Four-coloring P_6-free graphs.
I. Extending an excellent precoloring

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Abstract

This is the first paper in a series whose goal is to give a polynomial-time algorithm for the 4-COLORING PROBLEM and the 4-PRECOLORING EXTENSION problem restricted to the class of graphs with no induced six-vertex path, thus proving a conjecture of Huang. Combined with previously known results this completes the classification of the complexity of the 4-coloring problem for graphs with a connected forbidden induced subgraph.

In this paper we give a polynomial-time algorithm that determines if a special kind of precoloring of a P_6-free graph has a precoloring extension, and constructs such an extension if one exists. Combined with the main result of the second paper of the series, this gives a complete solution to the problem.

1 Introduction

All graphs in this paper are finite and simple. We use $[k]$ to denote the set $\{1, \dots, k\}$. Let G be a graph. A *k-coloring* of G is a function $f : V(G) \to [k]$. A k-coloring is *proper* if for every edge $uv \in E(G)$, $f(u) \neq f(v)$, and G is *k-colorable* if G has a proper k-coloring. The k-COLORING PROBLEM is the problem of deciding, given a graph G, if G is k-colorable. This problem is well-known to be NP-hard for all $k \geq 3$.

A function $L : V(G) \to 2^{[k]}$ that assigns a subset of $[k]$ to each vertex of a graph G is a *k-list assignment* for G. For a k-list assignment L, a function $f : V(G) \to [k]$ is an *L-coloring* if f is a k-coloring of G and $f(v) \in L(v)$ for all $v \in V(G)$. A graph G is *L-colorable* if G has a proper L-coloring. We denote by $X^0(L)$ the set of all vertices v of G with $|L(v)| = 1$. The k-LIST COLORING PROBLEM is the problem of deciding, given a graph G and a k-list assignment L, if G is L-colorable. Since this generalizes the k-coloring problem, it is also NP-hard for all $k \geq 3$.

Let G be a graph. For $X \subseteq V(G)$ we denote by $G|X$ the subgraph induced by G on X, and by $G \setminus X$ the graph $G|(V(G) \setminus X)$. If $X = \{x\}$, we write $G \setminus x$ to mean $G \setminus \{x\}$. A *k-precoloring* (G, X, f) of a graph G is a function $f : X \to [k]$ for a set $X \subseteq V(G)$ such that f is a proper k-coloring of $G|X$. Equivalently, a k-precoloring is a k-list assignment L in which $|L(v)| \in \{1, k\}$ for all $v \in V(G)$. A *k-precoloring extension* for (G, X, f) is a proper k-coloring g of G such that $g|_X = f|_X$, and the k-PRECOLORING EXTENSION PROBLEM is the problem of deciding, given a graph G and a k-precoloring (G, X, f), if (G, X, f) has a k-precoloring extension.

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We denote by P_t the path with t vertices. Given a path P, its *interior* is the set of vertices that have degree two in P. We denote the interior of P by P^*. A P_t *in a graph* G is a sequence $v_1 - \ldots - v_t$ of pairwise distinct vertices where for $i, j \in [t]$, v_i is adjacent to v_j if and only if $|i-j| = 1$. We denote by $V(P)$ the set $\{v_1, \ldots, v_t\}$, and if $a, b \in V(P)$, say $a = v_i$ and $b = v_j$ and $i < j$, then $a - P - b$ is the path $v_i - v_{i+1} - \ldots - v_j$. A graph is P_t*-free* if there is no P_t in G. Throughout the paper by "polynomial time" or "polynomial size" we mean that there exists a polynomial p such that the running time, or size, is bounded by $p(|V(G)|)$.

Since the k-COLORING PROBLEM and the k-PRECOLORING EXTENSION PROBLEM are NP-hard for $k \geq 3$, their restrictions to graphs with a forbidden induced subgraph have been extensively studied; see [2, 7] for a survey of known results. In particular, the following is known (given a graph H, we say that a graph G is H*-free* if no induced subgraph of G is isomorphic to H):

Theorem 1 ([7]). *Let H be a (fixed) graph, and let $k > 2$. If the k-COLORING PROBLEM can be solved in polynomial time when restricted to the class of H-free graphs, then every connected component of H is a path (assuming $P \neq NP$).*

Thus if we assume that H is connected, then the question of determining the complexity of k-coloring H-free graph is reduced to studying the complexity of coloring graphs with certain induced paths excluded, and a significant body of work has been produced on this topic. Below we list a few such results.

Theorem 2 ([1]). *The 3-COLORING PROBLEM can be solved in polynomial time for the class of P_7-free graphs.*

Theorem 3 ([5]). *The k-COLORING PROBLEM can be solved in polynomial time for the class of P_5-free graphs.*

Theorem 4 ([6]). *The 4-COLORING PROBLEM is NP-complete for the class of P_7-free graphs.*

Theorem 5 ([6]). *For all $k \geq 5$, the k-COLORING PROBLEM is NP-complete for the class of P_6-free graphs.*

The only cases for which the complexity of k-coloring P_t-free graphs is not known are $k = 4$, $t = 6$, and $k = 3$, $t \geq 8$. This is the first paper in a series of two. The main result of the series is the following:

Theorem 6. *The 4-PRECOLORING EXTENSION PROBLEM can be solved in polynomial time for the class of P_6-free graphs.*

In contrast, the 4-LIST COLORING PROBLEM restricted to P_6-free graphs is NP-hard as proved by Golovach, Paulusma, and Song [7]. As an immediate corollary of Theorem 6, we obtain that the 4-COLORING PROBLEM for P_6-free graphs is also solvable in polynomial time. This proves a conjecture of Huang [6], thus resolving the former open case above, and completes the classification of the complexity of the 4-COLORING PROBLEM for graphs with a connected forbidden induced subgraph.

Let G be a graph. For disjoint subsets $A, B \subset V(G)$ we say that A is *complete* to B if every vertex of A is adjacent to every vertex of B, and that A is *anticomplete* to B if every vertex of A is non-adjacent to every vertex of B. If $A = \{a\}$ we write a is complete (or anticomplete) to B to mean $\{a\}$ that is complete (or anticomplete) to B. If $a \notin B$ is not complete and not anticomplete to B, we say that a is *mixed* on B. Finally, if H is an induced subgraph of G and $a \in V(G) \setminus V(H)$, we say that a is *complete to, anticomplete to*, or *mixed on* H if a is complete to, anticomplete to, or mixed on $V(H)$, respectively. For $v \in V(G)$ we write $N_G(v)$ (or $N(v)$ when there is no danger of confusion) to mean the set of vertices of G that are adjacent to v. Observe that since G is simple, $v \notin N(v)$. For $A \subseteq V(G)$, an *attachment* of A is a vertex of $V(G) \setminus A$ complete to A. For $B \subseteq V(G) \setminus A$ we denote by $B(A)$ the set of attachments of A in B. If $F = G|A$, we sometimes write $B(F)$ to mean $B(V(F))$.

Given a list assignment L for G, we say that the pair (G, L) is colorable if G is L-colorable. For $X \subseteq V(G)$, we write $(G|X, L)$ to mean the list coloring problem where we restrict the domain of the list assignment L to X. Let $X \subseteq V(G)$ be such that $|L(x)| = 1$ for every $x \in X$, and let $Y \subseteq V(G)$. We say that a list assignment M is *obtained from L by updating Y from X* if $M(v) = L(v)$ for every $v \notin Y$, and $M(v) = L(v) \setminus \bigcup_{x \in N(v) \cap X} L(x)$ for every $v \in Y$. If $Y = V(G)$, we say that M is *obtained from L by updating from X*. If M is obtained from L by updating from $X^0(L)$, we say that M is *obtained from L by updating.* Let $L = L_0$, and for $i \geq 1$ let L_i be obtained from L_{i-1} by updating. If $L_i = L_{i-1}$, we say that L_i is *obtained* *from L by updating exhaustively.* Since $0 \leq \sum_{v \in V(G)} |L_j(v)| < \sum_{v \in V(G)} |L_{j-1}(v)| \leq 4|V(G)|$ for all $j < i$, it follows that $i \leq 4|V(G)|$ and thus L_i can be computed from L in polynomial time.

An *excellent starred precoloring* of a graph G is a six-tuple $P = (G, S, X_0, X, Y^*, f)$ such that

(A) $f : S \cup X_0 \to \{1,2,3,4\}$ is a proper coloring of $G|(S \cup X_0)$;

(B) $V(G) = S \cup X_0 \cup X \cup Y^*$;

(C) $G|S$ is connected and no vertex in $V(G) \setminus S$ is complete to S;

(D) every vertex in X has neighbors of at least two different colors (with respect to f) in S;

(E) no vertex in X is mixed on a component of $G|Y^*$; and

(F) for every component of $G|Y^*$, there is a vertex in $S \cup X_0 \cup X$ complete to it.

We call S the *seed* of P. We define two list assignments associated with P. First, define $L_P(v) = \{f(v)\}$ for every $v \in S \cup X_0$, and let $L_P(v) = \{1,2,3,4\} \setminus (f(N(v) \cap S))$ for $v \notin S \cup X_0$. Second, M_P is the list assignment obtained as follows. First, define M_1 to be the list assignment for $G|(X \cup X_0)$ obtained from $L_P|(X \cup X_0)$ by updating exhaustively; let $X_1 = \{x \in X \cup X_0 : |M_1(x_1)| = 1\}$. Now define $M_P(v) = L_P(v)$ if $v \notin X \cup X_0$, and $M_P(v) = M_1(v)$ if $v \in X \cup X_0$. Let $X^0(P) = X^0(M_P)$. Then $S \cup X_0 \subseteq X^0(P)$. A *precoloring extension of* P is a proper 4-coloring c of G such that $c(v) = f(v)$ for every $v \in S \cup X_0$; it follows that $M_P(v) = \{c(v)\}$ for every $v \in X^0(P)$. It will often be convenient to assume that $X_0 = X^0(P) \setminus S$, and this assumption can be made without loss of generality. Note that in this case, $M_P(v) = L_P(v)$ for all $v \in X$.

For an excellent starred precoloring P and a collection excellent starred $\mathcal{L}$ of precolorings, we say that $\mathcal{L}$ is an *equivalent collection* for P (or that P is *equivalent* to $\mathcal{L}$) if P has a precoloring extension if and only if at least one of the precolorings in $\mathcal{L}$ has a precoloring extension, and a precoloring extension of P can be constructed from a precoloring extension of a member of $\mathcal{L}$ in polynomial time.

We break the proof of Theorem 6 into two independent parts, each handled in a separate paper of the series. In the first part, we reduce the 4-PRECOLORING EXTENSION PROBLEM for P_6-free graphs to determining if an excellent starred precolorings of a P_6-free graph has a precoloring extension, and finding one if it exists. In fact, we restrict the problem further, by ensuring that there is a universal bound (that works for all 4-precolorings of all P_6-free graphs) on the size of the seed of the excellent starred precolorings that we need to consider. More precisely, we prove:

Theorem 7. *There exists an integer $C > 0$ and a polynomial-time algorithm with the following specifications.*

Input: *A 4-precoloring (G, X_0, f) of a P_6-free graph G.*

Output: *A collection $\mathcal{L}$ of excellent starred precolorings of G such that*

1. *If for every $P' \in \mathcal{L}$ we can in polynomial time either find a precoloring extension of P', or determine that none exists, then we can construct a 4-precoloring extension of (G, X_0, f) in polynomial time, or determine that none exists;*
2. *$|\mathcal{L}| \leq |V(G)|^C$; and*
3. *for every $(G', S', X'_0, X', Y^*, f') \in \mathcal{L}$,*
   - *$|S'| \leq C$;*
   - *$X_0 \subseteq S' \cup X'_0$;*
   - *G' is an induced subgraph of G; and*
   - *$f'|X_0 = f|X_0$.*

The proof of Theorem 7 is hard and technical, and we postpone it to the second paper of the series [3]. The second part of the proof of Theorem 6 is an algorithm that tests in polynomial time if an excellent starred precoloring (where the size of the seed is fixed) has a precoloring extension. The goal of the present paper is to solve this problem. We prove:

Theorem 8. *For every positive integer C there exists a polynomial-time algorithm with the following specifications:*

Input: *An excellent starred precoloring $P = (G, S, X_0, X, Y^*, f)$ of a P_6-free graph G with $|S| \leq C$.*

Output: *A precoloring extension of P or a statement that none exists.*

Clearly, Theorem 7 and Theorem 8 together imply Theorem 6. The proof of Theorem 8 consists of several steps. At each step we replace the problem that we are trying to solve by a polynomially sized collection of simpler problems, and the problems created in the last step can be encoded via 2-SAT (and therefore can be solved in polynomial time). Here is an outline of the proof. First we show that an excellent starred precoloring P of a P_6-free graph G can be replaced by a polynomially sized collection $\mathcal{L}$ of excellent starred precolorings of G that have an additional property (to which we refer as "being orthogonal") and P has a precoloring extension if and only if some member of $\mathcal{L}$ does. Thus in order to prove Theorem 8, it is enough to be able to test if an orthogonal excellent starred precoloring of a P_6-free graph has a precoloring extension. Our next step is an algorithm whose input is an orthogonal excellent starred precoloring P of a P_6-free graph G, and whose output is a "companion triple" for P. A companion triple consists of a graph H that may not be P_6-free, but certain important parts of it are, a list assignment L for H, and a correspondence function h that establishes the connection between H and P. Moreover, in order to test if P has a precoloring extension, it is enough to test if (H, L) is colorable.

The next step of the algorithm is replacing (H, L) by a polynomially sized collection $\mathcal{M}$ of list assignments for H, such that (H, L) is colorable if and only if there exists $L' \in \mathcal{L}$ such that (H, L') is colorable, and in addition for every $L' \in \mathcal{L}$ the pair (H, L') is "insulated". Being insulated means that H is the union of four induced subgraphs $H_1, \dots, H_4$, and in order to test if (H, L') is colorable, it is enough to test if (H_i, L') is colorable for each $i \in \{1, 2, 3, 4\}$. The final step of the algorithm is converting the problem of coloring each (H_i, L') into a 2-SAT problem, and solving it in polynomial time. Moreover, at each step of the proof, if a coloring exists, then we can find it, and convert in polynomial time into a precoloring extension of P.

This paper is organized as follows. In Section 2 we produce a collection $\mathcal{L}$ of orthogonal excellent starred precolorings. In Section 3 we construct a companion triple for an orthogonal precoloring. In Section 4 we start with a precoloring and its companion triple, and construct a collection $\mathcal{M}$ of lists L' such that every pair (H, L') is insulated. Finally, in Section 5 we describe the reduction to 2-SAT. Section 6 contains the proof of Theorem 8 and of Theorem 6.

2 From Excellent to Orthogonal

Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring. For $v \in X \cup Y^*$, the *type* of v is the set $N(v) \cap S$. Thus the number of possible types for a given precoloring is at most $2^{|S|}$. In this section we will prove several lemmas that allow us to replace a given precoloring by an equivalent polynomially sized collection of "nicer" precolorings, with the additional property that the size of the seed of each of the new precolorings is bounded by a function of the size of the seed of the precoloring we started with. Keeping the size of the seed bounded allows us to maintain the property that the number of different types of vertices of $X \cup Y^*$ is bounded, and therefore, from the point of view of running time, we can always consider each type separately.

For $T \subseteq S$ we denote by $L_P(T)$ the set $\{1, 2, 3, 4\} \setminus \bigcup_{v \in T} \{f(v)\}$. Thus if v is of type T, then $L_P(v) = L_P(T)$. For $T \subseteq S$ and $U \subseteq X \cup Y^*$ we denote by $U(T)$ the set of vertices of U of type T.

A subset Q of X is *orthogonal* if there exist $a \neq b \in \{1, 2, 3, 4\}$ such that for every $q \in Q$ either $M_P(q) = \{a, b\}$ or $M_P(q) = \{1, 2, 3, 4\} \setminus \{a, b\}$. We say that P is *orthogonal* if $N(y) \cap X$ is orthogonal for every $y \in Y^*$.

The goal of this section is to prove that for every excellent starred precoloring P of a P_6-free graph G, we can construct in polynomial time an equivalent collection $\mathcal{L}(P)$ of orthogonal excellent starred precolorings of G, where $|\mathcal{L}(P)|$ is polynomial.

We start with a few technical lemmas.

Lemma 1. *Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of a P_6-free graph G. Let $i \neq j \in \{1, 2, 3, 4\}$ and $k \in \{1, 2, 3, 4\} \setminus \{i, j\}$. Let T_i, T_j be types such that $L_P(T_i) = \{i, k\}$ and $L_P(T_j) = \{j, k\}$,* *and let $x_i, x_i' \in X(T_i)$ and $x_j, x_j' \in X(T_j)$. Suppose that $y_i, y_j \in Y^*$ are such that $i, j \in M_P(y_i) \cap M_P(y_j)$, where possibly $y_i = y_j$. Suppose further that the only possible edge among x_i, x_i', x_j, x_j' is $x_i x_j$, and y_i is adjacent to x_i' and not to x_i, and y_j is adjacent to x_j' and not to x_j. Then there does not exist $y \in Y^*$ with $i, j \in M_P(y)$ and such that y is complete to $\{x_i, x_j\}$ and anticomplete to $\{x_i', x_j'\}$.*

Proof. Suppose such y exists. Since no vertex of X is mixed on a component of $G|Y^*$, it follows that y is anticomplete to $\{y_i, y_j\}$. Since $x_i, x_i' \in X$ and $i, k \in L_P(T_i)$, it follows that there exists $s_j \in T_i$ with $L_P(s_j) = \{j\}$. Similarly, there exists $s_i \in T_j$ with $L_P(s_i) = \{i\}$. Since $i \in L_P(T_i)$ and $j \in L_P(T_j)$, it follows that s_i is anticomplete to $\{x_i, x_i'\}$ and s_j is anticomplete to $\{x_j, x_j'\}$.

Since $i, j \in M_P(y_i) \cap M_P(y_j) \cap M_P(y)$ it follows that $\{s_i, s_j\}$ is anticomplete to $\{y_i, y_j, y\}$. Since $x_i' - s_j - x_i - y - x_j - s_i - x_j'$ (possibly shortcutting through $x_i x_j$) is not a P_6 in G, it follows that s_i is adjacent to s_j. If y_i is non-adjacent to x_j', and y_j is non-adjacent to x_i', then $y_i \neq y_j$, and since P is excellent, y_i is non-adjacent to y_j, and so $y_i - x_i' - s_j - s_i - x_j' - y_j$ is a P_6, a contradiction, so we may assume that y_i is adjacent to x_j'. But now $x_j' - y_i - x_i' - s_j - x_i - y$ is a P_6, a contradiction. This proves Lemma 1. ☐

Lemma 2. *Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of a P_6-free graph G. Let $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Let T_i, T_j be types such that $L_P(T_i) = \{i, k\}$ and $L_P(T_j) = \{j, k\}$, and let $x_i, x_i' \in X(T_i)$ and $x_j, x_j' \in X(T_j)$. Let $y_i^i, y_j^i \in Y^*$ with $i, l \in M_P(y_i^i) \cap M_P(y_j^i)$, and let $y_i^j, y_j^j \in Y^*$ with $j, l \in M_P(y_i^j) \cap M_P(y_j^j)$, where possibly $y_i^i = y_j^i$ and $y_i^j = y_j^j$. Assume that*

- *some component C_i of $G|Y^*$ contains both y_i^i, y_i^j;*
- *some component C_j of $G|Y^*$ contains both y_j^i, y_j^j;*
- *for every $t \in \{i, j\}$ there is a path M in C_t from y_t^i to y_t^j with $l \in M_P(u)$ for every $u \in V(M)$;*
- *the only possible edge among x_i, x_i', x_j, x_j' is $x_i x_j$;*
- *y_i^i, y_i^j are adjacent to x_i' and not to x_i;*
- *y_j^i, y_j^j are adjacent to x_j' and not to x_j.*

Then there do not exist $y^i, y^j \in Y^*$ with $i, l \in M_P(y^i)$, $j, l \in M_P(y^j)$ and such that

- *some component C of $G|Y^*$ contains both y^i and y^j, and*
- *$l \in M_P(u)$ for every $u \in V(C)$, and*
- *$\{y^i, y^j\}$ is complete to $\{x_i, x_j\}$ and anticomplete to $\{x_i', x_j'\}$.*

Proof. Suppose such y^i, y^j exist. Since P is an excellent starred precoloring, no vertex of X is mixed on a component of $G|Y^*$, and therefore $V(C)$ is anticomplete to $V(C_i) \cup V(C_j)$. Since $x_i, x_i' \in X$ and $i, k \in L_P(T_i)$, it follows that there exists $s_j \in T_i$ with $L_P(s_j) = \{j\}$. Similarly, there exists $s_i \in T_j$ with $L_P(s_i) = \{i\}$. Since $i \in L_P(T_i)$ and $j \in L_P(T_j)$, it follows that s_i is anticomplete to $\{x_i, x_i'\}$ and s_j is anticomplete to $\{x_j, x_j'\}$. Since $i \in M_P(y^i) \cap M_P(y_i^i) \cap M_P(y_j^i)$, it follows that s_i is anticomplete to $\{y^i, y_i^i, y_j^i\}$, and similarly s_j is anticomplete to $\{y^j, y_i^j, y_j^j\}$.

First we prove that s_i is adjacent to s_j. Suppose not. Since $x_i' - s_j - x_i - x_j - s_i - x_j'$ is not a P_6 in G, it follows that x_i is non-adjacent to x_j. But now $x_i' - s_j - x_i - y^j - x_j - s_i$ or $x_i' - s_j - x_i - y^j - s_i - x_j'$ is a P_6 in G, a contradiction. This proves that s_i is adjacent to s_j.

If y_i^j is adjacent to x_j', then $x_j' - y_i^j - x_i' - s_j - x_i - y^j$ is a P_6, a contradiction. Therefore x_j' is non-adjacent to y_i^j, and therefore x_j' is anticomplete to C_i. Similarly, x_i' is anticomplete to C_j. In particular it follows that $C_i \neq C_j$.

Since $L_P(T_j) = \{j, k\}$ there exists $s_l \in S$ with $L_P(s_l) = \{l\}$ such that s_l is complete to $X(T_j)$. Since $l \in M_P(y)$ for every $y \in \{y_i^i, y_i^j, y_j^i, y_j^j, y^i, y^j\}$, it follows that s_l is anticomplete to $\{y_i^i, y_i^j, y_j^i, y_j^j, y^i, y^j\}$. Recall that $x_i, x_i' \in X(T_i)$, and so no vertex of S is mixed on $\{x_i, x_i'\}$. Similarly no vertex of S is mixed on $\{x_j, x_j'\}$. If s_l is anticomplete to $\{x_i, x_i'\}$, then one of $y_i^j - x_i' - s_j - s_l - x_j' - y_j^j$, $x_i' - s_j - x_i - y^j - x_j - s_l$, $x_i' - s_j - x_i - x_j - s_l - x_j'$ is a P_6, so s_l is complete to $\{x_i, x_i'\}$.

Since $y_i^i - x_i' - s_j - s_i - x_j' - y_j^j$ is not a P_6, it follows that either s_j is adjacent to y_i^i, or s_i is adjacent to y_j^j. We may assume that s_j is adjacent to y_i^i.

Let M be a path in C_i from y_i^j to y_i^i with $l \in M_P(u)$ for every $u \in V(M)$. Since s_j is adjacent to y_i^i and not to y_i^j, there is exist adjacent $a, b \in V(M)$ such that s_j is adjacent to a and not to b. Since $l \in M_P(u)$ for every $u \in V(M)$, it follows that s_l is anticomplete to $\{a, b\}$. But now if s_l is non-adjacent to s_j, then $b - a - s_j - x_i - s_l - x_j'$ is a P_6, and if s_l is adjacent to s_j, then $b - a - s_j - s_l - x_j' - y_j^j$ is a P_6; in both cases a contradiction. This proves Lemma 2. $\square$

Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of a P_6-free graph G. Let $S'' \subseteq X$, and let $X_0'' \subseteq X \cup Y^*$. Let $f' : S \cup X_0 \cup S'' \cup X_0'' \to \{1, 2, 3, 4\}$ be such that $f'|(S \cup X_0) = f|(S \cup X_0)$ and $(G, S \cup X_0 \cup S'' \cup X_0'', f')$ is a 4-precoloring of G. Let X'' be the set of vertices x of $X \setminus X_0''$ such that x has a neighbor $z \in S''$ with $f'(z) \in M_P(x)$. Let

$$S' = S \cup S''$$

$$X_0' = X_0 \cup X'' \cup X_0''$$

$$X' = X \setminus (X'' \cup S'' \cup X_0'')$$

$$Y^{*\prime} = Y^* \setminus X_0''.$$

We say that $P' = (G, S', X_0', X', Y^{*\prime}, f')$ is *obtained from P by moving S'' to the seed with colors $f'(S'')$, and moving X_0'' to X_0 with colors $f'(X_0'')$*. Sometimes we say that "we move S'' to S with colors $f'(S'')$, and X_0'' to X_0 with colors $f'(X_0'')$".

In the next lemma we show that this operation creates another excellent starred precoloring.

Lemma 3. *Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of a P_6-free graph G. Let $S'' \subseteq X$ and $X_0'' \subseteq X \cup Y^*$, and let $S', X_0', X', Y^{*\prime}, f'$ be as above. Then $P' = (G, S', X_0', X', Y^{*\prime}, f')$ is an excellent starred precoloring.*

Proof. We need to check the following conditions:

1. $f' : S' \cup X_0' \to \{1, 2, 3, 4\}$ is a proper coloring of $G|(S' \cup X_0')$;
2. $V(G) = S' \cup X_0' \cup X' \cup Y^{*\prime}$;
3. $G|S'$ is connected and no vertex in $V(G) \setminus S'$ is complete to S';
4. every vertex in X' has neighbors of at least two different colors (with respect to f') in S';
5. no vertex in X' is mixed on a component of $G|Y^{*\prime}$; and
6. for every component of $G|Y^{*\prime}$, there is a vertex in $S' \cup X_0' \cup X'$ complete to it.

Next we check the conditions.

1. holds by the definition of P'.
2. holds since $S' \cup X_0' \cup X' \cup Y^{*\prime} = S \cup X_0 \cup X \cup Y^*$.
3. $G|S'$ is connected since $G|S$ is connected, and every $z \in S''$ has a neighbor in S. Moreover, since no vertex of $V(G) \setminus S$ is complete to S, it follows that no vertex of $V(G) \setminus S'$ is complete to S'.
4. follows from the fact that $X' \subseteq X$.
5. follows from the fact that $Y^{*\prime} \subseteq Y^*$ and $X' \subseteq X$.
6. follows from the fact that $Y^{*\prime} \subseteq Y^*$ and $S \cup X_0 \subseteq S' \cup X_0'$.

$\square$

Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring. Let $i \neq j \in \{1, 2, 3, 4\}$. Write $X_{ij} = \{x \in X$ such that $M_P(x) = \{i, j\}\}$. For $y \in Y^*$ let $C_P(y)$ (or $C(y)$ when there is no danger of confusion) denote the vertex set of the component of $G|Y^*$ that contains y.

Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring, and let $\{i, j, k, l\} = \{1, 2, 3, 4\}$. We say that P is *kl-clean* if there does not exist $y \in Y^*$ with the following properties:

- $i, j \in M_P(y)$, and
- there is $u \in C(y)$ with $k \in M_P(u)$, and
- y has both a neighbor in X_{ik} and a neighbor in X_{jk}.

We say that P is *clean* if it is kl-clean for every $k \neq l \in \{1, 2, 3, 4\}$.

We say that P is *kl-tidy* if there do not exist vertices $y_i, y_j \in Y^*$ such that

- $i \in M_P(y_i)$, $j \in M_P(y_j)$, and
- $C(y_i) = C(y_j)$, and
- there is a path M from y_i to y_j in C such that $l \in M_P(u)$ for every $u \in V(M)$, and
- there is $u \in V(C)$ with $k \in M_P(u)$, and
- y_i has a neighbor in X_{ki} and a neighbor in X_{kj} (recall that since P is excellent y_i and y_j have the same neighbors in X).

Observe that since no vertex of X is mixed on an a component of $G|Y^*$, it follows that $N(y_i) \cap X_{ki}$ is precisely the set of vertices of X_{ki} that are complete to $C(y_i)$, and an analogous statement holds for X_{kj}. We say that P is *tidy* if it is kl-tidy for every $k \neq l \in \{1, 2, 3, 4\}$.

We say that P is *kl-orderly* if for every y in Y^* with $\{i, j\} \subseteq M_P(y)$, $N(y) \cap X_{ik}$ is complete to $N(y) \cap X_{jk}$. We say that P is *orderly* if it is kl-orderly for every $k \neq l \in \{1, 2, 3, 4\}$

Finally, we say that P is *kl-spotless* if no vertex y in Y^* with $\{i, j\} \subseteq M_P(y)$ has both a neighbor in X_{ik} and a neighbor in X_{jk}. We say that P is *spotless* if it is kl-spotless for every $k \neq l \in \{1, 2, 3, 4\}$

Our goal is to replace an excellent starred precoloring by an equivalent collection of spotless precolorings. First we prove a lemma that allows us to replace an excellent starred precoloring with an equivalent collection of clean precolorings.

Lemma 4. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph, and let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$*;*
- *every* $P' \in \mathcal{L}$ *is kl-clean for every* (k, l) *for which* P *is kl-clean;*
- *every* $P' \in \mathcal{L}$ *is* 14*-clean;*
- $\mathcal{L}$ *is an equivalent collection for* P*.*

Proof. Without loss of generality we may assume that $X_0 = X^0(P) \setminus S$. Thus $L_P(x) = M_P(x)$ for every $x \in X$. We may assume that P is not 14-clean for otherwise we may set $\mathcal{L} = \{P\}$. Let Y be the set of vertices of Y^* with $2, 3 \in M_P(y)$ and such that some $u \in C(y)$ has $1 \in M_P(u)$. Let $T_1, \ldots, T_p$ be the subsets of S with $L_P(T_s) = \{1, 2\}$ and $T_{p+1}, \ldots, T_m$ the subsets of S with $L_P(T_s) = \{1, 3\}$. Let $\mathcal{Q}$ be the collection of all m-tuples

$$((S_1, Q_1), (S_2, Q_2), \ldots, (S_m, Q_m))$$

where for every $r \in \{1, \ldots, m\}$

- $S_r \subseteq X(T_r)$ and $|S_r| \in \{0, 1\}$,

- if $S_r = \emptyset$, then $Q_r = \emptyset$
- if $S_r = \{x_r\}$ then $Q_r = \{y\}$ where $y \in Y \cap N(x_r)$.

For $Q \in \mathcal{Q}$ construct a precoloring P_Q as follows. Let $r \in \{1, \dots, m\}$. We may assume that $r \leq p$.

- Assume first that $S_r = \{x_r\}$. Then $Q_r = \{y_r\}$. Move $\{x_r\}$ to the seed with color 1, and for every $y \in Y$ such that $N(y) \cap X(T_r) \subseteq N(y_r) \cap X(T_r) \setminus \{x_r\}$, move $N(y) \cap X(T_r)$ to X_0 with the unique color of $L_P(T_r) \setminus \{1\}$.
- Next assume that $S_r = \emptyset$. Now for every $y \in Y$ move $N(y) \cap X(T_r)$ to X_0 with the unique color of $L_P(T_r) \setminus \{1\}$.

In the notation of Lemma 3, if the precoloring of $G|(X'_0 \cup S')$ thus obtained is not proper, remove Q from $\mathcal{Q}$. Therefore we may assume that the precoloring is proper. Repeatedly applying Lemma 3 we deduce that P_Q is an excellent starred precoloring. Observe that $Y^{*\prime} = Y^*$. Since $X' \subseteq X$ and $Y^{*\prime} = Y^*$, it follows that if P is kl-clean, then so is P_Q.

Now we show that P_Q is 14-clean. Let Y' be the set of vertices y of Y^* such that $2, 3 \in M_{P_Q}(y)$ and some vertex $u \in C(y)$ has $1 \in M_{P_Q}(u)$. Observe that $Y' \subseteq Y$. It is enough to check that no vertex of Y' has both a neighbor in X'_{12} and a neighbor in X'_{13}. Suppose this is false, and suppose that $y \in Y'$ has a neighbor $x_2 \in X'_{12}$ and a neighbor $x_3 \in X'_{13}$. Then $x_2 \in X_{12}$ and $x_3 \in X_{13}$. We may assume that $x_2 \in X(T_1)$ and $x_3 \in X(T_{p+1})$. Since $x_2, x_3 \notin X^0(P_Q)$, it follows that both $S_1 \neq \emptyset$ and $S_{p+1} \neq \emptyset$, and therefore $Q_1 \neq \emptyset$ and $Q_{p+1} \neq \emptyset$. Write $S_1 = \{x'_2\}$, $Q_1 = \{y_2\}$, $S_{p+1} = \{x'_3\}$ and $Q_{p+1} = \{y_3\}$. Since some $u \in C(y)$ has $1 \in M_{P_Q}(u)$, and since x'_2, x'_3 are not mixed on $C(y)$, it follows that y is anticomplete to $\{x'_2, x'_3\}$. Again since $x_2 \notin X^0(P_Q)$, it follows that $N(y) \cap X(T_1) \not\subseteq N(y_2) \cap X(T_1)$, and so we may assume that $x_2 \notin N(y_2)$. Similarly, we may assume that $x_3 \notin N(y_3)$. But now the vertices $x_2, x'_2, x_3, x'_3, y_2, y_3, y$ contradict Lemma 1. This proves that P_Q is 14-clean.

Since $S' = S \cup \bigcup_{i=1}^m S_i$, and since $m \leq 2^{|S|}$, it follows that $|S'| \leq |S| + m \leq |S| + 2^{|S|}$.

Let $\mathcal{L} = \{P_Q \;:\; Q \in \mathcal{Q}\}$. Then $|\mathcal{L}| \leq |V(G)|^{2m} \leq |V(G)|^{2^{|S|+1}}$. We show that $\mathcal{L}$ is an equivalent collection for P. Since every $P' \in \mathcal{L}$ is obtained from P by precoloring some vertices and updating, it is clear that if c is a precoloring extension of a member of $\mathcal{L}$, then c is a precoloring extension of P. To see the converse, let c be a precoloring extension of P. For every $i \in \{1, \dots, m\}$ define (S_i, S'_i, Q_j, Q'_j) as follows. If no vertex of Y has a neighbor $x \in X(T_i)$ with $c(x) = 1$, set $S_i = Q_i = \emptyset$. If some vertex of Y has neighbor $x \in X(T_i)$ with $c(x) = 1$, let y be a vertex with this property and in addition with $N(y) \cap X(T_i)$ minimal; let $x \in X(T_i) \cap N(y)$ with $c(x) = 1$; and set $Q_i = \{y\}$ and $S_i = \{x\}$. Let $Q = ((S_1, Q_1), \dots, (S_m, Q_m))$. We claim that c is a precoloring extension of P_Q. Write $P_Q = (G, S', X'_0, X', Y', f')$. We need to show that $c(v) = f'(v)$ for every $v \in S' \cup X'_0$. Since c is a precoloring extension of P, it follows that $c(v) = f(v) = f'(v)$ for every $v \in S \cup X_0$. Since $S' \setminus S = \bigcup_{s=1}^m S_s$ and $c(v) = f'(v) = 1$ for every $v \in \bigcup_{s=1}^m S_s$, we deduce that $c(v) = f'(v)$ for every $v \in S'$. Finally let $v \in X'_0 \setminus X_0$. It follows that $v \in X$, $f'(v)$ is the unique color of $M_P(v) \setminus \{1\}$, and there are three possibilities.

1. $1 \in M_P(v)$ and v has a neighbor in $\bigcup_{s=1}^m S_s$, or
2. there is $i \in \{1, \dots, m\}$ with $S_i = \{x_i\}$ and $Q_i = \{y_i\}$, and there is $y \in Y^*$ such that $N(y) \cap X(T_i) \subseteq (N(y_i) \cap X(T_i)) \setminus \{x_i\}$, and $v \in N(y) \cap X(T_i)$, or
3. there is $i \in \{1, \dots, m\}$ with $S_i = Q_i = \emptyset$, and there is $y \in Y^*$ such that $v \in N(y) \cap X(T_i)$.

We show that in all these cases $c(v) = f'(v)$.

1. Let $x \in \bigcup_{s=1}^m S_s$. Then $c(x) = 1$, and so $c(v) \neq 1$, and thus $c(v) = f'(v)$.
2. By the choice of y_i and since $N(y) \cap X(T_i) \subseteq (N(y_i) \cap X(T_i)) \setminus \{x_i\}$, it follows that $c(u) \neq 1$ for every $u \in N(y) \cap X(T_i)$, and therefore $c(v) = f'(v)$.
3. Since $S_i = \emptyset$, it follows that for every $y' \in Y^*$ and for every $u \in N(y') \cap X(T_i)$ we have that $c(u) \neq 1$, and again $c(v) = f'(v)$.

This proves that c is a precoloring extension of P_Q, and completes the proof of Lemma 4. $\square$

Repeatedly applying Lemma 4 and using symmetry, we deduce the following:

Lemma 5. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph. Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- $|\mathcal{L}| \le |V(G)|^{q(|S|)}$;
- $|S'| \le q(|S|)$ *for every* $P' \in \mathcal{L}$;
- *every* $P' \in \mathcal{L}$ *is clean;*
- $\mathcal{L}$ *is an equivalent collection for* P.

Next we show that a clean precoloring can be replaced with an equivalent collection of precolorings that are both clean and tidy.

Lemma 6. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph. Let $P = (G, S, X_0, X, Y^*, f)$ be a clean excellent starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- $|\mathcal{L}| \le |V(G)|^{q(|S|)}$;
- $|S'| \le q(|S|)$ *for every* $P' \in \mathcal{L}$;
- *every* $P' \in \mathcal{L}$ *is clean;*
- *every* $P' \in \mathcal{L}$ *is* kl*-tidy for every* k, l *for which* P *is* kl*-tidy;*
- *every* $P' \in \mathcal{L}$ *is* 14*-tidy;*
- $\mathcal{L}$ *is an equivalent collection for* P.

Proof. Without loss of generality we may assume that $X_0 = X^0(P) \setminus S$, and thus $L_P(x) = M_P(x)$ for every $x \in X$. We may assume that P is not 14-tidy for otherwise we may set $\mathcal{L} = \{P\}$. Let Y be the set of all pairs (y_2, y_3) with $y_2, y_3 \in Y^*$ such that

- $2 \in M_P(y_2)$, $3 \in M_P(y_3)$,
- y_2, y_3 are in the same component C of $G|Y^*$,
- there is a path M from y_2 to y_3 in C such that $4 \in M_P(u)$ for every $u \in V(M)$, and
- for some $u \in V(C)$, $1 \in M_P(u)$,

Let $T_1, \dots, T_p$ be the subsets of S with $L_P(T_s) = \{1, 2\}$ and let $T_{p+1}, \dots, T_m$ be the subsets of S with $L_P(T_s) = \{1, 3\}$. Let $\mathcal{Q}$ be the collection of all m-tuples

$$((S_1, Q_1), (S_2, Q_2), \dots, (S_m, Q_m))$$

where for $r \in \{1, \dots, m\}$

- $S_r \subseteq X(T_r)$ and $|S_r| \in \{0, 1\}$,
- if $S_r = \emptyset$, then $Q_r = \emptyset$
- $S_r = \{x_r\}$ then $Q_r = \{(y_2^r, y_3^r)\}$ where $(y_2^r, y_3^r) \in Y$ and x_r is complete to $\{y_2^r, y_3^r\}$.

For $Q \in \mathcal{Q}$ construct a precoloring P_Q as follows. For $r = 1, \dots, m$, we proceed as follows.

- Assume first that $S_r = \{x_r\}$. Then $Q_r = \{(y_2^r, y_3^r)\}$. Move x_r to the seed with color 1, and for every $(y_2, y_3) \in Y$ such that $N(y_2) \cap X(T_r) \subseteq N(y_2^r) \cap (X(T_r) \setminus \{x_r\})$, move $N(y_2) \cap X(T_r)$ to X_0 with the unique color of $L_P(T_r) \setminus \{1\}$.

- Next assume that $S_r = \emptyset$. Now for every $y \in Y$ move $N(y) \cap X(T_r)$ to X_0 with the unique color of $L_P(T_r) \setminus \{1\}$.

In the notation of Lemma 3, if the precoloring of $G|(X_0' \cup S')$ thus obtained is not proper, remove Q form $\mathcal{Q}$. Therefore we may assume that the precoloring is proper. Repeatedly applying Lemma 3 we deduce that P_Q is an excellent starred precoloring. Observe that $Y^{*\prime} = Y^*$, $M_{P_Q}(y) \subseteq M_P(y)$ for every $y \in Y^{*\prime}$, and $M_{P_Q}(x) = M_P(x)$ for every $x \in X' \setminus X^0(P_Q)$. It follows that P_Q is clean, and that if P is kl-tidy, then so is P_Q.

Now we show that P_Q is 14-tidy. Suppose that there exist $y_2, y_3 \in Y^Q$ that violate the definition of being 14-tidy. Let $x_2 \in X_{12}'$ and $x_3 \in X_{13}'$ be adjacent to y_2, say, and therefore complete to $\{y_2, y_3\}$. We may assume that $x_2 \in X(T_1)$ and $x_3 \in X(T_{p+1})$. Since $x_2, x_3 \notin X^0(P_Q)$, it follows that both $S_1 \neq \emptyset$ and $S_{p+1} \neq \emptyset$, and therefore $Q_1 \neq \emptyset$ and $Q_{p+1} \neq \emptyset$. Write $S_1 = \{x_2'\}$, $Q_1 = \{(y_2^2, y_3^2)\}$, $S_{p+1} = \{x_3'\}$ and $Q_{p+1} = \{y_2^3, y_3^3\}$.

Since there is a vertex u in the component of $G|(Y^*)'$ containing y_2, y_3 with $1 \in M_{P_Q}(u)$, and since no vertex of X is mixed on a component of Y^*, it follows that $\{y_2, y_3\}$ is anticomplete to $\{x_2', x_3'\}$. Since $x_2 \notin X^0(P_Q)$, it follows that $N(y_2) \cap X(T_1) \not\subseteq N(y_2^2) \cap (X(T_1) \setminus \{x_2'\})$, and so we may assume that $x_2 \notin N(y_2^2)$. Similarly, we may assume that $x_3 \notin N(y_2^3)$. But now, since no vertex of X is mixed on a component of Y^*, we deduce that the vertices $x_2, x_2', x_3, x_3', y_2^2, y_2^3, y_3^2, y_3^3, y_2, y_3$ contradict Lemma 2. This proves that P_Q is 14-tidy.

Since $S' = S \cup \bigcup_{i=1}^m S_i$, and since $m \leq 2^{|S|}$, it follows that $|S'| \leq |S| + m \leq |S| + 2^{|S|}$.

Let $\mathcal{L} = \{P_Q \ : \ Q \in \mathcal{Q}\}$. Then $|\mathcal{L}| \leq |V(G)|^{3m} \leq |V(G)|^{3 \times 2^{|S|}}$. We show that $\mathcal{L}$ is an equivalent collection for P. Since every $P' \in \mathcal{L}$ is obtained from P by precoloring some vertices and updating, it is clear that every precoloring extension of a member of $\mathcal{L}$ is a precoloring extension of P. To see the converse, suppose that P has a precoloring extension c. For every $i \in \{1, \ldots, m\}$ define S_i and Q_i as follows. If there does not exist $(y_2^2, y_2^3) \in Y$ such that some $x \in X(T_i)$ with $c(x) = 1$ is complete to $\{y_2^2, y_2^3\}$, set $S_i = Q_i = \emptyset$. If such a pair exists, let (y_2^2, y_2^3) be a pair with this property and subject to that with the set $N(y_2^2) \cap X(T_i)$ minimal; let $x \in X(T_i)$ be complete to $\{y_2^2, y_2^3\}$ and with $c(x) = 1$; and set $Q_i = \{(y_2^2, y_2^3)\}$ and $S_i = \{x\}$. Let $Q = ((S_1, Q_1), \ldots, (S_m, Q_m))$. We claim that c is a precoloring extension of P_Q. Write $P_Q = (G, S', X_0', X', Y', f')$. We need to show that $c(v) = f'(v)$ for every $v \in S' \cup X_0'$. Since c is a precoloring extension of P, it follows that $c(v) = f(v) = f'(v)$ for every $v \in S \cup X_0$. Since $S' \setminus S = \bigcup_{s=1}^m S_s$ and $c(v) = f'(v) = 1$ for every $v \in \bigcup_{s=1}^m S_s$, we deduce that $c(v) = f'(v)$ for every $v \in S'$. Finally let $v \in X_0' \setminus X_0$. Then $v \in X$, $f'(v)$ is the unique color of $M_P(v) \setminus \{1\}$, and there are three possibilities.

1. $1 \in M_P(v)$ and v has a neighbor in $\bigcup_{s=1}^m S_s$, or
2. there is $i \in \{1, \ldots, m\}$ with $S_i = \{x_i\}$ and $Q_i = \{(y_i^2, y_i^3)\}$, and there exists $(y_2, y_3) \in Y$ such $N(y_2) \cap X(T_i) \subseteq X(T_i) \cap (N(y_i^2) \setminus \{x_i\})$, or
3. there is $i \in \{1, \ldots, m\}$ with $S_i = Q_i = \emptyset$, and there exists $(y_2, y_3) \in Y$ such that $v \in X(T_i) \cap N(y_2)$.

We show that in all these cases $c(v) = f'(v)$.

1. Let $x \in \bigcup_{s=1}^m S_s$. Then $c(x) = 1$, and so $c(v) \neq 1$, and thus $c(v) = f'(v)$.
2. By the choice of y_i^2, y_i^3 and since $N(y_2) \cap X(T_i) \subseteq (N(y_i^2) \cap X(T_i)) \setminus \{x_i\})$, it follows that $c(u) \neq 1$ for every $u \in N(y_2) \cap X(T_i)$, and therefore $c(v) = f'(v)$.
3. Since $S_i = \emptyset$, it follows that for every $(y_2, y_3) \in Y$ and for every $u \in N(y_2) \cap X(T_i)$ we have $c(u) \neq 1$, and again $c(v) = f'(v)$.

This proves that c is an extension of P_Q, and completes the proof of Lemma 6. □

Repeatedly applying Lemma 6 and using symmetry, we deduce the following:

Lemma 7. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph. Let $P = (G, S, X_0, X, Y^*, f)$ be a clean excellent starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- *$|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;*
- *$|S'| \leq q(|S|)$ for every $P' \in \mathcal{L}$;*
- *every $P' \in \mathcal{L}$ is clean and tidy;*
- *$\mathcal{L}$ is an equivalent collection for P.*

Our next goal is to show that a clean and tidy precoloring can be replaced with an equivalent collection of orderly precolorings.

Lemma 8. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph. Let $P = (G, S, X_0, X, Y^*, f)$ be a clean, tidy starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- *$|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;*
- *$|S'| \leq q(|S|)$ for every $P' \in \mathcal{L}$;*
- *every $P' \in \mathcal{L}$ is clean and tidy;*
- *every $P' \in \mathcal{L}$ is kl-orderly for every (k, l) for which P is kl-orderly;*
- *every $P' \in \mathcal{L}$ is 14-orderly;*
- *P is equivalent to $\mathcal{L}$.*

Proof. Without loss of generality we may assume that $X_0 = X^0(P)$, and so $L_P(x) = M_P(x)$ for every $x \in X$. We may assume that P is not 14-orderly for otherwise we may set $\mathcal{L} = \{P\}$. Let $Y = \{y \in Y^*$ such that $\{2, 3\} \subseteq M_P(y)\}$. Let $T_1, \ldots, T_p$ be the types with $L(T_s) = \{1, 2\}$ and $T_{p+1}, \ldots, T_m$ the types with $L(T_s) = \{1, 3\}$. Let $\mathcal{Q}$ be the collection of all $p(m-p)$-tuples of quadruples (S_i, S_i', Q_j, Q_j') with $i \in \{1, \ldots, p\}$ and $j \in \{p+1, \ldots, m\}$, where

- $S_i, S_i', Q_j, Q_j' \subseteq Y$;
- $|S_i|, |S_i'|, |Q_j||Q_j'| \in \{0, 1\}$;
- $S_i' \neq \emptyset$ only if $S_i \neq \emptyset$ and $Q_j = \emptyset$
- $Q_j' \neq \emptyset$ only if $Q_j \neq \emptyset$ and $S_i = \emptyset$
- if $N(S_i) \cap X(T_i) = \emptyset$, then $S_i = \emptyset$;
- if $N(Q_j) \cap X(T_j) = \emptyset$, then $Q_j = \emptyset$;
- if $N(S_i') \cap (X(T_i) \setminus N(S_i)) = \emptyset$, then $S_i' = \emptyset$;
- if $N(Q_j') \cap (X(T_j) \setminus N(Q_j)) = \emptyset$, then $Q_j' = \emptyset$.

Please note that quadruples for distinct pairs (T_i, T_j) are selected independently. For $Q \in \mathcal{Q}$ construct a precoloring P_Q as follows. Let $i \in \{1, \ldots, p\}$ and $j \in \{p+1, \ldots, m\}$.

- Assume first that $S_i = \{y_i\}$ and $Q_j = \{y_j\}$. If there is an edge between $N(y_i) \cap X(T_i)$ and $N(y_j) \cap X(T_j)$, remove Q from $\mathcal{Q}$. Now suppose that $N(y_i) \cap X(T_i)$ is anticomplete to $N(y_j) \cap X(T_j)$. Move $T = (N(y_i) \cap X(T_i)) \cup (N(y_j) \cap X(T_j))$ into X_0 with color 1. For every $y \in Y$ complete to T and both with a neighbor in $X(T_i) \setminus T$ and a neighbor in $X(T_j) \setminus T$, proceed as follows: if $4 \in M_P(y)$, move y into X_0 with color 4; if $4 \notin M_P(y)$, remove Q from $\mathcal{Q}$.
- Next assume that exactly one of S_i, Q_j is non-empty. By symmetry we may assume that $S_i = \{y_i\}$ and $Q_j = \emptyset$. Move $T = N(y_i) \cap X(T_i)$ into X_0 with color 1. For every $y \in Y$ complete to T and both with a neighbor in $X(T_i) \setminus T$ and a neighbor in $X(T_j)$, proceed as follows: if $4 \in M_P(y)$, move y into X_0 with color 4; if $4 \notin M_P(y)$, remove Q from $\mathcal{Q}$.

- Suppose $S'_i \neq \emptyset$, write $S'_i = \{y'_i\}$. Let $T' = N(y'_i) \cap (X(T_i) \setminus T)$. If there is an edge between T and T', remove Q from $\mathcal{Q}$. Now we may assume that T' is anticomplete to T. Move T' into X_0 with color 1. For every $y \in Y$ complete to T' and both with a neighbor in $X(T_i) \setminus (T \cup T')$ and a neighbor in $X(T_j)$, proceed as follows: if $4 \in M_P(y)$, move y into X_0 with color 4; if $4 \notin M_P(y)$, remove Q from $\mathcal{Q}$.
   - Next suppose that $S'_i = \emptyset$. For every $y \in Y$ with a neighbor in $X(T_i) \setminus T$ and a neighbor in $X(T_j)$, proceed as follows: if $4 \in M_P(y)$, move y into X_0 with color 4; if $4 \notin M_P(y)$, remove Q from $\mathcal{Q}$.

- Finally assume that $S_i = Q_j = \emptyset$. For every $y \in Y$ with both a neighbor in $X(T_i)$ and a neighbor in $X(T_j)$, proceed as follows: if $4 \in M_P(y)$, move y to X_0 with color 4; if $4 \notin M_P(y)$, remove Q from $\mathcal{Q}$.

Let $Q \in \mathcal{Q}$, and let $P_Q = (G, S', X'_0, X', Y^{*\prime}, f')$. Since $X' \subseteq X$, $Y' \subseteq Y^*$ and $M_{P_Q}(v) \subseteq M_P(v)$ for every v, it follows that P_Q is excellent, clean, tidy, and that for $k \neq l \in \{1, 2, 3, 4\}$, if P is kl-orderly, then P_Q is kl-orderly.

Next we show that P_Q is 14-orderly. Suppose that some $y \in Y$ has a neighbor in $x_2 \in X'_{12}$ and a neighbor in $x_3 \in X'_{13}$ such that x_2 is non-adjacent to x_3. Then $x_2 \in X_{12}$ and $x_3 \in X_{13}$. We may assume that $x_2 \in X(T_1)$ and $x_3 \in X(T_{p+1})$. Since $x_2, x_3 \notin X^0(P_Q)$, it follows that at least one of $S_1, Q_{p+1} \neq \emptyset$.

Suppose first that both $S_1 \neq \emptyset$ and $Q_{p+1} \neq \emptyset$. Let $S_1 = \{y_2\}$ and $Q_{p+1} = \{y_3\}$. Since $x_2, x_3 \notin X^0(P_Q)$, it follows that y_2 is non-adjacent to x_2, and y_3 is non-adjacent to x_3. Since $y \notin X^0(P_Q)$, we may assume by symmetry that there is $x'_2 \in N(y_2) \cap X(T_1)$ such that y is non-adjacent to x'_2. Let $x'_3 \in N(y_3) \cap X(T_{p+1})$. Since $x_2, x_3, y \notin X^0(P_Q)$, it follows that $\{x'_2, x'_3\}$ is anticomplete to $\{x_2, x_3\}$. By the construction of P_Q, x'_2 is non-adjacent to x'_3. By Lemma 1, y is adjacent to x'_3. Since $L_P(T_1) = \{1, 2\}$, there is $s_3 \in S$ with $f(s_3) = 3$ complete to $X(T_1)$. Since $3 \in M_{P_Q}(y) \cap L_P(y_2) \cap L_P(y_3) \cap L_P(x'_3) \cap L_P(x_3)$, it follows that s_3 is anticomplete to $\{y, y_2, y_3, x_3, x'_3\}$. Similarly, since $L_P(T_{p+1}) = \{1, 3\}$, there is $s_2 \in S$ with $f(s_2) = 2$ complete to $X(T_{p+1})$. Since $2 \in M_{P_Q}(y) \cap L_P(y_2) \cap L_P(y_3) \cap L_P(x_2) \cap L_P(x'_2)$, it follows that s_2 is anticomplete to $\{y, y_2, y_3, x_2, x'_2\}$. Since $y_2 - x'_2 - s_3 - x_2 - y - t$ is not a P_6 for $t \in \{x_3, x'_3\}$, it follows that y_2 is complete to $\{x_3, x'_3\}$. Since $y_3 - x'_3 - y - x_2 - s_3 - x'_2$ is not a P_6, it follows that y_3 is adjacent to at least one of x_2, x'_2. Since the path $x_2 - y - x_3 - y_2 - x'_2$ cannot be extended to a P_6 via y_3, it follows that y_3 is complete to $\{x_2, x'_2\}$. But now $s_2 - x_3 - y - x_2 - y_3 - x'_2$ is a P_6, a contradiction.

Next suppose that exactly one of S_1, Q_{p+1} is non-empty. By symmetry we may assume that $S_1 = \{y_2\}$ and $Q_{p+1} = \emptyset$. Since $x_2, x_3 \notin X^0(P_Q)$, it follows that y_2 is non-adjacent to x_2. Since $y \notin X^0(P_Q)$, it follows that $S'_1 \neq \emptyset$. Write $S'_i = \{y'_2\}$; now x_2 is non-adjacent to y'_2. Since $y \notin X^0(P_Q)$, we may assume that there is $x'_2 \in N(y_2) \cap X(T_1)$ such that y is non-adjacent to x'_2, and $x''_2 \in N(y'_2) \cap (X(T_1) \setminus N(y_2))$ such that y is non-adjacent to x''_2. Since $x_2, x_3 \notin X^0(P_Q)$, it follows that $\{x'_2, x''_2\}$ is anticomplete to $\{x_2, x_3\}$. Since $L_P(T_1) = \{1, 2\}$, there is $s_3 \in S$ with $f(s_3) = 3$ complete to $X(T_1)$. Since $3 \in M_{P_Q}(y) \cap L_P(y_2) \cap L_P(y'_2) \cap L_P(x_3)$, it follows that s_3 is anticomplete to $\{y, y_2, y'_2, x_3\}$. Since $y_2 - x'_2 - s_3 - x_2 - y - x_3$, is not a P_6, it follows that x_3 is adjacent to y_2. But now $y - x_3 - y_2 - x'_2 - s_3 - x''_2$ is a P_6, a contradiction. This proves that P_Q is 14-orderly.

Observe that $S' = S$, and so $|S'| = |S|$. Observe also that $p(m-p) \leq (\frac{m}{2})^2$, and since $m \leq 2^{|S|}$, it follows that $p(m-p) \leq 2^{2|S|-2}$. Let $\mathcal{L} = \{P_Q \ : \ Q \in \mathcal{Q}\}$. Now $|\mathcal{L}| \leq |V(G)|^{4p(m-p)} \leq |V(G)|^{2^{2|S|}}$.

We show that $\mathcal{L}$ is an equivalent collection for P. Since every $P' \in \mathcal{L}$ is obtained from P by precoloring some vertices and updating, it is clear that if c is a precoloring extension of a member of $\mathcal{L}$, then c is a precoloring extension of P. To see the converse, suppose that P has a precoloring extension c. For every $i \in \{1, \ldots, p\}$ and $j \in \{p+1, \ldots, m\}$ define S_i, S'_i, Q_j and Q'_j as follows. If every vertex of Y has a neighbor $x \in X(T_i)$ with $c(x) \neq 1$, set $S_i = \emptyset$, and if every vertex of Y has a neighbor $x \in X(T_j)$ with $c(x) \neq 1$, set $Q_j = \emptyset$. If some vertex of Y has no neighbor $x \in X(T_i)$ with $c(x) \neq 1$, let y_i be a vertex with this property and in addition with $N(y_i) \cap X(T_i)$ maximal; set $S_i = \{y_i\}$. If some vertex of Y has no neighbor $x \in X(T_j)$ with $c(x) \neq 1$, let y_j be a vertex with this property and in addition with $N(y_j) \cap X(T_j)$ maximal; set $Q_j = \{y_j\}$. If $|S_i| = |Q_j|$, set $S'_i = Q'_j = \emptyset$. Next assume that $|S_i| \neq |Q_j|$; by symmetry we may assume that $S_i = \{y_i\}$ and $Q_j = \emptyset$. If every vertex of Y has a neighbor $x \in X(T_i) \setminus N(y_i)$ with $c(x) \neq 1$, set $S'_i = \emptyset$. If some vertex of Y has no neighbor $x \in X(T_i) \setminus N(y_i)$ with $c(x) \neq 1$, let y'_i be a vertex with this property and in addition with $N(y_i) \cap (X(T_i) \setminus N(y_i))$ maximal; set $S'_i = \{y'_i\}$.

We claim that c is a precoloring extension of P_Q. Write $P_Q = (G, S', X'_0, X', Y', f')$. We need to show that $c(v) = f'(v)$ for every $v \in S' \cup X'_0$. Since c is a precoloring extension of P, and since $S = S'$, it follows that $c(v) = f(v) = f'(v)$ for every $v \in S' \cup X_0$. Let $v \in X'_0 \setminus X_0$. It follows that either

1. $S_i = \{y_i\}$, $Q_j = \{y_j\}$, and $v \in X$ and $v \in (N(y_i) \cap X(T_i)) \cup (N(y_j) \cap X(T_j))$ and $f'(v) = 1$, or
2. $S_i = \{y_i\}$, $Q_j = \{y_j\}$, $v \in Y$, v is complete to $(N(y_i) \cap X(T_i)) \cup (N(y_j) \cap X(T_j))$, v has both a neighbor in $X(T_i) \setminus N(y_i)$ and a neighbor in $X(T_j) \setminus N(y_j)$, and $f'(v) = 4$, or
3. (possibly with the roles of i and j exchanged) $S_i = \{y_i\}$, $Q_j = \emptyset$, and $v \in X$ and $v \in N(y_i) \cap X(T_i)$, and $f'(v) = 1$, or
4. (possibly with the roles of i and j exchanged) $S_i = \{y_i\}$, $Q_j = \emptyset$, $v \in Y$, v is complete to $N(y_i) \cap X(T_i)$, v has both a neighbor in $X(T_i) \setminus N(y_i)$ and a neighbor in $X(T_j)$, and $f'(v) = 4$, or
5. (possibly with the roles of i and j exchanged) $S_i = \{y_i\}$, $S'_i = \{y'_i\}$, and $v \in X$ and $v \in N(y'_i) \cap (X(T_i) \setminus N(y_i))$, and $f'(v) = 1$, or
6. (possibly with the roles of i and j exchanged) $S_i = \{y_i\}$, $S'_i = \{y'_i\}$, $v \in Y$, v is complete to $N(y'_i) \cap (X(T_i) \setminus N(y_i))$, v has both a neighbor in $X(T_i) \setminus (N(y_i) \cup N(y'_i))$ and a neighbor in $X(T_j)$, and $f'(v) = 4$, or
7. (possibly with the roles of i and j exchanged) $S_i = \{y_i\}$, $Q_j = S'_i = \emptyset$, $v \in Y$, v has both a neighbor in $X(T_i) \setminus (N(y_i) \cup N(y'_i))$ and a neighbor in $X(T_j)$, and $f'(v) = 4$, or
8. $S_i = Q_j = \emptyset$, $v \in Y$, v has both a neighbor in $X(T_i)$ and a neighbor in $X(T_j)$, and $f'(v) = 4$.

We show that in all these cases $c(v) = f'(v)$.

1. By the choice of y_i, y_j, $c(u) = 1$ for every $u \in (N(y_i) \cap X(T_i)) \cup (N(y_j) \cap X(T_j))$, and so $c(v) = 1$.
2. It follows from the maximality of y_i, y_j that v has both a neighbor $x_2 \in X(T_i)$ with $c(x_2) = 2$ and a neighbor $x_3 \in X(T_j)$ with $c(x_3) = 3$. Since P is clean, it follows that $1 \notin M_P(v)$, and therefore $c(v) = 4$.
3. By the choice of y_i, $c(u) = 1$ for every $u \in N(y_i) \cap X(T_i)$, and so $c(v) = f'(v)$.
4. It follows from the maximality of y_i that v has a neighbor $x_2 \in X(T_i)$ with $c(x_2) = 2$. Since $Q_j = \emptyset$, v has a neighbor $x_3 \in X(T_j)$ with $c(x_3) = 3$. Since P is clean, it follows that $1 \notin M_P(v)$, and so $c(v) = 4$.
5. By the choice of y'_i, $c(u) = 1$ for every $u \in N(y'_i) \cap (X(T_i) \setminus N(y_i))$, and so $c(v) = 1$.
6. It follows from the maximality of y'_i that v has a neighbor $x_2 \in X(T_i)$ with $c(x_2) = 2$. Since $Q_j = \emptyset$, v has a neighbor $x_3 \in X(T_j)$ with $c(x_3) = 3$. Since P is clean, it follows that $1 \notin M_P(v)$, and so $c(v) = 4$.
7. Since $S'_i = \emptyset$, it follows that for every $y \in Y$ with a neighbor in $X(T_i) \setminus N(y_i)$, y has a neighbor u in $X(T_i) \setminus N(y_i)$ with $c(u) = 2$. Since $Q_j = \emptyset$, v has a neighbor $x_3 \in X(T_j)$ with $c(x_3) = 3$. Since P is clean, it follows that $1 \notin M_P(v)$, and so $c(v) = 4$.
8. Since $S_i = Q_j = \emptyset$, it follows that v has both a neighbor $x_2 \in X(T_i)$ with $c(x_2) = 2$, and a neighbor $x_3 \in X(T_j)$ with $c(x_3) = 3$. Since P is clean, it follows that $1 \notin M_P(v)$, and so $c(v) = 4$.

This proves that c is an extension of P_Q, and completes the proof of Lemma 8. □

Repeatedly applying Lemma 8 and using symmetry, we deduce the following:

Lemma 9. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph. Let $P = (G, S, X_0, X, Y^*, f)$ be a clean and tidy excellent starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$;
- *every* $P' \in \mathcal{L}$ *is clean, tidy and orderly;*

- *P is equivalent to $\mathcal{L}$.*

Next we show that a clear, tidy and orderly excellent starred precoloring can be replaced by an equivalent collection of spotless precolorings.

Lemma 10. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph. Let $P = (G, S, X_0, X, Y^*, f)$ be a clean, tidy and orderly excellent starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- *$|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;*
- *$|S'| \leq q(|S|)$ for every $P' \in \mathcal{L}$;*
- *every $P' \in \mathcal{L}$ is clean, tidy and orderly;*
- *every $P' \in \mathcal{L}$ is kl-spotless for every (k, l) for which P is kl-spotless;*
- *every $P' \in \mathcal{L}$ is 14-spotless;*
- *P is equivalent to $\mathcal{L}$.*

Proof. The proof is similar to the proof of Lemma 8. Without loss of generality we may assume that $X_0 = X^0(P)$, and so $L_P(x) = M_P(x)$ for every $x \in X$. We may assume that P is not 14-spotless for otherwise we may set $\mathcal{L} = \{P\}$. Let Y be the set of vertices $y \in Y^*$ such that $\{2, 3\} \subseteq M_P(y)$ and y has both a neighbor in X_{12} and a neighbor in X_{13}. Let $T_1, \ldots, T_p$ be the types with $L(T_s) = \{1, 2\}$ and $T_{p+1}, \ldots, T_m$ the types with $L(T_s) = \{1, 3\}$. Let $\mathcal{Q}$ be the collection of all $p(m-p)$-tuples (S_i, Q_j) with $i \in \{1, \ldots, p\}$ and $j \in \{p+1, \ldots, m\}$, where $S_i, Q_i \subseteq Y$ and $|S_i|, |Q_i| \in \{0, 1\}$.

For $Q \in \mathcal{Q}$ construct a precoloring P_Q as follows. Let $i \in \{1, \ldots, p\}$ and $j \in \{p+1, \ldots, m\}$.

- Assume first that $S_i = \{y_i\}$ $Q_j = \{y_j\}$. If there is an edge between $N(y_i) \cap X(T_i)$ and $N(y_j) \cap X(T_j)$, remove Q from $\mathcal{Q}$. Now suppose that $N(y_i) \cap X(T_i)$ is anticomplete to $N(y_j) \cap X(T_j)$. Move $T = (N(y_i) \cap X(T_i)) \cup (N(y_j) \cap X(T_j))$ into X_0 with color 1. For every $y \in Y$ complete to T and both with a neighbor in $X(T_i) \setminus T$ and a neighbor in $X(T_j) \setminus T$, proceed as follows: if $4 \in M_P(y)$, move y to X_0 with color 4; if $4 \notin M_P(y)$, remove Q from $\mathcal{Q}$.
- Next assume that exactly one of S_i, Q_j is non-empty. By symmetry we may assume that $S_i = \{y_i\}$ and $Q_j = \emptyset$. Move $T = N(y_i) \cap X(T_i)$ into X_0 with color 1. For every $y \in Y$ complete to T and both with a neighbor in $X(T_i) \setminus T$ and a neighbor in $X(T_j)$, proceed as follows. If $4 \in M_P(y)$, move y to X_0 with color 4; if $4 \notin M_P(y)$, remove Q from $\mathcal{Q}$.
- Finally assume that $S_i = S_j = \emptyset$. For every $y \in Y$ with both a neighbor in $X(T_i)$ and a neighbor in $X(T_j)$, proceed as follows: if $4 \in M_P(y)$, move y to X_0 with color 4; if $4 \notin M_P(y)$, remove Q from $\mathcal{Q}$.

Let $Q \in \mathcal{Q}$, and let $P_Q = (G, S', X'_0, X', Y^{*\prime}, f')$. If f' is not a proper coloring of $G|(S' \cup X'_0)$, remove Q from $\mathcal{Q}$. Since $X' \subseteq X$, $Y' \subseteq Y^*$ and $M_{P_Q}(v) \subseteq M_P(v)$ for every v, it follows that P_Q is excellent, clean, tidy and orderly, and that for $k \neq l \in \{1, 2, 3, 4\}$, if P is kl-spotless, then P_Q is kl-spotless.

Next we show that P_Q is 14-spotless. Suppose that some $y \in Y$ has a neighbor in $x_2 \in X'_{12}$ and a neighbor in $x_3 \in X'_{13}$. Then $x_2 \in X_{12}$ and $x_3 \in X_{13}$. We may assume that $x_2 \in X(T_1)$ and $x_3 \in X(T_{p+1})$. Since $x_2, x_3 \notin X^0(P_Q)$, we may assume that $S_1 \neq \emptyset$; let $S_1 = \{y_2\}$. Since P is orderly, $N(y_2) \cap X(T_{p+1})$ is complete to $N(y_2) \cap X(T_1)$, and consequently $M_{P_Q}(x) = 3$ for every $x \in N(y_2) \cap X(T_{p+1})$. Since $x_2, x_3 \notin X^0(P_Q)$, it follows that y_2 is anticomplete to $\{x_2, x_3\}$. Since $y \notin X^0(P_Q)$, we may assume (using symmetry if $Q_{p+1} \neq \emptyset$) that there is $x'_2 \in N(y_2) \cap X(T_1)$ such that y is non-adjacent to x'_2. Since $L_P(T_1) = \{1, 2\}$, there is $s_3 \in S$ with $f(s_3) = 3$ complete to $X(T_1)$. Since $3 \in M_{P_Q}(y) \cap L_P(y_2) \cap L_P(x_3)$, it follows that s_3 is anticomplete to $\{y, y_2, x_3\}$. Similarly there exists $s_2 \in S$ with $f(s_2) = 2$ complete to $X(T_{p+1})$. Since $2 \in M_{P_Q}(y) \cap L_P(y_2) \cap L_P(x_2)$, it follows that s_2 is anticomplete to $\{y, y_2, x_2\}$. If s_2 is non-adjacent to s_3, then $y_2 - x'_2 - s_3 - x_2 - x_3 - s_2$ is a P_6, a contradiction. Thus s_2 is adjacent to s_3. Now $y_2 - x'_2 - s_3 - s_2 - x_3 - y$ is a P_6, again a contradiction. This proves that P_Q is 14-spotless.

Observe that $S = S'$, and so $|S| = |S'|$. Observe also that also that $p(m-p) \leq (\frac{m}{2})^2$, and since $m \leq 2^{|S|}$, it follows that $p(m-p) \leq 2^{2|S|-2}$. Let $\mathcal{L} = \{P_Q \ : \ Q \in \mathcal{Q}\}$. Now $|\mathcal{L}| \leq |V(G)|^{2p(m-p)} \leq |V(G)|^{2^{2|S|-1}}$.

Next we show that $\mathcal{L}$ is an equivalent collection for P. Since every $P' \in \mathcal{L}$ is obtained from P by precoloring some vertices and updating, it is clear that if c is a precoloring extension of a member of $\mathcal{L}$, then P is a precoloring extension of P. To see the converse, suppose that P has a precoloring extension c. For every $i \in \{1, \dots, p\}$ and $j \in \{p+1, \dots, m\}$ define (S_i, Q_j) as follows. If every vertex of Y has a neighbor $x \in X(T_i)$ with $c(x) \neq 1$, set $S_i = \emptyset$, and if every vertex of Y has a neighbor $x \in X(T_j)$ with $c(x) \neq 1$, set $Q_j = \emptyset$. If some vertex of Y has no neighbor $x \in X(T_i)$ with $c(x) \neq 1$, let y_i be a vertex with this property and in addition with $N(y_i) \cap X(T_i)$ maximal; set $S_i = \{y_i\}$. If some vertex of Y has no neighbor $x \in X(T_j)$ with $c(x) \neq 1$, let y_j be a vertex with this property and in addition with $N(y_j) \cap X(T_j)$ maximal; set $Q_j = \{y_j\}$. We claim that c is a precoloring extension of P_Q. Write $P_Q = (G, S', X_0', X', Y', f')$. We need to show that $c(v) = f'(v)$ for every $v \in S' \cup X_0'$. Since c is a precoloring extension of P, and since $S = S'$, it follows that $c(v) = f(v) = f'(v)$ for every $v \in S' \cup X_0$. Let $v \in X_0' \setminus X_0$. It follows that either

1. $S_i = \{y_i\}$, $Q_j = \{y_j\}$, and $v \in X$ and $v \in (N(y_i) \cap X(T_i)) \cup (N(y_j) \cap X(T_j))$ and $f'(v) = 1$, or
2. $S_i = \{y_i\}$, $Q_j = \{y_j\}$, $v \in Y$, v is complete to $(N(y_i) \cap X(T_i)) \cup (N(y_j) \cap X(T_j))$, v has both a neighbor in $X(T_i) \setminus N(y_i)$ and a neighbor in $X(T_j) \setminus N(y_j)$, and $f'(v) = 4$, or
3. (possibly with the roles of i and j exchanged) $S_i = \{y_i\}$, $Q_j = \emptyset$, and $v \in X$ and $v \in N(y_i) \cap X(T_i)$, and $f'(v) = 1$, or
4. (possibly with the roles of i and j exchanged) $S_i = \{y_i\}$, $Q_j = \emptyset$, $v \in Y$, v is complete to $N(y_i) \cap X(T_i)$, v has both a neighbor in $X(T_i) \setminus N(y_i)$ and a neighbor in $X(T_j)$, and $f'(v) = 4$, or
5. $S_i = Q_j = \emptyset$, $v \in Y$, v has both a neighbor in $X(T_i)$ and a neighbor in $X(T_j)$, and $f'(v) = 4$.

We show that in all these cases $c(v) = f'(v)$.

1. By the choice of y_i, y_j, $c(u) = 1$ for every $u \in (N(y_i) \cap X(T_i)) \cup (N(y_j) \cap X(T_j))$, and so $c(v) = f'(v)$.
2. It follows from the maximality of y_i, y_j that v has both a neighbor $x_2 \in X(T_i)$ with $c(x_2) = 2$ and a neighbor $x_3 \in X(T_j)$ with $c(x_3) = 3$. Since P is clean it follows that $1 \notin M_P(v)$, and therefore $c(v) = 4$.
3. By the choice of y_i, $c(u) = 1$ for every $u \in N(y_i) \cap X(T_i)$, and so $c(v) = f'(v)$.
4. It follows from the maximality of y_i that v has a neighbor $x_2 \in X(T_i)$ with $c(x_2) = 2$. Since $Q_j = \emptyset$, v has a neighbor $x_3 \in X(T_j)$ with $c(x_3) = 3$. Since P is clean, it follows that $1 \notin M_P(v)$, and so $c(v) = 4$.
5. Since $S_i = Q_j = \emptyset$, it follows that v has both a neighbor $x_2 \in X(T_i)$ with $c(x_2) = 2$, and a neighbor $x_3 \in X(T_j)$ with $c(x_3) = 3$. Since P is clean, it follows that $1 \notin M_P(v)$, and so $c(v) = 4$.

This proves that P_Q has a precoloring extension, and completes the proof of Lemma 10. □

Observe that if an excellent starred precoloring is spotless, then it is clean and orderly. Repeatedly applying Lemma 10 and using symmetry, we deduce the following:

Lemma 11. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph. Let $P = (G, S, X_0, X, Y^*, f)$ be a clean, tidy and orderly excellent starred precoloring of G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$;
- *every* $P' \in \mathcal{L}$ *is tidy and spotless;*
- P *is equivalent to* $\mathcal{L}$.

We now summarize what we have proved so far. Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of a P_6-free graph G. We say that $y \in Y^*$ is *wholesome* if $|M_P(y)| \geq 3$. A component of $G|Y^*$ is *wholesome* if it contains a wholesome vertex. We say that P is *near-orthogonal* if for every wholesome $y \in Y^*$ either

- $N(y) \cap X$ is orthogonal, or
- there exist $\{i, j, k, l\} = \{1, 2, 3, 4\}$ such that
  - $N(y) \cap X \subseteq X_{ki} \cup X_{kj}$, and
  - For every $u \in C(y)$, $|M_P(u) \cap \{i, j\}| \leq 1$, and
  - if there is $v_i \in C(y)$ with $i \in M_P(v_i)$ and $v_j \in C(y)$ with $j \in M_P(v_j)$, then for some $u \in C(y)$, $l \notin M_P(u)$.

Lemma 12. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of a P_6-free graph G. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of excellent starred precolorings of G such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$*;*
- *every* $P' \in \mathcal{L}$ *is near-orthogonal;*
- P *is equivalent to* $\mathcal{L}$*.*

Proof. Let $\mathcal{L}_1$ be the collection of precolorings obtained by applying Lemma 5 to P. Let $\mathcal{L}_2$ be the union of the collections of precolorings obtained by applying Lemma 7 to each member of $\mathcal{L}_1$. Let $\mathcal{L}_3$ be the union of the collections of precolorings obtained by applying Lemma 9 to each member of $\mathcal{L}_2$. Let $\mathcal{L}$ be the union of the collections of precolorings obtained by applying Lemma 11 to each member of $\mathcal{L}_3$. Then $\mathcal{L}$ satisfies the first, second and fourth bullet in the statement of Lemma 12, and every $P' \in \mathcal{L}$ is tidy and spotless. Let $P' \in \mathcal{L}$, write $P' = (S', X'_0, X', Y', f')$. Suppose that P' is not near-orthogonal. Let $y \in Y'$ be wholesome, and assume that the neighbors of y are not orthogonal. We show that y satisfies the conditions in the definition of near-orthogonal. We may assume that y has a neighbor in X'_{12} and a neighbor in X'_{13}. Since P' is spotless, it follows that for every $u \in C(y)$, $|M_P(u) \cap \{2, 3\}| \leq 1$. Since y is wholesome, we may assume that $M_P(y) = \{1, 2, 4\}$. Since P' is spotless, it follows that $N(y) \cap X' \subseteq X'_{12} \cup X'_{13}$. Since P' is tidy and $1 \in M_P(y)$, it follows that if there is $v_2 \in C(y)$ with $2 \in M_P(v_2)$ and $v_3 \in C(y)$ with $3 \in M_P(v_3)$, then for some $u \in C(y)$ $4 \notin M_P(u)$. This proves that y satisfies the conditions in the definition of near orthogonal, and completes the proof of Lemma 12. $\square$

Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring. Let $\{i, j, k, l\} = \{1, 2, 3, 4\}$, let T^i be a type of X with $L_P(T^i) = \{i, k\}$ and let T^j be a type of X with $L_P(T^j) = \{j, k\}$. A *type A extension with respect to* (T^i, T^j) is a precoloring extension c of P such that there exists $y \in Y^*$ with $k, i \in M_P(y)$ and such that y has a neighbor $x_i \in X(T^i)$ and a neighbor $x_j \in X(T^j)$ with $c(x_i) = c(x_j) = k$.

Let $\mathcal{T}(P)$ be the set of all pairs (T^i, T^j) of types of X with $|L_P(T^j) \cap L_P(T^j)| = 1$. A precoloring extension of P is *good* if it is not of type A for any $T \in \mathcal{T}(P)$. We say that P is *smooth* if P has a good precoloring extension.

We say that an excellent starred precoloring $P' = (G, S', X'_0, X', Y^{*\prime}, f')$ is a *refinement* of P if for every type T' of X', there is a type T of X such that $X'(T') \subseteq X(T)$.

Lemma 13. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y^*, f)$ be a near-orthogonal excellent starred precoloring of a P_6-free graph G. There is an algorithm with running time $O(|V(G)|^{q(|S|)}$ that outputs a collection $\mathcal{L}$ of near-orthogonal excellent starred precolorings of G such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- $|S'| \leq q(|S|)$ *for every* $P' \in \mathcal{L}$*;*
- *a precoloring extension of a member of* $\mathcal{L}$ *is also a precoloring extension of* P*;*

- *if P has a precoloring extension, then some $P' \in \mathcal{L}$ is smooth.*

Proof. Let $\mathcal{T}(P) = \{(T_1, T_1'), \dots, (T_t, T_t')\}$. Let $\mathcal{Q}$ be the collection of t-tuples of triples $Q_{T_i,T_i'} = (Y_{T_i,T_i'}, A_{T_i,T_i'}, B_{T_i,T_i'})$ such that

- $|Y_{T_i,T_i'}| = |A_{T_i,T_i'}| = |B_{T_i,T_i'}| \leq 1$.
- $A_{T_i,T_i'} \subseteq X(T_i)$.
- $B_{T_i,T_i'} \subseteq X(T_i')$.
- $Y_{T_i,T_i'} \subseteq Y^*$ and if $Y_{T_i,T_i'} = \{y\}$, then $L_P(T_i) \subseteq M_P(y)$.
- $Y_{T_i,T_i'}$ is complete to $A_{T_i,T_i'} \cup B_{T_i,T_i'}$.
- $A_{T_i,T_i'}$ is anticomplete to $B_{T_i,T_i'}$.

For $Q = (Q_{T_i,T_i'})_{(T_i,T_i') \in \mathcal{T}(P)} \in \mathcal{Q}$, we construct a precoloring P_Q by moving $A_{T_i,T_i'} \cup B_{T_i,T_i'}$ to the seed with the unique color of $L_P(T_i) \cap L_P(T_i')$ for all $(T_i, T_i') \in \mathcal{T}(P)$. Let $P_Q = (G, S', X_0', X', Y', f')$. Since $X' \subseteq X$ and $Y' \subseteq Y^*$, and $M_{P'}(v) \subseteq M_P(v)$ for every $v \in V(G)$, it follows that P_Q is excellent, near-orthogonal and for every type T' of X', there is a type T of X such that $X'(T') \subseteq X(T)$.

Let $\mathcal{L} = \{P\} \cup \{P_Q : Q \in \mathcal{Q}\}$. Observe that there are at most $2^{|S|}$ types, and therefore $t \leq 2^{2|S|}$. Now $|S'| \leq |S| + 2t \leq |S| + 2^{2|S|+1}$ and $|\mathcal{L}| \leq |V(G)|^{3t} \leq |V(G)|^{3 \times 2^{2|S|}}$.

Since every member of $\mathcal{L}$ is obtained from P by precoloring some vertices and updating, it follows that every precoloring extension of a member of $\mathcal{L}$ is also a precoloring extension of P.

Now we prove the last assertion of Lemma 13. Suppose that P has a precoloring extension. We need to show that some $P' \in \mathcal{L}$ is smooth. Let c be a precoloring extension of P. For every $(T_i, T_i') \in \mathcal{T}(P)$ such that c is of type A with respect to (T_i, T_i'), proceed as follows. We may assume that $L_P(T_i) = \{1, 2\}$ and $L_P(T_i') = \{1, 3\}$. Let $y \in Y^*$ with $1, 2 \in M_P(y)$, $x_2 \in X(T_i)$ and $x_3 \in X(T_i')$ such that y is adjacent to x_2, x_3 and $c(x_2) = c(x_3) = 1$, and subject to the existence of such x_2, x_3, choose y with the set $\{x \in N(y) \cap X(T_i')$ such that $c(x) = 1\}$ minimal. Let $Q_{T_i,T_i'} = (\{y\}, \{x_2\}, \{x_3\})$. For every $(T_i, T_i') \in \mathcal{T}(P)$ such that c is not of type A with respect to (T_i, T_i'), set $Q_{T_i,T_i'} = (\emptyset, \emptyset, \emptyset)$. Let $Q = (Q_{T_i,T_i'})_{(T_i,T_i') \in \mathcal{P}}$; then $P_Q \in \mathcal{L}$.

We claim that c is a precoloring extension of P_Q that is not of type A for any $(T_i, T_i') \in \mathcal{T}(P_Q)$. Write $P_Q = (G, S', X_0', X', Y', f')$. Let $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Suppose that T^i is a type of X' with $L_{P_Q}(T^i) = \{i, k\}$ and T^j is a type of X' with $L_{P_Q}(T^j) = \{j, k\}$, and such that $(T^i, T^j) \in \mathcal{T}(P_Q)$, and $y' \in Y'$ with $i, k \in M_{P_Q}(y')$ has neighbor $x_i' \in X'(T^i)$ and $x_j' \in X'(T^j)$ with $c(x_i') = c(x_j') = k$. Let $(\tilde{T^i}, \tilde{T^j}) \in \mathcal{T}(P)$ be such that $X'(T^i) \subseteq X(\tilde{T^i})$ and $X'(T^j) \subseteq X(\tilde{T^j})$. Since $i, k \in M_P(y)$, it follows that c is of type A for $(\tilde{T^i}, \tilde{T^j})$, and therefore $|Y_{\tilde{T^i},\tilde{T^j}}| = |A_{\tilde{T^i},\tilde{T^j}}| = |B_{\tilde{T^i},\tilde{T^j}}| = 1$. Let $Y_{\tilde{T^i},\tilde{T^j}} = \{y\}$ $A_{\tilde{T^i},\tilde{T^j}} = \{x_i\}$ and $B_{\tilde{T^i},\tilde{T^j}} = \{x_j\}$. Since $k \in M_{P_Q}(y')$ it follows that y' is anticomplete to $\{x_i, x_j\}$. By the choice of y, it follows that y' has a neighbor $x' \in X(\tilde{T^j}) \setminus N(y)$ with $c(x') = k$, and so we may assume that x_j' is non-adjacent to y. Since $L_P(T_i') = \{j, k\}$ there exists $s_i \in S$ with $f(s_i) = i$ such that s_i is complete to $\{x_j, x_j'\}$. Since $i \in L_P(x_i) \cap L_P(y') \cap L_P(y)$, it follows that s_i is anticomplete to $\{x_i, y', y\}$. Since $c(x_i) = c(x_i') = c(x_j) = c(x_j')$, it follows that $\{x_i, x_i', x_j, x_j'\}$ is a stable set. But now $x_i - y - x_j - s_i - x_j' - y'$ is a P_6 in G, a contradiction. This proves that c is a good precoloring extension of P_Q, and completes the proof of Lemma 13. □

We are finally ready to construct orthogonal precolorings.

Lemma 14. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y^*, f)$ be a near-orthogonal excellent precoloring of a P_6-free graph G. There exist an induced subgraph G' of G and an orthogonal excellent starred precoloring $P' = (G', S', X_0', X', Y', f')$ of G', such that*

- *$S = S'$,*
- *if P is smooth, then P' has a precoloring extension, and*
- *if c is a precoloring extension of P', then a precoloring extension of P can be constructed from c in polynomial time.*

Moreover, P' can be constructed in time $O(|V(G)|^{q(|S|)})$.

Proof. We may assume that P is not orthogonal. We say that a component C of $G|Y^*$ is *troublesome* if C is wholesome, and the set of attachments of C in X are not orthogonal. Let W be the union of the vertex sets of the components of $G|Y^*$ that are not wholesome.

We construct a set Z, starting with $Z = \emptyset$. For every troublesome component C, proceed as follows. We may assume that C has attachments in X_{12} and in X_{13}. Since P is near-orthogonal, and C is wholesome, we may assume that C contains a vertex z with $M_P(z) = \{1, 2, 4\}$.

- If there is $y \in V(C)$ with $M_P(y) = \{1, 3\}$, move $N(y) \cap X_{12}$ to X_0 with color 2.
- Suppose that there is no y as in the first bullet. If $|V(C)| \geq 2$, or $V(C) = \{z\}$ and z has a neighbor v in X_0 with $f(v) = \{4\}$, move $N(z) \cap X_{13}$ to X_0 with color 3.
- If none of the first two conditions hold, add $V(C)$ to Z. Observe that in this case $V(C) = \{y\}$, y has no neighbors in $Z \setminus \{y\}$. Moreover, since P is near-orthogonal, $V(C)$ is anticomplete to $X \setminus (X_{12} \cup X_{13})$, and so for every $u \in N(y)$, $4 \notin L_P(u)$. In this case we call 4 the *free color* of y.

Let $P'' = (G, S', X_0', X'', Y'', f')$ be the precoloring we obtained after we applied the procedure above to all troublesome components. Let $G' = G \setminus Z$, and let $P' = (G', S', X_0', X', Y', f')$ where $Y' = Y'' \setminus (W \cup Z)$ and $X' = X'' \cup W$. Since no vertex of W is wholesome, It follows from the definition of M_P that every vertex of W has neighbors of at least two different colors in S' (with respect to f'). Since W is anticomplete to Y', $X' \setminus W \subseteq X$, and $Y' \subseteq Y^*$, we deduce that P' is excellent and orthogonal. It follows from the construction of Z that every precoloring extension of P' can be extended to a precoloring extension of P by giving each member of Z its free color.

It remains to show that if P is smooth, then P' has a precoloring extension. Suppose that P is smooth, and let c be a good precoloring extension of P. We claim that $c|V(G')$ is a precoloring extension of P'. We need to show that $c(v) = f'(v)$ for every $v \in S' \cup X_0'$. Since $S = S'$, and $f(v) = f'(v)$ for every $v \in X_0$, it is enough to show that $c(v) = f'(v)$ for every $v \in X_0' \setminus X_0$. Thus we may assume that there is a troublesome component C of $G|Y^*$ that has an attachment in X_{12} and an attachment in X_{13}, and $v \in X(C)$. Since P is near-orthogonal, we may assume that C contains a vertex z with $M_P(z) = \{1, 2, 4\}$, and $v \in X_{12} \cup X_{13}$. There are two possibilities.

1. There is $y \in V(C)$ with $M_P(y) = \{1, 3\}$, $v \in N(y) \cap X_{12}$ and $f'(v) = 2$, but $c(v) = 1$. We show that this is impossible. Since c is a good coloring, it follows that $c(u) = 3$ for every $u \in N(y) \cap X_{13}$, contrary to the fact that c is a coloring of G.
2. There is no y as in the first case, and either $|V(C)| \geq 2$, or $V(C) = \{z\}$ and z has a neighbor u in X_0 with $f(u) = 4$, and $v \in X_{13} \cap N(z)$, $f'(v) = 3$ but $c(v) = 1$. We show that this too is impossible. It follows that there is a vertex $y' \in V(C)$ with $c(y') \neq 4$. Choose such y' with $4 \notin M_P(y')$ if possible. Since P is excellent, y' is adjacent to v. Since c is a good coloring, it follows that $c(w) = 2$ for every $w \in X_{12} \cap N(y')$. This implies that $c(y') = 3$. Since P is near-orthogonal and $3 \in M_P(y')$, it follows that $2 \notin M_P(y')$. Since $M_P(y') \neq \{1, 3\}$, it follows that $4 \in M_P(y')$. Since $1, 4 \in M_P(y')$ and $3 \in M_P(y')$, and since P is near-orthogonal, it follows that there is $t \in V(C)$ such that $4 \notin M_P(t)$. Since $c(v) = 1$ and $c(w) = 2$ for every attachment w of $V(C)$ in X_{12}, it follows that $c(t) = 3$, contrary to the choice of y'.

Thus $c(v) = f'(v)$ for every $v \in S' \cup X_0'$, and so $c|V(G')$ is a precoloring extension of P'. This completes the proof of Lemma 14. □

We can now prove the main result of this section.

Theorem 9. *There is a function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let $P = (G, S, X_0, X, Y^*, f)$ be an excellent starred precoloring of a P_6-free graph G with $|S| \leq C$. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of orthogonal excellent starred precolorings of induced subgraphs of G such that:*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$;

- *$|S'| \leq q(|S|)$ for every $P' \in \mathcal{L}$, and*
- *P has a precoloring extension, if and only if some $P' \in \mathcal{L}$ has a precoloring extension;*
- *given a precoloring extension of a member of $\mathcal{L}$, a precoloring extension of P can be constructed in polynomial time.*

Proof. By Lemma 12 there exist a function $q_1 : \mathbb{N} \to \mathbb{N}$ and a polynomial-time algorithm that outputs a collection $\mathcal{L}_1$ of excellent starred precolorings of G such that:

- $|\mathcal{L}_1| \leq |V(G)|^{q_1(|S|)}$;
- $|S'| \leq q_1(|S|)$ for every $P' \in \mathcal{L}_1$;
- every $P' \in \mathcal{L}_1$ is near-orthogonal; and
- P is equivalent to $\mathcal{L}_1$.

Let $P' \in \mathcal{L}_1$. Write $P' = (G, S(P'), X_0(P'), X(P'), Y^*(P'), f_{P'})$. By Lemma 13 there exist a function $q_2 : \mathbb{N} \to \mathbb{N}$ and a polynomial-time algorithm that outputs a collection $\mathcal{L}(P')$ of near-orthogonal excellent starred precolorings of G such that:

- $|\mathcal{L}(\mathcal{P}')| \leq |V(G)|^{q_2(|S(P')|)}$;
- $|S''| \leq q_2(|S(P')|)$ for every $P'' \in \mathcal{L}(P')$;
- if P' has a precoloring extension, then some $P'' \in \mathcal{L}(P')$ is smooth; and
- a precoloring extension of a member of $\mathcal{L}(P')$ is also a precoloring extension of P'.

Let $\mathcal{L}_2 = \bigcup_{P' \in \mathcal{L}_1} \mathcal{L}(P')$.

Clearly $\mathcal{L}_2$ has the following properties:

- $|\mathcal{L}_2| \leq |V(G)|^{q_1(q_2(|S|))}$;
- $|S'| \leq q_1(q_2(|S(P)|))$ for every $P' \in \mathcal{L}_2$;
- if P has a precoloring extension, then some $P'' \in \mathcal{L}(P')$ is smooth; and
- given a precoloring extension of a member of $\mathcal{L}_2$, one can construct in polynomial time a precoloring extension of P.

Let $P'' \in \mathcal{L}_2$. Write $P'' = (G, S(P''), X_0(P'), X'(P'), Y^*(P''), f_{P''})$. By Lemma 14, there exists an induced subgraph G' of G and an orthogonal excellent starred precoloring $Orth(P'') = (G', S', X_0', X', Y', f')$ of G', such that

- $S(P'') = S'$;
- if P'' is smooth, then $Orth(P'')$ has a precoloring extension; and
- if c is a precoloring extension of $Orth(P'')$, then a precoloring extension of P'', and therefore of P, can be constructed from c in polynomial time.

Moreover, $Orth(P'')$ can be constructed in polynomial time.

Let $\mathcal{L} = \{Orth(P'') \ : \ P'' \in \mathcal{L}_2\}$. Now $\mathcal{L}$ has the following properties.

- $|\mathcal{L}| \leq |V(G)|^{q_1(q_2(|S|)}$;
- $|S'| \leq q_1(q_2(|S|))$ for every $P' \in \mathcal{L}$; and
- if c is a precoloring extension of $P' \in \mathcal{L}$, then a precoloring extension of P can be constructed from c in polynomial time.

- every $P' \in \mathcal{L}$ is orthogonal.

To complete the proof of the Theorem 9 we need to show that if P has a precoloring extension, then some $P' \in \mathcal{L}$ has a precoloring extension. So assume that P has a precoloring extension. Since $\mathcal{L}_1$ is equivalent to P, it follows that some $P_1 \in \mathcal{L}_1$ has a precoloring extension. This implies that some $P_2 \in \mathcal{L}(P_1) \subseteq \mathcal{L}_2$ is smooth. But now $Orth(P_2)$ has a precoloring extension, and $Orth(P_2) \in \mathcal{L}$. This completes the proof of Theorem 9. $\square$

3 Companion triples

In view of Theorem 9 we now focus on testing for the existence of a precoloring extension for an orthogonal excellent starred precoloring.

Let G be a P_6-free graph, and let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of G. We may assume that $X_0 = X^0(P)$. Let $\mathcal{C}(P)$ be the set of components of $G|Y^*$, and let $C \in \mathcal{C}(P)$. It follows that $X \setminus X(C)$ is anticomplete to $V(C)$, and we may assume (using symmetry) that $X(C) \subseteq X_{12} \cup X_{34}$. We now define the precoloring obtained from P by *contracting the* 12*-neighbors of* C, or, in short, by *neighbor contraction.* Suppose that $X_{12} \cap X(C) \neq \emptyset$, and let $x_{12} \in X_{12} \cap X(C)$. Let $\tilde{G}$ be the graph define as follows:

$$V(\tilde{G}) = G \setminus (X_{12} \cap X(C)) \cup \{x_{12}\}$$

$$\tilde{G} \setminus \{x_{12}\} = G \setminus (X_{12} \cap X(C))$$

$$N_{\tilde{G}}(x_{12}) = \bigcup_{x \in X_{12} \cap X(C)} N_G(x) \cap V(\tilde{G}).$$

Moreover, let

$$\tilde{X} = X \setminus (X_{12} \cap X(C)) \cup \{x_{12}\}.$$

Then $\tilde{P} = (\tilde{G}, X_0, \tilde{X}, Y^*, f)$ is an orthogonal excellent starred precoloring of $\tilde{G}$. We say that $\tilde{P}$ is *obtained from* P *by contracting the* 12*-neighbors of* C, or, in short, *obtained from* P *by neighbor contraction.* We call x_{12} the image of $X_{12} \cap X(C)$, and define $x_{12}(C) = x_{12}$. Observe that $x_{12} \in X$ (this fact simplifies notation later), and that $M_P(v) = M_{\tilde{P}}(v)$ for every $v \in V(\tilde{G})$. For every $i \neq j \in \{1, 2, 3, 4\}$ we define the precoloring obtained from P by *contracting the* ij*-neighbors of* C similarly.

For $i \neq j \in \{1, 2, 3, 4\}$ and $t \in X_0 \cup S$ let $\tilde{G}_{ij}(t) = \tilde{G}|(\tilde{X}_{ij} \cup Y^* \cup \{t\})$. We remind the reader that given a path P, its *interior* is the set of vertices that have degree two in P. We denote the interior of P by P^*. While graph $\tilde{G}$ may not be P_6-free, the following weaker statement holds:

Lemma 15. *Let P be an excellent orthogonal precoloring of a P_6-free graph G. Let $C \in \mathcal{C}(P)$ and assume that $X(C) \cap X_{12}$ is non-empty. Let $\tilde{P} = (\tilde{G}, X_0, \tilde{X}, Y^*, f)$ be obtained from P by contracting the* 12*-neighbors of C. Then $\tilde{G}_{ij}(t)$ is P_6-free for every $i \neq j \in \{1, 2, 3, 4\}$ and $t \in S \cup X_0$.*

Proof. If $\{i, j\} \neq \{1, 2\}$, then $\tilde{G}_{ij}(t)$ is an induced subgraph of G, and therefore it is P_6-free. So we may assume that $\{i, j\} = \{1, 2\}$. Suppose that $Q = q_1 - \ldots - q_6$ is a P_6 in $\tilde{G}_{ij}(t)$. Since $\tilde{G}_{ij}(t) \setminus x_{12}$ is an induced subgraph of G, it follows that $x_{12} \in V(Q)$. If the neighbors of x_{12} in Q have a common neighbor $n \in X(C) \cap X_{12}$, then $G|((V(Q) \setminus \{x_{12}\}) \cup \{n\})$ is a P_6 in G, a contradiction. It follows that x_{12} has two neighbors in Q, say a, b, each of a, b has a neighbor in $X_{12} \cap X(C)$, and no vertex of $X(C) \cap X_{12}$ is complete to $\{a, b\}$. Since $V(C)$ is complete to $X(C)$, it follows that $a, b \notin V(C)$, and so $a, b \in (X_{12} \setminus X(C)) \cup (Y^* \setminus V(C)) \cup \{t\}$. Since x_{12} has exactly two neighbors in Q, it follows that $V(Q) \cap V(C) = \emptyset$. Consequently, $V(Q) \setminus \{t, x_{12}\} \subseteq (X_{12} \setminus X(C)) \cup (Y^* \setminus V(C))$. Since $X_{12} \setminus \{x_{12}\}$ is anticomplete to $V(C)$, and since C is a component of $G|Y^*$, we deduce that $V(Q) \setminus \{t, x_{12}\}$ is anticomplete to $V(C)$. Let Q' be a shortest path from a to b with $Q'^* \subseteq X(C) \cup V(C)$. Since $V(Q) \setminus \{x_{12}, t\}$ is anticomplete to $V(C)$, and $V(Q) \setminus \{a, b\}$ is anticomplete to $X(C) \cap X_{12}$, it follows that $V(Q')$ is anticomplete to $V(Q) \setminus (\{x_{12}\} \cup \{a, b, t\})$. Moreover, if $t \neq a, b$ and $t \in V(Q)$, then t is anticomplete to $Q'^* \setminus V(C)$. If follows that if $t \notin V(Q) \setminus \{a, b, x_{12}\}$ or t is anticomplete to $V(Q') \cap V(C)$ then $q_1 - Q - a - Q' - b - Q - q_6$ is a path of length at least six in G, a contradiction; so $t \in V(Q) \setminus \{a, b, x_{12}\}$, and t has a neighbor in $V(Q') \cap V(C)$. Since $V(C)$ is complete to $X(C)$, it follows that $|V(C) \cap V(Q')| = 1$, and $|Q'^*| = 3$. Let $V(Q') \cap V(C) = \{q'\}$. We may assume that

b has a neighbor $c \in V(Q) \setminus \{x_{12}\}$, and if $a = q_i$ and $b = q_j$, then $i < j$. Since $a - Q' - b - c$ is not a P_6 in G, it follows that $t = c$. But now $q_1 - a - Q' - q' - t - Q - q_6$ is a P_6 in G, a contradiction. This proves Lemma 15. □

Let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring. Let H be a graph, and let L be a 4-list assignment for H. Recall that $X^0(L)$ is the set of vertices of H with $|L(x_0)| = 1$. Let M be the list assignment obtained from M_P by updating Y^* from X_0. We say that (H, L, h) is a *near-companion triple for P with correspondence h* if there is an orthogonal excellent starred precoloring $\tilde{P} = (\tilde{G}, S, X_0, \tilde{X}, Y^*, f)$ obtained from P by a sequence of neighbor contractions, and the following hold:

- $V(H) = \tilde{X} \cup Z$;
- $h : Z \to \mathcal{C}(P)$;
- for every $z \in Z$, $N(z) = \tilde{X}(V(h(z)))$;
- $H|(Z \cup \tilde{X}_{ij})$ is P_6-free for all i, j;
- Z is a stable set;
- for every $x \in \tilde{X}$, $L(x) \subseteq M_P(x) = M(x)$;
- for every $z \in Z$ such that $L(z) \neq \emptyset$, if $q \in \{1, 2, 3, 4\}$ and $q \notin L(z)$, then some vertex $V(h(z))$ has a neighbor $u \in S \cup X_0 \cup X^0(L)$ with $f(u) = q$; and
- for every $z \in Z$ and every $q \in L(z)$, there is $v \in V(h(z))$ with $q \in M(v)$, and no vertex $u \in S \cup X_0$ with $f(u) = q$ is complete to $V(h(z))$.

For $z \in Z$, we call $h(z)$ the *image of z*.

If (H, L, h) is a near-companion triple for P, and in addition

- $\tilde{P}$ has a precoloring extension if and only if (H, L) is colorable, and a coloring of (H, L) can be converted to a precoloring extension of P in polynomial time.

we say that (H, L, h) is a *companion triple* for P.

For $i \neq j \in \{1, 2, 3, 4\}$ and $t \in S \cup X_0$ let $H_{ij}(t)$ be the graph obtained from $H|(\tilde{X}_{ij} \cup Z)$ by adding the vertex t and making t adjacent to the vertices of $N_{\tilde{G}}(t) \cap \tilde{X}_{ij}$. The following is a key property of near-companion triples.

Lemma 16. *Let G be a P_6-free graph, let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of G, and let (H, L, h) be a near-companion triple for P. Let M be the list assignment obtained from M_P by updating Y^* from X_0. Assume that $L(v) \neq \emptyset$ for every $v \in V(H)$. Let $i, j \in \{1, 2, 3, 4\}$ and $t \in X_0 \cup S$, and let Q be a P_6 in $H_{ij}(t)$. Then $t \in V(Q)$, and there exists $q \in V(Q) \setminus N(t)$ such that $f(t) \notin M(q)$.*

Proof. Since $H|(\tilde{X}_{ij} \cup Z)$ is P_6-free, it follows that $t \in V(Q)$. Suppose that for every $q \in V(Q) \setminus N(t)$, $f(t) \in L(q)$. Let $z \in V(Q) \cap Z$. Since t is anticomplete to Z, it follows that $f(t) \in L(z)$. By the definition of a near-companion triple, there is a vertex $q(z) \in V(h(z))$ such that $f(t) \in M(q(z))$. Since M is obtained from M_P by updating Y^* from X_0, it follows that t is non-adjacent to $q(z)$. Now replacing z with $q(z)$ for every $z \in V(Q) \cap Z$, we get a P_6 in $\tilde{G}_{ij}(t)$ that contradicts Lemma 15. This proves Lemma 16. □

The following is the main result of this section.

Theorem 10. *Let G be a P_6-free graph and let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of G. Then there is a polynomial-time algorithm that outputs a companion triple for P.*

Proof. We may assume that $X_0 = X^0(P)$. Let M be the list assignment obtained from M_P by updating Y^* from X_0. Write $\mathcal{C} = \mathcal{C}(P)$. For $Q \subseteq \{1, 2, 3, 4\}$ and $C \in \mathcal{C}$, we say that a coloring c of (C, M) is a Q-coloring if $c(v) \in Q$ for every $v \in V(C)$. Given $Q \subseteq \{1, 2, 3, 4\}$, we say that Q is *good for C* if (C, M) admits a proper Q-coloring, and *bad for C* otherwise. By Theorem 2, for every Q with $|Q| \leq 3$, we can test in polynomial time if Q is good for C. Let $\mathcal{Q}(C)$ be the set of all inclusion-wise maximal bad subsets of $\{1,2,3,4\}$. Observe that if Q is bad, then all its subsets are bad.

Here is another useful property of $\mathcal{Q}(C)$.

(1) *Let $Q \in \mathcal{Q}(C)$, and let $i \in Q$ be such that no $u \in S \cup X_0$ with $f(u) = i$ has a neighbor in $V(C)$. Then for every $j \in \{1,2,3,4\} \setminus Q$, the set $(Q \setminus \{i\}) \cup \{j\}$ is bad.*

Suppose not. Let $Q' = Q \setminus \{i\} \cup \{j\}$. Let c be a proper Q'-coloring of (C, M). It follows from the definition of M that $i \in M(y)$ for every $y \in V(C)$. Recolor every vertex $u \in V(C)$ with $c(u) = j$ with color i. This gives a proper Q-coloring of (C, M), a contradiction. This proves (1).

First we describe a sequence of neighbor contractions to produce $\tilde{P}$ as in the definition of a companion triple. Let $C \in \mathcal{C}$ with $|V(C)| > 1$. We may assume (without loss of generality) that $X(C) \subseteq X_{12} \cup X_{34}$. If $X(C)$ meets both X_{12} and X_{34}, contract the $1,2$-neighbors of C, and the $3,4$-neighbors of C; observe that in this case $\tilde{X}(C) = \{x_{12}(C), x_{34}(C)\}$. If $X(C)$ meets exactly one of X_{12}, X_{34}, say $X(C) \subseteq X_{12}$, and $\{3,4\}$ is bad for C, contract the 12-neighbors of C. Repeat this for every $Q \in \mathcal{Q}(C)$; let $\tilde{P} = (\tilde{G}, S, X_0, \tilde{X}, Y^*, f)$ be the resulting precoloring. Observe that $\tilde{X} \subseteq X$.

(2) *P has a precoloring extension if and only if $\tilde{P}$ has a precoloring extension, and a precoloring extension of $\tilde{P}$ can be converted into a precoloring extension of P in polynomial time.*

Since $|\mathcal{C}(P)| \leq |V(G)|$, it is enough to show that the property of having a precoloring extension, and the algorithmic property, do not change when we perform one step of the construction above.

Let us say that we start with $P_1 = (G_1, S, X_0, X_1, Y^*, f)$ and finish with $P_2 = (G_2, S, X_0, X_2, Y^*, f)$. We claim that in all cases, each of the sets that is being contracted (that is, replaced by its image) is monochromatic in every precoloring extension of P.

Let $C \in \mathcal{C}(P_1)$ with $|V(C)| > 1$, such that P_2 is obtained from P_1 by contracting neighbors of C. Let $\{i,j,k,l\} = \{1,2,3,4\}$ and let $X_1(C) \subseteq X_{ij} \cup X_{kl}$. If $X_1(C)$ meets both X_{ij} and X_{kl}, then since $|V(C)| > 1$, each of the sets $X_1(C) \cap X_{ij}$, $X_1(C) \cap X_{kl}$ is monochromatic in every precoloring extension of P_1, as required. So we may assume that $X_1(C) \subseteq X_{ij}$. Now $X_1(C)$ is monochromatic in every precoloring extension of P_1 because the set $\{k,l\}$ is bad for C. This proves the claim.

Now suppose that a set A was contracted to produce its image a. If P_1 has a precoloring extension, we can give a the unique color that appears in A, thus constructing an extension of P_2. On the other hand, if P_2 has a precoloring extension, then every vertex of A can be colored with the color of a. This proves (2).

Next we define $L : \tilde{X} \to 2^{[4]}$. Start with $L(x) = M_{\tilde{P}}(x)$ for every $x \in \tilde{X}$. Again let $C \in \mathcal{C}$ with $|V(C)| > 1$, let $\{i,j,k,l\} = \{1,2,3,4\}$, and let $\tilde{X}(C) \subseteq X_{ij} \cup X_{kl}$. For every $Q \in \mathcal{Q}(C)$ such that $Q = \{1,2,3,4\} \setminus \{m\}$, update L by removing m from $L(x)$ for every $x \in X_{ij} \cap \tilde{X}(C)$.

Next assume that $X(C)$ meets both X_{ij}, X_{kl} , the sets $\{i,k\}, \{i,l\}$ are good for C, and the sets $\{j,k\}, \{j,l\}$ are bad for C. Update L by removing i from $L(x_{ij}(C))$.

Finally, assume that $X(C)$ meets both X_{ij}, X_{kl} , the set $\{i,k\}$ is good for C, and the sets $\{i,l\}, \{j,k\}, \{j,l\}$ are bad for C. Update L by removing i from $L(x_{ij}(C))$ and by removing k from $L(x_{kl}(C))$.

Now the following holds.

(3) *Let $\{1,2,3,4\} = \{i,j,k,l\}$ and let $C \in \mathcal{C}$ such that $X(C) \subseteq X_{ij} \cup X_{kl}$.*

1. *If $\{1,2,3,4\} \setminus \{i\} \in \mathcal{Q}(C)$, then $i \notin \bigcup_{x \in \tilde{X}(C)} L(x)$.*
2. *If $\tilde{X}(C)$ meets both X_{ij} and X_{kl} and $\{i,k\}, \{i,l\}$ are both good for C, and $\{j,k\}, \{j,l\}$ are both bad for C, then $i \notin L(x_{ij}(C)) \cup L(x_{kl}(C))$.*
3. *If $\tilde{X}(C)$ meets both X_{ij} and X_{kl} and $\{i,k\}$ is good for C, and $\{i,l\}, \{j,k\}, \{j,l\}$ are bad for C, then $i, k \notin L(x_{ij}(C)) \cup L(x_{kl}(C))$.*

Next we show that:

(4) *If c is a precoloring extension of $\tilde{P}$, then $c(x) \in L(x)$ for every $x \in \tilde{X}$.*

This is clear for x such that $L(x) = M(x)$, so let $x \in \tilde{X}$ be such that $L(x) \neq M(x)$. Then there exists $C \in \mathcal{C}$ with $|V(C)| > 1$, and $\{i, j, k, l\} = \{1, 2, 3, 4\}$ with $\tilde{X}(C) \subseteq X_{ij} \cup X_{kl}$, such that $x \in \tilde{X}(C)$. Suppose that $c(x) \in M(x) \setminus L(x)$. Observe that $c|V(C)$ is a coloring of (C, M). There are three possible situations in which $c(x)$ could have been removed from $M(x)$ to produce $L(x)$.

- $\{1, 2, 3, 4\} \setminus \{i\}$ is bad for C, and $x \in X_{ij}$, and $c(x) = i$. In this case, since (C, M) is not $\{1, 2, 3, 4\} \setminus \{i\}$-colorable, it follows that some $v \in V(C)$ has $c(v) = i$, but $V(C)$ is complete to $\tilde{X}(C)$, a contradiction.

- $\tilde{X}(C)$ meets both X_{ij} and X_{kl}, the sets $\{i, k\}, \{i, l\}$ are good for C, the sets $\{j, k\}, \{j, l\}$ are bad for C, $x = x_{ij}(C)$, and $c(x) = i$. Since $\tilde{X}(C) \cap X_{kl} \neq \emptyset$, it follows that $c(u) \in \{k, l\}$ for some $u \in \tilde{X}(C)$. Since the sets $\{j, k\}, \{j, l\}$ are bad for C and $|V(C)| > 1$, it follows that $c(v) = i$ for some $v \in V(C)$, but $x_{ij}(C)$ is complete to $V(C)$, a contradiction.

- $\tilde{X}(C)$ meets both X_{ij} and X_{kl}, the set $\{i, k\}$ is good for C, the sets $\{i, l\}, \{j, k\}, \{j, l\}$ are bad for C, and either $x = x_{ij}(C)$ and $c(x) = i$, or $x = x_{kl}(C)$ and $c(x) = k$. Since $\tilde{X}(C)$ meets both X_{ij} and X_{kl} and $|V(C)| > 1$, it follows that $|c(V(C)) \cap \{i, j\}| = 1$, and $|c(V(C)) \cap \{k, l\}| = 1$. Since $\{j, k\}, \{j, l\}$ are bad for C, it follows that for some $v \in V(C)$ has $v(c) = i$, and so $c(x_{ij}(C)) \neq i$. Since $\{i, l\}$ is bad for C, it follows that $c(V(C)) = \{i, k\}$, and so $c(x) \neq k$, in both cases a contradiction.

This proves (4).

Finally, for every $C \in \mathcal{C}$, we construct the set $h^{-1}(C)$ and define $L(v)$ for every $v \in h^{-1}(C)$.

If $|V(C)| = 1$, say $C = \{y\}$, let $h^{-1}(C) = \{y\}$, and let $L(y) = M(y)$.

Now assume $|V(C)| > 1$. We may assume that $\tilde{X}(C) \subseteq X_{12} \cup X_{34}$.

If all subsets of $\{1, 2, 3, 4\}$ of size three are bad, let z_C be a new vertex, and set $h^{-1}(C) = \{z_C\}$ and $L(z_C) = \emptyset$. From now on we assume that there is a good subset for C of size at most three.

If $\tilde{X}(C) \subseteq X_{12}$ or $\tilde{X}(C) \subseteq X_{34}$, set $h^{-1}(C) = \emptyset$.

So we may assume that $\tilde{X}(C)$ meets both X_{12} and X_{34}. If all sets of size two, except possibly $\{1, 2\}$ and $\{3, 4\}$, are bad for C, let z_C be a new vertex, and set $h^{-1}(C) = \{z_C\}$ and $L(z_C) = \emptyset$. Next let $Q \in \mathcal{Q}(C)$ with $|Q| = 2$; write $\{i, j, k, l\} = \{1, 2, 3, 4\}$, and say $Q = \{i, j\}$. We say that Q is *friendly* if there exist $u_i, u_j \in S \cup X_0$, both with neighbors in C, and with $f(u_i) = i$ and $f(u_j) = j$. For every friendly set Q, let $v(C, Q)$ be a new vertex, and let $h^{-1}(C)$ consist of all such vertices $v(C, Q)$. Set $L(v(C, Q)) = \{1, 2, 3, 4\} \setminus Q$.

Let $Z = \bigcup_{C \in \mathcal{C}} h^{-1}(C)$. Finally, define the correspondence function h by setting $h(z) = C$ for every $z \in h^{-1}(C)$ and $C \in \mathcal{C}$.

Now we define H. We set $V(H) = \tilde{X} \cup Z$, and $pq \in E(H)$ if and only if either

- $p, q \in \tilde{X}$ and $pq \in E(G)$, or

- there exists $C \in \mathcal{C}$ such that $p \in h^{-1}(C)$ and $q \in \tilde{X}(C)$.

The triple (H, L, h) that we have constructed satisfies the following.

- $\tilde{X} \subseteq V(H)$; write $Z = V(H) \setminus \tilde{X}$.

- $N(z) = \tilde{X}(V(h(x)))$ for every $z \in Z$.

- Z is a stable set.

- For every $x \in \tilde{X}$, $L(x) \subseteq M_P(x) = M(x)$.

- $h : Z \to \mathcal{C}(P)$.

- If $z \in Z$ with $L(z) \neq \emptyset$, and $q \in \{1, 2, 3, 4\} \setminus L(z)$, then some vertex $V(h(z))$ has a neighbor $u \in S \cup X_0$ with $f(u) = q$. (This is in fact stronger than what is required in the definition of a companion triple; we will relax this condition later.)

To complete the proof of Theorem 10, it remains to show the following

1. For every $z \in Z$ and every $q \in L(z)$, there is $v \in V(h(z))$ with $q \in M(v)$, and no vertex $u \in S \cup X_0$ with $f(u) = q$ is complete to $V(h(z))$.
2. for every $i \neq j \in \{1,2,3,4\}$, $H|(\tilde{X_{ij}} \cup Z)$ is P_6-free.
3. P has a precoloring extension if and only if (H, L) is colorable, and a proper coloring of (H, L) can be converted to a precoloring extension of P in polynomial time.

We prove the first statement first. Let $z \in Z$ and $q \in L(z)$, and suppose that for every $v \in V(h(z))$ $q \notin M(v)$, or some vertex $u \in S \cup X_0$ with $f(u) = q$ is complete to $V(h(z))$. It follows that $|V(h(z))| > 1$. Since $z \in Z$, it follows that there exists a set $\{i,j\} \in \mathcal{Q}(h(z))$ and $L(z) = \{1,2,3,4\} \setminus \{i,j\}$. But now it follows that $\{q,i,j\}$ is also bad for $h(z)$, contrary to the maximality of $\{i,j\}$. This proves the first statement.

Next we prove the second statement. By Lemma 15, $\tilde{G}|(\tilde{X_{ij}} \cup Y^*)$ is P_6-free for every $i \neq j \in \{1,2,3,4\}$. Suppose Q is a P_6 in H. Let $C \in \mathcal{C}(P)$. Since no vertex of $V(H) \setminus h^{-1}(C)$ is mixed on $h^{-1}(C)$, it follows that $|V(Q) \cap h^{-1}(C)| \leq 1$. Moreover, $\tilde{X}_{ij}(h^{-1}(C)) = \tilde{X}_{ij}(C)$. Let G' be obtained from $\tilde{G}$ by replacing each $C \in \mathcal{C}$ by a single vertex of C, choosing this vertex to be in $V(Q)$ if possible. Then G' is an induced subgraph of G, and Q is a P_6 in G', a contradiction. This proves the second statement.

Finally we prove the last statement. Let $\mathcal{C}_1 = \{C \in \mathcal{C} \;:\; |V(C)| = 1\}$, and let $Y = \bigcup_{C \in \mathcal{C}_1} V(C)$. Then $Y \subseteq Z$.

Suppose first that P has a precoloring extension. It is easy to see that $L(z) \neq \emptyset$ for all $z \in Z$. By (2), there exists a precoloring extension of $\tilde{P}$; denote it by c. By (4), $c|(\tilde{X} \cup Y)$ is a coloring of $(H|(X \cup Y), L)$. It remains to show that c can be extended to $Z \setminus Y$. Let $z \in Z$, and let $h(z) = C$. Then there is a friendly set $\{i,j\} \in \mathcal{Q}$ such that $z = v(C, Q)$. Since Z is a stable set, in order to show that c can be extended to $Z \setminus Y$, it is enough to show that

$$L(z) \not\subseteq c(\tilde{X}(C)).$$

Since $L(v(C,Q)) = \{1,2,3,4\} \setminus Q$, it is enough to show that

$$\{1,2,3,4\} \setminus c(\tilde{X}(C)) \not\subseteq Q.$$

But the latter statement is true because

$$c(V(C)) \subseteq \{1,2,3,4\} \setminus c(\tilde{X}(C))$$

and $c(V(C))$ is a good set, and therefore $c(V(C)) \not\subseteq Q$. This proves that if $\tilde{P}$ has a precoloring extension, then (H, L) is colorable.

Now let c be a proper coloring of (H, L). By (2) it is enough to show that $\tilde{P}$ has a precoloring extension. We define a precoloring extension $\tilde{c}$ of $\tilde{P}$. Set $\tilde{c}(v) = f(v)$ for every $v \in S \cup X_0$, and $\tilde{c}(x) = c(x)$ for every $x \in \tilde{X} \cup Y$. It follows from the definition of L that $\tilde{c}$ is a precoloring extension of $(\tilde{G} \setminus (Y^* \setminus Y), S, X_0, \tilde{X}, Y)$.

Let $C \in \mathcal{C}$ with $|V(C)| > 2$. We extend $\tilde{c}$ to C. We will show that for every $Q \in \mathcal{Q}(C)$, $\{1,2,3,4\} \setminus c(\tilde{X}(C)) \not\subseteq Q$. Consequently $T = \bigcup_{y \in V(C)} M_P(y) \setminus c(\tilde{X}(C))$ is good for C. Since some vertex of $S \cup X_0 \cup \tilde{X}$ is complete to $V(C)$, it follows that $|T| \leq 3$. Therefore we can define $\tilde{c} : V(C) \to \{1,2,3,4\}$ to be a proper T-coloring of (C, M), which can be done in polynomial time by Theorem 2.

So suppose that there is $Q \in \mathcal{Q}(C)$ such that $\{1,2,3,4\} \setminus c(\tilde{X}(C)) \subseteq Q$. Then $\{1,2,3,4\} \setminus Q \subseteq c(\tilde{X}(C))$. By (3.1), $|Q| < 3$.

We may assume that $\tilde{X}(C) \subseteq X_{12} \cup X_{34}$. Suppose first that $\tilde{X}(C)$ meets both X_{12} and X_{34}, and so $\tilde{X}(C) = \{x_{12}(C), x_{34}(C)\}$. Then $|c(\tilde{X}(C))| = 2$, and so $|Q| \neq 1$ (since $\{1,2,3,4\} \setminus c(\tilde{X}(C)) \subseteq Q$). Therefore we may assume that $|Q| = 2$. If Q is friendly, then $c(v(C,Q)) \notin Q$, and so $\{1,2,3,4\} \setminus Q \not\subseteq c(\tilde{X}(C))$, so we may assume that Q is not friendly. By symmetry, we may assume that $Q \in \{\{1,2\},\{1,3\}\}$. If $Q = \{1,2\}$, then since $L(x_{12}(C)) \subseteq \{1,2\}$, it follows that $\{1,2,3,4\} \setminus Q \not\subseteq c(\tilde{X}(C))$, so we may assume that $Q = \{1,3\}$.

Suppose first that for every $i \in Q$, there is no vertex $u \in S \cup X_0$ with $c(u) = i$ and such that u has a neighbor in $V(C)$. Now (1) implies that every set of size two is bad for C. Therefore $h^{-1}(C) = \{z\}$ and $L(z) = \emptyset$, contrary to the fact that c is a proper coloring of (H, L).

We may assume from symmetry that

- there is a vertex $u \in S \cup X_0$ with $c(u) = 1$ and such that u has a neighbor in $V(C)$.

- there is no vertex $u \in S \cup X_0$ with $c(u) = 3$ and such that u has a neighbor in $V(C)$.

Now by (1) all the sets sets $\{1, 2\}, \{1, 3\}, \{1, 4\}$ are bad. If the only good set is $\{3, 4\}$, then $L(z) = \emptyset$, contrary to the fact that c is a coloring of (H, L). Therefore, at least one of $\{2, 3\}, \{2, 4\}$ is good, and (3.2) and (3.3) imply that $2 \notin L(u)$ for every $u \in \tilde{X}(C)$, contrary to the fact that $2 \in \{1, 2, 3, 4\} \setminus Q \subseteq c(\tilde{X})$. This proves that not both $\tilde{X}(C) \cap X_{12}$ and $\tilde{X}(C) \cap X_{34}$ are non-empty.

We may assume that $\tilde{X}(C) \subseteq X_{12}$. Then $c(\tilde{X}(C)) \subseteq \{1, 2\}$, and so $3, 4 \in Q$. Since $|Q| < 3$, we have $Q = \{3, 4\}$. It follows from the construction of $\tilde{G}$ that $|\tilde{X}(C)| \leq 1$, contrary to the fact that $\{1, 2, 3, 4\} \setminus Q \subseteq \bigcup_{u \in X(C)} \{c(u)\}$. This completes the proof of the second statement, and Theorem 10 follows. □

4 Insulating cutsets

Our next goal is to transform companion triples further, restricting them in such a way that we can test colorability.

Let H be a graph and let L be a 4-list assignment for H. We say that $D \subseteq V(H)$ is a *chromatic cutset* in H if $V(H) = A \cup B \cup D$ (where A, B and D are pairwise disjoint), $A \neq \emptyset$, and $a \in A$ is adjacent to $b \in B$ only if $L(a) \cap L(b) = \emptyset$. For $i \neq j \in \{1, 2, 3, 4\}$ let $D_{ij} = \{d \in D \;:\; L(d) \subseteq \{i, j\}\}$. The set A is called the *far side* of the chromatic cutset. We say that a chromatic cutset D is 12-*insulating* if $D = D_{12} \cup D_{34}$ and for every $\{p, q\} \in \{\{1, 2\}, \{3, 4\}\}$ and every component $\tilde{D}$ of $H|D_{pq}$ the following conditions hold.

- $\tilde{D}$ is bipartite; let (D_1, D_2) be the bipartition.
- $|L(d)| = |L(d')|$ for every $d, d' \in D_1 \cup D_2$.
- There exists $a \in A$ with a neighbor in $\tilde{D}$ and with $L(a) \cap \{p, q\} \neq \emptyset$.
- Suppose that $|L(d)| = 2$ for every $d \in V(\tilde{D})$. Write $\{i, j\} = \{p, q\}$ and let $\{s, t\} = \{1, 2\}$. If $a \in A$ has a neighbor in $d \in D_s$ and $i \in L(a)$, and $b \in B$ has a neighbor in $\tilde{D}$, then
  - if b has a neighbor in D_s, then $j \notin L(b)$, and
  - if b has a neighbor in D_t, then $i \notin L(b)$.

Insulating cutsets are useful for the following reason. We say that a component $\tilde{D}$ of $H|D_{pq}$ is *complex* if $|L(d)| = 2$ for every $d \in V(\tilde{D})$.

Theorem 11. *Let D be a 12-insulating chromatic cutset in (H, L), and let A, B be as in the definition of an insulating cutset. Let D' be the union of the vertex sets of complex components of $H|D_{12}$ and of $H|D_{34}$, and let $D'' = D \setminus D'$. If $(H|(B \cup D''), L)$ and $(H \setminus B, L)$ are both colorable, then (H, L) is colorable. Moreover, given proper colorings of $(H|(B \cup D''), L)$ and $(H \setminus B, L)$, a proper coloring of (H, L) can be found in polynomial time.*

Proof. Let c_1 be a proper coloring of $(H|(B \cup D''), L)$ and let c_2 be a proper coloring of $(H \setminus B, L)$.

A *conflict* in c_1, c_2 is a pair of adjacent vertices u, v such that $c_1(u) = c_2(v)$. Since c_1, c_2 are both proper colorings and D is a chromatic cutset, and $|L(d)| = 1$ for every $d \in D''$, we deduce that every conflict involves one vertex of D' and one vertex of B. Below we describe a polynomial-time procedure that modifies c_2 to reduce the number of conflicts (with c_1 fixed).

Let $u \in D'$ and $v \in B$ be a conflict. Then $uv \in E(H)$ and $c_1(u) = c_2(v)$. Let $\tilde{D}$ be the component of $G|D$ containing u. Then $V(\tilde{D}) \subseteq D'$ and $\tilde{D}$ is bipartite; let (D_1, D_2) be the bipartition of $\tilde{D}$. We may assume that $u \in D_1$. We may also assume that $L(d) = \{1, 2\}$ for every $d \in V(\tilde{D})$, and that $c_1(u) = c_2(v) = 1$. Since $L(d) = \{1, 2\}$ for every $d \in V(\tilde{D})$, it follows that for every $i \in \{1, 2\}$ and $d \in D_i$, we have $c_2(d) = i$. Let c_3 be obtained from c_2 by setting $c_3(d) = 1$ for every $d \in D_2$; $c_3(d) = 2$ for every $d \in D_1$; and $c_3(d) = c_2(d)$ for every $w \in (A \cup D) \setminus (D_1 \cup D_2)$. (This modification can be done in linear time).

First we show that c_3 is a proper coloring of $(H \setminus B, L)$. Since $L(d) = \{1, 2\}$ for every $d \in V(\tilde{D})$, $c_3(v) \in L(v)$ for every $v \in A \cup D$. Suppose there exist adjacent $xy \in D \cup A$ such that $c_3(x) = c_3(y)$. Since $\tilde{D}$ is a component of $H|D$, we may assume that $x \in D_1 \cup D_2$ and $y \in A$. Suppose first that $x \in D_1$. Then $c_3(y) = c_3(x) = 2$, and so $2 \in L(y)$ and y has a neighbor in D_1. But $v \in B$ has a neighbor in D_1 and

$1 \in L(v)$, which is a contradiction. Thus we may assume that $x \in D_2$. Then $c_3(y) = c_3(x) = 1$, and so $1 \in L(y)$ and y has a neighbor in D_2. But $v \in B$ has a neighbor in D_1, and $1 \in L(b)$, again a contradiction. This proves that c_3 is a proper coloring of $(H \setminus B, L)$.

Clearly u, v is not a conflict in c_1, c_3. We show that no new conflict was created. Suppose that there is a new conflict, namely there exist adjacent $u' \in D'$ and $v' \in B$ such that $c_1(v') = c_3(u')$, but $c_1(v') \neq c_2(u')$. Then $u' \in V(\tilde{D})$. If $u' \in D_1$, then both v and v' have neighbors in D_1, and $1 \in L(v)$, and $2 \in L(v')$; if $u' \in D_2$, then v has a neighbor in D_1 and v' has a neighbor in D_2, and $1 \in L(v') \cap L(v)$; and in both cases we get a contradiction to the condition in the last bullet of the definition of a 12-insulating chromatic cutset. Thus the number of conflicts in c_1, c_3 was reduced.

Now applying this procedure at most $|V(G)|^2$ times we obtained a proper coloring c_1' of $(H|(B \cup D''), L)$ and a proper coloring c_2' of $(H \setminus B, L)$ such that there is no conflict in c_1', c_2'. Now define $c(v) = c_1'(v)$ if $v \in B \cup D''$ and $c(v) = c_2'(v)$ if $v \in V(H) \setminus B$; then c is a proper coloring of (H, L). This proves Theorem 11. $\square$

Let G be a P_6-free graph, let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of G, and let (H, L, h) be a companion triple for P. Let $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Let $Z^{ij} = \{z \in Z \,:\, N(z) \cap \tilde{X} \subseteq X_{ij} \cup X_{kl}\}$. It follows from the definition of a companion triple that $Z^{ij} = Z^{kl}$ and that $Z = \bigcup_{i,j \in \{1,2,3,4\}} Z^{ij}$. Next we prove a lemma that will allow us to replace a companion triple for P with a polynomially sized collection of near-companion triples for P, each of which has a useful insulating cutset. We will apply this lemma several times, and so we need to be able to apply it to near-companion triples for P, as well as to companion triples.

Lemma 17. *There is function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph, let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of G, and let (H, L, h) be a near-companion triple for P. Then there is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of 4-list assignments for H such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$*;*
- *if $L' \in \mathcal{L}$ and c is a proper coloring of (H, L'), then c is a proper coloring of (H, L); and*
- *if (H, L) is colorable, then there exists $L' \in \mathcal{L}$ such that (H, L') is colorable.*

Moreover, for every $L' \in \mathcal{L}$,

- $L'(v) \subseteq L(v)$ *for every $v \in V(H)$;*
- (H, L', h) *is a near companion triple for P;*
- *if for some $i \neq j \in \{1, 2, 3, 4\}$ (H, L) has an ij-insulating cutset D' with far side Z^{ij}, then D' is an ij-insulating cutset with far side Z^{ij} in (H, L', h); and*
- (H, L') *has a 12-insulating cutset $D \subseteq \tilde{X}$ with far side Z^{12}.*

Proof. Let $\tilde{P} = (\tilde{G}, S, X_0, \tilde{X}, Y^*, f)$ be as in the definition of a near-companion triple. Assume that $Z^{12} \neq \emptyset$. If one of the graphs $\tilde{G}|\tilde{X}_{12}$ and $\tilde{G}|\tilde{X}_{34}$ is not bipartite, set $\mathcal{L} = \emptyset$. From now on we assume that $\tilde{G}|\tilde{X}_{12}$ and $\tilde{G}|\tilde{X}_{34}$ are bipartite. We may assume that $X_0 = X^0(\tilde{P})$. Let $T_1, \ldots, T_m$ be types of $\tilde{X}$ with $|L_P(T_i)| = 2$ and such that $|L_P(T_i) \cap \{1, 2\}| = 1$. It follows that $|L_P(T_i) \cap \{3, 4\}| = 1$. Let $\mathcal{Q}$ be the set of all $2m$-tuples $Q = (Q_1, \ldots, Q_m, P_1, \ldots, P_m)$ such that

- $|Q_i| \leq 1$, $Q_i \subseteq \tilde{X}(T_i)$, and if $Q_i = \{q\}$, then $L(q) \cap \{1, 2\} \neq \emptyset$.
- $|P_i| \leq 1$, $P_i \subseteq \tilde{X}(T_i)$, and if $P_i = \{p\}$, then $L(p) \cap \{3, 4\} \neq \emptyset$.

For $x \in \tilde{X} \setminus (X_{12} \cup X_{34})$ and $z \in Z^{12}$ we say that z is a 12-*grandchild* of x if there is a component C of $\tilde{X}_{12}$ such that both x and z have neighbors in $V(C)$; a 34-*grandchild* is defined similarly. (Recall that $Z^{12} = Z^{34}$.) Let $G_{12}(x)$ be the set of 12-grandchildren of x; define $G_{34}(x)$ similarly.

We define a 4-list assignment L_Q' for H. Start with $L_Q' = L$. For every $i \in \{1, \ldots, m\}$, proceed as follows. If $|Q_i| = 1$, say $Q_i = \{q_i\}$, set $L_Q'(q_i)$ to be the unique element of $L(q_i) \cap \{1, 2\}$. For every $x \in \tilde{X}(T_i)$ such that $G_{12}(q_i) \subseteq G_{12}(x)$ and $G_{12}(x) \setminus G_{12}(q_i) \neq \emptyset$, update $L'_Q(x)$ by removing from it the unique element of $L(x) \cap \{1,2\}$. Next assume that $Q_i = \emptyset$. In this case, for every $x \in \tilde{X}(T_i)$ such that x has a 12-grandchild, update $L'_Q(x)$ by removing from it the unique element of $L(x) \cap \{1,2\}$.

If $|P_i| = 1$, say $P_i = \{p_i\}$, set $L'_Q(p_i)$ to be the unique element of $L(p_i) \cap \{3,4\}$. For every $x \in \tilde{X}(T_i)$ such that $G_{34}(p_i) \subseteq G_{34}(x)$ and $G_{34}(x) \setminus G_{34}(p_i) \neq \emptyset$, update $L'_Q(x)$ by removing from it the unique element of $L(x) \cap \{3,4\}$. Next assume that $P_i = \emptyset$. In this case, for every $x \in X(T_i)$ such that x has a 34-grandchild, update $L'_Q(x)$ by removing from it the unique element of $L(x) \cap \{3,4\}$.

If some vertex $z \in \tilde{X} \setminus \tilde{X}_{12}$ has neighbors on both sides of the bipartition of a component of $H|(\tilde{X}_{12})$, set $L'_Q(z) = L(z) \setminus \{1,2\}$. If some vertex $z \in \tilde{X} \setminus \tilde{X}_{34}$ has neighbors on both sides of the bipartition of a component of $H|(\tilde{X}_{34})$, set $L'_Q(z) = L(z) \setminus \{3,4\}$. Finally, set $L'_Q(v) = L(v)$ for every other $v \in V(H)$ not yet specified. Now let L_Q be obtained from L'_Q by updating exhaustively from $\bigcup_{i=1}^{m}(P_i \cup Q_i)$.

We need to check the following statements.

1. $L_Q(v) \subseteq L(v)$ for every $v \in V(H)$.

2. (H, L_Q, h) is a near-companion triple of P.

3. If for some $i \neq j \in \{1,2,3,4\}$ (H, L) has an ij-insulating cutset D' with far side Z^{ij}, then D' is an ij-insulating cutset with far side Z^{ij} in (H, L_Q).

4. (H, L_Q) has a 12-insulating cutset with far side Z^{12}.

Clearly $L_Q(v) \subseteq L(v)$ for every $v \in V(H)$, and consequently it is routine to check that the third statement holds, and that in order to prove the second statement it is sufficient to prove the following:

(5) *Set $f(x) = L_Q(x)$ for every $x \in X^0(L_Q)$. Then for every $z \in Z$ with $L(z) \neq \emptyset$ and $q \in \{1,2,3,4\}$ such that $q \notin L_Q(z)$, there is a vertex in $h(z)$ that has a neighbor $u \in S \cup X_0 \cup X^0(L_Q)$ with $f(u) = q$.*

We now prove this statement. Let $z \in Z$ and $q \in \{1,2,3,4\}$ such that $q \notin L_Q(z)$. We need to show that there is a vertex in $h(z)$ that has a neighbor $u \in S \cup X_0 \cup X^0(L')$ with $f(u) = q$. If $q \notin L(z)$, the claim follows from the fact that (H, L, h) is a near-companion triple for P, so we may assume that $q \in L(z)$, and therefore z has a neighbor u in $X^0(L_Q)$ with $f(u) = q$. Since Z is stable, it follows that $u \in \tilde{X}$, and therefore, by the definition of a companion triple, u is complete to $V(h(z))$. This proves (5).

Finally, we prove that (H, L_Q) has a 12-insulating cutset with far side Z^{12}. Let $D^1, \dots, D^t$ be the components of $H|\tilde{X}_{12}$ that contain a vertex x such that x has a neighbor z in Z^{12} with $L_Q(x) \cap L_Q(z) \neq \emptyset$. Let $F^1, \dots, F^s$ be defined similarly for $\tilde{X}_{34}$. Let $D = X^0(L_Q) \cup \bigcup_{i=1}^{t} V(D_i) \cup \bigcup_{j=1}^{w} V(F_j)$. We claim that D is the required cutset. Clearly D is a chromatic cutset, setting the far side to be Z^{12} and $B = V(H) \setminus (A \cup D)$, and the first two bullets of the definition of an insulating cutset are satisfied. Let $\tilde{D} \in \{D_1, \dots, D_t\}$ (the argument is symmetric for $F_1, \dots, F_s$). We need to check the following properties.

- *$\tilde{D}$ is bipartite.*
  This follows from the fact that $\tilde{G}|\tilde{X}_{ij} = H|\tilde{X}_{ij}$ is bipartite. Let (D_1, D_2) be the bipartition of $\tilde{D}$.

- *$|L(d)| = |L(d')|$ for every $d, d' \in D_1 \cup D_2$.*
  Since $L(d) \subseteq \{1,2\}$ for every $d \in V(\tilde{D})$, and since we have updated exhaustively, it follows that if $V(\tilde{D})$ meets $X^0(L_Q)$, then $V(\tilde{D}) \subseteq X^0(L_Q)$.

- *There exists $a \in A$ with a neighbor in $\tilde{D}$ and with $L(a) \cap \{1,2\} \neq \emptyset$.*
  This follows immediately from the definition of D.

- *Suppose that $|L(d)| = 2$ for every $d \in V(\tilde{D})$. We need to check that for $\{i,j\} = \{1,2\}$, if $a \in A$ has a neighbor in $d \in D_1$ and $i \in L_Q(a)$, and $b \in B$ has a neighbor in $\tilde{D}$, then*
  - *if b has a neighbor in D_1, then $j \notin L_Q(b)$, and*
  - *if b has a neighbor in D_2, then $i \notin L_Q(b)$.*

We now check the condition of the last bullet. Let $a \in A$ have a neighbor $d \in D_1$ and $1 \in L_Q(a)$. Suppose $b \in B$ has a neighbor in $D_1 \cup D_2$, and violates the conditions above. It follows from the definition of Z^{12} and B that $b \in \tilde{X}$ and $|L_Q(b)| = 2$. We may assume that $b \in T_1(X)$. Since $|L_Q(b)| = 2$, we deduce that $L_Q(b) = L(b) = M_P(b) = L_P(T_1)$. Since b exists, $Q_1 \neq \emptyset$. Since $|L(d)| = 2$ for every $d \in V(\tilde{D})$, it follows that q_1 is anticomplete to $D_1 \cup D_2$. Since $b \notin X^0(L_Q)$, there is a component D_0 of $H|\tilde{X}_{12}$ such that q_1 has a neighbor $d_0 \in V(D_0)$ and b is anticomplete to $V(D_0)$. Let $\{i\} = L_Q(b) \cap \{1,2\}$, and let $\{1,2\} \setminus \{i\} = \{j\}$. Then $j \notin L_Q(b) = M_P(b)$, and so $j \notin L_P(T_1)$. Consequently, there is $s \in S$ with $f(s) = j$, such that s is complete to $\tilde{X}(T_1)$. Since $V(\tilde{D}) \cup V(D_0) \subseteq X_{12}$, it follows that s is anticomplete to $V(\tilde{D}) \cup V(D_0)$.

Suppose first that $V(\tilde{D}) \neq \{d\}$. Since b is not complete to $D_1 \cup D_2$ (because $L_Q(b) \cap \{1,2\} \neq \emptyset$), there is an edge $d_1 d_2$ of $\tilde{D}$, such that b is adjacent to d_2 and not to d_1. Now $d_1 - d_2 - b - s - q_1 - d_0$ is a P_6 in $\tilde{G}_{12}(s)$, contrary to Lemma 15.

This proves that $V(\tilde{D}) = \{d\}$, and so b is adjacent to d, $i = 2$ and $j = 1$. Therefore $L_P(T_1) \cap \{1,2\} = \{2\}$, and so $L_Q(q_1) = c(q_1) = 2$. Since $d_0 \in \tilde{X}_{12}$, it follows that $L_Q(d_0) = 1$. Since $1 \in L_Q(a)$ and L_Q is obtained by exhaustive updating, we deduce that a is non-adjacent to d_0. But now since $1 \in L_Q(a)$ and $f(s) = 1$, we deduce that $a - d - b - s - q_0 - d_0$ is a path in $H_{12}(s)$ contradicting Lemma 16. This proves that (H, L_Q) has a 12-insulating cutset with far side Z^{12}.

Let $\mathcal{L} = \{L_Q \ ; \ Q \in \mathcal{Q}\}$. Then $|\mathcal{L}| \leq |V(G)|^{2m}$. Since $m \leq 2^{|S|}$, it follows that $|\mathcal{L}| \leq |V(G)|^{2^{|S|}}$. Since $L_Q(v) \subseteq L(v)$ for every $v \in V(H)$, it follows that every coloring of (H, L') is a coloring of (H, L).

Now suppose that (H, L) is colorable, and let c be a coloring. We show that some $L' \in \mathcal{L}$ is colorable. Let $i \in \{1, \ldots, m\}$. For a vertex $u \in \tilde{X}(T_i)$ define $val(u) = |G_{12}(u)|$. If some vertex u of $\tilde{X}(T_i)$ with a 12-grandchild has $c(u) \in L(u) \cap \{1,2\}$, let q_i be such a vertex with $val(q_i)$ maximum and set $Q_i = \{q_i\}$. If no such u exists, let $Q_i = \emptyset$.

Define $P_1, \ldots, P_m$ similarly replacing $\tilde{X}_{12}$ with $\tilde{X}_{34}$. Let

$$Q = (Q_1, \ldots, Q_m, P_1, \ldots, P_m).$$

We show that $c(v) \in L_Q(v)$ for every $v \in V(H)$, and so (H, L_Q) is colorable. Since L_Q is obtained from L'_Q by updating, it is enough to prove that $c(v) \in L'_Q(v)$. Suppose not. There are two possibilities (possibly replacing 12 with 34).

1. $v \in \tilde{X}(T_i)$, $Q_i \neq \emptyset$, $G_{12}(q_i)$ is a proper subset of $G_{12}(v)$, and $c(v) \in \{1,2\}$;
2. $v \in \tilde{X}(T_i)$, $Q_i = \emptyset$, $G_{12}(v) \neq \emptyset$, and $c(v) \in \{1,2\}$.

We show that in both cases we get a contradiction.

1. In this case $val(v) > val(q_i)$, contrary to the choice of q_i.
2. The existence of v contradicts the fact that $Q_i = \emptyset$.

This proves that (H, L_Q) is colorable and completes the proof of Theorem 17. □

Let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of a P_6-free graph G. We say that a near-companion triple (H, L, h) is *insulated* if for every $i \in \{2,3,4\}$ such that Z^{1i} is non-empty, (H, L) has a $1i$-insulating cutset $D \subseteq \tilde{X}$ with far side Z^{1i}. We can now prove the main result of this section.

Theorem 12. *There is function $q : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a P_6-free graph, let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of G, and let (H, L, h) be a near-companion triple for P. There is an algorithm with running time $O(|V(G)|^{q(|S|)})$ that outputs a collection $\mathcal{L}$ of 4-list assignments for H such that*

- $|\mathcal{L}| \leq |V(G)|^{q(|S|)}$.
- *If $L' \in \mathcal{L}$ and c is a proper coloring of (H, L'), then c is a proper coloring of (H, L).*
- *If (H, L) is colorable, there exists $L' \in \mathcal{L}$ such that (H, L') is colorable.*

Moreover, for every $L' \in \mathcal{L}$.

- $L'(v) \subseteq L(v)$ *for every* $v \in V(H)$.
- (H, L', h) *is insulated.*

Proof. Let $\mathcal{L}_2$ be as in Lemma 17. By symmetry, we can apply Lemma 17 with 12 replaced by 13 to (H, L', h) for every $L' \in \mathcal{L}_2$; let $\mathcal{L}_3$ be the union of all the collections of lists thus obtained. Again by symmetry, we can apply Lemma 17 with 12 replaced by 14 to (H, L', h) for every $L' \in \mathcal{L}_3$; let $\mathcal{L}_4$ be the union of all the collections of lists thus obtained. Now $\mathcal{L}_4$ is the required collection of lists. $\square$

5 Divide and Conquer

The main result of this section is the last piece of machinery that we need to solve the 4-precoloring-extension problem.

We need the following two facts.

Theorem 13. *[4] There is a polynomial-time algorithm that tests, for graph H and a list assignment L with $|L(v)| \leq 2$ for every $v \in V(H)$, if (H, L) is colorable, and finds a proper coloring if one exists.*

Theorem 14. *[8] The 2-SAT problem can be solved in polynomial time.*

We prove:

Lemma 18. *Let G be a P_6-free graph and let $P = (G, S, X_0, X, Y^*, f)$ be an orthogonal excellent starred precoloring of G. Let (H, L', h) be a companion triple for P, where $V(H) = \tilde{X} \cup Z$, as in the definition of a companion triple. Assume that $D \subseteq \tilde{X}$ is a 12-insulating chromatic cutset in (H, L') with far side Z^{12}. There is a polynomial-time algorithm that test if $(H|(Z^{12} \cup D), L')$ is colorable, and finds a proper coloring if one exists.*

Proof. We may assume that $X_0 = X^0(P)$. Let $\tilde{P} = (\tilde{G}, S, X_0, \tilde{X}, Y^*, f)$ be as in the definition of a companion triple, where $V(H) = \tilde{X} \cup Z$. By Theorem 13 we can test in polynomial time if $H|(D \cap \tilde{X}_{12}, L')$ and $H|(D \cap \tilde{X}_{34}, L')$ is colorable. If one of these pairs is not colorable, stop and output that $(H|(Z^{12} \cup D), L)$ is not colorable. So we may assume both the pairs are colorable, and in particular every component of $H|(D \cap \tilde{X}_{12})$ and $H|(D \cap \tilde{X}_{34})$ is bipartite.

We modify L' without changing the colorability property. First, let L'' be obtained from L' by updating exhaustively from $X^0(L')$. Next if $v \in V(H) \setminus \tilde{X}_{12}$ has a neighbor on both sides of the bipartition of a component of $H|\tilde{X}_{12}$, we remove both 1 and 2 from $L''(v)$, and the same for $\tilde{X}_{34}$; call the resulting list assignment L. (We have already done a similar modification while constructing list assignments L_Q in the proof of Lemma 17, but there we only modified lists of vertices in $\tilde{X}$, so this step is not redundant.) Set $f(u) = L(u)$ for every $u \in X^0(L)$. Clearly:

(6) *If $v \in V(H)$ is adjacent to $x \in X^0(L)$, then $L(v) \cap L(x) = \emptyset$.*

Next we prove:

(7) *Let $\{p, q\} \in \{\{1, 2\}, \{3, 4\}\}$ and let $z \in Z^{12}$ with $|L(z) \cap \{p, q\}| = 1$. Let $L(z) \cap \{p, q\} = \{i\}$ and $\{p, q\} \setminus L(z) = \{j\}$. Then there exists $y \in V(h(z))$ and $u \in S \cup X_0 \cup X^0(L)$ such that $f(u) = j$ and $uy \in E(\tilde{G})$.*

To prove (7) let $z \in Z$ with $L(z) \cap \{1, 2\} = \{1\}$ (the other cases are symmetric). Since $1 \in L(z)$, it follows that z does not have neighbors on both sides of the bipartition of a component of $H|\tilde{X}_{12}$, and therefore $L(z) = L''(z)$. If $2 \notin L'(z)$, then such u exists from the definition of a near-companion triple, so we may assume $2 \in L'(z)$. This implies that there is $u \in X^0(L)$ such that u is adjacent to z, and $f(u) = 2$. Since Z is stable, it follows that $u \in \tilde{X} \cup X_0 \cup S$, and so u is complete to $V(h(z))$, and (7) follows.

We define an instance I of the 2-SAT problem. The variables are the vertices of Z^{12}, and the clauses are as follows:

1. For every $z_1, z_2 \in Z^{12}$, if $L(z_i) \cap \{1,2\} = \{i\}$ for $i = 1,2$ and z_1, z_2 have neighbors on the same side of the bipartition of some component of $H|(D \cap \tilde{X}_{12})$, add the clause $(\neg z_1 \vee \neg z_2)$.

2. For every $z_1, z_2 \in Z^{12}$, if $L(z_1) \cap \{1,2\} = L(z_2) \cap \{1,2\} \in \{\{1\},\{2\}\}$ for $i = 1,2$ and z_1, z_2 have neighbors on opposite sides of the bipartition of some component of $H|(D \cap \tilde{X}_{12})$, add the clause $(\neg z_1 \vee \neg z_2)$.

3. For every $z_1, z_2 \in Z^{12}$, if z_1, z_2 have neighbors on the same side of the bipartition of some component of $H|(D \cap \tilde{X}_{12})$, and also z_1, z_2 have neighbors on opposite sides of the bipartition of some component of $H|(D \cap \tilde{X}_{12})$, add the clause $(\neg z_1 \vee \neg z_2)$.

4. For every $z_3, z_4 \in Z^{12}$, if $L(z_i) \cap \{3,4\} = \{i\}$ for $i = 3,4$ and z_3, z_4 have neighbors on the same side of the bipartition of some component of $H|(D \cap \tilde{X}_{34})$, add the clause $(z_3 \vee z_4)$.

5. For every $z_3, z_4 \in Z^{12}$, if $L(z_3) \cap \{3,4\} = L(z_4) \cap \{3,4\} \in \{\{3\},\{4\}\}$ for $i = 3,4$ and z_3, z_4 have neighbors on opposite sides of the bipartition of some component of $H|(D \cap \tilde{X}_{34})$, add the clause $(z_3 \vee z_4)$.

6. For every $z_3, z_4 \in Z^{12}$, if z_3, z_4 have neighbors on the same side of the bipartition of some component of $H|(D \cap \tilde{X}_{34})$, and also z_3, z_4 have neighbors on opposite sides of the bipartition of some component of $H|(D \cap \tilde{X}_{34})$, add the clause $(z_3 \vee z_4)$.

7. If $z \in Z^{12}$ and $L(z) \subseteq \{1,2\}$, add the clause $(z \vee z)$.

8. If $z \in Z$ and $L(z) \subseteq \{3,4\}$, add the clause $(\neg z \vee \neg z)$.

By Theorem 14 we can test in polynomial time if I is satisfiable.

We claim that I is satisfiable if and only if $(H|(Z^{12} \cup D), L)$ is colorable, and a proper coloring of $(H|(Z^{12} \cup D), L)$ can be constructed in polynomial time from a satisfying assignment for I.

Suppose first that $(H|(Z^{12} \cup D), L)$ is colorable, and let c be a proper coloring. For $z \in Z^{12}$, set $z = TRUE$ if $c(z) \in \{1,2\}$ and $z = FALSE$ if $c(z) \in \{3,4\}$. It is easy to check that every clause is satisfied.

Now suppose that I is satisfiable, and let g be a satisfying assignment. Let A' be the set of vertices $z \in Z^{12}$ with $g(z) = TRUE$, and let $B' = Z^{12} \setminus A'$. Let $A = A' \cup (D \cap \tilde{X}_{12})$ and $B = B' \cup (D \cap \tilde{X}_{34})$. For $v \in A$ let $L_A(v) = L'(v) \cap \{1,2\}$, and for $v \in B$ let $L_B(v) = L'(v) \cap \{3,4\}$. In order to show that $(H|(Z^{12} \cup D), L)$ is colorable and find a proper coloring, it is enough to prove that $(H|A, L_A)$ and $(H|B, L_B)$ are colorable, and find their proper colorings. We show that $(H|A, L_A)$ is colorable; the argument for $(H|B, L_B)$ is symmetric.

Since for every $z \in Z^{12}$ with $L(z) \subseteq \{3,4\}$ $(\neg z \vee \neg z)$ is a clause (of type 6) in I, it follows that $L(z) \cap \{1,2\} \neq \emptyset$ for every $z \in A$. Let $A_1 = \{v \in A : L_A(v) = \{1\}\}$, $A_2 = \{v \in A : L_A(v) = \{2\})$, and $A_3 = A \setminus (A_1 \cup A_2)$ Let F be a graph defined as follows. $V(F) = (A_3 \cup \{a_1, a_2\})$, where $F \setminus \{a_1, a_2\} = H|A_3$, $a_1a_2 \in E(F)$, and for $i = 1,2$ $v \in A_3$ is adjacent to a_i if and only if v has a neighbor in A_i in H.

We claim that $(H|A, L_A)$ is colorable if and only if F is bipartite; and if F is bipartite, then a proper coloring of $(H|A, L_A)$ can be constructed in polynomial time. Suppose F is bipartite and let (F_1, F_2) be the bipartition. We may assume $a_i \in F_i$. Let $i \in \{1,2\}$. For every $v \in (F_i \cup A_i) \setminus \{a_i\}$, we have that $i \in L_A(v)$, and so we can set $c(v) = i$. This proves that $(H|A, L_A)$ is colorable, and constructs a proper coloring. Next assume that $(H|A, L_A)$ is colorable. For $i = 1,2$, let F'_i be the set of vertices of A colored i. Then $A_i \subseteq F'_i$, and setting $F_i = (F'_i \setminus A_i) \cup \{a_i\}$, we get that (F_1, F_2) is a bipartition of F. This proves the claim.

Finally we show that F is bipartite. Recall that the pair $(H|(D \cap \tilde{X}_{12}), L)$ is colorable, and therefore $H|(D \cap \tilde{X}_{12})$ is bipartite. Since $L_A(v) \subseteq L(v)$ for every $v \in A_3$, and $L_A(v) \cap \{1,2\} \neq \emptyset$ for every $v \in A$, it follows that no vertex of $A \cap Z^{12}$ has a neighbor on two opposite sides of a bipartition of a component of $H|(D \cap \tilde{X}_{12})$. First we show that $H|A$ is bipartite. Suppose that there is an odd induced cycle C in $H|A$. Since by the fourth bullet in the defintion of a near-companion triple the graph $H|A$ is P_6-free, it follows that $|V(C)| = 5$. Since Z^{12} is stable, we deduce that $|C \cap Z^{12}| = 2$. But then some clause of type 3 or 6 is not satisfied, a contradiction. This proves that $H|A$ is bipartite.

Suppose that F is not bipartite. Then there is an odd cycle C in F, and so $V(C) \cap \{a_1, a_2\} \neq \emptyset$. In H this implies that there is a path $T = t_1 - \ldots - t_k$ with $\{t_2, \ldots, t_{k-1}\} \subseteq A_3$, such that either

- k is even, and for some $i \in \{1,2\}$ $t_1, t_k \in A_i$, or

- k is odd, $t_1 \in A_1$, and $t_k \in A_2$.

Since T is a path in $H|(Z \cup \tilde{X}_{12})$, it follows that $k \leq 5$. If $t_1 \in \tilde{X}_{12} \cap D$, then $t_1 \in X^0(L)$, and so by (6), $t_2 \in A_1 \cup A_2$, a contradiction. This proves that $t_1 \in Z^{12}$, and similarly $t_k \in Z^{12}$.

Suppose first that k is even. Since Z^{12} is stable, it follows that $k \neq 2$, and so $k = 4$. Since $t_1, t_4 \in Z^{12}$ and since Z^{12} is stable, it follows that $t_2, t_3 \in \tilde{X}_{12}$. But now $(\neg t_1 \vee \neg t_4)$ is a clause (of type 2) in I, and yet $g(t_1) = g(t_4) = TRUE$, a contradiction.

This proves that k is odd. If $k = 3$ then, since Z^{12} is stable, $t_2 \in \tilde{X}_{12}$, and so $(\neg t_1 \vee \neg t_3)$ is a clause (of type 1) in I, and yet $g(t_1) = g(t_3) = TRUE$, a contradiction. This proves that $k = 5$. Since Z^{12} is stable, it follows that $t_2, t_4 \in \tilde{X}_{12}$. If $t_3 \in \tilde{X}_{12}$, then $(\neg t_1 \vee \neg t_5)$ is a clause (of type 1) in I, contrary to the fact that both $g(t_1) = g(t_5) = TRUE$, a contradiction. Therefore $t_3 \in Z^{12}$. We may assume that $t_1 \in A_1$. By (7) there exist $u \in S \cup X_0 \cup X^0(L)$ and $y_1 \in V(h(t_1))$ such that $f(u) = 2$ and $uy_1 \in E(\tilde{G})$. Since $t_2 \in \tilde{X}$, it follows that t_2 is complete to $V(h(t_1))$, and in particular t_2 is adjacent to y_1. Since $X_0 = X^0(P)$, it follows that u is anticomplete to $\{t_2, t_4\}$. Let $i \in \{3, 5\}$. By the definition of a companion triple, since $2 \in L(t_i)$, there exists $y_i \in V(h(t_i))$ such that u is non-adjacent to y_i in $\tilde{G}$. Now since no vertex of $\tilde{X}$ is mixed on a component to $\tilde{G}|Y^*$, it follows that $u - y_1 - t_2 - y_3 - t_4 - y_5$ is a P_6 in $\tilde{G}_{12}(u)$, contrary to Lemma 15. This proves Lemma 18. □

6 The complete algorithm

First we prove Theorem 8, which we restate.

Theorem 8. *For every positive integer C there exists a polynomial-time algorithm with the following specifications:*

Input: *An excellent starred precoloring $P = (G, S, X_0, X, Y^*, f)$ of a P_6-free graph G with $|S| \leq C$.*

Output: *A precoloring extension of P or a statement that none exists.*

Proof. By Theorem 9 we can construct in polynomial time a collection $\mathcal{L}$ of orthogonal excellent starred precolorings of G, such that in order to determine if P has a precoloring extension (and find one if it exists), it is enough to check if each element of $\mathcal{L}$ has a precoloring extension, and find one if it exists. Thus let $P_1 \in \mathcal{L}$. By Theorem 10 we can construct in polynomial time a companion triple (H, L, h) for P_1, and it is enough to check if (H, L, h) is colorable.

Now proceed as follows. If $L(v) = \emptyset$ for some $v \in V(H)$, stop and output "no precoloring extension". So we may assume $L(v) \neq \emptyset$ for every $v \in V(H)$. Let $\mathcal{L}$ be a collection of lists as in Theorem 12. If $\mathcal{L} = \emptyset$, stop and output "no precoloring extension", so we may assume that $\mathcal{L} \neq \emptyset$. Let $L' \in \mathcal{L}$; then (H, L', h) is insulated. For every i let D^i be and insulating $1i$-cutset with far side Z^{1i}, and let $D^{i'} = \{d \in D_i \ : \ |L'(d)| = 2\}$. Let $H_i = H|(D^i \cup Z^{1i})$, and let $H_1 = H \setminus \bigcup_{i=2}^{4} (D^{i'} \cup Z^{1i})$. Observe that $V(H_1) \subseteq \tilde{X}$. By Lemma 18, we can check if each of the pairs (H_i, L') with $i \in \{2, 3, 4\}$ is colorable, and by Theorem 13, we can check if (H_1, L') is colorable and find a proper coloring if one exists. If one of these pairs is not colorable, stop and output "no precoloring extension". So we may assume that (H_i, L') is colorable for every $i \in \{1, \ldots, 4\}$. Observe that D^2 is an insulating 12-cutset in $(H|(V(H_1) \cup V(H_2)), L')$ with far side Z^{12}, D^3 is an insulating 13-cutset in $(H|(V(H_1) \cup V(H_2) \cup V(H_3)), L')$ with far side Z^{13}, and D^4 is an insulating 14-cutset in (H, L') with far side Z^{14}. Now three applications of Theorem 11 show that (H, L) is colorable, and produce a proper coloring. This proves 8. □

We can now prove the main result of the series, the following.

Theorem 15. *There exists a polynomial-time algorithm with the following specifications.*

Input: *A 4-precoloring (G, X_0, f) of a P_6-free graph G.*

Output: *A precoloring extension of (G, X_0, f) or a statement that none exists.*

Proof. Let $\mathcal{L}$ be as in Theorem 7. Then $\mathcal{L}$ can be constructed in polynomial time, and it is enough to check if each element of $\mathcal{L}$ has a precoloring extension, and find one if it exists. Now apply the algorithm of Theorem 8 to every element of $\mathcal{L}$. $\square$

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