A geometric investigation of non-regular separation applied to the bi-Helmholtz equation & its connection to symmetry operators

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The theory of non-regular separation is examined in its geometric form and applied to the bi-Helmholtz equation in the flat coordinate systems in 2-dimensions. It is shown that the bi-Helmholtz equation does not admit regular separation in any dimensions on any Riemannian manifold. It is demonstrated that the bi-Helmholtz equation admits non-trivial non-regular separation in the Cartesian and polar coordinate systems in $\mathbb{R}^2$ but does not admit non-trivial non-regular separation in the parabolic and elliptic-hyperbolic coordinate systems of $\mathbb{R}^2$. The results are applied to the study of small vibrations of a thin solid circular plate. It is conjectured that the reason as to why non-trivial non-regular separation occurs in the Cartesian and polar coordinate systems is due to the existence of first order symmetries (Killing vectors) in those coordinate systems. Symmetries of the bi-Helmholtz equation are examined in detail giving supporting evidence of the conjecture.

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Notations, Conventions and definitions

Einstein summation convention: $A^i B_i = \sum_i A^i B_i$

Metric tensor: g_{ij} Riemannian metric

Levi-Civita Connection: ∇

Christoffel symbols: $\Gamma^i{}_{jk} = \frac{1}{2} g^{il} \left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk} \right)$

Laplace-Beltrami operator: $\Delta = g^{ij} \partial_i \partial_j - g^{ij} \Gamma^k{}_{ij} \partial_k$

Riemann tensor: $R^i{}_{jkl}$

Ricci tensor: $R_{ij} = R^l{}_{ilj}$

Tensor symmetrization: $A_{(i_1 \cdots i_n)} = \frac{1}{n!} \sum\limits_{\sigma \in S_n} A_{\sigma(i_1) \cdots \sigma(i_n)}$

Tensor anti-symmetrization: $A_{[i_1 \cdots i_n]} = \frac{1}{n!} \sum\limits_{\sigma \in S_n} \mathrm{sgn}(\sigma) A_{\sigma(i_1) \cdots \sigma(i_n)}$

Partial differentiation: $u_{i_1 \cdots i_k} = \frac{\partial^k u}{\partial q^{i_1} \cdots \partial q^{i_k}}$

Valence p Killing tensor: $\nabla_{(l} K_{i_1 i_2 \cdots i_p)} = 0$

Chapter 1

Introduction

The method of separation of variables is a very useful and powerful tool for solving linear partial differential equations (PDEs). However, for high order equations it is not always possible to find separable solutions in the usual sense. In this thesis we investigate methods of separation applied to bi-Helmholtz equation $\Delta^2 u = \lambda u$ where Δ is the Laplace-Beltrami operator. Studying solutions to this equation shows to be extremely useful in elasticity theory which has direct applications in engineering. The bi-Helmholtz equation also has applications in more theoretical frameworks such as Hořava–Lifshitz gravity [7] where fourth order equations in space are a fundamental part of the theory representing the anisotropy between space and time at high energies.

Historically much work on the theory of separation of variables has been done in the context of the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(q^i, \frac{\partial S}{\partial q^i}, t\right) = 0, \tag{1.1}$$

where H is the Hamiltonian of some dynamical system which is assumed to be a smooth function of its arguments. This is generally a non-linear equation and the type of separable solution sought is of the form

$$S = \sum_i S_{(i)}(q^i) \tag{1.2}$$

and is called an *additively separable* solution; $S_{(i)}$ here are single variable functions of their corresponding coordinate.

Levi-Civita [10] in a 1904 paper gave necessary and sufficient conditions for the separability of the Hamilton-Jacobi (HJ) equation. However, this only gave a method of checking whether the HJ equation admits an additively separable solution in a coordinate system, but said nothing about which coordinate systems have this property. A particular HJ equation that was of interest to Levi-Civita was the geodesic HJ equation where the Hamiltonian function has the following form

$$H(q^i, p_j, t) = g^{ij}(q)p_i p_j, \tag{1.3}$$

where g^{ij} are the inverse components of some (pseudo-)Riemannian metric on a local patch of a (pseudo-)Riemannian manifold and p_i are the canonical momenta. The question then becomes, which coordinate systems admit a HJ equation which is additively separable?

Stäckel [13] provides us with a partial answer to this question. He develops the theory of orthogonal separation, i.e., separation in coordinate systems where the metric g^{ij} is diagonal. Stäckel gives us the general form of a metric for which the geodesic HJ equation is separable, that is that g^{ii} must be a row of the inverse of a Stäckel matrix. A Stäckel matrix $s_{ij}(q)$ is one such that $\partial_k s_{ij} = 0$ whenever $i \neq k$. This has become the standard theory for approaching additive separation of HJ equations. Robertson [12] further gave sufficient conditions for the multiplicative separability of Schrödinger equations

$$-\Delta\psi + V\psi = E\psi \tag{1.4}$$

based on the work done by Stäckel.

In a seminal paper, Eisenhart [6] demonstrated a deep geometrical insight into separation of variables. He shows that separable coordinates of the geodesic HJ equation are characterised by the existence of $n-1$ quadratic first integrals ($S = a^{ij}p_i p_j$) satisfying some conditions in orthogonal coordinates. Furthermore, he shows that his geometric characterization is equivalent to that of Stäckel, giving an invariant characterization of separation for the geodesic HJ equation. We discuss his results in more detail in the final chapter.

Benenti [1] simplified and generalized Eisenhart's result to show that the geodesic HJ equation is separable in orthogonal coordinates if and only if there exists a (valence two) Killing tensor K with pointwise real non-degenerate eigenvalues and orthogonally integrable eigenvectors.

Kalnins and Miller [9] generalize the work of Stäckel and others on separation of variables to a very general type of PDE of the form

$$\mathcal{H}(q^i, u, u_i, ..., u_{i_1 \cdots i_k}) = h, \tag{1.5}$$

where h here is a constant and $\mathcal{H}$ is in general a non-linear function of the coordinates q^i, u and its derivatives up to kth order. They give necessary and sufficient conditions for additive separability which we discuss in the coming chapters as well as an in depth discussion of a geometric generalization by Benenti, Chanu and Rastelli [2].

We now turn our attention to the study of linear PDEs and their separability.
Let $\mathcal{D}$ be a linear partial differential operator of order k on a manifold Q with local coordinates $\{q^i\}_{i=1}^n$. We say the equation $\mathcal{D}u = 0$ admits a *multiplicatively separable* solution if there exists a function

$$u(x) = \prod_{i=1}^{n} f_i(q^i) \tag{1.6}$$

for some smooth single variable functions f_i which satisfies $\mathcal{D}u = 0$. The solution of course, will depend on separation and integration constants. The maximal number of independent constants is $nk + 1$ where k is the degree of the differential operator $\mathcal{D}$. This is due to the fact that in the best case one obtains n separated k^{th} order ordinary differential equations each of which needs k constants, and the additional constant is a separation constant. If a solution has the maximal number of independent constants then we say the equation $\mathcal{D}u = 0$ admits *regular separation.* However, in the general case one does not always obtain the maximal number of independent constants when looking for separated solutions. In this case we say the PDE admits *non-regular separation.*

Kalnins and Miller [8] developed the theory of non-regular separation for equation (1.5) and gave conditions for regular separation. However, in this case the separation considered is additive separation

$$u = \sum_i S_{(i)}(q^i) \tag{1.7}$$

However, one can show that with a substitution $\psi = e^u$ this amounts to a multiplicatively separated solution of a different equation obtained by substituting $u = \ln(\psi)$ into the PDE. However, Kalnins and Miller did not give conditions for when non-regular separation can occur and the number of constants involved. This is done by Chanu [4] who also gives a geometric interpretation of non-regular separation which will be discussed the next chapter.
Given a linear equation $\mathcal{D}u = \lambda u$, which admits regular (multiplicative) separation of variables, one can build a new higher order equation $\mathcal{D}^2 u = \lambda^2 u$. One can clearly see that every solution to the lower order equation solves $\mathcal{D}^2 u = \lambda^2 u$, which hence admits non-regular separation with at least the same number of constants as the the original equation. We say the non-regular separation is *trivial* when the solution has the same number of constants, and in the case where the solution of $\mathcal{D}^2 u = \lambda^2 u$, admits more constants we say the equation admits *non-trivial non-regular separation.*

A symmetry $\mathcal{S}$ of a linear equation $\mathcal{D}u = 0$, is a linear differential operator which commutes with $\mathcal{D}$. This allows one to find new independent solutions by applying the symmetry operator to a given solution. Given a solution u of $\mathcal{D}u = 0$, $v = \mathcal{S}u$

also solves the equation since

$$\mathcal{D}v = \mathcal{D}\mathcal{S}u = \mathcal{S}\mathcal{D}u = 0$$

A natural question to ask would be: is there a way to characterize when trivial or non-trivial separation occurs? In the following chapters we will provide examples of trivial and non-trivial non-regular separation. These computations motivate the possible relation between existence of symmetries of a PDE and non-trivial non-regular separation.

Chapter 2

Geometric theory of separation for PDEs

2.1 General theory

Let Q be an n dimensional smooth manifold with local coordinates $\{q^i\}_{i=1}^n$ and u be a smooth function on Q. We consider a k-th order non-linear PDE of the form

$$\mathcal{H}(q^i, u, u_i, ..., u_{i_1 \cdots i_k}) = h \tag{2.1}$$

where $\mathcal{H}$ is some smooth function of the coordinates, u and its derivatives and $h \in \mathbb{R}$ a constant. Note that we use $u_{i_1 \cdots i_k}$ to denote $\frac{\partial^k u}{\partial q^{i_1} \cdots \partial q^{i_k}}$. We say that a function u is an additively separable solution of (2.1) if it solves the equation and has the form

$$u = \sum_i S_{(i)}(q^i, c_a), \tag{2.2}$$

where $S_{(i)}$ are functions of only the i-th coordinate q^i and depends on $nk + 1$ constant c_a. To guarantee independence of these constants the following completeness condition must hold [9]

$$\text{rank} \left[\frac{\partial u}{\partial c_a} \middle| \frac{\partial u_i}{\partial c_a} \middle| \ldots \middle| \frac{\partial u_i^{(l)}}{\partial c_a} \right] = nk + 1 \tag{2.3}$$

If u is additively separable then it is necessary for the mixed partial derivatives of all order to vanish, i.e.

$$u_i = S'_{(i)}, \qquad u_{ij} = S''_{(i)}\delta_{ij} = u_i^{(2)}\delta_{ij}, \qquad u_{ijk} = S'''_{(i)} = u_i^{(3)}\delta_{ij}\delta_{ik}, \quad ... \tag{2.4}$$

where we defined the compact notation $u_i^{(k)}$ to be the k^{th} derivative with respect to the i^{th} coordinate. When looking for additively separable solutions, (2.1) has the form

$$\mathcal{H}(q^i, u, u_i, ..., u_i^{(k)}) = h$$

Since $\mathcal{H}$ is constant we must have $d\mathcal{H} = 0$. In coordinates that is the following condition

$$\frac{\partial \mathcal{H}}{\partial q^i} + \frac{\partial \mathcal{H}}{\partial u} u_i + \frac{\partial \mathcal{H}}{\partial u_i} u_i^{(2)} + \cdots + \frac{\partial \mathcal{H}}{\partial u_i^{(k)}} u_i^{(k+1)} = 0 \tag{2.5}$$

Without loss of generality we assume that $\frac{\partial \mathcal{H}}{\partial u_i^{(k)}} \neq 0$. Then we can isolate for the highest order derivative

$$\begin{aligned} u_i^{(k+1)} &= -\left(\frac{\partial \mathcal{H}}{\partial u_i^{(k)}}\right)^{-1} \left(\frac{\partial \mathcal{H}}{\partial q^i} + \frac{\partial \mathcal{H}}{\partial u} u_i + \frac{\partial \mathcal{H}}{\partial u_i} u_i^{(2)} + \cdots + \frac{\partial \mathcal{H}}{\partial u_i^{(k-1)}} u_i^{(k)}\right) \\ &\stackrel{\text{def}}{=} R_i(q^j, u, ..., u_j^{(k)}) \end{aligned} \tag{2.6}$$

Define the following operators

$$D_i = \frac{\partial}{\partial q^i} + u_i \frac{\partial}{\partial u} + \cdots + u_i^{(k)} \frac{\partial}{\partial u_i^{(k-1)}} + R_i \frac{\partial}{\partial u_i^{(k)}} \tag{2.7}$$

The equation (2.5) can now be written as $D_i \mathcal{H} = 0$. A PDE of the form (2.5) is said to be *regularly/freely separable* if it admits a solution of the form (2.2) and satisfies the completeness condition (2.3). Kalnins and Miller [9] proved that regular separation occurs if and only if

$$D_i R_j = 0 \tag{2.8}$$

identically everywhere for $i \neq j$. Since we are treating the derivatives of our function as independent coordinates we need to identify the derivatives with the coordinates. Our system then becomes

$$\begin{aligned} \partial_i u &= u_i \\ \partial_i u_i &= u_{ii} \\ &\vdots \\ \partial_i u_i^{(k)} &= R_i \end{aligned} \tag{2.9}$$

2.2 Geometric theory

Benenti, Chanu and Rastelli [2] showed that if one has operators as in equation (2.7) without assuming assuming the form the functions R_i have in equation (2.6), then the conditions

$$[D_i, D_j] = 0 \qquad D_i\mathcal{H} = 0 \tag{2.10}$$

determine the functions R_i to be as in (2.6) and also give the condition $D_i R_j = 0$ for $i \neq j$. This means that regular separation is the same as the existence of n commuting symmetries of $\mathcal{H}$ as in (2.10).

Theorem. *(Benenti, Chanu, Rastelli [2]) A first-order differential system of the form* (2.9) *is completely integrable, i.e., it admits a local complete solution satisfying the completeness condition* (2.3) *if and only if the operators D_i commute.*

These operators can now be interpreted as vector fields in involution on a larger manifold M with local coordinates $\left\{q^i, u, u_i, , ..., u_i^{(k)}\right\}$. One can also view the space M as a formal trivial bundle over Q

$$M \xrightarrow{\pi} Q$$

of rank $nk + 1$.

Since D_i are a set of commuting smooth sections of TM, by Frobenius' theorem the distribution $\Delta = \text{Span}\{D_i\}$ is integrable giving an n dimensional foliation of M. The leaves of the foliation are then solutions of the PDE. This distribution can also be viewed as specifying a choice of horizontal subspaces in the bundle (Ehresmann connection), the condition $[D_i, D_j] = 0$ implies that the connection is flat. Boundary data is given by fixing values of u,u_i,...,$u_i^{(k)}$ at a point $q \in Q$ which in turn fixes the leaf which corresponds to the unique solution of the boundary value problem.

The figure below shows this foliation on M in the $k = 1$ and $n = 1$ case. Here Q is a one dimensional manifold which has two dimensional leaves which correspond values for $u(q)$ and $u'(q)$ at each point $q \in Q$.

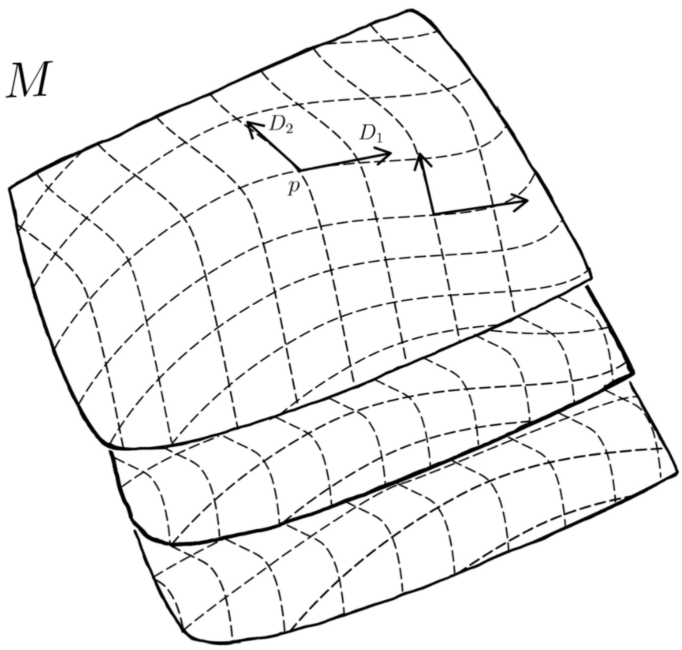

Figure 2.1: Leaves of the folation of M. The vector field D_i is shown tangent to the leaves. The point p which corresponds to $(q, u(q), u_i(q), ..., u_i^{(k)}(q))$ for some $q \in Q$. Q is transverse to the leaves of the foliation.

This completes the geometric classification of regular additive separation of a PDE.

Generally it is not always possible to find a separated solution which depends on the maximal number of independent constants N. However one might be able to find separated solutions which depend on a smaller number number of constants $N - r$. This occurs when the condition (2.8) is not satisfied identically everywhere and the separation is called *non-regular*. This means that generally the derivatives of u will depend on each other through relations other than $\mathcal{H} = h$.

In [4] Chanu gives a geometric interpretation of non-regular separation which goes as follows. Let N be a submanifold of M described implicitly by

$$f_a = 0 \qquad (a = 1, \ldots, r),$$

where f_a are functions on M. These functions represent the non-trivial relations between the derivatives of our solution that decrease the number of independent constants. If the condition $D_i R_j|_N = 0$ for $i \neq j$ is satisfied and the vectors D_i are tangent to N (in other words $D_i f_a = 0$), then the foliation can be restricted to N and the total number of independent constants will have decreased to $nk + 1 - r$. The completeness condition now has the form

$$\operatorname{rank}\left[\frac{\partial u}{\partial c_a}\middle|\frac{\partial u_i}{\partial c_a}\middle|\ldots\middle|\frac{\partial u_i^{(l)}}{\partial c_a}\right] = nk + 1 - r \tag{2.11}$$

Thus for non-regular separation to occur there must be n vector fields D_i as well as r functions f_a such that the following conditions are satisfied

$$D_i\mathcal{H} = 0, \qquad [D_i, D_j]\big|_N = 0, \qquad D_i f_a\big|_N = 0 \tag{2.12}$$

So we can interpret this geometrically as a foliation on the submanifold N due to the Frobenius' theorem. One implication which follows immediately is that if $D_i R_j$ does not vanish at any point in M then $\mathcal{H} = h$ does not admit any (additively) separated solutions.

We present the Helmholtz equation in Cartesian coordinates as an example of an equation that admits regular separation with the geometric considerations taken into account.

2.3 The Helmholtz equation in Cartesian coordinates

The Helmholtz equation (with no potential function) has the following form

$$\Delta\psi = \lambda\psi$$

where Δ is the Laplace-Beltrami operator. In Cartesian coordinates in n-dimensions we have

$$\sum_i \frac{\partial^2\psi}{\partial x_i^2} = \lambda\psi \tag{2.13}$$

Generally we know that this equation admits multiplicatively separated solutions. To apply the previously discussed geometric theory we use the following trick. Define $u = \ln\psi$, if ψ is multiplicatively separate then u is additively separate and vice versa.

Taking derivatives of $\psi = e^u$ gives us an associated differential equation to (2.13) which should admit additive separation.

$$\frac{\partial^2\psi}{\partial x_i^2} = \left(u_{ii} + (u_i)^2\right)\psi$$

This gives the associated differential equation for u by substituting into (2.13)

$$\mathcal{H} \stackrel{\text{def}}{=} \sum_i u_{ii} + (u_i)^2 = \lambda \tag{2.14}$$

In this special case we see that the function $\mathcal{H}$ a priori does not depend on mixed partial derivatives of u. This equation has a simple way of separating, let $c_1, ..., c_n \in \mathbb{R}$ such that

$$\sum_i c_i = \lambda$$

Then (2.14) additively separates into n ODEs of the form

$$u_{ii} + (u_i)^2 = c_i \tag{2.15}$$

The sections D_i have the form

$$D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ii} \frac{\partial}{\partial u_i} + R_i \frac{\partial}{\partial u_{ii}}$$

Recall that the condition $D_i \mathcal{H} = 0$ determines the functions R_i

$$D_i \mathcal{H} = 2u_{ii} u_i + R_i = 0$$

Which gives $R_i = -2u_i u_{ii}$, notice that in this case the function R_i only depends on u's corresponding derivatives (u_i and u_{ii}). Consequently, we have $D_i R_j = 0$ everywhere for $i \neq j$ and hence $[D_i, D_j] = 0$ everywhere and we have the maximal foliation corresponding to regular separation.

Regarding the number of independent constants, at first glance one might over-count the number as follows: $n-1$ independent separation constants c_i (since they obey one linear equation), and $2n$ integration constants from the n second order ODEs. Hence one counts $3n-1$ constants. However notice that the solutions of (2.15) are themselves defined up to a constant (since the ode only depends on derivatives of u). Recalling that for additively separable solutions we have $u = \sum_i S_{(i)}$, the previous implies that adding a constant α_i to each $S_{(i)}$ does not affect the solution as long as the constants add up to 0

$$\sum_i \left(S_{(i)} + \alpha_i\right) = \sum_i S_{(i)} + \sum_i \alpha_i = \sum_i S_{(i)} = u \tag{2.16}$$

This takes away $n-1$ independent constants from our count, since there are $n-1$ independent constants α_i. So we have $3n-1-(n-1) = 2n$ constants as expected. This redundancy can also be seen in the multiplicative formulation, due to the projective nature of the constants involved. If $\psi = \prod_i U_{(i)}$, the separated functions $U_{(i)}$ satisfy the following differential equations

$$U''_{(i)} = c_i U_{(i)}$$

It's easy to see that these solutions are defined upto a multiplicative constant α_i. So as long as those constants multiply to 1 the final solution ψ remains unchanged.

2.4 The Helmholtz equation in Liouville coordinates in 2-dimensions

Liouville coordinates are coordinates on $\mathbb{R}^n$, characterized by the following metric:

$$g = \left(\sum_{i=1}^{n} f_i(x^i) \right) g_0. \tag{2.17}$$

where f_i are positive functions depending only on their corresponding coordinate x^i, and g_0 is the standard flat metric on $\mathbb{R}^n$.

We will show that the Helmholtz equation is (regulary) separable in this coordinate system in 2-dimensions, and in fact this turns out to be the most general class of coordinate systems in 2-dimensions for which the Helmholtz equation admits regular separation (see chapter 5).

In two dimensions the Liouville metric has the following form:

$$g = (f(x) + g(y)) \left(dx^2 + dy^2\right) \tag{2.18}$$

This metric is conformally flat. Consequently, in two dimensions the Laplace-Beltrami operator associated to this metric has a simple form in terms of the flat Laplacian

$$\Delta = \frac{1}{f+g} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \frac{1}{f+g} \Delta_0 \tag{2.19}$$

So the Helmholtz equation becomes:

$$\Delta_0 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \lambda (f+g) \tag{2.20}$$

after multiplying both sides by $(f+g)$. Now this problem is quite similar to the Cartesian case, we do the same substitution $\psi = e^u$ to obtain

$$\mathcal{H} = u_{xx} + (u_x)^2 + u_{yy} + (u_y)^2 - \lambda(f+g) = 0 \tag{2.21}$$

Further we apply D_x and D_y to $\mathcal{H}$ to find the functions R_x and R_y:

$$R_x = \lambda f'(x) - 2u_{xx}u_x \qquad R_y = \lambda g'(y) - 2u_{yy}u_y \tag{2.22}$$

Once again we have that R_x depends only on x, u_x, u_{xx} and thus $D_y R_x = 0$ holds everywhere, and similarly for R_y. This means that this equation admits regular separation. This example will be useful for our investigations in the coming chapters.

Chapter 3

The bi-Helmholtz equation

At the end of the previous chapter we investigated the separability of the Helmholtz equation in different coordinate systems as examples of PDEs admitting regular separation. In this section we investigate the separability of the bi-Helmholtz equation

$$\Delta^2\psi = \lambda\psi \tag{3.1}$$

The bi-Helmholtz equation appears regularly in the subjects of continuum mechanics, solid mechanics, harmonic analysis and many others. We mainly investigate the bi-Helmholtz equation as a concrete example to further understand non-regular separation, when it occurs and the number of constants in those solutions.

Since we know the Helmholtz equation separates regularly in the Liouville coordinate systems, we should investigate the separability of the bi-Helmholtz equation in those coordinates since we are guaranteed to have non-regular separability (since every solution to the Helmholtz equation is a solution of the bi-Helmholtz equation). We will call a separated solution *non-trivial* if it is a solution to the bi-Helmholtz equation but not a solution of the Helmholtz equation.

In a general coordinate system the Laplace-Beltrami operator has the following form

$$\Delta = g^{ij}\partial_{ij} - g^{ij}\Gamma^k{}_{ij}\partial_k = g^{ij}\partial_{ij} - \Gamma^k\partial_k \tag{3.2}$$

where $\Gamma^k = g^{ij}\Gamma^k{}_{ij}$ (summation over repeated indices[1]). Squaring this operator we obtain the bi-Laplace operator, this has the following form

$$\begin{aligned} \Delta^2\psi &= g^{ij}\partial_{ij}(g^{hk}\partial_{hk}\psi - \Gamma^h\partial_h\psi) - \Gamma^i\partial_i(g^{hk}\partial_{hk}\psi - \Gamma^h\partial_h\psi) = \\ &= g^{ij}g^{hk}\partial_{ijhk}\psi + 2(g^{ij}\partial_j g^{hk} - g^{hk}\Gamma^i)\partial_{ihk}\psi + \\ &\quad (g^{ij}\partial_{ij}g^{hk} - 2g^{jk}\partial_j\Gamma^h - \Gamma^i\partial_i g^{hk} + \Gamma^h\Gamma^k)\partial_{hk}\psi + \\ &\quad (-g^{ij}\partial_{ij}\Gamma^h + \Gamma^i\partial_i\Gamma^h)\partial_h\psi, \end{aligned} \tag{3.3}$$

for the sake of brevity we will write the bi-Laplace operator in the following form

$$\Delta^2 = A^{ijkl}\partial_{ijkl} + B^{ijk}\partial_{ijk} + C^{ij}\partial_{ij} + D^i\partial_i \tag{3.4}$$

where

$$A^{ijkl} = g^{(ij}g^{kl)} \tag{3.5}$$

$$B^{ijk} = 2(g^{h(i}\partial_h g^{jk)} - g^{(ij}\Gamma^{k)}) \tag{3.6}$$

$$C^{ij} = g^{kl}\partial_{kl}g^{ij} - 2g^{k(i}\partial_k\Gamma^{j)} - \Gamma^k\partial_k g^{ij} + \Gamma^i\Gamma^j \tag{3.7}$$

$$D^i = -g^{jk}\partial_{jk}\Gamma^i + \Gamma^j\partial_j\Gamma^i = -\Delta\Gamma^i \tag{3.8}$$

The following is the a general result from our paper [5] about the separability of the bi-Helmholtz equation:

[1]In this chapter summation over repeated indices is implied unless otherwise specified

Theorem. *Let (M, g) be any Riemannian manifold. The bi-Helmholtz equation on M does not admit regularly separable solutions in any coordinate system.*

Proof. We begin with the same substitution $\psi = e^u$ and taking all the required derivatives we obtain the following:

$$\partial_i \psi = u_i \psi \tag{3.9}$$

$$\partial_{ij} \psi = (u_{ij} + u_i u_j)\, \psi \tag{3.10}$$

$$\begin{aligned} \partial_{ijk} \psi &= (u_{ijk} + u_{ki} u_j + u_{jk} u_i + u_{ij} u_k + u_i u_j u_k)\, \psi \\ &= \left(u_{ijk} + 3u_{(ij} u_{k)} + u_i u_j u_k\right) \psi \end{aligned} \tag{3.11}$$

$$\partial_{ijkl} \psi = \left(u_{ijkl} + 4u_{(i} u_{jkl)} + 3u_{(ij} u_{kl)} + 6u_{(i} u_j u_{kl)} + u_i u_j u_k u_l\right) \psi \tag{3.12}$$

Substituting these derivatives into the bi-Helmholtz equation and dividing by ψ we get our PDE for u

$$\begin{aligned} \mathcal{H} =& A^{ijkl} u_{ijkl} + (B^{ijk} + 4A^{ijkl} u_l) u_{ijk} + (C^{ij} + 3B^{ijk} u_k) u_{ij} + \\ & 3A^{ijkl} u_{ij} u_{kl} + 6A^{ijkl} u_i u_j u_{kl} + \\ & A^{ijkl} u_i u_j u_k u_l + B^{ijk} u_i u_j u_k + C^{ij} u_i u_j + D^i u_i = \lambda \end{aligned} \tag{3.13}$$

Now since we are looking for additively separable u, the mixed partial derivatives vanish and $\mathcal{H}$ depends only on derivatives of the form $u_i^{(k)}$:

$$\begin{aligned} \mathcal{H} =& (g^{ii})^2 u_i^{(4)} + (4g^{ii} g^{ij} u_j + B^{iii}) u_i^{(3)} (g^{ii} g^{jj} + 2(g^{ij})^2) u_i^{(2)} u_j^{(2)} \\ & + (2(g^{ii} g^{hj} + 2g^{ij} g^{ih}) u_j u_h + (B^{iij} + B^{iji} + B^{jii}) u_j + C^{ii}) u_i^{(2)} \\ & + g^{ij} g^{hk} u_i u_j u_h u_k + B^{ihk} u_i u_h u_k + C^{ij} u_i u_j + D^i u_i \end{aligned} \tag{3.14}$$

Since g is a Riemannian metric $g^{ii} \neq 0$, this is due to the fact that g is positive definite; $g^{-1}(dq^i, dq^i) = g^{ii} > 0$, where g^{-1} is the induced metric on the cotangent spaces.

Furthermore, this function is homogeneous in spatial derivatives, meaning that in every term the number of derivatives of u appearing in the term plus the number of derivatives of the metric is constant (in this case 4), this is due to the fact that B^{ijk}, C^{ij} and D^i contain first, second and third order derivatives of the metric respectively.

With all of this in mind we can start by computing R_i as in (2.6)

$$\begin{aligned}
R_i &= -\left(\frac{\partial \mathcal{H}}{\partial u_i^{(4)}}\right)\left(\frac{\partial \mathcal{H}}{\partial q^i} + \frac{\partial \mathcal{H}}{\partial u} u_i + \frac{\partial \mathcal{H}}{\partial u_i} u_i^{(2)} + \frac{\partial \mathcal{H}}{\partial u_{ii}} u_i^{(3)} + \frac{\partial \mathcal{H}}{\partial u_i^{(3)}} u_i^{(4)}\right) \\
&= -\frac{1}{g^{ii}}\left(\frac{\partial \mathcal{H}}{\partial q^i} + \frac{\partial \mathcal{H}}{\partial u_i} u_i^{(2)} + \frac{\partial \mathcal{H}}{\partial u_{ii}} u_i^{(3)} + \frac{\partial \mathcal{H}}{\partial u_i^{(3)}} u_i^{(4)}\right) \qquad (3.15)
\end{aligned}$$

with no sum over repeated indices in the equation and used the fact that $\mathcal{H}$ is independent of u. Now we will show that $D_i R_j$ cannot vanish everywhere.

$$D_i R_j = \partial_i R_j + u_i^{(2)} \frac{\partial R_j}{\partial u_i} + u_i^{(3)} \frac{\partial R_j}{\partial u_i^{(2)}} + u_i^{(4)} \frac{\partial R_i}{\partial u_i^{(3)}} + R_i \frac{\partial R_j}{\partial u_i^{(4)}} \qquad (3.16)$$

We begin by computing the last term $R_i \frac{\partial R_j}{\partial u_i^{(4)}}$

$$\begin{aligned}
R_i \frac{\partial R_j}{\partial u_i^{(4)}} &= -\frac{R_i}{(g^{jj})^2}\left(\frac{\partial^2 \mathcal{H}}{\partial q^j \partial u_i^{(4)}} + u_j^{(2)} \frac{\partial^2 \mathcal{H}}{\partial u_i^{(4)} \partial u_j} + u_j^{(3)} \frac{\partial^2 \mathcal{H}}{\partial u_i^{(4)} \partial u_j^{(2)}} + u_j^{(4)} \frac{\partial^2 \mathcal{H}}{\partial u_j^{(3)} \partial u_i^{(4)}}\right) \\
&= -\frac{R_i \partial_j (g^{ii})^2}{(g^{jj})^2} \qquad (3.17)
\end{aligned}$$

Next we have the term $u_i^{(4)} \frac{\partial R_i}{\partial u_i^{(3)}}$

$$\begin{aligned}
u_i^{(4)} \frac{\partial R_j}{\partial u_i^{(3)}} &= -\frac{u_i^{(4)}}{(g^{jj})^2}\left(\frac{\partial^2 \mathcal{H}}{\partial q^j \partial u_i^{(3)}} + u_j^{(2)} \frac{\partial^2 \mathcal{H}}{\partial u_i^{(3)} \partial u_j} + u_j^{(3)} \frac{\partial^2 \mathcal{H}}{\partial u_i^{(3)} \partial u_j^{(2)}} + u_j^{(4)} \frac{\partial^2 \mathcal{H}}{\partial u_j^{(3)} \partial u_i^{(3)}}\right) \\
&= -\frac{u_i^{(4)}}{(g^{jj})^2}\left(\partial_j B^{iii} + 4\partial_j (g^{ii} g^{ik}) u_k + 4 u_j^{(2)} g^{ii} g^{ij}\right) \qquad (3.18)
\end{aligned}$$

Notice that this contains a term of the form

$$-4 \frac{g^{ii} g^{ij}}{(g^{jj})^2} u_j^{(2)} u_i^{(4)} \qquad (3.19)$$

Then we have the term $u_i^{(3)}\frac{\partial R_j}{\partial u_i^{(2)}}$

$$u_i^{(3)}\frac{\partial R_j}{\partial u_i^{(2)}} = -\frac{u_i^{(3)}}{(g^{jj})^2}\left(\frac{\partial^2 \mathcal{H}}{\partial q^j \partial u_i^{(2)}} + u_j^{(2)}\frac{\partial^2 \mathcal{H}}{\partial u_i^{(2)}\partial u_j} + u_j^{(3)}\frac{\partial^2 \mathcal{H}}{\partial u_i^{(2)}\partial u_j^{(2)}} + u_j^{(4)}\frac{\partial^2 \mathcal{H}}{\partial u_j^{(3)}\partial u_i^{(2)}}\right)$$
$$= -\frac{u_i^{(3)}}{(g^{jj})^2}\left(\frac{\partial^2 \mathcal{H}}{\partial q^j \partial u_i^{(2)}} + u_j^{(2)}\frac{\partial^2 \mathcal{H}}{\partial u_i^{(2)}\partial u_j} + 2(g^{ii}g^{jj} + 2(g^{ij})^2)u_j^{(3)}\right) \tag{3.20}$$

We note that this contains the following term

$$\frac{-2}{(g^{jj})^2}(g^{ii}g^{jj} + 2(g^{ij})^2)u_j^{(3)}u_i^{(3)} \tag{3.21}$$

Finally we have the term $u_i^{(2)}\frac{\partial R_j}{\partial u_i}$

$$u_i^{(2)}\frac{\partial R_j}{\partial u_i} = -\frac{u_i^{(2)}}{(g^{jj})^2}\left(\frac{\partial^2 \mathcal{H}}{\partial q^j \partial u_i} + u_j^{(2)}\frac{\partial^2 \mathcal{H}}{\partial u_i\partial u_j} + u_j^{(3)}\frac{\partial^2 \mathcal{H}}{\partial u_i\partial u_j^{(2)}} + u_j^{(4)}\frac{\partial^2 \mathcal{H}}{\partial u_j^{(3)}\partial u_i}\right)$$
$$= -u_i^{(2)}\left(4\frac{g^{ij}u_j^{(4)}}{g^{jj}} + \text{Lower order derivatives of } u\right) \tag{3.22}$$

This contains the terms

$$-\frac{4g^{ij}u_j^{(4)}u_i^{(2)}}{g^{jj}} \tag{3.23}$$

These highlighted terms (3.19,3.21,3.23) are the only terms containing $u_l^{(4)}u_m^{(2)}$ and $u_l^{(3)}u_m^{(3)}$ in D_iR_j, and we can see by inspection that they do not cancel when added and none of them vanish since $g^{ii} > 0$. We conclude that D_iR_j does not vanish implying that the bi-Helmholtz equation does not admit regular separation in any coordinate system on any n-dimensional Riemannian manifold. □

3.1 Cartesian Coordinates

In Cartesian coordinates the bi-Laplacian takes the simple form:

$$\Delta^2 = (\partial_x^2 + \partial_y^2)^2 = \partial_x^4 + \partial_y^4 + 2\partial_x^2\partial_y^2 \tag{3.24}$$

Looking for a separable solution for the bi-Helmholtz equation $\Delta^2\psi = \lambda\psi$ for some $\lambda \in \mathbb{R}$, we assume $\psi(x, y) = f(x)g(y)$. Substituting this into the bi-Helmholtz equation we obtain

$$\begin{aligned} f^{(4)}g + fg^{(4)} + 2f''g'' &= \lambda fg \\ \frac{f^{(4)}}{f} + \frac{g^{(4)}}{g} + 2\frac{f''g''}{fg} &= \lambda \end{aligned} \tag{3.25}$$

A necessary condition for separation can be obtained applying ∂_{xy} to (3.25) which gives

$$\left(\frac{f''}{f}\right)'\left(\frac{g''}{g}\right)' = 0 \tag{3.26}$$

This condition immediately tells us that the separation of variables is non-regular since there are extra conditions on the functions for the solution to be separable. The previous equation gives three case: Either one of the terms or both terms in the above equation must vanish. We first consider the case

$$\frac{f''}{f} = C_1 \qquad \frac{g''}{g} = C_2$$

For some $C_1, C_2 \in \mathbb{R} \setminus \{0\}$ such that the following relation holds

$$C_1^2 + C_2^2 + 2C_1C_2 = \lambda = (C_1 + C_2)^2 \tag{3.27}$$

Since $f'' = C_1 f$ gives $f^{(4)} = C_1 f'' = C_1^2 f$ and similarly for g. So this case only works when $\lambda \geq 0$. Now we let $k_i = \sqrt{C_i}$ for $i = 1, 2$ where k_i can be imaginary, then we have

$$f = A_1 \cosh(k_1 x) + A_2 \sinh(k_1 x)$$
$$g = A_3 \cosh(k_2 y) + A_4 \sinh(k_2 y)$$

For some arbitrary constants $A_1, ..., A_4$ and the constants k_1 and k_2 need to satisfy (3.27). We can see that in this case the solution is what we previously called trivially separable since this is a solution to a Helmholtz equation $\Delta\psi = \sqrt{\lambda}\psi$.
The other sufficient conditions is when one of $\frac{f''}{f}$, $\frac{g''}{g}$ is constant and the other is not. In the case that $f'' = cf = \pm k^2 f$ and $\frac{g''}{g}$ is not constant we have that

$$f = A\cosh(kx) + B\sinh(kx) \qquad \text{or} \qquad f = A\cos(kx) + B\sin(kx) \tag{3.28}$$

Depending on the sign of c, where $A, B \in \mathbb{R}$. We also have that g satisfies

$$g^{(4)} + 2cg'' + (c^2 - \lambda)g = 0 \tag{3.29}$$

Which comes from (4.2) after subbing in $f'' = cf$. This has a characteristic equation

$$r^4 + 2cr^2 + c^2 = \lambda \tag{3.30}$$
$$(r^2 \pm k^2)^2 = \sigma^4 \tag{3.31}$$

We see that to have a separated solution we also need $\sigma^4 = \lambda \geq 0$ in these cases as well. From here we obtain all possible solutions by considering the different cases for the signs and magnitude of c. We see that in this case g satisfies a fourth order equation and hence our solution can never be a solution to a Helmholtz equation so this is an example of non-trivial non-regular separation. As a result of this the solution will contain 6 independent constants (4 for g and 2 for f) and 1 separation constant namely c. If the equation had regular separation then one would have $2 \times 4 + 1 = 9$ constants, and hence from the geometrical interpretation we expect there to be a six dimensional submanifold (defined by two constraint functions) of our bundle on which the non-regular separation occurs. We now show the geometrical treatment of this problem to compare.

After substituting $f = e^{u_1}$ and $g = e^{u_2}$ the equation (3.25) the only term which obstructs the additive separation is the term corresponding to $\frac{f''g''}{fg}$ which has the form

$$(u_1'' + u_1'^2)(u_2'' + u_2'^2) \tag{3.32}$$

and must be constant. Applying the mixed partial gives

$$(u_1^{(3)} + 2u_1'u_1'')(u_2^{(3)} + 2u_2'u_2'') = 0 \tag{3.33}$$

As before without loss of generality we assume the first term is zero which gives the following equations

$$u_1^{(3)} + 2u_1'u_1'' = 0 \qquad u_1^{(4)} + 2(u_1'')^2 + 2u_1^{(3)}u_1' = 0 \tag{3.34}$$

where the second equation is the derivative of the first and must also be identically zero for the separation to occur. We now have two candidate functions f_1 and f_2 that will define our submanifold on which the non-regular separation occurs. So let's take

$$f_1 = u_1^{(3)} + 2u_1'u_1'' \qquad f_2 = u_1^{(4)} + 2(u_1'')^2 + 2u_1^{(3)}u_1'$$

we now need to check the $D_i f_a|_N = 0$ on the submanifold N defined by $f_1 = f_2 = 0$.

3.2 Polar Coordinates

For polar coordinates we have $\Gamma^r = -\frac{1}{r}, \Gamma^\theta = 0$, so the Laplace-Beltrami operator has the form

$$\Delta = g^{ij}\partial_{ij} - \Gamma^i\partial_i = \partial_r^2 + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r}\partial_r$$

From this we have that the bi-Laplacian

$$\begin{aligned}\Delta^2\psi &= \Delta\left(\partial_r^2\psi + \frac{1}{r^2}\partial_\theta^2\psi + \frac{1}{r}\partial_r\psi\right)\\ &= \psi_{rrrr} + \frac{1}{r^4}\psi_{\theta\theta\theta\theta} + \frac{2}{r}\psi_{rrr} - \frac{1}{r^2}\psi_{rr} + \frac{1}{r^3}\psi_r + \frac{4}{r^4}\psi_{\theta\theta} - \frac{2}{r^3}\psi_{r\theta\theta} + \frac{2}{r^2}\psi_{rr\theta\theta} \quad (3.35)\end{aligned}$$

Looking for a separable solution $\psi(r,\theta) = R(r)\Theta(\theta)$, where we require the theta function to be 2π periodic. Plugging this into the bi-Helmholtz equation we have

$$\begin{aligned}\frac{\Delta^2\psi}{\psi} &= \frac{R^{(4)}}{R} + \frac{\Theta^{(4)}}{r^4\Theta} + \frac{2R^{(3)}}{rR} - \frac{R''}{r^2R} + \frac{R'}{r^3R} - 2\frac{R'\Theta''}{r^3R\Theta} + 2\frac{R''\Theta''}{r^2R\Theta} + 4\frac{\Theta''}{r^4\Theta} = \lambda\\ &\frac{r^4R^{(4)} + 2r^3R^{(3)} - r^2R'' + rR' - \lambda r^4R}{R} + \frac{\Theta^{(4)} + 2(r^2\frac{R''}{R} - r\frac{R'}{R} + 2)\Theta''}{\Theta} = \lambda \quad (3.36)\end{aligned}$$

We can see that the necessary condition for separation in (3.36) is

$$\left(r^2\frac{R''}{R} - r\frac{R'}{R} + 2\right)'\left(\frac{\Theta''}{\Theta}\right)' = 0 \quad (3.37)$$

This gives gives three cases as in the Cartesian case, the case where both terms vanish as before gives trivial non-regular separation. Given that our coordinate θ is a cyclic coordinate we need to insure that our function Θ is continuous. Thus we should consider the case where $\Theta'' = C_1\Theta$ where $C_1 < 0$ for our function to be periodic (continuous in the periodic variable). In this case we obtain a fourth order equation for R

$$r^4R^{(4)} + 2r^3R^{(3)} + (2C_1 - 1)r^2R'' + (1 - 2C_1)rR' + (C_1^2 + 4C_1 - \lambda r^4)R = 0 \quad (3.38)$$

Obtained by plugging in $\Theta'' = C_1\Theta$ and $\Theta^{(4)} = C_1\Theta'' = C_1^2\Theta$ into (3.36). Similar to the Cartesian case this case gives non-regular separation where R satisfies a fourth order equation and Θ satisfies a second order equation. This gives two constraint function in the additive formulation

$$f_1 = u_2^{(3)} + 2u_2u_2'' \qquad\qquad f_2 = u_2^{(4)} + 2(u_2'')^2 + 2u_2^{(3)}u_2', \tag{3.39}$$

where $\Theta = e^{u_2}$, in which case we also have $D_i f_j = 0$ on the surface defined by the functions.

3.3 Liouville Coordinates

Liouville coordinates are the most general set of coordinates in which the Helmholtz equation in 2-dimensions separates. The metric in this coordinate system has the form

$$ds^2 = (f(u) + g(v))\,(du^2 + dv^2)$$

Since the metric is conformally flat $\Gamma^i = 0$ in 2-dimensions and hence we have the Helmholtz equation has the form

$$\Delta\psi = \frac{1}{f+g}\left(\partial_u^2\psi + \partial_v^2\psi\right) = \lambda\psi \tag{3.40}$$

For a separable solution $\psi = U(u)V(v)$

$$\left(\frac{U''}{U} - \lambda f\right) + \left(\frac{V''}{V} - \lambda g\right) = 0, \tag{3.41}$$

which directly tells us that the equation separates for any smooth functions f, g. Since this case always admits separation for the Helmholtz equation we can use it to investigate which functions f, g give non-trivial non-regular separation for the bi-Helmholtz equation. A special case of Liouville coordinates on $\mathbb{R}^2$ are Cartesian, polar, parabolic and elliptic hyperbolic coordinates all obtained by appropriately changing f and g. The aforementioned cases all have vanishing Gaussian curvature which in this coordinate system has the form

$$K = -\frac{1}{2(f+g)^2}\left(f'' + g'' - \frac{f'^2 + g'^2}{f+g}\right) \tag{3.42}$$

From here on we will set $K = 0$ in order to consider the parabolic and elliptic-hyperbolic cases in which we also have that $f' \neq 0, g' \neq 0$. We use the Laplace-Beltrami operator as in (3.40) to write the bi-Laplace operator

$$\begin{aligned}\Delta^2\psi &= \frac{\Delta_0^2\psi}{(f+g)^2} - \frac{2}{(f+g)^3}(f'\partial_u\Delta_0\psi + g'\partial_v\Delta_0\psi) + \frac{1}{(f+g)^4}\left(f'^2 + g'^2 + 2(f+g)^3K\right)\Delta_0\psi \\ &= \frac{\Delta_0^2\psi}{(f+g)^2} - \frac{2}{(f+g)^3}(f'\partial_u\Delta_0\psi + g'\partial_v\Delta_0\psi) + \frac{f'' + g''}{(f+g)^3}\Delta_0\psi \end{aligned} \tag{3.43}$$

Where $\Delta_0 = \partial_u^2 + \partial_v^2$ is the flat Laplacian. The vanishing Gaussian curvature condition was used to simplify the form of Δ^2. Looking for separable solutions of the form $\psi = U(u)V(v)$ and subbing this into the bi-Helmholtz equation we get

$$\begin{aligned}
&f\frac{U^{(4)}}{U} + g\frac{V^{(4)}}{V} - 2f'\frac{U^{(3)}}{U} - 2g'\frac{V^{(3)}}{V} + f''\frac{U''}{U} + g''\frac{V''}{V} - \lambda f^3 - \lambda g^3 + \\
&+2(f+g)\frac{U''V''}{UV} + g\frac{U^{(4)}}{U} + f\frac{V^{(4)}}{V} - 2f'\frac{U'V''}{UV} - 2g'\frac{V'U''}{UV} + f''\frac{V''}{V} + g''\frac{U''}{U} \\
&-3\lambda f^2 g - 3\lambda f g^2 = 0.
\end{aligned} \tag{3.44}$$

We see that the equation does not admit regular separability as expected. However it might still admit non-regular separability. Applying $\partial_{u,v}$ to this once we obtain

$$\begin{aligned}
&f'\left(\frac{V^{(5)}}{V} - \frac{V^{(4)}V'}{V^2}\right) + g'\left(\frac{U^{(5)}}{U} - \frac{U^{(4)}U'}{U^2}\right) - 6\lambda(f+g)f'g' + \\
&+\left(\frac{V^{(3)}}{V} - \frac{V''V'}{V^2}\right)\left(f'' + 2f\frac{U''}{U} - 2f'\frac{U'}{U}\right)' \\
&+\left(\frac{U^{(3)}}{U} - \frac{U''U'}{U^2}\right)\left(g'' + 2g\frac{V''}{V} - 2g'\frac{V'}{V}\right)' = 0.
\end{aligned} \tag{3.45}$$

This condition once again does not separate. To obtain a simpler sufficient condition for separation, we divide by $f'g'$ and apply ∂_{uv} once again

$$\begin{aligned}
&\left(\frac{1}{g'}\left(\frac{V''}{V}\right)'\right)'\left(\frac{\left(f'' + 2f\frac{U''}{U} - 2f'\frac{U'}{U}\right)'}{f'}\right)' \\
&+\left(\frac{1}{f'}\left(\frac{U''}{U}\right)'\right)'\left(\frac{\left(g'' + 2g\frac{V''}{V} - 2g'\frac{V'}{V}\right)'}{g'}\right)' = 0.
\end{aligned} \tag{3.46}$$

From here we obtain several cases: (i) neither $\left(\frac{1}{f'}\left(\frac{U''}{U}\right)'\right)'$ nor $\left(\frac{1}{g'}\left(\frac{V''}{V}\right)'\right)'$ vanish, from which we obtain $\lambda = 0$ (ii) only one vanishes which also implies $\lambda = 0$ (iii) both vanish which gives us a solution to the Helmholtz equation in Liouville coordinates. Hence, we do not get any non-trivial separated solutions. The proof in case (i) is given below and in Appendix A while that for cases (ii) and (iii) is given in Appendix B.

Case(i): Dividing (3.46) by $\left(\frac{1}{f'}\left(\frac{U''}{U}\right)'\right)'\left(\frac{1}{g'}\left(\frac{V''}{V}\right)'\right)'$ we obtain a separable equation. Separating and integrating (see appendix A for this calculation) we find the following equations

$$(2g - C)\frac{V''}{V} - 2g'\frac{V'}{V} + g'' - C_1 g - C_2 = 0, \tag{3.47}$$

$$(2f + C)\frac{U''}{U} - 2f'\frac{U'}{U} + f'' - D_1 f - D_2 = 0, \tag{3.48}$$

where $C, C_1, C_2, D_1.D_2 \in \mathbb{R}$. We can use these conditions in (3.45) to obtain

$$\frac{1}{g'}\left(\frac{V^{(4)}}{V}\right)' + \frac{1}{f'}\left(\frac{U^{(4)}}{U}\right)' + \frac{D_1}{g'}\left(\frac{V''}{V}\right)' + \frac{C_1}{f'}\left(\frac{U''}{U}\right)' - 6\lambda(f+g) = 0. \tag{3.49}$$

Notice that this condition is now separable. And using this condition we can integrate back to get a simplified form of equation (3.44)

$$\begin{aligned} &f\frac{U^{(4)}}{U} + g\frac{V^{(4)}}{V} - 2f'\frac{U^{(3)}}{U} - 2g'\frac{V^{(3)}}{V} + f''\frac{U''}{U} + g''\frac{V''}{V} - \lambda f^3 - \lambda g^3 + \\ &+ f\frac{V^{(4)}}{V} + g\frac{U^{(4)}}{U} + (C_1 g + C_2)\frac{U''}{U} + (D_1 f + D_2)\frac{V''}{V} - 3\lambda f^2 g - 3\lambda f g^2 = 0. \end{aligned} \tag{3.50}$$

To make sure we account for all the constraints properly we also need to use the derivatives of equations (3.47) and (3.48). There are other constraints coming from the additional requirement $K = 0$, for the interest of brevity we will include these calculations Appendix A.

After using all the constraint equations we end up with the following

$$
\begin{aligned}
&\frac{(C_2 - D_2 - (C_1 + D_1)f)}{C_1 + D_1}\left(\alpha f + \beta + 3\lambda f^2 - \frac{(k + C_1)(D_1 - k)f + C_1(D_2 - D) + D(D_1 - k)}{2f + C}\right. \\
&\left. + 2\frac{f'^2}{(2f + C)^2}(D_1 - k) - \frac{(D_1 f + D_2)^2 - f''^2}{(2f + C)^2}\right) - 2\frac{(D_1 - k)}{2f + C}f'^2 \\
&+ ((k - C_1)f + C_2 + D)\left(\frac{(D_1 - k)f + D_2 - D}{2f + C}\right) + 2\lambda f^3 + \alpha f^2 + (\beta + \gamma)f = \delta,
\end{aligned}
\tag{3.51}
$$

where from the condition $K = 0$ we have $f'^2 = kf^2 + 2Df - k$ and $g'^2 = -kg^2 + 2Dg + k$. Thus the above equation simplifies to a polynomial in f after multiplication by $(2f+C)^2$. Furthermore, since $f' \neq 0$, the set $\{1, f, f^2, \ldots, f^n\}$ is linearly independent. This implies that the coefficients of the different powers of f must all vanish. The coefficient of the highest power of f the is 8λ the vanishing of which implies that $\lambda = 0$. In the case that $C_1 + D_1 = 0$ (20) is a polynomial in f with highest order term $3\lambda f^2$ which also implies $\lambda = 0$.

Chapter 4

Vibrating circular clamped plate

A vibrating plate with some constant thickness and constant density satisfies [11]

$$c^4 \Delta^2 u + \ddot{u} = 0, \tag{4.1}$$

on the region Ω in the interior of the plate, where u is the lateral elevation at every point of the plate and c^4 is some positive constant of the material which contains the density, Young's modulus and thickness. A derivation of this equation can be found in Rayleigh's book [11]. If the plate is clamped the boundary conditions on u are $u|_{\partial\Omega} = u_n|_{\partial\Omega} = 0$ where the subscript n denotes differentiating with respect to the normal coordinate to the boundary. For the case of a circular plate we have that Ω is a disk of some radius a. First we separate out the time variable $u = w(r, \theta)T(t)$

$$\frac{\Delta^2 w}{w} = -\frac{1}{c^4}\frac{\ddot{T}}{T} = k^4, \tag{4.2}$$

where we used a positive separation constant since we expect oscillatory behaviour in time. Notice that this separation is regular since there are no additional constraints on the separated functions. We define $\omega^2 = k^4 c^4$, so the time part will have the form

$$T(t) = G\cos(\omega t) + H\sin(\omega t) \tag{4.3}$$

The spatial part will satisfy a bi-Helmholtz equation,

$$\Delta^2 w = k^4 w \tag{4.4}$$

Written out explicitly in polar coordinates this has the form

$$w_{rrrr} + \frac{1}{r^4} w_{\theta\theta\theta\theta} + \frac{2}{r} w_{rrr} - \frac{1}{r^2} w_{rr} + \frac{1}{r^3} w_r + \frac{4}{r^4} w_{\theta\theta} - \frac{2}{r^3} w_{r\theta\theta} + \frac{2}{r^2} w_{rr\theta\theta} = k^4 w \tag{4.5}$$

Now to separate the spatial coordinates first we require our solution to be 2π periodic in θ. This means that we can write $w_n = R_n(r)\Theta_n(\theta)$ where $\Theta_n'' = -n^2\Theta$. Substituting this into the (4.5) we have the following equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} + k^2\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} - k^2\right) R_n = 0 \tag{4.6}$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} - k^2\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} + k^2\right) R_n = 0, \tag{4.7}$$

where we have explicitly written the equation in two ways to show that these operators commute. Rayleigh in his book [11] arrives at a similar result starting with a Fourier series ansatz. These are the Bessel and modified Bessel operators. Since we know a basis of solutions to the Bessel and modified Bessel equations which are linearly independent the radial function will be a linear combination of those solutions

$$R_n(r) = A_n J_n(kr) + B_n Y_n(kr) + C_n I_n(kr) + D_n K_n(kr), \tag{4.8}$$

where J_n, Y_n, I_n, K_n are the Bessel and modified Bessel functions of the first and second kind. The Bessel functions of the second kind are inadmissible since they are singular at the origin so we take $B_n = D_n = 0$. The angular function satisfies the second order DE and thus has the form

$$\Theta_n = E_n \cos(n\theta) + F_n \sin(n\theta)$$

One retrieves the solution to the vibrating membrane when also taking $C_n = 0$. The boundary conditions tell us that $R_n(a) = R'_n(a) = 0$.

$$R_n(a) = A_n J_n(ka) + C_n I_n(ka) = 0 \tag{4.9}$$

$$R'_n(a) = kA_n J'_n(ka) + kC_n I'_n(ka) = 0 \tag{4.10}$$

To solve (4.9) we either need $ka = j_{n,m}$ (the mth zero of J_n) and $C_n = 0$ in which case we retrieve the vibrating membrane solution, or $A_n = -\frac{I_n(ka)}{J_n(ka)} C_n$ (we can absorb C_n into the definition of E_n, F_n to simplify). Since we are interested in solutions other than the ones for the vibrating membrane we take the latter in which case (4.10) becomes

$$I'_n(ka) - \frac{I_n(ka)}{J_n(ka)} J'_n(ka) = 0 \tag{4.11}$$

$$\frac{I'_n(ka)}{I_n(ka)} - \frac{J'_n(ka)}{J_n(ka)} = 0 \tag{4.12}$$

We can find the roots numerically and get a condition $ka = l_{n,m}$ where $l_{n,m}$ is the mth root of equation (4.12) some fixed m

$$R_{n,m}(r) = C_n \left(I_n \left(\frac{l_{n,m} r}{a} \right) - \frac{I_n(l_{n,m})}{J_n(l_{n,m})} J_n \left(\frac{l_{n,m} r}{a} \right) \right) \tag{4.13}$$

So the general solution for u has the form

$$u = \sum_{m,n} \left(E_n \cos(n\theta) + F_n \sin(n\theta) \right) \left(G_{n,m} \cos(\omega_{n,m} t) + H_{n,m} \sin(\omega_{n,m} t) \right) R_{n,m}, \tag{4.14}$$

where $\omega_{n,m} = ck_{n,m} = c\frac{l_{n,m}}{a}$. This is an application of non-regular separation in a physical situation, as mentioned at the beginning of the section. Rayleigh approached this problem with a Fourier approach and has more details in his book [11].

Chapter 5

Symmetries of the bi-Laplace operator

The link between symmetries of a linear partial differential equation and separable coordinates has been a longstanding area of research. There has been good progress in understanding this link especially for the Helmholtz equation and other second order equations mainly due to Stäckel, Eisenhart, Miller and others [6, 8, 13]. The full picture of the relationship, however, has yet to be understood. In this section we present a new avenue in this area of study by trying to understand the relationship between symmetries and non-regular separation.

In the previous section we looked at the bi-Helmholtz equation in a special class of coordinate systems: those which we called Liouville coordinate systems. These coordinate systems are characterized by the existence of a second order symmetry (valence two Killing tensor); it is due to this fact that, as mentioned before, they are the most general coordinate systems in 2-dimensions for which the Helmholtz equation separates. There are four of these coordinate systems where the Gaussian curvature K vanishes, namely Cartesian, Polar, Parabolic and Elliptic-Hyperbolic coordinates.

The Cartesian and Polar coordinate systems also admit a first order symmetry operator (Killing vector); these are ∂_x, ∂_y for Cartesian and ∂_θ for the polar coordinates. In these coordinate systems we see that we obtain what we call non-trivial non-regular separation, non-regularly separable solutions to the bi-Helmholtz equation which are not solutions of the Helmholtz equation. The other two coordinate systems (Parabolic and Hyperbolic-Elliptic), do not admit first order symmetries. We see in our analysis that in these two cases we do not obtain non-trivial non-regular separation, i.e. if the solution is separable it was a solution of the Helmholtz equation.

This indicates that a coordinate system having a first order symmetry could be a sufficient condition for non-trivial non-regular separation of the bi-Helmholtz equation. We study the symmetries of the bi-Laplace operator Δ^2 (which are those of the bi-Helmholtz operator) in order to investigate this hypothesis between symmetry and non-trivial non-regular separation.

5.1 Preliminaries

A symmetry of a linear differential operator $\mathcal{D}$ is a linear differential operator L such that $[\mathcal{D}, L] = 0$. If u is a solution to $Du = 0$ then Lu is also a solution since

$$DLu = LDu = 0 \tag{5.1}$$

We are interested specifically in the case of the bi-Laplacian $D = \Delta^2$. In this case we have

$$\left[\Delta^2, L\right] = \Delta[\Delta, L] + [\Delta, L]\Delta = \{\Delta, [\Delta, L]\}\,, \tag{5.2}$$

where $\{\,,\}$ is the anticommutator bracket. We can immediately see that if L is a symmetry of the Laplacian it follows that it must be a symmetry of the bi-Laplacian.

We reformulate the problem in the following way for ease of computation, a symmetry L of the bi-Laplacian must satisfy

$$[\Delta, L] \stackrel{\text{def}}{=} S \tag{5.3}$$

$$\{\Delta, S\} = 0 \tag{5.4}$$

As mentioned earlier it is clear that if L is a symmetry of Δ it is a symmetry of Δ^2. Hence we will only be interested in non-zero operators S.

Let ∇ be the Levi-Civita connection on a given Riemannian manifold (M, g). Recall we can write the Laplacian as follows

$$\Delta = g^{ij}\nabla_i\nabla_j, \tag{5.5}$$

where $\nabla_i = \nabla_{\partial_i}$ and g^{ij} are the components of the inverse of the metric in a given coordinate basis $\{\partial_i\}_{i=1}^n$. For notational purposes we will denote

$$\nabla_i\nabla_j\cdots\nabla_k = \nabla_{ij\cdots k}$$

The computations in this chapter will rely on some identities from Riemmanian geometry. We recall and derive some of these identities before getting into the bulk of the calculation. In a coordinate basis we have

$$(\nabla_{ij} - \nabla_{ji})f = 0 \tag{5.6}$$

$$(\nabla_{ij} - \nabla_{ji})\,\omega_k = R^l{}_{kij}\omega_l \tag{5.7}$$

and for a general covariant tensor

$$(\nabla_{ij} - \nabla_{ji})\omega_{kl\cdots m} = R^n{}_{kij}\omega_{nl\cdots m} + R^n{}_{lij}\omega_{kn\cdots m} + \cdots + R^n{}_{mij}\omega_{kl\cdots n} \tag{5.8}$$

These identities will be useful in order to symmetrize covariant derivatives. The symmetrization is required in order to compare differential operators order by order. The symmetrized covariant derivatives $\nabla_{(ij\cdots k)}$ cannot be reduced to any lower order symmetric covariant derivatives by the Ricci identities.

Compare this with the un-symmetrized version of the covariant derivatives where we have for example

$$\nabla_{ijk} f = \nabla_{jik} f + R^l{}_{kij} \nabla_l f \tag{5.9}$$

$$\nabla_{ijkl} f = \nabla_{jikl} f + R^m{}_{kij} \nabla_{ml} f + R^m{}_{lij} \nabla_{km} f, \tag{5.10}$$

which come from applying (5.8) to $\omega = \nabla f = df$ and $\omega = \nabla\nabla f = \text{Hess} f$. This shows that the un-symmetrized covariant derivatives can be reduced down using the Ricci identity. Symmetrizing the above relations we have

$$\nabla_{ijk} f = \nabla_{(ijk)} f - \frac{2}{3} R^l{}_{(jk)i} \nabla_l f \tag{5.11}$$

$$\nabla_{ijkl} f = \nabla_{(ijkl)} f + \frac{3}{2} R^m{}_{(j|i|k} \nabla_{l)m} f - \frac{2}{3} (\nabla_i R^m{}_{(kl)j} \nabla_m f + R^m{}_{(kl)j} \nabla_{im} f) \tag{5.12}$$

$$\nabla_{ijklm} f = \nabla_{(ijklm)} f + \text{terms of order 3 and less} \tag{5.13}$$

It's generally always possible to symmetrize a covariant derivative of order n by obtaining terms which are of order $n-2$ or less by using the Ricci identity (5.8):

$$\nabla_{i_1 i_2 \cdots i_n} f = \nabla_{(i_1 i_2 \cdots i_n)} f + \text{unsymmetrized derivatives of order } n-2 \text{ or less.}$$

With this in mind we start investigating the symmetries of the bi-Laplace operator.

5.2 First order symmetries

We investigate non-trivial first order operators which commute with the bi-Laplace operator (on functions). Any such operator X can be written $X = X^i\nabla_i$. As described before we start by calculating the commutator with the Laplace-Beltrami operator and then set the anticommutator of the result with the Laplace-Beltrami operator to 0. Recall that the Laplace-Beltrami operator can be written as $\Delta = g^{ij}\nabla_{ij}$ in terms of the Levi-Civita connection, computing the commutator first we have

$$\begin{aligned}[\Delta, X]f &= g^{ij}\nabla_{ij}\left(X^k\nabla_k f\right) - X^i\nabla_i\left(g^{jk}\nabla_{jk}f\right)\\ &= g^{ij}\nabla_{ij}X^k\nabla_k f + g^{ij}\nabla_j X^k\nabla_{ik}f + g^{ij}\nabla_i X^k\nabla_{jk}f - X^i g^{jk}R^l{}_{kij}\nabla_l f\end{aligned}$$

The third order derivatives of f cancel. Grouping terms by order of covariant derivatives we have

$$[\Delta, X]f = 2\nabla^{(i}X^{j)}\nabla_{ij}f + \left(\Delta X^l - X^i g^{jk}R^l{}_{kij}\right)\nabla_l f \tag{5.14}$$

$$= (S^{ij}\nabla_{ij} + \sigma^l\nabla_l)f := Sf, \tag{5.15}$$

where we define $S^{ij} = 2\nabla^{(i}X^{j)}$, $\sigma^l = \left(\Delta X^l - X^i g^{jk}R^l{}_{kij}\right)$ and $S = S^{ij}\nabla_{ij} + \sigma^l\nabla_l$. Now we compute the anticommutator of S with Δ and set it equal to 0

$$\{\Delta, S\}f = (g^{ij}S^{kl} + g^{kl}S^{ij})\nabla_{ijkl}f + \text{lower order terms} \tag{5.16}$$

The highest order terms (4th order here) are the ones where all the derivatives act on f; these are the terms that are shown in the above equation. As mentioned before we can always symmetrize the derivatives to obtain lower order terms:

$$\nabla_{ijkl}f = \nabla_{(ijkl)}f + \text{lower order terms} \tag{5.17}$$

Specifically the lower order terms are second order in the covariant derivatives in this case. Substituting this into (5.16) we have

$$\{\Delta, S\}f = 2g^{(ij}S^{kl)}\nabla_{(ijkl)}f + \text{lower order terms} \tag{5.18}$$

So this gives us

$$g^{(ij}\nabla^k X^{l)} = 0 \tag{5.19}$$

This is of course is equivalent to the equation

$$g^{i(j}\nabla^k X^{l)} = 0$$

since g is symmetric. Contracting with g_{ij} and expanding the symmetrization brackets we have

$$\begin{aligned} g_{ij}(g^{ij}\nabla^k X^l + g^{il}\nabla^j X^k + g^{ik}\nabla^l X^j + g^{ik}\nabla^j X^l + g^{il}\nabla^k X^j + g^{ij}\nabla^l X^k) &= 0 \\ n\nabla^k X^l + \delta^l_j\nabla^j X^k + \delta^k_j\nabla^l X^j + \delta^k_j\nabla^j X^l + \delta^l_j\nabla^k X^j + n\nabla^l X^k &= 0 \\ n\nabla^k X^l + \nabla^l X^k + \nabla^l X^k + \nabla^k X^l + \nabla^k X^l + n\nabla^k X^l &= 0 \\ 4(n+2)\nabla^{(k}X^{l)} &= 0 \end{aligned}$$

Hence obtaining $\nabla^{(k}X^{l)} = 0$, and X is a Killing vector in which case the integrability condition $\Delta X^l - X^i g^{jk} R^l{}_{kij} = \sigma^l = 0$ is satisfied [14] and thus $[X, \Delta] = 0$. This means that X is a also a symmetry of the Lalplace-Beltrami operator. We conclude that any first order symmetry of the bi-Laplace operator is also a symmetry of the Laplace operator (a Killing vector). We note that if a metric admits a Killing vector, then it also admits a valence two Killing tensor obtained by taking the symmetric tensor product of the Killing vector with itself. This will be of relevance to the upcoming section.

5.3 Second order symmetries

We consider second order operators of the following form $Kf = \nabla_i(K^{ij}\nabla_j f)$ such that K^{ij} is symmetric. These are the operators which, when identified as endomorphisms of vectors, are self-adjoint with respect to the metric g [3]. Carter computes the commutator of these operators with the Laplace-Beltrami operator [3]:

$$\begin{aligned}[\Delta, K] =& 2\nabla^h K^{ij}\nabla_{(ijh)} + 3\nabla_h \nabla^{(h}K^{ij)}\nabla_{(ij)} \\ &+ \nabla_j \left(\frac{1}{2}g_{hk}(\nabla^j\nabla^{(i}K^{hk)} - \nabla^i\nabla^{(j}K^{hk)}) + \frac{4}{3}K_h^{[j}R^{i]h}\right)\nabla_i \\ &:= S\end{aligned} \tag{5.20}$$

Highlighting the first term as before we get

$$\begin{aligned}\{S, \Delta\}\, f &= \left(g^{ij}S^{klm} + g^{lm}S^{ijk}\right)\nabla_{ijklm}f + \text{terms of lower order derivatives} \\ &= 2g^{(ij}S^{klm)}\nabla_{(ijklm)}f + \text{terms of lower order derivatives,}\end{aligned} \tag{5.21}$$

where here $S^{klm} = \nabla^k K^{ij}$ is the leading order term in (5.20), and we have symmetrized to obtain lower order derivative terms and obtain an independent leading term. For the anti-commutator to vanish we must have

$$g^{(ij}S^{klm)} = g^{(ij}\nabla^k K^{lm)} = 0 \tag{5.22}$$

As before contracting with g_{ij} give us that K is a (valence two) Killing tensor $\nabla^{(k}K^{lm)} = 0$. Substituting this backing into (5.20) we have

$$S = [\Delta, K] = \frac{4}{3}\nabla_j\left(K_h^{[j}R^{i]h}\right)\nabla_i := C^i\nabla_i, \tag{5.23}$$

where we have defined $C^i = \frac{4}{3}\nabla_j\left(K_h^{[j}R^{i]h}\right)$. This can go back into the anti-commutator equation to see what happens to the lower order terms

We have

$$
\begin{aligned}
\{S, \Delta\} &= \left(g^{ij} C^l + g^{jl} C^i\right) \nabla_{ijl} + \text{lower order terms} \\
&= 2 g^{(ij} C^{l)} \nabla_{(ijl)} + \text{lower order terms},
\end{aligned}
$$

where as before we symmetrize the covariant derivative of the leading term to obtain lower order terms. Thus we have that $g^{(ij}C^{l)} = 0$ must be satisfied which when contracted with g_{ij} and expanded gives that $C^i = 0$. In other words

$$
\frac{4}{3} \nabla_j \left(K^{[j}_h R^{i]h}\right) = 0 \tag{5.24}
$$

must be satisfied for K to be a a symmetry of the bi-Laplace operator. This condition being satisfied makes (5.23) vanish identically. The above condition is called the Carter condition which must be satisfied by a second order symmetry of the Laplace-Beltrami operator. So we have once again that a symmetry of the bi-Laplace operator is a symmetry of the Laplace-Beltrami operator.

A special case of when the Carter condition (5.24) is satisfied is when our Riemannian manifold is Einstein, i.e we have $R^{ij} = \alpha g^{ij}$:

$$
\frac{4}{3} \alpha \nabla_j \left(K^{[j}_h g^{i]h}\right) = \frac{2}{3} \alpha \nabla_j \left(K^j{}_h g^{ih} - K^i{}_h g^{jh}\right) \tag{5.25}
$$

$$
= \frac{2}{3} \alpha \nabla_j \left(K^{ji} - K^{ij}\right) = 0 \tag{5.26}
$$

since K is symmetric. Hence indeed the Carter condition is satisfied. We will now briefly discuss how this relates to the specific set of coordinates which we have examined in the past chapters (the Liouville coordinate systems).

5.4 Separable coordinates from symmetry

In his 1934 paper, Eisenhart [6] investigated the link between symmetries and separable coordinate systems. We summarize some of his results in this section as well as apply it to our case study. Eisenhart was specifically interested in quadratic first integrals of the geodesic equation, i.e. if γ is a geodesic in our Riemannian manifold he was interested in constant quantities along the geodesic of the form

$$a(\dot{\gamma}, \dot{\gamma}) = a_{ij}\dot{\gamma}^i\dot{\gamma}^j = \text{const} \tag{5.27}$$

A sufficient and necessary condition for this to hold is that a_{ij} is a Killing tensor. This can be seen by taking the covariant derivative along $\dot{\gamma} = \frac{\mathrm{d}}{\mathrm{d}t}$ of both sides. a_{ij} can be taken to be a symmetric tensor without loss of generality. We deviate slightly from Einsenhart's work, since we are interested in the 2-dimensional case. We write our metric in conformal or isothermal coordinates which always exist locally in 2-dimensions.

$$g = H^2(dx^2 + dy^2) \tag{5.28}$$

In contrast, Eisenhart writes the Killing equations $\nabla_{(i}a_{jk)} = 0$ explicitly in an orthogonal coordinate system in which a_{ij} is diagonal. These coordinates are constructed by integrating the eigenvectors of a_{ij} when seen as a endomorphism of vector fields. Note that, since a_{ij} is symmetric its associated endomorphism will be self-adjoint with respect to our metric and hence the eigenvectors will be orthogonal given that the eigenvalues are distinct; any degenerate eigenspaces can be made orthogonal by the Gram-Schmidt procedure. Integrating these orthogonal vectors Eisenhart obtains a local orthogonal coordinate system in which both the metric and a_{ij} are diagonal. Thus we can assume in our isothermal coordinates that a_{ij} is diagonal. Eisenhart writes the Killing equations in orthogonal coordinates which for the case of $n = 2$ become

$$\frac{\partial \log\left(\sqrt{a_{ii}}\right)}{\partial x^i} = \frac{\partial \log H}{\partial x^i} \tag{5.29}$$

$$\frac{\partial a_{ii}}{\partial x^j} - 4a_{ii}\frac{\partial \log H}{\partial x^j} + \frac{a_{jj}}{H^2}\frac{\partial H^2}{\partial x^j} = 0, \qquad (i, j = 1, 2) \tag{5.30}$$

Integrating (5.29) we immediately obtain that $a_{ii} = \rho_i H^2$ where ρ_i does not depend on q^i (i.e. $\frac{\partial \rho_i}{\partial q^j} = 0$ for $i \neq j$). Combining this with (5.30) Eisenhart writes an equation for these multipliers ρ_i which in our coordinate system reads

$$\frac{\partial \rho_i}{\partial q^j} = (\rho_i - \rho_j)\frac{\partial \log H^2}{\partial q^j} \tag{5.31}$$

One integrability condition of this equation is obtained by applying $\frac{\partial}{\partial q^i}$ to both sides of the equation, keeping in mind that $\frac{\partial \rho_i}{\partial q^j} = 0$ for $i \neq j$

$$\begin{aligned} 0 &= -\frac{\partial \rho_j}{\partial q^i}\frac{\partial \log H^2}{\partial q^j} + (\rho_i - \rho_j)\frac{\partial^2 \log H^2}{\partial q^j \partial q^i} \\ &= (\rho_i - \rho_j)\,\frac{\partial \log H^2}{\partial q^i}\frac{\partial \log H^2}{\partial q^j} + (\rho_i - \rho_j)\frac{\partial^2 \log H^2}{\partial q^j \partial q^i} \\ &= (\rho_i - \rho_j)\left(\frac{\partial \log H^2}{\partial q^i}\frac{\partial \log H^2}{\partial q^j} + \frac{\partial^2 \log H^2}{\partial q^j \partial q^i}\right) \end{aligned} \tag{5.32}$$

For distinct ρ_i this implies

$$\frac{\partial^2 \log H^2}{\partial q^j \partial q^i} + \frac{\partial \log H^2}{\partial q^i}\frac{\partial \log H^2}{\partial q^j} = 0 \tag{5.33}$$

For our case with only two coordinates we obtain a single equation

$$\frac{\partial^2 \log H^2}{\partial x \partial y} + \frac{\partial \log H^2}{\partial x}\frac{\partial \log H^2}{\partial y} = 0 \tag{5.34}$$

It is an equation for the conformal factor of our metric. We now show that this conformal factor is of the Liouville type. Expanding the derivatives in (5.34) we obtain

$$\begin{aligned} 0 &= \frac{\partial}{\partial x}\left(\frac{\partial_y(H^2)}{H^2}\right) + \frac{\partial_x(H^2)\partial_y(H^2)}{H^4} \\ &= \frac{\partial_{xy}(H^2)}{H^2} - \cancel{\frac{\partial_x(H^2)\partial_y(H^2)}{H^4}} + \cancel{\frac{\partial_x(H^2)\partial_y(H^2)}{H^4}} \\ &\implies \partial_{xy}(H^2) = 0 \end{aligned} \tag{5.35}$$

Integrating this gives us back the conformal factor for the Liouville metric

$$H^2 = f(x) + g(y) \tag{5.36}$$

This insight tells us that the bi-Helmholtz equation admits at least trivial non-regular separation in the coordinate system defined by (5.36). This result is due to the fact that the Helmholtz equation separates in this coordinate system as shown in chapter 2. We've seen that the bi-Helmholtz equation admits non-trivial non-regular separation in some of these coordinate systems (in Euclidean space) namely the Cartesian and polar coordinate systems, and does not admit non-trivial regular separation in the elliptic-hyperbolic and parabolic coordinate systems all of which are special cases of the Liouville coordinate system. The Cartesian and polar coordinate systems both admit a first order symmetry (Killing vector), where the elliptic-hyperbolic and parabolic coordinate systems do not. This leads us to the following conjecture:

Conjecture. *If $\mathcal{D} = \mathcal{L}^2$ is a linear operator which is the square of another linear operator $\mathcal{L}$, the equation $\mathcal{D}u = 0$ admits non-trivial non-regular separation if $\mathcal{D}$ has a first order symmetry operator $\mathcal{S}$.*

Further work is needed to fully comprehend the link between non-regular separation and symmetry. Nonetheless, this analysis provides a starting point for further investigation into the link between non-regular separation and symmetries of the bi-Laplace and other globally defined linear differential operators.

Chapter 6

Conclusion

In previous chapters we investigated the method of non-regular separation and applied it to the bi-Helmholtz equation. The bi-Helmholtz equation provides a good example of a PDE admitting non-regular separation; it shows the intricacies of the dependence of non-regular separation on the coordinate system and its symmetries. In the final chapter we show that the coordinate systems examined in the previous chapters arise naturally from considering symmetries of the bi-Helmholtz equation. Furthermore, we conjecture that non-trivial non-regular separation occurs when the equation admits a first order symmetry. Thus, the previous analysis of the bi-Helmholtz equation serves as a stepping stone toward exploring the interesting and rich theory of separability of high order PDEs in relation to their symmetries.

In light of our analysis in this thesis, future investigations on this topic should address the bi-Helmholtz equation in higher dimensions, including spaces with constant scalar curvature. Our study of the symmetries of the bi-Helmholtz equation might also be extended to give an invariant characterization of non-trivial non-regular separation of this equation. This would be a step in a program towards studying invariant characterization of non-trivial non-regular separation for a general PDE.

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Appendices

Appendix A

Dividing (3.45) by $\left(\frac{U^{(3)}}{f'U} - \frac{U''U'}{g'U^2}\right)'\left(\frac{V^{(3)}}{g'V} - \frac{V''V'}{g'V^2}\right)'$, we see that we can separate as follows

$$\frac{\left(\frac{\left(g''+2g\frac{V''}{V}-2g'\frac{V'}{V}\right)'}{g'}\right)'}{\left(\frac{V^{(3)}}{g'V} - \frac{V''V'}{g'V^2}\right)'} = -\frac{\left(\frac{\left(f''+2f\frac{U''}{U}-2f'\frac{U'}{U}\right)'}{f'}\right)'}{\left(\frac{U^{(3)}}{f'U} - \frac{U''U'}{f'U^2}\right)'} = C, \tag{1}$$

where $C \in \mathbb{R}$. Separating and integrating

$$\frac{\left(g''+2g\frac{V''}{V}-2g'\frac{V'}{V}\right)'}{g'} = \frac{C}{g'}\left(\frac{V''}{V}\right)' + C_1 \tag{2}$$

$$\frac{\left(f''+2f\frac{U''}{U}-2f'\frac{U'}{U}\right)'}{f'} = -\frac{C}{f'}\left(\frac{U''}{U}\right)' + D_1 \tag{3}$$

The case where one of $\left(\frac{U^{(3)}}{f'U} - \frac{U''U'}{g'U^2}\right)'$, $\left(\frac{V^{(3)}}{g'V} - \frac{V''V'}{g'V^2}\right)'$ vanish corresponds to setting $C = 0$ in one of the above equations. Multiplying through by f' and g' respectively and integrating once again we obtain

$$g'' + 2g\frac{V''}{V} - 2g'\frac{V'}{V} = C\frac{V''}{V} + C_1 g + C_2 \tag{4}$$

$$f'' + 2f\frac{U''}{U} - 2f'\frac{U'}{U} = -C\frac{U''}{U} + D_1 f + D_2, \tag{5}$$

where $C_1, C_2, D_1.D_2 \in \mathbb{R}$. Or more compactly

$$(2g-C)\frac{V''}{V} - 2g'\frac{V'}{V} + g'' - C_1 g - C_2 = 0 \tag{6}$$
$$(2f+C)\frac{U''}{U} - 2f'\frac{U'}{U} + f'' - D_1 f - D_2 = 0 \tag{7}$$

We can now separate (3.49)

$$\frac{U^{(4)}}{U} + C_1\frac{U''}{U} - 3\lambda f^2 = \alpha f + \beta \tag{8}$$
$$\frac{V^{(4)}}{V} + D_1\frac{V''}{V} - 3\lambda g^2 = -\alpha g + \gamma \tag{9}$$

The condition $K = 0$ separates into

$$\frac{f^{(3)}}{f'} = -\frac{g^{(3)}}{g'} = k, \tag{10}$$

where $k \in \mathbb{R}$ is a separation constant. Integrating once

$$f'' = kf + D \qquad g'' = -kg + D \tag{11}$$

These are extra conditions that have to be taken into account. We also need to account for the derivatives of equation (6) and (7)

$$\frac{U''}{U} = \frac{2f'}{2f+C}\frac{U'}{U} + \frac{D_1 f + D_2 - f''}{2f+C} \tag{12}$$
$$\frac{U^{(3)}}{U} = \frac{D_1 f + D_2 + f''}{2f+C}\frac{U'}{U} + \frac{D_1 - k}{2f+C}f' \tag{13}$$
$$\frac{U^{(4)}}{U} = \frac{D_1}{2f+C}f'' + \frac{D_1 f + D_2 + f''}{2f+C}\frac{U''}{U} - \frac{2f'}{2f+C}\frac{U^{(3)}}{U} \tag{14}$$
$$= \frac{(D_1-k)f''}{2f+C} + \frac{(D_1 f + D_2)^2 - f''^2}{(2f+C)^2} - \frac{2f'^2}{(2f+C)^2}(D_1 - k) \tag{15}$$
$$\frac{V''}{V} = \frac{2g'}{2g-C}\frac{V'}{V} + \frac{C_1 g + C_2 + g''}{2g-C} \tag{16}$$
$$\frac{V^{(3)}}{V} = \frac{C_1 g + C_2 + g''}{2g-C}\frac{V'}{V} + \frac{C_1 + k}{2g-C}g' \tag{17}$$
$$\frac{V^{(4)}}{V} = \frac{C_1}{2g-C}g'' + \frac{C_1 g + C_2 - g''}{2g-C}\frac{V''}{V} - \frac{2g'}{2g-C}\frac{V^{(3)}}{V} \tag{18}$$
$$= \frac{(C_1+k)g''}{2g+C} + \frac{(C_1 g + C_2)^2 - g''^2}{(2g-C)^2} - \frac{2g'^2}{(2g-C)^2}(C_1 + k) \tag{19}$$

Using (12) to eliminate higher derivatives in (8)

$$(C_1+D_1)\frac{2f'}{2f+C}\frac{U'}{U}-2\frac{f'^2}{(2f+C)^2}(D_1-k)+(D_1-k)\frac{f''}{2f+C}+\frac{(D_1f+D_2)^2-f''^2}{(2f+C)^2}+$$
$$+C_1\frac{D_1f+D_2-f''}{2f+C}-3\lambda f^2=\alpha f+\beta \tag{20}$$
$$(C_1+D_1)\frac{2g'}{2g-C}\frac{V'}{V}-2\frac{g'^2}{(2g-C)^2}(C_1+k)+(C_1+k)\frac{g''}{2g-C}+\frac{(C_1g+C_2)^2-g''^2}{(2g-C)^2}+$$
$$+D_1\frac{C_1g+C_2-g''}{2g-C}-3\lambda g^2=-\alpha g+\gamma \tag{21}$$

Substituting (8) into (3.50) we have the following

$$-2f'\frac{U^{(3)}}{U}-2g'\frac{V^{(3)}}{V}+((k-C_1)f+C_2+D)\frac{U''}{U}+(D_2+D-(k+D_1)g)\frac{V''}{V}$$
$$+2\lambda(f^3+g^3)+\alpha(f^2-g^2)+(\beta+\gamma)(f+g)=0 \tag{22}$$

Separating this equation we have

$$-2f'\frac{U^{(3)}}{U}+((k-C_1)f+C_2+D)\frac{U''}{U}+2\lambda f^3+\alpha f^2+(\beta+\gamma)f=\delta \tag{23}$$
$$-2g'\frac{V^{(3)}}{V}+(-(k+D_1)g+D_2+D)\frac{V''}{V}+2\lambda g^3-\alpha g^2+(\beta+\gamma)g=-\delta \tag{24}$$

For some $\delta\in\mathbb{R}$. Eliminating the derivatives from (23) we have the following

$$2f'\frac{C_2-D_2-(C_1+D_1)f}{2f+C}\frac{U'}{U}-2\frac{(D_1-k)}{2f+C}f'^2 \tag{25}$$
$$+((k-C_1)f+C_2+D)\left(\frac{(D_1-k)f+D_2-D}{2f+C}\right)$$
$$+2\lambda f^3+\alpha f^2+(\beta+\gamma)f=\delta \tag{26}$$

Isolating for $\frac{2f'}{2f+C}\frac{U'}{U}$ from (20) assuming $C_1+D_1\neq 0$ and using this to eliminate derivatives in (25)

$$
\begin{aligned}
&\frac{(C_2 - D_2 - (C_1 + D_1)f)}{C_1 + D_1}\left(\alpha f + \beta + 3\lambda f^2 - \frac{(k + C_1)(D_1 - k)f + C_1(D_2 - D) + D(D_1 - k)}{2f + C}\right.\\
&\left. + 2\frac{f'^2}{(2f + C)^2}(D_1 - k) - \frac{(D_1 f + D_2)^2 - f''^2}{(2f + C)^2}\right) - 2\frac{(D_1 - k)}{2f + C}f'^2\\
&+ ((k - C_1)f + C_2 + D)\left(\frac{(D_1 - k)f + D_2 - D}{2f + C}\right) + 2\lambda f^3 + \alpha f^2 + (\beta + \gamma)f = \delta
\end{aligned}
\tag{27}
$$

When $C_1 + D_1 = 0$, (20) implies $\lambda = 0$.

Appendix B

Case (ii): In this case we have

$$\left(\frac{1}{f'}\left(\frac{U''}{U}\right)'\right)' = 0, \quad \left(\frac{1}{g'}\left(\frac{V''}{V}\right)'\right)' \neq 0. \tag{28}$$

Integrating we get

$$\frac{U''}{U} = \alpha_5 f + \alpha_6, \tag{29}$$

where α_5, α_6 are constants. Equation (3.46) implies that

$$2f\frac{U''}{U} - 2f'\frac{U'}{U} + f'' = \alpha_3 f + \alpha_4 \tag{30}$$

α_3, α_4 are constants. Substitution for $\frac{U''}{U}$ from (29) yields

$$2f'U' = (f'' + 2\alpha_5 f^2 + (2\alpha_6 - \alpha_3)f - \alpha_4)U. \tag{31}$$

Differentiation of the above equation followed substitution for U'' from (29) and $2f'U'$ from (31) yields after simplification

$$(2\alpha_5 f^2 + (2\alpha_6 - \alpha_3)f - \alpha_4^2)^2 - f''^2 + 2f'f^{(3)} + 4\alpha_5 ff'^2 - \alpha_3 f'^2 = 0 \tag{32}$$

Using the relations between f and its derivatives

$$f'' = kf + D, \quad f'^2 = kf^2 + 2Df + \Lambda \tag{33}$$

Equation (32) becomes a polynomial in f, the coefficient of the highest power of f is $4\alpha_5^2$ which implies that $\alpha_5 = 0$. In view of the above equation (3.46) separates, the compatibility of the separated equation for U with (29) and (31) gives us that $\lambda = \alpha_5^2$, thus we conclude that $\lambda = 0$ in this case as well.

Case (iii): both of the following conditions hold:

$$\left(\frac{1}{f'}\left(\frac{U''}{U}\right)'\right)' = 0 \tag{34}$$

$$\left(\frac{1}{g'}\left(\frac{V''}{V}\right)'\right)' = 0. \tag{35}$$

The solutions of (34) and (35) are given by

$$\frac{U''}{U} = \alpha_5 f + \alpha_6 \tag{36}$$

$$\frac{V''}{V} = (\beta_5 f + \beta_6) \tag{37}$$

Computing derivatives of the above equations:

$$U^{(3)} = (\alpha_5 f + \alpha_6)U' + \alpha_5 f' U \tag{38}$$

$$U^{(4)} = 2\alpha_5 f' U' + (\alpha_5 f'' + (\alpha_5 f + \alpha_6)^2)U \tag{39}$$

$$V^{(3)} = (\beta_5 g + \beta_6)V' + \beta_5 g' V \tag{40}$$

$$V^{(4)} = 2\beta_5 g' V' + (\beta_5 g'' + (\beta_5 g + \beta_6)^2)V \tag{41}$$

With the use of the above derivatives the integrability condition (3.45) separates to yield the following equations:

$$2(\beta_5 - \alpha_5)f'U' = ((\alpha_5 + \beta_5)f'' + \alpha_5(\alpha_5 + 2\beta_5)f^2 \tag{42}$$

$$+ (2\alpha_6(\alpha_5 + \beta_5) - \alpha)f - 3\lambda f^2 + \alpha_6^2 - \alpha_7)U, \tag{43}$$

$$2(\alpha_5 - \beta_5)g'V' = ((\alpha_5 + \beta_5)g'' + \beta_5(\alpha_5 + 2\beta_5)g^2 \tag{44}$$

$$+ (2\beta_6(\alpha_5 + \beta_5) + \alpha)g - 3\lambda g^2 + \alpha_6^2 - \alpha_7)V, \tag{45}$$

where α is the separation constant. If $\beta_5 = \alpha_5$, (42) and (44) imply that

$$\lambda = \alpha_5^2, \quad \beta_5 = \alpha_5, \quad \beta_6 = -\alpha_6 \tag{46}$$

We conclude that (46) implies that $\phi(u, v) = U(u)V(v)$ defines a separable solution of the Helmholtz equation. If $\beta_5 \neq \alpha_5$, one differentiates (42) and (44) and eliminates all derivatives of U and V. One obtains polynomial equations in f and g which imply that $\beta_5^2 = \alpha_5^2$. The case $\beta_5 = \alpha_5$ has already been considered. The case $\beta_5 = -\alpha_5$, yields $3\lambda = -\alpha_5^2$, which is un-physical. This completes the proof of Case (iii).