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ON NORM-LIMITS OF ALGEBRAIC QUASIDIAGONAL OPERATORS

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ABSTRACT. It is still an open question to know whether or not every quasidiagonal operator can be expressed as a norm-limit of algebraic quasidiagonal operators. In this note, we provide an alternative characterization of those operators which may be expressed as such limits, in the hope that this may lead to a solution of this problem.

KEYWORDS: *Quasidiagonal, algebraic operators, nilpotent operators, approximation.*

MSC (2010): Primary 47A66, 47A58; Secondary 47A05.

1. INTRODUCTION

1.1. A standard and elementary result in linear algebra asserts that if $T \in \mathbb{M}_n(\mathbb{C})$ is an $n \times n$ complex matrix with n *distinct* eigenvalues, then there exists a basis $\{v_k\}_{k=1}^n$ for $\mathbb{C}^n$ with respect to which the matrix for T is diagonal. In other words, there exists an invertible matrix $S \in \mathbb{M}_n(\mathbb{C})$ such that $S^{-1}TS$ is a diagonal matrix. It is an easy exercise to see that every matrix $A \in \mathbb{M}_n(\mathbb{C})$ may be approximated (say, in the operator norm $\|\cdot\|$ on $\mathbb{M}_n(\mathbb{C})$, thinking of the latter as the algebra of operators on the Hilbert space $\mathbb{C}^n$) arbitrarily well by a matrix with n distinct eigenvalues; indeed, one simply upper triangularizes A with respect to some orthonormal basis, and then perturbs the diagonal entries ever so slightly to produce n distinct eigenvalues. From this it immediately follows that every matrix $A \in \mathbb{M}_n(\mathbb{C})$ may be approximated by matrices of the form $S^{-1}DS$ where D is diagonal and S is invertible. Moreover, this notion extends to sequences of matrices: if $(A_n)_n$ is a sequence with $A_n \in \mathbb{M}_{k_n}(\mathbb{C})$ for some $k_n \geqslant 1$, and if $\varepsilon > 0$, then we can find sequences $(S_n)_n$ with $S_n \in \mathbb{M}_{k_n}(\mathbb{C})$ invertible and $(D_n)_n$ with $D_n \in \mathbb{M}_{k_n}(\mathbb{C})$ invertible for all $n \geqslant 1$ such that $\|A_n - S_n^{-1}D_nS_n\| < \varepsilon$ for all $n \geqslant 1$.

What is far less clear, however, is how well such approximations work if we start to impose restrictions on, say, the spectra of the D_n's, or on the condition numbers $\|S_n^{-1}\|\,\|S_n\|$ of the S_n's. That is, what can we say if we insist that there exists a $\mu > 0$ such that $\|S_n^{-1}\| \, \|S_n\| \leqslant \mu$ for all $n \geqslant 1$? It will be crucial in the problem that we examine below that our answer not depend upon the dimensions k_n, $n \geqslant 1$ of the underlying spaces. We shall demonstrate that if all of the A_n's satisfy a single, fixed (non-zero) polynomial equation $p(z) = 0$, then we can find such approximations with control over both the spectra of the D_n's and the condition numbers of the S_n's.

Our motivation for examining such questions stems from a problem in operator theory regarding so-called *quasidiagonal operators*. Before stating the question, we first require some definitions and some background.

Let $\mathcal{H}$ be a complex, separable, infinite-dimensional Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators acting on $\mathcal{H}$, and by $\mathcal{K}(\mathcal{H})$ the closed, two-sided ideal of all compact operators in $\mathcal{B}(\mathcal{H})$. We write π to denote the canonical map $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ from $\mathcal{B}(\mathcal{H})$ into the Calkin algebra. An element $T \in \mathcal{B}(\mathcal{H})$ is said to be *block-diagonal* (respectively *quasidiagonal*) if there exists an increasing sequence $(P_n)_{n=1}^{\infty}$ of finite-rank orthogonal projections in $\mathcal{B}(\mathcal{H})$ tending strongly to the identity operator I for which $P_n T = T P_n$ (respectively $\lim\limits_{n\to\infty} \|P_n T - T P_n\| = 0$). We write $T \in$ (BD) (respectively $T \in$ (QD)) to mean that T is block-diagonal (respectively T is quasidiagonal). By a result of Halmos [10], $T \in$ (QD) if and only if there exist $T_0 \in$ (BD), $K_0 \in \mathcal{K}(\mathcal{H})$ such that $T = T_0 + K_0$. What is more, if $\varepsilon > 0$ is specified in advance, then T_0 and K_0 can be chosen such that $\|K_0\| < \varepsilon$. Thus

$$(\mathrm{QD}) = \overline{(\mathrm{BD})} = (\mathrm{BD}) + \mathcal{K}(\mathcal{H}).$$

The set of quasidiagonal operators has been the focus of much study over the past forty years [14], [15], [16], [19] — indeed the notion of quasidiagonality was extended to *sets* of operators — for example, if $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ is a norm separable set of operators acting on a separable Hilbert space as above, we require the existence of a single increasing sequence $(P_n)_{n=1}^{\infty}$ of projections tending strongly to the identity as above for which $\lim\limits_{n\to\infty} \|P_n S - S P_n\| = 0$ for all $S \in \mathcal{S}$ — and there has been a great deal of interest in understanding C^*-algebras admitting (sometimes special) quasidiagonal representations [3], [4], [8], [9], [21].

In the article [10] cited above, Halmos also introduced the notion of quasitriangular operators. An operator T is said to be *triangular* and we write $T \in (\Delta)$ (respectively *quasitriangular* and we write $T \in$ (QT)) if there exists an increasing sequence $(P_n)_{n=1}$ of finite-rank orthogonal projections tending strongly to the identity operator I such that $TP_n - P_n T P_n = 0$ for all $n \geqslant 1$ (respectively $\lim\limits_{n} \|TP_n - P_n T P_n\| = 0$). Equivalently, T is triangular if there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathcal{H}$ with respect to which the matrix of T, $[T] = [t_{ij}]$ is upper triangular, i.e. $t_{ij} = \langle Te_j, e_i \rangle = 0$ if $i > j$. It is a deep and extremely useful result due to Apostol, Foiaş and Voiculescu [1] that an operator T is quasitriangular if and only if the *semi-Fredholm index* of $T - \lambda I$, namely $\mathrm{ind}\,(T - \lambda I) :=$

$\mathrm{nul}(T-\lambda I) - \mathrm{nul}\,(T-\lambda I)^*$ is greater than or equal to zero whenever $\pi(T-\lambda I)$ is either left or right invertible in the Calkin algebra.

Paralleling the results for quasidiagonality, we have that $T \in$ (QT) if and only if $T = T_0 + K_0$, where T_0 is triangular and K_0 is compact, and if $\varepsilon > 0$ is given, then we can choose T_0 triangular and K_0 compact with $\|K_0\| < \varepsilon$. Operators in the set (BQT) := (QT) $\cap$ (QT)* are said to be *biquasitriangular*. (From the result of Apostol, Foiaş and Voiculescu cited above, we see that T is biquasitriangular if and only if $\mathrm{ind}(T-\lambda I) = 0$ whenever $\pi(T-\lambda I)$ is either left or right invertible in the Calkin algebra.) We emphasize the fact that the sequence of projections implementing the quasitriangularity of a biquasitriangular operator T need not have anything to do with the sequence of projections implementing the quasitriangularity of T^*.

1.2. The study of quasitriangular operators took on special importance in relation to Halmos' seventh problem from [10], which asked for a characterization of those operators on $\mathcal{H}$ which can be expressed as limits of nilpotent operators. (Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *nilpotent of index* $k \geqslant 1$ if $T^k = 0 \neq T^{k-1}$.) Along the way to solving Halmos' seventh problem, two alternative descriptions of (BQT) were formulated. Let

$$(\mathrm{ALG}) = \{T \in \mathcal{B}(\mathcal{H}) : p(T) = 0 \text{ for some } 0 \neq p \in \mathbb{C}[z]\}, \quad \text{and}$$

$$(\mathrm{SN}) = \{T \in \mathcal{B}(\mathcal{H}) : T = S^{-1}NS \text{ for some } N \in \mathcal{B}(\mathcal{H}) \text{ normal and } S \in \mathcal{B}(\mathcal{H}) \text{ invertible}\}.$$

The acronyms (ALG) and (SN) refer to "algebraic" operators and to operators "similar to normal" operators, respectively. Combining the work of Voiculescu [22] and of Herrero [12], [13], we have that

$$\overline{(\mathrm{ALG})} = \overline{(\mathrm{SN})} = (\mathrm{BQT}).$$

Since, as is easily seen, (QD) $\subseteq$ (BQT), it follows that (QD) $\subseteq \overline{(\mathrm{ALG})}$, whence (QD) $= \overline{(\mathrm{BD})} = \overline{(\mathrm{BD})} \cap \overline{(\mathrm{ALG})}$. This led Davidson, Herrero and Salinas ([8], Problem 1.1) to ask: is it true that

$$(\mathrm{QD}) = \overline{(\mathrm{ALGQD})},$$

where (ALGQD) = (ALG) $\cap$ (QD)? As pointed out in [8], it follows from the work of Campbell and Gellar [5] that $\overline{(\mathrm{ALGQD})} = \overline{(\mathrm{ALGBD})}$, where (ALGBD) = (ALG) $\cap$ (BD). Thus the question may be rephrased as: is (QD) $= \overline{(\mathrm{ALGBD})}$? At this time, one of the best results along these lines is Theorem 2.4 of [8].

THEOREM 1.1 (Davidson–Herrero–Salinas). *Let $T \in$ (QD) and let $\varrho : C^*(\pi(T)) \to \mathcal{B}(\mathcal{H}_\varrho)$ be a unital $*$-representation onto a separable Hilbert space $\mathcal{H}_\varrho$. Suppose furthermore that:*

(i) *$\varrho(\pi(T)) \in \mathcal{B}(\mathcal{H}_\varrho)$ is quasidiagonal;*
(ii) *$\sigma(\pi(T)) = \sigma(\varrho(\pi(T)))$; and*
(iii) *$\sigma(\pi(T))$ does not disconnect the plane.*

Then $T \in \overline{(\mathrm{ALGBD})}$.

The hypothesis that $\varrho(\pi(T))$ be quasidiagonal nevertheless appears to be a rather strong one. As shown in [8], it implies the following: for each $\varepsilon > 0$ there exists an operator $T_\varepsilon \simeq R_\varepsilon \oplus B_\varepsilon^{(\infty)}$ with R_ε block-diagonal and B_ε acting upon a finite-dimensional space such that $\|T - T_\varepsilon\| < \varepsilon$. We propose the following as a *candidate* for a quasidiagonal operator for which such approximations may fail to exist.

Herrero and Szarek [17] have demonstrated the existence of a universal constant $\kappa > 0$ and of a sequence $M_j \in \mathbb{M}_{m_j}(\mathbb{C})$, $j \geqslant 1$ satisfying:

(i) $m_j < m_{j+1}$ for all $j \geqslant 1$;
(ii) $\|M_j\| = 1$ and $M_j^6 = 0$ for all $j \geqslant 1$; and
(iii) for all $j \geqslant 1$, $\operatorname{dist}(M_j, \operatorname{Red}(\mathbb{C}^{m_j})) \geqslant \kappa$.

Here, $\operatorname{Red}(\mathbb{C}^{m_j})$ denotes the set of *orthogonally reducible* operators on $\mathbb{C}^{m_j}$, that is, those operators $A \in \mathbb{M}_{m_j}(\mathbb{C})$ which can be expressed as an orthogonal direct sum $A = B \oplus C$ of two operators B and C, each acting on a non-trivial subspace of $\mathbb{C}^{m_j}$.

Observe that condition (ii) implies that $M := \bigoplus_{j=1}^{\infty} M_j \in (\text{ALGBD})$. Indeed, $M^6 = 0$. We suspect that there does not exist a unital $*$-representation ϱ of $C^*(\pi(M))$ onto a separable Hilbert space such that $\varrho(\pi(M))$ is quasidiagonal, though we have not yet been able to prove it. Regardless, the reader will observe that the potential non-existence of such a representation is not an impediment to M belonging to $\overline{(\text{ALGBD})}$.

1.3. In the next section, we shall prove an analogue for quasidiagonal operators of the results of Herrero and Voiculescu to the effect that $\overline{(\text{ALG})} = \overline{(\text{SN})}$, namely: setting

$$(\text{DSSN}) = \Big\{ T = \bigoplus_{n=1}^{\infty} T_n \in (\text{BD}) : T = S^{-1} D S \text{ for some } S = \bigoplus_n S_n \text{ invertible and } D = \bigoplus_n D_n \text{ diagonal} \Big\},$$

we shall prove that

$$\overline{(\text{ALGBD})} = \overline{(\text{DSSN})}.$$

Thus $T \in (\text{DSSN})$ implies that $T = \bigoplus_{n=1}^{\infty} T_n$ with $T_n = S_n^{-1} D_n S_n$ for all $n \geqslant 1$, subject to the condition that each D_n be normal and $\sup_{n \geqslant 1} \|S_n\| \|S_n^{-1}\| < \infty$. (The acronym (DSSN) is meant to refer to operators which are "Direct Sums of matrices Similar to Normal" matrices.) This opens a new approach to resolving the Davidson–Herrero–Salinas question which does not rely upon exhibiting quasidiagonal representations of the C^*-algebra generated by image of a quasidiagonal operator in the Calkin algebra. (We point out the fact that Wassermann [24] has proven that there exists a quasidiagonal C^*-algebra whose image in the Calkin algebra does not admit a faithful, quasidiagonal representation.) In light of this, the Davidson–Herrero–Salinas question becomes: is (QD) equal to $\overline{(\text{DSSN})}$?

In Section 3, we develop a few consequences of our results insofar as limits of block-diagonal nilpotent operators are concerned.

2. ALGEBRAIC QUASIDIAGONAL OPERATORS

2.1. NOTATION AND TERMINOLOGY. In order to improve the readability of the paper, we shall resort to a minor but common abuse of notation: given any complex Hilbert space $\mathcal{H}$ and $\alpha \in \mathbb{C}$, we shall also write α to denote the scalar operator αI, where $I \in \mathcal{B}(\mathcal{H})$ is the identity operator. Thus, for example, if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is the direct sum of two complex Hilbert spaces and $\alpha, \beta \in \mathbb{C}$, then $\alpha \oplus \beta \in \mathcal{B}(\mathcal{H})$ denotes the operator $\alpha I_1 \oplus \beta I_2$, where $I_k \in \mathcal{B}(\mathcal{H}_k)$ is the identity operator, $k = 1, 2$.

Given a non-zero polynomial $p = p(z) = \prod\limits_{s=1}^{r} (z - \beta_s)^{k_s} \in \mathbb{C}[z]$, we shall say that p is in *standard form* if $1 \leqslant s \neq t \leqslant r$ implies that $\beta_s \neq \beta_t$, and $k_s \geqslant 1$, $1 \leqslant s \leqslant r$. We then denote the *degree* of p by $\deg(p(z)) = \sum\limits_{s=1}^{r} k_s$, the *maximum multiplicity* of p by $\kappa(p(z)) := \max(k_1, k_2, \dots, k_r)$, and the zeros of p by $Z_p = \{\beta_1, \beta_2, \dots, \beta_r\}$.

If $\varnothing \neq F \subseteq \mathbb{C}$ is a finite set, we define $\text{DISP}(F) = \min\{|x - y| : x, y \in F, x \neq y\}$, and we shall refer to this as the *dispersion* of F.

Suppose that $\mathcal{H}$ is a complex Hilbert space, and that $0 \neq T \in \mathcal{B}(\mathcal{H})$ satisfies $p(T) = 0$. The minimal polynomial p_0 of T then divides p. We can (and will) assume without loss of generality (by reindexing the set Z_p if necessary) that $p_0(z) = \prod\limits_{s=1}^{r_0} (z - \beta_s)^{m_s}$ is in standard form, and note that $1 \leqslant r_0 \leqslant r$ and $1 \leqslant m_s \leqslant k_s$ for each $1 \leqslant s \leqslant r_0$.

Finally, for $w \in \mathbb{C}$ and $\varepsilon > 0$, we denote the open ball of radius ε centred at w by

$$B(w, \varepsilon) := \{z \in \mathbb{C} : |z - w| < \varepsilon\}.$$

2.2. We begin with a small technical lemma, whose purpose is as follows: suppose that $p(z) = \prod\limits_{s=1}^{r} (z - \beta_s)^{k_s} \in \mathbb{C}[z]$, where $i \neq j$ implies that $\beta_i \neq \beta_j$. We wish to approximate $p(z)$ by a polynomial $q(z)$ of the same degree, whose roots $\alpha_1, \alpha_2, \dots, \alpha_d$ ($d = \deg(p(z)) = \sum\limits_{s=1}^{r} k_s$) are all simple. Furthermore, we wish to do this in a manner that allows us to keep some control over the minimum distance between any two α_i's — that is, over the dispersion of the set $\{\alpha_1, \alpha_2, \dots, \alpha_d\}$. The usefulness of this will soon become apparent.

We mention in passing that there is a slight technicality that we must deal with in the case where $r = 1$, since in this case the dispersion of the set of roots of p has not been defined.

For this section of the paper, the connected set Γ introduced in Lemma 2.1 below may be taken to be $\Gamma = \mathbb{C}$. It will, however, become a useful technical device in the next section, when we will consider direct sums of algebraic quasidiagonal operators with normal operators.

LEMMA 2.1. *Let $\Gamma \subseteq \mathbb{C}$ be a connected set, $r \geqslant 1$ be an integer, and suppose that $\beta_1, \beta_2, \ldots, \beta_r$ are distinct elements of Γ. (If $r = 1$, we assume that Γ includes at least one point — any thus infinitely many points — not equal to β_1.) Suppose furthermore that $k_1, k_2, \ldots, k_r \geqslant 1$ are integers and set $\kappa := \max(k_1, k_2, \ldots, k_r)$. Finally,*

(i) *if $r = 1$, choose $\gamma_1 \in \Gamma \setminus \{\beta_1\}$ arbitrarily and define $\delta = |\gamma_1 - \beta_1| > 0$;*
(ii) *if $r \geqslant 2$, define $\delta = \mathrm{DISP}(\{\beta_1, \beta_2, \ldots, \beta_r\}) > 0$.*

Given $0 < \varepsilon < \frac{\delta}{3}$, there exists a set $A_\varepsilon := \{\alpha_s(t) : 1 \leqslant t \leqslant k_s, 1 \leqslant s \leqslant r\} \subseteq \Gamma$ of cardinality $d := \sum_{s=1}^{r} k_s$ so that:

(a) *$|\alpha_s(t) - \beta_s| < \varepsilon$ for all $1 \leqslant t \leqslant k_s$, $1 \leqslant s \leqslant r$, and*
(b) *$\mathrm{DISP}(A_\varepsilon) \geqslant \frac{\varepsilon}{2\kappa}$.*

Proof. Suppose that $0 < \varepsilon < \frac{\delta}{3}$. Observe that if $1 \leqslant s \leqslant r$, and $2 \leqslant t \leqslant 2\kappa$, then

$$\Big(B(\beta_s, \frac{t}{2\kappa}\varepsilon) \setminus \overline{B(\beta_s, \frac{t-1}{2\kappa}\varepsilon)}\Big) \cap \Gamma \neq \varnothing.$$

(Suppose otherwise for some $1 \leqslant s \leqslant r$ and $2 \leqslant t \leqslant \kappa$. Consider the disjoint open sets $B(\beta_s, \frac{t-1}{2\kappa}\varepsilon)$ and $\mathbb{C} \setminus \overline{B(\beta_s, \frac{t-1}{2\kappa}\varepsilon)}$. If $r = 1$, we see that β_1 lies in the first set, while γ_1 lies in the second, while for $r \geqslant 2$, we see that β_s lies in the first set, while β_{s_1} lies in the second for any $1 \leqslant s_1 \neq s \leqslant r$. Either way, this contradicts the connectedness of Γ.)

For $1 \leqslant t \leqslant k_s$, choose

$$\alpha_s(t) \in \Big(B(\beta_s, \frac{2t}{2\kappa}\varepsilon) \setminus \overline{B(\beta_s, \frac{2t-1}{2\kappa}\varepsilon)}\Big) \cap \Gamma.$$

Clearly $|\alpha_s(t) - \beta_s| < \frac{2t}{2\kappa}\varepsilon \leqslant \varepsilon, 1 \leqslant s \leqslant r, 1 \leqslant t \leqslant k_s$.

If $1 \leqslant s \leqslant r$ and $1 \leqslant t_1 < t_2 \leqslant k_s$, then

$$|\alpha_s(t_2) - \alpha_s(t_1)| \geqslant ||\alpha_s(t_2)| - |\alpha_s(t_1)|| \geqslant \frac{2t_2 - 1}{2\kappa}\varepsilon - \frac{2t_1}{2\kappa}\varepsilon \geqslant \frac{\varepsilon}{2\kappa}.$$

If $1 \leqslant s_1 \neq s_2 \leqslant r, 1 \leqslant t_1 \leqslant k_{s_1}, 1 \leqslant t_2 \leqslant k_{s_2}$, then

$$|\alpha_{s_1}(t_1) - \alpha_{s_2}(t_2)| \geqslant |\beta_{s_1} - \beta_{s_2}| - |\beta_{s_1} - \alpha_{s_1}(t_1)| - |\beta_{s_2} - \alpha_{s_2}(t_2)| \geqslant |\beta_{s_1} - \beta_{s_2}| - 2\varepsilon \geqslant \varepsilon.$$

Hence $\mathrm{DISP}(A_\varepsilon) \geqslant \frac{\varepsilon}{2\kappa}$. ∎

REMARK 2.2. In the following lemma, we shall require a function f of three positive parameters. The important thing for our purposes will not be the growth properties of this function, but rather the fact that it depends only upon these three parameters. Let $M, \delta > 0$ be positive numbers and $d \geqslant 1$ be an integer. If $d = 1$, we set $f(M, \delta, 1) = 1$.

Next, set $f(M, \delta, 2) = (1 + \delta^{-1}M)$, and for $d \geqslant 2$, define

$$f(M, \delta, d+1) = (1 + f(M, \delta, d))(1 + (\delta^{-1}M)(1 + f(M, \delta, d))).$$

It is clear that $f(M, \delta, d+1) \geqslant f(M, \delta, d)$ for all $d \geqslant 2$, and that $M_2 > M_1$ implies that $f(M_2, \delta, d) \geqslant f(M_1, \delta, d)$.

LEMMA 2.3. *Suppose that $A = \{\alpha_1, \alpha_2, \ldots, \alpha_d\} \subseteq \mathbb{C}$ is a set of cardinality d so that $\delta := \operatorname{DISP}(A) > 0$. Let $\mathcal{H}$ be a complex Hilbert space, and suppose that $T \in \mathcal{B}(\mathcal{H})$ is an algebraic operator which has*

$$q(z) = \prod_{s=1}^{d} (z - \alpha_s)$$

as its minimal polynomial. Then there exists a normal operator $D \in \mathcal{B}(\mathcal{H})$ with $\sigma(D) = A$ and an invertible operator $S \in \mathcal{B}(\mathcal{H})$ satisfying

$$\max(\|S\|, \|S^{-1}\|) \leqslant f(\|T\|, \delta, d)$$

for which $T = S^{-1}DS$.

Proof. The fact that q is the minimal polynomial for T implies that $\mathcal{H}$ admits a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_d$ with respect to which

$$T = \begin{bmatrix} \alpha_1 & T_{12} & T_{13} & \ldots & T_{1d} \\ & \alpha_2 & T_{23} & \ldots & T_{2d} \\ & & \alpha_3 & \ldots & T_{3d} \\ & & & \ddots & \vdots \\ & & & & \alpha_d \end{bmatrix}.$$

We shall argue by induction on d. If $d = 1$, we set $D = T = \alpha_1 I$ and $S = I$. There is nothing to prove.

Case 1. Let $d = 2$. Write $T = \begin{bmatrix} \alpha_1 & T_{12} \\ & \alpha_2 \end{bmatrix}$ relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Let $D = \begin{bmatrix} \alpha_1 & 0 \\ & \alpha_2 \end{bmatrix}$ relative to this same decomposition of $\mathcal{H}$.

Next, set

$$S = \begin{bmatrix} I & (\alpha_1 - \alpha_2)^{-1} T_{12} \\ & I \end{bmatrix},$$

so that $\|S\| \leqslant 1 + |\alpha_1 - \alpha_2|^{-1} \|T_{12}\| \leqslant 1 + \delta^{-1}\|T\| = f(\|T\|, \delta, 2)$. Note also that

$$S^{-1} = \begin{bmatrix} I & -(\alpha_1 - \alpha_2)^{-1} T_{12} \\ & I \end{bmatrix},$$

and thus a similar calculation to that above shows that

$$\|S^{-1}\| \leqslant f(\|T\|, \delta, 2).$$

A routine computation shows the following which completes the proof of this case:

$$T = S^{-1}DS.$$

Case 2. Let $d_0 \geqslant 2$ be an integer and suppose that the result holds for $d = 1, 2, \ldots, d_0$. We prove that it holds for $d = d_0 + 1$.

As we have seen, the hypothesis implies that there exists a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_{d_0+1}$ so that

$$T = \begin{bmatrix} \alpha_1 & T_{12} & T_{13} & \cdots & \cdots & T_{1d_0+1} \\ & \alpha_2 & T_{23} & \cdots & \cdots & T_{2d_0+1} \\ & & \alpha_3 & \cdots & \cdots & T_{3d_0+1} \\ & & & \ddots & & \vdots \\ & & & & \alpha_{d_0} & T_{d_0 d_0+1} \\ & & & & & \alpha_{d_0+1} \end{bmatrix}.$$

Let $D = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_{d_0+1} \in \mathcal{B}(\mathcal{H})$ relative to the above decomposition. Futhermore, set $\mathcal{H}_0 := \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_{d_0}$, and let $D_0 = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_{d_0} \in \mathcal{B}(\mathcal{H}_0)$, so that

$$D = \begin{bmatrix} D_0 & 0 \\ 0 & \alpha_{d_0+1} \end{bmatrix}, \quad T = \begin{bmatrix} T_0 & W \\ 0 & \alpha_{d_0+1} \end{bmatrix}$$

relative to $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_{d_0+1}$.

Our induction hypothesis guarantees the existence of an invertible operator $S_0 \in \mathcal{B}(\mathcal{H}_0)$ with $\max(\|S_0\|, \|S_0^{-1}\|) \leqslant f(\|T\|, \delta, d_0)$ so that $T_0 = S_0^{-1} D_0 S_0$.

Set $Z = (D_0 - \alpha_{d_0+1} I_{\mathcal{H}_0})^{-1} S_0 W \in \mathcal{B}(\mathcal{H}_{d_0+1}, \mathcal{H}_0)$, and observe that

$$\|Z\| \leqslant \|(D_0 - \alpha_{d_0+1} I_{\mathcal{H}_0})^{-1}\| \, \|S_0\| \, \|W\| \leqslant (\delta^{-1}\|T\|) f(\|T\|, \delta, d_0).$$

Define $R = \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix}$ relative to $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_{d_0+1}$, and let

$$S = R \begin{bmatrix} S_0 & \\ & I \end{bmatrix} = \begin{bmatrix} S_0 & Z \\ 0 & I \end{bmatrix}$$

relative to the same decomposition, so that

$$S^{-1} = \begin{bmatrix} S_0^{-1} & \\ & I \end{bmatrix} \begin{bmatrix} I & -Z \\ 0 & I \end{bmatrix} = \begin{bmatrix} S_0^{-1} & -S_0^{-1} Z \\ 0 & I \end{bmatrix}.$$

It follows that

$$\begin{aligned} \|S\| &\leqslant \|R\| \, \|S_0 \oplus I\| \leqslant (1 + \|Z\|)(1 + \|S_0\|) \\ &\leqslant (1 + (\delta^{-1}\|T\|) f(\|T\|, \delta, d_0))(1 + f(\|T\|, \delta, d_0)) = f(\|T\|, \delta, d_0 + 1). \end{aligned}$$

Similarly, $\|S^{-1}\| \leqslant \|R^{-1}\| \, \|S_0^{-1} \oplus I\| \leqslant f(\|T\|, \delta, d_0 + 1)$.

Again, a routine computation shows that $T = S^{-1}DS$, completing the induction step and the proof. ∎

REMARK 2.4. It is obvious that we have made no attempt to be efficient in estimating an upper bound for $\|S\|$ above. What is interesting and will prove useful, however, is that it does not depend upon the dimensions (finite or infinite) of the spaces $\mathcal{H}_k$, $1 \leqslant k \leqslant d$, nor upon the choice of T_{ij}, $1 \leqslant i < j \leqslant d$, so long as $\|T\| \leqslant M$.

LEMMA 2.5. *Let $p = p(z) = \prod_{s=1}^{r} (z - \beta_s)^{k_s} \in \mathbb{C}[z]$ be a polynomial in standard form, and $\Gamma \subseteq \mathbb{C}$ be an infinite, connected set containing Z_p. Let $\kappa = \kappa(p(z))$.*

If $r = 1$, choose $\gamma_1 \in \Gamma \setminus \{\beta_1\}$ and set $\delta := |\beta_1 - \gamma_1| > 0$. If $r \geqslant 2$, set $\delta := \mathrm{DISP}(Z_p)$.

Given $0 < \varepsilon < \frac{\delta}{3}$, there exists a finite set $A_\varepsilon = \{\alpha_s(t) : 1 \leqslant t \leqslant k_s,\ 1 \leqslant s \leqslant r\} \subseteq \Gamma$ of cardinality $d := \deg(p(z))$ with the following properties:

(i) *$|\alpha_s(t) - \beta_s| < \varepsilon$, $1 \leqslant t \leqslant k_s$, $1 \leqslant s \leqslant r$;*

(ii) *$\mathrm{DISP}(A_\varepsilon) \geqslant \frac{\varepsilon}{2\kappa}$; and*

(iii) *if $\mathcal{H}$ is a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ satisfies $p(T) = 0$, and if $p_0(z) = \prod_{s=1}^{r_0} (z - \beta_s)^{m_s}$ is the minimal polynomial of T, then there exists $T_0 \in \mathcal{B}(\mathcal{H})$ so that:*

(a) *$\|T - T_0\| < \varepsilon$, and*

(b) *$q(z) = \prod_{s=1}^{r_0} \prod_{t=1}^{m_s} (z - \alpha_s(t))$ is the minimal polynomial of T_0.*

Proof. Let $0 < \varepsilon < \frac{\delta}{3}$. The existence of a finite set $A_\varepsilon = \{\alpha_s(t) : 1 \leqslant t \leqslant k_s,\ 1 \leqslant s \leqslant r\} \subseteq \Gamma$ satisfying (i) and (ii) is the conclusion of Lemma 2.1.

Next, for $1 \leqslant t \leqslant m_s$, $1 \leqslant s \leqslant r_0$, set

$$\begin{aligned}\mathcal{H}_{m_1+m_2+\cdots+m_{s-1}+t} = &\ker (T-\beta_1)^{m_1}(T-\beta_2)^{m_2}\cdots(T-\beta_{s-1})^{m_{s-1}}(T-\beta_s)^{t} \\ &\ominus \ker (T-\beta_1)^{m_1}(T-\beta_2)^{m_2}\cdots(T-\beta_{s-1})^{m_{s-1}}(T-\beta_s)^{t-1},\end{aligned}$$

so that relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_{d_0}$, where $d_0 := \deg(q(z))$, we have

$$T = \begin{bmatrix} \beta_1 & T_{12} & \cdots & & & & & \cdots & T_{1\,d_0} \\ & \beta_1 & \ddots & & & \ddots & & & \vdots \\ & & \ddots & & & & T_{ij} & & \\ & & & \beta_1 & & & & \ddots & \\ & & & & \beta_2 & & & & \\ & & & & & \ddots & & & \\ & & & & & & \beta_{r_0} & & \\ & & & & & & & \ddots & T_{d_0-1,d_0} \\ & & & & & & & & \beta_{r_0} \end{bmatrix}.$$

In the matrix above, the diagonal scalar operator β_s appears exactly m_s times, $1 \leqslant s \leqslant r_0$.

Let T_0 be the operator obtained from T by replacing the m_s occurrences of β_s along the diagonal by the distinct elements $\alpha_s(t)$, $1 \leqslant t \leqslant m_s$. Thus

$$T_0 = \begin{bmatrix} \alpha_1(1) & T_{12} & \cdots & & & & & & T_{1\,d_0} \\ & \alpha_1(2) & \ddots & & & \ddots & & & \vdots \\ & & \ddots & & & & T_{ij} & & \\ & & & \alpha_1(m_1) & & & & \ddots & \\ & & & & \alpha_2(1) & & & & \\ & & & & & \ddots & & & \\ & & & & & & \alpha_{r_0}(1) & & \\ & & & & & & & \ddots & \\ & & & & & & & & \alpha_{r_0}(m_{r_0}) \end{bmatrix}.$$

It is readily verified that q is the minimal polynomial for T_0. Furthermore,

$$\|T - T_0\| = \max\{|\alpha_s^{(t)} - \beta_s| : 1 \leqslant s \leqslant r_0, 1 \leqslant t \leqslant m_s\} < \varepsilon. \quad ∎$$

REMARK 2.6. Once again, the important issue to note here is that with Γ fixed, the choice of the elements $\alpha_s^{(t)}$, $1 \leqslant t \leqslant k_s$, $1 \leqslant s \leqslant r$ was made to depend only upon the choice of $\varepsilon > 0$ and the choice of the polynomial p; more specifically, the values of $\beta_1, \beta_2, \ldots, \beta_r$ and their multiplicities. In particular, this choice is *independent* of the choice of the polynomial p_0 (subject to the condition that it divides p), of T_{ij}, $1 \leqslant i < j \leqslant d$, and of the dimensions of the spaces $\mathcal{H}_k$, $1 \leqslant k \leqslant d_0$ in the above decomposition.

PROPOSITION 2.7. *Let* $p(z) = \prod_{s=1}^{r} (z - \beta_s)^{k_s} \in \mathbb{C}[z]$ *be a polynomial in standard form, and suppose that* $\Gamma \subseteq \mathbb{C}$ *is an infinite, connected set with* $Z_p \subseteq \Gamma$. *Let* $\kappa = \max(k_1, k_2, \ldots, k_r)$.

If $r = 1$, *choose* $\gamma_1 \in \Gamma \setminus \{\beta_1\}$ *and set* $\delta := |\gamma_1 - \beta_1| > 0$. *If* $r \geqslant 2$, *set* $\delta := \mathrm{DISP}(Z_p)$.

Let $0 < \varepsilon < \min(1, \frac{\delta}{3})$, *and as in Lemma* 2.1, *choose a finite set* $A_\varepsilon = \{\alpha_s(t) : 1 \leqslant t \leqslant k_s, 1 \leqslant s \leqslant r\} \subseteq \Gamma$ *of cardinality* $d = \deg(p(z))$ *so that:*

(i) $|\alpha_s(t) - \beta_s| < \varepsilon$ *for all* $1 \leqslant t \leqslant k_s, 1 \leqslant s \leqslant r$, *and*

(ii) $\mathrm{DISP}(A_\varepsilon) \geqslant \frac{\varepsilon}{2\kappa}$.

Then for every complex Hilbert space $\mathcal{H}$ *and* $T \in \mathcal{B}(\mathcal{H})$ *satisfying* $p(T) = 0$, *there exists*

(a) *a normal operator* $D \in \mathcal{B}(\mathcal{H})$ *with* $\sigma(D) \subseteq A_\varepsilon$, *and*

(b) *an invertible operator* $S \in \mathcal{B}(\mathcal{H})$ *satisfying*

$$\max(\|S\|, \|S^{-1}\|) \leqslant f\Big(\|T\| + 1, \frac{\varepsilon}{2\kappa}, d\Big)$$

for which

$$\|T - S^{-1}DS\| < \varepsilon.$$

Proof. Having chosen $0 < \varepsilon < \min(1, \frac{\delta}{3})$, and A_ε as above, suppose that $\mathcal{H}$ is a Hilbert space, $T \in \mathcal{B}(\mathcal{H})$ and $p(T) = 0$. Let $p_0(z) = \prod_{s=1}^{r_0} (z - \beta_s)^{m_s}$ be the minimal polynomial for T, written in standard form.

By Lemma 2.5, there exists an operator $T_0 \in \mathcal{B}(\mathcal{H})$ satisfying $\|T - T_0\| < \varepsilon$ for which $q(z) = \prod_{s=1}^{r_0} \prod_{t=1}^{m_s} (z - \alpha_s(t))$ is the minimal polynomial of T_0. Clearly $\|T_0\| \leqslant \|T\| + \varepsilon \leqslant \|T\| + 1$. Furthermore, $\sigma(T_0) = \{\alpha_s(t) : 1 \leqslant t \leqslant m_s, 1 \leqslant s \leqslant r_0\} \subseteq A_\varepsilon$.

It then follows from Lemma 2.3 that there exists a normal operator $D \in \mathcal{B}(\mathcal{H})$ with $\sigma(D) = \sigma(T_0) \subseteq A_\varepsilon$ and an invertible $S \in \mathcal{B}(\mathcal{H})$ for which

$$\max(\|S\|, \|S^{-1}\|) \leqslant f(\|T_0\|, \delta(A_\varepsilon), d) \leqslant f\Big(\|T\| + 1, \frac{\varepsilon}{2\kappa}, d\Big),$$

and $T_0 = S^{-1}DS$. From this we see that $\|T - S^{-1}DS\| = \|T - T_0\| < \varepsilon$. ∎

THEOREM 2.8. *Let $p(z) = \prod_{s=1}^{r} (z - \beta_s)^{k_s} \in \mathbb{C}[z]$ be a polynomial in standard form, and suppose that $\Gamma \subseteq \mathbb{C}$ is an infinite, connected set with $Z_p \subseteq \Gamma$. Let $\kappa = \kappa(p(z))$.*

If $r = 1$, choose $\gamma_1 \in \Gamma \setminus \{\beta_1\}$ and set $\delta := |\gamma_1 - \beta_1| > 0$. If $r \geqslant 2$, set $\delta := \mathrm{DISP}(Z_p)$.

Let $(\mathcal{H}_\lambda)_\lambda$ be a family of complex Hilbert spaces, and $\mathcal{H} = \bigoplus_\lambda \mathcal{H}_\lambda$. Suppose that $T \in \mathcal{B}(\mathcal{H})$ admits a diagonal decomposition $T = \bigoplus_\lambda T_\lambda$ relative to this decomposition of $\mathcal{H}$, and that $p(T) = 0$.

If $0 < \varepsilon < \frac{\delta}{3}$, then there exists a normal operator $D = \bigoplus_\lambda D_\lambda$ in $\mathcal{B}(\mathcal{H})$ with finite spectrum contained in Γ and an invertible operator $S = \bigoplus_\lambda S_\lambda \in \mathcal{B}(\mathcal{H})$ satisfying

$$\max(\|S\|, \|S^{-1}\|) \leqslant f\Big(\|T\| + 1, \frac{\varepsilon}{2\kappa}, d\Big),$$

such that

$$\|T - S^{-1}DS\| \leqslant \varepsilon.$$

Proof. Having chosen $0 < \varepsilon < \frac{\delta}{3}$, choose $A_\varepsilon = \{\alpha_s(t) : 1 \leqslant t \leqslant k_s, 1 \leqslant s \leqslant r\} \subseteq \Gamma$ of cardinality $d := \deg(p(z))$ so that

(i) $|\alpha_s(t) - \beta_s| < \varepsilon$ for all $1 \leqslant t \leqslant k_s, 1 \leqslant s \leqslant r$, and

(ii) $\delta(A_\varepsilon) \geqslant \frac{\varepsilon}{2\kappa}$.

Observe that $p(T) = 0$ implies that $p(T_\lambda) = 0$ for all λ.

By Proposition 2.7 above, for each λ, we can find a normal operator $D_\lambda \in \mathcal{B}(\mathcal{H}_\lambda)$ with $\sigma(D_\lambda) \subseteq A_\varepsilon$ and an invertible operator $S_\lambda \in \mathcal{B}(\mathcal{H}_\lambda)$ satisfying

$$\max(\|S_\lambda\|, \|S_\lambda^{-1}\|) \leqslant f\Big(\|T_\lambda\| + 1, \frac{\varepsilon}{2\kappa}, d\Big),$$

for which $\|T_\lambda - S_\lambda^{-1} D_\lambda S_\lambda\| < \varepsilon$. Let $D = \bigoplus\limits_\lambda D_\lambda$ and $S = \bigoplus\limits_\lambda S_\lambda$. It is clear that D and S are bounded, as is $S^{-1} = \bigoplus\limits_\lambda S_\lambda^{-1}$. Furthermore, $\sigma(D) = \overline{\bigcup\limits_\lambda \sigma(D_\lambda)} \subseteq A_\varepsilon \subseteq \Gamma$, and

$$\max(\|S\|, \|S^{-1}\|) \leqslant \sup_\lambda \max(\|S_\lambda\|, \|S_\lambda^{-1}\|) \leqslant f\Big(\|T\| + 1, \frac{\varepsilon}{2\kappa}, d\Big).$$

Finally,

$$\|T - S^{-1} DS\| = \sup_\lambda \|T_\lambda - S_\lambda^{-1} D_\lambda S_\lambda\| \leqslant \varepsilon. \quad \blacksquare$$

THEOREM 2.9. $\overline{(\text{ALGQD})} = \overline{(\text{DSSN})}$.

Proof. First we show that $\overline{(\text{ALGQD})} \subseteq \overline{(\text{DSSN})}$. As noted in the introduction, it follows from a result of S.L. Campbell and R. Gellar [5] that (ALGBD) := (ALG) ∩ (BD) is norm-dense in (ALGQD). As such, it suffices to show that (ALGBD) $\subseteq \overline{(\text{DSSN})}$.

Let $T = \bigoplus\limits_n T_n \in$ (ALGBD) be the decomposition of T relative to the decomposition $\mathcal{H} = \bigoplus\limits_n \mathcal{H}_n$, where dim $\mathcal{H}_n < \infty$ for all $n \geqslant 1$. Let $p(z) = \prod\limits_{s=1}^{r} (z - \beta_s)^{k_s} \in \mathbb{C}[z]$ be the minimal polynomial of T, written in standard form, and suppose that $\Gamma = \mathbb{C}$. If $r = 1$, set $\delta = 1$, while if $r \geqslant 2$, set $\delta = \text{DISP}(Z_p)$.

By Theorem 2.8, for each $0 < \varepsilon < \frac{\delta}{3}$, there exist a normal operator $D = \bigoplus D_n$ with finite spectrum and an invertible operator $S = \bigoplus\limits_n S_n$ so that

$$\|T - S^{-1} DS\| \leqslant \varepsilon.$$

Since $S^{-1}DS = \bigoplus\limits_n S_n^{-1} D_n S_n \in$ (DSSN) and $\varepsilon > 0$ can be made arbitrarily small, the proof of this containment is complete.

Conversely, suppose that $T = \bigoplus\limits_n S_n^{-1} D_n S_n \in$ (DSSN), where $D = \bigoplus\limits_n D_n \in \mathcal{B}(\mathcal{H})$ is normal, $S = \bigoplus\limits_n S_n$ is invertible, and

$$M := \sup_n \max(\|S_n\|, \|S_n^{-1}\|) < \infty.$$

Let $\varepsilon > 0$. Now $\sigma(D)$ is compact, and so we can find a finite subset $A = \{\alpha_1, \alpha_2, \ldots, \alpha_d\}$ so that $\sigma(D) \subseteq \bigcup\limits_{i=1}^{d} B(\alpha_i, \frac{\varepsilon}{M^2})$.

For each $n \geqslant 1$, we can choose a basis for $\mathcal{H}_n \simeq \mathbb{M}_{m_n}(\mathbb{C})$ which diagonalizes D_n; i.e. $D_n = \text{diag}(\beta_1(n), \beta_2(n), \ldots, \beta_{m_n}(n))$. Relative to this basis, define a new diagonal operator

$$E_n = \mathrm{diag}(\gamma_1(n), \gamma_2(n), \ldots, \gamma_{m_n}(n)),$$

where for each $1 \leqslant t \leqslant m_n$, we have that $\gamma_t(n) \in A$ and $|\gamma_t(n) - \beta_t(n)| < \frac{\varepsilon}{M^2}$.

Letting $E = \bigoplus_n E_n$, we easily see that E is a bounded, normal operator with $\sigma(E) \subseteq A$. Thus E satisfies the polynomial $q(z) = \prod_{s=1}^{d} (z - \alpha_s)$. It follows that $R := SES^{-1} = \bigoplus_n S_n^{-1} E S_n \in$ (BD) also satisfies $q(R) = 0$, so that $R \in$ (ALGBD).

Finally, $\|T - R\| = \sup_n \|S_n^{-1}(D_n - E_n)S_n\| \leqslant \sup_n \frac{\varepsilon}{M^2} \|S_n^{-1}\| \|S_n\| \leqslant \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that $T \in \overline{(\text{ALGBD})}$. It follows that $\overline{(\text{DSSN})} \subseteq \overline{(\text{ALGBD})} = \overline{(\text{ALGQD})}$, which provides the reverse inclusion and completes the proof. ∎

2.3. We suspect, but we have been so far unable to prove, that (QD) $\neq \overline{(\text{DSSN})}$ (and therefore that (QD) $\neq \overline{(\text{ALGQD})}$). We now wish to propose a candidate for an operator R which we think may lie in (QD) $\setminus \overline{(\text{ALGQD})}$.

Let $n \geqslant 1$ be an integer. Recall first that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *n-normal* if $T \simeq [T_{ij}]_{i,j=1}^n$, where $\{T_{ij} : 1 \leqslant i, j \leqslant n\}$ is a commuting family of normal operators. We say that T is *algebraically n-normal* if $T \simeq \bigoplus_{m=1}^{n} T_m$, where each T_m is m-normal, $1 \leqslant m \leqslant n$. (This definition allows for some of the T_m's to act upon a trivial (i.e. 0-dimensional) Hilbert space.) It is well-known that if T is algebraically n-normal, then T can be approximated by block-diagonal operators $(B_k)_{k=1}^\infty$ with the property that each of the summands of each B_k, $k \geqslant 1$ acts upon a space of dimension at most n. It is a routine exercise to show that any such B_k is then a limit of algebraic, block-diagonal operators, and thus T itself is a limit of algebraic, block-diagonal operators.

It was shown by Voiculescu [23] that if a quasidiagonal operator $T = \lim_k B_k$ is a limit of algebraically n_k-normal operators (meaning that each approximating operator B_k is algebraically n_k-normal for some $n_k \geqslant 1$ depending upon k), then $C^*(T)$ must be *exact*; that is, the inclusion map $\iota : C^*(T) \to \mathcal{B}(\mathcal{H})$ must be a nuclear map. An equivalent formulation says that if $T = \lim_k B_k$, where $\dim C^*(B_k) < \infty$ for all $k \geqslant 1$, then $C^*(T)$ must be exact. (See [4] for a development of the theory of nuclear and exact C^*-algebras.) Brown [3] then proved that given $T \in$ (QD), this is the only possible obstruction; more precisely, a quasidiagonal operator T is a limit of operators B_k with $\dim C^*(B_k) < \infty$ for all $k \geqslant 1$ if and only if $C^*(T)$ is exact.

Arguing as in Example 2.3 of [3], suppose that $T \in$ (QD) and that there exists an orthonormal basis $\{e_n\}_{n \in \mathbb{Z}}$ for $\mathcal{H}$ with respect to which the matrix $[T] = [t_{ij}]$ of T has *finite band-width*; that is, there exists and integer $\nu \geqslant 1$ such that $|i - j| \geqslant \nu$ implies that $t_{ij} = 0$. Then $C^*(T)$ is contained in the crossed product algebra $\ell^\infty(\mathbb{Z}) \rtimes_\gamma \mathbb{Z}$, where the action γ of $\mathbb{Z}$ upon $\ell^\infty(\mathbb{Z})$ arises from the action of the bilateral shift $Ue_n = e_{n+1}$, $n \in \mathbb{Z}$ on $\ell^2(\mathbb{Z})$. Since the crossed product C^*-algebra $\ell^\infty(\mathbb{Z}) \rtimes_\gamma \mathbb{Z}$ is known to be nuclear, $C^*(T)$ is exact and thus T is a limit of algebraically n_k-normal operators B_k, $k \geqslant 1$. A fortiori, $T \in \overline{(\text{ALGQD})}$.

We have argued that any candidate for an operator lying in $(\text{QD}) \setminus \overline{(\text{ALGQD})}$ should not have finite band-width, or loosely speaking, should be far from "living close to the diagonal". With this in mind, for each $n \geqslant 1$, choose $\omega_n \in (0,1)$ such that if $R_n = [r_{ij}(n)] \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$r_{ij}(n) = \begin{cases} \omega_n & \text{if } i < j, \\ 0 & \text{otherwise,} \end{cases}$$

then $\|R_n\| = 1$. Our proposed candidate for an operator in $(\text{QD}) \setminus \overline{(\text{ALGQD})}$ is the operator $R = \bigoplus_n R_n$. (A first step would be to show that $C^*(R)$ fails to be exact. Although it does not have finite band-width with respect to the given orthonormal basis, it is not entirely obvious that there does not exist another orthonormal basis with respect to which it might have — or at least be the limit of operators which have — finite band-width.)

We finish this section by showing that $\overline{(\text{ALGQD})}$ is at least large enough to remain invariant under compact perturbations.

PROPOSITION 2.10. $\overline{(\text{ALGQD})} = \overline{(\text{ALGQD})} + \mathcal{K}(\mathcal{H})$. *In particular, the following is norm-closed:*

$$\overline{(\text{ALGQD})} + \mathcal{K}(\mathcal{H}).$$

Proof. That $\overline{(\text{ALGQD})} \subseteq \overline{(\text{ALGQD})} + \mathcal{K}(\mathcal{H})$ is obvious.

Suppose that $T = \bigoplus_n T_n \in (\text{ALGBD})$, where $T_n \in \mathcal{B}(\mathcal{H}_n)$, $n \geqslant 1$ and $K \in \mathcal{K}(\mathcal{H})$. Let $p_T(z) \in \mathbb{C}[z]$ be a non-zero polynomial such that $p_T(T) = 0$. Observe that $p_T(T_n) = 0$ for all $n \geqslant 1$.

Let P_n denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_n$, $n \geqslant 1$, and let $R_N = \sum_{n=1}^{N} P_n$, $N \geqslant 1$. Then $(R_N)_N$ is a sequence of projections tending strongly to the identity operator, and so

$$\lim_N \|R_N K R_N - K\| = 0.$$

Now $T + R_N K R_N = B_N \oplus \Big(\bigoplus_{n=N+1}^{\infty} T_n \Big) \in (\text{BD})$, where $B_N = \Big(\bigoplus_{n=1}^{N} T_n \Big) + R_N K R_N$.

Since B_N is finite-rank, there exists a polynomial $p_N(z) \in \mathbb{C}[z]$ such that $p_N(B_N) = 0$. Set $q_N(z) = p_N(z) p_T(z)$. Then

$$q_N(T + R_N K R_N) = p_N(B_N) p_T(B_N) \oplus \Big(\bigoplus_{n=N+1}^{\infty} p_N(T_n) p_T(T_n) \Big)$$

$$= 0\, p_T(B_N) \oplus \Big(\bigoplus_{n=N+1}^{\infty} p_N(T_n)\, 0 \Big) = 0,$$

so that $T + R_N K R_N \in (\text{ALGBD})$. But then $T + K \in \overline{(\text{ALGBD})} = \overline{(\text{ALGQD})}$.

That is, $\overline{(\text{ALGQD})} + \mathcal{K}(\mathcal{H}) \subseteq \overline{(\text{ALGQD})}$, completing the proof. ∎

3. LIMITS OF BLOCK-DIAGONAL NILPOTENTS

3.1. As mentioned in the introduction, Halmos' seventh problem from the paper [10] asks for a characterization of the norm-closure in $\mathcal{B}(\mathcal{H})$ of the set

$$(\text{NIL}) = \{T \in \mathcal{B}(\mathcal{H}) : \text{there exists } k \geqslant 1 \text{ for which } T^k = 0\}.$$

After a great deal of work on this problem by a great many authors over a period of roughly five years, the problem was finally solved by Apostol, Foiaş, and Voiculescu [2]. One of the key steps along the way to a complete solution was the characterization of those normal operators which belong to $\overline{(\text{NIL})}$, obtained by Herrero [11]. (We remind the reader that $\pi(T) \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ refers to the image of $T \in \mathcal{B}(\mathcal{H})$ in the Calkin algebra.)

THEOREM 3.1 (Herrero). *A normal operator $N \in \mathcal{B}(\mathcal{H})$ lies in $\overline{(\text{NIL})}$ if and only if the spectrum of N is connected and contains $\{0\}$.*

THEOREM 3.2 (Apostol–Foiaş–Voiculescu). *The operator $T \in \mathcal{B}(\mathcal{H})$ lies in $\overline{(\text{NIL})}$ if and only if T satisfies the following three conditions:*

(i) *$\sigma(T)$ is connected and $0 \in \sigma(T)$;*

(ii) *$\sigma(\pi(T))$ is connected and $0 \in \sigma(\pi(T))$; and*

(iii) $T \in (\text{BQT})$.

It is worth noting that the necessity of these three conditions is relatively straightforward: the necessity of conditions (i) and (ii) follows from the upper-semicontinuity of the spectrum combined with the fact that the set of invertible elements of a Banach algebra form an open set. The necessity of condition (iii) is a consequence of Voiculescu's result [22] that $\overline{(\text{ALG})} = (\text{BQT})$.

3.2. In his thesis, Williams [25] posed the question of characterizing the norm-closure of the set (BDN), where $(\text{BDN}) = (\text{BD}) \cap (\text{NIL})$. In particular, he asked whether or not $\overline{(\text{BDN})} = \overline{(\text{BD})} \cap \overline{(\text{NIL})} = (\text{QD}) \cap \overline{(\text{NIL})}$. That this is in fact *not* the case was shown by Herrero [16]. Nevertheless, Herrero established that a normal operator N (which always lies in (QD)) belongs to $\overline{(\text{NIL})}$ if and only if $N \in \overline{(\text{BDN})}$ [14]. Thus a normal operator N belongs to $\overline{(\text{BDN})}$ if and only if $\sigma(N)$ is connected and $0 \in \sigma(N)$.

Conjecture 1 from Herrero's paper [15] states that if $T \in (\text{QD})$ and $N \in \overline{(\text{BDN})}$ is a normal operator satisfying $\sigma(T) \subseteq \sigma(N)$, then $N \oplus T \in \overline{(\text{BDN})}$. Below we shall verify this conjecture in the case where $\sigma(\pi(T))$ does not disconnect the plane. Recall that two operators $A, B \in \mathcal{B}(\mathcal{H})$ are said to be *approximately unitarily equivalent*, and we write $A \simeq_a B$, if there exists a sequence $(U_n)_{n=1}^\infty$ of unitary operators such that $A = \lim_{n\to\infty} U_n^* B U_n$. It is known (see, for example [7]) that two normal operators N and M with connected spectra are approximately unitarily equivalent if and only if $\sigma(N) = \sigma(M)$. (More is true, but this suffices for our current purposes.)

PROPOSITION 3.3. *Let $T \in$ (ALGBD), and suppose that $N \in \overline{\text{(BDN)}}$ is normal. If $\sigma(T) \subseteq \sigma(N)$, then $N \oplus T \in \overline{\text{(BDN)}}$.*

Proof. First observe that if $\sigma(N)$ is a singleton set, then $\sigma(N) = \{0\}$ (since we know that it must be a connected set containing the origin), which in turn forces $\sigma(T) = \{0\}$. The result then follows from Theorem 5.5 of [16]. We suppose, therefore, that $\sigma(N)$ contains at least two (and therefore infinitely many) points, by virtue of its being connected.

We write $T = \bigoplus_n T_n$ relative to the decomposition $\mathcal{H} = \bigoplus_n \mathcal{H}_n$, where dim $\mathcal{H}_n < \infty$ for all $n \geqslant 1$. Let $p(z) = \prod_{s=1}^{r} (z - \beta_s)^{k_s}$ denote the minimal polynomial of T, written in standard form. Let $d = \deg(p(z))$ and $\kappa = \kappa(p(z))$ denote the maximum multiplicity of any root. Let $0 < \varepsilon < \frac{\delta}{3}$, and choose $A_\varepsilon = \{\alpha_s(t) : 1 \leqslant t \leqslant k_s, 1 \leqslant s \leqslant r\} \subseteq \sigma(N)$ so that

(i) $|\alpha_s(t) - \beta_s| < \varepsilon, 1 \leqslant t \leqslant k_s, 1 \leqslant s \leqslant r$; and
(ii) $\text{DISP}(A_\varepsilon) \geqslant \frac{\varepsilon}{2\kappa}$.

Note that $p(T) = 0$ implies that $p(T_n) = 0$ for all $n \geqslant 1$. Hence, by Proposition 2.7, for each $n \geqslant 1$, there exists a diagonal normal matrix D_n with $\sigma(D_n) \subseteq A_\varepsilon \subseteq \sigma(N)$, and an invertible matrix $S_n \in \mathcal{B}(\mathcal{H}_n)$ satisfying

$$\max(\|S_n\|, \|S_n^{-1}\|) \leqslant f\Big(\|T\| + 1, \frac{\varepsilon}{2\kappa}, d\Big)$$

so that

$$\|T_n - S_n^{-1} D_n S_n\| < \varepsilon.$$

Observe that since each D_n is normal with $\sigma(D_n) \subseteq \sigma(N)$, and since $\sigma(N)$ connected implies that $\sigma(N) = \sigma(\pi(N))$, we have that $N \simeq_a \bigoplus_n (N \oplus D_n)$, as argued above. Fix U_n unitary so that $\|(N \oplus D_n) - U_n^* N U_n\| < \varepsilon$. Also, choose $B \in$ (BDN) so that $\|N - B\| < \varepsilon$, and observe that $\|(N \oplus D_n) - U_n^* B U_n\| < 2\varepsilon$. Furthermore, if $B^q = 0$, then $(U_n^* B U_n)^q = 0$, and of course $U_n^* B U_n \in$ (BD).

Let $R_n := (I \oplus S_n)^{-1} U_n^* B U_n (I \oplus S_n)$, so that $R_n \in$ (QD) and $R_n^q = 0$ for all $n \geqslant 1$. By [15], we can find $X_n \in$ (BDN), so that $X_n^q = 0$ and $\|R_n - X_n\| < \varepsilon$, $n \geqslant 1$. Now, setting $W_n = I \oplus S_n$, $n \geqslant 1$, we find that

$$\begin{aligned}\|X_n - T_n\| &\leqslant \|X_n - R_n\| + \|R_n - (N \oplus T_n)\| \\ &\leqslant \varepsilon + \|R_n - W_n^{-1}(N \oplus D_n)W_n\| + \|W_n^{-1}(N \oplus D_n)W_n - N \oplus T_n\|\end{aligned}$$

$$< \varepsilon + \|W_n^{-1}\| \, \|W_n\| \, \|(N \oplus D_n) - U_n^* B U_n\| + \|0 \oplus (S_n^{-1} D_n S_n - T_n)\|$$
$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$

Let $X = \bigoplus_n X_n \in$ (BD). Then $X^q = 0$ and $\left\| X - \bigoplus_n (N \oplus T_n) \right\| \leqslant 3\varepsilon$.

But $\varepsilon > 0$ can be made arbitrarily small, so $N \oplus T \simeq_a \bigoplus_n (N \oplus T_n) \in \overline{\text{(BDN)}}$. ∎

COROLLARY 3.4. *Let $T \in \overline{\text{(ALGQD)}}$ and suppose that $N \in \overline{\text{(BDN)}}$ is normal with $\sigma(T) \subseteq \sigma(N)$. Then*

$$N \oplus T \in \overline{\text{(BDN)}}.$$

Proof. Let $\varepsilon > 0$. By the upper semicontinuity of the spectrum, there exists $0 < \delta < \varepsilon$ so that $\|X - T\| < \delta$ implies that $\sigma(X) \subseteq (\sigma(T))_\varepsilon = \{z \in \mathbb{C} :$ there exists $w \in \sigma(T)$ such that $|z - w| < \varepsilon\}$.

For this choice of δ, choose $R \in$ (ALGBD) so that

$$\|R - T\| < \delta.$$

Since R is algebraic, $\sigma(R) = \{\beta_1, \beta_2, \ldots, \beta_r\}$ is a finite set and $\text{dist}(\beta_s, \sigma(N)) < \varepsilon$, $1 \leqslant s \leqslant r$. Since $\sigma(N)$ is connected, we can easily find distinct elements $\{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subseteq \sigma(N)$ such that $|\beta_s - \alpha_s| < \varepsilon$, $1 \leqslant s \leqslant r$.

Let $p(z) = \prod_{s=1}^{r} (z - \beta_s)^{k_s}$ denote the minimal polynomial of R.

Write $R = \bigoplus_n R_n$ with R_n in upper triangular form, and for each $n \geqslant 1$, let $B_n \in \mathcal{B}(\mathcal{H}_n)$ be the matrix obtained from R_n by changing each diagonal occurrence of β_s to α_s, $1 \leqslant s \leqslant r$. If $q(z) = \prod_{s=1}^{r} (z - \alpha_s)^{k_s}$, then it is not hard to see that each $q(B_n) = 0$, whence $q(B) = 0$. Clearly $B = \bigoplus_n B_n \in$ (BD), and

$$\|T - B\| \leqslant \|T - R\| + \|R - B\| < \delta + \sup_{n \geqslant 1} \|R_n - B_n\|$$
$$\leqslant \delta + \max\{|\beta_s - \alpha_s| : 1 \leqslant s \leqslant r\} \leqslant \varepsilon + \varepsilon = 2\varepsilon.$$

By Proposition 3.3, $N \oplus B \in \overline{\text{(BDN)}}$, and so $\text{dist}(N \oplus T, \overline{\text{(BDN)}}) \leqslant 2\varepsilon$. But $\varepsilon > 0$ can be made arbitrarily small, so $N \oplus T \in \overline{\text{(BDN)}}$. ∎

Given a subset $\Omega \subseteq \mathbb{C}$, we denote by

$$\widehat{\Omega} = \left\{ z \in \mathbb{C} : |p(z)| \leqslant \max_{\xi \in \Omega} |p(\xi)| \text{ for all polynomials } p \right\}$$

the polynomially convex hull of L (see [6] for more details).

COROLLARY 3.5. *Let $T \in$ (QD), $N \in \overline{\text{(BDN)}}$ be a normal operator, and suppose that $\sigma(T) \cup \widehat{\sigma(\pi(T))} \subseteq \sigma(N)$. Then $N \oplus T \in \overline{\text{(BDN)}}$.*

Proof. Observe that $N \in \overline{(\text{BDN})}$ implies that $\sigma(N)$ is connected, and thus $\sigma(N) = \sigma(\pi(N))$. By Corollary 2.6 of [8], $N \oplus T \in \overline{(\text{ALGQD})}$. Also, the fact that $\sigma(T) \subseteq \sigma(N)$ by hypothesis implies that $\sigma(N \oplus T) \subseteq \sigma(N)$, and so by Corollary 3.4,

$$N \oplus (N \oplus T) \in \overline{(\text{BDN})}.$$

But $\sigma(N) = \sigma(\pi(N))$ implies that $N \simeq_{\text{a}} N \oplus N$ by the Weyl–von Neumann–Berg theorem (see, for example Theorem II.4.4 of [7]) whence

$$N \oplus T \simeq_{\text{a}} N \oplus (N \oplus T) \in \overline{(\text{BDN})}. \quad \blacksquare$$

COROLLARY 3.6. *Let* $T \in \mathcal{B}(\mathcal{H})$, $\operatorname{spr}(T) \leqslant 1$ *and* $N \in \mathcal{B}(\mathcal{H})$ *be a normal operator with* $\sigma(N) = \overline{\mathbb{D}}$. *If* $N \oplus T \in (\text{QD})$, *then* $N \oplus T \in \overline{(\text{BDN})}$.

Proof. The hypothesis that $N \oplus T \in (\text{QD})$, combined with the fact that $\sigma(N \oplus T) \subseteq \overline{\mathbb{D}} = \sigma(N)$ implies by Corollary 3.5 that $N \oplus (N \oplus T) \in \overline{(\text{BDN})}$. But $N \oplus T \simeq_{\text{a}} N \oplus (N \oplus T)$, from which the result follows. $\blacksquare$

REMARK 3.7. It is an interesting question to determine for which $T \in \mathcal{B}(\mathcal{H})$, $\|T\| \leqslant 1$ we have that $N \oplus T \in (\text{QD})$, where N is the normal operator from Corollary 3.6 above. In the case where T is a weighted shift operator, it was shown in [18] that this happens if and only if there exists a compact perturbation of T which is a direct sum of an essentially normal contraction and a block-diagonal operator. We conjecture that this characterization holds more generally.

EXAMPLE 3.8. Let $W \in \mathcal{B}(\mathcal{H})$ be a bilateral weighted shift operator with weight sequence

$$\Big\{ \dots, \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, \dots \Big\}.$$

Then W is periodic with period of length two. By a theorem of Smucker [20], $W \in (\text{QD})$. Furthermore, $\|W\| = 1$, and so $\operatorname{spr}(W) \leqslant 1$. If $N \in \mathcal{B}(\mathcal{H})$ is normal with $\sigma(N) = \overline{\mathbb{D}}$, then by Corollary 3.6 above, $N \oplus W \in \overline{(\text{BDN})}$.

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