_J = \bigcup_{n\in J} \bigcup_{m\in I^k_n} \mathcal{B}_{\epsilon_k}\big(y^k_m\big)$ and $O_J = \bigcup_{n\in J} \mathcal{B}_{\frac{\delta_k}{2}}\big(x^k_n\big)$ we have $\mathrm{F}[O_J] \subseteq V_J$. Further, if $O_J \supseteq \overline{V_J}$ then, $\overline{V_J}$ is a closed robust invariant set.*
3. *If C is a closed robust invariant set then there is a $k \in \mathbb{N}$ and a $J \subseteq [1, N_k] \cap \mathbb{N}$ such that*
$$V_J \subseteq \overline{\mathrm{F}[C]_{\epsilon_{k-3}}} \subseteq \operatorname{int}(C) \subseteq C \subseteq O_J \subseteq C_{\delta_k}.$$

Proof. 1. For any $k \in \mathbb{N}$ and $n \in [1, N_k] \cap \mathbb{N}$ we have $\mathrm{F}\big[x^k_n\big] \subseteq X \subseteq \bigcup_{m=1}^{M_k} \mathcal{B}_{\epsilon_k}\big(y^k_m\big)$. If $\ell \notin I^k_n$ then

$$\mathrm{F}\big[x^k_n\big] = \mathrm{F}\big[x^k_n\big] \setminus \mathcal{B}_{\epsilon_k}\big(y^k_\ell\big) \subseteq \bigcup_{m=1}^{M_k} \mathcal{B}_{\epsilon_k}\big(y^k_m\big) \setminus \mathcal{B}_{\epsilon_k}\big(y^k_\ell\big) \subseteq \bigcup_{m=1, m\neq\ell}^{M_k} \mathcal{B}_{\epsilon_k}\big(y^k_m\big).$$

Meaning we can remove all ℓ with $\ell \notin I^k_n$ from the union on the right and still have a covering of $\mathrm{F}\big[x^k_n\big]$; that is

$$\mathrm{F}\big[x^k_n\big] \subseteq \bigcup_{m\in I^k_n} \mathcal{B}_{\epsilon_k}\big(y^k_m\big).$$

From here we can see $\mathrm{F}\big[x_n^k\big]_{\epsilon_k} \subseteq \Big(\bigcup_{m\in I_n^k}\mathcal{B}_{\epsilon_k}\big(y_m^k\big)\Big)_{\epsilon_k} = \bigcup_{m\in I_n^k}\mathcal{B}_{\epsilon_{k-1}}\big(y_m^k\big)$.

Now suppose that $z \in \bigcup_{m\in I_n^k}\mathcal{B}_{\epsilon_{k-1}}\big(y_m^k\big)$, so for some $m \in I_m^k$ we have $\mathrm{d}\big(z, y_m^k\big) < \epsilon_{k-1}$. Since $m \in I_m^k$ we have $\mathrm{d}\big(y_m^k, y\big) < \epsilon_k$ for some $y \in \mathrm{F}\big[x_n^k\big]$. Thus

$$\mathrm{d}(z,y) \le \mathrm{d}\big(z, y_m^k\big) + \mathrm{d}\big(y_m^k, y\big) < \epsilon_{k-1} + \epsilon_k < \epsilon_{k-2}$$

meaning that $z \in \mathrm{F}\big[x_n^k\big]_{\epsilon_{k-2}}$. Hence, $\bigcup_{m\in I_n^k}\mathcal{B}_{\epsilon_{k-1}}\big(y_m^k\big) \subseteq \mathrm{F}\big[x_n^k\big]_{\epsilon_{k-2}}$.

2. We have assumed that $\mathrm{F}[\mathcal{B}_{\delta_k}(x)] \subseteq \mathrm{F}[x]_{\epsilon_k}$ for all $x \in X$; note we can do this since F is continuous and compact valued with X compact. It follows that $\mathrm{F}\Big[\mathcal{B}_{\frac{\delta_k}{2}}(x)\Big] \subseteq \mathrm{F}[x]_{\epsilon_k}$ and my item 1 we have for any $n \in [1, N_k] \cap \mathbb{N}$ and $k \in \mathbb{N}$

$$\mathrm{F}\Big[\mathcal{B}_{\frac{\delta_k}{2}}\big(x_n^k\big)\Big] \subseteq \mathrm{F}\big[x_n^k\big]_{\epsilon_k} \subseteq \bigcup_{m\in I_n^k}\mathcal{B}_{\epsilon_{k-1}}\big(y_m^k\big)$$

and for any $J \subseteq [1, N_k] \cap \mathbb{N}$

$$\mathrm{F}[O_J] = \bigcup_{n\in J}\mathrm{F}\Big[\mathcal{B}_{\frac{\delta_k}{2}}\big(x_n^k\big)\Big] \subseteq \bigcup_{n\in J}\bigcup_{m\in I_n^k}\mathcal{B}_{\epsilon_{k-1}}\big(y_m^k\big) = V_J.$$

If $J \subseteq [1, N_k] \cap \mathbb{N}$ has $\overline{V_J} \subseteq O_J$ as well then, $\mathrm{F}\big[\overline{V_J}\big] \subseteq \mathrm{F}[O_J] \subseteq V_J$. This shows that $\overline{V_J}$ is a closed robust invariant set.

3. Suppose that C is a compact robust invariant set then, by assumptions on F there is a $k \in \mathbb{N}$ with

$$\mathrm{int}(C) \supseteq \overline{\mathrm{F}[C]_{\epsilon_{k-3}}} \tag{3.1}$$

Now pick $J = \Big\{n \in [1, N_k] \cap \mathbb{N} : \mathcal{B}_{\frac{\delta_k}{2}}\big(x_n^k\big) \cap C \neq \emptyset\Big\}$, we claim that

$$C \subseteq \bigcup_{n\in J}\mathcal{B}_{\frac{\delta_k}{2}}\big(x_n^k\big) \subseteq C_{\delta_k}. \tag{3.2}$$

Let $O_J = \bigcup_{n\in J}\mathcal{B}_{\frac{\delta_k}{2}}\big(x_n^k\big)$. To prove the leftmost inclusion of equation 3.2, note that similarly to the proof of Item 1, any $n \notin J$ does not contribute to the covering $\bigcup_{n\in[1,N_k]\cap\mathbb{N}}\mathcal{B}_{\frac{\delta_k}{2}}\big(x_n^k\big) = O_J \cup O_{[1,N_k]\cap\mathbb{N}\setminus J} \supseteq C$. Hence $C \subseteq O_J$. On the other hand, let $z \in O_J$ and there is a $n \in J$ with $z \in \mathcal{B}_{\frac{\delta_k}{2}}\big(x_n^k\big)$ and a $c \in \mathcal{B}_{\frac{\delta_k}{2}}\big(x_n^k\big) \cap C$ and

$$\mathrm{d}(z,c) \le \mathrm{d}\big(z, x_n^k\big) + \mathrm{d}\big(x_n^k, c\big) < \frac{\delta_k}{2} + \frac{\delta_k}{2} = \delta_k.$$

Since $c \in C$, $z \in C_{\delta_k}$ which shows $O_J \subseteq C_{\delta_k}$.

Now take the union over $n \in J$ of item 1 to yield

$$V_J \subseteq \bigcup_{n \in J} \mathrm{F}\left[x_n^k\right]_{\epsilon_{k-2}}.$$

Recall that, by equation 3.2 we have for $n \in J$ that $x_n^k \in C_{\delta_k}$, so $V_J \subseteq \mathrm{F}[C_{\delta_k}]_{\epsilon_{k-2}}$ and by assumption on δ_k and the fact C is compact $\mathrm{F}[C_{\delta_k}] \subseteq \mathrm{F}[C]_{\epsilon_k}$. Keeping this and equation 3.1 in mind we see

$$\begin{aligned} V_J &\subseteq \mathrm{F}[C_{\delta_k}]_{\epsilon_{k-2}} \\ &\subseteq \left(\mathrm{F}[C]_{\epsilon_k}\right)_{\epsilon_{k-2}} \\ &= \mathrm{F}[C]_{\epsilon_{k-2}+\epsilon_k} \\ &\subseteq \overline{\mathrm{F}[C]_{\epsilon_{k-2}+\epsilon_{k-2}}} \\ &= \overline{\mathrm{F}[C]_{\epsilon_{k-3}}} \\ &\subseteq \mathrm{int}(C) \end{aligned}$$

and we can conclude

$$V_J \subseteq \overline{\mathrm{F}[C]_{\epsilon_{k-3}}} \subseteq \mathrm{int}(C) \subseteq C \subseteq O_J \subseteq C_{\delta_k}.$$

□

Lemma 3.3.1 is essentially a proof that Algorithm 1 will halt if there is a compact robust invariant set $C \neq \emptyset, X$ (both $\emptyset$ and X are vacuously compact robust invariant sets).

Algorithm 1: Detect Non-Trivial Closed Robust Invariant Set

Input: $\mathrm{F} : X \rightsquigarrow X$, $\delta : (0, \infty) \to (0, \infty)$
Where F is assumed to be compact valued and continuous, δ is a function satisfying $\mathrm{F}\left[\mathcal{B}_{\delta(\epsilon)}(x)\right] \subseteq \mathrm{F}[x]_\epsilon$ for any $\epsilon > 0$.
Output: O_J, V_J, k

1 **for** $k \in \mathbb{N}$ **do**
2 $\quad \epsilon_k = 2^{-k}$
3 $\quad \delta_k = \delta(\epsilon_k)$
4 $\quad$ Find points $\{y_m^k\}_{m=1}^{M_k}$ so that $X \subseteq \bigcup_{m=1}^{M_k} \mathcal{B}_{\epsilon_k}(y_m^k)$
5 $\quad$ Find points $\{x_n^k\}_{n=1}^{N_k}$ so that $X \subseteq \bigcup_{n=1}^{N_k} \mathcal{B}_{\frac{\delta_k}{2}}(x_n^k)$
6 $\quad$ **for** $n = 1, \ldots, N_k$ **do**
7 $\quad\quad I_n^k = \left\{m \in [1, M_k] \cap \mathbb{N} \middle| \mathrm{F}\left[x_n^k\right] \cap \mathcal{B}_{\epsilon_k}(y_m^k) \neq \emptyset\right\}$
8 $\quad$ **for** $J \subsetneq [1, N_k] \cap \mathbb{N}$ $J \neq \emptyset$ **do**
9 $\quad\quad V_J = \bigcup_{n \in J} \bigcup_{m \in I_n^k} \mathcal{B}_{\epsilon_k}(y_m^k)$
10 $\quad\quad O_J = \bigcup_{n \in J} \mathcal{B}_{\frac{\delta_k}{2}}(x_n^k)$
11 $\quad\quad$ **if** $O_J \supseteq \overline{V_J}$ **then**
12 $\quad\quad\quad$ **return** O_J, V_J, k

Algorithm 1 will halt if and only if X contains a closed robust invariant set of F that isn't X or $\emptyset$. This can be seen from Items 1 and 2 of Lemma 3.3.1. The use case of Algorithm 1 may seem limited and this largely true. However, Algorithm 1 can be easily modified for different use cases. For example, the requirement that $\mathrm{F} : X \rightsquigarrow X$ where X is compact already implies that X is a closed robust invariant set. A more realistic scenario is assuming $\mathrm{F} : \mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ and we wish to find a closed robust invariant set C in a compact set X. To solve this problem, on line 11 of Algorithm 1, add a check for $\overline{V_J} \subseteq \mathrm{int}(X)$. Another problem, of interest is using the robust invariant sets to over approximate the chain reachable set (and thus an over approximation of the reachable set). Specifically, given $x \in X$ we wish to find a (small) closed robust invariant set C with $\mathrm{F}[x] \subseteq \mathrm{int}(C)$. To achieve this we need only check that $\mathrm{F}[x] \subseteq V_J$ on line 11 of Algorithm 1.

The largest limitation of Algorithm 1, is that it requires this $\delta(\epsilon)$ function as input. Such information is typically not needed for numerical methods. But it is precisely this piece of information that allows this algorithm to make such strong conclusions–when Algorithm 1 halts a mathematical proof has been performed proving that $\overline{V_J}$ is a closed robust invariant set. It's not immediately clear how one would determine the function $\delta(\epsilon)$ and in general one would have to analytically figure it out ahead of time. In the case where $\mathrm{F}[x] = \bigcup_{u \in U} \{\mathrm{f}(x, u)\}$ and the functions $\mathrm{f} : X \times U \rightarrow X$ are Lipschitz with Lipschitz constant independent of U, say L, then one can take $\delta(\epsilon) = \frac{\epsilon}{L}$

Chapter 4

An everywhere computable reachable set is mostly unrealistic

Theorem 3.3.4 tells us that the closed reachable set is computable (at a set C) if and only if closed reachable set is robust (at a set C). Hence, it is natural to ask: when is the closed reachable set robust?

As a matter of practicality, we would want to be able to compute the closed reachable set everywhere. Therefore, we would to know when closed reachable set is pointwise robust, that is $\mathrm{clR}\,[x] = \mathrm{CR}(\{x\})$ for all $x \in X$. Initially, I was unable to find any non-trivial conditions which imply pointwise robustness. However, in my efforts to find such condition I found a number of startling necessary conditions. For example, every function from $\mathbb{R}$ to $\mathbb{R}$ with pointwise robust reachable set has a globally asymptotically stable fixed point, see Item 3 of Corollary 4.1.2.1. Similarly, for X being a connected metric space, a function $\mathrm{f} : X \to X$ with pointwise robust reachable set and a fixed point $\bar{x}$ then, $\bar{x}$ is globally asymptotically stable, see Item 2 of Corollary 4.1.2.1.

For this reason I spent a number of years, trying to prove that multifunctions with pointwise robust closed reachable sets are analogous to contractive maps, or even are contractive with respect to an appropriate metric.

Conjecture 4.0.1. *Suppose that (X, d) is a compact connected metric space and $\mathrm{F} : X \rightsquigarrow X$ is a compact valued continuous multifunction with pointwise robust closed reachable set.*

Then, there is a minimal invariant set $A \subseteq X$ of F, which is globally asymptotically stable; that is for all $x \in X$ we have that A is stable and $\omega[x] = A$.

Alas, Conjecture 4.0.1 is not true, see Example 4.0.1.

Example 4.0.1 (Conjecture 4.0.1 is false)**.** *Let* $X = [0,1] \subseteq \mathbb{R}$ *with the usual topology. Consider the multifunction* $\mathrm{F} : X \rightsquigarrow X$ *defined by*

$$\mathrm{F}[x] = [0, x^2]$$

$x \in [0,1]$, *assume* $[0,0] = \{0\}$. *Then,* F *is compact valued continuous multifunction. To see why* F *is l.s.c note that* $\mathrm{F}[x] = \bigcup_{\lambda \in [0,1]} \{\lambda x^2\}$ *and* λx^2 *is continuous for all* $\lambda \in [0,1]$ *so* F *is the union of continuous functions. One can see geometrically that* F *has closed graph, so* F *is o.s.c and since* X *is compact and Hausdorff* F *is also u.s.c.*

Note, $\mathrm{F}^{\circ n} \subseteq \mathrm{F}$ *for all* $n \in \mathbb{N}$, *so* $\mathrm{R}[x] = \mathrm{clR}[x] = \mathrm{F}[x]$ *for all* $x \in X$. *We claim that* F *is pointwise robust. First, we can see* $\mathrm{F}[1] = [0,1] = X$ *and* $\mathrm{CR}(\{1\}) \supseteq \mathrm{F}[1] = X$*; so* $\mathrm{CR}(\{1\}) = \mathrm{R}[1] = \mathrm{clR}[1]$ *which means* F *is robust at* $x = 1$.

Let $x \in [0,1)$ *and* O *be open with* $\mathrm{F}[x] \subseteq O$ *then, by compactness there is a* $\epsilon > 0$ *with* $\mathrm{F}[x] \subseteq \mathrm{F}[x]_\epsilon \subsetneq O$ *and in particular* $\mathrm{F}[x]_\epsilon = [0, x^2]_\epsilon = [0, x^2 + \epsilon)$. *Also,* $[0, x^2 + \epsilon]$ *is a roust invariant set since,* $\mathrm{F}[[0, x^2 + \epsilon]] = [0, (x+\epsilon)^2] \subseteq [0, x^2 + \epsilon)$. *As* O *was arbitrary, this shows* $\mathrm{CR}(\{x\}) = \mathrm{F}[x] = \mathrm{clR}[x]$ *(see Item 3 of Theorem 3.3.3 and note* $\mathrm{F}[x] = \bigcap_{\tau \ni O \supseteq \mathrm{F}[x]} \overline{O}$*) and so* F *is pointwise robust.*

Now we claim that $\{0\} = A$ *is the unique minimal invariant set in* X *but* $\omega[1] = [0,1] \neq A$. *Since,* $\mathrm{F}[0] = \{0\}$ *we have that* A *is a fixed set and by definition* $A \subseteq \mathrm{F}[x]$ *for all* $x \in X$, *it follows that* A *is the the unique minimal invariant set. But,* $\mathrm{F}[1] = [0,1]$ *and* $\mathrm{F}[[0,1]] = [0,1]$ *and for all* $n \in \mathbb{N}$ *we have* $\mathrm{F}^{\circ n}[1] = [0,1]$ *and it follows that* $\omega[1] = [0,1] \neq A$. *Therefore,* F *is pointwise robust but its minimal invariant set is not globally asymptotically stable.*

Despite the falsehood of Conjecture 4.0.1, I can show that pointwise robustness is a rather strong condition. One we cannot expect of most dynamical systems. Which is unfortunate as many people would like to compute the reachable set.

4.1 Pointwise robustness and continuity of the reachable set

Since we focussing on pointwise robustness, it makes sense to treat the chain reachable set as a multifunction. Thus, in an abuse of notation, we let $\mathrm{CR}[x] := \mathrm{CR}(\{x\})$ for all points x, whenever the chain reachable set is defined. Note that in this means that, $\mathrm{CR}[C] \subseteq \mathrm{CR}(C)$ for any set C. But $\mathrm{CR}[C] \subsetneq \mathrm{CR}(C)$ is possible for a set C.

As technical matter, we will often be assume that the space X is a regular topological space and the multifunction F is continuous. This is so that Item 1 of Theorem 3.3.3 holds; so $\mathrm{CR} = \mathrm{sCR} = s\rho\,\mathrm{R}$. Often, we will also assume that X is Hausdorff and locally compact, this will allow us to find compact robust invariant sets.

Theorem 4.1.1. *Let (X, τ) be a regular topological space, $\mathrm{F} : X \rightsquigarrow X$ be a continuous multifunction. The following hold:*

1. *The multifunction, CR is o.s.c.*
2. *Additionally, assume that X is Hausdorff and locally compact and F is compact valued.*
   - 2a. *If for $x \in X$ the set $\mathrm{CR}[x]$ is compact then, CR is u.s.c at x and the set $\mathrm{CR}[x]$ is stable (see Definition 3.2.4).*
   - 2b. *If F is pointwise robust and CR is compact valued then, clR is a stable compact valued continuous multifunction.*

Proof. Item 1: We seek to apply Item 1 of Theorem 3.3.3. Let $x \in X$ and $y \notin \mathrm{CR}[x]$ then, by Item 1 of Theorem 3.3.3 we have $y \notin \mathrm{CR}[x] = s\rho\,\mathrm{R}(\{x\})$. Now by Item 1 of Theorem 3.3.3 we have that there are open sets V, U with $V \ni y$, $U \supseteq C$ and $V \cap \mathrm{CR}[U] = \emptyset$. Therefore, we satisfy Item 5 of Theorem 2.3.8 and so CR is o.s.c.

Item 2a: We seek to apply Theorem 2.3.10 to show that CR is u.s.c. Let $O \supseteq \mathrm{CR}[x]$ then, by Item 3 of Proposition 3.3.1 there is an open cover $\mathcal{U}$ of X with $\overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, x]} \subseteq O$ and $\overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, x]}$ is compact. Since $\mathrm{int}(\overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, x]})$ is a robust invariant set (by Items 1 and 4 of Proposition 3.3.2) and $\mathrm{CR}[x] = \rho\,\mathrm{R}(\{x\})$ (by Item 3 of Theorem 3.3.3) we have that $\mathrm{CR}[x] \subseteq \mathrm{int}(\overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, x]})$. We also have, $\mathrm{F}[x] \subseteq \mathrm{CR}[x] \subseteq \mathrm{int}(\overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, x]})$, so define $U = \mathrm{F}^{+}\left[\mathrm{int}(\overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, x]})\right]$ then, U is open as F is u.s.c and $x \in U$. Given $z \in U$ we have that $\mathrm{F}[z] \subseteq \mathrm{int}(\overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, x]}) \in \rho o\mathcal{I}_{\mathrm{F}}$. Since, $\mathrm{CR} = \rho\,\mathrm{R}$ we have that $\mathrm{CR}[z] \subseteq \mathrm{int}(\overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, x]}) \subseteq$

$\overline{\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, x]}$. Therefore, $\mathrm{CR}\,[U] \subseteq \overline{\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, x]}$ which is compact. By Theorem 2.3.10 CR is u.s.c at x.

Also note that since $\mathrm{CR}\,[x] \subseteq \mathrm{int}(\overline{\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, x]}) \subseteq \overline{\mathrm{R}\,[\mathrm{F}_{\mathcal{U}}, x]} \subseteq O$, where O was an arbitrary open set with $\mathrm{CR}\,[x] \subseteq O$, we know that $\mathrm{CR}\,[x]$ is, by definition, stable.

Item 2b: Recall that clR is l.s.c by Item 3a of Proposition 3.1.11. Since F is pointwise robust we have that $\mathrm{clR}\,[x] = \mathrm{CR}\,[x]$ for all $x \in X$ and we have assumed that $\mathrm{CR}\,[x]$ is compact. So we can apply Item 2a to get that $\mathrm{CR} = \mathrm{clR}$ is u.s.c at x. And since CR is stable and compact valued the same is true for clR. □

We see that when we have sufficient compactness and pointwise robustness that the closed reachable set is continuous, by Item 2b of Theorem 4.1.1. This may seem unassuming, however this is a rather strong condition. For example, we can recall from Item 4 of Proposition 3.2.4 when clR is u.s.c then, every closed invariant set is stable. This precludes almost any kind of instability in the dynamics of F.

When the hypothesis of Theorem 3.3.4 are satisfied we will say that F has pointwise computable reachable set if the closed reachable set of F is pointwise robust and compact valued.

Corollary 4.1.1.1. *Let (X, τ) be a computable Hausdorff space which is a topological manifold and $\mathrm{F} : X \rightsquigarrow X$ be a continuous compact valued multifunction such that the closed reachable set of* F *is pointwise computable. Then,* clR *is a stable compact valued continuous multifunction. In particular, every closed invariant set of* F *is stable. Therefore, if $\mathrm{F}[x] = \{\mathrm{f}(x)\}$ for some $\mathrm{f} : X \to X$ also holds then,* f *cannot have any unstable fixed points or periodic orbits.*

Proof. Noting that a computable Hausdorff space, is regular, locally compact and Hausdorff we can apply Theorem 3.3.4 to conclude that $\mathrm{clR}\,[x] = \mathrm{CR}\,[x]$ is compact for all $x \in X$. So we can apply Item 2b of Theorem 4.1.1 to conclude that clR is a stable compact valued continuous multifunction. It follows from Item 4 of Proposition 3.2.4 that every closed invariant set of F is stable. By definition this means that minimal invariant sets of F are stable. Since fixed points and periodic orbits are minimal invariant sets, the last claim also holds. □

Already, we can see that many discrete time dynamical systems do not have computable reachable sets. For example, let X be the interval $[0, 1]$ and $\mathrm{f}(x) = x^2$ then, $\bar{x} = 1$ is an unstable fixed point of f. Hence, the closed reachable set of f is not computable. I find this a little absurd, after all it is not that hard to calculate the reachable set of f by hand.

But the absurdity of a pointwise computable reachable set gets worse. Not only are the fixed points of f guaranteed to be stable, if the space is connected and there is a fixed point then, that fixed point is unique and globally asymptotically stable.

To prove this, we need a number of intermediary results. But first, I should point out that: In [10], I showed that in a compact connected metric space X and a compact valued continuous multifunction $\mathrm{F} : X \rightsquigarrow X$ with a pointwise computable reachable set, either has a unique minimal invariant set (m.i.s) (see Definition 3.2.3) or infinitely many minimal invariant sets. In this Section we will sharpen this result, and conclude that there can only be a unique m.i.s.

Proposition 4.1.1. *Let (X, τ) be a topological space, $\mathrm{F} : X \rightsquigarrow X$ be a continuous multifunction and I be a robust invariant set of* F*. Then,*

$$\mathrm{clR}^-[I] = \mathrm{clR}^-[\mathrm{int}(I)] = \mathrm{R}^-[\mathrm{int}(I)] = \mathrm{R}^-[I].$$

Proof. Observe that since I is a robust invariant set

$$I \subseteq \mathrm{F}^+[\mathrm{int}(I)] \subseteq \mathrm{F}^-[\mathrm{int}(I)]$$

since $\mathrm{F}^+ \subseteq \mathrm{F}^-$. Sets satisfying $I \subseteq \mathrm{F}^-[\mathrm{int}(I)]$ are called robust viable sets. Breaking down the inclusion yields

$$\forall c \in I \quad \mathrm{F}[c] \cap \mathrm{int}(I) \neq \emptyset.$$

Note that if F, G are both multifunctions with $\mathrm{F}[x] \subseteq \mathrm{G}[x]$ for all $x \in X$ then we also have $\mathrm{F}^-[A] \subseteq \mathrm{G}^-[A]$ for all sets A. Also note that if $A \subseteq B$ then $\mathrm{F}[A] \subseteq \mathrm{F}[B]$.

To start we will prove $\mathrm{R}^-[\mathrm{int}(I)] = \mathrm{R}^-[I]$. We need only show $\mathrm{R}^-[\mathrm{int}(I)] \supseteq \mathrm{R}^-[I]$; since $\mathrm{int}(I) \subseteq I$ gives the other inclusion immediately. Let $x \in \mathrm{R}^-[I]$ then, by definition there is a $y \in \mathrm{R}\,[x] \cap I$. But $y \in I \subseteq \mathrm{F}^-[\mathrm{int}(I)]$ so

$$\emptyset \neq \mathrm{F}[y] \cap \mathrm{int}(I) \subseteq \mathrm{R}\,[x] \cap \mathrm{int}(I)$$

noting that $\mathrm{R}\,[x]$ is invariant and $y \in \mathrm{R}\,[x]$. This shows that $x \in \mathrm{R}^-[\mathrm{int}(I)]$; which proves $\mathrm{R}^-[\mathrm{int}(I)] = \mathrm{R}^-[I]$.

The inclusions $\mathrm{clR}^-[I] \supseteq \mathrm{clR}^-[\mathrm{int}(I)] \supseteq \mathrm{R}^-[\mathrm{int}(I)]$ are immediate. So we need only show that $\mathrm{clR}^-[I] \subseteq \mathrm{R}^-[\mathrm{int}(I)]$. Let $x \in \mathrm{clR}^-[I]$ then, there is a $y \in \mathrm{clR}[x] \cap I$. Much like in the previous case we can conclude that

$$\emptyset \neq \mathrm{F}[y] \cap \mathrm{int}(I) \subseteq \mathrm{clR}[x] \cap \mathrm{int}(I) \neq \emptyset \iff \mathrm{R}\,[x] \cap \mathrm{int}(I) \neq \emptyset$$

Therefore, $x \in \mathrm{R}^-[\mathrm{int}(I)]$, which proves the result. $\square$

Proposition 4.1.1 formalizes the idea that the boundary of a robust invariant set (or more generally a robust viable set) is insignificant, for the purposes of the reachability problem. Here the reachability problem is: find all x such that a trajectory of F reaches I, i.e. find $\{x|\,\mathrm{R}\,[x] \cap I \neq \emptyset\} = \mathrm{R}^{-}[I]$.

Lemma 4.1.1. *Let (X, τ) be a connected topological space, $\mathrm{F} : X \rightsquigarrow X$ be a continuous multifunction, I be a robust invariant set of F such that $I \neq \emptyset$. If clR (or R) is continuous then,*

$$X = \mathrm{R}^{-}[\mathrm{int}(I)].$$

Proof. By Proposition 4.1.1 we have

$$\mathrm{clR}^{-}[I] = \mathrm{clR}^{-}[\mathrm{int}(I)] = \mathrm{R}^{-}[\mathrm{int}(I)] = \mathrm{R}^{-}[I]. \tag{4.1}$$

In the case where clR is continuous, the equality $\mathrm{clR}^{-}[I] = \mathrm{clR}^{-}[\mathrm{int}(I)]$ tells us that both of these sets are open and closed. Indeed, by Item 4 of Theorem 2.3.4, the set $\mathrm{clR}^{-}[I]$ is closed since clR is u.s.c and $\mathrm{clR}^{-}[\mathrm{int}(I)]$ is open since clR is l.s.c. Thus all the sets in Equation 4.1 are both open and closed. Lastly, these sets are nonempty since by definition of a robust invariant set and the assumption of $I \neq \emptyset$ we have

$$\emptyset \neq I \subseteq \mathrm{F}^{+}[\mathrm{int}(I)] \subseteq \mathrm{F}^{-}[\mathrm{int}(I)] \subseteq \mathrm{R}^{-}[\mathrm{int}(I)];$$

Noting that $\mathrm{F}^{+}[A] \subseteq \mathrm{F}^{-}[A] \subseteq \mathrm{R}^{-}[A]$ for all sets A. Hence, $\mathrm{R}^{-}[\mathrm{int}(I)]$ is a nonempty open and closed set; by connectedness of the space, $\mathrm{R}^{-}[\mathrm{int}(I)] = X$. □

The equation, $\mathrm{R}^{-}[\mathrm{int}(I)] = X$ means that given *any* nonempty robust invariant set, it is possible for a trajectory to reach the interior of that robust invariant set (or more generally for a robust viable set). When we are dealing with a single valued function, this means every trajectory must reach the set. Of course, even with the multivalued case, it's nonsensical for there to be a pair of disjoint nonempty robust invariant sets, provided the conclusion of Lemma 4.1.1 holds.

For if I, C are two nonempty robust invariant sets and $X = \mathrm{R}^{-}[\mathrm{int}(I)] = \mathrm{R}^{-}[\mathrm{int}(C)]$ then we can take $x \in I \subseteq X = \mathrm{R}^{-}[\mathrm{int}(C)]$ so $\mathrm{R}\,[x] \cap \mathrm{int}(C) \neq \emptyset$. But I is a robust invariant so $\mathrm{R}\,[x] \subseteq \mathrm{int}(I)$ and so

$$\emptyset \neq \mathrm{R}\,[x] \cap \mathrm{int}(C) \subseteq \mathrm{int}(I) \cap \mathrm{int}(C).$$

Hence, the interior of any two nonempty robust invariant sets have nonempty intersection, whenever the hypothesises of Lemma 4.1.1 hold.

Now we can improve the results of [10].

Theorem 4.1.2. *Let* (X,τ) *be a a connected Hausdorff locally compact topological space and* $\mathrm{F} : X \rightsquigarrow X$ *be a continuous compact valued multifunction whose closed reachable set is pointwise robust and compact. Then* F *possess a stable small set.*

Proof. First we show that F possess a unique m.i.s. Since $\overline{\mathrm{R}[x]}$ is compact valued, by Theorem 3.2.1 F has a m.i.s. So suppose that A, C are both m.i.s of F then, by Item 2 of Theorem 3.2.3 and pointwise robustness we have: that for all $a \in A$, $c \in C$

$$A = \overline{\mathrm{R}[a]} = \mathrm{CR}[a]$$
$$C = \overline{\mathrm{R}[c]} = \mathrm{CR}[c].$$

Fix $a \in A$ and $c \in C$. By Item 3 of Theorem 3.3.3 there are open covers $\mathcal{U}_A$,$\mathcal{U}_C$ of X with $\mathrm{clR}[\mathrm{F}_{\mathcal{U}_A}, a], \mathrm{clR}[\mathrm{F}_{\mathcal{U}_C}, c]$ compact. Again, given open covers $\mathcal{U}, \mathcal{V}$ we write $\mathcal{U} \leq \mathcal{V}$ if $\mathrm{F}_{\mathcal{U}} \subseteq \mathrm{F}_{\mathcal{V}}$. Then, for all $x \in X$ and open covers of X, $\mathcal{U}$ with $\mathcal{U} \leq \mathcal{U}_A$ and $\mathcal{U} \leq \mathcal{U}_C$ we have $\mathrm{R}[\mathrm{F}_{\mathcal{U}}, x] \subseteq \mathrm{R}[\mathrm{F}_{\mathcal{U}_A}, x] \cap \mathrm{R}[\mathrm{F}_{\mathcal{U}_C}, x]$.

Now define,

$$K^{\mathcal{U}} = \mathrm{clR}[\mathrm{F}_{\mathcal{U}}, a] \cap \mathrm{clR}[\mathrm{F}_{\mathcal{U}}, c] \subseteq \mathrm{clR}[\mathrm{F}_{\mathcal{U}_A}, a]$$

where $\mathcal{U}$ is an open cover of X with $\mathcal{U} \leq \mathcal{U}_A$ and $\mathcal{U} \leq \mathcal{U}_C$. We seek to apply the finite intersection theorem to these sets, $K^{\mathcal{U}}$. Firstly, the sets $K^{\mathcal{U}}$ are closed subsets of the compact set $\mathrm{clR}[\mathrm{F}_{\mathcal{U}_A}, a]$. Secondly, the $K^{\mathcal{U}}$ are nonempty, since by Item 2b of Theorem 4.1.1 we have that clR is continuous. So we can apply Lemma 4.1.1 and the discussion thereafter to conclude that

$$\emptyset \neq \mathrm{int}(\mathrm{clR}[\mathrm{F}_{\mathcal{U}}, a]) \cap \mathrm{int}(\mathrm{clR}[\mathrm{F}_{\mathcal{U}}, c]) \subseteq K^{\mathcal{U}};$$

recalling that $\mathrm{clR}[\mathrm{F}_{\mathcal{U}}, x]$ is a robust invariant set from Item 4 of Proposition 3.3.2. Finally, one can prove (by similar reasoning) that the $K^{\mathcal{U}}$ have the finite intersection property, by noting that $K^{\mathcal{U}} \in \rho\mathcal{I}_{\mathrm{F}}$ holds by Item 2 of Proposition 3.3.2.

Let $\mathfrak{U}$ be the set of all open covers of X and $\mathfrak{U}_{A,C} = \{\mathcal{U} \in \mathfrak{U} : \mathcal{U} \leq \mathcal{U}_A, \mathcal{U} \leq \mathcal{U}_C\}$. One can prove that $\mathrm{CR}[x] = \bigcap_{\mathcal{U} \in \mathfrak{U}_{A,C}} \mathrm{clR}[\mathrm{F}_{\mathcal{U}}, x]$ for all $x \in X$. Then, by the finite intersection

Theorem,

$$\begin{aligned}\emptyset \neq \bigcap_{\mathcal{U}\in\mathfrak{U}_{A,C}} K^{\mathcal{U}} &= \bigcap_{\mathcal{U}\in\mathfrak{U}_{A,C}} (\mathrm{clR}\,[\mathrm{F}_{\mathcal{U}}, a] \cap \mathrm{clR}\,[\mathrm{F}_{\mathcal{U}}, c]) \\ &= \left(\bigcap_{\mathcal{U}\in\mathfrak{U}_{A,C}} \mathrm{clR}\,[\mathrm{F}_{\mathcal{U}}, a]\right) \cap \left(\bigcap_{\mathcal{U}\in\mathfrak{U}_{A,C}} \mathrm{clR}\,[\mathrm{F}_{\mathcal{U}}, c]\right) \\ &= \mathrm{CR}\,[a] \cap \mathrm{CR}\,[c] \\ &= \overline{\mathrm{R}\,[a]} \cap \overline{\mathrm{R}\,[c]} \\ &= A \cap C.\end{aligned}$$

We see that $\emptyset \neq A \cap C$ and by Item 1 of Proposition 3.2.2 we have that $A = C$. Therefore, F has a unique m.i.s, say A.

Now we show that A is the small set. Since for all $x \in X$, $\mathrm{clR}\,[x]$ is compact, we know that $\mathrm{clR}\,[x]$ contains a m.i.s (by Theorem 3.2.1) and by uniqueness of A we have that $\mathrm{clR}\,[x] \supseteq A$ for all $x \in X$. Let $I \in \mathrm{cl}\,\mathcal{I}_{\mathrm{F}}$ then, for $x \in I$ we have that $A \subseteq \mathrm{clR}\,[x] \subseteq I$. This shows that A is the small set of F. The set A is stable since it is closed and invariant; so we can apply Item 4 of Proposition 3.2.4. □

The unstated immediate corollary of Theorem 4.1.2 is that every F with pointwise computable closed reachable set, has a small set. This rules yet more F that can have pointwise computable closed reachable set.

Example 4.1.1. *Let $n \in \mathbb{N}$ and $X = \mathbb{R}^n$ with the usual topology then, define the identity map*

$$\mathrm{F}[x] = \{x\}.$$

Then, the closed reachable set of F *is not computable. Since the map* F *has two distinct fixed points.*

Corollary 4.1.2.1. *Let (X, τ) be a computable connected Hausdorff topological space which is a topological manifold and $\mathrm{f} : X \to X$ be a continuous function. Let $\mathrm{F}[x] = \{\mathrm{f}(x)\}$ for all $x \in X$. If the closed reachable set of* F *is pointwise computable then, the following hold:*

1. F *(and* f*) has a stable small set, A. Moreover, A is globally asymptotically stable (for all $x \in X$ we have that $\omega[x] = A$ and A is stable).*

2. *If $\bar{x} \in X$ is a fixed point of* f *then, $A = \{\bar{x}\}$ is the small set of* F *and $\bar{x}$ is globally asymptotically stable.*

3. *If* $X = \mathbb{R}$ *with the usual topology then,* f *has a globally asymptotically stable fixed point.*

Proof. Note that the multifunction F is continuous and compact valued.

Item 1: It follows from Theorem 4.1.2 that F has a stable small set, A. Let $x \in X$ then, for every $N \in \mathbb{N}$ we have that $\mathrm{clR}_N[\mathrm{F}, x]$ is a closed nonempty invariant set and we have $A \subseteq \mathrm{clR}_N[\mathrm{F}, x] = \overline{\bigcup_{n \in \mathbb{N}_N} \{\mathrm{f}^{\circ n}(x)\}}$ for every $N \in \mathbb{N}$. It follows that

$$A \subseteq \omega[x] = \bigcap_{N \in \mathbb{N}} \mathrm{clR}_N[\mathrm{F}, x] = \operatorname*{Ls}_{n \to \infty} \{\mathrm{f}^{\circ n}(x)\}.$$

Hence, the sequence $\{\mathrm{f}^{\circ n}(x)\}_{n \in \mathbb{N}}$ is a trajectory of F with $A \cap \mathrm{Ls}_{n \to \infty} \{\mathrm{f}^{\circ n}(x)\} \neq \emptyset$. By Item 1 of Proposition 3.2.4 we have that $\omega[x] = \mathrm{Ls}_{n \to \infty} \{\mathrm{f}^{\circ n}(x)\} \subseteq A$, noting that X must be regular.

Therefore, $A = \omega[x]$ for all $x \in X$ and since A is stable, we have that A is globally asymptotically stable.

Item 2: When $\bar{x}$ is a fixed point of f, we see that $\mathrm{F}[\bar{x}] = \{\bar{x}\}$. Since, $\{\bar{x}\}$ is a singleton and a fixed set of F, it must be a minimal invariant set of F. But by Item 1 there is a small set, A, which means that A is the only minimal invariant set. So $A = \{\bar{x}\}$. Also by Item 1, A is globally asymptotically stable and it follows that $\bar{x}$ is too.

Item 3: We claim that, when $X = \mathbb{R}$ and $\mathrm{clR}\,[\mathrm{f}, x]$ is pointwise compact then, f has a fixed point.

First we show that: if f does not have a fixed point then, $\mathrm{f}(x) < x$ for all $x \in \mathbb{R}$ or $\mathrm{f}(x) > x$ for all $x \in \mathbb{R}$. To see this suppose to the contrary that, there are $x, y \in \mathbb{R}$ with $\mathrm{f}(x) \leq x$ and $\mathrm{f}(y) \geq y$. If $x = \mathrm{f}(x)$ or $y = \mathrm{f}(y)$ then, we have a contradiction. So assume that $\mathrm{f}(x) < x$ and $\mathrm{f}(y) > y$. WLOG, assume that $x < y$ and define $\mathrm{g}(z) = \mathrm{f}(z) - z$, $z \in \mathbb{R}$, we have that $\mathrm{g}(x) < 0$ and $\mathrm{g}(y) > 0$. It follows from the Intermediate Value Theorem that $\mathrm{g}([x, y]) \ni 0$. Let $\bar{x} \in [x, y]$ with $\mathrm{g}(\bar{x}) = 0$ then, $0 = \mathrm{g}(\bar{x}) = \mathrm{f}(\bar{x}) - \bar{x}$ and $\mathrm{f}(\bar{x}) = \bar{x}$. Another contradiction.

So we will assume that $\mathrm{f}(x) < x$ for all $x \in \mathbb{R}$ (the case where $\mathrm{f}(x) > x$ for all $x \in \mathbb{R}$ is similar). We see that $\mathrm{f}(\mathrm{f}(x)) < \mathrm{f}(x) < x$ for all $x \in X$. An induction proof can show that for all $n \in \mathbb{N}$ we have $\mathrm{f}^{\circ n}(x) < \mathrm{f}^{\circ (n-1)}(x)$. Therefore, the sequence $\{\mathrm{f}^{\circ n}(x)\}_{n \in \mathbb{N}}$ is monotone decreasing. Since $\mathrm{clR}\,[\mathrm{f}, x]$ is compact, the sequence $\{\mathrm{f}^{\circ n}(x)\}_{n \in \mathbb{N}}$ is also bounded. By the Monotone Convergence Theorem, the sequence $\{\mathrm{f}^{\circ n}(x)\}_{n \in \mathbb{N}}$ converges, to say, $\bar{x}$. And we see that $\mathrm{f}(\bar{x}) = \lim_{n \to \infty} \mathrm{f}(\mathrm{f}^{\circ n}(x)) = \lim_{n \to \infty} \mathrm{f}^{\circ n+1}(x) = \bar{x}$. And f has a fixed point, another contradiction.

In all cases f must have fixed point, and we can apply Item 2 for the rest of the result. □

As shown in Example 4.0.1, Item 1 of Corollary 4.1.2.1 does not hold for a general multifunction.

It turns out that Item 1 of Corollary 4.1.2.1 is also sufficient for pointwise computability (for single valued functions, see Corollary 4.2.5.2). To prove this we need a lot more machinery.

4.2 Asymptotic stability for multifunctions

In this section, we present some notions of asymptotic stability for multifunctions then, attempt to find necessary/sufficient conditions to have these notions. This section can be thought of as a more advanced version of Section 3.2.

We begin by considering stronger notions of stability.

Definition 4.2.1. *Let (X, τ) be a topological space and $\mathrm{F} : X \rightsquigarrow X$ be a multifunction.*

1. *A set $W \subseteq X$ is said to be a weakly robust invariant set of F or a weak robust set of F if W is invariant and there is a $N \in \mathbb{N}$ with*

$$\mathrm{F}^{\circ N}\left[\overline{W}\right] \subseteq \mathrm{int}(W).$$

   Let $w\rho\mathcal{I}_{\mathrm{F}}$ be the set of all nonempty weakly robust sets of F. Let $w\rho\,\mathrm{cl}\,\mathcal{I}_{\mathrm{F}}$ be the set of all nonempty closed weakly robust sets of F. When there is no confusion of the multifunction F we may omit mentions of F.

2. *A set $A \subseteq X$ is said to be robustly F-stable, $\rho\mathcal{I}$-stable or simply robustly stable if for any open $O \supseteq A$ there is a $I \in \rho\mathcal{I}$ with*

$$A \subseteq \mathrm{int}(I) \subseteq I \subseteq O.$$

3. *A set $A \subseteq X$ is said to be weakly robustly F-stable, $w\rho\mathcal{I}$-stable or simply weakly robustly stable if for any open $O \supseteq A$ there is a $W \in w\rho\mathcal{I}$ with*

$$A \subseteq \mathrm{int}(W) \subseteq W \subseteq O.$$

4. *A set $A \subseteq X$ is said to be locally asymptotically stable if it is $\mathcal{I}$-stable and there is an open set $U \supseteq A$ such that for all $x \in U$ we have $A = \omega[x]$. The set A is called globally asymptotically stable if it is locally asymptotically stable where the open set $U = X$.*

5. *A set $A \subseteq X$ is said to be locally attractive, if there is an open set $U \supseteq A$ such that for all $x \in U$ we have $\omega[x] \subseteq A$. The set A is called globally attractive if it is locally attractive where the open set $U = X$.*

6. *A set $A \subseteq X$ is said to be locally attractive for trajectories, if there is an open set $U \supseteq A$ such that for all $x \in U$ and any trajectory, $\{x_n\}_{n\in\mathbb{N}_0}$ of F with $x_0 = x$ we have that $\mathrm{Ls}_{n\in\mathbb{N}_0}\{x_n\} \subseteq A$. The set A is called globally attractive for trajectories if it is locally attractive for trajectories where the open set $U = X$.*

We will examine the notions of $\mathcal{I}$-stability, $w\rho\mathcal{I}$-stability and $\rho\mathcal{I}$-stability primarily in the context of local attractiveness. Similarly, we are are most interested in local attractivity for stable sets.

Theorem 4.2.1. *Let $(X.\tau)$ be a Hausdorff locally compact topological space, $\mathrm{F} : X \rightsquigarrow X$ be a continuous compact valued multifunction and $A \subseteq X$ be a compact set which is locally attractive. The following are equivalent:*

1. *A is $\mathcal{I}$-stable.*
2. *A is $w\rho\mathcal{I}$-stable.*
3. *A is $\rho\mathcal{I}$-stable.*

Proof. It follows from the inclusions $\rho\mathcal{I} \subseteq w\rho\mathcal{I} \subseteq \mathcal{I}$ that: $3 \implies 2 \implies 1$.

We will begin with proving that $1 \implies 2$. Let O be an open set with $O \supseteq A$, WLOG we may assume that $\overline{O}$ is compact and that $\overline{O} \subseteq U$; where U is an open set with for all $x \in U$, $\omega[x] \subseteq A$. By $\mathcal{I}$-stability of A, regularity and compactness there are sets $I, J \in \operatorname{cl}\mathcal{I}$ with

$$A \subseteq \operatorname{int}(J) \subseteq J \subseteq \operatorname{int}(I) \subseteq I \subseteq O \subseteq \overline{O} \subseteq U.$$

We will show that I is weakly robust.

Notice that for all $x \in I$, the set net $\{\mathrm{F}^{\circ n}[x]\}_{n\in\mathbb{N}}$ converges to I in the upper Vietoris topology (u.v.t). One can see this from, $\mathrm{F}^{\circ n}[x] \subseteq \operatorname{cl}\mathrm{R}\,[x] \subseteq I$ for all $n \in \mathbb{N}$ and Item 2 of Proposition 2.2.4. As I is compact, the set net $\{\mathrm{F}^{\circ n}[x]\}_{n\in\mathbb{N}}$ converges to a compact set in the u.v.t, we can apply Item 3 of Proposition 2.2.6 to conclude that $\{\mathrm{F}^{\circ n}[x]\}_{n\in\mathbb{N}}$ converges to $\omega[x] = \operatorname{Ls}_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x]$ in the u.v.t. Hence, by local attractiveness we have $\omega[x] \subseteq A$ and Item 6 of Proposition 2.2.4, $\{\mathrm{F}^{\circ n}[x]\}_{n\in\mathbb{N}}$ also converges to $A \supseteq \omega[x]$ the u.v.t.

Therefore, by Item 2 of Proposition 2.2.4, for all $x \in I$ there is a $N_x \in \mathbb{N}$ such that for all $n \geq N_x$ we have $\mathrm{F}^{\circ n}[x] \subseteq \operatorname{int}(J)$. It follows that $I \subseteq \bigcup_{x\in I} \mathrm{F}^{+\circ N_x}[\operatorname{int}(J)]$; by compactness of I and u.s.c of F, we can find a find a finite subcover. So there is a $M \in \mathbb{N}$ and $x_m \in I$, $m = 1, \ldots, M$ with $I \subseteq \bigcup_{m=1}^{M} \mathrm{F}^{+\circ N_{x_m}}[\operatorname{int}(J)]$. Let $N = \max\{N_{x_m} : m = 1, \ldots, M\}$ then, for any $x \in I$, $x \in \mathrm{F}^{+\circ N_{x_m}}[\operatorname{int}(J)]$ for some $m = 1, \ldots, M$. So, $\mathrm{F}^{\circ N_{x_m}}[x] \subseteq \operatorname{int}(J)$, since $N \geq N_{x_m}$ and J is invariant we see that $\mathrm{F}^{\circ N}[x] \subseteq J \subseteq \operatorname{int}(I)$. Since this holds for all $x \in I$, we conclude $\mathrm{F}^{\circ N}[I] \subseteq \operatorname{int}(I)$. Therefore, I is weakly robust and 2 holds.

Now we prove $2 \implies 3$, so assume that Item 2 holds. Again, let O be an open set with $O \supseteq A$, WLOG we may assume that $\overline{O}$ is compact and that $\overline{O} \subseteq U$; where U is an open set with for all $x \in U$, $\omega[x] \subseteq A$. By $w\rho\mathcal{I}$-stability of A, regularity and compactness there is a set $W \in w\rho\,\mathrm{cl}\,\mathcal{I}$ with

$$A \subseteq \mathrm{int}(W) \subseteq W \subseteq O \subseteq \overline{O} \subseteq U.$$

We will construct an open set $I \in \rho\mathcal{I}$ with $W \subseteq I \subseteq \overline{I} \subseteq O$. To begin note that since W is weakly robust, there is a $N \in \mathbb{N}$ with $\mathrm{F}^{\circ N}[W] \subseteq \mathrm{int}(W)$. Also, W is invariant and compact, so for all $n \in \mathbb{N}$ we have that $\mathrm{F}^{\circ n}[W] \subseteq W \subseteq O$ and $\mathrm{F}^{\circ n}[W]$ is compact; as $\mathrm{F}^{\circ n}$ is compact valued and continuous for all $n \in \mathbb{N}$. If $N = 1$ then $W \in \rho\mathcal{I}$ and we can set $I = \mathrm{int}(W)$ to get the result. So we assume $N \geq 2$. We see that $\mathrm{F}^{\circ(N-1)}[W] \subseteq \mathrm{F}^{+}[\mathrm{int}(W)]$ and $\mathrm{F}^{\circ(N-1)}[W] \subseteq O$. So (again by regularity, compactness, and u.s.c of F) there is an open set V_{N-1} with

$$\mathrm{F}^{\circ(N-1)}[W] \subseteq V_{N-1} \subseteq \overline{V_{N-1}} \subseteq \mathrm{F}^{+}[\mathrm{int}(W)] \cap O.$$

Notably, $\overline{V_{N-1}} \subseteq O$ and has $\mathrm{F}\left[\overline{V_{N-1}}\right] \subseteq \mathrm{int}(W)$. As $N \geq 2$, we may do this again, and there is an open V_{N-2} with

$$\mathrm{F}^{\circ(N-2)}[W] \subseteq V_{N-2} \subseteq \overline{V_{N-2}} \subseteq \mathrm{F}^{+}[V_{N-1}] \cap O.$$

If $N = 2$ then, $\mathrm{F}^{\circ(N-2)}[W] = \mathrm{F}^{\circ(0)}[W] = W$, it follows that $\mathrm{int}(W) \subseteq W \subseteq V_0$. Let $I = V_0 \cup V_1$, we see that I is open, $\overline{I} \subseteq O$, $I \supseteq W$ and

$$\mathrm{F}\left[\overline{I}\right] = \mathrm{F}\left[\overline{V_0} \cup \overline{V_1}\right] = \mathrm{F}\left[\overline{V_0}\right] \cup \mathrm{F}\left[\overline{V_1}\right] \subseteq V_1 \cup \mathrm{int}(W) \subseteq I.$$

Therefore, I is a robust invariant set.

Therefore, we assert that an induction proof can construct open sets V_n $n = 0, \ldots, N-1$ with $V_N = \mathrm{int}(W)$, $\mathrm{F}\left[\overline{V_n}\right] \subseteq V_{n+1}$, $\overline{V_n} \subseteq O$ and $V_0 \supseteq W$. From here one can show that $I = \bigcup_{n=0}^{N-1} V_n$ is a robust invariant open set with $\overline{I} \subseteq O$ and $I \supseteq W$; as required for Item 3.

□

Given that $\mathcal{I}, w\rho\mathcal{I}, \rho\mathcal{I}$-stability are equivalent in the important case of local attractivity, one might wonder why bring up $w\rho\mathcal{I}, \rho\mathcal{I}$-stability at all? It's mostly for technical reasons. A $\rho\mathcal{I}$-stable set actually has a local base consisting of open invariant sets. This is shockingly convenient for analyzing long term behaviour.

The following result, allows more effective use of the ω limit set with robust invariant sets.

Lemma 4.2.1. *Suppose that* (X, τ) *is a topological space,* $\mathrm{F} : X \rightsquigarrow X$ *a l.s.c, o.s.c multifunction and* I *is a compact invariant set. Then, for all* $x \in I$ *we have*

$$\mathrm{F}[\omega[x]] = \omega[x].$$

Proof. Let $A_n = \mathrm{F}^{\circ n}[x]$ for $n \in \mathbb{N}$, since I is invariant we have $\bigcup_{n\in\mathbb{N}} A_n \subseteq I$. Also I is compact, so we apply Proposition 2.3.5 and see

$$\omega[x] = \operatorname*{Ls}_{n\to\infty} \mathrm{F}^{\circ n}[x] = \operatorname*{Ls}_{n\to\infty} \mathrm{F}^{\circ(n+1)}[x] = \operatorname*{Ls}_{n\to\infty} \mathrm{F}[A_n] \subseteq \mathrm{F}\Big[\operatorname*{Ls}_{n\to\infty} A_n\Big] = \mathrm{F}[\omega[x]].$$

So $\omega[x] \subseteq \mathrm{F}[\omega[x]]$, the other inclusion follows since $\omega[x]$ is an invariant set, by Item 3 of Proposition 3.2.1. □

A related idea to the ω limit set being a fixed set, is the idea of super-invariant sets and the largest fixed set.

Proposition 4.2.1. *Suppose that* (X, τ) *is a topological space,* $\mathrm{F} : X \rightsquigarrow X$ *is total multifunction, and* I *be a nonempty closed compact invariant set. The following hold:*

1. *Every super-invariant set in* I *is contained in* $\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I] \subseteq I$.
2. *If* X *is regular Hausdorff,* F *is compact valued and continuous then:*
   - 2a. $\lim_{n\to\infty} \mathrm{F}^{\circ n}[I] = \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ *is nonempty and compact, where the limit is taken in the Vietoris topology.*
   - 2b. $\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ *is the largest super-invariant set in* I *and largest fixed set in* I.
   - 2c. A *is the unique compact fixed set in* I *if and only if* A *is a m.i.s and* $A = \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$.

Proof. Item 1: Suppose that $B \subseteq I$ is super-invariant then, $B \subseteq \mathrm{F}[B]$ and by applying F to both sides of this inclusion we find $\mathrm{F}[B] \subseteq \mathrm{F}[\mathrm{F}[B]] = \mathrm{F}^{\circ 2}[B]$. Thus $\mathrm{F}[B]$ is super-invariant and by induction we see that, $\mathrm{F}^{\circ n}[B]$ is super-invariant. Further,

$$B \subseteq \mathrm{F}[B] \subseteq \mathrm{F}^{\circ 2}[B] \subseteq \cdots \subseteq \mathrm{F}^{\circ n}[B] \subseteq \dots$$

but more prominently this means that $B \subseteq \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[B]$. Recall that I is invariant and we see that,

$$B \subseteq \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[B] \subseteq \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I] \subseteq I$$

so $\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ contains every super-invariant set in I.

Item 2a: Since X is Hausdorff, F is compact valued and continuous one can tell that $\mathrm{F}^{\circ n}[I]$ is closed and compact for all $n \in \mathbb{N}$; this follows from Item 2 of Proposition 2.3.8, Theorem 2.3.11 and Proposition 2.3.6.

Since I is invariant, the sequence of compact sets $\{\mathrm{F}^{\circ n}[I]\}_{n\in\mathbb{N}}$ is decreasing, i.e $\mathrm{F}^{\circ(n+1)}[I] \subseteq \mathrm{F}^{\circ n}[I]$ for all $n \in \mathbb{N}$. Also the $\mathrm{F}^{\circ n}[I] \subseteq I$ for all $n \in \mathbb{N}$. It follows that $\mathrm{F}^{\circ n}[I] \to I$ in the u.v.t. Thus, by Item 3 of Proposition 2.2.6 we have that $\mathrm{F}^{\circ n}[I] \to \mathrm{Ls}_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I] = \omega[x]$ in the u.v.t. Since the set net is decreasing we have,

$$\operatorname*{KLim}_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I] = \bigcap_{n\in\mathbb{N}} \overline{\mathrm{F}^{\circ n}[I]} = \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I],$$

by Item 10 of Proposition 2.2.5. Since $\mathrm{KLim}_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ exists we have that $\mathrm{KLim}_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I] = \mathrm{Li}_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I] = \omega[x]$, it follows that $\mathrm{F}^{\circ n}[I] \to \mathrm{Li}_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ in the u.v.t. By Item 5 of Proposition 2.2.6 we have that $\mathrm{F}^{\circ n}[I] \to \mathrm{Li}_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I] = \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ in the Vietoris topology. Note that by Item 4b of Proposition 2.2.3, limits in the Vietoris topology, to closed sets, are unique.

Item 2b: Since F is continuous and compact valued we have that $\lim_{n\to\infty} \mathrm{F}[A_n] = \mathrm{F}[\lim_{n\to\infty} A_n]$ when A_n is a sequence of compact sets converging in the Vietoris topology (this can be seen from Item 8 of Theorem 2.3.6 and uniqueness of limits in the Vietoris topology, to closed sets). We apply this fact to Item 2a and

$$\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I] = \lim_{n\to\infty} \mathrm{F}^{\circ n}[I] = \lim_{n\to\infty} \mathrm{F}^{\circ n+1}[I] = \lim_{n\to\infty} \mathrm{F}[\mathrm{F}^{\circ n}[I]] = \mathrm{F}\left[\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]\right].$$

So $\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ is fixed. In particular it is super-invariant and by Item 1 every super-invariant in I is contained in $\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I] \subseteq I$, thus it is the largest such set. Similarly, every fixed set is super-invariant and thus is also contained in $\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$, since $\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ is fixed it is the largest fixed set in I.

Item 2c: We first assume that A is the unique compact fixed set of F. By 2b, $\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ is a fixed set in I and by uniqueness $A = \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$. Similarly, the set A is a m.i.s. Since, every m.i.s is fixed (by Item 2 of Theorem 3.2.1 noting that F is o.s.c) and there is at least one m.i.s in I by Theorem 3.2.1.

Suppose now that A is a m.i.s and $A = \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$. Then, $\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ is a m.i.s. Suppose that B is a nonempty compact fixed set then it is super-invariant. By 2b, $B \subseteq \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ but since $\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ is m.i.s and B is a nonempty closed invariant set, we have $B = \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ by minimality. Thus, A is the unique nonempty compact fixed set. □

Before getting to the truly interesting results on asymptomatic stability, we take a small digression to consider locally attractivity for trajectories.

Theorem 4.2.2. *Let $(X.\tau)$ be a Hausdorff locally compact topological space,* $\mathrm{F} : X \rightsquigarrow X$ *be a compact valued continuous multifunction, and A be a compact set. TFAE*

1. *A is $\mathcal{I}$-stable and locally attractive for trajectories.*
2. *A is $\mathcal{I}$-stable and there is a neighborhood of A, U for which all m.v.s in U are also in A.*

Proof. Item 1 $\implies$ Item 2: Suppose that Item 1 holds, and let $U \supseteq A$ be open such that any trajectory starting in U has all its limit points in A. If $Q \subseteq U$ is a m.v.s then, by Theorem 3.2.4 there is a $\{q_{n+1} \in \mathrm{F}[q_n]\}_{n\in\mathbb{N}_0}$ $q_0 \in Q$ with $Q = \mathrm{Ls}_{n\to\infty}\{q_n\}$. Hence by properties of U, $\mathrm{Ls}_{n\to\infty}\{q_n\} \subseteq A$ and $Q \subseteq A$. Stability is assumed outright, so Item 2 holds.

Item 2 $\implies$ Item 1: Let U be an open set for which all m.v.s in U are also in A. WLOG we may consider $\overline{U}$ to be compact. By stability, compactness and regularity, we can find a $I \in \mathrm{cl}\,\mathcal{I}$ with $A \subseteq \mathrm{int}(I) \subseteq I \subseteq U$.

Let $x \in \mathrm{int}(I)$ and $\{x_n\}_{n\in\mathbb{N}_0}$ be a trajectory with $x_0 = x$. The set $\{x_n : n \in \mathbb{N}_0\}$ has compact closure, so by Theorem 3.2.2 we have that $\mathrm{Ls}_{n\to\infty}\{x_n\}$ contains a m.v.s, Q, and by Item 2 we have $Q \subseteq A$. Therefore, $\mathrm{Ls}_{n\to\infty}\{x_n\} \cap A \neq \emptyset$ and by Item 1 of Proposition 3.2.4 we have that $\mathrm{Ls}_{n\to\infty}\{x_n\} \subseteq A$, as required.

□

The condition on the minimal viable sets in Item 2 of Theorem 4.2.2 prevents a trajectory starting near A but remain bounded away from A, see Example 4.2.1.

Example 4.2.1. *Let $X = \mathbb{R}$ with the usual topology and define*

$$\mathrm{F}_1[x] = \{0\} \quad \mathrm{F}_2[x] = \{0, x\}.$$

For both F_1 and F_2 the set $A = \{0\}$ is a $\mathcal{I}$-stable m.i.s. One can tell that A is locally attractive for trajectories with respect to F_1.

However, this is not the case for F_2; given any open set of A, say V, and a point $x \in V \setminus \{0\}$ then, $x \in \mathrm{F}_2[x]$. Which means that the constant sequence $\{x_n = x\}_{n\in\mathbb{N}}$ is a trajectory of F_2. This trajectory does not tend to A, so A is not locally attractive for trajectories with respect to F_2. In fact the set $\{x\}$ is a m.v.s of F_2. For the same reason $A = \{0\}$ is a $\mathcal{I}$-stable m.i.s for F_2 but not $\rho\mathcal{I}$-stable.

This example, is almost the generic case of why condition Item 2 of Theorem 4.2.2 on the m.v.ss is necessary. Note that the every point in a m.v.s, Q, can be recurred to, see Theorem 3.2.4, Item 4. So the condition Item 2 of Theorem 4.2.2 on the m.v.ss can be thought of as saying that an attractor A, must be bounded away from recurrent points outside of A.

We can now characterize locally asymptotically stability.

Theorem 4.2.3. *Let $(X.\tau)$ be a Hausdorff locally compact locally connected topological space, $\mathrm{F} : X \rightsquigarrow X$ be a continuous compact valued multifunction and $A \subseteq X$ be a nonempty compact set. The following are equivalent:*

1. *A is locally asymptotically stable.*
2. *A is a $\rho\mathcal{I}$-stable m.i.s and there is an open set $O \supseteq A$ where ω is l.s.c on O.*
3. *A is a $\rho\mathcal{I}$-stable m.i.s and A is locally attractive for trajectories.*
4. *A is a $\rho\mathcal{I}$-stable m.i.s and there is an open set $O \supseteq A$ and every m.v.s in O is also in A.*
5. *A is $\mathcal{I}$-stable, there is an open $U \supseteq A$ such that A is the unique nonempty compact fixed set (ie $\mathrm{F}[A] = A$) in U.*
6. *A is $\mathcal{I}$-stable m.i.s, there is a compact $I \in \mathrm{cl}\,\mathcal{I}$ with $\mathrm{int}(I) \supseteq A$ such that $A = \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ (A is the largest fixed set in I).*

Proof. Item 1 $\implies$ Item 2: Suppose that Item 1 holds. Then, there is an open set $U \supseteq A$ with $\omega[x] = A$ for all $x \in U$. Suppose that $a \in A$ then $a \in U$ and we have that $\omega[a] = A$, by Item 3 of Theorem 3.2.3 A is a m.i.s.

The multifunction ω is constant on U, it follows that ω is l.s.c on U.

Since A is locally asymptotically stable it is also locally attractive and $\mathcal{I}$-stable. And by Theorem 4.2.1 A is also $\rho\mathcal{I}$-stable, so Item 2 holds.

Item 1 $\implies$ Item 3: Suppose that Item 1 holds. Then, as in the last case, A is a $\rho\mathcal{I}$-stable m.i.s.

To see why A attracts trajectories, let $\{x_n\}_{n\in\mathbb{N}_0}$ be a trajectory with $x_0 \in U$, where U is the open set with $\omega[x] = A$ for all $x \in U$ and $U \supseteq A$. Then, by Item 2 of Proposition 3.2.1 we know that $\mathrm{Ls}_{n\in\mathbb{N}}\{x_n\} \subseteq \omega[x_0] \subseteq A$. Which means A attracts trajectories and Item 3 holds.

Item 2 $\implies$ Item 1: Suppose that Item 2 holds. Let U be an open set with $U \supseteq A$, WLOG we may take U to have both $\overline{U}$ is compact, ω is l.s.c on U and (by $\rho\mathcal{I}$-stability) clR is compact valued on U. Moreover, as X is locally connected the connected components of U are open (in X) and closed in U. As A is compact, we may assume that $U = \bigcup_{m=1}^{M} U_m$ where the U_m, $m = 1, \ldots, M$, are the connected components of U and $U_m \cap A \neq \emptyset$ for all $m = 1, \ldots, M$.

Let J be an open robust invariant set with

$$A \subseteq J \subseteq \overline{J} \subseteq U.$$

We may find such a J since X is regular, A is compact and Item 1 of Proposition 3.3.2. We claim that $\omega^+\left[\overline{J}\right] \cap U = \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J] \cap U$. Let $x \in \omega^+\left[\overline{J}\right] \cap U$ then, $\omega[x] \subseteq \overline{J}$. Since $x \in U$ we have that $\mathrm{clR}\,[x]$ is compact and $\mathrm{F}^{\circ n}[x] \subseteq \mathrm{clR}\,[x]$ for all $n \in \mathbb{N}$, we apply Item 3 of Proposition 2.2.6 to conclude that $\mathrm{F}^{\circ n}[x] \to \mathrm{Ls}_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[x] = \omega[x]$ in the u.v.t. Since $\overline{J}$ is compact and F is sufficiently continuous (see Theorem 2.3.9) we can apply Lemma 4.2.1. So by applying F to both sides of the inclusion $\omega[x] \subseteq \overline{J}$ we find

$$\omega[x] = \mathrm{F}[\omega[x]] \subseteq \mathrm{F}\left[\overline{J}\right] \subseteq J,$$

recalling that J is an open robust invariant set. By the convergence in u.v.t, we have that there is a $N \in \mathbb{N}$ such that for all $k \geq N$ we have

$$\mathrm{F}^{\circ k}[x] \subseteq J \iff x \in \mathrm{F}^{\circ k+}[J] \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J].$$

Hence, $x \in \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J] \cap U$. Conversely, let $x \in \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J] \cap U$. Then, for some $N \in \mathbb{N}$ we have that $\mathrm{F}^{\circ N}[x] \subseteq J$, as J is invariant we have that for all $n \geq N$, $\mathrm{F}^{\circ n}[x] \subseteq J$. It follows that $\omega[x] \subseteq \overline{J}$ and so $x \in \omega^+\left[\overline{J}\right] \cap U$.

However, $\omega^+\left[\overline{J}\right] \cap U$ is closed in U, since ω is l.s.c on U (see Item 4 of Theorem 2.3.1). And, $\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J] \cap U$ is open in U (and X) by u.s.c of the iterates of F. Therefore, $\omega^+\left[\overline{J}\right] \cap U$ is closed and open in U.

Notice that, $A \subseteq \omega^+\left[\overline{J}\right] \cap U$, since $\omega[a] \subseteq A \subseteq J$ for all $a \in A$, holds by the invariance of A (A is a m.i.s). Now consider, a connected component of U, U_m for $m = 1, \ldots, M$. Then, $U_m \cap (\omega^+\left[\overline{J}\right] \cap U) \neq \emptyset$ as we assumed that each $U_m \cap A \neq \emptyset$. Also, note that $U_m \cap (\omega^+\left[\overline{J}\right] \cap U)$ is closed in U, as both U_m, $\omega^+\left[\overline{J}\right] \cap U$ are closed in U. Moreover, $U_m \cap (\omega^+\left[\overline{J}\right] \cap U)$ is open, since both U_m, $\omega^+\left[\overline{J}\right] \cap U$ are open in X. Therefore, $U_m \cap (\omega^+\left[\overline{J}\right] \cap U)$ closed and open in U, it follows that $U_m \cap (\omega^+\left[\overline{J}\right] \cap U)$ is nonempty closed and open in U_m. Since U_m is a connected, we have that $U_m = U_m \cap (\omega^+\left[\overline{J}\right] \cap U)$. This holds for every $m = 1, \ldots, M$. It follows that, $U = \omega^+\left[\overline{J}\right] \cap U$.

In particular, $U \subseteq \omega^+\left[\overline{J}\right]$ holds for all open $J \in \rho\mathcal{I}$ with $A \subseteq J \subseteq \overline{J} \subseteq U$. We see that

$$U \subseteq \bigcap\left\{\omega^+\left[\overline{J}\right] : J \in \rho o\mathcal{I}, A \subseteq J \subseteq \overline{J} \subseteq U\right\}.$$

Let $x \in U$ then, $\omega[x] \subseteq \bigcap\left\{\overline{J} : J \in \rho o\mathcal{I}, A \subseteq J \subseteq \overline{J} \subseteq U\right\}$. It can be shown that $\bigcap\{\overline{J} : J \in \rho o\mathcal{I}, A \subseteq J \subseteq \overline{J} \subseteq U\} = A$ (we essentially proved this in the proof of Item 2 of Proposition 3.2.4). Therefore, for all $x \in U$ we have $\omega[x] \subseteq A$, since A is a m.i.s and $\omega[x]$ is a nonempty closed invariant set, we have that $\omega[x] = A$. So Item 1 holds.

Item 3 $\implies$ Item 4: This implication follows immediately from Theorem 4.2.2.

Item 4 $\implies$ Item 1: Suppose that Item 4 holds. Let U be an open set with $U \supseteq A$, WLOG we may take U to have both $\overline{U}$ is compact and every m.v.s in U is also in A. Since A is $\rho\mathcal{I}$-stable, there are open[1] $I, J \in \rho\mathcal{I}$ with

$$A \subseteq J \subseteq \overline{J} \subseteq I \subseteq \overline{I} \subseteq U.$$

Consider I to be fixed and consider J to be any open robust invariant which satisfies the above inclusion.

We claim that $\overline{I} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J]$ is a compact viable set. Suppose that $x \in \overline{I} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J]$ and since $\overline{I}$ is invariant we have $\mathrm{F}[x] \subseteq \overline{I}$. For the sake of contradiction, suppose that $\mathrm{F}[x] \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J]$. Note that since J is invariant we have that $J \subseteq \mathrm{F}^+[J]$, by applying F^+ to both sides of this inclusion, one can prove that the sets $\mathrm{F}^{\circ n+}[J]$, $n \in \mathbb{N}$ are open (by u.s.c), nested and increasing. Since F is compact valued there is a finite subcover of the $\left\{\mathrm{F}^{\circ n+}[J] : n \in \mathbb{N}\right\}$ which cover $\mathrm{F}[x]$. But since these set are nested, there is a $N \in \mathbb{N}$ with $\mathrm{F}[x] \subseteq \mathrm{F}^{\circ N+}[J]$ and so $\mathrm{F}^{\circ(N+1)}[x] \subseteq J$. But now $x \in \mathrm{F}^{\circ(N+1)+}[J] \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J]$, which is a contradiction. Therefore, $\mathrm{F}[x] \cap \overline{I} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J] \neq \emptyset$ and so $\overline{I} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J] \subseteq \mathrm{F}^-\left[\overline{I} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J]\right]$. And $\overline{I} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J]$ is viable by definition. Also note that $\overline{I}$ is compact and $\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J]$ is open; making $\overline{I} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J]$ compact.

For the sake of contradiction, suppose that $\overline{I} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J] \neq \emptyset$. Then, by Item 2 of Proposition 3.1.8 there is a trajectory, say $\{x_n\}_{n\in\mathbb{N}}$, with $\{x_n : n \in \mathbb{N}\} \subseteq \overline{I} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J]$. Also, $\overline{\{x_n : n \in \mathbb{N}\}}$ is compact, by Theorem 3.2.2 there is a m.v.s, Q, with

$$Q \subseteq \overline{\{x_n : n \in \mathbb{N}\}} \subseteq \overline{I} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J] \subseteq U.$$

This however is a contradiction, since by properties of U we have $Q \subseteq A$ but $A \subseteq J \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J]$ (which can be seen by the sets in the union being nested) and so $A \cap Q \subseteq A \cap \overline{I} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J] = \emptyset$.

[1] We may take J, I to be open by Item 1 of Proposition 3.3.2.

Therefore we must have that $\overline{I} \setminus \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J] = \emptyset$. It follows that $\overline{I} \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J]$. This holds for all $J \in \rho o\mathcal{I}$ with $A \subseteq J \subseteq \overline{J} \subseteq I$, we can see that

$$I \subseteq \overline{I} \subseteq \bigcap \left\{ \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J] : J \in \rho o\mathcal{I}, A \subseteq J \subseteq \overline{J} \subseteq I \right\}.$$

Much like in the proof of Item 2 $\implies$ Item 1 we can conclude that for all $x \in I$ we have that $\omega[x] \subseteq A$. Since A is a m.i.s and $\omega[x]$ is a nonempty closed invariant set, we have that $\omega[x] = A$. So Item 1 holds.

Item 6 $\implies$ Item 5: Suppose that Item 6 holds, so A is $\mathcal{I}$-stable m.i.s, there is a compact $I \in \mathrm{cl}\mathcal{I}$ with $\mathrm{int}(I) \supseteq A$ such that $A = \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$. We may apply Item 2c of Proposition 4.2.1 to conclude A is the unique fixed set in $\mathrm{int}(I) \supseteq A$. So Item 5 is true.

Item 5 $\implies$ Item 1: Assume that Item 5 holds. Then, A is $\mathcal{I}$-stable and there is an open $U \supseteq A$ such that A is the unique nonempty compact fixed set in U. As A is compact $\mathcal{I}$-stable, X is locally compact and Hausdorff, there is a compact $I \in \mathrm{cl}\mathcal{I}$ with $A \subseteq \mathrm{int}(I) \subseteq I \subseteq U$.

Then, by Lemma 4.2.1, for all $x \in \mathrm{int}(I)$ we have $\omega[x] \subseteq I \subseteq U$ is a fixed set of F. By assumption A is the unique nonempty compact fixed set in U, so it follows that $A = \omega[x]$ for all $x \in \mathrm{int}(I)$. It follows that Item 1 holds.

Item 1 $\implies$ Item 6: Note that at this point in the proof, Items 1 to 4 are all equivalent.

We will do a proof by contradiction. Assume that Item 4 is true but Item 6 is false. By Item 4 A is a $\rho\mathcal{I}$-stable m.i.s, it follows that: for all compact $I \in \mathcal{I}$ with $\mathrm{int}(I) \supseteq A$ we have that $A \neq \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$. Moreover, there is an open $O \supseteq A$ such that all m.v.s in O are in A. WLOG, assume that $\overline{O}$ is compact.

Let $I \in \mathcal{I}$ be compact with $\mathrm{int}(I) \supseteq A$ have $I \subseteq O$ (this can be done since A is $\rho\mathcal{I}$-stable) with $A \neq \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$. By Item 2b of Proposition 4.2.1 we now that every fixed set in I, is a subset of $\bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$. Since $A \subseteq I$ is a fixed set by Item 2 of Proposition 3.2.2 we have that $A \subsetneq \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$. One can show that $\mathrm{F}^{\circ n}[I] \not\to A$ in the u.v.t. This can be seen from, Item 2a of Proposition 4.2.1, noting $\mathrm{Ls}_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I] = \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ and $\mathrm{Ls}_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$ is the smallest closed set which $\{\mathrm{F}^{\circ n}[I]\}_{n\in\mathbb{N}}$ can converge to in the u.v.t (see Item 2 of Proposition 2.2.6).

Hence, there is an open $U \supseteq A$ with for all $N \in \mathbb{N}$ there is a $n \geq N$ with $\mathrm{F}^{\circ n}[I] \not\subseteq U$. As A is $\rho\mathcal{I}$-stable there is an open $V \in \rho\mathcal{I}$ with $\overline{V} \subseteq U$ and $A \subseteq V$. It follows that for all $n \in \mathbb{N}$, $\mathrm{F}^{\circ n}[I] \not\subseteq V$; since otherwise would have $\mathrm{F}^{\circ n}[I] \subseteq V$ for some V and by invariance for all $k \geq n$ we have $\mathrm{F}^{\circ k}[I] \subseteq V \subseteq U$, a contradiction. If follows that (noting $\mathrm{F}^{\circ n}[I] \subseteq I$)

$\mathrm{F}^{\circ n}[I] \cap I \setminus V \neq \emptyset$ for all $n \in \mathbb{N}$; equivalently this means that $\emptyset \neq I \cap \mathrm{F}^{-\circ n}[I \setminus V]$ for all $n \in \mathbb{N}$.

Now we claim that $\mathrm{Ls}_{n\in\mathbb{N}}\, I \cap \mathrm{F}^{-\circ n}[I \setminus V]$ is a nonempty viable set. It is nonempty, since for all $n \in \mathbb{N}$ we have $\emptyset \neq I \cap \mathrm{F}^{-\circ n}[I \setminus V] \subseteq I$ and I is compact, so we can apply Item 4 of Proposition 2.2.6. For viability, suppose that $x \notin \mathrm{F}^{-}\left[\mathrm{Ls}_{n\in\mathbb{N}}\, I \cap \mathrm{F}^{-\circ n}[I \setminus V]\right]$. Then, $\mathrm{F}[x] \subseteq X \setminus \mathrm{Ls}_{n\in\mathbb{N}}\, I \cap \mathrm{F}^{-\circ n}[I \setminus V]$. By definition of Ls, for all $y \in \mathrm{F}[x]$ there is $W_y \in \tau_y$ and a $N_y \in \mathbb{N}$ such that for all $n \geq N_y$ we have that $W_y \cap I \cap \mathrm{F}^{-\circ n}[I \setminus V] = \emptyset$. We see for all $n \geq N_y$ that

$$\begin{aligned}
W_y \cap I \subseteq I \setminus \mathrm{F}^{-\circ n}[I \setminus V] &= I \cap X \setminus \mathrm{F}^{-\circ n}[I \setminus V] \\
&= I \cap \mathrm{F}^{+\circ n}[X \setminus (I \setminus V)] \\
&= I \cap \mathrm{F}^{+\circ n}[V \cup X \setminus I] \\
&\subseteq \mathrm{F}^{+\circ n}[I] \cap \mathrm{F}^{+\circ n}[V \cup X \setminus I] \\
&= \mathrm{F}^{+\circ n}[I \cap (V \cup X \setminus I)] \\
&= \mathrm{F}^{+\circ n}[I \cap V] \\
&\subseteq \mathrm{F}^{+\circ n}[V],
\end{aligned}$$

recalling Items 5 and 6 of Proposition 2.3.1 and $I \subseteq \mathrm{F}^{+\circ n}[I]$ for all $n \in \mathbb{N}$, since I is invariant. By compactness of $\mathrm{F}[x]$ there are $y_1, \ldots, y_K$ with $\mathrm{F}[x] \subseteq \bigcup_{k=1}^{K} W_{y_k}$ and since $x \in I$ we have $\mathrm{F}[x] \subseteq I$. Thus, $\mathrm{F}[x] \subseteq \bigcup_{k=1}^{K} W_{y_k} \cap I$. Let $N = \max\{N_{y_k} : k = 1, \ldots, K\}$ then for all $n \geq N$ we have that $\mathrm{F}[x] \subseteq \mathrm{F}^{+\circ n}[V]$. And so, $x \in \mathrm{F}^{+\circ(N+1)}[V]$ since F is u.s.c there is an open set B with $x \in B \subseteq \mathrm{F}^{+\circ(N+1)}[V]$. It follows that $\mathrm{F}^{\circ(N+1)}[B] \subseteq V$ and since V is invariant we have that $\mathrm{F}^{\circ(n+1)}[B] \subseteq V$ for all $n \geq N$. Therefore, $B \subseteq \bigcap_{n\geq N} \mathrm{F}^{\circ+(n+1)}[V]$ and we see

$$\emptyset = B \cap X \setminus \bigcap_{n\geq N} \mathrm{F}^{\circ+(n+1)}[V] = B \cap \bigcup_{n\geq N} \mathrm{F}^{\circ-(n+1)}[X \setminus V] \supseteq B \cap \bigcup_{n\geq N} \mathrm{F}^{\circ-(n+1)}[I \setminus V].$$

And we can tell that $x \notin \overline{\bigcup_{n\geq N} \mathrm{F}^{\circ-(n+1)}[I \setminus V]}$, and it follows from Items 2 and 7 of Proposition 2.2.5 that $x \notin \mathrm{Ls}_{n\in\mathbb{N}}\, I \cap \mathrm{F}^{-\circ n}[I \setminus V]$. This shows that $\mathrm{Ls}_{n\in\mathbb{N}}\, I \cap \mathrm{F}^{-\circ n}[I \setminus V] \subseteq \mathrm{F}^{-}\left[\mathrm{Ls}_{n\in\mathbb{N}}\, I \cap \mathrm{F}^{-\circ n}[I \setminus V]\right]$ and so $\mathrm{Ls}_{n\in\mathbb{N}}\, I \cap \mathrm{F}^{-\circ n}[I \setminus V]$ is viable.

Finally, we can apply Theorem 3.2.2 (since $\mathrm{Ls}_{n\in\mathbb{N}}\, I \cap \mathrm{F}^{-\circ n}[I \setminus V]$ is a nonempty compact viable set) to conclude there is a m.v.s $Q \subseteq \mathrm{Ls}_{n\in\mathbb{N}}\, I \cap \mathrm{F}^{-\circ n}[I \setminus V]$. Notice $\mathrm{Ls}_{n\in\mathbb{N}}\, I \cap \mathrm{F}^{-\circ n}[I \setminus V] \subseteq I \setminus V$, since if $x \in I \cap \mathrm{F}^{-\circ n}[I \setminus V]$ for some $n \in \mathbb{N}$ and $x \in V$ then by invariance of V we have $\mathrm{F}^{\circ n}[x] \subseteq V$; But this contradicts $x \in \mathrm{F}^{-\circ n}[I \setminus V]$. Therefore, $\mathrm{Ls}_{n\in\mathbb{N}}\, I \cap \mathrm{F}^{-\circ n}[I \setminus V] \subseteq I \setminus V$ and so $Q \subseteq I \setminus V$. But this contradicts Item 4, as $Q \subseteq I \setminus V \subseteq O \setminus V$ and $A \subseteq V$; so Q is a m.v.s in O but not in A. This concludes the proof.

□

Let us consider how Theorem 4.2.3 applies to the simpler case of a single valued function $\mathrm{f} : X \to X$ and the set $A = \{\bar{x}\}$ where $\bar{x}$ is some point in X. If A is asymptotically stable then, its easy to show that $\bar{x}$ is a fixed point. Meaning that A is a m.i.s. The set A being asymptotically stable is the same thing as A being asymptotically stable for trajectories. So in this case Items 1 and 3 are exactly the same, more or less. Similarly, Item 4 is trivialized; for in this single valued case the m.i.s are m.v.s and vice-versa. This means that Item 4 says that $\bar{x}$ is a stable fixed point, which is bounded away from from other m.i.s, like periodic orbits or other fixed points. Even in this simple case, the necessity of Items 5 and 6 is not obvious. However, its not to hard to show that locally, the sequence of functions $\{\mathrm{f}^{\circ n}\}_{n\in\mathbb{N}}$ converges uniformly to the constant function $\mathrm{g}(x) = \bar{x}$. From here we know that any local compact set K should have $\mathrm{f}^{\circ n}(K) \to \{\bar{x}\}$ in the Vietoris topology. So a local compact fixed set (which is not $\{\bar{x}\}$) is impossible. Lastly, we have the weird one, Item 2. I have no great insight into this condition, even in this simple case.

In the general multivalued case, we can note that the presumption of local connectedness is only relevant for the implication Item 2 $\implies$ Item 1.

Remark 4.2.1. *Although Conjecture 4.0.1 is false, by Example 4.0.1, it could be the case that a multifunction with a pointwise computable reachable set has a locally asymptotically stable small set.*

This is also not the case. One can see this by modifying Example 4.0.1. Let $X = [0,1] \subseteq \mathbb{R}$ *with the usual topology. Consider the multifunction* $\mathrm{F} : X \rightsquigarrow X$ *defined by*

$$\mathrm{F}[x] = \begin{cases} \{1\} & x = 1 \\ \left[0, 2^{n+1}(x - 2^{-(n+1)})^2 + 2^{-(n+1)}\right] & x \in [2^{-(n+1)}, 2^{-n}), n \in \mathbb{N}_0 \\ \{0\} & x = 0. \end{cases}$$

By similar reasoning to Example 4.0.1, F *is a continuous compact valued multifunction with pointwise robust reachable set. But the small set* $A = \{0\}$ *is not locally asymptotically stable. Since the m.v.ss* $Q_n = \{2^{-n}\}$*,* $n \in \mathbb{N}$ *get arbitrarily close to* A*, but are not eventually contained in* A*. Hence, Item 4 of Theorem 4.2.3 does not hold, so* A *is not locally asymptotically stable.*

Armed with knowledge of local stability, it's not to hard to get global stability.

Theorem 4.2.4. *Let* $(X.\tau)$ *be a Hausdorff locally compact connected topological space,* $\mathrm{F} : X \rightsquigarrow X$ *be a continuous compact valued multifunction and* $A \subseteq X$ *be a nonempty compact set. Assume that* clR *is compact valued. The following are equivalent:*

1. *A is globally asymptotically stable.*

2. *A is the small set, A is $\rho\mathcal{I}$-stable and ω is l.s.c on X.*

3. *A is the small set, A is $\rho\mathcal{I}$-stable and A is globally attractive for trajectories.*

4. *A is the small set, A is $\rho\mathcal{I}$-stable and every m.v.s of F is contained in A.*

5. *A is $\mathcal{I}$-stable and A is the unique nonempty compact fixed set (ie $\mathrm{F}[A] = A$) in X.*

6. *A is the small set, A is $\mathcal{I}$-stable, for every compact $I \in \mathrm{cl}\,\mathcal{I}$ we have $A = \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$.*

Proof. Item 1 $\implies$ Item 2: Suppose that Item 1 holds. Let I be a nonempty closed invariant set then, for all $x \in I$ we have $I \supseteq \overline{\mathrm{R}\,[x]} \supseteq \omega[x] = A$. This shows that A is the smallest nonempty closed invariant set, and so A is the small set.

The multifunction ω is constant on X, it follows that ω is l.s.c on X.

Since A is globally asymptotically stable it is also locally attractive and $\mathcal{I}$-stable. And by Theorem 4.2.1 A is also $\rho\mathcal{I}$-stable, so Item 2 holds.

Item 1 $\implies$ Item 3: Suppose that Item 1 holds. Then, as in the last case, A is a $\rho\mathcal{I}$-stable and the small set.

To see why A attracts trajectories, let $\{x_n\}_{n\in\mathbb{N}_0}$ be a trajectory with $x_0 \in X$. Then, by Item 2 of Proposition 3.2.1 we know that $\mathrm{Ls}_{n\in\mathbb{N}}\{x_n\} \subseteq \omega[x_0] \subseteq A$. Which means A attracts trajectories and Item 3 holds.

Item 3 $\implies$ Item 4: If Q is a m.v.s then, by Theorem 3.2.4 there is a $\{q_{n+1} \in \mathrm{F}[q_n]\}_{n\in\mathbb{N}_0}$, $q_0 \in Q$ with $Q = \mathrm{Ls}_{n\to\infty}\{q_n\}$. When Item 3 holds, A is globally attractive for trajectories, $\mathrm{Ls}_{n\to\infty}\{q_n\} \subseteq A$ and $Q \subseteq A$.

Item 4 $\implies$ Item 1: This proof is very similar to the proof of Item 4 $\implies$ Item 1 of Theorem 4.2.3. So we will just provide a sketch of the proof. Let $x \in X$, then by assumption $\mathrm{clR}\,[x]$ is compact. Since A is $\rho\mathcal{I}$-stable, A is compact and X is locally compact, there is a $J \in \rho o\mathcal{I}$ with

$$A \subseteq J \subseteq \overline{J}.$$

From here one can show that $\mathrm{clR}\,[x] \setminus \bigcup_{n\in\mathbb{N}}[J]$ is a compact viable set. Then, one can argue that if $\mathrm{clR}\,[x] \setminus \bigcup_{n\in\mathbb{N}}[J] \neq \emptyset$ that $\mathrm{clR}\,[x] \setminus \bigcup_{n\in\mathbb{N}}[J]$ must contain a m.v.s outside of A. This would contradict Item 4. So $\mathrm{clR}\,[x] \subseteq \bigcup_{n\in\mathbb{N}}[J]$ for all $J \in \rho o\mathcal{I}$ with $A \subseteq J \subseteq \overline{J}$. From here an argument involving $\rho\mathcal{I}$-stability and A being the small set will give us Item 1.

Item 2 $\implies$ Item 1: This proof is also very similar to the proof of Item 2 $\implies$ Item 1 of Theorem 4.2.3. To sketch the proof, let $J \in \rho o \mathcal{I}$ with

$$A \subseteq J \subseteq \overline{J}.$$

From here we can argue that $\omega^+\left[\overline{J}\right] = \bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J]$ using the fact that $\mathrm{clR}\,[x]$ is compact for all $x \in X$. Then, $\omega^+\left[\overline{J}\right]$ is closed by l.s.c of ω and $\bigcup_{n\in\mathbb{N}} \mathrm{F}^{\circ n+}[J]$ is open by u.s.c of F. So, $\omega^+\left[\overline{J}\right]$ is closed and open, it is also nonempty since $A \subseteq \omega^+\left[\overline{J}\right]$. Therefore, $X = \omega^+\left[\overline{J}\right]$ since X is connected. From here an argument involving $\rho\mathcal{I}$-stability and A being the small set will give us Item 1.

Item 6 $\implies$ Item 5: Suppose that Item 6 holds, so A is $\mathcal{I}$-stable, A the small set, every compact $I \in \mathrm{cl}\,\mathcal{I}$ has $A = \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$. If B is a compact fixed then it is invariant. Thus,

$$A = \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[B] = \bigcap_{n\in\mathbb{N}} B = B.$$

So Item 5 holds.

Item 5 $\implies$ Item 1: Assume that Item 5 holds. Then, A is $\mathcal{I}$-stable and A is the unique compact fixed set in X.

Then, by Lemma 4.2.1, for all $x \in X$ we have $\omega[x]$ is a fixed set of F. By assumption A is the unique nonempty compact fixed set in X, so it follows that $A = \omega[x]$ for all $x \in X$. Therefore, Item 1 holds.

Item 1 $\implies$ Item 6: This proof is very similar to the proof of Item 1 $\implies$ Item 6 in Theorem 4.2.3. To sketch the proof assume that Item 6 is false but Item 1 (and therefore Item 4) is true.

We see that, there is a compact $I \in \mathcal{I}$ with $A \subsetneq \bigcap_{n\in\mathbb{N}} \mathrm{F}^{\circ n}[I]$. One can argue that $\mathrm{F}^{\circ n}[I] \not\to A$ in the u.v.t. It follows from this and by $\rho\mathcal{I}$-stability of A that there is an open $V \in \rho\mathcal{I}$ with $A \subseteq V$ and $\overline{V}$ compact. Then, one can argue that $\mathrm{Ls}_{n\in\mathbb{N}}\, I \cap \mathrm{F}^{-\circ n}[I \setminus V] \subseteq I \setminus V$ is a nonempty compact viable set contained in $I \setminus V$. But then there a m.v.s, Q in $I \setminus V$ which contradicts Item 4. □

Again note that the assumption of connectedness is only relevant for Item 2 of Theorem 4.2.4.

As a technical matter, it is hard to drop the assumption that clR is compact valued in Theorem 4.2.4, see Example 4.2.2.

Example 4.2.2. *Let $X = \mathbb{R}$ with the usual topology and define*

$$\mathrm{F}[x] = \begin{cases} \{0\} & x \in [-1,1] \\ \{0, -e^{x-1}+1\} & x \geq 1 \\ \left\{0, e^{-(x+1)}-1\right\} & x \leq -1 \end{cases}$$

Then, F *satisfies Items 2 and 4 of Theorem 4.2.4 but not Item 1 or Item 3 with $A = \{0\}$. It is possible to show that $\overline{\mathrm{R}\,[4]}$ is unbounded and so not compact.*

With Theorem 4.2.4 we are armed with knowledge of global asymptotically stability. Using these insights we strengthen Corollary 4.1.2.1 and find that a sufficient condition for computability of the reachable set is global asymptotically stability. To do this we first need the following result.

Lemma 4.2.2. *Let (X,τ) be a regular Hausdorff topological space, $\mathrm{F} : X \rightsquigarrow X$ be a continuous compact valued multifunction and $C \subseteq X$ be compact. We have that*

1. *For any $n \in \mathbb{N}$ and $O \supseteq \mathrm{F}^{\circ n}[C]$ there a open cover $\mathcal{U}_{O,n}$ of X with $\mathrm{F}^{\circ n}_{\mathcal{U}_{O,n}}[C] \subseteq O$.*

2. $\mathrm{sCR}(C) = \mathrm{R}\,[\mathrm{F}, C] \cup \bigcap \left\{ \overline{\mathrm{R}_N[\mathrm{F}_{\mathcal{U}}, C]} : N \in \mathbb{N}, \mathcal{U} \text{ is an open cover of } X \right\}.$

Proof. Let $\mathfrak{U}$ be the set of all open covers of X.

To prove Item 1 we use induction. When $n = 1$ and $O \supseteq \mathrm{F}[C]$, there is an open set V with $O \supseteq \overline{V} \supseteq \mathrm{F}[C]$; we can find such a V since X is regular Hausdorff and $\mathrm{F}[C]$ is compact. Let $\mathcal{U}_{O,1} = \left\{O, X \setminus \overline{V}\right\}$ then, $\mathrm{F}[C] \subseteq \overline{V} \iff \mathrm{F}[C] \cap X \setminus \overline{V} = \emptyset$ means that $\mathrm{F}_{\mathcal{U}_{O,1}}[C] \subseteq O$.

Suppose that for $n \in \mathbb{N}$ and any $O \supseteq \mathrm{F}^{\circ n}[C]$ there a $\mathcal{U}_{O,n} \in \mathfrak{U}$ with $\mathrm{F}^{\circ n}_{\mathcal{U}_{O,n}}[C] \subseteq O$. Suppose that $O \supseteq \mathrm{F}^{\circ(n+1)}[C]$ and again we can find an open V with $O \supseteq \overline{V} \supseteq V \supseteq \mathrm{F}^{\circ(n+1)}[C]$. We see that $\mathrm{F}^{\circ n}[C] \subseteq \mathrm{F}^+[V]$ and $\mathrm{F}^+[V]$ is open by u.s.c. Define

$$\mathcal{U}_{O,n+1} = \left\{U \cap O : U \in \mathcal{U}_{\mathrm{F}^+[V],n}\right\} \cup \left\{U \cap \left(X \setminus \overline{V}\right) : U \in \mathcal{U}_{\mathrm{F}^+[V],n}\right\}.$$

It can be shown that $\mathrm{F}_{\mathcal{U}_{O,n+1}} \subseteq \mathrm{F}_{\mathcal{U}_{\mathrm{F}^+[V],n}}$. By the inductive hypothesis we have

$$\mathrm{F}^+[V] \supseteq \mathrm{F}^{\circ n}_{\mathcal{U}_{\mathrm{F}^+[V],n}}[C] \supseteq \mathrm{F}^{\circ n}_{\mathcal{U}_{O,n+1}}[C]$$

and so $\mathrm{F} \circ \mathrm{F}^{\circ n}_{\mathcal{U}_{O,n+1}}[C] \subseteq V \subseteq \overline{V}$. By construction of $\mathcal{U}_{O,n+1}$ we see that $\mathrm{F}^{\circ(n+1)}_{\mathcal{U}_{O,n+1}}[C] \subseteq O$. Which proves 1.

Now for Item 2. Recall, by definition of sCR we have that

$$\mathrm{sCR}(C) = \bigcap\left\{\overline{\mathrm{R}[\mathrm{F}_{\mathcal{U}}, C]} : \mathcal{U} \in \mathfrak{U}\right\} = \bigcap\left\{\bigcup_{n=1}^{N} \overline{\mathrm{F}^{\circ n}_{\mathcal{U}}[C]} \cup \overline{\mathrm{R}_{N+1}[\mathrm{F}_{\mathcal{U}}, C]} : \mathcal{U} \in \mathfrak{U}\right\}$$

for all $N \in \mathbb{N}$. Let $y \in \mathrm{sCR}(C)$ then for all $\mathcal{U} \in \mathfrak{U}$ and all $N \in \mathbb{N}$ we have that $y \in \bigcup_{n=1}^{N} \overline{\mathrm{F}^{\circ n}_{\mathcal{U}}[C]} \cup \overline{\mathrm{R}_{N+1}[\mathrm{F}_{\mathcal{U}}, C]}$. We consider two cases.

Case 1: There is an $N \in \mathbb{N}$ for all $\mathcal{U} \in \mathfrak{U}$ we have that $y \in \bigcup_{n=1}^{N} \overline{\mathrm{F}^{\circ n}_{\mathcal{U}}[C]}$.

Now let $O \supseteq \bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[C]$. By Item 1, for $n = 1, \ldots, N$ there are open covers $\mathcal{U}_{O,n}$ with $\mathrm{F}^{\circ n}_{\mathcal{U}_{O,n}}[C] \subseteq O$. Now define the open cover

$$\mathcal{U} = \left\{\bigcap_{n=1}^{N} U_n : U_n \in \mathcal{U}_{O,n} \text{ for } n = 1, \ldots, N\right\}$$

and again one can show that $\mathrm{F}_{\mathcal{U}} \subseteq \mathrm{F}_{\mathcal{U}_{O,n}}$ for $n = 1, \ldots, N$. Then, we can see that for $n = 1, \ldots, N$ we have $\mathrm{F}^{\circ n}_{\mathcal{U}}[C] \subseteq \mathrm{F}^{\circ n}_{\mathcal{U}_{O,n}}[C] \subseteq O$. Therefore, for all $O \supseteq \bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[C]$ there is an open cover $\mathcal{U}$ with $\bigcup_{n=1}^{N} \mathrm{F}^{\circ n}_{\mathcal{U}} \subseteq O$.

Thus we can see,

$$\bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[C] \subseteq \bigcap_{\mathcal{U}\in\mathfrak{U}} \overline{\bigcup_{n=1}^{N} \mathrm{F}^{\circ n}_{\mathcal{U}}[C]} \subseteq \bigcap\left\{\overline{O} : O \in \tau, O \supseteq \bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[C]\right\}$$

and by regularity $\bigcap\left\{\overline{O} : O \in \tau, O \supseteq \bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[C]\right\} = \overline{\bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[C]} = \bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[C]$. Recalling that $\bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[C]$ is closed since each $\mathrm{F}^{\circ n}[C]$ is closed. Therefore, we can conclude that $y \in \bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[C] \subseteq \mathrm{R}\left[\mathrm{F}, C\right]$.

Case 2: For all $N \in \mathbb{N}$ there is a $\mathcal{U}_N \in \mathfrak{U}$ with $y \notin \bigcup_{n=1}^{N} \overline{\mathrm{F}^{\circ n}_{\mathcal{U}_N}[C]}$.

Let $N \in \mathbb{N}$ and $\mathcal{U}_N$ be as above and let $\mathcal{U} \in \mathfrak{U}$ be arbitrary then,

$$\mathcal{V} = \{U_1 \cap U_2 : U_1 \in \mathcal{U}_N, U_2 \in \mathcal{U}\} \in \mathfrak{U}$$

and $\mathrm{F}_{\mathcal{V}} \subseteq \mathrm{F}_{\mathcal{U}_N}, \mathrm{F}_{\mathcal{U}}$. Therefore, $y \notin \bigcup_{n=1}^{N} \overline{\mathrm{F}^{\circ n}_{\mathcal{V}}[C]} \subseteq \bigcup_{n=1}^{N} \overline{\mathrm{F}^{\circ n}_{\mathcal{U}_N}[C]} \cap \bigcup_{n=1}^{N} \overline{\mathrm{F}^{\circ n}_{\mathcal{U}}[C]}$.

In summery, (recalling that $y \in \mathrm{sCR}(C)$) for all $N \in \mathbb{N}$ and all $\mathcal{U} \in \mathfrak{U}$ there is a $\mathcal{V} \in \mathfrak{U}$ with $y \in \overline{\mathrm{R}_{N+1}[\mathrm{F}_{\mathcal{V}}, C]} \subseteq \overline{\mathrm{R}_{N+1}[\mathrm{F}_{\mathcal{U}}, C]}$. It follows that, $y \in \overline{\mathrm{R}_{N+1}[\mathrm{F}_{\mathcal{U}}, C]}$ for all $N \in \mathbb{N}$ and $\mathcal{U} \in \mathfrak{U}$; that is $y \in \bigcap\left\{\overline{\mathrm{R}_N[\mathrm{F}_{\mathcal{U}}, C]} : N \in \mathbb{N}, \mathcal{U} \in \mathfrak{U}\right\}$.

Together these cases show that $\mathrm{sCR}(C) \subseteq \mathrm{R}[\mathrm{F}, C] \cup \bigcap\left\{\overline{\mathrm{R}_N[\mathrm{F}_\mathcal{U}, C]} : N \in \mathbb{N}, \mathcal{U} \in \mathfrak{U}\right\}$. The other inclusion is a bit easier. By definitions one can see that $\mathrm{sCR}(C) \supseteq \overline{\mathrm{R}[\mathrm{F}, C]} \supseteq \mathrm{R}[\mathrm{F}, C]$. Similarly, for any $\mathcal{U} \in \mathfrak{U}$ and any $N \in \mathbb{N}$ we find that $\mathrm{R}[\mathrm{F}_\mathcal{U}, C] \supseteq \mathrm{R}_N[\mathrm{F}_\mathcal{U}, C]$. So it follows that $\mathrm{sCR}(C) \supseteq \bigcap\left\{\overline{\mathrm{R}_N[\mathrm{F}_\mathcal{U}, C]} : \mathcal{U} \in \mathfrak{U}, N \in \mathbb{N}\right\}$. □

Theorem 4.2.5. *Let $(X.\tau)$ be a Hausdorff locally compact topological space, $\mathrm{F} : X \rightsquigarrow X$ be a continuous compact valued multifunction with* clR *compact valued and $A \subseteq X$ be a nonempty compact set which is globally asymptotically stable.*

Then, clR *is pointwise robust and compact valued.*

Proof. Since A is globally asymptotically stable, we have for all $x \in X$ that $\mathrm{clR}[\mathrm{F}, x] = \mathrm{R}[\mathrm{F}, x] \cup \omega[\mathrm{F}, x] = \mathrm{R}[\mathrm{F}, x] \cup A$ (by Item 1 of Proposition 3.2.1). Since $\mathrm{clR}[x] \subseteq \mathrm{CR}[x]$ for all $x \in X$, Item 2 of Lemma 4.2.2 and Item 2 of Theorem 3.3.2 we have

$$\mathrm{clR}[x] \subseteq \mathrm{CR}[x] = \mathrm{sCR}(\{x\}) = \mathrm{R}[\mathrm{F}, x] \cup \bigcap\left\{\overline{\mathrm{R}_N[\mathrm{F}_\mathcal{U}, x]} : N \in \mathbb{N}, \mathcal{U} \text{ is an open cover of } X\right\}.$$

Thus, we need only show that $A \supseteq \bigcap\{\overline{\mathrm{R}_N[\mathrm{F}_\mathcal{U}, x]} : N \in \mathbb{N}, \mathcal{U} \text{ is an open cover of } X\}$.

By local compactness, let O be an open set with $O \supseteq A$, and $\overline{O}$ is compact. Since A is globally asymptotically stable, A is also $\mathcal{I}$-stable and locally attractive. By Theorem 4.2.1 we know that A is $\rho\mathcal{I}$-stable. By regularity and compactness there are $I, J \in \rho\,\mathrm{cl}\,\mathcal{I}$ with $\overline{\mathrm{int}(I)} = I$, $\overline{\mathrm{int}(J)} = J$ (see Item 1 of Proposition 3.3.2) and

$$A \subseteq \mathrm{int}(J) \subseteq J \subseteq \mathrm{int}(I) \subseteq I \subseteq O.$$

Since A is globally asymptotically stable and clR is compact valued, for all $x \in X$ we have that $\mathrm{F}^{\circ n}[x] \to A$ in the u.v.t (this follows from Item 3 of Proposition 2.2.6). Thus, there is a $N_J \in \mathbb{N}$ such that for all $n \geq N_J$ we have $\mathrm{F}^{\circ n}[x] \subseteq \mathrm{int}(J)$.

Let $n \geq N_J$ and by Item 1 of Lemma 4.2.2 there is an open cover $\mathcal{U}_{J,n}$ of X with $\mathrm{F}^{\circ n}_{\mathcal{U}_{J,n}}[x] \subseteq \mathrm{int}(J)$. Also $\overline{\mathrm{F}^{\circ n}_{\mathcal{U}_{J,n}}[x]} \subseteq \overline{\mathrm{int}(J)} = J \subseteq \mathrm{int}(I)$. By Item 3 of Proposition 3.3.2 there is an open cover $\mathcal{V}_I$ of X with $\mathrm{R}\left[\mathrm{F}_{\mathcal{V}_I}, \overline{\mathrm{F}^{\circ n}_{\mathcal{U}_{J,n}}[x]}\right] \subseteq \mathrm{int}(I)$. Now define the open cover

$$\mathcal{W}_{I,J,n} = \{V \cap U : V \in \mathcal{V}_I, U \in \mathcal{U}_{J,n}\}.$$

Again note that $\mathrm{F}_{\mathcal{W}_{I,J,n}} \subseteq \mathrm{F}_{\mathcal{V}_I}, \mathrm{F}_{\mathcal{U}_{J,n}}$. Which means we have

$$\mathrm{int}(I) \supseteq \mathrm{R}\left[\mathrm{F}_{\mathcal{V}_I}, \overline{\mathrm{F}^{\circ n}_{\mathcal{U}_{J,n}}[x]}\right] \supseteq \mathrm{R}\left[\mathrm{F}_{\mathcal{V}_I}, \mathrm{F}^{\circ n}_{\mathcal{U}_{J,n}}[x]\right] \supseteq \mathrm{R}\left[\mathrm{F}_{\mathcal{W}_{I,J,n}}, \mathrm{F}^{\circ n}_{\mathcal{W}_{I,J,n}}[x]\right] = \mathrm{R}_n[\mathrm{F}_{\mathcal{W}_{I,J,n}}, x].$$

So $\overline{\mathrm{R}_n[\mathrm{F}_{\mathcal{W}_{I,J,n}}, x]} \subseteq \overline{\mathrm{int}(I)} = I \subseteq O$.

Therefore, for all open $O \supseteq A$ and all large $n \in \mathbb{N}$ there is an open cover $\mathcal{W}_{O,n}$ of X with

$$\overline{\mathrm{R}_n[\mathrm{F}_{\mathcal{W}_{O,n}}, x]} \subseteq O.$$

Let $\mathfrak{U}$ be the set of all open covers of X and it follows that if $y \in \bigcap_{\mathcal{U}\in\mathfrak{U}} \bigcap_{N\in\mathbb{N}} \overline{\mathrm{R}_N[\mathrm{F}_{\mathcal{U}}, x]}$ then, for all $O \supseteq A$ and all large $n \in \mathbb{N}$ we have that $y \in \overline{\mathrm{R}_n[\mathrm{F}_{\mathcal{W}_{O,n}}, x]} \subseteq O$. By regularity, we see $y \in \bigcap_{O\in\tau, O\supseteq A} \overline{O} = \overline{A} = A$. We can conclude that $\bigcap_{\mathcal{U}\in\mathfrak{U}} \bigcap_{N\in\mathbb{N}} \overline{\mathrm{R}_N[\mathrm{F}_{\mathcal{U}}, x]} \subseteq A$, as required.

□

Corollary 4.2.5.1. *Let (X, τ) be a connected computable Hausdorff space which is a topological manifold and $\mathrm{F} : X \rightsquigarrow X$ be a continuous compact valued multifunction such that the closed reachable set of* F *is pointwise compact.*

If a nonempty compact set $A \subseteq X$ is globally asymptotically stable then, the closed reachable set is pointwise computable.

Proof. This result follows immediately from Theorem 3.3.4 and Theorem 4.2.5. □

Corollary 4.2.5.1 finds that globally asymptotically stability is a sufficient condition for pointwise computability. Intuitively, this makes sense, as we can over approximate $\mathrm{clR}\,[x]$ by picking some open $O \supseteq A$ then, iterating F until we see $\mathrm{F}^{\circ N+1}[x] \subseteq O$ for some $N \in \mathbb{N}$. Then, call $O \cup \bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[x]$ an over approximation of $\mathrm{clR}\,[x]$. Note that $O \cup \bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[x] \supseteq \mathrm{clR}\,[x]$ is true only when O is invariant. However, A is robustly stable by Theorem 4.2.4 and by using Algorithm 1 we can find an open robust invariant set V with $A \subseteq V$, $\overline{V} \subseteq O$. Therefore, $V \cup \bigcup_{n=1}^{N} \mathrm{F}^{\circ n}[x] \supseteq \mathrm{clR}\,[x]$ if $\mathrm{F}^{\circ N+1}[x] \subseteq V$. So it is possible to find a rigours over approximation of $\mathrm{clR}\,[x]$ in finite time.

In the single valued case the sufficiency of global asymptotic stability is also necessary.

Corollary 4.2.5.2 (Characterization of pointwise computability for single valued functions)**.** *Let (X, τ) be a connected computable Hausdorff space which is a topological manifold, $\mathrm{f} : X \to X$ be a continuous function. Define $\mathrm{F}[x] = \{\mathrm{f}(x)\}$ for all $x \in X$. The following are equivalent:*

1. *The closed reachable set of* F *is pointwise computable.*
2. *There is a nonempty compact set $A \subseteq X$ which is globally asymptotically stable for* F.

Proof. Item 1 $\implies$ Item 2: This implication follows from Item 1 of Corollary 4.1.2.1.

Item 2 $\implies$ Item 1: We show that when Item 2 holds then, $\mathrm{clR}\,[x]$ is compact. The space X is locally compact, A is compact and $\mathcal{I}$-stable. Thus, there is a compact $I \in \mathrm{cl}\,\mathcal{I}$ with $A \subseteq \mathrm{int}(I) \subseteq I$. Let $x \in X$ and as $A = \omega[x] = \mathrm{Ls}_{n\to\infty}\{\mathrm{f}^{\circ n}(x)\}$, for all $N \in \mathbb{N}$ there is a $n \geq N$ with $\mathrm{f}^{\circ n}(x) \in \mathrm{int}(I)$. But I is invariant, it follows that for all $k \geq n$ we have $\mathrm{f}^{\circ k}(x) \in I$. Which means $\mathrm{R}_n[x] \subseteq I$ and so $\mathrm{clR}_n[x] \subseteq I$. Therefore, $\mathrm{clR}\,[x] = (\bigcup_{k=1}^{n-1}\{\mathrm{f}^{\circ k}(x)\}) \cup \mathrm{clR}_n[x]$ is compact.

Hence, we can apply Corollary 4.2.5.1, to get Item 1. □

Corollary 4.2.5.3 (Characterization of pointwise computability for single valued real functions)**.** *Let $X = \mathbb{R}$ be the real line with the usual topology, $\mathrm{f} : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define $\mathrm{F}[x] = \{\mathrm{f}(x)\}$ for all $x \in \mathbb{R}$. The following are equivalent:*

1. *The closed reachable set of F is pointwise computable.*
2. *There is a fixed point $\bar{x}$ which is globally asymptotically stable for f.*

Proof. This result follows quickly from Corollary 4.2.5.2 and Item 3 of Corollary 4.1.2.1. □

As mentioned in the beginning of this chapter, I initially believed that all multifunctions with computable reachable set were essentially contraction maps. While this is not the case of multifunctions, Corollaries 4.2.5.2 and 4.2.5.3 tell us that, if f is a single valued function with pointwise computable reachable set then, f is essentially contraction map. And especially so when f is a real function.

We can actually make this idea a little more formal, by using converses to Banach's Fixed Point Theorem.

Theorem 4.2.6 (Banach's Fixed Point Theorem)**.** *Let (X, d) be a complete metric space and $\mathrm{f} : X \to X$ be a contraction map. That is f satisfies:*

$$\exists c \in [0, 1)\ \forall x, y \in X \quad \mathrm{d}(\mathrm{f}(x), \mathrm{f}(y)) < c\,\mathrm{d}(x, y).$$

Then, there is $\bar{x} \in X$ were $\bar{x}$ is the unique point of f, which is globally asymptotically stable.

Theorem 4.2.7 (Meyer's Converse to Banach's Fixed Point Theorem, see [14, 9])**.** *Let (X, d) be a complete metric space and $\mathrm{f} : X \to X$ is continuous. The following are equivalent:*

1. *There is a metric ρ on X, equivalent to d such that f is a contraction in (X, ρ).*

2. *There is* $\bar{x} \in X$ *were* $\bar{x}$ *is the unique point of* f*, which is globally asymptotically stable.*

Therefore, we can extend Corollary 4.2.5.3 slightly more.

Corollary 4.2.7.1. *Let* $X = \mathbb{R}$ *be the real line with the usual topology,* $\mathrm{f} : \mathbb{R} \to \mathbb{R}$ *be a continuous function. Define* $\mathrm{F}[x] = \{\mathrm{f}(x)\}$ *for all* $x \in \mathbb{R}$*. The following are equivalent:*

1. *The closed reachable set of* F *is pointwise computable.*
2. *There is a metric* ρ *on* $\mathbb{R}$*, which induces the usual topology on* $\mathbb{R}$*, such that* f *is a contraction in* $(\mathbb{R}, \rho)$*.*
3. *There is a fixed point* $\bar{x}$ *which is globally asymptotically stable for* f*.*

Chapter 5

Conclusions

We have shown that a multifunction with pointwise computable reachable set, in a connected computable Hausdorff space satisfies the following:

1. The multifunction possess a stable small set. (Theorem 4.1.2)
2. Every closed invariant set is stable. (Item 2b of Theorem 4.1.1)

In a practical situation, I think that it is unlikely that a given system satisfies the above conditions.

The only sufficient condition I found, for a multifunction with pointwise computable reachable set, is that the multifunction possess a globally asymptotically stable set (Corollary 4.2.5.1). When we apply this to single valued functions, possession of a globally asymptotically stable set is also necessary for a pointwise computable reachable set (Corollary 4.2.5.2). Furthermore, a single valued function of $\mathbb{R}$ has a pointwise computable reachable set if and only if the function is contractive with respect some metric (Corollary 4.2.7.1). I believe this sufficient condition is poor practical assumption, indeed I expect many systems do not satisfy it in practice.

To expose another feature of pointwise computability, we recall again the system

$$x_{n+1} = \mathrm{f}(x_n, u_n)$$

for $n \in \mathbb{N}$, $x_0 \in X$, $\{u_n \in U\}_{n \in \mathbb{N}}$ where $\mathrm{f} : X \times U \to X$. The natural multifunction to consider is $\mathrm{F}[x] = \mathrm{f}(\{x\} \times U) = \bigcup_{u \in U} \{\mathrm{f}(x, u)\}$ for all $x \in X$. It is possible for F to have a pointwise computable reachable set but for some fixed $y \in U$ the function f_y, defined by $\mathrm{f}_y[x] = \mathrm{f}(x, y)$ for all $x \in X$, does not have a pointwise computable reachable set.

For example, this occurs Example 4.0.1 where $X = U = [0,1] \subseteq \mathbb{R}$, $\mathrm{f}(x,u) = ux^2$ for $x \in X$ and $u \in U$. It was shown that F has pointwise computable reachable set, however the function $\mathrm{f}(x,1) = \mathrm{f}_1(x) = x^2$ has two fixed points, 0 and 1, so the reachable set of f_1 not pointwise computable. Of course, the reachable set of f_1 is (a rather simple) trajectory. More specifically, $\mathrm{clR}\,[\mathrm{f}_1, x]$ is only not computable at $x = 1$, as 0 is asymptotically stable for f_1, on $[0,1)$. Notably, $\mathrm{clR}\,[\mathrm{f}_1, 1] = \{1\}$. Indeed. the only trajectory of F, say $\{x_n\}_{n\in\mathbb{N}}$, with $1 \in \mathrm{Ls}_{n\in\mathbb{N}}\{x_n\}$ has controller $u_n = 1$ for all $n \in \mathbb{N}$ and $x_0 = 1$; so $x_n = \mathrm{f}_1^{\circ n}(1) = 1$ for all $n \in \mathbb{N}$. Therefore, despite the reachable set of F being computable, the unique trajectory of F with $1 \in \mathrm{Ls}_{n\in\mathbb{N}}\{x_n\}$ is not computable.

For the above reasons, I assert that pointwise computability of the reachable set is too restrictive of a condition to be practical. Therefore, in order for there to be broadly applicable, practically satisfactory conditions to approximate the reachable set we must discard type-II computability theory.

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Glossary

fixed set $\mathrm{F}[S] = S$, see Definition 3.1.4.

invariant set $\mathrm{F}[S] \subseteq S$ or equivalently $S \subseteq \mathrm{F}^{+}[S]$, see Definition 3.1.4.

lower semicontinuous (l.s.c) multifunction $\mathrm{F}^{-}[V]$ is open, for all open V, see Definition 2.3.5.

lower Vietoris topology (l.v.t) The topology generated by the sub-base $\{U^{-} : U \in \tau\}$, see Definition 2.2.1.

minimal invariant set (m.i.s) A minimal closed nonempty invariant set of a multifunction, see Definition 3.2.3.

minimal viable set (m.v.s) A minimal closed nonempty viable set of a multifunction, see Definition 3.2.3.

multifunction A function from X to 2^{Y}, see Definition 2.3.1.

outer semicontinuous (o.s.c) multifunction A multifunction with closed graph, see Definition 2.3.6.

robust invariant set $\mathrm{F}\left[\overline{S}\right] \subseteq \mathrm{int}(S)$, see Definition 3.3.4.

robust viable set $\overline{S} \subseteq \mathrm{F}^{-}[\mathrm{int}(S)]$, see Definition 3.3.4.

super-invariant set $S \subseteq \mathrm{F}[S]$, see Definition 3.1.4.

upper semicontinuous (u.s.c) multifunction $\mathrm{F}^{+}[V]$ is open, for all open V, see Definition 2.3.5.

upper Vietoris topology (u.v.t) The topology generated by the base $\{U^+ : U \in \tau\}$, see Definition 2.2.1.

viable set $S \subseteq \mathrm{F}^-[S]$, see Definition 3.1.4.

Vietoris topology (v.t) The topology generated by the sub-base $\{U^- : U \in \tau\} \cup \{U^+ : U \in \tau\}$, see Definition 2.2.1.