Controller and Observer Designs for Partial Differential-Algebraic Equations

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapter 4, Chapter 5, and Chapter 6 describe my novel contributions. The content of these chapters is largely the papers listed below, with the following exceptions. Chapter 4 differs by rewording some parts and including additional detail in the proofs. Chapter 5 differs by adding more detail to the proof of Theorem 5.2.2. Some parts were reworded in Chapter 6, and the notations were changed throughout the chapter. I am the primary author of these papers. Professor Kirsten Morris contributed as a supervisor by suggesting the problems and some possible approaches, checking proofs, and editing the papers.

1. *Ala' Alalabi and Kirsten Morris, Stabilization of a parabolic-elliptic system via backstepping, IEEE Conference on Decision and Control (CDC), 2023. (See Chapter 4)*

2. *Ala' Alalabi and Kirsten Morris, Boundary control and observer design via backstepping for a coupled parabolic-elliptic system, preprint, 2023. Submitted to Automatica. (See Chapter 4)*

3. *Ala' Alalabi and Kirsten Morris, Enhanced stability conditions associated with stabilization and estimator design for a coupled parabolic-elliptic system. Submitted to 2024 Conference on Decision and Control. (See Chapter 4)*

4. *Ala' Alalabi and Kirsten Morris, Finite-time LQ optimal control of partial differential-algebraic equations, preprint, 2024. Submitted to IEEE Transactions on Automatic Control. (See Chapter 5)*

5. *Ala' Alalabi and Kirsten Morris, Linear-quadratic control on a finite-horizon for higher-index differential-algebraic equations. To appear in the American Control Conference (ACC) in 2024. (See Chapter 6)*

Abstract

Partial differential-algebraic equations (PDAEs) arise in numerous situations, including the coupling between differential-algebraic equations (DAEs) and partial differential equations (PDEs). They also emerge from the coupling of partial differential equations where one of the equations is in equilibrium, as seen in parabolic-elliptic systems. Stabilizing PDAEs and achieving certain performance necessitate sophisticated controller designs. Although there are well-developed controllers for each of PDEs and DAEs, research into controllers for PDAEs remains limited. Discretizing PDAEs to DAEs or reducing PDAE systems to PDEs, when feasible, often results in undesirable outcomes or a loss of the physical meaning of the algebraic constraints. Consequently, this thesis concentrates on the direct design of controllers based on PDAEs, using two control techniques: linear-quadratic and boundary control.

The thesis first addresses the stabilization of coupled parabolic-elliptic systems, an important class of PDAEs with broad applications in fields such as biology, incompressible fluid dynamics, and electrochemical processes. Even when the parabolic equation is exponentially stable on its own, the coupling between the two equations can cause instability in the overall system. A backstepping approach is used to derive a boundary control input to stabilize the system, resulting in an explicit expression for the control law in a state-feedback form. Since the system state is not always available, exponentially convergent observers are designed to estimate the system state using boundary measurements. The observation error system is shown to be exponentially stable, again by employing a backstepping method. This leads to the design of observer gains in closed form. By integrating these observers with state feedback boundary control, the thesis also tackles the output feedback problem.

Next, the thesis considers finite-time linear-quadratic control of PDAEs that are radial with index 0; this corresponds to a nilpotency degree of 1. The well-known results for PDEs are generalized to this class of PDAEs. Here, the existence of a unique minimizing optimal control is established. In addition, a projection is used to derive a system of differential Riccati-like equation coupled with an algebraic equation, yielding the solution of the optimization problem in a feedback form. These equations, and hence the optimal control, can be calculated without constructing the projected PDAE.

Lastly, the thesis examines the linear-quadratic (LQ) control problem for linear DAEs of arbitrary index over a finite horizon. Without index reduction or a behavioral approach, it is shown that a certain projection can lead to the derivation of a differential Riccati equation, from which the optimal control is obtained. Numerical simulations are presented to illustrate the theoretical findings for each objective of the thesis.

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Lastly, but most importantly, to God for blessing me with all these incredible people and opportunities. We have less control over things than we think, and nothing would have proceeded without His will.

Dedication

To my parents Waddah Alalabi and Ghada Abu Hamad.

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Chapter 1

Introduction

Controller synthesis methods have a tremendous impact on the performance and stability of dynamical systems, playing a crucial role across various domains. For instance, in electrical power systems, control is necessary to regulate and maintain voltage and frequency stability. In thermo-acoustic combustion, lack of control can lead to combustion instabilities, manifesting in dangerous pressure oscillations that can damage the combustor structure or even cause explosions. A dramatic example of this occurred when a NASA rocket engine once partially destroyed itself due to combustion instabilities (*see Figure 1.1*) [92]. The significance of control extends to biological phenomena, such as chemotaxis phenomena, which describes the movement of cells in response to chemical stimuli (*see Figure 1.2*). Studies have shown that in the absence of control, chemotaxis can manifest instability in its dynamics, leading to spatially complex patterns in cell density and chemical concentration. These patterns can impact processes such as tissue development, tumor growth, and wound healing [93, 114]. Beyond stabilization, one of the primary control objectives is to enhance the system's response in a certain manner, such as optimally driving the system to equilibrium. This capability becomes crucial in scenarios where strategic outcomes are important, such as maximizing business profits with minimal energy input, achieving efficient orbital transfers with minimal fuel consumption, or optimizing the yield and purity of products in chemical processes. State estimation, on the other hand, plays a critical role in control systems and various applications, where information about a system's state is needed but not directly measurable. In electric vehicles, state estimation methods are used to determine the state of charge of lithium-ion cells. Further, accurate temperature estimation is essential during the development and operation of electric motors to prevent overheating, which can compromise the machine's condition and lead to early degradation and damage. State estimation is also used for weather prediction. All of the processes and

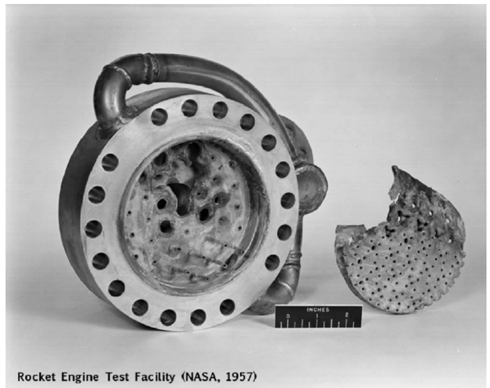


Figure 1.1: Rocket engine damaged due to combustion instabilities in the initial stages of the U.S. rocket program [92, Figure 1]. Such incidents highlight the need for controller designs to manage instabilities and enhance safety and efficiency in aerospace applications.

systems described previously rely fundamentally on mathematical models to simulate and study their behavior.

Partial differential equations (PDEs) are used to model systems where there is dependence on both space and time. They are vital to the mathematical study of diffusion processes, heat transfer and vibrations. Consequently, they are widely used across a diverse range of scientific disciplines. On the other hand, differential-algebraic equations (DAEs), also known as descriptor systems or implicit differential equations, are needed when modeling mechanical multi-body mechanisms and electrical circuits. DAE systems incorporate not just ordinary differential equations (ODEs), which describe the dynamic behavior of physical processes through rates of change with respect to one independent variable, typically time, but also algebraic equations that address constraints or conservation laws. Examples of such algebraic constraints include adherence to Kirchhoff's laws in electrical networks or mass points moving on constrained surfaces.

However, there are scenarios where neither PDEs nor DAEs alone can provide an accurate model. This limitation becomes apparent in the domain of electrical networks with semiconductor devices. Modeling the dynamics of these systems necessitates the coupling of differential-algebraic equations to describe the electrical networks and partial differential equations for modeling the semiconductor devices [3, 6, 5]. The interaction between pantograph and catenary [106], which is critical in railway trains, is also described by the coupling of a set of differential-algebraic equations (for the pantograph) and equations for strings and beams (for the catenary). Such systems are examples of partial differential-algebraic equations (PDAEs). In general, systems that involve both algebraic and partial

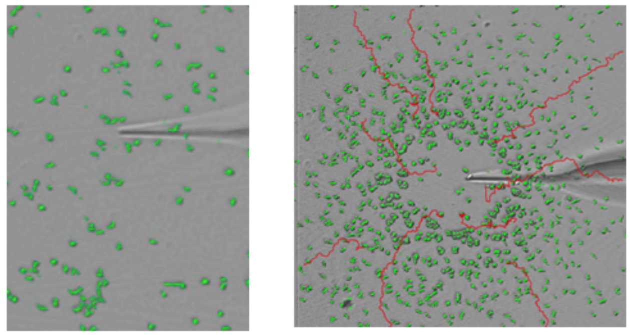


Figure 1.2: Description of chemotaxis phenomena: Cells are highlighted in green, with red lines depicting their trajectories in response to a chemical substance [54, Figure 2]. Chemotaxis is described by coupled parabolic-elliptic systems, where the parabolic equation models the cellular density while the elliptic equation models the concentration of a chemical attractant. Control is needed to avoid the development of complex spatial patterns that can affect many biological processes, such as tissue development and wound healing.

differential equations are known as partial differential-algebraic equations (PDAEs). Existing research has also referred to PDAEs as infinite-dimensional descriptors [99], singular distributed parameter systems [39, 41] or abstract DAEs [66]. Partial differential-algebraic equations can also emerge from the coupling of partial differential equations where one sub-system is in equilibrium, such as in coupled parabolic-elliptic systems. Here, elliptic equations serve as the algebraic constraints. As the name suggests, such coupled systems result from the coupling of both parabolic and elliptic equations. Parabolic equations depict time-dependent phenomena such as diffusion and heat transfer. Elliptic equations, on the other hand, model steady-state conditions like electrostatics and the flow of incompressible fluids. In general, coupling both equations is needed to describe dynamical models where transient and steady-state behaviors coexist, with applications ranging from modeling electro-chemical processes within lithium-ion cells [31, 132] to biological transport networks [69], piezo-electric beams with quasi-static magnetic effect [86] and Keller-Segel systems [45]. For instance, in electro-chemical modeling of lithium-ion cells [23], coupled parabolic-elliptic systems model the interaction between electrical potential (described by elliptic equations) and the dynamic movement of ions (described by parabolic equations). The Navier-Stokes equations in fluid dynamics with divergence-free flow is another example, where the requirement for the velocity field to be divergence-free (reflecting the medium's incompressibility) serves as an algebraic constraint [70]. Parabolic-elliptic systems are also used to model the fluid behavior in porous media within groundwater hydrodynamics, where the parabolic component addresses temporal variations in fluid density, and the elliptic component deals with spatial distributions influenced by gravitational and hydrodynamic forces [28]. Furthermore, such coupled systems also appear in ecological models for studying prey-predator dynamics and in semiconductor physics for modeling the interaction between charge carriers and electric potential in semiconductor devices [53, 82]. They also play a role in the research of various age-related diseases, including glaucoma, atherosclerosis, and Alzheimer's disease [15, 104]. Coupled parabolic-elliptic systems, offering a framework for a range of applications, may encounter instability issues. An illustration of such instability is given in [89]. Interestingly, coupling an exponentially stable parabolic equation with an elliptic equation may result in unstable dynamics. The instability can manifest in various forms, such as oscillations, divergence in solution, or sensitivity to initial and boundary conditions [131]. Addressing the instability in coupled parabolic-elliptic systems specifically, and in partial differential-algebraic equations more generally, is therefore a critical concern.

The development of controller designs for partial differential equations has been extensively explored e.g. [14, 32, 61, 67, 84] over the past decade. Similarly, research on differential-algebraic equations has achieved substantial advancements. Nevertheless, the area of controller synthesis for partial differential-algebraic equations remains notably underexplored with many open questions. This thesis is a step towards the exciting area of control for the general class of partial differential-algebraic equations, that also includes parabolic-elliptic systems.

Of particular interest in this thesis are two well-known controller designs for PDEs: the linear-quadratic (LQ) optimal control and state feedback boundary control. Optimal control involves designing a control strategy that achieves a desired objective by minimizing a defined system criterion. In infinite-dimensional control systems, research on optimal control began in the mid-1960s, and there has been a continuous interest in advancing this field ever since; see [32] and the references therein. An important aspect within optimal control theory is the linear-quadratic (LQ) approach, where system dynamics are represented by linear equations, and the objectives are defined by quadratic cost functions. Another significant control technique, feedback boundary stabilization, plays a crucial role in controlling dynamical systems by applying specific conditions at the system's boundaries to influence its internal state to achieve outcomes such as consistent temperature distribution, steady flow rates, or balanced chemical concentrations. State feedback boundary control requires the availability of the entire state or some parts of it. However, even with systems described by a single PDE, obtaining full state information for feedback is often impractical, as some states may not be directly measurable, and even when feasible, it can be costly. This issue has motivated the study of constructing an estimate of the state by designing an observer from boundary measurements. It also motivated combining both the state observer with the state feedback controller, giving rise to what is known as output feedback control.

One might consider simplifying a partial differential-algebraic equation into a partial differential equation by differentiating or solving the constraint equations to achieve explicit time derivative formulations. Even when possible, this might not always be the best approach. Simplifying the system could make it more sensitive to small changes or disturbances. Also, this process removes constraints from the equation, resulting in the loss of their physical significance. Numerical integration of the reduced system can introduce roundoff errors due to discretization, potentially leading to results that violate the original constraints. Historically, PDE controllers have been designed using finite-dimensional approximations of PDE models, a process known as early-lumping technique, with the hope that it will control the original system as intended. However, success is not always guaranteed. Similarly, when dealing with partial differential-algebraic equations, the dynamics can be indeed discretized into differential-algebraic equations, but this approach does not guarantee successful control of the original PDAE model. Therefore, a controller synthesis based directly on partial differential-algebraic equations is considered here.

To further complicate matters, there is a concept of index associated with PDAEs which serves as an indicator of the theoretical difficulty expected when working with these equations. Higher-index represents more constraints in the system. Unlike finite-dimensional DAEs, where various index definitions are equivalent, different definitions for PDAEs' indices exist in the literature and they are not equivalent. A notable difference from PDEs is the concept of consistent initialization [62]. This means that when applying the control, particularly to the algebraic part of the system, both the initial value and the control must meet a specific condition. Failure to satisfying this condition will lead to distributions in the solution of the PDAE. A straightforward way to understand this concept is to think of the simplest case on finite-dimensional space when a ODE is coupled with a single algebraic equation. At the initial time, both the control and state's initial value must satisfy the algebraic equation. Another challenge appears when dealing with the abstract formulation of PDAEs due to the non-invertibility of certain operators.

The aim of this thesis is to design controllers directly from the partial differential-algebraic equation system without first approximating it into a differential-algebraic equation model or reducing it to a partial differential equation (PDE) system. We approach this goal by first establishing a single feedback boundary control for a special class of PDAEs, that is, coupled parabolic-elliptic systems. We also design a state observer for these coupled systems using a single boundary measurement. Combining both the boundary controller and the state observer, we present an output feedback controller for this coupled system. Next, we extend the classical approach of linear-quadratic (LQ) control to index-1 PDAEs in Hilbert spaces. Finally, the natural progression of this research leads us to address higher-index systems, focusing particularly on those within finite-dimensional spaces.

The structure of this thesis is as follows. In Chapter 2, we recall some important definitions and concepts from system theory for PDEs. We also review some helpful mathematical tools in the context of LQ control for linear PDEs and a boundary control, namely, backstepping. In Chapter 3, we present a summary of the system theory for partial differential-algebraic equations in both finite and infinite-dimensional spaces. In particular, we present the framework that will be used to ensure the well-posedness of the class under study. The aim is for this chapter to serve as a comprehensive reference for readers interested in learning about PDAEs, providing valuable references. In Chapter 4, we present a single boundary controller and two observer designs for a class of parabolic-elliptic systems. In Chapter 5, we establish LQ controller design for a class of linear PDAEs, those with index-1. In Chapter 6, we study linear differential-algebraic equations without imposing any restrictions on the system's index. We present conditions for the solvability of optimization problems for such systems. Finally, we conclude the thesis with Chapter 7 by giving a brief summary and discussing on-going and possible future work related to the research in this thesis.

Chapter 2

System theory and control of linear partial differential equations

In this chapter, we consider the following abstract formulation of partial differential equations

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t), \tag{2.1a}$$

$$x(0) = x_i \in \mathcal{X}, \quad t \geq 0. \tag{2.1b}$$

Here, the state operator $A : D(A) \to \mathcal{X}$ and the input operator $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$. We also assume that the spaces $\mathcal{X}$, $\mathcal{U}$ are complex Hilbert spaces. The existence and uniqueness of solutions for system (2.1) have been well-established through the use of "C_0 semigroup" theory; see [32]. Conditions under which the operator A generates a C_0- semigroup were discussed. When it comes to designing controllers for partial differential equation models, many methods have emerged to achieve this objective [14, 32, 61, 67, 84]. The focus of this thesis will be on extending two methods, namely, linear-quadratic control and boundary state feedback control through backstepping to PDAEs. Therefore, this chapter is structured as follows: In Section 2.1, we introduce the concept of semigroups, which will be essential for understanding the subsequent sections. In Section 2.2, we discuss the main theoretical results for the well-posedness of system (2.1). In Section 2.3, we present some classical findings concerning linear-quadratic control for system (2.1). In Section 2.4, we present a brief overview of the backstepping approach for PDEs. In the last two sections of this chapter, Section 2.5 and Section 2.6, we consider a simple parabolic PDE and review controller and observer designs for this PDE using the backstepping method.

2.1 Semigroups of operators

In this section and the remaining parts of this thesis, we assume that the reader is familiar with the standard results of functional analysis, operator theory and state-space theory. We direct the reader to Appendix A for a brief overview of bounded linear operators. We refer the interested reader to [32] for additional information.

We begin this section by considering system (2.1) in a finite-dimensional setting, where A, B are complex matrices of appropriate dimensions and $\mathcal{X}$, $\mathcal{U}$ are finite-dimensional complex spaces. If $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times m}$, then assuming that $u(t) \in C([0,\infty);\mathbb{C}^m)$, the unique solution of the differential equation (2.1a) is

$$x(t) = e^{At}x_i + \int_0^t e^{A(t-s)}Bu(s)\,ds, \tag{2.2}$$

and $x(t)$ is a continuous function on $[0,\infty)$ with values in $\mathbb{C}^n$. Transitioning to infinite-dimensional spaces, semigroup theory becomes crucial in defining the concept of a "solution" for equation (2.1). Specifically, a "strongly continuous operator semigroup" serves as the natural extension of matrix exponential to infinite-dimensional systems. However, a key difference between the matrix exponential and a semigroup is that the operator A is not, in general, a bounded operator on the state space $\mathcal{X}$, nor is the domain of A the entire space. In what follows, we recall the definition of strongly continuous semigroup and some elementary properties. To set notation, we will denote the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{X}$ by $\mathcal{L}(\mathcal{X})$.

Definition 2.1.1 (Strongly continuous semigroup)**.** A strongly continuous semigroup (C_0-semigroup) is an operator-valued function $T(t)$ from $\mathbb{R}^0_+$ (the non-negative real line) to $\mathcal{L}(\mathcal{X})$ that satisfies the following properties

- $T(t+s) = T(t)T(s)$, for $t, s \geq 0$;
- $T(0) = I$;
- $\|T(t)x_i - x_i\| \to 0$ as $t \to 0^+$, for all $x_i \in \mathcal{X}$.

Definition 2.1.2. The infinitesimal generator A of a C_0-semigroup $\mathcal{X}$ is

$$Ax = \lim_{t\to 0^+} \frac{1}{t}(T(t) - I)x,$$

with domain $D(A) \subset \mathcal{X}$ being the set of elements $x \in \mathcal{X}$ for which the limit exists.

Theorem 2.1.1. *[32, Theorem 2.1.7] A strongly continuous semigroup $T(t)$ on a Hilbert space $\mathcal{X}$, possesses the following properties*

1. *$\|T(t)\|$ is bounded on every finite subinterval of $[0,\infty)$;*
2. *$T(t)$ is strongly continuous for all $t \in [0,\infty)$;*
3. *For all $x \in \mathcal{X}$, we have that $\dfrac{1}{t}\displaystyle\int_0^t T(s)x\,ds \to x$ as $t \to 0^+$;*
4. *If $\omega_0 = \inf\limits_{t>0}\left(\dfrac{1}{t}\log\|T(t)\|\right)$, then $\omega_0 = \lim_{t\to\infty}\left(\frac{1}{t}\log\|T(t)\|\right) < \infty$;*
5. *For all $\omega > \omega_0$, there exists a constant M_ω such that for all $t \geq 0$, $\|T(t)\| \leq M_\omega e^{\omega t}$.*

The resolvent operator, $(\lambda I - A)^{-1}$, associated with the infinitesimal generator A of a C_0-semigroup, is important in our applications. In fact, establishing specific resolvent estimates yields C_0- semigroup generation. These conditions are stated below in the Hille-Yosida theorem, which is vital for the characterization of infinitesimal generators.

Theorem 2.1.2. *(Hille–Yosida theorem) [32, Theorem 2.1.15] The closed, densely defined, linear operator $A : D(A) \subset \mathcal{X} \to \mathcal{X}$ is the generator of a C_0-semigroup $T(t)$ on $\mathcal{X}$ if and only if there exist real numbers M, ω such that for all $s > \omega,\ s \in \rho(A)$*

$$\left\|(sI - A)^{-k}\right\| \leq \frac{M}{(s-\omega)^k}, \quad k \geq 1.$$

In this case, $\|S(t)\| \leq Me^{\omega t}$.

Definition 2.1.3. The C_0-semigroup $T(t)$, for $t \geq 0$, is called a contraction semigroup if

$$\|S(t)\| \leq 1 \text{ for all } t \geq 0.$$

The Lumer-Phillips Theorem is a fundamental result in the theory of infinite-dimensional systems. It provides conditions that are more easily verifiable than those in the Hille-Yosida theorem.

Theorem 2.1.3 (Lumer-Phillips theorem)**.** *[32, Theorem 2.3.2] Let $A : D(A) \subset \mathcal{X} \to \mathcal{X}$ be a closed, densely defined operator on a Hilbert space $\mathcal{X}$. The operator A generates a contraction semigroup on $\mathcal{X}$ if and only if for all real $\omega > 0$*

$$\|(\omega I - A)x\| \geq \omega\|x\|, \text{ for all } x \in D(A),$$

and

$$\|(\omega I - A^*)x\| \geq \omega\|x\|, \text{ for all } x \in D(A^*),$$

where A^* denotes the adjoint of A.

2.2 Well-posedness of controlled partial differential equations

It is important to define what is meant by a solution to a linear controlled equation (2.1) and to establish what is meant by the term "well-posed".

Definition 2.2.1 (Classical Solution)**.** [32, 26, Definition 3.1.1] , [90, [Definition 3.1.1] A function $x : [0, t_f) \to \mathcal{X}$ is a classical solution to equation (2.1) on the interval $[0, t_f)$ if $x(t)$ is continuous on $[0, t_f)$, continuously differentiable on $(0, t_f)$, $x(t) \in D(A)$ for all $0 < t < t_f$, and satisfies equation (2.1).

Definition 2.2.2. If a unique solution to the differential equation (2.1) exists and it depends continuously on the initial condition, then the equation is said to be well-posed.

Establishing that the operator A generates a C_0-semigroup is equivalent to showing well-posedness of system (2.1).

Theorem 2.2.1. *[32, Theorem 5.1.3][90, Theorem 2.4] Let A be the infinitesimal generator of a C_0-semigroup $T(t)$ on $\mathcal{X}$. If $u(t) \in C^1([0, t_f]; \mathcal{X})$ and $x_i \in D(A)$, then*

$$x(t) = T(t)x_i + \int_0^t T(t-s)Bu(s)ds,$$

is the unique classical solution of (2.1)*.*

It is useful to consider a weaker definition of a solution that can be used for all initial conditions.

Definition 2.2.3 (Mild Solution)**.** [32, Definition 3.1.4] Let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on $\mathcal{X}$, with $x_i \in \mathcal{X}$ and $u(t) \in L_1([0, t_f]; \mathcal{U})$. The state trajectory $x(t)$

$$x(t) = T(t)x_i + \int_0^t T(t-s)Bu(s)\,ds,$$

is a mild solution to equation (2.1).

In practice, it is common to have $u(t) \in L_2([0, t_f]; \mathcal{U})$ which implies that $u(t) \in L_1([0, t_f]; \mathcal{U})$.

2.3 Linear-quadratic control

Linear-quadratic control is a common method in feedback controller design. This control is derived by minimizing a quadratic cost function that has penalties on the system's state and the control input. In this section, we present some classical results on finite-time linear-quadratic control for linear partial differential equations, where one is interested in controlling a process over a specified time-frame. These results are a direct generalization of the results for finite-dimensional linear-quadratic control [84]. The material in this section can be found in [32, Chapter 9].

Consider the linear system (2.1). Let A, with domain $D(A)$, be the generator of a C_0-semigroup $T(t)$ on $\mathcal{X}$. Let $R \in \mathcal{L}(\mathcal{U})$ be a coercive operator on $\mathcal{U}$, that is, R is self-adjoint, and $R \geq \epsilon I$ for some $\epsilon > 0$. Let $Q \in \mathcal{L}(\mathcal{X})$, $G \in \mathcal{L}(\mathcal{X})$ be non-negative, self-adjoint operators. For an arbitrary initial condition x_i, the objective of linear quadratic optimal control over a finite-time horizon is to minimize the cost function

$$J(x_i, u; t_f) = \langle x(t_f), Gx(t_f)\rangle_{\mathcal{X}} + \int_0^{t_f} \langle x(t), Qx(t)\rangle_{\mathcal{X}} + \langle u(t), Ru(t)\rangle_{\mathcal{U}}\, dt,$$

over the set of admissible controls $u(t) \in L_2([0, t_f]; \mathcal{U})$. The optimal control problem can be written concisely as

$$\min_{u \in L_2([0,t_f];\mathcal{U})} J(x_i, u; t_f), \tag{2.3}$$

subject to the dynamics of (2.1).

Lemma 2.3.1. *[32, Lemma 9.1.6] Let $u^{opt}(t)$ be the minimizing input function for the control problem (2.3) and let $x^{opt}(t)$ be its corresponding state trajectory. For $t \in [0, t_f]$, the following holds:*

$$u^{opt}(t) = -R^{-1}B^*\Big(T^*(t_f - t)Gx^{opt}(t) + \int_t^{t_f} T^*(s - t)Qx^{opt}(s)\, ds\Big).$$

For $t_1 \in [0, t_f]$, we use the notation $x^{opt}(t; x_i, t_1, t_f)$ to denote the state trajectory corresponding to the input function $u^{opt}(t; x_i, t_1, t_f)$ that minimizes the cost functional

$$J(x_i, u; t_1, t_f) = \langle x(t_f), Gx(t_f)\rangle_{\mathcal{X}} + \int_{t_1}^{t_f} \langle x(t), Qx(t)\rangle_{\mathcal{X}} + \langle u(t), Ru(t)\rangle_{\mathcal{U}}\, dt.$$

The next lemma will prove useful in obtaining the results in Chapter 5.

Lemma 2.3.2. *[32, Lemma 9.1.9] Let $x^{opt}(t)$ be the optimal state trajectory in Lemma 2.3.1. For a given $t \in [0, t_f]$, we define the following operator on $\mathcal{X}$*

$$\Pi(t)x_i = T^*(t_f - t)Gx^{opt}(t_f; x_i, t, t_f) + \int_t^{t_f} T^*(s-t)Qx^{opt}(s; x_i, t, t_f)\, ds.$$

This operator has the following properties:

(a) $\Pi(t) \in \mathcal{L}(\mathcal{X})$ *for all* $t \in [0, t_f]$.

(b) The following relationships hold between the optimal state and the optimal input trajectory:

$$u^{opt}(t) = -R^{-1}B^*\Pi(t)x^{opt}(t). \tag{2.4}$$

(c) The following relationship holds between the minimum cost and $\Pi(t_f)$*:*

$$J(x_i, u^{opt}; t_f) = \langle x_i, \Pi(t_f)x_i \rangle_{\mathcal{X}}.$$

(d) $\Pi(t_f)$ *is a self-adjoint, non-negative operator.*

(e) $\Pi(\cdot)$ *is strongly continuous from the right in* $[0, t_f]$*, i.e.,*

$$\lim_{h \downarrow 0} \Pi(t+h)x_i = \Pi(t)x_i$$

for all $x_i \in \mathcal{X}$ *and* $t \in [0, t_f]$.

As for finite-dimensional systems, the optimal control (2.4) is a time-varying feedback operator. The trajectory of a time-varying system requires defining the extension of a strongly continuous semigroup to a mild evolution operator.

Definition 2.3.1. [32, Definition 5.3.4] For $\tau > 0$, define $\Delta = \{(s,t) \,|\, 0 \leq s \leq t \leq \tau\}$. The operator-valued family $U(t,s) : \Delta \to \mathcal{L}(\mathcal{X})$ is a family of mild evolution operators if

- $U(s,s) = I$, for all $s \in [0, \tau]$;
- $U(t,r)U(r,s) = U(t,s)$, for all $s \leq r \leq t \leq \tau$;
- $U(\cdot, s)$ is continuous on $[s, \tau]$ and $U(t, \cdot)$ is continuous on $[0, t]$.

The optimal state trajectory $x^{opt}(t)$ is the mild solution to an abstract evolution equation, as stated below.

Corollary 2.3.1. *The operator $A - BR^{-1}B^*\Pi(\cdot)$ generates the mild evolution operator $U(t,s)$ on the set $\{(t,s) \mid 0 \le s \le t \le t_f\}$. Furthermore,*

$$x^{opt}(t; x_i, 0, t_f) = U(t,0)x_i.$$

The optimal control (2.4) is a feedback control. It can be obtained by solving a differential Riccati equation.

Theorem 2.3.1. *[32, Theorem 9.1.15, Corollary 9.2.11] Consider the differential Riccati equation*

$$\begin{aligned}
&\frac{d}{dt}\langle x_1, \Pi(t)x_2\rangle_{\mathcal{X}} + \langle x_1, A^*\Pi(t)x_2\rangle_{\mathcal{X}} + \langle x_1, \Pi(t)Ax_2\rangle_{\mathcal{X}} + \langle x_1, Qx_2\rangle_{\mathcal{X}} \\
&- \langle x_1, \Pi(t)BR^{-1}B^*\Pi(t)x_1\rangle_{\mathcal{X}} = 0, \quad x_1, x_2 \in D(A), \\
&\Pi(t_f) = G.
\end{aligned} \tag{2.5}$$

Equation (2.5) has a unique solution $\Pi(t) \in \mathcal{L}(\mathcal{X})$ in the class of continuous, self-adjoint operators on $\mathcal{X}$ such that $\langle x_1, \Pi(t)x_2\rangle_{\mathcal{X}}$ is differentiable for $x_1, x_2 \in D(A)$.

2.4 Backstepping for partial differential equations

Backstepping is an important technique in designing boundary control for partial differential equations. It is one of the few methods that yields an explicit control law for PDEs without first approximating the PDE. Backstepping relies on the use of transformations which are generally formulated as a Volterra operator, guaranteeing under weak conditions the invertibility of the transformation. This transformation converts the original system into a *"target"* system, which has desirable stability characteristics. Then, the controller design is derived through making sure that the boundary conditions of both original and target systems are consistent. To determine the backstepping transformation, it is necessary for the gain *"kernel"* functions of these transformations to satisfy certain partial differential equations. Meanwhile, the stabilizing state feedback controllers can be obtained by using the solutions of these kernel functions together with the full state information. Due to the invertibility and specific regularity conditions of the established transformations, the closed-loop control system with the inclusion of the derived feedback control, is guaranteed to exhibit stability properties similar to those of the target system. Lyapunov theory is often used to examine the stability the target system. This involves finding a suitable Lyapunov function, often selected as the system's energy function, to determine stability without the knowledge of the solution. Backstepping has proven effective not just for boundary stabilization of PDEs but also for observer design with boundary measurements, the dual problem of the boundary control.

Many challenges accompany controller design using backstepping. Primarily, for an equivalence relation to be established between the target and original systems, the spatial transformation must be invertible. The invertibility of a Volterra integral transformation of the second kind is relatively straightforward. Nonetheless, identifying an appropriate kernel, particularly in the context of coupled equations where one is static, poses a significant technical difficulty. This issue represents the principal technical challenge of this method and will be addressed in detail subsequently.

Originally used to study simple one-dimensional parabolic and hyperbolic PDEs [60, 107, 108], PDE's backstepping has been successfully extended to more complex systems such as the linearized Kortweg-de Vries and Kuramoto-Sivashinsky equations [22, 61], the Timoshenko beam [24, 59], and PDEs on arbitrary-dimensional balls [128]. The application of this method has expanded to include coupled systems of PDEs [8, 9, 18, 25, 26, 49, 126, 127] as well as PDE-ODE systems [33, 47, 57, 113]. Further applications of this method to parabolic PDEs with Volterra nonlinearities can be found in [122, 124]. For more detail on PDE's backstepping and Lyapunov theory, some good textbooks are [11, 56, 61].

2.5 Backstepping controller for parabolic PDEs

In this section, we present a backstepping controller design for a simple unstable parabolic partial differential equation, specifically the reaction-diffusion equation. This will provide the reader with a general understanding of the backstepping approach for PDEs. Further details and discussions on this topic can be found in [61, Chapter 4] and [107].

Consider the following unstable reaction-diffusion equation

$$w_t(x,t) = w_{xx}(x,t) + \rho w(x,t), \tag{2.6a}$$
$$w_x(0,t) = 0, \tag{2.6b}$$
$$w_x(1,t) = u(t), \tag{2.6c}$$

where ρ is an arbitrary constant and $u(t)$ is the control input. The open-loop system, i.e. $u(t) = 0$, is unstable for sufficiently large ρ. Since the term that causes instability in (2.6) is $\rho w(x,t)$, we seek a backstepping transformation that transforms system (2.6) into a target system where this destabilizing term is eliminated. A possible choice for the target system is the following.

Theorem 2.5.1. *For $c_1 + \rho > 0$, system*

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) - (c_1+\rho)\tilde{w}(x,t), \tag{2.7a}$$
$$\tilde{w}_x(0,t) = 0, \tag{2.7b}$$
$$\tilde{w}_x(1,t) = 0, \tag{2.7c}$$

is exponentially stable.

Proof. To show the exponential stability of system (2.7), we use Lyapunov theory. Define the following Lyapunov candidate

$$V(t) = \int_0^1 \tilde{w}^2(x,t)dx. \tag{2.8}$$

Taking the time derivative of $V(t)$,

$$\begin{aligned}\dot{V}(t) &= \int_0^1 \tilde{w}(x,t)[\tilde{w}_{xx}(x,t) - (c_1+\rho)\tilde{w}(x,t)]dx \\ &= \int_0^1 \tilde{w}(x,t)\tilde{w}_{xx}(x,t)dx - (c_1+\rho)\int_0^1 \tilde{w}^2(x,t)dx. \end{aligned}\tag{2.9}$$

Integrating by parts and using the boundary conditions (2.7b) and (2.7c), we obtain

$$\begin{aligned}\dot{V}(t) &= \tilde{w}(1,t)\tilde{w}_x(1,t) - \tilde{w}(0,t)\tilde{w}_x(0,t) - \int_0^1 \tilde{w}_x^2(x,t)dx - (c_1+\rho)\int_0^1 \tilde{w}^2(x,t)dx \\ &= -\int_0^1 \tilde{w}_x^2(x,t)dx - (c_1+\rho)\int_0^1 \tilde{w}^2(x,t)dx. \end{aligned}\tag{2.10}$$

Noting that $-\int_0^1 \tilde{w}_x^2(x,t)dx < 0$, we arrive at

$$\dot{V}(t) \le -(c_1+\rho)\int_0^1 \tilde{w}^2(x,t)dx, \tag{2.11}$$

and so

$$V(t) \le e^{-2(c_1+\rho)t}V(0). \tag{2.12}$$

If $c_1 > -\rho$, then $V(t)$ and $\|\tilde{w}(\cdot,t)\|$ decay exponentially with rate $2(c_1+\rho)$. □

We seek a backstepping transformation of the form

$$\tilde{w}(x,t) = w(x,t) - \int_0^x k^a(x,y)w(y,t)\,dy,$$

that will lead to the target system (2.7). To establish this transformation, we must show that the function $k^a(x,y)$, known as the gain kernel, is well-defined as the solution of an auxiliary PDE. The following notation will be useful.

$$k_x^a(x,x) := \frac{\partial}{\partial x}k^a(x,y)\bigg|_{y=x}, \tag{2.13a}$$

$$k_y^a(x,x) := \frac{\partial}{\partial y}k^a(x,y)\bigg|_{y=x}, \tag{2.13b}$$

$$\frac{d}{dx}k^a(x,x) = k_x^a(x,x) + k_y^a(x,x). \tag{2.13c}$$

The following lemma is essential for defining the backstepping transformation. Although the proof is similar to that found in [61, Chapter 4, Sections 4.3 & 4.4], it is included here for completeness.

Lemma 2.5.1. *Let $c_1 > 0$ be a real constant. For $(x,y) \in \mathcal{T} = \{x, y : 0 < y < x < 1\}$, the partial differential equation*

$$k_{xx}^a(x,y) - k_{yy}^a(x,y) - c_1 k^a(x,y) = 0, \tag{2.14a}$$

$$k_y^a(x,0) = 0, \quad k^a(x,x) = -\frac{1}{2}c_1 x, \tag{2.14b}$$

has a unique twice continuously differentiable solution, i.e., in $C^2(\mathcal{T})$,

$$k^a(x,y) = -c_1 x \frac{I_1\left(\sqrt{c_1(x^2-y^2)}\right)}{\sqrt{c_1(x^2-y^2)}}, \tag{2.15}$$

where $I_1(\cdot)$ is the modified Bessel function of first-order defined as

$$I_1(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{2m+1}}{m!(m+1)!}. \tag{2.16}$$

Proof. First, we convert equation (2.14a) into an integral equation. To do so, we introduce the change of variables

$$\xi = x + y, \qquad \eta = x - y, \tag{2.17}$$

and so the statements in (2.13) lead to

$$k^a(x,y) = G(\xi,\eta) = k^a\Big(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\Big), \tag{2.18a}$$

$$k^a_x = G_\xi + G_\eta, \tag{2.18b}$$

$$k^a_{xx} = G_{\xi\xi} + 2G_{\xi\eta} + G_{\eta\eta}, \tag{2.18c}$$

$$k^a_y = G_\xi - G_\eta, \tag{2.18d}$$

$$k^a_{yy} = G_{\xi\xi} - 2G_{\xi\eta} + G_{\eta\eta}. \tag{2.18e}$$

Thus, system (2.14) can be written as

$$G_{\xi\eta}(\xi,\eta) = \frac{c_1}{4}G(\xi,\eta), \tag{2.19}$$

$$G_\xi(\xi,\xi) = G_\eta(\xi,\xi), \tag{2.20}$$

$$G(\xi,0) = -\frac{c_1}{4}\xi, \tag{2.21}$$

where $(\xi,\eta) \in \{\xi,\eta : 0 < \xi < 2, 0 < \eta < \min(\xi, 2-\xi)\}$. Integrating equation (2.19) with respect to η from 0 to η and using (2.21),

$$\begin{aligned} G_\xi(\xi,\eta) &= G_\xi(\xi,0) + \int_0^\eta \frac{c_1}{4}G(\xi,s)\,ds \\ &= -\frac{c_1}{4} + \frac{c_1}{4}\int_0^\eta G(\xi,s)\,ds. \end{aligned} \tag{2.22}$$

Integrating (2.22) with respect to ξ from η to ξ to find $G(\xi,\eta)$, we obtain

$$G(\xi,\eta) = G(\eta,\eta) - \frac{c_1}{4}(\xi-\eta) + \frac{c_1}{4}\int_\eta^\xi \int_0^\eta G(\tau,s)\,ds\,d\tau. \tag{2.23}$$

It remains to find an expression for $G(\eta,\eta)$, which appears on the right-hand-side of (2.23). To do so, we refer to (2.20) to obtain

$$\begin{aligned} \frac{d}{d\xi}G(\xi,\xi) &= G_\xi(\xi,\xi) + G_\eta(\xi,\xi) \\ &= 2G_\xi(\xi,\xi). \end{aligned} \tag{2.24}$$

Using (2.22) with $\eta = \xi$, we can find an expression for $G_\xi(\xi,\xi)$. Then, equation (2.24) yields

$$\frac{d}{d\xi}G(\xi,\xi) = -\frac{c_1}{2} + \frac{c_1}{2}\int_0^\xi G(\xi,s)\,ds. \tag{2.25}$$

Integrating the previous equation from 0 to η with respect to ξ and using (2.21), we obtain

$$G(\eta,\eta) = -\frac{c_1}{2}\eta + \frac{c_1}{2}\int_0^{\eta}\int_0^{\tau} G(\tau,s)\,ds\,d\tau. \tag{2.26}$$

Substituting (2.26) into equation (2.23), we obtain

$$G(\xi,\eta) = -\frac{c_1}{2}\eta - \frac{c_1}{4}(\xi-\eta) + \frac{c_1}{2}\int_0^{\eta}\int_0^{\tau} G(\tau,s)\,dsd\tau + \frac{c_1}{4}\int_{\eta}^{\xi}\int_0^{\eta} G(\tau,s)\,ds\,d\tau. \tag{2.27}$$

Now, we solve the integral equation (2.27) using successive approximations. Let us start with an initial guess:

$$G^0(\xi,\eta) = 0, \tag{2.28}$$

and set up the recursive formula for (2.24) as follows

$$G^{n+1}(\xi,\eta) = -\frac{c_1}{4}(\xi+\eta) + \frac{c_1}{2}\int_0^{\eta}\int_0^{\tau} G^n(\tau,s)\,dsd\tau + \frac{c_1}{4}\int_0^{\xi}\int_0^{\eta} G^n(\tau,s)\,ds\,d\tau. \tag{2.29}$$

If this converges, we can write the solution $G(\xi,\eta)$ as

$$G(\xi,\eta) = \lim_{n\to\infty} G^n(\xi,\eta). \tag{2.30}$$

Denote the difference between two consecutive terms as

$$\Delta G^n(\xi,\eta) = G^{n+1}(\xi,\eta) - G^n(\xi,\eta). \tag{2.31}$$

Then,

$$\Delta G^{n+1}(\xi,\eta) = \frac{c_1}{2}\int_0^{\eta}\int_0^{\tau} \Delta G^n(\tau,s)\,dsd\tau + \frac{c_1}{4}\int_0^{\xi}\int_0^{\eta} \Delta G^n(\tau,s)\,ds\,d\tau, \tag{2.32}$$

and statement (2.30) can be alternatively written as

$$G(\xi,\eta) = \sum_{n=0}^{\infty} \Delta G^n(\xi,\eta). \tag{2.33}$$

Computing ΔG^n from (2.31) starting with

$$\Delta G^0 = G^1(\xi,\eta) = -\frac{c_1}{4}(\xi+\eta). \tag{2.34}$$

It can be proved by induction that

$$\Delta G^n(\xi,\eta) = -\frac{(\xi+\eta)\xi^n\eta^n}{n!(n+1)!}\left(\frac{c_1}{4}\right)^{n+1}. \tag{2.35}$$

Thus, referring to (2.33), the solution to the integral equation is

$$G(\xi,\eta) = -\sum_{n=0}^{\infty}\frac{(\xi+\eta)\xi^n\eta^n}{n!(n+1)!}\left(\frac{c_1}{4}\right)^{n+1}. \tag{2.36}$$

Comparing the expression of first-order modified Bessel function of the first kind (2.16) with (2.36),

$$G(\xi,\eta) = -\frac{c_1}{2}(\xi+\eta)\frac{I_1\left(\sqrt{c_1\xi\eta}\right)}{\sqrt{c_1\xi\eta}}. \tag{2.37}$$

Referring to (2.17) and (2.18a), it follows that $k^a(x,y)$ is given by (2.15). □

There are several key points to highlight here. First, as mentioned in Lemma 2.5.1, the gain kernel PDE operates within a specific domain. This domain is triangular, defined by $0 < y < x < 1$, and is depicted in Figure 2.1. The boundary conditions in (2.14b) are specified along two sides of this triangle. The hyperbolic PDE (2.14) possesses a unique solution that is twice continuously differentiable. The rationale for transforming the PDE into an integral equation is that the latter format simplifies analysis using certain tools and allows for the derivation of solutions in terms of the modified Bessel function of first-order.

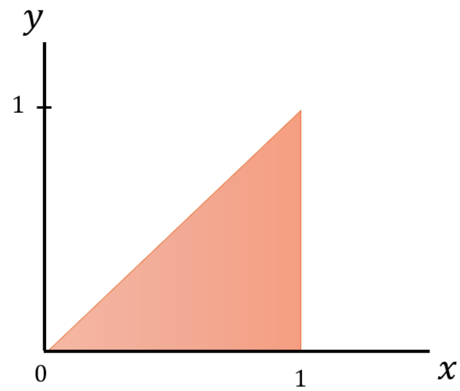


Figure 2.1: The figure presents the domain of the gain kernel PDE $k^a(x,y)$ that solves (2.14). This domain is triangular, defined by $0 < y < x < 1$, and the boundary conditions (2.14b) are specified along two sides of this triangle. The gain kernel $k^a(x,y)$ is twice continuously differentiable and will be used to define an invertible backstepping transformation. Further, the gain kernel $k^a(x,y)$ can be calculated using the modified Bessel function of first-order, and this facilitates obtaining an expression for the feedback boundary control that is easy to evaluate.

Using the gain kernel $k^a(x,y)$, we are now in a position to present the backstepping transformation that will transform the original system (2.6) into the target system (2.7). As a byproduct of applying the transformation on the boundaries of the system, we obtain an expression for the control $u(t)$, and in a feedback form. This is explained in detail in the next theorem.

Theorem 2.5.2. *Let $k^a(x,y)$ be the solution of system (2.14). If the feedback control is*

$$u(t) = \int_0^1 k_x^a(1,y)w(y,t)dy + k^a(1,1)w(1,t), \tag{2.38}$$

then the backstepping transformation

$$\tilde{w}(x,t) = w(x,t) - \int_0^x k^a(x,y)w(y,t)\,dy, \tag{2.39}$$

transforms system (2.6) into the exponentially stable target system (2.7).

Proof. It will prove useful to rewrite (2.39) as

$$w(x,t) = \tilde{w}(x,t) + \int_0^x k^a(x,y)w(y,t)dy. \tag{2.40}$$

We differentiate (2.40) with respect to x twice

$$w_x(x,t) = \tilde{w}_x(x,t) + \int_0^x k_x^a(x,y)w(y,t)dy + k^a(x,x)w(x,t), \tag{2.41}$$

$$\begin{aligned} w_{xx}(x,t) = \tilde{w}_{xx}(x,t) + \int_0^x k_{xx}^a(x,y)w(y,t)dy + k_x^a(x,x)w(x,t) + \frac{d}{dx}k^a(x,x)w(x,t) \\ + k^a(x,x)w_x(x,t), \end{aligned} \tag{2.42}$$

and with respect to t

$$\begin{aligned} w_t(x,t) &= \tilde{w}_t(x,t) + \int_0^x k^a(x,y)w_t(y,t)dy \\ &= \tilde{w}_t(x,t) + k^a(x,x)w_x(x,t) - \int_0^x k_y^a(x,y)w_y(y,t)dy - \rho\int_0^x k^a(x,y)w(y,t)dy \\ &= \tilde{w}_t(x,t) + k^a(x,x)w_x(x,t) - k_y^a(x,x)w(x,t) + k_y^a(x,0)w(0,t) \\ &+ \int_0^x k_{yy}^a(x,y)w(y,t)dy - \rho\int_0^x k^a(x,y)w(y,t)dy. \end{aligned} \tag{2.43}$$

Substituting (2.42) and (2.43) in (2.6a),

$$\begin{aligned} &\tilde{w}_t(x,t) + k^a(x,x)w_x(x,t) - k_y^a(x,x)w(x,t) + k_y^a(x,0)w(0,t) \\ &+ \int_0^x k_{yy}^a(x,y)w(y,t)dy - \rho\int_0^x k^a(x,y)w(y,t)dy \\ &= \tilde{w}_{xx}(x,t) + \int_0^x k_{xx}^a(x,y)w(y,t)dy + k_x^a(x,x)w(x,t) + \frac{d}{dx}k^a(x,x)w(x,t) \\ &+ k^a(x,x)w_x(x,t) - \rho w(x,t). \end{aligned} \tag{2.44}$$

Since $k_y^a(x,0) = 0$, then adding and subtracting $(c_1+\rho)\tilde{w}(x,t)$ to the right-hand-side of (2.44), we obtain

$$\begin{aligned} \tilde{w}_t(x,t) &= \tilde{w}_{xx}(x,t) - (c_1+\rho)\tilde{w}(x,t) \\ &+ (2\frac{d}{dx}k^a(x,x) + c_1)w(x,t) + \int_0^x [k_{xx}^a(x,y) - k_{yy}^a(x,y) - c_1k^a(x,y)]w(y,t)dy. \end{aligned}$$

Since $k^a(x,y)$ is given by (2.14), the previous equation reduces to (4.20a). Also,

$$\tilde{w}_x(0,t) = w_x(0,t) - k^a(0,0)w(0,t) = 0.$$

As for the other boundary condition on $\tilde{w}(x,t)$ at $x=1$, we have

$$w_x(1,t) = \tilde{w}_x(1,t) + \int_0^1 k_x^a(1,y)w(y,t)dy + k^a(1,1)w(1,t). \tag{2.45}$$

Rearranging terms in the previous equation and using the expression of the control $u(t)$ in

$$\begin{aligned}\tilde{w}_x(1,t) &= w_x(1,t) - \int_0^1 k_x^a(1,y)w(y,t)dy - k^a(1,1)w(1,t)\\ &= u(t) - \int_0^1 k_x^a(1,y)w(y,t)dy - k^a(1,1)w(1,t)\\ &= 0.\end{aligned}$$

□

The final step in the design is to ensure that the stability of the target system (2.7) implies the stability of the closed-loop system, i.e., the controlled system (2.6). This is demonstrated via the invertibility of the transformation (2.39). The invertibility holds due to the properties of Volterra transformations, the well-posedness of system (2.14), and the regularity of the solution $k^a(x,y)$, which is twice continuously differentiable. The inverse transformation of (2.39), useful for the results in Chapter 4, is presented below.

Lemma 2.5.2. *[61, Chapter 4, pages 36-37],[107] Assuming that $c_1 > 0$ and using $k^a(x,y)$ from Lemma 2.5.1, the inverse of the state transformation (2.39) is*

$$w(x,t) = \tilde{w}(x,t) + \int_0^x \ell^a(x,y)\tilde{w}(y,t)dy, \tag{2.46}$$

where $\ell^a(x,y)$ is the unique continuous solution of

$$\ell^a_{xx}(x,y) - \ell^a_{yy}(x,y) + c_1\ell^a(x,y) = 0, \quad 0 < y < x < 1, \tag{2.47a}$$

$$\ell^a_y(x,0) = 0, \quad \ell^a(x,x) = -\frac{1}{2}c_1 x. \tag{2.47b}$$

Further,

$$\ell^a(x,y) = -c_1 x \frac{J_1\left(\sqrt{c_1(x^2-y^2)}\right)}{\sqrt{c_1(x^2-y^2)}}, \tag{2.48}$$

where $J_1(\cdot)$ is the Bessel function of first-order defined as

$$J_1(x) = \sum_{m=0}^{\infty}(-1)^m \frac{(x/2)^{2m+1}}{m!(m+1)!}.$$

2.6 Backstepping state observer for parabolic PDEs

Since the backstepping controllers require knowledge of the state at every point within the domain, as demonstrated in equation (2.38), there is a need to develop state observers. In distributed parameter systems, it is uncommon for measurements to be available across the domain. Typically, sensors are installed at the boundaries. This section presents a backstepping-based observer for the reaction-diffusion equation, a simple parabolic PDE that was considered in the previous section. The available measurements are restricted to the boundary. The material of this section can be found in [108]. For more information about observer design using backstepping, we refer the reader to [61, Chapter 5].

Recall system (2.6). Suppose that $w(1,t)$ is available for measurement. In the following, we illustrate that, with this available measurement, designing a state observer that estimates the state is possible. The observer for system (2.6) has the form

$$\hat{w}_t(x,t) = \hat{w}_{xx}(x,t) + \rho\hat{w}(x,t) + p_1(x)(w(1,t) - \hat{w}(1,t)), \tag{2.49a}$$
$$\hat{w}_x(0,t) = 0, \tag{2.49b}$$
$$\hat{w}_x(1,t) = u(t) + p_2(w(1,t) - \hat{w}(1,t)). \tag{2.49c}$$

The function $p_1(x)$ and the constant p_2 are observer gains that will be determined such that $\hat{w}(x,t)$ converges to $w(x,t)$ as time goes to infinity. Notably, the structure of this observer mirrors the traditional design found in finite-dimensional systems, where a copy of the plant is added to an output injection. To see this, note that the observer for a system on a finite-dimension,

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t),$$

is

$$\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)),$$

where L is the observer gain. In system (2.49), the observer gains $p_1(x)$ and p_2 form an infinite-dimensional vector, similar to L in finite-dimensional systems.

The observer gains $p_1(x)$ and p_2 have to be chosen such that $\hat{w}(x,t)$ converges to $w(x,t)$ as time goes to infinity. To do so, define the state of the observation error as

$$e^w(x,t) = w(x,t) - \hat{w}(x,t), \tag{2.51}$$

then the system describing the observation error is

$$e_t^w(x,t) = e_{xx}^w(x,t) - \rho e^w(x,t) - p_1(x)e^w(1,t), \tag{2.52a}$$
$$e_x^w(0,t) = 0, \tag{2.52b}$$
$$e_x^w(1,t) = -p_2 e^w(1,t). \tag{2.52c}$$

As in the previous section, where the stabilizing boundary control was achieved by transforming the original system into an exponentially stable system via an invertible transformation, we design these observer gains by using an invertible transformation

$$e^w(x,t) = e^{\tilde{w}}(x,t) - \int_x^1 k^b(x,y)e^{\tilde{w}}(y,t)dy, \tag{2.53}$$

between the error dynamics (2.52) and the following target system

$$e_t^{\tilde{w}}(x,t) = e_{xx}^{\tilde{w}}(x,t) - (o_1+\rho)e^{\tilde{w}}(x,t), \tag{2.54a}$$
$$e_x^{\tilde{w}}(0,t) = 0, \tag{2.54b}$$
$$e_x^{\tilde{w}}(1,t) = 0, \tag{2.54c}$$

which was shown to be exponentially stable in Theorem 2.5.1 if $o_1 > -\rho$. We need to find the kernel $k^b(x,y)$ of the transformation (2.53). To do this, we take the spatial derivatives of (2.53) as follows.

$$e_x^w(x,t) = e_x^{\tilde{w}}(x,t) - \int_x^1 k_x^b(x,y)e^{\tilde{w}}(y,t)dy + k^b(x,x)e^{\tilde{w}}(x,t), \tag{2.55}$$

$$\begin{aligned} e_{xx}^w(x,t) = e_{xx}^{\tilde{w}}(x,t) &- \int_x^1 k_{xx}^b(x,y)e^{\tilde{w}}(y,t)dy + k_x^b(x,x)e^{\tilde{w}}(x,t) + \frac{d}{dx}k^b(x,x)e^{\tilde{w}}(x,t) \\ &+ k^b(x,x)e_x^{\tilde{w}}(x,t). \end{aligned} \tag{2.56}$$

Taking the time derivative of (2.53) and integrating by parts, we obtain

$$\begin{aligned} e_t^w(x,t) &= e_t^{\tilde{w}}(x,t) - \int_x^1 k^b(x,y)e_t^{\tilde{w}}(y,t)dy \\ &= e_t^{\tilde{w}}(x,t) - \int_x^1 k^b(x,y)[e_{yy}^{\tilde{w}}(y,t) - (o_1+\rho)e^{\tilde{w}}(y,t)]dy \\ &= e_t^{\tilde{w}}(x,t) + (o_1+\rho)\int_x^1 k^b(x,y)e^{\tilde{w}}(y,t)dy - k^b(x,1)e_x^{\tilde{w}}(1,t) \\ &+ k^b(x,x)e_x^{\tilde{w}}(x,t) + k_y^b(x,1)e^{\tilde{w}}(1,t) - k_y^b(x,x)e^{\tilde{w}}(x,t) \\ &- \int_x^1 k_{yy}^b(x,y)e^{\tilde{w}}(y,t)dy. \end{aligned} \tag{2.57}$$

Substituting (2.56) and (2.57) in equation (2.52a),

$$
\begin{aligned}
&e_t^{\tilde{w}}(x,t) + (o_1+\rho)\int_x^1 k^b(x,y)e^{\tilde{w}}(y,t)dy - k^b(x,1)e_x^{\tilde{w}}(1,t) \\
&+ k^b(x,x)e_x^{\tilde{w}}(x,t) + k_y^b(x,1)e^{\tilde{w}}(1,t) - k_y^b(x,x)e^{\tilde{w}}(x,t) - \int_x^1 k_{yy}^b(x,y)e^{\tilde{w}}(y,t)dy \\
&= e_{xx}^{\tilde{w}}(x,t) - \int_x^1 k_{xx}^b(x,y)e^{\tilde{w}}(y,t)dy + k_x^b(x,x)e^{\tilde{w}}(x,t) + \frac{d}{dx}k^b(x,x)e^{\tilde{w}}(x,t) \\
&+ k^b(x,x)e_x^{\tilde{w}}(x,t) - \rho e^{\tilde{w}}(x,t) + \rho\int_x^1 k^b(x,y)e^{\tilde{w}}(y,t)dy - p_1(x)e^{\tilde{w}}(1,t).
\end{aligned}
$$

Rearranging terms in the previous equation,

$$
\begin{aligned}
&e_t^{\tilde{w}}(x,t) - e_{xx}^{\tilde{w}}(x,t) + (o_1+\rho)e^{\tilde{w}}(x,t) \\
&= \Big(o_1 + \frac{d}{dx}k^b(x,x) + k_x^b(x,x) + k_y^b(x,x)\Big)e^{\tilde{w}}(x,t) + k^b(x,1)e_x^{\tilde{w}}(1,t) \\
&+ \int_x^1 [k_{yy}^b(x,y) - k_{xx}^b(x,y) - o_1 k^b(x,y)]e^{\tilde{w}}(y,t)dy - (k_y^b(x,1) + p_1(x))e^{\tilde{w}}(1,t).
\end{aligned}
$$

Since $e_x^{\tilde{w}}(1,t) = 0$ and $k_x^b(x,x) + k_y^b(x,x) = \frac{d}{dx}k^b(x,x)$, the previous equation yields

$$
\begin{aligned}
&e_t^{\tilde{w}}(x,t) - e_{xx}^{\tilde{w}}(x,t) + (o_1+\rho)e^{\tilde{w}}(x,t) \\
&= \Big(o_1 + 2\frac{d}{dx}k^b(x,x)\Big)e^{\tilde{w}}(x,t) + \int_x^1 [k_{yy}^b(x,y) - k_{xx}^b(x,y) - o_1 k^b(x,y)]e^{\tilde{w}}(y,t)dy \\
&- (k_y^b(x,1) + p_1(x))e^{\tilde{w}}(1,t).
\end{aligned}
\tag{2.58}
$$

For equation (2.58) to hold, the following three conditions must be satisfied

$$
k_{yy}^b(x,y) - k_{xx}^b(x,y) - o_1 k^b(x,y) = 0, \tag{2.59}
$$

$$
\frac{d}{dx}k^b(x,x) = -\frac{1}{2}o_1, \tag{2.60}
$$

$$
p_1(x) = -k_y^b(x,1). \tag{2.61}
$$

Further, the boundary condition (2.52b) together with (2.55) lead to

$$
k^b(0,0) = 0, \tag{2.62}
$$

$$
k_x^b(0,y) = 0. \tag{2.63}
$$

Similarly, (2.52c) and (2.55) imply that

$$p_2 = -k^b(1,1). \tag{2.64}$$

We integrate (2.60) from 0 to x, and use (2.63) to solve for $k^b(x,x)$, then

$$k^b(x,x) = -\frac{1}{2}o_1 x. \tag{2.65}$$

The previous discussion implies that for the transformation between the original error dynamics (2.52) and the target system (2.54) to hold, the gain kernel $k^b(x,y)$ must solve the following partial differential equation.

Lemma 2.6.1. *[108] Let $o_1 > 0$, be a real constant. For $(x,y) \in \mathcal{T} = \{x, y : 0 < x < y < 1\}$, the partial differential equation*

$$k^b_{yy}(x,y) - k^b_{xx}(x,y) - o_1 k^b(x,y) = 0, \tag{2.66a}$$

$$k^b_x(0,y) = 0, \quad k^b(x,x) = -\frac{1}{2}o_1 x, \tag{2.66b}$$

has a unique twice continuously differentiable solution, i.e. $k^b(x,y) \in C^2(\mathcal{T})$,

$$k^b(x,y) = -o_1 y \frac{I_1\left(\sqrt{o_1(y^2 - x^2)}\right)}{\sqrt{o_1(y^2 - x^2)}}.$$

Proof. We make a change of variables

$$\bar{x} = y, \quad \bar{y} = x, \quad \bar{k}^b(\bar{x},\bar{y}) = k^b(x,y). \tag{2.67}$$

Then equation (2.66) leads to

$$\bar{k}^b_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \bar{k}^b_{\bar{y}\bar{y}}(\bar{x},\bar{y}) - o_1 \bar{k}^b(\bar{x},\bar{y}) = 0, \tag{2.68a}$$

$$\bar{k}^b_{\bar{y}}(\bar{x},0) = 0, \quad \bar{k}^b(\bar{x},\bar{x}) = -\frac{o_1}{2}\bar{x}. \tag{2.68b}$$

The well-posedness of PDE (2.68) was shown in Theorem 2.5.1. Hence,

$$\begin{aligned}\bar{k}^b(\bar{x},\bar{y}) &= -o_1 \bar{x} \frac{I_1\left(\sqrt{o_1(\bar{x}^2 - \bar{y}^2)}\right)}{\sqrt{o_1(\bar{x}^2 - \bar{y}^2)}} \\ &= -o_1 y \frac{I_1\left(\sqrt{o_1(y^2 - x^2)}\right)}{\sqrt{o_1(y^2 - x^2)}}.\end{aligned}$$

□

The observer gains $p_1(x)$ and p_2 are obtained using (2.61)and (2.64), respectively.

As in Section 2.5, the final step in the design is to ensure that the stability of the target system (2.54) implies the stability of the error dynamics (2.52). This follows from the invertibility of the transformation (2.53).

Chapter 3

System theory for partial differential-algebraic equations

Consider PDAEs of the class

$$\frac{d}{dt}Ex(t) = Ax(t) + Bu(t), \quad t \geq 0, \tag{3.1a}$$

$$Ex(0) = Ex_i, \tag{3.1b}$$

where $E \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$, $A : D(A) \subset \mathcal{X} \to \mathcal{Z}$ is densely defined and closed, $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$, and $x_i \in \mathcal{X}$. Also, $\mathcal{X}, \mathcal{U}, \mathcal{Z}$ are Hilbert spaces. When the operator E is invertible, equation (3.1a) reduces to a standard PDE. This thesis focuses on scenarios when E is non-invertible, and /or unbounded on $\mathcal{Z}$. Just as with partial differential equations, establishing the well-posedness of system (3.1) is fundamental before proceeding with the development of control designs. Thus, the primary objective of this chapter is to provide a comprehensive overview of theory developed in this direction.

This chapter is organized as follows: We begin in Section 3.1 by briefly reviewing the well-posedness theory of system (3.1) within a finite-dimensional space, which gives rise to differential-algebraic equations. Next, in Section 3.2, we discuss the mathematical theoretical framework that will be adopted in this thesis to ensure the well-posedness of system (3.1) in an infinite-dimensional space. Finally, in Section 3.3, we provide examples of systems in the form (3.1) to illustrate practical applications.

3.1 Theory on finite-dimensional spaces

This section presents the basic existence and uniqueness theory for linear differential-algebraic equations (DAEs) with constant coefficients. The discussion in this section will prove useful for our results in Chapter 6. The material in this section can be found in [62, Chapter 2].

Consider systems whose dynamics are given by

$$\frac{d}{dt}Ex(t) = Ax(t) + Bu(t), \quad x(0) = x_i, \tag{3.2}$$

where $E,\ A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$. For each $t \geq 0$, $x(t)$ has values in $\mathbb{R}^n$, and $u(t)$ has values in $\mathbb{R}^r$. In this discussion, we focus on particular class of DAE systems; the regular systems.

Definition 3.1.1. [62, Definition 2.5]
Let $E,\ A \in \mathbb{R}^{n\times n}$. System (3.2) is called regular if the characteristic polynomial

$$\rho(\lambda) = \det(\lambda E - A),$$

is not identically zero. In this case, the matrix pair (E, A) is called regular.

The following definition of equivalence is needed to allow the transformation of a given system into a desired form.

Definition 3.1.2. [62, Definition 2.1] Two pairs of matrices $(E_i, A_i),\ E_i, A_i \in \mathbb{R}^{m\times n},\ i = 1, 2$, are equivalent if there exist nonsingular matrices $\mathcal{P} \in \mathbb{R}^{m\times m}$ and $\mathcal{Q} \in \mathbb{R}^{n\times n}$ such that:

$$E_2 = \mathcal{P}E_1\mathcal{Q}, \qquad A_2 = \mathcal{P}A_1\mathcal{Q}.$$

We can write $(E_1, A_1) \sim (E_2, A_2)$.

The following theorem displays that regular differential-algebraic equations can be written in a simpler form. The proof of this relies on transforming (3.2) to Jordan normal form.

Theorem 3.1.1. *[62, Theorem 2.7]*
Let $E, A \in \mathbb{R}^{n\times n}$. If the pair (E, A) is regular, then

$$(E, A) \sim \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \right), \tag{3.3}$$

where A_1 is a matrix in Jordan canonical form and N is a nilpotent matrix with degree of nilpotency $\nu \geq 1$, i.e. for $\nu > 1$ $N^\nu = 0$ and $N^{\nu-1} \neq 0$; $\nu = 1$ when $N = 0$.

When the matrix E is invertible, equation (3.2) simplifies to an ordinary differential equation

$$\frac{d}{dt}x(t) = E^{-1}Ax(t) + E^{-1}Bu(t), \quad x(0) = x_i. \tag{3.4}$$

Then, the theorem above simplifies to transforming $E^{-1}A$ into its Jordan canonical form A_1.

Thus, for regular DAEs (3.2), there exist nonsingular matrices $\mathcal{P}$ and $\mathcal{Q}$ such that

$$\mathcal{P}E\mathcal{Q} = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad \mathcal{P}A\mathcal{Q} = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}, \quad PB = \begin{bmatrix} B_1 \\ B_0 \end{bmatrix}.$$

Setting

$$\mathcal{Q}^{-1}x(t) = \begin{bmatrix} x_1(t) \\ x_0(t) \end{bmatrix}, \quad \mathcal{Q}^{-1}x_i = \begin{bmatrix} (x_i)_1 \\ (x_i)_0 \end{bmatrix},$$

we write system (3.2) into the *Weierstrass-Kronecker* form

$$\frac{d}{dt}x_1(t) = A_1x_1(t) + B_1u(t), \quad x_1(0) = (x_i)_1, \tag{3.5a}$$

$$\frac{d}{dt}Nx_0(t) = x_0(t) + B_0u(t), \quad x_0(0) = (x_i)_0. \tag{3.5b}$$

It is well-known from the theory of ordinary differential equations that initial value problems of the form (3.5a) are uniquely solvable for $u(t) \in C(\mathbb{I}, \mathbb{R}^r)$ where $\mathbb{I} \subset \mathbb{R}$. To handle system (3.5b), we have the following lemma which is a straightforward consequence of the nilpotency of N.

Lemma 3.1.1. *Let N be a nilpotent matrix with nilpotency-index $\nu \geq 1$. Let D be the linear differential operator which maps a continuously differentiable function to its derivative. Then, $I - ND$ is invertible and*

$$(I - ND)^{-1} = \sum_{j=0}^{\nu-1} (ND)^j.$$

Proof. Since N is nilpotent, it follows from [34, Corollary A.5] that $(I - ND)$ is invertible. Using Neumann series, we obtain

$$(I - ND)^{-1} = \sum_{n=0}^{\infty} (ND)^n = I + ND + (ND)^2 + \cdots = \sum_{j=0}^{\nu-1} (ND)^j.$$

□

The next theorem presents the solutions of system (3.5b) for a given control input $u(t)$ and initial condition x_i.

Theorem 3.1.2. *[34, Corollary 3.2] Let $\mathbb{I} \subset \mathbb{R}$. Considering equation (3.5b), let ν be the nilpotency degree of the matrix N with $u(t) \in C^{\nu}(\mathbb{I}, \mathbb{R}^r)$. When $N \neq 0$, equation (3.5b) has a unique solution*

$$x_0(t) = x_0^{normal}(t) + x_0^{pulse}(t), \tag{3.6}$$

where

$$x_0^{normal}(t) = -\sum_{j=0}^{\nu-1} N^j B_0 \frac{d^j}{dt^j} u(t), \tag{3.7}$$

$$x_0^{pulse}(t) = -\sum_{j=1}^{\nu-1} N^j \delta^{(j-1)}(t) \Big((x_i)_0 + \sum_{j=0}^{\nu-1} N^j B_0 \frac{d^j}{dt^j} u(0) \Big). \tag{3.8}$$

Here $\frac{d^j}{dt^j}u(t)$ denotes the jth derivative of $u(t)$ with respect to t, and $\delta(t)$ is the Dirac delta distribution. When $N = 0$, if $(x_i)_0 \neq -B_0 u(0)$, an impulse behavior is manifested at $t = 0$, illustrated by

$$x_0(t) = -B_0 u(t) + \delta(t) \left((x_i)_0 + B_0 u(0) \right).$$

The previous lemma implies that an arbitrary initial value on the state may lead to a distributional solution for the regular DAE (3.2). In order to avoid distributions in solutions, for $N \neq 0$ and for a given x_i, we must have that

$$(x_i)_0 = -\sum_{j=0}^{\nu-1} N^j B_0 \frac{d^j}{dt^j} u(0). \tag{3.9}$$

and when $N = 0$, we must have $(x_i)_0 = -B_0 u(0)$. This introduces the principle of consistent initialization for differential-algebraic equations when the initial and control inputs fulfill statement (3.9).

Definition 3.1.3. An initial condition x_i is consistent if

$$(x_i)_0 = -\sum_{j=0}^{\nu-1} N^j B_0 \frac{d^j}{dt^j} u(0).$$

Further, in the special case when $N = 0$, the initial condition x_i is consistent if $(x_i)_0 = -B_0 u(0)$.

Consistent initial conditions ensure that the starting point of a system modeled by differential-algebraic equations (DAEs) adheres to constraints, such as conservation laws for mass, energy, or momentum, which are often represented by the algebraic part of the DAE. Ensuring these conditions are met is crucial for preventing non-physical behaviors in the mathematical model. If the initial conditions are inconsistent, the resulting numerical solutions may deviate from physical behaviors. For more information, we refer the interested reader to [34, Section 3.5.1].

The literature frequently addresses the concept of consistency by assuming that the initial condition satisfies equation (3.9); see [34, 62]. Since the initial condition is typically known or measured in most scenarios, we instead make sure that the control input is assigned in a manner that ensures statement (3.9). This approach introduces additional complexity as extra care must be taken to guarantee that the control does indeed satisfy this consistency statement.

The parameter ν, frequently mentioned in previous discussions, represents the degree of nilpotency of the nilpotent matrix N. In fact, this parameter is also referred to as the nilpotency-index of differential-algebraic equations (DAEs). The concept of the index plays a significant role in the study of DAEs. Adding more constraints to the system dynamics usually results in a set of higher-index DAEs [19]. The index of DAEs can be characterized in several ways, including the differentiation-index, perturbation-index, nilpotency-index, and resolvent-index, all of which are equivalent for DAEs [62, 78]. We are interested in the nilpotency-index, as defined previously, and the differentiation-index, which describes the number of differentiations needed to transform the DAE into an ordinary differential equation (ODE).

In the remainder of this section, we provide illustrative examples of differential-algebraic equations with different nilpotency and differentiation indices. The notation $\dot{x}(t)$ denotes the derivative of $x(t)$ with respect to time. Also, the superscript A^* is used to indicate the transpose of a matrix A.

Example 3.1.1. *Consider*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} x(t) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u(t), \tag{3.10a}$$

$$x(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^*. \tag{3.10b}$$

This system is written in the Weierstrass-Kronecker form, where

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ A_1 = \begin{bmatrix} -1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 1 \end{bmatrix}, \ B_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^*. \tag{3.11}$$

This system has nilpotency-index $\nu = 3$ since $N^3 = 0$ and $N^2 \neq 0$.

Example 3.1.2. *A simple example of a differential-algebraic equation is a description of the interaction between two populations*

$$\frac{d}{dt}x_1(t) = ax_1(t) + bx_2(t), \tag{3.12a}$$

$$0 = x_1(t) + x_2(t) - 1, \tag{3.12b}$$

Here, $x_1(t)$ and $x_2(t)$ represent populations of two different species. This system can be rewritten in linear form as:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E} \frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -1 \end{bmatrix}}_{Bu(t)}. \tag{3.13}$$

for arbitrary real constants a, b. This system has a differentiation-index 1, *since one has to differentiate* (3.12b) *with respect to time once to obtain an ordinary differential equation for $x_1(t)$. Equations of the form* (3.12) *are also referred to as semi-explicit index 1 DAE [46, Chapter 1].*

Example 3.1.3. *[62, Example 1.2] Differential-algebraic equations are used to model the charging of a capacitor via a resistor. For $j = 1, 2, 3$, let x_j be the potential with each node of the circuit, as shown in Figure 3.1. The voltage source increases the potential from x_3 to x_1 by U, resulting in the equation $x_1 - x_3 - U = 0$. By Kirchhoff's first law, the sum of the currents vanishes at each node. Hence, assuming ideal electronic units for the second node, we obtain the following:*

$$C(\dot{x}_3 - \dot{x}_2) + \frac{x_1 - x_2}{R} = 0,$$

where R is the resistance of the resistor and C is the capacitance of the capacitor. By choosing the zero potential as $x_3 = 0$, we obtain

$$x_1 - x_3 - U = 0, \tag{3.14}$$

$$C(\dot{x}_3 - \dot{x}_2) + \frac{x_1 - x_2}{R} = 0, \tag{3.15}$$

$$x_3 = 0. \tag{3.16}$$

This simple system can be solved for x_3 and x_1 to obtain an ordinary differential equation for x_2 only, combined with algebraic equations for x_1, x_3. This system has a differentiation-index of one since one will need to differentiate the system above one time to obtain a differential equation for $\frac{d}{dt}x(t)$ where $x(t) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^*$.

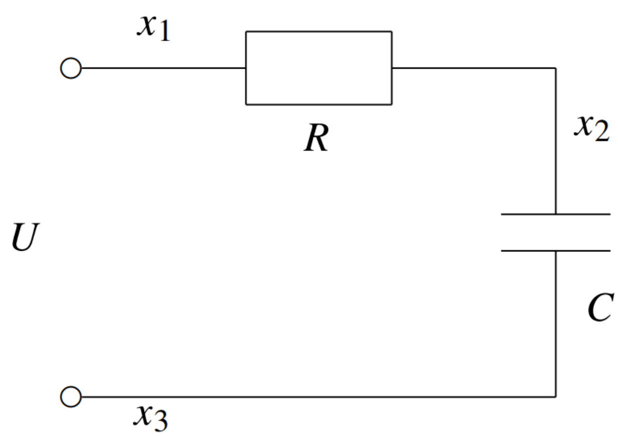


Figure 3.1: A simple electrical network

Finally, we present an example of a DAE with a higher differentiation-index.

Example 3.1.4. *Consider*

$$\dot{x}_1(t) = -x_3(t) + u_1(t), \tag{3.17a}$$
$$\dot{x}_2(t) = -x_1(t) + u_2(t), \tag{3.17b}$$
$$0 = -x_2(t) + u_3(t). \tag{3.17c}$$

Setting

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix},$$

then system (3.17) can be written as

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E} \frac{d}{dt}x(t) = \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}}_{u(t)}. \tag{3.18}$$

This equation has differentiation-index 3. The first time-derivatives of $x_1(t)$ and $x_2(t)$ appear in (3.17a) and (3.17b), respectively. To find the first time-derivative of $x_3(t)$, assuming that $u(t)$ is smooth enough, we differentiate (3.17a) with respect to time,

$$\ddot{x}_1(t) = -\dot{x}_3(t) + \dot{u}(t), \tag{3.19}$$

and so, we must find $\ddot{x}_1(t)$. To do so, we differentiate (3.17b) twice to obtain

$$\dddot{x}_2(t) = -\ddot{x}_1(t) + \ddot{u}_2(t). \tag{3.20}$$

To obtain $\dddot{x}_2(t)$, we must differentiate (3.17c) three times to obtain

$$\dddot{x}_2(t) = \dddot{u}_3(t). \tag{3.21}$$

Finally, we substitute (3.21) into (3.20) and then solve (3.20) for $\ddot{x}_1(t)$. We obtain $\dot{x}_3(t)$ by referring back to (3.19). Therefore, system (3.17) has a differentiation index of 3, since we differentiated (3.17) three times to obtain $\dot{x}(t)$.

3.2 State-decoupling and solvability of linear PDAEs

Similar to differential-algebraic equations, the concept of the index plays a significant role for partial differential-algebraic equations (PDAEs). Not all the indices of PDAEs are equivalent. A comparison between different indices can be found in [36]. Interested readers can also refer to [4, 21, 27, 72, 73, 76, 77, 98, 109, 130], which are beyond the scope of this thesis. In this section, we define an index known as the radiality-index [35, 50, 111], which plays a crucial role in demonstrating the well-posedness of systems (3.1).

The solvability of partial differential-algebraic equations has been a subject of intensive research over the past decades; see [35, 42, 50, 79, 99, 101, 111, 115, 118, 119]. We will discuss in more detail the results from [35, 50, 111]. In [111], a theory called (E, p)-radiality was introduced. This theory enables a decomposition of the state-space similar to the *Weierstrass-Kronecker* form, and provides Hille-Yosida type estimates essential for semigroup generation in Banach spaces. This theory forms the basis for the discussions in Chapter 5, and therefore will be presented in detail. Recently, dealing with Hilbert spaces, the assumptions of this theory have been relaxed. As our discussion in this thesis primarily focuses on Hilbert spaces, we will concentrate on these weaker assumptions.

We start by presenting definitions for the resolvents of (3.1).

Definition 3.2.1. The resolvent of system (3.1) is $(sE - A)^{-1}$, and the corresponding resolvent set is

$$\rho(E, A) = \{s \in \mathbb{C} : (sE - A)^{-1} \in \mathcal{L}(\mathcal{Z}, \mathcal{X})\}.$$

Furthermore, the right and left $E-$resolvents of A are

$$R_s^E(A) = (sE - A)^{-1}E, \quad L_s^E(A) = E(sE - A)^{-1}. \tag{3.22}$$

Note that if $s = 0 \in \rho(E, A)$, then

$$R_0^E(A) = A^{-1}E, \quad L_0^E(A) = EA^{-1}.$$

Definition 3.2.2. Let $s_k \in \rho(E, A)$ for $k = 0, \ldots, p$. The right and left $(E, p)-$resolvents are

$$R_s^E(p, A) = \prod_{k=0}^{p} R_{s_k}^E(A) = (s_0E - A)^{-1}E \ldots (s_pE - A)^{-1}E, \tag{3.23a}$$

$$L_s^E(p, A) = \prod_{k=0}^{p} L_{s_k}^E(A) = E(s_0E - A)^{-1} \ldots E(s_pE - A)^{-1}. \tag{3.23b}$$

It is straightforward to observe that when $p = 0$, the right and left $(E, 0)-$resolvents reduce to the right and left $E-$resolvents (3.22), respectively. We now define $(E, p)-$radial operators.

Definition 3.2.3. [111, Definition 2.2.1., page 21] The operator A is (E, p)-radial if

- There exists $a \in \mathbb{R}$ such that $s \in \rho(E, A)$ for all real $s > a$;
- There exists $K > 0$ such that $\forall n \in \mathbb{N}$ and for all real $s_k > 0$ with $k = 0, \ldots, p$,

$$\|R_s^E(p, A)\|_{\mathcal{L}(\mathcal{X})} = \|\Big((s_0E - A)^{-1}E \ldots (s_pE - A)^{-1}E\Big)^n\|_{\mathcal{L}(\mathcal{X})} \leq \frac{K}{\prod_{k=0}^{p}(s_k - a)^n} \tag{3.24}$$

$$\|L_s^E(p, A)\|_{\mathcal{L}(\mathcal{Z})} = \|\Big(E(s_0E - A)^{-1} \ldots E(s_pE - A)^{-1}\Big)^n\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K}{\prod_{k=0}^{p}(s_k - a)^n}. \tag{3.25}$$

The definition above of $(E,p)-$radial allows for the consideration of the resolvent in a broader right-hand-plane setting, with $s > a$. Note that if $E = I$ and $p = 0$, then inequalities (3.24) and (3.25) become equivalent. Also, when $E = I$, either of these resolvent estimates implies the generation of a C_0-semigroup via the Hille-Yosida theorem; see Theorem (2.1.2).

Definition 3.2.4. System (3.1) has *radiality-index* p if $s \in \rho(E,A)$ for all real $s > a$, and statements (3.24)-(3.25) hold with $n = 1$.

If A is an $(E,p)-$radial operator, it has *radiality-index* p. The converse holds if $K \leq 1$.

Recall the definition of right and left $(E,p)-$resolvents in (3.23). Let $\alpha \in \rho(E,A)$ and $\alpha_k \in \rho(E,A)$ for $k = 0,\dots,p$,

$$\begin{aligned} \mathcal{X}_0 &= \ker R^E_\alpha(p,A), & \mathcal{Z}_0 &= \ker L^E_\alpha(p,A),\\ \mathcal{X}_1 &= \overline{\text{ran } R^E_\alpha(p,A)}, & \mathcal{Z}_1 &= \overline{\text{ran } L^E_\alpha(p,A)}. \end{aligned}$$

These spaces are independent of the choice of α [111, Lemma 2.1.2, page 18]. It is easy to show that $\mathcal{X}_0 = \ker\ E$. Also, when $p = 0$, $z \in \mathcal{Z}_0$ if and only if $x = (\alpha E - A)^{-1} z \in \ker\ E$. In what follows, we present some useful identities for these resolvents; see [111, Chapter 2].

Lemma 3.2.1. *[111, Lem. 2.2.6 , pg 23] If system (3.1) has radiality-index p, then*

$$\lim_{s\to\infty} (sR^E_s(p,A))^{p+1}x = x, \quad \text{for all } x \in \mathcal{X}_1, \tag{3.26}$$

$$\lim_{s\to\infty} (sL^E_s(p,A))^{p+1}z = z, \quad \text{for all } z \in \mathcal{Z}_1. \tag{3.27}$$

Theorem 3.2.1. *[111, Theorem 2.5.1.] If system (3.1) has radiality-index p, then*

$$\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1, \qquad \mathcal{Z} = \mathcal{Z}_0 \oplus \mathcal{Z}_1. \tag{3.28}$$

If system (3.1) has *radiality-index* p, then the operator $P^{\mathcal{X}_1} : \mathcal{X} \to \mathcal{X}$, given by

$$P^{\mathcal{X}_1}x := \lim_{s\to\infty} (sR^E_s(p,A))^{p+1}x \tag{3.29}$$

is a projection onto $\mathcal{X}_1$ with $\ker P^{\mathcal{X}_1} = \mathcal{X}_0$, ran $P^{\mathcal{X}_1} = \mathcal{X}_1$. Similarly, the operator $P^{\mathcal{Z}_1} : \mathcal{Z} \to \mathcal{Z}$ defined by

$$P^{\mathcal{Z}_1}z := \lim_{s\to\infty} (sL^E_s(p,A))^{p+1}z \tag{3.30}$$

is a projection onto $\mathcal{Z}_1$ with $\ker P^{\mathcal{Z}_1} = \mathcal{Z}_0$, ran $P^{\mathcal{Z}_1} = \mathcal{Z}_1$. If system (3.1) has *radiality-index* p, then $P^{\mathcal{X}_1}$ and $P^{\mathcal{Z}_1}$ are bounded operators. This follows from (3.24) and (3.25); see [50].

We define restrictions of the operators E and A as follows.

$$E_0 = E|_{\mathcal{X}_0}, \quad A_0 = A|_{D(A_0)}, \quad D(A_0) = \mathcal{X}_0 \cap D(A),$$

$$E_1 = E|_{\mathcal{X}_1}, \quad A_1 = A|_{D(A_1)}, \quad D(A_1) = \mathcal{X}_1 \cap D(A).$$

For convenience of notation, we also set

$$B_0 = P^{\mathcal{Z}_0} B, \qquad B_1 = P^{\mathcal{Z}_1} B. \tag{3.31}$$

The assumption that system (3.1) has *radiality-index* p implies that these operators have certain properties.

Lemma 3.2.2. *[111, Lemma 2.2.1, page 20] The operator $E_0 \in L(\mathcal{X}_0, \mathcal{Z}_0)$, and $A_0 : D(A_0) \to \mathcal{Z}_0$.*

Lemma 3.2.3. *[111, Lemma 2.2.4, page 22] The operator A_0 is boundedly invertible, and $A_0^{-1} \in \mathcal{L}(\mathcal{Z}_0, \mathcal{X}_0)$.*

Lemma 3.2.4. *[111, Lemma 2.2.5, page 22] The operators $A_0^{-1}E_0$, $E_0A_0^{-1}$, are nilpotent operators with nilpotency-index $\nu \leq p+1$, that is, for $\nu > 1$, $N^\nu = 0$ and $N^{\nu-1} \neq 0$; when $\nu = 1$ we have that $N = 0$.*

With strong assumptions, the following proposition was proved in [111, Corollary 2.5.1, page 38]. Later, these assumptions were relaxed in [50, Proposition 1] for systems with a radiality-index 0, and in [35, Theorem 3.3] for any radiality-index p.

Proposition 3.2.1. *[35, Theorem 3.3] If system (3.1) has radiality-index p, then*

- *for all $x \in D(A)$, $P^{\mathcal{X}_1}x \in D(A)$ and $AP^{\mathcal{X}_1}x = P^{\mathcal{Z}_1}Ax$;*
- *for all $x \in \mathcal{X}$, $EP^{\mathcal{X}_1}x = P^{\mathcal{Z}_1}Ex$.*

Corollary 3.2.1. *[35, Theorem 3.3] If system (3.1) has radiality-index p, then*

- $E_1 \in \mathcal{L}(\mathcal{X}_1, \mathcal{Z}_1)$;
- *$A_0 : D(A_0) \subset \mathcal{X}_0 \to \mathcal{Z}_0$ is densely defined, closed and boundedly invertible;*

- $A_1 : D(A_1) \subset \mathcal{X}_1 \to \mathcal{Z}_1$ *is densely defined and closed.*

The following proposition was also shown in [111, Theorem 2.5.3, page 40] with stronger assumptions. These assumptions were relaxed in [35, Theorem 3.3].

Proposition 3.2.2. *[35, Theorem 3.3] If system (3.1) has radiality-index p and $\operatorname{ran} E$ is closed in $\mathcal{Z}$, then $E_1 \in L(\mathcal{X}_1, \mathcal{Z}_1)$ is boundedly invertible.*

We are now in a position to write system (3.1) in a form similar to the *Weierstrass-Kronecker* form for DAEs, (3.5). Define

$$P^{\mathcal{X}_0} = I^{\mathcal{X}} - P^{\mathcal{X}_1}, \quad P^{\mathcal{Z}_0} = I^{\mathcal{Z}} - P^{\mathcal{Z}_1}, \tag{3.32}$$

where $I^{\mathcal{X}}$ and $I^{\mathcal{Z}}$ denote the identity operator on the spaces $\mathcal{X}$ and $\mathcal{Z}$, respectively. Then using the non-orthogonal projections $P^{\mathcal{X}_1}$ and $P^{\mathcal{X}_0}$, we define

$$\tilde{P}^{\mathcal{X}} = \begin{bmatrix} P^{\mathcal{X}_1} \\ P^{\mathcal{X}_0} \end{bmatrix} \in \mathcal{L}(\mathcal{X}, \mathcal{X}_1 \times \mathcal{X}_0), \quad \tilde{P}^{\mathcal{Z}} = \begin{bmatrix} P^{\mathcal{Z}_1} \\ P^{\mathcal{Z}_0} \end{bmatrix} \in \mathcal{L}(\mathcal{Z}, \mathcal{Z}_1 \times \mathcal{Z}_0), \tag{3.33}$$

$$(\tilde{P}^{\mathcal{X}})^{-1} = \begin{bmatrix} I^{\mathcal{X}_1} & I^{\mathcal{X}_0} \end{bmatrix} \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_0, \mathcal{X}), \tag{3.34}$$

$$(\tilde{P}^{\mathcal{Z}})^{-1} = \begin{bmatrix} I^{\mathcal{Z}_1} & I^{\mathcal{Z}_0} \end{bmatrix} \in \mathcal{L}(\mathcal{Z}_1 \times \mathcal{Z}_0, \mathcal{Z}). \tag{3.35}$$

For ease of notation, we set

$$N = A_0^{-1} E_0, \qquad \tilde{A}_1 = E_1^{-1} A_1, \tag{3.36a}$$

$$\tilde{B}_0 = A_0^{-1} B_0, \qquad \tilde{B}_1 = E_1^{-1} B_1. \tag{3.36b}$$

The next theorem presents the *Weierstrass-Kronecker* form for PDAEs (3.1). Since this form will be a key element for our subsequent findings, a detailed proof is provided below.

Theorem 3.2.2. *Let $\begin{bmatrix} x_1(t) \\ x_0(t) \end{bmatrix} = \tilde{P}^{\mathcal{X}} x(t)$. If $\operatorname{ran} E$ is closed in $\mathcal{Z}$, and system (3.1) has radiality-index p, then system (3.1) can be decomposed into two sub-systems*

$$\frac{d}{dt} x_1(t) = \tilde{A}_1 x(t) + \tilde{B}_1 u(t), \tag{3.37a}$$

$$\frac{d}{dt} N x_0(t) = x_0(t) + \tilde{B}_0 u(t), \tag{3.37b}$$

where N is a nilpotent operator with nilpotency-index $\nu \le p + 1$.

Proof. Let $x(t) \in D(A)$. Using the operators $\tilde{P}^{\mathcal{X}}$ and $(\tilde{P}^{\mathcal{X}})^{-1}$, we rewrite system (3.1) as

$$\frac{d}{dt}E(\tilde{P}^{\mathcal{X}})^{-1}\begin{bmatrix} x_1(t) \\ x_0(t) \end{bmatrix} = A(\tilde{P}^{\mathcal{X}})^{-1}\begin{bmatrix} x_1(t) \\ x_0(t) \end{bmatrix} + Bu(t).$$

Pre-multiplying by $\tilde{P}^{\mathcal{Z}}$,

$$\frac{d}{dt}\tilde{P}^{\mathcal{Z}}E(\tilde{P}^{\mathcal{X}})^{-1}\begin{bmatrix} x_1(t) \\ x_0(t) \end{bmatrix} = \tilde{P}^{\mathcal{Z}}A(\tilde{P}^{\mathcal{X}})^{-1}\begin{bmatrix} x_1(t) \\ x_0(t) \end{bmatrix} + \tilde{P}^{\mathcal{Z}}Bu(t).$$

The previous equation yields

$$\frac{d}{dt}\tilde{P}^{\mathcal{Z}}E(\tilde{P}^{\mathcal{X}})^{-1}\begin{bmatrix} x_1(t) \\ x_0(t) \end{bmatrix} = \tilde{P}^{\mathcal{Z}}A(\tilde{P}^{\mathcal{X}})^{-1}\begin{bmatrix} x_1(t) \\ x_0(t) \end{bmatrix} + \tilde{P}^{\mathcal{Z}}Bu(t).$$

Now, since the operators A_0, A_1 and E_1 are invariant with respect to the projected spaces,

$$\tilde{P}^{\mathcal{Z}}E(\tilde{P}^{\mathcal{X}})^{-1} = \begin{bmatrix} E_1 & 0 \\ 0 & E_0 \end{bmatrix}, \quad \tilde{P}^{\mathcal{Z}}E(\tilde{P}^{\mathcal{X}})^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_0 \end{bmatrix}.$$

Referring to the definitions of B_1 and B_0 in (3.31), and using that both E_1 and A_0 are boundedly invertible, we write (3.1) as

$$\frac{d}{dt}\begin{bmatrix} I & 0 \\ 0 & A_0^{-1}E_0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_0(t) \end{bmatrix} = \begin{bmatrix} E_1^{-1}A_1 & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_0(t) \end{bmatrix} + \begin{bmatrix} E_1^{-1}B_1 \\ A_0^{-1}B_0 \end{bmatrix}u(t). \tag{3.38}$$

Then, using the definition in (3.36), system (3.38) leads to (3.37). □

Recall from Chapter 2 the connection between the generation of a semigroup and well-posedness for PDEs. Now that we have established sufficient conditions for the existence of a Weierstrass-Kronecker form for PDAEs, the final step in demonstrating the well-posedness of system (3.1) is to ensure semigroup generation.

Theorem 3.2.3. *[50, Theorem 1] If $A - \alpha E$ is $(E, 0)-$radial and $\operatorname{ran} E$ is closed, then the operator $\tilde{A}_1$ with domain $D(A) \cap \mathcal{X}_1$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on $\mathcal{X}_1$ with bound $Ke^{\alpha t}$. The component on $\mathcal{X}_0$ is identically zero.*

The result below was proved in [35, Corollary 3.10] for any p. It established a connection between the existence of Weierstrass-Kronecker form for PDAEs and the generation of integrated semigroups [88].

Corollary 3.2.2. *[35, Corollary 3.10] If system (3.1) has the Weierstrass-Kronecker form as in (3.37), then $\tilde{A}_1$ generates an integrated semigroup.*

Theorem 3.2.4. *[35, Corollary 3.7] It system (3.1) has a Weierstrass-Kronecker form as in (3.37) and $\tilde{A}_1$ generates a C_0-semigroup $T(t)$ on $\mathcal{X}_1$, then the radiality-index p and the nilpotency-index ν both exist and $\nu = p + 1$.*

If the operator $\tilde{A}_1$ generates a C_0-semigroup $T(t)$ on $\mathcal{X}_1$, it follows from (3.37) that if $u(t) \in C([0, t_f]; \mathcal{U})$, the mild solution of (3.37a) is

$$x_1(t) = T(t)(x_i)_1 + \int_0^t T(t-s)\tilde{B}_1 u(s)\, ds. \tag{3.39}$$

For the solution of sub-system (3.37b), we have

$$x_0(t) = -\sum_{j=0}^{p} N^j \tilde{B}_0 \frac{d^j}{dt^j} u(t) - \sum_{j=1}^{p} N^j \delta^{(j-1)}(t) \left((x_i)_0 + \sum_{k=0}^{p} N^k \tilde{B}_0 \frac{d^k}{dt^k} u(0) \right). \tag{3.40}$$

If the control input is chosen consistently, then the solution of system (3.1) is

$$x(t) = T(t)(x_i)_1 + \int_0^t T(t-s)\tilde{B}_1 u(s)\, ds - \sum_{j=0}^{p} N^j \tilde{B}_0 \frac{d^j}{dt^j} u(t). \tag{3.41}$$

3.3 Examples

This section will provide examples of systems modeled by equation (3.1), and can be written in the Weierstrass-Kronecker form (3.37).

Example 3.3.1. *The class of coupled systems below was demonstrated to be well-posed through the application of $(E, 0)-$radiality [50, Section 5], where the Schur complement was used. A specific example employing eigenfunction expansion was discussed in [116, Section 5]. We will now provide a detailed presentation using the methodology from [50]. Consider coupled equations of the form*

$$\frac{d}{dt}\tilde{x}(t) = A_1 \tilde{x}(t) + A_2 y(t), \tag{3.42a}$$

$$0 = A_3 \tilde{x}(t) + A_4 y(t). \tag{3.42b}$$

Here, for $i = 1, \dots, 4$, the operators $A_i : D(A_i) \subset Z \to Z$ are closed and densely defined. Let $\mathcal{X} = \mathcal{Z} = Z \times Z$ and $D(A) = (D(A_1) \cap D(A_3)) \times (D(A_2) \cap D(A_4))$. Then, setting

$$x(t) = \begin{bmatrix} \tilde{x}(t) \\ \tilde{y}(t) \end{bmatrix}, \tag{3.43}$$

these coupled equations can be written in the form (3.1) as

$$\frac{d}{dt} \underbrace{\begin{bmatrix} I^Z & 0 \\ 0 & 0 \end{bmatrix}}_{E} x(t) = \underbrace{\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}}_{A} x(t), \quad t > 0. \tag{3.44}$$

We now illustrate that system (3.42) has radiality-index 0, and is $(E, 0)-$radial. We shall make some assumptions that will ensure well-posedness. The notation $\overline{A}$ denotes the closure of an operator.

Assumption 1:

- *Let A_4 have a bounded inverse, $D(A_4) \subset D(A_2)$ and $D(A_4^*) \subset D(A_3^*)$. Then, the operator $A_2 A_4^{-1} A_3 : D(A_3) \to Z$ is well-defined; see [116, Remark 2.2.315]. We also assume that $\overline{A_2 A_4^{-1} A_3} \in L(Z)$.*
- *The operator A_1 generates a C_0-semigroup on Z, so there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that for every $s > \omega$, $s \in \rho(A_1)$ and*

$$\|(sI - A_1)^{-n}\| \leq \frac{M}{(s - \omega)^n}, \quad s > \omega, \ n \in \mathbb{N}.$$

For $s > \omega$, we also define the Schur complement $S_1(s) : D(A_1) \subset Z \to Z$ as

$$S_1(s) := sI - A_1 + \overline{A_2 A_4^{-1} A_3}. \tag{3.45}$$

Since A_1 is closed and densely defined, it follows that the Schur complement $S_1(s)$ is closed and densely defined.

Proposition 3.3.1. *[116, Theorem 2.3.3] The operator A is closable. Let $\omega_0 = \omega I + M\|A_2 A_4^{-1} A_3\|$, then for every $s > \omega_0$, $s \in \rho(E, A)$ and*

$$\begin{aligned}
(sE - \overline{A})^{-1} &= \begin{bmatrix} sI - A_1 & -A_2 \\ -A_3 & -A_4 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} I & 0 \\ -\overline{A_4^{-1} A_3} & I \end{bmatrix} \begin{bmatrix} S_1(s)^{-1} & 0 \\ 0 & -A_4^{-1} \end{bmatrix} \begin{bmatrix} I & -A_2 A_4^{-1} \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} S_1(s)^{-1} & -S_1(s)^{-1} A_2 A_4^{-1} \\ -\overline{A_4^{-1} A_3} S_1(s)^{-1} & \overline{A_4^{-1} A_3} S_1(s)^{-1} A_2 A_4^{-1} - A_4^{-1} \end{bmatrix}.
\end{aligned}$$

Theorem 3.3.1. *The operator $A - \omega_0 E$ is $(E, 0)-$radial.*

Proof. We first rewrite the Schur complement (3.45) as

$$S_1(s) = (sI - A_1)\Big(I + (sI - A_1)^{-1}\overline{A_2 A_4^{-1} A_3}\Big).$$

Using Neumann series, it can be shown that the operator $S_1(s)$ is invertible for $s > \omega_0 = \omega I + M\|A_2 A_4^{-1} A_3\|$ and

$$\begin{aligned} \|S_1(s)^{-n}\| &\leq \|(sI - A_1)^{-n}\| \| \left(I + (sI - A_1)^{-1}\, \overline{A_2 A_4^{-1} A_3}\right)^{-1} \|^n \\ &\leq \frac{M}{(sI - \omega)^n \Big(1 - \frac{M\|\overline{A_2 A_4^{-1} A_3}\|}{(s-\omega)}\Big)^n} \\ &= \frac{M}{(sI - \omega_0)^n}. \end{aligned} \tag{3.46}$$

Now, with the help of Proposition 3.3.1, we obtain

$$\begin{aligned} (sE - A)^{-1}E)^n &= \begin{bmatrix} S_1(s)^{-n} & 0 \\ \overline{-A_4^{-1} A_3 S_1(s)^{-n}} & 0 \end{bmatrix}, \\ &= \begin{bmatrix} I & 0 \\ \overline{-A_4^{-1} A_3} & 0 \end{bmatrix} \begin{bmatrix} S_1(s)^{-n} & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} (E(sE - A)^{-1})^n &= \begin{bmatrix} S_1(s)^{-n} & -S_1(s)^{-n} A_2 A_4^{-1} \\ 0 & 0 \end{bmatrix}, \\ &= \begin{bmatrix} S_1(s)^{-n} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & -A_2 A_4^{-1} \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

Referring to Definition 3.2.4, we use the calculations above together with inequality (3.46) to conclude that $A - \omega_0 E$ is $(E, 0)-$radial. Since ran E is closed, Theorem 3.2.3 implies that system (3.42) is also well-posed. □

We now calculate the projections $P^{\mathcal{X}_1}$ and $P^{\mathcal{Z}_1}$. Recall from assumption 3.3.1 that A_1 generates a C_0-semigroup,

$$\lim_{s \to \infty} s(S_1(s))^{-1} z = z.$$

Hence, recalling (3.29) *and* (3.30)*,*

$$P^{\mathcal{X}_1}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} = \lim_{s\to\infty} s(sE-A)^{-1}\begin{bmatrix}\tilde{x}\\ 0\end{bmatrix} = \lim_{s\to\infty}\left[\frac{sS_1(s)^{-1}\tilde{x}}{-A_4^{-1}A_3 sS_1(s)^{-1}\tilde{x}}\right]$$
$$= \left[\frac{I}{-A_4^{-1}A_3}\ \begin{matrix}0\\0\end{matrix}\right]\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} \tag{3.47}$$

and similarly

$$P^{\mathcal{Z}_1}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}I & -A_2A_4^{-1}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix}. \tag{3.48}$$

It follow that the Weierstrass-Kronecker form (3.37) *of system* (3.42) *is*

$$\frac{d}{dt}x_1(t) = (A_1 - \overline{A_2A_4^{-1}A_3})x_1(t), \tag{3.49}$$
$$0 = x_0(t). \tag{3.50}$$

Here $x_1(t) = \tilde{x}(t)$ *and* $x_0(t) = A_3\tilde{x}(t) + A_4\tilde{y}(t)$*.*

Now, let us consider coupled systems of the form (3.42) *with distributed control,*

$$\frac{d}{dt}\tilde{x}(t) = A_1\tilde{x}(t) + A_2y(t) + B_{1u}u(t), \tag{3.51a}$$
$$0 = A_3\tilde{x}(t) + A_4y(t) + B_{2u}u(t). \tag{3.51b}$$

Here, for $i = 1,\ 2$ *the operator* $B_{iu} \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$*. We write system* (3.51) *as*

$$\frac{d}{dt}\underbrace{\begin{bmatrix}I^Z & 0\\ 0 & 0\end{bmatrix}}_{E}x(t) = \underbrace{\begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}}_{A}x(t) + \underbrace{\begin{bmatrix}B_{1u}\\ B_{0u}\end{bmatrix}}_{B}u(t), \quad t > 0. \tag{3.52}$$

Corollary 3.3.1. *Let*

$$x_1(t) = \tilde{x}(t), \quad x_0(t) = A_3\tilde{x}(t) + A_4\tilde{y}(t), \tag{3.53a}$$
$$\tilde{A}_1 = (A_1 - A_2\overline{A_4^{-1}A_3}), \quad \tilde{A}_0 = I^Z \tag{3.53b}$$
$$\tilde{B}_1 = (B_{1u} - A_2A_4^{-1}B_{0u}), \quad \tilde{B}_0 = B_{0u}. \tag{3.53c}$$

System (3.51) *can be written in the Weierstrass-Kronecker form* (3.37) *as follows.*

$$\frac{d}{dt}x_1(t) = \tilde{A}_1x_1(t) + \tilde{B}_1u(t), \tag{3.54}$$
$$0 = x_0(t) + \tilde{B}_0u(t). \tag{3.55}$$

Proof. The proof relies on applying the projections $P^{\mathcal{X}_1}$ and $P^{\mathcal{Z}_1}$ obtained previously in (3.47) and (3.48), respectively, on equation (3.52). The calculations are tedious but will be given for completeness. First, it will prove useful to compute $P^{\mathcal{Z}_0}$ from (3.32) as follows

$$P^{\mathcal{Z}_0} = I^{\mathcal{Z}} - P^{\mathcal{Z}_1} = \begin{bmatrix} 0 & A_2 A_4^{-1} \\ 0 & I \end{bmatrix}, \tag{3.56}$$

and so

$$P^{\mathcal{Z}_0} B = \begin{bmatrix} A_2 A_4^{-1} B_{0u} \\ B_{0u} \end{bmatrix}, \qquad P^{\mathcal{Z}_1} B = \begin{bmatrix} B_{1u} - A_2 A_4^{-1} B_{0u} \\ 0 \end{bmatrix}. \tag{3.57}$$

Also,

$$AP^{\mathcal{X}_0} = \begin{bmatrix} A_2 \overline{A_4^{-1} A_3} & A_2 \\ A_3 & A_4 \end{bmatrix}, \qquad AP^{\mathcal{X}_1} = \begin{bmatrix} A_1 - A_2 \overline{A_4^{-1} A_3} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.58}$$

and

$$EP^{\mathcal{X}_0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad EP^{\mathcal{X}_1} = \begin{bmatrix} I^{Z} & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.59}$$

Using the calculations above, the sub-system

$$\frac{d}{dt} EP^{\mathcal{X}_1} x(t) = P^{\mathcal{X}_1} x(t) + P^{\mathcal{Z}_1} Bx(t),$$

yields

$$\frac{d}{dt} \begin{bmatrix} I^{Z} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{y}(t) \end{bmatrix} = \begin{bmatrix} A_1 - A_2 \overline{A_4^{-1} A_3} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{y}(t) \end{bmatrix} + \begin{bmatrix} B_{1u} - A_2 A_4^{-1} B_{0u} \\ 0 \end{bmatrix} u(t),$$

which implies that

$$\frac{d}{dt} \tilde{x}(t) = (A_1 - A_2 \overline{A_4^{-1} A_3}) \tilde{x}(t) + (B_{1u} - A_2 A_4^{-1} B_{0u}) u(t). \tag{3.60}$$

Similarly, the sub-system

$$\frac{d}{dt} EP^{\mathcal{X}_0} x(t) = P^{\mathcal{X}_0} x(t) + P^{\mathcal{Z}_0} Bu(t),$$

yields

$$\frac{d}{dt} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{y}(t) \end{bmatrix} = \begin{bmatrix} A_2 \overline{A_4^{-1} A_3} & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{y}(t) \end{bmatrix} + \begin{bmatrix} A_2 A_4^{-1} B_{0u} \\ B_{0u} \end{bmatrix} u(t),$$

which leads to

$$A_2\tilde{y}(t) = -A_2\overline{A_4^{-1}A_3}\tilde{x}(t) - A_2A_4^{-1}B_{0u}u(t), \tag{3.61}$$
$$0 = A_3\tilde{x}(t) + A_4\tilde{y}(t) + B_{0u}u(t). \tag{3.62}$$

The conclusion follows from equations (3.60) and (3.62). Equation(3.61) presents the connection between $\tilde{y}(t)$ and $\tilde{x}(t)$, which also follows directly form (3.53) since A_4 is assumed to be invertible. $\square$

Example 3.3.2. *Consider the following parabolic-elliptic system*

$$\frac{\partial}{\partial t}w(\xi,t) = \frac{\partial^2}{\partial \xi^2}w(\xi,t) - \rho w(\xi,t) + \alpha v(\xi,t), \tag{3.63}$$
$$0 = \frac{\partial^2}{\partial \xi^2}v(\xi,t) - \gamma v(\xi,t) + \beta w(\xi,t), \tag{3.64}$$

where $\xi \in \Omega = (0,1)$, $t \geq 0$, and the parameters ρ, α, β, γ are real-valued. The boundary conditions are

$$\frac{\partial}{\partial \xi}w(0,t) = \frac{\partial}{\partial \xi}w(1,t) = 0,$$
$$\frac{\partial}{\partial \xi}v(0,t) = \frac{\partial}{\partial \xi}v(1,t) = 0.$$

Define

$$\mathcal{X} = \mathcal{W} \times \mathcal{V} = L_2(\Omega) \times L_2(\Omega), \quad \mathcal{Z} = L_2(\Omega) \times L_2(\Omega),$$

and set the operator A_d as

$$A_d(w(\xi)) = \frac{d^2}{d\xi^2}w(\xi),$$

with

$$D(A_d) = \{w(\xi) \in H^2(\Omega) : \frac{\partial}{\partial \xi}w(0) = \frac{\partial}{\partial \xi}w(1) = 0\},$$

System (3.63)-(3.64) can be written as

$$\frac{d}{dt}\underbrace{\begin{bmatrix} I_w & 0 \\ 0 & 0 \end{bmatrix}}_{E}\underbrace{\begin{bmatrix} w(t) \\ v(t) \end{bmatrix}}_{x(t)} = \underbrace{\begin{bmatrix} A_d - \rho I_w & -\alpha I_v \\ \beta I_w & A_d - \gamma I_v \end{bmatrix}}_{A}\begin{bmatrix} w(t) \\ v(t) \end{bmatrix}, \tag{3.65}$$

where $I_w \in \mathcal{L}(\mathcal{W})$, $I_v \in \mathcal{L}(\mathcal{V})$ *are the injections operators on* $\mathcal{W}$ *and* $\mathcal{V}$, *respectively. Let* $\{\phi_k : k = 0, 1, \dots\}$ *be the orthonormal eigenfunctions of the operator* $A_d w = \frac{d^2}{d\xi^2} w$, *and let* $\{\eta_k : k = 0, 1, \dots\}$ *be the corresponding eigenvalues. It can be shown that* $\phi_k(\xi) = \cos(k\pi\xi)$, *and* $\eta_k = -(k\pi)^2$.

Theorem 3.3.2. *Consider system* (3.65).

1. *If* $\gamma \neq \eta_k$, *then* A *is* $(E, 0)-$*radial.*
2. *If* $\gamma = \eta_k$, *then* A *is* $(E, 1)-$*radial.*

Proof. **1.** The first statement is a consequence of Example 3.3.1.

2. The proof is similar to the one given in [120, Theorem 5]. Using Fourier series expansion in the basis $\phi_k : k = 0, 1, \dots$,

$$(\lambda E - A)^{-1} = \begin{bmatrix} \displaystyle\sum_{k=1}^{\infty} \frac{(-\eta_k + \gamma) < ., \phi_k > \phi_k}{\lambda\gamma - (\lambda + \gamma + \rho)\eta_k + \rho\gamma + \eta_k^2 - \beta\alpha} & \displaystyle\sum_{k=1}^{\infty} \frac{\alpha < ., \phi_k > \phi_k}{\lambda\gamma - (\lambda + \gamma + \rho)\eta_k + \rho\gamma + \eta_k^2 - \beta\alpha} \\ \displaystyle\sum_{k=1}^{\infty} \frac{\beta < ., \phi_k > \phi_k}{\lambda\gamma - (\lambda + \gamma + \rho)\eta_k + \rho\gamma + \eta_k^2 - \beta\alpha} & \displaystyle\sum_{k=1}^{\infty} \frac{(\lambda + \rho - \eta_k) < ., \phi_k > \phi_k}{\lambda\gamma - (\lambda + \gamma + \rho)\eta_k + \rho\gamma + \eta_k^2 - \beta\alpha} \end{bmatrix}.$$

Define

$$\mu_k = \frac{-(\gamma + \rho)\eta_k + \rho\gamma + \eta_k^2 - \beta\alpha}{(\gamma - \eta_k)},$$

it follows that

$$(\lambda E - A)^{-1} = \begin{bmatrix} \displaystyle\sum_{\eta_k \neq \gamma} \frac{< ., \phi_k > \phi_k}{(\lambda - \mu_k)} & \displaystyle\sum_{\eta_k \neq \gamma} \frac{\alpha < ., \phi_k > \phi_k}{(\gamma - \eta_k)(\lambda - \mu_k)} - \sum_{\eta_k = \gamma} \beta < ., \phi_k > \phi_k \\ \displaystyle\sum_{\eta_k \neq \gamma} \frac{\beta < ., \phi_k > \phi_k}{(\gamma - \eta_k)(\lambda - \mu_k)} - \sum_{\eta_k = \gamma} \alpha < ., \phi_k > \phi_k & \displaystyle\sum_{\eta_k \neq \gamma} \frac{(\lambda + \rho - \eta_k) < ., \phi_k > \phi_k}{(\gamma - \eta_k)(\lambda - \mu_k)} - \frac{(\lambda + \rho - \gamma)}{\beta\alpha} \sum_{\eta_k = \gamma} < ., \phi_k > \phi_k \end{bmatrix},$$

Define $\mathbb{K} = \{k \in \mathbb{N} : \eta_k \neq \gamma\}$, then

$$a = \max_{k \in \mathbb{K}} \mu_k < \infty,$$

as the set $\{\eta_k : k \in \mathbb{K}\}$ is bounded on the right. Thus, there exists $a \in \mathbb{R}$ such that $(a, \infty) \subset \rho^E(A)$. Also,

$$R^E_\lambda(A) = (\lambda E - A)^{-1}E = \begin{bmatrix} \displaystyle\sum_{\eta_k \neq \gamma} \frac{< ., \phi_k > \phi_k}{(\lambda - \mu_k)} & 0 \\ \displaystyle\sum_{\eta_k \neq \gamma} \frac{\beta < ., \phi_k > \phi_k}{(\gamma - \eta_k)(\lambda - \mu_k)} - \sum_{\eta_k = \gamma} \alpha < ., \phi_k > \phi_k & 0 \end{bmatrix},$$

$$R^E_{\lambda_0}(A) R^E_{\lambda_1}(A) = \begin{bmatrix} \displaystyle\sum_{\eta_k \neq \gamma} \frac{< ., \phi_k > \phi_k}{(\lambda_0 - \mu_k)(\lambda_1 - \mu_k)} & 0 \\ \displaystyle\sum_{\eta_k \neq \gamma} \frac{< ., \phi_k > \phi_k}{(\gamma - \eta_k)(\lambda_0 - \mu_k)(\lambda_1 - \mu_k)} & 0 \end{bmatrix},$$

$$\begin{aligned} L^E_\lambda(A) &= E(\lambda E - A)^{-1} \\ &= \begin{bmatrix} \displaystyle\sum_{\eta_k \neq \gamma} \frac{< ., \phi_k > \phi_k}{(\lambda - \mu_k)} & \displaystyle\sum_{\eta_k \neq \gamma} \alpha \frac{< ., \phi_k > \phi_k}{(\gamma - \eta_k)(\lambda - \mu_k)} - \sum_{\eta_k = \gamma} \beta < ., \phi_k > \phi_k \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

$$L^E_{\lambda_0}(A) L^E_{\lambda_1}(A) = \begin{bmatrix} \displaystyle\sum_{\eta_k \neq \gamma} \frac{< ., \phi_k > \phi_k}{(\lambda_0 - \mu_k)(\lambda_1 - \mu_k)} & \displaystyle\sum_{\eta_k \neq \gamma} \frac{\alpha < ., \phi_k > \phi_k}{(\gamma - \eta_k)(\lambda_0 - \mu_k)(\lambda_1 - \mu_k)} \\ 0 & 0 \end{bmatrix},$$

Taking

$$C = \max_{k \in \mathbb{K}} \left\{ 1,\ \left| \frac{1}{1 + \eta_k} \right|,\ \left| \frac{1}{(1 + \eta_k)^2} \right| \right\},$$

then for $\mu,\ \lambda_0,\ \lambda_1 \in \rho^E(A)$, we obtain

$$\max\left\{\|(R^E_{(\lambda,1)}(A))^n\|_{\mathcal{L}(\mathcal{X})}, \|(L^E_{(\lambda,1)}(A))^n\|_{\mathcal{L}(\mathcal{Z})}\right\} \leq \frac{C}{|\lambda_0 - a||\lambda_1 - a|},$$

Thus, the operator A is $(E, 1)-$radial.

□

Example 3.3.3. *Consider the linearized Navier–Stokes system*

$$\frac{\partial}{\partial t}v(x,t) = \nu\Delta v(x,t) - \nabla p, \quad (x,t) \in \Omega \times \mathbb{R}_+, \tag{3.66a}$$

$$\nabla \cdot v = 0, \quad (x,t) \in \Omega \times \mathbb{R}_+. \tag{3.66b}$$

$$v(x,0) = v_0(x), \quad x \in \Omega, \tag{3.66c}$$

$$v(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{3.66d}$$

Here, ∇p is the pressure gradient, $\nu > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$ of class C^∞. System (3.66) was shown to be $(E, 1)-$radial in [37, Section 7].

We present another example for an equation that falls under the form (3.1); see [38, section 6] and [40, section 5]. The following definition is needed.

Definition 3.3.1. (E-eigenvalue and E-eigenvector with respect to A)
$\lambda \in \mathbb{C}$ is said to be E-eigenvalue of the operator A if there exists a vector $z \neq 0$ such that $\lambda E z = Az$. In that case, such vector z is called the E-eigenvector of the operator A corresponding to the E-eigenvalue λ.

Example 3.3.4. *Consider the following equation*

$$\frac{\partial}{\partial t}\left(1 + \frac{\partial^2}{\partial \xi^2}\right)x(\xi,t) = \left(\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^4}{\partial \xi^4}\right)x(\xi,t) + u(t), \tag{3.67}$$

$\xi \in (0,\pi)$ and $(0,+\infty)$, subject to the following initial and boundary conditions:

$$x(0,t) = \frac{\partial^2 x}{\partial \xi^2}(0,t) = 0,$$

$$x(\pi,t) = \frac{\partial^2 x}{\partial \xi^2}(\pi,t) = 0,$$

$$x(\xi,0) = x_0(\xi).$$

Defining

$$\begin{aligned}
&\mathcal{X} = \{x \in H^2(0,\pi) : x(0) = x(\pi) = 0\}, \quad \mathcal{Z} = L_2(0,\pi),\\
&E = I + \frac{d^2}{d\xi^2}, \quad A = \frac{d^2}{d\xi^2} + 2\frac{d^4}{d\xi^4},\\
D(A) &= \{x \in H^4(0,\pi) : x(0) = \frac{d^2}{d\xi^2}x(0) = x(\pi) = \frac{d^2}{d\xi^2}x(\pi) = 0\},\\
Bu(x) &= u, \qquad u \in \mathcal{U} = \mathbb{R},
\end{aligned}$$

We rewrite equation (3.67) *as a PDAE of the form*(3.1),

$$\frac{d}{dt}\underbrace{\left(I + \frac{d^2}{d\xi^2}\right)}_{E} x(t) = \underbrace{\left(\frac{d^2}{d\xi^2} + 2\frac{d^4}{d\xi^4}\right)}_{A} x(t) + \underbrace{I}_{B} u(t), \quad t > 0. \tag{3.68}$$

Clearly, $\ker E \neq 0$*. The operator* A *is* $(E,0)-$*radial. We must show that the conditions in Definition 3.2.3 hold. Define the operator* $\Delta = \frac{d^2}{d\xi^2}$ *where*

$$D(\Delta) = \{x \in H^2(0,\pi) : x(0) = x(\pi) = 0\}.$$

Let $\phi_k : k = 1,\ 2, \ldots$ *be the eigenfunctions of the Laplacian operator* Δ*, which are orthonormal in the sense of scalar product in* $\langle \cdot,\cdot \rangle$ *and let* $\{\eta_k : k \in \mathbb{N}\}$ *be the corresponding eigenvalues. One can show that* $\phi_k(x) = sin(kx),\ \eta_k = -k^2$ *where* $k \in \mathbb{N}$*. Recalling Definition 3.3.1, and since*

$$\begin{aligned}
&Esin(x) = 0, \quad Asin(x) = sin(x),\\
&Esin(kx) = (1-k^2)sin(kx),\ (k = 2,\ 3, \ldots),\\
&Asin(kx) = (2k^4 - k^2)sin(kx),\ (k = 2,\ 3, \ldots),
\end{aligned}$$

it is evident that ϕ_k *is also an E-eigenvector of the operator* A *corresponding to the eigenvalue* $\mu_k = \frac{2k^4-k^2}{1-k^2} = -k^2(1+\frac{k^2}{k^2-1}),\ k = 2,\ 3,\ 4, \ldots$*. The set* $\{\mu_k,\ k \neq 1\}$ *form the spectrum set* $\sigma^E(A)$*. The spectrum of* A *is real and approaches* $-\infty$ *as* $k \to \infty$ *and so is the spectrum of* A *with respect to* E*. Therefore, there exists* $a \geq 0$ *such that* $\sigma^E(A) \subset \{\mu \in \mathbb{C} : Re\,\mu \leq a\}$*. Hence, there exists* $a > 0$ *such that* $(a,\infty) \subset \rho^E(A)$*.*

Let $\lambda > a$*,* $z \in \mathcal{Z}$ *and* $x \in \mathcal{X}$ *such that* $(\lambda E - A)x = z$*, recalling that* Δ *is the Laplacian operator with eigenvalues* η_k *and that*

$$E = I + \Delta, \qquad A = \Delta^2 + 2\Delta^4,$$

we use a Fourier series expansion as follows.

$$\begin{aligned}(\lambda E - A)x &= \sum_{k=1}^{\infty}(\lambda(1+\eta_k) - (\eta_k^2 + 2\eta_k^4)) < x, \phi_k > \phi_k \\ &= z \\ &= \sum_{k=1}^{\infty} < z, \phi_k > \phi_k,\end{aligned}$$

and so

$$< x, \phi_k >= \frac{< z, \phi_k >}{(\lambda(1+\eta_k) - (\eta_k^2 + 2\eta_k^4))}.$$

Thus,

$$\begin{aligned}(\lambda E - A)^{-1} z = x &= \sum_{k=1}^{\infty} < x, \phi_k > \phi_k \\ &= \sum_{k=1}^{\infty} \frac{< z, \phi_k > \phi_k}{(\lambda(1+\eta_k) - (\eta_k^2 + 2\eta_k^4))},\end{aligned}$$

and

$$\begin{aligned}E(\lambda E - A)^{-1} = (\lambda E - A)^{-1} E = &= \sum_{k=1}^{\infty} \frac{(1+\eta_k) < \cdot, \phi_k > \phi_k}{(\lambda(1+\eta_k) - (\eta_k^2 + 2\eta_k^4))} \\ &= \sum_{k|\eta_k \neq -1} \frac{< \cdot, \phi_k > \phi_k}{(\lambda - \frac{(\eta_k^2 + 2\eta_k^4)}{(1+\eta_k)})}.\end{aligned}$$

Also,

$$\Big(R_\lambda^E(0,A)\Big)^n = \Big((\lambda E - A)^{-1} E\Big)^n = \sum_{k|\eta_k \neq -1} \frac{< \cdot, \phi_k > \phi_k}{\Big(\lambda - \frac{(\eta_k^2 + 2\eta_k^4)}{(1+\eta_k)}\Big)^n}. \tag{3.69}$$

Similarly,

$$\Big(L_\lambda^E(0,A)\Big)^n = \Big(E(\lambda E - A)^{-1}\Big)^n = \sum_{k|\eta_k \neq -1} \frac{< \cdot, \phi_k > \phi_k}{\Big(\lambda - \frac{(\eta_k^2 + 2\eta_k^4)}{(1+\eta_k)}\Big)^n}. \tag{3.70}$$

Recall that the eigenvalues of the Laplacian operator Δ *are* $\eta_k = -k^2$*. We use statements* (3.69), (3.70)*, with* $\eta_k \neq -1$*, i.e., the natural number* $k \neq 1$*, and we set*

$$a = \frac{(\eta_k^2 + 2\eta_k^4)}{(1 + \eta_k)} = \frac{2k^4 - k^2}{1 - k^2}, \qquad K = 1,$$

in the definition of the $(E, 0)-$*radiality. It follows that* A *is* $(E, 0)-$*radial.*

We now calculate the projections $P^{\mathcal{X}_1}$ *and* $P^{\mathcal{Z}_1}$*. Since* $p = 0$*, statement* (3.30) *implies that*

$$\begin{aligned} P^{\mathcal{Z}_1} &= \lim_{\lambda\to\infty} (\lambda L_\lambda^E(A))^{p+1} \\ &= \lim_{\lambda\to\infty} \lambda E(\lambda E - A)^{-1} \\ &= \lim_{\lambda\to\infty} \sum_{k\neq 1} \frac{\lambda < ., \phi_k > \phi_k}{\lambda - \mu_k} \\ &= \sum_{k\neq 1} < ., \phi_k > \phi_k. \end{aligned}$$

Similarly, statement (3.29) *leads to*

$$P^{\mathcal{X}_1} = \sum_{k\neq 1} < ., \phi_k > \phi_k.$$

Define the sub-spaces $\mathcal{X}_1$*,* $\mathcal{X}_0$*,* $\mathcal{Z}_1$ *and* $\mathcal{Z}_0$ *as follows*

$$\begin{aligned} \mathcal{X}_1 = \mathcal{Z}_1 = \overline{span\{sin(kx) : k \neq 1\}}, \\ \mathcal{X}_0 = \mathcal{Z}_0 = span\{sin(kx) : k = 1\}. \end{aligned}$$

It is clear that the subspaces $\mathcal{X}_0$*,* $\mathcal{Z}_0$ *are one-dimensional. Equation* (3.68) *can be transformed into the Weierstrass-Kronecker form, decomposing the system into two sub-systems. One sub-system evolves in subspace* $\mathcal{X}_1$*, excluding the eigenvector where the operator* E *fails to be invertible, converting system* (3.67) *into a standard PDE. The other sub-system is defined on* $\mathcal{X}_0$*, which includes the eigenvector where the operator* E *is non-invertible. Let* E_i *and* A_i *be the restrictions of* E *and* A *on* $\mathcal{X}_i$ *for* $i = 0,\ 1$*. Then,*

$$\begin{aligned} &\tilde{A}_1\ sin(kx) = E_1^{-1} A_1 sin(kx) = \left(\frac{2k^4 - k^2}{1 - k^2}\right) sin(kx), \ \ k = 2,\ 3, \ldots \\ &dom(\tilde{A}_1) = span\{sin(kx) : k = 2,\ 3,\ \ldots\}. \end{aligned}$$

$$
\begin{aligned}
B_1 &= E_1^{-1}\sum_{k=1}^{\infty}\frac{< 1, sin(kx) >_{L^2(0,\pi)}}{< sin(kx), sin(kx) >_{L^2(0,\pi)}} sin(kx)\\
&= E_1^{-1}(1-\frac{4}{\pi}sin(x))\\
&= \sum_{k=2}^{\infty}\frac{4}{(2k-1)\pi}E_1^{-1}(sin((2k-1)x))\\
&= \sum_{k=2}^{\infty}\frac{4}{[1-(2k-1)^2](2k-1)\pi}E_1^{-1}(sin((2k-1)x)),
\end{aligned}
$$

$$
\begin{aligned}
B_0 &= A_0^{-1}\frac{< 1, sin(x) >_{L^2(0,\pi)}}{< sin(x), sin(x) >_{L^2(0,\pi)}} sin(x) = \frac{4}{\pi}sin(x),\\
N &= A_0^{-1}E_0 = 0.
\end{aligned}
$$

Hence, equation (3.67) can be decomposed into

$$
\begin{aligned}
\frac{d}{dt}x_1(t) &= A_1x_1(t) + B_1u(t),\\
0 &= x_0(t) + B_0u(t),
\end{aligned}
$$

where $\begin{bmatrix} x_1(t) \\ x_0(t) \end{bmatrix} \in \mathcal{X}_1 \times \mathcal{X}_0$.

The well-posedness of system (3.67) was shown in [50, section 5] and [38, Section 6].

Example 3.3.5. *[36, Example 5.3] Consider the following system*

$$
\rho\frac{\partial^2 v(\xi,t)}{\partial t^2} = \alpha\frac{\partial}{\partial\xi}\left(\frac{\partial v(\xi,t)}{\partial\xi}\right) - \gamma\beta\frac{\partial}{\partial\xi}\left(\frac{\partial p(\xi,t)}{\partial\xi}\right), \tag{3.71a}
$$

$$
\mu\frac{\partial^2 p(\xi,t)}{\partial t^2} = \beta\frac{\partial}{\partial\xi}\left(\frac{\partial p(\xi,t)}{\partial\xi}\right) - \gamma\beta\frac{\partial}{\partial\xi}\left(\frac{\partial v(\xi,t)}{\partial\xi}\right), \tag{3.71b}
$$

with boundary conditions

$$
v(a,t) = 0, \tag{3.71c}
$$

$$
p(a,t) = 0, \tag{3.71d}
$$

$$
\beta\frac{\partial p(b,t)}{\partial\xi} - \gamma\beta\frac{\partial v(b,t)}{\partial\xi} = 0, \tag{3.71e}
$$

$$
\alpha\frac{\partial v(b,t)}{\partial\xi} - \gamma\beta\frac{\partial p(b,t)}{\partial\xi} = 0. \tag{3.71f}
$$

These equations model the dynamics of undamped beam, fixed at one end and free at the other, over an interval $[a, b]$*. Here* $\xi \in [a, b)$*,* $t > 0$*,* $v(\xi, t)$ *is the longitudinal displacement, and* $p(\xi, t)$ *is the electric charge. The material parameters include magnetic permeability, denoted by* μ*, which is non-negative. The strictly positive parameters are material density* ρ*, elastic stiffness* α_1*, impermittivity* β*, and piezoelectric coefficient* γ*. We also define* $\alpha = \alpha_1 + \gamma^2\beta$*. The total energy of the system is given by:*

$$H(t) = \frac{1}{2}\int_a^b \Big[\rho\left(\frac{\partial v(\xi,t)}{\partial t}\right)^2 + \alpha_1\left(\frac{\partial v(\xi,t)}{\partial \xi}\right)^2 + \mu\left(\frac{\partial p(\xi,t)}{\partial t}\right)^2 + \beta\left(\frac{\partial p(\xi,t)}{\partial \xi} - \gamma\frac{\partial v(\xi,t)}{\partial \xi}\right)^2\Big]d\xi.$$

This model was shown to be well-posed in [86] associated with a contraction semigroup. A different choice of state variable was used in [36], that is,

$$x(\xi,t) = \begin{bmatrix} \frac{\partial v(\xi,t)}{\partial \xi} \\ \sqrt{\rho}\frac{\partial v(\xi,t)}{\partial t} \\ \frac{\partial p(\xi,t)}{\partial \xi} \\ \sqrt{\mu}\frac{\partial p(\xi,t)}{\partial t} \end{bmatrix}.$$

We write system (3.71) *as*

$$\frac{\partial}{\partial t}\underbrace{\begin{bmatrix} \sqrt{\rho} & 0 & 0 & 0 \\ 0 & \sqrt{\rho} & 0 & 0 \\ 0 & 0 & \sqrt{\mu} & 0 \\ 0 & 0 & 0 & \sqrt{\mu} \end{bmatrix}}_{E} x(\xi,t) = \underbrace{\begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}\frac{\partial}{\partial \xi}\begin{bmatrix} \alpha & 0 & -\gamma\beta & 0 \\ 0 & I & 0 & 0 \\ -\gamma\beta & 0 & \beta & 0 \\ 0 & 0 & 0 & I \end{bmatrix}}_{A} x(\xi,t). \quad (3.72)$$

When μ *is very small, it is often considered to be zero, yielding in a quasi-static piezoelectric beam model. Under this assumption, the fourth state variable becomes zero. Equation* (3.72) *simplifies to,*

$$\frac{\partial}{\partial t}\begin{bmatrix} \sqrt{\rho} & 0 & 0 & 0 \\ 0 & \sqrt{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} \frac{\partial v(\xi,t)}{\partial \xi} \\ \sqrt{\rho}\frac{\partial v(\xi,t)}{\partial t} \\ \frac{\partial p(\xi,t)}{\partial \xi} \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}\frac{\partial}{\partial \xi}\begin{bmatrix} \alpha & 0 & -\gamma\beta & 0 \\ 0 & I & 0 & 0 \\ -\gamma\beta & 0 & \beta & 0 \\ 0 & 0 & 0 & I \end{bmatrix}\begin{pmatrix} \frac{\partial v(\xi,t)}{\partial \xi} \\ \sqrt{\rho}\frac{\partial v(\xi,t)}{\partial t} \\ \frac{\partial p(\xi,t)}{\partial \xi} \\ 0 \end{pmatrix}.$$

Since the fourth variable is identically zero, the matrix E *is reduced by removing the fourth* *row and column, yielding*

$$\frac{\partial}{\partial t}\begin{bmatrix}\sqrt{\rho} & 0 & 0\\ 0 & \sqrt{\rho} & 0\\ 0 & 0 & 0\end{bmatrix}\begin{pmatrix}\frac{\partial v(\xi,t)}{\partial \xi}\\ \sqrt{\rho}\frac{\partial v(\xi,t)}{\partial t}\\ \frac{\partial p(\xi,t)}{\partial \xi}\end{pmatrix} = \begin{bmatrix}0 & I & 0\\ I & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\frac{\partial}{\partial \xi}\begin{bmatrix}\alpha & 0 & -\gamma\beta\\ 0 & I & 0\\ -\gamma\beta & 0 & \beta\end{bmatrix}\begin{pmatrix}\frac{\partial v(\xi,t)}{\partial \xi}\\ \sqrt{\rho}\frac{\partial v(\xi,t)}{\partial t}\\ \frac{\partial p(\xi,t)}{\partial \xi}\end{pmatrix}.$$

This PDAE is $(E,0)-$radial, as shown in Example 3.3.1.

Example 3.3.6. *[36, Example 6.5] Let*

$$Fz = z'', \quad D(F) = \{w \in H^2(0,1) \mid z'(0) = z'(1) = 0\}.$$

F generates a C_0-semigroup on $L^2(0,1)$, and denote the growth bound of the semigroup by ω. Define the operators $Bu = u$ where $u \in \mathbb{C}$, and $Cz = \langle z, 1\rangle$ for any $z \in L^2(0,1)$. Consider the PDAE on $Z = L^2(0,1)\times\mathbb{C}$

$$\frac{d}{dt}\underbrace{\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}}_{E}x(t) = \underbrace{\begin{bmatrix}F & B\\ C & 0\end{bmatrix}}_{A}x(t), \quad t \geq 0. \tag{3.73}$$

Define $G(s) = C(sI - F)^{-1}B$, and the operators

$$\begin{aligned} R_{1,s} &= (sI - F)^{-1}(I - BG(s)^{-1}C(sI - F)^{-1}),\\ R_{2,s} &= G(s)^{-1}C(sI - F)^{-1},\\ L_{2,s} &= (sI - F)^{-1}BG(s)^{-1}. \end{aligned}$$

Then,

$$(sE - A)^{-1}E = \begin{pmatrix}R_{1,s} & 0\\ R_{2,s} & 0\end{pmatrix}, \qquad E(sE - A)^{-1} = \begin{pmatrix}R_{1,s} & L_{2,s}\\ 0 & 0\end{pmatrix},$$

$$(sI - F)^{-1}Bu = \frac{1}{s}u, \qquad C(sI - F)^{-1}z = \frac{1}{s}\langle z, 1\rangle,$$
$$G(s) = C(sI - F)^{-1}B = \frac{1}{s}.$$

Hence,

$$R_{1,s}z = (sI - F)^{-1}z - \frac{\langle z, 1\rangle}{s}, \qquad R_{2,s}z = \langle z, 1\rangle,$$
$$L_{2,s}u = u.$$

Since $R_{2,s}$ is independent of s, the radiality degree must be larger than 0. Define the projection onto $W_1 := \ker C \subset W$,

$$Q_c z = z - \frac{\langle z, c\rangle}{\langle c, c\rangle} c.$$

Then, with $W_2 := \operatorname{span}\{c\}$, Q_c splits W into $W_1 \oplus W_2$. For $\alpha_1 \in \operatorname{span}\{c\}$, we have that

$$R_{1,s}\alpha_1 = 0,$$

and for $z \in \ker C$

$$R_{1,s} z = (sI - F)^{-1} z \in \ker C.$$

Thus, for $w = \alpha z_1 \in W_1 \oplus W_2$

$$R_s^E(A) R_\mu^E(A) \begin{pmatrix} z + \alpha_1 \\ u \end{pmatrix} = \begin{pmatrix} R_{1,s}R_{1,\mu} & 0 \\ R_{2,s}R_{1,\mu} & 0 \end{pmatrix} \begin{pmatrix} z + \alpha_1 \\ u \end{pmatrix} = \begin{pmatrix} R_{1,s}R_{1,\mu} z \\ 0 \end{pmatrix},$$

$$L_s^E(A) L_\mu^E(A) \begin{pmatrix} z + \alpha_1 \\ u \end{pmatrix} = \begin{pmatrix} R_{1,s}R_{1,\mu} & R_{2,s}L_{2,\mu} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z + \alpha_1 \\ u \end{pmatrix} = \begin{pmatrix} R_{1,s}R_{1,\mu} z \\ 0 \end{pmatrix}.$$

Since F is the generator of a contraction semigroup,

$$\|R_{1,s}R_{1,\mu}\| \le \frac{1}{s\mu}, \quad s, \mu > 0.$$

Hence, A is $(E, 1)-$radial. Furthermore, (E, A) can be written into a Weierstrass-Kronecker form with nilpotency index $\nu = 2$; see [36, Example 6.5]. A generalization of this example is given in [35, Example 3.5].

More examples for systems that can be written in the form (3.1) can be found in [36, 37] and the references therein.

Chapter 4

Backstepping controller and observer design for a coupled parabolic-elliptic system

Many physical processes are governed by the dynamics of coupled parabolic-elliptic systems [1, 31, 69, 81, 87]. These systems fall under a specific category of linear partial differential-algebraic equations (PDAEs), as demonstrated in Section 3.3. Given that our ultimate goal is to develop control strategies for linear PDAEs, a logical initial step is to study the stabilization of these coupled systems.

Stabilization through boundary control for coupled linear parabolic PDEs has been extensively explored [9, 51, 55, 71, 105, 127]. For instance, the backstepping method was used to stabilize the dynamics of linear coupled reaction-diffusion systems with constant coefficients [9]. Later, the same problem was extended to systems with variable coefficients in [127]. There are a few papers addressing the stabilization of coupled parabolic-elliptic systems. In [30], the stabilization of such coupled systems appeared in the boundary control of linearized Navier-Stokes channel flow. Moreover, some previous works have considered specific parabolic-elliptic stabilization problems such as 2-D convection loop using singular perturbation theory and backstepping approach [121], or fluid flows where the pressure elliptical equation is considered [123]. In [61, Chapter 10], stabilization of parabolic-elliptic systems arose in the context of stabilizing boundary control of linearized Kuramoto-Sivashinsky and Korteweg-de Vries equations. Therein, the controller required the presence of two Dirichlet control inputs. More recently, the boundary control of unstable parabolic-elliptic systems with input delay was examined in [89].

The first contribution of this chapter is the design of a single feedback control law that exponentially stabilizes the dynamics of the two coupled equations. The control law is directly designed on the system of partial differential equations without approximation by finite-dimensional systems. Explicit calculation of the eigenfunctions is not required. This will be done by using a backstepping approach [61] (see Section 2.4). When using backstepping, it is typical to determine the destabilizing terms in the system, find a suitable exponentially stable target system where the destabilizing terms are eliminated, and look for an invertible state transformation of the original system into the exponentially stable target system. This requires finding a kernel of the Volterra operator and also showing that the kernel is well-defined as the solution of an auxiliary PDE. One possible approach for stabilizing a parabolic-elliptic system is to convert the coupled system into one equation in terms of the parabolic state. However, this will result in the presence of a Fredholm operator, which makes it difficult to establish a suitable kernel for the backstepping transformation. Another approach would be a vector-valued approach, which is well-documented in the literature for its efficacy in handling coupled equations; see, for instance, [9]. This approach requires the use of two control inputs to achieve the desired stabilization. However, our work is an extension of existing literature where two control inputs have already been used to stabilize similar systems [61, chap. 10] [123]. Our contribution is in showing that, under certain conditions on the system parameters, stabilization of the coupled system can be achieved through a single control input. To do so, we take a different approach. We use a single transformation previously used for a parabolic equation [61](see Section 2.5). Properties of the kernel of this transformation have already been established. This leads to an unusual target system in a parabolic-elliptic form. An explicit expression for the controller is obtained as a byproduct of the transformation. Then, the remaining step is to establish the stability of the target system obtained from the transformation, which will imply the stability of the original coupled system via the invertible transformation. The limitation of using a single control input, as opposed to two, is that the system parameters must meet a strict criterion to ensure the exponential stability of the controlled system. Therefore, we develop a weaker sufficient condition for the parameters. Instead of analyzing the stability of the target system, we leverage Lyapunov theory to directly examine the controlled coupled system while incorporating the same control input obtained through the backstepping state transformation.

The second contribution of this chapter concerns the observer design for the same parabolic-elliptic system under study. In [52], authors designed a state observer for a coupled parabolic-elliptic system by requiring a two-sided boundary input for the observer. In [125], observer design with two measurements for a parabolic-elliptic system appeared within the context of boundary observer for output-feedback stabilization of thermal-fluid convection loop. Observer design for coupled parabolic-elliptic systems also appeared when considering the observer design for the linearized Navier-Stokes Channel Flow [129]. Using a different approach from previous works, we first design an observer for the coupled parabolic-elliptic system when two boundary measurements are available. We use a backstepping transformation that was previously used for a class of parabolic PDEs in [108](see Section 2.6). The exponential stability of the observation error dynamics is shown. In parallel to our work for stabilization with a single input, we design an observer using a single boundary measurement. As for stabilization, the stability of the error dynamics requires a constraint on the system parameters. Therefore, we establish a less strict condition than the previous one to ensure stable error dynamics. This is done by means of using Lyapunov theory once again to examine the stability of the original observation error dynamics instead of the corresponding target system.

We finally combine the state feedback and observer designs to obtain an output feedback controller for the coupled parabolic-elliptic system. The result is a controller that depends on using only the available measurements to stabilize the system. Because a backstepping approach is used, no approximation of the PDAE is required. Numerical simulations are given to illustrate the theoretical results.

This chapter is organized as follows: Section 4.1 presents the well-posedness of the parabolic-elliptic systems under consideration. Stability analysis for the uncontrolled system is also described. Section 4.2 includes the first main result, which is the use of a backstepping method to design a boundary controller for the coupled system. The design of a state observer for the coupled system with two measurements is presented in Section 4.3, while the design with a single measurement is given in Section 4.4. The output feedback design is presented in Section 4.5.

4.1 Well-posedness and stability of a coupled parabolic-elliptic system

We study parabolic-elliptic systems of the form

$$w_t(x,t) = w_{xx}(x,t) - \rho w(x,t) + \alpha v(x,t), \tag{4.1}$$

$$0 = v_{xx}(x,t) - \gamma v(x,t) + \beta w(x,t), \tag{4.2}$$

$$w_x(0,t) = 0, \quad w_x(1,t) = u(t), \tag{4.3}$$

$$v_x(0,t) = 0, \quad v_x(1,t) = 0. \tag{4.4}$$

where $x \in [0,1]$, $t \geq 0$, and $v(x,0) = v_0$ and $w(x,0) = w_0$. The parameters ρ, α, β, γ are all real, with α, β both nonzero. With the notation $\Delta v(x) = \frac{d^2 v}{d^2 x}$, we define the operator $A^\gamma : D(A^\gamma) \to L_2(0,1)$

$$\begin{aligned} &A^\gamma v(x) = (\gamma I - \Delta) v(x) = w(x), \\ &D(A^\gamma) = \{v \in H^2(0,1),\ v'(0) = v'(1) = 0\}. \end{aligned}$$

For values of γ that are not eigenvalues of A^γ, that is $\gamma \neq -n\pi^2$ with $n = 0, \dots,$ the inverse operator of A^γ, $(A^\gamma)^{-1} : L_2(0,1) \to D(A^\gamma)$ exists. In this situation, the uncontrolled system (4.1)-(4.4) is well-posed [50]. It will henceforth be assumed that $\gamma \neq -(n\pi)^2$.

Alternatively, define

$$\begin{aligned} &A = \Delta - \rho I + \alpha\beta(\gamma I - \Delta)^{-1}, \\ &D(A) = \{w \in H^2(0,1) :\ w'(0) = w'(1) = 0\}. \end{aligned} \tag{4.5}$$

System (4.1)-(4.4) can be reformulated as

$$\dot{w}(t) = Aw(t),$$

$$w_x(0,t) = 0, \quad w_x(1,t) = u(t).$$

The theorem presented next establishes the well-posedness of the controlled system (4.1)-(4.4). This result is a critical foundation for the developments and findings discussed in the subsequent sections of this chapter.

Theorem 4.1.1. *If $\gamma \neq -(n\pi)^2$, the operator A generates a C_0- semigroup and system (4.1)-(4.4) with observation $w(0,t)$ is well-posed on the state-space $L_2(0,1)$. It is similarly well-posed with control instead at $x = 0$, and/or observation at $x = 1$.*

Proof. With $\alpha = 0$ the control system is the heat equation with Neumann boundary control. This control system is well-known to be well-posed on $L_2(0,1)$. Since the operator A is a bounded perturbation of Δ, then the conclusion of the theorem follows by referring to the classical results in [32, Section 5.3]. □

Lemma 4.1.1. *Let $u(t) \equiv 0$. The eigenvalues of system (4.1)-(4.4) are*

$$\lambda_n = -\rho + \frac{\alpha\beta}{\gamma + (n\pi)^2} - (n\pi)^2, \quad n = 0,\ 1,\ \dots \tag{4.6}$$

Proof. The analysis is standard but given for completeness. Let $\{\phi_j\}_{j\geq 0} \subset \mathcal{C}^4(0,1)$ be the eigenfunctions of the operator A as given in (4.5), corresponding to the eigenvalues λ_j. Then, setting $\beta(\gamma I - \partial_{xx})^{-1}\phi_j = e_j$,

$$\lambda_j \phi_j(x) = \phi_j^{''}(x) - \rho\phi_j(x) + \alpha e_j(x) \tag{4.7}$$

$$0 = e_j^{''}(x) - \gamma e_j(x) + \beta\phi_j(x) \tag{4.8}$$

$$\phi_j^{'}(0) = \phi_j^{'}(1) = 0 \tag{4.9}$$

$$e_j^{'}(0) = e_j^{'}(1) = 0. \tag{4.10}$$

Solving (4.7) for $e_j(x)$

$$e_j(x) = \frac{\rho + \lambda_j}{\alpha}\phi_j(x) - \frac{1}{\alpha}\phi_j^{''}(x). \tag{4.11}$$

Substituting for $e_j(x)$ in (4.8), we obtain the fourth-order differential equation

$$\phi_j^{''''}(x) - (\lambda_j + \rho + \gamma)\phi_j^{''}(x) + (\gamma(\lambda_j + \rho) - \alpha\beta)\phi_j(x) = 0, \tag{4.12}$$

with the boundary conditions

$$\phi_j^{'}(0) = \phi_j^{'}(1) = \phi_j^{'''}(0) = \phi_j^{'''}(1) = 0. \tag{4.13}$$

Solving system (4.12)-(4.13) for ϕ_j yields that $\phi_j = \cos(j\pi x)$ for $j = 0,\ 1, \ldots$. Substituting for ϕ_j in (4.12) and solving for λ_j leads to (4.6). □

Corollary 4.1.1. *Let $u(t) \equiv 0$. System (4.1)-(4.4) is exponentially stable if and only if*

$$\rho > \frac{\alpha\beta}{\gamma}, \tag{4.14}$$

and the decay rate in that case is bounded by the maximum eigenvalue $\rho - \frac{\alpha\beta}{\gamma}$.

Proof. The operator Δ with domain $D(A)$ is a Riesz-spectral operator, then since A is a bounded perturbation, it is also a spectral operator. Alternatively, we note that A is a self-adjoint operator with a compact inverse, and hence it is Riesz-spectral [32, section 3]. Thus, A generates a C_0-semigroup with growth bound determined by its eigenvalues. □

Thus, even in the case when the parabolic equation is exponentially stable, coupling with the elliptic system can cause the uncontrolled system to be unstable. In the remainder of this chapter, we shall assume consistent initialization, defined by

$$v_0 = \beta(\gamma I - d_{xx})w_0. \tag{4.15}$$

4.2 Boundary control for a coupled parabolic-elliptic system

As shown in the previous section, the coupled system(4.1)-(4.4) may display instability in its dynamics. To address this issue, we design a single stabilizing boundary control law $u(t)$. The first step towards the control design is to apply the invertible state transformation

$$\tilde{w}(x,t) = w(x,t) - \int_0^x k^a(x,y)w(y,t)\,dy, \tag{4.16}$$

on the parabolic state $w(x,t)$, while the elliptic state $v(x,t)$ is unchanged. Here, the kernel of the transformation $k^a(x,y)$ is given by the well-posed hyperbolic PDE in Lemma 2.5.1, that is,

$$k^a_{xx}(x,y) - k^a_{yy}(x,y) - c_1 k^a(x,y) = 0, \tag{4.17a}$$

$$k^a_y(x,0) = 0, \quad k^a(x,x) = -\frac{1}{2}c_1 x. \tag{4.17b}$$

Recalling Lemma 2.5.2, the inverse of transformation (4.16) is

$$w(x,t) = \tilde{w}(x,t) + \int_0^x \ell^a(x,y)\tilde{w}(y,t)\,dy, \tag{4.18}$$

where $\ell^a(x,y)$ solves the hyperbolic PDE in Lemma 2.5.2. Throughout this section and the rest of the chapter, we denote by $c_1 > 0$ a positive real constant that will be required to fulfill specific conditions for the system's stability later on.

Proposition 4.2.1. *If the control signal is*

$$u(t) = \int_0^1 k^a_x(1,y)w(y,t)dy + k^a(1,1)w(1,t), \tag{4.19}$$

then transformation (4.16), with $k^a(x,y)$ given by system (4.17), converts the parabolic-elliptic system (4.1)-(4.4) into the target system

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) - (c_1+\rho)\tilde{w}(x,t) + \alpha v(x,t) - \alpha\int_0^x k^a(x,y)v(y,t)dy, \tag{4.20a}$$

$$0 = v_{xx}(x,t) - \gamma v(x,t) + \beta\tilde{w}(x,t) + \beta\int_0^x \ell^a(x,y)\tilde{w}(y,t)dy, \tag{4.20b}$$

$$\tilde{w}_x(0,t) = 0, \quad \tilde{w}_x(1,t) = 0, \tag{4.20c}$$

$$v_x(0,t) = 0, \quad v_x(1,t) = 0. \tag{4.20d}$$

Proof. It will prove useful to rewrite (4.16) as

$$w(x,t) = \tilde{w}(x,t) + \int_0^x k^a(x,y)w(y,t)dy. \tag{4.21}$$

We differentiate (4.21) with respect to x twice

$$\begin{aligned} w_{xx}(x,t) = \tilde{w}_{xx}(x,t) + \int_0^x k^a_{xx}(x,y)w(y,t)dy + k^a_x(x,x)w(x,t) + \frac{d}{dx}k^a(x,x)w(x,t) \\ + k^a(x,x)w_x(x,t), \end{aligned} \tag{4.22}$$

and with respect to t

$$\begin{aligned} w_t(x,t) &= \tilde{w}_t(x,t) + \int_0^x k^a(x,y)w_t(y,t)dy \\ &= \tilde{w}_t(x,t) + k^a(x,x)w_x(x,t) - \int_0^x k^a_y(x,y)w_y(y,t)dy - \rho\int_0^x k^a(x,y)w(y,t)dy \\ &+ \alpha\int_0^x k^a(x,y)v(y,t)dy \\ &= \tilde{w}_t(x,t) + k^a(x,x)w_x(x,t) - k^a_y(x,x)w(x,t) + k^a_y(x,0)w(0,t) \\ &+ \int_0^x k^a_{yy}(x,y)w(y,t)dy - \rho\int_0^x k^a(x,y)w(y,t)dy + \alpha\int_0^x k^a(x,y)v(y,t)dy. \end{aligned} \tag{4.23}$$

Here,

$$\begin{aligned} k^a_x(x,x) =& \frac{\partial}{\partial x}k^a(x,y)|_{x=y}, \; k^a_y(x,x) = \frac{\partial}{\partial y}k^a(x,y)|_{x=y}, \\ \frac{d}{dx}k^a(x,x) =& k^a_x(x,x) + k^a_y(x,x). \end{aligned}$$

Substituting (4.22) and (4.23) into (4.1), we obtain

$$\begin{aligned} &\tilde{w}_t(x,t) + k^a(x,x)w_x(x,t) - k^a_y(x,x)w(x,t) + k^a_y(x,0)w(0,t) \\ &+ \int_0^x k^a_{yy}(x,y)w(y,t)dy - \rho\int_0^x k^a(x,y)w(y,t)dy + \alpha\int_0^x k^a(x,y)v(y,t)dy \\ &= \tilde{w}_{xx}(x,t) + \int_0^x k^a_{xx}(x,y)w(y,t)dy + k^a_x(x,x)w(x,t) + \frac{d}{dx}k^a(x,x)w(x,t) \\ &+ k^a(x,x)w_x(x,t) - \rho w(x,t) + \alpha v(x,t). \end{aligned} \tag{4.24}$$

Since $k^a_y(x,0) = 0$, then adding and subtracting $(c_1+\rho)\tilde{w}(x,t)$ to the right-hand-side of (4.24) yields

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) - (c_1+\rho)\tilde{w}(x,t) + \alpha v(x,t) - \alpha\int_0^x k^a(x,y)v(y,t)dy$$
$$+ (2\frac{d}{dx}k^a(x,x) + c_1)w(x,t) + \int_0^x [k^a_{xx}(x,y) - k^a_{yy}(x,y) - c_1 k^a(x,y)]w(y,t)dy.$$

Since $k^a(x,y)$ is given by (4.17), the previous equation reduces to (4.20a). Also,

$$\tilde{w}_x(0,t) = w_x(0,t) - k^a(0,0)w(0,t) = 0,$$

and the other boundary condition on $\tilde{w}(x,t)$ at $x = 1$ holds by using (4.19). Equation (4.20b) can be obtained by using the inverse transformation (4.18). □

Next, we provide conditions that ensure the exponential stability of the target system. First, we need the following lemma, which provides bounds on the induced $L_2([0,1]\times[0,1])$-norms of the kernel functions $k^a(x,y)$ and $\ell^a(x,y)$.

Lemma 4.2.1. *Writing* $erfi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{\xi^2}d\xi$, $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}d\xi$, *the* L_2*-norms of* k *and* ℓ *are bounded by*

$$\|k^a\| \leq \sqrt{\frac{c_1\pi}{8}}\left(erfi(\sqrt{\frac{c_1}{2}})erf(\sqrt{\frac{c_1}{2}})\right)^{\frac{1}{2}}, \tag{4.25a}$$

$$\|\ell^a\| \leq \sqrt{\frac{c_1\pi}{8}}\left(erfi(\sqrt{\frac{c_1}{2}})erf(\sqrt{\frac{c_1}{2}})\right)^{\frac{1}{2}}. \tag{4.25b}$$

Proof. To prove relation (4.25a), we recall the expression for the kernel $k^a(x,y)$ given previously in (2.15), i.e, $k^a(x,y) = -c_1 x\frac{I_1\left(\sqrt{c_1(x^2-y^2)}\right)}{\sqrt{c_1(x^2-y^2)}}$. We set $z = \sqrt{c_1(x^2-y^2)}$, then

$$k^a(x,y) = \frac{-c_1}{z}x\sum_{m=0}^{\infty}\left(\frac{z}{2}\right)^{2m+1}\frac{1}{m!m+1!}$$
$$= \frac{-c_1}{2}x\sum_{m=0}^{\infty}\frac{(z^2/4)^m}{m!}\frac{1}{m+1!}$$
$$\leq \frac{-c_1}{2}x\sum_{m=0}^{\infty}\frac{(z^2/4)^m}{m!}.$$

Thus, the induced L_2-norm of $k^a(x,y)$ is bounded as follows

$$\begin{aligned}\|k^a(x,y)\| &\leq \frac{c_1}{2}\|x\|\|e^{\frac{z^2}{4}}\| \\ &\leq \frac{c_1}{2}\|x\|\|e^{\frac{c_1x^2}{4}}\|\|e^{\frac{-c_1y^2}{4}}\| \\ &\leq \sqrt{\frac{c_1\pi}{8}}\left(erfi(\sqrt{\frac{c_1}{2}})erf(\sqrt{\frac{c_1}{2}})\right)^{\frac{1}{2}}.\end{aligned}$$

Similarly, one can prove (4.25b) by referring back to (2.48), i.e, $\ell^a(x,y) = -\frac{c_1xJ_1\left(\sqrt{c_1(x^2-y^2)}\right)}{\sqrt{c_1(x^2-y^2)}}$,

$$\begin{aligned}\ell^a(x,y) &= \frac{-c_1}{z}x\sum_{m=0}^{\infty}(-1)^m\left(\frac{z}{2}\right)^{2m+1}\frac{1}{m!m+1!} \\ &\leq \frac{c_1}{z}x\sum_{m=0}^{\infty}\left(\frac{z}{2}\right)^{2m+1}\frac{1}{m!m+1!},\end{aligned}$$

and the L_2-norm of $l(x,y)$ is bounded by

$$\|\ell^a(x,y)\| \leq \sqrt{\frac{c_1\pi}{8}}\left(erfi(\sqrt{\frac{c_1}{2}})erf(\sqrt{\frac{c_1}{2}})\right)^{\frac{1}{2}}.$$

□

The next lemma gives a relation between the L_2-norm of the parabolic and elliptic states of the target system (4.20).

Lemma 4.2.2. *Let $\gamma > 0$. The states of the target system (4.20a)-(4.20d) satisfy*

$$\|v(x,t)\| \leq \frac{|\beta|}{\gamma}(1+\|\ell^a\|)\|\tilde{w}(x,t)\|. \tag{4.26}$$

Proof. Multiply equation (4.20b) with $v(x,t)$ and integrate from 0 to 1,

$$\begin{aligned}0 = &\int_0^1 v_{xx}(x,t)v(x,t)dx - \gamma\int_0^1 v^2(x,t)dx + \beta\int_0^1 \tilde{w}(x,t)v(x,t)dx \\ &+ \beta\int_0^1 v(x,t)\int_0^x \ell^a(x,y)\tilde{w}(y,t)dydx.\end{aligned}$$

Thus

$$\gamma \int_0^1 v^2(x,t)dx \le \beta \int_0^1 \tilde{w}(x,t)v(x,t)dx + \beta \int_0^1 v(x,t) \int_0^x \ell^a(x,y)\tilde{w}(y,t)dydx.$$

Using the Cauchy-Schwarz inequality, we can bound the terms on the right-hand side of the previous inequality, leading to (4.26). □

The next theorem develops conditions for which the target system (4.20a)-(4.20d) is exponentially stable.

Theorem 4.2.1. *If*

$$c_1 + \rho > \frac{|\alpha\beta|}{\gamma} \left[1 + \sqrt{\frac{c_1\pi}{8}} \left(erfi(\sqrt{\frac{c_1}{2}}) erf(\sqrt{\frac{c_1}{2}}) \right)^{\frac{1}{2}} \right]^2, \tag{4.27}$$

then the target system (4.20a) − (4.20d) *is exponentially stable on the space* $L_2[0,1]$ *with decay rate* $2c_2$ *where*

$$c_2 = c_1 + \rho - \frac{|\alpha\beta|}{\gamma} \left[1 + \sqrt{\frac{c_1\pi}{8}} \left(erfi(\sqrt{\frac{c_1}{2}}) erf(\sqrt{\frac{c_1}{2}}) \right)^{\frac{1}{2}} \right]^2. \tag{4.28}$$

Proof. Recall that the operator $(\gamma I - \partial_{xx})$ has a bounded inverse. It follows from equation (4.20b) that

$$v(x,t) = \beta(\gamma I - \partial_{xx})^{-1} \left(\tilde{w}(x,t) + \int_0^x \ell^a(x,y)\tilde{w}(y,t)dy \right). \tag{4.29}$$

Substituting for $v(x,t)$ in (4.20a) leads to a single equation in terms of $\tilde{w}(x,t)$, that is,

$$\begin{aligned}\tilde{w}_t(x,t) =& \tilde{w}_{xx}(x,t) - (c_1+\rho)\tilde{w}(x,t) + \alpha\beta(\gamma I - \partial_{xx})^{-1} \left(\tilde{w}(x,t) + \int_0^x \ell^a(x,y)\tilde{w}(y,t)dy \right) \\ &- \alpha\beta \int_0^x k^a(x,y)(\gamma I - \partial_{yy})^{-1} \left(\tilde{w}(y,t) + \int_0^y \ell^a(y,z)\tilde{w}(z,t)dz \right) dx.\end{aligned}$$

Hence, the exponential stability of the coupled system follows from the exponential stability of the equation above.

Define the Lyapunov function candidate

$$V(t) = \frac{1}{2} \int_0^1 \tilde{w}^2(x,t)dx = \frac{1}{2}\|\tilde{w}(x,t)\|^2. \tag{4.30}$$

Taking the time derivative of $V(t)$,

$$\dot{V}(t) \le -(c_1+\rho)\int_0^1 \tilde{w}^2(x,t)dx + \alpha\int_0^1 \tilde{w}(x,t)v(x,t)dx$$
$$- \alpha\int_0^1 \tilde{w}(x,t)\int_0^x k^a(x,y)v(y,t)dydx. \tag{4.31}$$

Using Cauchy-Schwarz inequality, we estimate the terms on the right-hand-side of inequality (4.31) as follows.

$$\alpha\int_0^1 \tilde{w}(x,t)v(x,t)dx \le \frac{|\alpha\beta|}{\gamma}(1+\|\ell^a\|)\|\tilde{w}\|^2, \tag{4.32}$$

and

$$-\alpha\int_0^1 \tilde{w}(x,t)\int_0^x k^a(x,y)v(y,t)dydx \le |\alpha|\|k^a\|\|\tilde{w}\|\|v\|$$
$$\le \frac{|\alpha\beta|}{\gamma}\|k^a\|(1+\|\ell^a\|)\|\tilde{w}\|^2. \tag{4.33}$$

Substituting (4.32) and (4.33) into (4.31), and using Lemma 4.2.1

$$\dot{V}(t) \le -\left(c_1+\rho-\frac{|\alpha\beta|}{\gamma}(1+\|\ell^a\|)(1+\|k^a\|)\right)\|\tilde{w}\|^2$$
$$\le -\Big(c_1+\rho-\frac{|\alpha\beta|}{\gamma}\left[1+\sqrt{\frac{c_1\pi}{8}}\left(erfi(\sqrt{\frac{c_1}{2}})erf(\sqrt{\frac{c_1}{2}})\right)^{\frac{1}{2}}\right]^2\Big)\|\tilde{w}\|^2.$$

Defining c_2 as in (4.28), this shows that

$$V(t) \le e^{-2c_2t}V(0). \tag{4.34}$$

If the parameter c_1 is chosen such that (4.27) is satisfied, then $V(t)$ and $\|\tilde{w}(\cdot,t)\|$ decay exponentially with rate $2c_2$. Equation (4.29) implies that there is a constant M such that $\|v(\cdot,t)\| \le M\|\tilde{w}(\cdot,t)\|$ and so v converges to the steady-state solution with the same decay rate. □

The main result of this section is now immediate.

Theorem 4.2.2. *If $k^a(x,y)$ is given by system (4.17), where parameter c_1 satisfies (4.27), then the control signal (4.19) exponentially stabilizes system (4.1)-(4.4) on the space $L_2(0,1)$ with the convergence rate*

$$\|w(\cdot,t)\| \le c\|w(\cdot,0)\|e^{-2c_2 t}, \tag{4.35}$$
$$\|v(\cdot,t)\| \le m\|w(\cdot,t)\|. \tag{4.36}$$

Here, c and m are positive constants independent of $\|w(\cdot,0)\|$, and the initial conditions of w and v satisfy (4.15).

Proof. Since c_1 is given by (4.27), we deduce from Theorem (4.2.1) that the target system (4.20a)-(4.20d) is exponentially stable. Proposition 4.2.1 with $u(t)$ given by (4.19) implies that there is an invertible state transformation between the original system (4.1)-(4.4) and the exponentially stable target system (4.20a)-(4.20d). It then follows from [107] that we can derive similar convergence properties for the original system (4.1)-(4.4) as the ones for the target system since the backstepping transformation is invertible. Note that the kernel functions $k^a(x,y)$ and $\ell^a(x,y)$ are bounded, then a straightforward generalization of [107, Theorem 4] yields that there is an equivalence of norms of $w(x,t)$ and $\tilde{w}(x,t)$ in $L_2(0,1)$, ensuring the existence of a positive constant c independent of $\tilde{w}(\cdot,0)$ such that (4.35) holds. Following this, inequality (4.36) can also be derived by using the equivalence of norms of $w(x,t)$ and $\tilde{w}(x,t)$ in $L_2(0,1)$, combined with inequality (4.26). This completes the proof. □

Figure 4.1 illustrates the restrictiveness of the criterion (4.27). This figure gives a comparison between the right-hand-side of inequality (4.27) and different straight lines $c_1+\rho$ for various values of ρ while setting $\gamma=\beta=1$ and $\alpha=0.5$. The dashed line describes the right-hand-side of inequality (4.27), whereas the straight lines present straight lines $c_1+\rho$, for different values of ρ. For some ρ, if values of c_1 are such that the dashed line (- - -) is below the straight line $c_1+\rho$, bound (4.27) is fulfilled, and hence stability of the target system (4.20a) − (4.20d) follows.

The solutions of system (4.1)-(4.4), both controlled and uncontrolled, were simulated numerically using a finite-element approximation in COMSOL Multiphysics software. The finite-element method (FEM) with linear splines was used to approximate the coupled equations by a system of DAEs. The spatial interval was divided into 27 subintervals. Also, time was discretized by a time-stepping algorithm called generalized alpha. We set $\gamma=\frac{1}{4}$, $\rho=\frac{1}{3}$, $\alpha=\frac{1}{4}$ and $\beta=\frac{1}{2}$. For these parameter values, the system is unstable.

Figure 4.2 presents the dynamics of the states $w(x,t), v(x,t)$ in the absence of the control with initial condition $w_0=\frac{1}{2}(1-\cos(2\pi x)),\ v_0=\beta(\gamma I-d_{xx})^{-1}w_0$. The system was controlled with the controller resulting from the choice of parameter $c_1 = 1.2$ which satisfies inequality (4.27) and thus stability of the controlled system is guaranteed. As predicted by the theory, the dynamics of the system decay to zero with time.

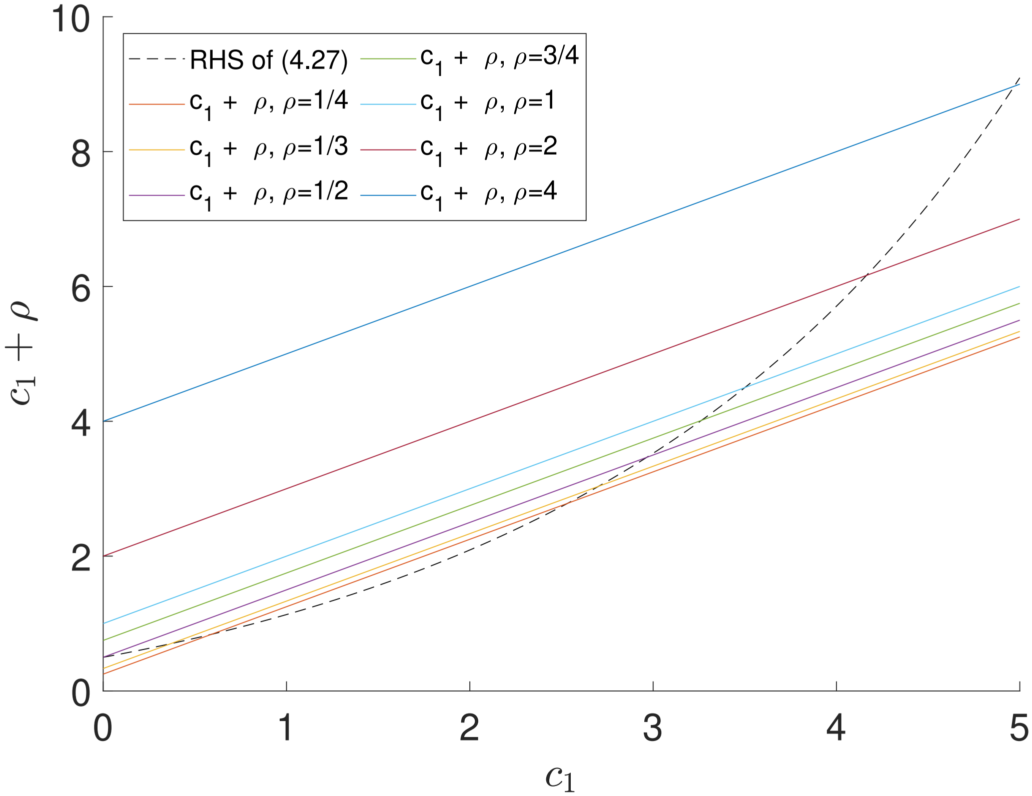
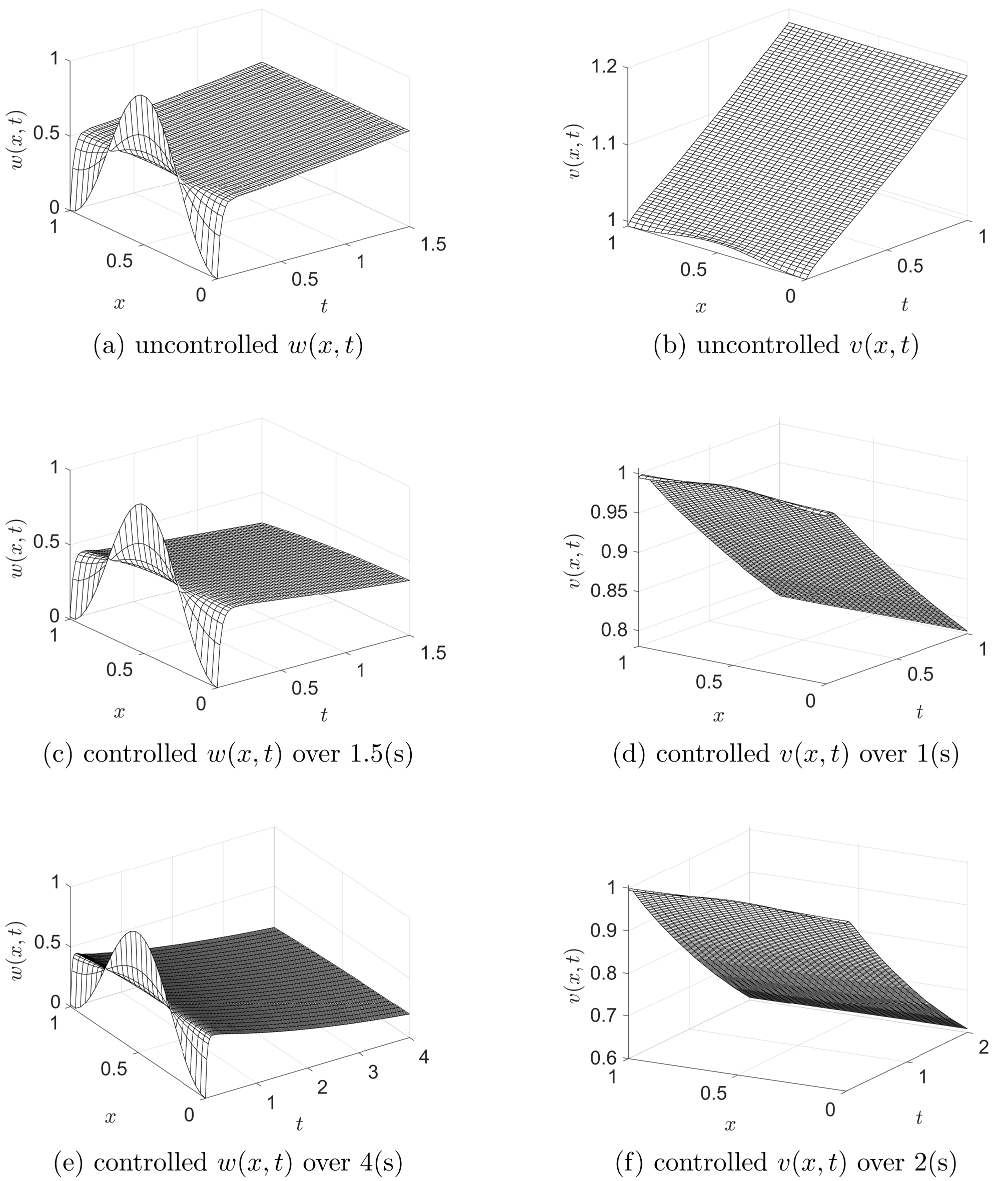


Figure 4.1: A comparison between the right-hand-side of (4.27) as a function of c_1 against several straight lines $c_1 + \rho$ for different values of ρ, where the other parameters are fixed as $\beta = \gamma = 1$, $\alpha = 0.5$. The right-hand-side of (4.27) is described using a dashed line(- - -). The target system (4.20a)-(4.20d) is exponentially stable for values of c_1 at which the straight line $c_1 + \rho$, for some ρ, is above the dashed line(- - -). The figure showcases the restrictive nature associated with condition (4.27).



(a) uncontrolled $w(x,t)$

(b) uncontrolled $v(x,t)$

(c) controlled $w(x,t)$ over 1.5(s)

(d) controlled $v(x,t)$ over 1(s)

(e) controlled $w(x,t)$ over 4(s)

(f) controlled $v(x,t)$ over 2(s)

Figure 4.2: A 3D landscape of the dynamics of a coupled parabolic-elliptic system (4.1)-(4.4) without and with control. Here, $w_0 = \frac{1}{2}(1-\cos(2\pi x))$, $v_0 = \beta(\gamma I - d_{xx})^{-1} w_0$, and $\gamma = \frac{1}{4}$, $\rho = \frac{1}{3}$, $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$. The uncontrolled system is unstable with this choice of parameters. However, with control gain $c_2 = 1.2$, the controlled system's solutions converge to a steady-state as $t \to \infty$.

Let us briefly recall that the results presented previously in this section relied on analyzing the stability of the target system. This is typical when using backstepping approach since, of course, the stability of the original coupled system will follow from the invertible state transformation between the original and the target systems. However, this specific step resulted in a limiting criterion on the permissible system parameters that must be met to achieve stability, illustrated in Figure 4.1. Therefore, in the remainder of this section, we introduce a weaker sufficient condition that guarantees the system's stability. We examine the stability of the controlled coupled system while incorporating the control input (4.19). The following lemma is needed.

Lemma 4.2.3. *Consider system (2.5.1). The L_2-norm of $k_x^a(1,y)$ is bounded by*

$$\|k_x^a(1,y)\| \leq \frac{c_1}{2}(1+\frac{c_1}{2})e^{\frac{c_1}{4}}. \tag{4.37}$$

Proof. Relation (4.37) can be shown by noting that the solution of system (4.17) is

$$k^a(x,y) = -c_1 x \frac{I_1(\sqrt{c_1(x^2-y^2)})}{\sqrt{c_1(x^2-y^2)}}.$$

After straightforward mathematical steps, we arrive at

$$k_x^a(x,y) = -\,c_1 \frac{I_1(\sqrt{c_1(x^2-y^2)})}{\sqrt{c_1(x^2-y^2)}} - c_1 x \frac{I_2(\sqrt{c_1(x^2-y^2)}}{(x^2-y^2))}, \tag{4.38}$$

where we have used that $\frac{d}{dx}I_1(x) = \frac{I_1(x)}{x} + I_2(x)$. Setting $z = \sqrt{c_1(x^2-y^2)}$ and using the definition of the modified Bessel function, equation (4.38) can be written as

$$\begin{aligned} k_x^a(x,y) &= -c_1\frac{I_1(z)}{z} - c_1^2 x\frac{I_2(z)}{z^2} \\ &= -\frac{c_1}{z}\sum_{m=0}^{\infty}\left(\frac{z}{2}\right)^{2m+1}\frac{1}{m!m+1!} - \frac{c_1^2}{z^2}x\sum_{m=0}^{\infty}\left(\frac{z}{2}\right)^{2m+2}\frac{1}{m!m+2!} \\ &= -\frac{c_1}{2}\sum_{m=0}^{\infty}\left(\frac{z^2}{4}\right)^{m}\frac{1}{m!m+1!} - \frac{c_1^2}{4}x\sum_{m=0}^{\infty}\left(\frac{z^2}{4}\right)^{m}\frac{1}{m!m+2!} \\ &\leq \frac{c_1}{2}\sum_{m=0}^{\infty}\left(\frac{z^2}{4}\right)^{m}\frac{1}{m!m+1!} + \frac{c_1^2}{4}x\sum_{m=0}^{\infty}\left(\frac{z^2}{4}\right)^{m}\frac{1}{m!m+2!}. \end{aligned} \tag{4.39}$$

To find a bound on the induced L_2-norm of $k_x^a(1,y)$, with $x = 1$ the variable z becomes $z = \sqrt{c_1(1-y^2)}$ where $0 < y < 1$. Equation (4.39) leads to

$$\begin{aligned} \|k_x^a(1,y)\| &\leq \frac{c_1}{2}\sum_{m=0}^{\infty}\frac{(z^2/4)^m}{m!} + \frac{c_1^2}{4}x\sum_{m=0}^{\infty}\frac{(z^2/4)^m}{m!} \\ &\leq \frac{c_1}{2}\|e^{\frac{z^2}{4}}\| + \frac{c_1^2}{4}\|e^{\frac{z^2}{4}}\| = \frac{c_1}{2}(1+\frac{c_1}{2})\|e^{\frac{z^2}{4}}\| \\ &\leq \frac{c_1}{2}(1+\frac{c_1}{2})e^{\frac{c_1}{4}}, \end{aligned} \tag{4.40}$$

where we used that $z = \sqrt{c_1(1-y^2)}$ so the term $e^{\frac{z^2}{4}}$ has it maximum value at $y = 0$. □

The next theorem is the second main contribution of the this section.

Theorem 4.2.3. *If $k^a(x,y)$ is given by system (4.17), where parameter c_1 satisfies*

$$c_1 + 2\rho > 2\|k_x^a(1,y)\| + 2|\frac{\alpha\beta}{\gamma}|, \tag{4.41}$$

such that $c_1 \leq \frac{\pi^2}{2}(1-\frac{\|k_x^a(1,y)\|}{4})$, then the control signal (4.19) exponentially stabilizes system (4.1)-(4.4) on the space $L_2(0,1)$.

Proof. Recall that γ is such that the operator $(\gamma I - \partial_{xx})$ has a bounded inverse. It follows from equation (4.2) that

$$v(x,t) = \beta(\gamma I - \partial_{xx})^{-1}w(x,t). \tag{4.42}$$

This implies that there is a constant M such that

$$\|v(\cdot,t)\| \leq M\|w(\cdot,t)\|.$$

Substituting (4.42) in (4.1), the coupled system (4.1)-(4.4) can be written as

$$\begin{aligned} w_t(x,t) =& w_{xx}(x,t) - \rho w(x,t) + \alpha\beta(\gamma I - \partial_{xx})^{-1}w(x,t), \\ w_x(0,t) =& 0, \qquad w_x(1,t) = u(t), \end{aligned} \tag{4.43}$$

and so the exponential stability of system (4.1)-(4.4) can be demonstrated by showing the exponential stability of system (4.43) only.

Define the Lyapunov function candidate

$$V(t) = \frac{1}{2}\int_0^1 w^2(x,t)dx = \frac{1}{2}\|w(x,t)\|^2. \tag{4.44}$$

Taking the time derivative of $V(t)$,

$$\begin{aligned}\dot{V}(t) &= \int_0^1 w(x,t)w_t(x,t)dx \\ &= \int_0^1 w(x,t)[w_{xx}(x,t) - \rho w(x,t) + \alpha v(x,t)]dx \\ &= \int_0^1 w(x,t)w_{xx}(x,t)dx - \rho \int_0^1 w^2(x,t)dx + \alpha \int_0^1 w(x,t)v(x,t)dx. \end{aligned} \tag{4.45}$$

Integrating by parts and using the boundary conditions in (4.3), along with the expression of the control signal in (4.19)

$$\begin{aligned}\int_0^1 w(x,t)w_{xx}(x,t)dx &= w(1,t)w_x(1,t) - w(0,t)w_x(0,t) - \|w_x\|^2 \\ &= w(1,t)w_x(1,t) - \|w_x\|^2 \\ &= -\frac{1}{2}c_1 w^2(1,t) + \int_0^1 k_x^a(1,y)w(y,t)dy w(1,t) - \|w_x\|^2. \end{aligned} \tag{4.46}$$

Using a variation of Wirtinger's inequality (See [61, inequality 2.31] and [112, Theorem 2 & Corollary 3]), it follows that

$$-w^2(1,t) \leq -\int_0^1 w^2(x,t)dx + \frac{4}{\pi^2}\int_0^1 w_x^2(x,t)dx. \tag{4.47}$$

Also, we bound the term $\int_0^1 k_x^a(1,y)w(y,t)dy\, w(1,t)$ using Cauchy-Schwarz inequality as follows

$$\begin{aligned}\int_0^1 k_x^a(1,y)w(y,t)dy\, w(1,t) &\leq \|\int_0^1 k_x^a(1,y)w(y,t)dy\| \max_{x\in[0,1]} w(x,t) \\ &\leq \|k_x^a(1,y)\|\|w\|\|w\|_\infty. \end{aligned}$$

Applying Agmon's inequality [117] on the right-hand-side of the previous inequality leads to

$$\int_0^1 k_x^a(1,y)w(y,t)dy\, w(1,t) \leq \|k_x^a(1,y)\|\|w\|\|w\|^{1/2}\|w\|_{H^1}^{1/2}.$$

By invoking Young's inequality ($\|ab\| \leq \frac{\|a\|^2}{2} + \frac{\|b\|^2}{2}$) on the right-hand-side of the previous inequality, we obtain

$$\begin{aligned}\int_0^1 k_x^a(1,y)w(y,t)dy\, w(1,t) &\leq \|k_x^a(1,y)\|\Big(\frac{\|w\|^2}{2} + \frac{\|w\|\|w\|_{H^1}}{2}\Big)\\ &\leq \|k_x^a(1,y)\|\Big(\frac{\|w\|^2}{2} + \frac{1}{2}(\frac{\|w\|^2}{2} + \frac{\|w\|_{H^1}^2}{2})\Big)\end{aligned}$$

Using that $\|w\|_{H^1} = (\|w\|^2 + \|w_x\|^2)^{\frac{1}{2}}$, we arrive at

$$\int_0^1 k_x^a(1,y)w(y,t)dy\, w(1,t) \leq \|k_x^a(1,y)\|(\|w\|^2 + \frac{\|w_x\|^2}{4}). \tag{4.48}$$

Substituting (4.47) and (4.48) into (4.46) implies that

$$\begin{aligned}\int_0^1 w(x,t)w_{xx}(x,t)dx \leq -\frac{1}{2}c_1\|w\|^2 + \frac{2c_1}{\pi^2}\|w_x\|^2 + \|k_x^a(1,y)\|\|w\|^2\\ + \frac{\|k_x^a(1,y)\|}{4}\|w_x\|^2 - \|w_x\|^2.\end{aligned} \tag{4.49}$$

We can bound the right-hand-side of (4.45) using (4.49) and Cauchy-Schwarz as follows,

$$\begin{aligned}\dot{V}(t) \leq -\frac{1}{2}c_1\|w\|^2 - (1 - \frac{\|k_x^a(1,y)\|}{4} - \frac{2c_1}{\pi^2})\|w_x\|^2 - \rho\|w\|^2\\ + \|k_x^a(1,y)\|\|w\|^2 + |\frac{\alpha\beta}{\gamma}|\|w\|^2.\end{aligned} \tag{4.50}$$

Hence,

$$\dot{V}(t) \leq -\left(\frac{1}{2}c_1 + \rho - \|k_x^a(1,y)\| - |\frac{\alpha\beta}{\gamma}|\right)\|w\|^2 - (1 - \frac{\|k_x^a(1,y)\|}{4} - \frac{2c_1}{\pi^2})\|w_x\|^2. \tag{4.51}$$

Choosing c_1 such that $c_1 \leq \frac{\pi^2}{2}(1 - \frac{\|k_x^a(1,y)\|}{4})$, inequality (4.51) leads to

$$\dot{V}(t) \leq -\left(\frac{1}{2}c_1 + \rho - \|k_x^a(1,y)\| - |\frac{\alpha\beta}{\gamma}|\right)\|w\|^2. \tag{4.52}$$

If c_1 satisfies (4.41), then defining $c_2 = c_1 + 2\rho - 2\|k_x^a(1,y)\| - 2|\frac{\alpha\beta}{\gamma}| > 0$, inequality (4.52) implies

$$V(t) \leq e^{-2c_2 t}V(0). \tag{4.53}$$

Thus, $\|w(x,t)\|$ decays exponentially. □

The stability condition (4.41) improves on the condition (4.27). This improvement is demonstrated in Figure 4.3, where we compare the right-hand-side of (4.41) with the lines defined by $c_1 + 2\rho$, keeping other parameters constant ($\alpha = \gamma = 4$, $\beta = 4.6$). For the values of c_1 where the line $c_1 + 2\rho$, for a given ρ, lies above the dashed line, the controlled coupled system described by (4.1)-(4.4) is shown to be exponentially stable. By comparing Figure 4.1 with Figure 4.3, it is clear that analyzing the stability of the original system with the inclusion of the control input results in a more lenient restriction on the system parameters. Specifically, the new criterion accommodates a broader range for the coupling factor $\alpha\beta$ than what was permissible under the previously established condition in (4.27).

The solutions of system (4.1)-(4.4), both controlled and uncontrolled, were simulated numerically using a finite-element approximation in COMSOL Multiphysics software. The finite-element method (FEM) with linear splines was used to approximate the coupled equations by a system of DAEs. With $\gamma = 10$, $\rho = 9.97$, $\alpha = 10$ and $\beta = 10$, system (4.1)-(4.4) is unstable. Selecting $c_1 = 0.9$, the new stability condition in (4.41) is satisfied for this set of parameters. Figure 4.4 indicates that the control (4.19) is stabilizing the system.

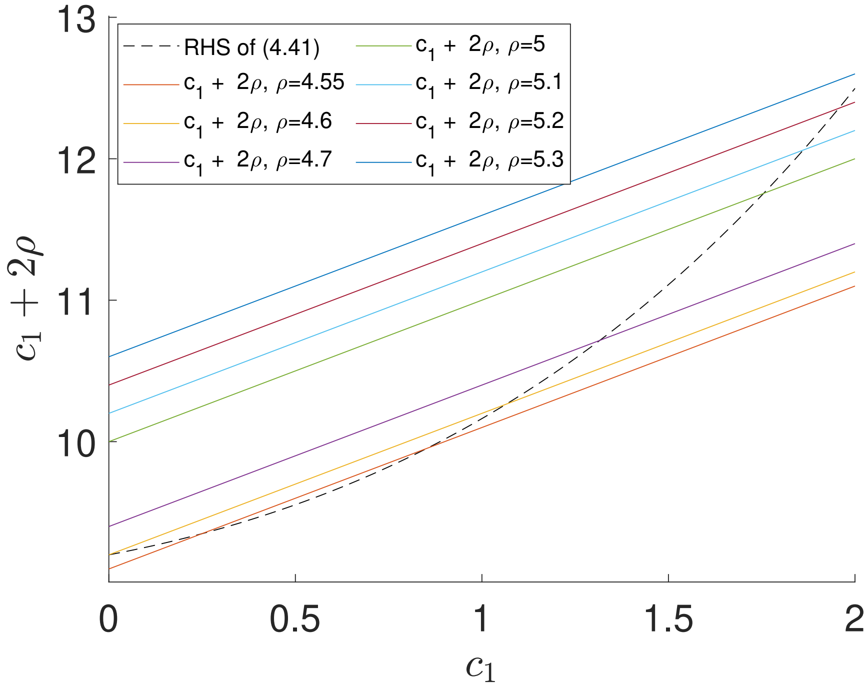


Figure 4.3: A comparison between the right-hand-side of (4.41) as a function of c_1 against several straight lines $c_1 + 2\rho$ for different values of ρ, where the other parameters are fixed as $\alpha = \gamma = 4$, $\beta = 4.6$. The dashed line represents (4.41)'s right-hand side. If $c_1 + 2\rho$ is above this line, the controlled system (4.1)-(4.4) with control (4.19) is exponential stability. This figure indicates that condition (4.41) allows for larger coupling factors $\alpha\beta$ compared to condition (4.27).

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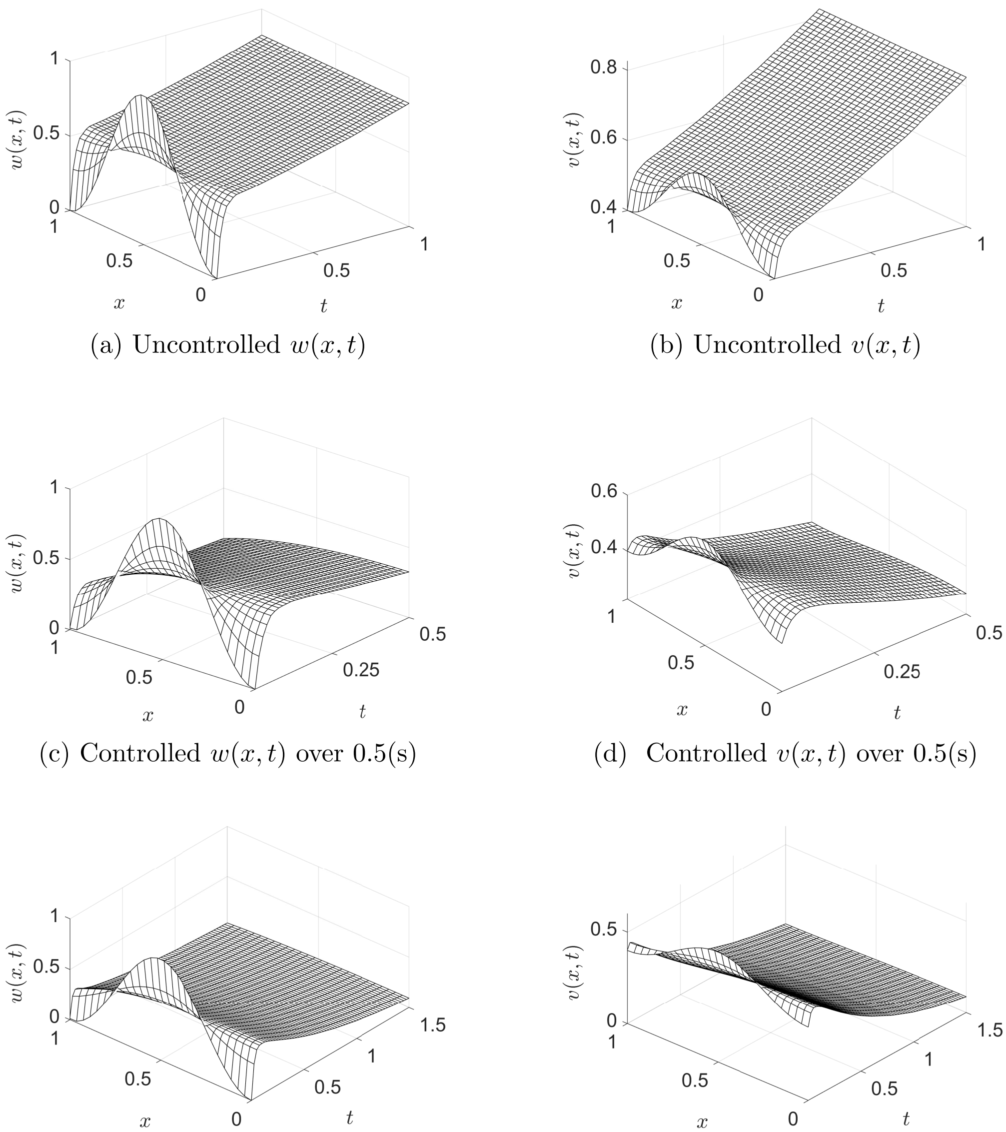


(a) Uncontrolled $w(x,t)$ (b) Uncontrolled $v(x,t)$

(c) Controlled $w(x,t)$ over 0.5(s) (d) Controlled $v(x,t)$ over 0.5(s)

(e) Controlled $w(x,t)$ over 1.5(s) (f) Controlled $v(x,t)$ over 1.5(s)

Figure 4.4: A 3D landscape of the dynamics of an unstable coupled parabolic-elliptic system (4.1)-(4.4), where $\gamma = 10$, $\rho = 9.97$, $\alpha = 10$, $\beta = 10$ and $w_0 = \frac{1}{2}(1 - \cos(2\pi x))$, $v_0 = \beta(\gamma I - d_{xx})^{-1} w_0$. Applying the control (4.19), with $c_1 = 0.9$, forces the solutions to decay exponentially to the steady state solution. The chosen parameters satisfy inequality (4.41). However, they do not meet condition (4.27).

4.3 Observer design for a coupled parabolic-elliptic system with two measurements available

In the previous section, the control input was designed based on the assumption that the state of system (4.1)-(4.4) is known. The objective is to design an observer when the available measurements of system (4.1)-(4.4) are $w(1,t)$ and $v(1,t)$. We propose the following observer for system (4.1)-(4.4):

$$\hat{w}_t(x,t) = \hat{w}_{xx}(x,t) - \rho\hat{w}(x,t) + \alpha\hat{v}(x,t) + \eta_1(x)[w(1,t) - \hat{w}(1,t)], \tag{4.54a}$$

$$0 = \hat{v}_{xx}(x,t) - \gamma\hat{v}(x,t) + \beta\hat{w}(x,t) + \eta_2(x)[v(1,t) - \hat{v}(1,t)], \tag{4.54b}$$

$$\hat{w}_x(0,t) = 0, \quad \hat{w}_x(1,t) = u(t) + \eta_3[w(1,t) - \hat{w}(1,t)], \tag{4.54c}$$

$$\hat{v}_x(0,t) = 0, \quad \hat{v}_x(1,t) = \eta_4[v(1,t) - \hat{v}(1,t)]. \tag{4.54d}$$

Two in-domain output injection functions $\eta_1(x)$ and $\eta_2(x)$, and two boundary injections values η_3 and η_4 are to be designed. Define the error states

$$e^w(x,t) = w(x,t) - \hat{w}(x,t), \tag{4.55a}$$

$$e^v(x,t) = v(x,t) - \hat{v}(x,t). \tag{4.55b}$$

Then, the observer error dynamics satisfy

$$e^w_t(x,t) = e^w_{xx}(x,t) - \rho e^w(x,t) + \alpha e^v(x,t) - \eta_1(x)e^w(1,t), \tag{4.56a}$$

$$0 = e^v_{xx}(x,t) - \gamma e^v(x,t) + \beta e^w(x,t) - \eta_2(x)e^v(1,t), \tag{4.56b}$$

$$e^w_x(0,t) = 0, \quad e^w_x(1,t) = -\eta_3 e^w(1,t), \tag{4.56c}$$

$$e^v_x(0,t) = 0, \quad e^v_x(1,t) = -\eta_4 e^v(1,t). \tag{4.56d}$$

A backstepping approach is used to find $\eta_1(x)$, $\eta_2(x)$, η_3, η_4 so that the error system (4.56) is exponentially stable. For a real constant $o_1 > 0$, we introduce the target system

$$e^{\tilde{w}}_t(x,t) = e^{\tilde{w}}_{xx}(x,t) - (o_1 + \rho)e^{\tilde{w}}(x,t) + \alpha e^{\tilde{v}}(x,t), \tag{4.57a}$$

$$0 = e^{\tilde{v}}_{xx}(x,t) - (o_1 + \gamma)e^{\tilde{v}}(x,t) + \beta e^{\tilde{w}}(x,t), \tag{4.57b}$$

$$e^{\tilde{w}}_x(0,t) = 0, \qquad e^{\tilde{w}}_x(1,t) = 0, \tag{4.57c}$$

$$e^{\tilde{v}}_x(0,t) = 0, \qquad e^{\tilde{v}}_x(1,t) = 0. \tag{4.57d}$$

A pair of state transformations

$$e^w(x,t) = e^{\tilde{w}}(x,t) - \int_x^1 k_1(x,y)e^{\tilde{w}}(y,t)dy, \tag{4.58a}$$

$$e^v(x,t) = e^{\tilde{v}}(x,t) - \int_x^1 k_2(x,y)e^{\tilde{v}}(y,t)dy, \tag{4.58b}$$

that transform the target system (4.57) into (4.56) are needed. The kernels $k_1(x,y)$ and $k_2(x,y)$ shall be determined.

Proposition 4.3.1. *If $k_1(x,y) = k_2(x,y) = k^b(x,y)$ where $k^b(x,y)$ satisfies Lemma 2.6.1, that is, $k^b(x,y)$ solves*

$$k^b_{yy}(x,y) - k^b_{xx}(x,y) - o_1 k^b(x,y) = 0, \tag{4.59a}$$

$$k^b_x(0,y) = 0, \quad k^b(x,x) = -\frac{1}{2}o_1 x, \tag{4.59b}$$

and if the output injections are

$$\eta_1(x) = \eta_2(x) = -k^b_y(x,1), \tag{4.60a}$$

$$\eta_3 = \eta_4 = -k^b(1,1), \tag{4.60b}$$

then transformations (4.58a) and (4.58b) convert the target system (4.57) into the original error dynamics (4.56).

Proof. We first take the spatial derivatives of (4.58a)

$$e^w_x(x,t) = e^{\tilde{w}}_x(x,t) - \int_x^1 k_{1x}(x,y)e^{\tilde{w}}(y,t)dy + k_1(x,x)e^{\tilde{w}}(x,t), \tag{4.61}$$

$$e^w_{xx}(x,t) = e^{\tilde{w}}_{xx}(x,t) - \int_x^1 k_{1xx}(x,y)e^{\tilde{w}}(y,t)dy + k_{1x}(x,x)e^{\tilde{w}}(x,t) + \frac{d}{dx}k_1(x,x)e^{\tilde{w}}(x,t) + k_1(x,x)e^{\tilde{w}}_x(x,t). \tag{4.62}$$

Taking the time derivative of (4.58a) and integrating by parts

$$\begin{aligned} e^w_t(x,t) &= e^{\tilde{w}}_t(x,t) - \int_x^1 k_1(x,y)e^{\tilde{w}}_t(y,t)dy \\ &= e^{\tilde{w}}_t(x,t) - \int_x^1 k_1(x,y)[e^{\tilde{w}}_{yy}(y,t) - (o_1+\rho)e^{\tilde{w}}(y,t) + \alpha e^{\tilde{v}}(y,t)]dy \\ &= e^{\tilde{w}}_t(x,t) + (o_1+\rho)\int_x^1 k_1(x,y)e^{\tilde{w}}(y,t)dy - k_1(x,1)e^{\tilde{w}}_x(1,t) \\ &- \alpha\int_x^1 k_1(x,y)e^{\tilde{v}}(y,t)dy + k_1(x,x)e^{\tilde{w}}_x(x,t) + k_{1y}(x,1)e^{\tilde{w}}(1,t) - k_{1y}(x,x)e^{\tilde{w}}(x,t) \\ &- \int_x^1 k_{1yy}(x,y)e^{\tilde{w}}(y,t)dy. \end{aligned} \tag{4.63}$$

We rewrite the right-hand-side of the parabolic equation (4.56a) of the error dynamics as

$$e_t^w(x,t) - e_{xx}^w(x,t) + \rho e^w(x,t) - \alpha e^v(x,t) + \eta_1(x) e^w(1,t) = 0. \tag{4.64}$$

Substituting (4.62) and (4.63) in (4.64), then the left-hand-side and the right-hand-side of (4.64) are

$$\begin{aligned}
&(\text{L.H.S})_1 = e_t^{\tilde{w}}(x,t) - e_{xx}^{\tilde{w}}(x,t) - \int_x^1 k_{1yy}(x,y) e^{\tilde{w}}(y,t)dy + (o_1+\rho)\int_x^1 k_1(x,y) e^{\tilde{w}}(y,t)dy \\
&+ \int_x^1 k_{1xx}(x,y) e^{\tilde{w}}(y,t)dy - k_{1y}(x,x)e^{\tilde{w}}(x,t) - k_{1x}(x,x)e^{\tilde{w}}(x,t) - \frac{d}{dx}k_1(x,x)e^{\tilde{w}}(x,t) \\
&- k_1(x,x)e_x^{\tilde{w}}(x,t) + k_1(x,x)e_x^{\tilde{w}}(x,t) + \rho e^{\tilde{w}}(x,t) - \alpha e^{\tilde{v}}(x,t) - \rho\int_x^1 k_1(x,y)e^{\tilde{w}}(y)dy \\
&+ \eta_1(x)e^{\tilde{w}}(1,t) + k_{1y}(x,1)e^{\tilde{w}}(1,t) - k_1(x,1)e_x^{\tilde{w}}(1,t) - \alpha\int_x^1 k_1(x,y)e^{\tilde{v}}(y,t)dy \\
&+ \alpha\int_x^1 k_2(x,y)e^{\tilde{v}}(y,t)dy, \qquad (4.65) \\
&(\text{R.H.S})_1 = 0. \qquad (4.66)
\end{aligned}$$

Adding and subtracting the term $(o_1+\rho)e^{\tilde{w}}(x,t)$ to the right-hand-side of (4.65)

$$\begin{aligned}
&(\text{L.H.S})_1 = e_t^{\tilde{w}}(x,t) - e_{xx}^{\tilde{w}}(x,t) + o_1 e^{\tilde{w}}(x,t) - \alpha e^{\tilde{v}}(x,t) \\
&- \int_x^1 [-o_1 k_1(x,y) - k_{1xx}(x,y) + k_{1yy}(x,y)] e^{\tilde{w}}(y,t)dy \\
&- k_1(x,1)e_x^{\tilde{w}}(1,t) - (2\frac{d}{dx}k_1(x,x) + o_1)e^{\tilde{w}}(x,t) + (\eta_1(x) + k_{1y}(x,1))e^{\tilde{w}}(1,t) \\
&- \alpha\int_x^1 k_1(x,y)e^{\tilde{v}}(y,t)dy + \alpha\int_x^1 k_2(x,y)e^{\tilde{v}}(y,t)dy.
\end{aligned} \tag{4.67}$$

Using the boundary condition $e_x^{\tilde{w}}(1,t) = 0$, if $k_1(x,y) = k_2(x,y) = k^b(x,y)$ then equation (4.67) reduces to

$$\begin{aligned}
(\text{L.H.S})_1 = {} & e_t^{\tilde{w}}(x,t) - e_{xx}^{\tilde{w}}(x,t) + (o_1+\gamma)e^{\tilde{w}}(x,t) - \alpha e^{\tilde{v}}(x,t) \\
& - \int_x^1 [-o_1 k^b(x,y) - k_{xx}^b(x,y) + k_{yy}^b(x,y)] e^{\tilde{w}}(y,t)dy - (2\frac{d}{dx}k^b(x,x) + o_1)e^{\tilde{w}}(x,t) \\
& + (\eta_1(x) + k_y^b(x,1))e^{\tilde{w}}(1,t).
\end{aligned} \tag{4.68}$$

If $k^b(x,y)$ satisfies system (4.59) and $\eta_1(x) = k^b_y(x,1)$, then (4.68) becomes

$$(\text{L.H.S})_1 = e^{\tilde{w}}_t(x,t) - e^{\tilde{w}}_{xx}(x,t) + (o_1+\gamma)e^{\tilde{w}}(x,t) - \alpha e^{\tilde{v}}(x,t).$$

Referring to (4.57a) and (4.66),

$$(\text{L.H.S})_1 = 0 = (\text{R.H.S})_1.$$

Hence, the state transformation (4.58) transforms the parabolic equation (4.57a) into (4.56a). Referring to (4.61), we apply transformation (4.58a) to the boundary conditions (4.57c)

$$e^w_x(0,t) = e^{\tilde{w}}_x(0,t) - \int_0^1 k^b_x(0,y)e^{\tilde{w}}(y,t)dy + k^b(0,0)e^{\tilde{w}}(0,t) = 0,$$

where the previous step was obtained by using (2.66b) and $e^{\tilde{w}}(0,t) = e^w(0,t)$. Thus, we obtain the boundary condition in (4.56c) at $x = 0$. Similarly,

$$\begin{aligned} e^w_x(1,t) =& e^{\tilde{w}}_x(1,t) - \int_1^1 k^b_x(1,y)e^{\tilde{w}}(y,t)dy + k^b(1,1)e^{\tilde{w}}(1,t) \\ =& k^b(1,1)e^{\tilde{w}}(1,t) = k^b(1,1)e^w(1,t). \end{aligned}$$

If $\eta_3 = -k^b(1,1)$ then we obtain the boundary condition in (4.56c) at $x = 1$. We perform similar calculations on the elliptic equation (4.57c). First, we take the spatial derivative of (4.58b),

$$\begin{aligned} e^v_{xx}(x,t) = e^{\tilde{v}}_{xx}(x,t) - \int_x^1 k_{2xx}(x,y)e^{\tilde{v}}(y,t)dy + k_{2x}(x,x)e^{\tilde{v}}(x,t) + \frac{d}{dx}k_2(x,x)e^{\tilde{v}}(x,t) \\ + k_2(x,x)e^{\tilde{v}}_x(x,t). \end{aligned} \tag{4.69}$$

Substituting (4.69) in the right-hand-side of elliptic equation (4.56b),

$$\begin{aligned} (\text{R.H.S})_2 = e^{\tilde{v}}_{xx}(x,t) - \int_x^1 k_{2xx}(x,y)e^{\tilde{v}}(y,t)dy + k_{2x}(x,x)e^{\tilde{v}}(x,t) + \frac{d}{dx}k_2(x,x)e^{\tilde{v}}(x,t) \\ + k_2(x,x)e^{\tilde{v}}_x(x,t) - \gamma e^{\tilde{v}}(x,t) + \gamma\int_x^1 k_2(x,y)e^{\tilde{v}}(y,t)dy + \beta e^{\tilde{w}}(x,t) \\ - \beta\int_x^1 k_1(x,y)e^{\tilde{w}}(y,t)dy - \eta_2(x)e^{\tilde{v}}(1,t) \end{aligned} \tag{4.70}$$

$$(\text{L.H.S})_2 = 0. \tag{4.71}$$

Rewriting the last term of (4.70) as follows

$$\beta \int_x^1 k_1(x,y)e^{\tilde{w}}(y,t)dy = -\int_x^1 k_1(x,y)e^{\tilde{v}}_{yy}(y,t)dy + (o_1+\gamma)\int_x^1 k_1(x,y)e^{\tilde{v}}(y,t)dy,$$

which can be obtained by referring to the elliptic equation of (4.57b), then (4.70) yields

$$\begin{aligned}(\text{R.H.S})_2 = {} & e^{\tilde{v}}_{xx}(x,t) - \int_x^1 k_{2xx}(x,y)e^{\tilde{v}}(y,t)dy + k_{2x}(x,x)e^{\tilde{v}}(x,t) + \frac{d}{dx}k_2(x,x)e^{\tilde{v}}(x,t) \\ & + k_2(x,x)e^{\tilde{v}}_x(x,t) - \gamma e^{\tilde{v}}(x,t) + \gamma \int_x^1 k_2(x,y)e^{\tilde{v}}(y,t)dy + \beta e^{\tilde{w}}(x,t) \\ & + \int_x^1 k_1(x,y)e^{\tilde{v}}_{yy}(y,t)dy - \eta_2(x)e^{\tilde{v}}(1,t) - (o_1+\gamma)\int_x^1 k_1(x,y)e^{\tilde{v}}(y,t)dy. \end{aligned} \tag{4.72}$$

Since $k_1(x,y) = k_2(x,y) = k^b(x,y)$, then (4.72) leads to

$$\begin{aligned}(\text{R.H.S})_2 = {} & e^{\tilde{v}}_{xx}(x,t) - \int_x^1 k^b_{xx}(x,y)e^{\tilde{v}}(y,t)dy + k^b_x(x,x)e^{\tilde{v}}(x,t) + \frac{d}{dx}k^b(x,x)e^{\tilde{v}}(x,t) \\ & + k^b(x,x)e^{\tilde{v}}_x(x,t) - \gamma e^{\tilde{v}}(x,t) + \gamma \int_x^1 k^b(x,y)e^{\tilde{v}}(y,t)dy + \beta e^{\tilde{w}}(x,t) \\ & - (o_1+\gamma)\int_x^1 k^b(x,y)e^{\tilde{v}}(y,t)dy - \eta_2(x)e^{\tilde{v}}(1,t) + e^{\tilde{v}}_x(1,t)k^b(x,1) \\ & - e^{\tilde{v}}_x(x,t)k^b(x,x) - e^{\tilde{v}}(1,t)k^b_y(x,1) + e^{\tilde{v}}(x,t)k^b_y(x,x) + \int_x^1 k^b_{yy}(x,y)e^{\tilde{v}}(y,t)dy. \end{aligned}$$

Adding and subtracting the term $o_1e^{\tilde{v}}(x,t)$ and incorporating $e^{\tilde{v}}_x(1,t) = 0$,

$$\begin{aligned}(\text{R.H.S})_2 = {} & e^{\tilde{v}}_{xx}(x,t) - (o_1+\gamma)e^{\tilde{v}}(x,t) + \beta e^{\tilde{w}}(x,t) \\ & + \int_x^1 [-k^b_{xx}(x,y) + k^b_{yy}(x,y) - o_1k^b(x,y)]e^{\tilde{v}}(y,t)dy \\ & + (2\frac{d}{dx}k^b(x,x) + o_1)e^{\tilde{v}}(x,t) - (k^b_y(x,1) + \eta_2(x))e^{\tilde{v}}(1,t). \end{aligned} \tag{4.73}$$

Since $k^b(x,y)$ is given by (4.59) and $\eta_2(x) = -k^b_y(x,1)$, then referring to (4.57b) and (4.71), it follows that

$$(\text{L.H.S})_2 = 0 = (\text{R.H.S})_2.$$

Thus the state transformation (4.58) transforms the elliptic equation (4.57b) into (4.56b). We apply the transformation to the boundary conditions (4.57d),

$$\begin{aligned} e_x^v(0,t) &= e_x^{\tilde{v}}(0,t) - \int_0^1 k_x^b(0,y)e^{\tilde{v}}(y,t)dy + k^b(0,0)e^{\tilde{v}}(0,t) \\ &= 0, \end{aligned}$$

by means of using (4.59b) and that $e_x^{\tilde{v}}(0,t) = 0$. We obtain the boundary condition at $x = 0$ in (4.56d). Similarly,

$$\begin{aligned} e_x^v(1,t) =& e_x^{\tilde{v}}(1,t) - \int_1^1 k_x^b(1,y)e^{\tilde{v}}(y,t)dy + k^b(1,1)e^{\tilde{v}}(1,t) \\ =& k^b(1,1)e^{\tilde{v}}(1,t) = k^b(1,1)e^v(1,t), \end{aligned}$$

where the previous step was obtained by noting that $e^{\tilde{v}}(0,t) = e^v(0,t)$. If $\eta_4 = -k^b(1,1)$, then we obtain the second boundary condition in (4.56d) at $x = 1$. The conclusion of the theorem follows. □

The next result, which follows from Proposition 4.3.1, is the main contribution of this section.

Theorem 4.3.1. *Let $k^b(x,y)$ be the solution of system (4.59). The error dynamics (4.56) with output injections η_j, $j = 1,\ldots,4$ defined as given in (4.60a)-(4.60b) are exponentially stable if and only if the parameter o_1 satisfies*

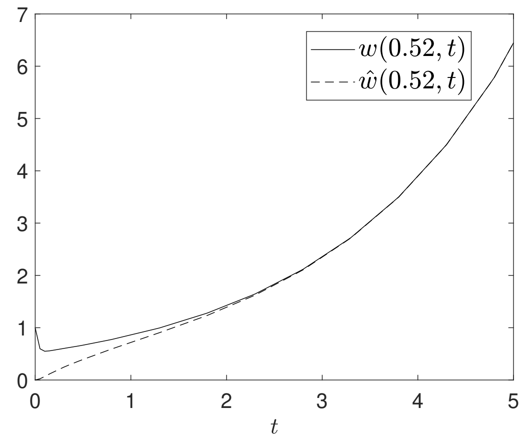
$$(o_1 + \rho)(o_1 + \gamma) > \alpha\beta, \tag{4.74}$$

and $o_1 + \gamma \neq -(n\pi)^2$.

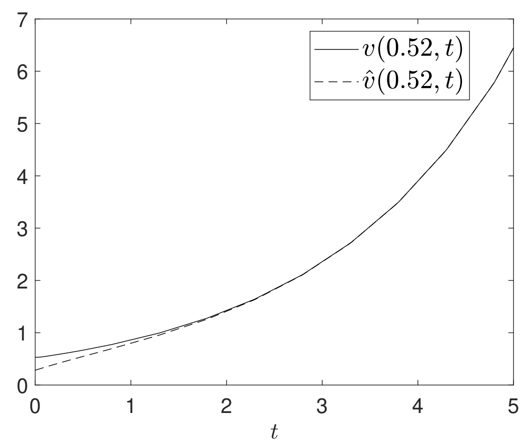
Proof. If o_1 is given by (4.74) such that $o_1 + \gamma \neq -(n\pi)^2$, the target system (4.57) has a unique solution and is exponentially stable due to the criteria for stability of parabolic-elliptic systems established previously in Corollary 4.1.1. Finally, the exponential stability of the error dynamics (4.56) follows by referring to Theorem 4.3.1 and using the invertibility of transformation (4.58). This concludes the proof. □

We conducted numerical simulations for the dynamics of both the coupled system (4.1)-(4.4) and the state observer (4.54) in the situation where two measurements, $w(1,t)$ and $v(1,t)$, are available. The simulations were performed using COMSOL Multiphysics software. We used linear splines to approximate the coupled equations by a system of

DAEs. The spatial interval was divided into 27 subintervals. Time was discretized by a time-stepping algorithm called generalized alpha.



(a) $w(0.52, t)$ and $\hat{w}(0.52, t)$



(b) $v(0.52, t)$ and $\hat{v}(0.52, t)$

Figure 4.5: A comparison between the states of the coupled system (4.1)-(4.4) versus the estimated states using observer (4.54) at $x = 0.52$. System parameters are $\gamma = 1$, $\rho = 0.5$, $\alpha = 1$, $\beta = 1$, $o_1 = 5$ with initial conditions $w_0 = \sin(\pi x)$, $v_0 = \beta(\gamma I - d_{xx})^{-1} w_0$ and $\hat{w}_0 = 0$, $\hat{v}_0 = \beta(\gamma I - d_{xx})^{-1} \hat{w}_0$.

Observer designs were done for system (4.1)-(4.4) with $u(t) \equiv 0$. The chosen parameters were $\gamma = 1$, $\rho = 0.5$, $\alpha = 1$ and $\beta = 1$. With these parameters, the system is unstable. With $o_1 = 5$, the sufficient condition (4.74) for the error dynamics to be exponentially stable is satisfied. The true and estimated states at $x = 0.52$ are shown in Figure 4.5.

4.4 Observer design for a coupled parabolic-elliptic system with a single measurement

The objective of this section is to design an exponentially convergent observer for (4.1)-(4.4) given only a single measurement $w(1,t)$. We present the following observer:

$$\hat{w}_t(x,t) = \hat{w}_{xx}(x,t) - \rho\hat{w}(x,t) + \alpha\hat{v}(x,t) + \eta_1(x)[w(1,t) - \hat{w}(1,t)], \tag{4.75a}$$

$$0 = \hat{v}_{xx}(x,t) - \gamma\hat{v}(x,t) + \beta\hat{w}(x,t), \tag{4.75b}$$

$$\hat{w}_x(0,t) = 0, \quad \hat{w}_x(1,t) = u(t) + \eta_2[w(1,t) - \hat{w}(1,t)], \tag{4.75c}$$

$$\hat{v}_x(0,t) = 0, \quad \hat{v}_x(1,t) = 0, \tag{4.75d}$$

where $\eta_1(x)$ and η_2 are output injections to be designed. Defining the states of the error dynamics as in (4.55), the system describing the observation error satisfies

$$e_t^w(x,t) = e_{xx}^w(x,t) - \rho e^w(x,t) + \alpha e^v(x,t) - \eta_1(x)e^w(1,t), \tag{4.76a}$$

$$0 = e_{xx}^v(x,t) - \gamma e^v(x,t) + \beta e^w(x,t), \tag{4.76b}$$

$$e_x^w(0,t) = 0, \quad e_x^w(1,t) = -\eta_2 e^w(1,t), \tag{4.76c}$$

$$e_x^v(0,t) = 0, \quad e_x^v(1,t) = 0. \tag{4.76d}$$

Both $\eta_1(x)$ and η_2 have to be chosen so that exponential stability of error dynamics is achieved. Following a backstepping approach, we define the transformation

$$e^{\tilde{w}}(x,t) = e^w(x,t) - \int_0^x k^a(x,y)e^w(y,t)dy, \tag{4.77}$$

where $k^a(x,y)$ is given by system (4.17) with c_1 replaced by o_1; see Lemma 2.5.1. The inverse transformation is

$$e^w(x,t) = e^{\tilde{w}}(x,t) + \int_0^x \ell^a(x,y)e^{\tilde{w}}(y,t)dy, \tag{4.78}$$

where $\ell^a(x,y)$ satisfies the hyperbolic PDE in Lemma 2.5.2.

Proposition 4.4.1. *If the output injections are*

$$\eta_1(x) = 0, \tag{4.79}$$

$$\eta_2 = -k^a(1,1), \tag{4.80}$$

where $k^a(x,y)$ solves (4.17) with c_1 being replaced by o_2, then transformation (4.82) converts the error dynamics (4.76) into the target system

$$e_t^{\tilde{w}}(x,t) = e_{xx}^{\tilde{w}}(x,t) - (o_1+\rho)e^{\tilde{w}}(x,t) + \alpha e^v(x,t) - \alpha \int_0^x k^a(x,y)e^v(y,t)dy, \tag{4.81a}$$

$$0 = e_{xx}^v(x,t) - \gamma e^v(x,t) + \beta e^{\tilde{w}}(x,t) + \beta \int_0^x \ell^a(x,y)e^{\tilde{w}}(y,t)dy, \tag{4.81b}$$

$$e_x^{\tilde{w}}(1,t) = -\int_0^1 k_x^a(1,y)e^{\tilde{w}}(y,t)dy - \int_0^1 k_x^a(1,y)\int_0^y \ell^a(y,z)e^{\tilde{w}}(z,t)dzdy, \tag{4.81c}$$

$$e_x^{\tilde{w}}(0,t) = 0, \quad e_x^v(0,t) = 0, \quad e_x^v(1,t) = 0. \tag{4.81d}$$

Proof. It will be useful to rewrite (4.77) as

$$e^w(x,t) = e^{\tilde{w}}(x,t) + \int_0^x k^a(x,y)e^w(y,t)dy. \tag{4.82}$$

We take the spatial and the time derivatives of (4.82),

$$\begin{aligned} e_{xx}^w(x,t) = {} & e_{xx}^{\tilde{w}}(x,t) + \int_0^x k_{xx}^a(x,y)e^w(y,t)dy + k_x^a(x,x)e^w(x,t) + \frac{d}{dx}k^a(x,x)e^w(x,t) \\ & + k^a(x,x)e_x^w(x,t), \end{aligned} \tag{4.83}$$

$$\begin{aligned} e_t^w(x,t) &= e_t^{\tilde{w}}(x,t) + \int_0^x k^a(x,y)e_t^w(y,t)dy \\ &= e_t^{\tilde{w}}(x,t) - \rho\int_0^x k^a(x,y)e^w(y,t)dy + \alpha\int_0^x k^a(x,y)e^v(y,t)dy + k^a(x,x)e_x^w(x,t) \\ &\quad - k^a(x,0)e_x^w(0,t) - k_y^a(x,x)e^w(x,t) + k_y^a(x,0)e^w(0,t) + \int_0^x k_{yy}^a(x,y)e^w(y,t)dy \\ &\quad - e^w(1,t)\int_0^x k^a(x,y)\eta_1(y)dy. \end{aligned} \tag{4.84}$$

Substituting (4.83) and (4.84) in the parabolic equation (4.76a), and using $e_x^w(0,t) = 0$

and $k^a_y(x,0) = 0$,

$$e^{\tilde{w}}_t(x,t) = e^{\tilde{w}}_{xx}(x,t) + \alpha e^v(x,t) + \left(k^a_y(x,x) + k^a_x(x,x)\right) e^w(x,t) + (-\rho + \frac{d}{dx}k^a(x,x))e^w(x,t)$$
$$+ k^a(x,x)e^w_x(x,t) - k^a(x,x)e^w_x(x,t) - \alpha \int_0^x k^a(x,y)e^v(y,t)dy$$
$$- \eta_1(x)e^w(1,t) + \int_0^x [k^a_{xx}(x,y) - k^a_{yy}(x,y) + \rho k^a(x,y)]e^w(y,t)dy$$
$$+ e^w(1,t)\int_0^x k^a(x,y)\eta_1(y)dy. \tag{4.85}$$

We add and subtract the term $(o_1 + \rho)e^w(x,t)$ to the right-hand-side of equation (4.85), where

$$(o_1 + \rho)e^w(x,t) = (o_1 + \rho)e^{\tilde{w}}(x,t) + (o_1 + \rho)\int_0^x k^a(x,y)e^w(y,t)dy,$$

then after simplifying and rearranging terms, we obtain

$$e^{\tilde{w}}_t(x,t) = e^{\tilde{w}}_{xx}(x,t) + \alpha e^v(x,t) - (o_1 + \rho)e^{\tilde{w}}(x,t) + (o_1 + 2\frac{d}{dx}k^a(x,x))e^w(x,t)$$
$$- \alpha \int_0^x k^a(x,y)e^v(y,t)dy + e^w(1,t)\int_0^x k^a(x,y)\eta_1(y)dy$$
$$- \eta_1(x)e^w(1,t) + \int_0^x [k^a_{xx}(x,y) - k^a_{yy}(x,y) - o_1 k^a(x,y)]e^w(y,t)dy.$$

Since $k^a(x,y)$ is given by (4.17), the previous equation yields

$$e^{\tilde{w}}_t(x,t) = e^{\tilde{w}}_{xx}(x,t) - (o_1 + \rho)e^{\tilde{w}}(x,t) + \alpha e^v(x,t) - \alpha \int_0^x k^a(x,y)e^v(y,t)dy$$
$$- e^w(1,t)\int_0^x k^a(x,y)\eta_1(y)dy - \eta_1(x)e^w(1,t).$$

If $\eta_1(x) = 0$, we obtain the parabolic equation (4.81a). We now apply transformation (4.82) on the boundary conditions (4.76c), using statement (4.17b)

$$e^{\tilde{w}}_x(0,t) = e^w_x(0,t) - \int_0^0 k^a_x(1,y)w(y)dy - k^a(0,0)w(1,t) = 0,$$

and

$$
\begin{aligned}
e_x^{\tilde{w}}(1,t) &= e_x^w(1,t) - \int_0^1 k_x^a(1,y)e^w(y,t)dy - k^a(1,1)e^w(1,t) \\
&= -(\eta_2 + k^a(1,1))e^w(1,t) - \int_0^1 k_x^a(1,y)e^w(y,t)dy \\
&= -\int_0^1 k_x^a(1,y)e^w(y,t)dy. \\
&= -\int_0^1 k_x^a(1,y)e^{\tilde{w}}(y,t)dy - \int_0^1 k_x^a(1,y)\int_0^y \ell^a(y,z)e^{\tilde{w}}(z,t)dzdy.
\end{aligned}
$$

The previous equation holds true via using (4.80) and using the inverse transformation (4.78). The elliptic equation (4.81b) can be obtained via using the inverse transformation (4.78). □

Theorem 4.4.1. *If*

$$
\begin{aligned}
o_1 + \rho > \frac{1}{2} &+ \left(1 + \sqrt{\frac{o_1\pi}{8}}\left(erfi(\sqrt{\frac{o_1}{2}})erf(\sqrt{\frac{o_1}{2}})\right)^{\frac{1}{2}}\right)^2 \\
&\times \left(\frac{|\alpha\beta|}{\gamma} + \frac{1}{2}\left[\frac{o_1}{2}(1 + \frac{o_1}{2})e^{\frac{o_1}{4}}\right]^2\right), \qquad (4.86)
\end{aligned}
$$

then the target system (4.81) *is exponentially stable on the space* $L_2[0,1]$ *with decay rate* $2o_2$ *where*

$$
\begin{aligned}
o_2 = &o_1 + \rho - \left(1 + \sqrt{\frac{o_1\pi}{8}}\left(erfi(\sqrt{\frac{o_1}{2}})erf(\sqrt{\frac{o_1}{2}})\right)^{\frac{1}{2}}\right)^2 \\
&\times \left(\frac{|\alpha\beta|}{\gamma} + \frac{1}{2}\left[\frac{o_1}{2}(1 + \frac{o_1}{2})e^{\frac{o_1}{4}}\right]^2\right) - \frac{1}{2}. \qquad (4.87)
\end{aligned}
$$

Proof. With a parallel line of reasoning as the one used to prove Theorem 4.2.1, it follows that the stability of the target system (4.81) follows from the exponential decay of the state $e^{\tilde{w}}(x,t)$. Define the Lyapunov function candidate

$$
V(t) = \frac{1}{2}\int_0^1 (e^{\tilde{w}}(x,t))^2 dx = \frac{1}{2}\|e^{\tilde{w}}(x,t)\|^2. \qquad (4.88)
$$

Taking the time derivative of $V(t)$

$$\dot{V}(t) = \int_0^1 e^{\tilde{w}}(x,t)e^{\tilde{w}}_{xx}(x,t)dx - (o_1+\rho)\int_0^1 (e^{\tilde{w}}(x,t))^2 dx$$
$$+ \alpha \int_0^1 e^{\tilde{w}}(x,t)e^{v}(x,t)dx - \alpha \int_0^1 e^{\tilde{w}}(x,t)\int_0^x k^a(x,y)e^v(y,t)dydx. \qquad (4.89)$$

Integrating the term $\int_0^1 e^{\tilde{w}}(x,t)e^{\tilde{w}}_{xx}(x,t)dx$ by parts, and using the boundary conditions (4.81c)-(4.81d)

$$\int_0^1 e^{\tilde{w}}(x,t)e^{\tilde{w}}_{xx}(x,t)dx = -\int_0^1 k^a_x(1,y)e^w(y,t)dy\, e^{\tilde{w}}(1,t) - \|e^{\tilde{w}}_x\|^2. \qquad (4.90)$$

To bound the term $-\int_0^1 k^a_x(1,y)e^w(y,t)dy\, e^{\tilde{w}}(1,t)$ in (4.90), we use Cauchy-Schwarz inequality

$$-\int_0^1 k^a_x(1,y)e^w(y,t)dye^{\tilde{w}}(1,t) \leq \|k^a_x(1,y)\|\|e^w\|\|e^{\tilde{w}}\|_\infty.$$

Invoking Agmon's inequality [117] on the right-hand-side of the previous inequality leads to

$$-\int_0^1 k^a_x(1,y)e^w(y,t)dye^{\tilde{w}}(1,t) \leq \|k^a_x(1,y)\|\|e^w\|\|e^{\tilde{w}}\|^{1/2}\|e^{\tilde{w}}\|^{1/2}_{H^1}.$$

By invoking Young's inequality ($\|ab\| \leq \frac{\|a\|^2}{2} + \frac{\|b\|^2}{2}$) on the right-hand-side of the previous inequality, we obtain

$$-\int_0^1 k^a_x(1,y)e^w(y,t)dye^{\tilde{w}}(1,t) \leq \frac{\|k^a_x(1,y)\|^2\|e^w\|^2}{2} + \frac{\|e^{\tilde{w}}\|\|e^{\tilde{w}}\|_{H^1}}{2}$$
$$\leq \frac{\|k^a_x(1,y)\|^2\|e^w\|^2}{2} + \frac{1}{2}\Big(\frac{\|e^{\tilde{w}}\|^2}{2} + \frac{\|e^{\tilde{w}}\|^2_{H^1}}{2}\Big).$$

Using that $\|e^{\tilde{w}}\|_{H^1} = (\|e^{\tilde{w}}\|^2 + \|e^{\tilde{w}}_x\|^2)^{\frac{1}{2}}$, we arrive at

$$-\int_0^1 k^a_x(1,y)e^w(y,t)dye^{\tilde{w}}(1,t) \leq \Big(\frac{\|k^a_x(1,y)\|^2\|e^w\|^2}{2} + \frac{\|e^{\tilde{w}}\|^2}{2} + \frac{\|e^{\tilde{w}}_x\|^2}{4}\Big).$$

Referring to the inverse transformation (4.78),

$$-\int_0^1 k^a_x(1,y)e^w(y,t)dye^{\tilde{w}}(1,t) \leq \frac{(1+\|\ell^a\|)^2\|k^a_x(1,y)\|^2+1}{2}\|e^{\tilde{w}}\|^2 + \frac{1}{4}\|e^{\tilde{w}}_x\|^2. \qquad (4.91)$$

Combining (4.90) and (4.91), we obtain

$$\int_0^1 e^{\tilde{w}}(x,t)e^{\tilde{w}}_{xx}(x,t)dx \leq \frac{(1+\|\ell^a\|)^2\|k^a_x(1,y)\|^2+1}{2}\|e^{\tilde{w}}\|^2. \tag{4.92}$$

Bounding the other on the right-hand side of (4.89) using Cauchy-Schwarz inequality and following steps to the one taken in Lemma 4.2.2,

$$\begin{aligned}\alpha\int_0^1 e^{\tilde{w}}(x,t)e^{v}(x,t)dx &\leq |\alpha|\|e^{\tilde{w}}\|\|e^{v}\| \\ &\leq \frac{|\alpha\beta|}{\gamma}(1+\|\ell^a\|)\|e^{\tilde{w}}\|^2. \end{aligned}\tag{4.93}$$

Similarly,

$$-\alpha\int_0^1 e^{\tilde{w}}(x,t)\int_0^x k^a(x,y)e^{v}(y,t)dydx \leq \frac{|\alpha\beta|}{\gamma}(1+\|\ell^a\|)\|k^a\|\|e^{\tilde{w}}\|^2. \tag{4.94}$$

Using inequalities (4.92), (4.93) and (4.94), equation (4.89) leads to

$$\dot{V}(t) \leq -\Big(o_1+\rho-\frac{(1+\|\ell^a\|)^2\|k^a_x(1,y)\|^2+1}{2}-\frac{|\alpha\beta|}{\gamma}(1+\|\ell^a\|)(1+\|k^a\|)\Big)\|e^{\tilde{w}}\|^2,$$

and so using the bounds on $\|\ell^a\|$, $\|k^a\|$ and $\|k^a_x(1,y)\|$ in Lemma 4.2.1 and Lemma 4.2.3, respectively, we arrive at

$$\begin{aligned}\dot{V}(t) \leq -\Bigg(o_1+\rho-&\left(1+\sqrt{\frac{o_1\pi}{8}}\left(erfi(\sqrt{\frac{o_1}{2}})erf(\sqrt{\frac{o_1}{2}})\right)^{\frac{1}{2}}\right)^2 \\ &\times\left(\frac{|\alpha\beta|}{\gamma}+\frac{1}{2}\left[\frac{o_1}{2}(1+\frac{o_1}{2})e^{\frac{o_1}{4}}\right]^2\right)-\frac{1}{2}\Bigg)\|e^{\tilde{w}}\|^2.\end{aligned}$$

Setting the parameter o_2 as given in (4.87), we obtain

$$\dot{V}(t) \leq -2o_2V(t),$$

where we used equation (4.88). If the parameter o_1 is chosen such that (4.86) is satisfied, then $V(t)$ decays exponentially as $t\to\infty$. Thus, $\|e^{\tilde{w}}(x,t)\|$ decays exponentially. □

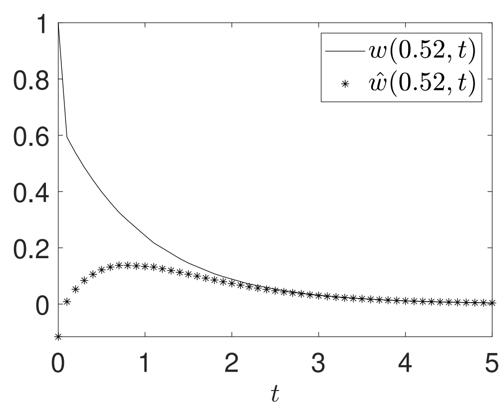
Assuming consistent initialization on system (4.76) (see equation (4.15)), the following result now follows from Proposition 4.4.1 and Theorem 4.4.1.

Theorem 4.4.2. *Let $k^a(x,y)$ be the solution of system (4.17) where c_1 is replaced by o_1, and o_1 satisfies inequality (4.86). The error dynamics (4.76) with output injections defined as given in (4.79)-(4.80) are exponentially stable with convergence rate*

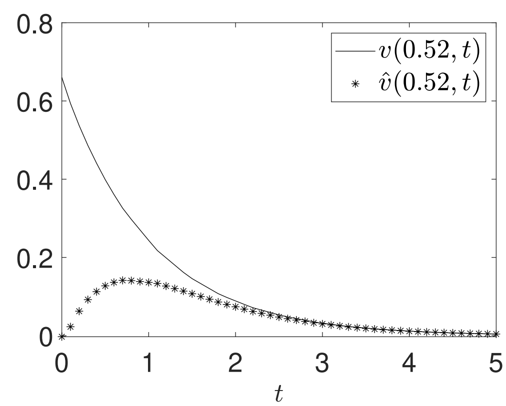
$$\|e^w(\cdot,t)\| \leq c\|e^w(\cdot,0)\|e^{-2o_2 t}, \qquad \|e^v(\cdot,t)\| \leq m\|e^w(\cdot,t)\|.$$

Here, o_2 is defined in (4.87), and c and m are positive constants that are independent of the initial conditions.

Numerical simulations were also conducted to study the observer (4.75) when a single measurement $w(1,t)$ is available, and using the parameter values $\gamma = 1$, $\rho = 1$, $\alpha = 0.5$, $\beta = 0.5$. With these parameter values, the system is stable. Also, we set $o_1 = 0.5$ so the stability condition for the observation error dynamics, i.e., (4.86) is satisfied. The control $u(t)$ was set identically to zero. The initial conditions were $w_0 = \sin(\pi x)$ and $\hat{w}_0 = \sin(2\pi x)$. The true and estimated states at $x = 0.52$ are given in Figure 4.6.



(a) $w(0.52,t)$ and $\hat{w}(0.52,t)$



(b) $v(0.52,t)$ and $\hat{v}(0.52,t)$

Figure 4.6: A comparison between the states of the coupled system (4.1)-(4.4) versus the estimated states using observer (4.75) at $x = 0.52$. Here $\rho = 1$, $\gamma = 0.5$, $\alpha = 0.5$, $\beta = 0.5$, $o_1 = 0.5$ with $w_0 = \sin(\pi x)$, $v_0 = \beta(\gamma I - d_{xx})^{-1}w_0$ and $\hat{w}_0 = \sin(2\pi x)$, $\hat{v}_0 = \beta(\gamma I - d_{xx})^{-1}\hat{w}_0$.

Condition (4.86), concerning the stability of the observation error dynamics, imposes restrictions on the permissible choices of system parameters. This observation is demonstrated in Figure 4.7, where we present a comparison between the right-hand-side of inequality (4.86) as a function of o_1, and several straight lines $o_1 + \rho$ varying with different ρ. The other parameters are fixed as $\beta = \gamma = 1,\ \alpha = 0.5$. The dashed line in Figure 4.7

represents the right-hand-side of (4.86). The observation error dynamics (4.76) is exponentially stable if the values of o_1 are such that the dashed line in Figure 4.7 is beneath the straight lines, for different ρ.

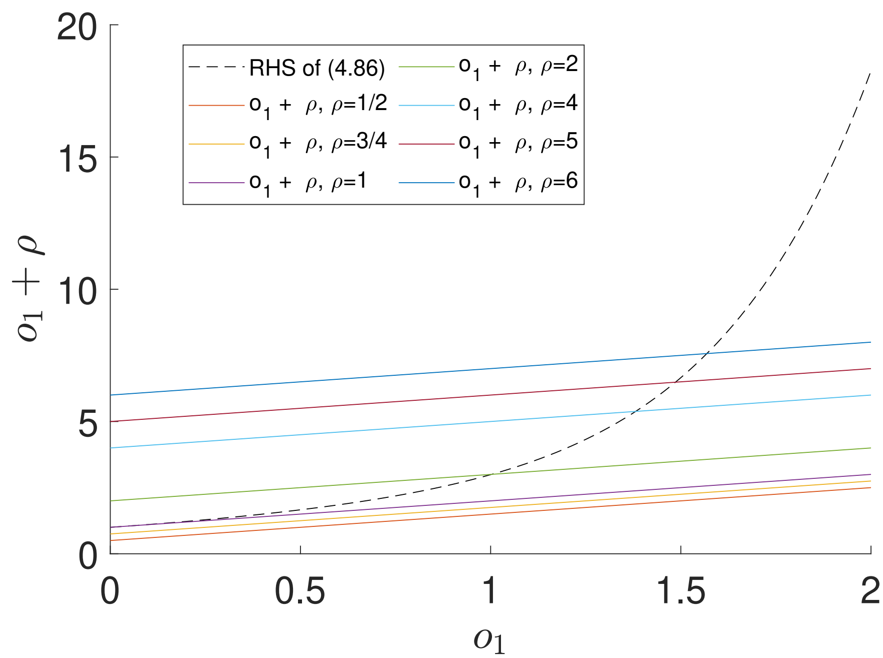


Figure 4.7: A comparison between the right-hand-side of inequality (4.86) as a function of o_1 against several straight lines $o_1 + \rho$ for different values of ρ, while the other parameters are fixed as $\beta = \gamma = 1$, $\alpha = 0.5$. The right-hand-side of (4.86) is described using a dashed line(- - -). For any ρ, the error dynamics (4.81) is exponentially stable if o_1 is such that the dashed line(- - -) is beneath the straight line $o_1 + \rho$.

As for stabilization with a single control, we now establish a less restrictive sufficient condition on the system parameters for the stability of the error dynamics (4.76). To do so, we study system (4.76) while using the values of the output injections $\eta_1(x)$ and η_2 as given in (4.79) and (4.80), respectively.

Theorem 4.4.3. *The error dynamics* (4.76)*, with* $\eta_1(x) = 0$ *and* $\eta_2 = -k^a(1, 1)$*, is expo-*

nentially stable if

$$o_1 + 2\rho > 2|\frac{\alpha\beta}{\gamma}|, \tag{4.95}$$

and $\frac{\pi^2}{2} \geq o_1$.

Proof. As in the proof of Theorem 4.2.3, the exponential stability of system (4.76) follows from the exponential decay of the state $e^w(x,t)$.

Define the Lyapunov function candidate

$$V(t) = \frac{1}{2}\int_0^1 (e^w(x,t))^2 dx = \frac{1}{2}\|e^w(x,t)\|^2.$$

Taking the time derivative of $V(t)$ and noting that $\eta_1(x) = 0$,

$$\begin{aligned}\dot{V}(t) &= \int_0^1 e^w(x,t)e_t^w(x,t)dx \\ &= \int_0^1 e^w(x,t)[e_{xx}^w(x,t) - \rho e^w(x,t) + \alpha e^v(x,t)]dx \\ &= \int_0^1 e^w(x,t)e_{xx}^w(x,t)dx - \rho\int_0^1 (e^w(x,t))^2 dx + \alpha\int_0^1 e^w(x,t)e^v(x,t)dx. \end{aligned} \tag{4.96}$$

Integrating by parts, using the boundary conditions in (4.76c), and noting that $e_x^w(1,t) = -\eta_2 e^w(1,t) = \frac{1}{2}o_1 e^w(1,t)$, we obtain

$$\begin{aligned}\int_0^1 e^w(x,t)e_{xx}^w(x,t)dx &= e^w(1,t)e_x^w(1,t) - e^w(0,t)e_x^w(0,t) - \|e_x^w\|^2 \\ &= e^w(1,t)e_x^w(1,t) - \|e_x^w\|^2 \\ &= -\frac{1}{2}o_1(e^w(1,t))^2 - \int_0^1 (e_x^w(x,t))^2 dx. \end{aligned} \tag{4.97}$$

Using a variation of Wirtinger's inequality [61, inequality 2.31],

$$-(e^w(1,t))^2 \leq -\int_0^1 (e^w(x,t))^2 dx + \frac{4}{\pi^2}\int_0^1 (e_x^w(x,t))^2 dx, \tag{4.98}$$

then (4.97) implies that

$$\int_0^1 e^w(x,t)e^w_{xx}(x,t)dx \le -\frac{1}{2}o_1\int_0^1 (e^w(x,t))^2dx + \frac{2o_1}{\pi^2}\int_0^1 (e^w_x(x,t))^2dx - \int_0^1 (e^w_x(x,t))^2dx. \tag{4.99}$$

We can bound the right-hand-side of (4.96) using (4.99) and Cauchy-Schwarz as follows.

$$\dot V(t) \le -\frac{1}{2}o_1\|e^w\|^2 - (1-\frac{2o_1}{\pi^2})\|e^w_x\|^2 - \rho\|e^w\|^2 + |\frac{\alpha\beta}{\gamma}|\|e^w\|^2. \tag{4.100}$$

Hence

$$\dot V(t) \le -(\frac{1}{2}o_1 + \rho - |\frac{\alpha\beta}{\gamma}|)\|e^w\|^2 - (1-\frac{2o_1}{\pi^2})\|e^w_x\|^2. \tag{4.101}$$

If o_1 is chosen such that $o_1 \le \frac{\pi^2}{2}$ such that (4.95) is satisfied, then setting $o_2 = o_1 + 2\rho - 2|\frac{\alpha\beta}{\gamma}| > 0$, (4.101) leads to

$$V(t) \ \le \ e^{-2o_2t}V(0).$$

$V(t)$ decays exponentially as $t \to \infty$. Thus, $\|e^w(x,t)\|$ decays exponentially. □

If the parameters in system (4.1)-(4.4) satisfy inequality (4.95) for some $\frac{\pi^2}{2} > o_1 > 0$, then in the absence of disturbances the observation error obtained from estimating the state using observer (4.75) will exhibit exponential decay.

Figure 4.8 presents the right-hand-side of (4.95) versus straight lines $o_1 + 2\rho$, for some values of ρ while fixing the remaining parameters as $\gamma = \alpha = 4,\ \beta = 4.6$. A comparison between Figure 4.7 and Figure 4.8 indicates that the new sufficient condition, given by inequality (4.95), is weaker than the condition in (4.86). This is apparent by the dashed line (- - -) lying below the straight lines for a wider range of the parameter ρ. We simulated the solutions of the true system (4.1)-(4.4) and the observer (4.75) numerically using a finite-element approximation in COMSOL Multiphysics software. The simulations were carried with parameter values $\gamma = 1,\ \rho = 2,\ \alpha = 1,\ \beta = 1$. We also set $o_1 = 2$ ensuring the stability condition for the observation error dynamics (i.e. (4.95)) is met. The initial conditions are $w_0 = \sin(\pi x)(1+\cos^2(\pi x))$, $v_0 = \beta(\gamma I - d_{xx})^{-1}w_0$. Also, the initial conditions on the estimator (4.75) are $\hat w_0 = \hat v_0 = 0$. The true and estimated states at $x = 0.52$ are given in Figure 4.9.

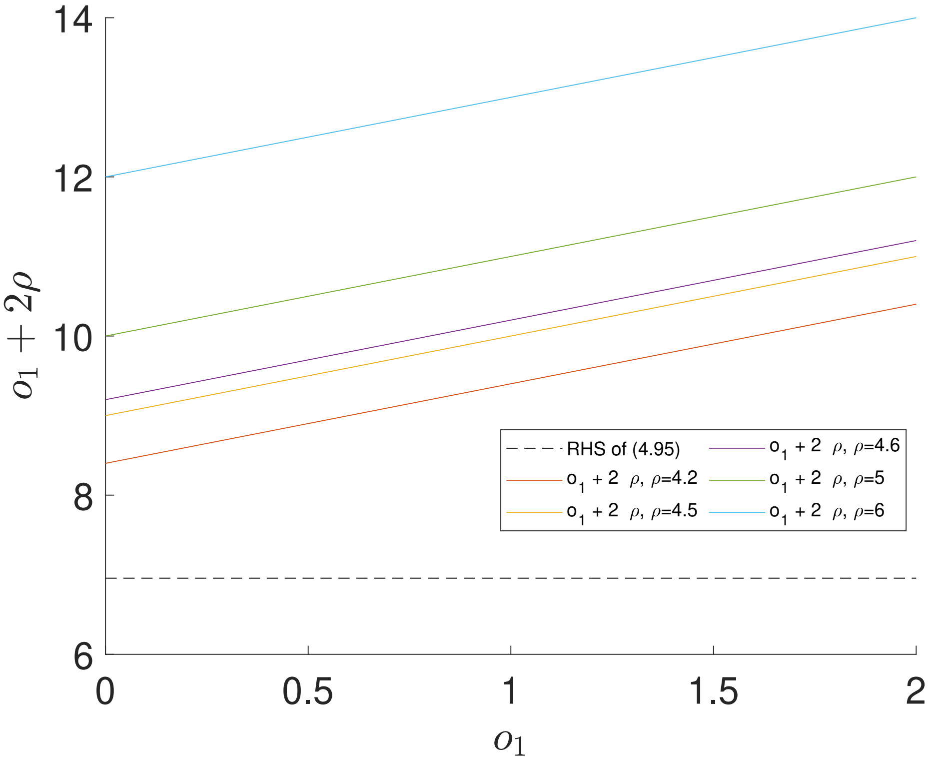
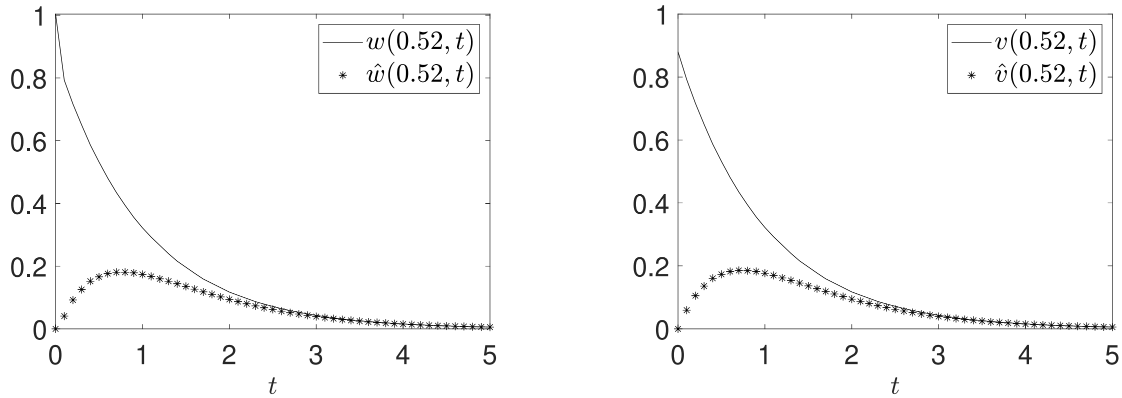


Figure 4.8: A comparison between the right-hand-side of (4.95) as a function of o_1 against several straight lines $o_1 + 2\rho$ for different values of ρ, where the other parameters are fixed as $\alpha = \gamma = 4$, $\beta = 4.6$. The right-hand-side of (4.95) is described using a dashed line (- - -). The observation error (4.76) is exponentially stable for values of o_1 at which the straight line $o_1 + 2\rho$, for some ρ, is above the dashed line (- - -). A bigger range of parameter combinations meets the criteria of inequality (4.95) compared to inequality (4.86). This indicates that the bound in (4.95) is less restrictive than the condition presented in (4.86).



(a) Comparison between $w(0.52,t)$ and $\hat{w}(0.52,t)$ (b) Comparison between $v(0.52,t)$ and $\hat{v}(0.52,t)$

Figure 4.9: A comparison between the states of the coupled system (4.1)-(4.4) versus the estimated states using estimator (4.75) at $x = 0.52$. System's parameters are $\gamma = 1$, $\rho = 2$, $\alpha = 1$, $\beta = 1$, $o_1 = 2$ with initial conditions $w_0 = \sin(\pi x)(1 + \cos^2(\pi x))$, $v_0 = \beta(\gamma I - d_{xx})^{-1} w_0$. Also, the initial conditions on the estimator (4.75) are $\hat{w}_0 = \hat{v}_0 = 0$.

4.5 Output feedback control for a coupled parabolic-elliptic system

In general, the full state is not available for control. Output feedback is based on using only the available measurements to stabilize the system. A common approach to output feedback is to combine a stabilizing state feedback with an observer. The estimated state from the observer is used to replace the state feedback Kz by $K\hat{z}$ where z is the true state, and $\hat{z}$ is the estimated state.

In the situation considered here, if there are two measurements, this leads to the output feedback controller consisting of the observer (4.75) combined with the state feedback

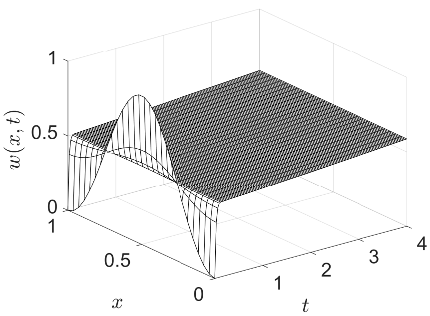
$$u(t) = \int_0^1 k_x^a(1,y)\hat{w}(y,t)dy + k^a(1,1)\hat{w}(1,t), \tag{4.102}$$

where $k^a(x,y)$ is the solution of system (2.14) with c_1 satisfying the bound (4.27). Since the original system (4.1)-(4.4) is a well-posed control system and the observer combined with the state feedback is a well-posed system, the following result follows immediately from the results in [83, Section 3].

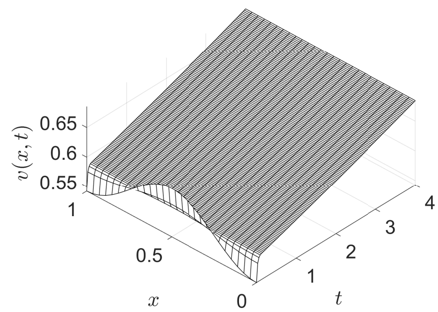
Theorem 4.5.1. *For any consistent initial conditions* w_0, $\hat{w}_0 \in L_2(0,1)$*, the closed-loop system consisting of* (4.1)-(4.4)*, together with the observer dynamics* (4.75) *and control input* (4.102) *has a unique classical solution in* $C^{2,1}((0,1)\times(0,\infty))$*. In addition, the system is exponentially stable at the origin in* $L_2(0,1)$*.*

Note that when using output feedback, the parameters have to satisfy both the stability condition associated with the control problem (4.27) or (4.41), and also the bound of the stability of the observation error (4.86) or (4.95).

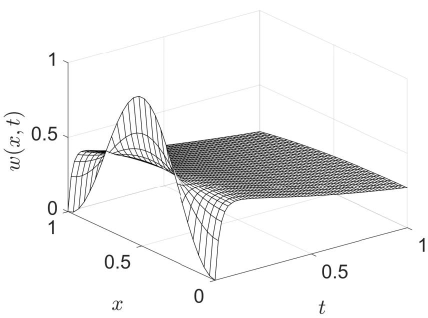
We illustrated the efficacy of control (4.102) by conducting simulations of the controlled coupled system (4.1)-(4.4). This was done using linear finite-elements in COMSOL Multiphysics software. The system parameters were set to $\gamma = 3$, $\rho = 3.47$, $\alpha = 3$, and $\beta = 3.5$, which means the uncontrolled system is unstable. Choosing $o_1 = 2$ and control gain $c_1 = 0.9$, we ensure the control and observation error dynamics' stability criteria are met. The previous bounds (4.27) and (4.86) are not satisfied for these parameters. The controller $u(t)$ in (4.102), was approximated using 16 elements. This approach, known as *late-lumping*, postpones the approximation of the controller until the last step in the design, see [7] for a comparison of late and early lumping. The initial conditions were set to $w_0 = \frac{1}{2}(1-\cos(2\pi x))$, $v_0 = \beta(\gamma I - d_{xx})^{-1}w_0$, with the initial estimates $\hat{w}_0 = \hat{v}_0 = 0$. The system was simulated with 27 elements, more than the number of elements used to approximate the controller, to mimic some of the higher-order modes in the infinite-dimensional state. The trajectories of the controlled system (4.1)-(4.4) under the applied control (4.102) are shown in Figure 4.10.



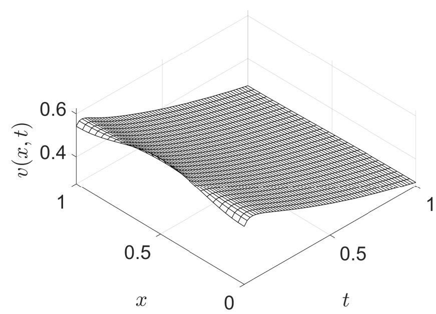
(a) Uncontrolled $w(x,t)$



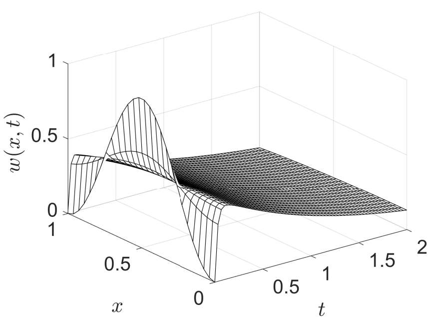
(b) Uncontrolled $v(x,t)$



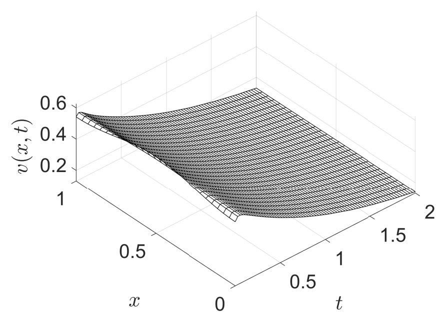
(c) Controlled $w(x,t)$ over 1(s)



(d) Controlled $v(x,t)$ over 1(s)



(e) Controlled $w(x,t)$ over 2(s)



(f) Controlled $v(x,t)$ over 2(s)

Figure 4.10: A 3D landscape of the dynamics for the controlled coupled parabolic-elliptic system (4.1)–(4.4), after applying output feedback (4.102). Here, $\gamma = 3$, $\rho = 3.46$, $\alpha = 3$, and $\beta = 3.5$. Also, $w_0 = \frac{1}{2}(1 - \cos(2\pi x))$ and $v_0 = \beta(\gamma I - d_{xx})^{-1} w_0$. The observer (4.75) starts from $\hat{w}_0 = \hat{v}_0 = 0$. With control and observation gains $c_1 = 0.9$ and $o_1 = 2$, the output feedback law (4.102) stabilizes the coupled system (4.1)-(4.4).

4.6 Summary

Stabilization of systems composed of coupled parabolic and elliptic equations presents considerable challenges. The first part of this chapter considers the boundary stabilization of a linear coupled parabolic-elliptic system. Previous work has shown that the coupling between the two equations can result in an unstable system. Stabilization via two boundary control inputs was used in [61, 121]. In this chapter, we use a single control input to stabilize both equations. One approach for designing the controller is to rewrite the coupled system into one equation in terms of the parabolic state. However, the appearance of a Fredholm operator makes it difficult to establish a suitable kernel for the backstepping transformation. Using separate transformations for each of the parabolic and elliptic states would require two control inputs. Therefore, in this chapter, we transform only the parabolic part of the system, which simplifies the calculations. This enables the reuse of a previously calculated transformation for simple parabolic equations. However, the drawback is that this transformation maps the original coupled system into an unusual target system. Lyapunov theory provides a sufficient condition for the stability of the obtained target system, which implies the stability of the original controlled systems. This stability condition imposes a strict requirement on the system's parameters. Therefore, rather than focusing solely on the stability of the target system, we establish a new stability criterion by directly analyzing the controlled coupled system.

The second part of this chapter focuses on the observer design problem. Several designs are presented depending on the available measurements. Output injections are chosen so that the exponential stability of the observation error dynamics is ensured. Again, instead of looking for a new state transformation that maps the original error dynamics into an exponentially stable target system, well-known transformations in the literature are employed. Then, the exponential stability of the original error dynamics is shown by establishing suitable sufficient conditions for the stability of the target system. The key to obtaining a stability condition is again to use Lyapunov theory. As with controller design, the technical conditions for observer design depend on the number of available measurements. When measurements for both states are provided, two transformations are applied to both parabolic and elliptic states of the error dynamics. A total of four filters, two throughout the domain and two at the boundary, are needed. On the other hand, when a single measurement for the parabolic state is given, one boundary filter is designed for the parabolic equation. However, in the latter case, a more restrictive condition for the stability of the error dynamics is obtained. Observer design with a single sensor parallels to a great extent that of stabilization via one control signal. Determining a sufficient stability condition by studying the original error dynamics, where the obtained output injections are incorporated, gives a more relaxed criterion.

In the final part of this chapter, the controller and observer designs are combined to obtain an output feedback controller. The results are again illustrated with simulations.

Chapter 5

Linear-quadratic control for a class of linear partial differential-algebraic equations

Designing controllers for linear partial differential-algebraic equations is crucial, as they play a vital role in enhancing system stability and/or achieving a desired performance. One valuable technique in controller design is linear-quadratic (LQ) control. In [32], the linear quadratic (LQ) control problem for partial differential equations (PDEs) was addressed. On a finite horizon, it was proven therein that a unique minimizing control exists, characterized by a time-dependent state feedback. Moreover, the process of determining this optimal control was linked to solving a differential Riccati equation. Nonetheless, there has been a notable gap in the literature regarding the examination of partial differential-algebraic equations (PDAEs) within this context, especially concerning the derivation of appropriate differential Riccati equations to solve for the optimal control.

For finite-dimensional DAEs, researchers have made progress towards establishing LQ controller design for DAEs, addressing nilpotency-index 1 systems [12, 29, 68, 80] as well as general higher-index systems [93, 102]. For instance, for time-invariant matrices G, R, S, Q, Bender and Laub [12] considered the cost functional

$$J(x_i, u; t_f) = \frac{1}{2}(Ex(t_f))^* G E x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \begin{bmatrix} x^*(t) & u^*(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \tag{5.1}$$

where the notation $\cdot^*$ denotes the transpose of the matrix, $R = R^* > 0$ and

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \geq 0,$$

subject to the dynamics of a nilpotency-index 1 DAE

$$\begin{aligned} Ex(t) &= Ax(t) + Bu(t), \\ Ex(t_0) &= Ex_i, \end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and E is a square matrix of rank $r \leq n$. Additionally, the pair (E, A) was assumed to be regular. Using calculus of variations, the authors studied this optimization problem on finite and infinite-horizons by transforming the system into a singular value decomposition (SVD) coordinate. To solve for the optimal control on a finite horizon, Bender and Laub [12] have derived several differential Riccati equations, all of which require the knowledge of the matrices in the (SVD) form. Using once again singular value decomposition, Mehrmann [80] studied the same optimization problem above, and considered the situation when the matrix R is singular. Mehrmann [80] showed that the existence of a unique continuous optimal control depends on the solvability of a two-point boundary value problem. In addition, this optimal control was assumed to ensure consistent initialization. Decomposing Q and S into

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \quad \begin{bmatrix} S_1 \\ S_2 \end{bmatrix},$$

respectively, Mehrmann [80] also derived a differential Riccati equation. However, among many assumptions, a strict one was made, namely, that $Q_{22} = -S_2 R^{-1} S_2^*$. This suggests that if $S = 0$ in the cost function (5.1), the penalty-weight on the algebraic sub-state, Q_{22}, disappears. On the other hand, Reis and Voigt [102] used a behavior-based approach to study optimal control for DAEs with arbitrary index. Petreczky and Zhuk [91] also used behaviors to study optimal control for linear DAEs that are not regular.

There are very few studies on optimal control for PDAEs. Grenkin et al. [44] tackled the boundary optimal control problem for a heat transfer model consisting of coupled transient and steady-state heat equations. They showed the existence of weak solutions to this optimization problem under certain assumptions but without taking into consideration the initial condition's consistency. More recently, Gernandt and Reis [43] studied PDAEs with resolvent-index one of the form

$$\begin{aligned} \frac{d}{dt}Ex(t) =& Ax(t) + Bu(t), \\ y(t) =& Cx(t), \\ Ex(0) =& Ex_i, \end{aligned}$$

where $E \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$, $A : D(A) \subset \mathcal{X} \to \mathcal{Z}$ is closed and densely defined on $\mathcal{X}$, and $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$. The initial condition $x(0) = x_i \in \mathcal{X}$. The authors studied this system in a pseudo-resolvent sense by considering the mild solutions of the system. Gernandt and Reis [43] considered the cost functional

$$J(u, y) = \int_0^{t_f} \langle \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}, \begin{bmatrix} Q(t) & S^*(t) \\ S(t) & R(t) \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \rangle_{\mathcal{Y}\times\mathcal{U}} dt, \tag{5.3}$$

where the time-varying weights $Q(\cdot) \in L_\infty([0, t_f]; \mathbb{R}^{n\times n})$, $R(\cdot) \in L_\infty([0, t_f]; \mathbb{R}^{n\times n})$ were assumed to be symmetric almost everywhere, and $S(\cdot) \in L_\infty([0, t_f]; \mathbb{R}^{n\times n})$. Defining a certain Popov operator with certain properties, Gernandt and Reis [43] showed the existence of a unique minimizing control in the space of square-integrable functions by assuming that the initial algebraic sub-state of the system adheres to a certain algebraic constraint. The optimal cost was shown to be determined by a bounded Riccati operator.

This chapter studies the LQ control problem over a finite-horizon for a class of linear PDAEs, those with *radiality-index* 0; see Section 3.2. Many equations arising in applications, such as parabolic-elliptic systems and also the equation used to model the free surface of seepage liquid, have *radiality-index* 0 [50]. With proofs different from those for finite-dimensional systems [80, 102], a fixed-point argument is used to show that there is a unique continuous optimal control and that this control can be written in feedback form. In many practical scenarios, the initial condition, such as initial temperature, velocity, or concentration, is typically known or measured. Therefore, we do not impose any assumption on the initial values such as been done in [43]. Instead, by defining a set of admissible control signals for the optimization problem, we demand that the control signal maintain the consistency of the initial conditions, thereby preventing distributions in the solutions [40]. Considering that the algebraic components of the PDAE often have physical interpretations, maintaining consistent control inputs is crucial to avoid non-physical behaviors. Inconsistencies in control can lead to numerical solutions that diverge from real-world physics, thereby reducing the model's accuracy.

Next, we derive a coupled system consisting of a differential Riccati-like equation and an algebraic equation that leads to the optimal control. For this purpose, our approach draws inspiration from the works of Heinkenschloss [48] and Stykel [110], where they developed a Lyapunov equation within the study of balanced truncation model reduction for specific finite-dimensional systems. Similarly, Duan [34] derived a Lyapunov equation for a particular class of DAEs with differentiation-index 1. Breiten et al. [16] studied optimal control of the linearized Navier-Stokes equations and derived a certain Riccati equation. Our work marks a first effort towards deriving differential Riccati-like equations for a general class of differential-algebraic equations. It is important to note that although certain projections are involved in the proof process, once the state weight's decomposition is established, there is no need to use these projections in computing the optimal control.

Additionally, the derived Riccati-like equation allows for scenarios when there is penalization on the algebraic state, even though the associated cost function does not include cross terms. This weakens the assumptions stated previously in [80].

This chapter is divided into three main sections: In Section 5.1, we formulate the problem and define the class of PDAEs under study. Section 5.2 establishes the existence of a unique optimal control for the finite-horizon optimization problem. The derivation of a differential Riccati equation, essential for determining this optimal control, is elaborated in Section 5.3. Finally, in Section 5.4, numerical simulations are given to illustrate the theoretical results, by designing of an LQ-optimal control for an unstable coupled parabolic-elliptic system.

5.1 Problem statement

Let $\mathcal{X}$, $\mathcal{Z}$ and $\mathcal{U}$ be Hilbert spaces. Consider a system modeled by

$$\frac{d}{dt}Ex(t) = Ax(t) + Bu(t), \tag{5.4a}$$

$$Ex(0) = Ex_i, \tag{5.4b}$$

where $E \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$, $A : D(A) \subset \mathcal{X} \to \mathcal{Z}$ is closed and densely defined on $\mathcal{X}$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$. The initial condition $x(0) = x_i \in \mathcal{X}$. Systems of the form (5.4) reduce to classical infinite-dimensional systems [32] when E is invertible. The situation where E is non-invertible, and/or unbounded on $\mathcal{Z}$ is of particular interest in this chapter. It will be assumed throughout this chapter that the PDAE (5.4) is *radial with degree 0*, corresponding to having nilpotency-index 1. Therefore, the PDAE (5.4) can be written in *Weierstrass-Kronecker* form (see Section 3.2). In particular, we have a certain space decomposition

$$\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_0, \quad \mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_0. \tag{5.5}$$

We can also define certain bounded projections $P^{\mathcal{X}_1}$ and $P^{\mathcal{Z}_1}$ with ranges $\mathcal{X}_1$ and $\mathcal{Z}_1$, such that the following statements hold:

$$\text{For all } x \in \mathcal{X},\ P^{\mathcal{Z}_1}Ex = EP^{\mathcal{X}_1}x, \tag{5.6a}$$

$$\text{For all } x \in D(A),\ P^{\mathcal{X}_1}x \in D(A) \text{ and } P^{\mathcal{Z}_1}Ax = AP^{\mathcal{X}_1}x. \tag{5.6b}$$

We define the following operators

$$E_0 = E|_{\mathcal{X}_0}, \qquad E_1 = E|_{\mathcal{X}_1}, \qquad A_0 = A|_{\mathcal{X}_0 \cap D(A)}, \qquad A_1 = A|_{\mathcal{X}_1 \cap D(A)}, \tag{5.7a}$$

$$B_0 = P^{\mathcal{Z}_0}B, \qquad B_1 = P^{\mathcal{Z}_1}B. \tag{5.7b}$$

Since the PDAE (5.4) is *radial with degree 0*, it follows that operator $A_0 : D(A_0) \to \mathcal{Z}_0$ has a bounded inverse: $A_0^{-1} \in \mathcal{L}(\mathcal{Z}_0, \mathcal{X}_0)$, and the operator E_0 is the zero operator. Also, $E_1 \in \mathcal{L}(\mathcal{X}_1, \mathcal{Z}_1)$ maps into $\mathcal{Z}_1$, and has a bounded inverse: $E_1^{-1} \in \mathcal{L}(\mathcal{Z}_1, \mathcal{X}_1)$. The operator $A_1 : D(A) \cap \mathcal{X}_1 \to \mathcal{Z}_1$ is closed and densely defined, and $E_1^{-1}A_1$ generates a C_0-semigroup operator $T(t)$ on $\mathcal{X}_1$. We refer the reader to Section 3.2 for more detail.

Letting $I^{\mathcal{X}}$ and $I^{\mathcal{Z}}$ indicate the identity operator on the space $\mathcal{X}$ and $\mathcal{Z}$, respectively, we also define

$$P^{\mathcal{X}_0} = I^{\mathcal{X}} - P^{\mathcal{X}_1}, \quad P^{\mathcal{Z}_0} = I^{\mathcal{Z}} - P^{\mathcal{Z}_1},$$

and

$$\tilde{P}^{\mathcal{X}} = \begin{bmatrix} P^{\mathcal{X}_1} \\ P^{\mathcal{X}_0} \end{bmatrix} \in \mathcal{L}(\mathcal{X}, \mathcal{X}_1 \times \mathcal{X}_0), \quad \tilde{P}^{\mathcal{Z}} = \begin{bmatrix} P^{\mathcal{Z}_1} \\ P^{\mathcal{Z}_0} \end{bmatrix} \in \mathcal{L}(\mathcal{Z}, \mathcal{Z}_1 \times \mathcal{Z}_0), \tag{5.8}$$

$$(\tilde{P}^{\mathcal{X}})^{-1} = \begin{bmatrix} I^{\mathcal{X}_1} & I^{\mathcal{X}_0} \end{bmatrix} \in \mathcal{L}(\mathcal{X}_1 \times \mathcal{X}_0, \mathcal{X}), \tag{5.9}$$

$$(\tilde{P}^{\mathcal{Z}})^{-1} = \begin{bmatrix} I^{\mathcal{Z}_1} & I^{\mathcal{Z}_0} \end{bmatrix} \in \mathcal{L}(\mathcal{Z}_1 \times \mathcal{Z}_0, \mathcal{Z}). \tag{5.10}$$

For notational convenience, we set

$$\tilde{A}_1 = E_1^{-1}A_1, \quad \tilde{B}_1 = E_1^{-1}B_1, \quad \tilde{B}_0 = A_0^{-1}B_0, \tag{5.11}$$

and

$$P^{\mathcal{X}_1}x_i = (x_i)_1, \quad P^{\mathcal{X}_0}x_i = (x_i)_0.$$

Pre-multiplying system (5.4) with the operator $\tilde{P}^{\mathcal{Z}}$ and using (5.6a)-(5.6b), we obtain

$$\frac{d}{dt}x_1(t) = \tilde{A}_1 x_1(t) + \tilde{B}_1 u(t), \quad x_1(0) = (x_i)_1, \tag{5.12a}$$

$$0 = x_0(t) + \tilde{B}_0 u(t), \quad x_0(0) = (x_i)_0, \tag{5.12b}$$

For continuous control $u(t) \in C([0, t_f]; \mathcal{U})$, the mild solution of (5.12a) is

$$x_1(t) = T(t)(x_i)_1 + \int_0^t T(t-s)\tilde{B}_1 u(s) ds. \tag{5.13}$$

For system (5.12b), the solution is

$$x_0(t) = -\tilde{B}_0 u(t) + \delta(t)\Big((x_i)_0 + \tilde{B}_0 u(0)\Big), \quad t \geq 0, \tag{5.14}$$

where $\delta(t)$ is the Dirac delta distribution. If $(x_i)_0 = -\tilde{B}_0 u(0)$, then the distributional component of the solution (5.14) is eliminated. This equation is known as the consistency condition on the initial condition in the DAE literature [62, Theorem 2.12]. With consistent initial conditions, we have

$$x_0(t) = -\tilde{B}_0 u(t), \quad t \geq 0, \tag{5.15}$$

and so the solution of the PDAE (5.4) is

$$x(t) = -\tilde{B}_0 u(t) + T(t)(x_i)_1 + \int_0^t T(t-s)\tilde{B}_1 u(s) ds. \tag{5.16}$$

The optimal control problem is to minimize, for arbitrary initial condition x_i, the quadratic performance criterion

$$J(x_i, u; t_f) = \langle x(t_f), Gx(t_f)\rangle_{\mathcal{X}} + \int_0^{t_f} \langle x(s), Qx(s)\rangle_{\mathcal{X}} + \langle u(s), Ru(s)\rangle_{\mathcal{U}} ds. \tag{5.17}$$

As usual, R is assumed to be coercive; that is, R is self-adjoint and $R \geq \epsilon I$ for some $\epsilon > 0$. Also, G and Q are self-adjoint non-negative operators. The notation $\langle \cdot, \cdot \rangle_{\mathcal{U}}$ stands for the inner product on some Hilbert space $\mathcal{U}$.

To avoid distributions in the solution, the set of admissible controls for minimization of the cost $J(x_i, u; t_f)$ is

$$\mathcal{U}_a = \{u(t) \in C([0, t_f]; \mathcal{U}) : \ (x_i)_0 = -\tilde{B}_0 u(0)\}. \tag{5.18}$$

This leads to a formal definition of the optimal control problem as

$$\inf_{u \in \mathcal{U}_a} J(x_i, u; t_f), \tag{5.19}$$

subject to $x(t)$ that solves (5.4).

The radiality of system (5.4) also implies the existence of projections $P^{\mathcal{Z}_1 *}$ and $P^{\mathcal{Z}_0 *}$,

$$P^{\mathcal{Z}_1 *} : \mathcal{Z} \to \mathcal{Z}, \qquad P^{\mathcal{X}_1 *} : \mathcal{X} \to \mathcal{X},$$

which are the adjoint operators of $P^{\mathcal{Z}_1}$ and $P^{\mathcal{Z}_0}$, respectively, such that

$$\text{For all } z \in \mathcal{Z}, \ P^{\mathcal{X}_1 *} E^* z = E P^{\mathcal{Z}_1 *} z, \tag{5.20a}$$

$$\text{For all } z \in D(A^*), \ P^{\mathcal{Z}_1 *} z \in D(A^*) \text{ and } P^{\mathcal{X}_1^*} A^* z = A^* P^{\mathcal{Z}_1 *} z. \tag{5.20b}$$

We also set

$$P^{\mathcal{X}_0*} = I^{\mathcal{X}} - P^{\mathcal{X}_1*}, \qquad P^{\mathcal{Z}_0*} = I^{\mathcal{Z}} - P^{\mathcal{Z}_1*}.$$

Throughout this chapter, it is assumed that the final cost vanishes on the subspace $\mathcal{X}_0$,

$$P^{\mathcal{X}_1*}GP^{\mathcal{X}_0} = P^{\mathcal{X}_0*}GP^{\mathcal{X}_1} = P^{\mathcal{X}_0*}GP^{\mathcal{X}_0} = 0. \tag{5.21}$$

Since in many applications $G = 0$, this assumption is, in practice, not difficult to meet. It is also assumed that

$$P^{\mathcal{X}_0*}QP^{\mathcal{X}_1} = P^{\mathcal{X}_1*}QP^{\mathcal{X}_0} = 0. \tag{5.22}$$

We also define the following restrictions on the subspaces $\mathcal{X}_1$ and $\mathcal{X}_0$,

$$G_1 = G|_{\mathcal{X}_1}, \quad Q_1 = Q|_{\mathcal{X}_1}, \quad Q_0 = Q|_{\mathcal{X}_0}.$$

Note that the operator G_1 maps to $\mathcal{X}_1$ due to assumption (5.21) and so $G_1 \in \mathcal{L}(\mathcal{X}_1)$. Similarly, (5.22) implies that $Q_1 \in \mathcal{L}(\mathcal{X}_1)$ and $Q_0 \in \mathcal{L}(\mathcal{X}_0)$.

5.2 Existence of the optimal control

Given the control problem defined in the previous section, particularly the definition of the set of admissible controls in (5.18), we consider the set of admissible variations $\mathcal{V}_a$. This set includes all variations v such that for any control $u \in \mathcal{U}_a$, the sum $u + v$ also belongs to $\mathcal{U}_a$,

$$\mathcal{V}_a = \{h(t) \in C([0, t_f]; \mathcal{U}) : \ 0 = -\tilde{B}_0 h(0)\}. \tag{5.23}$$

Let the spaces $\mathcal{U}$, $\mathcal{X}$ be real Hilbert space. The next proposition shows that an arbitrary variation in the control leads to a change in cost that is a sum of quadratic and linear terms.

Proposition 5.2.1. *Consider any $u(t) \in \mathcal{U}_a$, $x_i \in \mathcal{X}$ and $h(t) \in \mathcal{V}_a$. Define, letting $T^*(t)$ indicate the C_0-semigroup on $\mathcal{Z}_1$ generated by $\tilde{A}_1^*$,*

$$\begin{aligned} z_u(t) = & 2\Big[\tilde{B}_1^* T^*(t_f - t) G_1 x_1(t_f) + \tilde{B}_1^* \int_t^{t_f} T^*(r - t) Q_1 x_1(r) dr \\ & - \tilde{B}_0^* Q_0 x_0(t) + Ru(t)\Big], \end{aligned} \tag{5.24}$$

$$\phi(t)h(t) = \int_0^t T(t-s)\tilde{B}_1 h(s)ds - \tilde{B}_0 h(t). \tag{5.25}$$

Then,

$$\begin{aligned} J(x_i, u+h; t_f) - J(x_i, u; t_f) = &\int_0^{t_f} \langle z_u(s), h(s)\rangle_{\mathcal{U}} ds \\ &+ \langle \phi(t_f)h(t_f), G\phi(t_f)h(t_f)\rangle_{\mathcal{X}} + \int_0^{t_f} \langle \phi(s)h(s), Q\phi(s)h(s)\rangle_{\mathcal{X}} \\ &+ \langle h(s), Rh(s)\rangle_{\mathcal{U}} ds. \end{aligned} \tag{5.26}$$

Proof. The proof is similar to the classical argument in $LQ-$optimal control for ODEs, e.g., [84]. Let $u(t) \in \mathcal{U}_a$ and $h(t) \in \mathcal{V}_a$. Since $(u+h)(t) \in C([0, t_f]; \mathcal{U})$, and $-\tilde{B}_0(u+h)(0) = (x_i)_0$, it follows that $(u+h)(t) \in \mathcal{U}_a$. Let $x(t)$ indicate the trajectory corresponding to $u(t)$, then statement (5.16) implies that

$$x(t) = -\tilde{B}_0 u(t) + T(t)(x_i)_1 + \int_0^t T(t-s)\tilde{B}_1 u(s)ds. \tag{5.27}$$

With the definition of $\phi(t)h(t)$ in (5.25), the trajectory corresponding to $(u+h)(t)$ is

$$x_{u+h}(t) = x(t) + \phi(t)h(t). \tag{5.28}$$

Thus,

$$\begin{aligned} J(x_i, u+h; t_f) - J(x_i, u; t_f) = &\, 2\langle x(t_f), G\phi(t_f)h(t_f)\rangle_{\mathcal{X}} + 2\Big(\int_0^{t_f} \langle u(s), Rh(s)\rangle_{\mathcal{U}} \\ &+ \langle x(s), Q\phi(s)h(s)\rangle_{\mathcal{X}} ds\Big) + \langle \phi(t_f)h(t_f), G\phi(t_f)h(t_f)\rangle_{\mathcal{X}} \\ &+ \int_0^{t_f} \langle \phi(s)h(s), Q\phi(s)h(s)\rangle_{\mathcal{X}} + \langle h(s), Rh(s)\rangle_{\mathcal{U}} ds. \end{aligned} \tag{5.29}$$

Setting

$$\begin{aligned} \nabla J(u, h, t_f) = &\, 2\langle x(t_f), G\phi(t_f)h(t_f)\rangle_{\mathcal{X}} + 2\Big(\int_0^{t_f} \langle x(s), Q\phi(s)h(s)\rangle_{\mathcal{X}} \\ &+ \langle u(s), Rh(s)\rangle_{\mathcal{U}} ds\Big), \end{aligned} \tag{5.30}$$

we rewrite (5.29) as

$$J(x_i, u+h; t_f) - J(x_i, u; t_f) = \nabla J(u, h, t_f) + \langle \phi(t_f)h(t_f), G\phi(t_f)h(t_f)\rangle_{\mathcal{X}} \\ + \int_0^{t_f} \langle \phi(s)h(s), Q\phi(s)h(s)\rangle_{\mathcal{X}} + \langle h(s), Rh(s)\rangle_{\mathcal{U}} ds. \quad (5.31)$$

In the following steps, we compute a $z_u \in L_2([0,t_f];\mathcal{U})$ in a way that

$$\nabla J(u, h, t_f) = \int_0^{t_f} \langle z_u(s), h(s)\rangle_{\mathcal{U}} ds. \quad (5.32)$$

Using the definition of $\phi(t)h(t)$ and assumptions (5.21) and (5.22) on the operators G and Q, respectively, we write

$$\nabla J(u, h, t_f) = 2\Big[\langle x_1(t_f), G_1 \int_0^{t_f} T(t_f - s)\tilde{B}_1 h(s) ds\rangle_{\mathcal{X}} + \int_0^{t_f} \langle u(s), Rh(s)\rangle_{\mathcal{U}} \\ + \langle x_1(s), Q_1 \int_0^{s} T(s-r)\tilde{B}_1 h(r) dr\rangle_{\mathcal{X}} - \langle x_0(s), Q_0 \tilde{B}_0 h(s)\rangle_{\mathcal{X}} ds\Big]. \quad (5.33)$$

We simplify the right-hand-side of (5.33). Direct calculations lead to

$$\langle x_1(t_f), G_1 \int_0^{t_f} T(t_f - s)\tilde{B}_1 h(s) ds\rangle_{\mathcal{X}} + \int_0^{t_f} \langle x_1(s), Q_1 \int_0^{s} T(s-r)\tilde{B}_1 h(r) dr\rangle_{\mathcal{X}} ds \\ = \int_0^{t_f} \langle \tilde{B}_1^* T^*(t_f - s) G_1 x_1(t_f) + \tilde{B}_1^* \int_s^{t_f} T^*(r-s) Q_1 x_1(r) dr, h(s)\rangle_{\mathcal{U}} ds, \quad (5.34)$$

and

$$-2\int_0^{t_f} \langle x_0(s), Q_0 \tilde{B}_0 h(s)\rangle_{\mathcal{X}} ds = -2\int_0^{t_f} \langle \tilde{B}_0^* Q_0 x_0(s), h(s)\rangle_{\mathcal{U}} ds. \quad (5.35)$$

Substituting (5.34) and (5.35) in (5.33), we obtain

$$\nabla J(u, h, t_f) = \int_0^{t_f} \Big[\langle 2\tilde{B}_1^* T^*(t_f - s) G_1 x_1(t_f) \\ + 2\tilde{B}_1^* \int_s^{t_f} T^*(r-s) Q_1 x_1(r) dr - 2\tilde{B}_0^* Q_0 x_0(s) + 2Ru(s), h(s)\rangle_{\mathcal{U}}\Big] ds. \quad (5.36)$$

Defining $z_u(t)$ as given in (5.24), the conclusion follows by combining (5.31) and (5.36). □

The next theorem shows that the optimal control problem (5.19) has a solution.

Theorem 5.2.1. *The control input*

$$u^{opt}(t) = - R^{-1}\Big[\tilde{B}_1^* T^*(t_f - t) G_1 x_1^{opt}(t_f) + \tilde{B}_1^* \int_t^{t_f} T^*(r-t) Q_1 x_1^{opt}(r) dr - \tilde{B}_0^* Q_0 x_0^{opt}(t)\Big], \tag{5.37}$$

is the unique solution of the optimization problem (5.19). Here $x^{opt}(t)$ is the system state corresponding to control $u^{opt}(t)$, and $x^{opt}(t) = x_1^{opt}(t) + x_0^{opt}(t)$.

Proof. The proof takes two main steps. First, we show that $u^{opt}(t)$ is the unique solution in $C([0,t_f];\mathcal{U})$ solving equation

$$z_u(t) = 0. \tag{5.38}$$

We rewrite (5.38) in terms of a fixed-point operator. Define the operator

$$(V_1 u)(t) = T(t)(x_i)_1 + \int_0^t T(t-s)\tilde{B}_1 u(s) ds. \tag{5.39}$$

It is straightforward to show that $V_1 u(t) : C([0,t_f];\mathcal{U}) \to C([0,t_f];\mathcal{U})$, by using the fact that $T(t)$ is a C_0-semigroup, and following a similar argument as the one given in [32, Lemma 5.1.5]. Next, referring to (5.24), we rewrite equation (5.38) by using (5.13) and (5.15)

$$Ru(t) = -\tilde{B}_1^* T^*(t_f - t) G_1 x_1(t_f) - \tilde{B}_1^* \int_t^{t_f} T^*(r-t) Q_1 \Big(T(r)(x_i)_1 + \int_0^r T(r-s)\tilde{B}_1 u(s) ds\Big) dr - \tilde{B}_0^* Q_0 A_0^{-1} B_0 u(t).$$

The previous equation can now be written with the help of operator V_1 as

$$Ru(t) = -\tilde{B}_1^* T^*(t_f - t) G_1 x_1(t_f) - \tilde{B}_1^* \int_t^{t_f} T^*(r-t) Q_1 (V_1 u)(r) dr - \tilde{B}_0^* Q_0 A_0^{-1} B_0 u(t),$$

that is

$$(R + \tilde{B}_0^* Q_0 \tilde{B}_0) u(t) = -\tilde{B}_1^* T^*(t_f - t) G_1 x_1(t_f) \\ - \tilde{B}_1^* \int_t^{t_f} T^*(r-t) Q_1 (V_1 u)(r) dr. \tag{5.40}$$

Since the operator $\tilde{B}_0^* Q_0 \tilde{B}_0$ is positive semi-definite and R is coercive, it follows that the operator $R + \tilde{B}_0^* Q_0 \tilde{B}_0$ is also coercive. Consequently, $R + \tilde{B}_0^* Q_0 \tilde{B}_0$ has a bounded inverse. For convenience of notation, we set

$$\tilde{R} = R + \tilde{B}_0^* Q_0 \tilde{B}_0. \tag{5.41}$$

For $u(t) \in C([0, t_f]; \mathcal{U})$, define $F : C([0, t_f]; \mathcal{U}) \to C([0, t_f]; \mathcal{U})$ as

$$(Fu)(t) = -(\tilde{R})^{-1} \tilde{B}_1^* T^*(t_f - t) G_1 x_1(t_f) \\ - (\tilde{R})^{-1} \tilde{B}_1^* \int_t^{t_f} T^*(r-t) Q_1 (V_1 u)(r) dr. \tag{5.42}$$

With this definition, equation (5.40) can be written as

$$u(t) = (Fu)(t). \tag{5.43}$$

It will now be shown that F^n is a contraction for large enough n. For $u_1,\ u_2 \in C([0, t_f]; \mathcal{U})$,

$$\begin{aligned} &|(Fu_1)(t) - (Fu_2)(t)| \\ &= |(\tilde{R})^{-1} \tilde{B}_1^* \int_t^{t_f} T^*(r-t) Q_1 \Big[(V_1 u_1)(r) - (V_1 u_2)(r) \Big] dr| \\ &\le \|(\tilde{R})^{-1} \tilde{B}_1^*\| \int_t^{t_f} \|T^*(r-t)\| \|Q_1\| |(V_1 u_1)(r) - (V_1 u_2)(r)| dr \\ &\le \|(\tilde{R})^{-1} \tilde{B}_1^*\| \int_t^{t_f} \|T^*(r-t)\| \|Q_1\| \int_0^r \|T(r-s)\| \|\tilde{B}_1\| |u_1(s) - u_2(s)| ds dr. \end{aligned}$$

Note that $T(t)$ is bounded on every finite subinterval of $[0, \infty)$; see [32, Theorem 2.1.7 a], and so is $T^*(t)$. The operators $(\tilde{R})^{-1} \tilde{B}_1^* = (R + \tilde{B}_0^* Q_0 \tilde{B}_0)^{-1} \tilde{B}_1^* \in \mathcal{L}(\mathcal{X}_1, \mathcal{U})$. Also, $Q_1,\ \tilde{B}_1$ are bounded linear operators. Hence, there is a constant $M > 0$ such that

$$|(Fu_1)(t) - (Fu_2)(t)| \le M \int_t^{t_f} \int_0^r |u_1(s) - u_2(s)| ds dr. \tag{5.44}$$

Thus,

$$
\begin{aligned}
|(Fu_1)(t) - (Fu_2)(t)| &\le M\|u_1 - u_2\|_\infty \int_t^{t_f} \int_0^r dsdr \\
&\le t_f(t_f - t)M\|u_1 - u_2\|_\infty. \qquad (5.45)
\end{aligned}
$$

Replacing u_1 and u_2 in the inequalities above with Fu_1 and Fu_2, respectively, we obtain

$$
\begin{aligned}
|(F^2u_1)(t) - (F^2u_2)(t)| &\le M \int_t^{t_f} \int_0^r |(Fu_1)(s) - (Fu_2)(s)|dsdr \\
&\le M^2 t_f \|u_1 - u_2\|_\infty \int_t^{t_f} \int_0^r (t_f - s)dsdr \\
&= M^2 t_f \|u_1 - u_2\|_\infty \int_t^{t_f} \frac{(t_f - r)^2}{2} dr \\
&\le M^2 t_f^2 \|u_1 - u_2\|_\infty \frac{(t_f - t)^2}{2}. \qquad (5.46)
\end{aligned}
$$

It will now be shown by induction that

$$
|(F^n u_1)(t) - (F^n u_2)(t)| \le M^n t_f^n \frac{(t_f - t)^n}{n!} \|u_1 - u_2\|_\infty. \qquad (5.47)
$$

First, it is clear from inequality (5.45) that statement (5.47) holds for $n = 1$. Now, assuming that (5.47) holds for some n, we use (5.44) to obtain

$$
\begin{aligned}
|(F^{n+1}u_1)(t) - (F^{n+1}u_2)(t)| &\le M \int_t^{t_f} \int_0^r |F^n u_1(s) - F^n u_2(s)|dsdr \\
&\le M^{n+1} t_f^n \|u_1 - u_2\|_\infty \int_t^{t_f} \int_0^r \frac{(t_f - s)^n}{n!} dsdr \\
&\le M^{n+1} t_f^n \|u_1 - u_2\|_\infty \int_t^{t_f} \frac{(t_f - r)^{n+1}}{(n+1)!} dr \\
&\le M^{n+1} t_f^{n+1} \frac{(t_f - t)^{n+1}}{(n+1)!} \|u_1 - u_2\|_\infty, \qquad (5.48)
\end{aligned}
$$

and so statement (5.47) also holds for $n + 1$, which completes the proof by induction. Defining $a = (Mt_f^2)$, it follows that

$$
\|(F^n u_1)(t) - (F^n u_2)(t)\|_\infty \le \frac{a^n}{n!} \|u_1 - u_2\|_\infty. \qquad (5.49)
$$

For sufficiently large n, we have that $\frac{a^n}{n!} < 1$ and hence F^n is a contraction for sufficiently large n. It follows from [58, Lemma 5.4-3] that F has a unique fixed point. Hence, there exists a unique control in $C([0,t_f];\mathcal{U})$ that solves equation (5.43). It is also the unique solution of equation (5.38).

The control $u^{opt}(t) \in C([0,t_f];\mathcal{U})$, and is the unique control leading to $z_u = 0$. Since equation (5.26) was derived by allowing for the variation $h(t) \in \mathcal{V}_a$, ensuring that $(u+h)(t)$ satisfies the consistency condition, it follows that $u^{opt}(t)$ ensures the consistency of the initial condition, that is, $u^{opt}(t) \in \mathcal{U}_a$.

Referring to (5.26), since $R > 0$, $Q \geq 0$, $G \geq 0$, for any admissible variation h,

$$J(x_i, u^{opt} + h; t_f) - J(x_i, u^{opt}; t_f) > 0, \tag{5.50}$$

and so $u^{opt}(t)$ is the unique control minimizing the cost (5.17). □

5.3 Derivation of differential Riccati equations

Consider the cost functional (5.17) with a variable initial time t_0, $0 \leq t_0 \leq t_f$,

$$\begin{aligned} J(x_i, u; t_0, t_f) =& \langle x(t_f), Gx(t_f)\rangle_{\mathcal{X}} + \int_{t_0}^{t_f} \langle x(s), Qx(s)\rangle_{\mathcal{X}} \\ &+ \langle u(s), Ru(s)\rangle_{\mathcal{U}} ds. \end{aligned} \tag{5.51}$$

Since the governing equations are time-invariant, assuming $u(t_0)$ is consistent with $x(t_0)$, the previous section's results apply. Thus, there is unique input that minimizes the cost functional (5.51) for trajectories of (5.4) with initial condition $x(t_0) = x_i$. The control that minimizes cost functional (5.51) is denoted $u^{opt}(\cdot; x_i, t_0, t_f)$, and its corresponding optimal state trajectory is $x^{opt}(\cdot; x_i, t_0, t_f)$. The control that minimizes (5.17) shall be denoted by $u^{opt}(t)$ or $u^{opt}(\cdot; x_i, 0, t_f)$, and its corresponding state trajectory is $x^{opt}(t)$ or $x^{opt}(\cdot; x_i, 0, t_f)$.

The following result holds due to the uniqueness of the optimal control. It is an extension of the principle of optimality from linear PDEs to linear PDAEs, and for each of the sub-states $x_1(t)$ and $x_0(t)$.

Lemma 5.3.1. *Let $0 \leq t_0 \leq t_f$. For all $s \in [t_0, t_f]$,*

$$x^{opt}(s; x_i, 0, t_f) = x^{opt}(s; x^{opt}(t_0, x_i, 0, t_f), t_0, t_f). \tag{5.52}$$

In addition, each of the dynamical and the algebraic sub-states satisfy the principle of optimality, that is,

$$x_0^{opt}(s;(x_i)_0,0,t_f) = x_0^{opt}(s;x_0^{opt}(t_0,(x_i)_0,0,t_f),t_0,t_f),$$

and

$$x_1^{opt}(s;(x_i)_1,0,t_f) = x_1^{opt}(s;x_1^{opt}(t_0,(x_i)_1,0,t_f),t_0,t_f). \tag{5.53}$$

Proof. Since the optimal control $u^{opt}(t)$ is unique, we can use a similar line of reasoning as the one presented in [32, Lemma 9.1.7] to establish equation (5.52) for all $s \in [t_0,t_f]$. Referring to equation (5.12b) and using once again the uniqueness of the optimal control, it follows that

$$\begin{aligned} x_0^{opt}(s;(x_i)_0,0,t_f) &= \tilde{B}_0 u^{opt}(s;x_i,0,t_f) \\ &= \tilde{B}_0 u^{opt}(s;x^{opt}(t_0,x_i,0,t_f),t_0,t_f) \\ &= x_0^{opt}(s;x_0^{opt}(t_0,(x_i)_0,0,t_f),t_0,t_f), \end{aligned} \tag{5.54}$$

for any $s \in [t_0,t_f]$. Thus, the algebraic sub-state of the PDAE conforms to the principle of optimality. The trajectory of system (5.4) on $[0,t_f]$ with initial condition x_i is

$$x^{opt}(s;x_i,0,t_f) = x_1^{opt}(s;(x_i)_1,0,t_f) + x_0^{opt}(s;(x_i)_0,0,t_f), \tag{5.55}$$

and the trajectory on $[t_0,t_f]$ with initial condition $x^{opt}(t_0,x_i,0,t_f)$ is

$$\begin{aligned} x^{opt}(s;x^{opt}(t_0,x_i,0,t_f),t_0,t_f) = {} & x_1^{opt}(s;x_1^{opt}(t_0,(x_i)_1,0,t_f),t_0,t_f) \\ & + x_0^{opt}(s;x_0^{opt}(t_0,(x_i)_0,0,t_f),t_0,t_f). \end{aligned} \tag{5.56}$$

Using (5.52), we deduce that the right-hand-side of equations (5.55) and (5.56) are equal. It then follows from (5.54) that the dynamical sub-state of the PDAE, i.e. $x_1^{opt}(t)$, also satisfies the principle of optimality (5.53). □

The next proposition demonstrates that at any given time $t \in [0,t_f]$, the optimal control (5.37) can be written as a feedback of the dynamical state $x_1^{opt}(t)$.

Proposition 5.3.1. *The optimization problem* (5.19) *reduces to minimizing the cost functional*

$$\begin{aligned} J((x_i)_1,u;t_f) = & \langle x_1(t_f), G_1 x_1(t_f)\rangle_{\mathcal{X}} + \int_0^{t_f} \langle x_1(s), Q_1 x_1(s)\rangle_{\mathcal{X}} \\ & + \langle u(s), \tilde{R} u(s)\rangle_{\mathcal{U}} ds, \end{aligned} \tag{5.57}$$

over the set of admissible control $\mathcal{U}_a$, where $x_1(t)$ solves system (5.12a).

The minimizing optimal control $u^{opt}(t)$ can be rewritten as

$$u^{opt}(t) = -\tilde{R}^{-1}\tilde{B}_1^*\big[T^*(t_f - t)G_1 x_1^{opt}(t_f) + \int_t^{t_f} T^*(r-t)Q_1 x_1^{opt}(r)dr\big]. \tag{5.58}$$

Proof. Rewrite the cost functional (5.17) as

$$\begin{aligned} J(x_i, u; t_f) =& \langle x_1(t_f), G_1 x_1(t_f)\rangle_{\mathcal{X}} + \int_0^{t_f} \langle x_1(s), Q_1 x_1(s)\rangle_{\mathcal{X}} + \langle x_0(s), Q_0 x_0(s)\rangle_{\mathcal{X}} \\ &+ \langle u(s), Ru(s)\rangle_{\mathcal{U}} ds. \end{aligned} \tag{5.59}$$

Since the optimization is over $\mathcal{U}_a$, we use that $x_0^{opt}(t) = -\tilde{B}_0 u^{opt}(t)$ to obtain the cost (5.57). Note that the existence of unique optimizing control for cost functional (5.57) over the set $\mathcal{U}_a$ follows from the results in Section 5.2 concerning the equivalent optimization problem (5.37). To prove statement (5.58), we first recall that $\tilde{R} = R + \tilde{B}_0^* Q_0 \tilde{B}_0$. From (5.37), it follows that

$$\begin{aligned} u^{opt}(t) =& -R^{-1}\Big[\tilde{B}_1^* T^*(t_f - t)G_1 x_1^{opt}(t_f) + \tilde{B}_1^* \int_t^{t_f} T^*(r-t)Q_1 x_1^{opt}(r)dr \\ &+ \tilde{B}_0^* Q_0 \tilde{B}_0 u^{opt}(t)\Big]. \end{aligned}$$

Now, the previous equation can be rewritten as

$$(R + \tilde{B}_0^* Q_0 \tilde{B}_0)u^{opt}(t) = -\tilde{B}_1^*\Big[T^*(t_f - t)G_1 x_1^{opt}(t_f) + \int_t^{t_f} T^*(r-t)Q_1 x_1^{opt}(r)dr\Big].$$

Since the operator $\tilde{R} = R + \tilde{B}_0^* Q_0 \tilde{B}_0$ is coercive, we arrive at equation (5.58). □

The optimization problem (5.19) reduces to a standard LQ problem on $\mathcal{X}_1$. This simplification enables the application of well-known results for finite-time LQ control of PDEs; see [32, Chapter 9].

Lemma 5.3.2. *[32, Lemma 9.1.9] For any $t \in [0, t_f]$ and any $x_1 \in \mathcal{X}_1$, define the operator $\Pi_1(t)$ on $\mathcal{X}_1$*

$$\Pi_1(t)x_1 = T^*(t_f - t)G_1 x_1^{opt}(t_f; x_1, t, t_f) + \int_t^{t_f} T^*(r-t)Q_1 x_1^{opt}(r; x_1, t, t_f)dr; \tag{5.60}$$

$\Pi_1(t) \in \mathcal{L}(\mathcal{X}_1)$ *for all* $t \in [0, t_f]$. *The optimal control* (5.58) *can be written as*

$$u^{opt}(t; x_i, 0, t_f) = -\tilde{R}^{-1}\tilde{B}_1^*\Pi_1(t)x_1^{opt}(t; (x_i)_1, 0, t_f), \tag{5.61}$$

and the minimum cost is

$$J(x_i, u^{opt}; t_f) = \langle (x_i)_1, \Pi_1(0)(x_i)_1\rangle_{\mathcal{X}}. \tag{5.62}$$

The optimal dynamical state $x_1^{opt}(t)$ is the mild solution to an abstract evolution equation. This is stated in the corollary below.

Corollary 5.3.1. *The optimal sub-state* $x_1^{opt}(t)$ *is the mild solution of the abstract evolution equation*

$$\begin{aligned} \frac{d}{dt}x_1^{opt}(t) &= (E_1^{-1}A_1 - \tilde{B}_1\tilde{R}^{-1}\tilde{B}_1^*\Pi_1(t))x_1^{opt}(t), \\ x_1^{opt}(0) &= (x_i)_1. \end{aligned} \tag{5.63}$$

Also, the operator $E_1^{-1}A_1 - \tilde{B}_1\tilde{R}^{-1}\tilde{B}_1^*\Pi_1(t)$ *generates the mild evolution operator* $U(t,s)$ *on the set* $\{(t,s); 0 \leq s \leq t \leq t_f\}$, *so*

$$x_1^{opt}(t; (x_i)_1, t_0, t_f) = U(t, t_0)(x_i)_1. \tag{5.64}$$

Proof. This statement follows directly by using the expression of the optimal control in (5.58) and applying the results in [32, Corollary 9.1.10]. □

As a matter of fact, the operator-valued function $\Pi_1(t)$ is the unique solution of a standard differential Riccati equation.

Lemma 5.3.3. *[32, Lemma 4.3.2,Theorem 9.1.11] The operator-valued function* $\Pi_1(t)$ *solves the following differential Riccati equation*

$$\begin{aligned} \frac{d}{dt}\Pi_1(t)x_1 + \Pi_1(t)E_1^{-1}A_1x_1 + A_1^*E_1^{-*}\Pi_1(t)x_1 \\ -\Pi_1(t)\tilde{B}_1\tilde{R}^{-1}\tilde{B}_1^*\Pi_1(t)x_1 + Q_1x_1 = 0, \qquad \forall x_1 \in D(A_1), \end{aligned} \tag{5.65}$$

$$\tilde{\Pi}_1(t_f)x_1 = G_1x_1. \tag{5.66}$$

The operator-valued function $\Pi_1(t)$ *is the unique solution of this equation in the class of strongly continuous, self-adjoint operators in* $\mathcal{L}(\mathcal{X}_1)$ *such that* $\langle x_1^a, \Pi_1(t)x_1^b\rangle_{\mathcal{X}}$ *is differentiable for* $t \in (0, t_f)$ *and* $x_1^a,\ x_1^b \in D(A_1)$.

This leads to the main result of this section: a characterization of the optimal control (5.37) without calculating the restrictions of operators A, E, B on the subspace $\mathcal{X}_1$ or $\mathcal{X}_0$.

Theorem 5.3.1. *Define the operator*

$$\Pi_0 = -A_0^{-*}Q_0; \tag{5.67}$$

$\Pi_0 \in \mathcal{L}(\mathcal{X}_0, \mathcal{Z}_0)$. *Also, recalling the operator-valued function* (5.60), *define*

$$\tilde{\Pi}_1(t) = P^{\mathcal{Z}_1 *} E_1^{-*} \Pi_1(t) E_1^{-1} P^{\mathcal{Z}_1}, \tag{5.68a}$$

$$\tilde{\Pi}_0 = P^{\mathcal{Z}_0 *} \Pi_0 P^{\mathcal{X}_0}. \tag{5.68b}$$

1. *The solution of the optimization problem* (5.19) *is*

$$u^{opt}(t) = -\, R^{-1} B^* (\tilde{\Pi}_0 + \tilde{\Pi}_1(t) E) x^{opt}(t). \tag{5.69}$$

2. *The operator* $\tilde{\Pi}_0$ *solves the algebraic equation*

$$A^* \tilde{\Pi}_0 x = -Q P^{\mathcal{X}_0} x, \quad \forall x \in \mathcal{X}, \tag{5.70}$$

and is uniquely defined on $\mathcal{X}_0$.

3. *The operator-valued function* $\tilde{\Pi}_1(t) \in C([0, t_f]; \mathcal{L}(\mathcal{Z}))$ *solves*

$$\frac{d}{dt} E^* \tilde{\Pi}_1(t) E x + E^* \tilde{\Pi}_1(t) A x + A^* \tilde{\Pi}_1(t) E x - E^* \tilde{\Pi}_1(t) B R^{-1} B^* \tilde{\Pi}_1(t) E x$$
$$- E^* \tilde{\Pi}_1(t) B R^{-1} B^* \tilde{\Pi}_0 x + Q P^{\mathcal{X}_1} x = 0, \qquad \forall x \in D(A), \tag{5.71a}$$

$$E^* \tilde{\Pi}_1(t_f) E x = G x, \tag{5.71b}$$

such that $\langle E x^a, \tilde{\Pi}_1(t) E x^b \rangle_{\mathcal{Z}}$ *is differentiable for* $t \in (0, t_f)$ *and* $x^a, x^b \in D(A)$.

4. *The minimum cost is*

$$J(x_i, u^{opt}; t_f) = \langle E x_i, \tilde{\Pi}_1(0) E x_i \rangle_{\mathcal{Z}}. \tag{5.72}$$

Proof. **1.** Using the definitions of the operators $\tilde{B}_0$ and $\tilde{R}$, we write the optimal control (5.61) as follows:

$$(R + B_0^* A_0^{-*} Q_0 \tilde{B}_0) u^{opt}(t) = -\, \tilde{B}_1^* \Pi_1(t) x_1^{opt}(t),$$

and so

$$u^{opt}(t) = -R^{-1}\Big(\tilde{B}_1^*\Pi_1(t)x_1^{opt}(t) + B_0^*A_0^{-*}Q_0\tilde{B}_0u^{opt}(t)\Big).$$

Since $u^{opt}(t) \in \mathcal{U}_a$, then $x_0^{opt}(t) = -\tilde{B}_0u^{opt}(t)$. Using the definition of operator Π_0 in (5.67), we rewrite the previous equation as

$$\begin{aligned} u^{opt}(t) &= -R^{-1}\Big(\tilde{B}_1^*\Pi_1(t)x_1^{opt}(t) - B_0^*A_0^{-*}Q_0x_0^{opt}(t)\Big) \\ &= -R^{-1}\Big(\tilde{B}_1^*\Pi_1(t)x_1^{opt}(t) + B_0^*\Pi_0x_0^{opt}(t)\Big), \end{aligned} \tag{5.73}$$

From (5.7) and (5.6)a, recall that for any $x(t) \in \mathcal{X}$

$$E_1x_1(t) = EP^{\mathcal{X}_1}x(t) = P^{\mathcal{Z}_1}Ex(t), \tag{5.74a}$$

$$B_1 = P^{\mathcal{Z}_1}B, \quad B_0 = P^{\mathcal{Z}_0}B. \tag{5.74b}$$

Using the statements above and that $\tilde{B}_1 = E_1^{-1}B_1$, we rewrite (5.73) as

$$u^{opt}(t) = -R^{-1}B^*\Big(P^{\mathcal{Z}_1*}E_1^{-*}\Pi_1(t)E_1^{-1}EP^{\mathcal{X}_1}x^{opt}(t) + P^{\mathcal{Z}_0*}\Pi_0P^{\mathcal{X}_0}x^{opt}(t)\Big).$$

Equation (5.69) now follows from the definition of $\tilde{\Pi}_1(t)$ and $\tilde{\Pi}_0$ in (5.68).

2. From (5.67), for any $x_0 \in \mathcal{X}_0$,

$$A_0^*\Pi_0x_0 = -Q_0x_0. \tag{5.75}$$

Writing $P^{\mathcal{X}_0}x = x_0$, and using (5.7) and (5.6b), the previous equation implies that

$$A^*P^{\mathcal{Z}_0*}\Pi_0P^{\mathcal{X}_0}x + QP^{\mathcal{X}_0}x = 0. \tag{5.76}$$

Referring to (5.68b), we find that the operator $\tilde{\Pi}_0$ solves equation (5.70). Deducing that $\tilde{\Pi}_0$ is the unique solution of (5.70) on $\mathcal{X}_0$ is straightforward.

3. If $x_i \in D(A)$, then $(x_i)_1 \in D(A_1)$. It follows from equation (5.64) that $x_1^{opt}(t) \in D(A_1)$, and equation (5.65) implies that

$$\begin{aligned} \frac{d}{dt}\Pi_1(t)x_1^{opt}(t) + \Pi_1(t)E_1^{-1}A_1x_1^{opt}(t) + A_1^*E_1^{-*}\Pi_1(t)x_1^{opt}(t) \\ -\Pi_1(t)\tilde{B}_1\tilde{R}^{-1}\tilde{B}_1^*\Pi_1(t)x_1^{opt}(t) + Q_1x_1^{opt}(t) = 0. \end{aligned} \tag{5.77}$$

Since the right-hand-sides of equations (5.61) and (5.73) are equal,

$$-\tilde{R}^{-1}\tilde{B}_1^*\Pi_1(t)x_1^{opt}(t) = -R^{-1}\Big(\tilde{B}_1^*\Pi_1(t)x_1^{opt}(t) + B_0^*\Pi_0 x_0^{opt}(t)\Big),$$

and so

$$-\Pi_1(t)\tilde{B}_1\tilde{R}^{-1}\tilde{B}_1^*\Pi_1(t)x_1^{opt}(t) = -\Pi_1(t)\tilde{B}_1 R^{-1}\Big(\tilde{B}_1^*\Pi_1(t)x_1^{opt}(t) + B_0^*\Pi_0 x_0^{opt}(t)\Big). \tag{5.78}$$

Substituting (5.78) in equation (5.77), we obtain

$$\begin{aligned}&\frac{d}{dt}\Pi_1(t)x_1^{opt}(t) + \Pi_1(t)E_1^{-1}A_1x_1^{opt}(t) + A_1^*E_1^{-*}\Pi_1(t)x_1^{opt}(t)\\ &-\Pi_1(t)\tilde{B}_1R^{-1}\Big(\tilde{B}_1^*\Pi_1(t)x_1^{opt}(t) + B_0^*\Pi_0x_0^{opt}(t)\Big) + Q_1x_1^{opt}(t) = 0\end{aligned} \tag{5.79}$$

From (5.6b), recall that for any $x_1(t) \in D(A_1)$

$$A_1x_1(t) = AP^{\mathcal{X}_1}x(t) = P^{\mathcal{Z}_1}Ax(t). \tag{5.80}$$

Hence, using the statement above along with (5.74), we rewrite equation (5.79) as

$$\begin{aligned}&\frac{d}{dt}E^*P^{\mathcal{Z}_1*}E_1^{-*}\Pi_1(t)E_1^{-1}P^{\mathcal{Z}_1}Ex^{opt}(t) + E^*P^{\mathcal{Z}_1*}E_1^{-*}\Pi_1(t)E_1^{-1}P^{\mathcal{Z}_1}Ax^{opt}(t)\\ &+ A^*P^{\mathcal{Z}_1*}E_1^{-*}\Pi_1(t)E_1^{-1}P^{\mathcal{Z}_1}Ex^{opt}(t)\\ &- E^*P^{\mathcal{Z}_1*}E_1^{-*}\Pi_1(t)E_1^{-1}P^{\mathcal{Z}_1}BR^{-1}B^*P^{\mathcal{Z}_1*}E_1^{-*}\Pi_1(t)E_1^{-1}P^{\mathcal{Z}_1}Ex^{opt}(t)\\ &- E^*P^{\mathcal{Z}_1*}E_1^{-*}\Pi_1(t)E_1^{-1}P^{\mathcal{Z}_1}BR^{-1}B^*P^{\mathcal{Z}_0*}\Pi_0P^{\mathcal{X}_0}x^{opt}(t) + QP^{\mathcal{X}_1}x^{opt}(t) = 0.\end{aligned}$$

Using the definitions of $\tilde{\Pi}_1(t)$ and $\tilde{\Pi}_0$ in (5.68), the previous equation yields

$$\begin{aligned}&\Big(\frac{d}{dt}E^*\tilde{\Pi}_1(t)E + E^*\tilde{\Pi}_1(t)A + A^*\tilde{\Pi}_1(t)E - E^*\tilde{\Pi}_1(t)BR^{-1}B^*\tilde{\Pi}_1(t)E\\ &\quad - E^*\tilde{\Pi}_1(t)BR^{-1}B^*\tilde{\Pi}_0 + QP^{\mathcal{X}_1}\Big)x_{opt}(t) = 0.\end{aligned} \tag{5.81}$$

Since the initial condition x_i was arbitrary in $D(A)$, and equation (5.81) holds for all $t \geq 0$, it follows that $\tilde{\Pi}_1(t)$ solves (5.71a), where $\tilde{\Pi}_0$ solves (5.70).

The final condition in (5.71b) is obtained from (5.66) with the help of (5.6) and (5.21).

3. To obtain the minimum cost (5.72), we refer to statement (5.62). For any $x_i \in \mathcal{X}$,

$$J(x_i, u^{opt}; t_f) = \langle E_1(x_i)_1, E_1^{-*}\Pi_1(0)E_1^{-1}E_1(x_i)_1\rangle_{\mathcal{Z}}. \tag{5.82}$$

Using (5.68a), the previous equation yields

$$J(x_i, u^{opt}; t_f) = \langle Ex_i, \tilde{\Pi}_1(0)Ex_i\rangle_{\mathcal{Z}}. \tag{5.83}$$

□

The natural question we now address is whether the differential equation (5.71) has a unique solution. The following theorem shows that the solution to this equation is unique in the range of E.

Theorem 5.3.2. *Recall the operator $\tilde{\Pi}_0$ in (5.68b) and the differential equation (5.71a), that is,*

$$\begin{aligned}\frac{d}{dt}E^*Z(t)Ex + E^*Z(t)Ax + A^*Z(t)Ex - E^*Z(t)BR^{-1}B^*Z(t)Ex \\ -E^*Z(t)BR^{-1}B^*\tilde{\Pi}_0 x + QP^{\mathcal{X}_1}x = 0, \qquad \forall x \in D(A), \qquad (5.84a)\\ E^*Z(t_f)Ex = Gx. \qquad (5.84b)\end{aligned}$$

1. *Let $\tilde{\Pi}_1(t) \in C([0,t_f];\mathcal{L}(\mathcal{Z}))$ be as defined by (5.68a), where $\langle Ex^a, \tilde{\Pi}_1(t)Ex^b\rangle_{\mathcal{X}}$ is differentiable for all $t \in (0,t_f)$ and $x^a,\ x^b \in D(A)$. Also, let*

   $$Z_2(t) \in C([0,t_f];\mathcal{L}(\mathcal{Z}_0,\mathcal{Z}_1)), \quad Z_4(t) \in C([0,t_f];\mathcal{L}(\mathcal{Z}_0)),$$

   where $Z_4(t)$ is arbitrary and $Z_2(t)$ solves

   $$Z_2(t)(A_0 - B_0R^{-1}B_0^*\tilde{\Pi}_0) = Z_1(t)B_1R^{-1}B_0^*\tilde{\Pi}_0. \tag{5.85}$$

   The general solution to equation (5.84) is

   $$Z(t) = \tilde{\Pi}_1(t) + P^{\mathcal{Z}_1 *}Z_2(t)P^{\mathcal{Z}_0} + P^{\mathcal{Z}_0 *}Z_4(t)P^{\mathcal{Z}_0}. \tag{5.86}$$

2. *For any solution $Z(t)$ of the equation (5.84), the operator-valued function $Z(t)E \in \mathcal{L}(\mathcal{X},\mathcal{Z})$ is unique and leads to the optimal control (5.69)*

   $$u^{opt}(t) = -\,R^{-1}B^*(\tilde{\Pi}_0 + Z(t)E)x^{opt}(t). \tag{5.87}$$

   The minimum cost is

   $$J(x_i, u^{opt}; t_f) = \langle Ex_i, Z(0)Ex_i\rangle_{\mathcal{Z}}. \tag{5.88}$$

Proof. Decomposing equation (5.84a) with the projections $\tilde{P}^{\mathcal{Z}}$ and $\tilde{P}^{\mathcal{X}}$ in (5.8), we write an arbitrary solution $Z(t)$

$$Z(t) = \begin{bmatrix} Z_1(t) & Z_2(t) \\ Z_3(t) & Z_4(t) \end{bmatrix},$$

where

$$Z_1(t) \in \mathcal{L}(\mathcal{Z}_1), \qquad Z_2(t) \in \mathcal{L}(\mathcal{Z}_1, \mathcal{Z}_0),$$
$$Z_3(t) \in \mathcal{L}(\mathcal{Z}_0, \mathcal{Z}_1), \qquad Z_4(t) \in \mathcal{L}(\mathcal{Z}_0).$$

Using the expression of $\tilde{\Pi}_0$ in (5.68b) and that Q satisfies (5.22), we obtain

$$\begin{aligned}
&\frac{d}{dt}\begin{bmatrix} E_1^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Z_1(t) & Z_2(t) \\ Z_3(t) & Z_4(t) \end{bmatrix}\begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} E_1^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Z_1(t) & Z_2(t) \\ Z_3(t) & Z_4(t) \end{bmatrix}\begin{bmatrix} A_1 & 0 \\ 0 & A_0 \end{bmatrix} \\
&+ \begin{bmatrix} A_1^* & 0 \\ 0 & A_0^* \end{bmatrix}\begin{bmatrix} Z_1(t) & Z_2(t) \\ Z_3(t) & Z_4(t) \end{bmatrix}\begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix} \\
&- \begin{bmatrix} E_1^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Z_1(t) & Z_2(t) \\ Z_3(t) & Z_4(t) \end{bmatrix}\begin{bmatrix} B_1 \\ B_0 \end{bmatrix} R^{-1}\begin{bmatrix} B_1^* & B_0^* \end{bmatrix}\begin{bmatrix} Z_1(t) & Z_2(t) \\ Z_3(t) & Z_4(t) \end{bmatrix}\begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix} \\
&- \begin{bmatrix} E_1^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Z_1(t) & Z_2(t) \\ Z_3(t) & Z_4(t) \end{bmatrix}\begin{bmatrix} B_1 \\ B_0 \end{bmatrix} R^{-1}\begin{bmatrix} 0 & B_0^*\Pi_0 P^{\mathcal{Z}_0} \end{bmatrix} + \begin{bmatrix} QP^{\mathcal{X}_1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned} \tag{5.89}$$

Similarly, we use the assumption on G in (5.21) to decompose the final condition (5.84b) as

$$\begin{aligned}
\begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} E_1^* & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Z_1(t_f) & Z_2(t_f) \\ Z_3(t_f) & Z_4(t_f) \end{bmatrix}\begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} E_1^* Z_1(t_f) E_1 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned} \tag{5.90}$$

The resulting four equations yield $Z_3(t) \equiv 0$, $Z_4(t)$ is arbitrary, $Z_2(t)$ solves (5.85). Also, recalling that $\Pi_1(t)$ is the unique solution of (5.65), it follows that

$$Z_1(t) = E_1^{-*}\Pi_1(t)E_1^{-1}. \tag{5.91}$$

2. This assertion follows by substituting the general solution of (5.71), which is (5.86), into the right-hand-side of equation (5.87). Using that $P^{\mathcal{Z}_0}E = 0$, we arrive at the expression of the optimal control (5.69). In a similar way, we can find the minimum cost from (5.88). □

Equations (5.12) and the projections $P^{\mathcal{X}}$, $P^{\mathcal{Z}}$ were used to derive (5.71). However, once the restrictions of Q on the subspaces $\mathcal{X}_1$ and $\mathcal{X}_0$ are found, solving system (5.71) requires no knowledge of the projections. When comparing system (5.71) with the optimality equations derived in finite-dimensional space, it is apparent that system (5.71) differs from the ones derived in [2, 12] since the knowledge of the restrictions such as E_1, A_1, etc. is not required here. Assuming no penalization on the L_2-norm of the algebraic state, i.e., $QP^{\mathcal{X}_0} = 0$, system (5.71) reduces to a single differential Riccati equation that mirrors the one derived for finite-dimensional DAEs through the use of behaviors [102].

5.4 Numerical simulations

In order to illustrate the theoretical results, we study the following class of coupled systems

$$w_t(\xi,t) = w_{\xi\xi}(\xi,t) - \rho w(\xi,t) + \alpha v(\xi,t) + u(t), \tag{5.92a}$$

$$0 = v_{\xi\xi}(\xi,t) - \gamma v(\xi,t) + \beta w(\xi,t) + u(t), \tag{5.92b}$$

with the boundary conditions

$$w_\xi(0,t) = 0, \quad w_\xi(1,t) = 0, \tag{5.92c}$$

$$v_\xi(0,t) = 0, \quad v_\xi(1,t) = 0, \tag{5.92d}$$

where $\xi \in [0,1]$ and $t \geq 0$. The system's parameters are $\rho = 1$, $\gamma = \alpha = \beta = 2$. For $n = 0, 1, \ldots$, $\gamma \neq -(n\pi)^2$, and so γ is not an eigenvalue of the Laplacian operator ∂_{xx} [32, Example 8.1.8]. Consequently, defining $\mathrm{Z} = H_2(0,1)$ and I to be the identity operator on Z, the operator $\gamma I - \partial_{\xi\xi}$ is invertible. Also, $w(\xi,0) = \sin(\pi\xi)$ and $v(\xi,0) = \beta(\gamma I - d_{\xi\xi})^{-1}\sin(\pi\xi)$. It will also prove useful to define

$$\Delta w = \frac{d^2}{d\xi^2}w, \quad D(\Delta) = \{w \in \mathrm{Z} : w_\xi(0) = w_\xi(1) = 0\},$$

and

$$x(t) = \begin{bmatrix} w(\xi,t) \\ v(\xi,t) \end{bmatrix} \in \mathcal{X} = \mathrm{Z} \times \mathrm{Z}.$$

System (5.92) can now be written in the form (5.4)

$$\frac{d}{dt}\underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{E} x(t) = \underbrace{\begin{bmatrix} \Delta - \rho I & \alpha I \\ \beta I & \Delta - \gamma I \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} I \\ I \end{bmatrix}}_{B} u(t),$$

$$x(0) = \begin{bmatrix} \frac{1}{2}\big(1 - \cos(2\pi\xi)\big) & \beta(\frac{\gamma}{2}I - d_{\xi\xi})^{-1}\big(1 - \cos(2\pi\xi)\big) \end{bmatrix}^*.$$

System (5.92) was shown to be radial of degree 0 in Section 3.3. To approximate this coupled system, we use finite-element method with linear splines to obtain a system of DAEs. The spatial interval $[0, 1]$ is subdivided into 27 equal intervals. The dynamics of the parabolic and elliptic states without control (i.e., $u(t) \equiv 0$) are in Figure 5.2 (a & b).

Define the cost functional

$$J(x_i, u; 6) = \int_0^6 \langle x(s), \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x(s)\rangle_{\mathrm{Z}} + \langle u(s), u(s)\rangle_{\mathcal{U}} ds. \tag{5.93}$$

Comparing the previous cost with (5.17), it is clear that $t_f = 6$, $Q = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Note that projections $P^{\mathcal{X}_1}$ and $P^{\mathcal{X}_0}$, which were calculated in Section 3.3 (Example 3.3.1), can be used used to obtain

$$QP^{\mathcal{X}_1} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad QP^{\mathcal{X}_0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.94}$$

To demonstrate the findings in the previous sections, we solve the equations in (5.71) for the optimal control (5.69). Since equations (5.71) consist of operators on an infinite-dimensional Hilbert space, they cannot be solved exactly. Therefore, the control is calculated using a finite-dimensional approximation. The convergence of the approximation method to the true optimal one and closed-loop performance can be discussed by doing calculations of different approximation orders and following a similar approach as in [85, Chapter 4]. This task will be addressed in future work. Without the need to decompose the state $x(t)$ or calculate the operators E_1, A_1, etc., we now solve the finite-dimensional approximation of equations (5.71). This is done by using "ode15s", which is based on a backward differential formula (BDF). Consequently, we obtain an approximation of the optimal control (5.69). The approximated control signal at $\xi = 0$ (5.69) is given in Figure 5.1. Note that the control $u(t)$ ensures the consistency statement on the initial conditions, which is

$$\begin{aligned} u(\xi, 0) &= \frac{(d_{\xi\xi} - \gamma I)}{\beta} v(\xi, 0) - \beta w(\xi, 0) \\ &= 0. \end{aligned}$$

Figure 5.2 (c, d, e & f) illustrates the dynamics of the coupled system throughout the first 3(s) and 6(s) after applying the controller (5.69).

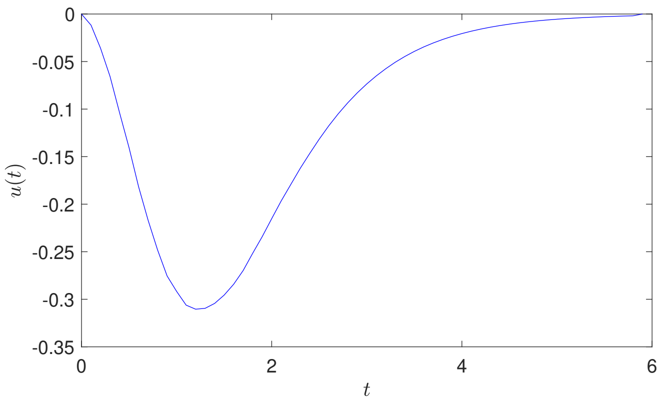
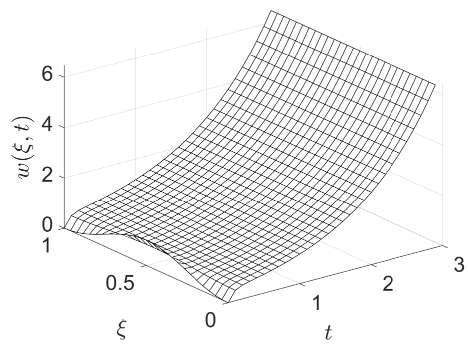
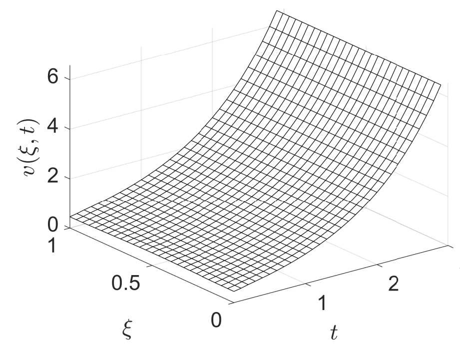


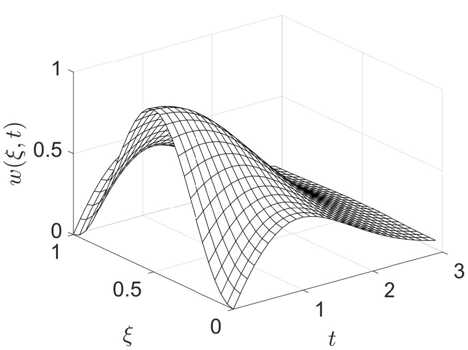
Figure 5.1: LQ-optimal feedback control at $\xi = 0$. This control signal minimizes the cost functional (5.93) and is derived by solving system (5.71) after discretization.



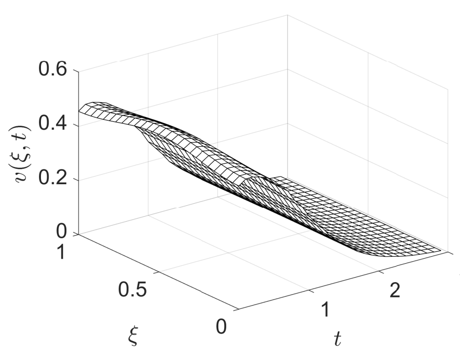
(a) Uncontrolled $w(\xi,t)$



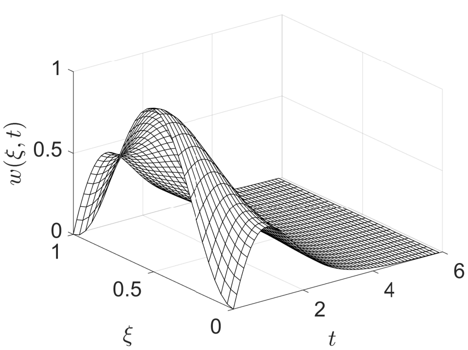
(b) Uncontrolled $v(\xi,t)$



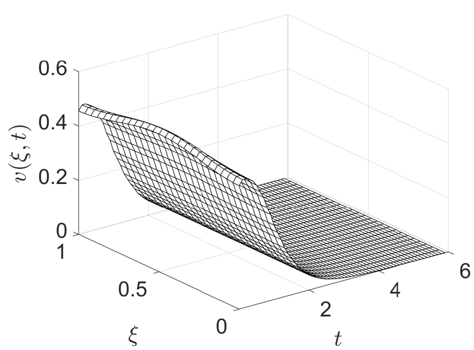
(c) Controlled $w(\xi,t)$ over 3(s)



(d) Controlled $v(\xi,t)$ over 3(s)



(e) Controlled $w(\xi,t)$ over 6(s)



(f) Controlled $v(\xi,t)$ over 6(s)

Figure 5.2: A 3D landscape of the dynamics of the coupled parabolic-elliptic system (5.92) without and with control (5.69). The initial conditions are $w_0 = \frac{1}{2}\big(1-\cos(2\pi\xi)\big)$, $v_0 = -\beta(d_{\xi\xi}-\gamma I)^{-1}w_0$, and $\rho = 1$, $\gamma = \alpha = \beta = 2$. The uncontrolled system is unstable, but the use of LQ feedback control causes the states to decay towards zero over this time interval. The simulations are conducted over the first 3 and 6 seconds.

5.5 Summary

This chapter extends the classical finite-time linear quadratic control problem for finite-dimensional DAEs into the infinite-dimensional case. We showed the existence of a continuous optimal control that ensures the consistency of the initial conditions while minimizing the cost functional. Decomposing the PDAE into a Weierstrass-Kronecker form was crucial in the proofs. However, the optimal control can be calculated from the differential Riccati-like equation without projecting any operators except for the state weight.

Chapter 6

Linear-quadratic control for higher-index differential-algebraic equations

In this chapter, we extend our study of linear quadratic (LQ) optimal control of differential-algebraic equations (DAEs). In Chapter 5, we established the existence of a unique solution to an optimization problem for partial differential-algebraic equations (PDAEs) of radiality-index 0. We also derived an associated differential Riccati-like equation to solve for the optimal control. Here, we focus on the same equations within finite-dimensional spaces and with arbitrary differentiation-index. The differentiation-index represents the minimum number of times that all or part of the differential-algebraic equation must be differentiated to transform it into an ordinary differential equation. A differentiation-index of 1 corresponds to a nilpotency-index 1 for DAEs, and radiality-index 0 for PDAEs. In this case, the algebraic state can be explicitly defined in terms of the control input (see Section 3.1). This chapter, while focused on finite-dimensional DAEs, presents an initial step towards achieving LQ-optimal control for partial differential-algebraic equations with arbitrary radiality-index.

The extension of linear-quadratic control to DAEs with differentiation-index 1 has been addressed in previous work [12, 29, 68, 80]. Furthermore, differential Riccati equations whose solutions yield the optimal control, as in the ODE situation, were obtained in [12] and [80]. In recent years, efforts have been dedicated to studying optimal control for high-index DAEs [20, 74, 75, 94, 95, 100]. In [94, 95], Pytlak studied a class of semi-explicit DAEs that can be reduced to a DAE with differentiation-index 1. The approach involved establishing first-order approximations of the functionals describing the optimal control and deriving adjoint equations for these functionals. In [96, 97], Pytlak et al. developed a numerical method for solving optimal control problems for differentiation-index 3 DAEs. Benner et al. [13] studied the infinite-horizon LQ control problem for Hessenberg-index 2 DAEs. Therein, a projected Riccati equation was derived. There are also contributions tackling linear-quadratic optimal control for linear DAEs with varying coefficients [10, 63, 64, 65]. On the other hand, Reis and Voigt [102] used a behavior-based approach to study infinite-time optimal control for DAEs with arbitrary index. Petreczky and Zhuk [91] also used behaviors to study optimal control for linear DAEs that are not regular. However, a behavior-based approach differs from the traditional state-space or input-output formulations commonly found in systems and control theory [84].

Index-reduction has drawbacks since this approach involves reducing the index through differentiation and eliminations; see [62]. It can be computationally demanding. Such transformations often have numerical errors leading to equations that violate the original constraints.

To our knowledge, there has been no consideration of the existence of LQ control for higher-index DAEs without using index-reduction or behaviors. Also, deriving a general differential Riccati equation that can be used to obtain the optimal control in a feedback form for such systems has not been obtained without incorporating behaviors. In this chapter, we consider linear DAEs without any restrictions on the system's index. Behaviors are not used. This optimal control is shown to satisfy a differential Riccati equation. The solution of this equation can be used to obtain the optimal control input, through an initial projection. When the nilpotent part is zero, the resulting equation in this chapter reduces to that obtained in [80]. The chapter is divided as follows: In Section 6.1, we formulate our problem. We show that the existence of an optimal control depends on the solvability of a particular two-point boundary problem in Section 6.2. Section 6.3 includes the derivation of a differential Riccati equation that can be solved to determine the optimal control.

6.1 Problem statement

We consider systems whose dynamics are described by the following class of DAEs

$$\frac{d}{dt}Ex(t) = Ax(t) + Bu(t),\ t \geq 0, \tag{6.1a}$$

$$x(0) = x_i, \tag{6.1b}$$

where $x(t)$ is a vector-function with values in $\mathbb{R}^n$ and the matrices $E,\ A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. When E is invertible, system (6.1) can be transformed into a system of ordinary differential equations. Thus, we are primarily interested in the case when the matrix E is singular. We assume throughout the chapter that system (6.1) is regular, meaning that $\det(sE - A) \neq 0$. Then, by virtue of the results in Section 3.1, there exist two invertible matrices $P_1,\ P_2 \in \mathbb{R}^{n\times n}$ such that

$$\tilde{E} = P_1 E P_2 = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad \tilde{A} = P_1 A P_2 = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}, \tag{6.2a}$$

$$\tilde{B} = P_1 B = \begin{bmatrix} B_1 \\ B_0 \end{bmatrix}, \quad \tilde{x}(t) = P_2^{-1} x(t) = \begin{bmatrix} x_1(t) \\ x_0(t) \end{bmatrix}, \tag{6.2b}$$

where I is the identity matrix of proper dimensions. Also, $x_1(t)$ and $x_0(t)$ are vector-functions with values in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_0}$, respectively, where $n = n_1 + n_0$. The matrix N is nilpotent with degree of nilpotency $\nu \geq 1$, i.e. for $\nu > 1$ $N^\nu = 0$ and $N^{\nu-1} \neq 0$; $\nu = 1$ when $N = 0$. With the transformation (6.2), we can decompose system (6.1) into

$$\frac{d}{dt} x_1(t) = A_1 x_1(t) + B_1 u(t), \qquad x_1(0) = (x_i)_1, \tag{6.3a}$$

$$\frac{d}{dt} N x_0(t) = x_0(t) + B_0 u(t), \qquad x_0(0) = (x_i)_0. \tag{6.3b}$$

This decomposition is known as the *Weierstrass-Kronecker* form; see Section 3.1. Assuming consistent initialization, more precisely that the control input $u(t)$ satisfies

$$(x_i)_0 = -\sum_{j=0}^{\nu-1} N^j B_0 \frac{d^j}{dt^j} u(0). \tag{6.4}$$

We use classical theory of ODEs and [62, Lemma 2.8] to find the trajectory corresponding to the control input $u(t)$ and an initial condition x_i, so

$$P_2^{-1} x(t) = \begin{bmatrix} x_1(t) \\ x_0(t) \end{bmatrix} = \begin{bmatrix} e^{A_1 t}(x_i)_1 + \int_0^t e^{A_1(t-s)} B_1 u(s) ds \\ -\sum_{j=0}^{\nu-1} N^j B_0 \frac{d^j}{dt^j} u(t) \end{bmatrix}, \tag{6.5}$$

where $\frac{d^j}{dt^j} u(t)$ indicates the jth derivative of $u(t)$ with respect to time. The parameter ν indicates the degree of nilpotency of the matrix N. This parameter also denotes the differentiation-index of the DAE. The control input must have a certain degree of smoothness. Otherwise, the solution becomes a distribution. Throughout this chapter, it is assumed that the control input $u(t)$ is $(2\nu)-$times differentiable. The reason for requiring this degree of smoothness is as follows. The expression for $x_0(t)$ in (6.5) requires the control input to be ν-times continuously differentiable to prevent the solution from being a distribution and to ensure that $x_0(t)$ itself is continuously differentiable. However, we are assuming an additional degree of smoothness, namely that the control input is in $C^{2\nu}([0,t_f];\mathbb{R}^m)$, to guarantee the existence of classical solutions for the adjoint system of (6.1), which will be needed later in Section 6.2.

We seek a control input to minimize the quadratic performance criterion

$$J(x_i,u;t_f) = x^*(t_f)Gx(t_f) + \int_0^{t_f} x^*(s)Qx(s) + u^*(s)Ru(s)ds. \tag{6.6}$$

Here the matrices $G,\ Q \in \mathbb{R}^{n\times n}$ are non-negative and symmetric. The notation $\cdot^*$ denotes the transpose of the matrix. The matrix $R \in \mathbb{R}^{m\times m}$ is positive-definite and assumed symmetric. In order to avoid distributions in the solution, the cost functional is minimized over the following set of admissible control inputs

$$\mathcal{U}_{\mathrm{a}} = \left\{ u(t) \in C^{2\nu}([0,t_f];\mathbb{R}^m) : (x_i)_0 = -\sum_{j=0}^{\nu-1} N^j B_0 \frac{d^j}{dt^j}u(0) \right\}, \tag{6.7}$$

where $x(t)$ solves the descriptor system (6.1). We define the set of admissible control as given in (6.7) to avoid distributions in the solution. This optimal control problem can be written as

$$\inf_{u(t)\in\mathcal{U}_a} J(x_i,u;t_f), \tag{6.8}$$

where $x(t)$ solves equation (6.1).

We define

$$\tilde{G} = P_2^* G P_2 = \begin{bmatrix} \tilde{G}_1 & \tilde{G}_2 \\ \tilde{G}_3 & \tilde{G}_4 \end{bmatrix}, \tag{6.9}$$

$$\tilde{Q} = P_2^* Q P_2 = \begin{bmatrix} \tilde{Q}_1 & \tilde{Q}_2 \\ \tilde{Q}_3 & \tilde{Q}_4 \end{bmatrix}. \tag{6.10}$$

The cost functional (6.6) can be written using the *Weierstrass-Kronecker* form as follows,

$$J(x_i,u;t_f) = \tilde{x}^*(t_f)\tilde{G}\tilde{x}(t_f) + \int_0^{t_f} \tilde{x}^*(s)\tilde{Q}\tilde{x}(s) + u^*(s)Ru(s)ds. \tag{6.11}$$

We assume $\tilde{Q}_2 = 0$, which implies that $\tilde{Q}_3 = 0$ due to the symmetry of Q. Similarly, we set $\tilde{G}_2 = \tilde{G}_3 = 0$. In addition, we define the adjoint system of (6.1)

$$-E^* \frac{d}{dt} z(t) = A^* z(t) + Qx(t), \tag{6.12a}$$

$$E^* z(t_f) = Gx(t_f), \tag{6.12b}$$

where $z(t)$ is a vector-function with values in $\mathbb{R}^n$. Since (E^*, A^*) is regular, setting

$$P_1^{-*} z(t) = \begin{bmatrix} z_1(t) \\ z_0(t) \end{bmatrix},$$

we arrive at the following decomposition for the adjoint system (6.12),

$$-\frac{d}{dt} z_1(t) = A_1^* \, z_1(t) + \tilde{Q}_1 x_1^{opt}(t), \tag{6.13a}$$

$$-\frac{d}{dt} N^* z_0(t) = z_0(t) + \tilde{Q}_4 x_0^{opt}(t). \tag{6.13b}$$

Also, the final condition (6.12b) leads to

$$z_1(t_f) = \tilde{G}_1 x_1^{opt}(t_f), \tag{6.13c}$$

$$N^* z_0(t_f) = \tilde{G}_4 x_0^{opt}(t_f). \tag{6.13d}$$

Thus

$$z_1(t) = e^{A_1^*(t_f - t)} \tilde{G}_1 x_1^{opt}(t_f) + \int_t^{t_f} e^{A_1^*(s-t)} \tilde{Q}_1 x_1^{opt}(s) ds, \tag{6.14}$$

$$z_0(t) = -\sum_{j=0}^{\nu-1} (-N^*)^j \tilde{Q}_4 \frac{d^j}{dt^j} x_0^{opt}(t). \tag{6.15}$$

Note that (6.13d) and (6.15) pose a constraint on $\tilde{G}_4$ and on the final condition of $z_0(t_f)$, respectively. To obtain non-distributional classical solutions for system (6.12), it is clear from (6.15) that $x_0(t)$ has to be in $C^\nu([0, t_f]; \mathbb{R}^n)$. This is ensured by referring to the solutions of system (6.3b) and recalling that the control input $u(t)$ is assumed to be in $C^{2\nu}([0, t_f]; \mathbb{R}^m)$.

6.2 Conditions for the existence of a unique minimizing optimal control input

The next theorem shows that the existence of a unique optimal control depends on the solvability of a certain two-point boundary problem.

Theorem 6.2.1. *Let $u^{opt}(t) \in \mathcal{U}_a$ and let $x^{opt}(t)$ indicate the state of the system with control $u^{opt}(t)$. Define $(z(t), x^{opt}(t))$ to be the solution of*

$$\begin{bmatrix} E & 0 \\ 0 & -E^* \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x^{opt} \\ z \end{bmatrix} (t) = \begin{bmatrix} A & 0 \\ Q & A^* \end{bmatrix} \begin{bmatrix} x^{opt} \\ z \end{bmatrix} (t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u^{opt}(t), \tag{6.16a}$$

$$x^{opt}(0) = x_i, \quad E^* z(t_f) = G x^{opt}(t_f). \tag{6.16b}$$

If

$$u^{opt}(t) = -R^{-1} B^* z(t) \in \mathcal{U}_a, \tag{6.17}$$

then $u^{opt}(t)$ minimizes the cost functional (6.6) over $\mathcal{U}_a$.

Proof. The proof is similar to the one used in the context of optimal control for DAEs with index 1 [80]. We first define the set of admissible variations $\mathcal{V}_a$ as

$$\mathcal{V}_a = \left\{ v(t) \in C^{2\nu-2}([0, t_f]; \mathbb{R}^m) : \ 0 = -\sum_{j=0}^{\nu-1} N^j B_0 \frac{d^j}{dt^j} v(0) \right\}, \tag{6.18}$$

which consists of all functions $v(t)$ such that, if $u^{opt}(t) \in \mathcal{U}_a$ and $v(t) \in \mathcal{V}_a$, then $u^{opt}(t)+v(t)$ also belongs to $\mathcal{U}_a$. Consider a first-order admissible variation of $u^{opt}(t)$

$$u^p(t) = u^{opt}(t) + \epsilon v(t),$$

where $v(t) \in \mathcal{V}_a$. Thus, if $u^{opt}(t)$ is an admissible control input, i.e., belongs to $\mathcal{U}_a$, then $u^p(t) \in \mathcal{U}_a$ and so

$$x_0(0) = -\sum_{j=0}^{\nu-1} N^j B_0 \frac{d^j}{dt^j} u^{opt}(0) = -\sum_{j=0}^{\nu-1} N^j B_0 \frac{d^j}{dt^j} u^p(0).$$

Referring to (6.5), the trajectory corresponding to $u^{opt}(t)$ is

$$x^{opt}(t) = P_2 \begin{bmatrix} e^{A_1 t}(x_i)_1 + \int_0^t e^{A_1(t-s)} B_1 u^{opt}(s) ds \\ -\sum_{j=0}^{\nu-1} N^j B_0 \frac{d^j}{dt^j} u^{opt}(t) \end{bmatrix}.$$

Define

$$\phi(t)v = P_2 \begin{bmatrix} \int_0^t e^{A_1(t-s)} B_1 v(s) ds \\ -\sum_{j=0}^{\nu-1} N^j B_0 \frac{d^j}{dt^j} v(t) \end{bmatrix},$$

then the trajectory corresponding to $u^p(t)$ is

$$x^p(t) = x^{opt}(t) + \epsilon\phi(t)v,$$

with $\phi(0)v = 0$. We rewrite the cost functional (6.6) as

$$\begin{aligned} J(x_i, u; t_f) = x^*(t_f)Gx(t_f) + \int_0^{t_f} & x^*(s)Qx(s) + u^*(s)Ru(s) \\ & + z^*(s)\Big(Ax(s) + Bu(s)\Big) + \Big(Ax(s) + Bu(s)\Big)^* z(s) \\ & - z^*(s)E\frac{d}{ds}x(s) - (E\frac{d}{ds}x(s))^* z(s) ds, \end{aligned} \tag{6.19}$$

for $z(t) \in \mathbb{R}^n$ that solve (6.16a). To simplify notation, we define the Hamiltonian function

$$\begin{aligned} H(x, z, u)(t) = {} & x^*(t)Qx(t) + u^*(t)Ru(t) \\ & + z^*(t)\Big(Ax(t) + Bu(t)\Big) + \Big(Ax(t) + Bu(t)\Big)^* z(t). \end{aligned} \tag{6.20}$$

Rewriting (6.19) in terms of the Hamiltonian function, we obtain

$$\begin{aligned} J(x_i, u^p; t_f) - J(x_i, u^{opt}; t_f) = {} & (x^p(t_f))^* G x^p(t_f) - (x^{opt}(t_f))^* G x^{opt}(t_f) \\ & + \int_0^{t_f} H(x^p, z, u^p) - H(x^{opt}, z, u^{opt}) ds \\ & + 2\int_0^{t_f} z^*(s)E\frac{d}{ds}x^{opt}(s) - z^*(s)E\frac{d}{ds}x^p(s) ds. \end{aligned} \tag{6.21}$$

Computing the terms on the right-hand-side of (6.21),

$$(x^p(t_f))^* G x^p(t_f) - (x^{opt}(t_f))^* G x^{opt}(t_f) = \epsilon (x^{opt}(t_f))^* G \phi(t_f) v \\ + \epsilon(\phi(t_f)v)^* G x^{opt}(t_f) + \epsilon^2 (\phi(t_f)v)^* G\phi(t_f) v, \tag{6.22}$$

$$\int_0^{t_f} H(x^p,\eta,u^p)(s) - H(x^{opt},\eta,u^{opt})(s) ds = 2\epsilon \int_0^{t_f} (x^{opt}(s))^* Q\phi(s) v + (u^{opt}(s))^* R v(s) \\ + z^*(s) A\phi(s) v + z^*(s) B v(s) ds \\ + \epsilon^2 \int_0^{t_f} (\phi(s)v)^* Q\phi(s) v + v^*(s) R v(s) ds, \tag{6.23}$$

$$2\int_0^{t_f} z^*(s) E \frac{d}{ds} x^{opt}(s) - z^*(s) E \frac{d}{ds} x^p(s) ds = -2\epsilon z^*(t_f) E\phi(t_f) v \\ + 2\epsilon \int_0^{t_f} \frac{d}{dt} z^*(s) E \phi(s) v ds. \tag{6.24}$$

Combining (6.22)-(6.24) and rearranging terms using the assumptions that R, Q, G are symmetric, equation (6.21) yields

$$\begin{aligned} J(x_i, u^p; t_f) - J(x_i, u^{opt}; t_f) \\ &= \epsilon^2 \Big((\phi(t_f)v)^* G\phi(t_f) v + \int_0^{t_f} (\phi v(s))^* Q \phi(s) v + v^*(s) R v(s) ds \Big) \\ &+ \epsilon \Big(2(G x^{opt}(t_f) - E^* z(t_f))^* \phi(t_f) v \Big) \\ &+ \epsilon \Big(\int_0^{t_f} (E^* \frac{d}{ds} z(s) + A^* z(s) + Q x^{opt}(s))^* \phi(s) v ds \Big) \\ &+ \epsilon \Big(\int_0^{t_f} (R u(s) + B^* z(s))^* v(s) ds \Big). \end{aligned} \tag{6.25}$$

For $z(t)$, $x^{opt}(t)$ that solve (6.16a),(6.16b), if the control $u^{opt}(t)$ is given by (6.17), then the coefficients of ϵ in (6.25) vanish. Since R is positive-definite, for all admissible non-zero variations, it follows that

$$J(x_i, u^p; t_f) - J(x_i, u^{opt}; t_f) > 0.$$

Therefore, $u^{opt}(t) = -R^{-1} B^* z(t)$ is the minimizing optimal control for (6.6). □

Theorem 6.2.1 indicates that if system (6.16) has a unique solution, then the optimization problem (6.8) has a unique solution. For the sequel findings, we assume that the two-point value problem (6.16) is uniquely solvable.

6.3 Derivation of differential Riccati equation with no penalty on the algebraic state

In this section, we present a differential Riccati equation that can be used to solve for the optimal control (6.17). Besides the assumptions given in Section 6.1, we shall also assume that there is no penalization on the algebraic sub-state $x_0(t)$, that is, $\tilde{Q}_4 = 0$ in (6.10). It follows from (6.13b) and (6.15) that

$$z_0(t) = 0. \tag{6.26}$$

Referring to (6.13d), we must also have that $\tilde{G}_4 = 0$. The optimization problem (6.8) reduces to minimizing the cost functional

$$J(x_i, u; t_f) = x_1^*(t_f)\tilde{G}_1 x_1(t_f) + \int_0^{t_f} x_1^*(s)\tilde{Q}_1 x_1(s) + u^*(s)Ru(s)ds, \tag{6.27}$$

over $\mathcal{U}_a$, where $x_1(t)$ solves (6.3a). This control problem falls within the framework of a standard linear-quadratic control problem for linear ordinary differential equations(ODEs) [84]. The associated differential Riccati equation, which is identical to the one used for ODEs, is presented for completeness in the following theorem.

Theorem 6.3.1. *Assume that system* (6.1) *is regular with nilpotency-index* $\nu \geq 1$. *Equation* (6.13a) *defines a mapping from* $x_1^{opt}(t)$ *to* $z_1(t)$,

$$z_1(t) = \Pi_1(t) x_1^{opt}(t). \tag{6.28}$$

If

$$\tilde{Q}_2 = \tilde{Q}_3 = \tilde{Q}_4 = 0$$
$$\tilde{G}_2 = \tilde{G}_3 = \tilde{G}_4 = 0,$$

then the following statements hold true.

1. *The optimal control that minimizes (6.8) is*

$$u^{opt}(t) = -R^{-1}\tilde{B}_1^*\Pi_1(t)x_1^{opt}(t), \tag{6.29}$$

where $\Pi_1(t)$ *solves*

$$\frac{d}{dt}\Pi_1(t) + \Pi_1(t)A_1 + A_1^*\Pi_1(t) - \Pi_1(t)B_1R^{-1}B_1^*\Pi_1(t) + \tilde{Q}_1 = 0, \tag{6.30a}$$

$$\Pi_1(t_f) = \tilde{G}_1. \tag{6.30b}$$

2. *The optimal cost is*

$$J(x_i, u^{opt}; t_f) = (x_1(0))^*\Pi_1(0)(x_1(0)) \tag{6.31}$$

Proof. Since the differential equations (6.3a) and (6.13a) are uniquely solvable, it is straightforward to deduce that the mapping (6.28) is well-defined.

(1) Since $\tilde{Q}_4 = 0$, then $z_0(t) = 0$. Hence, referring to the expression of $u^{opt}(t)$ in (6.17), we obtain

$$\begin{aligned} u^{opt}(t) &= - R^{-1}B^*P_1^*(P_1^*)^{-1}z(t) \\ &= - R^{-1}(B_1^*z_1(t) + B_0^*z_0(t)) \\ &= - R^{-1}B_1^*\Pi_1(t)x_1^{opt}(t). \end{aligned} \tag{6.32}$$

To derive equation (6.30a), we take the derivative of (6.28) with respect to time

$$\frac{d}{dt}\Pi_1(t)x_1^{opt}(t) + \Pi_1(t)A_1x_1^{opt}(t) + \Pi_1(t)B_1u^{opt}(t) + A_1^*\Pi_1(t)x_1^{opt}(t) + \tilde{Q}_1x_1^{opt}(t) = 0. \tag{6.33}$$

Using (6.32), we substitute for $u^{opt}(t)$ in the previous equation. We obtain

$$\Big(\frac{d}{dt}\Pi_1(t) + \Pi_1(t)A_1 - \Pi_1(t)B_1R^{-1}B_1^*\Pi_1(t) + A_1^*\Pi_1(t) + \tilde{Q}_1\Big)x_1^{opt}(t) = 0. \tag{6.34}$$

Since equation (6.34) holds true for all $x_1^{opt}(t)$, we arrive at equation (6.30a). The final condition (6.30b) follows immediately from (6.13c). This concludes the first part of the theorem.

2. Substituting the optimal control $u^{opt}(t)$ into the cost (6.27), we obtain

$$J(x_i, u^{opt}; t_f) = (x_1^{opt}(t_f))^*\tilde{G}_1x_1^{opt}(t_f) + \int_0^{t_f}(x_1^{opt}(s))^*\tilde{Q}_1x_1^{opt}(s) + (u^{opt}(s))^*Ru^{opt}(s)ds. \tag{6.35}$$

Recalling that $\tilde{G}_1 x_1^{opt}(t_f) = z_1(t_f) = \Pi_1(t)x_1^{opt}(t)$, the previous equation implies that

$$\begin{aligned} J(x_i, u^{opt}; t_f) &= (x_1^{opt}(0))^* \Pi_1(0) x_1^{opt}(0) \\ &+ \int_0^{t_f} \frac{d}{ds}\left((x_1^{opt}(s))^* \Pi_1(s) x_1^{opt}(s) \right) + (x_1^{opt}(s))^* \tilde{Q}_1 x_1^{opt}(s) + (u^{opt}(s))^* R u^{opt}(s) ds. \end{aligned}$$

Since $\Pi_1(t)$ solves equation (6.30a) and the optimal control is (6.29), the previous equation leads to the optimal cost in (6.31). □

The differential Riccati equation (6.30) relies on knowing the projections P_1, P_2; see (6.2). Therefore, we derive a differential Riccati equation that leads to the optimal control $u^{opt}(t)$ without the need for projecting the DAE (6.1).

Theorem 6.3.2. *The optimal control that minimizes (6.8) is*

$$u^{opt}(t) = -R^{-1} B^* \Pi(t) x^{opt}(t), \tag{6.36}$$

where $\Pi(t)$ *solves*

$$E^*\Pi(t)E + E^*\Pi(t)A + A^*\Pi(t)E - E^*\Pi(t)BR^{-1}B^*\Pi(t)E + Q = 0, \tag{6.37a}$$

$$E^*\Pi_1(t_f)E = G. \tag{6.37b}$$

The optimal cost is

$$J(x_i, u^{opt}; t_f) = x_i^* \Pi(0) x_i. \tag{6.38}$$

Proof. Define

$$\Pi(t) = \begin{bmatrix} \Pi_1(t) & 0 \\ 0 & 0 \end{bmatrix}. \tag{6.39}$$

We write (6.30a) as

$$\begin{aligned} &\frac{d}{dt} \begin{bmatrix} I & 0 \\ 0 & N^* \end{bmatrix} \begin{bmatrix} \Pi_1(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \\ &+ \begin{bmatrix} I & 0 \\ 0 & N^* \end{bmatrix} \begin{bmatrix} \Pi_1(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} A_1^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Pi_1(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \\ &+ \begin{bmatrix} I & 0 \\ 0 & N^* \end{bmatrix} \begin{bmatrix} \Pi_1(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 \\ B_0 \end{bmatrix} R^{-1} \begin{bmatrix} B_1^* & B_0^* \end{bmatrix} \begin{bmatrix} \Pi_1(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} + \begin{bmatrix} \tilde{Q}_1 & 0 \\ 0 & 0 \end{bmatrix} = 0. \end{aligned} \tag{6.40}$$

Recall from (6.2) that

$$E = P_1^{-1} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} P_2^{-1}, \quad A = P_1^{-1} \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} P_2^{-1}, \quad B = P_1^{-1} \begin{bmatrix} B_1 \\ B_0 \end{bmatrix}. \tag{6.41}$$

Also, equation (6.10) together with $\tilde{Q}_2 = \tilde{Q}_3 = \tilde{Q}_4 = 0$ imply that

$$Q = P_2^{-*} \begin{bmatrix} \tilde{Q}_1 & 0 \\ 0 & 0 \end{bmatrix} P_2^{-1}. \tag{6.42}$$

We write

$$\Pi(t) = P_1^* \begin{bmatrix} \Pi_1(t) & 0 \\ 0 & 0 \end{bmatrix} P_1. \tag{6.43}$$

Then, multiplying (6.40) with P_2^{-*} from the right and with P_2^{-1} from the left, we use (6.41) to obtain equation (6.37a). Since $\tilde{G}_2 = \tilde{G}_3 = \tilde{G}_4 = 0$, the final condition (6.37b) follows immediately from (6.30b). Similarly, we arrive at the optimal cost (6.38) by referring to (6.31) and using the definition of $\Pi(t)$ in (6.43).

□

Although mentioned previously in Chapter 5, we now recall some of the results obtained in [80, page 44]. There a differential Riccati equation for DAEs (6.1) of nilpotency-index 1 was derived, incorporating a specific cost functional, that is

$$J(x_i, u; t_f) = \frac{1}{2}(Ex(t_f))^* GEx(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \begin{bmatrix} x^*(t) & u^*(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt,$$

where the matrix $S \in \mathbb{R}^{n\times m}$ has been decomposed as $\tilde{S} = P_2^* S = \begin{bmatrix} \tilde{S}_1 & \tilde{S}_2 \end{bmatrix}$, and the condition

$$Q_4 = -\tilde{S}_2 R^{-1} \tilde{S}_2^*$$

must be satisfied. Note that $\tilde{Q}_4 = 0$ when $S = 0$, which is what we have in the cost (5.17). Hence, the results in this section are an extension of these earlier findings for an arbitrary nilpotency (or differentiation) index.

In this context, we also mention works that inspired our approach for deriving a differential Riccati equation without needing the projections for calculating the optimal control. The studies in [48] and [110] derived a Lyapunov equation for balanced truncation model reduction of specific DAE systems, without the need for projections to solve the equation. Furthermore, in [34] a Lyapunov equation was derived for a class of DAEs with a differentiation-index 1.

6.4 Summary

We presented an approach to linear-quadratic (LQ) control on a finite-horizon for a class of differential-algebraic equations (DAEs) with arbitrary differentiation-index. Solutions for the optimization problem depend on the solvability of a particular two-point system. Then, assuming that this two-point system is uniquely solvable and that there is no weight on the algebraic state, we derived a differential Riccati equation. The solution of this equation leads to obtaining the optimal control in feedback form. The *Weierstrass-Kronecker* form and associated transformations were fundamental. When the nilpotency-index is one, the obtained differential Riccati equation reduces to the results obtained [80].

Chapter 7

Conclusions and future research

7.1 Concluding remarks

This thesis was driven by the need to control systems described by partial differential-algebraic equations (PDAEs). Such equations arise from the coupling of partial differential equations (PDEs) and differential-algebraic equations (DAEs), and also from the coupling of partial differential equations where one equation is in equilibrium. The control of these systems is vital due to their significance in modeling a variety of practical applications. While the control of each type of system, PDE models and DAE systems, has been extensively studied and established as broad areas of mathematical research, less research has been conducted on systems where these equations are combined. A specific example of PDAEs that has been of particular interest in this thesis is the coupled parabolic-elliptic system, where the elliptic equation can be viewed as an algebraic constraint. Such coupled systems attract significant interest in technical sectors due to their role in the mathematical modeling of various complex phenomena. Examples include the dynamics of electrochemical lithium-ion cells, the Navier-Stokes equations with the incompressibility constraint (divergence-free velocity field), and chemotaxis phenomena. Such coupled systems can exhibit instability in their dynamics, leading to physical consequences.

The limitations of simplifying a partial differential-algebraic equation to a partial differential equation, by solving the algebraic constraints, mean that a direct design approach based on PDAEs is needed. While it is true that any resulting infinite-dimensional controller must be approximated by a finite-dimensional one before practical implementation, we were interested in direct controller designs for PDAEs where discretization to DAEs may be used as a final step.

Our first objective was initially motivated by the stabilization of a coupled parabolic-elliptic system. We designed a single control input that stabilizes both equations. This was done by using a backstepping approach. This approach relies on establishing state transformations, usually Volterra transformations of the second kind, to convert the system into a target system with more desirable characteristics. We obtained a single stabilizing control input in a state feedback form. Eventually this led to designing state observers that construct estimates for the state, from some partial boundary measurements.

While linear-quadratic control has been widely applied to systems described by PDEs [32] and DAEs [12, 20, 80, 102], its application to PDAEs is still developing. We extended the application of linear-quadratic (LQ) control from linear partial differential equations to a specific class of partial differential-algebraic equations in Hilbert spaces, specifically those with a radiality-index 0. To avoid distributions, we only allowed control inputs that ensure consistent initialization. Employing a fixed point argument, we established that a unique continuous optimal control input exists. The next step was to derive a differential-like equation that determines the optimal control. This step involved using projections with certain properties coming from the system's 0-index radiality. The projections facilitated the derivation but they are not required to obtain the optimal control.

The concept of index is a crucial aspect for differential-algebraic equations, and naturally, the question arises whether linear-quadratic controllers can be extended to arbitrary higher index systems. To address this, we studied DAEs with arbitrary index on finite-dimensional spaces. We did not employ index reduction or behavioral approach, which are tools previously used in the literature for this purpose. Our results demonstrate that the existence of a unique optimal control is related to solving a two-point boundary problem. In addition, we derived a differential Riccati equation whose solution yields the optimal control. Our approach extends the work presented in [80] for index-1 DAEs.

In this thesis, we established two different controller designs for partial-differential algebraic equations: backstepping controller and linear-quadratic controller. To compare both designs, we begin by emphasizing that the backstepping controller operates at the system's boundary and does not require the calculation of eigenfunctions. In fact, backstepping does not rely on a specific placement of the PDE's eigenvalues. Instead, it achieves Lyapunov stabilization by collectively shifting all eigenvalues in a favorable direction within the complex plane. Backstepping differs from optimal control methods as it avoids the need to solve differential Riccati equations, which can be challenging for infinite-dimensional systems. Instead, backstepping requires solving specific kernel equations to obtain the control in a feedback form. Backstepping is often classified as a direct controller, sometimes referred to as a "late lumping" controller. This designation arises because the controller is approximated as a finite-dimensional or lumped parameter system only at the final stage of its design. On the other hand, the linear quadratic control described in this thesis operates as a distributed control and applies to a broader range of systems. It allows for the adjustment of weights on the functional to achieve certain performance. However, this design necessitates solving a differential Riccati-like equation to derive the optimal feedback control. Consequently, the dynamics of the PDAE must be approximated to solve this equation and implement the controller, a process known as “early-lumping”.

7.2 Current and future directions

The research presented in this thesis introduces several interesting questions for further study, some of which are currently being addressed. In many physical problems, the coupled parabolic-elliptic equations are nonlinear. Hence, the design of a boundary control in Chapter 4 when some nonlinear terms are present in the system is a point of interest. Furthermore, an extension of the work to the case when the coefficients are spatially-variant will be studied.

Another immediate question is whether the standard results concerning the infinite-time LQ-control problem for PDEs carry over to the class of PDAEs with radiality-index 0. This question is the subject of current research.

The class of partial differential-algebraic equations considered in Chapter 5 was limited to those with a radiality-index of 0. We are working on extending these results to include higher-index PDAEs. Inspired by our approach in Chapter 6 and the weak formulation of solutions in [43], we are using the weak solutions of the PDAEs under study, and assuming that the initial conditions are consistent.

Finally, a very interesting open question is whether there is a duality between Kalman filter design and LQ control for PDAEs. This is a subject of future work.

References

[1] Jaewook Ahn and Changwook Yoon. Global well-posedness and stability of constant equilibria in parabolic–elliptic chemotaxis systems without gradient sensing. *Nonlinearity*, 32(4):1327, 2019.

[2] Ala' Alalabi and Kirsten Morris. Linear quadratic control problem on a finite-horizon for a class of differential-algebraic equations. In *the proceedings of the 2024 American Control Conference*, 2024.

[3] Giuseppe Alì, Andreas Bartel, Michael Günther, and Caren Tischendorf. Elliptic partial differential-algebraic multiphysics models in electrical network design. *Mathematical Models and Methods in Applied Sciences*, 13(09):1261–1278, 2003.

[4] Giuseppe Alì and Nella Rotundo. On the tractability index of a class of partial differential-algebraic equations. *Acta Applicandae Mathematicae*, 122(1):3–17, 2012.

[5] Giuseppe Alì. PDAE models of integrated circuits. *Mathematical and Computer Modelling*, 51(7):915–926, 2010.

[6] Giuseppe Alì and Nella Rotundo. An existence result for elliptic partial differential–algebraic equations arising in semiconductor modeling. *Nonlinear Analysis*, 72(12):4666–4681, 2010.

[7] Jean Auriol, Kirsten Morris, and Florent Di Meglio. Late-lumping backstepping control of partial differential equations. *Automatica*, 100:247–259, 2019.

[8] Antonello Baccoli and Alessandro Pisano. Anticollocated backstepping observer design for a class of coupled reaction-diffusion PDEs. *Journal of Control Science and Engineering*, 2015:53–53, 2015.

[9] Antonello Baccoli, Alessandro Pisano, and Yury Orlov. Boundary control of coupled reaction–diffusion processes with constant parameters. *Automatica*, 54:80–90, 2015.

[10] Katalin Balla, Galina A Kurina, and Roswitha März. Index criteria for differential-algebraic equations arising from linear-quadratic optimal control problems. *Journal of Dynamical and Control Systems*, 12(3):289–311, 2006.

[11] Georges Bastin and Jean-Michel Coron. *Stability and boundary stabilization of 1-D hyperbolic systems*, volume 88. Springer, 2016.

[12] Douglas Bender and Alan Laub. The linear-quadratic optimal regulator for descriptor systems. *IEEE Transactions on Automatic Control*, 32(8):672–688, 1987.

[13] Peter Benner, Matthias Heinkenschloss, Jens Saak, and Heiko K Weichelt. Efficient solution of large-scale algebraic Riccati equations associated with index-2 DAEs via the inexact low-rank Newton-ADI method. *Applied Numerical Mathematics*, 152:338–354, 2020.

[14] Alain Bensoussan, Giuseppe Da Prato, Michel C Delfour, and Sanjoy K Mitter. *Representation and control of infinite-dimensional systems*, volume 1. Springer, 2007.

[15] Lorena Bociu, Boris Muha, and Justin T Webster. Mathematical effects of linear visco-elasticity in quasi-static Biot models. *Journal of Mathematical Analysis and Applications*, 527(2):127462, 2023.

[16] Tobias Breiten, Karl Kunisch, and Laurent Pfeiffer. Feedback stabilization of the two-dimensional Navier–Stokes equations by value function approximation. *Applied Mathematics & Optimization*, 80:599–641, 2019.

[17] Haim Brézis. *Functional analysis, Sobolev spaces and partial differential equations*, volume 2. Springer, 2011.

[18] Leobardo Camacho-Solorio, Rafael Vázquez, and Miroslav Krstić. Boundary observers for coupled diffusion–reaction systems with prescribed convergence rate. *Systems & Control Letters*, 135:104586, 2020.

[19] Stephen Campbell. High-index differential-algebraic equations. *Journal of Structural Mechanics*, 23(2):199–222, 1995.

[20] Stephen Campbell and Peter Kunkel. Solving higher-index DAE optimal control problems. *Numerical Algebra, Control and Optimization*, 6(4):447–472, 2016.

[21] Stephen Campbell and Wieslaw Marszalek. The index of an infinite-dimensional implicit system. *Mathematical and Computer Modelling of Dynamical Systems*, 5(1):18–42, 1999.

[22] Eduardo Cerpa and Jean-Michel Coron. Rapid stabilization for a Korteweg-de Vries equation from the left Dirichlet boundary condition. *IEEE Transactions on Automatic Control*, 58(7):1688–1695, 2013.

[23] Nalin A Chaturvedi, Reinhardt Klein, Jake Christensen, Jasim Ahmed, and Aleksandar Kojic. Algorithms for advanced battery-management systems. *IEEE Control Systems Magazine*, 30(3):49–68, 2010.

[24] Guangwei Chen, Rafael Vázquez, and Miroslav Krstić. Rapid stabilization of timoshenko beam by PDE backstepping. *IEEE Transactions on Automatic Control*, 2023.

[25] Guangwei Chen, Rafael Vázquez, Zhitao Liu, and Hongye Su. Backstepping control of mixed hyperbolic-parabolic PDE system with multiple coupling terms. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 2584–2589, 2021.

[26] Guangwei Chen, Rafael Vázquez, Zhitao Liu, and Hongye Su. Backstepping control of an underactuated hyperbolic-parabolic coupled PDE system. *IEEE Transactions on Automatic Control*, 2023.

[27] Kurt Chudej, Hans Josef Pesch, and Joachim Rang. Index analysis of models. *Molten Carbonate Fuel Cells: Modeling, Analysis, Simulation, and Control*, pages 63–74, 2007.

[28] Ph Clément, CJ Van Duijn, and Shuanhu Li. On a nonlinear elliptic-parabolic partial differential equation system in a two-dimensional groundwater flow problem. *SIAM Journal on Mathematical Analysis*, 23(4):836–851, 1992.

[29] Daniel Cobb. Descriptor variable systems and optimal state regulation. *IEEE Transactions on Automatic Control*, 28(5):601–611, 1983.

[30] Jennie Cochran, Rafael Vázquez, and Miroslav Krstić. Backstepping boundary control of Navier-Stokes channel flow: Explicit gain formulae in 3D. In *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 5329–5334. IEEE, 2006.

[31] Luisa Consiglieri. Weak solutions for multiquasilinear elliptic–parabolic systems: Application to thermoelectrochemical problems. *Boletín de la Sociedad Matemática Mexicana*, 26(2):535–562, 2020.

[32] Ruth Curtain and Hans Zwart. *Introduction to infinite-dimensional systems theory: A state-space approach*, volume 71. Springer Nature, 2020.

[33] Gustavo Artur de Andrade, Rafael Vázquez, and Daniel Juan Pagano. Backstepping stabilization of a linearized ODE–PDE Rijke tube model. *Automatica*, 96:98–109, 2018.

[34] Guang-Ren Duan. *Analysis and design of descriptor linear systems*, volume 23. Springer Science & Business Media, 2010.

[35] Mehmet Erbay, Birgit Jacob, and Kirsten Morris. On the Weierstraß form of infinite-dimensional differential-algebraic equations. *ArXiv preprint arXiv:2402.17560*, 2024.

[36] Mehmet Erbay, Birgit Jacob, Kirsten Morris, Timo Reis, and Caren Tischendorf. Index concepts for linear differential-algebraic equations in finite and infinite-dimensions. *ArXiv preprint arXiv:2401.01771*, 2024.

[37] V Fedorov and O Ruzakova. On solvability of perturbed sobolev type equations. *St. Petersburg Mathematical Journal*, 20(4):645–664, 2009.

[38] Vladimir E Fedorov and Benzion Shklyar. Exact null controllability of degenerate evolution equations with scalar control. *Sbornik: Mathematics*, 203(12):1817, 2012.

[39] Zhaoqiang Ge and Dexing Feng. Well-posed problem of nonlinear singular distributed parameter systems and nonlinear GE-semigroup. *Science China Information Sciences*, 56(12):1–14, 2013.

[40] Zhaoqiang Ge, Xiaochi Ge, and Jifeng Zhang. Approximate controllability and approximate observability of singular distributed parameter systems. *IEEE Transactions on Automatic Control*, 65(5):2294–2299, 2019.

[41] ZhaoQiang Ge, GuangTian Zhu, and DeXing Feng. Exact controllability for singular distributed parameter system in Hilbert space. *Science in China Series F: Information Sciences*, 52(11):2045–2052, 2009.

[42] Hannes Gernandt and Timo Reis. A pseudo-resolvent approach to abstract differential-algebraic equations. *ArXiv preprint arXiv:2312.02303*, 2023.

[43] Hannes Gernandt and Timo Reis. Linear-quadratic optimal control for abstract differential-algebraic equations. *ArXiv preprint arXiv:2402.08762*, 2024.

[44] Gleb V Grenkin, Alexander Yu Chebotarev, Andrey E Kovtanyuk, Nikolai D Botkin, and Karl-Heinz Hoffmann. Boundary optimal control problem of complex heat transfer model. *Journal of Mathematical Analysis and Applications*, 433(2):1243–1260, 2016.

[45] Bao-Zhu Guo and Liang Zhang. Local null controllability for a chemotaxis system of parabolic–elliptic type. *Systems & Control Letters*, 65:106–111, 2014.

[46] Ernst Hairer, Christian Lubich, and Michel Roche. *The numerical solution of differential-algebraic systems by Runge-Kutta methods*, volume 1409. Springer, 2006.

[47] Agus Hasan, Ole Morten Aamo, and Miroslav Krstić. Boundary observer design for hyperbolic PDE–ODE cascade systems. *Automatica*, 68:75–86, 2016.

[48] Matthias Heinkenschloss, Danny C Sorensen, and Kai Sun. Balanced truncation model reduction for a class of descriptor systems with application to the Oseen equations. *SIAM Journal on Scientific Computing*, 30(2):1038–1063, 2008.

[49] Long Hu, Florent Di Meglio, Rafael Vázquez, and Miroslav Krstić. Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs. *IEEE Transactions on Automatic Control*, 61(11):3301–3314, 2015.

[50] Birgit Jacob and Kirsten Morris. On solvability of dissipative partial differential-algebraic equations. *IEEE Control Systems Letters*, 6:3188–3193, 2022.

[51] Chunting Ji and Zhengqiang Zhang. Adaptive boundary observer design for coupled parabolic PDEs with different diffusions and parameter uncertainty. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 69(7):3037–3047, 2022.

[52] Yushan Jiang, Chao Liu, Qingling Zhang, and Tieyu Zhao. Two side observer design for singular distributed parameter systems. *Systems & Control Letters*, 124:112–120, 2019.

[53] Ansgar Jüngel. *Transport equations for semiconductors*, volume 773. Springer, 2009.

[54] Yoichiro Kamimura, Yukihiro Miyanaga, and Masahiro Ueda. Heterotrimeric G-protein shuttling via Gip1 extends the dynamic range of eukaryotic chemotaxis. *Proceedings of the National Academy of Sciences*, 113(16):4356–4361, 2016.

[55] Simon Kerschbaum and Joachim Deutscher. Backstepping control of coupled linear parabolic PDEs with space and time dependent coefficients. *IEEE Transactions on Automatic Control*, 65(7):3060–3067, 2020.

[56] Hassan K Khalil. *Control of nonlinear systems*. Prentice Hall, New York, NY, 2002.

[57] Shumon Koga, Mamadou Diagne, Shuxia Tang, and Miroslav Krstić. Backstepping control of the one-phase Stefan problem. In *2016 American Control Conference (ACC)*, pages 2548–2553, 2016.

[58] Erwin Kreyszig. *Introductory functional analysis with applications*, volume 17. John Wiley & Sons, (1991).

[59] Miroslav Krstić, Antranik A Siranosian, and Andrey Smyshlyaev. Backstepping boundary controllers and observers for the slender Timoshenko beam: Part I-Design. In *2006 American Control Conference*, pages 2412–2417. IEEE, 2006.

[60] Miroslav Krstić and Andrey Smyshlyaev. Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays. *Systems & Control Letters*, 57(9):750–758, 2008.

[61] Miroslav Krstić and Andrey Smyshlyaev. *Boundary control of PDEs: A course on backstepping designs*. SIAM, 2008.

[62] Peter Kunkel. *Differential-algebraic equations: Analysis and numerical solution*, volume 2. European Mathematical Society, 2006.

[63] Peter Kunkel and Volker Mehrmann. The linear quadratic optimal control problem for linear descriptor systems with variable coefficients. *Mathematics of Control, Signals and Systems*, 10(3):247–264, 1997.

[64] Galina A Kurina and Roswitha März. On linear-quadratic optimal control problems for time-varying descriptor systems. *SIAM Journal on Control and Optimization*, 42(6):2062–2077, 2004.

[65] Galina A Kurina and Roswitha März. Feedback solutions of optimal control problems with DAE constraints. *SIAM journal on control and optimization*, 46(4):1277–1298, 2007.

[66] René Lamour, Roswitha März, and Caren Tischendorf. *Differential-algebraic equations: A projector based analysis*. Springer Science & Business Media, 2013.

[67] Irena Lasiecka and Roberto Triggiani. *Control theory for partial differential equations: Continuous and approximation theories*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2000.

[68] Frank L Lewis. Optimal control for singular systems. In *1985 24th IEEE Conference on Decision and Control*, pages 266–272. IEEE, 1985.

[69] Bin Li. Global existence and decay estimates of solutions of a parabolic–elliptic–parabolic system for ion transport networks. *Results in Mathematics*, 75(2):45, 2020.

[70] Ping Lin. A sequential regularization method for time-dependent incompressible Navier–Stokes equations. *SIAM Journal on Numerical Analysis*, 34(3):1051–1071, 1997.

[71] Bai-Nan Liu, Driss Boutat, and Da-Yan Liu. Backstepping observer-based output feedback control for a class of coupled parabolic PDEs with different diffusions. *Systems & Control Letters*, 97:61–69, 2016.

[72] Wenfried Lucht and Karl Strehmel. Discretization based indices for semilinear partial differential-algebraic equations. *Applied Numerical Mathematics*, 28(2-4):371–386, 1998.

[73] Wenfried Lucht and Karl Strehmel. *Sensitivity of solutions of linear PDAEs with respect to perturbations*. CiteSeer, 2000.

[74] Harleigh C Marsh, Mark Karpenko, and Qi Gong. A pseudospectral approach to high index DAE optimal control problems. *ArXiv preprint arXiv:1811.12582*, 2018.

[75] Björn Martens and Matthias Gerdts. Error analysis for the implicit Euler discretization of linear-quadratic control problems with higher-index DAEs and bang–bang solutions. In *Progress in Differential-Algebraic Equations II*, pages 291–330. Springer, 2020.

[76] Wade S Martinson and Paul I Barton. A differentiation index for partial differential-algebraic equations. *SIAM Journal on Scientific Computing*, 21(6):2295–2315, 2000.

[77] Wade S Martinson and Paul I Barton. Index and characteristic analysis of linear PDAE systems. *SIAM Journal on Scientific Computing*, 24(3):905–923, 2003.

[78] Volker Mehrmann. Index concepts for differential-algebraic equations. *Encyclopedia of Applied and Computational Mathematics*, 1:676–681, 2015.

[79] Volker Mehrmann and Hans Zwart. Abstract dissipative Hamiltonian differential-algebraic equations are everywhere. *ArXiv preprint arXiv:2311.03091*, 2023.

[80] Volker Ludwig Mehrmann. *The autonomous linear quadratic control problem: Theory and numerical solution*. Springer, 1991.

[81] Hannes Meinlschmidt, Christian Meyer, and Joachim Rehberg. Optimal control of the thermistor problem in three spatial dimensions, part 1: Existence of optimal solutions. *SIAM Journal on Control and Optimization*, 55(5):2876–2904, 2017.

[82] Michael S Mock. An initial value problem from semiconductor device theory. *SIAM Journal on Mathematical Analysis*, 5(4):597–612, 1974.

[83] Kirsten Morris. State feedback and estimation of well-posed systems. *Mathematics of Control, Signals and Systems*, 7(4):351–388, 1994.

[84] Kirsten Morris. *Introduction to feedback control*. Academic Press, Inc., 2000.

[85] Kirsten Morris. *Controller design for distributed parameter systems*. Springer, 2020.

[86] Kirsten Morris and Ahmet Ozkan Ozer. Modeling and stabilizability of voltage-actuated piezoelectric beams with magnetic effects. *SIAM Journal on Control and Optimization*, 52(4):2371–2398, 2014.

[87] Mihaela Negreanu, Jose Ignacio Tello, and Antonio Manuel Vargas. On a parabolic-elliptic chemotaxis system with periodic asymptotic behavior. *Mathematical Methods in the Applied Sciences*, 42(4):1210–1226, 2019.

[88] Frank Neubrander. Integrated semigroups and their applications to the abstract Cauchy problem. *Pacific Journal of Mathematics*, 135(1):111–155, 1988.

[89] Hugo Parada, Eduardo Cerpa, and Kirsten Morris. Feedback control of an unstable parabolic-elliptic system with input delay. *preprint*, 2020.

[90] Amnon Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44. Springer Science & Business Media, 2012.

[91] Mihály Petreczky and Sergiy Zhuk. Solutions of differential–algebraic equations as outputs of LTI systems: Application to LQ control problems. *Automatica*, 84:166–173, 2017.

[92] Thierry Poinsot. Prediction and control of combustion instabilities in real engines. *Proceedings of the Combustion Institute*, 36(1):1–28, 2017.

[93] Andrey A Polezhaev, Ruslan A Pashkov, Alexey I Lobanov, and Igor B Petrov. Spatial patterns formed by chemotactic bacteria Escherichia coli. *International Journal of Developmental Biology*, 50(2-3):309–314, 2003.

[94] Radosław Pytlak. Optimal control of differential–algebraic equations of higher index, part 1: First-order approximations. *Journal of Optimization Theory and Applications*, 134:61–75, 2007.

[95] Radosław Pytlak. Optimal control of differential–algebraic equations of higher index, part 2: Necessary optimality conditions. *Journal of Optimization Theory and Applications*, 134:77–90, 2007.

[96] Radosław Pytlak and Damian Suski. On solving hybrid optimal control problems with higher-index DAEs. *Optimization Methods and Software*, 32(4):940–962, 2017.

[97] Radosław Pytlak and Tomasz Zawadzki. On solving optimal control problems with higher index differential-algebraic equations. *Optimization Methods and Software*, 29(5):1139–1162, 2014.

[98] Joachim Rang and Lutz Angermann. Perturbation index of linear partial differential-algebraic equations. *Applied Numerical Mathematics*, 53(2-4):437–456, 2005.

[99] Timo Reis. Controllability and observability of infinite-dimensional descriptor systems. *IEEE Transactions on Automatic Control*, 53(4):929–940, 2008.

[100] Timo Reis, Sara Grundel, and Sebastian Schöps. *Progress in Differential-Algebraic Equations II*. Springer, 2020.

[101] Timo Reis and Caren Tischendorf. Frequency domain methods and decoupling of linear infinite-dimensional differential-algebraic systems. *Journal of Evolution Equations*, 5(3):357–385, 2005.

[102] Timo Reis and Matthias Voigt. Linear-quadratic optimal control of differential-algebraic systems: The infinite-time horizon problem with zero terminal state. *SIAM Journal on Control and Optimization*, 57(3):1567–1596, 2019.

[103] James Robinson and Paul Glendinning. *From finite to infinite-dimensional dynamical systems. Proceedings of the NATO Advanced Study Institute on From Finite to Infinite-Dimensional Dynamic Systems*, volume 19. Springer Science & Business Media, 2001.

[104] Riccardo Sacco, Giovanna Guidoboni, and Aurelio Giancarlo Mauri. *A comprehensive physically based approach to modeling in bioengineering and life sciences*. Academic press, 2019.

[105] Yuanchao Si, Chengkang Xie, Zhiyuan Zhen, and Ailiang Zhao. Local stabilization of coupled nonlinear parabolic equations by boundary control. *Journal of the Franklin Institute*, 355(13):5592–5612, 2018.

[106] B Simeon and Maik Arnold. Coupling DAEs and PDEs for simulating the interaction of pantograph and catenary. *Mathematical and Computer Modelling of Dynamical Systems*, 6(2):129–144, 2000.

[107] Andrey Smyshlyaev and Miroslav Krstić. Closed-form boundary state feedbacks for a class of 1-D partial integro-differential equations. *IEEE Transactions on Automatic Control*, 49(12):2185–2202, 2004.

[108] Andrey Smyshlyaev and Miroslav Krstić. Backstepping observers for a class of parabolic PDEs. *Systems & Control Letters*, 54(7):613–625, 2005.

[109] Kati Sternberg, Kurt Chudej, and Hans Josef Pesch. Suboptimal control of a 2D molten carbonate fuel cell PDAE model. *Mathematical and Computer Modelling of Dynamical Systems*, 13(5):471–485, 2007.

[110] Tatjana Stykel. Balanced truncation model reduction for semidiscretized Stokes equation. *Linear Algebra and its Applications*, 415(2-3):262–289, 2006.

[111] Georgy A Sviridyuk and Vladimir E Fedorov. *Linear Sobolev type equations and degenerate semigroups of operators*. De Gruyter, 2003.

[112] CA Swanson. Wirtinger's inequality. *SIAM Journal on Mathematical Analysis*, 9(3):484–491, 1978.

[113] Shuxia Tang and Chengkang Xie. State and output feedback boundary control for a coupled PDE–ODE system. *Systems & Control Letters*, 60(8):540–545, 2011.

[114] Youshan Tao and Michael Winkler. Boundedness vs. blow-up in a two-species chemotaxis system with two chemicals. *Discrete & Continuous Dynamical Systems-B*, 20(9):3165–3183, 2015.

[115] Bernd Thaller and Sigrid Thaller. Factorization of degenerate Cauchy problems: The linear case. *Journal of Operator Theory*, pages 121–146, 1996.

[116] Christiane Tretter. *Spectral theory of block operator matrices and applications*. World Scientific, 2008.

[117] Roberto Triggiani. On the stabilizability problem in Banach space. *Journal of Mathematical Analysis and Applications*, 52(3):383–403, 1975.

[118] Sascha Trostorff. Semigroups associated with differential-algebraic equations. In *Conference on Semigroups of Operators: Theory and Applications*, pages 79–94. Springer, 2018.

[119] Sascha Trostorff. Semigroups associated with differential-algebraic equations. In *Semigroups of Operators–Theory and Applications*, pages 79–94. Springer, 2020.

[120] Anna Viktorovna Urazaeva and Vladimir Evgen'evich Fedorov. On the well-posedness of the prediction-control problem for certain systems of equations. *Mathematical Notes*, 85:426–436, 2009.

[121] Rafael Vázquez and Miroslav Krstić. Explicit integral operator feedback for local stabilization of nonlinear thermal convection loop PDEs. *Systems & control letters*, 55(8):624–632, 2006.

[122] Rafael Vázquez and Miroslav Krstić. Boundary control laws for parabolic PDEs with Volterra nonlinearities, part II: Examples. *IFAC Proceedings Volumes*, 40(12):330–335, 2007.

[123] Rafael Vázquez and Miroslav Krstić. A closed-form feedback controller for stabilization of the linearized 2-D Navier–Stokes Poiseuille system. *IEEE Transactions on Automatic Control*, 52(12):2298–2312, 2007.

[124] Rafael Vázquez and Miroslav Krstić. Control of 1-D parabolic PDEs with Volterra nonlinearities, part I: Design. *Automatica*, 44(11):2778–2790, 2008.

[125] Rafael Vázquez and Miroslav Krstić. Boundary observer for output-feedback stabilization of thermal-fluid convection loop. *IEEE Transactions on Control Systems Technology*, 18(4):789–797, 2009.

[126] Rafael Vázquez and Miroslav Krstić. Marcum Q-functions and explicit kernels for stabilization of 2×2 linear hyperbolic systems with constant coefficients. *Systems & Control Letters*, 68:33–42, 2014.

[127] Rafael Vázquez and Miroslav Krstić. Boundary control of coupled reaction-advection-diffusion systems with spatially-varying coefficients. *IEEE Transactions on Automatic Control*, 62:2026–2033, 2016.

[128] Rafael Vázquez and Miroslav Krstić. Explicit output-feedback boundary control of reaction-diffusion PDEs on arbitrary-dimensional balls. *ESAIM: Control, Optimisation and Calculus of Variations*, 22(4):1078–1096, 2016.

[129] Rafael Vázquez, Eugenio Schuster, and Miroslav Krstić. A closed-form observer for the 3D inductionless MHD and Navier-Stokes channel flow. In *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 739–746. IEEE, 2006.

[130] Yvonne Wagner. A further index concept for linear PDAEs of hyperbolic type. *Mathematics and computers in simulation*, 53(4-6):287–291, 2000.

[131] Yulan Wang, Michael Winkler, and Zhaoyin Xiang. The fast signal diffusion limit in Keller–Segel (-fluid) systems. *Calculus of Variations and Partial Differential Equations*, 58(6):196, 2019.

[132] Jinbiao Wu, Jinchao Xu, and Henghui Zou. On the well-posedness of a mathematical model for lithium-ion battery systems. *Methods and Applications of Analysis*, 13(3):275–298, 2006.

APPENDICES

Appendix A

Functional analysis

Here, the reader is presumed to have a foundational understanding of normed linear spaces and Hilbert space theory. The results in this section are well-established concepts in functional analysis. We refer the interested reader to [17, 58] for more information. To set notation, let $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. We begin by defining linear operators.

Definition A.1. A linear operator A from $\mathcal{X}$ to $\mathcal{Y}$ is a map $A : D(A) \subset \mathcal{X} \to \mathcal{Y}$, such that $D(A)$ is a subspace of $\mathcal{X}$, and for all $x_1, x_2 \in D(A)$ and any scalar α, it holds that

$$\begin{aligned} A(x_1 + x_2) &= Ax_1 + Ax_2, \\ A(\alpha x) &= \alpha Ax. \end{aligned}$$

Definition A.2. A linear operator $A : \mathcal{X} \to \mathcal{Y}$ is a bounded linear operator if a positive constant $c > 0$ exists satisfying

$$\|Ax\|_{\mathcal{Y}} \leq c\|x\|_{\mathcal{X}}, \quad \forall x \in \mathcal{X}.$$

Definition A.3. If $\mathcal{X}$ and $\mathcal{Y}$ are normed linear spaces, we define the normed linear space $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ to be the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. If $\mathcal{X} = \mathcal{Y}$, we denote $\mathcal{L}(\mathcal{X}, \mathcal{X})$ by $\mathcal{L}(\mathcal{X})$.

Theorem A.1 (Contraction Mapping Theorem)**.** Let M be a closed subset of $\mathcal{X}$, and let A be a mapping from $\mathcal{X}$ to $\mathcal{X}$, $n \in \mathbb{N}$, and $\alpha < 1$. Suppose that A satisfies

$$\|A^n(x_1) - A^n(x_2)\| \leq \alpha\|x_1 - x_2\|,$$

for all $x_1, x_2 \in M$. Then there exists a unique $x^* \in M$ such that $A(x^*) = x^*$. The point x^* is called the fixed point of A. Furthermore, for any $x_0 \in A$, the sequence $\{x_n, n \geq 1\}$ defined by $x_n := A^n(x_0)$ converges to x^* as $n \to \infty$.

Proposition A.1 (Young's Inequality)**.** [103, Lemma 5.40]
If $a, b \geq 0$, then

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2,$$

where $\varepsilon > 0$ is any positive constant in $\mathbb{R}$.

Proposition A.2 (Cauchy-Schwarz Inequality)**.** Let $f(x)$ and $g(x)$ be two functions such that $f(x),\ g(x) \in L_2(0,1)$, then

$$\left(\int_0^1 f(x)g(x)\,dx\right)^2 \leq \int_0^1 f(x)^2\,dx \cdot \int_0^1 g(x)^2\,dx.$$