

LIST-3-COLORING ORDERED GRAPHS WITH A FORBIDDEN INDUCED SUBGRAPH

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ABSTRACT. The LIST-3-COLORING PROBLEM is to decide, given a graph G and a list $L(v) \subseteq \{1, 2, 3\}$ of colors assigned to each vertex v of G , whether G admits a proper coloring ϕ with $\phi(v) \in L(v)$ for every vertex v of G , and the 3-COLORING PROBLEM is the LIST-3-COLORING PROBLEM on instances with $L(v) = \{1, 2, 3\}$ for every vertex v of G . The LIST-3-COLORING PROBLEM is a classical NP-complete problem, and it is well-known that while restricted to H -free graphs (meaning graphs with no induced subgraph isomorphic to a fixed graph H), it remains NP-complete unless H is isomorphic to an induced subgraph of a path. However, the current state of art is far from proving this to be sufficient for a polynomial time algorithm; in fact, the complexity of the 3-COLORING PROBLEM on P_8 -free graphs (where P_8 denotes the eight-vertex path) is unknown. Here we consider a variant of the LIST-3-COLORING PROBLEM called the ORDERED GRAPH LIST-3-COLORING PROBLEM, where the input is an *ordered graph*, that is, a graph along with a linear order on its vertex set. For ordered graphs G and H , we say G is H -free if H is not isomorphic to an induced subgraph of G with the isomorphism preserving the linear order. We prove, assuming H to be an ordered graph, a nearly complete dichotomy for the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to H -free ordered graphs. In particular, we show that the problem can be solved in polynomial time if H has at most one edge, and remains NP-complete if H has at least three edges. Moreover, in the case where H has exactly two edges, we give a complete dichotomy when the two edges of H share an end, and prove several NP-completeness results when the two edges of H do not share an end, narrowing the open cases down to three very special types of two-edge ordered graphs.

1. INTRODUCTION

Graphs in this paper are finite and simple. We denote the set of positive integers by \mathbb{N} , and for every integer $k \in \mathbb{N}$, we denote by $[k]$ the set of all positive integers which are smaller than or equal to k . Let $G = (V(G), E(G))$ be a graph. For every X which is either a vertex or a subset of vertices of G , we write $G \setminus X$ for the graph obtained from G by removing X . Also, we denote by $G[X]$ the *subgraph of G induced by X* , that is, $G \setminus (V(G) \setminus X) = (X, \{e \in E(G) : e \subseteq X\})$. For graphs G and H , we say H is an *induced subgraph* of G if H is isomorphic to $G[X]$ for some $X \subseteq V(G)$, and otherwise we say G is H -free. For all $t \in \mathbb{N}$, we use P_t to denote the path on t vertices.

Let G be a graph and $k \in \mathbb{N}$. By a k -coloring of G , we mean a function $\phi : V(G) \rightarrow [k]$. A coloring ϕ of G is said to be *proper* if $\phi(u) \neq \phi(v)$ for every edge $uv \in E(G)$. In other words, ϕ is proper if and only if for every $i \in [k]$, $\phi^{-1}(i)$ is a stable set in G . We say G is k -colorable if G has a proper k -coloring. For fixed $k \in \mathbb{N}$, the k -COLORING PROBLEM asks, given graph G , whether G is k -colorable.

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A k -list-assignment of G is a map $L : V(G) \rightarrow 2^{[k]}$. For $v \in V(G)$, we refer to $L(v)$ as the *list of v* . Also, for every $i \in [k]$, we define $L^{(i)} = \{v \in V(G) : i \in L(v)\}$. An L -coloring of G is a proper k -coloring ϕ of G with $\phi(v) \in L(v)$ for all $v \in V(G)$. We say G is L -colorable if it admits an L -coloring. For example, if $L(v) = \emptyset$ for some $v \in V(G)$, then G admits no L -coloring. Also, if $V(G) = \emptyset$, then G vacuously admits an L -coloring for every k -list-assignment L . For fixed $k \in \mathbb{N}$, the LIST- k -COLORING PROBLEM is to decide, given an instance (G, L) consisting of a graph G and a k -list-assignment L of G , whether G is L -colorable. Note that the k -coloring problem is in fact the LIST- k -COLORING PROBLEM restricted to instances (G, L) where $L(v) = [k]$ for every $v \in V(G)$.

The LIST- k -COLORING PROBLEM is famously known to be NP-complete for all $k \geq 3$ [19], and understanding the complexity of this problem while restricted to graphs with a fixed forbidden induced subgraph, that is, H -free graphs for some fixed graph H , is of enormous interest. The following two results, combined, show that the NP-hardness persists unless H is an induced subgraph of a path.

Theorem 1 (Kamiński and Lozin [18], see also [10]). *For all $k \geq 3$, the k -COLORING problem restricted to H -free graphs is NP-complete if H contains a cycle.*

Theorem 2 (Holyer [17], Leven and Galil [20]). *For all $k \geq 3$, the k -COLORING PROBLEM restricted to H -free graphs is NP-complete if H contains a claw (a vertex with three pairwise nonadjacent neighbors).*

Of course, for each $k \geq 3$, it is most desirable to completely distinguish graphs H for which the LIST- k -COLORING PROBLEM on H -free graphs is polynomial-time solvable from those for which the problem remains NP-complete. Despite several attempts [3, 5, 7, 16], no such value of k had been known until very recently, when the authors of the present paper settled the case $k = 5$:

Theorem 3 (Hajebi, Li, Spirkl [15]). *Assuming $P \neq NP$, the LIST-5-COLORING PROBLEM restricted to H -free graphs can be solved in polynomial time if and only if H is an induced subgraph of a graph H' of one of the following two types.*

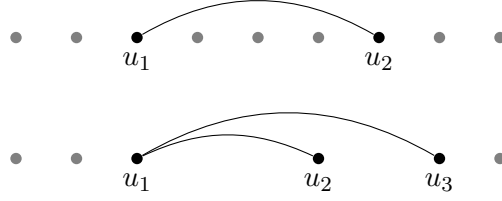
- Each component of H' is isomorphic to P_3 .
- One component of H' is isomorphic to P_5 and all other components of H are isomorphic to P_1 .

To the best of our knowledge (and curiously enough), $k = 5$ is currently the only value of k for which a complete dichotomy has been found. Among open cases, $k = 3$ has attracted a great deal of attention, thanks to the special place of the 3-COLORING PROBLEM in complexity theory as a fundamental NP-complete problem. The following theorem summarizes how successful attempts in this direction have been so far.

Theorem 4. *The LIST-3-COLORING PROBLEM restricted to H -free graphs can be solved in polynomial time if H is an induced subgraph of a graph H' of one of the following two types.*

- H' is isomorphic to P_7 (Bonomo, Chudnovsky, Maceli, Schaudt, Stein and Zhong [3]).
- One component of H' is isomorphic to P_6 and all other components of H' are isomorphic to P_3 (Chudnovsky, Huang, Spirkl and Zhong [5]).

In particular, determining the complexity of the 3-COLORING PROBLEM on P_t -free graphs (for each fixed $t \geq 8$) is a central open problem of both a structural and an algorithmic flavour. On a broader scale, graphs excluding a fixed path as an induced subgraph seem to have played a significant role in the development of modern structural graph theory. Examples include the theory of χ -boundedness (see [25]) and the great body of work built around the notoriously difficult Erdős-Hajnal conjecture (see, for instance, [6],) the simplest open case of which concerns P_5 -free graphs. A common approach to problems in these areas [1, 23, 26], as well as several

FIGURE 1. The cases of H which are polynomial-time solvable.

other problems in structural and extremal and graph theory [2, 21, 22, 24], has been to study their variants on graphs with additional specifications, such as an orientation of edges or an ordering of vertices. The goal of this paper is to look at list-3-coloring H -free graphs from the same perspective.

Let us provide formal definitions. An *ordered graph* G is a triple (V, E, φ) such that (V, E) is a graph with vertex set V and edge set E , and $\varphi : V \rightarrow \mathbb{R}$ is an injective function. We say φ is the *ordering* of G . For an ordered graph $G = (V, E, \varphi)$, we define $V(G) = V$, $E(G) = E$, and $\varphi_G = \varphi$. For convenience, we sometimes write the ordered graph (V, E, φ) as (G, φ) where $G = (V, E)$ is a graph. For $X \subseteq V$, we define $G[X] = (X, \{e \in E : e \subseteq X\}, \varphi|_X)$, and $G \setminus X = G[V(G) \setminus X]$. We also define $-G = (V, E, v \mapsto -\varphi_G(v))$.

Given an ordered graph $G = (V, E, \varphi)$, an ordered graph $G' = (V', E', \varphi')$ is *isomorphic* to G if there exists a bijective function $f : V' \rightarrow V$ such that for any two vertices v and w in V' , $\varphi'(v) < \varphi'(w)$ if and only if $\varphi(f(v)) < \varphi(f(w))$, and $vw \in E'$ if and only if $f(v)f(w) \in E$. We denote G and G' being isomorphic by $G' \cong G$. An ordered graph H is an *induced subgraph* of G if there exists a set $X \subseteq V(G)$ such that $H \cong G[X]$; otherwise G is *H -free*.

Let $G = (V, E, \varphi)$ be an ordered graph and let $k \in \mathbb{N}$. By a *k -coloring* and a *k -list-assignment* of G , we mean a k -coloring and a k -list-assignment of the graph (V, E) . Also, for a k -list-assignment L of G , an *L -coloring* of G is an L -coloring of (V, E) . We say G is *L -colorable* if (V, E) is L -colorable. For a fixed $k \in \mathbb{N}$, the ORDERED GRAPH LIST-3-COLORING PROBLEM is to decide, given an instance (G, L) consisting of an ordered graph G and a k -list-assignment L of G , whether G admits an L -coloring.

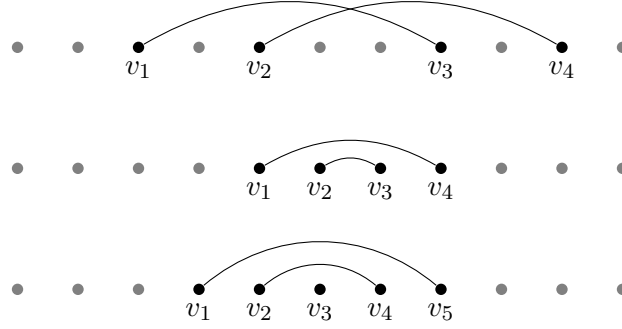
Let us emphasize that, as opposed to our notion of “induced subgraph,” our notion of “coloring” for ordered graphs ignores the ordering of vertices. In particular, the ORDERED GRAPH LIST-3-COLORING PROBLEM is NP-complete, as so is the usual LIST-3-COLORING PROBLEM. We establish a nearly complete dichotomy for the complexity of the ORDERED GRAPH LIST-3-COLORING PROBLEM on H -free ordered graphs, where H is a fixed ordered graph. Our first two theorems, below, give a complete dichotomy for the case $|E(H)| \neq 2$.

Theorem 5. *The ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to H -free ordered graphs is polynomial-time solvable if H has at most one edge.*

Theorem 6. *The ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to H -free ordered graphs is NP-complete if H has at least three edges.*

In order to state our results concerning the case $|E(H)| = 2$, we need to define some special ordered graphs, each with exactly two edges. The strange naming is due to the remaining ordered graphs that are to be defined in Section 5. Let $U' = \{u_1, u_2, u_3, u_4, u_5\}$ and $U = U' \setminus \{u_5\}$, the ordering $\varphi' : U' \rightarrow \mathbb{R}$ with $u_i \mapsto i$ for $i \in [5]$. Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $\varphi : V \rightarrow \mathbb{R}$ with $v_i \mapsto i$ for $i \in [6]$.

- Let $J_9 = (U, \{u_1u_2, u_3u_4\}, \varphi'|_U)$.
- Let $J_{16} = (U \setminus \{u_4\}, \{u_1u_2, u_1u_3\}, \varphi'|_{U \setminus \{u_4\}})$.
- Let $M_1 = (V, \{v_1v_6, v_2v_5\}, \varphi)$.
- Let $M_5 = (V \setminus \{v_6\}, \{v_1v_5, v_2v_3\}, \varphi|_{V \setminus \{v_6\}})$.

FIGURE 2. The cases of H which are still open.

- Let $M_6 = (V \setminus \{v_5, v_6\}, \{v_1v_3, v_2v_4\}, \varphi|_{V \setminus \{v_5, v_6\}})$.
- Let $M_7 = (V \setminus \{v_5, v_6\}, \{v_1v_4, v_2v_3\}, \varphi|_{V \setminus \{v_5, v_6\}})$.
- Let $M_8 = (V \setminus \{v_6\}, \{v_1v_5, v_2v_4\}, \varphi|_{V \setminus \{v_6\}})$.

Also, given an ordered graph $G = (V, E, \varphi)$ and $k, l \in \mathbb{N} \cup \{0\}$, we define $G(k, l)$ as the ordered graph obtained from G by adding $k + l$ isolated vertices $\{a_i, b_j : i \in [k], j \in [l]\}$ to G and extending φ from V to $V \cup \{a_i, b_j : i \in [k], j \in [l]\}$ by defining $\varphi(a_i) = \min_{v \in V} \varphi(v) - (k + 1 - i)$ for each $i \in [k]$ and $\varphi(b_i) = \max_{v \in V} \varphi(v) + i$ for each $i \in [l]$. The following theorem is a full dichotomy for the case where the two edges of H share an end.

Theorem 7. *Let H be an ordered graph with $|E(H)| = 2$ where the two edges of H share an end. Then the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to H -free ordered graphs is polynomial-time solvable if H is isomorphic to $J_{16}(k, l)$ or $-J_{16}(k, l)$ for some $k, l \in \mathbb{N} \cup \{0\}$, and NP-complete otherwise.*

Figure 1 depicts ordered graphs H for which we prove, in Theorems 5 and 7, that the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to H -free ordered graphs can be solved in polynomial time (gray vertices represent arbitrarily many isolated vertices.)

Let us now consider ordered graphs H with $|E(H)| = 2$ where the two edges of H do not share an end. In this case, we prove the following hardness result.

Theorem 8. *Let H be an ordered graph containing an induced subgraph isomorphic to J_9 , M_1 , M_5 or $-M_5$. Then the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to H -free ordered graphs is NP-complete.*

Note that Theorems 5, 6, 7 and 8, together, determine the complexity of the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to H -free ordered graphs for all H , except for the following three cases, which remain open (see Figure 2 for a depiction, where gray vertices represent arbitrarily many isolated vertices.)

- H has exactly two edges, and the ordered graph obtained from H by removing isolated vertices is isomorphic to M_6 .
- H is isomorphic to $M_7(k, l)$ for some $k, l \in \mathbb{N} \cup \{0\}$.
- H is isomorphic to $M_8(k, l)$ for some $k, l \in \mathbb{N} \cup \{0\}$.

The organization of this paper is as follows. In Section 2, we set some notation and terminology to be used throughout. In section 3, we prove Theorem 5. Section 4 contains the polynomial-time algorithm promised in Theorem 7. Finally, Section 5 is devoted to our hardness results that is, Theorems 6 and 8, and the NP-completeness assertion in Theorem 7.

2. PRELIMINARIES

Here we introduce some notation and terminology. Let G be a (ordered) graph. By a *clique* in G we mean a set of pairwise adjacent vertices, and a *stable set* in G is a set of pairwise nonadjacent vertices. We say two disjoint sets $U, W \subseteq V(G)$ are *anticomplete* if no vertex in U has a neighbor in W . For $x, y \in \mathbb{R} \cup \{-\infty, \infty\}$ with $x < y$, we define $G(x : y) = \{v \in V : x < \varphi_G(v) \leq y\}$, and $G[x : y], G[x : y]$ and $G(x : y)$ are defined similarly. For every $\rho \in \mathbb{N}$ and every vertex $v \in V(G)$, we denote by $N_G^\rho(v)$ the set of all vertices in G at distance ρ from v , and by $N_G^\rho[v]$ the set of all vertices in G at distance at most ρ from v . In particular, we write $N_G(v)$ for $N_G^1(v)$, which is the set of neighbors of v in G , and $N_G[v]$ for $N_G^1[v] = N_G(v) \cup \{v\}$. Also, the set of *forward neighbors* of v is defined as $N_G^+(v) = \{u \in N_G(v) : \varphi(v) < \varphi(u)\}$, and the set of *backward neighbors* of v is $N_G^-(v) = \{u \in N_G(v) : \varphi(v) > \varphi(u)\}$. Moreover, for every $X \subseteq V(G)$, we define $N_G^\rho[X] = \bigcup_{x \in X} N_G^\rho[x]$ and $N_G^\rho(X) = N_G^\rho[X] \setminus X$. Again, we write $N_G(X)$ for $N_G^1(X)$ and $N_G[X]$ for $N_G^1[X]$. Throughout, we sometimes omit the subscript G from the latter notations if there is no ambiguity.

Let (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM. An instance (G', L') is a (G, L) -refinement if G' is an induced subgraph of G and for all $v \in V(G')$, $L'(v) \subseteq L(v)$. A (G, L) -refinement (G', L') is *spanning* if $G' = G$. A (G, L) -profile \mathcal{L} is a set of (G, L) -refinements. A (G, L) -profile is *spanning* if all its elements are spanning. Two list assignments L and L' are *equivalent* for G if for every coloring c of G , c is an L -coloring if and only if it is an L' -coloring.

3. EXCLUDING AN ORDERED GRAPH WITH AT MOST ONE EDGE

The goal of this section is to prove Theorem 5. For every $w \in \mathbb{N}$, let J_w denote the ordered graph with $V(J_w) = \{v_1, \dots, v_{3w+2}\}$, $E(J_w) = \{v_{w+1}v_{2w+2}\}$ and $\varphi_{J_w}(v_i) = i$ for all $i \in [3w+2]$. Note that every ordered graph with at most one edge is isomorphic to an induced subgraph of J_w for some $w \in \mathbb{N}$. Therefore, in order to prove Theorem 5, it suffices to prove the following.

Theorem 9. *For every fixed $w \in \mathbb{N}$, the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to J_w -free ordered graphs can be solved in polynomial time.*

The proof of Theorem 9 is involved, but roughly speaking, the algorithm we give is based on dynamic programming on a decomposition of the input ordered graph using its “maximal” edges. A concrete presentation of the algorithm, though, consists of several steps. To begin with, let $w \in \mathbb{N}$, (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM, and $e = uv \in E(G)$. We set the notations $\text{Under}_G(e) = G[\min\{\varphi_G(u), \varphi_G(v)\} : \max\{\varphi_G(u), \varphi_G(v)\}]$ and $\text{Left}_G(e) = G(-\infty : \min\{\varphi_G(u), \varphi_G(v)\})$. Also, we define $\Gamma_w(G, L, e)$ to be the set of all pairs (S, σ) where S is a subset of $\text{Under}_G(e)$ containing both u and v , $\sigma : S \rightarrow [3]$ is an $L|_S$ -coloring of $G[S]$, and for each $i \in [3]$, we have $|\sigma^{-1}(i)| \leq 27w^2 + 3$. From this definition, one may easily deduce the following.

Lemma 10. *Given $w \in \mathbb{N}$, an instance (G, L) of the ORDERED GRAPH LIST-3-COLORING PROBLEM and $e \in E(G)$, we have $|\Gamma_w(G, L, e)| \leq \mathcal{O}(|V(G)|^{81w^2+7})$ and one can compute $\Gamma_w(G, L, e)$ in time $\mathcal{O}(|V(G)|^{81w^2+7})$.*

For an instance (G, L) of the ORDERED GRAPH LIST-3-COLORING PROBLEM and $X, Y \subseteq V(G)$, an $L|_X$ -coloring ϕ_1 of $G[X]$ and an $L|_Y$ -coloring ϕ_2 of $G[Y]$ are said to be *compatible* if $\phi_1|_{X \cap Y} = \phi_2|_{X \cap Y}$. Let $e = uv \in E(G)$ and $g = (S, \sigma) \in \Gamma_w(G, L, e)$. Let $X \subseteq V(G)$ and $\phi : X \rightarrow [3]$ be an $L|_X$ -coloring of $G[X]$. We say (ϕ, g) has *property X* if

- (X1) ϕ and σ are compatible, and;
- (X2) $\sigma \cup \phi$ is an $L|_{S \cup X}$ -coloring of $G[S \cup X]$.

Also, we say (ϕ, g) has property Y if

- (Y1) ϕ and σ are compatible, and;
 (Y2) for each $i \in [3]$, every $x \in \text{Left}_G(e)$ with a neighbor in $\phi^{-1}(i) \cap \text{Under}_G(e)$ has a neighbor in $\sigma^{-1}(i)$.

The proof of the following lemma is straightforward and we omit it.

Lemma 11. *Let (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM, $e = uv \in E(G)$ and $g = (S, \sigma) \in \Gamma_w(G, L, e)$. Let $X \subseteq V(G)$ and $\phi : X \rightarrow [3]$ be an $L|_X$ -coloring of $G[X]$. Then for every $X' \subseteq X$, if (ϕ, g) has property X (resp. Y), then $(\phi|_{X'}, g)$ has property X (resp. Y), as well.*

To prove our next lemma, we need the following classical result of Erdős and Szekeres [12].

Theorem 12 (Erdős and Szekeres [12]). *For all $n \geq 0$ and every sequence a_1, \dots, a_{n+1} of reals, there exists an increasing injection $\alpha : [n+1] \rightarrow [n^2+1]$ such that either $a_{\alpha(1)} < \dots < a_{\alpha(n+1)}$ or $a_{\alpha(1)} > \dots > a_{\alpha(n+1)}$.*

Lemma 13. *Let $w \in \mathbb{N}$ and let (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM where G is J_w -free. Let ϕ be an L -coloring of G . Then for every edge $e = uv \in E(G)$, there exists $g_e \in \Gamma_w(G, L, e)$ such that (ϕ, g_e) has both properties X and Y .*

Proof. Let $i \in [3]$ be arbitrary. Let U_i be the set of all vertices in $\text{Left}_G(e)$ with at least one neighbor in $\phi^{-1}(i) \cap \text{Under}_G(e)$, and S_i be a minimal subset of $\phi^{-1}(i) \cap \text{Under}_G(e)$ such that every vertex in U_i has a neighbor in S_i . Let $|S_i| = k_i$ and $S_i = \{s_1^i, \dots, s_{k_i}^i\}$ where $\varphi_G(s_1^i) < \dots < \varphi_G(s_{k_i}^i)$. It follows from the minimality of S_i that there are k_i distinct vertices $t_1^i, \dots, t_{k_i}^i \in U_i$ such that for each $j \in [k_i]$, we have $N_G(t_j^i) \cap S_i = \{s_j^i\}$. We further deduce:

- (1) For each $i \in [3]$, we have $k_i \leq 27w^2 + 2$.

Suppose for a contradiction that $k_i \geq 3(9w^2 + 1)$ for some $i \in [3]$. Since ϕ is a 3-coloring of G , there exists an increasing injection from $\alpha : [9w^2 + 1] \rightarrow [k_i]$ such that $\{t_{\alpha(j)}^i : j \in [9w^2 + 1]\}$ is a stable set in G . In addition, $\{s_{\alpha(j)}^i : j \in [9w^2 + 1]\} \subseteq S_i \subseteq \phi^{-1}(i)$ is also a stable set of G . As a result, we have $E(G[\{s_{\alpha(j)}^i, t_{\alpha(j)}^i : j \in [9w^2 + 1]\}]) = \{t_{\alpha(j)}^i s_{\alpha(j)}^i : j \in [9w^2 + 1]\}$. By Theorem 12 applied to the sequence $\varphi_G(t_{\alpha(1)}^i), \dots, \varphi_G(t_{\alpha(9w^2+1)}^i)$ of integers, there exists an increasing injection $\beta : [3w + 1] \rightarrow \alpha([9w^2 + 1])$ such that either $\varphi_G(t_{\beta(1)}^i) < \dots < \varphi_G(t_{\beta(3w+1)}^i)$ or $\varphi_G(t_{\beta(1)}^i) > \dots > \varphi_G(t_{\beta(3w+1)}^i)$. Recall also that, since both α and β are increasing, it follows that $\varphi_G(s_{\beta(1)}^i) < \dots < \varphi_G(s_{\beta(3w+1)}^i)$. Now, in the former case, let

$$W = \{s_{\beta(1)}^i, \dots, s_{\beta(2w+1)}^i\} \cup \{t_{\beta(w+1)}^i\} \cup \{t_{\beta(2w+2)}^i, \dots, t_{\beta(3w+1)}^i\}.$$

Then one may readily observe that $E(G[W]) = \{s_{\beta(w+1)}^i t_{\beta(w+1)}^i\}$, and so $G[W]$ is isomorphic to J_w . Also, in the latter case, let

$$W = \{s_{\beta(1)}^i, \dots, s_{\beta(w)}^i\} \cup \{s_{\beta(2w+1)}^i, \dots, s_{\beta(3w+1)}^i\} \cup \{t_{\beta(w+1)}^i, \dots, t_{\beta(2w+1)}^i\}.$$

Then it straightforward to check that $E(G[W]) = \{s_{\beta(2w+1)}^i t_{\beta(2w+1)}^i\}$, and so $G[W]$ is isomorphic to J_w . The last two conclusions violate the assumption that G is J_w -free, hence proving (1).

Now, let $S = S_1 \cup S_2 \cup S_3 \cup \{u, v\}$, $\sigma = \phi|_S$, and $g_e = (S, \sigma)$. Note that S is a subset of $\text{Under}_G(e)$ with $u, v \in S$, $\sigma : S \rightarrow [3]$ is an $L|_S$ -coloring of $G[S]$, and by (1), for each $i \in [3]$,

we have $|\sigma^{-1}(i)| \leq |S_i| + 1 = k_i + 1 \leq 27w^2 + 3$. Therefore, we have $g_e \in \Gamma_w(G, L, e)$. Also, it follows directly from the definition of g_e that (ϕ, g_e) has property X. In particular, ϕ and σ are compatible. Moreover, for each $i \in [3]$, by the definition of U_i and S_i , every vertex in $\text{Left}_G(e)$ with a neighbor in $\phi^{-1}(i) \cap \text{Under}_G(e)$ has a neighbor in $S_i \subseteq \sigma^{-1}(i)$. Hence, (ϕ, σ) has property Y. This completes the proof of Lemma 13. \blacksquare

We continue with a few more definitions. For an instance (G, L) of the ORDERED GRAPH LIST-3-COLORING PROBLEM, an edge $e = uv \in E(G)$ is said to be *maximal* if there is no edge $u'v' \in E(G)$ with $\varphi_G(u') \leq \varphi_G(u)$ and $\varphi_G(v) \leq \varphi_G(v')$. We denote the set of all maximal edges of G by $\text{MAX}(G)$. Let $u_1v_1, u_2v_2 \in \text{MAX}(G)$ be two distinct maximal edges of G with $\varphi_G(u_1) < \varphi_G(v_1)$ and $\varphi_G(u_2) < \varphi_G(v_2)$. Then we have $u_1 \neq u_2$ and $v_1 \neq v_2$, and if $\varphi_G(u_1) < \varphi_G(u_2)$, then $\varphi_G(v_1) < \varphi_G(v_2)$. As a result, φ_G induces a natural ordering on the elements of $\text{MAX}(G)$. In accordance, for distinct $u_1v_1, u_2v_2 \in \text{MAX}(G)$, we say u_1v_1 is *before* (resp. *after*) u_2v_2 if $\varphi_G(u_1) < \varphi_G(u_2)$ (resp. $\varphi_G(u_2) < \varphi_G(u_1)$), and we say u_1v_1 is *immediately before* (resp. *immediately after*) u_2v_2 if u_1v_1 is before (resp. after) u_2v_2 and there is no $u_3v_3 \in \text{MAX}(G)$ with $\varphi_G(u_1) < \varphi_G(u_3) < \varphi_G(u_2)$ (resp. $\varphi_G(u_2) < \varphi_G(u_3) < \varphi_G(u_1)$). By the *first* maximal edge of G , we mean the one that is before every other maximal edge of G . Note that for every vertex $x \in V(G)$, either x is isolated or $x \in \text{Under}_G(e)$ for some maximal edge $e \in E(G)$. We leave the proof of the following lemma to the reader.

Lemma 14. *Given an instance (G, L) of the ORDERED GRAPH LIST-3-COLORING PROBLEM, we have $|\text{MAX}(G)| \leq |V(G)| - 1$ and one can compute $\text{MAX}(G)$ in time $\mathcal{O}(|V(G)|^4)$.*

Next we introduce the key notion of “successfulness”, which has a recursive definition. Let (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM. For an edge $e \in \text{MAX}(G)$ and $g = (S, \sigma) \in \Gamma_w(G, L, e)$, we say g is *successful* if either e is the first maximal edge of G , or, having the successful elements of $\Gamma_w(G, L, e')$ defined for the edge $e' \in \text{MAX}(G)$ immediately before e , there exists a successful pair $g' = (T, \tau) \in \Gamma_w(G, L, e')$, such that

(S) $G[\text{Under}_G(e')]$ admits an $L|_{\text{Under}_G(e')}$ -coloring ψ for which both (ψ, g) and (ψ, g') have both properties X and Y.

The following two lemmas examine how successfulness interacts with properties X and Y.

Lemma 15. *Let $w \in \mathbb{N}$ and let (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM, where G is J_w -free and $L(v) \neq \emptyset$ for all $v \in V(G)$. Let $e = uv \in \text{MAX}(G)$ and $g = (S, \sigma) \in \Gamma_w(G, L, e)$ be successful. Then $G[\text{Left}_G(e)]$ admits an $L|_{\text{Left}_G(e)}$ -coloring ϕ such that (ϕ, g) has property X.*

Proof. Suppose not. Going through the natural ordering of the elements of $\text{MAX}(G)$ induced by φ_G , let e be the first edge violating Lemma 15. Note that if e is the first maximal edge of G , then $\text{Left}_G(e) = \emptyset$, and so e vacuously satisfies Lemma 15, a contradiction. Therefore, we may choose $e' = u'v'$ distinct from e to be the maximal edge of G immediately before e . Since g is successful, there exists a successful pair $g' = (T, \tau) \in \Gamma_w(G, L, e')$ such that e, e', g and g' satisfy (S) with ψ as in (S). Also, from $g' \in \Gamma_w(G, L, e')$ being successful, and the choice of e , it follows that there exists an $L|_{\text{Left}_G(e')}$ -coloring ϕ' of $G[\text{Left}_G(e')]$ such that (ϕ', g') has property X. Note that every vertex in the set $\text{Left}_G(e)$ is either isolated or belongs to the disjoint union of $\text{Left}_G(e')$ and $\text{Under}_G(e') \setminus \text{Under}_G(e)$. So we can define the map $\phi : \text{Left}_G(e) \rightarrow [3]$ as $\phi(x) = \phi'(x)$ for all $x \in \text{Left}_G(e')$, $\phi(x) = \psi(x)$ for all $x \in \text{Under}_G(e') \setminus \text{Under}_G(e)$, and $\phi(x) \in L(x)$ is chosen arbitrarily for every isolated vertex $x \in \text{Left}_G(e)$ (the latter is possible because $L(x) \neq \emptyset$.) We deduce the following.

(2) *The map ϕ is an $L|_{\text{Left}_G(e)}$ -coloring of $G|_{\text{Left}_G(e)}$.*

Suppose not. Then since ϕ' is an $L|_{\text{Left}_G(e')}$ -coloring of $G[\text{Left}_G(e')]$ and ψ is an $L|_{\text{Under}_G(e')}$ -coloring of $G[\text{Under}_G(e')]$, there exists an edge $xy \in E(G)$ with $x \in \text{Left}_G(e')$ and $y \in \text{Under}(e')_G \setminus \text{Under}_G(e)$ such that $\phi'(x) = \phi(x) = \phi(y) = \psi(y)$. Now, by (S), (ψ, g') has property Y, which together with (Y2) for ψ and τ , implies that x has a neighbor $z \in \tau^{-1}(\psi(y))$. In other words, x has a neighbor $z \in T$ with $\tau(z) = \psi(y) = \phi'(x)$. But this shows that ϕ' and τ do not satisfy (X2), which violates our assumption that (ϕ', g') has property X. This proves (2).

(3) *The pair (ϕ, g) has property X.*

Note that from $\text{Left}_G(e) \cap S = \emptyset$, it follows readily that (ϕ, σ) satisfies (X1). So it remains to show that (X2) holds for (ϕ, σ) , that is, $\sigma \cup \phi$ is an $L|_{S \cup \text{Left}_G(e)}$ -coloring of $G[(S \cup \text{Left}_G(e))]$. Suppose not. By (2), ϕ is an $L|_{\text{Left}_G(e)}$ -coloring of $G[\text{Left}_G(e)]$ and σ is an $L|_S$ -coloring of $G[S]$. Thus, there exists an edge $xy \in E(G)$ with $x \in \text{Left}_G(e)$ and $y \in S$ such that $\phi(x) = \sigma(y)$. First, assume that $x \in \text{Left}_G(e')$, and so $\phi'(x) = \phi(x) = \sigma(y)$. Then since e' is a maximal edge of G , we have $y \in \text{Under}_G(e') \cap S$. On the other hand, by (S), (ψ, g) has property X, and so by (X1), ψ and σ are compatible. Thus, we have $\sigma(y) = \psi(y)$, and so $\phi'(x) = \psi(y)$. Now, by (S), (ψ, g') has property Y, which along with (Y2) for ψ and τ implies that x has a neighbor $z \in \tau^{-1}(\psi(y))$. In other words, x has a neighbor $z \in T$ with $\tau(z) = \psi(y) = \phi'(x)$. But this implies that ϕ' and τ do not satisfy (X2), which violates our assumption that (ϕ', g') has property X. It follows that $x \notin \text{Left}_G(e')$, that is, $x \in \text{Under}_G(e') \setminus \text{Under}_G(e)$, and so $\psi(x) = \phi(x) = \sigma(y)$. But this shows that $\sigma \cup \psi$ is not an $L|_{S \cup \text{Under}_G(e')}$ -coloring of $G[(S \cup \text{Under}_G(e'))]$, and so by (X2), (ψ, g) does not have property X, which in turn contradicts (S). This proves (3).

Finally, from (2) and (3), we conclude that $G[\text{Left}_G(e)]$ admits an $L|_{\text{Left}_G(e)}$ -coloring ϕ such that (ϕ, g) has property X, a contradiction. This completes the proof of Lemma 15. \blacksquare

Lemma 16. *Let $w \in \mathbb{N}$ and let (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM where G is J_w -free. Let ϕ be an L -coloring of G . Assume that for every $e \in \text{MAX}(G)$, there exists $g_e \in \Gamma_w(G, L, e)$ such that (ϕ, g_e) has both properties X and Y. Then for every $e \in \text{MAX}(G)$, g_e is successful.*

Proof. Suppose not. Going through the natural ordering of the elements of $\text{MAX}(G)$ induced by φ_G , let e be the first edge where $g_e = (S, \sigma)$ is not successful. So e is not the first maximal edge of G , and we may choose $e' = u'v'$ distinct from e to be the maximal edge of G immediately before e . It follows that $g_{e'} = (T, \tau)$ is successful. Since both (ϕ, g_e) and $(\phi, g_{e'})$ have property X, by (X1), ϕ and σ are compatible, and ϕ and τ are compatible, as well. It follows that $\sigma = \phi|_S$ and $\tau = \phi|_T$. Now, from $\sigma = \phi|_S$, Lemma 11 and the fact that $(\phi, g_{e'})$ has property X, it follows that:

(4) *The pair $(\sigma, g_{e'})$ has property X, and so $\sigma \cup \tau$ is an $L|_{S \cup T}$ -coloring of $G[(S \cup T)]$.*

Also, from (4), $\sigma = \phi|_S$ and $\tau = \phi|_T$, it follows that $\sigma \cup \tau = \phi|_{(S \cup T)}$. This, along with Lemma 11 and the fact that $(\phi, g_{e'})$ has property Y, implies that

(5) *The pair $(\sigma \cup \tau, g_{e'})$ has property Y.*

Finally, let $\psi = \phi|_{\text{Under}_G(e')}$. Then by Lemma 11 and the fact that both (ϕ, g_e) and $(\phi, g_{e'})$ have both properties X and Y, we have

(6) *ψ is an $L|_{\text{Under}_G(e')}$ -coloring of $G[\text{Under}_G(e')]$ such that both (ψ, g_e) and $(\psi, g_{e'})$ have both properties X and Y.*

But then by (4), (5) and (6), e, e', g_e and $g_{e'}$ satisfy (S), and so g_e is successful, a contradiction. This completes the proof of Lemma 16. \blacksquare

Our last two lemmas use the following definition. For an instance (G, L) of the ORDERED GRAPH LIST-3-COLORING PROBLEM, we define the instance (G^*, L^*) as follows. Let G^* be the ordered graph with $V(G^*) = V(G) \cup \{q_1, q_2\}$ for two new vertices q_1 and q_2 and $E(G^*) = E(G) \cup \{q_1q_2\}$, where $\varphi_{G^*}|_{V(G)} = \varphi_G$ and $\varphi_{G^*}(q_i) = i + \max_{v \in V(G)} \varphi_G(v)$ for each $i \in \{1, 2\}$. Also, let $L^*|_{V(G)} = L$ and $L^*(q_i) = \{i\}$ for every $i \in \{1, 2\}$. Note that if G is J_w -free, then G^* is J_{w+1} -free. Moreover, we have $\text{MAX}(G^*) = \text{MAX}(G) \cup \{q_1q_2\}$. We deduce the following from Lemmas 13, 15 and 16.

Lemma 17. *Let $w \in \mathbb{N}$ and let (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM where G is J_w -free and $L(v) \neq \emptyset$ for all $v \in V(G)$. Then G admits an L -coloring if and only if the set $\Gamma_{w+1}(G^*, L^*, q_1q_2)$ contains a successful element.*

Proof. For the “only if” implication, assume that ϕ is an L -coloring of G . Then we may extend ϕ to an L^* -coloring ϕ of G^* by defining $\phi(q_i) = i$ for each $i \in \{1, 2\}$. Now, since G^* is J_{w+1} -free, by Lemma 13 applied to (G^*, L^*) and ϕ , for every $e \in \text{MAX}(G^*)$, there exists $g_e \in \Gamma_{w+1}(G^*, L^*, e)$ such that (ϕ, g_e) has both properties X and Y. It follows from Lemma 16 applied to (G^*, L^*) , ϕ and these g_e 's, that for every $e \in \text{MAX}(G^*)$, g_e is successful, and in particular, $g_{q_1q_2} \in \Gamma_{w+1}(G^*, L^*, q_1q_2)$ is successful. For the “if” implication, let $g \in \Gamma_{w+1}(G^*, L^*, q_1q_2)$ be successful. By Lemma 15 applied to (G^*, L^*) and g , $G^*[\text{Left}_{G^*}(q_1q_2)]$ admits an $L^*|_{\text{Left}_{G^*}(q_1q_2)}$ -coloring. Also, by the definition of (G^*, L^*) , we have $G^*[\text{Left}_{G^*}(q_1q_2)] = G$ and $L^*|_{\text{Left}_{G^*}(q_1q_2)} = L$. Hence, G admits an L -coloring. This completes the proof of Lemma 17. ■

We need some preparation for proving our last lemma. The following theorem was proved in [15] for unordered graphs (which, of course, does not interfere with the correctness of the following.)

Theorem 18 (Hajebi, Li and Spirkl [15]). *Let (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM. Then there exists a (G, L) -refinement (\hat{G}, \hat{L}) with the following specifications.*

- (\hat{G}, \hat{L}) can be computed from (G, L) in time $\mathcal{O}(|V(G)|^2)$.
- $|\hat{L}(v)| \neq 1$ for all $v \in V(\hat{G})$.
- If G admits an L -coloring if and only if \hat{G} admits an \hat{L} -coloring.

The following can be proved via a reduction to the 2SAT PROBLEM, and has been discovered independently by many authors [9, 11, 27].

Theorem 19 (Edwards [9]). *Let $k \in \mathbb{N}$ be fixed and (G, L) be an instance of the LIST- k -COLORING PROBLEM with $|L(v)| \leq 2$ for every $v \in V(G)$. Then it can be decided in time $\mathcal{O}(|V(G)|^2)$ whether G admits an L -coloring.*

Our use of Theorem 19 though is through the following straightforward corollary of it. We leave the proof to the reader.

Corollary 20. *Let (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM and $c \in \mathbb{N} \cup \{0\}$. Then the following hold.*

- If $|\{x \in V(G) : |L(x)| = 3\}| \leq c$, then one can decide in time $\mathcal{O}(|V(G)|^2)$ whether G admits an L -coloring.
- One can decide in time $\mathcal{O}(|V(G)|^{c+4})$ whether G admits an L -coloring ξ with $|\xi^{-1}(i)| < c$ for some $i \in [3]$.

The following is the penultimate step toward a proof of Theorem 9.

Lemma 21. *Let $w \in \mathbb{N}$ and let (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM where G is J_w -free. Then one of the following holds.*

- G contains a clique on four vertices.

- G admits an L -coloring ξ with $|\xi^{-1}(i)| < 2w$ for some $i \in [3]$.
- There exists a (G, L) -profile $\Sigma(G, L)$ with the following specifications.
 - $|\Sigma(G, L)| \leq (|V(G)|^{6w})$ and $\Sigma(G, L)$ can be computed from (G, L) in time $\mathcal{O}(|V(G)|^{6w+2})$.
 - G admits an L -coloring if and only if for some $(G', L') \in \Sigma(G, L)$, G' admits an L' -coloring.
 - Given $(G', L') \in \Sigma(G, L)$ with $L(v) \neq \emptyset$ for all $v \in V(G')$, one can compute the set of all successful elements of $\Gamma_{w+1}(G'^*, L'^*, q_1 q_2)$ in time $\mathcal{O}(|V(G)|^{162w^2+18})$.

Proof. Assume that the first two bullets of Lemma 21 do not hold. We define \mathcal{A} to be the set of all 6-tuples $\alpha = (X_1, X_2, X_3, Y_1, Y_2, Y_3)$ where for each $i \in [3]$, X_i and Y_i are stable subsets of $L^{(i)}$ of cardinality w such that for every choice of $x \in X_i$ and $y \in Y_i$, we have $\varphi_G(x) < \varphi_G(y)$, and X_1, X_2, X_3, Y_1, Y_2 and Y_3 are pairwise disjoint. It is readily observed that:

(7) We have $|\mathcal{A}| \leq \mathcal{O}(|V(G)|^{6w})$ and \mathcal{A} can be computed from (G, L) in time $\mathcal{O}(|V(G)|^{6w})$.

Next, for every $\alpha = (X_1, X_2, X_3, Y_1, Y_2, Y_3) \in \mathcal{A}$, we define $L_\alpha : V(G) \rightarrow 2^{[3]}$ as follows. For every vertex $v \in V(G)$, if $v \in X_i \cup Y_i$ for some $i \in [3]$, then let $L_\alpha(v) = \{i\}$, and otherwise, let

$$L_\alpha(v) = \{i \in [3] : \max_{x \in X_i} \varphi_G(x) < \varphi_G(v) < \min_{y \in Y_i} \varphi_G(y)\} \setminus \{i \in [3] : N_G(v) \cap (X_i \cup Y_i) \neq \emptyset\}.$$

Then for every $\alpha \in \mathcal{A}$, (G, L_α) is a spanning (G, L) -refinement. For every $\alpha \in \mathcal{A}$, let $(\hat{G}, \hat{L}_\alpha)$ be the (G, L_α) -refinement as in Theorem 18, and $\Sigma(G, L) = \{(\hat{G}, \hat{L}_\alpha) : \alpha \in \mathcal{A}\}$. Then $\Sigma(G, L)$ is a (G, L) -profile. We deduce:

(8) $|\Sigma(G, L)| \leq \mathcal{O}(|V(G)|^{6w})$ and $\Sigma(G, L)$ can be computed from (G, L) in time $\mathcal{O}(|V(G)|^{6w+2})$.

The first assertion is immediate from (7) and the fact that $|\Sigma(G, L)| = |\mathcal{A}|$. The second assertion follows directly from the combination of (7), the fact that for all $\alpha \in \mathcal{A}$ and $v \in V(G)$, $L_\alpha(v)$ can be computed from (G, L) and α in constant time, and the first bullet of Theorem 18. This proves (8).

(9) G admits an L -coloring if and only if for some $\alpha \in \mathcal{A}$, \hat{G} admits an \hat{L}_α -coloring.

For the “if” implication, let \hat{G} admit an \hat{L}_α -coloring for some $\alpha \in \mathcal{A}$. Then by the last bullet of Theorem 18, (G, L_α) admits an L_α -coloring, and so since (G, L_α) is a spanning (G, L) -refinement, it follows that G admits an L -coloring. To see the “only if” implication, assume that G admits an L -coloring ξ . Since the second bullet of Lemma 21 does not hold, we have $|\xi^{-1}(i)| \geq 2w$ for all $i \in [3]$. Thus, with respect to the ordering of $V(G)$ inherited from φ_G , for each $i \in [3]$, we may define X_i to be the set of first w vertices in $\xi^{-1}(i)$ and Y_i to be the set of last w vertices in $\xi^{-1}(i)$, where $X_i \cap Y_i \neq \emptyset$. We conclude immediately from this definition that $\alpha = (X_1, X_2, X_3, Y_1, Y_2, Y_3) \in \mathcal{A}$. Now, for each $i \in [3]$ and all $v \in X_i \cup Y_i$, we have $\xi(v) = i \in \{i\} = L_\alpha(v)$. Also, for every $v \in V(G) \setminus \cup_{i=1}^3 (X_i \cup Y_i)$, since ξ is an L -coloring of G and by the choice of X_i, Y_i for $i \in \{1, 2, 3\}$, we have $\max_{x \in X_{\xi(v)}} \varphi_G(x) < \varphi_G(v) < \min_{y \in Y_{\xi(v)}} \varphi_G(y)$ and $N_G(v) \cap (X_{\xi(v)} \cup Y_{\xi(v)}) = \emptyset$, which in turn implies that $\xi(v) \in L_\alpha(v)$. Therefore, we have $\xi(v) \in L_\alpha(v)$ for all $v \in V(G)$, and so ξ is an L_α -coloring of G . But then by the third bullet of Theorem 18, \hat{G} admits an \hat{L}_α -coloring. This proves (9).

(10) Assume that $(\hat{G}, \hat{L}_\alpha) \in \Sigma(G, L)$ with $\hat{L}_\alpha(v) \neq \emptyset$ for all $v \in V(\hat{G})$, and $e, e' \in \text{MAX}(\hat{G}^*)$ such that e' is immediately before e . Let $g' = (T, \tau) \in \Gamma_{w+1}(\hat{G}^*, \hat{L}_\alpha^*, e')$ be successful, and $g = (S, \sigma) \in \Gamma_{w+1}(\hat{G}^*, \hat{L}_\alpha^*, e)$. Then one can decide in time $\mathcal{O}(|V(G)|^3)$ whether e, e', g and g' satisfy (S).

Let $\alpha = (X_1, X_2, X_3, Y_1, Y_2, Y_3)$. Let $e' = u'v'$, and by symmetry, assume that $\phi_G(u') \leq \phi_G(v')$. Then we have $u', v' \in T$ and $\{u_1, u_2\} \cap \{q_1, q_2\} = \emptyset$, with q_1, q_2 as in the definition of $(\hat{G}^*, \hat{L}_\alpha^*)$. Also, we have $\hat{L}_\alpha^*(u'), \hat{L}_\alpha^*(v') \neq \emptyset$, and so by the second bullet of Theorem 18, we have $|L_\alpha^*(u')|, |L_\alpha^*(v')| \geq 2$. As a result, we may pick $i_0 \in \hat{L}_\alpha^*(u') \cap \hat{L}_\alpha^*(v')$. Note that by the definition of \hat{L}_α , $\{u', v'\}$ is anticomplete to $X_{i_0} \cup Y_{i_0}$ and we have $\varphi_G(x) < \varphi_G(u') < \varphi_G(v') < \varphi_G(y)$ for every choice of $x \in X_{i_0}$ and $y \in Y_{i_0}$. For every $x \in \text{Under}_{\hat{G}^*}(e')$, we define $C_x \subseteq [3]$ as follows. If $x \in \text{Under}_{\hat{G}^*}(e') \setminus \text{Under}_{\hat{G}^*}(e)$, then let C_x be the set of all $i \in [3]$ for which x has a neighbor in $\text{Left}_{\hat{G}^*}(e')$ that is anticomplete to $\tau^{-1}(i)$, and if $x \in \text{Under}_{\hat{G}^*}(e') \cap \text{Under}_{\hat{G}^*}(e)$, then let C_x be the set of all $i \in [3]$ for which either x has a neighbor in $\text{Left}_{\hat{G}^*}(e')$ that is anticomplete to $\tau^{-1}(i)$, or x has a neighbor in $\text{Left}_{\hat{G}^*}(e)$ that is anticomplete to $\sigma^{-1}(i)$. Also, for every $x \in \text{Under}_{\hat{G}^*}(e')$, let $D_x = C_x \cup \sigma(N_G(x) \cap S) \cup \tau(N_G(x) \cap T) \subseteq [3]$. It is straightforward to show that for every $x \in \text{Under}_{\hat{G}^*}(e')$, one can compute D_x from $(\hat{G}, \hat{L}_\alpha)$ in time $\mathcal{O}(|V(G)|^2)$. Having the latter definition, we define $\tilde{L}_\alpha : \text{Under}_{\hat{G}^*}(e') \rightarrow 2^{[3]}$ as follows. For every $x \in \text{Under}_{\hat{G}^*}(e')$,

- if $x \in T \setminus S$, then let $\tilde{L}_\alpha(x) = \{\tau(x)\} \setminus D_x$,
- if $x \in (S \cap \text{Under}_{\hat{G}^*}(e')) \setminus T$, then let $\tilde{L}_\alpha(x) = \{\sigma(x)\} \setminus D_x$,
- if $x \in S \cap T$, then let $\tilde{L}_\alpha(x) = (\{\sigma(x)\} \cap \{\tau(x)\}) \setminus D_x$, and;
- if $x \in \text{Under}_{\hat{G}^*}(e') \setminus (S \cup T)$, then let $\tilde{L}_\alpha(x) = \hat{L}_\alpha(x) \setminus D_x$.

Now, one can easily show that

- (i) $(\hat{G}^*[\text{Under}_{\hat{G}^*}(e')], \tilde{L}_\alpha)$ is a $(\hat{G}^*, \hat{L}_\alpha^*)$ -refinement which can be computed from $(\hat{G}, \hat{L}_\alpha), e, e', g$ and g' in time $\mathcal{O}(|V(G)|^3)$, and;
- (ii) we have that e, e', g and g' satisfy (S) if and only if $\hat{G}^*[\text{Under}_{\hat{G}^*}(e')]$ admits an \tilde{L}_α -coloring.

On the other hand, it follows from the definition of \tilde{L}_α that $\{x \in \text{Under}_{\hat{G}^*}(e') : |\tilde{L}_\alpha(x)| = 3\}$ is disjoint from and anticomplete to $X_{i_0} \cup Y_{i_0} \cup \{u', v'\}$. Thus, if $G[\{x \in \text{Under}_{\hat{G}^*}(e') : |\tilde{L}_\alpha(x)| = 3\}]$ has a stable set I of size w , then $G[I \cup X_{i_0} \cup Y_{i_0} \cup \{u', v'\}]$ is isomorphic to J_w , which is impossible. So $G[\{x \in \text{Under}_{\hat{G}^*}(e') : |\tilde{L}_\alpha(x)| = 3\}]$ has no stable set of size w . This, together with the assumption that the first bullet of Lemma 21 does not hold, implies that $|\{x \in \text{Under}_{\hat{G}^*}(e') : |\tilde{L}_\alpha(x)| = 3\}| < R(w, 4)$, where $R(w, 4)$ is the Ramsey number. But then by first bullet of Theorem 20, one can decide in time $\mathcal{O}(|V(G)|^2)$ whether $G[\text{Under}_{\hat{G}^*}(e')]$ admits an \tilde{L}_α -coloring. Hence, by (i) and (ii), we conclude that given $(\hat{G}, \hat{L}_\alpha), e, e', g$ and g' , one can decide in time $\mathcal{O}(|V(G)|^3)$ whether e, e', g and g' satisfy (S). This proves (10).

From (10) and Lemma 10, we deduce:

(11) Assume that $(\hat{G}, \hat{L}_\alpha) \in \Sigma(G, L)$ with $\hat{L}_\alpha(v) \neq \emptyset$ for all $v \in V(\hat{G})$, and $e, e' \in \text{MAX}(\hat{G}^*)$ such that e' is immediately before e . Assume further that the set of all successful elements of $\Gamma_{w+1}(\hat{G}^*, \hat{L}_\alpha^*, e')$ is given. Then one can compute the set of all successful elements of $\Gamma_{w+1}(\hat{G}^*, \hat{L}_\alpha^*, e)$ in time $\mathcal{O}(|V(G)|^{162w^2+17})$.

As a result,

(12) Given $(\hat{G}, \hat{L}_\alpha) \in \Sigma(G, L)$ with $\hat{L}_\alpha(v) \neq \emptyset$ for all $v \in V(\hat{G})$, one can compute the set of all successful elements of $\Gamma_{w+1}(\hat{G}^*, \hat{L}_\alpha^*, q_1q_2)$ in time $\mathcal{O}(|V(G)|^{162w^2+18})$.

First, we compute $\text{MAX}(\hat{G}^*)$, which by Lemma 14 is doable in time $\mathcal{O}(|V(G)|^4)$. Then we sort the elements of $\text{MAX}(\hat{G}^*)$ according to the natural ordering induced by φ_G , which is immediately observed to be doable in time $\mathcal{O}(|V(G)|^2)$. Now, for the first maximal edge e_0 of \hat{G}^* , all elements of $\Gamma_w(\hat{G}^*, \hat{L}_\alpha^*, e_0)$ are successful. Also, by Lemma 14, we have $|\text{MAX}(\hat{G}^*)| \leq |V(G)| - 1$. Therefore, going through the ordering of the elements of $\text{MAX}(\hat{G}^*)$ starting at the maximal edge

immediately after e_0 and applying (11), we can compute the set of all successful elements of $\Gamma_w(\hat{G}^*, \hat{L}_\alpha^*, q_1 q_2)$ in time $\mathcal{O}(|V(G)|^{162w^2+18})$. This proves (12).

Finally, by (8), (9) and (12), $\Sigma(G, L)$ is a (G, L) -refinement satisfying the third bullet of Lemma 21. This completes the proof. \blacksquare

Now we are in a position to prove Theorem 9.

Proof of Theorem 9. We propose the following algorithm.

Step 1: Determine whether G has a clique on four vertices. If so, return “ G does not admit an L -coloring.” Otherwise, go to the next step.

Step 2: Determine whether G admits an L -coloring ξ with $|\xi^{-1}(i)| < 2w$ for some $i \in [3]$. If so, return “ G admits an L -coloring.” Otherwise, go to the next step.

Step 3: Compute the (G, L) -profile $\Sigma(G, L)$ described in the third bullet of Lemma 21.

Step 4: Compute the set $\Sigma^b(G, L)$ of all (G, L) -refinements $(G', L') \in \Sigma(G, L)$ with $L'(v) \neq \emptyset$ for all $v \in V(G')$. If $\Sigma^b(G, L) = \emptyset$, then return “ G does not admit an L -coloring.” Otherwise, go to the next step.

Step 5: For each $(G', L') \in \Sigma^b(G, L)$, compute the set of all successful elements of $\Gamma_{w+1}(G'^*, L'^*, q_1 q_2)$. If this set is non-empty for some $(G', L') \in \Sigma^b(G, L)$, the return “ G admits an L -coloring,” otherwise, return “ G does not admit an L -coloring.”

First we examine the correctness. If the algorithm stops at step 1 or 2, then its correctness is trivial. Also, if it stops at step 4, then the correctness follows from the second dash of the third bullet of Lemma 21. Moreover, if the algorithm stops at step 5, then its correctness is implied by Lemma 17 combined with the second dash of the third bullet of Lemma 21.

It remains to evaluate the running time of the algorithm. Step 1 runs in time $\mathcal{O}(|V(G)|^4)$. By the second bullet of Corollary 18, step 2 runs in time $\mathcal{O}(|V(G)|^{2w+4})$. By the first dash of the third bullet of Lemma 21, step 3 runs in time $\mathcal{O}(|V(G)|^{6w+2})$, and, having completed step 3, step 4 runs in time $\mathcal{O}(|V(G)|^{6w+1})$. Finally, by the third dash of the third bullet of Lemma 21, step 5 runs in time $\mathcal{O}(|V(G)|^{162w^2+18})$, and so the entire algorithm runs in time $\mathcal{O}(|V(G)|^{162w^2+18})$. This completes the proof of Theorem 9. \blacksquare

4. EXCLUDING $J_{16}(k, l)$

In this section, we present the polynomial-time algorithm claimed in Theorem 7.

Theorem 22. *For every fixed $k, l \in \mathbb{N} \cup \{0\}$, the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to $J_{16}(k, l)$ -free ordered graphs can be solved in polynomial time. Consequently, for every $H \in \{J_{16}(k, l), -J_{16}(k, l)\}$, the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to H -free ordered graphs can be solved in polynomial time.*

The main idea of the algorithm is the following. A graph G is *chordal* if in G , every cycle of length at least 4 has an edge connecting two vertices of the cycle but not in the cycle. Equivalently, every induced cycle in G is a triangle.

An ordered graph is J_{16} -free if and only if it is chordal and its ordered is a *perfect elimination ordering* of the chordal graph, that is, an ordering such that for each vertex, its forward neighbors are a clique.

Our goal will be to “guess” the first k and last l vertices in G of each color (Lemma 25). After updating lists of their neighbors, we will be able to show that in the induced subgraph of vertices whose lists still contain at least two colors, each vertex has at most two forward neighbors (but they may not be adjacent). Now, within this subgraph, we guess the colors of the first k vertices and their forward neighbors, as well as colors of the last $3l + 6$ vertices (Lemma 27). This gives us enough “padding” on both ends of the ordering that in the “middle”, where there are vertices

whose colors are not yet determined, we do not have a copy of J_{16} ; thus these vertices form a chordal graph (and with no clique of size four, for otherwise, a coloring does not exist).

There is an old known result derived from [8] that the treewidth of a chordal graph can be computed in polynomial-time. Indeed, the treewidth of a chordal graph is bounded by its clique number minus 1, and what we compute is the clique number. We also know that the LIST- k -COLORING PROBLEM restricted to graphs with bounded treewidth is polynomial-time solvable with respect to the input size and the treewidth [13]. Since graphs with a clique of size $k + 1$ do not have a list- k -coloring, it follows that:

Theorem 23. *For every fixed k , the LIST- k -COLORING PROBLEM restricted to chordal graphs is polynomial-time solvable.*

Throughout this section, let (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM.

Lemma 24. *There exists a spanning (G, L) -refinement (G, L') such that for all $uv \in E(G)$ with $|L(v)| = 1$, $L(u) \cap L(v) = \emptyset$, and L and L' are equivalent for G . Moreover, L' can be computed from L in time $\mathcal{O}(n^3)$.*

Proof. We define a sequence of lists recursively. Let $L_0 = L$. Suppose that we have defined L_i . If there is an edge $uv \in E(G)$ with $|L_i(v)| = 1$ and $L_i(u) \cap L_i(v) \neq \emptyset$, let $L_{i+1}(u) = L_i(u) \setminus L_i(v)$, and $L_{i+1}(w) = L_i(w)$ for all $w \in V(G) \setminus \{u\}$. Otherwise stop and let $L' = L_i$.

This terminates within at most $3n$ steps, as $\sum_{w \in V(G)} |L_{i+1}(w)| \leq \sum_{w \in V(G)} |L_i(w)| - 1$, and $\sum_{w \in V(G)} |L_0(w)| \leq 3n$. In each step, finding an edge $uv \in E(G)$ with $|L(v)| = 1$ and $L(u) \cap L(v) \neq \emptyset$ takes time at most $\mathcal{O}(n^2)$ and constructing a new list L_{i+1} takes time $\mathcal{O}(n)$. Thus L' can be computed from L in time $\mathcal{O}(n^3)$.

Since $L_0 = L$, G has an L -coloring if and only if G has an L_0 -coloring. For all L_i -colorings c of G and for all edges $uv \in E(G)$, $c(v) \in L_i(v)$ and $c(u) \neq c(v)$. Thus c is an L_{i+1} -coloring of G . For all L_{i+1} -colorings c' of G , since $L_{i+1}(w) \subseteq L_i(w)$ for all $w \in V(G)$, c' is an L_i -coloring of G . Thus, L and L' are equivalent for G . ■

Lemma 25. *Let $k, l \in \mathbb{N}$ be fixed, and (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to $J_{16}(k, l)$ -free ordered graphs. There is a spanning (G, L) -profile \mathcal{L}'_1 such that:*

- $|\mathcal{L}'_1| \leq \mathcal{O}(n^{3(k+l)})$, and \mathcal{L}'_1 can be constructed from L in time $\mathcal{O}(n^{3(k+l)+4})$.
- For all $(G, L') \in \mathcal{L}'_1$, let $X' = \{v \in V(G) : |L'(v)| \geq 2\}$. Then in the graph $G[X']$, every vertex has at most 2 forward neighbors.
- If there is an L -coloring c of G with $|c^{-1}(i)| \geq k + l$ for all $i \in [3]$, then there exists $(G, L') \in \mathcal{L}'_1$ such that c is an L' -coloring.

Proof. Let \mathcal{Q} be the set of all 6-tuples $Q = (A_1, A_2, A_3, B_1, B_2, B_3)$ of disjoint subsets of $V(G)$ such that for all $i \in [3]$, $|A_i| = k$ and $|B_i| = l$, $i \in L(v)$ for all $v \in A_i \cup B_i$, and $A_i \cup B_i$ are stable. For each $q \in \mathcal{Q}$, we construct a (G, L) -refinement (G, L^Q) as follows.

The list L_0^Q is defined as follows. For each vertex $v \in V(G)$, we let $L_0^Q(v) = \{i\}$ if $v \in A_i \cup B_i$ for some $i \in [3]$, otherwise let $L_0^Q(v) = L(v)$. For each $i \in [3]$, let $m_i = \max\{\varphi_G(a) : a \in A_i\}$, $n_i = \min\{\varphi_G(b) : b \in B_i\}$. Then, for all $i \in [3]$, remove i from $L_0^Q(v)$ for every $v \in V(G(-\infty : m_i)) \setminus A_i$ and every $v \in V(G[n_i : \infty)) \setminus B_i$. By Lemma 24, the list L_0^Q such that for all $uv \in E(G)$ with $|L_0^Q(v)| = 1$, $L_0^Q(u) \cap L_0^Q(v) = \emptyset$, can be constructed from L_0^Q in polynomial time.

The list L^Q is constructed recursively. Starting from the list L_0^Q , we construct a sequence of equivalent list assignments L_1^Q, L_2^Q, \dots until some L_s^Q satisfies the second property of this lemma. For convenience, every time we define L_t^Q for $0 \leq t \leq s$, we also define the following

sets. For $\{i, j\} \subseteq [3]$, let $X_t^{ij} = \{v \in V(G) : L_t^Q(v) = \{i, j\}\}$, and let $X_t^{123} = \{v \in V(G) : L_t^Q(v) = \{1, 2, 3\}\}$. Let $X_t = X_t^{12} \cup X_t^{13} \cup X_t^{23} \cup X_t^{123}$.

If in the graph $G[X_t]$, every vertex has at most 2 forward neighbors, then let $L^Q = L_t^Q$. Otherwise, there is a vertex v with at least 3 forward neighbors in the graph $G[X_t]$. Notice that if G contains K_4 as a subgraph, then G is not L' -colorable for any $L' : V(G) \rightarrow 2^{[3]}$. We return $\mathcal{L}'_1 = \emptyset$ in this case. Thus, we may assume that G contains no K_4 from now on.

Also, we may assume that

(13) *The vertex v does not have two distinct nonadjacent forward neighbors u, w such that $L_t^Q(v) \cap L_t^Q(u) \cap L_t^Q(w) \neq \emptyset$.*

Suppose not. Consider a color $i \in L_t^Q(v) \cap L_t^Q(u) \cap L_t^Q(w)$ for two distinct nonadjacent forward neighbors u, w of v . From the construction of L_t^Q , we know that v, u, w are not adjacent to any vertex from $A_i \cup B_i$. A_i and B_i are disjoint and $A_i \cup B_i$ is a stable set. The vertices u, w are nonadjacent forward neighbors of v . For every $x \in A_i, y \in \{u, v, w\}$ and $z \in B_i$, $\varphi(x) < \varphi(y) < \varphi(z)$. From the constructions of L_0^Q , we have $G[A_i \cup B_i \cup \{u, v, w\}] \cong J_{16}(k, l)$, which contradicts the fact that G is $J_{16}(k, l)$ -free. This proves (13).

If $v \in X_t^{123}$, then for every two forward neighbors u, w of v in X_t , $L_t^Q(v) \cap L_t^Q(u) \cap L_t^Q(w) \neq \emptyset$. So by (13), u and w are adjacent. But then, since v has at least 3 forward neighbors, there exists a K_4 as a subgraph of G , which is a contradiction. Thus, this case is impossible.

It remains to consider the case $v \in X_t^{ij}$. Since G has no K_4 , there are two forward neighbors u, w of v in X_t with $uw \notin E$. By (13), it follows that $L_t^Q(u) \cap L_t^Q(v) \cap L_t^Q(w) = \emptyset$. By symmetry, we may assume that $L_t^Q(u) = \{i, m\}, L_t^Q(w) = \{j, m\}$ where $\{i, j, m\} = [3]$. Let x be a forward neighbor of v in X_t different from u and w . We have the following subcases.

- $L_t^Q(x) \supseteq \{i, j\}$.

Then $ux, wx \in E(G)$ by (13). Since u and w have two adjacent neighbors in common, it follows that $c(u) = c(w)$ for every 3-coloring of G . We let $L'_{t+1}(u) = L'_{t+1}(w) = \{m\}$, $L'_{t+1}(y) = L_t^Q(y) \setminus \{m\}$ for $y \in N(u) \cup N(w)$, and $L'_{t+1}(y) = L_t^Q(y)$ for all $y \in V(G) \setminus (N[u] \cup N[w])$.

- $L_t^Q(x) = \{i, m\}$. (The case $\{j, m\}$ follows from symmetry.)

By (13), we have $ux \in E(G)$. But now v has two adjacent neighbors with list $\{i, m\}$. So in every L_t^Q -coloring c of G , we have $c(v) = j$. We let $L'_{t+1}(v) = \{j\}$, $L'_{t+1}(y) = L_t^Q(y) \setminus \{j\}$ for $y \in N(v)$, and $L'_{t+1}(y) = L_t^Q(y)$ for all $y \in V(G) \setminus N[v]$.

At the end of each step, by applying Lemma 24, we replace the list L'_{t+1} by an equivalent list L_{t+1}^Q such that for all $uv \in E(G)$ with $|L_{t+1}^Q(v)| = 1$, we have $L_{t+1}^Q(u) \cap L_{t+1}^Q(v) = \emptyset$, in time $\mathcal{O}(n^3)$.

For all t , $|X_{t+1}| \leq |X_t| - 1$, and $|X_0| \leq n$. Thus the algorithm above terminates in at most n steps. In each step t , finding the vertex v with at least 3 forward neighbors in $G[X_t]$ takes time $\mathcal{O}(n)$, constructing the list L'_{t+1} takes time $\mathcal{O}(n)$, and constructing the list L_{t+1}^Q takes time $\mathcal{O}(n^3)$. So L^Q can be constructed in time $\mathcal{O}(n^4)$. We let $\mathcal{L}'_1 = \{L^Q : Q \in \mathcal{Q}\}$. There are at most $\binom{n}{k} \cdot \binom{n-k}{k} \cdot \binom{n-2k}{k} = \mathcal{O}(n^{3k})$ different choices of the triple (A_1, A_2, A_3) , and at most $\mathcal{O}(n^{3l})$ different choices of the triple (B_1, B_2, B_3) . For each 6-tuple Q we add at most one list to \mathcal{L}'_1 . Thus, $|\mathcal{L}'_1| \leq \mathcal{O}(n^{3(k+l)})$. Therefore, \mathcal{L}_1 can be constructed from L in time $\mathcal{O}(n^{3(k+l)+4})$.

Finally, let c be an L -coloring of G with $|c^{-1}(i)| \geq k + l$ for all $i \in [3]$. Define $A'_i \subseteq c^{-1}(i)$ to be the set of vertices such that $|A'_i| = k$, and $\varphi(v) > \varphi(u)$ for all $v \in c^{-1}(i) \setminus A'_i$ and $u \in A'_i$, that is, A'_i is the set of first k vertices colored i in c . Similarly, for all $i \in [3]$, define $B'_i \subseteq c^{-1}(i)$ to be the set of vertices such that $|B'_i| = l$, and $\varphi(v) < \varphi(u)$ for any $v \in c^{-1}(i) \setminus B'_i$ and $u \in B'_i$. Let

$Q' = (A'_1, A'_2, A'_3, B'_1, B'_2, B'_3)$. It follows that $Q' \in \mathcal{Q}$. Thus, the corresponding (G, L) -refinement $(G, L^{Q'})$ is in \mathcal{L}'_1 .

We want to show that c is also an $L^{Q'}$ -coloring. We will prove this by induction on t . For every vertex $v \in V(G)$, we have $c(v) \in L_0^{Q'}(v)$ from the choice of Q' . Thus, c is an $L_0^{Q'}$ -coloring. Suppose c is an $L_t^{Q'}$ -coloring. Then for $t+1$, from our construction, $c(v) \in L_{t+1}^{Q'}(v)$ for all vertex v . So c is an $L_{t+1}^{Q'}$ -coloring of G . By Lemma 24, c is also an $L_{t+1}^{Q'}$ -coloring. Thus, the L -coloring c is an $L^{Q'}$ -coloring of G . \blacksquare

Lemma 26. *Let $k, l \in \mathbb{N}$ be fixed, and (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to $J_{16}(k, l)$ -free ordered graphs. There is a spanning (G, L) -profile \mathcal{L}'_2 such that:*

- $|\mathcal{L}'_2| \leq 3 \cdot n^{k+l}$, and \mathcal{L}'_2 can be constructed from L in time $\mathcal{O}(n^{k+l+1})$.
- For all $(G, L') \in \mathcal{L}'_2$, let $X = \{v \in V(G) : |L'(v)| \geq 2\}$. Then $|L'(v)| = 2$ and $L'(u) = L'(v)$ for all $u, v \in X$.
- If c is an L -coloring of G with $|c^{-1}(i)| < k + l$ for some $i \in [3]$, then there exists $(G, L') \in \mathcal{L}'_2$ such that c is an L' -coloring.

Proof. Let \mathcal{P} be a set of all pairs $P = (i, A_i)$ such that $i \in [3]$ and $A_i \subseteq V(G)$ with $|A_i| < k + l$, A_i stable and $i \in L(v)$ for all $v \in A_i$. For each $P \in \mathcal{P}$, we construct a (G, L) -refinement (G, L^P) as follows.

Let $L^P(v) = \{i\}$ for all $v \in A_i$, and $L^P(v) = L(v) \setminus \{i\}$ otherwise. It follows that $L^P(v) = [3] \setminus \{i\}$ for all $v \in V(G)$ with $|L^P(v)| \geq 2$.

The set \mathcal{P} is of size at most $3 \cdot n^{k+l}$. For each pair $P \in \mathcal{P}$ we add at most one refinement to \mathcal{L}'_2 . Thus, $|\mathcal{L}'_2| \leq 3 \cdot n^{k+l}$. Constructing the list L^P takes time $\mathcal{O}(n)$. Thus, \mathcal{L}'_2 can be constructed from L in time $\mathcal{O}(n^{k+l+1})$.

Let c be an L -coloring of G with $|c^{-1}(i)| < k + l$ for some $i \in [3]$. The pair $P' = (i, c^{-1}(i))$ satisfies the property that $|c^{-1}(i)| < k + l$, $c^{-1}(i)$ is stable and $i \in L(v)$ for all $v \in c^{-1}(i)$. Thus, the corresponding (G, L) -refinement $(G, L^{P'})$ is in \mathcal{L}'_2 . By the construction of $L^{P'}$, c is an $L^{P'}$ -coloring. \blacksquare

Lemma 27. *Let $k, l \in \mathbb{N}$ be fixed, and (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to $J_{16}(k, l)$ -free ordered graphs. Let $X = \{v \in V(G) : |L(v)| \geq 2\}$ and let us assume that every vertex in X has at most two forward neighbors in $G[X]$ and that $|X| \geq 3k + 3l + 6$. There is a spanning (G, L) -profile \mathcal{L}_1 such that:*

- $|\mathcal{L}_1| = \mathcal{O}(1)$, and \mathcal{L}_1 can be constructed in time $\mathcal{O}(n^3)$.
- For all $(G, L^*) \in \mathcal{L}_1$, let $X^* = \{v \in V(G) : |L^*(v)| \geq 2\}$. Then the graph $G[X^*]$ is chordal.
- If c is an L -coloring of G , then there exists $(G, L^*) \in \mathcal{L}_1$ such that c is an L^* -coloring of G .

Proof. First, we define two sets $C' \subseteq C \subseteq X$ as follows. We start with $C' = C = \emptyset$. In each step, we take the vertex $v \in X \setminus C$ with the smallest $\varphi(v)$. Add v and its forward neighbors in $G[X]$ to C , and add v to C' . We repeat this k times. Since every vertex in X has at most two forward neighbors in $G[X]$, $|C| \leq 3k$. By construction, C' is a stable set of size k . Moreover, no vertex in C' is adjacent to a vertex in $X \setminus C$. Define $D \subseteq X$ to be the set of vertices such that $|D| = 3l + 6$, and $\varphi(v) < \varphi(u)$ for all $v \in X \setminus (C \cup D)$ and $u \in D$, that is, D is the set of last $3l + 6$ vertices in $X \setminus C$. Since $|X| \geq 3k + 3l + 6$, it follows that C, C' and D are well-defined.

Let \mathcal{F} be the set of all functions $f : C \cup D \rightarrow [3]$ such that f is an L -coloring of $G[C \cup D]$. For every $f \in \mathcal{F}$, we construct a (G, L) -refinement (G, L^f) such that $L^f(v) = \{f(v)\}$ if $v \in C \cup D$, and $L^f(v) = L(v)$ otherwise. By Lemma 24, there is an equivalent list L^f of L^f such that for all $uv \in E(G)$ with $|L^f(v)| = 1$, $L^f(u) \cap L^f(v) = \emptyset$. Let $\mathcal{L}_1 = \{(G, L^f) : f \in \mathcal{F}\}$. There are

at most $3^{3k+3l+6} = \mathcal{O}(1)$ possible choices of f . Thus, $|\mathcal{L}_1| = \mathcal{O}(1)$. Constructing the set C and D takes time $\mathcal{O}(1)$. Each L'^f can be constructed in time $\mathcal{O}(n)$. Each L^f can be constructed in time $\mathcal{O}(n^3)$. So \mathcal{L}_1 can be constructed in time $\mathcal{O}(n^3)$.

Now let $(G, L^*) \in \mathcal{L}_1$. Every non-chordal ordered graph contains a vertex with two nonadjacent forward neighbors. To be more precise, the vertex with the smallest order in an induced cycle of size at least 4 is a desired vertex. Now we want to show that $G[X^*]$ is chordal using this property. Suppose for a contradiction that in $G[X^*]$, there is a vertex v_1 with two non-adjacent forward neighbors v_2, v_3 . There is a stable set $D' \subseteq D$ of size at least l such that D' is anticomplete to $\{v_1, v_2, v_3\}$. That is because $X^* \subseteq X$ and every vertex in X has at most 2 forward neighbors in $G[X]$, so $D \setminus N(\{v_1, v_2, v_3\})$ is of size at least $3l$. Since D has a 3-coloring by construction, there is a stable set $D' \subseteq D \setminus N(\{v_1, v_2, v_3\})$ of size at least l and which is anticomplete to $\{v_1, v_2, v_3\}$. From the construction above, the sets $\{v_1, v_2, v_3\}$, C' and D' are disjoint. Moreover, for every $x \in C'$, $y \in \{v_1, v_2, v_3\}$ and $z \in D'$, $\varphi(x) < \varphi(y) < \varphi(z)$. So $G[C' \cup D' \cup \{v_1, v_2, v_3\}] \cong J_{16}(k, l)$, which is a contradiction. Therefore, $G[X^*]$ is chordal.

Finally, let c be an L -coloring of G . Take the coloring $c' = c|_{C \cup D}$ and consider the corresponding (G, L) -refinement $(G, L^{c'})$ and $(G, L^{c'})$ defined above. Since we have covered all possible colorings f of $G[C \cup D]$, $(G, L^{c'})$ is in \mathcal{L}_1 . We can verify that $c(v) \in L^{c'}(v)$ for all vertices $v \in V(G)$. Thus c is also an $L^{c'}$ -coloring. \blacksquare

Lemma 28. *Let $k, l \in \mathbb{N}$ be fixed, and (G, L) be an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to $J_{16}(k, l)$ -free ordered graphs. Let $X = \{v \in V(G) : |L(v)| \geq 2\}$ and let us assume that $|X| < 3k + 3l + 6$. There is a spanning (G, L) -profile \mathcal{L}_2 such that:*

- $|\mathcal{L}_2| = \mathcal{O}(1)$, and \mathcal{L}_2 can be constructed in time $\mathcal{O}(n^3)$.
- For any $(G, L^*) \in \mathcal{L}_2$, $|L^*(v)| \leq 1$ for all $v \in V(G)$.
- If c is an L -coloring of G , then there exists $(G, L^*) \in \mathcal{L}_2$ such that c is an L^* -coloring of G .

Proof. Let \mathcal{F} be the set of all functions $f : X \rightarrow [3]$ such that f is an L -coloring of $G[X]$. For every possible function $f \in \mathcal{F}$, we construct a list L^f such that $L^f(v) = \{f(v)\}$ for all $v \in X$, and $L^f(v) = L(v)$ otherwise. Let $\mathcal{L}_2 = \{(G, L^f) : f \in \mathcal{F}\}$.

For every $(G, L^f) \in \mathcal{L}_2$ and for every $v \in V(G)$, if $v \in X$ then $|L^f(v)| \leq 1$; otherwise by the definition of X , we have $|L^f(v)| \leq |L^f(v)| \leq 1$. Thus $|L^f(v)| \leq 1$ for all $v \in V(G)$.

Since there are at most $3^{3k+3l+6} = \mathcal{O}(1)$ possible choices of f , $|\mathcal{L}_2| = \mathcal{O}(1)$. Each L^f can be constructed in time $\mathcal{O}(n)$, and L^f can be constructed from L'^f in time $\mathcal{O}(n^3)$. So \mathcal{L}_2 can be constructed in time $\mathcal{O}(n^3)$.

Finally, let c be an L -coloring of G . Let $c' = c|_X$ and consider the corresponding (G, L) -refinements $(G, L^{c'})$ and $(G, L^{c'})$ defined above. Since we have covered all possible L -colorings $f : X \rightarrow [3]$, $(G, L^{c'}) \in \mathcal{L}_2$. By the construction of c' and $L^{c'}$, c is an $L^{c'}$ -coloring thus is an $L^{c'}$ -coloring. \blacksquare

Theorem 29. *For fixed $k, l \in \mathbb{N}$, there is an algorithm with the following specifications:*

- *Input:* (G, L) , which is an instance of the ORDERED GRAPH LIST-3-COLORING PROBLEM and G is $J_{16}(k, l)$ -free.
- *Output:* one of
 - an L -coloring of G ;
 - a determination that G is not L -colorable;
 - a spanning (G, L) -profile \mathcal{L} with $|\mathcal{L}| \leq \mathcal{O}(n^{3(k+l)})$ such that for every $(G, L^*) \in \mathcal{L}$, if $X^{L^*} = \{v \in V(G) : |L^*(v)| \geq 2\}$, then $G[X^{L^*}]$ is chordal.
- *Running time:* $\mathcal{O}(n^{3(k+l+1)})$.

Proof. Let \mathcal{L}'_1 be as in Lemma 25. Let \mathcal{L}'_2 be as in Lemma 26. By Theorem 19, every (G, L) -refinement $(G, L') \in \mathcal{L}'_2$ can be solved in time $\mathcal{O}(n^2)$. If this finds an L -coloring of G , we just output the coloring instead of processing with the other things.

For every (G, L) -refinement $(G, L') \in \mathcal{L}'_1$, let $X = \{v \in V(G) : |L'(v)| \geq 2\}$. If $|X| \geq 3k + 3l + 6$, then there is a spanning (G, L) -profile $\mathcal{L}^{L'}$ which satisfies the properties in Lemma 27. If $|X| < 3k + 3l + 6$, then there is a spanning (G, L) -profile $\mathcal{L}^{L'}$ which satisfies the properties in Lemma 28. Finally, let $\mathcal{L} = \cup_{(G, L') \in \mathcal{L}'_1} \mathcal{L}^{L'}$. From the constructions, for every $(G, L^*) \in \mathcal{L}$, $G[X^{L^*}]$ is chordal. Since $|\mathcal{L}^{L'}| = \mathcal{O}(1)$ and $|\mathcal{L}'_1| \leq \mathcal{O}(n^{3(k+l)})$, $|\mathcal{L}| \leq \mathcal{O}(n^{3(k+l)})$. The collection $\mathcal{L}^{L'}$ can be constructed from L' in time $\mathcal{O}(n^3)$, and $|\mathcal{L}'_1| \leq \mathcal{O}(n^{3(k+l)})$. Thus, \mathcal{L} can be constructed from L in time $\mathcal{O}(n^{3(k+l+1)})$. ■

Now we are ready to prove Theorem 22.

Proof of Theorem 22. Let G be a $J_{16}(k, l)$ -free ordered graph and L be a 3-list-assignment for G . We apply the algorithm from Theorem 29 to (G, L) . If the output is an L -coloring of G or a determination that G is not L -colorable, then we are done; so we may assume that the output is a (G, L) -profile \mathcal{L} . For each $(G, L^*) \in \mathcal{L}$, we let $X^{L^*} = \{v \in V(G) : |L^*(v)| \geq 2\}$ as in Theorem 29. By Theorem 24, we may assume that $L^*(u) \cap L^*(v) = \emptyset$ for all $uv \in E(G)$ such that $|L^*(u)| = 1$. If $L^*(u) = \emptyset$ for some $u \in V(G)$, then G has no L^* -coloring and we continue. Otherwise, since Theorem 29 guarantees that $G[X^{L^*}]$ is chordal, and by Theorem 23, we can check in polynomial time if $G[X^{L^*}]$ is L^* -colorable. If this returns a coloring f , then by Theorem 24, we obtain an L -coloring of G as follows:

- for $x \in X^{L^*}$, let $c(x) = f(x)$;
- for all other $x \in V(G)$, let $c(x)$ be the unique color in $L^*(x)$.

If there is no $(G, L^*) \in \mathcal{L}$ such that this returns a coloring of G , then, from the definition of a (G, L) -profile, it follows that G is not L -colorable. This concludes the proof. ■

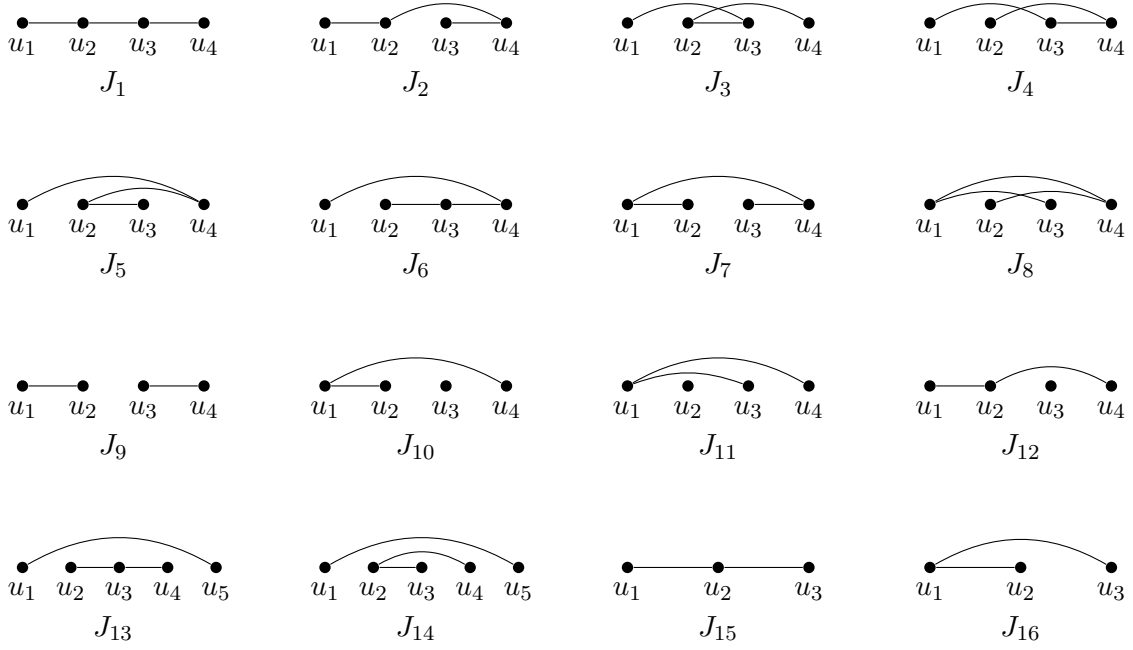
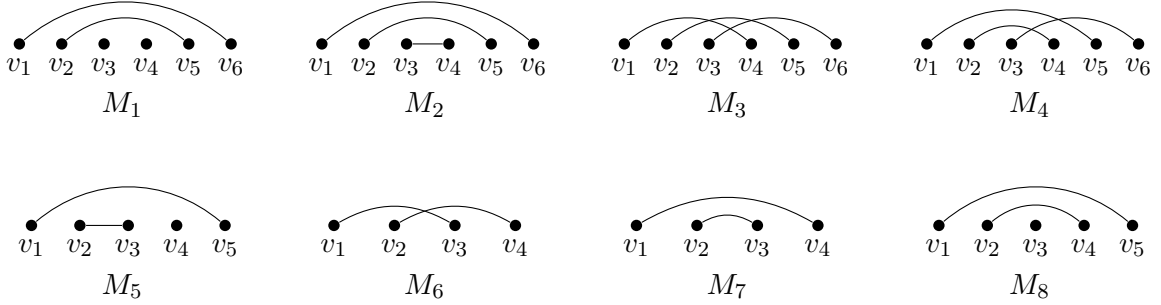
5. NP-COMPLETENESS RESULTS

Let us begin with giving the full list of all ordered graphs we will use. Let $U' = \{u_1, u_2, u_3, u_4, u_5\}$ and $U = U' \setminus \{u_5\}$, the ordering $\varphi' : U' \rightarrow \mathbb{R}$ with $u_i \mapsto i$ for $i \in [5]$.

- Let $J_1 = (U, \{u_1u_2, u_2u_3, u_3u_4\}, \varphi'|_U)$.
- Let $J_2 = (U, \{u_1u_2, u_2u_4, u_3u_4\}, \varphi'|_U)$.
- Let $J_3 = (U, \{u_1u_3, u_2u_3, u_2u_4\}, \varphi'|_U)$.
- Let $J_4 = (U, \{u_1u_3, u_2u_4, u_3u_4\}, \varphi'|_U)$.
- Let $J_5 = (U, \{u_1u_4, u_2u_3, u_2u_4\}, \varphi'|_U)$.
- Let $J_6 = (U, \{u_1u_4, u_2u_3, u_3u_4\}, \varphi'|_U)$.
- Let $J_7 = (U, \{u_1u_2, u_1u_4, u_3u_4\}, \varphi'|_U)$.
- Let $J_8 = (U, \{u_1u_3, u_1u_4, u_2u_4\}, \varphi'|_U)$.
- Let $J_9 = (U, \{u_1u_2, u_3u_4\}, \varphi'|_U)$.
- Let $J_{10} = (U, \{u_1u_2, u_1u_4\}, \varphi'|_U)$.
- Let $J_{11} = (U, \{u_1u_3, u_1u_4\}, \varphi'|_U)$.
- Let $J_{12} = (U, \{u_1u_2, u_2u_4\}, \varphi'|_U)$.
- Let $J_{13} = (U', \{u_1u_5, u_2u_3, u_3u_4\}, \varphi')$.
- Let $J_{14} = (U', \{u_1u_5, u_2u_3, u_2u_4\}, \varphi')$.
- Let $J_{15} = (U \setminus \{u_4\}, \{u_1u_2, u_2u_3\}, \varphi'|_{U \setminus \{u_4\}})$.
- Let $J_{16} = (U \setminus \{u_4\}, \{u_1u_2, u_1u_3\}, \varphi'|_{U \setminus \{u_4\}})$.

Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $\varphi : V \rightarrow \mathbb{R}$ with $v_i \mapsto i$ for $i \in [6]$.

- Let $M_1 = (V, \{v_1v_6, v_2v_5\}, \varphi)$.
- Let $M_2 = (V, \{v_1v_6, v_2v_5, v_3v_4\}, \varphi)$.

FIGURE 3. The ordered graphs J_i for $i \in [16]$.FIGURE 4. The ordered graphs M_i for $i \in [8]$.

- Let $M_3 = (V, \{v_1v_4, v_2v_5, v_3v_6\}, \varphi)$.
- Let $M_4 = (V, \{v_1v_5, v_2v_4, v_3v_6\}, \varphi)$.
- Let $M_5 = (V \setminus \{v_6\}, \{v_1v_5, v_2v_3\}, \varphi|_{V \setminus \{v_6\}})$.
- Let $M_6 = (V \setminus \{v_5, v_6\}, \{v_1v_3, v_2v_4\}, \varphi|_{V \setminus \{v_5, v_6\}})$.
- Let $M_7 = (V \setminus \{v_5, v_6\}, \{v_1v_4, v_2v_3\}, \varphi|_{V \setminus \{v_5, v_6\}})$.
- Let $M_8 = (V \setminus \{v_6\}, \{v_1v_5, v_2v_4\}, \varphi|_{V \setminus \{v_6\}})$.

The main theorem of this section is the following, which implies Theorem 6, the hardness assertion in Theorem 7, and the entire Theorem 8.

Theorem 30. *If H is an ordered graph such that at least one of the following holds:*

- H has at least three edges;
- H has a vertex of degree at least 2 and is not isomorphic to $J_{16}(k, l)$ or $-J_{16}(k, l)$ for any k, l ;
- H contains J_9 , M_1 or M_5 as induced ordered subgraph;

then the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to (H, φ) -free ordered graphs is NP-complete.

In order to show Theorem 30, we will show the following two theorems.

Theorem 31. *The ORDERED GRAPH LIST-3-COLORING PROBLEM is NP-complete when restricted to the class of J_j -free ordered graphs, for $j \in [15]$.*

Theorem 32. *The ORDERED GRAPH LIST-3-COLORING PROBLEM is NP-complete when restricted to the class of M_j -free ordered graphs, for $j \in [5]$.*

Moreover, as a consequence of Theorem 31, we also have the following result, which will be proved later:

Theorem 33. *Let H be a graph and $\varphi : V(H) \rightarrow \mathbb{Z}$. The ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to (H, φ) -free ordered graphs is NP-complete if H contains a copy of P_4 or $P_3 + P_2$ as an induced subgraph.*

Before we start the proof, let us first point out the following observation:

Corollary 34. *If the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to H -free ordered graphs is NP-complete, then the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to $-H$ -free ordered graphs is NP-complete.*

Proof of Theorem 30 assuming Theorems 31, 32 and 33. If H is a graph with at least three edges, then H contains at least one of the following induced subgraph: a cycle, a claw, P_4 , $P_3 + P_2$ or $3P_2$. NP-hardness in the first two cases follow from Theorems 1 and 2, respectively. The third and fourth case follow from Theorem 33. Finally, if H contains $3P_2$, the result follows from the cases of J_9 of Theorem 31 and $M_2, M_3, M_4, -M_4$ and M_5 of Theorem 32.

If H has a vertex of degree at least 2 and is not isomorphic to $J_{16}(k, l)$ or $-J_{16}(k, l)$ for all k, l , then either H has at least three edges, or H contains a P_3 and H contains $J_{10}, -J_{10}, J_{11}, -J_{11}$, or J_{15} ; now the result follows from Theorem 31. ■

We will use two constructions to show the ORDERED GRAPH LIST-3-COLORING PROBLEM is NP-complete when restricted to the class of J_i -free ordered graphs, for every $i \in [15]$. Then with these proofs, we will show Theorem 33.

In the first construction, we will use the following result.

Theorem 35 (Chlebík and Chlebíková [4]). *The LIST-3-COLORING PROBLEM restricted to bipartite graphs is NP-complete.*

Theorem 36. *The ORDERED GRAPH LIST-3-COLORING PROBLEM restricted to $J_1, J_2, J_4, J_6, J_7, J_9, J_{12}, J_{13}$ or J_{15} -free ordered graphs is NP-complete.*

Proof. Given a bipartite graph G with bipartition (X, Y) and its list assignment L , we construct an ordered graph (G, τ_5) as follows. We enumerate the set $X = \{x_1, \dots, x_s\}$ and $Y = \{y_1, \dots, y_t\}$. Let $\tau_5 : V(G) \rightarrow \mathbb{Z}$ be a function with $\tau_5(x_i) = i$ for $i \in [s]$ and $\tau_5(y_j) = s + j$ for $j \in [t]$.

Clearly, the ordered graph (G, τ_5) can be computed in time $\mathcal{O}(n)$, and (G, τ_5) is list-3-colorable if and only if G is list-3-colorable. The ordered graph (G, τ_5) is J_7 -free, J_9 -free and J_{15} -free, as for every edge $zw \in E(G)$, without loss of generality, we have $z \in X$ and $w \in Y$. The fact that (G, τ_5) is J_{15} -free implies that (G, τ_5) does not contain $J_1, J_2, J_4, J_6, J_{12}$ and J_{13} . ■

The second construction is a reduction from NAE3SAT. Given an instance I consisting of n Boolean variables and m clauses, each of which contains 3 literals, the NOT-ALL-EQUAL-3-SATISFIABILITY PROBLEM (NAE3SAT) is to decide whether there exists a truth assignment for every variable such that every clause contains at least one true literal and one false literal. We say I is *satisfiable* if it admits such an assignment. *Monotone* NAE3SAT is a NAE3SAT restricted to instances with no negated literals.

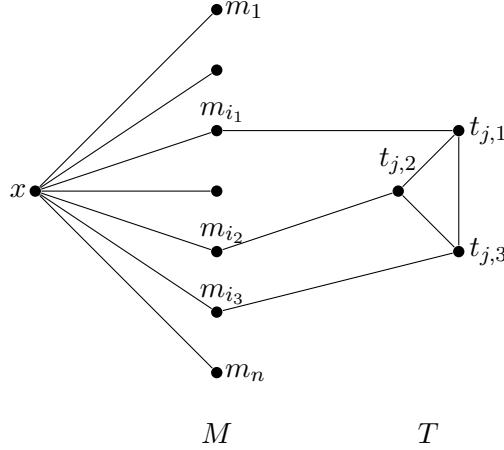


FIGURE 5. Construction of H_1 from Theorem 38, with M corresponding to variables and T corresponding to clauses.

Theorem 37 (Garey and Johnson [14]). *Monotone NAE3SAT is NP-complete.*

Notice that we will prove a stronger result: In the following theorem, we actually prove the NP-completeness of the ORDERED GRAPH 3-COLORING PROBLEM instead of the ORDERED GRAPH LIST-3-COLORING PROBLEM within specific classes of graphs.

Theorem 38. *Given a monotone NAE3SAT instance I , there is a graph H_1 such that:*

- (1) *The graph H_1 can be computed from I in time $\mathcal{O}(m + n)$, where m is the number of clauses of I and n is the number of variables of I ;*
- (2) *The graph H_1 is 3-colorable if and only if I is satisfiable;*
- (3) *There is an injective function $\tau_1 : V(H_1) \rightarrow \mathbb{Z}$ such that (H_1, τ_1) is J_3 and $-J_{11}$ -free, and τ_1 can be computed from H_1 in time $\mathcal{O}(m + n)$;*
- (4) *There is an injective function $\tau_2 : V(H_1) \rightarrow \mathbb{Z}$ such that (H_1, τ_2) is J_5 and J_8 -free, and τ_2 can be computed from H_1 in time $\mathcal{O}(m + n)$;*
- (5) *There is an injective function $\tau_3 : V(H_1) \rightarrow \mathbb{Z}$ such that (H_1, τ_3) is J_{10} and J_{14} -free, and τ_3 can be computed from H_1 in time $\mathcal{O}(m + n)$.*

Therefore, the ORDERED GRAPH 3-COLORING PROBLEM is NP-complete when restricted to the class of J_3 , J_5 , J_8 , J_{10} , $-J_{11}$ or J_{14} -free ordered graphs.

Proof. First we create a vertex x . For every variable x_i of I , we create a vertex m_i , and denote the set of such vertices as M . For every clause C_j of I , we create three vertices $t_{j,k}$ for $k \in [3]$, and denote the set of such vertices as T . Let $V(H_1) = \{x\} \cup M \cup T$.

For every vertex $m_i \in M$, we add an edge xm_i . For every clause C_j of I , if the variables in C_j are $x_{i_1}, x_{i_2}, x_{i_3}$ with $1 \leq i_1 < i_2 < i_3 \leq n$, we add edges $m_{i_k}t_{j,k}$ for $k \in [3]$ and edges $t_{j,1}t_{j,2}$, $t_{j,2}t_{j,3}$, $t_{j,1}t_{j,3}$. Let $E(H_1)$ be the set of all defined edges. We deduced directly from the construction of H_1 that:

(14) $H_1[M]$ is stable and $H_1[T]$ is disjoint union of triangles.

(15) The graph H_1 can be computed from I in time $\mathcal{O}(m + n)$.

Also, we have:

(16) The graph H_1 is 3-colorable if and only if I is satisfiable.

Let $f : V(H_1) \rightarrow [3]$ be a 3-coloring of H_1 . Without loss of generality, we assume $f(x) = 1$. Since every vertex in M is adjacent to x , we have $f(m_i) \in \{2, 3\}$ for every $i \in [n]$. We claim that if the variables in C_j are $x_{i_1}, x_{i_2}, x_{i_3}$ with $1 \leq i_1 < i_2 < i_3 \leq n$, then at least one of $f(m_{i_1})$, $f(m_{i_2})$ and $f(m_{i_3})$ has value 2 and at least one of them has value 3. Suppose for a contradiction, without loss of generality, that $f(m_{i_k}) = 2$ for every $k \in [3]$. Then $f(t_{j,k}) \in \{1, 3\}$ for every $k \in [3]$, at least two of the three vertices receive the same color. But by (14), the vertices $t_{j,1}$, $t_{j,2}$ and $t_{j,3}$ form a triangle. which leads to a contradiction as desired. Thus, by assigning true to the variable x_i if $f(m_i) = 2$ and false otherwise, we get a valid truth assignment to the monotone NAE3SAT instance I .

If there is a valid truth assignment to I , we define a 3-coloring $g : V(H_1) \rightarrow [3]$ as follows. For every $i \in [n]$, let $g(m_i) = 2$ if the variable v_i is true in this truth assignment, otherwise let $g(m_i) = 3$. Let $g(x) = 1$. For each clause C_j , we denote the variables in C_j as $x_{i_1}, x_{i_2}, x_{i_3}$ with $1 \leq i_1 < i_2 < i_3 \leq n$. Let $g(t_{j,1}) \in \{2, 3\} \setminus \{g(m_{i_1})\}$. Since $g|_M$ is constructed from a valid truth assignment of I , at least one of the $g(m_{i_1})$, $g(m_{i_2})$ and $g(m_{i_3})$ has value 2 and at least one of them has value 3. If $g(m_{i_2}) \neq g(m_{i_1})$, then let $g(t_{j,2}) \in \{2, 3\} \setminus \{g(m_{i_2})\}$ and $g(t_{j,3}) = 1$, otherwise let $g(t_{j,3}) \in \{2, 3\} \setminus \{g(m_{i_3})\}$ and $g(t_{j,2}) = 1$. To verify this is a valid 3-coloring, we simply go through and check every edge in $E(H_1)$.

(17) *There is an injective function $\tau_1 : V(H_1) \rightarrow \mathbb{Z}$ such that (H_1, τ_1) is J_3 and $-J_{11}$ -free, and τ_1 can be computed from H_1 in time $\mathcal{O}(m+n)$.*

The function $\tau_1 : V(H_1) \rightarrow \mathbb{Z}$ is defined as follows. Let $\tau_1(x) = 1$. Let $\tau_1(m_i) = i + 1$ for every $i \in [n]$. For every $j \in [m]$ and $k \in [3]$, let $\tau_1(t_{j,k}) = n + 3j + k - 2$. The function τ_1 can be constructed in time $\mathcal{O}(m+n)$ as we go through every vertex once.

From the construction we have $\tau_1|_M$ and $\tau_1|_T$ are injective, and $\tau_1(x) < \tau_1(m_i) < \tau_1(t_{j,k})$ for every $i \in [n]$, $j \in [m]$ and $k \in [3]$. So the function τ_1 is injective.

Suppose for a contradiction that (H_1, τ_1) contains an induced path $w_1 w_3 w_2 w_4$ with $\tau_1(w_1) < \tau_1(w_2) < \tau_1(w_3) < \tau_1(w_4)$. Since the vertex x does not have any backward neighbor and every $m_i \in M$ has only one backward neighbor x , we have $w_3 \in T$. At most one of w_1 and w_2 is in T as $w_1, w_2 \in N(w_3)$ and $w_1 w_2 \notin E(H_1)$. Thus we have $w_1 \in M$, $w_2 \in T \cup M$ and $w_4 \in T$. If $w_2 \in T$, then w_3 is also adjacent to w_4 since $w_3, w_4 \in N(w_2)$, which is a contradiction. If $w_2 \in M$, then w_3 has two neighbors in M , which is a contradiction. Thus, we have proved (H_1, τ_1) is J_3 -free.

Suppose (H_1, τ_1) contains an induced subgraph $(\{w_1, w_2, w_3, w_4\}, \{w_1 w_4, w_2 w_4\})$ with $\tau_1(w_1) < \tau_1(w_2) < \tau_1(w_3) < \tau_1(w_4)$. Since w_4 has two backward neighbors, we have $w_4 \in T$. For the two backward neighbors w_1, w_2 of w_4 , since $w_1 w_2 \notin E(H_1)$ and $\tau_1(w_1) < \tau_1(w_2)$, we have $w_1 \in M$ and $w_2 \in T$. From the construction of τ_1 , since $w_2 w_4 \in E(H_1)$ and $\tau_1(w_2) < \tau_1(w_3) < \tau_1(w_4)$, we have $w_3 \in T$ and $w_2 w_3, w_3 w_4 \in E(H_1)$, which is a contradiction. Thus, we have proved (H_1, τ_1) is $-J_{11}$ -free.

(18) *There is an injective function $\tau_2 : V(H_1) \rightarrow \mathbb{Z}$ such that (H_1, τ_2) is J_5 and J_8 -free, and τ_2 can be computed from I in time $\mathcal{O}(m+n)$.*

The function $\tau_2 : V(H_1) \rightarrow \mathbb{Z}$ is defined as follows. Let $\tau_2(m_i) = i$ for every $i \in [n]$. Let $\tau_2(x) = n + 1$. For every $j \in [m]$ and $k \in [3]$, let $\tau_2(t_{j,k}) = n + 3j + k - 2$. The function τ_2 can be constructed in time $\mathcal{O}(m+n)$ as we go through every vertex once.

From the construction we have $\tau_2|_M$ and $\tau_2|_T$ are injective, and $\tau_2(m_i) < \tau_2(x) < \tau_2(t_{j,k})$ for every $i \in [n]$, $j \in [m]$ and $k \in [3]$. So the function τ_2 is injective.

Suppose for a contradiction that (H_1, τ_2) contains an induced path $w_1 w_4 w_2 w_3$ with $\tau_2(w_1) < \tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$. Since the vertex x does not have any forward neighbor, we have $w_1, w_2 \neq x$. Since every vertex in M has no backward neighbor, we have $w_3, w_4 \notin M$. If $w_2 \in T$, then we have $w_3, w_4 \in T$ as $\tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$. But from the construction of H_1 and τ_2 ,

we also have $w_3w_4 \in E(H_1)$ as $w_2w_4 \in E(H_1)$, which is a contradiction. Thus we have $w_2 \in M$, which implies $w_1 \in M$. Also since w_2 has two forward neighbors and $\tau_2(x) < \tau_2(t_{j,k})$ for every $j \in [m]$ and $k \in [3]$, we have $w_4 \in T$. But then w_4 has two neighbors $w_1, w_2 \in M$, which is a contradiction. Thus, we have proved (H_1, τ_2) is J_5 -free.

Suppose for a contradiction that (H_1, τ_2) contains an induced path $w_3w_1w_4w_2$ with $\tau_2(w_1) < \tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$. Since the vertex x does not have any forward neighbor, we have $w_1, w_2 \neq x$. Since every vertex in M has no backward neighbor, we have $w_3, w_4 \notin M$. Since w_1 has two nonadjacent forward neighbors w_3, w_4 with $\tau_2(w_3) < \tau_2(w_4)$, we have $w_1 \in M$ and $w_4 \in T$. Thus we have $w_2 \in T$, as w_4 has exactly one neighbor in M . But from the construction of H_1 and τ_2 , we also have $w_3 \in T$ and so $w_2w_3, w_3w_4 \in E(H_1)$ as $w_2w_4 \in E(H_1)$ and $\tau_2(w_2) < \tau_2(w_3) < \tau_2(w_4)$, which is a contradiction. Thus, we have proved (H_1, τ_2) is J_8 -free.

(19) *There is an injective function $\tau_3 : V(H_1) \rightarrow \mathbb{Z}$ such that (H_1, τ_3) is J_{10} and J_{14} -free, and τ_3 can be computed from H_1 in time $\mathcal{O}(m+n)$.*

The function $\tau_3 : V(H_1) \rightarrow \mathbb{Z}$ is defined as follows. Let $\tau_3(x) = 1$. Let $\tau_3(m_i) = i + 1$ for every $i \in [n]$. For every $j \in [m]$ and $k \in [3]$, let $m_i \in M$ be the vertex such that $t_{j,k} \in N(m_i)$, then we set $\tau_3(t_{j,k}) = n + 2 + \sum_{i'=1}^{i-1} (\deg(m_{i'}) - 1) + |\{t_{j',k'} \in N(m_i) : j' < j\}|$. The function τ_3 can be constructed in time $\mathcal{O}(m+n)$ as we go through every vertex once.

From the construction we have $\tau_3|_M$ and $\tau_3|_T$ are injective, and $\tau_3(x) < \tau_3(m_i) < \tau_3(t_{j,k})$ for every $i \in [n]$, $j \in [m]$ and $k \in [3]$. So the function τ_3 is injective.

Suppose that (H_1, τ_3) contains an induced subgraph $(\{w_1, w_2, w_3, w_4\}, \{w_1w_2, w_1w_4\})$ with $\tau_3(w_1) < \tau_3(w_2) < \tau_3(w_3) < \tau_3(w_4)$. Now we consider the vertex w_1 . Since $w_1 = x$ implies $w_1w_3 \in E(H_1)$ as $w_2, w_4 \in N(w_1)$, and $w_1 \in T$ implies $w_2w_4 \in E(H_1)$, we have $w_1 \in M$. But from the construction of τ_3 , we also have $w_1w_3 \in E(H_1)$, which is a contradiction. Thus, we have proved (H_1, τ_3) is J_{10} -free.

Suppose that (H_1, τ_3) contains an induced subgraph $(\{w_1, w_2, w_3, w_4, w_5\}, \{w_1w_5, w_2w_3, w_2w_4\})$ with $\tau_3(w_1) < \tau_3(w_2) < \tau_3(w_3) < \tau_3(w_4) < \tau_3(w_5)$. Now we consider the vertex w_1 . If $w_1 = x$, then $w_1w_2 \in E(H_1)$ as $w_5 \in N(w_1)$, a contradiction. If $w_1 \in T$, then $w_2, w_3, w_4 \in T$, which implies $w_3w_4 \in E(H_1)$, again a contradiction. So we have $w_1 \in M$. But from the construction of τ_3 , if $w_2 \in M$, we have $\tau_3(w_3) > \tau_3(w_5)$, which is a contradiction. If $w_2 \in T$, then we have $w_3w_4 \in E(H_1)$ as a contradiction. Thus, we have proved (H_1, τ_3) is J_{14} -free. ■

Remark 39. *We can verify that the ordered graph (H_1, τ_1) is J_6 -free and the ordered graph (H_1, τ_2) is $J_1, J_2, -J_4$ and J_{12} -free. So the ORDERED GRAPH 3-COLORING PROBLEM is also NP-complete when restricted to the class of $J_1, J_2, -J_4, J_6$ or J_{12} -free ordered graphs.*

Remark 40. *The following theorem gives another proof that the ORDERED GRAPH 3-COLORING PROBLEM is also NP-complete when restricted to the class of J_7, J_{13} or J_{14} -free ordered graphs. Therefore, the only two cases that we do not know whether the ORDERED GRAPH 3-COLORING PROBLEM restricted to the class of J_j -free ordered graphs is NP-complete are J_9 and J_{15} .*

Theorem 41. *Given a monotone NAE3SAT instance I , there is an ordered graph (H_2, τ_4) such that*

- (1) *The ordered graph (H_2, τ_4) can be computed from I in time $\mathcal{O}(m+n)$;*
- (2) *The graph H_2 is 3-colorable if and only if I is satisfiable;*
- (3) *The ordered graph (H_2, τ_4) is J_7, J_{13} and J_{14} -free.*

Therefore, the ORDERED GRAPH 3-COLORING PROBLEM is NP-complete when restricted to the class of J_7, J_{13} or J_{14} -free ordered graphs.

Proof. First we create a vertex x . For every variable x_i of I , we create a vertex m_i and add edge xm_i . We denote the set of such vertices m_i as M . For every clause C_j of I , if the variables in

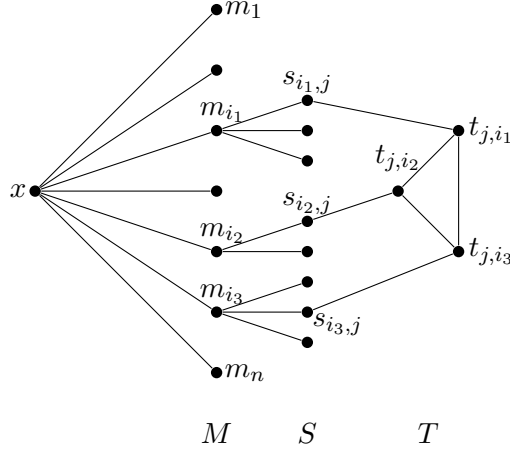


FIGURE 6. Construction of H_2 from Theorem 41 omitting edges between x and all vertices in S .

C_j are $x_{i_1}, x_{i_2}, x_{i_3}$ with $1 \leq i_1 < i_2 < i_3 \leq n$, we create six vertices t_{j,i_k} and $s_{i_k,j}$ for $k \in [3]$. We add edges $t_{j,i_k}t_{j,i_{k'}}$ for $\{k, k'\} \subseteq [3]$, and $xs_{i_k,j}$, $m_{i_k}s_{i_k,j}$ and $s_{i_k,j}t_{j,i_k}$ for $k \in [3]$. We denote the set of vertices t_{j,i_k} as T , and the set of vertices $s_{i_k,j}$ as S . Finally, let $V(H_2) = \{x\} \cup M \cup S \cup T$ and $E(H_2)$ be the set of all edges defined above.

The function $\tau_4 : V(H_2) \rightarrow \mathbb{Z}$ is defined as follows. Let $\tau_4(x) = 1$ and $\tau_4(m_i) = i + 1$ for every $i \in [n]$. For every $s_{i,j} \in S$, let $\tau_4(s_{i,j}) = n + 2 + \sum_{i'=1}^{i-1} (\deg(m_{i'}) - 1) + |\{s_{i',j'} \in N(m_{i'}) : j' < j\}|$. For every $t_{j,i} \in T$, let $\tau_4(t_{j,i}) = \tau_4(s_{i,j}) + 3m$.

(20) *The ordered graph (H_2, τ_4) can be computed from I in time $\mathcal{O}(m + n)$.*

It takes time $\mathcal{O}(m + n)$ to compute both the set $V(H_2)$ and $E(H_2)$. The function τ_4 can be constructed in time $\mathcal{O}(m + n)$, since $|V(H_2)| = n + 6m + 1$ and we go through every vertex once.

(21) *The graph H_2 is 3-colorable if and only if I is satisfiable.*

Let $f : V(H_2) \rightarrow [3]$ be a 3-coloring of H_2 . Without loss of generality, we assume $f(x) = 1$. Since every vertex in $M \cup S$ is adjacent to x , we have $f(y) \in \{2, 3\}$ for every $y \in M \cup S$.

We claim that if the variables in C_j are $x_{i_1}, x_{i_2}, x_{i_3}$ with $1 \leq i_1 < i_2 < i_3 \leq n$, then at least one of $f(m_{i_1})$, $f(m_{i_2})$ and $f(m_{i_3})$ has value 2 and at least one of them has value 3. Suppose for a contradiction, without loss of generality, that $f(m_{i_k}) = 2$ for every $k \in [3]$. We have $f(s_{i_k,j}) = 3$ for every $k \in [3]$. Then $f(t_{j,i_k}) \in \{1, 2\}$ for every $k \in [3]$, at least two of the three vertices receive the same color. But from the construction, the vertices t_{j,i_1} , t_{j,i_2} and t_{j,i_3} form a triangle, which leads to a contradiction as desired. Thus, by assigning true to the variable x_i if $f(m_i) = 2$ and false otherwise, we get a valid truth assignment to the monotone NAE3SAT instance I .

If there is a valid truth assignment to I , we define a 3-coloring $g : V(H_2) \rightarrow [3]$ as follows. For every $i \in [n]$, let $g(m_i) = 2$ if the variable v_i is true in this truth assignment, otherwise let $g(m_i) = 3$. Let $g(x) = 1$. For every vertex $s_{i,j} \in S$, we define $g(s_{i,j}) \in \{2, 3\} \setminus \{g(m_i)\}$.

For each clause C_j , we denote the variables in C_j as $x_{i_1}, x_{i_2}, x_{i_3}$ with $1 \leq i_1 < i_2 < i_3 \leq n$. Let $g(t_{j,i_1}) \in \{2, 3\} \setminus \{g(s_{i_1,j})\}$. Since $g|_M$ is constructed from a valid truth assignment of I , at least one of the $g(m_{i_1})$, $g(m_{i_2})$ and $g(m_{i_3})$ has value 2 and at least one of them has value 3. If $g(m_{i_2}) \neq g(m_{i_1})$, then let $g(t_{j,i_2}) \in \{2, 3\} \setminus \{g(s_{i_2,j})\}$ and $g(t_{j,i_3}) = 1$, otherwise let $g(t_{j,i_3}) \in \{2, 3\} \setminus \{g(s_{i_3,j})\}$ and $g(t_{j,i_2}) = 1$.

To verify this is a valid 3-coloring, we simply go through every edge yz in $E(H_1)$. If without loss of generality $y = x$ and $z \in M \cup S$, then $g(y) = 1$ and $g(z) \in \{2, 3\}$. So $g(y) \neq g(z)$. If

$y \in M$ and $z \in S$, then from the construction $g(z) \in \{2, 3\} \setminus \{g(y)\}$, so $g(z) \neq g(y)$. If $y \in S$ and $z \in T$, then from the construction either $g(z) = 1$ or $g(z) \in \{2, 3\} \setminus \{g(y)\}$, so $g(y) \neq g(z)$. If $y, z \in T$, we have $g(y) \neq g(z)$ as we use all three colors to color the vertices whose corresponding variables are in the same clause.

(22) *The ordered graph (H_2, τ_4) is J_7 , J_{13} and J_{14} -free.*

Suppose for a contradiction that (H_2, τ_4) contains an induced path $w_2w_1w_4w_3$ with $\tau_4(w_1) < \tau_4(w_2) < \tau_4(w_3) < \tau_4(w_4)$. Now we consider the vertex w_1 . Since w_1 has two nonadjacent forward neighbors, we know that $w_1 \notin S \cup T$. If $w_1 = x$, then $w_2, w_4 \in M \cup S$. But from the construction of τ_4 , we also have $w_3 \in M \cup S$, which implies $w_3 \in S$ and so $w_1w_3 \in E(H_2)$ as a contradiction. If $w_1 \in M$, then $w_2, w_4 \in S$, which implies $w_1w_3 \in E(H_2)$ as a contradiction. Thus, we have proved (H_2, τ_4) is J_7 -free.

Suppose (H_2, τ_4) contains an induced subgraph $(\{w_1, w_2, w_3, w_4, w_5\}, \{w_1w_5, w_2w_3, w_3w_4\})$ with $\tau_4(w_1) < \tau_4(w_2) < \tau_4(w_3) < \tau_4(w_4) < \tau_4(w_5)$. Now we consider the vertex w_1 . Since $w_1w_5 \in E(H_2)$ and $w_1w_2 \notin E(H_2)$, we have $w_1 \notin \{x\}$. If $w_1 \in M$, we have $w_5 \in S$, which causes $w_2 \in M$ and $w_3 \in S$. But then w_4 has no place to go, which is a contradiction. If $w_1 \in T$, we know that $w_2, w_3, w_4 \in T$. But then from the construction of H_2 and τ_4 , we have $w_2w_4 \in E(H_2)$, which is a contradiction. If $w_1 \in S$, then $w_5 \in T$. Since $w_2w_4 \notin E(H_2)$, at least one of w_2, w_3, w_4 is in S . From the construction of τ_4 , we know that $w_2 \in S$. So the forward neighbor w_3 of w_2 is in T . But from the construction of τ_4 , the inequality $\tau_4(w_1) < \tau_4(w_2)$ implies $\tau_4(w_5) < \tau_4(w_3)$, which is a contradiction. Thus, we have proved (H_2, τ_4) is J_{13} -free.

A similar argument holds for the case that (H_2, τ_4) is J_{14} -free. ■

Theorem 36, 38 and Corollary 34 together give us Theorem 31. Now we are ready to prove Theorem 33.

Proof of Theorem 33. Let (H, φ) be an ordered graph. If H contains a copy of P_4 , say Q_1 , then since $(Q_1, \varphi|_{Q_1}) \in \{J_i : i \in [8]\} \cup \{-J_i : i \in [8]\}$, by Theorems 36, 38 and Corollary 34, we have the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted $(Q_1, \varphi|_{Q_1})$ -free ordered graphs is NP-complete.

If H contains a copy of $P_3 + P_2$, we denote it $Q_2 = (\{v_1, v_2, v_3, v_4, v_5\}, \{v_1v_2, v_2v_3, v_4v_5\})$. By symmetry, we assume without loss of generality that $\varphi(v_4) < \varphi(v_5)$. Then we consider $\min\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\}$ and $\max\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\}$. If $\max\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\} < \varphi(v_4)$ or $\min\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\} > \varphi(v_5)$, then Theorem 36 indicates the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted $(Q_2, \varphi|_{Q_2})$ -free ordered graphs is NP-complete.

If $\varphi(v_4) < \min\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\}$ and $\max\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\} < \varphi(v_5)$, then Q_2 either contains a copy of J_{13} or $-J_{13}$ as induced subgraph, or a copy of J_{14} or $-J_{14}$. By Theorems 36 and 38, respectively, the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted $(Q_2, \varphi|_{Q_2})$ -free ordered graphs is NP-complete.

If $\min\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\} < \varphi(v_i) < \max\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\}$ for some $i \in \{4, 5\}$, then Q_2 either contains a copy of J_{10} as induced subgraph, or a copy of J_{11} , or a copy of J_{12} , or $-J_{10}$, $-J_{11}$, $-J_{12}$. By Theorem 36, 38 and Corollary 34, the ORDERED GRAPH LIST-3-COLORING PROBLEM restricted $(Q_2, \varphi|_{Q_2})$ -free ordered graphs is NP-complete. ■

Before we start proving Theorem 32, let us prove the following lemma:

Lemma 42. *Given a graph G , let H be a graph satisfying the following conditions:*

- (1) $V(G) \subseteq V(H)$.
- (2) *For every edge $uv \in E(G)$, we have $uv \notin E(H)$ and there are three vertex disjoint uv -paths P_1^{uv} , P_2^{uv} and P_3^{uv} in H of length at least 3. Moreover, for all edges $uv, st \in E(G)$ and $i, j \in [3]$, we have $V(P_i^{uv}) \cap V(P_j^{st}) = \{u, v\} \cap \{s, t\}$.*

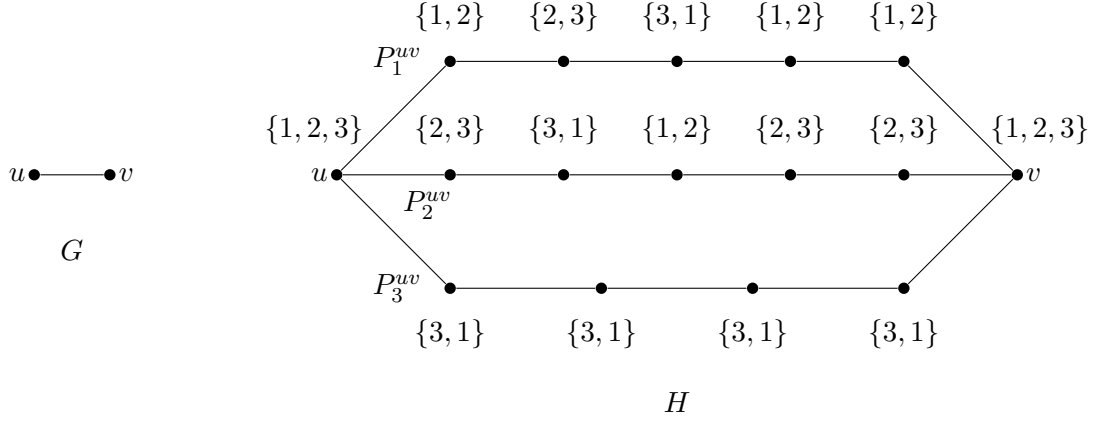


FIGURE 7. An example of a realization (H, L) of G . Each vertex is labeled with its list.

- (3) The correspondence between every edge $uv \in E(G)$ and its paths P_1^{uv} , P_2^{uv} and P_3^{uv} in H is given.
- (4) Every edge in H is contained in some P_i^e , for $i \in [3]$ and $e \in E(G)$.

Then there is a list assignment $L : V(H) \rightarrow 2^{[3]}$ such that:

- (1) The list assignment L can be computed from H in time $\mathcal{O}(|E(H)|)$.
- (2) For every vertex $u \in V(G)$, we have $L(u) = [3]$.
- (3) For every L -coloring f of H , the function $f|_{V(G)}$ is a 3-coloring of G .
- (4) For every 3-coloring g of G , there is a corresponding L -coloring g' of H with $g'|_{V(G)} = g$.

Note:

- We say a pair (H, L) as in Lemma 42 a realization of G .
- As the 3-COLORING PROBLEM is NP-complete, to decide whether H is L -colorable is also NP-complete.

Proof. For every edge $uv \in E(G)$, let the three vertex disjoint uv -paths be P_1^{uv} , P_2^{uv} and P_3^{uv} of length at least 4. For convenience, in this proof, we read every color modulo 3 (so if this would assign color 4, we assign color 1 instead). We define the list assignment $L : V(H) \rightarrow 2^{[3]}$ as follows. Let $L(u) = [3]$ for every vertex $u \in V(G)$. Then we take a path P_i^{uv} , $i \in [3]$ and denote $P_i^{uv} = uw_1w_2 \dots w_tv$. If t is even, we set $L(w_j) = \{i, i+1\}$ for all $i \in [t]$. If t is odd, we set $L(w_j) = \{i+j-1, i+j\}$ for $j \in [3]$, and $L(w_j) = \{i, i+1\}$ for $i \in \{4, \dots, t\}$.

- (23) The list assignment L can be computed from H in time $\mathcal{O}(|E(H)|)$, if given the correspondence between every edge $uv \in E(G)$ and its paths P_1^{uv} , P_2^{uv} and P_3^{uv} in H .

For a given path P_i^e , defining $L|_{V(P_i^e)}$ takes time $|E(P_i^e)|$. So the running time is $\sum_{e \in E(G)} \sum_{i=1}^3 |E(P_i^e)| = |E(H)|$.

- (24) For every vertex $u \in V(G)$, we have $L(u) = [3]$.

This holds immediately from the construction.

- (25) For every L -coloring f of H , the function $f|_{V(G)}$ is a 3-coloring of G .

Suppose not, then there is an L -coloring f of H such that there is an edge $uv \in E(G)$ with $f(u) = f(v) = i$ for some $i \in [3]$. We consider the path $P_i^{uv} = uw_1w_2 \dots w_tv$ in H . If t is even, then from the construction we have that $f(w_j) = i+1$ if j is odd, and $f(w_j) = i$ if j is

even. But then we have $f(w_t) = f(v)$, which leads to a contradiction. If t is odd, then from the construction we have $f(w_1) = i + 1$, $f(w_2) = i + 2$, $f(w_3) = i$, and for $j \in \{4, \dots, t\}$, $f(w_j) = i + 1$ if j is even and $f(w_j) = i$ if j is odd. But then $f(w_t) = f(v) = i$, which is a contradiction. Thus $f|_{V(G)}$ is a 3-coloring of G .

(26) For every 3-coloring g of G , there is a corresponding L -coloring g' of H with $g'|_{V(G)} = g$.

For every $u \in V(G)$, let $g'(u) = g(u)$. For every edge $uv \in V(G)$, we denote $g(u) = a$ and $g(v) = b$, $\{a, b\} \subseteq [3]$. Then we consider the path $P_i^{uv} = uw_1w_2 \dots w_tv$. We may assume that $a \in L(w_1)$ and $b \in L(w_t)$, for otherwise P_i^{uv} is L -colorable.

If t is even, let $g'(w_1) \in L(w_1) \setminus \{a\} \neq \emptyset$, $g'(w_2) = a$, and $g'(w_j) = g'(w_1)$ if $j \in \{3, \dots, t\}$ is odd, and $g'(w_j) = g'(w_2)$ if $j \in \{3, \dots, t\}$ is even. Notice that we have $g'(w_t) = g'(w_2) = a$. Since $a \neq b$, we have $g'(v) \neq g'(w_t)$.

So let us assume t is odd. If $a = i$, or $a = i + 2$ and $b = i + 1$, then we set $g'(w_1) = i + 1$, $g'(w_2) = i + 2$, $g'(w_3) = i$, and $g'(w_j) = i$ if $j \in \{4, \dots, t\}$ odd, $g'(w_j) = i + 1$ if $j \in \{4, \dots, t\}$ even. Thus $g'(w_t) = i \neq b = g'(v)$. If $a = i + 1$, or $a = i + 2$ and $b = i$, then we set $g'(w_1) = i$, $g'(w_2) = i + 1$, $g'(w_3) = i + 2$, and $g'(w_j) = i + 1$ if $j \in \{4, \dots, t\}$ is odd, $g'(w_j) = i$ if $j \in \{4, \dots, t\}$ is even. Thus $g'(w_t) = i + 1 \neq b = g'(v)$.

Therefore, we have defined an L -coloring g' of H with $g'|_{V(G)} = g$. ■

The proof of Theorem 32 is divided into three constructions, all of which use Lemma 42 as a helper method. Intuitively, we want to construct a realization H of G replacing edges of G by long paths in such a way that the interference between the paths is well-controlled. To do so, we create “levels,” ordering vertices in H by their distance to “original” vertices of G in H . The levels are ordered from left to right. All edges are either within a level or between two consecutive levels; and between two consecutive levels, the edges usually form a matching (except edges from the first to the second level; and edges that “close” a path). Then in each “level”, the ordering is defined according the function f of its closest vertex in $V(G)$, either the same or the reverse of f . This means that the matchings between consecutive levels will either consist of edges that all pairwise cross, or all pairwise do not cross (depending on whether the ordering stays the same or is reversed). Moreover, within each level, we define a way to “close” some path. The key points in our constructions are: (1) whether the ordering is reversed between consecutive levels; (2) how to “close the path”; (3) where to put the vertices used to “close” paths.

Theorem 43. *Given a graph G , there is a graph H_3 and two injective functions $\tau_5, \tau_6 : V(G) \rightarrow \mathbb{R}$ such that:*

- (1) *There is a list assignment $L_1 : V(H_3) \rightarrow 2^{[3]}$ such that the pair (H_3, L_1) is a realization of G .*
- (2) *The ordered graphs (H_3, τ_5) and (H_3, τ_6) can be constructed from G in time $\mathcal{O}(m^2)$, where $m = |E(G)|$.*
- (3) *The ordered graph (H_3, τ_5) is M_1 and M_2 -free.*
- (4) *The ordered graph (H_3, τ_6) is M_3 -free.*

Therefore, the ORDERED GRAPH LIST-3-COLORING PROBLEM is NP-complete when restricted to the class of M_1 , M_2 or M_3 -free ordered graphs.

Proof. We denote $|V(G)| = n$ and $|E(G)| = m$. Let $f : E(G) \rightarrow [m]$ be an ordering of $E(G)$, and $g : V(G) \rightarrow [n]$ be an ordering of $V(G)$. Intuitively, in each level, we use a path with very short edges to connect two given vertices, and put the vertices used to “close paths” in each level. For τ_5 , the ordering within each level is the same as f , which creates more crossing edges. For τ_6 , the ordering within each level is either the same or the reverse of f depending on the

parity, which creates more parallel (nested) edges. Formally, we construct the ordered graphs (H_3, τ_5) and (H_3, τ_6) as follows.

For every edge $uv \in E(G)$, we create 6 vertices $w_1(u, v, j)$ and $w_1(v, u, j)$ for $j \in \{3f(uv) - 2, 3f(uv) - 1, 3f(uv)\}$, and add edges $uw_1(u, v, j)$ and $vw_1(v, u, j)$ for $j \in \{3f(uv) - 2, 3f(uv) - 1, 3f(uv)\}$. Let W_1 be the set of such vertices $w_1(u, v, j)$. Suppose now W_{i-1} has been defined. We create a new vertex $w_i(u, v, j)$ if $w_{i-1}(u, v, j) \in W_{i-1}$ and $j \geq i$, and add an edge $w_{i-1}(u, v, j)w_i(u, v, j)$. Let W_i be the set of such vertices $w_i(u, v, j)$. For convenience, we also denote $W_0 = V(G)$.

We define $\tau_5(v) = \tau_6(v) = g(v)$ for every $v \in V(G)$. For every $i \in \{1, \dots, 3m\}$, we define $\tau_5(w_i(u, v, j)) = n + (\sum_{i'=1}^{i-1} |W_{i'}|) + |\{w_i(x, y, k) \in W_i : g(x) < g(u)\}| + |\{w_i(u, y, k) \in W_i : g(y) < g(v)\}| + j + 3 - 3f(uv)$ for every $w_i(u, v, j) \in W_i$.

Let us consider the vertices $w_i(u', v', i)$ and $w_i(v', u', i)$ in W_i . Without loss of generality we may assume that $\tau_5(w_i(u', v', i)) < \tau_5(w_i(v', u', i))$. For every vertex $w_i(u, v, j) \in W_i$ with $\tau_5(w_i(u', v', i)) < \tau_5(w_i(u, v, j)) < \tau_5(w_i(v', u', i))$, we add one new vertex $z_i(u, v, j)$. Let Z_i be the set of such vertices $z_i(u, v, j)$. Let $\tau_5(z_i(u, v, j)) = \tau_5(w_i(u, v, j)) + \frac{1}{2}$. For every $z_i(u, v, j), z_i(u^*, v^*, j^*) \in Z_i$ with $\tau_5(z_i(u, v, j)) < \tau_5(z_i(u^*, v^*, j^*))$, we add edges $z_i(u, v, j)z_i(u^*, v^*, j^*)$ if $\tau_5(z_i(u, v, j)) = \tau_5(z_i(u^*, v^*, j^*)) - 1$, and edges $w_i(u', v', i)z_i(u, v, j)$ if $\tau_5(z_i(u, v, j)) = \tau_5(w_i(u', v', i)) + \frac{3}{2}$, and $w_i(v', u', i)z_i(u^*, v^*, j^*)$ if $\tau_5(z_i(u^*, v^*, j^*)) = \tau_5(w_i(v', u', i)) - \frac{1}{2}$.

We define $\tau_6(v)$ as follows. Let $\tau_6(v) = \tau_5(v)$ for every $v \in V(G)$. For every $i \in \{1, \dots, 3m\}$ and $w_i(u, v, j) \in W_i$, we define $\tau_6(w_i(u, v, j)) = \tau_5(w_i(u, v, j))$ if i is even, and $\tau_6(w_i(u, v, j)) = n + (\sum_{i'=1}^{i-1} |W_{i'}|) + |W_i| + 1 - (|\{w_i(x, y, k) \in W_i : g(x) < g(u)\}| + |\{w_i(u, y, k) \in W_i : g(y) < g(v)\}|) + j + 3 - 3f(uv)$ if i is odd. For every $i \in \{1, \dots, 3m\}$ and $z_i(u, v, j) \in Z_i$, let $\tau_6(z_i(u, v, j)) = \tau_6(w_i(u, v, j)) + \frac{1}{2}$ if i odd, and $\tau_6(z_i(u, v, j)) = \tau_6(w_i(u, v, j)) - \frac{1}{2}$ if i is even.

Let $V(H_3) = V(G) \cup \bigcup_{i=1}^{3m} (W_i \cup Z_i)$ and $E(H_3)$ be the set of all edges defined above. From the construction, the functions τ_5 and τ_6 are orderings of H_3 .

We then let P_i^{uv} consist of vertices

$$u, v, w_1(u, v, 3f(uv) + i - 3), w_2(u, v, 3f(uv) + i - 3), \dots, w_{3f(uv)+i-3}(u, v, 3f(uv) + i - 3), \\ w_1(v, u, 3f(uv) + i - 3), w_2(v, u, 3f(uv) + i - 3), \dots, w_{3f(uv)+i-3}(v, u, 3f(uv) + i - 3),$$

as well as all vertices in $Z_{3f(uv)+i-3}$, for $uv \in E(G)$ and $i \in [3]$. The graph H_3 and the paths P_i^{uv} satisfy the condition of Lemma 42. Thus, letting L_1 be as in Lemma 42, we have:

(27) *The pair (H_3, L_1) is a realization of G .*

Also, we deduce

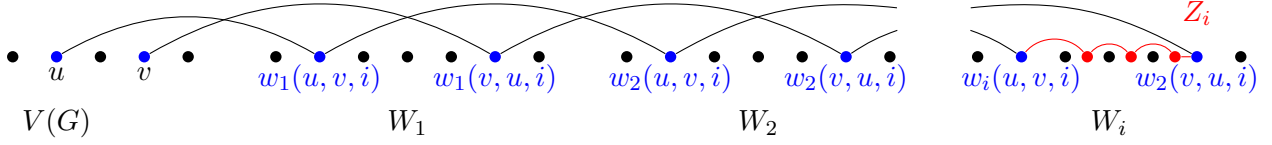
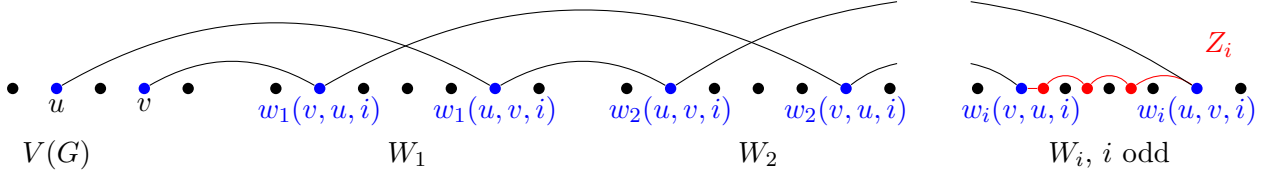
(28) *The ordered graphs (H_3, τ_5) and (H_3, τ_6) can be computed from G in time $\mathcal{O}(m^2)$.*

It takes time $\mathcal{O}(m)$ and $\mathcal{O}(n)$ to get the functions f and g , respectively. The set W_1 can be computed from G and f in time $\mathcal{O}(m)$. For $i \in \{2, \dots, 3m\}$, the set W_i can be computed from W_{i-1} in time $\mathcal{O}(m)$. And the function $\tau_5|_{W_i}$ can be computed in time $\mathcal{O}(m)$ for $i \in [3m]$. The set Z_i and the function $\tau_5|_{Z_i}$ can be computed from W_i and $\tau_5|_{W_i}$ in time $\mathcal{O}(m)$. Finally, the function τ_6 can be computed in time $\mathcal{O}(m^2)$. Thus, the ordered graphs (H_3, τ_5) and (H_3, τ_6) can be computed from G in time $\mathcal{O}(m^2)$.

From the construction defined above, we notice that:

(29) *Given $k \in \{5, 6\}$, for each edge $uv \in E(H_3)$ with $\tau_k(u) < \tau_k(v)$ and $u \in Z_i$ or $v \in Z_i$ for some $i \in \{1, \dots, 3m\}$, there is at most one vertex w with $\tau_k(u) < \tau_k(w) < \tau_k(v)$.*

With this observation, we are ready to prove the remaining two properties.

FIGURE 8. Construction of (H_3, τ_5) from Theorem 43.FIGURE 9. Construction of (H_3, τ_6) from Theorem 43.

(30) *The ordered graph (H_3, τ_5) is M_1 and M_2 -free.*

Suppose (H_3, τ_5) contains a copy of M_1 or M_2 as ordered induced subgraph. We enumerate the vertices v_i of M_1 or M_2 in increasing order with respect to the ordering τ_5 . Let us consider the vertex v_1 . Because of the vertices v_3, v_4 , by (29) we have $v_1, v_2, v_5, v_6 \notin Z_i$ for every $i \in [3m]$. If $v_1 \in V(G)$, then we have $v_6 \in W_1$. So $v_2 \in V(G)$ and $v_5 \in W_1$. But then either $\tau_5(v_2) < \tau_5(v_1)$ or $\tau_5(v_5) > \tau_5(v_6)$, both of which cause a contradiction. If $v_1 \in W_i$ for some $i \in [3m]$, then $v_6 \in W_{i+1}$, and $v_2 \in W_i$, $v_5 \in W_{i+1}$, which is a contradiction. Thus, we have proved (H_3, τ_5) is M_1 -free and M_2 -free.

(31) *The ordered graph (H_3, τ_6) is M_3 -free.*

Suppose (H_3, τ_6) contains a copy of M_3 as ordered induced subgraph. We enumerate the vertices v_i of M_3 in increasing order with respect to the ordering τ_6 . Let us consider the vertex v_1 . Because of the vertices v_2, v_3 , by (29) we have $v_1, v_4 \notin Z_i$ for every $i \in [3m]$. Similarly, we have $v_t \notin Z_i$ for every $t \in [6]$ and $i \in [3m]$. Let $v_1 \in W_i$ for some $i \in \{0, 1, \dots, 3m\}$, then $v_4 \in W_{i+1}$. Then let us consider the vertex v_2 . For every vertex $x \in W_i$ with $\tau_6(x) > \tau_6(v_1)$ and for every vertex $y \in N(x)$ with $\tau_6(y) > \tau_6(x)$ and $y \notin Z_i$, we have $\tau_6(y) < \tau_6(v_4)$. Thus, the vertex v_2 is in W_{i+1} . But then for every vertex $x \in W_{i+1}$ with $\tau_6(v_2) < \tau_6(x) < \tau_6(v_4)$ and for every vertex $y \in N(x)$ with $\tau_6(y) > \tau_6(x)$ and $y \notin Z_i$, we have $\tau_6(y) < \tau_6(v_5)$, which means the vertex v_3 has nowhere to go. Thus, we have proved (H_3, τ_6) is M_3 -free. ■

Theorem 44. *Given a graph G , there is an ordered graph (H_4, τ_7) and a list assignment $L_2 : V(H_4) \rightarrow 2^{[3]}$ such that:*

- (1) *The pair (H_4, L_2) is a realization of G .*
- (2) *The ordered graph (H_4, τ_7) can be constructed from G in time $\mathcal{O}(m^2)$.*
- (3) *The ordered graph (H_4, τ_7) is M_4 -free.*

Therefore, the ORDERED GRAPH LIST-3-COLORING PROBLEM is NP-complete when restricted to the class of M_4 -free ordered graphs.

Proof. We denote $|V(G)| = n$ and $|E(G)| = m$. Let $f : E(G) \rightarrow [m]$ be an ordering of $E(G)$, and $g : V(G) \rightarrow [n]$ be an ordering of $V(G)$. In this construction, the ordering within each level is the same as f . We “close the path” directly using one extra vertex. The key idea is, the

vertices used to “close paths” are put at last. Formally, we construct the ordered graph (H_4, τ_7) as follows.

For every edge $uv \in E(G)$, we create 6 vertices $w_1(u, v, j)$ and $w_1(v, u, j)$ for $j \in \{3f(uv) - 2, 3f(uv) - 1, 3f(uv)\}$, and add edges $uw_1(u, v, j)$ and $vw_1(v, u, j)$ for $j \in \{3f(uv) - 2, 3f(uv) - 1, 3f(uv)\}$. Let W_1 be the set of such vertices $w_1(u, v, j)$. Suppose now W_{i-1} has been defined. We create a new vertex $w_i(u, v, j)$ if $w_{i-1}(u, v, j) \in W_{i-1}$ and $j \geq i$, and add an edge $w_{i-1}(u, v, j)w_i(u, v, j)$. Let W_i be the set of such vertices $w_i(u, v, j)$. For convenience, we also denote $W_0 = V(G)$.

We define $\tau_7(v) = g(v)$ for every $v \in V(G)$. For every $i \in \{1, \dots, 3m\}$, we define $\tau_7(w_i(u, v, j)) = n + \sum_{i'=1}^{i-1} |W_{i'}| + |\{w_i(x, y, k) \in W_i : g(x) < g(u)\}| + |\{w_i(u, y, k) \in W_i : g(y) < g(v)\}| + j + 3 - 3f(uv)$ for every $w_i(u, v, j) \in W_i$.

Let us consider the vertices $w_i(u', v', i)$ and $w_i(v', u', i)$ in W_i . For every $i \in \{1, \dots, 3m\}$, we add one new vertex z_i and edges $w_i(u', v', i)z_i$ and $w_i(v', u', i)z_i$, and let $\tau_7(z_i) = n + 3m(3m + 1) + (3m - i + 1)$.

Let $V(H_4) = V(G) \cup (\bigcup_{i=1}^{3m} W_i \cup \{z_i\})$ and $E(H_4)$ be the set of all edges defined above. From the construction, the function $\tau_7 : V(H_4) \rightarrow \mathbb{R}$ is an ordering of H_4 .

We then let P_i^{uv} consist of vertices

$$u, v, w_1(u, v, 3f(uv) + i - 3), w_2(u, v, 3f(uv) + i - 3), \dots, w_{3f(uv)+i-3}(u, v, 3f(uv) + i - 3), \\ w_1(v, u, 3f(uv) + i - 3), w_2(v, u, 3f(uv) + i - 3), \dots, w_{3f(uv)+i-3}(v, u, 3f(uv) + i - 3),$$

and $z_{3f(uv)+i-3}$, for every $uv \in E(G)$ and $i \in [3]$. The graph H_4 and the paths P_i^{uv} satisfy the condition of Lemma 42. Thus, letting L_2 be as in Lemma 42, we have:

(32) *The pair (H_4, L_2) is a realization of G .*

Also, we deduce:

(33) *The ordered graph (H_4, τ_7) can be computed from G in time $\mathcal{O}(m^2)$.*

It takes time $\mathcal{O}(m)$ and $\mathcal{O}(n)$ to get the functions f and g , respectively. The set W_1 can be computed from G and f in time $\mathcal{O}(m)$. For $i \in \{2, \dots, 3m\}$, the set W_i can be computed from W_{i-1} in time $\mathcal{O}(m)$. And the function $\tau_7|_{W_i}$ can be computed in time $\mathcal{O}(m)$ for $i \in [3m]$. Thus, the ordered graph (H_4, τ_7) can be computed from G in time $\mathcal{O}(m^2)$.

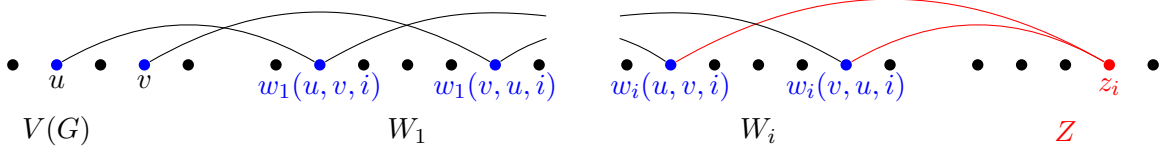
(34) *The ordered graph (H_2, τ_7) is M_4 -free.*

Suppose that (H_4, τ_7) contains a copy of M_4 as ordered induced subgraph. We enumerate the vertices v_i of M_4 in increasing order with respect to the ordering τ_7 . Because of the edges of M_4 , we have $v_1, v_2, v_3 \notin \{z_1, \dots, z_{3m}\}$, and $v_1, v_2 \notin W_{3m}$. Let $v_1 \in W_i$ for some $i \in \{0, \dots, 3m - 1\}$. If $v_5 \in W_{i+1}$, then we have $v_4 \notin \{z_1, \dots, z_{3m}\}$. For every vertex x with $\tau_7(v_1) < \tau_7(x) < \tau_7(v_5)$ and for every vertex $y \in N(x) \setminus \{z_1, \dots, z_{3m}\}$ with $\tau_7(y) > \tau_7(x)$, we have $\tau_7(y) > \tau_7(v_5)$. Thus, we conclude that $v_5 \in \{z_1, \dots, z_{3m}\}$. But then for every vertex $x \in \{z_1, \dots, z_{3m}\}$ with $\tau_7(x) > \tau_7(v_5)$ and for every vertex $y \in N(x)$, we have $\tau_7(y) < \tau_7(v_1)$, which is a contradiction. Thus, we have proved (H_4, τ_7) is M_4 -free. \blacksquare

Theorem 45. *Given a graph G , there is an ordered graph (H_5, τ_8) and a list assignment $L_3 : V(H_5) \rightarrow 2^{[3]}$ such that:*

- (1) *The pair (H_5, L_3) is a realization of G .*
- (2) *The ordered graph (H_5, τ_8) can be constructed from G in time $\mathcal{O}(m^3)$.*
- (3) *The ordered graph (H_5, τ_8) is M_5 -free.*

Therefore, the ORDERED GRAPH LIST-3-COLORING PROBLEM is NP-complete when restricted to the class of M_5 -free ordered graphs.

FIGURE 10. Construction of (H_4, τ_7) from Theorem 44.

Proof. We denote $|V(G)| = m$ and $|E(G)| = n$. Let $f : E(G) \rightarrow [m]$ be an ordering of $E(G)$, and $g : V(G) \rightarrow [n]$ be an ordering of $V(G)$. The key point in this construction is the way we “close path.” Instead of adding new vertices, one vertex “switches” (in the ordering) with its previous vertex until the two given vertices are consecutive. Formally, we construct the ordered graph (H_5, τ_8) as follows.

For every edge $uv \in E(G)$, we create 6 vertices $w_1(u, v, j)$ and $w_1(v, u, j)$ for $j \in \{3f(uv) - 2, 3f(uv) - 1, 3f(uv)\}$, and add edges $uw_1(u, v, j)$ and $vw_1(v, u, j)$ for $j \in \{3f(uv) - 2, 3f(uv) - 1, 3f(uv)\}$. Let W_1 be the set of such vertices $w_1(u, v, j)$. Suppose now W_{i-1} has been defined. We create a new vertex $w_i(u, v, j)$ if $w_{i-1}(u, v, j) \in W_{i-1}$ and $j \geq i$. Let W_i be the set of such vertices $w_i(u, v, j)$. For convenience, we also denote $W_0 = V(G)$.

We define $\tau_8(v) = g(v)$ for every $v \in V(G)$. For every $i \in \{1, \dots, 3m\}$, we define $\tau_8(w_i(u, v, j)) = 36m^2(i-1) + n + \sum_{i'=1}^{i-1} |W_{i'}| + |\{w_i(x, y, k) \in W_i : g(x) < g(u)\}| + |\{w_i(u, y, k) \in W_i : g(y) < g(v)\}| + j + 3 - 3f(uv)$ for every $w_i(u, v, j) \in W_i$.

Let us consider the vertices $w_i(u', v', i)$ and $w_i(v', u', i)$ in W_i . Without loss of generality we may assume that $\tau_8(w_i(u', v', i)) = \tau_8(w_i(v', u', i)) - a_i$ for some $a_i \in \mathbb{N}$. We create a sequence of sets $X_k^i = \{x_k^i(u, v, j) : w_i(u, v, j) \in W_i\}$, for $k \in [a_i - 1]$. For convenience, we also denote $X_{a_i}^i = W_{i+1} = X_0^{i+1}$. For every $i \in [3m]$, we add edges $x_k^i(u, v, j)x_{k+1}^i(u, v, j)$ for every $k \in \{0, \dots, a_i - 1\}$ and $x_{k+1}^i(u, v, j) \in X_{k+1}^i$, and edge $x_{a_i-1}^i(u', v', i)x_{a_i-1}^i(v', u', i)$.

For every $k \in [a_i - 1]$, let $\tau_8(x_k^i(v', u', i)) = \tau_8(w_i(v', u', i)) + 6mk - k - \frac{1}{2}$, and $\tau_8(x_k^i(u, v, j)) = \tau_8(w_i(u, v, j)) + 6mk$ otherwise. Notice that this implies $\tau_8(x_{k+1}^i(v', u', i)) - \tau_8(x_k^i(v', u', i)) = \tau_8(x_{k+1}^i(u, v, j)) - \tau_8(x_k^i(u, v, j)) - 1$ for every $k \in \{0, \dots, a_i - 1\}$ and $\{u, v\} \neq \{u', v'\}$ and $x_{k+1}^i(u, v, j) \in X_{k+1}^i$.

Let $V(H_5) = V(G) \cup (\bigcup_{i=1}^{3m} W_i \cup (\bigcup_{k=1}^{a_i-1} X_k^i))$ and $E(H_5)$ be the set of all edges defined above. From the construction, the function $\tau_8 : V(H_5) \rightarrow \mathbb{R}$ is an ordering of H_5 .

For $uv \in E(G)$ and $i \in [3]$, we then let P_i^{uv} consist of vertices

$$u, v, w_1(u, v, 3f(uv) + i - 3), w_2(u, v, 3f(uv) + i - 3), \dots, w_{3f(uv)+i-3}(u, v, 3f(uv) + i - 3), \\ w_1(v, u, 3f(uv) + i - 3), w_2(v, u, 3f(uv) + i - 3), \dots, w_{3f(uv)+i-3}(v, u, 3f(uv) + i - 3),$$

as well as all vertices $x_k^j(u, v, 3f(uv) + i - 3)$ and $x_k^j(v, u, 3f(uv) + i - 3)$ for $j \in [3f(uv) + i - 3]$ and $k \in [a_j - 1]$. The graph H_5 and the paths P_i^{uv} satisfy the conditions of Lemma 42. Thus, letting L_3 be as in Lemma 42, we have:

(35) *The pair (H_5, L_3) is a realization of G .*

Also, we deduce:

(36) *The ordered graph (H_5, τ_8) can be computed from G in time $\mathcal{O}(m^3)$.*

The set W_1 can be computed from G and f in time $\mathcal{O}(m)$. For $i \in \{2, \dots, 3m\}$, the set W_i can be computed from W_{i-1} in time $\mathcal{O}(m)$. And the function $\tau_8|_{W_i}$ can be computed in time $\mathcal{O}(m)$ for $i \in [3m]$. The set X_k^i and the function $\tau_8|_{X_k^i}$ can be computed from W_i and $\tau_8|_{W_i}$ in

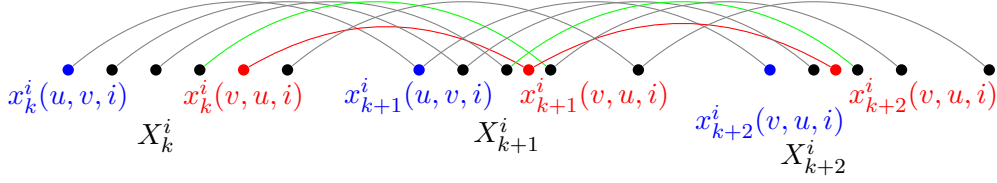


FIGURE 11. Construction of X_k^i , X_{k+1}^i and X_{k+2}^i in (H_5, τ_8) from Theorem 45.

time $\mathcal{O}(m)$ for $k \in [a_i - 1]$. And $a_i \leq |W_i| = \mathcal{O}(m)$, $|X_k^i| = |W_i|$ for $k \in [a_i - 1]$. Thus, the ordered graph (H_5, τ_8) can be computed from G in time $\mathcal{O}(m^3)$.

(37) *The ordered graph (H_5, τ_8) is M_5 -free.*

Suppose (H_5, τ_8) contains a copy of M_5 as ordered induced subgraph. We enumerate the vertices v_i of M_5 in increasing order with respect to the ordering τ_8 . Let us consider the edges v_1v_5 and v_2v_3 . We claim that $v_2 = x_k^i(v', u', i)$ and $v_3 = x_{k+1}^i(v', u', i)$ for some $i \in [3m]$ and $k \in [a_i - 1]$, and u' and v' being the vertices in $V(H_5)$ such that $g(u') < g(v')$ and $f(u'v') = \lceil \frac{i}{3} \rceil$. This is because otherwise from the construction of τ_8 , the condition $\tau_8(v_1) < \tau_8(v_2)$ implies $\tau_8(y) < \tau_8(z)$ for every $y \in N^+(v_1)$ and $z \in N^+(v_2)$.

Since v_1v_5 is an edge, we have $v_1 \in X_k^i$ and $v_5 \in X_{k+1}^i$. Moreover, from the construction of τ_8 , for every two distinct $x, x' \in X_{k+1}^i \setminus \{x_k^i(v', u', i)\}$, we have $|\tau_8(x) - \tau_8(x')| \geq 1$. Since $\tau_8(x_{k+1}^i(v', u', i)) - \tau_8(x_k^i(v', u', i)) = \tau_8(x_{k+1}^i(u, v, i)) - \tau_8(x_k^i(u, v, i)) - 1$, there is no vertex in X_{k+1}^i which could be v_4 , a contradiction. Thus, we have proved (H_5, τ_8) is M_5 -free. ■

Thus, combining Theorem 43, 44 and 45, we have proved Theorem 32.

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