Classification of Nilpotent Lie Algebras of Dimension 7
(Over Algebraically Closed Fields and $\mathbf{R}$)

by

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Abstract

This thesis is concerned with the classification of 7-dimensional nilpotent Lie algebras. Skjelbred and Sund have published in 1977 their method of constructing all nilpotent Lie algebras of dimension n given those algebras of dimension $< n$, and their automorphism groups. By using this method, we construct all nonisomorphic 7-dimensional nilpotent Lie algebras in the following two cases: (1) over an algebraically closed field of arbitrary characteristic except 2; (2) over the real field $\mathbf{R}$.

We have compared our lists with three of the most recent lists (those of Seeley, Ancochea-Goze, and Romdhani). While our list in case (1) over $\mathbf{C}$ differs greatly from that of Ancochea-Goze, which contains too many errors to be usable, it agrees with that of Seeley apart from a few corrections that should be made in his list, Our list in case (2) over $\mathbf{R}$ contains all the algebras on Romdhani's list, which omits many algebras.

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Contents

Introduction

This thesis is concerned with the classification of 7-dimensional nilpotent Lie algebras. Skjelbred and Sund have published in 1977 their method of constructing all nilpotent Lie algebras of dimension n given those algebras of dimension $< n$, and their automorphism groups. By using this method, we construct all nonisomorphic 7-dimensional nilpotent Lie algebras in the following two cases: (1) over an algebraically closed field of arbitrary characteristic except 2; (2) over the real field $\mathbf{R}$. Our lists are given in Chapter 4.

Many attempts have been made on this topic, and a number of lists have been published. To mention just a few: The earliest list is given by Umlauf (1891) [37] in dimensions ≤ 6 over complex field. Later on Dixmier (1958) [8] gives a complete list in dimensions ≤ 5 over a commutative field.

In dimension 6, there are various lists obtained by Morozov (1958, over a field of characteristic 0) [20], Shedler (1964, over any field) [34],Vergne (1966, over $\mathbf{C}$) [38], Skjelbred and Sund (1978, over $\mathbf{R}$) [36], Beck and Kolman (1981, over $\mathbf{R}$) [3]. Nielsen (1983) [22] compares the tables of Morozov, Vergne, Skjelbred and Sund, and Umlauf and gives for the first time a complete and nonredundant list for nilpotent Lie algebras of dimension 6 over the real field.

In dimension 7, there are also several lists available: Safiullina (1964, over $\mathbf{C}$) [26], [27], Romdhani (1985, over $\mathbf{R}$ and $\mathbf{C}$) [24] [25], Seeley (1988, over $\mathbf{C}$) [31], Ancochea and Goze (1989, over $\mathbf{C}$) [2]. The lists above are obtained using different invariants. By introducing a new invariant – the weight system, Carles (1989) [6] compares the lists of Safiullina, Romdhani and Seeley, and has identified omissions and some mistakes in all of them. Later on in 1993, basing on his own thesis, by incorporating all the previous results, Seeley [33] published his list over $\mathbf{C}$.

There are also other partial classifications concerning some particular properties of nilpotent Lie algebras. Among them are: Favre (1973) [10] for nilpotent Lie algebras of maximal rank; Scheuneman (1967) [30], Gauger (1973) [11] and Revoy (1980) [23] for two-step nilpotent algebras; Ancochea and Goze (1988) [1] for filiform Lie algebras.

Various tactics have been implemented. Morozov's classification depends heavily on the property that a nilpotent Lie algebra of dimension n contains a maximal Abelian ideal of dimension $m \geq 1/2((8n+1)^{1/2} - 1)$ and a classification of the representations by nilpotent transformations of a low dimensional Lie algebra. Safiullina's list is obtained by using this approach. Magnin (1986) [18] introduces a different approach, to enlarge a smaller algebra by adjoining a derivation. He uses this method to construct all nilpotent Lie algebras of dimension ≤ 7 having a fixed Lie algebra of codimension 1, and also obtains among others results, a new classification of 6-dimensional nilpotent Lie algebras over **R** (same as Morozov's). For algebraically closed fields, Favre [10] and Gauger [11] give another method by regarding all nilpotent Lie algebras as quotients of some "free nilpotent Lie algebras". Later Santharoubane (1979) [28][29] further generalizes this idea and establishes a link between nilpotent Lie algebras and Kac-Moody Lie algebras. Skjelbred and Sund (1978) [36] reduce the classification of nilpotent algebras in a given dimension to the study of orbits under the action of a group on the space of second degree cohomology of a smaller Lie algebra with coefficients in a trivial module. Seeley assumes knowledge of algebras in dimensions less than seven, and considers the upper central series dimensions of a nilpotent algebra as an invariant, which are usually shared by many non-isomorphic algebras. So he also identifies some further invariants for each typical upper central series dimensions in order to sort out various possibilities and resorts to many kinds of techniques trying to get all the algebras without redundancy. So essentially, we might say, and to put in his own words, Seeley obtains his list "without machinery, taking the attitude that no reduction in the amount of hard work would result." ([31], pp. vi).

One phenomenon worth mentioning is: there are only finitely many isomorphism classes of nilpotent Lie algebras of dimension less than or equal to 6, whereas in higher dimensions there are infinite families of pairwise nonisomorphic nilpotent Lie algebras. In dimension 7, each infinite family can be parametrized by a single parameter. Seeley (1992) [32] has tackled the problem of determining the number F_n of parameters needed to classify the laws of n-dimensional complex nilpotent Lie algebras, and comes up with the estimate that $F_{2n+2} \geq n(n-1)(n+4)/6 - 3$. In particular, for dimensions 8 and 10, the number of parameters involved will be respectively ≥ 4 and ≥ 13, which makes it very difficult to give a complete list (as for dimension ≤ 7). Therefore it becomes all the more desirable to have a complete and nonredundant list for 7-dimensional nilpotent Lie algebras.

We use the Skjelbred-Sund method to construct all the 7-dimensional algebras. From our point of view, this is the best method, as it provides a systematic approach to construct all the algebras, as the readers will see in the following chapters. But before our project is carried out, it should be noted that many people think otherwise. In talking about this method, Seeley [31] [33] said "it is difficult to use in practice". Magnin [18] even claimed that "le calcul des orbites présentant des difficultés, elle ne semble pas pouvoir être actuellement utilisée pour la classification des algèbres de dimension 7".

A detailed illustration of this method will be given in Chapter 2. Unlike many of the previous 7-dimensional lists, where "trial and error and good guesswork came into play" ([31], pp. vii), we come up with all the necessary mathematical details that everyone can follow and check — both for completeness and nonredundancy. Naturally we follow Seeley's labelling of algebras by using central series dimensions. There are two reasons: firstly because his list is the most reliable one, and secondly also due to the method we use, which regards all the algebras as central extensions of smaller dimensional Lie algebras.

We have compared our list with that of Seeley over $\mathbb{C}$. It turns out that, although Seeley's list is almost perfect, there are still some errors – some of them Seeley himself has also been aware of. The following four corrections should be made:

- $1,3,7_B$: $[b,c]=g$ should be replaced by $[b,d]=g$, otherwise it is isomorphic to $1,3,7_A$.

- $1,3,4,5,7_H$: Not a Lie algebra, since $\mathrm{Jac}(a,b,c)=[a,[b,c]]+[b,[c,a]]+[c,[a,b]]\neq 0$. Should be deleted.

- $1,4,7_E$: One should impose a further restriction on the parameter: $\xi\neq 0,1$. If $\xi=0$, or 1, the center has dimension 2, and the algebra is isomorphic to $2,4,7_P$.

- $1,3,5,7_S$: One should impose a further restriction on the parameter: $\xi\neq 1$. If $\xi=1$, the center has dimension 2, and the algebra is isomorphic to $2,3,5,7_D$.

These corrections are necessary, as people still refer to Seeley's list without being aware of some of these errors. In a recent paper by Cairns, Jessup and Pitkethly [5] in 1997, they give the Betti numbers of nilpotent Lie algebras of dimensions at most 7, where they also provide the Betti numbers for $1,3,4,5,7_H$, which, according to above, should not be there at all.

We have also compared Seeley's list (as corrected above) with that of Ancochea and Goze's. Unfortunately, Ancochea and Goze's list turns out to contain too many errors to be usable, with a lot omissions, and among those being listed, many of them are not Lie algebras at all, and others occur more than once.

Before our work, only Romdhani [24] [25] has provided a list for the real case. A comparison with his list of real algebras reveals that he has also missed many algebras.

Maple V© plays a decisive role in our classification, and especially in our comparisons with all the other lists. It is totally unimaginable to carry out this project without something like Maple, and we do hope that the readers, while reading through the proofs, will appreciate the power of this interactive computer algebra system, which has been used in the computation of (1) the Jacobi identities; (2) the cocycles; (3) the orbits of normalized cocycles under the automorphism group; (4) the isomorphism between two algebras, and as a special case, the automorphism groups; (5) the derivation algebras; (6) solving all kinds of equations, etc., among many other things.

Now we mention briefly the layout of the thesis.

In Chapter 1, we introduce some of the basic definitions of nilpotent Lie algebras which are used throughout the thesis.

In Chapter 2, we describe the method of Skjelbred and Sund, and include some basic introduction to cohomology theory of nilpotent Lie algebras.

In Chapter 3, we present the list of all six-dimensional nilpotent Lie algebras over an arbitrary algebraically closed field, followed by the proof that the list is complete and nonredundant. Included in the list are the weight system and the generic automorphism for each algebra, as we need all this information for our construction of 7-dimensional nilpotent algebras.

In Chapter 4, we present our lists of all indecomposable 7-dimensional nilpotent Lie algebras over algebraically closed fields of arbitrary characteristic except 2, and also over $\mathbf{R}$.

In Chapter 5, we construct all indecomposable two-step nilpotent Lie algebras (i.e., central extensions of Abelian algebras), both for the real field and for algebraically closed fields.

In Chapter 6, we give the proof for the case when the ground field is algebraically closed of characteristic not 2.

In Chapter 7, we give the proof for the case when the ground field is real.

In Appendix A, we establish the correpondence between our list and Nielsen's list for indecomposable six-dimensional real nilpotent Lie algebras.

In Appendix B, we compare Seeley's (corrected) list with that of Ancochea-Goze's for all the indecomposable 7-dimensional nilpotent Lie algebras over $\mathbf{C}$.

In Appendix C, we compare our list of indecomposable 7-dimensional nilpotent real algebras with that of Romdhani's.

In Appendix D, we give a summary of all the 7-dimensional indecomposable nilpotent Lie algebras as they arise from those of dimensions ≤ 6 in our construction. The readers may easily identify the central quotients of all the seven-dimensional algebras with this list, and locate the details of the corresponding proofs if they wish.

In Appendix E, we provide some of the main Maple programs that have been used in our computation.

Chapter 1

Some Concepts of Lie Algebras

In this chapter we introduce some basic definitions and notations that are used throughout the thesis. Most of them can be found in any standard books on Lie algebras [15] [19].

1.1 Basic Definitions

Definition 1.1 *Let* $\mathfrak{g}$ *be a Lie algebra over a field* $\mathbf{F}$. *Let* $D^0\mathfrak{g} = \mathfrak{g}$, $C^0\mathfrak{g} = \mathfrak{g}$, $C_0\mathfrak{g} = \{0\}$, $D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}]$, $C^{i+1}\mathfrak{g} = [C^i\mathfrak{g}, \mathfrak{g}]$, *and* $C_{i+1}(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, \mathfrak{g}] \subset C_i(\mathfrak{g})\}$ *for any* i. *We call*

$$\mathfrak{g} = D^0\mathfrak{g} \supset D^1\mathfrak{g} \supset \cdots \supset D^k\mathfrak{g} \supset \cdots$$

the derived series of $\mathfrak{g}$,

$$\mathfrak{g} = C^0\mathfrak{g} \supset C^1\mathfrak{g} \supset \cdots \supset C^k\mathfrak{g} \supset \cdots$$

the lower central series of $\mathfrak{g}$, *and*

$$\{0\} = C_0(\mathfrak{g}) \subset C_1(\mathfrak{g}) \subset \cdots \subset C_k(\mathfrak{g}) \subset \cdots$$

the upper central series of $\mathfrak{g}$. *We also call respectively*

$$\dim C^0\mathfrak{g}, \dim C^1\mathfrak{g}, \cdots, \dim C^k\mathfrak{g}, \cdots$$

and

$$\dim C_1(\mathfrak{g}), \dim C_2(\mathfrak{g}), \cdots, \dim C_k(\mathfrak{g}), \cdots$$

the lower central series dimensions of $\mathfrak{g}$ *and the upper central series dimensions of* $\mathfrak{g}$. *We will simply denote them by* $(\dim C^0\mathfrak{g}, \dim C^1\mathfrak{g}, \cdots)$ *and* $(\dim C_1(\mathfrak{g}), \dim C_2(\mathfrak{g}), \cdots)$.

Definition 1.2 *A Lie algebra* $\mathfrak{g}$ *of dimension* n *is called filiform if*

$$\dim C^k\mathfrak{g} = n - k - 1 \quad for \quad k \geq 1.$$

Definition 1.3 *A nilpotent Lie algebra* $\mathfrak{g}$ *is called* *two-step nilpotent* *(or metabelian) if it satisfies* $C^2\mathfrak{g} = \{0\}$.

Definition 1.4 *The Heisenberg algebra* H_p *of dimension* $2p+1$ *is defined by the brackets:*

$$[x_1, x_2] = [x_3, x_4] = \cdots = [x_{2p-1}, x_{2p}] = x_{2p+1},$$

and all other brackets $[x_i, x_j]$ *are* 0*, where* $x_1, \cdots, x_{2p+1}$ *is a basis for* H_p.

Definition 1.5 *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra and* Der $\mathfrak{g}$ *its derivation algebra. The Lie algebra* $\mathfrak{g}$ *is called* *characteristically nilpotent* *if every* $f \in$ Der $\mathfrak{g}$ *is a nilpotent endomorphism of* $\mathfrak{g}$.

Definition 1.6 *Let* $\mathcal{F}$ *be the free Lie algebra on g-generators* $y_1, \cdots, y_g$ *([15], p.167). Let* $\mathcal{F}_n$ *denote the subspace of* $\mathcal{F}$ *generated by all elements of the type* $[y_{i_1}, y_{i_2}, \cdots, y_{i_{n-1}}, y_{i_n}] = [\cdots, [[y_{i_1}, y_{i_2}], \cdots, y_{i_{n-1}}], y_{i_n}]$ *where* $i_j \in \{1, 2, \cdots, g\}$. $\mathcal{F}$ *is graded with* $\mathcal{F}_n$ *as the homogeneous componet of degree n, and furthermore* $\mathcal{F}^n = \bigoplus_{j \geq n} \mathcal{F}_j$. *We call* $N(l, g) = \mathcal{F}/\mathcal{F}^{l+1}$ *a free nilpotent Lie algebra of class l on g generators.*

1.2 Weight Systems and Decomposability

Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $\mathbf{F}$ of characteristic 0. Denote by Der $\mathfrak{g}$ and Aut $\mathfrak{g}$ its Lie algebra of derivations and the group of automorphisms. Let T by a commutative subalgebra of Der $\mathfrak{g}$ consisting of semi-simple endomorphisms. T is called torus on $\mathfrak{g}$. A torus T on $\mathfrak{g}$ is called maximal if it is not contained in any other torus of larger dimension. A torus T on $\mathfrak{g}$ defines naturally a representation in $\mathfrak{g}$, and the elements of T can be diagonalized simultaneously. Therefore $\mathfrak{g}$ can be decomposed as a direct sum of weight spaces, i.e.,

$$\mathfrak{g} = \oplus_{\alpha \in T^*} \mathfrak{g}^\alpha$$

where T^* is the dual space of T, and

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} | t(x) = \alpha(t)x, \forall t \in T\}.$$

Over algebraically closed fields, the conjugacy theorem of Mostow [21] shows that the weight system associated with a maximal torus is invariant up to a permutation by isomorphim. We define the rank of $\mathfrak{g}$ to be the common dimension of maximal tori over $\mathfrak{g}$, and denote it by $\mathrm{rank}(\mathfrak{g})$.

Let T be a maximal torus on $\mathfrak{g}$, and

$$R(T) = \{\alpha \in T^* | \dim \mathfrak{g}^\alpha > 0\}.$$

Let $W(T)$ be the set of all the pairs $(\alpha, d\alpha)$, where $\alpha \in R(T)$ and $d\alpha$ the multiplicity of α, that is,

$$W(T) = \{(\alpha, d\alpha) | \alpha \in R(T), \quad d\alpha = \dim g^\alpha\}.$$

Definition 1.7 *The set $W(T)$ is called the weight system associated to $\mathfrak{g}$, or we may say that a weight system is just the set of weights together with their multiplicities.*

Definition 1.8 *Two weight systems $W(T)$ and $W'(T')$ are said to be equivalent if $\dim T = \dim T'$ and the linear representation of T in $\mathfrak{g}$ is equivalent to that of T' in $\mathfrak{g}'$.*

Theorem 1.1 *[10] The equivalence class of a weight system of a Lie algebra $\mathfrak{g}$ is an invariant of $\mathfrak{g}$.*

Let B be the set of all the weights corresponding to $\mathfrak{g}/C^2\mathfrak{g}$.

Definition 1.9 *A path in $R(T)$ is a sequence $\beta_1, \cdots, \beta_l$ of points in $R(T)$ such that $\beta_{i+1} - \beta_i$ or $\beta_i - \beta_{i+1}$ are in B for all $1 \leq i \leq l-1$. A connected component of $R(T)$ is an arcwise connected component.*

Theorem 1.2 *[10, 17] Let* $\mathfrak{g}$ *be a nilpotent Lie algebra.*

1). If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ *(direct ideal sum), then* $\text{rank}(\mathfrak{g}) = \text{rank}(\mathfrak{g}_1) + \text{rank}(\mathfrak{g}_2)$*;*

2). If $\mathfrak{g}$ *is indecomposable, then* $R(T)$ *is connected;*

3). If $R_1, \cdots, R_l$ *are the connected components of* $R(T)$ *and let* $\mathfrak{g}_i = \oplus_{\alpha \in R_i} \mathfrak{g}^\alpha$*, then each* $\mathfrak{g}_i$ *is an ideal of* $\mathfrak{g}$*, and* $\mathfrak{g}$ *is a direct product of* $\mathfrak{g}_i$*:* $\mathfrak{g} = \prod_{1 \le i \le l} \mathfrak{g}_i$*. Furthermore,* $\mathfrak{g}_i$ *is indecomposable, i.e. it cannot be decomposed into the product of two nonzero Lie algebras.*

Therefore we may use the weight system to determine the decomposability of an algebra over an algebraically closed field of characteristic 0. Using Carles's work on weight systems for nilpotent Lie algebras [6], this has been made quite straightforward.

Chapter 2

The Skjelbred-Sund Method

2.1 Cohomology of Nilpotent Lie Algebras

We will introduce some basic definitions and properties of the cohomology of nilpotent Lie algebras in this section. Readers may refer to [7] [13] [15] for details.

Let $\mathfrak{g}$ be a Lie algebra, $\mathbf{F}$ a field, and consider $\mathbf{F}^k$ as a trivial $\mathfrak{g}$-module.

Definition 2.1 *A mapping $f : \mathfrak{g} \times \cdots \times \mathfrak{g}$(i times) $\rightarrow \mathbf{F}^k$ is called an i-linear mapping if f sends an i-tuple $(x_1, \cdots, x_i)$, $x_q \in \mathfrak{g}$, into $f(x_1, \cdots, x_i) \in \mathbf{F}^k$ in such a way that for fixed values of $x_1, \cdots, x_{q-1}, x_{q+1}, \cdots, x_i$ the mapping $x_q \rightarrow f(x_1, \cdots, x_i)$ is a linear mapping of $\mathfrak{g}$ into $\mathbf{F}^k$.*

Definition 2.2 *An i-linear mapping is skew symmetric or alternating if f takes value 0 when any two of the x_q are the same.*

Definition 2.3 *An i-dimensional $\mathbf{F}^k$-cochain (or simply "an i-cochain") for $\mathfrak{g}$ is a skew symmetric i-linear mapping of $\mathfrak{g} \times \cdots \times \mathfrak{g}$ (i times) into $\mathbf{F}^k$.*

The set $C^i(\mathfrak{g}, \mathbf{F}^k)$ of all i-cochains is a vector space relative to the usual definitions of addition and scalar multiplication of functions.

Definition 2.4 *If f is an i-cochain, $i \geq 1$, f determines an $(i+1)$-dimensional cochain df, called coboundary of f, defined by the formula*

$$(df)(x_1, \cdots, x_{i+1}) = \sum_{m<l} (-1)^{m+l} f(x_1, \cdots, \hat{x}_m, \cdots, \hat{x}_l, \cdots, x_{i+1}, [x_m, x_l]),$$

where the $\hat{}$ over an argument means that this argument is omitted. If $i = 0$, we set $df = 0$. d maps $C^i(\mathfrak{g}, \mathbf{F}^k)$ linearly into $C^{i+1}(\mathfrak{g}, \mathbf{F}^k)$ and is called the coboundary operator.

Definition 2.5 *An i-cochain f is called a* cocycle *if $df = 0$ and a* coboundary *if $f = dg$ for some $(i-1)$-cochain g.*

The set $Z^i(\mathfrak{g}, \mathbf{F}^k)$ of i-cocycles is the kernel of the homomorphism d of C^i into C^{i+1}, so it is a subspace of C^i. Similarly, the set $B^i(\mathfrak{g}, \mathbf{F}^k)$ of i-coboundaries is a subspace of C^i since it is the image under d of C^{i-1}. When $i = 0$, we define $B^0(\mathfrak{g}, \mathbf{F}^k) = 0$. Due to a well-known result in cohomology theory, i.e., $d^2 = 0$, the coboundaries form a subspace of the cocycles.

Definition 2.6 *We call the factor space, denoted by $H^i(\mathfrak{g}, \mathbf{F}^k) = Z^i(\mathfrak{g}, \mathbf{F}^k)/B^i(\mathfrak{g}, \mathbf{F}^k)$,* the i-dimensional cohomology group of $\mathfrak{g}$ *(with coefficients in $\mathbf{F}^k$).*

Now we shall look at some properties of $H^i(\mathfrak{g}, \mathbf{F})$ for $i \leq 2$. For $i = 0$ we have $Z^0 = C^0 = \mathbf{F}$ and $B^0 = 0$ so that

$$H^0(\mathfrak{g}, \mathbf{F}) = \mathbf{F}.$$

For $i = 1$ we have $B^1 = 0$ so that $H^1 = Z^1$. If $f \in C^1(\mathfrak{g}, \mathbf{F})$, then $(df)(x_1, x_2) = -f([x_1, x_2])$. Therefore f is a 1-cocycle if and only if it vanishes on $[\mathfrak{g}, \mathfrak{g}]$. Hence

Lemma 2.1 *$H^1(\mathfrak{g}, \mathbf{F})$ is isomorphic to the dual space of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.*

For $i = 2$, if $f \in C^2(\mathfrak{g}, \mathbf{F})$, then

$$(df)(x_1, x_2, x_3) = -f(x_3, [x_1, x_2]) + f(x_2, [x_1, x_3]) - f(x_1, [x_2, x_3]).$$

Therefore, $df = 0$ or $f \in Z^2$ if and only if the Jacobi identity holds:

$$\mathrm{Jac}(x_1, x_2, x_3) = f([x_1, x_2], x_3) + f([x_2, x_3], x_1) + f([x_3, x_1], x_2) = 0.$$

Let $B^2(\mathfrak{g}, \mathbf{F})$ be the set of all 2-coboundaries, i.e. elements f for which there exists $g \in \mathrm{Hom}(\mathfrak{g}, \mathbf{F})$ such that $f(x, y) = g([x, y])$ for any $x, y \in \mathfrak{g}$. An immediate consequence follows:

Lemma 2.2 $\dim B^2(\mathfrak{g}, \mathbf{F}) = \dim[\mathfrak{g}, \mathfrak{g}]$.

2.2 The Method

In this part, we will explain the method described by Skjelbred and Sund [35] for constructing nilpotent Lie algebras of fixed finite dimension from those of smaller dimensions.

Firstly, we need to introduce some notations and definitions ([28, 29]).

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbf{F}$. For each $B \in C^2(\mathfrak{g}, \mathbf{F}^k)$ and $\phi \in \mathrm{Aut}\ \mathfrak{g}$, the automorphism group of $\mathfrak{g}$, we define $B^\phi \in C^2(\mathfrak{g}, \mathbf{F}^k)$ by $B^\phi(x, y) = B(\phi x, \phi y)$ for any $x, y \in \mathfrak{g}$.

Since $Z^2(\mathfrak{g}, \mathbf{F}^k)$ and $B^2(\mathfrak{g}, \mathbf{F}^k)$ are invariant under this action, we can define the action of Aut $\mathfrak{g}$ on $H^2(\mathfrak{g}, \mathbf{F}^k)$ as well. For $B \in Z^2(\mathfrak{g}, \mathbf{F})$, we denote $\bar{B}$ as its corresponding element in $H^2(\mathfrak{g}, \mathbf{F})$, then we may write the action of Aut $\mathfrak{g}$ on $\widetilde{B}$ as $\widetilde{B}^\phi = \widetilde{B^\phi}$.

For $B \in C^2(\mathfrak{g}, \mathbf{F}^k)$, the <u>kernel</u> of B will be defined as $\mathfrak{g}_B^\perp$, with

$$\mathfrak{g}_B^\perp = \{x \in \mathfrak{g} : B(x, \mathfrak{g}) = 0\}.$$

Note that

$$C^2(\mathfrak{g}, \mathbf{F}^k) = C^2(\mathfrak{g}, \mathbf{F})^k, \;\; H^2(\mathfrak{g}, \mathbf{F}^k) = H^2(\mathfrak{g}, \mathbf{F})^k.$$

So for any $B \in C^2(\mathfrak{g}, \mathbf{F}^k)$, we may write

$$B = (B_1, \cdots, B_k) \in C^2(\mathfrak{g}, \mathbf{F})^k,$$

and we have $\mathfrak{g}_B^\perp = \mathfrak{g}_{B_1}^\perp \bigcap \cdots \bigcap \mathfrak{g}_{B_k}^\perp$.

Define $G_k(H^2(\mathfrak{g}, \mathbf{F}))$ to be the Grassmannian of subspaces of dimension k in $H^2(\mathfrak{g}, \mathbf{F})$. There is a natural action of Aut $\mathfrak{g}$ on this Grassmannian. Let $\widetilde{B}_1\mathbf{F} \oplus \cdots \oplus \widetilde{B}_k\mathbf{F} \in G_k(H^2(\mathfrak{g}, \mathbf{F})$, then $\phi(\widetilde{B}_1\mathbf{F} \oplus \cdots \oplus \widetilde{B}_k\mathbf{F}) = \widetilde{B_1^\phi}\mathbf{F} \oplus \cdots \oplus \widetilde{B_k^\phi}\mathbf{F}$. It is well-defined ([28, 29]).

Denote the center of $\mathfrak{g}$ by $Z(\mathfrak{g})$, and if $\widetilde{B}_1\mathbf{F} \oplus \cdots \oplus \widetilde{B}_k\mathbf{F} \in G_k(H^2(\mathfrak{g}, \mathbf{F})$, write $B = (B_1, \cdots, B_k)$. Then

$$U_k(\mathfrak{g}) = \{\widetilde{B}_1\mathbf{F} \oplus \cdots \oplus \widetilde{B}_k\mathbf{F} \in G_k(H^2(\mathfrak{g}, \mathbf{F})) : \mathfrak{g}_B^\perp \bigcap Z(\mathfrak{g}) = 0\}$$

is well-defined, and is also Aut $\mathfrak{g}$ stable ([28, 29]).

Let $U_k(\mathfrak{g})/\text{Aut } \mathfrak{g}$ be the set of (Aut $\mathfrak{g}$)-orbits of $U_k(\mathfrak{g})$.

Theorem 2.1 *[35] Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbf{F}$. The isomorphism classes of Lie algebras $\widetilde{\mathfrak{g}}$ with center $\widetilde{\mathfrak{z}}$ of dimension k, $\widetilde{\mathfrak{g}}/\widetilde{\mathfrak{z}} \cong \mathfrak{g}$, and without Abelian direct factors, are in bijective correspondence with the elements in $U_k(\mathfrak{g})/\text{Aut } \mathfrak{g}$.*

By this theorem, we may construct all the nilpotent Lie algebras of dimension n, given those algebras of dimension less than n, by central extension.

We carry out the procedure for constructing 6 and 7-dimensional nilpotent Lie algebras in the following way:

(1) For a given algebra of smaller dimension, we list at first its center (or the generators of its center), to help us identify the 2-cocycles satifying $\mathfrak{g}_B^\perp \bigcap Z(\mathfrak{g}) = 0$.

(2) We also list its derived algebra (or the generators of the derived algebra), which is needed in computing the coboundaries $B^2(\mathfrak{g}, \mathbf{F})$.

(3) Then we compute all the 2-cocycles $Z^2(\mathfrak{g}, \mathbf{F})$. For each fixed algebra $\mathfrak{g}$ with given base $\{x_1, x_2, \cdots, x_n\}$, we may represent a 2-cocycle B by a skew symmetric matrix $B =$

$\sum_{1\leq i<j\leq n} C_{ij}\Delta_{ij}$, where Δ_{ij} is the $n\times n$ matrix with (i,j) element being 1, (j,i) element being -1 and all the others 0. When computing the 2-cocycles, we will just list all the constraints on the elements C_{ij} of the skew symmetric matrix B.

(4) We have $Z^2(\mathfrak{g},\mathbf{F}) = B^2(\mathfrak{g},\mathbf{F})\oplus W$, where W is a subspace of $Z^2(\mathfrak{g},\mathbf{F})$, complementary to $B^2(\mathfrak{g},\mathbf{F})$, and $B^2(\mathfrak{g},\mathbf{F}) = \{df | f\in C^1(\mathfrak{g},\mathbf{F}) = \mathfrak{g}^*\}$ (d is the coboundary operator). One easy way to obtain W is as follows. When a nilpotent Lie algebra $\mathfrak{g}$ of dimension $n = r+s$ has a basis in the form $\{x_1,\cdots,x_r,x_{r+1},\cdots,x_{r+s}\}$, where $\{x_1,\cdots,x_r\}$ are the generators, and $\{x_{r+1},\cdots,x_{r+s}\}$ forms a basis for the derived algebra $[\mathfrak{g},\mathfrak{g}]$, with $x_{r+t} = [x_{i_t},x_{j_t}]$, where $1\leq i_t < j_t < r+t$ and $1\leq t\leq s$.

Consider $C^1(\mathfrak{g},\mathbf{F}) = \mathfrak{g}^*$ generated by the dual basis

$$< f_1,\cdots,f_r,g_1,\cdots,g_s >$$

of

$$< x_1,\cdots,x_r,x_{r+1},\cdots,x_{r+s} > .$$

Then

$$B^2(\mathfrak{g},\mathbf{F}) = \{dh | h\in\mathfrak{g}^*\} =< df_1,\cdots,df_r,dg_1,\cdots,dg_s > .$$

Since $df_i(x,y) = -f_i([x,y]) = 0$, we have $B^2(\mathfrak{g},\mathbf{F}) =< dg_1,\cdots,dg_s >$. Now we have

$$Z^2(\mathfrak{g},\mathbf{F}) =< dg_1,\cdots,dg_s > \oplus W.$$

For $B\in W$, we may assume that $B(x_{i_t},x_{j_t}) = 0$, $t = 1,\cdots,s$, otherwise, if $B(x_{i_t},x_{j_t}) = u_{i_tj_t}\neq 0$, we choose $B + u_{i_tj_t}dg_t$ instead. When we carry out the group action on W, we do it as if it were done in $H^2(\mathfrak{g},\mathbf{F})$, and may identify $H^2(\mathfrak{g},\mathbf{F})$ with W, by calling all the nonzero elements in W the <u>normalized 2-cocycles</u>.

(5) We also list the dimension of the second cohomology group.

(6) For a fixed basis $\{x_1,x_2,\cdots,x_n\}$ of $\mathfrak{g}$, a basis for W in (4) is given, and we will simply regard it as a basis for $H^2(\mathfrak{g},\mathbf{F})$ without causing any confusion.

(7) An arbitrary element in the second cohomology group is given, together with the action of the generic automorphism on it. Keep in mind that, though the elements are chosen from $W\subset Z^2(\mathfrak{g},\mathbf{F})$, we regard them as elements from $H^2(\mathfrak{g},\mathbf{F})$. The group action on these elements is carried out as if they were in $H^2(\mathfrak{g},\mathbf{F})$.

(8) We determine all the representatives of the orbits in the Grassmanian $G_k(H^2(\mathfrak{g},\mathbf{F}))$ under the action of the automorphism group that satisfy the condition mentioned in (1).

(9) With the representatives obtained in (8), we give the list of nonisomorphic central extension algebras of $\mathfrak{g}$ without Abelian factors, i.e., if B is a representative obtained, then we can define a Lie algebra structure on $\mathfrak{g}(B) = \mathfrak{g}\oplus\mathbf{F}^k$ by letting

$$[(x,u),(y,v)] = ([x,y],B(x,y)).$$

We also have the following theorem describing the automorphism group of the new algebra $\mathfrak{g}(B) = \mathfrak{g} \oplus \mathbf{F}^k$ as obtained by the above method through a 2-cocycle B from $\mathfrak{g}$:

Theorem 2.2 *[35] Let $\mathfrak{g}$ be a nilpotent Lie algebra, and $\alpha_0 \in \mathrm{Aut}\ \mathfrak{g}$. Let $B \in H^2(\mathfrak{g}, \mathbf{F}^k)$ and $\mathfrak{g}_B^{\perp} \bigcap Z(\mathfrak{g}) = 0$. The the automorphism group* Aut $\mathfrak{g}(B)$ *of the extended algebra $\mathfrak{g}(B)$ consists of all linear operators of the matrix form*

$$\alpha = \begin{pmatrix} \alpha_0 & 0 \\ \phi & \psi \end{pmatrix}, \text{ where } \alpha_0 \in \mathrm{Aut}\ \mathfrak{g},\ \psi \in Gl_k,\ \phi \in \mathrm{Hom}(\mathfrak{g}, \mathbf{F}^k),$$

and

$$B(\alpha_0 X, \alpha_0 Y) = \psi B(X, Y) + \phi[X, Y], \text{ all } X, Y \in \mathfrak{g}.$$

This is a very useful theorem, which will be used in our computation of the automorphism groups.

We also like to point out that, from the method we described above, it is possible to get decomposable Lie algebras (without Abelian factors, but could be the product of two or more indecomposable nilpotent Lie algebras) by central extensions.

Fortunately, we have the following lemma by Seeley [31] [33]:

Lemma 2.3 *In a decomposition of a finite-dimensional Lie algebra as a direct sum of indecomposable ideals, the isomorphism classes of the ideals are unique. If $L = A_1 \oplus \cdots \oplus A_r$ and $L = C_1 \oplus \cdots \oplus C_s$ are two such decompositions, then $r = s$; after reordering the indices the derived parts $D^1(A_i)$ and $D^1(C_i)$ are equal, $A_i \cong C_i$, and a set of of generators for A_i equals a set of generators for C_i modulo adding to each generator a vector in $Z(L)$.*

Seeley has also observed that there are 31 decomposable nilpotent Lie algebras in dimension 7. All except one have an Abelian summand. Therefore it becomes a fairly easy job for us to check the indecomposability — we just need to take care of the exceptional case, which corresponds to the upper central series dimension (257), and can be done through the comparison of the orbits.

2.3 The Examples

We will illustrate the Skjelbred-Sund method through the following 6 examples. For the labelling of the algebras, and also their automorphism groups, please refer to Chapter 3 or 4. We will explain our notations and conventions along the way. Make sure that you read this part first before you dig into the proofs in the subsequent chapters. Please be reminded that whenever we talk about central extensions, we always refer to those extensions that are without Abelian factors.

Example 1 *Find the central extensions without Abelian factors of dimension* 6 *over any algebraically closed field of the algebra* $\mathfrak{g} = N_{5,2,2}$ *with basis* x_i, $1 \leq i \leq 5$, *and nonzero brackets* $[x_1, x_2] = x_4, [x_1, x_4] = [x_2, x_3] = x_5$.

The center of $N_{5,2,2}$ is $\mathbf{F}x_5$, we will simply write later on in our proofs that $Z(\mathfrak{g}) : x_5$.

The derived algebra $[\mathfrak{g}, \mathfrak{g}]$ is generated by x_4, x_5. Later on we will just write "Derived Algebra: x_4, x_5" or $[\mathfrak{g}, \mathfrak{g}] : x_4, x_5$.

Now we need to determine all the 2-cocycles $B = \sum_{1\leq i<j\leq n} C_{ij}\Delta_{ij}$, by using the basis $\{x_1, \cdots, x_5\}$, as described in (3) of Section 2.2. By checking the Jacobi identity, we can easily get the following constraints for B to be a 2-cocycle: $\{C_{25} = C_{35} = C_{45} = 0, C_{34}+C_{15} = 0\}$, and we will write "Cocycle: $C_{25} = C_{35} = C_{45} = 0, C_{34} + C_{15} = 0$" or $Z^2(\mathfrak{g}) : C_{25} = C_{35} = C_{45} = 0, C_{34} + C_{15} = 0$.

Since $[x_1, x_2] = x_4, [x_1, x_4] = x_5$, and the derived algebra has dimension 2, we may normalize 2-cocycles by requiring $B(x_1, x_2) = B(x_1, x_4) = 0$, as described in (4) of Section 2.2, which will give us the following two extra constraints on B: $C_{12} = C_{14} = 0$, and we will write "Normalization: $C_{12} = C_{14} = 0$" or $W(H^2) : C_{12} = C_{14} = 0$.

From the above, it is easy to see that the dimension of $H^2(\mathfrak{g}, \mathbf{F})$ is 4, and we will write "$\dim H^2 : 4$".

Now we can get a basis for W as in (4) of Section 2.2, regarded also as a basis for $H^2(\mathfrak{g}, \mathbf{F})$, and write "Basis: Δ_{13}, $\Delta_{15} - \Delta_{34}, \Delta_{23}, \Delta_{24}$".

In this case, we are considering the 1-dimensional central extensions of $\mathfrak{g}$. We need to find a set of representatives of the orbits of 1-dimensional subspaces of $H^2(\mathfrak{g}, \mathbf{F})$ under the action of the automorphism group Aut $\mathfrak{g}$. With the chosen basis, we may denote an arbitrary element in $H^2(\mathfrak{g}, \mathbf{F})$ by $x := [a, b, c, d] = a\Delta_{13} + b(\Delta_{15} - \Delta_{34}) + c\Delta_{23} + d\Delta_{24}$. When a generic element g in Aut $\mathfrak{g}$ acts on x, we get $g \cdot x = a'\Delta_{13} + b'(\Delta_{15} - \Delta_{34}) + c'\Delta_{23} + d'\Delta_{24}$ mod $B^2(\mathfrak{g}, \mathbf{F})$, We will simply write $a \to a'$, $b \to b'$, $c \to c'$ and $d \to d'$. In this example we have

$a \to aa_{11}^3 + ba_{11}(a_{53} + a_{21}a_{31} + a_{11}a_{41}) + ca_{11}^2a_{21} - da_{11}a_{21}^2$;

$b \to ba_{11}^3a_{22}$;

$c \to ca_{11}^2a_{22} + 2ba_{11}a_{22}a_{31} - 2da_{11}a_{21}a_{22}$;

$d \to da_{11}a_{22}^2 - ba_{11}a_{22}a_{32}$.

As x_5 is in the center, we must have $b \neq 0$ to ensure that the 2-cocycle does not have x_5 in its kernel. Since a_{11} and a_{22} are not 0, b will remain nonzero throughout.

By taking $a_{11} = a_{22} = 1$, $a_{21} = a_{41} = 0$, $a_{32} = d/b$, $a_{53} = -a/b$ (and ensuring at the same time that the matrix of g is nonsingular), we make $a \to 0$, $d \to 0$.

With these new values for coefficients, the above formulae take simpler form:

$a = 0 \to ba_{11}(a_{53} + a_{21}a_{31} + a_{11}a_{41}) + ca_{11}^2a_{21}$;

$b \to ba_{11}^3a_{22}$;

$c \to ca_{11}^2a_{22} + 2ba_{11}a_{22}a_{31}$;

$d = 0 \to -ba_{11}a_{22}a_{32}$.

Now we need to take into consideration the characteristic χ of $\mathbf{F}$.

Case 1: $\chi \neq 2$. Set $a_{11} = a_{22} = 1$, $a_{21} = a_{32} = a_{41} = a_{53} = 0$, $a_{31} = -c/(2b)$, we obtain the representative (1) $[a, b, c, d] = [0, 1, 0, 0]$.

Case 2: $\chi = 2$. We now have $c \to ca_{11}^2a_{22}$. If $c = 0$, then we get the representative (2) $[0, 1, 0, 0]$. If $c \neq 0$, taking $a_{21} = a_{31} = a_{32} = a_{41} = a_{53} = 0$, we have

$$[a, b, c, d] \to [0, ba_{11}^3a_{22}, ca_{11}^2a_{22}, 0].$$

Make $ba_{11}^2a_{22} = ca_{11}^2a_{22}$ by taking $a_{22} = 1$ and $a_{11} = c/b$ to get the representative (3) $[0, 1, 1, 0]$.

(1) and (2) give us the same algebra, denoted by $N_{6,2,3}$ in Chapter 3. (3) gives us another algebra, denoted by (B), which only exists over the field of $\chi = 2$. It can be easily seen that, when $\chi \neq 2$, it is isomorphic to $N_{6,2,3}$. It is obvious that (B) and $N_{6,2,3}$ are not isomorphic when $\chi = 2$, as the corresponding orbits are different.

Therefore the central extensions of $N_{5,2,2}$ of dimension 6 are:

$N_{6,2,3}$:	$[x_1, x_2] = x_4$,	$[x_1, x_4] = x_5$,	$[x_1, x_5] = x_6$,
	$[x_2, x_3] = x_5$,	$[x_3, x_4] = -x_6$.	
(B)	(for $\chi = 2$ only)		
	$[x_1, x_2] = x_4$,	$[x_1, x_4] = x_5$,	$[x_1, x_5] = x_6$,
	$[x_2, x_3] = x_5 + x_6$,	$[x_3, x_4] = -x_6$.	

Example 2 *Find the central extensions without Abelian factors of dimension 7 over an algebraically closed field of $\chi \neq 2$ and the real field $\mathbf{R}$ of the algebra $\mathfrak{g} = N_{5,2,2}$ with basis x_i, $1 \leq i \leq 5$, and nonzero brackets $[x_1, x_2] = x_4, [x_1, x_4] = [x_2, x_3] = x_5$.*

From Example 1, we have

$Z(\mathfrak{g}) : x_5$; $[\mathfrak{g},\mathfrak{g}] : x_4, x_5$; $Z^2(\mathfrak{g}) : C_{25} = C_{35} = C_{45} = 0$, $C_{15} + C_{34} = 0$; $W(H^2) : C_{12} = C_{24} = 0$; $\dim H^2 : 4$; Basis: $\Delta_{13}, \Delta_{15} - \Delta_{34}, \Delta_{23}, \Delta_{24}$.

According to Theorem 2.1, we need to find the representatives of the orbits of 2-dimensional subspaces of $H^2(\mathfrak{g}, \mathbf{F})$, i.e., orbits in $G_2(H^2(\mathfrak{g}, \mathbf{F}))$. Up to a scalar, we can always identify a 2-dimensional subspace with the wedge product of two vectors $A, B \in H^2(\mathfrak{g}, \mathbf{F})$, i.e. $A \wedge B$.

Let $A = [a, b, c, d]$ and $B = [a_1, b_1, c_1, d_1]$ in $H^2(\mathfrak{g}, \mathbf{F})$. As we require that the kernel of (A, B) does not contain any central elements, we have one of $b, b_1 \neq 0$. Therefore we may assume $A = [a, 1, c, d]$, which is always doable – if $b = 0$, then $b_1 \neq 0$, switch A and B so $b \neq 0$ in A, then multiply A by b^{-1} to get the above form for A. Bear in mind that we can multiply any of our vectors by a scalar, as we are dealing essentially with the subspaces, instead of the vectors.

From the discussions in Example 1, as $\chi \neq 2$, a representative for A can be chosen as $A = [0, 1, 0, 0]$.

Once we get A, we may assume $B = [a_1, 0, c_1, d_1]$ because we can replace B by a linear combination $B + \lambda A$. Although the original $B = [a_1, b_1, c_1, d_1]$ is different from the new $B = [a_1, 0, c_1, d_1]$, we still use the same notation B to denote it, same thing for a_1, c_1 and d_1. This is the convention throughout the proofs in our Thesis. The readers will be able to tell the differences, without causing any confusion.

In the following we will mainly discuss the case when the ground field is $\mathbf{R}$. The algebraically closed case can be easily obtained with minor adjustments.

Now to fix A (up to a scalar), we require $a_{31} = a_{32} = 0$ and $a_{53} = -a_{11}a_{41}$.

For B, we have $a_1 \to a_1a_{11}^3 + c_1a_{11}^2a_{21} - d_1a_{11}a_{21}^2$; $b_1 = 0 \to 0$; $c_1 \to c_1a_{11}^2a_{22} - 2d_1a_{11}a_{21}a_{22}$; $d_1 \to d_1a_{11}a_{22}^2$.

Case 1: $d_1 \neq 0$. As the first step, we make $c_1 = 0$ by solving for a_{21}, which can be done by taking $a_{11} = a_{22} = 1$ and $a_{21} = c_1/(2d_1)$. With these new values, the formulae above become $a_1 \to a_1a_{11}^3 - d_1a_{11}a_{21}^2$; $b_1 \to 0$; $c_1 = 0 \to -2d_1a_{11}a_{21}a_{22}$; $d_1 \to d_1a_{11}a_{22}^2$.

In the second step, to keep $c_1 = 0$, we require $a_{21} = 0$, which in turn makes $a_1 \to a_1a_{11}^3$, $b_1 = c_1 = 0$ and $d_1 \to d_1a_{11}a_{22}^2$, or $B = [a_1a_{11}^3, 0, 0, d_1a_{11}a_{22}^2]$.

Case 1.1: $a_1 = 0$. We obtain our first representative $B_1 = [0, 0, 0, 1]$, and $A \wedge B_1$ corresponds to the 7-dimensional algebra (2357C).

Case 1.2: $a_1 \neq 0$.

Subcase 1.2.1: $a_1d_1 > 0$. Make $a_1a_{11}^3 = d_1a_{11}a_{22}^2$ in B by solving for a_{22} and multiply B by a scalar, which can be done by taking $a_{11} = 1$ and $a_{22} = \sqrt{a_1/d_1}$, and multiply by a_1^{-1}, we will obtain our second representative $B_2 = [1, 0, 0, 1]$, and $A \wedge B_2$ corresponds to the 7-dimensional algebra (2357D).

Subcase 1.2.2: $a_1d_1 < 0$. Make $a_1a_{11}^3 = -d_1a_{11}a_{22}^2$ in B and multiply it by a scalar, which can be done by taking $a_{11} = 1$ and $a_{22} = \sqrt{-a_1/d_1}$, and multiply it by a_1^{-1}, we will have $B_3 = [1,0,0,-1]$, $A \wedge B_3$ corresponds to the 7-dimensional algebra (2357D$_1$).

(If the ground field is algebraically closed, then B_2 and B_3 are in the same orbit.)

Case 2: $d_1 = 0$. Since $b_1 = d_1 = 0$ in B, we have $a_1 \to a_1a_{11}^3 + c_1a_{11}^2a_{21}$; $c_1 \to c_1a_{11}^2a_{22}$.

Subcase 2.1: $c_1 = 0$. We obtain $B_4 = [1,0,0,0]$, and $A \wedge B_4$ corresponds to (2357B).

Subcase 2.2: $c_1 \neq 0$. Taking $a_{11} = 1$ and $a_{21} = -a_1/c_1$, we make $a_1 = 0$ to obtain $B_5 = [0,0,1,0]$, and $A \wedge B_5$ corresponds to (2357A).

Now we have obtained 5 algebras:(2357A-D, D$_1$). We need also to show that they are mutually nonisomorphic, which can be done by comparing their corresponding orbits. As an example, we will show that (2357D) and (2357D$_1$) are nonisomorphic over $\mathbf{R}$ but are isomorphic over an algebraically closed field of $\chi \neq 2$.

For (2357D), we have $A = [0,1,0,0]$ and $B_2 = [1,0,0,1]$. Under the group action,

$$A \to [a_{11}a_{53} + a_{11}a_{21}a_{31} + a_{11}^2a_{41}, a_{11}^3a_{22}, 2a_{11}a_{22}a_{31}, -a_{11}a_{22}a_{32}]$$

and

$$B \to [a_{11}^3 - a_{11}a_{21}^2, 0, -2a_{11}a_{21}a_{22}, a_{11}a_{22}^2].$$

Then

$$\begin{aligned} A \wedge B \to \quad & -a_{11}^4a_{22}(a_{11}^2 - a_{21}^2)\Delta_{13} \wedge (\Delta_{15} - \Delta_{34}) \\ & -2a_{11}^2a_{22}((a_{53} + a_{21}a_{31} + a_{11}a_{41})a_{21} + a_{31}(a_{11}^2 - a_{21}^2))\Delta_{13} \wedge \Delta_{23} \\ & +a_{11}^2a_{22}((a_{53} + a_{21}a_{31} + a_{11}a_{41})a_{22} + a_{32}(a_{11}^2 - a_{21}^2))\Delta_{13} \wedge \Delta_{24} \\ & -2a_{11}^4a_{22}^2a_{21}(\Delta_{15} - \Delta_{34}) \wedge \Delta_{23} \\ & +a_{11}^4a_{22}^3(\Delta_{15} - \Delta_{34}) \wedge \Delta_{24} \\ & +2a_{11}^2a_{22}^2(a_{31}a_{22} - a_{32}a_{21})\Delta_{23} \wedge \Delta_{24} \end{aligned}$$

Compare with (2357D$_1$), where $A = [0,1,0,0]$ and $B_3 = [1,0,0,-1]$, in which case

$$A \wedge B_3 = (\Delta_{15} - \Delta_{34}) \wedge (\Delta_{13} - \Delta_{24}).$$

If it is in the same orbit of (2357D), then the coefficients of $(\Delta_{15} - \Delta_{34}) \wedge \Delta_{23}$ and $\Delta_{23} \wedge \Delta_{24}$ are zero, which give $a_{21} = a_{31} = 0$. As the coefficients of $\Delta_{13} \wedge (\Delta_{15} - \Delta_{34})$ and $(\Delta_{15} - \Delta_{34}) \wedge \Delta_{24}$ must be equal, i.e., $-a_{11}^4a_{22}a_{11}^2 = a_{11}^4a_{22}^3$ or $a_{11}^2 = -a_{22}^2$, which has no solution over the real field $\mathbf{R}$ but do have solutions over algebraically closed field. Therefore (2357D) and (2357D$_1$) are distinct over $\mathbf{R}$ and are isomorphic over an algebraically closed field of $\chi \neq 2$. All the other nonisomorphisms can be proved similarly.

In some cases, we may also use some other invariants, like minimal numbers, as used by Seeley, to separate the algebras (see Chapter 5 for the definition of minimal numbers and for examples).

Therefore the corresponding central extensions of $N_{5,2,2}$ of dimension 7 over $\mathbf{R}$ are:

(2357A):	$[x_1, x_2] = x_4$,	$[x_1, x_4] = x_5$,	$[x_1, x_5] = x_7$,
	$[x_2, x_3] = x_5 + x_6$,	$[x_3, x_4] = -x_7$;	
(2357B):	$[x_1, x_2] = x_4$,	$[x_1, x_3] = x_6$,	$[x_1, x_4] = x_5$,
	$[x_1, x_5] = x_7$,	$[x_2, x_3] = x_5$,	$[x_3, x_4] = -x_7$;
(2357C):	$[x_1, x_2] = x_4$,	$[x_1, x_4] = x_5$,	$[x_1, x_5] = x_7$,
	$[x_2, x_3] = x_5$,	$[x_2, x_4] = x_6$,	$[x_3, x_4] = -x_7$;
(2357D):	$[x_1, x_2] = x_4$,	$[x_1, x_3] = x_6$,	$[x_1, x_4] = x_5$,
	$[x_1, x_5] = x_7$,	$[x_2, x_3] = x_5$,	$[x_2, x_4] = x_6$,
	$[x_3, x_4] = -x_7$;		
($2357D_1$):	$[x_1, x_2] = x_4$,	$[x_1, x_3] = x_6$,	$[x_1, x_4] = x_5$,
	$[x_1, x_5] = x_7$,	$[x_2, x_3] = x_5$,	$[x_2, x_4] = -x_6$,
	$[x_3, x_4] = -x_7$.		

When the ground field $\mathbf{F}$ is algebraically closed and $\chi \neq 2$, then the central extensions of $N_{5,2,2}$ are (2357A-D), with (2357D)$\cong$ ($2357D_1$) in this case.

Remarks: (1) Throughout the computation above, although we may assign different values to the entries of the matrix of $g \in$ Aut $\mathfrak{g}$, we always ensure that the nonsingularity is maintained; (2) Quite often, care is required so as not to disturb previous assumptions, for example, when we assume $b \neq 0$, then we preserve it throughout the simplification procedure, even though we may not point it out explicitly; (3) By abuse of terminology, we refer to the elements in $H^2(\mathfrak{g}, \mathbf{R})$ as 2-cocycles (causing no confusion); (4) On some occasions, we may provide an isomorphism between two algebras, and write it as $x_i \to ax_1 + bx_2 + \cdots$, etc. Then the x_i before the arrow is an element of the basis for the first algebra (or the "old one"), and the x_i's after the arrow are the elements of the basis for the second algebra (or the "new one").

Example 3 *Find the central extensions of dimension 7 without Abelian factors of $N_{6,3,2}$: $[x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_4, x_5] = x_6$.*

$Z(\mathfrak{g})$: x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_6; $Z^2(\mathfrak{g})$: $C_{16} = C_{26} = C_{34} = C_{35} = C_{36} = C_{46} = C_{56} = 0$;

It is obvious that $N_{6,3,2}$ has no central extension of the desired type, as all the 2-cocycles have x_6 in their kernels.

Example 4 *Find the central extensions of dimension 7 - over an algebraically closed field of $\chi \neq 2$ and $\mathbf{R}$ - without Abelian factors of $N_{6,1,1}$: $[x_1, x_i] = x_{i+1}$, $2 \leq i \leq 5$, $[x_2, x_i] = x_{i+2}$, $i = 3, 4$.*

$Z(\mathfrak{g})$: x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_4, x_5, x_6; $Z^2(\mathfrak{g})$: $C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = C_{56} = 0$, $C_{34} + C_{25} = C_{16}$, $C_{15} = C_{24}$; $W(H^2)$: $C_{12} = C_{13} = C_{14} = C_{15} = 0$; $\dim H^2$: 3; Basis: Δ_{23}, $\Delta_{16} +$

Δ_{25}, $\Delta_{16}+\Delta_{34}$;

Group Action: $a\Delta_{23}+b(\Delta_{16}+\Delta_{25})+c(\Delta_{16}+\Delta_{34})$:

$a \to aa_{11}^5 + c(a_{11}a_{32}^2 - 2a_{11}^3a_{42})$;

$b \to ba_{11}^7$; $c \to ca_{11}^7$;

We must have $b+c \neq 0$ to ensure that the 2-cocycles do not contain x_6 in their kernel.

Case 1: $b=0$. Then $c \neq 0$, and make $c=1$, $a=0$ (solving for a_{42}) to obtain $A_1=[0,0,1]$;

Case 2: $b \neq 0$. Assume first that $c=0$. When $a=0$, we get $A_2=[0,1,0]$. When $a \neq 0$, we get $[aa_{11}^5, ba_{11}^7, 0]$. If $\mathbf{F}$ is a algebraically closed, we can make it to be $A_3=[1,1,0]$. If $\mathbf{F}=\mathbf{R}$, depending on the signs of a,b, we get two representatives $A_4=[1,1,0]$ if $ab>0$ and $A_5=[1,-1,0]$ if $ab<0$.

Next let $c \neq 0$. Then make $a=0$ by solving for a_{42}, and get the representative $[0, ba_{11}^7, ca_{11}^7]$. Because we are dealing with the subspaces of $H^2(\mathfrak{g},\mathbf{F})$, we can multiply the representative by a nonzero scalar as we like, and keep in mind that $b+c \neq 0$. So we obtain $A_6=[0,\lambda,1-\lambda]$ (with $\lambda \neq 0,1$).

It is easy to see that if we allow $\lambda=0,1$, we can include A_1 and A_2 in A_6 as special cases.

Now A_3 and A_4 correspond to the same algebra, written as (123457H). A_5 corresponds to (123457H$_1$), which only exists when the ground field is $\mathbf{R}$, and is isomorphic to (123457H) when the ground field is algebraically closed, and A_6 corresponds to (123457I). It is obvious that (123457H,H$_1$,I) are distinct, the corresponding orbits being different.

Therefore, the central extensions of $N_{6,1,1}$ of dimension 7 over $\mathbf{R}$ are:

(123457H):	$[x_1,x_i]=x_{i+1}$, $2 \le i \le 5$,	$[x_1,x_6]=x_7$,	$[x_2,x_3]=x_5+x_7$,
	$[x_2,x_4]=x_6$,	$[x_2,x_5]=x_7$;	
(123457H$_1$):	$[x_1,x_i]=x_{i+1}$, $2 \le i \le 5$,	$[x_1,x_6]=-x_7$,	$[x_2,x_3]=x_5+x_7$,
	$[x_2,x_4]=x_6$,	$[x_2,x_5]=-x_7$;	
(123457I):	One parameter family.		
	$[x_1,x_i]=x_{i+1}$, $2 \le i \le 5$,	$[x_1,x_6]=x_7$,	$[x_2,x_i]=x_{i+2}$, $i=3,4$,
	$[x_2,x_5]=\lambda x_7$,	$[x_3,x_4]=(1-\lambda)x_7$.	

When $\mathbf{F}$ is algebraically closed and $\chi \neq 2$, the central extensions of $N_{6,1,1}$ are (123457H,I), with (123457H)$\cong$(123457H$_1$).

Example 5 *Find the central extensions of dimension* 6 *over any algebraically closed fields, without Abelian factors, of* $\mathfrak{g}=N_{5,2,1}$: $[x_1,x_i]=x_{i+1}$, $i=2,3,4$.

$Z(\mathfrak{g})$: x_5; $[\mathfrak{g},\mathfrak{g}]$: x_3, x_4, x_5; $Z^2(\mathfrak{g})$: $C_{24}=C_{35}=C_{45}=0, C_{25}+C_{34}=0$; $W(H^2)$: $C_{12}=C_{13}=C_{14}=0$; $\dim H^2$: 3; Basis: Δ_{15}, Δ_{23}, $\Delta_{25}-\Delta_{34}$;

Group Actions: $a\Delta_{15} + b\Delta_{23} + c(\Delta_{25} - \Delta_{34})$:

$a \to aa_{11}^4 a_{22} + ca_{11}^3 a_{21} a_{22}$;

$b \to ba_{11} a_{22}^2 + 2ca_{11} a_{22} a_{42} - ca_{11} a_{32}^2$;

$c \to ca_{11}^3 a_{22}^2$;

One of $a, c \neq 0$ (due to the reason that we require the kernel of the desired 2-cocycles does not contain any central elements, this is also a requirement in all the proofs of the subsequent chapters, and from time to time, we will use this assumption without further explanation).

When $c \neq 0$, make $a = 0$ by solving for a_{21} and $b = 0$ by a_{32} to get $[0, 0, 1]$ (corresponding to $N_{6,2,2}$).

When $c = 0$, then $a \neq 0$. Get two representatives depending on whether $b = 0$ or not, i.e., $[1, 0, 0]$ (corresponding to $N_{6,2,1}$) or $[1, 1, 0]$ (corresponding to $N_{6,1,3}$);

So the central extensions of $N_{5,2,1}$ of dimension 6 over any algebraically closed fields are:

$N_{6,1,3}$:	$[x_1, x_i] = x_{i+1}, 2 \leq i \leq 5,$	$[x_2, x_3] = x_6;$	
$N_{6,2,1}$:	$[x_1, x_i] = x_{i+1}, 2 \leq i \leq 5;$		
$N_{6,2,2}$:	$[x_1, x_i] = x_{i+1}, i = 2, 3, 4,$	$[x_2, x_5] = x_6,$	$[x_3, x_4] = -x_6.$

Example 6 *Find the central extensions of dimension 7 over an algebraically closed field of $\chi \neq 2$, without Abelian factors, of $\mathfrak{g} = N_{5,2,1}$: $[x_1, x_i] = x_{i+1}, i = 2, 3, 4$.*

$Z(\mathfrak{g})$, $[\mathfrak{g}, \mathfrak{g}]$, $W(H^2)$, $\dim H^2$, Basis and group actions can all be found in Exmaple 5.

As we are considering the 2-dimensional central extensions of $N_{5,2,1}$, we need to find a set of representatives of orbits of the 2-dimensional subspaces of $H^2(\mathfrak{g}, \mathbf{F})$, or representatives of the form $A \wedge B$, where A and B are elements in $H^2(\mathfrak{g}, \mathbf{F})$.

Let $A = [a, b, c]$ and $B = [a_1, b_1, c_1]$. One of $a, c, a_1, c_1 \neq 0$. According to the discussions of Example 5, WLOG, we may let A be (1) $[1, 0, 0]$, (2) $[1, 1, 0]$ or (3) $[0, 0, 1]$.

Case 1: $A = [1, 0, 0]$. Then $B = [0, b_1, c_1]$. Fix A (up to a scalar, which put no restrictions on the entries of the automorphism group at all).

Subcase 1.1: $c_1 \neq 0$. Make $c_1 = 1$ (by multiplying a scalar) to get $B = [0, b_1, 1]$. Consider the group action on B:

$$B = [a_{11}^3 a_{21} a_{22}, b_1 a_{11} a_{22}^2 + 2a_{11} a_{22} a_{42} - a_{11} a_{32}, a_{11}^3 a_{22}^2].$$

By fixing A, we can always make $a_1 = 0$ by linear combination. Make further $b_1 = 0$ by solving for a_{32} to get $B = [0, 0, 1]$, with $A \wedge B$ corresponding to (23457C).

Subcase 1.2: $c_1 = 0$. Then $b_1 \neq 0$, and get $B = [0, 1, 0]$, with $A \wedge B$ corresponding to (23457A).

Case 2: $A = [1, 1, 0]$. Then assume $B = [0, b_1, c_1]$. To fix A (up to a scalar), we require $a_{22} = a_{11}^3$.

Subcase 2.1: $c_1 \neq 0$. Make $c_1 = 1$ to get $B = [0, b_1, 1]$. From Subcase 1.1, we have

$$B = [a_{11}^6 a_{21}, b_1 a_{11}^7 + 2a_{11}^4 a_{42} - a_{11} a_{32}, a_{11}^9].$$

Can make both $a_1 = b_1 = 0$ by solving for a_{21} and a_{42} respectively to get $B = [0, 0, 1]$, with $A \wedge B$ corresponding to (23457D).

Subcase 2.2: $c_1 = 0$. Then $b_1 \neq 0$ to get $B = [0, 1, 0]$. But $A \wedge B$ will become Subcase 1.2. So we omit it.

Case 3: $A = [0, 0, 1]$. Then $B = [a_1, b_1, 0]$. To fix A (up to a scalar), we may set $a_{21} = a_{32} = a_{42} = 0$. Consider the group action on B:

$$B = [a_1 a_{11}^4 a_{22}, b_1 a_{11} a_{22}^2, 0].$$

Subcase 3.1: $a_1 \neq 0$. If $b_1 = 0$, then $B = [1, 0, 0]$, and $A \wedge B$ is the same as Subcase 1.1, omit it; If $b_1 \neq 0$, we have $B = [1, 1, 0]$, and $A \wedge B$ is the same as Subcase 2.1, omit it.

Subcase 3.2: $a_1 = 0$. Then $b_1 \neq 0$ to get $B = [0, 1, 0]$, with $A \wedge B$ correponding to (23457B).

To prove that (23457A–D) are distinct, we let V_1 be the subspace generated by Δ_{15} and Δ_{23}, and V the space generated by Δ_{15}, Δ_{23} and $\Delta_{25} - \Delta_{34}$. Then V_1 is a submodule under Aut $\mathfrak{g}$. Now it becomes obvious that (23457A) is different from all the other three algebras in that only its corresponding 2-cocycles (i.e., A and B) are in V_1.

To show that (23457B) is different from (23457C,D), we just need to compare their orbits. For (23457B), we have $A = [0, 0, 1]$ and $B = [0, 1, 0]$. Under the group action, we have $A \to [a_{11}^3 a_{21} a_{22}, 2a_{11} a_{22} a_{42} - a_{11} a_{32}^2, a_{11}^3 a_{22}^2]$ and $B \to [0, a_{11} a_{22}^2, 0]$. Then

$$A \wedge B \to a_{11}^4 a_{21} a_{22}^3 \Delta_{15} \wedge \Delta_{23} + a_{11}^4 a_{22}^4 (\Delta_{25} - \Delta_{34}) \wedge \Delta_{23}.$$

It is obvious that (23457C,D) cannot be in the same orbit. Therefore (23457B) is not isomorphic to (23457C,D).

Similarly we can prove that (23457C) and (23457D) are distinct.

Therefore the central extensions of $N_{5,2,1}$ of dimensions 7 over an algebraically closed field ($\chi \neq 2$) are:

(23457A):	$[x_1, x_i] = x_{i+1}, i = 2, 3, 4,$	$[x_1, x_5] = x_6,$	$[x_2, x_3] = x_7;$
(23457B):	$[x_1, x_i] = x_{i+1}, i = 2, 3, 4,$	$[x_2, x_3] = x_7,$	$[x_2, x_5] = x_6,$
	$[x_3, x_4] = -x_6;$		
(23457C):	$[x_1, x_i] = x_{i+1}, i = 2, 3, 4,$	$[x_1, x_5] = x_6,$	$[x_2, x_5] = x_7,$
	$[x_3, x_4] = -x_7;$		
(23457D):	$[x_1, x_i] = x_{i+1}, i = 2, 3, 4,$	$[x_1, x_5] = x_6,$	$[x_2, x_3] = x_6,$
	$[x_2, x_5] = x_7,$	$[x_3, x_4] = -x_7;$	

Chapter 3

Nilpotent Lie Algebras of Dimension ≤ 6

3.1 Notations

In this chapter, we will give a complete list of all the nilpotent Lie algebras of dimension 6 over an algebraically closed field $\mathbf{F}$ of any characteristic χ.

We will firstly present the list, including all the algebras of dimension ≤ 5, which was obtained by Dixmier [8], together with their types, ranks, weight systems and automorphism groups, and we will follow by providing the details of the proof for the classification of the 6-dimensional nilpotent Lie algebras.

As pointed out by Dixmier, for algebras of dimension less than 6, their structure constants can be chosen to be independent of the characteristic χ of the ground field. For dimension 6, we find that the only exception is when the characteristic equals 2.

Shedler [34] has obtained a list of all the 6-dimensional nilpotent Lie algebras for any field But his work has never been published, and his proof also contains many errors. Here we reconstruct all the 6-dimensional nilpotent algebras over algebraically closed fields. Our list agrees with that of Shedler's when $\chi \neq 2$. When $\chi = 2$, Shedler has missed one algebra, i.e., (B) of Example 1 in Section 2.3.

We will first give the list for all the algebras over algebraically closed field of characteristic $\chi \neq 2$, and then follow by those of characteristic $\chi = 2$.

The algebras have been ordered by the increasing lexicographic order of their types: the dimension of the algebra, its rank, the sequence of dimensions of its upper and lower central series.

We now explain our notations:

– $N_{i,j,k}$: The k-th algebra of dimension i and rank j, and when there is only one algebra with the specific dimension and rank, we simply denote it by $N_{i,j}$.

–$(i, j, \cdots/m, n, \cdots,)$, where $i, j, \cdots$ and $m, n, \cdots$ are respectively the dimensions of the upper and lower central series.

– $[\alpha, \beta, \gamma, \cdots]$: The weight system of the corresponding Lie algebra L with respect to a maximal torus of the automorphism group of L. More precisely, the basis vectors x_i are weight vectors, α is the weight of x_1, β of x_2, etc.

– CQ: The central quotient algebra L/Z where Z is the center of L.

– Aut L: The automorphism group of L. We use our Maple package to compute the generic element of this automorphism group, except for the case $N_{5,3,1}$. In general, it is easy to figure out which maximal torus of Aut L is used when we consider the weight system mentioned above. When Aut L is not connected, its identity component is denoted by Aut_0 L and σ is a representative of the other component (as all the groups here have at most two components).

– R_u and S: The unipotent radical R_u and the Levi factor S of Aut L. By GL_1^m we denote the direct product of m copies of GL_1.

– $\mathfrak{a}_i$: i-dimensional Abelian Lie algebra.

The matrices of the automorphisms are of course nonsingular, which imposes some obvious restrictions that are not stated explicitly.

3.2 The List

3.2.1 Algebras of Dimensions ≤ 5

INDECOMPOSABLE ALGEBRAS

Dimension 1

$N_{1,1}:\ a_1$;

— Aut $N_{1,1}: \mathrm{GL}_1$.

Dimension 2

None.

Dimension 3

$N_{3,2}:\ [x_1, x_2] = x_3$;

— a Heisenberg Lie algebra, free nilpotent of class 2 with 2 generators;

— $(1, 3/3, 1)$;

— $[\alpha, \beta, \alpha+\beta]$;

— CQ: $N_{2,2}$;

— Aut $N_{3,2}$:

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix},$$

with dim $R_u = 2$ and $S = \mathrm{GL}_2$.

Dimension 4

$N_{4,2}:\ [x_1, x_i] = x_{i+1},\ i = 2, 3$;

— $(1, 2, 4/4, 2, 1)$;

— $[\alpha, \beta, \alpha+\beta, 2\alpha+\beta]$;

— CQ: $N_{3,2}$;

— Aut $N_{4,2}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{11}\,a_{22} & 0 \\ a_{41} & a_{42} & a_{11}\,a_{32} & {a_{11}}^2 a_{22} \end{bmatrix},$$

with dim $R_u = 5$ and $S = \mathrm{GL}_1^2$.

<u>Dimension</u> 5

$N_{5,1}$: $[x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4$, $[x_2, x_3] = x_5$;

— $(1, 2, 3, 5/5, 3, 2, 1)$;

— $[\alpha, 2\alpha, 3\alpha, 4\alpha, 5\alpha]$;

— CQ: $N_{4,2}$;

— Aut $N_{5,1}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & {a_{11}}^2 & 0 & 0 & 0 \\ a_{31} & a_{32} & {a_{11}}^3 & 0 & 0 \\ a_{41} & a_{42} & a_{11}\,a_{32} & {a_{11}}^4 & 0 \\ a_{51} & a_{52} & u & v & {a_{11}}^5 \end{bmatrix},$$

where $u = a_{11}a_{42} + a_{21}a_{32} - a_{11}^2 a_{31}$, $v = a_{11}^3 a_{32} + a_{21}a_{11}^3$, dim $R_u = 7$ and $S = \mathrm{GL}_1$.

$N_{5,2,1}$: $[x_1, x_i] = x_{i+1}$, for $i = 2, 3, 4$;

— $(1, 2, 3, 5/5, 3, 2, 1)$;

— $[\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta]$;

— CQ: $N_{4,2}$;

— Aut $N_{5,2,1}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}\,a_{22} & 0 & 0 \\ a_{41} & a_{42} & a_{11}\,a_{32} & {a_{11}}^2 a_{22} & 0 \\ a_{51} & a_{52} & a_{11}\,a_{42} & {a_{11}}^2 a_{32} & {a_{11}}^3 a_{22} \end{bmatrix},$$

with dim $R_u = 7$ and $S = \mathrm{GL}_1^2$.

$N_{5,2,2}:\ [x_1,x_2]=x_4,\ [x_1,x_4]=[x_2,x_3]=x_5;$

— $(1,3,5/5,2,1)$;

— $[\alpha,\beta,2\alpha,\alpha+\beta,2\alpha+\beta]$;

— CQ: $N_{4,3}$;

— Aut $N_{5,2,2}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & {a_{11}}^2 & 0 & 0 \\ a_{41} & a_{42} & -a_{21}\,a_{11} & a_{11}\,a_{22} & 0 \\ a_{51} & a_{52} & a_{53} & u & {a_{11}}^2 a_{22} \end{bmatrix},$$

where $u=a_{11}\,a_{42}+a_{21}\,a_{32}-a_{22}\,a_{31}$, $\dim R_u=8$ and $S=\mathrm{GL}_1^2$.

$N_{5,2,3}:\ [x_1,x_i]=x_{i+1},\text{for } i=2,3,\ [x_2,x_3]=x_5;$

— Free nilpotent Lie algebra of class 3 with 2 generators;

— $(2,3,5/5,3,2)$;

— $[\alpha,\beta,\alpha+\beta,2\alpha+\beta,\alpha+2\beta]$;

— CQ: $N_{3,2}$;

— Aut $N_{5,2,3}$:

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & e & 0 & 0 \\ a_{41} & a_{42} & a_{11}\,a_{32}-a_{12}\,a_{31} & a_{11}\,e & a_{12}\,e \\ a_{51} & a_{52} & a_{21}\,a_{32}-a_{22}\,a_{31} & a_{21}\,e & a_{22}\,e \end{bmatrix},$$

where $e=a_{11}a_{22}-a_{12}a_{21}$, $\dim R_u=6$ and $S=\mathrm{GL}_2$.

$N_{5,3,1}:\ [x_1,x_2]=[x_3,x_4]=x_5;$

— a Heisenberg Lie algebra;

— $(1,5/5,1)$;

— $[\alpha,\gamma-\alpha,\beta,\gamma-\beta,\gamma]$;

— CQ: $N_{4,4}$;

— Aut $N_{5,3,1}$:

$$\begin{bmatrix} \lambda A & 0 \\ u & \lambda^2 \end{bmatrix}, \ \lambda \neq 0,$$

where A satisfies $A^t J A = J$, with

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

u is an arbitrary 4-dimensional vector, $\dim R_u = 4$, and $S = (\mathrm{GL}_1 \times \mathrm{Sp}_4)/\mathbf{Z}_2$.

$N_{5,3,2}$: $[x_1, x_2] = x_4$, $[x_1, x_3] = x_5$;

— $(2, 5/5, 2)$;

— $[\alpha, \beta, \gamma, \alpha+\beta, \alpha+\gamma]$;

— CQ: $N_{3,3}$;

— Aut $N_{5,3,2}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{11}a_{22} & a_{11}a_{23} \\ a_{51} & a_{52} & a_{53} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix},$$

with $\dim R_u = 8$ and $S = \mathrm{GL}_1 \times \mathrm{GL}_2$.

DECOMPOSABLE ALGEBRAS

Dimension 2

$N_{2,2}$: $\mathfrak{a}_2$;

— Aut $N_2 = \mathrm{GL}_2$;

Dimension 3

$N_{3,3}$: $\mathfrak{a}_3$;

— Aut $N_{3,3} = \mathrm{GL}_3$.

Dimension 4

$N_{4,3}$: $N_{3,2} \times \mathfrak{a}_1$; or $[x_1, x_2] = x_3$;

— Aut $N_{4,3}$:

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} - a_{12}a_{21} & a_{34} \\ a_{41} & a_{42} & 0 & a_{44} \end{bmatrix},$$

with $\dim R_u = 5$, $S = \mathrm{GL}_1 \times \mathrm{GL}_2$.

$N_{4,4}$: $\mathfrak{a}_4$;

— Aut $N_{4,4} = \mathrm{GL}_4$.

Dimension 5

$N_{5,3,3}$: $N_{4,2} \times \mathfrak{a}_1$;

— Aut $N_{5,3,3}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} & 0 & 0 \\ a_{41} & a_{42} & a_{11}a_{32} & a_{11}{}^2a_{22} & a_{45} \\ a_{51} & a_{52} & 0 & 0 & a_{55} \end{bmatrix},$$

with $\dim R_u = 8$, $S = \mathrm{GL}_1^3$.

$N_{5,4}$: $N_{3,2} \times \mathfrak{a}_2$;

— Aut $N_{5,4}$:

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} - a_{12}a_{21} & a_{34} & a_{35} \\ a_{41} & a_{42} & 0 & a_{44} & a_{45} \\ a_{51} & a_{52} & 0 & a_{54} & a_{55} \end{bmatrix},$$

with $\dim R_u = 8$, $S = \mathrm{GL}_2^2$.

$N_{5,5}$: $\mathfrak{a}_5$;

— Aut $N_{5,5} = \mathrm{GL}_5$.

3.2.2 Algebras of Dimension 6 over Algebraically Closed Fields of $\chi \neq 2$

INDECOMPOSABLE ALGEBRAS

$N_{6,1,1}$: $[x_1, x_i] = x_{i+1},\ 2 \le i \le 5,\ [x_2, x_i] = x_{i+2},\ i = 3, 4;$

— $(1, 2, 3, 4, 6/6, 4, 3, 2, 1)$;

— $[\alpha, 2\alpha, 3\alpha, 4\alpha, 5\alpha, 6\alpha]$;

— CQ: $N_{5,1}$;

— Aut $N_{6,1,1}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{11}^2 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}^3 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{11}a_{32} & a_{11}^4 & 0 & 0 \\ a_{51} & a_{52} & u & a_{11}^2 a_{32} & a_{11}^5 & 0 \\ a_{61} & a_{62} & v & w & a_{11}^3 a_{32} & a_{11}^6 \end{bmatrix},$$

where $u = a_{11}a_{42} - a_{11}^2 a_{31}$, $v = a_{11}a_{52} - a_{11}^2 a_{41}$, $w = a_{11}^2 a_{42} - a_{11}^3 a_{31}$, $\dim R_u = 8$, and $S = \mathrm{GL}_1$.

$N_{6,1,2}$: $[x_1, x_i] = x_{i+1},\ i = 2, 3, 4,\ [x_2, x_3] = x_5,\ [x_2, x_5] = x_6,\ [x_3, x_4] = -x_6.$

—$(1, 2, 3, 4, 6/6, 4, 3, 2, 1)$;

— $[\alpha, 2\alpha, 3\alpha, 4\alpha, 5\alpha, 7\alpha]$;

— CQ: $N_{5,1}$;

— Aut $N_{6,1,2}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{11}^2 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}^3 & 0 & 0 & 0 \\ a_{41} & \frac{a_{32}^2}{2a_{11}^2} & a_{11}a_{32} & a_{11}^4 & 0 & 0 \\ a_{51} & a_{52} & u & a_{11}^2 a_{32} & a_{11}^5 & 0 \\ a_{61} & a_{62} & v & w & -a_{31}a_{11}^4 & a_{11}^7 \end{bmatrix},$$

where $u = a_{11}a_{32}^2/(2a_{11}^2) - a_{11}^2 a_{31}$, $v = a_{32}a_{41} - a_{11}^2 a_{51} - a_{31}a_{32}^2/(2a_{11}^2)$, $w = a_{11}^3 a_{41} - a_{31}a_{11}a_{32}$, $\dim R_u = 7$, and $S = \mathrm{GL}_1$;

$N_{6,1,3}:\ [x_1, x_i] = x_{i+1},\ 2 \le i \le 5,\ [x_2, x_3] = x_6;$

— $(1,2,3,4,6/6,4,3,2,1)$;

— $[\alpha, 3\alpha, 4\alpha, 5\alpha, 6\alpha, 7\alpha]$;

— CQ: $N_{5,2,1}$;

— Aut $N_{6,1,3}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & {a_{11}}^3 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & {a_{11}}^4 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{11}\,a_{32} & {a_{11}}^5 & 0 & 0 \\ a_{51} & a_{52} & a_{11}\,a_{42} & {a_{11}}^2 a_{32} & {a_{11}}^6 & 0 \\ a_{61} & a_{62} & u & v & {a_{11}}^3 a_{32} & {a_{11}}^7 \end{bmatrix},$$

where $u = a_{11}a_{52} + a_{21}a_{32} - a_{11}^3 a_{31}$, $v = a_{11}^2 a_{42} + a_{21}a_{11}^4$, $\dim R_u = 9$ and $S = \mathrm{GL}_1$.

$N_{6,1,4}:\ [x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_6,\ [x_2, x_3] = x_6,\ [x_2, x_5] = x_6;$

— $(1,3,4,6/6,3,2,1)$;

— $[\alpha, 2\alpha, 3\alpha, 4\alpha, 3\alpha, 5\alpha]$;

— CQ: $N_{5,3,3} = N_{4,2} \times a_1$;

— Aut $N_{6,1,4}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & {a_{11}}^2 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & {a_{11}}^3 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{11}a_{32} & {a_{11}}^4 & -a_{21}a_{11}^2 & 0 \\ a_{51} & a_{52} & 0 & 0 & {a_{11}}^3 & 0 \\ a_{61} & a_{62} & u & v & a_{65} & {a_{11}}^5 \end{bmatrix},$$

where $u = a_{11}a_{42} + a_{21}(a_{32} + a_{52}) - a_{11}^2(a_{31} + a_{51})$, $v = a_{11}^2 a_{32} + a_{21}a_{11}^3$, $\dim R_u = 10$ and $S = \mathrm{GL}_1$.

$N_{6,2,1}:\ [x_1, x_i] = x_{i+1},\ 2 \le i \le 5;$

— $(1,2,3,4,6/6,4,3,2,1)$;

— $[\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 4\alpha+\beta]$;

— CQ: $N_{5,2,1}$;

— Aut $N_{6,2,1}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}\,a_{22} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{11}\,a_{32} & {a_{11}}^2 a_{22} & 0 & 0 \\ a_{51} & a_{52} & a_{11}\,a_{42} & {a_{11}}^2 a_{32} & {a_{11}}^3 a_{22} & 0 \\ a_{61} & a_{62} & a_{11}\,a_{52} & {a_{11}}^2 a_{42} & {a_{11}}^3 a_{32} & {a_{11}}^4 a_{22} \end{bmatrix},$$

with $\dim R_u = 9$ and $S = \mathrm{GL}_1^2$.

$N_{6,2,2}$: $[x_1, x_i] = x_{i+1}$, $i = 2, 3, 4$, $[x_2, x_5] = x_6$, $[x_3, x_4] = -x_6$;

— $(1, 2, 3, 4, 6/6, 4, 3, 2, 1)$;

— $[\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta]$;

— CQ: $N_{5,2,1}$;

— Aut $N_{6,2,2}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}\,a_{22} & 0 & 0 & 0 \\ a_{41} & \frac{{a_{32}}^2}{2\,a_{22}} & a_{11}\,a_{32} & {a_{11}}^2 a_{22} & 0 & 0 \\ a_{51} & a_{52} & \frac{a_{11}\,{a_{32}}^2}{2\,a_{22}} & {a_{11}}^2 a_{32} & {a_{11}}^3 a_{22} & 0 \\ a_{61} & a_{62} & u & v & -a_{31}\,{a_{11}}^2 a_{22} & {a_{11}}^3 {a_{22}}^2 \end{bmatrix},$$

where $u = a_{32}a_{41} - a_{22}a_{51} - a_{31}a_{32}^2/(2a_{22})$, $v = a_{11}a_{22}a_{41} - a_{31}a_{11}a_{32}$, $\dim R_u = 7$ and $S = \mathrm{GL}_1^2$;

$N_{6,2,3}$: $[x_1, x_2] = x_4$, $[x_1, x_i] = x_{i+1}$, $i = 4, 5$, $[x_2, x_3] = x_5$, $[x_3, x_4] = -x_6$;

— $(1, 2, 4, 6/6, 3, 2, 1)$;

— $[\alpha, \beta, 2\alpha, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta]$;

— CQ: $N_{5,2,2}$;

— Aut $N_{6,2,3}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & {a_{11}}^2 & 0 & 0 & 0 \\ a_{41} & a_{42} & -a_{21}\,a_{11} & a_{11}\,a_{22} & 0 & 0 \\ a_{51} & a_{52} & -a_{11}\,a_{41} & a_{11}\,a_{42} & {a_{11}}^2 a_{22} & 0 \\ a_{61} & a_{62} & a_{63} & a_{11}\,a_{52} & {a_{11}}^2 a_{42} & {a_{11}}^3 a_{22} \end{bmatrix},$$

with $\dim R_u = 8$ and $S = \mathrm{GL}_1^2$.

$N_{6,2,4}$: $[x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_6,\ [x_2, x_5] = x_6;$

— $(1, 3, 4, 6/6, 3, 2, 1)$;

— $[\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha, 3\alpha+\beta]$;

— CQ: $N_{5,3,3} = N_{4,2} \times \mathfrak{a}_1$;

— Aut $N_{6,2,4}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}\,a_{22} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{11}\,a_{32} & {a_{11}}^2 a_{22} & -a_{21}\,{a_{11}}^2 & 0 \\ a_{51} & a_{52} & 0 & 0 & {a_{11}}^3 & 0 \\ a_{61} & a_{62} & u & {a_{11}}^2 a_{32} & a_{65} & {a_{11}}^3 a_{22} \end{bmatrix},$$

where $u = a_{11}a_{42} + a_{21}a_{52} - a_{22}a_{51}$, $\dim R_u = 10$, and $S = \mathrm{GL}_1^2$.

$N_{6,2,5}$: $[x_1, x_i] = x_{i+1},\ i = 2, 3, 5,\ [x_2, x_j] = x_{j+2},\ j = 3, 4;$

— $(1, 3, 4, 6/6, 4, 3, 1)$;

— $[\alpha, \beta, \alpha+\beta, 2\alpha+\beta, \alpha+2\beta, 2\alpha+2\beta]$;

— CQ: $N_{5,2,3}$;

— Aut $N_{6,2,5}$:

$$\mathrm{Aut}_0 : \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}\,a_{22} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{11}\,a_{32} & {a_{11}}^2 a_{22} & 0 & 0 \\ a_{51} & a_{52} & -a_{22}\,a_{31} & 0 & a_{11}\,{a_{22}}^2 & 0 \\ a_{61} & a_{62} & u & v & w & {a_{11}}^2 {a_{22}}^2 \end{bmatrix},$$

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus [-1] \oplus \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \oplus [-1],$$

where $u = a_{11}a_{52} - a_{22}a_{41}$, $v = -a_{11}a_{22}a_{31}$, $w = a_{11}a_{22}a_{32}$, $\dim R_u = 8$ and $S = \mathrm{GL}_1^2 \times \mathrm{Z}_2$.

$N_{6,2,6}:\ [x_1, x_2] = x_4,\ [x_1, x_3] = x_5,\ [x_1, x_4] = x_6,\ [x_3, x_5] = x_6;$

— $(1, 3, 6/6, 3, 1)$;

— $[\alpha, -\alpha + 2\beta, \beta, 2\beta, \alpha + \beta, \alpha + 2\beta]$;

— CQ: $N_{5,3,2}$;

— Aut $N_{6,2,6}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & \frac{{a_{33}}^2}{a_{11}} & -\frac{a_{31}\,a_{33}}{a_{11}} & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & {a_{33}}^2 & -a_{31}\,a_{33} & 0 \\ a_{51} & 0 & a_{53} & 0 & a_{11}\,a_{33} & 0 \\ a_{61} & a_{62} & a_{63} & a_{11}\,a_{42} & u & a_{11}\,{a_{33}}^2 \end{bmatrix},$$

where $u = a_{11}a_{43} + a_{31}a_{53} - a_{33}a_{51}$, $\dim R_u = 10$ and $S = \mathrm{GL}_1^3$.

$N_{6,2,7}:\ [x_1, x_i] = x_{i+1},\ i = 2, 3, 4,\ [x_2, x_3] = x_6;$

— $(2, 3, 4, 6/6, 4, 3, 2)$;

— $[\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, \alpha + 2\beta]$;

— CQ: $N_{4,2}$;

— Aut $N_{6,2,7}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}\,a_{22} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{11}\,a_{32} & {a_{11}}^2 a_{22} & 0 & 0 \\ a_{51} & a_{52} & a_{11}\,a_{42} & {a_{11}}^2 a_{32} & {a_{11}}^3 a_{22} & 0 \\ a_{61} & a_{62} & u & a_{11}\,a_{21}\,a_{22} & 0 & a_{11}\,{a_{22}}^2 \end{bmatrix},$$

where $u = a_{21}a_{32} - a_{22}a_{31}$, $\dim R_u = 9$, and $S = \mathrm{GL}_1^2$.

$N_{6,2,8}$: $[x_1, x_2] = x_3$, $[x_1, x_3] = x_5$, $[x_1, x_4] = x_6$, $[x_2, x_4] = x_5$;

— $(2, 4, 6/6, 3, 1)$;

— $[\alpha, \beta, \alpha + \beta, 2\alpha, 2\alpha + \beta, 3\beta]$;

— CQ: $N_{4,3}$;

— Aut $N_{6,2,8}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}\,a_{22} & a_{34} & 0 & 0 \\ a_{41} & a_{42} & 0 & {a_{11}}^2 & 0 & 0 \\ a_{51} & a_{52} & u & a_{54} & {a_{11}}^2 a_{22} & v \\ a_{61} & a_{62} & a_{11}\,a_{42} & a_{64} & 0 & {a_{11}}^3 \end{bmatrix},$$

where $u = a_{11}a_{32} + a_{21}a_{42} - a_{22}a_{41}$ and $v = a_{11}a_{34} + a_{11}^2 a_{21}$, $\dim R_u = 12$ and $S = \mathrm{GL}_1^2$.

$N_{6,2,9}$: $[x_1, x_2] = x_3$, $[x_1, x_3] = x_5$, $[x_2, x_3] = x_6$, $[x_2, x_4] = x_6$;

— $(2, 4, 6/6, 3, 2)$;

— $[\alpha, \beta, \alpha + \beta, \alpha + \beta, 2\alpha + \beta, \alpha + 2\beta]$;

— CQ: $N_{4,3} = N_{3,2} \times \mathfrak{a}_1$;

— Aut $N_{6,2,9}$:

$$\mathrm{Aut}_0 : \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}\,a_{22} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & a_{11}\,a_{22} & 0 & 0 \\ a_{51} & a_{52} & a_{11}\,a_{32} & a_{54} & {a_{11}}^2 a_{22} & 0 \\ a_{61} & a_{62} & u & a_{64} & 0 & a_{11}\,{a_{22}}^2 \end{bmatrix},$$

and

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

with $u = -a_{22}\ (a_{31} + a_{41})$, and $\dim R_u = 10$, and $S = \mathrm{GL}_1^2 \times \mathbf{Z}_2$.

$N_{6,2,10}$: $[x_1, x_2] = x_3$, $[x_1, x_3] = x_5$, $[x_2, x_3] = x_6$, $[x_2, x_4] = x_5$;

— $(2, 4, 6/6, 3, 2)$;

— $[\alpha, \beta, \alpha+\beta, 2\alpha, 2\alpha+\beta, \alpha+2\beta]$;

— CQ: $N_{4,3} = N_{3,2} \times \mathfrak{a}_1$;

— Aut $N_{6,2,10}$:

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}\,a_{22} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & {a_{11}}^2 & 0 & 0 \\ a_{51} & a_{52} & u & a_{54} & {a_{11}}^2 a_{22} & a_{11}\,a_{22}\,a_{12} \\ a_{61} & a_{62} & -a_{22}\,a_{31} & a_{64} & 0 & a_{11}\,{a_{22}}^2 \end{bmatrix},$$

where $u = a_{11}a_{32} - a_{12}a_{31} - a_{22}a_{41}$, $\dim R_u = 11$ and $S = \mathrm{GL}_1^2$.

$N_{6,3,1}$: $[x_1, x_i] = x_{i+2}$, $i = 2, 3$, $[x_2, x_5] = [x_3, x_4] = x_6$;

— $(1, 3, 6/6, 3, 1)$;

— $[\alpha, \beta, \gamma, \alpha+\beta, \alpha+\gamma, \alpha+\beta+\gamma]$;

— CQ: $N_{5,3,2}$;

— Aut $N_{6,3,1}$:

$$\mathrm{Aut}_0 : \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{11}\,a_{22} & 0 & 0 \\ a_{51} & a_{52} & \frac{a_{33}\,a_{42}}{a_{22}} & 0 & a_{11}\,a_{33} & 0 \\ a_{61} & a_{62} & a_{63} & -a_{22}\,a_{51} & -a_{33}\,a_{41} & a_{11}\,a_{22}\,a_{33} \end{bmatrix},$$

and

$$\sigma = \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus [1],$$

with $\dim R_u = 8$ and $S = \mathrm{GL}_1^3 \times \mathbf{Z}_2$.

$N_{6,3,2}$: $[x_1, x_2] = x_3$, $[x_1, x_3] = x_6$, $[x_4, x_5] = x_6$;

— $(1, 4, 6/6, 2, 1)$;

— $[\alpha, \beta+\gamma-2\alpha, \beta+\gamma-\alpha, \beta, \gamma, \beta+\gamma]$;

— CQ: $N_{5,4} = N_{3,2} \times \mathfrak{a}_2$;

— Aut $N_{6,3,2}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & \frac{e}{a_{11}^2} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & \frac{e}{a_{11}} & \frac{f}{a_{11}} & \frac{g}{a_{11}} & 0 \\ a_{41} & 0 & 0 & a_{44} & a_{45} & 0 \\ a_{51} & 0 & 0 & a_{54} & a_{55} & 0 \\ a_{61} & a_{62} & a_{11}\,a_{32} & a_{64} & a_{65} & e \end{bmatrix},$$

where $e = a_{44}a_{55} - a_{45}a_{54}$, $f = a_{44}a_{51} - a_{41}a_{54}$, $g = a_{45}a_{51} - a_{41}a_{55}$, $\dim R_u = 9$ and $S = \mathrm{GL}_1 \times \mathrm{GL}_2$.

$N_{6,3,3}$: $[x_1, x_2] = x_3$, $[x_1, x_4] = x_6$, $[x_2, x_3] = x_5$;

— $(2, 4, 6/6, 3, 1)$;

— $[\alpha, \beta, \alpha+\beta, \gamma, \alpha+2\beta, \alpha+\gamma]$;

— CQ: $N_{4,3} = N_{3,2} \times \mathfrak{a}_1$;

— Aut $N_{6,3,3}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}\,a_{22} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & a_{44} & 0 & 0 \\ a_{51} & a_{52} & -a_{22}\,a_{31} & a_{54} & a_{11}\,{a_{22}}^2 & 0 \\ a_{61} & a_{62} & a_{11}\,a_{42} & a_{64} & 0 & a_{44}\,a_{11} \end{bmatrix},$$

with $\dim R_u = 10$ and $S = \mathrm{GL}_1^3$.

$N_{6,3,4}$: $[x_1, x_2] = x_3$, $[x_2, x_3] = x_5$, $[x_2, x_4] = x_6$;

— $(2, 4, 6/6, 3, 1)$;

— $[\alpha, \beta, \alpha + \beta, \gamma, \alpha + 2\beta, \beta + \gamma]$;

— CQ: $N_{4,3} = N_{3,2} \times \mathfrak{a}_1$;

— Aut $N_{6,3,4}$:

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}\,a_{22} & a_{34} & 0 & 0 \\ a_{41} & a_{42} & 0 & a_{44} & 0 & 0 \\ a_{51} & a_{52} & -a_{22}\,a_{31} & a_{54} & a_{11}\,{a_{22}}^2 & a_{22}\,a_{34} \\ a_{61} & a_{62} & -a_{22}\,a_{41} & a_{64} & 0 & a_{44}\,a_{22} \end{bmatrix}$$

with $\dim R_u = 12$ and $S = \mathrm{GL}_1^3$.

$N_{6,3,5}$: $[x_1, x_2] = x_5$, $[x_1, x_4] = x_6$, $[x_2, x_3] = x_6$;

— $(2, 6/6, 2)$;

— $[\alpha, \beta, \gamma - \beta, \gamma - \alpha, \alpha + \beta, \gamma]$;

— CQ: $N_{4,4}$;

— Aut $N_{6,3,5}$:

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & \frac{a_{11}a_{66}}{e} & -\frac{a_{12}a_{66}}{e} & 0 & 0 \\ a_{41} & a_{42} & -\frac{a_{21}a_{66}}{e} & \frac{a_{22}a_{66}}{e} & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & e & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & u & a_{66} \end{bmatrix},$$

where $e = a_{11}a_{22} - a_{12}a_{21}$, $u = a_{11}a_{42} + a_{21}a_{32} - a_{12}a_{41} - a_{22}a_{31}$, $\dim R_u = 12$ and $S = \mathrm{GL}_2 \times \mathrm{GL}_1$.

$N_{6,3,6}$: $[x_1, x_2] = x_4$, $[x_1, x_3] = x_5$, $[x_2, x_3] = x_6$;

— Free nilpotent Lie algebra of class 2 with 3 generators;

— $(3, 6/6, 3)$;

— $[\alpha, \beta, \gamma, \alpha+\beta, \alpha+\gamma, \beta+\gamma]$;

— CQ: $N_{3,3}$;

— Aut $N_{6,3,6}$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & r & s & t \\ a_{51} & a_{52} & a_{53} & u & v & w \\ a_{61} & a_{62} & a_{63} & x & y & z \end{bmatrix},$$

where

$$\begin{aligned} r &= a_{11}a_{22} - a_{12}a_{21}, & s &= a_{11}a_{23} - a_{13}a_{21}, & t &= a_{12}a_{23} - a_{13}a_{22}, \\ u &= a_{11}a_{32} - a_{12}a_{31}, & v &= a_{11}a_{33} - a_{13}a_{31}, & w &= a_{12}a_{33} - a_{13}a_{32}, \\ x &= a_{21}a_{32} - a_{22}a_{31}, & y &= a_{21}a_{33} - a_{23}a_{31}, & z &= a_{22}a_{33} - a_{23}a_{32}, \end{aligned}$$

with $\dim R_u = 9$ and $S = \mathrm{GL}_3$.

DECOMPOSABLE ALGEBRAS

$N_{6,2,11}$: $N_{5,1} \times \mathfrak{a}_1$;

— Aut $N_{6,2,11}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & {a_{11}}^2 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & {a_{11}}^3 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{11}\,a_{32} & {a_{11}}^4 & 0 & 0 \\ a_{51} & a_{52} & u & v & {a_{11}}^5 & a_{56} \\ a_{61} & a_{62} & 0 & 0 & 0 & a_{66} \end{bmatrix},$$

where $u = a_{11}\,a_{42} + a_{21}\,a_{32} - {a_{11}}^2 a_{31}$, $v = {a_{11}}^2 a_{32} + a_{21}\,{a_{11}}^3$, $\dim R_u = 10$ and $S = \mathrm{GL}_1^2$.

$N_{6,3,7}$: $N_{5,2,1} \times \mathfrak{a}_1$;

— Aut $N_{6,3,7}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}\,a_{22} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{11}\,a_{32} & {a_{11}}^2 a_{22} & 0 & 0 \\ a_{51} & a_{52} & a_{11}\,a_{42} & {a_{11}}^2 a_{32} & {a_{11}}^3 a_{22} & a_{56} \\ a_{61} & a_{62} & 0 & 0 & 0 & a_{66} \end{bmatrix},$$

with $\dim R_u = 10$ and $S = \mathrm{GL}_1^2$.

$N_{6,3,8}$: $N_{5,2,2} \times \mathfrak{a}_1$;

— Aut $N_{6,3,8}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & {a_{11}}^2 & 0 & 0 & 0 \\ a_{41} & a_{42} & -a_{21}\,a_{11} & a_{11}\,a_{22} & 0 & 0 \\ a_{51} & a_{52} & a_{53} & u & {a_{11}}^2 a_{22} & a_{56} \\ a_{61} & a_{62} & a_{63} & 0 & 0 & a_{66} \end{bmatrix},$$

with $u = a_{11}\,a_{42} + a_{21}\,a_{32} - a_{22}\,a_{31}$, $\dim R_u = 12$ and $S = \mathrm{GL}_1^3$.

$N_{6,3,9}$: $N_{5,2,3} \times \mathfrak{a}_1$;

— Aut $N_{6,3,9}$:

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & u & 0 & 0 & 0 \\ a_{41} & a_{42} & v & a_{11}\,u & a_{12}\,u & a_{46} \\ a_{51} & a_{52} & w & a_{21}\,u & a_{22}\,u & a_{56} \\ a_{61} & a_{62} & 0 & 0 & 0 & a_{66} \end{bmatrix}$$

where $u = a_{11}\,a_{22} - a_{12}\,a_{21}$, $v = a_{11}\,a_{32} - a_{12}\,a_{31}$, $w = a_{21}\,a_{32} - a_{22}\,a_{31}$, $\dim R_u = 10$, $S = \mathrm{GL}_2 \times \mathrm{GL}_1$.

$N_{6,4,1}$: $N_{5,3,1} \times \mathfrak{a}_1$;

— Aut $N_{6,4,1}$: $\dim R_u = 9$, and $S = S' \times \mathrm{GL}_1$, with $S' = (\mathrm{GL}_1 \times \mathrm{Sp}_4)/\mathbf{Z}_2$.

$N_{6,4,2}$: $N_{5,3,2} \times \mathfrak{a}_1$;

— Aut $N_{6,4,2}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{11}\,a_{22} & a_{11}\,a_{23} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{11}\,a_{32} & a_{11}\,a_{33} & a_{56} \\ a_{61} & a_{62} & a_{63} & 0 & 0 & a_{66} \end{bmatrix},$$

with $\dim R_u = 13$, and $S = \mathrm{GL}_1^2 \times \mathrm{GL}_2$.

$N_{6,4,3}$: $N_{4,2} \times \mathfrak{a}_2$;

— Aut $N_{6,4,3}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}\,a_{22} & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{11}\,a_{32} & {a_{11}}^2 a_{22} & a_{45} & a_{46} \\ a_{51} & a_{52} & 0 & 0 & a_{55} & a_{56} \\ a_{61} & a_{62} & 0 & 0 & a_{65} & a_{66} \end{bmatrix},$$

with $\dim R_u = 11$, and $S = \mathrm{GL}_1^2 \times \mathrm{GL}_2$.

$N_{6,4,4}$: $N_{3,2} \times N_{3,2}$; or $[x_1, x_2] = x_5$, $[x_3, x_4] = x_6$;

— Aut $N_{6,4,4}$:

$$\text{Aut}_0 : \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & u & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & 0 & v \end{bmatrix},$$

$$\sigma = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where $u = a_{11}a_{22} - a_{21}a_{12}$, $v = a_{33}a_{44} - a_{34}a_{43}$, $\dim R_u = 8$, and $S = \mathrm{GL}_2^2 \times \mathbf{Z}_2$.

$N_{6,5}$: $N_{3,2} \times \mathfrak{a}_3$;

— Aut $N_{6,5}$:

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{11}a_{22} - a_{12}a_{21} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & 0 & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & 0 & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & 0 & a_{64} & a_{65} & a_{66} \end{bmatrix},$$

with $\dim R_u = 11$, and $S = \mathrm{GL}_2 \times \mathrm{GL}_3$.

$N_{6,6}$: $\mathfrak{a}_6$;

—Aut $N_{6,6} = \mathrm{GL}_6$.

3.2.3 Algebras of Dimension 6 over Algebraically Closed Fields of $\chi = 2$

In addition to the algebras over algebraically closed fields of $\chi \neq 2$, we have the following 5 extra indecomposable algebras for $\chi = 2$:

(A): $[x_1, x_i] = x_{i+1}$, $2 \leq i \leq 5$, $[x_2, x_3] = x_5 + x_6$, $[x_2, x_4] = x_6$;

— $(1, 2, 3, 4, 6/6, 4, 3, 2, 1)$;

— Characteristically nilpotent Lie algebra;

— CQ: $N_{5,1}$;

Remarks: (1) This algebra was first observed by Bratzlavsky [4], which turns out to be the only characteristically nilpotent Lie algebra in dimension 6 when $\chi = 2$; (2) When $\chi \neq 2$, this algebra is isomorphic to $N_{6,1,1}$, which is not characteristically nilpotent. In fact, in this case, the smallest algebra which is characteristically nilpotent is of dimension 7 (see Favre [9]).

(B): $[x_1, x_2] = x_4,\ [x_1, x_4] = x_5,\ [x_1, x_5] = x_6,\ [x_2, x_3] = x_5 + x_6,\ [x_3, x_4] = -x_6$;

— $(1, 2, 4, 6/6, 3, 2, 1)$;

— CQ: $N_{5,2,2}$;

Remark: When $\chi \neq 2$, this algebra is isomorphic to $N_{6,2,3}$.

(C): $[x_1, x_i] = x_{i+1},\ i = 2, 3, 5,\ [x_2, x_3] = x_5,\ [x_2, x_4] = [x_2, x_5] = x_6$;

— $(1, 3, 4, 6/6, 4, 3, 1)$;

— CQ: $N_{5,2,3}$;

Remark: When $\chi \neq 2$, this algebra is isomorphic to $N_{6,2,5}$.

(D): $[x_1, x_2] = x_4,\ [x_1, x_3] = x_5,\ [x_2, x_5] = x_6,\ [x_3, x_4] = x_6,\ [x_3, x_5] = x_6$;

— $(1, 3, 6/6, 3, 1)$;

— CQ: $N_{5,3,2}$;

Remark: When $\chi \neq 2$, it is isomorphic to $N_{6,3,1}$.

(E): $[x_1, x_2] = x_3,\ [x_1, x_3] = x_5,\ [x_1, x_4] = x_6, [x_2, x_3] = x_6,\ [x_2, x_4] = x_5$;

— $(2, 4, 6/6, 3, 2)$;

— CQ: $N_{4,3}$;

Remark: When $\chi \neq 2$, it is isomorphic to $N_{6,2,10}$.

3.3 The Proof for 6-Dimensional Algebras

From the description of Skjelbred and Sund's method, it is easy to see that, if $\mathfrak{g}'$ is a central extension of $\mathfrak{g}$ without Abelian factors , then the dimension of $Z(\mathfrak{g}')$ cannot exceed $\dim H^2(\mathfrak{g}, \mathbf{F})$. So an indecomposable 6 dimensional nilpotent Lie algebra cannot be a central extension of any 2-dimensional nilpotent Lie algebra. Furthermore the only 3 dimensional nilpotent Lie algebra that has nontrivial central extensions of dimension 6 is $N_{3,3}$, the Abelian algebra. So we will start from considering its central extensions. The 5-dimensional Abelian algebra $N_{5,5}$ has no 6 dimensional central extensions either, as all the skew symmetric bilinear maps are singular on a 5 dimensional vector space.

Revoy [23] has obtained a complete list for all the 2-step nilpotent Lie algebras of dimension ≤ 7 with the number of generators ≤ 4. There are 3 such algebras of dimension 6, i.e., $L_{6,1}$, $L_{6,2}$ and $L_{6,3}$ in his list, which are the central extenisons of 3 or 4 dimensional Abelian Lie algebras.

3.3.1 Extensions of 3-Dimensional Algebras

Central extensions of $N_{3,3}$:

$Z(\mathfrak{g})$: x_1, x_2, x_3; $[\mathfrak{g}, \mathfrak{g}]$: 0; $\dim H^2$: 3; Basis: $\Delta_{12}, \Delta_{13}, \Delta_{23}$;

There is only one 3-dimensional subspace, therefore the only representative for $G_3(H^2(\mathfrak{g}, \mathbf{F}))$ can be chosen to be $A_1 = [1,0,0], A_2 = [0,1,0], A_3 = [0,0,1]$, corresponding to $N_{6,3,6}$.

So the central extension of $N_{3,3}$ of dimension 6 is:

$$\boxed{N_{6,3,6}: \quad [x_1, x_2] = x_4, \quad [x_1, x_3] = x_5 \quad [x_2, x_3] = x_6.}$$

Remark: Revoy [23] has also obtained this algebra ($L_{6,1}$). We can see that the Skjelbred-Sund method works quite well in this case.

3.3.2 Extensions of 4-Dimensional Algebras

Central extensions of $N_{4,2}$:

$Z(\mathfrak{g})$: x_4; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_4; $Z^2(\mathfrak{g})$: $C_{24} = C_{34} = 0$; $W(H^2)$: $C_{12} = C_{13} = 0$; $\dim H^2$: 2; Basis: Δ_{14}, Δ_{23};

There is only one 2-dimensional subspace in $H^2(\mathfrak{g}, \mathbf{F})$. Then the only representative in $G_2(H^2(\mathfrak{g}, \mathbf{F}))$ can be chosen to be $A = [1,0]$ and $B = [0,1]$, corresponding to $N_{6,2,7}$.

So the central extension of $N_{4,2}$ of dimension 6 is

$$\boxed{N_{6,2,7}: \quad [x_1, x_i] = x_{i+1}, i = 2,3,4, \quad [x_2, x_3] = x_6.}$$

Central extensions of $N_{4,3}$:

$Z(\mathfrak{g})$: x_3, x_4; $[\mathfrak{g},\mathfrak{g}]$: x_3; $Z^2(\mathfrak{g})$: $C_{34}=0$; $W(H^2)$: $C_{12}=0$; $\dim H^2$: 4; Basis: $\Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}$;

Group Action: $a\Delta_{13}+b\Delta_{14}+c\Delta_{23}+d\Delta_{24}$, let $\delta := a_{11}a_{22}-a_{12}a_{21}$, then

$a \to aa_{11}\delta + ca_{21}\delta$; $b \to aa_{11}a_{34}+ba_{11}a_{44}+ca_{21}a_{34}+da_{21}a_{44}$; $c \to aa_{12}\delta + ca_{22}\delta$; $d \to aa_{12}a_{34}+ba_{12}a_{44}+ca_{22}a_{34}+da_{22}a_{44}$.

Let V_0 be the subspace generated by Δ_{14} and Δ_{24}, it is a submodule under the group action.

Let L be any two-dimensional subspace of $H^2(\mathfrak{g},\mathbf{F})$. Denote $L = A\wedge B$, where $A, B \in H^2(\mathfrak{g},\mathbf{F})$.

Case 1: $L\bigcap V_0 \neq 0$. Then we assume that in A, both $a=c=0$. As one of $b, d \neq 0$, we can always make $b=1$ and $d=0$ to assume $A=[0,1,0,0]$. Fixing A, we require $a_{12}=0$ and $a_{11}a_{44}=1$. Now assume $B=[a,0,c,d]$ and one of $a,c\neq 0$. We have $a \to aa_{11}\delta + ca_{21}\delta$; $b=0 \to aa_{11}a_{34}+ca_{21}a_{34}+da_{21}a_{44}$; $c \to ca_{22}\delta$; $d \to ca_{22}a_{34}+da_{22}a_{44}$.

If $c\neq 0$, then make $a=d=0$ by solving for a_{21} and a_{34} respectively to get (1) $B=[0,0,1,0]$, with $A\wedge B$ corresponding to $N_{6,3,3}$.

If $c=0$, then $a\neq 0$, depending on whether $d=0$ or not, we get two representatives for B: (2) $B=[1,0,0,0]$ (which can be easily showed to be in the same orbit as $A\wedge B$, where $A=[0,0,0,1]$ and $B=[0,0,1,0]$, corresponding to $N_{6,3,4}$) and (3) $B=[1,0,0,1]$ ($A\wedge B$ corresponding to $N_{6,2,8}$).

Case 2: $L\bigcap V_0 = 0$. Then at least one of a, c in both A and B are nonzero. Assume $A=[1,b,c,d]$. Make $b=c=0$ in A, and depending on whether $d=0$ or not, we get two representatives for $A=[1,0,0,0]$ and $A=[1,0,0,1]$.

For $A=[1,0,0,0]$, assume $B=[0,b,c,d]$. Then $c\neq 0$ and one of $b,d\neq 0$. Fix A (up to a scalar), we require $a_{12}=a_{34}=0$. Now in B, we have

$a=0\to 0$ (by subtracting a multiple of A from B); $b\to ba_{11}a_{44}+da_{21}a_{44}$; $c\to ca_{22}\delta$; $d\to da_{22}a_{44}$;

If $d\neq 0$, make $b=0$ and get (4) $B=[0,0,1,1]$ ($A\wedge B$ corresponding to $N_{6,2,9}$).

If $d=0$, then $b\neq 0$ and get (5) $B=[0,1,1,0]$ ($A\wedge B$ corresponding to $N_{6,2,10}$).

For $A=[1,0,0,1]$, assume $B=[0,b,c,d]$, with $c\neq 0$. To fix A (up to a scalar), we require $a_{44}=a_{11}^2$, $a_{12}=0$, $a_{34}=-a_{21}a_{11}$. Then

$a=0\to ca_{21}a_{11}a_{22}$; $b\to ba_{11}a_{44}+ca_{21}a_{34}+da_{21}a_{44}$; $c\to ca_{22}\delta$; $d\to ca_{22}a_{11}a_{21}+da_{22}a_{44}$;

Subcase 2.1: $\chi\neq 2$. We can make $a=d$ by solving for a_{21} and subtract a multiple of A from B to make both $a=d=0$. Then depending on $b=0$ or not, we get two representatives for B: (4') $B=[0,1,1,0]$ ($A\wedge B$ corresponding to $N_{6,2,9}$) and (5') $B=[0,0,1,0]$ ($A\wedge B$ corresponding to $N_{6,2,10}$).

We prove at first that the following two pairs are isomorphic: (4) and (4'), (5) and (5').

For (4) and (4'), take $x_1 \to x_1 - x_2$, $x_2 \to x_1 + x_2$, $x_3 \to 2x_3$, $x_4 \to x_3 + x_4$, $x_5 \to 2x_5 - 2x_6$, $x_6 \to 2x_5 + 2x_6$.

For (5') and (5), take $x_1 \to x_2$, $x_2 \to x_1$, $x_3 \to -x_3$, $x_4 \to -x_4$, $x_5 \to -x_6$ and $x_6 \to -x_5$.

To show the nonisomorphism between the algebras, we just need to compare the algebras among the same group as follows:

Group 1: (1), (2) and (3);

Group 2: (4) and (5').

In Group 1, take (1) as an example. To show (1) is not isomorphic to (2) and (3), we just compare their orbits. In (1), $A = [0,1,0,0]$ and $B = [0,0,1,0]$. Under the group action, we have $A \to [0, a_{11}a_{44}, 0, a_{12}a_{44}]$ and $B \to [a_{21}\delta, a_{21}a_{34}, a_{22}\delta, a_{22}a_{34}]$.

So the wedge product is

$$\begin{aligned} A \wedge B \to \quad & a_{21}\delta a_{11}a_{44}\Delta_{14} \wedge \Delta_{13} + \delta a_{44}a_{34}\Delta_{14} \wedge \Delta_{24} + a_{11}a_{44}a_{22}\delta\Delta_{14} \wedge \Delta_{23} \\ & + a_{12}a_{44}a_{21}\delta\Delta_{24} \wedge \Delta_{13} + a_{12}a_{44}a_{22}\delta\Delta_{24} \wedge \Delta_{23} \end{aligned}$$

Now compare with (2) and (3), we know the coefficients of $\Delta_{14} \wedge \Delta_{23}$, $\Delta_{24} \wedge \Delta_{13}$ and $\Delta_{24} \wedge \Delta_{23}$ are zero, i.e., $a_{11}a_{22}a_{44} = a_{12}a_{21}a_{44} = a_{12}a_{22}a_{44} = 0$, as $a_{44} \neq 0$, we must have $a_{22} = 0$, and $a_{12}a_{21} = 0$, which is impossible, so (1) cannot be isomorphic to (2) or (3).

Similarly we can prove the distinctness between all the other algebras.

Subcase 2.2: $\chi = 2$. Now we consider two subcases:

Subcase 2.2.1: $d = 0$. Then we can make $a = d = 0$ by taking $a_{21} = 0$. And depending on $b = 0$ or not, we get two representatives for B: (6) $B = [0,1,1,0]$ ($A \wedge B$ corresponding to (E)) and (5") $B = [0,0,1,0]$ ($A \wedge B$ corresponding to $N_{6,2,10}$, which can be seen easily from the isomorphim given between (5) and (5')).

Subcase 2.2.2: $d \neq 0$. Then depending on $b = 0$ or not, we get two representatives for B: (4"): $B = [0,0,1,1]$ or (7) $B = [0,1,1,1]$ ($A \wedge B$ corresponding to $N_{6,2,9}$).

We can prove that (4") is isomorphic to (4): $x_1 \to -x_2$, $x_2 \to x_1 + x_2$, $x_3 \to x_3$, $x_4 \to x_3 - x_4$, $x_5 \to x_5 + x_6$, $x_7 \to -x_7$.

From Subcase 2.1, we know that (6) is isomorphic to (4) when $\chi \neq 2$. Now we compare the orbits of (4) with both (6) and (7) under the condition that $\chi = 2$.

In (4), $A = [1,0,0,0]$ and $B = [0,0,1,1]$. Then under the automorphism group, $A \to$

$[a_{11}\delta, a_{11}a_{34}, a_{12}\delta, a_{12}a_{34}]$ and $B \to [a_{21}\delta, a_{21}a_{34} + a_{21}a_{44}, a_{22}\delta, a_{22}a_{34} + a_{22}a_{44}]$. Then

$$\begin{aligned} A \wedge B \to\ & (a_{11}\delta a_{21}a_{44})\Delta_{13} \wedge \Delta_{14} \\ & +(a_{11}\delta a_{22}\delta - a_{12}\delta a_{21}\delta)\Delta_{13} \wedge \Delta_{23} \\ & +(a_{11}\delta(a_{22}a_{34} + a_{22}a_{44}) - a_{12}a_{34}a_{21}\delta)\Delta_{13} \wedge \Delta_{24} \\ & +(a_{11}a_{34}a_{22}\delta - a_{12}\delta(a_{21}a_{34} + a_{21}a_{44}))\Delta_{14} \wedge \Delta_{23} \\ & +(a_{11}a_{34}(a_{22}a_{34} + a_{22}a_{44}) - a_{12}a_{34}a_{22}\delta)\Delta_{14} \wedge \Delta_{24} \\ & +(a_{12}\delta a_{22}a_{44})\Delta_{23} \wedge \Delta_{24} \end{aligned}$$

Compare with (6), if (6) and (4) are in the same orbit, we would require that the coefficients of $\Delta_{13} \wedge \Delta_{24}$ and $\Delta_{14} \wedge \Delta_{23}$ to be zero, which give us

$$a_{34} = \frac{-a_{11}a_{22}a_{44}}{\delta} = \frac{a_{12}a_{21}a_{44}}{\delta}.$$

and leads to the singularitiy of the automorphism group. Therefore (4) and (6) cannot be isomorphic when $\chi = 2$.

Compare with (7), if (7) and (4) are in the same orbit, then the coefficients of $\Delta_{13} \wedge \Delta_{14}$, $\Delta_{13} \wedge \Delta_{23}$, $\Delta_{13} \wedge \Delta_{24}$, $\Delta_{14} \wedge \Delta_{23}$, $\Delta_{24} \wedge \Delta_{14}$ and $\Delta_{24} \wedge \Delta_{23}$ are all equal (nonzero), while the coefficient of $\Delta_{14} \wedge \Delta_{23}$ is 0. A simple computation shows that we can indeed find a set of solutions while maintaining the nonsingularity of the automorphism group. Therefore (7) and (4) are isomorphic.

So the central extensions of $N_{4,3}$ of dimension 6 are:

$N_{6,2,8}$:	$[x_1, x_2] = x_3$,	$[x_1, x_3] = x_5$,	$[x_1, x_4] = x_6$,
	$[x_2, x_4] = x_5$;		
$N_{6,2,9}$:	$[x_1, x_2] = x_3$,	$[x_1, x_3] = x_5$,	$[x_2, x_3] = x_6$,
	$[x_2, x_4] = x_6$;		
$N_{6,2,10}$:	$[x_1, x_2] = x_3$,	$[x_1, x_3] = x_5$,	$[x_2, x_3] = x_6$,
	$[x_2, x_4] = x_5$;		
$N_{6,3,3}$:	$[x_1, x_2] = x_3$,	$[x_1, x_4] = x_6$,	$[x_2, x_3] = x_5$;
$N_{6,3,4}$:	$[x_1, x_2] = x_3$,	$[x_2, x_3] = x_5$,	$[x_2, x_4] = x_6$;
(E):	(for $\chi = 2$ only)		
	$[x_1, x_2] = x_3$,	$[x_1, x_3] = x_5$,	$[x_1, x_4] = x_6$,
	$[x_2, x_3] = x_6$,	$[x_2, x_4] = x_5$.	

Central extensions of $N_{4,4}$:

According to Revoy [23], the central extension of $N_{4,4}$ of dimension 6 is:

$N_{6,3,5}$:	$[x_1, x_2] = x_5$,	$[x_1, x_4] = x_6$,	$[x_2, x_3] = x_6$;

3.3.3 Extensions of 5-Dimensional Algebras

Central extensions of $N_{5,1}$:

$Z(\mathfrak{g})$: x_5; $[\mathfrak{g},\mathfrak{g}]$: x_3, x_4, x_5; $Z^2(\mathfrak{g})$: $C_{15} - C_{24} = 0$, $C_{34} + C_{25} = 0, C_{35} = C_{45} = 0$; $W(H^2)$: $C_{12} = C_{13} = C_{14} = 0$; $\dim H^2$: 3; Basis: $\Delta_{15} + \Delta_{24}, \Delta_{23}, \Delta_{25} - \Delta_{34}$;

Group Action: $a(\Delta_{15} + \Delta_{24}) + b\Delta_{23} + c(\Delta_{25} - \Delta_{34})$

$a \to aa_{11}^6 + ca_{11}^5 a_{21}$; $b \to -2aa_{11}^4 a_{21} + ba_{11}^5 + ca_{11}(2a_{11}^2 a_{42} - a_{32}^2 - a_{11}^2 a_{21}^2)$; $c \to ca_{11}^7$;

One of $a, c \neq 0$. When $c \neq 0$, make $a = 0$ by solving for a_{21}, and $b = 0$ by solving for a_{32}, and get the representative $[0,0,1]$ (corresponding to $N_{6,1,2}$).

When $c = 0$, then $a \neq 0$, get $a \to aa_{11}^6$ and $b \to -2aa_{11}^4 a_{21} + ba_{11}^5$. If $\chi \neq 2$, make $b = 0$ by solving for a_{21} and get $[1,0,0]$ (corresponding to $N_{6,1,1}$); If $\chi = 2$, then we have $b \to ba_{11}^5$, and get two representatives $[1,0,0]$ (corresponding to $N_{6,1,1}$) for $b = 0$ and $[1,1,0]$ (corresponding to (A)) for $b \neq 0$.

So the central extensions of $N_{5,1}$ of dimension 6 are:

(A):	(for $\chi = 2$ only)		
	$[x_1, x_i] = x_{i+1}, 2 \leq i \leq 5$,	$[x_2, x_3] = x_5 + x_6$,	$[x_2, x_4] = x_6$;
$N_{6,1,1}$:	$[x_1, x_i] = x_{i+1}, 2 \leq i \leq 5$,	$[x_2, x_i] = x_{i+2}, i = 3, 4$;	
$N_{6,1,2}$:	$[x_1, x_i] = x_{i+1}, i = 2, 3, 4$,	$[x_2, x_3] = x_5$,	
	$[x_2, x_5] = x_6$,	$[x_3, x_4] = -x_6$.	

The central extensions of $N_{5,2,1}$ can be found in chapter 2, Example 5.

The central extensions of $N_{5,2,2}$ can be found in chapter 2, Example 1.

Central extensions of $N_{5,2,3}$:

$Z(\mathfrak{g})$: x_4, x_5; $[\mathfrak{g},\mathfrak{g}]$: x_3, x_4, x_5; $Z^2(\mathfrak{g})$: $C_{15} - C_{24} = 0$, $C_{34} = C_{35} = C_{45} = 0$; $W(H^2)$: $C_{12} = C_{13} = C_{23} = 0$; $\dim H^2$: 3; Basis: $\Delta_{14}, \Delta_{15} + \Delta_{24}, \Delta_{25}$;

Group Action: $a\Delta_{14} + b(\Delta_{15} + \Delta_{24}) + c\Delta_{25}$

Let $\delta := a_{11}a_{22} - a_{12}a_{21}$.

$a \to (aa_{11}^2 + 2ba_{11}a_{21} + ca_{21}^2)\delta$; $b \to (aa_{11}a_{12} + ba_{11}a_{22} + ba_{12}a_{21} + ca_{21}a_{22})\delta$; $c \to (aa_{12}^2 + 2ba_{12}a_{22} + ca_{22}^2)\delta$;

Case 1: $\chi \neq 2$.

Subcase 1: $b^2 - ac \neq 0$. Assume $b \neq 0$. Then we can make $a = c = 0$ by solving for a_{21} and a_{22}, i.e.

$$a_{21} = \frac{-b \pm \sqrt{b^2 - ac}}{c} a_{11}, \; a_{22} = \frac{-b \pm \sqrt{b^2 - ac}}{c} a_{12}.$$

Choose appropriately a_{21} and a_{22} to ensure that the automorphism is nonsingular, we get the representative $[0,1,0]$, corresponding to $N_{6,2,5}$.

Subcase 2: $b^2 - ac = 0$. Assume $c \neq 0$. Making $a = 0$ by solving for a_{12}, we get $a_{22} = -ba_{12}/c$. Plug in the formula for b, we have $b \to (aa_{11}a_{12} - b^2 a_{11}a_{12}/c + ba_{12}a_{21} - ba_{12}a_{21}) = 0$, since $b^2 = ac$. So we have the representative $[0,0,1]$, which contains the central element x_4 in its kernel. Therefore we omit it.

Case 2: $\chi = 2$. Then we have $a \to (aa_{11}^2 + ca_{21}^2)\delta$, $b \to (aa_{11}a_{12} + ba_{11}a_{22} + ba_{12}a_{21} + ca_{21}a_{22})\delta$, $c \to (aa_{12}^2 + ca_{22}^2)\delta$. If both $a, c = 0$, we get $[0,1,0]$ (corresponding to $N_{6,2,5}$), and if one of $a, c \neq 0$, make $c = 1$ and $a = 0$, get $[0,1,1]$ (corresponding to (C)).

So the central extensions of $N_{5,2,3}$ of dimension 6 are:

(C):	(for $\chi = 2$ only)			
	$[x_1, x_i] = x_{i+1},\ i = 2,3,5$	$[x_2, x_3] = x_5$	$[x_2, x_4] = x_6$,	
	$[x_2, x_5] = x_6$;			
$N_{6,2,5}$:	$[x_1, x_i] = x_{i+1},\ i = 2,3,5$	$[x_2, x_3] = x_5$	$[x_2, x_4] = x_6$.	

Central extension of $N_{5,3,1}$:

$Z(\mathfrak{g})$: x_5; $[\mathfrak{g},\mathfrak{g}]$: x_5; $Z^2(\mathfrak{g})$: $C_{15} = 0$, $C_{25} = C_{35} = C_{45} = 0$; $W(H^2)$: $C_{12} = 0$; $\dim H^2$: 5; Basis: $\Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{34}$;

It is obvious from this basis that all the elements in $H^2(\mathfrak{g}, \mathbf{F})$ have x_5 in its kernel, so it does not have any central extension without Abelian factors.

Central extensions of $N_{5,3,2}$:

$Z(\mathfrak{g})$: x_4, x_5; $[\mathfrak{g},\mathfrak{g}]$: x_4, x_5; $Z^2(\mathfrak{g})$: $C_{45} = 0$, $C_{25} - C_{34} = 0$; $W(H^2)$: $C_{12} = C_{13} = 0$; $\dim H^2$: 6; Basis: $\Delta_{14}, \Delta_{15}, \Delta_{23}, \Delta_{24}, \Delta_{25} + \Delta_{34}, \Delta_{35}$;

Group Action: $a\Delta_{14} + b\Delta_{15} + c\Delta_{23} + d\Delta_{24} + e(\Delta_{25} + \Delta_{34}) + f\Delta_{35}$;

$a \to aa_{11}^2 a_{22} + ba_{11}^2 a_{32} + da_{11}a_{21}a_{22} + ea_{11}(a_{21}a_{32} + a_{22}a_{31}) + fa_{11}a_{32}a_{31}$;

$b \to aa_{11}^2 a_{23} + ba_{11}^2 a_{33} + da_{11}a_{21}a_{23} + ea_{11}(a_{21}a_{33} + a_{23}a_{31}) + fa_{11}a_{33}a_{31}$;

$c \to c(a_{22}a_{33} - a_{32}a_{23}) + d(a_{22}a_{43} - a_{42}a_{23}) + e(a_{22}a_{53} - a_{52}a_{23}) + e(a_{32}a_{43} - a_{42}a_{33}) + f(a_{32}a_{53} - a_{52}a_{33})$;

$d \to da_{11}a_{22}^2 + 2ea_{11}a_{22}a_{32} + fa_{11}a_{32}^2$;

$e \to da_{11}a_{22}a_{23} + ea_{11}(a_{22}a_{33} + a_{23}a_{32}) + fa_{11}a_{32}a_{33}$;

$f \to da_{11}a_{23}^2 + 2ea_{11}a_{23}a_{33} + fa_{11}a_{33}^2$.

We have $e \neq 0$ or when $e = 0$, one of $a, d \neq 0$ and one of $b, f \neq 0$.

Case 1. $\chi \neq 2$. As one of $d, e, f \neq 0$, otherwise the 2-cocyles will contain some nontrivial elements from the center in their kernel.

When $e^2 - df \neq 0$, we can always make $e = 1$. Then we further make $d = f = 0$ by solving for a_{22} and a_{23} respectively to get

$$a_{22} = \frac{-e \pm \sqrt{e^2 - df}}{d} a_{32}, \quad a_{23} = \frac{-e \pm \sqrt{e^2 - df}}{d} a_{33}.$$

Choose appropriately a_{22} and a_{23} to ensure that the automorphism is nonsingular, we will get the representative $[0, 0, 0, 0, 1, 0]$ (corresponding to $N_{6,3,1}$).

When $e^2 = df$, we can make $d = e = 0$ and $f = 1$ instead. Since $e^2 = df$, by solving for a_{22} to make $d = 0$, we get $a_{22} = -ea_{32}/d$. Plug in the expression for e, we have

$$\begin{aligned} &da_{11}a_{22}a_{23} + ea_{11}(a_{22}a_{33} + a_{23}a_{32}) + fa_{11}a_{32}a_{33} \\ &= a_{11}(-ea_{32}a_{23}/d + e(-ea_{32}a_{33}/d + a_{23}a_{32}) + fa_{32}a_{33}) = 0. \end{aligned}$$

So we have $d = e = 0$, and assume $f = 1$. Now we can further make $b = c = 0$ by solving for a_{52} and a_{31}. Now we need $a \neq 0$ and get a representative $B = [1, 0, 0, 0, 0, 1]$ (corresponding to $N_{6,2,6}$)

Case 2. $\chi = 2$. Then $d \to da_{11}a_{22}^2 + fa_{11}a_{32}^2$; $f \to da_{11}a_{23}^2 + fa_{11}a_{33}^2$.

When both $d, f = 0$, then $e \neq 0$, we can make $a = b = c = 0$ and get a representative $[0, 0, 0, 0, 1, 0]$ (corresponding to $N_{6,3,1}$). When one of $d, f \neq 0$, make $d = 0$ and $f = 1$. If $e \neq 0$, then make $a = b = c = 0$ and get the representative $[0, 0, 0, 0, 1, 1]$ (correponding to (D)), If $e = 0$, Then make $b = c = 0$, and require $a \neq 0$ to get $[1, 0, 0, 0, 0, 1]$ (corresponding to $N_{6,2,6}$).

So the central extensions of $N_{5,3,2}$ of dimension 6 are:

(D):	(for $\chi = 2$ only)		
	$[x_1, x_2] = x_4$,	$[x_1, x_3] = x_5$,	$[x_2, x_5] = x_6$,
	$[x_3, x_4] = x_6$,	$[x_3, x_5] = x_6$;	
$N_{6,2,6}$:	$[x_1, x_2] = x_4$,	$[x_1, x_3] = x_5$,	$[x_1, x_4] = x_6$,
	$[x_3, x_5] = x_6$;		
$N_{6,3,1}$:	$[x_1, x_2] = x_4$,	$[x_1, x_3] = x_5$,	$[x_2, x_5] = x_6$,
	$[x_3, x_4] = x_6$.		

Central extensions of $N_{5,3,3}$:

$Z(\mathfrak{g})$: x_4, x_5; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_4; $Z^2(\mathfrak{g})$: $C_{24} = C_{34} = C_{35} = C_{45} = 0$; $W(H^2)$: $C_{12} = C_{13} = 0$; $\dim H^2$: 4; Basis: $\Delta_{14}, \Delta_{15}, \Delta_{23}, \Delta_{25}$;

Group Action: $a\Delta_{14} + b\Delta_{15} + c\Delta_{23} + d\Delta_{25}$;

$a \to aa_{11}^3a_{22}$; $b \to aa_{11}a_{45} + ba_{11}a_{55} + da_{21}a_{55}$; $c \to ca_{11}a_{22}^2$; $d \to da_{22}a_{55}$;

Both $a, d \neq 0$. Make $b = 0$, and get two representatives depending on whether $c = 0$ or not, i.e., $[1, 0, 0, 1]$ (corresponding to $N_{6,2,4}$) or $[1, 0, 1, 1]$ (corresponding to $N_{6,1,4}$).

So the central extensions of $N_{5,3,3}$ of dimension 6 are:

$N_{6,1,4}:$	$[x_1, x_i] = x_{i+1},\ i = 2, 3$	$[x_1, x_4] = x_6,$	
	$[x_2, x_3] = x_6,$	$[x_2, x_5] = x_6;$	
$N_{6,2,4}:$	$[x_1, x_i] = x_{i+1},\ i = 2, 3$	$[x_1, x_4] = x_6,$	$[x_2, x_5] = x_6;$

Central extensions of $N_{5,4}$:

$Z(\mathfrak{g})$: x_3, x_4, x_5; $[\mathfrak{g}, \mathfrak{g}]$: x_3; $Z^2(\mathfrak{g})$: $C_{34} = C_{35} = 0$; $W(H^2)$: $C_{12} = 0$; $\dim H^2$: 7; Basis: $\Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{23}, \Delta_{24}, \Delta_{25}, \Delta_{45}$;

Group Action: $a\Delta_{13} + \Delta_{14} + b\Delta_{15} + c\Delta_{23} + d\Delta_{24} + e\Delta_{25} + f\Delta_{45}$;

Let $\delta := a_{11}a_{22} - a_{12}a_{21}$, then

$a \to (aa_{11} + da_{21})\delta$;

$b \to aa_{11}a_{34} + ba_{11}a_{44} + ca_{11}a_{54} + da_{21}a_{34} + ea_{21}a_{44} + fa_{21}a_{54} + g(a_{41}a_{54} - a_{51}a_{44})$;

$c \to aa_{11}a_{35} + ba_{11}a_{45} + ca_{11}a_{55} + da_{21}a_{35} + ea_{21}a_{45} + fa_{21}a_{55} + g(a_{41}a_{55} - a_{51}a_{45})$;

$d \to (aa_{12} + da_{22})\delta$;

$e \to aa_{12}a_{34} + ba_{12}a_{44} + ca_{12}a_{54} + da_{22}a_{34} + ea_{22}a_{44} + fa_{22}a_{54} + g(a_{42}a_{54} - a_{52}a_{44})$;

$f \to aa_{12}a_{35} + ba_{12}a_{45} + ca_{12}a_{55} + da_{22}a_{35} + ea_{22}a_{45} + fa_{22}a_{55} + g(a_{42}a_{55} - a_{52}a_{45})$;

$g \to g(a_{44}a_{55} - a_{54}a_{45})$;

One of $a, d \neq 0$. Can always make $a = 1$ and $d = 0$. Make $b = 0$ by solving for a_{34}, $c = 0$ for a_{35}. Now fix a, b, c, d and we have

$a \to a_{11}\delta = 1$; $b \to a_{11}a_{34} + ea_{21}a_{44} + fa_{21}a_{54} + g(a_{41}a_{54} - a_{51}a_{44}) = 0$; $c \to a_{11}a_{35} + ea_{21}a_{45} + fa_{21}a_{55} + g(a_{41}a_{55} - a_{51}a_{45}) = 0$; $d \to a_{12}\delta = 0$; $e \to a_{12}a_{34} + ea_{22}a_{44} + fa_{22}a_{54} + g(a_{42}a_{54} - a_{52}a_{44})$; $f \to a_{12}a_{35} + ea_{22}a_{45} + fa_{22}a_{55} + g(a_{42}a_{55} - a_{52}a_{45})$; $g \to g(a_{44}a_{55} - a_{54}a_{45})$;

So we have $a_{12} = 0$, and we can solve for a_{34} and a_{35} to keep $b = c = 0$. If $e \neq 0$, we can solve for a_{44} (let $a_{52} = 0$) to make it 0. Then we require $g \neq 0$ (to ensure that the kernel of the cocyles do not contain x_4). With $g \neq 0$, we can make further $f = 0$ by solving for a_{42}. So we get the representative $[1, 0, 0, 0, 0, 0, 1]$, corresponding to $N_{6,3,2}$.

So the central extension of $N_{5,4}$ of dimension 6 is:

$N_{6,3,2}:$	$[x_1, x_2] = x_3$	$[x_1, x_3] = x_6,$	$[x_4, x_5] = x_6.$

Chapter 4

List of 7-Dimensional Nilpotent Lie Algebras

In this chapter we list the presentations of all nonisomorphic indecomposable 7-dimensional nilpotent Lie algebras in the following two cases: (1) over algebraically closed fields of characteristic $\neq 2$ and (2) over the real field. A multiplication table for each algebra is given, with nonzero brackets only.

Over the algebraically closed fields, there are 6 one parameter continuous families, and 119 isolated algebras in total (when $\chi = 3$, there are 120). Over the real field, in addition to the algebras in the first list, we find 3 one parameter continuous families and 21 isolated algebras, which makes it a total of 9 one parameter continuous families and 140 isolated algebras in this case.

We follow Seeley's labelling of algebras when $\mathbf{F}$ is algebraically closed, i.e., each algebra is labelled by its upper central series dimensions plus an additional letter to distinguish nonisomorphic algebras. For example, the algebras having a center Z of dimension 3, and a second center Z^2 of dimension 7 are listed as (37A), (37B), (37C),and (37D) — in total 4 algebras. The algebra in our list having the same label as an algebra in Seeley's list are always isomorphic. However, our presentations of the Lie algebras may be different from those of Seeley's. If this is the case, then an explicit isomorphism can be always found in the proof where the algebras arise, for example, see (27A) and (27B) in Chapter 5.

When the ground field is $\mathbf{R}$, we may get more algebras. In this case, we will use L_i to denote those algebras that are isomorphic to L over $\mathbf{C}$. For example, if we consider all the algebras with the upper central series dimension (37) over $\mathbf{R}$, we get two more algebras, denoted by ($37\mathrm{B}_1$) and ($37\mathrm{D}_1$), meaning that over $\mathbf{C}$, these two algebras are isomorphic to (37B) and (37D) respectively.

For the one-parameter continuous families, a variable λ is used to denote a structure constant that may take on arbitrary values (with some exceptions) in $\mathbf{F}$. An invariant $I(\lambda)$ is given for each family in which multiple values of λ yield isomorphic algebras, i.e., if $I(\lambda_1) = I(\lambda_2)$, then the two corresponding algebras are isomorphic and conversely.

The 6 one-parameter continuous families over algebraically closed fields ($\chi \neq 2$) are:

(123457I): λ arbitrary.
(12457N): λ arbitrary, with invariant $I(\lambda) = \lambda + \lambda^{-1}$.
(1357M): $\lambda \neq 0$.
(1357N): λ arbitrary.
(1357S): $\lambda \neq 1$.
(147E): $\lambda \neq 0, 1$, with invariant $I(\lambda) = \frac{(1-\lambda+\lambda^2)^3}{\lambda^2(\lambda-1)^2}$.

Unlike in Seeley's list over **C**, we no longer list separately those algebras which are just special cases of the families of the same upper central series dimensions. To be exact, as (123457G), (12457M), (1357K) and (147C) in Seeley's list are respectively the special cases of (123457I) by taking $\lambda = 1$, (12457N) by taking $\lambda = 0$, (1357M) by taking $\lambda = 1/2$, and (147E) by taking $\lambda = 1/2$, instead of listing them separately, we include them in (123457I), (1357M) and (147E) respectively as special cases. That is why our list has 119 (for $\chi \neq 3$) isolated algebras while Seeley's has 124, with the merging of the above 4 algebras and also the deleting of (13457H), which is not a Lie algebra at all.

We also want to point out that in our list, (147E) becomes (247P) if we let $\lambda = 0$ or 1, (1357S) becomes (2357D) if $\lambda = 1$, (1357M) becomes (2357B) if $\lambda = 0$. Although it is more natural to include all these special cases in the corresponding continuous families, we still list them separately, due to their different upper central series dimensions.

Over **R**, there are 3 additional one-parameter families:

($12457N_2$): $\lambda \geq 0$.
($1357QRS_1$): $\lambda \neq 0$, with invariant $I(\lambda) = \lambda + \lambda^{-1}$.
($147E_1$): $\lambda > 1$.

The reason we use the notation ($1357QRS_1$) is that because over **C**, if $\lambda = 1$, this algebra is isomorphic to (1357Q); if $\lambda = -1$, it is isomorphic to (1357R), and for all other $\lambda \neq 0$, it is isomorphic to (1357S,$\lambda > 0$). When $\lambda = 0$, it becomes (2357D).

Some special features are: (i) Except in the case when $\chi = 3$, where we obtain an extra algebra (147F), the structure constants of all the algebras can be chosen to be integers and independent of the characteristic of the ground field; (ii) When the ground field is changed from **C** to **R**, we may get more algebras. The only algebra that has three different real forms is (1357Q). All the other algebras have at most two nonisomorphic real forms.

Carles [6] obtains a table giving the union of the tables of [24], [26] and [31] according to the weight systems. He considers in particular the limit points of all the one parameter continuous families. Readers may refer to [6] for more details. We want to mention that the basis for each algebra in our list has been chosen so that it also diagonalizes a maximal torus.

4.1 List of 7-Dimensional Indecomposable Nilpotent Lie Algebras over Algebraically Closed Fields ($\chi \neq 2$)

Upper Central Series Dimensions (37)

(37A): $[x_1, x_2] = x_5,\quad [x_2, x_3] = x_6,\quad [x_2, x_4] = x_7;$
(37B): $[x_1, x_2] = x_5,\quad [x_2, x_3] = x_6,\quad [x_3, x_4] = x_7;$
(37C): $[x_1, x_2] = x_5,\quad [x_2, x_3] = x_6,\quad [x_2, x_4] = x_7,$
$[x_3, x_4] = x_5;$
(37D): $[x_1, x_2] = x_5,\quad [x_1, x_3] = x_6,\quad [x_2, x_4] = x_7,$
$[x_3, x_4] = x_5.$

Upper Central Series Dimensions (357)

(357A): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_5,\quad [x_1, x_4] = x_7,$
$[x_2, x_4] = x_6;$
(357B): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_5,\quad [x_1, x_4] = x_7,$
$[x_2, x_3] = x_6;$
(357C): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_5,\quad [x_1, x_4] = x_7,$
$[x_2, x_3] = x_6,\quad [x_2, x_4] = x_5;$

Upper Central Series Dimensions (27)

(27A): $[x_1, x_2] = x_6,\quad [x_1, x_4] = x_7,\quad [x_3, x_5] = x_7;$
(27B): $[x_1, x_2] = x_6,\quad [x_1, x_5] = x_7,\quad [x_2, x_3] = x_7,$
$[x_3, x_4] = x_6.$

Upper Central Series Dimensions (257)

(257A): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_6,\quad [x_1, x_5] = x_7,$
$[x_2, x_4] = x_6;$
(257B): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_6,\quad [x_1, x_4] = x_7,$
$[x_2, x_5] = x_7;$
(257C): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_6,\quad [x_2, x_4] = x_6,$
$[x_2, x_5] = x_7;$
(257D): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_6,\quad [x_1, x_4] = x_7,$
$[x_2, x_4] = x_6,\quad [x_2, x_5] = x_7;$

(257E): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_6,\quad [x_2, x_4] = x_7,$
$[x_4, x_5] = x_6;$

(257F): $[x_1, x_2] = x_3,\quad [x_2, x_3] = x_6,\quad [x_2, x_4] = x_7,$
$[x_4, x_5] = x_6;$

(257G): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_6,\quad [x_1, x_5] = x_7,$
$[x_2, x_4] = x_7,\quad [x_4, x_5] = x_6;$

(257H): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_6,\quad [x_2, x_4] = x_6,$
$[x_4, x_5] = x_7;$

(257I): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_6,\quad [x_1, x_4] = x_6,$
$[x_1, x_5] = x_7,\quad [x_2, x_3] = x_7;$

(257J): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_6,\quad [x_1, x_5] = x_7,$
$[x_2, x_3] = x_7,\quad [x_2, x_4] = x_6;$

(257K): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_6,\quad [x_2, x_3] = x_7,$
$[x_4, x_5] = x_7;$

(257L): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_6,\quad [x_2, x_3] = x_7,$
$[x_2, x_4] = x_6,\quad [x_4, x_5] = x_7;$

Upper Central Series Dimensions (247)

(247A): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4, 5;$

(247B): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4\quad [x_3, x_5] = x_7;$

(247C): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4\quad [x_1, x_5] = x_7,\quad [x_3, x_5] = x_6.$

(247D): $[x_1, x_i] = x_{i+2}, i = 2, 3,\quad [x_1, x_4] = x_6,\quad [x_2, x_5] = x_7,$
$[x_3, x_4] = x_7;$

(247E): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4\quad [x_1, x_5] = x_6,\quad [x_2, x_5] = x_7,$
$[x_3, x_4] = x_7;$

(247F): $[x_1, x_i] = x_{i+2}, i = 2, 3,\quad [x_2, x_4] = x_6,\quad [x_2, x_5] = x_7,$
$[x_3, x_4] = x_7,\quad [x_3, x_5] = x_6;$

(247G): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4\quad [x_1, x_5] = x_6,\quad [x_2, x_4] = x_6,$
$[x_2, x_5] = x_7,\quad [x_3, x_4] = x_7,\quad [x_3, x_5] = x_6;$

(247H): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4\quad [x_2, x_4] = x_6,\quad [x_2, x_5] = x_7,$
$[x_3, x_4] = x_7,\quad [x_3, x_5] = x_6;$

(247I): $[x_1, x_i] = x_{i+2}, i = 2, 3,\quad [x_2, x_5] = x_6,\quad [x_3, x_4] = x_6,$
$[x_3, x_5] = x_7;$

(247J): $[x_1, x_i] = x_{i+2}, i = 2, 3\quad [x_1, x_5] = x_6,\quad [x_2, x_5] = x_7,$
$[x_3, x_4] = x_7,\quad [x_3, x_5] = x_6;$

(247K): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4\quad [x_2, x_5] = x_7,\quad [x_3, x_4] = x_7,$
$[x_3, x_5] = x_6;$

(247L): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4, 5\quad [x_2, x_3] = x_6;$

(247M): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4$ $[x_2, x_3] = x_6$, $[x_3, x_5] = x_7$;
(247N): $[x_1, x_i] = x_{i+2}, i = 2, 3$, $[x_1, x_5] = x_6$, $[x_2, x_3] = x_7$,
$[x_2, x_4] = x_6$;
(247O): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4$ $[x_1, x_5] = x_7$, $[x_2, x_3] = x_7$,
$[x_3, x_5] = x_6$;
(247P): $[x_1, x_i] = x_{i+2}, i = 2, 3$, $[x_2, x_3] = x_6$, $[x_2, x_5] = x_7$,
$[x_3, x_4] = x_7$;
(247Q): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4$, $[x_2, x_3] = x_6$, $[x_2, x_5] = x_7$,
$[x_3, x_4] = x_7$;
(247R): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4$ $[x_1, x_5] = x_6$, $[x_2, x_3] = x_6$,
$[x_2, x_5] = x_7$, $[x_3, x_4] = x_7$;

Upper Central Series Dimensions (2457)

(2457A): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_i] = x_{i+2}, i = 4, 5$;
(2457B): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_4] = x_7$, $[x_2, x_5] = x_6$;
(2457C): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_i] = x_{i+2}, i = 4, 5$, $[x_2, x_5] = x_6$;
(2457D): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_i] = x_{i+2}, i = 4, 5$, $[x_2, x_3] = x_6$,
$[x_2, x_5] = x_6$;
(2457E): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_4] = x_7$, $[x_2, x_3] = x_6$,
$[x_2, x_5] = x_6$;
(2457F): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_i] = x_{i+2}, i = 4, 5$, $[x_2, x_3] = x_6$;
(2457G): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_4] = x_7$, $[x_1, x_5] = x_6$,
$[x_2, x_3] = x_6$;
(2457H): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_4] = x_7$, $[x_2, x_3] = x_6$,
$[x_2, x_5] = x_7$;
(2457I): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_4] = x_6$, $[x_2, x_3] = x_6$,
$[x_2, x_5] = x_7$;
(2457J): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_4] = x_6$, $[x_2, x_3] = x_6 + x_7$,
$[x_2, x_5] = x_7$;
(2457K): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_4] = x_7$, $[x_1, x_5] = x_6$,
$[x_2, x_3] = x_6$, $[x_2, x_5] = x_7$;
(2457L): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_4] = x_6$, $[x_1, x_5] = x_7$,
$[x_2, x_3] = x_5$, $[x_2, x_4] = x_7$, $[x_2, x_5] = x_6$;
(2457M): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_4] = x_7$, $[x_1, x_5] = x_6$,
$[x_2, x_3] = x_5$, $[x_2, x_4] = x_6$;

Upper Central Series Dimensions (2357)

(2357A): $[x_1, x_2] = x_4,\quad [x_1, x_4] = x_5,\quad [x_1, x_5] = x_7,$
$[x_2, x_3] = x_5 + x_6,\quad [x_3, x_4] = -x_7;$

(2357B): $[x_1, x_2] = x_4,\quad [x_1, x_3] = x_6,\quad [x_1, x_4] = x_5,$
$[x_1, x_5] = x_7,\quad [x_2, x_3] = x_5,\quad [x_3, x_4] = -x_7;$

(2357C): $[x_1, x_2] = x_4,\quad [x_1, x_4] = x_5,\quad [x_1, x_5] = x_7,$
$[x_2, x_3] = x_5,\quad [x_2, x_4] = x_6,\quad [x_3, x_4] = -x_7;$

(2357D): $[x_1, x_2] = x_4,\quad [x_1, x_3] = x_6,\quad [x_1, x_4] = x_5,$
$[x_1, x_5] = x_7,\quad [x_2, x_3] = x_5,\quad [x_2, x_4] = x_6,$
$[x_3, x_4] = -x_7;$

Upper Central Series Dimensions (23457)

(23457A): $[x_1, x_i] = x_{i+1}, i = 2, 3, 4,\quad [x_1, x_5] = x_6,\quad [x_2, x_3] = x_7;$

(23457B): $[x_1, x_i] = x_{i+1}, i = 2, 3, 4,\quad [x_2, x_3] = x_7,\quad [x_2, x_5] = x_6,$
$[x_3, x_4] = -x_6;$

(23457C): $[x_1, x_i] = x_{i+1}, i = 2, 3, 4,\quad [x_1, x_5] = x_6,\quad [x_2, x_5] = x_7,$
$[x_3, x_4] = -x_7;$

(23457D): $[x_1, x_i] = x_{i+1}, i = 2, 3, 4,\quad [x_1, x_5] = x_6,\quad [x_2, x_3] = x_6,$
$[x_2, x_5] = x_7,\quad [x_3, x_4] = -x_7;$

(23457E): $[x_1, x_i] = x_{i+1}, i = 2, 3, 4,\quad [x_1, x_5] = x_6,\quad [x_2, x_3] = x_5 + x_7,$
$[x_2, x_4] = x_6;$

(23457F): $[x_1, x_i] = x_{i+1}, i = 2, 3, 4,\quad [x_2, x_3] = x_5 + x_7,\quad [x_2, x_5] = x_6,$
$[x_3, x_4] = -x_6;$

(23457G): $[x_1, x_i] = x_{i+1}, i = 2, 3, 4,\quad [x_1, x_5] = x_6,\quad [x_2, x_3] = x_5,$
$[x_2, x_4] = x_6,\quad [x_2, x_5] = x_7,\quad [x_3, x_4] = -x_7;$

Upper Central Series Dimensions (17)

(17): $[x_1, x_2] = x_7,\quad [x_3, x_4] = x_7,\quad [x_5, x_6] = x_7;$

Upper Central Series Dimensions (157)

(157): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_7,\quad [x_2, x_4] = x_7,$
$[x_5, x_6] = x_7;$

Upper Central Series Dimensions (147)

(147A): $[x_1, x_2] = x_4$, $[x_1, x_3] = x_5$, $[x_1, x_6] = x_7$,
$[x_2, x_5] = x_7$, $[x_3, x_4] = x_7$;

(147B): $[x_1, x_2] = x_4$, $[x_1, x_3] = x_5$, $[x_1, x_4] = x_7$,
$[x_2, x_6] = x_7$, $[x_3, x_5] = x_7$;

(147D): $[x_1, x_2] = x_4$, $[x_1, x_3] = -x_6$, $[x_1, x_5] = x_7$,
$[x_1, x_6] = x_7$, $[x_2, x_3] = x_5$, $[x_2, x_6] = x_7$,
$[x_3, x_4] = -2x_7$.

(147E): One parameter family, with invariant $I(\lambda) = \frac{(1-\lambda+\lambda^2)^3}{\lambda^2(\lambda-1)^2}$, $\lambda \neq 0, 1$
$[x_1, x_2] = x_4$, $[x_1, x_3] = -x_6$, $[x_1, x_5] = -x_7$,
$[x_2, x_3] = x_5$, $[x_2, x_6] = \lambda x_7$, $[x_3, x_4] = (1-\lambda)x_7$.
When $\lambda = 0$ or 1, it is isomorphic to (247P).

(147F): (for $\chi = 3$ only)
$[x_1, x_2] = x_4$, $[x_1, x_3] = -x_6$, $[x_1, x_5] = x_7$,
$[x_1, x_6] = x_7$, $[x_2, x_3] = x_5$, $[x_2, x_4] = x_7$,
$[x_2, x_6] = x_7$, $[x_3, x_4] = x_7$.

Remark: (147C) in Seeley's list is a special case of (147E) by taking $\lambda = 1$.

Upper Central Series Dimensions (1457)

(1457A): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_4] = x_7$, $[x_5, x_6] = x_7$;

(1457B): $[x_1, x_i] = x_{i+1}, i = 2, 3$, $[x_1, x_4] = x_7$, $[x_2, x_3] = x_7$,
$[x_5, x_6] = x_7$;

Upper Central Series Dimensions (137)

(137A): $[x_1, x_2] = x_5$, $[x_1, x_5] = x_7$, $[x_3, x_4] = x_6$,
$[x_3, x_6] = x_7$;

(137B): $[x_1, x_2] = x_5$, $[x_1, x_5] = x_7$, $[x_2, x_4] = x_7$,
$[x_3, x_4] = x_6$, $[x_3, x_6] = x_7$;

(137C): $[x_1, x_2] = x_5$, $[x_1, x_4] = x_6$, $[x_1, x_6] = x_7$,
$[x_2, x_3] = x_6$, $[x_3, x_5] = -x_7$;

(137D): $[x_1, x_2] = x_5$, $[x_1, x_4] = x_6$, $[x_1, x_6] = x_7$,
$[x_2, x_3] = x_6$, $[x_2, x_4] = x_7$ $[x_3, x_5] = -x_7$;

(1357A): $[x_1, x_2] = x_4$, $[x_1, x_4] = x_5$, $[x_1, x_5] = x_7$, $[x_2, x_3] = x_5$, $[x_2, x_6] = x_7$, $[x_3, x_4] = -x_7$;

(1357B): $[x_1, x_2] = x_4$, $[x_1, x_4] = x_5$, $[x_1, x_5] = x_7$, $[x_2, x_3] = x_5$, $[x_3, x_4] = -x_7$, $[x_3, x_6] = x_7$;

(1357C): $[x_1, x_2] = x_4$, $[x_1, x_4] = x_5$, $[x_1, x_5] = x_7$, $[x_2, x_3] = x_5$, $[x_2, x_4] = x_7$, $[x_3, x_4] = -x_7$, $[x_3, x_6] = x_7$;

(1357D): $[x_1, x_2] = x_3$, $[x_1, x_6] = x_7$, $[x_2, x_i] = x_{i+2}, i = 3, 4$, $[x_2, x_5] = x_7$, $[x_3, x_4] = x_7$;

(1357E): $[x_1, x_2] = x_3$, $[x_2, x_i] = x_{i+2}, i = 3, 4$, $[x_2, x_5] = x_7$, $[x_4, x_6] = x_7$;

(1357F): $[x_1, x_2] = x_3$, $[x_1, x_3] = x_7$, $[x_2, x_i] = x_{i+2}, i = 3, 4$, $[x_2, x_5] = x_7$, $[x_4, x_6] = -x_7$;

(1357G): $[x_1, x_2] = x_3$, $[x_1, x_4] = x_6$, $[x_1, x_6] = x_7$, $[x_2, x_3] = x_5$, $[x_2, x_5] = x_7$;

(1357H): $[x_1, x_2] = x_3$, $[x_1, x_4] = x_6$, $[x_1, x_6] = x_7$, $[x_2, x_3] = x_5$, $[x_2, x_5] = x_7$, $[x_2, x_6] = x_7$, $[x_3, x_4] = -x_7$;

(1357I): $[x_1, x_2] = x_3$, $[x_1, x_4] = x_6$, $[x_2, x_3] = x_5$, $[x_2, x_5] = x_7$, $[x_4, x_6] = x_7$;

(1357J): $[x_1, x_2] = x_3$, $[x_1, x_3] = x_7$, $[x_1, x_4] = x_6$, $[x_2, x_3] = x_5$, $[x_2, x_5] = x_7$, $[x_4, x_6] = x_7$;

(1357L): $[x_1, x_2] = x_3$, $[x_1, x_i] = x_{i+2}, i = 3, 4, 5$, $[x_2, x_3] = x_7$, $[x_2, x_4] = x_5$, $[x_2, x_6] = \frac{1}{2}x_7$, $[x_3, x_4] = \frac{1}{2}x_7$;

(1357M): One parameter family, with $\lambda \neq 0$
$[x_1, x_2] = x_3$, $[x_1, x_i] = x_{i+2}, i = 3, 4, 5$, $[x_2, x_4] = x_5$, $[x_2, x_6] = \lambda x_7$, $[x_3, x_4] = (1 - \lambda)x_7$;
When $\lambda = 0$, it is isomorphic to (2357B).

(1357N): One parameter family.
$[x_1, x_2] = x_3$, $[x_1, x_i] = x_{i+2}, i = 3, 4, 5$, $[x_2, x_3] = \lambda x_7$, $[x_2, x_4] = x_5$, $[x_3, x_4] = x_7$, $[x_4, x_6] = x_7$;

(1357O): $[x_1, x_2] = x_3$, $[x_1, x_3] = x_5$, $[x_1, x_6] = x_7$, $[x_2, x_3] = x_6$, $[x_2, x_4] = x_5$, $[x_2, x_5] = x_7$;

(1357P): $[x_1, x_2] = x_3$, $[x_1, x_i] = x_{i+2}, i = 3, 5$, $[x_2, x_3] = x_6$, $[x_2, x_4] = x_5$, $[x_2, x_6] = x_7$, $[x_3, x_4] = x_7$;

(1357Q): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_5,\quad [x_1, x_5] = x_7,$
$[x_2, x_3] = x_6,\quad [x_2, x_4] = x_6,\quad [x_2, x_6] = x_7;$

(1357R): $[x_1, x_2] = x_3,\quad [x_1, x_3] = x_5,\quad [x_1, x_6] = x_7,$
$[x_2, x_3] = x_6,\quad [x_2, x_4] = x_6,\quad [x_2, x_5] = x_7,$
$[x_3, x_4] = x_7;$

(1357S): One parameter family, with $\lambda \neq 1$
$[x_1, x_2] = x_3,\quad [x_1, x_3] = x_5,\quad [x_1, x_5] = x_7,$
$[x_1, x_6] = x_7,\quad [x_2, x_3] = x_6,\quad [x_2, x_4] = x_6,$
$[x_2, x_5] = x_7,\quad [x_2, x_6] = \lambda x_7,\quad [x_3, x_4] = x_7;$
When $\lambda = 1$, it is isomorphic to (2357D).

Remark: (1357K) in Seeley's list is a special case of (1357M) by taking $\lambda = 1/2$.

Upper Central Series Dimensions (13457)

(13457A): $[x_1, x_i] = x_{i+1},\ i = 2, 3, 4,\quad [x_1, x_5] = x_7,\quad [x_2, x_6] = x_7;$

(13457B): $[x_1, x_i] = x_{i+1},\ i = 2, 3, 4,\quad [x_1, x_5] = x_7,\quad [x_2, x_3] = x_7,$
$[x_2, x_6] = x_7;$

(13457C): $[x_1, x_i] = x_{i+1},\ i = 2, 3, 4,\quad [x_1, x_6] = x_7,\quad [x_2, x_5] = x_7,$
$[x_3, x_4] = -x_7;$

(13457D): $[x_1, x_i] = x_{i+1}, i = 2, 3, 4,\quad [x_1, x_5] = x_7,\quad [x_2, x_3] = x_5,$
$[x_2, x_4] = x_7,\quad [x_2, x_6] = x_7;$

(13457E): $[x_1, x_i] = x_{i+1}, i = 2, 3, 4,\quad [x_1, x_6] = x_7,\quad [x_2, x_3] = x_5,$
$[x_2, x_5] = x_7,\quad [x_3, x_4] = -x_7;$

(13457F): $[x_1, x_i] = x_{i+1},\ i = 2, 3, 4,\quad [x_1, x_5] = x_7,\quad [x_2, x_3] = x_6,$
$[x_2, x_6] = x_7;$

(13457G): $[x_1, x_i] = x_{i+1},\ i = 2, 3, 4,\quad [x_1, x_6] = x_7,\quad [x_2, x_3] = x_6,$
$[x_2, x_4] = x_7,\quad [x_2, x_5] = x_7,\quad [x_3, x_4] = -x_7;$

(13457I): $[x_1, x_i] = x_{i+1},\ i = 2, 3, 4,\quad [x_1, x_5] = x_7,\quad [x_2, x_3] = x_6,$
$[x_2, x_5] = x_7,\quad [x_2, x_6] = x_7,\quad [x_3, x_4] = -x_7.$

Remark: (13457H) in Seeley's list is not a Lie algebra, should be deleted.

Upper Central Series Dimensions (12457)

(12457A): $[x_1, x_i] = x_{i+1},\ i = 2,3,$ $[x_1, x_4] = x_6,$ $[x_1, x_6] = x_7,$
$[x_2, x_5] = x_6,$ $[x_3, x_5] = x_7;$

(12457B): $[x_1, x_i] = x_{i+1},\ i = 2,3,$ $[x_1, x_4] = x_6,$ $[x_1, x_6] = x_7,$
$[x_2, x_5] = x_6 + x_7,$ $[x_3, x_5] = x_7;$

(12457C): $[x_1, x_i] = x_{i+1},\ i = 2,3,$ $[x_1, x_4] = x_6,$ $[x_2, x_5] = x_6,$
$[x_2, x_6] = x_7,$ $[x_3, x_4] = -x_7;$

(12457D): $[x_1, x_i] = x_{i+1},\ i = 2,3,$ $[x_1, x_i] = x_{i+2}, i = 4,5,$ $[x_2, x_5] = x_6,$
$[x_2, x_6] = x_7,$ $[x_3, x_4] = -x_7;$

(12457E): $[x_1, x_i] = x_{i+1}, i = 2,3,$ $[x_1, x_4] = x_6,$ $[x_1, x_6] = x_7,$
$[x_2, x_3] = x_6,$ $[x_2, x_4] = x_7$ $[x_2, x_5] = x_6,$
$[x_3, x_5] = x_7;$

(12457F): $[x_1, x_i] = x_{i+1}, i = 2,3,$ $[x_1, x_4] = x_6,$ $[x_2, x_3] = x_6$
$[x_2, x_i] = x_{i+1}, i = 5,6,$ $[x_3, x_4] = -x_7;$

(12457G): $[x_1, x_i] = x_{i+1}, i = 2,3,$ $[x_1, x_4] = x_6,$ $[x_1, x_5] = x_7,$
$[x_2, x_3] = x_6,$ $[x_2, x_i] = x_{i+1}, i = 5,6$ $[x_3, x_4] = -x_7;$

(12457H): $[x_1, x_i] = x_{i+1},\ i = 2,3,5,6$ $[x_2, x_j] = x_{j+2}, j = 3,4,$ $[x_3, x_4] = x_7;$

(12457I): $[x_1, x_i] = x_{i+1},\ i = 2,3,5,6$ $[x_2, x_j] = x_{j+2}, j = 3,4,$ $[x_2, x_5] = x_7,$
$[x_3, x_4] = x_7;$

(12457J): $[x_1, x_i] = x_{i+1},\ i = 2,3,5,6$ $[x_1, x_4] = x_7,$ $[x_2, x_3] = x_5,$
$[x_2, x_4] = x_6,$ $[x_2, x_5] = x_7,$ $[x_3, x_4] = x_7;$

(12457K): $[x_1, x_i] = x_{i+1},\ i = 2,3,5,6$ $[x_1, x_4] = x_7,$ $[x_2, x_3] = x_5,$
$[x_2, x_4] = x_6,$ $[x_3, x_4] = x_7;$

(12457L): $[x_1, x_i] = x_{i+1},\ i = 2,3,5,6$ $[x_2, x_j] = x_{j+2}, j = 3,4,$ $[x_2, x_6] = x_7,$
$[x_3, x_4] = x_7,$ $[x_3, x_5] = -x_7;$

(12457N): One parameter family, with invariant $I(\lambda) = \lambda + \lambda^{-1}$.
$[x_1, x_i] = x_{i+1},\ i = 2,3,5,6$ $[x_1, x_4] = x_7,$ $[x_2, x_3] = x_5,$
$[x_2, x_4] = x_6,$ $[x_2, x_5] = \lambda x_7,$ $[x_2, x_6] = x_7,$
$[x_3, x_4] = x_7,$ $[x_3, x_5] = -x_7;$

Remark: (12457M) in Seeley's list is just a special case of (12457N) by taking $\lambda = 0$.

Upper Central Series Dimensions (12357)

(12357A): $[x_1, x_2] = x_4,$ $[x_1, x_i] = x_{i+1},\ i = 4,5,6,$ $[x_2, x_3] = x_5,$
$[x_3, x_4] = -x_6,$ $[x_3, x_5] = -x_7;$

(12357B): $[x_1, x_2] = x_4,$ $[x_1, x_i] = x_{i+1},\ i = 4,5,6,$ $[x_2, x_3] = x_5 + x_7,$
$[x_3, x_4] = -x_6,$ $[x_3, x_5] = -x_7;$

(12357C): $[x_1, x_2] = x_4,$ $[x_1, x_i] = x_{i+1},\ i = 4,5,6,$ $[x_2, x_3] = x_5,$
$[x_2, x_4] = x_7,$ $[x_3, x_4] = -x_6,$ $[x_3, x_5] = -x_7;$

Upper Central Series Dimensions (123457)

(123457A): $[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 6.$

(123457B): $[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 6,\quad [x_2, x_3] = x_7.$

(123457C): $[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 6,\quad [x_2, x_5] = x_7,\quad [x_3, x_4] = -x_7.$

(123457D): $[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 5,\quad [x_1, x_6] = x_7,\quad [x_2, x_3] = x_6,$
$[x_2, x_4] = x_7;$

(123457E): $[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 5,\quad [x_1, x_6] = x_7,\quad [x_2, x_3] = x_6 + x_7,$
$[x_2, x_4] = x_7;$

(123457F): $[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 5,\quad [x_1, x_6] = x_7,\quad [x_2, x_3] = x_6,$
$[x_2, x_4] = [x_2, x_5] = x_7,\quad [x_3, x_4] = -x_7.$

(123457H): $[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 5,\quad [x_1, x_6] = x_7,\quad [x_2, x_3] = x_5 + x_7,$
$[x_2, x_4] = x_6,\quad [x_2, x_5] = x_7;$

(123457I): One parameter family.
$[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 5,\quad [x_1, x_6] = x_7,\quad [x_2, x_3] = x_5,$
$[x_2, x_4] = x_6,\quad [x_2, x_5] = \lambda x_7,\quad [x_3, x_4] = (1 - \lambda)x_7.$

Remark: (123457G) in Seeley's list is a special case of (123457I) with $\lambda = 1$.

4.2 List of 7-Dimensional Indecomposable Nilpotent Lie Algebras over the Real Field

Each of the algebras in the list of Section 4.1 can be interpreted as a Lie algebra over **R**. In the case of infinte families, we have to restrict the parameter λ to take real values. The exceptional algebra which occurs in the case $\chi = 3$ should be omitted. In addition to these algebras, we have the following 24 extra indecomposable algebras over the real field **R**.

Upper Central Series Dimensions (37)

(37B$_1$): $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_1, x_4] = x_7$,
$[x_2, x_4] = x_6$, $[x_3, x_4] = -x_5$;

(37D$_1$): $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_1, x_4] = x_7$,
$[x_2, x_3] = -x_7$, $[x_2, x_4] = x_6$, $[x_3, x_4] = -x_5$;

Upper Central Series Dimensions (257)

(257J$_1$): $[x_1, x_2] = x_3$, $[x_1, x_3] = x_6$, $[x_1, x_4] = x_6$,
$[x_1, x_5] = x_7$, $[x_2, x_3] = x_7$, $[x_2, x_5] = x_6$;

Upper Central Series Dimensions (247)

(247E$_1$): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4$ $[x_2, x_4] = x_7$, $[x_3, x_5] = x_7$;

(247F$_1$): $[x_1, x_i] = x_{i+2}, i = 2, 3,$ $[x_2, x_4] = x_6$, $[x_2, x_5] = x_7$,
$[x_3, x_4] = x_7$, $[x_3, x_5] = -x_6$;

(247H$_1$): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4$ $[x_2, x_4] = x_6$, $[x_2, x_5] = x_7$,
$[x_3, x_4] = x_7$, $[x_3, x_5] = -x_6$;

(247P$_1$): $[x_1, x_i] = x_{i+2}, i = 2, 3,$ $[x_2, x_3] = x_6$,
$[x_2, x_4] = x_7$, $[x_3, x_5] = x_7$;

(247R$_1$): $[x_1, x_i] = x_{i+2}, i = 2, 3, 4$ $[x_2, x_3] = x_6$,
$[x_2, x_4] = x_7$, $[x_3, x_5] = x_7$;

Upper Central Series Dimensions (2457)

(2457L$_1$): $[x_1, x_i] = x_{i+1}, i = 2, 3,$ $[x_1, x_4] = x_6$, $[x_1, x_5] = x_7$,
$[x_2, x_3] = x_5$, $[x_2, x_4] = x_7$, $[x_2, x_5] = -x_6$;

Upper Central Series Dimensions (2357)

(2357D$_1$): $[x_1, x_2] = x_4$, $[x_1, x_3] = x_6$, $[x_1, x_4] = x_5$,
$[x_1, x_5] = x_7$, $[x_2, x_3] = x_5$, $[x_2, x_4] = -x_6$,
$[x_3, x_4] = -x_7$;

Upper Central Series Dimensions (147)

(147A$_1$): $[x_1, x_2] = x_4$, $[x_1, x_3] = x_5$, $[x_1, x_6] = x_7$,
$[x_2, x_4] = x_7$, $[x_3, x_5] = x_7$;

(147E$_1$): One parameter family, with $\lambda > 1$
$[x_1, x_2] = x_4$, $[x_2, x_3] = x_5$, $[x_1, x_3] = -x_6$,
$[x_1, x_6] = -\lambda x_7$, $[x_2, x_5] = \lambda x_7$, $[x_2, x_6] = 2x_7$,
$[x_3, x_4] = -2x_7$.

Upper Central Series Dimensions (137)

(137A$_1$): $[x_1, x_3] = x_5$, $[x_1, x_4] = x_6$, $[x_1, x_5] = x_7$,
$[x_2, x_3] = -x_6$, $[x_2, x_4] = x_5$, $[x_2, x_6] = x_7$;

(137B$_1$): $[x_1, x_3] = x_5$, $[x_1, x_4] = x_6$, $[x_1, x_5] = x_7$,
$[x_2, x_3] = -x_6$, $[x_2, x_4] = x_5$, $[x_2, x_6] = x_7$,
$[x_3, x_4] = x_7$;

Upper Central Series Dimensions (1357)

(1357F$_1$): $[x_1, x_2] = x_3$, $[x_1, x_3] = x_7$, $[x_2, x_i] = x_{i+2}, i = 3, 4$,
$[x_2, x_5] = x_7$, $[x_4, x_6] = x_7$;

(1357P$_1$): $[x_1, x_2] = x_3$, $[x_1, x_i] = x_{i+2}, i = 3, 5$, $[x_2, x_3] = x_6$,
$[x_2, x_4] = x_5$, $[x_2, x_6] = -x_7$, $[x_3, x_4] = x_7$;

(1357Q$_1$): $[x_1, x_2] = x_3$, $[x_1, x_3] = x_5$, $[x_1, x_5] = x_7$,
$[x_2, x_3] = x_6$, $[x_2, x_4] = x_6$, $[x_2, x_6] = -x_7$;

(1357QRS$_1$): One parameter family, with invariant $I(\lambda) = \lambda + \lambda^{-1}$ and $\lambda \neq 0$.
$[x_1, x_2] = x_3$ $[x_1, x_3] = x_5$, $[x_1, x_4] = x_6$
$[x_1, x_5] = x_7$, $[x_2, x_3] = -x_6$ $[x_2, x_4] = x_5$,
$[x_2, x_6] = \lambda x_7$ $[x_3, x_4] = (1 - \lambda)x_7$.
When $\lambda = 1$, (1357QRS$_1$)$\cong$ (1357Q) over **C**;
When $\lambda = -1$, (1357QRS$_1$)$\cong$(1357R) over **C**.
(1357QRS$_1$, $\lambda \neq 0, \pm 1$) becomes (1357S, $\lambda > 1$) over **C**.
When $\lambda = 0$, it becomes (2357D).

Upper Central Series Dimensions (12457)

(12457J$_1$): $[x_1, x_i] = x_{i+1},\ i = 2,3,5,6$ $\quad [x_1, x_4] = x_7,\quad [x_2, x_j] = x_{j+2}, j = 3,4,$
$[x_2, x_5] = -x_7,\quad [x_3, x_4] = x_7;$

(12457L$_1$): $[x_1, x_i] = x_{i+1},\ i = 2,3,\quad [x_1, x_4] = -x_6,\quad [x_1, x_6] = x_7,$
$[x_2, x_3] = x_5,\quad [x_2, x_5] = -x_6,\quad [x_3, x_5] = -x_7;$

(12457N$_1$): $[x_1, x_i] = x_{i+1},\ i = 2,3,\quad [x_1, x_4] = -x_6,\quad [x_1, x_6] = x_7,$
$[x_2, x_3] = x_5,\quad [x_2, x_5] = -x_6 + x_7,\quad [x_3, x_5] = -x_7;$
It is isomorphic to (12457N, $\lambda = 1$) over **C**.

(12457N$_2$): One parameter family, with $\lambda \geq 0$.
$[x_1, x_i] = x_{i+1},\ i = 2,3,\quad [x_1, x_4] = -x_6,\quad [x_1, x_5] = x_7,$
$[x_1, x_6] = x_7,\quad [x_2, x_3] = x_5,\quad [x_2, x_4] = x_7,$
$[x_2, x_5] = -x_6 + \lambda x_7,\quad [x_3, x_5] = -x_7;$
It is isomorphic to (12457N,$\lambda \neq 1$) over **C**.

Upper Central Series Dimensions (12357)

(12357B$_1$): $[x_1, x_2] = x_4,\quad [x_1, x_i] = x_{i+1},\ i = 4,5,6,\quad [x_2, x_3] = x_5 - x_7,$
$[x_3, x_4] = -x_6,\quad [x_3, x_5] = -x_7;$

Upper Central Series Dimensions (123457)

(123457H$_1$): $[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 5,\quad [x_1, x_6] = -x_7,\quad [x_2, x_3] = x_5 + x_7,$
$[x_2, x_4] = x_6,\quad [x_2, x_5] = -x_7;$

Chapter 5

Two-Step Nilpotent Lie Algebras

In this chapter, we consider all the central extensions of Abelian algebras – $N_{6,6}$, $N_{5,5}$ and $N_{4,4}$ – over both algebraically closed fields of characteristic $\neq 2$, and over the real field $\mathbf{R}$.

Central extensions of $N_{6,6} = \mathfrak{a}_6$:

Basis: Δ_{ij}, $1 \leq i < j \leq 6$.

In this case, we have Aut $N_{6,6} = GL_6$. To make sure that the central extension does not have any Abelian direct factors, we require that the skew-symmetric matrix corresponding to the 2-cocycles to be nonsingular, therefore by a classical result on the canonical form for nonsingular skew-symmetric matrices (see [16] for example), we can immediately obtain a representation in $U_1(\mathfrak{g})/\text{Aut}\ \mathfrak{g}$ as $\Delta_{12} + \Delta_{34} + \Delta_{56}$, which corresponds to the algebra (17).

Therefore the corresponding central extension of $N_{6,6}$ of dimension 7 over any field is:

$$\boxed{(17): \quad [x_1, x_2] = x_7, \quad [x_3, x_4] = x_7, \quad [x_5, x_6] = x_7.}$$

Central extensions of $N_{5,5} = \mathfrak{a}_5$:

Basis: $\Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{23}, \Delta_{24}, \Delta_{25}, \Delta_{34}, \Delta_{35}, \Delta_{45}$.

Group action: $a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{15} + e\Delta_{23} + f\Delta_{24} + g\Delta_{25} + h\Delta_{34} + i\Delta_{35} + j\Delta_{45}$:

$a \to a_{11}(aa_{22} + ba_{32} + ca_{42} + da_{52}) + a_{21}(-aa_{12} + ea_{32} + fa_{42} + ga_{52}) + a_{31}(-ba_{12} - ea_{22} + ha_{42} + ia_{52}) + a_{41}(-ca_{12} - fa_{22} - ha_{42} + ja_{52}) + a_{51}(-da_{12} - ga_{22} - ia_{32} - ja_{42})$;

$b \to a_{11}(aa_{23} + ba_{33} + ca_{43} + da_{53}) + a_{21}(-aa_{13} + ea_{33} + fa_{43} + ga_{53}) + a_{31}(-ba_{13} - ea_{23} + ha_{43} + ia_{53}) + a_{41}(-ca_{13} - fa_{23} - ha_{33} + ja_{53}) + a_{51}(-da_{13} - ga_{23} - ia_{33} - ja_{43})$;

$c \to a_{11}(aa_{24} + ba_{34} + ca_{44} + da_{54}) + a_{21}(-aa_{14} + ea_{34} + fa_{44} + ga_{54}) + a_{31}(-ba_{14} - ea_{24} + ha_{44} + ia_{54}) + a_{41}(-ca_{14} - fa_{24} - ha_{34} + ja_{54}) + a_{51}(-da_{14} - ga_{24} - ia_{34} - ja_{44})$;

$d \to a_{11}(aa_{25} + ba_{35} + ca_{45} + da_{55}) + a_{21}(-aa_{15} + ea_{35} + fa_{45} + ga_{55}) + a_{31}(-ba_{15} - ea_{25} + ha_{45} + ia_{55}) + a_{41}(-ca_{15} - fa_{25} - ha_{35} + ja_{55}) + a_{51}(-da_{15} - ga_{25} - ia_{35} - ja_{45})$;

$e \to a_{12}(aa_{23}+ba_{33}+ca_{43}+da_{53})+a_{22}(-aa_{13}+ea_{33}+fa_{43}+ga_{53})+a_{32}(-ba_{13}-ea_{23}+ha_{43}+ia_{53})+a_{42}(-ca_{13}-fa_{23}-ha_{33}+ja_{53})+a_{52}(-da_{13}-ga_{23}-ia_{33}-ja_{43});$

$f \to a_{12}(aa_{24}+ba_{34}+ca_{44}+da_{54})+a_{22}(-aa_{14}+ea_{34}+fa_{44}+ga_{54})+a_{32}(-ba_{14}-ea_{24}+ha_{44}+ia_{54})+a_{42}(-ca_{14}-fa_{24}-ha_{34}+ja_{54})+a_{52}(-da_{14}-ga_{24}-ia_{34}-ja_{44});$

$g \to a_{12}(aa_{25}+ba_{35}+ca_{45}+da_{55})+a_{22}(-aa_{15}+ea_{35}+fa_{45}+ga_{55})+a_{32}(-ba_{15}-ea_{25}+ha_{45}+ia_{55})+a_{42}(-ca_{15}-fa_{25}-ha_{35}+ja_{55})+a_{52}(-da_{15}-ga_{25}-ia_{35}-ja_{45});$

$h \to a_{13}(aa_{24}+ba_{34}+ca_{44}+da_{54})+a_{23}(-aa_{14}+ea_{34}+fa_{44}+ga_{54})+a_{33}(-ba_{14}-ea_{24}+ha_{44}+ia_{54})+a_{43}(-ca_{14}-fa_{24}-ha_{34}+ja_{54})+a_{53}(-da_{14}-ga_{24}-ia_{34}-ja_{44});$

$i \to a_{13}(aa_{25}+ba_{35}+ca_{45}+da_{55})+a_{23}(-aa_{15}+ea_{35}+fa_{45}+ga_{55})+a_{33}(-ba_{15}-ea_{25}+ha_{45}+ia_{55})+a_{43}(-ca_{15}-fa_{25}-ha_{35}+ja_{55})+a_{53}(-da_{15}-ga_{25}-ia_{35}-ja_{45});$

$j \to a_{14}(aa_{25}+ba_{35}+ca_{45}+da_{55})+a_{24}(-aa_{15}+ea_{35}+fa_{45}+ga_{55})+a_{34}(-ba_{15}-ea_{25}+ha_{45}+ia_{55})+a_{44}(-ca_{15}-fa_{25}-ha_{35}+ja_{55})+a_{54}(-da_{15}-ga_{25}-ia_{35}-ja_{45});$

Assume $a \neq 0$. Choose $a_{21}=a_{31}=a_{41}=a_{51}=0$. Then make $b=c=d=0$ by solving for a_{23}, a_{24} and a_{25} respectively. To fix b, c, d, we require that $a_{23}=a_{24}=a_{25}=0$.

Choose $a_{12}=a_{32}=a_{42}=a_{52}=0$, we can make $e=f=g=0$ by solving for a_{13}, a_{14} and a_{15} respectively. To fix e, f, g, we require that $a_{13}=a_{14}=a_{15}=0$.

Now

$h \to a_{33}(ha_{44}+ia_{54})+a_{43}(-ha_{34}+ja_{54})+a_{53}(ia_{34}-ja_{44});$

$i \to a_{33}(ha_{45}+ia_{55})+a_{43}(-ha_{35}+ja_{55})+a_{53}(-ia_{35}-ja_{45});$

$j \to a_{34}(ha_{45}+ia_{55})+a_{44}(-ha_{35}+ja_{55})+a_{54}(-ia_{35}-ja_{45});$

If one of $h, i, j \neq 0$, then make $h \neq 0$, and $i=j=0$ and get get case 1: $a \neq 0$, $h \neq 0$, while $b=c=d=e=f=g=i=j=0$, i.e. $A_1=[1,0,0,0,0,0,0,1,0,0]$.

If all $h=i=j=0$, then we get case 2: only $a \neq 0$, and all the others are zero, or $A_2=[1,0,0,0,0,0,0,0,0,0]$.

Case 1: $A_1=[1,0,0,0,0,0,0,1,0,0]$. Choose $a_{13}=a_{14}=a_{15}=a_{23}=a_{24}=a_{25}=a_{31}=a_{32}=a_{35}=a_{41}=a_{42}=a_{45}=0$, we will fix $b=c=d=e=f=g=i=j=0$.

Assume $B=[0,b,c,d,e,f,g,h,i,j]$. One of $d,g,i,j \neq 0$.

$a=0 \to a_{11}da_{52}+a_{21}ga_{52}+a_{51}(-da_{12}-ga_{22});$

$b \to a_{11}(ba_{33}+ca_{43}+da_{53})+a_{21}(ea_{33}+fa_{43}+ga_{53})+a_{51}(-ia_{33}-ja_{43});$

$c \to a_{11}(ba_{34}+ca_{44}+da_{54})+a_{21}(ea_{34}+fa_{44}+ga_{54})+a_{51}(-ia_{34}-ja_{44});$

$d \to a_{11}da_{55}+a_{21}ga_{55};$

$e \to a_{12}(ba_{33}+ca_{43}+da_{53})+a_{22}(ea_{33}+fa_{43}+ga_{53})+a_{52}(-ia_{33}-ja_{43});$

$f \to a_{12}(ba_{34}+ca_{44}+da_{54})+a_{22}(ea_{34}+fa_{44}+ga_{54})+a_{52}(-ia_{34}-ja_{44});$

$g \to a_{12}da_{55} + a_{22}ga_{55}$;

$h \to a_{33}(ha_{44} + ia_{54}) + a_{43}(-ha_{34} + ja_{54}) + a_{53}(-ia_{34} - ja_{44})$;

$i \to a_{33}ia_{55} + a_{43}ja_{55}$;

$j \to a_{34}ia_{55} + a_{44}ja_{55}$;

Subcase 1.1: One of $d, g \neq 0$. Make $d \neq 0$ and $g = 0$. Fix $g = 0$, we require that $a_{12} = 0$. Assume $a_{21} = a_{51} = 0$. Make $b = c = 0$ by solving for a_{53}, and a_{54} respectively. Make $a = h$ by solving for a_{52}, and further make them to be zero by subtracting a multiple of A from B.

Subcase 1.1.1: One of $i, j \neq 0$. Make $i \neq 0$ and $j = 0$.

Now by taking $a_{12} = a_{34} = a_{52} = a_{53} = a_{54} = 0$, $a = 0 \to 0$; $b = 0 \to 0$; $c = 0 \to 0$; $d \to a_{11}da_{55}$; $e \to a_{22}(ea_{33} + fa_{43})$; $f \to a_{22}fa_{44}$; $g = 0 \to 0$; $h = 0 \to 0$; $i \to a_{33}ia_{55}$; $j = 0 \to 0$.

Now if $f \neq 0$, make $e = 0$ by solving for a_{43}, to get a representative:

$$B_1 = [0, 0, 0, 1, 0, 1, 0, 0, 1, 0].$$

If $f = 0$, depending on $e = 0$ or not, we would have two representatives

$$B_2 = [0, 0, 0, 1, 0, 0, 0, 0, 1, 0]$$

and

$$B_3 = [0, 0, 0, 1, 1, 0, 0, 0, 1, 0].$$

Subcase 1.1.2: Both $i = j = 0$. Taking $a_{12} = a_{52} = a_{53} = a_{54} = 0$, we have $a = 0 \to 0$; $b = 0 \to 0$; $c = 0 \to 0$; $d \to a_{11}da_{55}$; $e \to a_{22}(ea_{33} + fa_{43})$; $f \to a_{22}(ea_{34} + fa_{44})$; $g = 0 \to 0$; $h = 0 \to 0$; $i = 0 \to 0$; $j = 0 \to 0$.

If one of $e, f \neq 0$, make $e = 1$ and $f = 0$ to get a representative

$$B_4 = [0, 0, 0, 1, 1, 0, 0, 0, 0, 0].$$

If both $e, f = 0$, then get $B_5 = [0, 0, 0, 1, 0, 0, 0, 0, 0, 0]$.

Subcase 1.2: Both $d = g = 0$. Then one of $i, j \neq 0$. Make $i \neq 0$ and $j = 0$. Taking $a_{34} = 0$, we have $a = 0 \to 0$; $b \to a_{11}(ba_{33} + ca_{43}) + a_{21}(ea_{33} + fa_{43}) + a_{51}(-ia_{33})$; $c \to a_{11}(ca_{44}) + a_{21}(fa_{44})$; $d \to 0$; $e \to a_{12}(ba_{33} + ca_{43}) + a_{22}(ea_{33} + fa_{43}) + a_{52}(-ia_{33})$; $f \to a_{12}(ca_{44}) + a_{22}fa_{44}$; $g \to 0$; $h \to a_{33}(ha_{44} + ia_{54})$; $i \to a_{33}ia_{55}$; $j = 0 \to 0$.

Make $b = e = h = 0$ by solving for a_{51}, a_{52} and a_{54}. If one of $c, f \neq 0$, then make $c = 1$ and $f = 0$, we get a representative $B_5 = [0, 0, 1, 0, 0, 0, 0, 0, 1, 0]$. If both $c = f = 0$, then we have $B_6 = [0, 0, 0, 0, 0, 0, 0, 0, 1, 0]$.

Case 2: $A_2 = [1,0,0,0,0,0,0,0,0,0]$. To fix A up to a scalar, we require that $a_{13} = a_{14} = a_{15} = a_{23} = a_{24} = a_{25} = 0$. Let $B = [0,b,c,d,e,f,g,h,i,j]$, we have

$a = 0 \to a_{11}(ba_{32}+ca_{42}+da_{52})+a_{21}(ea_{32}+fa_{42}+ga_{52})+a_{31}(-ba_{12}-ea_{22}+ha_{42}+ia_{52})+a_{41}(-ca_{12}-fa_{22}-ha_{42}+ja_{52})+a_{51}(-da_{12}-ga_{22}-ia_{32}-ja_{42})$;

$b \to a_{11}(ba_{33}+ca_{43}+da_{53})+a_{21}(ea_{33}+fa_{43}+ga_{53})+a_{31}(+ha_{43}+ia_{53})+a_{41}(-ha_{33}+ja_{53})+a_{51}(-ia_{33}-ja_{43})$;

$c \to a_{11}(ba_{34}+ca_{44}+da_{54})+a_{21}(ea_{34}+fa_{44}+ga_{54})+a_{31}(ha_{44}+ia_{54})+a_{41}(-ha_{34}+ja_{54})+a_{51}(-ia_{34}-ja_{44})$;

$d \to a_{11}(ba_{35}+ca_{45}+da_{55})+a_{21}(ea_{35}+fa_{45}+ga_{55})+a_{31}(ha_{45}+ia_{55})+a_{41}(-ha_{35}+ja_{55})+a_{51}(-ia_{35}-ja_{45})$;

$e \to a_{12}(ba_{33}+ca_{43}+da_{53})+a_{22}(ea_{33}+fa_{43}+ga_{53})+a_{32}(ha_{43}+ia_{53})+a_{42}(-ha_{33}+ja_{53})+a_{52}(-ia_{33}-ja_{43})$;

$f \to a_{12}(ba_{34}+ca_{44}+da_{54})+a_{22}(ea_{34}+fa_{44}+ga_{54})+a_{32}(ha_{44}+ia_{54})+a_{42}(-ha_{34}+ja_{54})+a_{52}(-ia_{34}-ja_{44})$;

$g \to a_{12}(ba_{35}+ca_{45}+da_{55})+a_{22}(ea_{35}+fa_{45}+ga_{55})+a_{32}(ha_{45}+ia_{55})+a_{42}(-ha_{35}+ja_{55})+a_{52}(-ia_{35}-ja_{45})$;

$h \to a_{33}(ha_{44}+ia_{54})+a_{43}(-ha_{34}+ja_{54})+a_{53}(-ia_{34}-ja_{44})$;

$i \to a_{33}(ha_{45}+ia_{55})+a_{43}(-ha_{35}+ja_{55})+a_{53}(-ia_{35}-ja_{45})$;

$j \to a_{34}(ha_{45}+ia_{55})+a_{44}(-ha_{35}+ja_{55})+a_{54}(-ia_{35}-ja_{45})$;

One of $h,i,j \neq 0$. Make $i \neq 0$ and $h = j = 0$. We would have $h \to a_{33}ia_{54}+a_{53}(-ia_{34}) = 0$; $i \to a_{33}(ia_{55})+a_{53}(-ia_{35})$; $j \to a_{34}(ia_{55})+a_{54}(-ia_{35}) = 0$.

We require that $a_{34} = a_{54} = 0$ to have $h = j = 0$. Now

$a = 0 \to a_{11}(ba_{32}+ca_{42}+da_{52})+a_{21}(ea_{32}+fa_{42}+ga_{52})+a_{31}(-ba_{12}-ea_{22}+ia_{52})+a_{41}(-ca_{12}-fa_{22})+a_{51}(-da_{12}-ga_{22}-ia_{32})$;

$b \to a_{11}(ba_{33}+ca_{43}+da_{53})+a_{21}(ea_{33}+fa_{43}+ga_{53})+a_{31}(ia_{53})+a_{51}(-ia_{33})$;

$c \to a_{11}(ca_{44})+a_{21}(fa_{44})$;

$d \to a_{11}(ba_{35}+ca_{45}+da_{55})+a_{21}(ea_{35}+fa_{45}+ga_{55})+a_{31}(ha_{45}+ia_{55})+a_{51}(-ia_{35})$;

$e \to a_{12}(ba_{33}+ca_{43}+da_{53})+a_{22}(ea_{33}+fa_{43}+ga_{53})+a_{32}(ia_{53})+a_{52}(-ia_{33})$;

$f \to a_{12}(ca_{44})+a_{22}(fa_{44})$;

$g \to a_{12}(ba_{35}+ca_{45}+da_{55})+a_{22}(ea_{35}+fa_{45}+ga_{55})+a_{32}(ia_{55})a_{52}(-ia_{35})$;

Make $g = 0$ by solving for a_{32}, $e = 0$ by a_{52}, $d = 0$ by a_{31}, $b = 0$ by a_{51}. Now one of $c, f \neq 0$, make $c = 1$ and $f = 0$. We can always make $a = 0$ by subtracting a multiple of A from B. So we get the representative for B as $B_7 = [0,0,1,0,0,0,0,0,1,0]$

Now we are going to prove that

(1) The representatives $A_1 \wedge B_1$, $A_1 \wedge B_3$, and $A_1 \wedge B_4$ are in the same orbit, and correponding to (27B).

Consider the corresponding algebras:

$$\begin{aligned}
(1.1)\quad & [x_1, x_2] = x_6, [x_1, x_5] = x_7, [x_3, x_4] = x_6, \\
& [x_2, x_4] = x_7, [x_3, x_5] = x_7; \\
(1.2)\quad & [x_1, x_2] = x_6, [x_1, x_5] = x_7, [x_3, x_4] = x_6, \\
& [x_2, x_3] = x_7, [x_3, x_5] = x_7; \\
(1.3)\quad & [x_1, x_2] = x_6, [x_1, x_5] = x_7, [x_3, x_4] = x_6, \\
& [x_2, x_3] = x_7.
\end{aligned}$$

Then

$(1.3) \cong (1.1)$: Taking $x_1 \to x_1$, $x_2 \to x_2 + x_4$, $x_3 \to -x_4$, $x_4 \to -x_1 + x_3$, $x_5 \to -x_5$, $x_6 \to x_6$, $x_7 \to -x_7$;

$(1.3) \cong (1.2)$: Taking $x_1 \to x_3$, $x_2 \to -x_2 + x_4$, $x_3 \to x_1$, $x_4 \to x_2 + x_5$, $x_5 \to x_4$, $x_6 \to x_6 + x_7$, $x_7 \to x_6$.

(2) The representatives $A_1 \wedge B_2$, $A_1 \wedge B_5$, $A_1 \wedge B_6$ and $A_2 \wedge B_7$ are all in the same orbit, corresponding to (27A).

Consider the corresponding algebras:

$$\begin{aligned}
(2.1)\quad & [x_1, x_2] = x_6, [x_1, x_5] = x_7, [x_3, x_4] = x_6, \\
& [x_3, x_5] = x_7; \\
(2.2)\quad & [x_1, x_2] = x_6, [x_1, x_5] = x_7, [x_3, x_4] = x_6, \\
(2.3)\quad & [x_1, x_2] = x_6, [x_3, x_5] = x_7, [x_3, x_4] = x_6. \\
(2.4)\quad & [x_1, x_2] = x_6, [x_1, x_4] = x_7, [x_3, x_5] = x_7.
\end{aligned}$$

Then

$(2.4) \cong (2.1)$: Taking $x_1 \to -x_1$, $x_2 \to x_5$, $x_3 \to -x_1 + x_3$, $x_4 \to x_2 + x_4$, $x_5 \to -x_4$, $x_6 \to -x_7$, $x_7 \to -x_6$;

$(2.4) \cong (2.2)$: Taking $x_1 \to x_1$, $x_2 \to x_5$, $x_3 \to x_3$, $x_4 \to x_2$, $x_5 \to x_4$, $x_6 \to x_7$, $x_7 \to x_6$;

$(2.4) \cong (2.3)$: Taking $x_1 \to x_3$, $x_2 \to x_5$, $x_3 \to x_2$, $x_4 \to -x_4$, $x_5 \to x_1$, $x_6 \to x_7$, $x_7 \to -x_6$.

Now all we need to prove is that (1.3) and (2.4), which correspond to (27B) and (27A) respectively, are not isomorphic. We can compare their orbits again, but here instead, we use the ad hoc argument used by Seeley [31] to compare the so called minimal numbers.

For a given algebra, we consider all the nonzero elements in $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ and look for an ordered basis $\{x_1 + [\mathfrak{g}, \mathfrak{g}], \cdots, x_s + [\mathfrak{g}, \mathfrak{g}]\}$ with the smallest

$$(\dim \mathrm{Im}(x_1), \cdots, \dim \mathrm{Im}(x_s))$$

(called <u>the minimal number</u>) in lexicographic order, where $\mathrm{Im}(a)$ is the image of the adjoint image of a. This is obviously an invariant for Lie algebra, and it offers us a very effective way to distinguish two algebras.

In (27A), the basis of $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is $\{x_2, x_3, x_4, x_5, x_1\}$. We have

$$\dim \mathrm{Im}(x_2) = \dim \mathrm{Im}(x_3) = \dim \mathrm{Im}(x_4) = \dim \mathrm{Im}(x_5) = 1,$$

and $\dim \mathrm{Im}(x_1) = 2$, and we can prove that

$$(\dim \mathrm{Im}(x_2), \dim \mathrm{Im}(x_3), \dim \mathrm{Im}(x_4), \dim \mathrm{Im}(x_5), \dim \mathrm{Im}(x_1)) = (1,1,1,1,2)$$

is the minimal number. It is easy to see that the first 4 numbers $\dim \mathrm{Im}(x_2), \dim \mathrm{Im}(x_3), \dim \mathrm{Im}(x_4), \dim \mathrm{Im}(x_5)$ are already the smallest, being 1.

Consider the image of $x = x_1 + ax_2 + bx_3 + cx_4 + dx_5$.

$$\mathrm{Im}(x) = <[x, x_2], [x, x_3], [x, x_4], [x, x_5]> = <x_6, dx_7, x_7, bx_7> = <x_6, x_7>,$$

therefore any element containing properly x_1 will have an image of dimension 2. So $(1,1,1,1,2)$ is the minimal number.

In (27B), the minimal number is going to be $(1,1,1,2,2)$, with the corresponding ordered basis as $\{x_3, x_4, x_5, x_1, x_2\}$.

Hence (27A) and (27B) are not isomorphic.

Therefore the central extensions of $N_{5,5}$ of dimension 7 over any field (not necessarily algebraically closed) are:

(27A):	$[x_1, x_2] = x_6$,	$[x_1, x_4] = x_7$,	$[x_3, x_5] = x_7$;
(27B):	$[x_1, x_2] = x_6$,	$[x_1, x_5] = x_7$,	$[x_3, x_4] = x_6$,
	$[x_2, x_3] = x_7$.		

Remark: The correspondence between the above and the algebra in Seeley's list are: (27A)$\to$ 2,7A: $x_1 \to -e$, $x_2 \to b$, $x_3 \to c$, $x_4 \to a$, $x_5 \to d$, $x_6 \to g$ and $x_7 \to f$; (27B) $\to$ 2,7B: $x_1 \to d + e$, $x_2 \to 3a + b - c$, $x_3 \to -e$, $x_4 \to 2a + 2b - c - d - e$, $x_5 \to -a$, $x_6 \to -f + g$ and $x_7 \to f$.

Central extensions of $N_{4,4}$:

Basis: $\Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{34}$;

Group Action: $a\Delta_{12} + b\Delta_{13} + c\Delta_{14} + d\Delta_{23} + e\Delta_{24} + f\Delta_{34}$;

Let $\Sigma_{ij}^{st} = a_{is}a_{jt} - a_{it}a_{js}$, for $1 \le i,j,s,t \le 4$. Then

$a \to a\Sigma_{12}^{12} + b\Sigma_{13}^{12} + c\Sigma_{14}^{12} + d\Sigma_{23}^{12} + e\Sigma_{24}^{12} + f\Sigma_{34}^{12}$;

$b \to a\Sigma_{12}^{13} + b\Sigma_{13}^{13} + c\Sigma_{14}^{13} + d\Sigma_{23}^{13} + e\Sigma_{24}^{13} + f\Sigma_{34}^{13}$;

$c \to a\Sigma_{12}^{14} + b\Sigma_{13}^{14} + c\Sigma_{14}^{14} + d\Sigma_{23}^{14} + e\Sigma_{24}^{14} + f\Sigma_{34}^{14};$

$d \to a\Sigma_{12}^{23} + b\Sigma_{13}^{23} + c\Sigma_{14}^{23} + d\Sigma_{23}^{23} + e\Sigma_{24}^{23} + f\Sigma_{34}^{23};$

$e \to a\Sigma_{12}^{24} + b\Sigma_{13}^{24} + c\Sigma_{14}^{24} + d\Sigma_{23}^{24} + e\Sigma_{24}^{24} + f\Sigma_{34}^{24};$

$f \to a\Sigma_{12}^{34} + b\Sigma_{13}^{34} + c\Sigma_{14}^{34} + d\Sigma_{23}^{34} + e\Sigma_{24}^{34} + f\Sigma_{34}^{34};$

Now let $A = [a, b, c, d, e, f]$. It is obvious that one of a, b, c, d, e, f is nonzero. Make $a = 0$ and $b = 1$ to get $A = [0, 1, c, d, e, f]$.

Let $a_{21} = a_{41} = a_{42} = a_{43} = 0$, we can make $c = d = f = 0$ by solving for a_{34}, a_{12}, a_{14} respectively, and get $A = [0, 1, 0, 0, e, 0]$.

Now $a = 0 \to \Sigma_{13}^{12} + e\Sigma_{24}^{12}$; $b = 1 \to \Sigma_{13}^{13} + e\Sigma_{24}^{13}$; $c = 0 \to \Sigma_{13}^{14} + e\Sigma_{24}^{14}$; $d = 0 \to \Sigma_{13}^{23} + e\Sigma_{24}^{23}$; $e \to \Sigma_{13}^{24} + e\Sigma_{24}^{24}$; $f = 0 \to \Sigma_{13}^{34} + e\Sigma_{24}^{34}$.

Depending on $e = 0$ or not, we can obtain the two representatives $A_1 = [0, 1, 0, 0, 0, 0]$ and $A_2 = [0, 1, 0, 0, 1, 0]$. It is easy to check that A_1 and A_2 are indeed in different orbits.

Case 1: $A = [0, 1, 0, 0, 0, 0]$. To fix A, we require that $a = 0 \to \Sigma_{13}^{12} = 0$; $b = 1 \to \Sigma_{13}^{13} = 1$; $c = 0 \to \Sigma_{13}^{14} = 0$; $d = 0 \to \Sigma_{13}^{23} = 0$; $e = 0 \to \Sigma_{13}^{24} = 0$; $f = 0 \to \Sigma_{13}^{34} = 0$.

We may just choose $a_{12} = a_{14} = a_{32} = a_{33} = a_{34} = 0$ and $a_{13} = -1/a_{31}$.

Now assume that $B = [a, 0, c, d, e, f]$, under the group action,

$a \to a\Sigma_{12}^{12} + c\Sigma_{14}^{12} + d\Sigma_{23}^{12} + e\Sigma_{24}^{12} + fa_{31}a_{42};$

As one of $a, c, d, e, f \neq 0$. We may assume $a = 1$ in B, hence

$$A = [0, 1, 0, 0, 0, 0],\ B = [1, 0, c, d, e, f].$$

Now in B, we have $a \to a_{11}a_{22} + ca_{11}a_{42} + d(-a_{22}a_{31}) + e\Sigma_{24}^{12} + fa_{31}a_{42}$; $b = 0 \to 0$ (by subtracting a multiple of A from B); $c \to a_{11}a_{24} + ca_{11}a_{44} + d(-a_{24}a_{31}) + e\Sigma_{24}^{14} + fa_{31}a_{44}$; $d \to (-a_{13}a_{22}) + c(-a_{13}a_{42}) + e(a_{22}a_{43} - a_{23}a_{42})$; $e \to e\Sigma_{24}^{24}$; $f \to a_{13}a_{24} + ca_{13}a_{44} + e(a_{23}a_{44} - a_{24}a_{43})$.

If $e \neq 0$, then make $a = c = d = f = 0$ by taking $a_{24} = a_{42} = 1$, $a_{22} = a_{41} = a_{44} = 0$, and solve for a_{21}, a_{11}, a_{23} and a_{43} to get the representative

Case 1.1: $A = [0, 1, 0, 0, 0, 0], B = [0, 0, 0, 0, 1, 0]$.

If $e = 0$, then $a \to a_{11}a_{22} + ca_{11}a_{42} + d(-a_{22}a_{31}) + fa_{31}a_{42}$; $b = 0 \to 0$ (by subtracting a multiple of A from B); $c \to a_{11}a_{24} + ca_{11}a_{44} + d(-a_{24}a_{31}) + fa_{31}a_{44}$; $d \to (-a_{13}a_{22}) + c(-a_{13}a_{42})$; $e = 0 \to 0$; $f \to a_{13}a_{24} + ca_{13}a_{44}$.

We cannot make both $d = f = 0$, for otherwise the automorphism group is going to be singular. Take $a_{44} = 0$ and $a_{24} = a_{42} = 1$, make $d = 0$ by solving for a_{22}, make $c = 0$ by solving for a_{11}. Then by taking $a_{11} = a_{22} = a_{44} = 0$, we have $a \to fa_{31}a_{42}$; $b = 0 \to 0$ (by subtracting a multiple of A from B); $c = 0 \to 0$; $d = 0 \to 0$; $e = 0 \to 0$; $f \to a_{13}a_{24}$.

Depending on $f = 0$ or not, we get two representatives for B: Case 1.2:$B = [0,0,0,0,0,1]$; and Case 1.3:$B = [1,0,0,0,0,1]$.

Case 1.1: $A = [0,1,0,0,0,0]$ and $B = [0,0,0,0,1,0]$. To fix A and B, we require that $a_{12} = a_{14} = a_{21} = a_{23} = a_{32} = a_{34} = a_{41} = a_{43} = a_{44} = 0$, $a_{11} = (a_{13}a_{31} + 1)/a_{33}$, and $a_{24} = -1/a_{42}$.

Now assume $C = [a,0,c,d,0,f]$, under the group action, we have $a \to aa_{11}a_{22} + ca_{11}a_{42} + d(-a_{22}a_{31}) + fa_{31}a_{42}$; $b \to 0$; $c \to aa_{11}a_{24} + d(-a_{31}a_{24})$; $d \to a(-a_{13}a_{22}) + c(-a_{13}a_{42}) + da_{22}a_{33} + f(-a_{42}a_{33})$; $e \to 0$; $f \to aa_{13}a_{42} + d(-a_{24}a_{33})$.

We may assume $d = 0$, for otherwise we can solve for a_{22} to make $d = 0$. Then $a \to aa_{11}a_{22} + ca_{11}a_{42} + fa_{31}a_{42}$; $b = 0 \to 0$; $c \to aa_{11}a_{24}$; $d \to a(-a_{13}a_{22}) + c(-a_{13}a_{42}) + f(-a_{33}a_{42})$; $e \to 0$; $f \to aa_{13}a_{24}$.

One of $a, c \neq 0$, we may assume $a = 0$, for otherwise we can solve for a_{22} to make it to be zero. So $c \neq 0$. Set $a_{13} = 0$ to get $a \to ca_{11}a_{42}$; $b = 0 \to 0$; $c \to 0$; $d \to 0$; $e \to 0$; $f \to 0$.

Then we have the representative $C = [1,0,0,0,0,0]$, with (1) $A \wedge B \wedge C$ corresponding to (37B).

Case 1.2: $A = [0,1,0,0,0,0]$, and $B = [0,0,0,0,0,1]$. To fix A and B (up to a nonzero scalar), we require that $a_{12} = a_{31} = a_{32} = a_{34} = a_{41} = a_{42} = 0$, , $a_{11}a_{22}a_{33}a_{44} \neq 0$.

Now consider $C = [a,0,c,d,e,0]$. Under the group action, $a \to aa_{11}a_{22}$; $b = 0 \to= 0$ (By subtracting a multiple of A from C); $c \to a(a_{11}a_{24} - a_{14}a_{21}) + ca_{11}a_{44} + ea_{21}a_{44}$; $d \to a(-a_{13}a_{22}) + c(-a_{13}a_{42}) + da_{22}a_{33} + ea_{22}a_{43}$; $e \to a(-a_{14}a_{22}) + ea_{22}a_{44}$; $f = 0 \to 0$ (By subtracting a multiple of B from C).

One of $a, d, e \neq 0$. If $a \neq 0$, taking $a_{21} = a_{43} = 0$ and make $c = d = e = 0$ by solving for a_{24}, a_{13} and a_{14} respectively to get a representative of C: $C_1 = [1,0,0,0,0,0]$, with (2) $A \wedge B \wedge C_1$ corresponding to (37B).

If $a = 0$ and $e \neq 0$, then we can make $c = d = 0$ by solving for a_{21} and a_{43}, and get the representative $C = [0,0,0,0,1,0]$, with (3) $A \wedge B \wedge C$ corresponding to (37B)

If $a = e = 0$, then $d \neq 0$, depending on $c = 0$ or not, we may obtain two representatives $C_1 = [0,0,0,1,0,0]$ and $C_2 = [0,0,1,1,0,0]$, with (4) $A \wedge B \wedge C_1$ corresponding to (37A) and (5) $A \wedge B \wedge C_2$ corresponding to (37C).

Case 1.3: $A = [0,1,0,0,0,0]$ and $B = [1,0,0,0,0,1]$. To fix A and B (up to a nonzero scalar) we require that $a_{12} = a_{14} = a_{24} = a_{31} = a_{32} = a_{34} = 0$, $a_{11} = a_{33}a_{44})/a_{22}$, $a_{13} = -a_{33}a_{42}/a_{22}$.

Now consider $C = [a,0,c,d,e,0]$. Under the group action, we have $a \to aa_{33}a_{44} + ca_{33}a_{42}a_{44}/a_{22} + e\Sigma^{12}_{24}$; $b = 0 \to 0$ (By subtracting a multiple of A from C); $c \to ca_{33}a_{44}^2/a_{22} + ea_{21}a_{44}$; $d \to aa_{33}a_{42} + c(a_{33}a_{42}^2/a_{22}) + da_{22}a_{33} + e(a_{22}a_{43} - a_{23}a_{42})$; $e \to e(a_{22}a_{44} - a_{24}a_{42})$; $f = 0 \to c(-a_{33}a_{42}a_{44}/a_{22}) + ea_{23}a_{44}$.

If $e = 0$, then one of $a, c, d \neq 0$. If $c \neq 0$. Make $a = f$ by solving for a_{42}, and further make them to be zero by subtracting a multiple of B from C. Then taking $a_{42} = 0$ and depending on whether $d = 0$ or not, we may get two representatives: (6) $C = [0,0,1,0,0,0]$ ($A \wedge B \wedge C$ corresponding to (37C)) and (7) $C = [0,0,1,1,0,0]$ ($A \wedge B \wedge C$ corresponding to ($37B_1$)).

If $c = 0$, then we may assume $a = 0$, for otherwise $a \neq 0$, make $d = 0$ to get $C = [1,0,0,0,0,0]$, which is in the same orbit as (2). If $a = 0$, then $d \neq 0$ to get (8) $C = [0,0,0,1,0,0]$ (corresponding to (37C)).

If $e \neq 0$, make $a = f$ by solving for a_{23}, and further make them to be zero by subtracting a multiple of B from C. Make $c = 0$ by solving for a_{21}, $d = 0$ by a_{43}. Then we get the representative, $C = [0,0,0,0,1,0]$, with (9) $A \wedge B \wedge C$ corresponding to (37B).

Case 2: $A = [0,1,0,0,1,0]$. To fix A (up to a scalar), We may choose $a_{23} = a_{33} = a_{32} = a_{43} = a_{44} = a_{34} = 0$, $a_{24} = -1/a_{42}$, $a_{13} = -1/a_{31}$, $a_{41} = a_{14}a_{31}a_{42}$, $a_{21} = a_{31}(a_{12} + a_{14}a_{42}a_{22})/a_{42}$.

Now assume $B = [a,0,c,d,e,f]$. After fixing A, we have

$a \to a\Sigma_{12}^{12} + c\Sigma_{14}^{12} + d(-a_{22}a_{31}) + e\Sigma_{24}^{12} + fa_{31}a_{42}$; $b = 0 \to a(a_{12} + a_{14}a_{42}a_{22})/a_{42} + ca_{14}a_{42}$; $c \to a\Sigma_{12}^{14} + c(-a_{14}^2a_{31}a_{42}) + d/(a_{42}a_{31}) + e(a_{41}/a_{42})$; $d \to a(a_{22}/a_{31}) + ca_{42}/a_{31}$; $e \to a\Sigma_{12}^{24} + c(-a_{14}a_{42}) - e$; $f \to a/(a_{31}a_{42})$.

One of $a, c, d, e, f \neq 0$, we can make $a = 1$. Taking $a_{14} = 0$, then $a_{41} = 0$ and $a_{21} = a_{12}a_{31}$. Make $c = 0$ by choosing $a_{11} = 0$.Then by fixing $a_{11} = a_{14} = a_{22} = 0$, we have $a = 1 \to \Sigma_{12}^{12} + ea_{42}a_{12}a_{31} + fa_{31}a_{42}$; $b = 0 \to a_{12}/a_{42}$; $c = 0 \to 0$; $d \to 0$; $e \to -a_{12}/a_{42} - e$; $f \to 1/(a_{31}a_{42})$.

Make $b = e$ by sloving for a_{12}, and further make them to be zero by subtracting a multiple of A from B. Now taking $a_{11} = a_{12} = a_{14} = a_{22} = 0$, and get $a = 1 \to fa_{31}a_{42}$; $b = 0 \to 0$; $c = 0 \to 0$; $d = 0 \to 0$; $e = 0 \to 0$; $f \to 1/(a_{31}a_{42})$.

Depending on whether $f = 0$ or not, we have the following three representatives: (i) $f = 0$, then $a = 0$ and make $f = 1$, $B = [0,0,0,0,0,1]$; (ii) $f < 0$, make $a = -f = 1$, $B = [1,0,0,0,0,-1]$; (iii) $f > 0$, make $a = f = 1$, $B = [1,0,0,0,0,1]$.

Subcase (i): It can be easily shown that it is in the same orbit as Subcase 1.3. So we just omit it.

Subcase (ii): $A = [0,1,0,0,1,0]$, $B = [1,0,0,0,0,-1]$. To fix A and B (up to a scalar), we require that $a_{11} = a_{21} = a_{31} = a_{22} = a_{24} = a_{33} = a_{34} = a_{44} = 0$, $a_{12} = -a_{43}$, $a_{13} = a_{42}$, $a_{14} = -1/a_{32}$, $a_{23} = -a_{32}$, $a_{41} = 1/a_{32}$.

Now consider $C = [a,b,c,d,0,0]$. Under the group action, we have $a \to ca_{43}/a_{32}$; $b \to c(-a_{42}/a_{32})$; $c \to c$; $d \to aa_{32}a_{43} + b(-a_{42}a_{32}) + c\Sigma_{14}^{23} + da_{32}^2$; $e \to b + ca_{42}/a_{32}$; $f \to -a + ca_{43}/a_{32}$.

If $c \neq 0$, make $a = -f$ by solving for a_{43}, and further make $a = f = 0$ by subtracting a multiple of B from C. Similarly, make $b = e$ by solving for a_{42}, and further make $b = e = 0$ by subtracting a multiple of A from C. Then taking $a_{43} = a_{42} = 0$, we have $a = 0 \to 0$; $b = 0 \to 0$; $c \to c$; $d \to da_{32}^2$; $e = 0 \to 0$; $f = 0 \to 0$.

Then we get the following representatives for C: (10) if $d = 0$, then $C = [0,0,1,0,0,0]$, with $A \wedge B \wedge C$ corresponding to ($37B_1$); (11) if $cd > 0$, then $C = [0,0,1,1,0,0]$, with $A \wedge B \wedge C$ corresponding to (37D); (12) if $cd < 0$, then $C = [0,0,1,-1,0,0]$, with $A \wedge B \wedge C$ corresponding to ($37D_1$).

If $c = 0$, then $a = -f = b = e = c = 0$, and $d \neq 0$. We get the representative $C = [0,0,0,1,0,0]$, with (13) $A \wedge B \wedge C$ corresponding to ($37B_1$).

Subcase (iii): $A = [0,1,0,0,1,0]$, $B = [1,0,0,0,0,1]$. To fix A, B (up to a scalar), we may choose $a_{21} = a_{31} = a_{24} = a_{34} = 0$ and $a_{12} = -a_{43}$, $a_{13} = -a_{42}$, $a_{14} = a_{41}$, $a_{22} = a_{33} = a_{11}a_{32}/a_{41}$, $a_{23} = a_{32}$, $a_{44} = a_{11}$.

Consider $C = [a,b,c,d,0,0]$. Then $a \to aa_{11}^2a_{32}/a_{41} + ba_{11}a_{32} + c(a_{11}a_{42} + a_{41}a_{43})$; $b \to aa_{11}a_{32} + ba_{11}^2a_{32}/a_{41} + c(a_{11}a_{43} + a_{41}a_{42})$; $c \to c(a_{11}^2 - a_{41}^2)$; $d \to a(-a_{32}a_{43} + a_{11}a_{32}a_{42}/a_{41}) + b(-a_{11}a_{32}a_{43}/a_{41} + a_{32}a_{42}) + c(a_{42}^2 - a_{43}^2) + d(a_{11}^2a_{32}^2/a_{41}^2 - a_{32}^2)$; $e = 0 \to a(-a_{11}a_{32}) - ba_{41}a_{32} + c(-a_{11}a_{43} - a_{41}a_{42})$; $f = 0 \to -aa_{41}a_{32} + b(-a_{11}a_{32}) + c(-a_{11}a_{42} - a_{41}a_{43})$;

If $c = 0$, then when at least one of $a, b \neq 0$, make $d = 0$ by solving for a_{42}. Now depending on the values of a, b, we can make either $a = f$ or $b = e$, and further reduce them to be zero, and by choosing properly the values of a_{11}, we may obtain the two representatives $C = [0,0,0,0,1,0]$, which is the same as (9), so omit it; and (14) $C = [0,0,0,0,0,1]$. When $a = b = 0$, then $d \neq 0$, we get $C = [0,0,0,1,0,0]$, which is the same as (13), so omit it. It can be shown that (14) corresponds to (37D).

If $c \neq 0$, make $a = f$ and $b = e$ by solving for a_{43} and a_{42} respectively, and further reduce them to be zero by linear combination. Then take $a_{42} = a_{43} = 0$, if $d = 0$, we get the representative $C = [0,0,1,0,0,0]$, which is the same as (10), so omit it. If $d \neq 0$, depending on whether $cd > 0$ or $cd < 0$, we obtain two representatives $C = [0,0,1,1,0,0]$ and $C = [0,0,1,-1,0,0]$, which are the same as (11) and (12) respectively, so omit them.

Now we consider all the 14 algebras:

(1): $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_2, x_4] = x_7$;
(2): $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_3, x_4] = x_7$;
(3): $[x_1, x_3] = x_5$, $[x_2, x_4] = x_6$, $[x_3, x_4] = x_7$;
(4): $[x_1, x_3] = x_5$, $[x_2, x_3] = x_6$, $[x_3, x_4] = x_7$;
(5): $[x_1, x_3] = x_5$, $[x_1, x_4] = x_6$, $[x_2, x_3] = x_6$, $[x_3, x_4] = x_7$;
(6): $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_1, x_4] = x_7$, $[x_3, x_4] = x_5$;
(7): $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_1, x_4] = x_7$, $[x_2, x_3] = x_7$, $[x_3, x_4] = x_5$;
(8): $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_2, x_3] = x_7$, $[x_3, x_4] = x_5$;
(9): $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_2, x_4] = x_7$, $[x_3, x_4] = x_5$;
(10): $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_1, x_4] = x_7$, $[x_2, x_4] = x_6$, $[x_3, x_4] = -x_5$;
(11): $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_1, x_4] = x_7$, $[x_2, x_3] = x_7$, $[x_2, x_4] = x_6$, $[x_3, x_4] = -x_5$;
(12): $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_1, x_4] = x_7$, $[x_2, x_3] = -x_7$, $[x_2, x_4] = x_6$, $[x_3, x_4] = -x_5$;
(13): $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_2, x_3] = x_7$, $[x_2, x_4] = x_6$, $[x_3, x_4] = -x_5$;
(14): $[x_1, x_2] = x_5$, $[x_1, x_3] = x_6$, $[x_2, x_4] = x_6$, $[x_3, x_4] = -x_5 + x_7$;

A) We will show that (1), (2), (3) are all isomorphic to (37B).

(1)$\cong$ (37B): $x_1 \to x_2$, $x_2 \to x_3$, $x_3 \to x_1$, $x_4 \to x_4$, $x_5 \to x_6$, $x_6 \to -x_5$, $x_7 \to x_7$;

(2)$\cong$ (37B): $x_1 \to x_2$, $x_2 \to x_1$, $x_3 \to x_3$, $x_4 \to x_4$, $x_5 \to -x_5$, $x_6 \to x_6$, and $x_7 \to x_7$.

(3)$\cong$ (37B): $x_1 \to x_4$, $x_2 \to x_1$, $x_3 \to x_3$, $x_4 \to x_2$, $x_5 \to -x_7$, $x_6 \to x_5$ and $x_7 \to -x_6$.

B) (4) is isomorphic to (37A).

(4)$\cong$ (37A): $x_1 \to x_1$, $x_2 \to x_3$, $x_3 \to x_2$, $x_4 \to x_4$, $x_5 \to x_5$, $x_6 \to -x_6$, and $x_7 \to x_7$.

C) (5), (6) and (8) are isomorphic to (37C).

(5) $\cong$ (37C): $x_1 \to x_3$, $x_2 \to x_1$, $x_3 \to x_2$, $x_4 \to x_4$, $x_5 \to -x_6$, $x_6 \to x_5$, $x_7 \to x_7$;

(6) $\cong$ (37C): $x_1 \to x_2$, $x_2 \to -x_1$, $x_3 \to x_3$, $x_4 \to x_4$, $x_5 \to x_5$, $x_6 \to x_6$, $x_7 \to x_7$;

(8) $\cong$ (37C): $x_1 \to x_3$, $x_2 \to x_4$, $x_3 \to x_2$, $x_4 \to -x_1$, $x_5 \to x_5$, $x_6 \to -x_6$ and $x_7 \to -x_7$;

D) We will show that (9), (11) and (14) are isomorphic to (37D); (12) is also isomorphic to (37D) over algebraically closed fields. Let α be a root of the equation $x^2+1=0$. Then

(9)$\cong$ (37D): obvious

(11)$\cong$ (37D): $x_1 \to -x_1+x_2$, $x_2 \to x_1+x_2$, $x_3 \to x_3+x_4$, $x_4 \to -x_3+x_4$, $x_5 \to -2x_5$, $x_6 \to -x_6+x_7$, and $x_7 \to x_6+x_7$.

(14)$\cong$ (37D): $x_1 \to x_4$, $x_2 \to x_2$, $x_3 \to x_3$, $x_4 \to x_1$, $x_5 \to -x_7$, $x_6 \to -x_5$, and $x_7 \to -x_6-x_7$;

(12)$\cong$ (11): $x_1 \to \alpha x_2$, $x_2 \to x_4$, $x_3 \to \alpha x_3$, $x_4 \to x_1$, $x_5 \to \alpha x_6$, $x_6 \to -x_7$, and $x_7 \to -\alpha x_5$.

E) We will show that (7), (10)=(37B$_1$) and (13) are isomorphic to (37B) over algebraically closed fields.

(10) $\cong$ (7): $x_1 \to x_1$, $x_2 \to x_4$, $x_3 \to x_2$, $x_4 \to -x_3$, $x_5 \to x_7$, $x_6 \to x_5$ and $x_7 \to x_6$;

(10)$\cong$ (37B): $x_1 \to \alpha x_2-\alpha x_3$, $x_2 \to x_1+\alpha x_4$, $x_3 \to \alpha x_1+x_4$, $x_4 \to x_2+x_3$, $x_5 \to -\alpha x_5+x_7$, $x_6 \to x_5-\alpha x_7$, and $x_7 \to 2\alpha x_6$.

(13) $\cong$ (10): $x_1 \to x_3$, $x_2 \to -x_4$, $x_3 \to x_1$, $x_4 \to -x_2$, $x_5 \to x_5$, $x_6 \to -x_6$, and $x_7 \to x_7$.

To show that (37A), (37B), (37B$_1$), (37C), (37D) and (37D$_1$) are distinct, we compare the minimal numbers again. We have

(37A): minimal number $(1,1,1,3)$, corresponding to the ordered basis $\{x_1, x_3, x_4, x_2\}$;

(37B): minimal number $(1,1,2,2)$, corresponding to the ordered basis $\{x_1, x_4, x_2, x_3\}$;

(37B$_1$): When the field is $\mathbf{R}$, the minimal number is $(2,2,3,3)$, corresponding to the ordered basis $\{x_2, x_3, x_1, x_4\}$; when the field is algebraically closed, it is the same as (37B);

(37C): minimal number $(1,2,2,3)$, corresponding to the ordered basis $\{x_1, x_3, x_4, x_2\}$;

(37D): minimal number $(2,2,2,2)$, corresponding to the ordered basis $\{x_1, x_2, x_3, x_4\}$.

(37D$_1$): When the field is $\mathbf{R}$, the minimal number is $(3,3,3,3)$, corresponding to the ordered basis $\{x_1, x_2, x_3, x_4\}$; when the field is algebraically closed, it is the same as (37D).

Therefore the central extensions of $N_{4,4}$ of dimensional 7 are:

(37A):	$[x_1, x_2] = x_5,$	$[x_2, x_3] = x_6,$	$[x_2, x_4] = x_7;$
(37B):	$[x_1, x_2] = x_5,$	$[x_2, x_3] = x_6,$	$[x_3, x_4] = x_7;$
(37B$_1$):	$[x_1, x_2] = x_5,$	$[x_1, x_3] = x_6,$	$[x_1, x_4] = x_7$
	$[x_2, x_4] = x_6,$	$[x_3, x_4] = -x_5;$	
(37C):	$[x_1, x_2] = x_5,$	$[x_2, x_3] = x_6,$	$[x_2, x_4] = x_7,$
	$[x_3, x_4] = x_5;$		
(37D):	$[x_1, x_2] = x_5,$	$[x_1, x_3] = x_6,$	$[x_2, x_4] = x_7,$
	$[x_3, x_4] = x_5;$		
(37D$_1$):	$[x_1, x_2] = x_5,$	$[x_1, x_3] = x_6,$	$[x_1, x_4] = x_7$
	$[x_2, x_3] = -x_7,$	$[x_2, x_4] = x_6,$	$[x_3, x_4] = -x_5;$

Chapter 6

Algebras over Algebraically Closed Fields

In this chapter we will consider the central extensions over algebraically closed fields of characteristics $\neq 2$. For those algebras whose central extensions give rise to new algebras over the real field, their proofs can be found in Chapter 7.

Some of the algebras we obtain have different presentations from those of Seeley, in that case, an isomorphism is provided.

6.1 Extensions of 4-Dimensional Algebras

All the 7-dimensional nilpotent Lie algebras without any Abelian factors have at most a 3-dimensional center (considering the dimension of $H^2(\mathfrak{g}, \mathbf{F})$). So we just consider the central extensions of algebras of dimension at least 4.

Central extensions of $N_{4,2}$:

$N_{4,2} : [x_1, x_i] = x_{i+1}, i = 2, 3;$

$Z(\mathfrak{g})$: x_4; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_4; $Z^2(\mathfrak{g})$: $C_{24} = C_{34} = 0$; $W(H^2)$: $C_{12} = C_{13} = 0$; $\dim H^2$: 2;

As the cohomology group is of dimension 2, then $G_3(H^2(\mathfrak{g}, \mathbf{F})) = 0$, hence $N_{4,2}$ has no central extensions without Abelian factors of dimension 7.

Central extensions of $N_{4,3}$:

$Z(\mathfrak{g})$: x_3, x_4; $[\mathfrak{g}, \mathfrak{g}]$: x_3; $Z^2(\mathfrak{g})$: $C_{34} = 0$; $W(H^2)$: $C_{12} = 0$; $\dim H^2$: 4; Basis: Δ_{13}, Δ_{14}, Δ_{23}, Δ_{24};

Group action: $a\Delta_{13} + b\Delta_{14} + c\Delta_{23} + d\Delta_{24}$;

Let $\delta := a_{11}a_{22} - a_{12}a_{21}$.

$a \to (aa_{11} + ca_{21})\delta$;

$b \to a_{11}(aa_{34} + ba_{44}) + a_{21}(ca_{34} + da_{44})$;

$c \to (aa_{12} + ca_{22})\delta$;

$d \to a_{12}(aa_{34} + ba_{44}) + a_{22}(ca_{34} + da_{44})$;

One of $a, c \neq 0$ and one of $b, d \neq 0$ in $A \wedge B \wedge C$, where A, B and C are of the form $a\Delta_{13} + b\Delta_{14} + c\Delta_{23} + d\Delta_{24}$.

Let $A = [a, b, c, d]$, $B = [a_1, b_1, c_1, d_1]$, $C = [a_2, b_2, c_2, d_2]$.

As one of $a, c \neq 0$, we can assume that $a = 1$. By subtracting scalar multiples of A from B and C, we may let $a_1 = a_2 = 0$.

We may assume that at least one of b_1 and b_2 is not zero. For otherwise, one of c_1, c_2 is nonzero, we can make b_1 or $b_2 \neq 0$, as $b_i \to c_i a_{21} a_{34} + d_i a_{21} a_{44}$.

Now we may assume $b_1 \neq 0$ (simply by switching B and C if necessary) and make $b_2 = 0$ (by subtracting a scalar multiple of B from C).

Case 1: $c_2 \neq 0$. By making $a = b_1 = c_2 = 1$, we may let, with respect to the wedge product,

$$A = [1, 0, 0, d],\ B = [0, 1, 0, d_1],\ C = [0, 0, 1, d_2].$$

Considering the action of the group on A, B, C, we have

$$\begin{aligned} A =& \ [a_{11}\delta, a_{11}a_{34} + da_{21}a_{44}, a_{12}\delta, a_{12}a_{34} + da_{22}a_{44}], \\ B =& \ [0, a_{11}a_{44} + d_1 a_{21}a_{44}, 0, a_{12}a_{44} + d_1 a_{22}a_{44}], \\ C =& \ [a_{21}\delta, a_{21}a_{34} + d_2 a_{21}a_{44}, a_{22}\delta, a_{22}a_{34} + d_2 a_{22}a_{44}]. \end{aligned}$$

We can make $a_2 = b_2 = 0$ by letting $a_{21} = 0$ and $d_2 = 0$ by solving for a_{34}. Also make $d_1 = 0$ by solving for a_{12}, and what is left can be changed into

$$A = [1, 0, 0, d],\ B = [0, 1, 0, 0],\ C = [0, 0, 1, 0].$$

And depending on whether $d = 0$ or not, we get two representatives for $A \wedge B \wedge C$: (1) $A = [1, 0, 0, 0]$, $B = [0, 1, 0, 0]$, $C = [0, 0, 1, 0]$ (corresponding to (357B)); and (2) $A = [1, 0, 0, 1]$, $B = [0, 1, 0, 0]$, $C = [0, 0, 1, 0]$ (corresponding to (357C)).

Case 2: $c_2 = 0$. Then $d_2 \neq 0$. And get, WLOG,

$$A = [1, 0, c, 0],\ B = [0, 1, c_1, 0],\ C = [0, 0, 0, 1].$$

Acting the automorphism group on $A \wedge B \wedge C$, we have

$$\begin{aligned} A = &\ [a_{11}\delta + ca_{21}\delta, a_{11}a_{34} + a_{21}a_{34}, a_{12}\delta + ca_{22}\delta, a_{12}a_{34} + ca_{22}a_{34}], \\ B = &\ [c_1a_{21}\delta, a_{11}a_{44} + c_1a_{21}a_{34}, c_1a_{22}\delta, a_{12}a_{44} + c_1a_{22}a_{34}], \\ C = &\ [0, a_{21}a_{44}, 0, da_{22}a_{44}]. \end{aligned}$$

Let $a_{21} = 0$, we make $b_2 = 0$, Also make $c = 0$ by solving for a_{12}, and get,

$$A = [1,0,0,0],\ B = [0,1,c_1,0],\ C = [0,0,0,1].$$

Depending on whether $c_1 = 0$ or not, we get the following two representatives for $A \wedge B \wedge C$: (3) $A = [1,0,0,0]$, $B = [0,1,0,0]$ and $C = [0,0,0,1]$ (corresponding to (357A)), and (4) $A = [1,0,0,0]$, $B = [0,1,1,0]$ and $C = [0,0,0,1]$. But we can show that (4) and (2) are in the same orbit.

By comparing the orbit, all the algebras (357A,B,C) can be easily showed to be distinct.

Therefore the corresponding central extensions of $N_{4,3}$ are:

(357A):	$[x_1, x_2] = x_3$,	$[x_1, x_3] = x_5$,	
	$[x_1, x_4] = x_7$,	$[x_2, x_4] = x_6$;	
(357B):	$[x_1, x_2] = x_3$,	$[x_1, x_3] = x_5$,	
	$[x_1, x_4] = x_7$,	$[x_2, x_3] = x_6$;	
(357C):	$[x_1, x_2] = x_3$,	$[x_1, x_3] = x_5$,	$[x_1, x_4] = x_7$,
	$[x_2, x_3] = x_6$,	$[x_2, x_4] = x_5$;	

6.2 Extensions of 5-Dimensional Algebras

Central extensions of $N_{5,1}$:

$Z(\mathfrak{g})$: x_5; $[\mathfrak{g},\mathfrak{g}]$: $x_3,\ x_4,\ x_5$; $Z^2(\mathfrak{g})$: $C_{35}=C_{45}=0,\ C_{15}-C_{24}=0,\ C_{25}+C_{34}=0$; $W(H^2)$: $C_{12}=C_{13}=C_{14}=0$; $\dim H^2$: 3; Basis: $\Delta_{15}+\Delta_{24},\ \Delta_{25}-\Delta_{34},\ \Delta_{23}$;

Group Action: $a(\Delta_{15}+\Delta_{24})+b(\Delta_{25}-\Delta_{34})+c\Delta_{23}$:

$a\rightarrow aa_{11}^6+ba_{11}^5a_{21}$;

$b\rightarrow ba_{11}^7$;

$c\rightarrow ca_{11}^5+2ba_{11}^3a_{42}-ba_{11}a_{32}^2-2aa_{11}^4a_{21}-ba_{11}^3a_{21}^2$;

Consider the wedge product of $A=[a,b,c]$ and $B=[a_1,b_1,c_1]$.

One of a,b,a_1,b_1 is nonzero, can always choose $a\neq 0$ (for example, if both $a=a_1=0$, then b or $b_1\neq 0$. Make a or $a_1\neq 0$, and switching A and B if necessary). So assume $A=[1,b,c]$, and by subtracting from B a multiple of A to get $B=[0,b_1,c_1]$.

Case 1: $b_1\neq 0$. Then take $B=[0,1,c_1]$ and $A=[1,0,c]$. Observe the group action on B, we have

$$B=[a_{11}^5a_{21},a_{11}^7,c_1a_{11}^5+2a_{11}^3a_{42}-a_{11}a_{32}^2].$$

Make both $a_1=c_1=0$ by solving for a_{21} and a_{42} to get $B=[0,1,0]$. Consider again the group action on both A and B, we have

$$\begin{aligned}A= &\ [a_{11}^6,0,ca_{11}^5-2a_{11}^4a_{21}]\\ B= &\ [a_{11}^5a_{21},a_{11}^7,2a_{11}^3a_{42}-a_{11}a_{32}^2-a_{11}^3a_{21}^2]\end{aligned}$$

Now we can make $c=c_1=0$ by solving for a_{21} and a_{42}. By subtracting a mulitiple of A from B, we can also make $a_1=0$ and get

$$A=[1,0,0],\ B=[0,1,0],$$

corresponding to (23457G).

Case 2: $b_1=0$. Then $c_1\neq 0$ and get $B=[0,0,1]$ and $A=[1,b,0]$. Consider the group action on both A and B, we have

$$\begin{aligned}A= &\ [a_{11}^6+ba_{11}^5a_{21},ba_{11}^7,2ba_{11}^3a_{42}-ba_{11}a_{32}^2-2a_{11}^4a_{21}-ba_{11}^3a_{21}^2],\\ B= &\ [0,0,a_{11}^5],\end{aligned}$$

Now depending $b=0$ or not, we can get

$$A=[1,0,0],\ B=[0,0,1],$$

or

$$A=[0,1,0],\ B=[0,0,1],$$

as when $b \neq 0$, we can make $a = 0$, corresponding respectively to (23457E) and (23457F). The non-isomorphism between all the algebras can be easily proven by comparing their orbits.

Therefore the corresponding central extensions of $N_{5,1}$ are:

(23457E):	$[x_1, x_i] = x_{i+1}, i = 2, 3, 4,$	$[x_1, x_5] = x_6,$
	$[x_2, x_3] = x_5 + x_7,$	$[x_2, x_4] = x_6;$
(23457F):	$[x_1, x_i] = x_{i+1}, i = 2, 3, 4,$	$[x_2, x_3] = x_5 + x_7,$
	$[x_2, x_5] = x_6,$	$[x_3, x_4] = -x_6;$
(23457G):	$[x_1, x_i] = x_{i+1}, i = 2, 3, 4,$	$[x_1, x_5] = x_6,$
	$[x_2, x_3] = x_5,$	$[x_2, x_4] = x_6,$
	$[x_2, x_5] = x_7,$	$[x_3, x_4] = -x_7;$

The central extensions of $N_{5,2,1}$ can be found in chapter 2, Example 6.

The central extensions of $N_{5,2,2}$ can be found in chapter 2, Example 1. By switching x_3 with x_4, and x_6 with x_7 in (2357A,B,C), and in (2357D) by taking $x_1 \to 2x_1 + x_2$, $x_2 \to x_2$, $x_3 \to -2x_3 + 4x_4$, $x_4 \to 2x_3$, $x_5 \to 4x_5 + 2x_7$, $x_6 \to 2x_7$, $x_7 \to 8x_6$, we get exactly the same representation as in Seeley's paper.

The central extensions of $N_{5,2,3}$ can be found in chapter 7.

Central extensions of $N_{5,3,1}$:

$Z(\mathfrak{g})$: x_5; $[\mathfrak{g}, \mathfrak{g}]$: x_5; $Z^2(\mathfrak{g})$: $C_{15} = C_{25} = C_{35} = C_{45} = 0$; $W(H^2)$: $C_{12} = C_{34} = 0$;

It is obvious that all the elements in $H^2(\mathfrak{g}, \mathbf{F})$ have x_5 in their kernels. So there is no desired central extension.

The central extensions of $N_{5,3,2}$ can be found in chapter 7. By the following transformations, we can get the exactly the same presentations as in Seeley's paper. In (247C), switch x_6 and x_7; In (247D), take $x_1 \to a$, $x_2 \to c$, $x_3 \to b$, $x_4 \to e$, $x_5 \to d$, $x_6 \to g$ and $x_7 \to f$; In (247E), switch x_6 and x_7; In (247F), take $x_1 \to a$, $x_2 \to -b + c$, $x_3 \to b + c$, $x_4 \to -d + e$, $x_5 \to d + e$, $x_6 \to f + g$, and $x_7 \to -f + g$; In (247G), $x_1 \to a + b$, $x_2 \to b + c$, $x_3 \to b - c$, $x_4 \to d + e$, $x_5 \to d - e$, $x_6 \to f + g$ and $x_7 \to f - g$; In (247H), take $x_1 \to a + b + c$, $x_2 \to 2(b + c)$, $x_3 \to -2(b - c)$, $x_4 \to 2(d + e)$, $x_5 \to -2(d - e)$, $x_6 \to 4(f + g)$ and $x_7 \to -4(f - g)$; In (247J), take $x_1 \to -a + c$, $x_2 \to b$, $x_3 \to c$, $x_4 \to -d$, $x_5 \to -e$, $x_6 \to -g$ and $x_7 \to -f$; In (247K), switch x_6 and x_7; In (247M), $x_1 \to a$, $x_2 \to c$, $x_3 \to -b$, $x_4 \to e$, $x_5 \to -d$, $x_6 \to f$ and $x_7 \to g$; In (247N), switch x_6 and x_7; In (247O), take $x_1 \to -a$, $x_2 \to -c$, $x_3 \to b$, $x_4 \to e$, $x_5 \to -d$, $x_6 \to -g$ and $x_7 \to f$; In (247Q), take $x_1 \to a$, $x_2 \to c$, $x_3 \to -b$, $x_4 \to e$, $x_5 \to -d$, $x_6 \to f$ and $x_7 \to -g$.

Central extensions of $N_{5,3,3}$:

$Z(\mathfrak{g})$: x_4, x_5; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_4; $Z^2(\mathfrak{g})$: $C_{45} = 0, C_{24} = 0, C_{34} = 0, C_{35} = 0$; $W(H^2)$: $C_{12} = C_{13} = 0$; $\dim H^2$: 4; Basis: $\Delta_{14}, \Delta_{15}, \Delta_{23}, \Delta_{25}$;

Group action: $a\Delta_{14} + b\Delta_{15} + c\Delta_{23} + d\Delta_{25}$;

$a \to aa_{11}^3a_{22}$; $b \to aa_{11}a_{45} + ba_{11}a_{55} + da_{21}a_{55}$; $c \to ca_{11}a_{22}^2$; $d \to da_{22}a_{55}$.

Let $A = [a, b, c, d]$ and $B = [a_1, b_1, c_1, d_1]$. WLOG, we assume that $a \neq 0$, and let $A = [1, b, c, d]$ and $B = [0, b_1, c_1, d_1]$.

Case 1: $d_1 \neq 0$. Then assume $B = [0, b_1, c_1, 1]$. We have $a_1 = 0 \to 0$; $b_1 \to b_1a_{11}a_{55} + a_{21}a_{55} = 0$; (Solve for a_{21}.) $c_1 \to c_1a_{11}a_{22}^2$; $d_1 = 1 \to a_{22}a_{55} = 1$.

Depending on whether $c_1 = 0$ or not, we get $B_1 = [0, 0, 0, 1]$ and $B_2 = [0, 0, 1, 1]$.

Subcase 1.1: With $B_1 = [0, 0, 0, 1]$, we assume $A = [1, b, c, 0]$. Then $a = 1 \to a_{11}^3a_{22} = 1$; $b \to a_{11}a_{45} + ba_{11}a_{55} = 0$ (Solve for a_{45}); $c \to ca_{11}a_{22}^2$; $d \to 0$.

Depending on whether $c = 0$ or not, we get $A_1 = [1, 0, 0, 0]$ ($A_1 \wedge B_1$ corresponding to (2457B)) and $A_2 = [1, 0, 1, 0]$ ($A_2 \wedge B_1$ corresponding to (2457I)).

Subcase 1.2: With $B_2 = [0, 0, 1, 1]$, assume $A = [1, b, c, 0]$. Similar discussions would lead to $A_1 = [1, 0, 0, 0]$ ($A_1 \wedge B_2$ corresponding to (2457E)) and $A_2 = [1, 0, 1, 0]$ ($A_2 \wedge B_2$ corresponding to (2457J)).

Case 2: $d_1 = 0$. Then $B = [0, b_1, c_1, 0]$.

Subcase 2.1: If $c_1 \neq 0$, then depending on whether $b_1 = 0$ or not, we can get two representatives Subcase (2.1.1) $B_1 = [0, 0, 1, 0]$ and Subcase (2.1.2) $B_2 = [0, 1, 1, 0]$.

Subcase 2.1.1: With $B_1 = [0, 0, 1, 0]$, we may let $A = [1, b, 0, d]$, and make $b = 0$, then $d \neq 0$ to get a representative $A = [1, 0, 0, 1]$ ($A \wedge B_1$ corresponding to (2457H)).

Subcase 2.1.2: With $B_2 = [0, 1, 1, 0]$, we may assume that $A = [1, b, 0, d]$, then make $b = 0$, and depending on whether $d = 0$ or not, we get two representatives for A: $A_1 = [1, 0, 0, 0]$ ($A_1 \wedge B_2$ corresponding to (2457G)) and $A_2 = [1, 0, 0, 1]$ ($A_2 \wedge B_2$ corresponding to (2457K)).

Subcase 2.2: If $c_1 = 0$, then $b_1 \neq 0$. Assume $B = [0, 1, 0, 0]$ and $A = [1, 0, c, d]$. Now $a = 1 \to a_{11}^3a_{22} = 1$; $b = 0 \to a_{11}a_{45} + da_{21}a_{55}$; $c \to ca_{11}a_{22}^2$; $d \to da_{22}a_{55}$.

We easily get four representatives for A: $A_1 = [1, 0, 0, 0]$ ($A_1 \wedge B$ corresponding to (2457A)), $A_2 = [1, 0, 0, 1]$ ($A_2 \wedge B$ corresponding to (2457C)), $A_3 = [1, 0, 1, 0]$ ($A_3 \wedge B$ corresponding to (2457F)) and $A_4 = [1, 0, 1, 1]$ ($A_4 \wedge B$ corresponding to (2457D)).

It is fairly straightforward to show that all the algebras are distinct by comparing their orbits.

Therefore the corresponding central extensions of $N_{5,3,3}$ are:

(2457A):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_i] = x_{i+2}, i = 4, 5;$
(2457B):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_4] = x_7,$
	$[x_2, x_5] = x_6$	
(2457C):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_i] = x_{i+2}, i = 4, 5,$
	$[x_2, x_5] = x_6$	
(2457D):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_i] = x_{i+2}, i = 4, 5,$
	$[x_2, x_3] = x_6,$	$[x_2, x_5] = x_6;$
(2457E):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_4] = x_7,$
	$[x_2, x_3] = x_6,$	$[x_2, x_5] = x_6;$
(2457F):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_i] = x_{i+2}, i = 4, 5,$
	$[x_2, x_3] = x_6;$	
(2457G):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_4] = x_7,$
	$[x_1, x_5] = x_6,$	$[x_2, x_3] = x_6;$
(2457H):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_4] = x_7,$
	$[x_2, x_3] = x_6,$	$[x_2, x_5] = x_7;$
(2457I):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_4] = x_6,$
	$[x_2, x_3] = x_6,$	$[x_2, x_5] = x_7;$
(2457J):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_4] = x_6,$
	$[x_2, x_3] = x_6 + x_7,$	$[x_2, x_5] = x_7;$
(2457K):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_4] = x_7,$
	$[x_1, x_5] = x_6,$	$[x_2, x_3] = x_6,$
	$[x_2, x_5] = x_7.$	

Remark: By taking $x_1 \to a$, $x_2 \to b$, $x_3 \to c$, $x_4 \to d$, $x_5 \to -e$, $x_6 \to f + g$ and $x_7 \to -g$, we will get the exact presentation of (2457J) as in Seeley's paper.

The central extensions of $N_{5,4}$ can be found in Chapter 7.

6.3 Extensions of 6-Dimensional Algebras

The central extensions of $N_{6,1,1}$ can be found in Chapter 2, Example 3. Notice that (123457G) of Seeley is just a special case of (123457I) by taking $\lambda = 1$.

Central extensions of $N_{6,1,2}$:

$Z(\mathfrak{g})$: x_6; $[\mathfrak{g},\mathfrak{g}]$: x_3, x_4, x_5, x_6; $Z^2(\mathfrak{g})$: $C_{16} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = C_{56} = 0$, $C_{34} + C_{25} = 0$, $C_{15} = C_{24}$; $W(H^2)$: $C_{12} = C_{13} = C_{14} = C_{25} = 0$; $\dim H^2$: 2; Basis: $\Delta_{15} + \Delta_{24}, \Delta_{23}$;

It is obvious that all the cocycles have x_6 in its kernel. So there is no central extension of $N_{6,1,2}$ at all.

Central extensions of $N_{6,1,3}$:

$Z(\mathfrak{g})$: x_6; $[\mathfrak{g},\mathfrak{g}]$: x_3, x_4, x_5, x_6; $Z^2(\mathfrak{g})$: $C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = C_{56} = 0$, $C_{34} + C_{25} = 0$, $C_{16} = C_{24}$; $W(H^2)$: $C_{12} = C_{13} = C_{14} = C_{15} = 0$; $\dim H^2$: 3; Basis: $\Delta_{16} + \Delta_{24}$, Δ_{23}, $\Delta_{25} - \Delta_{34}$;

Group action: $a(\Delta_{16} + \Delta_{24}) + b\Delta_{23} + c(\Delta_{25} - \Delta_{34})$;

$a \to aa_{11}^8$; $b \to ba_{11}^7 + c(2a_{11}^4 a_{42} - a_{11}a_{32}^2 - a_{11}^6 a_{21})$; $c \to ca_{11}^9$;

We have $a \neq 0$.

Case 1: $c = 0$. Then b goes to ba_{11}^7, get $[aa_{11}^8, ba_{11}^7, 0]$. So if $b = 0$, we get $[1,0,0]$, correspondig to (123457D); And if $b \neq 0$, we get $[1,1,0]$, corresponding to (123457E);

Case 2: $c \neq 0$. Make $b = 0$ by solving for a_{42} and get $[aa_{11}^8, 0, ca_{11}^9]$, and make it to $[1,0,1]$, corresponding to (123457F).

Therefore the corresponding central extensions of $N_{6,1,3}$ are:

(123457D):	$[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 5,$	$[x_1, x_6] = x_7,$
	$[x_2, x_3] = x_6,$	$[x_2, x_4] = x_7;$
(123457E):	$[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 5,$	$[x_1, x_6] = x_7,$
	$[x_2, x_3] = x_6 + x_7,$	$[x_2, x_4] = x_7;$
(123457F):	$[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 5,$	$[x_1, x_6] = x_7,$
	$[x_2, x_3] = x_6,$	$[x_2, x_4] = x_7,$
	$[x_2, x_5] = x_7,$	$[x_3, x_4] = -x_7.$

Central extensions of $N_{6,1,4}$:

$Z(\mathfrak{g})$: x_6; $[\mathfrak{g},\mathfrak{g}]$: x_3, x_4, x_6; $Z^2(\mathfrak{g})$: $C_{16} = C_{24} = C_{35}$, $C_{26} + C_{34} = 0$, $C_{36} = C_{45} = C_{46} = C_{56} = 0$; $W(H^2)$: $C_{12} = C_{13} = C_{23} = 0$; $\dim H^2$: 5; Basis: $\Delta_{14}, \Delta_{15}, \Delta_{16} + \Delta_{24} + \Delta_{35}, \Delta_{25}, \Delta_{26} - \Delta_{34}$;

Group action: $a\Delta_{14} + b\Delta_{15} + c(\Delta_{16} + \Delta_{24} + \Delta_{35}) + d\Delta_{25} + e(\Delta_{26} - \Delta_{34})$:

$a \to aa_{11}^5 + c(2a_{21}a_{11}^4 + a_{52}a_{11}^3) + e(a_{11}^3 a_{21}^2 - 2a_{42}a_{11}^3 - a_{11}^2 a_{21}a_{52} + a_{11}^4 a_{51} + a_{32}^2 a_{11})$;

$b \to -aa_{21}a_{11}^3 + ba_{11}^4 + c(a_{11}a_{65} - a_{11}^2 a_{21}^2 + a_{11}^3 a_{31}) + da_{21}a_{11}^3 + e(a_{21}a_{65} + a_{21}a_{11}^2 a_{31})$;

$c \to ca_{11}^6 + ea_{11}^5 a_{21}$;

$d \to c(-a_{21}a_{11}^4 + a_{11}^3 a_{52}) + da_{11}^5 + e(a_{11}^2 a_{65} - 2a_{42}a_{11}^3 - a_{11}^2 a_{21}a_{52} + a_{31}a_{11}^4 + a_{11}^4 a_{51} + a_{32}^2 a_{11})$;

$e \to ea_{11}^7$;

One of $c, e \neq 0$.

Case 1: $e \neq 0$. We can make $a = c = d = 0$ by solving for a_{51}, a_{21} and a_{65} respectively. What is left is b, and it goes to ba_{11}^4 after we fix a, c, d. So we have two different representatives in this case: $[0,0,0,0,1]$ $(b = 0)$ (corresponding to (12457F)) and $[0,1,0,0,1]$ (corresponding to (12457G)) $(b \neq 0)$;

Case 2: $e = 0$. We should have $c \neq 0$. Can make $c = 1$, and $a = b = d = 0$ by solving for a_{52}, a_{65} and a_{21} respectively. And we have iii) $[0,0,1,0,0]$, corresponding to (12457E).

Therefore the corresponding central extensions of $N_{6,1,4}$ are:

(12457E):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_4] = x_6,$
	$[x_1, x_6] = x_7,$	$[x_2, x_3] = x_6,$
	$[x_2, x_4] = x_7$	$[x_2, x_5] = x_6,$
	$[x_3, x_5] = x_7;$	
(12457F):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_4] = x_6,$
	$[x_2, x_3] = x_6$	$[x_2, x_i] = x_{i+1}, i = 5, 6,$
	$[x_3, x_4] = -x_7;$	
(12457G):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_4] = x_6,$
	$[x_1, x_5] = x_7,$	$[x_2, x_3] = x_6,$
	$[x_2, x_i] = x_{i+1}, i = 5, 6$	$[x_3, x_4] = -x_7;$

Central extensions of $N_{6,2,1}$:

$Z(\mathfrak{g})$: x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_4, x_5, x_6; $Z^2(\mathfrak{g})$: $C_{24} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = C_{56} = 0$, $C_{25} + C_{34} = 0$; $W(H^2)$: $C_{12} = C_{13} = C_{14} = C_{15} = 0$; $\dim H^2$: 3; Basis: $\Delta_{16}, \Delta_{23}, \Delta_{25} - \Delta_{34}$;

Group action: $a\Delta_{16} + b\Delta_{23} + c(\Delta_{25} - \Delta_{34})$:

$a \to aa_{11}^5 a_{22}$; $b \to ba_{11}a_{22}^2 + c(2a_{11}a_{22}a_{42} - a_{11}a_{32}^2)$; $c \to ca_{11}^3 a_{22}^2$;

We have $a \neq 0$. Make $a = 1$.

Case 1: $c = 0$. We can get representatives $[1,0,0]$ (when $b = 0$) (corresponding to (123457A)) and $[1,1,0]$ (when $b \neq 0$) (corresponding to (123457B)).

Case 2: $c \neq 0$. Make $b = 0$ by solving for a_{42} and get $[1, 0, 1]$ (corresponding to (123457C)).

Therefore the corresponding central extensions of $N_{6,2,1}$ are:

(123457A):	$[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 6;$		
(123457B):	$[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 6,$	$[x_2, x_3] = x_7;$	
(123457C):	$[x_1, x_i] = x_{i+1},\ 2 \leq i \leq 6,$	$[x_2, x_5] = x_7,$	$[x_3, x_4] = -x_7.$

Central extensions of $N_{6,2,2}$:

$Z(\mathfrak{g})$: x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_4, x_5, x_6; $Z^2(\mathfrak{g})$: $C_{16} = C_{24} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = C_{56} = 0$, $C_{25} + C_{34} = 0$; $W(H^2)$: $C_{12} = C_{13} = C_{14} = C_{25} = 0$; $\dim H^2$: 2; Basis: Δ_{15}, Δ_{23};

It is obvious that there is no central extension.

The central extensions of $N_{6,2,3}$ can be found in Chapter 7. By taking $x_1 \to a$, $x_2 \to b$, $x_3 \to d$, $x_4 \to c$, $x_5 \to e$, $x_6 \to f$ and $x_6 \to g$, we can get the exact presentations of (12357A), (12357B) and (12357C) as in Seeley's paper.

Central extensions of $N_{6,2,4}$:

$Z(\mathfrak{g})$: x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_4, x_6; $Z^2(\mathfrak{g})$: $C_{24} = C_{36} = C_{45} = C_{46} = C_{56} = 0$, $C_{26}+C_{34} = 0$, $C_{16} - C_{35}$; $W(H^2)$: $C_{12} = C_{13} = C_{14} = 0$; $\dim H^2$: 5; Basis: $\Delta_{15}, \Delta_{16} + \Delta_{35}, \Delta_{23}, \Delta_{25}, \Delta_{26} - \Delta_{34}$;

Group action: $a\Delta_{15} + b(\Delta_{16} + \Delta_{35}) + c\Delta_{23} + d\Delta_{25} + e(\Delta_{26} - \Delta_{34})$:

$a \to aa_{11}^4 + b(a_{11}a_{65} + a_{11}^3 a_{31}) + da_{11}^3 a_{21} + e(a_{21}a_{65} + a_{11}^2 a_{21} a_{31})$;

$b \to ba_{11}^4 a_{22}^2 + ea_{11}^3 a_{21} a_{22}$;

$c \to ca_{11}a_{22}^2 - ba_{11}a_{22}a_{52} + e(2a_{11}a_{22}a_{42} + a_{21}a_{22}a_{52} - a_{22}^2 a_{51} - a_{11}a_{32}^2)$;

$d \to da_{11}^3 a_{22} + e(a_{22}a_{65} + a_{11}^2 a_{22} a_{31})$;

$e \to ea_{11}^3 a_{22}^2$;

One of $b, e \neq 0$.

Case 1: $e = 0$. Then $b \neq 0$. Make $a = c = 0$ by solving for a_{65} and a_{52}, get two representatives $[0, 1, 0, 0, 0]$ (when $d = 0$) (corresponding to(12457A)) and $[0, 1, 0, 1, 0]$ (when $d \neq 0$) (corresponding to (12457B)).

Case 2: $e \neq 0$. Make $b = c = d = 0$ by solving for a_{21}, a_{42} and a_{65}, get two representatives $[0, 0, 0, 0, 1]$ (when $a = 0$) (corresponding to (12457C)) and $[1, 0, 0, 0, 1]$ (when $a \neq 0$) (corresponding to (12457D)).

Therefore the corresponding central extensions of $N_{6,2,4}$ are:

(12457A):	$[x_1, x_i] = x_{i+1},\ i = 2, 3,$	$[x_1, x_4] = x_6,$
	$[x_1, x_6] = x_7,$	$[x_2, x_5] = x_6,$
	$[x_3, x_5] = x_7;$	
(12457B):	$[x_1, x_i] = x_{i+1},\ i = 2, 3,$	$[x_1, x_4] = x_6,$
	$[x_1, x_6] = x_7,$	$[x_2, x_5] = x_6 + x_7,$
	$[x_3, x_5] = x_7;$	
(12457C):	$[x_1, x_i] = x_{i+1},\ i = 2, 3,$	$[x_1, x_4] = x_6,$
	$[x_2, x_5] = x_6,$	$[x_2, x_6] = x_7,$
	$[x_3, x_4] = -x_7;$	
(12457D):	$[x_1, x_i] = x_{i+1},\ i = 2, 3,$	$[x_1, x_i] = x_{i+2}, i = 4, 5,$
	$[x_2, x_5] = x_6,$	$[x_2, x_6] = x_7,$
	$[x_3, x_4] = -x_7.$	

The central extensions of $N_{6,2,5}$ can be found in Chapter 7.

Central extensions of $N_{6,2,6}$:

$Z(\mathfrak{g})$: x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_4, x_5, x_6; $Z^2(\mathfrak{g})$: $C_{16} = C_{26} = C_{36} = C_{45} = C_{46} = C_{56} = 0$, $C_{34} - C_{25} = 0$; $W(H^2)$: $C_{12} = C_{13} = C_{14} = 0$; $\dim H^2$: 5; Basis: $\Delta_{15}, \Delta_{23}, \Delta_{24}, \Delta_{25} + \Delta_{34}, \Delta_{35}$;

It is obvious that all the elements in $H^2(\mathfrak{g}, \mathbf{F})$ have x_6 in its kernel. Therefore there is no central extension of $N_{6,2,6}$.

Central extensions of $N_{6,2,7}$:

$Z(\mathfrak{g})$: x_5, x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_4, x_5, x_6; $Z^2(\mathfrak{g})$: $C_{35} = C_{36} = C_{45} = C_{46} = C_{56} = 0$, $C_{34} + C_{25} = 0$, $C_{16} - C_{24} = 0$; $W(H^2)$: $C_{12} = C_{13} = C_{14} = C_{23} = 0$; $\dim H^2$: 4; Basis: $\Delta_{15}, \Delta_{16} + \Delta_{24}, \Delta_{25} - \Delta_{34}, \Delta_{26}$;

Group action: $a\Delta_{15} + b(\Delta_{16} + \Delta_{24}) + c(\Delta_{25} - \Delta_{34}) + d\Delta_{26}$;

$a \to aa_{11}^4 a_{22} + ca_{11}^3 a_{21} a_{22}$; $b \to ba_{11}^2 a_{22}^2 + da_{11} a_{22}^2 a_{21}$; $c \to ca_{11}^3 a_{22}^2$; $d \to da_{11} a_{22}^3$;

One of $\{a, c\}$ and one of $\{b, d\}$ are nonzero. If $a = 0$ (or $b = 0$), then $b \neq 0$ (or $a \neq 0$). If $c = 0$ (or $d = 0$), then $d \neq 0$ (or $c \neq 0$).

Case 1: $d \neq 0$. Make $b = 0$ by solving for a_{21}. Then $a \neq 0$, and obtain two representatives $[1, 0, 0, 1]$ (when $c = 0$) (corresponding to (13457F)) and $[1, 0, 1, 1]$ (when $c \neq 0$) (corresponding to (13457I)).

Case 2: $d = 0$. Then $b \neq 0$ and $c \neq 0$. Make $a = 0$ by solving for a_{21} to get the representative $[0, 1, 1, 0]$, corresponding to (13457G).

Therefore the corresponding central extensions of $N_{6,2,7}$ are:

(13457F):	$[x_1, x_i] = x_{i+1},\ i = 2,3,4,$	$[x_1, x_5] = x_7,$
	$[x_2, x_3] = x_6,$	$[x_2, x_6] = x_7;$
(13457G):	$[x_1, x_i] = x_{i+1},\ i = 2,3,4,$	$[x_1, x_6] = x_7,$
	$[x_2, x_3] = x_6,$	$[x_2, x_4] = x_7,$
	$[x_2, x_5] = x_7,$	$[x_3, x_4] = -x_7;$
(13457I):	$[x_1, x_i] = x_{i+1},\ i = 2,3,4,$	$[x_1, x_5] = x_7,$
	$[x_2, x_3] = x_6,$	$[x_2, x_5] = x_7,$
	$[x_2, x_6] = x_7,$	$[x_3, x_4] = -x_7.$

Central extensions of $N_{6,2,8}$:

$Z(\mathfrak{g})$: x_5, x_6; $[\mathfrak{g},\mathfrak{g}]$: x_3, x_5, x_6; $Z^2(\mathfrak{g})$: $C_{25} = C_{35} = C_{36} = C_{45} = C_{56} = 0$, $C_{15} - C_{34} - C_{26} = 0$; $W(H^2)$: $C_{12} = C_{13} = C_{14} = 0$; $\dim H^2$: 6; Basis: $\Delta_{15} + \Delta_{34}, \Delta_{15} + \Delta_{26}, \Delta_{16}, \Delta_{23}, \Delta_{24}, \Delta_{46}$;

Group action: $a(\Delta_{15} + \Delta_{34}) + b(\Delta_{15} + \Delta_{26}) + c\Delta_{16} + d\Delta_{23} + e\Delta_{24} + f\Delta_{46}$;

$a \to aa_{11}^3a_{22} - fa_{11}^3a_{42}$;

$b \to ba_{11}^3a_{22} + fa_{11}^3a_{42}$;

$c \to ba_{11}^3a_{21} + ca_{11}^4 + fa_{11}^3a_{41}$;

$d \to -aa_{11}a_{22}a_{42} + ba_{11}a_{22}a_{42} + da_{11}a_{22}^2 + fa_{11}a_{42}^2$;

$e \to a(2a_{11}a_{22}a_{41} - a_{42}a_{34} - a_{11}a_{21}a_{42}) + b(a_{22}a_{64} + a_{11}a_{22}a_{41} - a_{11}^2a_{32} - 2a_{11}a_{21}a_{42}) - ca_{11}^2a_{42} + d(a_{22}a_{34} - a_{11}a_{21}a_{22}) + ea_{11}^2a_{22} + f(a_{42}a_{64} - a_{11}^2a_{62} - a_{11}a_{41}a_{42})$;

$f \to fa_{11}^5$;

Then one of $\{a, b\}$ and one of $\{b, c, f\}$ are nonzero. And one of $\{b, f\}$ is also nonzero.

Case 1: $f \neq 0$. Then make $b = c = e = 0$ by solving for a_{42}, a_{41}, a_{62} and $a \neq 0$. Make $a = f = 1$, we may get the orbit $[a_{11}^3a_{22}, 0, 0, da_{11}a_{22}^2, 0, a_{11}^5]$. This will give us a one parameter representative $[1, 0, 0, \lambda, 0, 1]$, corresponding to (1357N).

Case 2: $f = 0$. Then $b \neq 0$. Make $c = e = 0$ by solving for a_{21} and a_{64}. We may get the following orbit $[aa_{11}^3a_{22}, ba_{11}^3a_{22}, 0, -aa_{11}a_{22}a_{42} + ba_{11}a_{22}a_{42} + da_{11}a_{22}^2, 0, 0]$. If $a \neq b$, then make $d = 0$, and get the orbit $[aa_{11}^3a_{22}, ba_{11}^3a_{22}, 0, 0, 0, 0]$ (as now we require that $a + b \neq 0$), which can be reduced to a one parameter representative $[1-\lambda, \lambda, 0, 0, 0, 0]$ (with $\lambda \neq 0$), corresponding to (1357M). If $a = b$, then depending on whether $d = 0$ or not, we get two representatives $[1, 1, 0, 0, 0, 0]$ and $[1, 1, 0, 1, 0, 0]$, corresponding to (1357L). And the representative $[1, 1, 0, 0, 0, 0]$ is just a special case of the one parameter representative.

Therefore the corresponding central extensions of $N_{6,2,8}$ are:

(1357L):	$[x_1,x_2]=x_3$,	$[x_1,x_i]=x_{i+2}, i=3,4,5$,
	$[x_2,x_3]=x_7$,	$[x_2,x_4]=x_5$,
	$[x_2,x_6]=\frac{1}{2}x_7$,	$[x_3,x_4]=\frac{1}{2}x_7$;
(1357M):	One parameter family, with $\lambda\neq 0$	
	$[x_1,x_2]=x_3$,	$[x_1,x_i]=x_{i+2}, i=3,4,5$,
	$[x_2,x_4]=x_5$,	$[x_2,x_6]=\lambda x_7$,
	$[x_3,x_4]=(1-\lambda)x_7$;	
(1357N): λ		
	$[x_1,x_2]=x_3$,	$[x_1,x_i]=x_{i+2}, i=3,4,5$,
	$[x_2,x_3]=\lambda x_7$,	$[x_2,x_4]=x_5$,
	$[x_3,x_4]=x_7$,	$[x_4,x_6]=x_7$.

Remark: (1357K) of Seeley's is just a special case of (1357M) by taking $\lambda=1/2$.

The central extensions of $N_{6,2,9}$ can be found in Chapter 7. Notice that (1) By taking $x_1\to a$, $x_2\to -b$, $x_3\to -c-d$, $x_4\to -c$, $x_5\to -e$, $x_6\to f$, $x_7\to -g$, we can get the exact presentation of (1357Q) as in Seeley's paper; (2) By taking $x_1\to a$, $x_2\to b$, $x_3\to c+d$, $x_4\to c$, $x_5\to e$, $x_6\to f$, $x_7\to g$, we can get the exact presentations of (1357R) and (1357S) as in Seeley's paper.

The central extensions of $N_{6,2,10}$ can be found in Chapter 7. By taking $x_1\to b$, $x_2\to a$, $x_3\to -c$, $x_4\to -d$, $x_5\to -f$, $x_6\to -e$, $x_7\to -g$, we can get the exact presentations of (1357O) and (1357P) as in Seeley's paper.

Central extensions of $N_{6,2,11}$:

$Z(\mathfrak{g})$: x_5, x_6; $[\mathfrak{g},\mathfrak{g}]$: x_3, x_4, x_5; $Z^2(\mathfrak{g})$: $C_{35}=C_{36}=C_{45}=C_{46}=C_{56}=0$, $C_{15}=C_{24}, C_{25}+C_{34}=0$; $W(H^2)$: $C_{12}=C_{13}=C_{14}=0$; $\dim H^2$: 5; Basis: $\Delta_{15}+\Delta_{24}, \Delta_{16}, \Delta_{23}, \Delta_{25}-\Delta_{34}, \Delta_{26}$;

Group action: $a(\Delta_{15}+\Delta_{24})+b\Delta_{16}+c\Delta_{23}+d(\Delta_{25}-\Delta_{34})+e\Delta_{26}$;

$a\to aa_{11}^6+da_{11}^5a_{21}$;

$b\to aa_{11}a_{56}+ba_{11}a_{66}+da_{21}a_{56}+ea_{21}a_{66}$;

$c\to ca_{11}^5+2da_{11}^3a_{42}-a_{11}a_{32}^2-a_{11}^3a_{21}^2-2aa_{11}^4a_{21}$;

$d\to da_{11}^7$;

$e\to da_{11}^2a_{56}+ea_{11}^2a_{66}$;

One of $\{a,d\}$ and one of $\{b,e\}$ are nonzero. One of $\{a,b\}$ and one of $\{d,e\}$ are nonzero.

Case 1: $d\neq 0$. Make $a=c=e=0$ by solving for a_{21}, a_{42}, a_{56} respectively. Then $b\neq 0$, and get a representative $[0,1,0,1,0]$, corresponding to (13457E);

Case 2: $d=0$. So $a\neq 0$ and $e\neq 0$. Make $b=c=0$ by solving for a_{56} and a_{21} respectively to get a representative $[1,0,0,0,1]$, corresponding to (13457D).

Therefore the corresponding central extensions of $N_{6,2,11}$ are:

(13457D):	$[x_1,x_i]=x_{i+1}, i=2,3,4,$	$[x_1,x_5]=x_7,$	$[x_2,x_3]=x_5,$
	$[x_2,x_4]=x_7,$	$[x_2,x_6]=x_7;$	
(13457E):	$[x_1,x_i]=x_{i+1}, i=2,3,4,$	$[x_1,x_6]=x_7,$	$[x_2,x_3]=x_5,$
	$[x_2,x_5]=x_7,$	$[x_3,x_4]=-x_7;$	

Central extensions of $N_{6,3,1}$:

$Z(\mathfrak{g})$: x_6; $[\mathfrak{g},\mathfrak{g}]$: $x_4,\ x_5,\ x_6$; $Z^2(\mathfrak{g})$: $C_{26}=C_{36}=C_{46}=C_{56}=0,\ C_{16}+C_{45}=0,\ C_{16}-C_{45}=0$; $W(H^2)$: $C_{12}=C_{13}=C_{25}=0$;

It is obvious to see that $N_{6,3,1}$ has no central extension.

The central extensions of $N_{6,3,2}$ can be found in Chapter 2, Example 3.

Central extensions of $N_{6,3,3}$:

$Z(\mathfrak{g})$: x_5, x_6; $[\mathfrak{g},\mathfrak{g}]$: $x_3,\ x_5,\ x_6$; $Z^2(\mathfrak{g})$: $C_{15}=C_{35}=C_{36}=C_{45}=C_{56}=0, C_{34}+C_{26}=0$; $W(H^2)$: $C_{12}=C_{14}=C_{23}=0$; $\dim H^2$: 6; Basis:$\Delta_{13},\Delta_{16},\Delta_{24},\Delta_{25},\Delta_{26}-\Delta_{34},\Delta_{46}$;

Group action: $a\Delta_{13}+b\Delta_{16}+c\Delta_{24}+d\Delta_{25}+e(\Delta_{26}-\Delta_{34})+f\Delta_{46}$;

$a\to aa_{11}^2a_{22}+ba_{11}^2a_{42}+ea_{11}a_{22}a_{41}+fa_{11}a_{41}a_{42}$;

$b\to ba_{11}^2a_{44}+fa_{11}a_{44}a_{41}$;

$c\to ca_{22}a_{44}+da_{22}a_{54}+e(a_{22}a_{64}-a_{32}a_{44})+f(a_{42}a_{64}-a_{62}a_{44})$;

$d\to da_{11}a_{22}^3$;

$e\to ea_{11}a_{22}a_{44}+fa_{11}a_{44}a_{42}$;

$f\to fa_{11}a_{44}^2$;

One of $\{b,f\}$ is nonzero, and also $d\neq 0$. Can always make $c=0$.

Case 1: $f=0$. Then $b\neq 0$. Make $a=0$ by solving for a_{42} to get two representatives $[0,1,0,1,0,0]$ (when $e=0$, corresponding to (1357G)) and $[0,1,0,1,1,0]$ (when $e\neq 0$, corresponding to (1357H));

Case 2: $f\neq 0$. Make $b=e=0$ by solving for a_{41} and a_{42} respectively to get two representatives $[0,0,0,1,0,1]$ (when $a=0$, corresponding to (1357I)) and $[1,0,0,1,0,1]$ (when $a\neq 0$, corresponding to (1357J)).

Therefore the corresponding central extensions of $N_{6,3,3}$ are:

(1357G):	$[x_1, x_2] = x_3$,	$[x_1, x_4] = x_6$,	$[x_1, x_6] = x_7$,
	$[x_2, x_3] = x_5$,	$[x_2, x_5] = x_7$;	
(1357H):	$[x_1, x_2] = x_3$,	$[x_1, x_4] = x_6$,	$[x_1, x_6] = x_7$,
	$[x_2, x_3] = x_5$,	$[x_2, x_5] = x_7$,	$[x_2, x_6] = x_7$,
	$[x_3, x_4] = -x_7$;		
(1357I):	$[x_1, x_2] = x_3$,	$[x_1, x_4] = x_6$,	$[x_2, x_3] = x_5$,
	$[x_2, x_5] = x_7$,	$[x_4, x_6] = x_7$;	
(1357J):	$[x_1, x_2] = x_3$,	$[x_1, x_3] = x_7$,	$[x_1, x_4] = x_6$,
	$[x_2, x_3] = x_5$,	$[x_2, x_5] = x_7$,	$[x_4, x_6] = x_7$.

Remark: By taking $x_1 \to b$, $x_2 \to a$, $x_3 \to -c$, $x_4 \to d$, $x_5 \to -e$, $x_6 \to f$, $x_7 \to g$ in all the four algebras above, we can get the exact presentations as in Seeley's paper.

The central extensions of $N_{6,3,4}$ can be found in Chapter 7. By taking $x_1 \to b$, $x_2 \to a$, $x_3 \to -c$, $x_4 \to -d$, $x_5 \to -e$, $x_6 \to -f$, $x_7 \to -g$ for all the algebras there, we can get the exact presentations as in Seeley's paper.

Central extensions of $N_{6,3,5}$:

$Z(\mathfrak{g})$: x_5, x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_5, x_6; $Z^2(\mathfrak{g})$: $C_{36} = C_{46} = C_{56} = 0, C_{16} + C_{35} = 0, C_{26} - C_{45} = 0$; $W(H^2)$: $C_{12} = C_{14} = 0$; $\dim H^2$: 8; Basis:$\Delta_{13}, \Delta_{15}, \Delta_{16}-\Delta_{35}, \Delta_{23}, \Delta_{24}, \Delta_{25}, \Delta_{26}+\Delta_{45}, \Delta_{34}$;

Group action: $a\Delta_{13} + b\Delta_{15} + c(\Delta_{16} - \Delta_{35}) + d\Delta_{23} + e\Delta_{24} + f\Delta_{25} + g(\Delta_{26} + \Delta_{45}) + h\Delta_{34}$;

Let $\Delta := a_{11}a_{22} - a_{12}a_{21}$.

$a \to \{aa_{11}^2a_{66} + ba_{11}a_{53}\,\Delta + c(a_{11}a_{63} - a_{31}a_{53})\,\Delta + ca_{11}a_{66}a_{51} + da_{11}a_{21}a_{66} - ea_{21}^2a_{66} + fa_{21}a_{53}\,\Delta + g(a_{21}a_{63}\,\Delta + a_{41}a_{53}\,\Delta + a_{21}a_{51}a_{66}) - h(a_{21}a_{31}a_{66} + a_{11}a_{41}a_{66})\}\Delta^{-1}$;

$b \to ba_{11}\Delta + c(a_{11}^2a_{42} + a_{11}a_{21}a_{32} - a_{11}a_{12}a_{41} - 2a_{11}a_{22}a_{31} + a_{12}a_{21}a_{31}) + fa_{21}\Delta + g(a_{11}a_{21}a_{42} + a_{21}^2a_{32} - 2a_{12}a_{21}a_{41} - a_{21}a_{22}a_{31} + a_{11}a_{22}a_{41})$;

$c \to ca_{11}a_{66} + ga_{21}a_{66}$;

$d \to \{2aa_{11}a_{12}a_{66} + b\,\Delta\,(a_{12}a_{53} - a_{11}a_{54}) + c\,\Delta\,(a_{63}a_{12} - a_{11}a_{64}) + c\,\Delta\,(a_{31}a_{54} - a_{32}a_{53}) + ca_{66}(a_{52}a_{11} + a_{12}a_{51}) + da_{66}(a_{11}a_{22} + a_{12}a_{21}) - 2ea_{21}a_{22}a_{66} + f\,\Delta\,(a_{53}a_{22} - a_{54}a_{21}) + g\,\Delta\,(a_{22}a_{63} - a_{21}a_{64}) + g\,\Delta\,(a_{42}a_{53} - a_{41}a_{54}) + ga_{66}(a_{21}a_{52} + a_{22}a_{51}) - ha_{66}(a_{21}a_{32} + a_{22}a_{31} + a_{11}a_{42} + a_{12}a_{41})\}\Delta^{-1}$;

$e \to \{-aa_{12}^2a_{66} + ba_{12}a_{54}\,\Delta + ca_{12}a_{64}\,\Delta - ca_{32}a_{54}\,\Delta - ca_{12}a_{52}a_{66} - da_{12}a_{22}a_{66} + ea_{22}^2a_{66} + fa_{22}a_{54}\,\Delta + ga_{64}a_{22}\,\Delta + ga_{42}a_{54}\,\Delta - ga_{22}a_{66}a_{52} + ha_{22}a_{32}a_{66} + ha_{12}a_{42}a_{66}\}\Delta^{-1}$;

$f \to ba_{12}\,\Delta + ca_{11}a_{12}a_{42} + 2ca_{12}a_{21}a_{32} - ca_{12}^2a_{41} - ca_{12}a_{22}a_{31} - ca_{11}a_{22}a_{32} + fa_{22}\Delta + 2g\, a_{11}a_{22}a_{42} + ga_{21}a_{22}a_{32} - ga_{12}a_{22}a_{41} - ga_{22}^2a_{31} - ga_{12}a_{21}a_{42}$;

$g \to ca_{12}a_{66} + ga_{22}a_{66}$;

$h \to -a_{66}\{ca_{11}a_{54} + ca_{12}a_{53} + ga_{22}a_{53} + ga_{21}a_{54} - ha_{66}\}\Delta^{-1}$;

One of $\{c, g\}$ is nonzero. Can always make $c \neq 0$ and $g = 0$. To fix $g = 0$, we require that $a_{12} = 0$. Let $a_{21} = a_{31} = a_{41} = a_{51} = a_{52} = a_{53} = a_{61} = a_{62} = 0$, we have

$a \to \{aa_{11}^2 a_{66} + ca_{11}a_{63}\}\Delta^{-1}$; $b \to ba_{11}\ \Delta + ca_{11}^2 a_{42}$; $c \to ca_{11}a_{66}$; $d \to \{b\ \Delta\ (-a_{11}a_{54}) + c\ \Delta\ (-a_{11}a_{64}) + da_{66}a_{11}a_{22} - ha_{66}a_{11}a_{42}\}\Delta^{-1}$; $e \to \{-ca_{32}a_{54}\ \Delta + ea_{22}^2 a_{66} + fa_{22}a_{54}\ \Delta + ha_{22}a_{32}a_{66}\}\Delta^{-1}$; $f \to -ca_{11}a_{22}a_{32} + fa_{22}\Delta$; $g \to 0$; $h \to -a_{66}\{ca_{11}a_{54} - ha_{66}\}\Delta^{-1}$;

Make $h = 0$ by solving for a_{54}, $f = 0$ for a_{32}, $a = 0$ for a_{63}, $b = 0$ for a_{42}, $d = 0$ for a_{64}. Now take $a_{63} = a_{42} = a_{54} = a_{32} = 0$ also, and get $a \to 0$; $b \to 0$; $c \to ca_{11}a_{66}$; $d \to 0$; $e \to ea_{22}^2 a_{66}\Delta^{-1}$; $f \to 0$; $g \to 0$; $h \to 0$;

Depending on whether $e = 0$ or not, we get two representatives $[0, 0, 1, 0, 0, 0, 0, 0]$ (when $e = 0$), corresponding to (137C), and $[0, 0, 1, 0, 1, 0, 0, 0]$ (when $e \neq 0$), corresponding to (137D).

Therefore the corresponding central extensions $N_{6,3,5}$ are:

(137C):	$[x_1, x_2] = x_5$,	$[x_1, x_4] = x_6$,	$[x_1, x_6] = x_7$,
	$[x_2, x_3] = x_6$,	$[x_3, x_5] = -x_7$;	
(137D):	$[x_1, x_2] = x_5$,	$[x_1, x_4] = x_6$,	$[x_1, x_6] = x_7$,
	$[x_2, x_3] = x_6$,	$[x_2, x_4] = x_7$,	$[x_3, x_5] = -x_7$;

Remark: (1) By taking $x_1 \to a + \frac{1}{2}d$, $x_2 \to b + c$, $x_3 \to d$, $x_4 \to b$, $x_5 \to \frac{1}{2}e + f$, $x_6 \to e$ and $x_7 \to g$, we may get the exact presentation of (137C) as in Seeley's paper; (2) By taking $x_1 \to -a$, $x_2 \to -c$, $x_3 \to d$, $x_4 \to b$, $x_5 \to f$, $x_6 \to -e$, $x_7 \to g$, we can get the exact presentation of (137D) as in Seeley's paper.

The central extensions of $N_{6,3,6}$ can be found in Section 6.4.

Central extensions of $N_{6,3,7}$:

$Z(\mathfrak{g})$: x_5, x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_4, x_5; $Z^2(\mathfrak{g})$: $C_{24} = C_{35} = C_{36} = C_{45} = C_{46} = C_{56} = 0, C_{25} + C_{34} = 0$; $W(H^2)$: $C_{12} = C_{13} = C_{14} = 0$; $\dim H^2$: 5; Basis:$\Delta_{15}, \Delta_{16}, \Delta_{23}, \Delta_{25} - \Delta_{34}, \Delta_{26}$;

Group action: $a\Delta_{15} + b\Delta_{16} + c\Delta_{23} + d(\Delta_{25} - \Delta_{34}) + e\Delta_{26}$;

$a \to aa_{11}^4 a_{22} + da_{11}^3 a_{21} a_{22}$;

$b \to aa_{11}a_{56} + ba_{11}a_{66} + da_{21}a_{56} + ea_{21}a_{66}$;

$c \to ca_{11}a_{22}^2 + 2da_{11}a_{22}a_{42} - da_{11}a_{32}^2$;

$d \to da_{11}^3 a_{22}^2$;

$e \to da_{22}a_{56} + ea_{22}a_{66}$;

In each of the four sets $\{a, d\}, \{b, e\}, \{a, b\}$ and $\{d, e\}$, at least one element is nonzero.

Case 1: $d \neq 0$. Then make $a = c = e = 0$ by solving for a_{21}, a_{42} and a_{56} respectively. Then $b \neq 0$, we get a representative $[0, 1, 0, 1, 0]$, corresponding to (13457C);

Case 2: $d = 0$. Then $a \neq 0$ and $e \neq 0$. Make $b = 0$ by solving for a_{21} and get two representatives depending on whether $c = 0$ or not, i.e., $[1,0,0,0,1]$ (when $c = 0$, corresponding to (13457A)) and $[1,0,1,0,1]$ (when $c \neq 0$, corresponding to (13457B)).

Therefore the corresponding central extensions $N_{6,3,7}$ are:

(13457A):	$[x_1, x_i] = x_{i+1},\ i = 2,3,4,$	$[x_1, x_5] = x_7,$
	$[x_2, x_6] = x_7;$	
(13457B):	$[x_1, x_i] = x_{i+1},\ i = 2,3,4,$	$[x_1, x_5] = x_7,$
	$[x_2, x_3] = x_7,$	$[x_2, x_6] = x_7;$
(13457C):	$[x_1, x_i] = x_{i+1},\ i = 2,3,4,$	$[x_1, x_6] = x_7,$
	$[x_2, x_5] = x_7,$	$[x_3, x_4] = -x_7;$

Central extensions of $N_{6,3,8}$:

$Z(\mathfrak{g})$: x_5, x_6; $[\mathfrak{g},\mathfrak{g}]$: $x_4,\ x_5$; $Z^2(\mathfrak{g})$: $C_{25} = C_{35} = C_{45} = C_{46} = C_{56} = 0, C_{15} + C_{34} = 0$; $W(H^2)$: $C_{12} = C_{14} = 0$; $\dim H^2$: 7; Basis: $\Delta_{13}, \Delta_{15} - \Delta_{34}, \Delta_{16}, \Delta_{23}, \Delta_{24}, \Delta_{26}, \Delta_{36}$;

Group action: $a\Delta_{13} + b(\Delta_{15} - \Delta_{34}) + c\Delta_{16} + d\Delta_{23} + e\Delta_{24} + f\Delta_{26} + g\Delta_{36}$;

$a \to aa_{11}^3 + b(a_{11}a_{53} + a_{11}a_{21}a_{31} + a_{11}^2a_{41}) + ca_{11}a_{63} + da_{11}^2a_{21} - ea_{11}a_{21}^2 + fa_{21}a_{63} + ga_{31}a_{63} - ga_{11}^2a_{61}$;

$b \to ba_{11}^3a_{22}$;

$c \to ba_{11}a_{56} + ca_{11}a_{66} + fa_{21}a_{66} + ga_{31}a_{66}$;

$d \to 2ba_{31}a_{11}a_{22} + da_{11}^2a_{22} - 2ea_{11}a_{22}a_{21} + fa_{22}a_{63} + g(a_{32}a_{63} - a_{11}^2a_{62})$;

$e \to -ba_{11}a_{22}a_{32} + ea_{11}a_{22}^2$;

$f \to fa_{22}a_{66} + ga_{32}a_{66}$;

$g \to ga_{11}^2a_{66}$;

One always have $b \neq 0$ and one of $\{f, g\}$ is nonzero. Since $b \neq 0$, make $a = c = 0$ by solving for a_{53} and a_{56} respectively.

Case 1: $g = 0$. Then $f \neq 0$. Make $e = 0$ by solving for a_{32} and make $d = 0$ by solving for a_{63} and get a representative $[0,1,0,0,0,1,0]$, corresponding to (1357A);

Case 2: $g \neq 0$. Make $d = f = 0$ by solving for a_{62} and a_{32} respectively and get representatives $[0,1,0,0,0,0,1]$ (when $e = 0$, corresponding to (1357B)) and $[0,1,0,0,1,0,1]$ (when $e \neq 0$, corresponding to (1357C)).

Therefore the corresponding central extensions of $N_{6,3,8}$ are:

(1357A):	$[x_1, x_2] = x_4$,	$[x_1, x_4] = x_5$,	$[x_1, x_5] = x_7$,
	$[x_2, x_3] = x_5$,	$[x_2, x_6] = x_7$,	$[x_3, x_4] = -x_7$;
(1357B):	$[x_1, x_2] = x_4$,	$[x_1, x_4] = x_5$,	$[x_1, x_5] = x_7$,
	$[x_2, x_3] = x_5$,	$[x_3, x_4] = -x_7$,	$[x_3, x_6] = x_7$;
(1357C):	$[x_1, x_2] = x_4$,	$[x_1, x_4] = x_5$,	$[x_1, x_5] = x_7$,
	$[x_2, x_3] = x_5$,	$[x_2, x_4] = x_7$,	$[x_3, x_4] = -x_7$,
	$[x_3, x_6] = x_7$;		

Remark: By switching x_3 and x_4 in all the algebras above, we can get the exact presentations as in Seeley's paper.

Central extensions of $N_{6,3,9}$:

$Z(\mathfrak{g})$: x_4, x_5, x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_4, x_5; $Z^2(\mathfrak{g})$: $C_{34} = C_{35} = C_{36} = C_{45} = C_{46} = C_{56} = 0, C_{15} - C_{24} = 0$; $W(H^2)$: $C_{12} = C_{13} = C_{23} = 0$; $\dim H^2$: 5; Basis: $\Delta_{14}, \Delta_{15} + \Delta_{24}, \Delta_{16}, \Delta_{25}, \Delta_{26}$;

A little bit of calculation will show that any element in H^2 has none trivial kernel in the center of $N_{6,3,9}$. So $N_{6,3,9}$ does not have the desired central extension.

Central extensions of $N_{6,4,1}$:

$Z(\mathfrak{g})$: x_5, x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_5; $Z^2(\mathfrak{g})$: $C_{15} = C_{25} = C_{35} = C_{45} = C_{56} = 0$; $W(H^2)$: $C_{12} = 0$; $\dim H^2$: 9; Basis: $\Delta_{13}, \Delta_{14}, \Delta_{16}, \Delta_{23}, \Delta_{24}, \Delta_{26}, \Delta_{34}, \Delta_{36}, \Delta_{46}$;

It is obvious that all the elements in $H^2(\mathfrak{g}, \mathbf{F})$ have x_5 in the kernel, so $N_{6,4,1}$ has no central extension.

The central extensions of $N_{6,4,2}$ can be found in Chapter 7.

Central extensions of $N_{6,4,3}$:

$Z(\mathfrak{g})$: x_4, x_5, x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_4; $Z^2(\mathfrak{g})$: $C_{24} = C_{34} = C_{35} = C_{36} = C_{45} = C_{46} = 0$; $W(H^2)$: $C_{12} = C_{13} = 0$; $\dim H^2$: 7; Basis: $\Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{23}, \Delta_{25}, \Delta_{26}, \Delta_{56}$;

Group action: $a\Delta_{14} + b\Delta_{15} + c\Delta_{16} + d\Delta_{23} + e\Delta_{25} + f\Delta_{26} + g\Delta_{56}$;

$a \to a a_{11}^3 a_{22}$;

$b \to a a_{11} a_{45} + b a_{11} a_{55} + c a_{11} a_{65} + e a_{21} a_{55} + f a_{21} a_{65} + g(a_{51} a_{65} - a_{55} a_{61})$;

$c \to a a_{11} a_{46} + b a_{11} a_{56} + c a_{11} a_{66} + e a_{21} a_{56} + f a_{21} a_{66} + g(a_{51} a_{66} - a_{61} a_{56})$;

$d \to d a_{11} a_{22}^2$;

$e \to e a_{22} a_{55} + f a_{22} a_{65} + g(a_{52} a_{65} - a_{55} a_{62})$;

$f \to e a_{22} a_{56} + f a_{22} a_{66} + g(a_{52} a_{66} - a_{62} a_{56})$;

$g \to g(a_{55} a_{66} - a_{65} a_{56})$;

We have $a \neq 0$ and $g \neq 0$. Make $b = c = e = f = 0$ by solving for a_{45}, a_{46}, a_{62} and a_{52}

respectively (letting $a_{65} = a_{56} = 0$), and get two representatives $[1,0,0,0,0,0,1]$ (when $d = 0$) (corresponding to (1457A)) and $[1,0,0,1,0,0,1]$ (when $d \neq 0$) (corresponding to (1457B));

Therefore the corresponding central extensions of $N_{6,4,3}$ are:

(1457A):	$[x_1, x_i] = x_{i+1}, i = 2,3,$	$[x_1, x_4] = x_7,$	$[x_5, x_6] = x_7;$
(1457B):	$[x_1, x_i] = x_{i+1}, i = 2,3,$	$[x_1, x_4] = x_7,$	
	$[x_2, x_3] = x_7,$	$[x_5, x_6] = x_7.$	

Central extensions of $N_{6,4,4}$:

$Z(\mathfrak{g})$: x_5, x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_5, x_6; $Z^2(\mathfrak{g})$: $C_{16} = C_{26} = C_{35} = C_{45} = C_{56} = 0$; $W(H^2)$: $C_{12} = C_{34} = 0$; $\dim H^2$: 8; Basis: $\Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{23}, \Delta_{24}, \Delta_{25}, \Delta_{36}, \Delta_{46}$;

Group action: $a\Delta_{13} + b\Delta_{14} + c\Delta_{15} + d\Delta_{23} + e\Delta_{24} + f\Delta_{25} + g\Delta_{36} + h\Delta_{46}$;

The automorphism group of $N_{6,4,4}$ has two components, therefore we have

(1):

$a \to aa_{11}a_{33} + ba_{11}a_{43} + ca_{11}a_{53} + da_{21}a_{33} + ea_{21}a_{43} + fa_{21}a_{53} - ga_{33}a_{61} - ha_{43}a_{61}$;

$b \to aa_{11}a_{34} + ba_{11}a_{44} + ca_{11}a_{54} + da_{21}a_{34} + ea_{21}a_{44} + fa_{21}a_{54} - ga_{61}a_{34} - ha_{44}a_{61}$;

$c \to c(a_{11}^2 a_{22} - a_{11}a_{12}a_{21}) + f(a_{11}a_{21}a_{22} - a_{21}^2 a_{22})$;

$d \to aa_{12}a_{33} + ba_{12}a_{43} + ca_{12}a_{53} + da_{22}a_{33} + ea_{22}a_{43} + fa_{22}a_{53} - ga_{33}a_{62} - ha_{43}a_{62}$;

$e \to aa_{12}a_{34} + ba_{12}a_{44} + ca_{12}a_{54} + da_{22}a_{34} + ea_{22}a_{44} + fa_{22}a_{54} - ga_{34}a_{62} - ha_{44}a_{62}$;

$f \to ca_{12}(a_{11}a_{22} - a_{12}a_{21}) + fa_{22}(a_{11}a_{22} - a_{12}a_{21})$;

$g \to (ga_{33} + ha_{43})(a_{33}a_{44} - a_{34}a_{43})$;

$h \to (ga_{34} + ha_{44})(a_{33}a_{44} - a_{34}a_{43})$;

(2): $a \to -a$, $b \to -d$, $c \to g$, $d \to -b$, $e \to -e$, $f \to h$, $g \to b + c$, $h \to e + f$;

One of $\{c, f\}$ and one of $\{g, h\}$ are nonzero. We can always make $c \neq 0$, $g \neq 0$ and $f = h = 0$. Make $a = b = d = 0$ by solving for a_{53}, a_{54} and a_{62} respectively. Now by taking $a_{12} = a_{34} = a_{53} = a_{63} = a_{43} = a_{54} = a_{21} = 0$, we can make $a = b = d = f = h = 0$, and depending on whether $e = 0$ or not, we may obtain two representatives $[0,0,1,0,0,0,1,0]$ (when $e = 0$) (corresponding to (137A)) and $[0,0,1,0,1,0,1,0]$ (when $e \neq 0$), corresponding to (137B);

Therefore the central extensions of $N_{6,4,4}$ are:

(137A):	$[x_1, x_2] = x_5,$	$[x_1, x_5] = x_7,$	
	$[x_3, x_4] = x_6,$	$[x_3, x_6] = x_7;$	
(137B):	$[x_1, x_2] = x_5,$	$[x_1, x_5] = x_7,$	$[x_2, x_4] = x_7,$
	$[x_3, x_4] = x_6,$	$[x_3, x_6] = x_7;$	

Remark: There is an error in Seeley's paper about (137B), instead of having $[x_2, x_4] = x_7$, he had $[x_2, x_3] = x_7$, which was not a Lie algebra at all.

Central extensions of $N_{6,5}$:

$Z(\mathfrak{g})$: x_3, x_4, x_5, x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_3; $Z^2(\mathfrak{g})$: $C_{34} = C_{35} = C_{36} = 0$; $W(H^2)$: $C_{12} = 0$; $\dim H^2$: 11; Basis: $\Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{23}, \Delta_{24}, \Delta_{25}, \Delta_{26}, \Delta_{45}, \Delta_{46}, \Delta_{56}$;

Group action: $a\Delta_{13}+b\Delta_{14}+c\Delta_{15}+d\Delta_{16}+e\Delta_{23}+f\Delta_{24}+g\Delta_{25}+h\Delta_{26}+i\Delta_{45}+j\Delta_{46}+k\Delta_{56}$;

$a \to (aa_{11} + ea_{21})(a_{11}a_{22} - a_{12}a_{21})$;

$b \to aa_{11}a_{34}+ba_{11}a_{44}+ca_{11}a_{54}+da_{11}a_{64}+ea_{21}a_{34}+fa_{21}a_{44}+ga_{21}a_{54}+ha_{21}a_{64}+ia_{41}a_{54}+ ja_{41}a_{64} - ia_{44}a_{51} + ka_{51}a_{64} - ja_{44}a_{61} - ka_{61}a_{54}$;

$c \to aa_{11}a_{35}+ba_{11}a_{45}+ca_{11}a_{55}+da_{11}a_{65}+ea_{21}a_{35}+fa_{21}a_{45}+ga_{21}a_{55}+ha_{21}a_{65}+ia_{41}a_{55}+ ja_{41}a_{65} - ia_{45}a_{51} + ka_{51}a_{65} - ja_{45}a_{61} - ka_{55}a_{61}$;

$d \to aa_{11}a_{36}+ba_{11}a_{46}+ca_{11}a_{56}+da_{11}a_{66}+ea_{21}a_{36}+fa_{21}a_{46}+ga_{21}a_{56}+ha_{21}a_{66}+ia_{41}a_{56}+ ja_{41}a_{66} - ia_{46}a_{51} + ka_{51}a_{66} - ja_{46}a_{61} - ka_{56}a_{61}$;

$e \to (aa_{12} + ea_{22})(a_{11}a_{22} - a_{12}a_{21})$;

$f \to aa_{12}a_{34}+ba_{12}a_{44}+ca_{12}a_{54}+da_{12}a_{64}+ea_{22}a_{34}+fa_{22}a_{44}+ga_{22}a_{54}+ha_{22}a_{64}+ia_{42}a_{54}+ ja_{42}a_{64} - ia_{44}a_{52} + ka_{52}a_{64} - ja_{44}a_{62} - ka_{62}a_{54}$;

$g \to aa_{12}a_{35}+ba_{12}a_{45}+ca_{12}a_{55}+da_{12}a_{65}+ea_{22}a_{35}+fa_{22}a_{45}+ga_{22}a_{55}+ha_{22}a_{65}+ia_{42}a_{55}+ ja_{42}a_{65} - ia_{45}a_{52} + ka_{52}a_{65} - ja_{45}a_{62} - ka_{55}a_{62}$;

$h \to aa_{12}a_{36}+ba_{12}a_{46}+ca_{12}a_{56}+da_{12}a_{66}+ea_{22}a_{36}+fa_{22}a_{46}+ga_{22}a_{56}+ha_{22}a_{66}+ia_{42}a_{56}+ ja_{42}a_{66} - ia_{46}a_{52} + ka_{52}a_{66} - ja_{46}a_{62} - ka_{56}a_{62}$;

$i \to i(a_{44}a_{55} - a_{45}a_{54}) + j(a_{44}a_{65} - a_{64}a_{45}) + k(a_{54}a_{65} - a_{64}a_{55})$;

$j \to i(a_{44}a_{56} - a_{46}a_{54}) + j(a_{44}a_{66} - a_{64}a_{46}) + k(a_{54}a_{66} - a_{64}a_{56})$;

$k \to i(a_{45}a_{56} - a_{46}a_{55}) + j(a_{45}a_{66} - a_{65}a_{46}) + k(a_{55}a_{66} - a_{65}a_{56})$;

One of $\{a, e\}$ is nonzero. We can make $a \neq 0$ and $e = 0$, then by taking $a_{21} = a_{41} = a_{51} = a_{61} = a_{54} = a_{64} = 0$, we can make $b = c = d = 0$ by solving for a_{34}, a_{35} and a_{36} respectively. One of $\{i, j, k\}$ is nonzero. We can always make $k \neq 0$ and $i = j = 0$, as the coefficients of i, j, k are just the the second compound matrix of a nonsingular matrix. And we need $f \neq 0$.

Now take $a_{12} = a_{21} = a_{34} = a_{35} = a_{36} = a_{41} = a_{51} = a_{61} = a_{54} = a_{64} = a_{56} = a_{62} = a_{65} = a_{52} = 0$, we can fix $b = c = d = e = i = j = 0$ and $g \to fa_{22}a_{45} + ga_{22}a_{55}$;

$h \to fa_{22}a_{46} + ha_{22}a_{66}$;

Make $g = h = 0$ by solving for a_{45} and a_{46} and get representative $[1,0,0,0,0,1,0,0,0,0,1]$.

Therefore the central extensions of $N_{6,5}$ are:

$$\boxed{\begin{array}{lll} (157): & [x_1, x_2] = x_3, & [x_1, x_3] = x_7, \\ & [x_2, x_4] = x_7, & [x_5, x_6] = x_7; \end{array}}$$

6.4 Extensions of $N_{6,3,6}$

Although we can use the same procedure as we do to all the other algebras to get the desired central extensions, we find it very difficult to manipulate the parameters involved. So instead we use an ad hoc method to deal with this case, which will give us a slightly different invariant than the one used by Seeley.

Let V be a vector space of dimension 3 with a basis $\{a, b, c\}$. Because $N_{6,3,6}$ is a free nilpotent Lie algebra, by some standard arguments [11] [23], we have $N_{6,3,6} \cong V \oplus \wedge^2 V$. And isomorphically, $N_{6,3,6}$ can be written as:

$N_{6,3,6}$: $[a, b] = d = a \wedge b, [b, c] = e = b \wedge c, [c, a] = f = c \wedge a$.

Center: d, e, f;

$[\mathfrak{g}, \mathfrak{g}]$: d, e, f;

To find 2-cocycles, we need to find all $\phi : V \oplus \wedge^2 V \times V \oplus \wedge^2 V \to \mathbf{F}$ such that they satisfy the Jacobi identity

$$\mathrm{Jac}(x, y, z) = \phi([x, y], z) + \phi([y, z], x) + \phi([z, x], y) = 0, \ \forall x, y, z.$$

As $Z(\mathfrak{g}) = \wedge^2 V$, it is obvious that for ϕ to be a cocycle, ϕ must vanish on $\wedge^2 V \times \wedge^2 V$.

By normalizing the cocycles, we require that $\phi(a, b) = \phi(b, c) = \phi(c, a) = 0$, which means that $\phi(V, V) = 0$.

So for $\phi : V \oplus \wedge^2 V \times V \oplus \wedge^2 V \to \mathbf{F}$ to be a cocycle, we only need to check that the restriction $\phi : V \times \wedge^2 V \to \mathbf{F}$ satisfies the Jacobi identity.

For $x, y, z \in V$, we define $\det(x, y, z) \in \mathbf{F}$ by

$$x \wedge y \wedge z = \det(x, y, z) \cdot a \wedge b \wedge c.$$

Explicitly, if

$$\begin{aligned} x = &\ \alpha_1 a + \beta_1 b + \gamma_1 c \\ y = &\ \alpha_2 a + \beta_2 b + \gamma_2 c \\ z = &\ \alpha_3 a + \beta_3 b + \gamma_3 c \end{aligned}$$

then

$$\det(x, y, z) = \det \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}.$$

By direct computation, we have

$$\begin{aligned} &\phi(x, y \wedge z) + \phi(y, z \wedge x) + \phi(z, x \wedge y) \\ &= \det(x, y, z) \cdot (\phi(a, b \wedge c) + \phi(b, c \wedge a) + \phi(c, a \wedge b)) = 0. \end{aligned}$$

Therefore ϕ is a normalized cocycle if and only if for $\phi : V \times \wedge^2 V \to \mathbf{F}$,

$$\phi(a, b \wedge c) + \phi(b, c \wedge a) + \phi(c, a \wedge b) = 0.$$

The Levi factor of the automorphism group of $N_{6,3,6}$ is $G = GL(V)$, and its unipotent radical R_u acts trivially on $H^2(\mathfrak{g}, \mathbf{F})$, i.e., $\phi(\sigma(x), \sigma(y) \wedge \sigma(z)) = \phi(x, y \wedge z)$.

Because the set of all the bilinear maps from $V \times \wedge^2 V$ to $\mathbf{F}$ is isomorphic to the dual space $(V \otimes \wedge^2 V)^*$ of $V \otimes \wedge^2 V$, we can show that, taking into account the previous statement, $H^2(\mathfrak{g}, \mathbf{F})$ is isomorphic as a G–module to a submodule of $(V \times \wedge^2 V)^*$, which will be denoted by $(V \otimes \wedge^2 V)^*_0$, and where $G = GL(V)$.

Denote by $(E, *)$ the G-module $E = \mathrm{End}_{\mathbf{F}}(V)$ with the action

$$g * T = \det(g) g \circ T \circ g^{-1}.$$

We define a map $\varepsilon := V \otimes \wedge^2 V \to (E, *)$ by

$$\varepsilon(x \otimes (y \wedge z))(v) = \det(v, y, z) x$$

is an isomorphism of G-modules, as we have for any $g \in G$,

$$\begin{aligned} g \cdot \varepsilon(x \otimes (y \wedge z))(v) = & \det(g) g \varepsilon(x \otimes (y \wedge z))(g^{-1}(v)) \\ = & \det(g) \det(g^{-1}(v), y, z) g(x) \\ = & \det(v, g(y), g(z)) g(x) \\ = & \varepsilon(g(x) \otimes (g(y) \wedge g(z)))(v) \\ = & \varepsilon(g(x \otimes (y \wedge z)))(v) \end{aligned}$$

or

$$g \cdot \varepsilon(x \otimes (y \wedge z)) = \varepsilon(g(x \otimes (y \wedge z))).$$

It follows easily that ε acts on the basis of $V \otimes \wedge^2 V$ as

$$a \otimes (b \wedge c) \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a \otimes (c \wedge a) \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$a \otimes (a \wedge b) \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b \otimes (b \wedge c) \to \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$b \otimes (c \wedge a) \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad b \otimes (a \wedge b) \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$c \otimes (b \wedge c) \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad c \otimes (c \wedge a) \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$c \otimes (a \wedge b) \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The dual G-module of $(E, *)$ is isomorphic to the G-module $(E, \Box)$ where

$$g \Box T = \det(g)^{-1} g \circ T \circ g^{-1}.$$

Indeed the map

$$\Phi : (E, \Box) \to (E, *)^*$$

defined by

$$\Phi(T)(S) := \operatorname{tr}(TS)$$

is an isomorphism, since

$$\begin{aligned} (g \cdot \Phi(T))(S) = & \ \Phi(T)(g^{-1} * S) \\ = & \ \operatorname{tr}(T \circ (g^{-1} * S)) \\ = & \ \operatorname{tr}(T \circ \det(g)^{-1} g^{-1} \circ S \circ g) \\ = & \ \operatorname{tr}(\det(g)^{-1} g \circ T \circ g^{-1} \circ S) \\ = & \ \operatorname{tr}((g \Box T) \circ S) \\ = & \ \Phi(g \Box T)(S) \end{aligned}$$

Hence

$$g \cdot \Phi(T) = \Phi(g \Box T).$$

Let $v = \alpha a + \beta b + \gamma c \in V$. Then

$$\varepsilon(a \otimes (b \wedge c))(v) = \det(v, b, c) a = \alpha a.$$

Hence $\varepsilon(a \otimes (b \wedge c))$ = projector on $\mathbf{F}a$ with kernel $\mathbf{F}b + \mathbf{F}c$.

The operators $\varepsilon(b \otimes (c \wedge a))$ and $\varepsilon(c \otimes (a \wedge b))$ have similar descriptions.

In particular,

$$\varepsilon(a \otimes (b \wedge c)) + \varepsilon(b \otimes (c \wedge a)) + \varepsilon(c \otimes (a \wedge b)) = \mathrm{id}_V.$$

It follows that the transpose map

$$\varepsilon^t : (E, *)^* \longmapsto (V \otimes \wedge^2 V)^*$$

induces an isomorphism of the G-module of linear functions that vanish on id_V with the G-modules $(V \otimes \wedge^2 V)_0^*$.

We have

$$\Phi(T)(\mathrm{id}_V) = \mathrm{tr}(T \circ \mathrm{id}_V) = \mathrm{tr}(T)$$

and so $\varepsilon^t \circ \Phi$ induces an isomorphism of G-modules

$$(E_0, \Box) \longmapsto (V \otimes \wedge^2 V)_0^*$$

where

$$E_0 = \ker(\mathrm{tr}) = \{T \in E : \ \mathrm{tr}(T) = 0\}.$$

So we have proved the following

Theorem 6.1 *The G-module $H^2(\mathfrak{g}, \mathbf{F})$ is isomorphic to $(E_0, \Box)$.*

It is an easy fact that any element in $(E_0, \Box)$ is in the same orbit as one of the following three elements:

(i)

$$\begin{bmatrix} \xi & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & -\xi - \eta \end{bmatrix};$$

(ii)

$$\begin{bmatrix} \xi & 1 & 0 \\ 0 & \xi & 1 \\ 0 & 0 & \xi \end{bmatrix};$$

(iii)

$$\begin{bmatrix} \xi & 1 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & -2\xi \end{bmatrix}.$$

Now we try to find the corresponding elements in $H^2(\mathfrak{g}, \mathbf{F})$ for (i), (ii) and (iii).

For an arbitrary element T in $(E, \Box)$, we have the G-module isomorphism $\varepsilon^t \circ \Phi : (E, \Box) \mapsto (V \otimes \wedge^2 V)^*$ with

$$\varepsilon^t \circ \Phi(T)(v) = \Phi(T)(\varepsilon(v)) = \mathrm{tr}(T\varepsilon(v)). \tag{6.2}$$

where $v \in V \otimes \wedge^2 V$.

In (i), we have

$$T = \begin{bmatrix} \xi & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & -\xi - \eta \end{bmatrix}.$$

Therefore from (6.2), if the diagonal elements of $\varepsilon(v)$ are α, β, γ, we have

$$\varepsilon^t \circ \Phi(T)(v) = \Phi(T)(\varepsilon(v)) = \mathrm{tr}(T\varepsilon(v)) = \xi\alpha + \eta\beta + \zeta\gamma. \tag{6.3}$$

Let $\Psi = (b \wedge c) \otimes a + \eta(c \wedge a) \otimes b + \zeta(a \wedge b) \otimes c \in V^* \otimes (\wedge^2 V)^*$. Then

$$\Psi(v) = \xi\alpha + \eta\beta + \zeta\gamma. \tag{6.4}$$

Combining (6.3) and (6.4), we have

$$\varepsilon^t \circ \Phi(T)(v) = \Psi(v),$$

or

$$\varepsilon^t \circ \Phi(T) = (b \wedge c) \otimes a + \eta(c \wedge a) \otimes b + \zeta(a \wedge b) \otimes c.$$

It is easy to check that

$$\Psi(a, b \wedge c) = \xi, \Psi(b, c \wedge a) = \eta, \Psi(c, a \wedge b) = \zeta = -\xi - \eta,$$

and all the other combinations are zero, which in turn will give us the algebra

$$(147E): \quad \begin{array}{ll} [a,b] = d, & [b,c] = e, \\ [c,a] = f, & [a,e] = \xi g, \\ [b,f] = \eta g, & [c,d] = (-\xi - \eta)g. \end{array}$$

We may assume that none of $\xi, \eta, \zeta = -\xi - \eta$ equals 0, otherwise Ψ would have some nonzero element of $Z(\mathfrak{g})$ in its kernel.

By taking $\xi = -1, \eta = \lambda$ and $\zeta = -\xi - \eta = 1 - \lambda$, we get exactly the same family as in Seeley's paper, i.e., (147E) :

$$[a,b]=d,\quad [b,c]=e,$$
$$[a,c]=-f,\quad [a,e]=-g,$$
$$[b,f]=\lambda g,\quad [c,d]=(1-\lambda)g.$$

Let $\zeta=-\xi-\eta$, and

$$\begin{aligned} e_1 &= \xi+\eta+\zeta=0,\\ e_2 &= \xi\eta+\xi\zeta+\eta\zeta,\\ e_3 &= \xi\eta\zeta. \end{aligned}$$

In (i), for two elements diag(ξ,η,ζ) and diag(ξ',η',ζ') to be in the same subspace, we should have $\xi'=\tau\xi,\eta'=\tau\eta,\zeta'=\tau\zeta$, where $\zeta'=-\xi'-\eta'$. Let

$$I=-\frac{e_2^3}{e_3^2}$$

and it is obvious that I is an invariant.

Write $t=-\frac{\eta}{\xi}$, then $\eta=-t\xi,\zeta=-\xi-\eta=(t-1)\xi$. Then

$$\begin{aligned} e_2 &= \xi\eta+\xi\zeta+\eta\zeta,\\ &= \xi^2(-t-t(t-1)+t-1)\\ &= \xi^2(-t^2+t-1)\\ e_3 &= \xi\eta\zeta=\xi^3(-t)(t-1)\\ I &= -\frac{e_2^3}{e_3^2}=\frac{(1-t+t^2)^3}{t^2(t-1)^2}. \end{aligned}$$

Therefore $I(\lambda)=-\frac{e_2^3}{e_3^2}=\frac{(1-\lambda+\lambda^2)^3}{\lambda^2(\lambda-1)^2}$ is an invariant for (147E), with $\lambda\neq 0,1$. It is obvious that (147C) is just a special case of (147E), by letting $\lambda=1/2$.

It is interesting to observe that, up to a constant factor, this invariant has the same expression as the so called j-invariant of the elliptic curve $y^2=x(x-1)(x_\lambda)$ (see [14], pp.83). Seeley uses a somewhat different expression for his invariant in this case.

In (ii), when $\chi\neq 3$, as $3\xi=0$, we have $\xi=0$, then it is easy to see that the corresponding cocycle will contain a nonzero element of $Z(\mathfrak{g})$ in its kernel. So we just consider the case when $\chi=3$ and $\xi\neq 0$. Then we have

$$\begin{bmatrix} \xi & 1 & 0\\ 0 & \xi & 1\\ 0 & 0 & \xi \end{bmatrix}\sim\begin{bmatrix} 1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1 \end{bmatrix}.$$

And its corresponding cocycle is

$$\Psi=(b\wedge c)\otimes(a+b)+(c\wedge a)\otimes(b+c)+(a\wedge b)\otimes c.$$

It is easy to check that

$$\begin{array}{lll} \Psi(a, a\wedge b) = 0, & \Psi(a, b\wedge c) = 1, & \Psi(a, c\wedge a) = 1, \\ \Psi(b, a\wedge b) = 1, & \Psi(b, b\wedge c) = 0, & \Psi(b, c\wedge a) = 1, \\ \Psi(c, a\wedge b) = 1, & \Psi(c, b\wedge c) = 0, & \Psi(c, c\wedge a) = 0. \end{array}$$

The corresponding algebra is for $\chi = 3$ only:

$$\begin{array}{lll} (147F): & (\text{for } \chi = 3 \text{ only}) & \\ & [a, b] = d, & [b, c] = e, \\ & [a, c] = -f, & [a, e] = g, \\ & [a, f] = g, & [b, d] = g, \\ & [b, f] = g, & [c, d] = g. \end{array}$$

In (iii), when $\xi = 0$, the corresponding cocycle will contain a nonzero element of $Z(\mathfrak{g})$ in its kernel. And when $\xi \neq 0$, we have

$$\begin{bmatrix} \xi & 1 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & -2\xi \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Its corresponding cocycle is

$$\Psi = (b\wedge c)\otimes(a+b) + (c\wedge a)\otimes b - 2(a\wedge b)\otimes c.$$

It is easy to check that

$$\begin{array}{lll} \Psi(a, a\wedge b) = 0, & \Psi(a, b\wedge c) = 1, & \Psi(a, c\wedge a) = 1, \\ \Psi(b, a\wedge b) = 0, & \Psi(b, b\wedge c) = 0, & \Psi(b, c\wedge a) = 1, \\ \Psi(c, a\wedge b) = -2, & \Psi(c, b\wedge c) = 0, & \Psi(c, c\wedge a) = 0. \end{array}$$

Its corresponding algebra is

$$\begin{array}{llll} & [a, b] = d, & [b, c] = e, & [a, c] = -f, \\ (1): & [a, e] = g, & [a, f] = g, & [b, f] = g, \\ & [c, d] = -2g. & & \end{array}$$

which is isomorphic to (147D) of Seeley's paper, an isomorphism from (1) to (147D) can be given as: $a \to 1/2c$, $b \to b$, $c \to a$, $d \to -1/2e$, $e \to -d$, $f \to -1/2f$ and $g \to -1/4g$.

Therefore the central extensions of $N_{6,3,6}$ of dimension 7 are:

(147D):		
	$[a,b]=d,$	$[a,c]=-f,$
	$[a,e]=g,$	$[a,f]=g,$
	$[b,c]=e,$	$[b,f]=g,$
	$[c,d]=-2g.$	
(147E):	$I(\xi)=\frac{(1-\xi+\xi^2)^3}{\xi^2(\xi-1)^2}$, $\xi\neq 0,1$	($\xi=1/2$ gives (147C))
	$[a,b]=d,$	$[a,c]=-f,$
	$[a,e]=-g,$	$[b,c]=e,$
	$[b,f]=\xi g,$	$[c,d]=(1-\xi)g.$
(147F):	(for $\chi=3$ only)	
	$[a,b]=d,$	$[a,c]=-f,$
	$[a,e]=g,$	$[a,f]=g,$
	$[b,c]=e,$	$[b,d]=g,$
	$[b,f]=g,$	$[c,d]=g.$

Remark: (147C) is a special case of (147E) by taking $\xi=1/2$.

Chapter 7

Algebras over the Real Field

In this chapter, we will consider the central extensions of the algebras of dimensions ≤ 6 over the real field. We only provide the proofs for those cases where some new algebras arised due to the change of the ground field. Our proofs also apply to the case when the fields are algebraically closed with $\chi \neq 2$, with some minor modifications.

As we have discussed in Chapter 5, two new algebras arise from the central extensions of the 3-dimensional Abelian Lie algebras. No new algebra arises from the central extensions of 4-dimensional algebras. Therefore we start by considering the central extensions of 5-dimensional nilpotent Lie algebras.

7.1 Extensions of 5-Dimensional Algebras

The central extensions of $N_{5,2,2}$ over $\mathbf{R}$ can be found in chapter 2, Example 2.

Central extensions of $N_{5,2,3}$:

$Z(\mathfrak{g})$: x_4, x_5; $[\mathfrak{g},\mathfrak{g}]$: x_3, x_4, x_5; $Z^2(\mathfrak{g})$: $C_{34} = C_{35} = C_{45} = 0, C_{15} - C_{24} = 0$; $W(H^2)$: $C_{12} = C_{13} = C_{23} = 0$; $\dim H^2$: 3; Basis: Δ_{14}, $\Delta_{15} + \Delta_{24}$, Δ_{25}.

Group action: $a\Delta_{14} + b(\Delta_{15} + \Delta_{24}) + c\Delta_{25}$:

Let $\Delta := a_{11}a_{22} - a_{12}a_{21}$. Then $a \to (aa_{11}^2 + 2ba_{11}a_{21} + ca_{21}^2)\Delta$; $b \to (aa_{11}a_{12} + b(a_{11}a_{22} + a_{12}a_{21}) + ca_{21}a_{22})\Delta$; $c \to (aa_{12}^2 + 2ba_{12}a_{22} + ca_{22}^2)\Delta$.

Let $A = [a, b, c]$ and $B = [a_1, b_1, c_1]$. It is obvious that in A one of $a, b, c \neq 0$. May assume $a = 1$ and have $A = [1, b, c]$ and $B = [0, b_1, c_1]$. By taking $a_{21} = 0$ will ensure that $a = 1$ in A.

Now in B one of $b_1, c_1 \neq 0$ and (as $a_{21} = 0$) $a_1 = 0 \to 0$; $b_1 \to b_1a_{11}a_{22}$; $c_1 \to (2b_1a_{12}a_{22} + c_1a_{22}^2)\Delta$.

If $b_1 \neq 0$, make $c_1 = 0$ by solving for a_{12}, and obtain the representative for B: $B_1 = [0,1,0]$.

If $b_1 = 0$, then $c_1 \neq 0$ and obtain another representative for B: $B_2 = [0,0,1]$.

Case 1: $B_1 = [0,1,0]$. Then $A = [1,0,c]$. To fix B_1 (up to a scalar), we require $a_{12} = a_{21} = 0$. Consider the group action on A: $a = 1 \to a_{11}^2\triangle$; $b = 0 \to 0$; $c \to ca_{22}^2\triangle$.

Subcase 1.1: $c = 0$. we obtain the representative for A: $A_1 = [1,0,0]$, with $A_1 \wedge B_1$ corresponding to (2457M).

Subcase 1.2: $c \neq 0$. Then $A \to [a_{11}^2, 0, ca_{22}^2]$. If $c > 0$, then we obtain the representative $A_2 = [1,0,1]$, with $A_2 \wedge B_1$ corresponding to (2457L); If $c < 0$, then we obtain the representative $A_3 = [1,0,-1]$, with $A_3 \wedge B_1$ corresponding to (2457L$_1$).

Case 2: $B_2 = [0,0,1]$. Then $A = [1,b,0]$. To fix B_2 (up to a scalar), we require $a_{21} = 0$. With B_2 being fixed, we can always make $c = 0$ in A by linear combination.

Consider the group action on A: $a \to a_{11}^2\triangle$; $b \to (a_{11}a_{12} + ba_{11}a_{22})\triangle$.

We can obviously make $b = 0$ by solving for a_{12} and obtain the representative $A = [1,0,0]$, with $A \wedge B_2$ also corresponding to (2457L).

At first we show that (1) $A_2 \wedge B_1$ and (2) $A \wedge B_2$ are in the same orbit, as both of them are corresponding to (2457L). Compare the corresponding algebras: (1) $[x_1, x_i] = x_{i+1}, i = 2,3$, $[x_1,x_4] = x_7$, $[x_1,x_5] = [x_2,x_4] = x_6$, $[x_2,x_3] = x_5$, $[x_2,x_5] = x_7$; and (2) $[x_1,x_i] = x_{i+1}, i = 2,3$, $[x_1,x_4] = x_6$, $[x_2,x_3] = x_5$, $[x_2,x_5] = x_7$.

By taking $x_1 \to x_1 + x_2$, $x_2 \to -x_1 + x_2$, $x_3 \to 2x_3$, $x_4 \to 2x_4 + 2x_5$, $x_5 \to -2x_4 + 2x_5$, $x_6 \to -2x_6 + 2x_7$, $x_7 \to 2x_6 + 2x_7$, we map (1) to (2).

To prove the non-isomorphism among (2457L, 2457L$_1$, 2457M), we show that they are in different orbits.

Consider (2457L), i.e., $A \wedge B_2$, under the group action, $A = [1,0,0] \to \triangle[a_{11}^2, a_{11}a_{12}, a_{12}^2]$, and $B_2 = [0,0,1] \to \triangle[a_{21}^2, a_{21}a_{22}, a_{22}^2]$. Then

$$
\begin{aligned}
A \wedge B_2 \to \quad & \{(a_{11}^2a_{21}a_{22} - a_{11}a_{12}a_{21}^2)\Delta_{14} \wedge (\Delta_{15} + \Delta_{24}) \\
& + (a_{11}^2a_{22}^2 - a_{12}^2a_{21}^2)\Delta_{14} \wedge \Delta_{25} \\
& + (a_{11}a_{12}a_{22}^2 - a_{12}^2a_{21}a_{22})(\Delta_{15} + \Delta_{24}) \wedge \Delta_{25}\}\triangle^2
\end{aligned}
$$

Compare (2457L) and (2457M), then the coefficients of $\Delta_{14} \wedge \Delta_{25}$ and $(\Delta_{15} + \Delta_{24}) \wedge \Delta_{25}$ are zero. Then we have $a_{12}a_{22} = 0$ and $a_{11}a_{22} + a_{12}a_{21} = 0$, which will lead to the singularity of $\triangle$.

Compare (2457L) and (2457L$_1$), we would have

$$a_{11}a_{21}\triangle^3 = a_{12}a_{22}\triangle^3 \neq 0,$$

and $(a_{11}a_{22} + a_{12}a_{21})\triangle^3 = 0$.

Simplification would lead to $a_{21}^2 + a_{22}^2 = 0$, which has a solution over algebraically closed fields (and at the same time maintain the nonsingularity of the automorphism group), but not over the real field.

Therefore, as real Lie algebras, (2457L) and (2457L$_1$) are not isomorphic.

Similarly we can prove that (2457L$_1$) and (2457M) are not isomorphic.

Therefore the corresponding central extensions of $N_{5,2,3}$ are:

(2457L):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_4] = x_6,$
	$[x_1, x_5] = x_7,$	$[x_2, x_3] = x_5,$
	$[x_2, x_4] = x_7,$	$[x_2, x_5] = x_6;$
(2457L$_1$):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_4] = x_6,$
	$[x_1, x_5] = x_7,$	$[x_2, x_3] = x_5,$
	$[x_2, x_4] = x_7,$	$[x_2, x_5] = -x_6;$
(2457M):	$[x_1, x_i] = x_{i+1}, i = 2, 3,$	$[x_1, x_4] = x_7,$
	$[x_1, x_5] = x_6,$	$[x_2, x_3] = x_5,$
	$[x_2, x_4] = x_6;$	

Central extensions of $N_{5,3,2}$:

$Z(\mathfrak{g})$: x_4, x_5; $[\mathfrak{g}, \mathfrak{g}]$: x_4, x_5; $Z^2(\mathfrak{g})$: $C_{45} = 0, C_{25} - C_{34} = 0$; $W(H^2)$: $C_{12} = C_{13} = 0$; $\dim H^2$: 6; Basis: Δ_{14}, Δ_{15}, Δ_{23}, Δ_{24}, $\Delta_{25} + \Delta_{34}$, Δ_{35};

Group action: $a\Delta_{14} + b\Delta_{15} + c\Delta_{23} + d\Delta_{24} + e(\Delta_{25} + \Delta_{34}) + f\Delta_{35}$:

$a \to aa_{11}^2a_{22} + ba_{11}^2a_{32} + da_{11}a_{22}a_{21} + ea_{11}(a_{21}a_{32} + a_{22}a_{31}) + fa_{11}a_{31}a_{32}$;

$b \to aa_{11}^2a_{23} + ba_{11}^2a_{33} + da_{11}a_{21}a_{23} + ea_{11}(a_{21}a_{33} + a_{31}a_{23}) + fa_{11}a_{31}a_{33}$;

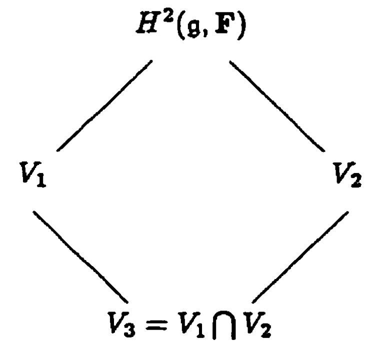
$c \to c(a_{22}a_{33} - a_{32}a_{23}) + d(a_{22}a_{43} - a_{23}a_{42}) + e(a_{22}a_{53} - a_{52}a_{23}) + e(a_{32}a_{43} - a_{42}a_{33}) + f(a_{32}a_{53} - a_{52}a_{33})$;

$d \to da_{11}a_{22}^2 + 2ea_{11}a_{22}a_{32} + fa_{11}a_{32}^2$;

$e \to da_{11}a_{22}a_{23} + ea_{11}(a_{22}a_{33} + a_{23}a_{32}) + fa_{11}a_{32}a_{33}$;

$f \to da_{11}a_{23}^2 + 2ea_{11}a_{23}a_{33} + fa_{11}a_{33}^2$;

Let V_1 be the subspace generated by $\Delta_{14}, \Delta_{15}, \Delta_{23}$, V_2 the subspace generated by Δ_{14}, Δ_{15}, $\Delta_{24}, \Delta_{25}+\Delta_{34}, \Delta_{35}$, and $V_3 = V_1 \bigcap V_2$. It is easy to see that all V_1, V_2 and V_3 are submodules under the group action.



Let L be any two-dimensional subspace of $H^2(\mathfrak{g}, \mathbf{F})$. Assume $L = A \wedge B$, with A and B of the form $a\Delta_{14} + b\Delta_{15} + c\Delta_{23} + d\Delta_{24} + e(\Delta_{25} + \Delta_{34}) + f\Delta_{35}$.

Among A and B, we have the following restrictions: (1) $e \neq 0$; or (2) one of $a, d \neq 0$; or (3) one of $b, f \neq 0$.

Case 1. $L \not\subset V_1$, or at least one of $d, e, f \neq 0$ in A. When $e^2 - df > 0$, we can make $e = 1$ and $b = f = 0$, and also $a = b = c = 0$ to get subcase (a) $A = [0, 0, 0, 0, 1, 0]$. When $e^2 - df = 0$, make $d = e = 0$ and $f = 1$. We can make further $b = c = 0$ by solving for a_{31} and a_{52} respectively. Depending on whether $a = 0$ or not, we get two subcases: (b) $A = [0, 0, 0, 0, 0, 1]$ and (c) $A = [1, 0, 0, 0, 0, 1]$. When $e^2 - df < 0$, we can make $e = 0$ and $d = f = 1$. We can further make $a = b = c = 0$ by solving for a_{21}, a_{31} and a_{43} respectively to get (d) $A = [0, 0, 0, 1, 0, 1]$.

Subcase 1.1: $L \subset V_2$. Or $c = 0$ in both A and B.

Subcase 1.1.1: $L \bigcap V_3 = 0$, or we have at least one of $d, e, f \neq 0$ in B, consider the following cases:

Subcase 1.1.1.1: $A = [0, 0, 0, 0, 1, 0]$. To fix A, we require $a_{21} = a_{31} = 0$, $a_{42} = a_{53}$, $a_{22}a_{32} = 0$, $a_{23}a_{33} = 0$, and $a_{11}(a_{22}a_{33} + a_{23}a_{32}) = 1$.

Assume $B = [a, b, c, d, 0, f]$. Then if $a_{23} = a_{32} = 0$, then $a \to aa_{11}^2a_{22}$; $b \to ba_{11}^2a_{33}$; $c \to ca_{22}a_{33} + da_{22}a_{43} + f(-a_{52}a_{33})$; $d \to da_{11}a_{22}^2$; $e = 0 \to 0$; $f \to fa_{11}a_{33}^2$.

If instead $a_{22} = a_{33} = 0$, we have $a \to ba_{11}^2a_{32}$; $b \to aa_{11}^2a_{23}$; $c \to c(-a_{32}a_{23}) + da_{23}a_{42} + fa_{32}a_{53}$; $d \to fa_{11}a_{32}^2$; $e = 0 \to 0$; $f \to da_{11}a_{23}^2$.

As one of $f, d \neq 0$, we can always assume $f \neq 0$ by the group action above. And if both $f, d \neq 0$, we can always make $a = 1$ when one of $a, b \neq 0$. Make $c = 0$. Depending on the values of a, b, d, we get all the following representatives for B: (1) $B = [0, 0, 0, 0, 0, 1]$ (when $a = b = d = 0$, $A \wedge B$ corresponds to (247I)); (2) $B = [0, 0, 0, 1, 0, 1]$ (when $a = b = 0$, $df > 0$, $A \wedge B$ corresponds to (247F)); (3) $B = [0, 0, 0, 1, 0, -1]$ (when $a = b = 0$, $df < 0$, $A \wedge B$ corresponds to ($247F_1$)); (4) $B = [0, 1, 0, 0, 0, 1]$ ($a = d = 0$, $b \neq 0$, $A \wedge B$ corresponds to (247J)); (5) $B = [1, 0, 0, 0, 0, 1]$ ($b = d = 0$, $a \neq 0$, $A \wedge B$ corresponds to (247K)); (5')

$B = [1, 1, 0, 0, 0, 1]$ ($ab \neq 0$, $d = 0$, $A \wedge B$ corresponds to (247K)); (6) $B = [1, 0, 0, 1, 0, 1]$ (one of $a, b \neq 0$, $df > 0$, $A \wedge B$ corresponds to (247H)); (7) $B = [1, 0, 0, 1, 0, -1]$ (one of $a, b \neq 0$, $df < 0$, $A \wedge B$ corresponds to ($247H_1$)); (8) $B = [1, 1, 0, 1, 0, 1]$ ($abdf \neq 0$, $f = 1$, $A \wedge B$ corresponds to (247G)); (9) $B = [1, 1, 0, 1, 0, f]$ ($abdf \neq 0, f \neq 0, 1$). We will show that when $f < 0$, it will become ($247H_1$) and when $f > 0$ and $f \neq 1$, it will becomes (247H).

It is obvious that each pair of (247F) and ($247F_1$), (247H) and ($247H_1$) are isomorphic over the algebraically closed field. We will prove later that they are different over the real field.

The isomorphism between (5) and (5') will be shown later on.

Subcase 1.1.1.2: $A = [0, 0, 0, 0, 0, 1]$. Fix A, we require $a_{31} = a_{32} = a_{52} = 0$ and $a_{11}a_{33}^2 = 1$.

Assume $B = [a, b, c, d, e, 0]$. We must have $e = 0$, for otherwise we can change it into Subcase 1.1.1.1, as $e^2 - df = e^2 > 0$. Now with $e = 0$ and $d \neq 0$, make $d = -1$ by multiplying B by $-1/d$, then $A + B = [*, *, *, -1, 0, 1]$, which can be changed into Subcase 1.1.1.1 as well. So we omit this case.

Subcase 1.1.1.3: $A = [1, 0, 0, 0, 0, 1]$. To fix A, we require $a_{32} = a_{52} = 0$, $a_{31} = -a_{11}a_{23}/a_{33}$, $a_{11}^2 a_{22} = a_{11}a_{33}^2 = 1$.

Assume $B = [a, b, c, d, e, 0]$. We will also omit this case, as by exactly the same argument as in Subcase 1.1.1.2, it can be changed into Subcase 1.1.1.1.

Subcase 1.1.1.4: $A = [0, 0, 0, 1, 0, 1]$. To fix A, we may let $a_{21} = a_{31} = 0$, $a_{22} = a_{33}$, and $a_{11} = a_{22} = a_{33} = 1$, $a_{42} = a_{43} = a_{52} = a_{53} = 0$. Now consider $B = [a, b, c, d, e, 0]$. We must have $e = 0$, for otherwise we can change it into Subcase 1.1.1.1, as $e^2 - df = e^2 > 0$. Now with $e = 0$ and $d \neq 0$, make $d = 2$ by multiplying B by $2/d$, then subtracting from A by B, $A - B = [*, *, *, -1, 0, 1]$, which can be changed into Subcase 1.1.1.1 as well. So we omit this case.

Subcase 1.1.2: Now consider the case when $L \bigcap V_3 \neq 0$, which means $B \in V_3$.

Subcase 1.1.2.1: $A = [0, 0, 0, 0, 1, 0]$. To fix A, we require $a_{21} = a_{31} = 0$, $a_{42} = a_{53}$, $a_{22}a_{32} = 0$, $a_{23}a_{33} = 0$, and $a_{11}(a_{22}a_{33} + a_{23}a_{32}) = 1$.

Assume $B = [a, b, 0, 0, 0, 0]$. Then if $a_{23} = a_{32} = 0$, then $a \rightarrow aa_{11}^2 a_{22}$; $b \rightarrow ba_{11}^2 a_{33}$.

If instead $a_{22} = a_{33} = 0$, we have $a \rightarrow ba_{11}^2 a_{32}$; $b \rightarrow aa_{11}^2 a_{23}$.

As one of $a, b \neq 0$, we can always assume $a \neq 0$ and get two representatives for B: (10) $B = [1, 0, 0, 0, 0, 0]$ ($A \wedge B$ corresponds to (247D)) and (11) $B = [1, 1, 0, 0, 0, 0]$ ($A \wedge B$ corresponds to (247E)).

Subcase 1.1.2.2: $A = [0, 0, 0, 0, 0, 1]$. Fix A, we require $a_{31} = a_{32} = a_{52} = 0$ and $a_{11}a_{33}^2 = 1$.

Assume $B = [a, b, 0, 0, 0, 0]$.

$a \rightarrow aa_{11}^2 a_{22}$;

$b \to aa_{11}^2 a_{23} + ba_{11}^2 a_{33}$;

As $a \neq 0$, make $b = 0$ to get a representative for B: (12) $B = [1, 0, 0, 0, 0, 0]$, corresponding to (247B).

Subcase 1.1.2.3: $A = [1, 0, 0, 0, 0, 1]$. To fix A, we require $a_{32} = a_{52} = 0$, $a_{31} = -a_{11}a_{23}/a_{33}$, $a_{11}^2 a_{22} = a_{11}a_{33}^2 = 1$. Assume $B = [a, b, 0, 0, 0, 0]$. Then $a \to aa_{11}^2 a_{22}$; $b \to aa_{11}^2 a_{23} + ba_{11}^2 a_{33}$.

If $a \neq 0$, make $b = 0$, and we will get exactly the same algebra as (12). Therefore we assume $a = 0$ and $b \neq 0$ to get (13) $B = [0, 1, 0, 0, 0, 0]$, corresponding to (247C).

Subcase 1.1.2.4: $A = [0, 0, 0, 1, 0, 1]$. To fix A, we may choose $a_{22} = a_{33}$, $a_{23} = -a_{32}$, $a_{21} = a_{31} = 0$, $a_{52}a_{33} = (a_{33}a_{43} - a_{23}a_{42} + a_{32}a_{53})$.

Then for $B = [a, b, 0, 0, 0, 0]$, we have $a \to aa_{11}^2 a_{22} + ba_{11}^2 a_{32}$; $b \to -aa_{11}^2 a_{32} + ba_{11}^2 a_{33}$.

One of $a, b \neq 0$, we can make $a = 1$ and $b = 0$ to get (14) $B = [1, 0, 0, 0, 0, 0]$, with $A \wedge B$ corresponding to (247E_1).

Subcase 1.2: $L \not\subset V_2$, which means $c \neq 0$ in B.

Subcase 1.2.1: $A = [0, 0, 0, 0, 1, 0]$. Let $B = [a, b, c, d, 0, f]$. Compare with the computation as in Subcase 1.1.1.1, may assume $d = f = 0$. If one of $a, b \neq 0$, we can similarly assume $a \neq 0$. Depending on the values of b, we get the following representatives for B: (15) $B = [1, 0, 1, 0, 0, 0]$, corresponding to (247Q). and (16) $B = [1, 1, 1, 0, 0, 0]$, corresponding to (247R).

If both $a = b = 0$, then get the representative: (17) $B = [0, 0, 1, 0, 0, 0]$, corresponding to (247P).

Subcase 1.2.2: $A = [0, 0, 0, 0, 0, 1]$. Assume $B = [a, b, c, d, e, 0]$ with $c \neq 0$. Compare with the computation as in Subcase 1.1.1.2, may assume $d = e = 0$ and get $a \to aa_{11}^2 a_{22}$; $b \to aa_{11}^2 a_{23} + ba_{11}^2 a_{33}$; $c \to ca_{22}a_{33}$.

As $a \neq 0$, make $b = 0$ to get a representative for B (18) $B = [1, 0, 1, 0, 0, 0]$, corresponding to (247M).

Subcase 1.2.3: $A = [1, 0, 0, 0, 0, 1]$. Assume $B = [a, b, c, d, e, 0]$. Compare with the computation as in Subcase 1.1.1.3, we may assume that $d = e = f = 0$. Then $a \to aa_{11}^2 a_{22}$; $b \to aa_{11}^2 a_{23} + ba_{11}^2 a_{33}$; $c \to ca_{22}a_{33}$.

If $a \neq 0$, make $b = 0$ to get a representative for B: (18') $B = [1, 0, 1, 0, 0, 0]$, corresponding to (247M).

If $a = 0$, then depending on the values of b, we get subcase (19) $B = [0, 0, 1, 0, 0, 0]$ (corresponding to (247N)) and (20) $B = [0, 1, 1, 0, 0, 0]$ (corresponding to (247O)).

Subcase 1.2.4: $A = [0, 0, 0, 1, 0, 1]$. Let $B = [a, b, c, d, e, 0]$. Compare with Subcase 1.1.1.4, we may assume that $d = e = 0$ to get $B = [a, b, c, 0, 0, 0]$. Compare with 1.1.2.4, we have

$a \to aa_{11}^2 a_{22} + ba_{11}^2 a_{32}$; $b \to -aa_{11}^2 a_{32} + ba_{11}^2 a_{33}$; $c \to c(a_{22}^2 + a_{32}^2)$.

If one of $a, b \neq 0$, make $a = 1$ and $b = 0$ to get (21) $B = [1, 0, 1, 0, 0, 0]$, corresponding to (247R_1). and if both $a = b = 0$, then we have (22) $B = [0, 0, 1, 0, 0, 0]$, corresponding to (247P_1).

Case 2. $L \subset V_1$, or $d = e = f = 0$ in both A, B. Then one of $a, b \neq 0$. Make $a = 1$ and $b = 0$ in A. Depending on whether $c = 0$ or not, we get two cases: Subcase (2.1) $A = [1, 0, 0, 0, 0, 0]$ and Subcase (2.2) $A = [1, 0, 1, 0, 0, 0]$.

Subcase 2.1: $L \subset V_3$, or both $A, B \in V_3$. For $A = [1, 0, 0, 0, 0, 0]$, we require $b \neq 0$ in B and get (23) $B = [0, 1, 0, 0, 0, 0]$, corresponding to (247A).

Subcase 2.2: $L \not\subset V_3$. For $A = [1, 0, 1, 0, 0, 0]$, to fix it, we require $a_{23} = 0$, $a_{11}^2 a_{22} = 1$, and $a_{22} a_{33} = 1$. Assume $B = [a, b, 0, 0, 0, 0]$, with $b \neq 0$. Now $a \to aa_{11}^2 a_{22} + ba_{11}^2 a_{32}$; $b \to ba_{11}^2 a_{33}$.

Make $a = 0$ and get (21) $B = [0, 1, 0, 0, 0, 0]$, corresponding to (247L). If we consider $A = [1, 0, 0, 0, 0, 0]$, we get a representative which is in the same orbit as (24). So we omit this case.

At first we will show that the following pairs are isomorphic: (5) and (5'), (18) and (18'). We will prove this by providing an isomorphism between the two algebras:

To show that (5) and (5') are in the same orbit, we may take $a_{11} = a_{22} = a_{33} = a_{42} = a_{53} = 1$, $a_{23} = a_{31} = 1/2$, and $a_{21} = 1/4$, and this will map $A \wedge B$ of (5) to that of (5').

For the isomorphism between (18) and (18'), we can actually establish an isomorphim between the two algebras: $x_1 \to x_1$, $x_2 \to -x_2 + x_5$, $x_3 \to x_3$, $x_4 \to -x_4$, $x_5 \to x_5$, $x_6 \to -x_6 - x_7$ and $x_7 \to x_7$.

For (9), when $f > 0$ and $f \neq 1$, an isomorphism between (9) and (247H) is (let α be a solution to the equation $f = \left(\frac{x^3+1}{x^3-1}\right)^2$): $x_1 \to \frac{\alpha^2}{\alpha^3+1}x_1 + \frac{\alpha^4}{\alpha^3+1}x_2 + \frac{1}{\alpha^3+1}x_3$, $x_2 \to \alpha x_2 + x_3$, $x_3 \to \frac{\alpha(\alpha^3+1)}{\alpha^3-1}x_2 - \frac{\alpha^3+1}{\alpha^3-1}x_3$, $x_4 \to \frac{\alpha^3}{\alpha^3+1}x_4 + \frac{\alpha^2}{\alpha^3+1}x_5$, $x_5 \to \frac{\alpha^3}{\alpha^3-1}x_4 - \frac{\alpha^2}{\alpha^3-1}x_5$, $x_6 \to \frac{\alpha^2}{\alpha^3+1}x_6 + \frac{\alpha^4}{\alpha^3+1}x_7$ and $x_7 \to -\frac{\alpha^2}{\alpha^3-1}x_6 + \frac{\alpha^4}{\alpha^3-1}x_7$; when $f < 0$, an isomorphim between (9) and (247H_1) is (let α be a solution to the equation $f = -x^2\left(\frac{x^2-3}{3x^2-1}\right)^2$): $x_1 \to \frac{(\alpha^2+1)^2}{\alpha(\alpha^2-3)}x_1 - 4\frac{\alpha}{\alpha^2-3}x_2 + 2\frac{\alpha^2-1}{\alpha^2-3}x_3$, $x_2 \to \alpha x_2 + x_3$, $x_3 \to \frac{\alpha(\alpha^2-3)}{3\alpha^2-1}x_2 - \frac{\alpha^2(\alpha^2-3)}{3\alpha^2-1}x_3$, $x_4 \to \frac{(\alpha^2+1)^2}{\alpha^2-3}x_4 + \frac{(\alpha^2+1)^2}{\alpha(\alpha^2-3)}x_5$, $x_5 \to \frac{(\alpha^2+1)^2}{3\alpha^2-1}x_4 - \frac{\alpha(\alpha^2+1)^2}{3\alpha^2-1}x_5$, $x_6 \to \frac{(\alpha^2-1)(\alpha^2+1)^2}{\alpha(\alpha^2-3)}x_6 + 2\frac{(\alpha^2+1)^2}{\alpha^2-3}x_7$, and $x_7 \to 2\frac{\alpha(\alpha^2+1)^2}{3\alpha^2-1}x_6 - \frac{(\alpha^2+1)^2(\alpha^2-1)}{3\alpha^2-1}x_7$.

We can also show that over the the algebraically closed field, (247E) and (247E_1), (247P) and (247P_1), (247R) and (247R_1) are isomorphic. (As the isomorphism between (247F, F_1, H, H_1) can be read off easily from the proof.) Let α be a root of $x^2 + 1$ and β be the root of $x^4 + 1$. Then

(247E$_1$)$\cong$ (247E): $x_1 \to x_1$, $x_2 \to x_2 + x_3$, $x_3 \to -\alpha x_2 + \alpha x_3$, $x_4 \to x_4 + x_5$, $x_5 \to -\alpha x_4 + \alpha x_5$, $x_6 \to 2x_6$, $x_7 \to 2x_7$;

(247P$_1$)$\cong$ (247P): $x_1 \to x_1$, $x_2 \to \alpha(x_2 - x_3)$, $x_3 \to x_2 + x_3$, $x_4 \to \alpha(x_4 - x_5)$, $x_5 \to x_4 + x_5$, $x_6 \to 2\alpha x_6$, and $x_7 \to 2x_7$;

(247R$_1$)$\cong$ (247R): $x_1 \to \beta x_1$, $x_2 \to x_2 + x_3$, $x_3 \to \beta^2(-x_2 + x_3)$, $x_4 \to \beta(x_4 + x_5)$, $x_5 \to \beta^3(-x_4 + x_5)$, $x_6 \to 2\beta^2 x_6$, and $x_7 \to 2\beta x_7$.

To show that all the algebras (247A-R), (247E$_1$), (247F$_1$), (247H$_1$), (247P$_1$), and (247R$_1$) are distinct, we just need to compare the algebras among the same groups as follows:

Group 1: (247A); Group 2: (247L); Group 3: (247B, C, D, E, E$_1$); Group 4: (247F,F$_1$, G, H,H$_1$, I,J, K); Group 5: (247M, N,O,P,Q,R, P$_1$, R$_1$).

Take (247F) as an example. We will prove that it is distinct from all the other algebras in Group 4. We have in this case $A = [0,0,0,0,1,0]$ and $B = [0,0,0,1,0,1]$. Then under the group action,

$A \to [a_{11}(a_{21}a_{32}+a_{22}a_{31}), a_{11}(a_{21}a_{33}+a_{31}a_{23}), a_{22}a_{53}-a_{52}a_{23}+a_{32}a_{43}-a_{42}a_{33}, 2a_{11}a_{22}a_{32}, a_{11}(a_{22}a_{33}+a_{23}a_{32}), 2a_{11}a_{23}a_{33}]$

and

$B \to [a_{11}(a_{21}a_{22}+a_{32}a_{31}), a_{11}(a_{21}a_{23}+a_{31}a_{33}), a_{22}a_{43}-a_{42}a_{23}+a_{32}a_{53}-a_{52}a_{33}, a_{11}(a_{22}^2 + a_{32}^2), a_{11}(a_{22}a_{23}+a_{33}a_{32}), a_{11}(a_{23}^2+a_{33}^2)]$.

Consider the wedge product $A \wedge B$, and the corresponding coefficients are: (let $\delta = a_{22}a_{33} - a_{23}a_{32}$)

$$
\begin{array}{ll}
\Delta_{14} \wedge \Delta_{15}: & -a_{11}^2(a_{21}^2 - a_{31}^2)\delta \\
\Delta_{14} \wedge \Delta_{24}: & -a_{11}^2(a_{22}^2 - a_{32}^2)(a_{21}a_{32} - a_{22}a_{31}) \\
\Delta_{14} \wedge (\Delta_{25}+\Delta_{34}): & -a_{11}^2(a_{22}^2 - a_{32}^2)(a_{21}a_{33} - a_{23}a_{31}) \\
\Delta_{15} \wedge (\Delta_{25}+\Delta_{34}): & a_{11}^2(a_{33}^2 - a_{23}^2)(a_{21}a_{32} - a_{22}a_{31}) \\
\Delta_{15} \wedge \Delta_{35}: & a_{11}^2(a_{33}^2 - a_{23}^2)(a_{21}a_{33} - a_{23}a_{31}) \\
\Delta_{24} \wedge (\Delta_{25}+\Delta_{34}): & -a_{11}^2(a_{22}^2 - a_{32}^2)\delta \\
\Delta_{24} \wedge \Delta_{35}: & -2a_{11}^2(a_{22}a_{23} - a_{33}a_{32})\delta \\
(\Delta_{25}+\Delta_{34}) \wedge \Delta_{35}: & a_{11}^2(a_{33}^2 - a_{23}^2)\delta
\end{array}
$$

Compare the coefficients of $A \wedge B$ with that of (247F$_1$). If they are isomorphic, then the coefficients of $\Delta_{24} \wedge (\Delta_{25}+\Delta_{34})$ and $(\Delta_{25}+\Delta_{34}) \wedge \Delta_{35}$ are equal (nonzero), while all the others are zero. Then $a_{22}a_{23} - a_{33}a_{32} = 0$ and $a_{22}^2 - a_{32}^2 = a_{23}^2 - a_{33}^2$. It is easy to see that when it is over R, then it has no solution (otherwise the automorphism group is singular, or, $\delta = 0$). (Notice that if it is over an algebraically closed field of $\chi \neq 2$, then it has a solution.)

Compare with (247I). Then the only coefficients that is nonzero is $a_{11}^2(a_{33}^2 - a_{23}^2)\delta \neq 0$, and all the others are zero, which include that $a_{22}^2 - a_{32}^2 = a_{22}a_{23} - a_{33}a_{32} = 0$, and leads to $a_{23}^2 = a_{33}^2$, contradiction. Therefore (247I) and (247F) are not isomorphic.

All the other algebras can be proved similarly.

Therefore the corresponding central extensions of $N_{5,3,2}$ are:

(247A):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4, 5;$		
(247B):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4$	$[x_3, x_5] = x_7;$	
(247C):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4$	$[x_1, x_5] = x_7,$	$[x_3, x_5] = x_6.$
(247D):	$[x_1, x_i] = x_{i+2}, i = 2, 3,$	$[x_1, x_4] = x_6,$	
	$[x_2, x_5] = x_7,$	$[x_3, x_4] = x_7;$	
(247E):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4$	$[x_1, x_5] = x_6,$	
	$[x_2, x_5] = x_7,$	$[x_3, x_4] = x_7;$	
(247E$_1$):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4$	$[x_2, x_4] = x_7,$	$[x_3, x_5] = x_7;$
(247F):	$[x_1, x_i] = x_{i+2}, i = 2, 3,$	$[x_2, x_4] = x_6,$	$[x_2, x_5] = x_7,$
	$[x_3, x_4] = x_7,$	$[x_3, x_5] = x_6;$	
(247F$_1$):	$[x_1, x_i] = x_{i+2}, i = 2, 3,$	$[x_2, x_4] = x_6,$	$[x_2, x_5] = x_7,$
	$[x_3, x_4] = x_7,$	$[x_3, x_5] = -x_6;$	
(247G):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4$	$[x_1, x_5] = x_6,$	$[x_2, x_4] = x_6,$
	$[x_2, x_5] = x_7,$	$[x_3, x_4] = x_7,$	$[x_3, x_5] = x_6;$
(247H):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4$	$[x_2, x_4] = x_6,$	$[x_2, x_5] = x_7,$
	$[x_3, x_4] = x_7,$	$[x_3, x_5] = x_6;$	
(247H$_1$):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4$	$[x_2, x_4] = x_6,$	$[x_2, x_5] = x_7,$
	$[x_3, x_4] = x_7,$	$[x_3, x_5] = -x_6;$	
(247I):	$[x_1, x_i] = x_{i+2}, i = 2, 3,$	$[x_2, x_5] = x_6,$	$[x_3, x_4] = x_6,$
	$[x_3, x_5] = x_7;$		
(247J):	$[x_1, x_i] = x_{i+2}, i = 2, 3$	$[x_1, x_5] = x_6,$	$[x_2, x_5] = x_7,$
	$[x_3, x_4] = x_7,$	$[x_3, x_5] = x_6;$	
(247K):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4$	$[x_2, x_5] = x_7,$	
	$[x_3, x_4] = x_7,$	$[x_3, x_5] = x_6;$	
(247L):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4, 5$	$[x_2, x_3] = x_6;$	
(247M):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4$	$[x_2, x_3] = x_6,$	$[x_3, x_5] = x_7;$
(247N):	$[x_1, x_i] = x_{i+2}, i = 2, 3,$	$[x_1, x_5] = x_6,$	
	$[x_2, x_3] = x_7,$	$[x_2, x_4] = x_6;$	
(247O):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4$	$[x_1, x_5] = x_7,$	
	$[x_2, x_3] = x_7,$	$[x_3, x_5] = x_6;$	

(247P):	$[x_1, x_i] = x_{i+2}, i = 2, 3,$	$[x_2, x_3] = x_6,$	
	$[x_2, x_5] = x_7,$	$[x_3, x_4] = x_7;$	
(247P$_1$):	$[x_1, x_i] = x_{i+2}, i = 2, 3,$	$[x_2, x_3] = x_6,$	
	$[x_2, x_4] = x_7,$	$[x_3, x_5] = x_7;$	
(247Q):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4,$	$[x_2, x_3] = x_6,$	$[x_2, x_5] = x_7,$
	$[x_3, x_4] = x_7;$		
(247R):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4$	$[x_1, x_5] = x_6,$	$[x_2, x_3] = x_6,$
	$[x_2, x_5] = x_7,$	$[x_3, x_4] = x_7;$	
(247R$_1$):	$[x_1, x_i] = x_{i+2}, i = 2, 3, 4$	$[x_2, x_3] = x_6,$	$[x_2, x_4] = x_7,$
	$[x_3, x_5] = x_7.$		

Central extensions of $N_{5,4}$:

$Z(\mathfrak{g})$: x_3, x_4, x_5; $[\mathfrak{g}, \mathfrak{g}]$: x_3; $Z^2(\mathfrak{g})$: $C_{34} = 0, C_{35} = 0$; $W(H^2)$: $C_{12} = 0$; $\dim H^2$: 7; Basis: $\Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{23}, \Delta_{24}, \Delta_{25}, \Delta_{45}$;

Group action: $a\Delta_{13} + b\Delta_{14} + c\Delta_{15} + d\Delta_{23} + e\Delta_{24} + f\Delta_{25} + g\Delta_{45}$;

Let $\delta := a_{11}a_{22} - a_{12}a_{21}$.

$a \to aa_{11}\delta + da_{21}\delta$;

$b \to a_{11}(aa_{34} + ba_{44} + ca_{54}) + a_{21}(da_{34} + ea_{44} + fa_{54}) + g(a_{41}a_{54} - a_{51}a_{44})$;

$c \to a_{11}(aa_{35} + ba_{45} + ca_{55}) + a_{21}(da_{35} + ea_{45} + fa_{55}) + g(a_{41}a_{55} - a_{51}a_{45})$;

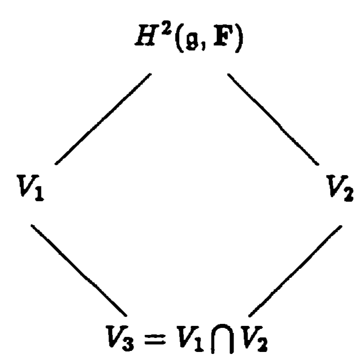
$d \to aa_{12}\delta + da_{22}\delta$;

$e \to a_{12}(aa_{34} + ba_{44} + ca_{54}) + a_{22}(da_{34} + ea_{44} + fa_{54}) + g(a_{42}a_{54} - a_{52}a_{44})$;

$f \to a_{12}(aa_{35} + ba_{45} + ca_{55}) + a_{22}(da_{35} + ea_{45} + fa_{55}) + g(a_{42}a_{55} - a_{52}a_{45})$;

$g \to g(a_{44}a_{55} - a_{54}a_{45})$;

Let V_1 be the subspace of $H^2(\mathfrak{g}, \mathbf{F})$ generated by $\Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{23}, \Delta_{24}, \Delta_{25}$, V_2 the subspace generated by $\Delta_{14}, \Delta_{15}, \Delta_{24}, \Delta_{25}, \Delta_{45}$, and V_3 the intersection of V_1 and V_2. By the group action above, we know that all V_1, V_2 and V_3 are submodules of $H^2(\mathfrak{g}, \mathbf{F})$.



Let L be any two-dimensional subspace of $H^2(\mathfrak{g}, \mathbf{F})$ with the desired property. Then it is obvious that $L \bigcap V_1 \neq 0$ and $L \not\subset V_2$. Denote $L = A \wedge B$, where $A, B \in H^2(\mathfrak{g}, \mathbf{F})$.

Case 1: $L \not\subset V_1$, or $g \neq 0$ in A. We may assume $A = [a, b, c, d, e, f, 1]$ and $B = [a_1, b_1, c_1, d_1, e_1, f_1, 0]$. To fix A, we require that $g = 1 \to a_{44}a_{55} - a_{54}a_{45} \neq 0$, which is always true, as $a_{44}a_{55} - a_{54}a_{45}$ is a factor of the determinant of the automorphism group Aut$N_{5,4}$.

Subcase 1.1: $L \bigcap V_3 = 0$, or at least one of $a_1, d_1 \neq 0$. Make $a_1 = 1$ and $d_1 = 0$. Then for $B = [1, b_1, c_1, 0, e_1, f_1, 0]$, we have $a_1 = 1 \to a_{11}\delta$; $b_1 \to a_{11}(a_{34} + b_1 a_{44} + c_1 a_{54}) + a_{21}(e_1 a_{44} + f_1 a_{54})$; $c_1 \to a_{11}(a_{35} + b_1 a_{45} + c_1 a_{55}) + a_{21}(e_1 a_{45} + f_1 a_{55})$; $d_1 \to a_{12}\delta$; $e_1 \to a_{12}(a_{34} + b_1 a_{44} + c_1 a_{54}) + a_{22}(e_1 a_{44} + f_1 a_{54})$; $f_1 \to a_{12}(a_{35} + b_1 a_{45} + c_1 a_{55}) + a_{22}(e_1 a_{45} + f_1 a_{55})$; $g_1 = 0 \to 0$.

Choose $a_{12} = 0$. We can make $b_1 = c_1 = 0$ by solving for a_{34} and a_{35} respectively. Assume that $a_{12} = 0$, we now have $a_1 = 1 \to a_{11}\delta = 1$; $b_1 = 0 \to a_{11}a_{34} + a_{21}(e_1 a_{44} + f_1 a_{54}) = 0$ (solve for a_{34}); $c_1 = 0 \to a_{11}a_{35} + a_{21}(e_1 a_{45} + f_1 a_{55}) = 0$ (solve for a_{35}); $d_1 = 0 \to 0$; $e_1 \to a_{22}(e_1 a_{44} + f_1 a_{54})$; $f_1 \to a_{22}(e_1 a_{45} + f_1 a_{55})$; $g_1 = 0 \to 0$.

If at least one of $e_1, f_1 \neq 0$, then make $e_1 = 1$ and $f_1 = 0$ to get $B_1 = [1, 0, 0, 0, 1, 0, 0]$. If both $e_1 = f_1 = 0$, then get $B_2 = [1, 0, 0, 0, 0, 0, 0]$.

Subcase 1.1.1: $L \bigcap V_2 \neq 0$, or $A \in V_2$. For $B_1 = [1, 0, 0, 0, 1, 0, 0]$, we may assume $A = [0, b, c, 0, e, f, 1]$. To fix B_1, we require $a_{12} = a_{35} = a_{45} = 0$, $a_{34} = -a_{21}a_{44}/a_{11}$, $a_{11}\delta = 1$ and $a_{12}a_{34} + a_{22}a_{44} = 1$. Now consider A. By taking also $a_{54} = 0$, we have $a = 0 \to 0$; $b \to a_{11}ba_{44} + a_{21}ea_{44} - a_{51}a_{44}$; $c \to a_{11}ca_{55} + a_{21}fa_{55} + a_{41}a_{55}$; $d = 0 \to 0$; $e \to a_{22}ea_{44} - a_{52}a_{44}$; $f \to a_{22}fa_{55} + a_{42}a_{55}$; $g = 1 \to a_{44}a_{55}$.

We make $a = b = c = e = f = 0$ by taking $a_{21} = a_{51} = a_{41} = a_{52} = a_{42} = 0$ and get a representative for A: (1.1.1a) $A = [0, 0, 0, 0, 0, 0, 1]$ ($A \wedge B$ corresponds to (257H)).

For $B_2 = [1, 0, 0, 0, 0, 0, 0]$, we assume $A = [0, b, c, 0, e, f, 1]$. To fix B_2, we require $a_{12} = a_{34} = a_{35} = 0$ and $a_{11}^2 a_{22} = 1$. Now consider A, $a = 0 \to 0$; $b \to a_{11}(ba_{44} + ca_{54}) + a_{21}(ea_{44} + fa_{54}) + (a_{41}a_{54} - a_{51}a_{44})$; $c \to a_{11}(ba_{45} + ca_{55}) + a_{21}(ea_{45} + fa_{55}) + (a_{41}a_{55} - a_{51}a_{45})$; $d \to 0$; $e \to a_{22}(ea_{44} + fa_{54}) + (a_{42}a_{54} - a_{52}a_{44})$; $f \to a_{22}(ea_{45} + fa_{55}) + (a_{42}a_{55} - a_{52}a_{45})$; $g \to (a_{44}a_{55} - a_{54}a_{45})$.

By taking $a_{45} = a_{54} = 0$, we can make $b = c = e = f = 0$ by solving for a_{51}, a_{41}, a_{52} and a_{42} respectively to get $A = [0, 0, 0, 0, 0, 0, 1]$. But this gives us a decomposable algebra (1.1.1b) $N_{4,2} \times N_{3,2}$: $[x_1, x_2] = x_3$, $[x_1, x_3] = x_6$, $[x_4, x_5] = x_7$.

Subcase 1.1.2: $L \bigcap V_2 = 0$, or $A \notin V_2$, i.e., one of $a, d \neq 0$.

For $B_1 = [1, 0, 0, 0, 1, 0, 0]$, we may assume $A = [0, b, c, d, e, f, 1]$. To fix B_1, we require $a_{12} = a_{35} = a_{45} = 0$, $a_{34} = -a_{21}a_{44}/a_{11}$, $a_{11}\delta = 1$ and $a_{22}a_{44} = 1$. Now consider A. By taking also $a_{54} = 0$, we have $a = 0 \to da_{21}\delta$; $b \to a_{11}ba_{44} + a_{21}(da_{34} + ea_{44}) - a_{51}a_{44}$; $c \to a_{11}ca_{55} + a_{21}fa_{55} + a_{41}a_{55}$; $d \to da_{22}\delta$; $e \to a_{22}(da_{34} + ea_{44}) - a_{52}a_{44})$; $f \to a_{22}fa_{55} + a_{42}a_{55}$;

$g = 1 \to a_{44}a_{55}$.

We make $a = b = c = e = f = 0$ by taking $a_{21} = a_{51} = a_{41} = a_{52} = a_{42} = 0$. Because $d \neq 0$, we get a representative for A: (1.1.2a) $A = [0,0,0,1,0,0,1]$ ($A \wedge B_1$ corresponds to (257L)).

For $B_2 = [1,0,0,0,0,0,0]$, we assume $A = [0, b, c, d, e, f, 1]$.

To fix B_2, we require $a_{12} = a_{34} = a_{35} = 0$ and $a_{11}^2 a_{22} = 1$. Now consider A, $a = 0 \to 0$ (By subtracting a multiple of B_2 from A, we can always make $a = 0$); $b \to a_{11}(ba_{44} + ca_{54}) + a_{21}(ea_{44} + fa_{54}) + (a_{41}a_{54} - a_{51}a_{44})$; $c \to a_{11}(ba_{45} + ca_{55}) + a_{21}(ea_{45} + fa_{55}) + (a_{41}a_{55} - a_{51}a_{45})$; $d \to da_{22}\delta$; $e \to a_{22}(ea_{44} + fa_{54}) + (a_{42}a_{54} - a_{52}a_{44})$; $f \to a_{22}(ea_{45} + fa_{55}) + (a_{42}a_{55} - a_{52}a_{45})$; $g \to (a_{44}a_{55} - a_{54}a_{45})$.

By taking $a_{45} = a_{54} = 0$, we can make $b = c = e = f = 0$ by solving for a_{51}, a_{41}, a_{52} and a_{42} respectively. As $d \neq 0$, we get (1.1.2b) $A = [0,0,0,1,0,0,1]$ ($A \wedge B_2$ corresponds to (257K)).

Subcase 1.2: $L \bigcap V_3 \neq 0$, or $a_1 = d_1 = 0$ in B. In this case, it is obvious that we also have $L \not\subset V_2$, or $A \notin V_2$ or one of $a, d \neq 0$ in A.

Then $B = [0, b_1, c_1, 0, e_1, f_1, 0]$ and one of $b_1, c_1, e_1, f_1 \neq 0$. May assume $e_1 = 1$. $a_1 = 0 \to 0$; $b_1 \to a_{11}(b_1a_{44} + c_1a_{54}) + a_{21}(a_{44} + f_1a_{54})$; $c_1 \to a_{11}(b_1a_{45} + c_1a_{55}) + a_{21}(a_{45} + f_1a_{55})$; $d_1 = 0 \to 0$; $e_1 = 1 \to a_{12}(b_1a_{44} + c_1a_{54}) + a_{22}(a_{44} + f_1a_{54}) = 1$; $f_1 \to a_{12}(b_1a_{45} + c_1a_{55}) + a_{22}(a_{45} + f_1a_{55})$; $g_1 = 0 \to 0$.

Make $b_1 = f_1 = 0$ by solving for a_{21} and a_{45}. Now we get $a_1 = 0 \to 0$; $b_1 = 0 \to a_{11}c_1a_{54} + a_{21}a_{44} = 0$; ($a_{21} = -c_1a_{11}a_{54}/a_{44}$) $c_1 \to a_{11}c_1a_{55} + a_{21}a_{45}$; $d_1 = 0 \to 0$; $e_1 = 1 \to a_{12}c_1a_{54} + a_{22}a_{44} = 1$; $f_1 = 0 \to a_{12}c_1a_{55} + a_{22}a_{45} = 0$; ($a_{45} = -c_1a_{12}a_{55}/a_{22}$) $g_1 = 0 \to 0$.

Substitute a_{21}, a_{45} into c_1, combining with the fact that $a_{12}c_1a_{54} + a_{22}a_{44} = 1$, we get

$$\begin{aligned} c_1 \to \;& c_1a_{11}a_{55} + \tfrac{c_1^2 a_{11}a_{54}a_{12}a_{55}}{a_{22}a_{44}} \\ =\;& c_1a_{11}a_{55}\left(1 + \tfrac{c_1a_{54}a_{12}}{a_{22}a_{44}}\right) \\ =\;& c_1a_{11}a_{55}\tfrac{c_1a_{54}a_{12} + a_{22}a_{44}}{a_{22}a_{44}} \\ =\;& c_1\tfrac{a_{11}a_{55}}{a_{22}a_{44}} \end{aligned}$$

Depending on whether $c_1 = 0$ or not, we get two representatives for B: $B_1 = [0,0,0,0,1,0,0]$ and $B_2 = [0,0,1,0,1,0,0]$.

Subcase 1.2.1: For $B_1 = [0,0,0,0,1,0,0]$, we assume $A = [a, b, c, d, 0, f, 1]$. To fix B_1, we require $a_{21} = a_{45} = 0$ and $a_{22}a_{44} = 1$. Now consider A. $a \to aa_{11}\delta$; $b \to a_{11}(aa_{34} + ba_{44} + ca_{54}) + (a_{41}a_{54} - a_{51}a_{44})$; $c \to a_{11}(aa_{35} + ca_{55}) + a_{41}a_{55}$; $d \to aa_{12}\delta + da_{22}\delta$; $e = 0 \to a_{12}(aa_{34} + ba_{44} + ca_{54}) + a_{22}(da_{34} + fa_{54}) + (a_{42}a_{54} - a_{52}a_{44}) = 0$ (By subtracting a multiple of B_1); $f \to a_{12}(aa_{35} + ca_{55}) + a_{22}(da_{35} + fa_{55}) + a_{42}a_{55}$; $g = 1 \to a_{44}a_{55} = 1$.

Make $b = c = f = 0$ by solving for a_{51}, a_{41} and a_{42} respectively. Now if $a \neq 0$, make $a = 1$ and $d = 0$ by solving for a_{12} to get (1.2.1a) $A_1 = [1,0,0,0,0,0,1]$ ($A_1 \wedge B_1$ corresponds to

(257E)). If $a = 0$, then $d \neq 0$, and get (1.2.1b) $A_2 = [0,0,0,1,0,0,1]$ ($A_2 \wedge B_1$ corresponds to (257F)).

Subcase 1.2.2: For $B_2 = [0,0,1,0,1,0,0]$, we may assume $A = [a,b,c,d,0,f,1]$. To fix B_2, we require that

$$a_{21}a_{44} + a_{11}a_{54} = 0, \quad a_{11}a_{55} + a_{21}a_{45} = 1,$$
$$a_{12}a_{54} + a_{22}a_{44} = 1, \quad a_{12}a_{55} + a_{22}a_{45} = 0,$$

or $a_{44} = a_{11}/\delta$, $a_{55} = a_{22}/\delta$, $a_{45} = -a_{12}/\delta$ and $a_{54} = -a_{21}/\delta$.

Now consider A. $a \to aa_{11}\delta + da_{21}\delta$; $b \to a_{11}(aa_{34} + ba_{44} + ca_{54}) + a_{21}(da_{34} + fa_{54}) + (a_{41}a_{54} - a_{51}a_{44})$; $c \to a_{11}(aa_{35} + ba_{45} + ca_{55}) + a_{21}(da_{35} + fa_{55}) + (a_{41}a_{55} - a_{51}a_{45})$; $d \to aa_{12}\delta + da_{22}\delta$; $e \to a_{12}(aa_{34} + ba_{44} + ca_{54}) + a_{22}(da_{34} + fa_{54}) + (a_{42}a_{54} - a_{52}a_{44})$; $f \to a_{12}(aa_{35} + ba_{45} + ca_{55}) + a_{22}(da_{35} + fa_{55}) + (a_{42}a_{55} - a_{52}a_{45})$; $g = 1 \to (a_{44}a_{55} - a_{54}a_{45})$.

Taking $a_{12} = a_{21} = a_{45} = a_{54} = 0$, we make $b = c = e = f = 0$ by solving for a_{51}, a_{41}, a_{52} and a_{42} respectively. Now for A, we have

$$\begin{aligned} a \to\ & aa_{11}\delta + da_{21}\delta \\ =\ & (aa_{11} + d(-a_{11}a_{54}/a_{44}))\delta \\ =\ & a_{11}(aa_{44} - da_{54})\delta/a_{44} \\ d \to\ & aa_{12}\delta + da_{22}\delta \\ =\ & (a(-a_{22}a_{45}/a_{55}) + da_{22})\delta \\ =\ & a_{22}(-aa_{45} + da_{55})\delta/a_{55} \end{aligned}$$

As one of $a, d \neq 0$, we can make $a = 1$ and $d = 0$ to get (1.2.2) $A = [1,0,0,0,0,0,1]$ ($A \wedge B_2$ corresponds to (257G)).

Case 2: $L \subset V_1$, or $g = 0$ in A, B. One of $a, d \neq 0$. Make $a = 1$ and assume that $A = [1,b,c,d,e,f,0]$ and $B = [0,b_1,c_1,d_1,e_1,f_1,0]$.

Bearing in mind that to fix A, we require $a_{11} + da_{21} \neq 0$.

Subcase 2.1: $L \bigcap V_3 = 0$. Since $A \in V_1$, this case is the same as $B \notin V_3$, or $d_1 \neq 0$ in B. Assume $d_1 = 1$, $B = [0,b_1,c_1,1,e_1,f_1,0]$. Now consider B, by taking $a_{21} = 0$, $a_1 = 0 \to 0$; $b_1 \to a_{11}(b_1a_{44} + c_1a_{54})$; $c_1 \to a_{11}(b_1a_{45} + c_1a_{55})$; $d_1 = 1 \to a_{22}\delta = 1$; $e_1 \to a_{12}(b_1a_{44} + c_1a_{54}) + a_{22}(a_{34} + e_1a_{44} + f_1a_{54})$; $f_1 \to a_{12}(b_1a_{45} + c_1a_{55}) + a_{22}(a_{35} + e_1a_{45} + f_1a_{55})$; $g_1 \to 0$.

Make $a_1 = e_1 = f_1 = 0$ by solving for a_{21}, a_{34} and a_{35}. Now $B = [0,b_1,c_1,1,0,0,0]$. Taking $a_{21} = a_{12} = a_{34} = a_{35} = 0$, $a_1 = 0 \to 0$; $b_1 \to a_{11}(b_1a_{44} + c_1a_{54})$; $c_1 \to a_{11}(b_1a_{45} + c_1a_{55})$; $d_1 = 1 \to a_{22}\delta = 1$; $e_1 = 0 \to 0$; $f_1 = 0 \to 0$; $g_1 \to 0$.

If one of $b_1, c_1 \neq 0$, then make $c_1 = 1$ and $b_1 = 0$ to get $B_1 = [0,0,1,1,0,0,0]$. If both $b_1 = c_1 = 0$, then get $B_2 = [0,0,0,1,0,0,0]$.

Subcase 2.1.1: For $B_1 = [0,0,1,1,0,0,0]$, we assume $A = [1,b,c,0,e,f,0]$. To fix B_1, we require $a_{21} = a_{34} = a_{54} = 0$, $a_{35} = -a_{12}a_{55}/a_{22}$, $a_{11}a_{55} = a_{11}a_{22}^2 = 1$. Consider A, we have

$a=1\to a_{11}\delta$; $b\to a_{11}ba_{44}$; $c\to a_{11}(a_{35}+ba_{45}+ca_{55})$; $d\to a_{12}\delta$; $e\to a_{12}ba_{44}+a_{22}ea_{44}$; $f\to a_{12}(a_{35}+ba_{45}+ca_{55})+a_{22}(ea_{45}+fa_{55})=a_{12}(-a_{12}a_{55}/a_{22}+ba_{45}+ca_{55})+a_{22}(ea_{45}+fa_{55})$; $g\to 0$.

One of $b,e\neq 0$. If $b\neq 0$, then assume $b=1$ and make $e=0$ by solving for a_{12}, make $c=d$ by solving for a_{45} and further make both $c=d=0$ by subtracting a multiple of B_1 from A. Now taking $a_{12}=a_{35}=a_{45}=0$, $a=1\to a_{11}\delta=1$; $b=1\to a_{11}a_{44}=1$; $c\to 0$; $d\to 0$; $e=0\to 0$; $f\to fa_{22}a_{55}$; $g\to 0$.

Now if $f=0$, we obtain the representative: Subcase (2.1.1a) $A_1=[1,1,0,0,0,0,0]$ ($A_1\wedge B_1$ corresponds to (257I)); If $f\neq 0$, and $f>0$, we have Subcase (2.1.1b) $A_2=[1,1,0,0,0,1,0]$ ($A_2\wedge B_1$ corresponds to ($257J_1$)); if $f<0$, $A_3=[1,1,0,0,0,-1,0]$ (we will omit this case, as we can show that it is in the same orbit as (2.1.1c) below, which has a simpler form).

If $b=0$, then assume $e=1$. Now, taking $a_{21}=0$, we have $a=1\to a_{11}\delta$; $b=0\to 0$; $c\to a_{11}(-a_{12}a_{55}/a_{22}+ca_{55})$; $d\to a_{12}\delta$; $e\to a_{22}a_{44}=1$; $f\to a_{12}(a_{35}+ba_{45}+ca_{55})+a_{22}(a_{45}+fa_{55})=a_{12}(-a_{12}a_{55}/a_{22}+ca_{55})+a_{22}(a_{45}+fa_{55})=0$ (Solve for a_{45}); $g\to 0$.

We can make $c=d$ and subtracting a multiple of B from A to make $c=d=0$ and get (2.1.1c) $A_4=[1,0,0,0,1,0,0]$ ($A_4\wedge B_1$ corresponds to (257J)). As it turns out to be in the same orbit as (2.1.1b), and because (2.1.1c) has a simpler form, so we omit (2.1.1b) instead.

Subcase 2.1.2: For $B_2=[0,0,0,1,0,0,0]$, we assume $A=[1,b,c,0,e,f,0]$. To fix B_2, we need $a_{21}=a_{34}=a_{35}=0$ and $a_{11}a_{22}^2=1$. Now consider A, $a\to a_{11}\delta$; $b\to a_{11}(ba_{44}+ca_{54})$; $c\to a_{11}(ba_{45}+ca_{55})$; $d\to a_{12}\delta$; $e\to a_{12}(ba_{44}+ca_{54})+a_{22}(ea_{44}+fa_{54})$; $f\to a_{12}(ba_{45}+ca_{55})+a_{22}(ea_{45}+fa_{55})$; $g\to 0$.

One of $b,c\neq 0$, for otherwise the 2-cocycles will contain some none trivial elements of the center in the kernel. Make $b=1$ and $c=0$. Make $e=0$ by solving for a_{12}. Then $f\neq 0$, and get $A=[1,1,0,0,0,1,0]$ ($A\wedge B_2$ corresponds to (257I)). And it can be easily shown that $A\wedge B_2$ is in the same orbit as (2.1.1a), so we omit it.

Subcase 2.2: $L\bigcap V_3\neq 0$. Or $B\in V_3$, or $d_1=0$ in B. Then $B=[0,b_1,c_1,0,e_1,f_1,0]$. One of $b_1,c_1,e_1,f_1\neq 0$, assume $f_1=1$ to get $B=[0,b_1,c_1,0,e_1,1,0]$ and $A=[1,b,c,d,e,0,0]$. Consider B, $a_1\to 0$; $b_1\to a_{11}(b_1a_{44}+c_1a_{54})+a_{21}(e_1a_{44}+a_{54})$; $c_1\to a_{11}(b_1a_{45}+c_1a_{55})+a_{21}(e_1a_{45}+a_{55})$; $d_1\to 0$; $e_1\to a_{12}(b_1a_{44}+c_1a_{54})+a_{22}(e_1a_{44}+a_{54})=0$; $f_1=1\to a_{12}(b_1a_{45}+c_1a_{55})+a_{22}(e_1a_{45}+a_{55})=1$; $g_1\to 0$.

Let $a_{21}=0$. When both $b_1=c_1=0$, we can easily get a representative for B: Subcase (2.2.1)

$$B_1=[0,0,0,0,0,1,0].$$

When one of $b_1,c_1\neq 0$, then if $b_1\neq e_1c_1$, we can always make $b_1=1$, $c_1=0$, $e_1=0$ and keep $f_1=1$ to get Subcase (2.2.2) $B_2=[0,1,0,0,0,1,0]$. If $b_1=e_1c_1$, then $b_1\to c_1a_{11}(e_1a_{44}+a_{54})$; $c_1\to c_1a_{11}(e_1a_{45}+a_{55})$; $e_1\to c_1a_{12}(e_1a_{44}+a_{54})+a_{22}(e_1a_{44}+a_{54})=0$; $f_1=1\to c_1a_{12}(e_1a_{45}+a_{55})+a_{22}(e_1a_{45}+a_{55})=1$.

Make $c_1 = 1$ and $b_1 = e_1 = 0$. Then make $f_1 = 1$ to get Subcase (2.2.3)

$$B_3 = [0,0,1,0,0,1,0].$$

Subcase 2.2.1: With $B_1 = [0,0,0,0,0,1,0]$, we assume $A = [1,b,c,d,e,0,0]$. To fix B_1, we need $a_{21} = a_{54} = 0$ and $a_{22}a_{55} = 1$. Consider A, $a = 1 \to a_{11}\delta$; $b \to a_{11}(a_{34} + ba_{44})$; $c \to a_{11}(a_{35} + ba_{45} + ca_{55})$; $d \to a_{12}\delta + da_{22}\delta$; $e \to a_{12}(a_{34} + ba_{44}) + a_{22}(da_{34} + ea_{44})$; $f = 0 \to a_{12}(a_{35} + ba_{45} + ca_{55}) + a_{22}(da_{35} + ea_{45})$ (By subtracting a multiple of B, we can always make $f = 0$); $g = 0 \to 0$.

We can make $b = c = d = 0$ by solving for a_{34}, a_{35} and a_{12} respectively. Now $e \neq 0$. So we make $e = 1$ and get subcase (2.2.1) $A = [1,0,0,0,1,0,0]$ ($A \wedge B_1$ corresponds to (257C)).

Subcase 2.2.2: With $B_2 = [0,1,0,0,0,1,0]$, we assume $A = [1,b,c,d,e,0,0]$. To fix B_2, we require

$$a_{11}a_{44} + a_{21}a_{54} = 1, \quad a_{11}a_{45} + a_{21}a_{55} = 0,$$
$$a_{12}a_{44} + a_{22}a_{54} = 0, \quad a_{12}a_{45} + a_{22}a_{55} = 1$$

or $a_{44} = a_{22}/\delta$, $a_{45} = -a_{21}/\delta$, $a_{54} = -a_{12}/\delta$, $a_{55} = a_{11}/\delta$.

We have $a = 1 \to a_{11}\delta + da_{21}\delta$; $b \to a_{11}(a_{34} + ba_{44} + ca_{54}) + a_{21}(da_{34} + ea_{44})$; $c \to a_{11}(a_{35} + ba_{45} + ca_{55}) + a_{21}(da_{35} + ea_{45})$; $d \to a_{12}\delta + da_{22}\delta$; $e \to a_{12}(a_{34} + ba_{44} + ca_{54}) + a_{22}(da_{34} + ea_{44})$; $f \to a_{12}(a_{35} + ba_{45} + ca_{55}) + a_{22}(da_{35} + ea_{45})$; $g \to 0$.

By taking $a_{21} = a_{45} = 0$, we can make $c = d = 0$ and $b = f$ by solving for a_{35}, a_{12} and a_{34} respectively. Then by subtracting a multiple of B from A, we can make $b = f = 0$. Now depending on $e = 0$ or not, we may obtain the following two representatives for A: Subcase (2.2.2a) $A_1 = [1,0,0,0,0,0,0]$ ($A_1 \wedge B_2$ corresponds to (257B)) and Subcase (2.2.2b) $A_2 = [1,0,0,0,1,0,0]$ ($A_2 \wedge B_2$ corresponds to (257D)).

Subcase 2.2.3: With $B_3 = [0,0,1,0,0,1,0]$, we assume $A = [1,b,c,d,e,0,0]$. To fix B_3, we require $a_{54} = 0$ and $(a_{11} + a_{21})a_{55} = 1$ and $(a_{12} + a_{22})a_{55} = 1$. Then for A, we have $a = 1 \to a_{11}\delta + da_{21}\delta$; $b \to a_{11}(a_{34} + ba_{44}) + a_{21}(da_{34} + ea_{44})$; $c \to a_{11}(a_{35} + ba_{45} + ca_{55}) + a_{21}(da_{35} + ea_{45})$; $d \to a_{12}\delta + da_{22}\delta = (a_{12} + da_{22})\delta$; $e \to a_{12}(a_{34} + ba_{44}) + a_{22}(da_{34} + ea_{44})$; $f \to a_{12}(a_{35} + ba_{45} + ca_{55}) + a_{22}(da_{35} + ea_{45})$; $g \to 0$.

If $d = 1$, then make $a = d = 1$. Taking $a_{21} = 0$, we can make $c = f$, by subtracting a multiple of B_3 from A, we have $c = f = 0$. Then we need $e \neq 0$ to get the desired representative: $A_1 = [1,0,0,1,1,0,0]$, $A_1 \wedge B_3$ correponding to (257A).

If $d \neq 1$, we can make $b = d = 0$ and $c = f$ by solving for a_{34}, a_{12} and a_{35}. Then we need $e \neq 0$ to get $A_2 = [1,0,0,0,1,0,0]$, with $A_2 \wedge B_3$ corresponding to (257C), hence it is in the same orbit as (2.2.1), we omit it.

Now we get all the possible representatives for the desired orbits, with the correponding algebras:

(1.1.1a) $\Delta_{45} \wedge (\Delta_{13} + \Delta_{24}) \to$ (257H);

(1.1.2a) $(\Delta_{45}+\Delta_{23})\wedge(\Delta_{13}+\Delta_{24}) \to$ (257L);

(1.1.2b) $(\Delta_{45}+\Delta_{23})\wedge\Delta_{13} \to$ (257K);

(1.2.1a) $(\Delta_{45}+\Delta_{13})\wedge\Delta_{24} \to$ (257E);

(1.2.1b) $(\Delta_{45}+\Delta_{23})\wedge\Delta_{24} \to$ (257F);

(1.2.2) $(\Delta_{45}+\Delta_{13})\wedge(\Delta_{15}+\Delta_{24}) \to$ (257G);

(2.1.1a) $(\Delta_{13}+\Delta_{14})\wedge(\Delta_{15}+\Delta_{23}) \to$ (257I);

(2.1.1b) $(\Delta_{13}+\Delta_{14}+\Delta_{25})\wedge(\Delta_{15}+\Delta_{23}) \to$ (257J_1);

(2.1.1c) $(\Delta_{13}+\Delta_{24})\wedge(\Delta_{15}+\Delta_{23}) \to$ (257J);

(2.2.1) $(\Delta_{13}+\Delta_{24})\wedge\Delta_{25} \to$ (257C);

(2.2.2a) $\Delta_{13}\wedge(\Delta_{14}+\Delta_{25}) \to$ (257B);

(2.2.2b) $(\Delta_{13}+\Delta_{24})\wedge(\Delta_{14}+\Delta_{25}) \to$ (257D);

(2.2.3a) $(\Delta_{13}+\Delta_{23}+\Delta_{24})\wedge(\Delta_{15}+\Delta_{25}) \to$ (257A);

To prove that all the algebras are dintinct, we consider the following four groups of algebras:

Group 1: (257A-D): $L\subset V_1$ and $L\bigcap V_3\neq 0$;

Group 2: (257I,J, J_1): $L\subset V_1$ and $L\bigcap V_3=0$;

Group 3: (257E,F,G): $L\not\subset V_1$, $L\bigcap V_3\neq 0$ and $L\not\subset V_2$;

Group 4: (257K,L): $L\not\subset V_1$, $L\bigcap V_3=0$ and $L\bigcap V_2=0$.

Group 5: (257H): $L\not\subset V_1$, $L\bigcap V_3=0$ and $L\bigcap V_2\neq 0$.

We just need to prove that all the algebras among the same group are distinct.

In Group 1, (257B) corresponds to $\Delta_{13}\wedge(\Delta_{14}+\Delta_{25})$, under the group action, it will be mapped to $(a_{11}\delta\Delta_{13}+a_{11}a_{34}\Delta_{14}+a_{11}a_{35}\Delta_{15}+a_{12}\delta\Delta_{23}+a_{12}a_{34}\Delta_{24}+a_{12}a_{35}\Delta_{25})\wedge((a_{11}a_{44}+a_{21}a_{54})\Delta_{14}+(a_{11}a_{45}+a_{21}a_{55})\Delta_{15}+(a_{12}a_{44}+a_{22}a_{54})\Delta_{24}+(a_{12}a_{45}+a_{22}a_{55})\Delta_{25})$, or

$$\begin{aligned}
&a_{11}\delta(a_{11}a_{44}+a_{21}a_{54})\Delta_{13}\wedge\Delta_{14}\\
&+a_{11}\delta(a_{11}a_{45}+a_{21}a_{55})\Delta_{13}\wedge\Delta_{15}\\
&+a_{11}\delta(a_{12}a_{44}+a_{22}a_{54})\Delta_{13}\wedge\Delta_{24}\\
&+a_{11}\delta(a_{12}a_{45}+a_{22}a_{55})\Delta_{13}\wedge\Delta_{25}\\
&+(a_{11}a_{34}(a_{11}a_{45}+a_{21}a_{55})-(a_{11}a_{44}+a_{21}a_{54})a_{11}a_{35})\Delta_{14}\wedge\Delta_{15}
\end{aligned}$$

$$
\begin{aligned}
&+(a_{11}a_{34}(a_{12}a_{44}+a_{22}a_{54})-(a_{11}a_{44}+a_{21}a_{54})a_{12}a_{34})\Delta_{14}\wedge\Delta_{24}\\
&+(a_{11}a_{34}(a_{12}a_{45}+a_{22}a_{55})-(a_{11}a_{44}+a_{21}a_{54})a_{12}a_{35})\Delta_{14}\wedge\Delta_{25}\\
&+(a_{11}a_{35}(a_{12}a_{44}+a_{22}a_{54})-a_{12}a_{34}(a_{11}a_{45}+a_{21}a_{55})\Delta_{15}\wedge\Delta_{24}\\
&+(a_{11}a_{35}(a_{12}a_{45}+a_{22}a_{55})-a_{12}a_{35}(a_{11}a_{45}+a_{21}a_{55})\Delta_{15}\wedge\Delta_{25}\\
&+a_{12}\delta(a_{11}a_{44}+a_{21}a_{54})\Delta_{23}\wedge\Delta_{14}\\
&+a_{12}\delta(a_{11}a_{45}+a_{21}a_{55})\Delta_{23}\wedge\Delta_{15}\\
&+a_{12}\delta(a_{12}a_{44}+a_{22}a_{54})\Delta_{23}\wedge\Delta_{24}\\
&+a_{12}\delta(a_{12}a_{45}+a_{22}a_{55})\Delta_{23}\wedge\Delta_{25}\\
&+(a_{12}a_{34}(a_{12}a_{45}+a_{22}a_{55})-(a_{12}a_{44}+a_{22}a_{54})a_{12}a_{35})\Delta_{24}\wedge\Delta_{25}.
\end{aligned}
$$

If (257B) could me mapped to (257A), then the coefficients of $\Delta_{13}\wedge\Delta_{14}$, $\Delta_{13}\wedge\Delta_{24}$, $\Delta_{23}\wedge\Delta_{14}$, $\Delta_{23}\wedge\Delta_{24}$ are zero, i.e.,

$$
\begin{aligned}
a_{11}\delta(a_{11}a_{44}+a_{21}a_{54})=0 \quad a_{11}\delta(a_{12}a_{44}+a_{22}a_{54})=0\\
a_{12}\delta(a_{11}a_{44}+a_{21}a_{54})=0 \quad a_{12}\delta(a_{12}a_{44}+a_{22}a_{54})=0.
\end{aligned}
$$

It is easy to see that there is no solution to the above system of equations, for otherwise the automorphism group is going to be singular. Hence (257A) and (257B) are not isomorphic.

Compare (257B) with (257C), exactly the same thing will happen. So they are not isomorphic.

Compare (257B) with (257D), the coefficients of $\Delta_{23}\wedge\Delta_{14}$, $\Delta_{23}\wedge\Delta_{15}$, $\Delta_{23}\wedge\Delta_{24}$, $\Delta_{23}\wedge\Delta_{25}$ will be zero, while that of $\Delta_{14}\wedge\Delta_{24}$ is not. It is obvious that a_{12} must be zero. As the coefficients of $\Delta_{13}\wedge\Delta_{14}$ and $\Delta_{13}\wedge\Delta_{25}$ are not zero, we have $a_{11}a_{44}+a_{21}a_{54}\neq 0$ and $a_{12}a_{45}+a_{22}a_{55}\neq 0$. As the coefficient of $\Delta_{14}\wedge\Delta_{25}$ equals 0, we will have $a_{34}=0$ since $a_{12}=0$ and $a_{11}\neq 0$. Similarly we can prove that $a_{35}=0$ by considering the coefficient of $\Delta_{15}\wedge\Delta_{24}$. The fact that both $a_{34}=a_{35}=0$ will make the coefficient of $\Delta_{14}\wedge\Delta_{24}$ to be zero, a contradiction. Therefore (257B) and (257D) are not isomorphic.

Now we need to check whether (257B) is decomposable or not. Take (1.1.1b), then $A=[0,0,0,0,0,0,1]$ and $B=[1,0,0,0,0,0,0]$. By simply looking at the coefficients of $A\wedge B$, it is obvious that (257B) and (1.1.1b) are not isomorphic, i.e., (257B) is indecomposable.

Similarly, we can prove that all the other algebras are distinct and indecomposable.

In order that the basis for (257A) will also diagonalize a maximal torus, we make the following basis transformation: $x_1\to x_1$, $x_2\to x_1+x_2$, $x_3\to x_3$, $x_4\to x_4$, $x_5\to x_5$, $x_6\to x_6$ and $x_7\to x_7$.

Therefore the corresponding central extensions of $N_{5,4}$ are:

(257A):	$[x_1, x_2] = x_3,$	$[x_1, x_3] = x_6,$	$[x_1, x_5] = x_7,$
	$[x_2, x_4] = x_6;$		
(257B):	$[x_1, x_2] = x_3,$	$[x_1, x_3] = x_6,$	
	$[x_1, x_4] = x_7,$	$[x_2, x_5] = x_7;$	
(257C):	$[x_1, x_2] = x_3,$	$[x_1, x_3] = x_6,$	
	$[x_2, x_4] = x_6,$	$[x_2, x_5] = x_7;$	
(257D):	$[x_1, x_2] = x_3,$	$[x_1, x_3] = x_6,$	$[x_1, x_4] = x_7,$
	$[x_2, x_4] = x_6,$	$[x_2, x_5] = x_7;$	
(257E):	$[x_1, x_2] = x_3,$	$[x_1, x_3] = x_6,$	
	$[x_2, x_4] = x_7,$	$[x_4, x_5] = x_6;$	
(257F):	$[x_1, x_2] = x_3,$	$[x_2, x_3] = x_6,$	
	$[x_2, x_4] = x_7,$	$[x_4, x_5] = x_6;$	
(257G):	$[x_1, x_2] = x_3,$	$[x_1, x_3] = x_6,$	$[x_1, x_5] = x_7,$
	$[x_2, x_4] = x_7,$	$[x_4, x_5] = x_6;$	
(257H):	$[x_1, x_2] = x_3,$	$[x_1, x_3] = x_6,$	
	$[x_2, x_4] = x_6,$	$[x_4, x_5] = x_7;$	
(257I):	$[x_1, x_2] = x_3,$	$[x_1, x_3] = x_6,$	$[x_1, x_4] = x_6,$
	$[x_1, x_5] = x_7,$	$[x_2, x_3] = x_7;$	
(257J):	$[x_1, x_2] = x_3,$	$[x_1, x_3] = x_6,$	$[x_1, x_5] = x_7,$
	$[x_2, x_3] = x_7,$	$[x_2, x_4] = x_6;$	
(257J$_1$):	$[x_1, x_2] = x_3,$	$[x_1, x_3] = x_6,$	$[x_1, x_4] = x_6,$
	$[x_1, x_5] = x_7,$	$[x_2, x_3] = x_7,$	$[x_2, x_5] = x_6;$
(257K):	$[x_1, x_2] = x_3,$	$[x_1, x_3] = x_6,$	
	$[x_2, x_3] = x_7,$	$[x_4, x_5] = x_7;$	
(257L):	$[x_1, x_2] = x_3,$	$[x_1, x_3] = x_6,$	$[x_2, x_3] = x_7,$
	$[x_2, x_4] = x_6,$	$[x_4, x_5] = x_7;$	

Remark: To get Seeley's presentations, (1) In (257B), by switching x_4 and x_5; (2) In (257F), by taking $x_1 \to -b$, $x_2 \to a+b$; (3) In (257L), by taking $x_1 \to b$, $x_2 \to a$, $x_3 \to -c$, $x_4 \to -d$, $x_5 \to e$, $x_6 \to -g$ and $x_7 \to -f$.

7.2 Extensions of 6-Dimensional Algebras

The central extensions of $N_{6,1,1}$ can be found in chapter 2, Example 4.

Central extensions of $N_{6,2,3}$:

$Z(\mathfrak{g})$: x_6; $[\mathfrak{g},\mathfrak{g}]$: x_4, x_5, x_6; $Z^2(\mathfrak{g})$: $C_{25} = C_{26} = C_{36} = C_{45} = C_{46} = C_{56} = 0$, $C_{15} + C_{34} = 0$, $C_{16} + C_{35}$;$W(H^2)$: $C_{12} = C_{14} = C_{15} = 0$; $\dim H^2$: 4; Basis: $\Delta_{13}, \Delta_{16} - \Delta_{35}, \Delta_{23}, \Delta_{24}$;

Group action: $a\Delta_{13} + b(\Delta_{16} - \Delta_{35}) + c\Delta_{23} + d\Delta_{24}$:

$a \to aa_{11}^3 + b(a_{11}a_{63} + a_{11}^2 a_{51}) + ca_{11}^2 a_{21} - da_{11}a_{21}^2$;

$b \to ba_{11}^4 a_{22}$;

$c \to ca_{11}^2 a_{22} - 2da_{11}a_{21}a_{22}$;

$d \to da_{11}a_{22}^2$;

Then we have $b \neq 0$. Make $a = 0$ by solving for a_{63}.

Case 1: $d = 0$. Then we obtain the representatives $[0,1,0,0]$ (when $c = 0$, corresponding to (12357A)), $[0,1,1,0]$ (when $c \neq 0$ and $bc > 0$,corresponding to (12357B)) and $[0,1,-1,0]$ (when $c \neq 0$ and $bc < 0$, corresponding to (12357B$_1$)).

Case 2: $d \neq 0$. Then make $c = 0$ by solving for a_{21} and get a representative $[0,1,0,1]$ (corresponding to (12357C));

Therefore the corresponding central extensions of $N_{6,2,3}$ are:

(12357A):	$[x_1,x_2] = x_4$,	$[x_1,x_i] = x_{i+1}$, $i = 4,5,6$,	$[x_2,x_3] = x_5$,
	$[x_3,x_4] = -x_6$,	$[x_3,x_5] = -x_7$;	
(12357B):	$[x_1,x_2] = x_4$,	$[x_1,x_i] = x_{i+1}$, $i = 4,5,6$,	$[x_2,x_3] = x_5 + x_7$,
	$[x_3,x_4] = -x_6$,	$[x_3,x_5] = -x_7$;	
(12357B$_1$):	$[x_1,x_2] = x_4$,	$[x_1,x_i] = x_{i+1}$, $i = 4,5,6$,	$[x_2,x_3] = x_5 - x_7$,
	$[x_3,x_4] = -x_6$,	$[x_3,x_5] = -x_7$;	
(12357C):	$[x_1,x_2] = x_4$,	$[x_1,x_i] = x_{i+1}$, $i = 4,5,6$,	$[x_2,x_3] = x_5$,
	$[x_2,x_4] = x_7$,	$[x_3,x_4] = -x_6$,	$[x_3,x_5] = -x_7$;

Central extensions of $N_{6,2,5}$:

$Z(\mathfrak{g})$: x_6; $[\mathfrak{g},\mathfrak{g}]$: x_3, x_4, x_5, x_6; $Z^2(\mathfrak{g})$: $C_{36} = C_{45} = C_{46} = C_{56} = 0$, $C_{15} - C_{24} = 0$, $C_{16} - C_{34}$, $C_{26} + C_{35} = 0$; $W(H^2)$: $C_{12} = C_{13} = C_{15} = C_{23} = 0$; $\dim H^2$: 4; Basis: $\Delta_{14}, \Delta_{16} + \Delta_{34}, \Delta_{25}, \Delta_{26} - \Delta_{35}$;

Group action: $a\Delta_{14} + b(\Delta_{16} + \Delta_{34}) + c\Delta_{25} + d(\Delta_{26} - \Delta_{35})$;

Notice that the automorphism group of $N_{6,2,5}$ has two components, we have respectively

(1): $a \to aa_{11}^3 a_{22}$; $b \to ba_{11}^3 a_{22}^2$; $c \to ca_{11}a_{22}^3$; $d \to da_{11}^2 a_{22}^3$;

and

(2): $a \longleftrightarrow -c$ and $b \longleftrightarrow -d$ simultaneously.

One of $b, d \neq 0$. Because of (2), we may always assume that $b \neq 0$.

Case 1: $d = 0$. We can get five representatives $[0,1,0,0]$ (when $a = c = 0$) (corresponding to (12457H)), $[0,1,1,0]$ when ($a = 0, c \neq 0$) (corresponding to (12457I)), $[1,1,0,0]$ (when $a \neq 0, c = 0$) (corresponding to (12457K)), $[1,1,1,0]$ (when $ac > 0$) (corresponding to (12457J)) and $[1,1,-1,0]$ (when $ac < 0$) (corresponding to (12457J$_1$));

Case 2: $d \neq 0$. We can get representatives $[0,1,0,1]$ (when $a = c = 0$) (corresponding to (12457L)), $[1,1,0,1]$ (when one of $a, c \neq 0$, we may assume $a \neq 0$ using (2)) (corresponding to (12457M)); And when both $a \neq 0, c \neq 0$, we can get a one parameter representative $[1,1,\lambda,1]$ (corresponding to (12457N)) for any $\lambda \neq 0$. Combining (1) and (2), we can show that $[1,1,\lambda^{-1},1]$ is in the same orbit. So we may introduce the invariant $I(\lambda) := \lambda + \lambda^{-1}$ for this representative.

Therefore the corresponding central extensions of $N_{6,2,5}$ are:

(12457H):	$[x_1, x_i] = x_{i+1},\ i = 2,3,5,6$	$[x_2, x_j] = x_{j+2}, j = 3,4,$	$[x_3, x_4] = x_7;$
(12457I):	$[x_1, x_i] = x_{i+1},\ i = 2,3,5,6$	$[x_2, x_j] = x_{j+2}, j = 3,4,$	$[x_2, x_5] = x_7,$
	$[x_3, x_4] = x_7;$		
(12457J):	$[x_1, x_i] = x_{i+1},\ i = 2,3,5,6$	$[x_1, x_4] = x_7,$	$[x_2, x_3] = x_5,$
	$[x_2, x_4] = x_6,$	$[x_2, x_5] = x_7,$	$[x_3, x_4] = x_7;$
(12457J$_1$):	$[x_1, x_i] = x_{i+1},\ i = 2,3,5,6$	$[x_1, x_4] = x_7,$	$[x_2, x_3] = x_5,$
	$[x_2, x_4] = x_6,$	$[x_2, x_5] = -x_7,$	$[x_3, x_4] = x_7;$
(12457K):	$[x_1, x_i] = x_{i+1},\ i = 2,3,5,6$	$[x_1, x_4] = x_7,$	$[x_2, x_3] = x_5,$
	$[x_2, x_4] = x_6,$	$[x_3, x_4] = x_7;$	
(12457L):	$[x_1, x_i] = x_{i+1},\ i = 2,3,5,6$	$[x_2, x_j] = x_{j+2}, j = 3,4,$	$[x_2, x_6] = x_7,$
	$[x_3, x_4] = x_7,$	$[x_3, x_5] = -x_7;$	
(12457N):	One parameter family, with invariant $I(\lambda) = \lambda + \lambda^{-1}$		
	$[x_1, x_i] = x_{i+1},\ i = 2,3,5,6$	$[x_1, x_4] = x_7,$	$[x_2, x_3] = x_5,$
	$[x_2, x_4] = x_6,$	$[x_2, x_5] = \lambda x_7,$	$[x_2, x_6] = x_7,$
	$[x_3, x_4] = x_7,$	$[x_3, x_5] = -x_7;$	
	(12457M) in Seeley's list is a special case of (12457N) with $\lambda = 0$.		

Central extensions of $N_{6,2,9}$:

$Z(\mathfrak{g})$: x_5, x_6; $[\mathfrak{g},\mathfrak{g}]$: $x_3,\ x_5,\ x_6$; $Z^2(\mathfrak{g})$: $C_{35} = C_{36} = C_{45} = C_{46} = C_{56} = 0,\ C_{16} = C_{25} = C_{34}$; $W(H^2)$: $C_{12} = C_{13} = C_{23} = 0$; $\dim H^2$: 5; Basis: $\Delta_{14}, \Delta_{15}, \Delta_{16} + \Delta_{25} + \Delta_{34}, \Delta_{24}, \Delta_{26}$;

Group action: $a\Delta_{14} + b\Delta_{15} + c(\Delta_{16} + \Delta_{25} + \Delta_{34}) + d\Delta_{24} + e\Delta_{26}$;

As the automorphism group of $N_{6,2,9}$ has two components, we have

(1):

$e \to ba_{11}a_{22}a_{12}^2 + 2ca_{11}a_{12}a_{22}^2 + ea_{11}a_{22}^3$;

One of $\{b, c\}$ and one of $\{c, e\}$ are nonzero.

Case 1: $b \neq 0$. Make $a = c = d = 0$ by solving for a_{31}, a_{12}, and a_{41} to get the representatives $[0, 1, 0, 0, 1]$ (when $be > 0$), corresponding to (1357P) and $[0, 1, 0, 0, -1]$ (when $be < 0$), corrsponding to (1357P$_1$).

Case 2: $b = 0$. Then $c \neq 0$. Make $a = d = e = 0$ by solving for a_{64}, a_{54}, a_{12} to get the representative $[0, 0, 1, 0, 0]$, corresponding to (1357O).

Therefore the corresponding central extensions of $N_{6,2,10}$ are:

(1357O):	$[x_1, x_2] = x_3$,	$[x_1, x_3] = x_5$,	$[x_1, x_6] = x_7$,
	$[x_2, x_3] = x_6$,	$[x_2, x_4] = x_5$,	$[x_2, x_5] = x_7$;
(1357P):	$[x_1, x_2] = x_3$,	$[x_1, x_i] = x_{i+2}, i = 3, 5$,	$[x_2, x_3] = x_6$,
	$[x_2, x_4] = x_5$,	$[x_2, x_6] = x_7$,	$[x_3, x_4] = x_7$;
(1357P$_1$):	$[x_1, x_2] = x_3$,	$[x_1, x_i] = x_{i+2}, i = 3, 5$,	$[x_2, x_3] = x_6$,
	$[x_2, x_4] = x_5$,	$[x_2, x_6] = -x_7$,	$[x_3, x_4] = x_7$;

Central extensions of $N_{6,3,4}$:

$Z(\mathfrak{g})$: x_5, x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_5, x_6; $Z^2(\mathfrak{g})$: $C_{15} = C_{35} = C_{36} = C_{45} = C_{56} = 0, C_{16} - C_{34} = 0$; $W(H^2)$: $C_{12} = C_{23} = C_{24} = 0$; $\dim H^2$: 6; Basis:$\Delta_{13}, \Delta_{14}, \Delta_{16} + \Delta_{34}, \Delta_{25}, \Delta_{26}, \Delta_{46}$;

Group action: $a\Delta_{13} + b\Delta_{14} + c(\Delta_{16} + \Delta_{34}) + d\Delta_{25} + e\Delta_{26} + f\Delta_{46}$;

$a \to aa_{11}^2a_{22} - 2ca_{11}a_{22}a_{41} - fa_{22}a_{41}^2$;

$b \to aa_{11}a_{34} + ba_{11}a_{44} + c(a_{11}a_{64} + a_{31}a_{44} - a_{34}a_{41}) + f(a_{41}a_{64} - a_{44}a_{61})$;

$c \to ca_{11}a_{22}a_{44} + fa_{22}a_{41}a_{44}$;

$d \to da_{11}a_{22}^3$;

$e \to ca_{12}a_{22}a_{44} + da_{22}^2a_{34} + ea_{22}^2a_{44} + fa_{22}a_{42}a_{44}$;

$f \to fa_{22}a_{44}^2$;

One of $\{c, f\}$ is nonzero, and also $d \neq 0$. Can always make $e = 0$.

Case 1: $f = 0$. So $c \neq 0$ and make $a = b = 0$ to get a representative $[0, 0, 1, 1, 0, 0]$, corresponding to (1357D).

Case 2: $f \neq 0$. Make $b = c = 0$ and get the representatives $[0, 0, 0, 1, 0, 1]$ (when $a = 0$, corresponding to (1357E)), $[1, 0, 0, 1, 0, 1]$ (when $a \neq 0$, and $af > 0$, corresponding to (1357F$_1$)) and $[1, 0, 0, 1, 0, -1]$ (when $a \neq 0$, and $af < 0$, corresponding to (1357F)).

Therefore the corresponding central extensions of $N_{6,3,4}$ are:

(1357D):	$[x_1,x_2]=x_3$,	$[x_1,x_6]=x_7$,	$[x_2,x_i]=x_{i+2}, i=3,4$,
	$[x_2,x_5]=x_7$,	$[x_3,x_4]=x_7$;	
(1357E):	$[x_1,x_2]=x_3$,	$[x_2,x_i]=x_{i+2}, i=3,4$,	
	$[x_2,x_5]=x_7$,	$[x_4,x_6]=x_7$;	
(1357F):	$[x_1,x_2]=x_3$,	$[x_1,x_3]=x_7$,	$[x_2,x_i]=x_{i+2}, i=3,4$,
	$[x_2,x_5]=x_7$,	$[x_4,x_6]=-x_7$;	
($1357F_1$):	$[x_1,x_2]=x_3$,	$[x_1,x_3]=x_7$,	$[x_2,x_i]=x_{i+2}, i=3,4$,
	$[x_2,x_5]=x_7$,	$[x_4,x_6]=x_7$;	

Central extensions of $N_{6,4,2}$:

$Z(\mathfrak{g})$: x_4, x_5, x_6; $[\mathfrak{g},\mathfrak{g}]$: x_4, x_5; $W(H^2)$: $C_{12} = C_{13} = 0$; $Z^2(\mathfrak{g})$: $C_{45} = C_{46} = C_{56} = 0, C_{25} - C_{34} = 0$; dim H^2: 9; Basis: $\Delta_{14}, \Delta_{15}, \Delta_{16}, \Delta_{23}, \Delta_{24}, \Delta_{25}+\Delta_{34}, \Delta_{26}, \Delta_{35}, \Delta_{36}$;

Group action: $a\Delta_{14} + b\Delta_{15} + c\Delta_{16} + d\Delta_{23} + e\Delta_{24} + f(\Delta_{25}+\Delta_{34}) + g\Delta_{26} + h\Delta_{35} + i\Delta_{36}$;

$a \to aa_{11}^2a_{22} + ba_{11}^2a_{32} + ea_{11}a_{22}a_{21} + fa_{11}a_{21}a_{32} + fa_{11}a_{22}a_{31} + ha_{11}a_{31}a_{32}$;

$b \to aa_{11}^2a_{23} + ba_{11}^2a_{33} + ea_{11}a_{21}a_{23} + fa_{11}a_{21}a_{33} + fa_{11}a_{31}a_{23} + ha_{11}a_{31}a_{33}$;

$c \to aa_{11}a_{46} + ba_{11}a_{56} + ca_{11}a_{66} + ea_{21}a_{46} + f(a_{21}a_{56} + a_{31}a_{46}) + ga_{21}a_{66} + ha_{31}a_{56} + ia_{31}a_{66}$;

$d \to d(a_{22}a_{33} - a_{32}a_{23}) + e(a_{22}a_{43} - a_{42}a_{23}) + f(a_{22}a_{53} + a_{32}a_{43} + a_{32}a_{43} - a_{42}a_{33} - a_{52}a_{23}) + g(a_{22}a_{63} - a_{62}a_{23}) + h(a_{32}a_{53} - a_{52}a_{33}) + i(a_{32}a_{63} - a_{62}a_{33})$;

$e \to ea_{11}a_{22}^2 + 2fa_{11}a_{22}a_{32} + ha_{11}a_{32}^2$;

$f \to ea_{11}a_{22}a_{23} + f(a_{11}a_{22}a_{33} + a_{11}a_{32}a_{23}) + ha_{11}a_{32}a_{33}$;

$g \to ea_{22}a_{46} + f(a_{22}a_{56} + a_{32}a_{46}) + ga_{22}a_{66} + ha_{32}a_{56} + ia_{32}a_{66}$;

$h \to ea_{11}a_{23}^2 + 2fa_{11}a_{23}a_{33} + ha_{11}a_{33}^2$;

$i \to ea_{23}a_{46} + f(a_{23}a_{56} + a_{33}a_{46}) + ga_{23}a_{66} + ha_{33}a_{56} + ia_{33}a_{66}$;

In each of the sets $\{a,e,f\}, \{b,f,h\}, \{c,g,i\}$, at least one element must be nonzero.

One of $e, f, h \neq 0$, for otherwise any element in the orbit will have none trivial kernel in the center of $N_{6,4,2}$.

Case 1: $f^2 - eh \neq 0$.

Subcase 1.1: $f^2 - eh > 0$. We may assume that $f \neq 0$, for otherwise $eh \neq 0$, then we can make $f \neq 0$ by using the group action. If $eh = 0$, then it is easy to make $e = h = 0$ by solving for a_{23} and a_{32}. And if $eh \neq 0$, then make $h = e = 0$ by solving for a_{23}, a_{32}, say

$$a_{23} = a_{33}\frac{-f+\sqrt{f^2-eh}}{e} \text{ and } a_{32} = a_{22}\frac{-f+\sqrt{f^2-eh}}{h}.$$

we can ensure that $a_{22}a_{33} - a_{23}a_{32} \neq 0$, i.e., the nonsingularity of the automorphism group.

Let $a_{23} = a_{32} = 0$, then make $a = b = d = g = i = 0$ by solving for $a_{31}, a_{21}, a_{53}, a_{56}, a_{46}$ respectively. So $c \neq 0$. Now we have $a = 0 \to fa_{11}a_{21}a_{32} + fa_{11}a_{22}a_{31}$; $b = 0 \to fa_{11}a_{21}a_{33} +$

$fa_{11}a_{31}a_{23}$; $c \to ca_{11}a_{66} + f(a_{21}a_{56} + a_{31}a_{46})$; $d = 0 \to f(a_{22}a_{53} + a_{32}a_{43} + a_{32}a_{43} - a_{42}a_{33} - a_{52}a_{23})$; $e = 0 \to 2fa_{11}a_{22}a_{32}$; $f \to f(a_{11}a_{22}a_{33} + a_{11}a_{32}a_{23})$; $g = 0 \to f(a_{22}a_{56} + a_{32}a_{46})$; $h = 0 \to 2fa_{11}a_{23}a_{33}$; $i = 0 \to f(a_{23}a_{56} + a_{33}a_{46})$.

Take $a_{23} = a_{32} = a_{31} = a_{21} = a_{46} = a_{53} = a_{56} = 0$ and solve for a_{11} and a_{66}, we can get the representative $[0,0,1,0,0,1,0,0,0]$, corresponding to (147A).

Subcase 1.2: $f^2 - eh < 0$. Then $eh > 0$, and we cannot make either e or $h = 0$. We may assume $f = 0$, for example, let $a_{23} = 0$ and solve for a_{32} will make $f = 0$.

Then take $a_{23} = a_{32} = 0$, we may further make $a = b = d = g = i = 0$ by solving for a_{21}, a_{31}, a_{53}, a_{46} and a_{56} respectively. Then we are left with $c \to ca_{11}a_{66}$; $e \to ea_{11}a_{22}^2$; $h \to ha_{11}a_{33}^2$.

Then since $eh > 0$, we can get the representative $[0,0,1,0,1,0,0,1,0]$, corresponding to ($147A_1$).

Case 2: $f^2 - eh = 0$. Then one of $e, h \neq 0$. Assume that $h \neq 0$, then make $e = 0$, which will automatically result in $f = 0$. Now $a \neq 0$. Make $b = c = d = i = 0$ by solving for a_{23}, a_{46}, a_{52} and a_{56} respectively. Then $g \neq 0$, and get a representative $[1,0,0,0,0,0,1,1,0]$, corresponding to (147B).

Therefore the central extensions of $N_{6,4,2}$ are:

(147A):	$[x_1, x_2] = x_4$,	$[x_1, x_3] = x_5$,	$[x_1, x_6] = x_7$,
	$[x_2, x_5] = x_7$,	$[x_3, x_4] = x_7$;	
($147A_1$):	$[x_1, x_2] = x_4$,	$[x_1, x_3] = x_5$,	$[x_1, x_6] = x_7$,
	$[x_2, x_4] = x_7$,	$[x_3, x_5] = x_7$;	
(147B):	$[x_1, x_2] = x_4$,	$[x_1, x_3] = x_5$,	$[x_1, x_4] = x_7$,
	$[x_2, x_6] = x_7$,	$[x_3, x_5] = x_7$;	

7.3 Extensions of $N_{6,3,6}$

The discussion is the same as that of complex case. And we also have Theorem 6.1 as in Chapter 6. The difference arises only when we consider the number of orbits in $(E_0, \square)$. In the real case we will have 4 orbits instead, as follows:

When all the three eigenvalues are real, we have

(i)

$$\begin{bmatrix} \xi & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & -\xi-\eta \end{bmatrix}.$$

(ii)

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(iii)

$$\begin{bmatrix} \xi & 1 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & -2\xi \end{bmatrix}.$$

When there are nonreal eigenvalues, assume they are ξ and $\bar{\xi}$, then it must be in the same orbit as

$$\begin{bmatrix} 0 & -|\xi|^2 & 0 \\ 1 & 2\mathrm{Re}\xi & 0 \\ 0 & 0 & -2\mathrm{Re}\xi \end{bmatrix};$$

which can be replaced by (because $\mathrm{Re}\xi \neq 0$)

(iv)

$$\begin{bmatrix} 0 & -\xi & 0 \\ \xi & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

with $\xi > 1$.

To find the corresponding elements in $H^2(\mathfrak{g}, \mathbf{R})$ for (i)–(iv), we may use the same argument as in the algebraically closed case.

For (i), let $\Psi = (b \wedge c) \otimes a + \eta(c \wedge a) \otimes b + \zeta(a \wedge b) \otimes c \in V^* \otimes (\wedge^2 V)^*$. Then

$$\varepsilon^t \circ \Phi(T)(v) = \Psi(v),$$

or

$$\varepsilon^t \circ \Phi(T) = (b \wedge c) \otimes a + \eta(c \wedge a) \otimes b + \zeta(a \wedge b) \otimes c,$$

and

$$\Psi(a, b\wedge c)=\xi, \Psi(b, c\wedge a)=\eta, \Psi(c, a\wedge b)=\zeta=-\xi-\eta,$$

with all the other combinations are zero, which in turn will give us the algebra

$$(147E):\quad \begin{array}{ll} [a,b]=d, & [b,c]=e, \\ [c,a]=f, & [a,e]=\xi g, \\ [b,f]=\eta g, & [c,d]=(-\xi-\eta)g. \end{array}$$

Or

$$(147E):\quad \begin{array}{ll} [a,b]=d, & [b,c]=e, \\ [a,c]=-f, & [a,e]=-g, \\ [b,f]=\lambda g, & [c,d]=(1-\lambda)g, \end{array}$$

with the invariant $I(\lambda)=-\frac{e_2^3}{e_3^2}=\frac{(1-\lambda+\lambda^2)^3}{\lambda^2(\lambda-1)^2}$ and $\lambda\neq 0,1$ as in the complex case.

It is obvious that (147C) is just a special case of (147E), by letting $\lambda=1/2$.

In (ii), it is easy to see that the corresponding cocycle will contain a nonzero element of $Z(\mathfrak{g})$ in its kernel. So we just omit it.

In (iii), when $\xi=0$, the corresponding cocycle will contain a nonzero element of $Z(\mathfrak{g})$ in its kernel. And when $\xi\neq 0$, we have

$$\begin{bmatrix} \xi & 1 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & -2\xi \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Its corresponding cocycle is

$$\Psi=(b\wedge c)\otimes(a+b)+(c\wedge a)\otimes b-2(a\wedge b)\otimes c.$$

And it is trivial to check that

$$\begin{array}{lll} \Psi(a, a\wedge b)=0, & \Psi(a, b\wedge c)=1, & \Psi(a, c\wedge a)=1, \\ \Psi(b, a\wedge b)=0, & \Psi(b, b\wedge c)=0, & \Psi(b, c\wedge a)=1, \\ \Psi(c, a\wedge b)=-2, & \Psi(c, b\wedge c)=0, & \Psi(c, c\wedge a)=0. \end{array}$$

And its corresponding algebra is

$$(1):\quad \begin{array}{lll} [a,b]=d, & [b,c]=e, & [a,c]=-f, \\ [a,e]=g, & [a,f]=g, & [b,f]=g, \\ [c,d]=-2g. & & \end{array}$$

which is isomorphic to (147D) of Seeley's paper, an isomorphism from (1) to (147D) can be given as: $a \to 1/2c$, $b \to b$, $c \to a$, $d \to -1/2e$, $e \to -d$, $f \to -1/2f$ and $g \to -1/4g$.

In (iv), its corresponding cocyle is

$$\Psi = (b \wedge c) \otimes (-\xi a) + (c \wedge a) \otimes (\xi a + 2b) + (a \wedge b) \otimes (-2c).$$

And it is easy to check that

$$\begin{array}{lll} \Psi(a, a \wedge b) = 0, & \Psi(a, b \wedge c) = 0, & \Psi(a, c \wedge a) = -\xi, \\ \Psi(b, a \wedge b) = 0, & \Psi(b, b \wedge c) = \xi, & \Psi(b, c \wedge a) = 2, \\ \Psi(c, a \wedge b) = -2, & \Psi(c, b \wedge c) = 0, & \Psi(c, c \wedge a) = 0. \end{array}$$

And its corresponding algebra is

$$\begin{array}{lll} [a, b] = d, & [b, c] = e, & [a, c] = -f, \\ [a, f] = -\xi g, & [b, e] = \xi g & [b, f] = 2g, \\ [c, d] = -2g. & & \end{array}$$

with $\xi > 1$, and corresponds to (147E$_1$).

Therefore the central extensions of $N_{6,3,6}$ of dimension 7 are:

(147D):		
	$[a, b] = d,$	$[a, c] = -f,$
	$[a, e] = g,$	$[a, f] = g,$
	$[b, c] = e,$	$[b, f] = g,$
	$[c, d] = -2g.$	
(147E):	$I(\lambda) = \frac{(1-\lambda+\lambda^2)^3}{\lambda^2(\lambda-1)^2}$, $\lambda \neq 0, 1$	($\lambda = 1/2$ gives (147C))
	$[a, b] = d,$	$[a, c] = -f,$
	$[a, e] = -g,$	$[b, c] = e,$
	$[b, f] = \lambda g,$	$[c, d] = (1 - \lambda)g.$
(147E$_1$):	$(\lambda > 1)$	
	$[a, b] = d,$	$[a, c] = -f,$
	$[a, f] = -\lambda g,$	$[b, c] = e,$
	$[b, e] = \lambda g,$	$[b, f] = 2g,$
	$[c, d] = -2g.$	

7.4 Four More Real Algebras and Their Extensions

7.4.1 The Four Algebras

In the real field $\mathbf{R}$, apart from all the algebras already listed over $\mathbf{C}$, we have 4 more algebras, which we will list in the following, with their corresponding automorphism groups. The notation L_a means that, as a Lie algebras over $\mathbf{R}$, L_a and L are nonisomorphic algebras, but are isomorphic over the complex field $\mathbf{C}$.

Our real 6-dimensional list is based on Nielson's list, and the correpondence between these two lists can be found in Appendix A.

$N_{6,2,5a}$: $[x_1, x_i] = x_{i+1}$, $i = 2, 3$, $[x_1, x_4] = -x_6$, $[x_2, x_3] = x_5$, $[x_2, x_5] = -x_6$.

— $(1, 3, 4, 6/6, 4, 3, 1)$;

— CQ: $N_{5,2,3}$;

— Aut $N_{6,2,5a}$:

$$\mathrm{Aut}_0 : \begin{bmatrix} a_{11} & -a_{12} & 0 & 0 & 0 & 0 \\ a_{12} & a_{11} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & \tau & 0 & 0 & 0 \\ a_{41} & a_{42} & x & a_{11}\tau & -a_{12}\tau & 0 \\ a_{51} & a_{52} & y & a_{12}\tau & a_{11}\tau & 0 \\ a_{61} & a_{62} & u & v & w & \tau^2 \end{bmatrix},$$

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus [-1] \oplus \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \oplus [-1],$$

where $\tau = a_{11}^2 + a_{12}^2$, $x = a_{11}a_{32} + a_{12}a_{31}$, $y = a_{12}a_{32} - a_{11}a_{31}$, $u = -a_{11}a_{42} - a_{12}a_{52} - a_{12}a_{41} + a_{11}a_{51}$, $v = -a_{11}^2 a_{32} - a_{12}^2 a_{32}$, $w = a_{11}^2 a_{31} + a_{12}^2 a_{31}$.

$N_{6,2,9a}$: $[x_1, x_2] = x_3$, $[x_1, x_i] = x_{i+2}$, $i = 3, 4$, $[x_2, x_3] = -x_6$, $[x_2, x_4] = x_5$;

— $(2, 4, 6/6, 3, 2)$;

— CQ: $N_{4,3} = N_{3,2} \times a_1$;

— Aut $N_{6,2,9a}$:

$$\mathrm{Aut}_0 : \begin{bmatrix} a_{11} & -a_{12} & 0 & 0 & 0 & 0 \\ a_{12} & a_{11} & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & \tau & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & \tau & 0 & 0 \\ a_{51} & a_{52} & x & a_{54} & a_{11}\tau & a_{12}\tau \\ a_{61} & a_{62} & u & a_{64} & -a_{12}\tau & a_{11}\tau \end{bmatrix},$$

and

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus [-1] \oplus [1] \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

with $\tau = a_{11}^2 + a_{12}^2$, $x = a_{11}a_{32} + a_{12}a_{42} + a_{12}a_{31} - a_{11}a_{41}$, $u = a_{11}a_{42} - a_{12}a_{32} + a_{12}a_{41} + a_{11}a_{31}$.

$N_{6,3,1a}$: $[x_1, x_i] = x_{i+2}$, $i = 2, 3$, $[x_2, x_4] = [x_3, x_5] = x_6$;

— $(1, 3, 6/6, 3, 1)$;

— CQ: $N_{5,3,2}$;

— Aut $N_{6,3,1a}$:

$$\mathrm{Aut}_0 : \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & a & -b & 0 & 0 & 0 \\ 0 & b & a & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{11}a & -a_{11}b & 0 \\ a_{51} & x & y & a_{11}b & a_{11}a & 0 \\ a_{61} & a_{62} & a_{63} & u & v & a_{11}\, a_{22}{}^2 \end{bmatrix},$$

with $a = a_{22}\cos\theta$, $b = a_{22}\sin\theta$, $x = a_{43} + a_{53}\sin\theta$, $y = -a_{42} + a_{53}\cos\theta$, $u = -a_{22}a_{41}\cos\theta - a_{22}a_{51}\sin\theta, a_{11}$, $v = a_{22}\sin\theta - a_{22}a_{51}\cos\theta$.

$N_{6,4,4a}$: $[x_1, x_3] = [x_2, x_4] = x_5$, $[x_1, x_4] = -[x_2, x_3] = x_6$.

— Aut $N_{6,4,4a}$:

$$\mathrm{Aut}_0 : \begin{bmatrix} a_{22} & a_{12} & 0 & 0 & 0 & 0 \\ -a_{12} & a_{22} & a_{23} & -a_{13} & 0 & 0 \\ a_{31} & a_{32} & 0 & -a_{43} & 0 & 0 \\ a_{32} & -a_{31} & a_{43} & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & u & v \\ a_{61} & a_{62} & a_{63} & a_{64} & x & y \end{bmatrix},$$

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus [1] \oplus [-1] \oplus \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

where $u = -a_{12}a_{43} - a_{13}a_{31} - a_{23}a_{32}$, $v = -a_{22}a_{43} + a_{13}a_{32} - a_{23}a_{31}$, $x = a_{22}a_{43} + a_{23}a_{31} - a_{13}a_{32}$, and $y = -a_{12}a_{43} - a_{13}a_{31} - a_{23}a_{32}$.

7.4.2 The Extensions

Central extensions of $N_{6,2,5a}$:

$Z(\mathfrak{g})$: x_6; $[\mathfrak{g},\mathfrak{g}]$: x_3, x_4, x_5, x_6; $Z^2(\mathfrak{g})$: $C_{36} = C_{45} = C_{46} = C_{56} = 0$, $C_{16} + C_{35} = 0$, $C_{34} = C_{26}$, $C_{15} = C_{24}$; $W(H^2)$: $C_{12} = C_{13} = C_{14} = C_{23} = 0$; $\dim H^2$: 4; Basis: $\Delta_{15} + \Delta_{24}$, $\Delta_{16} - \Delta_{35}$, Δ_{25}, $\Delta_{26} + \Delta_{34}$.

Group action: $a(\Delta_{15} + \Delta_{24}) + b(\Delta_{16} - \Delta_{35}) + c\Delta_{25} + d(\Delta_{26} + \Delta_{34})$:

(1):

$a \to a(a_{11}^4 - a_{12}^4) + ca_{11}a_{12}(a_{11}^2 + a_{12}^2)$;

$b \to ba_{11}(a_{11}^2 + a_{12}^2)^2 + da_{12}(a_{11}^2 + a_{12}^2)^2$;

$c \to -4aa_{11}a_{12}(a_{11}^2 + a_{12}^2) + c(a_{11}^4 - a_{12}^4)$;

$d \to -ba_{12}(a_{11}^2 + a_{12}^2)^2 + da_{11}(a_{11}^2 + a_{12}^2)^2$;

(2): $a \to -a$, $b \to -d$, $c \to c$, $d \to -b$;

One of $b, d \neq 0$. Make $b = 1$ and $d = 0$ to get $A = [a, 1, c, 0]$. Set $a_{12} = 0$, then we have $a \to aa_{11}^4$; $b = 1 \to a_{11}^5$; $c \to ca_{11}^4$; $d = 0 \to 0$.

Case 1: $c \neq 0$. When $a = 0$, we get a representative for A: $A = [0, 1, 1, 0]$, corresponding to ($12457N_1$) (the reason we use this notation is because it is isomorphic to ($12457N, \lambda = 1$) over C). When $a \neq 0$, then we get $A = [aa_{11}^4, a_{11}^5, ca_{11}^4, 0]$. Now it is easy to see that we get a parameter λ in A: $[1, 1, \lambda, 0]$ for $\lambda \neq 0$, corresponding to ($12457N_2$). By the group action (2), we may change A to $[-1, 0, \lambda, -1]$, and by (1) again and letting $a_{11} = 0$, we would get $[a_{12}^2, -a_{12}^3, -a_{12}^2\lambda, 0]$, which is in the same orbit as $[1, 1, -\lambda, 0]$ if we take $a_{12} = -1$. Therefore an invariant for this parameter could be chosen as $K(\lambda) = |\lambda|$.

Case 2: $c = 0$. Now depending on whether $a = 0$ or not, we get two representatives for A: $A = [0, 1, 0, 0]$, corresponding to ($12457L_1$), and $A = [1, 1, 0, 0]$, which can be included in ($12457N_2$) as a special case by choosing $\lambda = 0$.

Therefore the central extensions of $N_{6,2,5a}$ are:

(12457L$_1$):	$[x_1, x_i] = x_{i+1},\ i = 2, 3,$	$[x_1, x_4] = -x_6,$
	$[x_1, x_6] = x_7$	$[x_2, x_3] = x_5,$
	$[x_2, x_5] = -x_6,$	$[x_3, x_5] = -x_7;$
(12457N$_1$):	$[x_1, x_i] = x_{i+1},\ i = 2, 3,$	$[x_1, x_4] = -x_6,$
	$[x_1, x_6] = x_7,$	$[x_2, x_3] = x_5,$
	$[x_2, x_5] = -x_6 + x_7,$	$[x_3, x_5] = -x_7;$
(12457N$_2$):	One parameter family, with invariant $K(\lambda) = \lvert\lambda\rvert$	
	$[x_1, x_i] = x_{i+1},\ i = 2, 3,$	$[x_1, x_4] = -x_6,$
	$[x_1, x_5] = x_7,$	$[x_1, x_6] = x_7,$
	$[x_2, x_3] = x_5,$	$[x_2, x_4] = x_7,$
	$[x_2, x_5] = -x_6 + \lambda x_7,$	$[x_3, x_5] = -x_7;$

Central extensions of $N_{6,2,9a}$:

$Z(\mathfrak{g})$: x_5, x_6; $[\mathfrak{g}, \mathfrak{g}]$: x_3, x_5, x_6; $Z^2(\mathfrak{g})$: $C_{35} = C_{36} = C_{45} = C_{46} = C_{56} = 0$, $C_{16} + C_{25} = 0$, $C_{15} - C_{34} - C_{26} = 0$; $W(H^2)$: $C_{12} = C_{13} = C_{14} = 0$; $\dim H^2$: 5; Basis: $\Delta_{15} + \Delta_{26}$, $\Delta_{15} + \Delta_{34}$, $\Delta_{16} - \Delta_{25}$, Δ_{23}, Δ_{24};

Group action: $a(\Delta_{15} + \Delta_{26}) + b(\Delta_{15} + \Delta_{34}) + c(\Delta_{16} - \Delta_{25}) + d\Delta_{23} + e\Delta_{24}$:

(1):

$a \to a(a_{11}^4 - a_{12}^4) - ba_{12}^2(a_{11}^2 + a_{12}^2) - 2ca_{11}a_{12}(a_{11}^2 + a_{12}^2);$

$b \to b(a_{11}^2 + a_{12}^2)^2;$

$c \to 2aa_{11}a_{12}(a_{11}^2 + a_{12}^2) + ba_{11}a_{12}(a_{11}^2 + a_{12}^2) + c(a_{11}^4 - a_{12}^4);$

$d \to a(-2a_{11}a_{12}a_{32} - a_{12}^2a_{42} - a_{12}^2a_{31} + 2a_{11}a_{12}a_{41} + a_{11}^2a_{42} + a_{11}^2a_{31}) + b(-a_{11}a_{12}a_{32} - 2a_{12}^2a_{42} - a_{12}^2a_{31} + a_{11}a_{12}a_{41} - a_{11}^2a_{42}) + c(-2a_{11}a_{12}a_{42} + a_{12}^2a_{32} - a_{12}^2a_{41} - 2a_{11}a_{12}a_{31} - a_{11}^2a_{32} + a_{11}^2a_{41});$

$e \to a(-a_{12}a_{54} + a_{11}a_{64}) + b(-a_{12}a_{54} + a_{11}^2a_{32} + a_{12}^2a_{32}) + c(-a_{12}a_{64} - a_{11}a_{54}) + ea_{11}(a_{11}^2 + a_{12}^2);$

(2): $a \to a + b$, $b \to -b$, $c \to -c$, $d \to e$ and $e \to d$.

One of $a, c \neq 0$, and when $c = 0$, then $a \neq 0$ and $a + b \neq 0$. Make $a \neq 0$ and $c = 0$,which is always possible over $\mathbf{R}$. Fix $c = 0$ and let $a_{12} = 0$, we get $a \to aa_{11}^4$; $b \to ba_{11}^4$; $c = 0 \to 0$; $d \to a(a_{11}^2a_{42} + a_{11}^2a_{31}) + b(-a_{11}^2a_{42})$; $e \to aa_{11}a_{64} + ba_{11}^2a_{32} + ea_{11}^3$.

Make $d = e = 0$ by solving for a_{31} and a_{64}, depending on whether $b = 0$ or not, we get two representatives for A: $A = [1, 0, 0, 0, 0]$, and $A = [\lambda, 1 - \lambda, 0, 0, 0]$, with $\lambda \neq 0, 1$. Combining these two, we may just assume λ to be any nonzero real numbers. Using (2), we know that A is in the same orbit as $[1, \lambda - 1, 0, 0, 0]$. Since $\lambda \neq 0$ in A, we can multiply A by $1/\lambda$ and get $[1, \frac{1}{\lambda} - 1, 0, 0, 0]$, now it is easy to see that $K(\lambda) = \lambda + 1/\lambda$ can be used as an invariant for λ. This new algebra will be denoted by (1357QRS$_1$), since over $\mathbf{C}$, when $\lambda = 1$, it is isomorphic to (1357Q); when $\lambda = -1$, it is isomorphic to (1357R); and for all the other $\lambda \neq 0$, it becomes (1357S, $\lambda > 0$, $\lambda \neq 1$).

$a_{23}a_{63}) + d(-a_{23}a_{43}) + e(a_{13}a_{43}) + fa_{43}^2 + g(-a_{43}a_{64} + a_{53}a_{43}) + h(a_{43}a_{54} + a_{43}a_{63})$;

$g \to b(-a_{12}a_{13}a_{43} - a_{13}^2a_{31} - 2a_{13}a_{23}a_{32} + a_{22}a_{23}a_{43} + a_{23}^2a_{31}) + c(a_{13}a_{22}a_{43} + 2a_{13}a_{23}a_{31} - a_{13}^2a_{32} + a_{12}a_{23}a_{43} + a_{23}^2a_{32}) + g(-a_{22}a_{43}^2 - a_{43}a_{23}a_{31} + a_{43}a_{13}a_{32}) + h(-a_{12}a_{43}^2 - a_{43}a_{13}a_{31} - a_{43}a_{23}a_{32})$;

$h \to b(-a_{13}a_{22}a_{43} - a_{13}^2a_{32} - 2a_{13}a_{23}a_{31} - a_{12}a_{23}a_{43} - a_{23}^2a_{32}) + c(-a_{12}a_{13}a_{43} - a_{13}^2a_{31} - 2a_{13}a_{23}a_{32} + a_{22}a_{23}a_{43} + a_{31}a_{23}^2) + g(a_{12}a_{43}^2 + a_{43}a_{13}a_{31} + a_{43}a_{23}a_{32}) + h(-a_{22}a_{43}^2 + a_{43}a_{13}a_{32} - a_{43}a_{23}a_{31})$;

(2): $a \to -a$, $b \to -b$, $c \to c$, $d \to -d$, $e \to e$, $f \to -f$, $g \to -h$, $h \to -g$.

One of $b, c, g, h \neq 0$. Make $b \neq 0$ and $c = g = h = 0$. Take $a_{31} = a_{32} = 0$, $a_{51} = a_{52}$ and $a_{12} = a_{22}$, then we have

$a \to a(a_{22}^2 + a_{12}^2) + b(a_{22}a_{52} - a_{12}a_{62} - a_{12}a_{51} - a_{22}a_{61})$;

$b \to b(-2a_{22}^2a_{43})$;

$c = 0 \to 0$;

$d \to a(-a_{13}a_{22} + a_{12}a_{23} - a_{12}a_{13} - a_{22}a_{23}) + b(2a_{22}a_{63} + a_{12}a_{53} - a_{62}a_{23} - a_{13}a_{52} - a_{22}a_{53} + a_{13}a_{51} + a_{23}a_{61}) + e(a_{22}a_{43} + a_{12}a_{43})$;

$e \to a(a_{13}a_{22} - a_{12}a_{23} - a_{12}a_{13} - a_{22}a_{23}) + b(2a_{22}a_{64} + a_{12}a_{54} + a_{13}a_{62} - a_{22}a_{54} - a_{13}a_{61}) + d(-a_{22}a_{43} - a_{12}a_{43})$;

$f \to a(-a_{13}^2 - a_{23}^2) + b(a_{13}a_{54} + a_{23}a_{64} - a_{23}a_{53} + a_{13}a_{63}) + d(-a_{23}a_{43}) + e(a_{13}a_{43}) + fa_{43}^2$;

$g = 0 \to b(-a_{12}a_{13}a_{43} + a_{22}a_{23}a_{43})$;

$h = 0 \to b(-a_{13}a_{22}a_{43} - a_{12}a_{23}a_{43})$;

Now make $a = d = e = g = h = 0$ by solving for a_{52}, a_{63}, a_{64}, a_{13} and a_{23} respectively. Then choose further $a_{13} = a_{23} = a_{51} = a_{52} = a_{53} = a_{54} = a_{61} = a_{62} = a_{63} = a_{64} = 0$, to get $a = 0 \to 0$; $b \to b(-2a_{22}^2a_{43})$; $c = 0 \to 0$; $d = 0 \to 0$; $e = 0 \to 0$; $f \to fa_{43}^2$; $g = 0 \to 0$; $h = 0 \to 0$.

Depending on whether $f = 0$ or not, we get two representatives $[0, 1, 0, 0, 0, 0, 0, 0]$, corresponding to (137A$_1$), and $[0, 1, 0, 0, 0, 1, 0, 0]$, corresponding to (137B_1).

Therefore the central extensions of $N_{6,4,4a}$ are:

(137A$_1$):	$[x_1, x_3] = x_5$,	$[x_1, x_4] = x_6$,	$[x_1, x_5] = x_7$,
	$[x_2, x_3] = -x_6$,	$[x_2, x_4] = x_5$,	$[x_2, x_6] = x_7$;
(137B$_1$):	$[x_1, x_3] = x_5$,	$[x_1, x_4] = x_6$,	$[x_1, x_5] = x_7$,
	$[x_2, x_3] = -x_6$,	$[x_2, x_4] = x_5$,	$[x_2, x_6] = x_7$,
	$[x_3, x_4] = x_7$;		

Appendix A

Comparison with Nielsen's List

For 6-dimensional nilpotent Lie algebras, Nielson [22] presents a list of 24 indecomposable non-isomorphic algebras over the real field **R** and calculates a corresponding connected and simply-connected Lie group and its coadjoint orbits, and related data for each algebra. He also compares his list with those of Morozov [20], Skjelbred and Sund [35], Umlauf [37] and Vergne [38].

In this part, we indicate the correspondence between our list and Nielsen's list:

$N_{6,1,1} \cong G_{6,13}$;	$N_{6,1,2} \cong G_{6,14}$;
$N_{6,1,3} \cong G_{6,11}$;	$N_{6,1,4} \cong G_{6,3}$;
$N_{6,2,1} \cong G_{6,10}$;	$N_{6,2,2} \cong G_{6,12}$;
$N_{6,2,3} \cong G_{6,7}$;	$N_{6,2,4} \cong G_{6,2}$;
$N_{6,2,5} \cong G_{6,9}$;	$N_{6,2,5a} \cong G_{6,8}$;
$N_{6,2,6} \cong G_{6,5}$;	$N_{6,2,7} \cong G_{6,24}$;
$N_{6,2,8} \cong G_{6,20}$;	$N_{6,2,9} \cong G_{6,22}$;
$N_{6,2,9a} \cong G_{6,23}$;	$N_{6,2,10} \cong G_{6,21}$;
$N_{6,3,1} \cong G_{6,4}$;	$N_{6,3,1a} \cong G_{6,6}$;
$N_{6,3,2} \cong G_{6,1}$;	$N_{6,3,3} \cong G_{6,19}$;
$N_{6,3,4} \cong G_{6,18}$;	$N_{6,3,5} \cong G_{6,16}$;
$N_{6,3,6} \cong G_{6,15}$;	$N_{6,4,4a} \cong G_{6,17}$.

Appendix B

Comments on Ancochea-Goze List

In this appendix, we discuss the list of indecomposable complex nilpotent Lie algebras of dimension 7 obtained by Ancochea-Bermudez and Goze [2] . The list was originally published in Arch. Math. in 1989, which missed a lot of algebras and also contained many errors. Later on the list was incorporated as part of the book "Nilpotent Lie Algebras" by Goze and Khakimdjanov [12], with some adjustments and more algebras. This book was published in 1996, three years after Seeley's paper [33] appeared in Trans. AMS. We have compared all the indecomposable algebras in Seeley's list with this one, and as it turns out, Ancochea-Goze's list still misses many algebras, while some are not Lie algebras at all, and others are included more than once.

Below we will present the results of our comparison concerning Ancochea-Goze's list: (1) At first we will point out those that are not Lie algebras at all, by providing 3 elements which fail the Jacobi identity. We make no efforts in correcting the mistakes; (2) Secondly, we will list all the algebras which have been included more than once, together with an isomorphism between them; (3) Thirdly, we point out the correspondences between the two lists by using the upper central series dimensions as our invariant, also mentioned are the algebras that are missing from Ancochea-Goze list.

B.1 Decomposable or non-Lie Algebras

In this section, we will point out those algebras which are decomposable or not a Lie algebra at all. In total, we found two decomposable algebras, and ten classes which are not Lie algebras, including an infinite family.

n_7^{117}: Decomposable, $n_7^{117} = < x_2, x_3, x_4 > \times < x_1 - x_2, x_3 + x_7, x_5, x_6 >$.

n_7^{128}: Decomposable, $\mathbf{F}(x_2 - x_3)$ is an Abelian direct factor.

n_7^{6}: Not a Lie algebra, with $\mathrm{Jac}(x_1, x_6, x_7) \neq 0$.

n_7^{9}: Not a Lie algebra, it has obviously a typo, with $[x_6, x_7] = \frac{1}{2}x_3 + \frac{1}{2}x_3$.

n_7^{17}: Not a Lie algebra, with $\mathrm{Jac}(x_1, x_5, x_7) \neq 0$.

$n_7^{62,\alpha}$: Not a Lie algebra, with $\mathrm{Jac}(x_1, x_5, x_7) \neq 0$.

n_7^{81}: Not a Lie algebra, with $\mathrm{Jac}(x_1, x_5, x_7) \neq 0$.

n_7^{97}: Not a Lie algebra, with $\mathrm{Jac}(x_1, x_4, x_7) \neq 0$.

n_7^{98}: Not a Lie algebra, with $\mathrm{Jac}(x_1, x_4, x_7) \neq 0$.

n_7^{100}: Not a Lie algebra, with $\mathrm{Jac}(x_1, x_4, x_7) \neq 0$.

n_7^{120}: Not a Lie algebra, with $\mathrm{Jac}(x_1, x_2, x_4) \neq 0$.

n_7^{122}: Not a Lie algebra, with $\mathrm{Jac}(x_1, x_4, x_7) \neq 0$.

B.2 Algebras That Occur More Than Once

In this section, we list all the algebras that have appeared more than once. For those algebras with different presentations, we also provide an isomorphism between them. When we write $A \cong B$, it means that A and B are isomorphic but of different presentations, then the isomorphism given is from A to B. If the algebras are of exactly the same presentation, we simply write $A = B$.

$n_7^{37} = n_7^{38}$.

$n_7^{19} \cong n_7^{18}$: Taking $x_1 \to \frac{1}{2}x_1 + \frac{1}{2}x_7$, $x_2 \to \frac{1}{4}x_2 - \frac{1}{2} + \frac{1}{4}x_4$, $x_3 \to -\frac{1}{4}x_3$, $x_4 \to \frac{1}{4}x_3 - \frac{1}{2}x_4$, $x_5 \to \frac{1}{4}x_2 - \frac{1}{2}x_3 + \frac{1}{4}x_4 - \frac{1}{2}x_5$, $x_6 \to -\frac{1}{2}x_6$, $x_7 \to -x_7$.

$n_7^{29} = n_7^{32}$.

$n_7^{91} \cong n_7^{94}$: By taking $x_1 \to x_1 + \frac{1}{2}x_7$, $x_2 \to \frac{1}{2}x_2 + \frac{1}{2}x_5$, $x_3 \to x_3 + x_6$, $x_4 \to x_1 + x_4 + \frac{3}{2}x_7$, $x_5 \to x_2$, $x_6 \to x_3$ and $x_7 \to x_4$.

$n_7^{106} = n_7^{124}$.

$n_7^{108} \cong n_7^{121}$: By taking $x_1 \to x_1 - x_2 + x_4$, $x_2 \to x_4$, $x_3 \to -x_3$, $x_4 \to x_2$, $x_5 \to x_5$, $x_6 \to x_5 + x_6$, $x_7 \to x_7$.

$n_7^{118} \cong n_7^{126}$: By taking $x_1 \to x_1 - x_2 + x_4$, $x_2 \to x_4 + x_7$, $x_3 \to -x_3 - x_6$, $x_4 \to x_2$, $x_5 \to x_5$, $x_6 \to x_6$, $x_7 \to x_7$.

B.3 Comparison of Ancochea-Goze's and Seeley's Lists

In this section, we establish the correspondence between these two lists. We compare, of course, the corrected Seeley's, which in the case $\mathbf{F} = \mathbf{C}$ is identical to our list in Chapter 4 (also see the Introduction for comments) with the modified and updated version of the Ancochea-Goze list as presented in the book [12]. Also mentioned are the algebras that are missing from Ancochea-Goze list. We use the upper central series dimensions as the invariant.

(37):

$\boldsymbol{n}_7^{145} \cong$ (37A); $\boldsymbol{n}_7^{143} \cong$ (37B); $\boldsymbol{n}_7^{144} \cong$ (37C); $\boldsymbol{n}_7^{142} \cong$ (37D).
Missing: none.

(357):

$\boldsymbol{n}_7^{102} \cong$ (357A); $\boldsymbol{n}_7^{104} \cong$ (357B); $\boldsymbol{n}_7^{103} \cong$ (357C).
Missing: none.

(27):

$\boldsymbol{n}_7^{147} \cong$ (27A); $\boldsymbol{n}_7^{146} \cong$ (27B).
Missing: none.

(257):

$\boldsymbol{n}_7^{111} \cong$ (257A); $\boldsymbol{n}_7^{107} \cong$ (257B); $\boldsymbol{n}_7^{113} \cong$ (257C); $\boldsymbol{n}_7^{123} \cong$ (257D); $\boldsymbol{n}_7^{112} \cong$ (257E); $\boldsymbol{n}_7^{110} \cong$ (257F); $\boldsymbol{n}_7^{109} \cong$ (257G); $\boldsymbol{n}_7^{108} \cong$ (257H); $\boldsymbol{n}_7^{116} \cong$ (257I); $\boldsymbol{n}_7^{119} \cong$ (257K);
Missing: (257J), (257L).

(247):

$\boldsymbol{n}_7^{87} \cong$ (247A); $\boldsymbol{n}_7^{101} \cong$ (247B); $\boldsymbol{n}_7^{88} \cong$ (247C); $\boldsymbol{n}_7^{96} \cong$ (247E); $\boldsymbol{n}_7^{93} \cong$ (247F); $\boldsymbol{n}_7^{91} \cong$ (247G); $\boldsymbol{n}_7^{89} \cong$ (247H); $\boldsymbol{n}_7^{90} \cong$ (247I); $\boldsymbol{n}_7^{92} \cong$ (247J); $\boldsymbol{n}_7^{85} \cong$ (247L); $\boldsymbol{n}_7^{106} \cong$ (247N); $\boldsymbol{n}_7^{86} \cong$ (247O); $\boldsymbol{n}_7^{95} \cong$ (247Q); $\boldsymbol{n}_7^{99} \cong$ (247R).
Missing: (247D), (247K), (247M), (247P).

(2457):

$\boldsymbol{n}_7^{84} \cong$ (2457A); $\boldsymbol{n}_7^{77} \cong$ (2457B); $\boldsymbol{n}_7^{83} \cong$ (2457C); $\boldsymbol{n}_7^{64} \cong$ (2457E); $\boldsymbol{n}_7^{82} \cong$ (2457F); $\boldsymbol{n}_7^{67} \cong$ (2457G); $\boldsymbol{n}_7^{50} \cong$ (2457H); $\boldsymbol{n}_7^{76} \cong$ (2457I); $\boldsymbol{n}_7^{66} \cong$ (2457J); $\boldsymbol{n}_7^{65} \cong$ (2457K); $\boldsymbol{n}_7^{60} \cong$ (2457L); $\boldsymbol{n}_7^{61} \cong$ (2457M).
Missing: (2457D).

(2357):

$\boldsymbol{n}_7^{74} \cong$ (2357A); $\boldsymbol{n}_7^{80} \cong$ (2357B); $\boldsymbol{n}_7^{63} \cong$ (2357D).
Missing: (2357C).

(23457):

$n_7^{24} \cong$ (23457A);$n_7^{26} \cong$ (23457B); $n_7^{12} \cong$ (23457C);$n_7^{11} \cong$ (23457D); $n_7^{25} \cong$ (23457E); $n_7^{22} \cong$ (23457F); $n_7^{10} \cong$ (23457G).

Missing: none.

(17):

$n_7^{152} \cong$ (17).

Missing: none.

(157):

$n_7^{137} \cong$ (157).

Missing: none.

(147):

$n_7^{114} \cong$ (147B);

$n_7^{105} \cong$ (147E) by taking λ to be a root of $x^2 - x + 1$;

$n_7^{127,\alpha} \cong (147E)_\xi$ (Compare the invariant for ξ in (147E)).

Missing: (147A), (147D). (Notice that (147C) in Seeley is a special case of (147E))

(1457):

$n_7^{52} \cong$ (1457A);$n_7^{51} \cong$ (1457B).

Missing: none.

(137):

$n_7^{118} \cong$ (137A); $n_7^{125} \cong$ (137B); $n_7^{115} \cong$ (137C).

Missing: (137D).

(1357):

$n_7^{49} \cong$ (1357A); $n_7^{48} \cong$ (1357B); $n_7^{47} \cong$ (1357C);

$n_7^{72} \cong$ (1357E); $n_7^{71} \cong$ (1357F); $n_7^{75} \cong$ (1357G); $n_7^{73} \cong$ (1357H);$n_7^{69} \cong$ (1357I); $n_7^{68} \cong$ (1357J); $n_7^{79} \cong$ (1357L); $n_7^{78,\alpha} \cong (1357M)_\lambda$; $n_7^{70,\alpha} \cong (1357N)_\lambda$;

$n_7^{62,\alpha} \cong$ (1357S), in the original A-G list, $n_7^{62,\alpha}$ is not a Lie algebra, but after $[x_5, x_6] = x_2$ is replaced by $[x_5, x_6] = -\alpha x_2$, we have the above isomorphism.

Missing: (1357D), (1357O), (1357P), (1357Q), (1357R). Notice that (1357K) in Seeley is a special case of (1357M).

(13457):

$n_7^{41} \cong$ (13457A);$n_7^{40} \cong$ (13457B);$n_7^{29} \cong$ (13457C);

$n_7^{39} \cong$ (13457D); $n_7^{31} \cong$ (13457E);$n_7^{20} \cong$ (13457F); $n_7^{21} \cong$ (13457G); $n_7^{18} \cong$ (13457I).

Missing: none. Notice that (13457H) in Seeley is not a Lie algebra.

(12457):

$n_7^{15} \cong$ (12457A); $n_7^{37} \cong$ (12457B); $n_7^{30} \cong$ (12457C); $n_7^{28} \cong$ (12457D); $n_7^{36} \cong$ (12457E); $n_7^{27} \cong$ (12457G); $n_7^{16} \cong$ (12457K); $n_7^{23} \cong$ (12457L);

$n_7^{13,\alpha} \cong (12457N)_\lambda$. In A-G list, there is no restriction on α at all, compare λ in (12457N).

$n_7^{14} \cong (12457N, \lambda = -1)$.

Missing: (12457F), (12457H), (12457I),(12457J). Notice that (12457M) in Seeley is just a special case of (12457N) by taking $\lambda = 0$.

(12357):

$n_7^{35} \cong$ (12357A); $n_7^{34} \cong$ (12357B); $n_7^{33} \cong$ (12357C).

Missing: none.

(123457):

$n_7^{8} \cong$ (123457A);$n_7^{7} \cong$ (123457B); $n_7^{5} \cong$ (123457E);

$n_7^{4} \cong$ (123457F); $n_7^{3} \cong$ (123457H); $n_7^{1,\alpha} \cong (123457I)_\lambda$; n_7^{2} a special case of (123457I), with $\lambda = 1$;

Missing: (123457C), (123457D). Notice that (123457G) in Seeley is a special case of (123457I).

Appendix C

Comments on Romdhani's List

In this appendix, we discuss the list of indecomposable real nilpotent Lie algebras of dimension 7 obtained by Romdhani [24][25]. Carles [6] has compared Seeley's list with Romdhani's over the complex field. Readers who are interested in more details should refer to [6]. Carles has a very nice discussion especially about the behaviour of the six continuous families there.

Here we compare our list of 7-dimensional indecomposable real nilpotent Lie algebras with that of Romdhani [24][25]. Also mentioned are the algebras that are missing from his list, which are many in numbers. We use the upper central series dimensions as our invariant. Our purpose is more on the correspondence between the two lists, hence we make no effort in making corrections or providing the details of the isomorphism.

(37):

$\mathfrak{g}_{7,127} \cong$ (37A); $\mathfrak{g}_{7,126} \cong \mathfrak{g}_{7,128} \cong$ (37B); $\mathfrak{g}_{7,124} \cong$ (37D); $\mathfrak{g}_{7,125} \cong$ (37D_1).
Missing: (37C), (37B_1).

(357):

$\mathfrak{g}_{7,98} \cong$ (357A).
Missing: (357B),(357C).

(27):

$g_{7,131} \cong$ (27A); $g_{7,130} \cong$ (27B).
Missing: None.

(257):

$g_{7,121} \cong$ (257A); $g_{7,119} \cong$ (257B); $g_{7,123} \cong$ (257C); $g_{7,122} \cong$ (257E);
$g_{7,120} \cong$ (257F); $g_{7,118} \cong$ (257G); $g_{7,117} \cong$ (257H); $g_{7,106} \cong$ (257I);
$g_{7,105} \cong$ (257K); $g_{7,104} \cong$ (257L).
Missing: (257D), (257J), ($257J_1$).

(247):

$g_{7,92} \cong$ (247E); $g_{7,91} \cong$ ($247E_1$); $g_{7,83} \cong$ (247F); $g_{7,82} \cong$ (247G); $g_{7,81} \cong$ (247H); $g_{7,86} \cong$ (247I); $g_{7,85} \cong$ (247J); $g_{7,84} \cong$ (247K); $g_{7,90} \cong g_{7,96} \cong g_{7,97} \cong$ (247P); $g_{7,89} \cong$ ($247P_1$); $g_{7,87} \cong$ ($247R_1$); $g_{7,88} \cong$ (247R).
Missing: (247A-D),($247F_1$, H_1), (247L-O,Q).

(2457):

$g_{7,78} \cong$ (2457A); $g_{7,80} \cong$ (2457B); $g_{7,77} \cong$ (2457C); $g_{7,58} \cong$ (2457E);
$g_{7,76} \cong$ (2457F); $g_{7,61} \cong$ (2457G); $g_{7,60} \cong$ (2457H); $g_{7,79} \cong$ (2457I);
$g_{7,57} \cong$ (2457J); $g_{7,59} \cong$ (2457K); $g_{7,33} \cong$ (2457L); $g_{7,32} \cong$ (2457M).
Missing: (2457D),($2457L_1$).

(2357):

$g_{7,73} \cong$ (2357A); $g_{7,75} \cong$ (2357B); $g_{7,56} \cong$ (2357C), $g_{7,54} \cong$ (2357D); $g_{7,55} \cong$ ($2357D_1$).
Missing: None.

(23457):

$g_{7,31} \cong$ (23457A); $g_{7,28} \cong$ (23457B); $g_{7,13} \cong$ (23457C); $g_{7,12} \cong$ (23457D);
$g_{7,30} \cong$ (23457E); $g_{7,27} \cong$ (23457F); $g_{7,11} \cong$ (23457G).
Missing: None.

(17):

$g_{7,132} \cong$ (17).
Missing: None.

(157):

$\mathfrak{g}_{7,129} \cong (157)$.
Missing: None.

(147):

$\mathfrak{g}_{7,113} \cong \mathfrak{g}_{7,115} \cong (147A_1)$;$\mathfrak{g}_{7,114} \cong \mathfrak{g}_{7,116} \cong (147A)$; $\mathfrak{g}_{7,112} \cong (147B)$;$\mathfrak{g}_{7,95} \cong (147D)$; $\mathfrak{g}_{7,93}^{\lambda\neq 0} \cong (147E)$; $\mathfrak{g}_{7,94}^{\lambda\neq 0} \cong (147E_1)$.
Missing: None. (Notice that (147C) in Seeley is a special case of (147E))

(1457):

$\mathfrak{g}_{7,103} \cong (1457A)$; $\mathfrak{g}_{7,102} \cong (1457B)$.
Missing: None.

(137):

$\mathfrak{g}_{7,108} \cong (137A)$; $\mathfrak{g}_{7,109} \cong \mathfrak{g}_{7,110} \cong (137A_1)$; $\mathfrak{g}_{7,107} \cong (137B)$; $\mathfrak{g}_{7,111} \cong (137C)$.
Missing: $(137B_1)$, (137D).

(1357):

$\mathfrak{g}_{7,101} \cong (1357A)$; $\mathfrak{g}_{7,100} \cong (1357B)$; $\mathfrak{g}_{7,99} \cong (1357C)$; $\mathfrak{g}_{7,69} \cong (1357D)$; $\mathfrak{g}_{7,67} \cong \mathfrak{g}_{7,68} \cong (1357E)$; $\mathfrak{g}_{7,65} \cong (1357F_1)$; $\mathfrak{g}_{7,66} \cong (1357F)$; $\mathfrak{g}_{7,74} \cong (1357G)$; $\mathfrak{g}_{7,72} \cong (1357H)$;
$\mathfrak{g}_{7,71} \cong (1357I)$; $\mathfrak{g}_{7,70} \cong (1357J)$; $\mathfrak{g}_{7,63} \cong (1357L)$; $\mathfrak{g}_{7,64}^{\lambda\neq 0} \cong (1357M)$; $\mathfrak{g}_{7,62}^{\lambda} \cong (1357N)$; $\mathfrak{g}_{7,52}^{\lambda=0} \cong (1357P)$;$\mathfrak{g}_{7,53}^{\lambda=0} \cong (1357P_1)$; $\mathfrak{g}_{7,52}^{\lambda=1} \cong (1357Q)$; $\mathfrak{g}_{7,53}^{\lambda=1} \cong (1357R)$; $\mathfrak{g}_{7,52}^{\lambda=-1} \cong (1357QRS_1, \lambda = -1)$; $\mathfrak{g}_{7,52}^{\lambda>0} \cong (1357S, \lambda > 1)$; $\mathfrak{g}_{7,53}^{\lambda>0} \cong (1357S, \lambda < 1)$; $\mathfrak{g}_{7,52}^{\lambda<0,\lambda\neq -1} \cong (1357QRS_1, \lambda < 0, \lambda \neq -1)$;
Missing: (1357O), $(1357Q_1)$, $(1357QRS_1, \lambda = 1)$. Notice that (1357K) in Seeley's is a special case of (1357M).

(13457):

$\mathfrak{g}_{7,51} \cong (13457A)$; $\mathfrak{g}_{7,50} \cong (13457B)$; $\mathfrak{g}_{7,39} \cong (13457C)$; $\mathfrak{g}_{7,49} \cong (13457D)$;
$\mathfrak{g}_{7,38} \cong (13457E)$; $\mathfrak{g}_{7,29} \cong (13457F)$; $\mathfrak{g}_{7,26} \cong (13457G)$; $\mathfrak{g}_{7,25} \cong (13457I)$.
Missing: None. Notice that (13457H) in Seeley is not a Lie algebra.

(12457):

$\mathfrak{g}_{7,48} \cong$ (12457A); $\mathfrak{g}_{7,47} \cong$ (12457B); $\mathfrak{g}_{7,37} \cong$ (12457C); $\mathfrak{g}_{7,35} \cong$ (12457D); $\mathfrak{g}_{7,46} \cong$ (12457E); $\mathfrak{g}_{7,36} \cong$ (12457F); $\mathfrak{g}_{7,34} \cong$ (12457G); $\mathfrak{g}_{7,24} \cong$ (12457H); $\mathfrak{g}_{7,22} \cong$ (12457I); $\mathfrak{g}_{7,20} \cong$ (12457J); $\mathfrak{g}_{7,21} \cong$ (12457J$_1$); $\mathfrak{g}_{7,23} \cong$ (12457K); $\mathfrak{g}_{7,19} \cong$ (12457L); $\mathfrak{g}_{7,18} \cong$ (12457L$_1$);$\mathfrak{g}_{7,16} \cong$ (12457N, $\lambda = 1$); $\mathfrak{g}_{7,17} \cong$ (12457N$_1$); $\mathfrak{g}^{\lambda}_{7,15} \cong$ (12457N); $\mathfrak{g}^{\lambda}_{7,14} \cong$ (12457N$_2$).
Missing: None. Notice that (12457M) in Seeley is special case of (12457N) by choosing $\lambda = 0$.

(12357):

$\mathfrak{g}_{7,45} \cong$ (12357A); $\mathfrak{g}_{7,43} \cong$ (12357B); $\mathfrak{g}_{7,44} \cong$ (12357B$_1$); $\mathfrak{g}_{7,40} \cong \mathfrak{g}_{7,41} \cong \mathfrak{g}_{7,42} \cong$ (12357C).
Missing: None.

(123457):

$\mathfrak{g}_{7,10} \cong$ (123457A); $\mathfrak{g}_{7,9} \cong$ (123457B); $\mathfrak{g}_{7,3} \cong$ (123457C); $\mathfrak{g}_{7,8} \cong$ (123457D); $\mathfrak{g}_{7,7} \cong$ (123457E);$\mathfrak{g}_{7,2} \cong$ (123457F);$\mathfrak{g}_{7,4} \cong$ (123457H); $\mathfrak{g}_{7,5} \cong$ (123457H$_1$); $\mathfrak{g}_{7,6} \cong$ (123457I,$\lambda = 1$); $\mathfrak{g}^{\lambda \neq 1}_{7,1} \cong$ (123457I,$\lambda \neq 1$).
Missing: None.

Appendix D

An Overview of the Construction of the 7-Dimensional Algebras

Here we give the summary of all the 7-dimensional indecomposable nilpotent Lie algebras as they arise from those of dimensions ≤ 6 in our construction over algebraically closed fields of $\chi \neq 2$. The readers may easily identify the central quotients of all the 7-dimensional algebras with this list.

With regard to the number of algebras: Over the algebraically closed fields, there are 6 one parameter continuous families, together with 119 isolated algebras when $\chi \neq 3$ or 120 isolated algebras when $\chi = 3$ (the extra algebra is (147F)),

Over the real field, there are, in addition, 3 one parameter continuous families and 21 isolated algebras.

D.1 Algebras over Algebraically Closed Fields

Abelian Algebras and Their Extensions

$N_{6,6}$: (17).
$N_{5,5}$: (27A,B).
$N_{4,4}$: (37A–D).

Four-Dimensional Algebras and Their Extensions

$N_{4,2}$: None.
$N_{4,3}$: (357A–C).

Five-Dimensional Algebras and Their Extensions

$N_{5,1}$: (23457E–G).
$N_{5,2,1}$: (23457A–D).
$N_{5,2,2}$: (2357A–D).
$N_{5,2,3}$: (2457L, M).
$N_{5,3,1}$: None.
$N_{5,3,2}$: (247A–R).
$N_{5,3,3}$: (2457A–K).
$N_{5,4}$: (257A–L).

Six-Dimensional Algebras and Their Extensions

$N_{6,1,1}$: (123457H, I). (123457G) in Seeley's list is just a special case of (123457I) by taking $\lambda = 1$.
$N_{6,1,2}$: None.
$N_{6,1,3}$: (123457D-F).
$N_{6,1,4}$: (12457E-G).

$N_{6,2,1}$: (123457A-C).
$N_{6,2,2}$: None.
$N_{6,2,3}$: (12357A–C).
$N_{6,2,4}$: (12457A–D).
$N_{6,2,5}$: (12457H–L, N). (12457M) is just a special case of (12457N) by taking $\lambda = 0$.

$N_{6,2,6}$: None.
$N_{6,2,7}$: (13457F, G,I). (13457H) in Seeley's list is not a Lie algebra and should be deleted.
$N_{6,2,8}$: (1357L–N). (1357K) in Seeley's list is just a special case of (1357M) by taking $\lambda = 1/2$.

$N_{6,2,9}$: (1357Q–S).
$N_{6,2,10}$: (1357O, P).
$N_{6,2,11}$: (13457D,E).

$N_{6,3,1}$: None.
$N_{6,3,2}$: None.
$N_{6,3,3}$: (1357G–J).

$N_{6,3,4}$: (1357D–F).

$N_{6,3,5}$: (137C,D).

$N_{6,3,6}$: (147D,E) (and also(147F) if $\chi = 3$). (147C) in Seeley's list is a special case of (147E) by taking $\lambda = 1/2$.

$N_{6,3,7}$: (13457A–C).

$N_{6,3,8}$: (1357A–C).

$N_{6,3,9}$: None.

$N_{6,4,1}$: None.

$N_{6,4,2}$: (147A, B).

$N_{6,4,3}$: (1457A,B).

$N_{6,4,4}$: (137A,B).

$N_{6,5}$: (157).

D.2 Algebras over the Real Field

In addition to the above algebras over algebraically closed fields of $\chi \neq 2$, we have the following indecomposable algebras over **R**.

Abelian Algebras and Their Extensions

$N_{4,4}$: $(37\mathrm{B}_1, \mathrm{D}_1)$.

Five-Dimensional Algebras and Their Extensions

$N_{5,2,2}$: $(2357\mathrm{D}_1)$.
$N_{5,2,3}$: $(2457\mathrm{L}_1)$.
$N_{5,3,2}$: $(247\mathrm{E}_1, \mathrm{F}_1, \mathrm{H}_1, \mathrm{P}_1, R_1)$.
$N_{5,4}$: $(257\mathrm{J}_1)$.

Six-Dimensional Algebras and Their Extensions

$N_{6,1,1}$: $(123457\mathrm{H}_1)$.
$N_{6,2,3}$: $(12357\mathrm{B}_1)$.
$N_{6,2,5}$: $(12457\mathrm{J}_1)$.
$N_{6,2,5a}$: $(12457\mathrm{L}_1, \mathrm{N}_1, \mathrm{N}_2)$.

$N_{6,2,9}$: $(1357\mathrm{Q}_1)$.
$N_{6,2,9a}$: $(1357\mathrm{QRS}_1)$. The reason we use this notation is because over **C**, if $\lambda = 1$, $(1357\mathrm{QRS}_1) \cong (1357\mathrm{Q})$; if $\lambda = -1$, $(1357\mathrm{QRS}_1) \cong (1357\mathrm{R})$; and for other λ, it corresponds to (1357S).
$N_{6,2,10}$: $(1357\mathrm{P}_1)$.

$N_{6,3,1a}$: None.
$N_{6,3,4}$: $(1357\mathrm{F}_1)$.
$N_{6,3,6}$: $(147\mathrm{E}_1)$
$N_{6,4,2}$: $(147\mathrm{A}_1)$.
$N_{6,4,4a}$: $(137\mathrm{A}_1, \mathrm{B}_1)$.

Appendix E

Maple Programs

In this part we provide the main Maple V programs that we have used in our computation.

E.1 Introduction

A Lie algebra is uniquely determined by its structural constants, which can be naturally regarded as a 3-dimensional matrix in Maple V. Therefore we may expect that the computational systems such as Maple V are going to play a more and more important role in the research of Lie algbras and related topics.

All of our routines are to be used together with the Linear Algebra Package provided by Maple V, through the command `with(linalg)`.

For example, the Heisenberg Lie algebra

$$N_{5,3,1}: \quad [x_1, x_2] = [x_3, x_4] = x_5$$

can be denoted in Maple V as

```
N5,3,1=:array(sparse,1..5,1..5,1..5,[(1,2,5)=1, (2,1,5)=-1, (3,4,5)=1,
                                    (4,3,5)=-1]):
```

The procedures available are for the computation of:

- the Lie algebra conditions (including the Jacobi identity and the anticommutativity);
- the cocycles;
- the group actions;
- the isomorphism between two algebras (including automorphism groups);
- derivation algebras.

E.2 The Programs

E.2.1 Lie Algebra Conditions

Calling Sequence:

check_lie(*Algebra, Dimension*)

Parameters:

Algebra — An algebra in the form of a 3-dimensional matrix

Dimension – The dimension of the given algebra

Synopsis:

– To check whether an algebra is a Lie algebra or not by checking the Jacobi identity and the anticommutativity.

– Input is an algebra and its dimension.

– If the algebra is NOT a Lie algebra, then the output will specify the vectors where the anticommutativity or the Jacobi identity fails; If the algebra is a Lie algebra, the output will give a confirmation.

Procedure:

```
check_lie:=proc(A,n)

local i,j,k, l, m;

    for i from 1 by 1 to n do
       for j from 1 by 1 to n do
           for k from 1 by 1 to n do

             if A[i, i, k]<>0 then
               RETURN('Input is NOT a Lie algebra (',i,i,k,')=',
                           A[i,i,k], ' is not zero');
             elif A[i,j,k]+A[j,i,k]<>0 then
               RETURN('Input is NOT a Lie algebra, (',i,j,k ,')+
                 (',j,i,k,')=', A[i,j,k]+A[j,i,k],'is not zero');

              else
                for l from 1 by 1 to n do
                 if
                     simplify(sum(A[i,j,m]*A[m,l,k]+A[j,l,m]*A[m,i,k]
```

```
                                  +A[l,i,m]*A[m,j,k],    m=1..n))<>0
                    then
                     RETURN('Input is NOT a Lie algebra---the Jac(',
                                  i,j,l, ') is not zero');

                    fi;
                   od;
                  fi;
                od;
             od;
        od;
       print('Yes, input IS a Lie algebra');
   end:
```

E.2.2 Cocycles

Calling Sequence:

`cocycle(`*Lie_Algebra, Dimension*`)`

Parameters:

Lie_Algebra — An Lie algebra in the form of a 3-dimensional matrix

Dimension – The dimension of the given algebra

Synopsis:

– To compute the cocycles of a given Lie algebra.

– Input is the given Lie algebra and its dimension.

– Output is the set of constraints on the entries of the cocycles expressed as antisymmetric matrices.

Procedure:

```
cocycle:=proc(L,n)

local i,j,k,h, v,u,w,C,eqns,e,f,g;

 v:=vector(n);
 eqns:= { };
 u:=vector(n);
```

```
 w:=vector(n);
 C:=array(antisymmetric,1..n,1..n,[]);

for i to n do
 for j from i+1 to n do
  for k from j+1 to n do
    for h to n do
      v[h]:=L[i,j,h];
      u[h]:=L[j,k,h];
      w[h]:=L[k,i,h];
    od:
    e:=array(sparse, 1..n, [k=1]);
    f:=array(sparse,1..n, [i=1]);
    g:=array(sparse,1..n,[j=1]);

    eqns:=eqns union  multiply(transpose(e),multiply(C,v))+
     multiply(transpose(f),multiply(C,u))+
       multiply(transpose(g),multiply(C,w));
   od:
  od:
od:
print('The cocycles are', eqns);
end:
```

Comments: The output will give us some constraints on the entries of the antisymmetric matrix regarded as cocycles.

E.2.3 Isomorphisms

Calling Sequence:

isom(*Lie_Algebra_1, Lie_Algebra_2, Dimension*)

Parameters:

Lie_Algebra_1, Lie_Algebra_2 — Two given Lie algebras

Dimension – The common dimension of the two given algebras

Synopsis:

– To compute the isomorphism between two algebras (automorphism group can be obtained when the two algebras are identical).

– Input are two given algebras and their common dimension.

– Output is the isomorphism between the two given Lie algebras (or the automorphism group when the two algebras are identical).

Procedure:

```
isom:=proc(A,B,n)

local i,j,k,s,r,eqns,t,TEST, Andre,sols,l,S1,S2,C;

 C:=matrix(n,n);
 Andre:=matrix(n,n);
 TEST:=0;
 eqns:={ };

 for i to n-1 do
   for j from i+1 to n do
        for l to n do

          S1:=sum('A[i,j,k]*C[l,k]','k'=1..n);

          S2:=sum(C[r,i]*sum(C[s,j]*B[r,s,l],s=r+1..n),r=1..n-1)-
                sum(C[r,j]*sum(C[s,i]*B[r,s,l],s=r+1..n),r=1..n-1);

          eqns:=eqns union S1-S2=0;

        od:
    od:
 od:

 sols:=[solve(eqns)];
 t:=nops(sols);
 for i to t do
  for j to n do
      for k to n do
          Andre[j,k]:=subs(sols[i],C[j,k]);
       od:
   od:

   if simplify(det(Andre))<>0 then
    print(Andre);
    print(`The det is `, simplify(det(Andre)));
```

```
    TEST:=1;
   fi:
 od:
 if TEST=0 then
 print('These two algebras are not isomorphic');
 fi:
 end:
```

Comments: In some cases Maple V may give some error info, and not be able to find the automorphism. Then we need to use the automorphism group theorem given by Skjelbred and Sund to compute it.

E.2.4 Group Actions

Calling Sequence:

orbit(*Automorphism_Group, Dimension, Element from* $H^2(\mathfrak{g}, \mathbf{F})$)

Parameters:

Automorphism_Group — The generic automorphism for the given algebra in the form of a 2-dimensional matrix

Dimension – The dimension of the given algebra

Element from $H^2(\mathfrak{g}, \mathbf{F})$ – An element of $H^2(\mathfrak{g}, \mathbf{F})$, written as a linear combination of the basis vectors

Synopsis:

– To compute the group actions on an arbitrary element in $H^2(\mathfrak{g}, \mathbf{F})$.

– Input is the automorphism group of the given Lie algebra, the dimension of the algebra and an element from $H^2(\mathfrak{g}, \mathbf{F})$.

– Output are the corresponding entries under the group action.

Procedure:

```
orbit:=proc(aut,n,a,i,j,b,p,q,c,r,s,d,u,v,e,w,z)

local x,B,y;

  B:=array(sparse,1..n,1..n,[(i,j)=a,(j,i)=-a,(p,q)=b,(q,p)=-b,
    (r,s)=c,(s,r)=-c,(u,v)=d,(v,u)=-d,(w,z)=e,(z,w)=-e]);
```

```
  x:=transpose(aut);
  y:=multiply(x,multiply(B,aut));
   print(y);
 end:
```

Comments: This program applies to the case when $\dim H^2(\mathfrak{g}, \mathbf{F}) = 5$, and the antisymmetric element from $H^2(\mathfrak{g}, \mathbf{F})$ has nonzero values a, b, c, d, e at $(i, j), (p, q), (r, s), (u, v)$ and (w, z). The above program can be adjusted according to the different dimensions of the $H^2(\mathfrak{g}, \mathbf{F})$. Refer to Chapter 2 for the computation of normalized cocycles.

E.2.5 Derivation Algebras

Calling Sequence:

derivation(*Lie_Algebra, Dimension*)

Parameters:

Lie_Algebra — A given Lie algebras

Dimension – The dimension of the given algebra

Synopsis:

- To compute the derivation of a given Lie algebra.
- Input is a given algebra and its dimension.
- Output is the derivation algebra.

Procedure:

```
derivation:=proc(A,n)

  local i,j,k, t, s1,s2,l,D, sols,eqns, Andre;
  eqns:={ };
  D:=matrix(n,n);
  Andre:=matrix(n,n);

      for i  to n-1 do
        for j from i+1 to n do
            for l  to n do
              s1:=sum(A[i,j,k]*D[k,l],k=1..n);
              s2:=sum(A[k,j,l]*D[i,k]+A[i,k,l]*D[j,k],k=1..n);
              eqns:=eqns union s1=s2;
```

```
            od:
         od:
       od:

   sols:=[solve(eqns)];
   t:=nops(sols);
   for i to t do
       for j to n do
          for k to n do
           Andre[j,k]:=subs(sols[i],D[j,k]);
          od:
        od:
         print(Andre);
   od:
end:
```

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