Modeling of Multidimensional Linear Systems

by

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Abstract

This thesis deals with modeling and classification of multidimensional linear systems. In a behavioural framework, two representations of such systems are used: AR representations and ARMA representations. Three first-degree ARMA representations: Dual Pencil, Pencil, and Descriptor representations are defined and recasting methods between them are given. Several rank conditions which allow these recasting methods to result in equivalent representations with fewer auxiliary variables are found. With respect to each first-degree model a definition of order is given and some necessary rank conditions which allow reduction of order are derived. All AR representations of a given behaviour are associated with a vector space generated by their row spaces. A definition of order for each AR representation associated with this vector space is given and it is shown how to obtain a minimal order AR representation from any given AR representation using primary decomposition of polynomial equations and their p-adic valuations. A survey of the existing work that shows its limitations and extent is given.

Acknowledgements

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Contents

Chapter 1

Introduction

Multidimensional (n-D) system analysis has gained increasing importance over the last two decades. but the modeling of such systems is still one of the fundamental developments to be completed. One approach has been to generalize 1-D state space concepts. However, due to some major differences between the algebraic structure of 1-D and n-D systems, these generalizations and their corresponding results have been somewhat limited.

Inherent in the classical approaches is the concept of past, present and future in the operating time domain. In the n-D case. there is no natural analogue to this. Consequently, different possibilities. sometimes infinite. exist for state-updating equations. Additionally, any imposed past will have several *present* states at any given "time". In the 2-D case the 'present' is given by a line; in n-D, a hyperplane of dimension $n-1$. Thus two notions of state can be distinguished: each of these hyperplanes (of dimension n-1) constitutes a *global state*, analogous to the 1-D state variable, whereas each point in the hyperplane constitutes a *local state*. Therefore two kinds of system properties, local and global, can be characterized.

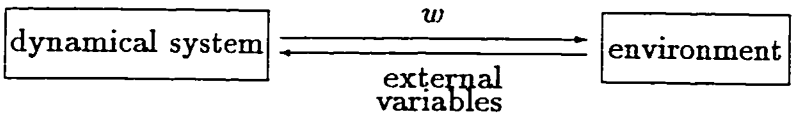

Figure 1.1: Behavioural modeling of a dynamical system

Many results in 1-D system theory over a field K are related to the Euclidean character of $K[z]$. These tools are no longer available in the n-D case which really belongs to the study of commutative Noetherian rings. These differences. among others. indicate that another approach may be desirable.

In this thesis I adapt to n-D systems an approach that was originally introduced by J.C. Willems in [55] in the study of 1-D cases, and was generalized to 2-D by Rocha [47]. A dynamical system is viewed as an entity which interacts with its environment through variables called *external variables.* (see Fig.1.1) The system laws govern the relationships between the external variables of the system and give rise to a family of admissible trajectories for the external variables . The set of all admissible system trajectories is called the system *behaviour.*

A mathematical description of a system by means of equations constitutes a *representation* of the system. In many situations it is convenient to work with representations which contain auxiliary variables in addition to the external variables. These additional variables will be called *internal variables* (sometimes referred to as *latent variables* [57]).

A typical example of internal variables is given by the state variables which are normally introduced in order to write the dynamical equations of a given system as first-degree equations.

Some interesting differences between the classical models and the behavioural models should be noted:

- In some situations, it is not convenient or possible to have *a priori* knowledge of the division of the system external variables into inputs and outputs. The behavioural approach can deal with these situations, as well as the classical context where such a priori knowledge is given or assumed.
- A dynamical system is characterized by a *set* (the system behaviour), rather than a *function* such as an input-output map.

The above idea will be illustrated by examples:

Example 1 *In seismology it is often of interest to measure the signal velocities in x, y, and z directions, since seismic waves penetrate formations of different characteristics, with different velocities. Let a 3-D velocity filter be such that it passes the signals whose velocities fall within a given cone-shaped region in the (f, k) space, where f is frequency and k is a 2-D wave number having two components k_x and k_y and given by $k = (k_x^2 + k_y^2)^{1/2}$ [26]. Then the behaviour of the system, determined by the geophysical structure, is the set of velocities $\mathbf{v} = (v_x, v_y, v_z)$ which are compatible with the geophysical laws and given by*

$$\mathcal{B} = \{(v_x, v_y, v_z) : \mathbb{R}^3 \to \mathbb{R}^3 | \mathbf{v} \text{ falls within the given cone-shaped region}\}. \quad (1.1)$$

Example 2 *A 2-D discrete modeling of natural self-purification process of a river is given be the following equations [13]:*

$$\begin{aligned} \beta((h+1)\Delta t, (k+1)\Delta l) &= (1 - k_1\Delta t)(\beta(h\Delta t, k\Delta l) + in_\beta(h\Delta t, k\Delta l)), \quad (1.2) \\ \delta((h+1)\Delta t, (k+1)\Delta l) &= k_1\Delta t\beta(h\Delta t, k\Delta l) + \\ & \quad (1 - k_2\Delta t)(\delta(h\Delta t, k\Delta l) + in_\delta(h\Delta t, k\Delta l)), \quad (1.3) \end{aligned}$$

where Δl *is length equal to* $v.\Delta t$. *for the time step* Δt *and the river velocity* v. $\beta(t.l)$, *is the concentration of oxygen needed for complete biochemical oxidation (BOD) of the pollutants ,* $\delta(t.l)$ *is the dissolved oxygen (DO) concentration deficit with respect to the saturation level,* $in_\beta(t,l)$, *and* $in_\delta(t,l)$ *are BOD and DO sources respectively, and* k_1 *and* k_2 *are two constants reflecting the self-purification process. due to the degradation of the originally discharged pollutants by bacteria. and the reaeration process, taking place at the water/atmosphere surface.*

Then the behaviour of the system is the set of all possible concentration levels of BOD and DO which satisfy (1.2) and (1.3), that is

$$\mathcal{B} = \{(in_\beta, in_\delta) : \mathbb{Z}^2 \to \mathbb{R}^2 | \text{ such that (1.2) and (1.3) are satisfied}\}. \quad (1.4)$$

Example 3 *[47]: A mathematical description of a discretized monochromatic image on a plane can be given by a function* $g : \mathbb{Z}^2 \to \mathbb{R}^2$ *which associates with each pixel at position* $(t_1, t_2) \in \mathbb{Z}^2$ *the corresponding gray-level* $g(t_1, t_2)$. *After simplifying the situation by restricting attention to edges with a pre-specified orientation. say vertical edges. enhancement of an image with gray-level function* g_1 *is given by the construction of a second image whose gray-level function* g_2 *is obtained from* g_1 *by the following equation:*

$$\begin{aligned} g_2(t_1, t_2) = {} & g_1(t_1+1, t_2+1) - g_1(t_1-1, t_2+1) + g_1(t_1+1, t_2 \\ & -g_1(t_1-1, t_2) + g_1(t_1+1, t_2-1) - g_1(t_1-1, t_2-1). \end{aligned} \quad (1.5)$$

The 2-D dynamical system corresponding to this process has g_1 *and* g_2 *as system variables and is characterized by the behaviour*

$$\mathcal{B} = \{(g_1, g_2) : \mathbb{Z}^2 \to \mathbb{R}^2 | \text{ such that equation (1.5) is satisfied }\}. \quad (1.6)$$

1.1 Outline of thesis

An outline of this thesis is as follows:

Chapter 2 contains a literature survey. Two commonly used n-D state space models, namely those of Roesser and Fornasini-Marchesini, are listed, as well as implicit representations of linear systems by Aplevich, and a definition of minimal bases for polynomial-matrix models of multivariable linear systems by Forney is given. Definitions of order and minimality with respect to these models are quoted, and order reduction algorithms and necessary and sufficient conditions suggested in the literature are closely examined and their restrictions and limitations are stated. Some minor new results relating to these models will be given. This will also allow us to place these models in the context of the work presented in this thesis.

In chapter 3, the behavioral models of multidimensional dynamical systems are formally defined. After giving a self-contained exposition of behavioural models of dynamical systems, some of the possible representations of these models will be stated: specifically the AR, ARMA, and MA models. In particular, a special type of ARMA model which is first-degree in the internal variables and zero-th degree in the external variables will be described. Different representations of this form will be described and a systematic way to obtain an equivalent representation of a different form will be given. This will also include the embedding of n-D Roesser and Fornasini-Marchesini models in the given first-degree form.

In chapter 4, definitions of order and minimality with respect to this order for the ARMA representations described in chapter 2 is given. Some necessary conditions under which a representation is minimal will be derived.

In chapter 5, some basic algebraic definitions and results with respect to polynomial equations are given: multivariate polynomial factorization, p-adic valuation, and then a definition of order for AR representations will be given. It will be shown how to find a minimal basis for polynomial equations describing an AR representation. It will be shown that all AR representations with the same external behaviour can be related to the AR representation given by this minimal basis.

Chapter 6 contains conclusions and suggestions for further work.

For brevity in exposition, throughout this thesis the 2-D case will be emphasized. In cases where brevity is not sacrificed, the results will be stated in the general n-D format. Situations where the generalization from the 2-D case to the general n-D case is not immediate, or in fact not possible, will be noted.

In the appendix, n-D ($n > 2$) generalizations of a known 2-D result, namely the relationship between n-Dimensional Roesser and Fornasini-Marchesini models is given.

An outline of new contributions made in this thesis is as follows:

- In this thesis, since only linear systems are analyzed, the term first-degree will refer to the degree of the operators used in polynomial equations describing a system. In chapter 2, a survey of the existing first-degree models is given and their corresponding definitions of minimality are also placed into their context. Some new results which show the extent and limitations of previous conditions for minimality of previous models are also given.

- In chapter 3, three new first-degree ARMA representations are defined. Recasting methods between the given first-degree ARMA representations are developed. Rank conditions which allow these recasting methods to result in an equivalent first-degree representation with a smaller number of internal variables are given. It is also shown how to recast a given AR representation into the given first-degree ARMA representations and vice versa.

- In chapter 4, new definitions of order for first-degree ARMA representations are given. Necessary rank conditions which characterize a minimal order first-degree ARMA representation are found.

- In chapter 5, a new definition of order with respect to AR representations of a given dynamical system is given and primary decomposition of polynomial equations and their p-adic valuations is used to find a least order AR representation for a given system. It is shown how all AR representations and a minimal order AR representation of a given system correspond to different bases of the same vector space.

- In the appendix, the relationship between two commonly used n-D state space models of Roesser and Fornasini-Marchesini for $n > 2$ is derived.

Chapter 2

Previous work

New first-degree ARMA models will be described in the next chapter. Chapter 4 contains certain minimality results. In order to place these results into the context of the existing first-degree ARMA models, in this chapter I will give a literature survey of first-degree n-D models, namely the Roesser model and the implicit model. I have also included Fornasini-Marchesini models, despite the fact that they are not in first-degree form, since they are commonly used in practice. Definitions of order and minimality with respect to these models are quoted, and order reduction algorithms and necessary and sufficient conditions suggested in the literature are closely examined and their restrictions and limitations are illustrated by examples.

In section 2.1, the Roesser model for n-D systems is given. This is followed in subsection 2.1.1, with the definition of a minimal order Roesser realization for a single-input, single output (SISO) system with strictly proper transfer function. Some of the differences that set minimality of n-D systems apart from their 1-D counterparts, and some of the suggested conditions that characterize minimality are given. This subsection is concluded by showing that there is no known method to obtain a minimal Roesser model from an arbitrary transfer function. with the only exception being the causal SISO system with a transfer function which is separable in the denominator or numerator. One of the methods that achieves a minimal realization for such systems, the Hinamoto and Fairman method. is given.

In section 2.2, a survey of definitions and results, similar to the ones given in section 2.1. is given for Fornasini-Marchesini models. This includes the generalization of Fornasini-Marchesini models to behavioural models by Rocha. In subsection 2.2.1, after giving a definition of minimality and stating the difference between the definition of order for Roesser models and Fornasini-Marchesini models. a survey of attempts by Fornasini-Marchesini to obtain a minimal model realization is given. Of particular interest are the method of obtaining minimality associated with a non-commutative power series using the B. L. Ho algorithm. and attempts to solve the minimal realization problem for the commutative case using a corresponding non-commutative power series and an inverse map.

In section 2.3, implicit models of linear systems. definition of order and minimality with respect to the defined order. and necessary conditions for minimality suggested by Aplevich are given. An illustrative example shows that the suggested conditions are not sufficient.

Another result presented in this thesis is the classification of minimal order AR representations which will be given in chapter 5. The main result in chapter 5 draws on Forney's work on polynomial-matrix models in 1-D . In section 2.4. I will give a summary of the result by Forney which will be used.

2.1 The Roesser Model

The Roesser model [49] allows the generalization of certain analysis. structure. and design results for one-dimensional state space systems by allowing the local state of its model to be divided into an *horizontal* and a *vertical* state which are propagated, respectively, horizontally and vertically by first-order difference equations. Extensions of the Roesser model to n-D can be found in [4]. [31]. [28]. and [22]. Also. with the growing interest in singular systems, 2-D (n-D) Roesser models are generalized as follows([27] [36], [37]):

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i,j) \quad (2.1)$$

$$y(i,j) = [C_1 \quad C_2] \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Du(i,j). \quad (2.2)$$

Example 4 *Consider the hyperbolic partial differential equation [40]*

$$\frac{\partial T(x,t)}{\partial x} = -\frac{\partial T(x,t)}{\partial t} - T(x,t) + u(t) \quad (2.3)$$

with initial and boundary value conditions:

$$T(x,0) = f_1(x). \qquad T(0,t) = f_2(t) \quad (2.4)$$

Let

$$T(i,j) = T(i\Delta x, j\Delta t). \qquad u(j) = u(j\Delta t) \quad (2.5)$$

$$\frac{\partial T(x,t)}{\partial x} \simeq \frac{T(i,j) - T(i-1,j)}{\Delta x}, \qquad \frac{\partial T(x,t)}{\partial t} \simeq \frac{T(i,j+1) - T(i,j)}{\Delta t}. \quad (2.6)$$

If we define $x^h(i,j) = T(i-1,j)$ *and* $x^v(i,j) = T(i,j)$, *then by rewriting (2.3) in the form*

$$T(i,j+1) = a_1 T(i,j) + a_2 T(i-1,j) + Bu(j) \quad (2.7)$$

where $a_1 = 1 - \frac{\Delta t}{\Delta x} - \Delta t$, *and* $a_2 = \frac{\Delta t}{\Delta x}$, *and* $B = \Delta t$; *we obtain the Roesser model*

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_2 & a_1 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u(j). \tag{2.8}$$

The suggested definition of *order* for a Roesser model realization of a given input-output behaviour is the dimension of the vector x in (2.1) and (2.2). In the following subsection, we investigate minimal order Roesser realizations and necessary and sufficient conditions for such realizations, as well as differences between 1-D and n-D cases.

2.1.1 Minimality

In 1-D state-space representations, the importance of minimal representations is due to factors such as uniqueness of system identification, economy, and more importantly, describing the input-output behaviour via concepts of controllability and observability. The original Roesser model [49] was defined for 2-D systems with quarter-plane causal transfer functions. Given a SISO 2-D causal system, let the highest powers of two operators in the denominator of its transfer function be, say n and m. Then a commonly used definition of *minimal* order is a Roesser model with state vector of dimension $n+m$ [34], however as will be shown in the following pages minimality in n-D ($n \geq 2$) has a field dependent nature. To the authors's knowledge, there has not been a definition of minimality for SISO systems with improper transfer functions. Furthermore, despite the reduction schemes for multi-input, multi-output (MIMO) Roesser realizations (for example, see [46] and [45]) the definition of minimality for such systems is unclear.

Before investigating necessary and sufficient conditions suggested for minimality of SISO 2-D Roesser models, several distinctions should be made between 2-D (n-D) minimal systems and 1-D systems:

1. It has been proven that all minimal realizations of a given 1-D transfer function are equal modulo a bijection on the state-space (i.e. a change of basis). However, as the following example will show, this is not the case for n-D systems:

Example 5 *Consider the transfer function*

$$H(z_1, z_2) = \frac{z_1 - z_2}{z_1 z_2 - 1} \tag{2.9}$$

Then

$$\begin{aligned} \begin{bmatrix} x_1(i+1, j) \\ x_2(i, j+1) \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(i, j) \\ x_2(i, j) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(i, j) && (2.10) \\ y(i, j) &= [-1 \quad 1] \begin{bmatrix} x_1(i, j) \\ x_2(i, j) \end{bmatrix} && (2.11) \end{aligned}$$

and

$$\begin{aligned} \begin{bmatrix} x'_1(i+1, j) \\ x'_2(i, j+1) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x'_1(i, j) \\ x'_2(i, j) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(i, j) && (2.12) \\ y(i, j) &= [-1 \quad 1] \begin{bmatrix} x'_1(i, j) \\ x'_2(i, j) \end{bmatrix} && (2.13) \end{aligned}$$

are minimal realizations of the same transfer function (2.9), but they are not algebraically equivalent, in the sense that we cannot find a non-singular matrix T such that

$$\begin{bmatrix} x_1(i, j) \\ x_2(i, j) \end{bmatrix} = T \begin{bmatrix} x'_1(i, j) \\ x'_2(i, j) \end{bmatrix}.$$

2. The dimension of a minimal realization depends on the field in which the entries of the matrices in equations (2.1) and (2.2) are chosen.

Example 6 *[34] Consider the transfer function*

$$H(z_1, z_2) = \frac{z_1 + z_2}{z_1 z_2 - 1} \tag{2.14}$$

It will be shown that although it is possible to get a Roesser realization of order 2 for the above system if the constants are taken over field of complex numbers. the smallest realization over real numbers is of order 3.

Assume to the contrary, that there exists a realization of order 2 over reals in the Roesser form for

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \; C = [\, c_1 \quad c_2 \,], \; \text{and } D. \tag{2.15}$$

Then we have

$$\det \begin{bmatrix} z_1 - a_{11} & -a_{12} \\ -a_{21} & z_2 - a_{22} \end{bmatrix} = z_1 z_2 - 1. \tag{2.16}$$

or

$$z_1 z_2 - a_{22} z_1 - a_{11} z_2 + a_{11} a_{21} - a_{12} a_{21} = z_1 z_2 - 1. \tag{2.17}$$

which implies that

$$a_{22} = a_{11} = 0. \; a_{12} a_{21} = 1. \text{ or } a_{21} = \frac{1}{a_{12}}.$$

Let $a_{21} = \alpha \Rightarrow a_{12} = \alpha^{-1}$. *Then we have*

$$[\, c_1 \quad c_2 \,] \begin{bmatrix} z_2 & -\alpha \\ -\alpha^{-1} & z_1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + D(z_1 z_2 - 1) = z_1 + z_2 \tag{2.18}$$

$$[\, c_1 \quad c_2 \,] \begin{bmatrix} b_1 z_2 - \alpha b_2 \\ -\alpha^{-1} b_1 + b_2 z_1 \end{bmatrix} = z_1 + z_2 \quad \text{and } D = 0. \tag{2.19}$$

or

$$c_1 b_1 z_2 - c_1 \alpha b_2 - c_2 \alpha^{-1} b_1 + c_2 b_2 z_1 = z_1 + z_2,$$

which implies

$$\begin{aligned} c_1 b_1 &= 1 \Rightarrow b_1 = \frac{1}{c_1} \text{ and } b_1 \neq 0 \\ c_2 b_2 &= 1 \\ 0 &= c_1 b_2 \alpha + c_2 b_1 \alpha^{-1} \end{aligned}$$

or

$$\frac{b_2}{b_1}\alpha + \frac{b_1}{b_2}\alpha^{-1} = 0.$$

Multiply both side of the above equation by $\frac{b_2}{b_1}\alpha$ *to get the equation*

$$\frac{b_2^2}{b_1^2}\alpha^2 + 1 = 0,$$

which clearly has no real solutions.

However, It is also easy to show that a possible Roesser realization over reals is given by

$$A = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, C = [1, \quad 0, \quad 1]. \qquad (2.20)$$

This shows that a Roesser realization of order 2 is only possible if constants are taken over the complex field, whereas realization over reals has to be at least of order 3.

3. Unlike 1-D systems, where a state-space realization is minimal if and only if it is controllable and observable, it has been shown that local controllability and observability [49] are neither necessary nor sufficient conditions for minimality. Definitions of local controllability and observability are given by Roesser [49] to be:

Definition 1 : *A state* $x_0 = x(i_1, i_2)$ *is controllable in the rectangle* $[(0,0),(n_1,n_2)]$ *if and only if when all boundary conditions are zero, there exists some pair* $(0,0) \leq (h,k) \leq (n_1,n_2)$ *and some input pattern such that* $x(h,k) = x_0$. *A Roesser realization is called* locally controllable *if all the states are controllable. This was proved to be equivalent to having* controllability matrix Q *to have full rank, where* Q *is defined to be*

$$Q = [M(0,0), M(0,1), \cdots, M(0,n_2), M(1,0), \cdots, M(n_1,n_2-1)], \qquad (2.21)$$

where

$$M(i,j) = A^{i-1,j}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} + A^{i,j-1}\begin{bmatrix} 0 \\ B_2 \end{bmatrix} \quad \text{where } A^{i,j} \text{ is the transition matrix}$$

and the transition matrix is defined to be

$$A^{i,j} = A^{1,0}A^{i-1,j} + A^{0,1}A^{i,j-1} \qquad (i,j) > (0,0)$$

$$A^{0,0} = I, \qquad A^{0,1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \qquad A^{1,0} = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

$$A^{-i,j} = A^{i,-j} = 0. \qquad \text{for} \quad j \geq 1, i \geq 1.$$

Definition 2 *: A state $x_0 = x(i_1, i_2)$ is observable $[(0,0),(n_1,n_2)]$ if and only if there is no non-zero initial state such that for zero boundary conditions and zero inputs, the output is also zero. A Roesser realization is called* locally observable *if all the states are observable. This was proved to be equivalent to having to* observability matrix *O to have full rank. where O is defined to be*

$$O = [C^T, C^T(A^{1,0})^T, \cdots, C^T(A^{n_1,n_2-1})^T]^T. \tag{2.22}$$

Example 7 *[34] Consider the transfer function*

$$H(z_1, z_2) = \frac{z_1 - z_2}{z_1 z_2 - 1} \tag{2.23}$$

The Roesser realization of this transfer function

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, C = [-1 \quad 1 \quad 0] \tag{2.24}$$

is locally controllable and observable. However, it is also easy to verify that another Roesser realization of $H(z_1, z_2)$ of smaller order is given by

$$A' = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, B' = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, C' = [1 \quad 1], \tag{2.25}$$

exhibiting the fact that local controllability and observability do not imply minimality. Also, for the transfer function

$$H(z_1, z_2) = \frac{z_2(z_1 + z_2 - 1)}{(z_2 - 1)(z_1 z_2 - z_1 - z_2)} \tag{2.26}$$

the realization

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \; C = [1 \quad 1 \quad 1] \tag{2.27}$$

is clearly minimal. However this realization is not locally observable since the observability matrix does not have full rank.

Different definitions of controllability and observability have been suggested to establish a connection between these concepts and minimality. Some of these are the notions of separate local controllability and observability, global controllability and observability, and modal controllability and observability [34]:

Definition 3 *Partition matrices Q and O in the previous definitions to be*

$$O = [\overbrace{O^h}^{n_1}, \vdots, \overbrace{O^v}^{n_2}], \quad Q = \begin{bmatrix} Q^h \\ \cdots \\ Q^v \end{bmatrix} \begin{matrix} \}n_1 \\ \\ \}n_2 \end{matrix}.$$

Then, $x^h(i,j)$ and $x^v(i,j)$ are separately locally observable (separately locally controllable) if and only if O^h and O^v (Q^h and Q^v) are separately of full rank.

Definition 4 *Define the* controllability map, *C to be*

$$\mathcal{C} : \mathcal{U} \rightarrow \mathcal{X}_{0,0}$$

and the observability map, $\mathcal{O}$ *to be*

$$\mathcal{O} : \mathcal{X}_{0,0} \rightarrow \mathcal{Y}$$

where $\mathcal{U}$ *is the space of past inputs, and* $\mathcal{X}_{0,0} = (x^h(0,j), x^v(i,0)), i, j \in \mathcal{Z}^+$ *is the global initial state, and* $\mathcal{Y}$ *is the space of future outputs. Then the Roesser model is is globally controllable if and only if the controllability map* C *is a surjective map, and is globally observable if and only if the observability map* $\mathcal{O}$ *is an injective map.*

By using the notions of separate local controllability and observability Kung *et al* [34] were able to show that separate local controllability and observability are necessary conditions for minimality. However, it was shown that neither separate local controllability and observability nor global controllability and observability are sufficient conditions for minimality. To tackle this problem, Kung *et al* suggested the following definition and proposition for 2-D systems based on notion of coprimeness. A detailed treatment of algorithms testing for relative coprimeness of n-variate polynomials can be found in [7].

Definition 5 *: C, A in (2.1)-(2.2) are* modally *observable if*

$$C, \begin{bmatrix} z_1 I_{n_1} & 0 \\ 0 & z_2 I_{n_2} \end{bmatrix} - A \tag{2.28}$$

are right coprime.

Definition 6 *:A, B in (2.1) are* modally *controllable if*

$$\begin{bmatrix} z_1 I_{n_1} & 0 \\ 0 & z_2 I_{n_2} \end{bmatrix} - A, B \tag{2.29}$$

are left coprime.

Conjecture 1 *The realization of a dynamical system given by the equations (2.1) and (2.2) is minimal if and only if A, B are modally controllable and C, A are modally observable.*

The major difficulty with such a conjecture is that the existence of such realizations cannot be guaranteed on the field of rationals and even their existence on the complex field has only been proven when the degree of z_1 and z_2 in the denominator of the transfer function are equal to one.

It should be noted again that the definition of minimal order realization in the above is only given for regular SISO Roesser models and is not applicable to many systems which cannot be realized in this form. A simple example of a system which does not have a Roesser model realization is one where input-output behaviour is given by the equation $z_1 y = z_2 u$.

The difficulty with the conjecture posed by Proposition 1, as well as close examination of other suggested conditions of minimality have shown that there is no known method of obtaining a minimal Roesser model from an arbitrary given transfer function. The only exception is the causal SISO systems with a transfer function which is separable in the denominator or numerator. In the remainder of this subsection, the Hinamoto and Fairman method of obtaining a minimal order Roesser realization from a non-minimal Roesser realization or directly from the input-output data for regular, separable SISO 2-D transfer functions will be shown.

In the 2-D case, where the denominator (or numerator) of the causal transfer function is separable, that is the denominator (or numerator) can be written as the product of two 1-D polynomials, it has been shown that a corresponding Roesser realization can be reduced to have a minimal order. It is also possible to obtain a realization of this order using a generalization of the B. L. Ho algorithm. It is easy to see that separability of the denominator implies that the matrices A_{12} or A_{21} in (2.1) are null matrices, since the denominator of the transfer function will be given by $(z_1 I - A_{11})(z_2 I - A_{22})$. Roesser [49] had shown transformations in the class of state space realizations given by

$$T = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix} \tag{2.30}$$

where T_{11} and T_{22} are non-singular matrices and 0 is the null matrix of appropriate dimension, will not affect the transfer function.

Let a regular Roesser realization of form (2.1)-(2.2) be given for a strictly proper SISO transfer function with highest degree in z_1 of m and in z_2 of n. Hinamoto and Fairman [25] showed that a realization of order $m+n$ can be obtained from a given realization using a similarity transformation of the form (2.30) for

$$T_{11} = [B_1, A_{11}B_1, \cdots, A_{11}^{n-1}B_1]^{-1} \tag{2.31}$$

$$T_{22} = [C_2^T, (C_2A_{22})^T, \cdots, (C_2A_{22}^{m-1})^T]^T. \tag{2.32}$$

If $(\tilde{A}, \tilde{B}, \tilde{C})$ represent a Roesser realization of order $m+n$, then we have

$$\tilde{A}_{11} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \alpha_1 \\ 1 & 0 & \cdots & 0 & \alpha_2 \\ 0 & 1 & \cdots & 0 & \alpha_3 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & \alpha_n \end{bmatrix}, \quad \tilde{A}_{21} = \begin{bmatrix} w_{11} & w_{21} & \cdots & w_{n1} \\ w_{12} & w_{22} & \cdots & w_{n2} \\ w_{13} & w_{23} & \cdots & w_{n3} \\ \vdots & \vdots & \ddots & \vdots \\ w_{1m} & w_{2m} & \cdots & w_{nm} \end{bmatrix},$$

$$\tilde{A}_{22} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \\ \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_m \end{bmatrix},$$

and

$$\begin{aligned}\tilde{B}_1 &= [1, 0, \cdots, 0]^T, \quad \tilde{B}_2 = [w_{01}, w_{02}, \cdots, w_{0m}]^T \\ \tilde{C}_2 &= [1, 0, \cdots, 0], \quad \tilde{C}_1 = [w_{10}, w_{20}, \cdots, w_{n0}]\end{aligned}$$

where the α_i's and β_j's are real numbers satisfying

$$\begin{aligned}A_{11}^n B_1 &= \alpha_1 B_1 + \alpha_2 A_{11} B_1 + \cdots + \alpha_n A_{11}^{n-1} B_1 \\ C_2 A_{22}^m &= \beta_1 C_2 + \beta_2 C_2 A_{22} + \cdots + \beta_m C_2 A_{22}^{m-1},\end{aligned}$$

and the two-dimensional Markov parameters, w_{ij} are given by

$$\begin{aligned}w_{i0} &= C_1 A_{11}^{i-1} B_1 \\ w_{0j} &= C_2 A_{22}^{j-1} B_2 \\ w_{ij} &= C_2 A_{22}^{j-1} A_{21} A_{11}^{i-1} B_1 \quad \text{for } (i,j) > (0,0).\end{aligned}$$

The canonical representation can also be found directly from the Markov parameters using a generalization of the B. L. Ho algorithm. The Markov parameters w_{ij} are defined such that for zero boundary conditions the following equation holds:

$$y(i,j) = \sum \sum_{(0,0)<(h,k)\leq(i,j)} w_{hk} u(i-h, j-k). \tag{2.33}$$

Define a sequence of Markov parameters, W_i to be

$$W_i = \{w_{i0}, w_{i1}, \cdots, w_{ii}, w_{i-1,i}, \cdots, w_{1i}, w_{0i}\}.$$

Assume that some upper bound, say N, is known for the sequence $\{W_i\}$. It will follow that the obtained realization will always have an order less than or equal to

$2N$. Define horizontal and vertical Hankel matrices to be $\bar{H}$ and $\tilde{H}$ respectively. as

$$\bar{H}_{NN}^{(N)} = \begin{bmatrix} \bar{W}_1^{(N)} & \bar{W}_2^{(N)} & \cdots & \bar{W}_{N-1}^{(N)} & \bar{W}_N^{(N)} \\ \bar{W}_2^{(N)} & \bar{W}_3^{(N)} & \cdots & \bar{W}_N^{(N)} & * \\ \vdots & \vdots & & * & * \\ \bar{W}_{N-1}^{(N)} & \bar{W}_N^{(N)} & * & & * \\ \bar{W}_N^{(N)} & * & * & * & * \end{bmatrix} \quad (2.34)$$

$$\tilde{H}_{NN}^{(N)} = \begin{bmatrix} \tilde{W}_1^{(N)} & \tilde{W}_2^{(N)} & \cdots & \tilde{W}_{N-1}^{(N)} & \tilde{W}_N^{(N)} \\ \tilde{W}_2^{(N)} & \tilde{W}_3^{(N)} & \cdots & \tilde{W}_N^{(N)} & * \\ \vdots & \vdots & & * & * \\ \tilde{W}_{N-1}^{(N)} & \tilde{W}_N^{(N)} & * & & * \\ \tilde{W}_N^{(N)} & * & * & * & * \end{bmatrix} \quad (2.35)$$

where the $*$'s indicate the corresponding entries that do not effect the rank of either $\bar{H}$ or $\tilde{H}$ matrices. It was shown that

$$n = \sum_{i=1}^{N} \operatorname{rank} \bar{H}_{N-i+1,i}^{(N)} - \sum_{i=1}^{N-1} \operatorname{rank} \bar{H}_{N-i,i}^{(N)} \quad (2.36)$$

$$m = \sum_{i=1}^{N} \operatorname{rank} \tilde{H}_{N-i+1,i}^{(N)} - \sum_{i=1}^{N-1} \operatorname{rank} \tilde{H}_{N-i,i}^{(N)} \quad (2.37)$$

where $\bar{H}_{ij}^{(N)}$ and $\tilde{H}_{ij}^{(N)}$ are defined as in (2.34)-(2.35). It is easy to show then that a canonical realization corresponding to the Markov parameters is given by $(\tilde{A}, \tilde{B}, \tilde{C})$ of page 20.

2.2 The Fornasini-Marchesini Model

The Fornasini-Marchesini approach to the algebraic realization problem of 2-D systems is to use input-output maps to obtain state space representations by Nerode equivalence classes of inputs (for details see [14]). Howe r, such (Nerode) representations are usually infinite dimensional and pose difficulties in describing the dynamics of systems in terms of a recursive 'updating equation'. Fornasini-Marchesini were the first to overcome these difficulties by introducing the notion of *local state* vs. *global state* in the 2-D case. To explain these notions, let $\mathcal{P}$ be a partially ordered set. A *cross-cut* $\mathcal{C} \subset \mathcal{P}$ is a set of points such that if $i \in \mathcal{P}$ exactly one of the following holds true: [42]

(i) $i < j$ for some $j \in \mathcal{C}$

(ii) $i \in \mathcal{C}$

(iii) $i > j$ for some $j \in \mathcal{C}$

Thus, a cross-cut $\mathcal{C}$ partitions $\mathcal{P}$ into three distinct sets of points, identified by (i), (ii) and (iii) which we will call *past*, *present* and *future* respectively, with respect to $\mathcal{C}$. There are usually infinitely many possibilities for selecting the set $\mathcal{C}$.

Define the following partial ordering for integer pairs:

$(h,k) \leq (i,j)$ if and only if $h \leq i$ and $k \leq j$

$(h,k) = (i,j)$ if and only if $h = i$ and $k = j$

$(h,k) < (i,j)$ if and only if $(h,k) \leq (i,j)$ and $(h,k) \neq (i,j)$

In $\mathbb{Z} \times \mathbb{Z}$, using the above partial ordering, a cross-cut $\mathcal{C}$ has the following characteristic properties: [17]

(i) if $h > i,\ k > j,\ (h,k)$ and (i,j) cannot belong simultaneously to $\mathcal{C}$

(ii) if $(h,k) \in \mathcal{C}$, then $\mathcal{C}$ intersects the sets $\{(h-1,k),(h,k+1),(h-1,k+1)\}$ and $\{(h+1,k),(h,k-1),(h+1,k-1)\}$ and does not contain the set $\{(h+1,k),(h,k+1)\}$

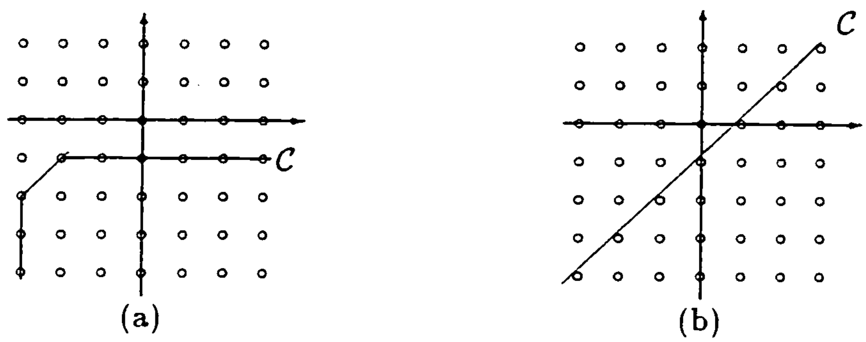

Figure 2.1: Examples of cross-cuts of $\mathbb{Z} \times \mathbb{Z}$

(iii) for any (i, j) in $\mathbb{Z} \times \mathbb{Z}$, the relation $(h, k) \leq (i, j)$ cannot be satisfied by infinitely many elements (h, k) in C.

Figure 2.1 illustrates two examples of cross-cuts of $\mathbb{Z} \times \mathbb{Z}$. A finite dimensional *local state* x is assigned to each point (h, k) of the plane. The *global state* X_C on the separation set C is defined as follows:

$$X_C = \{x(h, k) : (h, k) \in C\}$$

A distinction between global and local state in 2-D should be made. since unlike the 1-D case where the separation between past and future is given by a *point* (i.e. the present). in the 2-D case of planar domains such as $\mathbb{Z} \times \mathbb{Z}$. for the simplest case. the *present* will be given by a *line*. The global state is then defined on the lines which constitute the propagation front analogously to 1-D state variables. A local state is assigned to each single point of the domain and the collection of all its values along a propagation front will constitute a global state.

Assuming that the system is *linear, and shift invariant*, Fornasini-Marchesini proposed the two following models for updating the finite dimensional local state space. First [14]

$$\begin{aligned} x(h+1,k+1) &= A_0x(h,k) + A_1x(h+1,k) \\ &+ A_2x(h,k+1) + Bu(h,k) \qquad (2.38) \\ y(h,k) &= Cx(h,k) \qquad (2.39) \end{aligned}$$

referred to as the first kind[1], or [17] the second kind

$$\begin{aligned} x(h+1,k+1) &= A_1x(h,k+1) + A_2x(h+1,k) \\ &+ B_1u(h,k+1) + B_2u(h+1,k) \qquad (2.40) \\ y(h,k) &= Cx(h,k) \qquad (2.41) \end{aligned}$$

where $(h,k) \in \mathbb{Z} \times \mathbb{Z}$ partially ordered by the product of ordering

$x(h,k) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^n$ is a map whose value at time (h,k) is called the local state at time (h,k),

$u(h,k) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is the input with value $u(h,k)$ at time (h,k)

$y(h,k) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is the output with value $y(h,k)$ at time (h,k)

A_0, A_1, A_2, B_1, B_2 are real matrices of appropriate dimensions.

A generalization of the two proposed models was given by Kurek [35] :

$$\begin{aligned} x(h+1,k+1) &= A_0x(h,k) + A_1x(h,k+1) + A_2x(h+1,k) \qquad (2.42) \\ &+ B_0u(h,k) + B_1u(h,k+1) + B_2u(h+1,k) \\ y(h,k) &= Cx(h,k) + Du(h,k) \qquad (2.43) \end{aligned}$$

The n-D implicit generalization of the above is given by Kaczorek [28]:

$$Ex_{i_1+1,i_2+1,\cdots,i_n+1} = \qquad (2.44)$$

[1]Sometimes this is referred to the case where A_0 is identically zero [16].

$$
\begin{aligned}
& A_0 x_{i_1,i_2,\cdots,i_n} + \sum_{j=1}^{n} A_j x_{i_1,\cdots,i_{j-1},i_j+1,i_{j+1},\cdots,i_n} + \\
& + \sum_{1\le j<k\le n} A_{jk} x_{i_1,\cdots,i_{j-1},i_j+1,i_{j+1},\cdots,i_{k-1},i_k+1,i_{k+1},\cdots,i_n} + \cdots + \\
& + \sum_{j=1}^{n} A_{1,\cdots,j-1,j+1,\cdots,n} x_{i_1+1,\cdots,i_{j-1}+1,i_j,i_{j+1}+1,\cdots,i_n+1} + \\
& + B_0 u_{i_1,i_2,\cdots,i_n} + \sum_{j=1}^{n} B_j u_{i_1,\cdots,i_{j-1},i_j+1,i_{j+1},\cdots,i_n} + \\
& + \sum_{1\le j<k\le n} B_{jk} u_{i_1,\cdots,i_{j-1},i_j+1,i_{j+1},\cdots,i_{k-1},i_k+1,i_{k+1},\cdots,i_n} + \cdots + \\
& + \sum_{j=1}^{n} B_{1,\cdots,j-1,j+1,\cdots,n} u_{i_1+1,\cdots,i_{j-1}+1,i_j,i_{j+1}+1,\cdots,i_n+1} \\
y_{i_1,i_2,\cdots,i_n} &= C x_{i_1,i_2,\cdots,i_n} + D u_{i_1,i_2,\cdots,i_n} \qquad (2.45)
\end{aligned}
$$

Example 8 *[41]: Consider the following system for $u(x,t), v(x,t), f(x,t)$ and $x \in (0,1)$ $t \geq 0$.*

$$\frac{\partial u}{\partial x} = v \qquad (2.46)$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x} + f \qquad (2.47)$$

Discretizing the equations above, we have

$$\frac{u_{i+1,j+1} - u_{i,j+1}}{\Delta x} = v_{i,j+1}, \qquad (2.48)$$

$$\frac{u_{i+1,j+1} - u_{i+1,j}}{\Delta t} = \frac{v_{i+1,j} - v_{i,j+1}}{\Delta x} + f_{i,j} \qquad (2.49)$$

Equations (2.48) and (2.49) are in the form (2.42) and (2.43)for

$$A_0 = 0, \qquad B_1 = 0, \qquad B_2 = 0$$

and

$$E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & \Delta x \\ 0 & \frac{-\Delta t}{\Delta x} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 1 & \frac{\Delta t}{\Delta x} \end{bmatrix}, B_0 = \begin{bmatrix} 0 \\ \Delta t \end{bmatrix}. \qquad (2.50)$$

In the following chapter, it will be shown how the first-degree Roesser model of the previous section and the implicit models of the following section are special cases of first-degree ARMA representations. Regular and singular Fornasini-Marchesini models were also generalized to behavioural models by Rocha [47]: In the first model, equations describing the model is given by

$$Ex + Fz_1x + Gz_2x + Hz_1z_2x = 0 \tag{2.51}$$

$$Mw + Nx = 0, \tag{2.52}$$

where

$$E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \; F = \begin{bmatrix} F_1 \\ 0 \\ F_3 \end{bmatrix}, \; G = \begin{bmatrix} 0 \\ G_2 \\ G_3 \end{bmatrix}, \; H = \begin{bmatrix} 0 \\ 0 \\ H_3 \end{bmatrix}.$$

E_i, F_i, G_i, H_i, M, and N are constant matrices (over the given field) of appropriate dimensions, M is full row rank, and

i $[E \quad F \quad G \quad H]$ has full row rank

ii $\mathrm{im}(E) \cap \mathrm{im}(H) = \{0\} = \mathrm{im}\,(F) \cap \mathrm{im}\,(G)$

iii Any 1-D polynomials M_1, M_2 such that

$$M_1 \begin{bmatrix} E_1 + F_1z_1 \\ G_2 \\ G_3 + H_3z_1 \end{bmatrix} = 0, \quad M_2 \begin{bmatrix} E_1 + F_1z_1 \\ E_2 \\ E_3 + F_3z_1 \end{bmatrix} = 0.$$

will also satisfy

$$M_1 \begin{bmatrix} 0 \\ E_2 \\ E_3 + F_3z_1 \end{bmatrix} = K_1(E_1 + F_1z_1), \quad M_2 \begin{bmatrix} 0 \\ G_2 \\ G_3 + H_3z_1 \end{bmatrix} = K_2(E_1 + F_1z_1),$$

for some 1-D polynomial matrices K_1, K_2 in z_1, z_1^{-1}.

iv Any 1-D polynomials M_3, M_4 such that

$$M_3 \begin{bmatrix} E_2 + G_2 z_2 \\ F_1 \\ F_3 + H_3 z_2 \end{bmatrix} = 0, \quad M_4 \begin{bmatrix} E_2 + G_2 z_2 \\ E_1 \\ E_3 + G_3 z_2 \end{bmatrix} = 0.$$

will also satisfy

$$M_3 \begin{bmatrix} 0 \\ E_1 \\ E_3 + G_3 z_2 \end{bmatrix} = K_3(E_2 + G_2 z_2), \quad M_4 \begin{bmatrix} 0 \\ F_1 \\ F_3 + H_3 z_2 \end{bmatrix} = K_4(E_2 + G_2 z_2).$$

for some 1-D polynomial matrices K_3, K_4 in z_2, z_2^{-1}.

Such a representation does not require any preferred direction of propagation for the state values. However, because of the static nature of the relationship between the state variables x and the external variable w, the model may require a larger number of state variables than other models.

In the second model, a free auxiliary variable, referred to as the *driving variable* v, is used to obtain the following realization:

$$S(z)x = 0 \tag{2.53}$$

$$z_1 x = (A_1 z + A_2)x + (B_1 z + B_2)v \tag{2.54}$$

$$w = Cx + Dv \tag{2.55}$$

where $z = z_1 z_2^{-1}$ and A_i's, B_i's, C, and D are constant matrices and $S(z)$ is a 1-D polynomial matrix in z.

This model can be treated as a generalization of the regular Fornasini-Marchesini model, and hence it is not hard to show that not every AR 2-D system representation has an ARMA realization of this form (for example, an AR system described by $z_1 w_1 = z_2 w_2$).

The definition of order for either model is not given: but one may suggest the dimension of the state variable x to be used as the system order. Also. it may be possible to use the conditions on the matrices E, F, G, H. and M of the first model to establish some reductive algorithms.

2.2.1 Minimality

In this subsection. after giving a definition of order for Fornasini-Marchesini models. a survey of attempts to obtain a minimal order Fornasini-Marchesini model for a given system will be given. This is achieved by associating a non-commutative power series with the transfer function of the system. solving the minimal realization problem for the non-commutative case using a generalization of B. L. Ho algorithm. and then using an inverse map to obtain a realization for the commutative case.

The *order* of a given Fornasini-Marchesini model is defined to be the dimension of the local state space and hence a *minimal* order realization is the one in which the smallest dimension for the state vector is used. It should be noted that a minimal order realization no longer corresponds to a realization where the dimension of local state x is the sum of the sum of the highest power of z_1 and z_2 in the denominator of the transfer function describing the system. as the following example will show.

Example 9 *We already showed in example 7 that the smallest Roesser realization obtainable is of order two. whereas a generalized Fornasini-Marchesini realization of the form (2.42)-(2.43) of order one is given by* $A_0 = 1$. $A_1 = A_2 = 0$, $B_0 = 0$, $B_1 = -1$, *and* $B_2 = 1$. *However. note that the state vector is not updated using only first order equations.*

Similar arguments to those made in the earlier section with respect to minimality can be made with the following modifications:

- Fornasini-Marchesini [15] showed that every minimal realization is locally reachable and locally observable.

- Global reachability and observability are sufficient but not necessary conditions for minimality. [18]

It has already been shown in the previous section how the B. L. Ho algorithm can be used to obtain a minimal Roesser realization for the case where the denominator of the transfer function is separable. Let $\mathcal{U}$ be the space of past inputs and $\mathcal{Y}$ the space of future outputs. Also, let the input-output map be characterized by the linear map s, which is defined by

$$s : u \in \mathcal{U} \Rightarrow \sum_{i+j>0} (s(u), z_1^i z_2^j) z_1^i z_2^j \in \mathcal{Y} \tag{2.56}$$

where $(s(u), z_1^i z_2^j)$ represent the coefficient of $z_1^i z_2^j$ in the formal power series s. In this framework we have assumed commutativity between z_1 and z_2. Define the (infinite) Hankel matrix $\mathcal{H}(s)$ associated with the power series σ to be :

$$\mathcal{H}(s) = \begin{bmatrix} (s, 1) & (s, z_1) & (s, z_2) & (s, {z_1}^2) & (s, z_1 z_2) & \cdots \\ (s, z_1) & (s, {z_1}^2) & (s, z_1 z_2) & (s, {z_1}^3) & \cdots & \cdots \\ (s, z_2) & (s, z_2 z_1) & (s, {z_2}^2) & (s, z_2 {z_1}^2) & \cdots & \cdots \\ (s, {z_1}^2) & (s, {z_1}^3) & (s, {z_1}^2 z_2) & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{2.57}$$

Fliess [11] proved that the rank of $\mathcal{H}(s)$ is finite if and only if s is a power series representing a rational function with separable denominator. It should be noted that

(a) We have already shown that there are many systems with non-separable transfer function which have finite realization (see example 9).

(b) The rank of the Hankel matrix as defined above does not always correspond to the order of a minimal realization as following example will illustrate.

Example 10 *Consider a 2-D system described by the transfer function* $s = \frac{z_1 - z_2}{1+z_1+z_2+z_1 z_2} = \frac{z_1-z_2}{(1+z_1)(1+z_2)}$. *A generalized Fornasini-Marchesini model of order one can be given by* $A_0 = A_1 = A_2 = -1$, $B_0 = 0$, $B_1 = 1$, *and* $B_2 = -1$. *However,*

$$\begin{aligned} \frac{z_1 - z_2}{1 + z_1 + z_2 + z_1 z_2} &= (z_1 - z_2) \sum_{k=0}^{\infty} (-1)^k (z_1 + z_2 + z_1 z_2)^k \\ &= z_1 + z_2 - z_1^2 - 2 z_1 z_2 - z_2^2 + \cdots. \end{aligned}$$

Then the Hankel matrix is given by

$$\mathcal{H}(s) = \begin{bmatrix} 0 & 1 & 1 & -1 & -2 & -1 & \cdots \\ 1 & -1 & -2 & 1 & 0 & \cdots & \\ 1 & -2 & -1 & 0 & \cdots & & \\ -1 & 1 & \cdots & & & & \\ \cdots & & & & & & \end{bmatrix}. \tag{2.58}$$

Clearly rank $\mathcal{H}(s) > 1$, *so the rank of the Hankel matrix does not correspond to the order of a minimal realization.*

The Fornasini-Marchesini approach to constructing 2-D minimal realizations was to first associate a given commutative power series s to a noncommutative power series σ via a map, say ϕ. Then, an extension of earlier works of Fliess to noncommutative formal power series was used to obtain a minimal realization for σ. Finally the inverse map of ϕ was used to obtain a realization for s.[16].

To obtain a minimal realization, given a noncommutative power series σ, Fornasini[12] proposed a generalization of the B.L. Ho algorithm as follows:

Define *words* to be n-tuples $\eta = (w_{i_1}, w_{i_2}, \cdots, w_{i_n}), 0 \leq n \leq \infty$ where $w_{i_j} \in \{z_1, z_2\}, \forall i, j$ and the *length* of η to be i_n. If $\eta = (w_{i_1}, w_{i_2}, \cdots, w_{i_n})$ and $\xi = (u_{j_1}, u_{j_2}, \cdots, u_{j_m})$ are two possible words, define their product $\eta\xi$ to be their concatenation

$$\eta\xi = (w_{i_1}, w_{i_2}, \cdots, w_{i_n}, u_{j_1}, u_{j_2}, \cdots, u_{j_m}) \tag{2.59}$$

Writing w_i in place of the 1-tuple (w_i), a word η can be written as

$$\eta = w_{i_1} w_{i_2} \cdots w_{i_n} \qquad n > 0. \tag{2.60}$$

Then a noncommutative ring of formal power series $K \ll \xi_1, \xi_2 \gg$ in two variables ξ_1, ξ_2 and with coefficients in a given field K can be defined such that $\sigma \in K \ll \xi_1, \xi_2 \gg$ can be written as

$$\sigma := \sum_{\text{all possible words } w} (\sigma, w)w \qquad (\sigma, w) \in K. \tag{2.61}$$

Now we can form the (infinite) Hankel matrix $\mathcal{H}(\sigma)$ associated with the power series σ:

$$\mathcal{H}(\sigma) = \begin{bmatrix} (\sigma, 1) & (\sigma, \xi_1) & (\sigma, \xi_2) & (\sigma, {\xi_1}^2) & (\sigma, \xi_1\xi_2) & \cdots \\ (\sigma, \xi_1) & (\sigma, {\xi_1}^2) & (\sigma, \xi_1\xi_2) & (\sigma, {\xi_1}^3) & (\sigma, {\xi_1}^2\xi_2) & \cdots \\ (\sigma, \xi_2) & (\sigma, \xi_2\xi_1) & (\sigma, {\xi_2}^2) & (\sigma, \xi_2{\xi_1}^2) & \cdots & \\ (\sigma, {\xi_1}^2) & (\sigma, {\xi_1}^3) & (\sigma, {\xi_1}^2\xi_2) & & & \\ \cdots & & & & & \end{bmatrix} \tag{2.62}$$

The element in the (v, w) position of this matrix is simply given by (σ, vw). Also, ordering in this lexicographical manner allows us to talk about, say, the ith row (or column) of $\mathcal{H}(\sigma)$. Partition $\mathcal{H}(\sigma)$ in row and column blocks indexed by capital letters. For example, $\mathcal{H}_{M,N}(\sigma)$ is the composition of the Mth row and the$N$th column where the Mth row (the$N$th column) block includes all rows (columns) of

$\mathcal{H}(\sigma)$ whose indices are words of length $M-1(N-1)$. These block matrices are of finite dimension and contain $2^{(M-1)+(N-1)}$ elements. Also denote by $\mathcal{H}_{M\times N}(\sigma)$:

$$\mathcal{H}_{M\times N}(\sigma) = \begin{bmatrix} \mathcal{H}_{1,1}(\sigma) & \mathcal{H}_{1,2}(\sigma) & \cdots & \mathcal{H}_{1,N}(\sigma) \\ \mathcal{H}_{2,1}(\sigma) & \mathcal{H}_{2,2}(\sigma) & \cdots & \mathcal{H}_{2,N}(\sigma) \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{H}_{M,1}(\sigma) & \mathcal{H}_{M,2}(\sigma) & \cdots & \mathcal{H}_{M,N}(\sigma) \end{bmatrix} \tag{2.63}$$

Then the rank of the Hankel matrix $\mathcal{H}(\sigma)$, n_σ is

$$n_\sigma = \sup_{M,N} \text{rank}\mathcal{H}_{M\times N}(\sigma). \tag{2.64}$$

This rank, whenever finite, provides the dimension of the minimal realization of Fornasini-Marchesini form of σ.

Denote by $\mathcal{H}^r(\sigma)$ the infinite matrix whose element in the (w,v) position is given by (σ, wrv) for all possible words w and v. Note that in general $\mathcal{H}^r(\sigma)$ does not constitute a Hankel matrix, except for $r=1$. Apply the same partitioning to $\mathcal{H}^r(\sigma)$ as those of the Hankel matrix (i.e. $\mathcal{H}^r_{M,N}(\sigma)$, $\mathcal{H}^r_{M\times N}(\sigma)$).

Assume that some upper bound on the dimension of the minimal realization n_σ is known and n_σ =rank $\mathcal{H}(\sigma) < \infty$. If Denoting the row length of $\mathcal{H}(\sigma)$ by L' and the column length of $\mathcal{H}(\sigma)$ by L'', then the following steps lead to a minimal realization of σ of the form $\sigma = C(I - A_1\xi_1 - A_2\xi_2)^{-1}B$:

1. Find nonsingular matrices P and Q such that

$$P\mathcal{H}_{L'\times L''}(\sigma)Q = \left[\begin{array}{c|c} I_{n_\sigma} & 0 \\ \hline 0 & 0 \end{array}\right] \tag{2.65}$$

2. Compute

$$A_i = \left[\begin{array}{c|c} I_{n_\sigma} & 0 \end{array}\right] P\mathcal{H}^{\xi_i}_{L'\times L''}(\sigma)Q \left[\begin{array}{c} I_{n_\sigma} \\ \hline 0 \end{array}\right], i = 1,2 \tag{2.66}$$

$$B = \left[\begin{array}{c|c} I_{n_\sigma} & 0 \end{array} \right] P\mathcal{H}_{L'\times L''}(\sigma) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.67}$$

$$C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \mathcal{H}_{L'\times L''}(\sigma) Q \left[\frac{I_{n_\sigma}}{0} \right]. \tag{2.68}$$

Next, let $K[[z_1, z_2]]$ be the commutative ring of formal power series variables z_1, z_2. Define homomorphism ϕ to be $\phi : K \ll \xi_1, \xi_2 \gg \to K[[z_1, z_2]]$ such that $\phi(k) = k$, $\forall k \in K$, $\phi(\xi_1) = z_1$, and $\phi(\xi_2) = z_2$.

Consider a given (commutative) transfer function $s \in K[[z_1, z_2]]$. Pick any $\sigma \in K \ll \xi_1, \xi_2 \gg$ such that $\phi(\sigma) = s$. Then a minimal realization of σ can be related to a realization of s since

$$\phi(\sigma) = \phi(C - (I - A_1\xi_1 - A_2\xi_2)^{-1}B) = C - (I - A_1 z_1 - A_2 z_2)^{-1}B = s.$$

Finding minimal realization of σ, subject to the constraint $\phi(\sigma) = s$, requires solving several nonlinear equations corresponding to the ranks of some finite submatrices of the Hankel matrix associated with σ:

Let the highest powers of z_1 and z_2 in s be n_1 and n_2 respectively, and assume an upper bound on the rank of $\mathcal{H}(\sigma)$ is known and is equal to say m. Also, let $\bar{P} = m + \lceil \frac{n_1+n_2}{2} \rceil$ where $\lceil x \rceil$ is equal to the smallest integer n such that $n > x$. Define Γ to be the set of words w in ξ_1 and ξ_2 such that

$$\Gamma = \{w : |w| \leq 2\bar{P},\ w \neq {\xi_1}^h {\xi_2}^k\ \forall h, k\},$$

where $|w|$ is defined to be the length of w and is equal to n if $w = (\xi_{i_1}, \cdots, \xi_{i_n})$.

Let K^Γ be the space of Γ indexed sequences of elements of K. For every sequence $\{k^w\} \in K^\Gamma$, define a corresponding polynomial $\pi(\{k^w\})$ to be

$$\pi(\{k^w\}) = \sum_{w\in\Gamma} k^w w + \sum_{i+j\leq 2\bar{P}} (\sigma, \xi_1^i\xi_2^j)\xi_1^i\xi_2^j - \sum_{w\in\Gamma\cap\Xi^{i,j}} k^w {\xi_1}^i{\xi_2}^j,$$

where $\Xi^{i,j}$ denotes the set of words made of i ξ_1's and j ξ_2's respectively.

Then the corresponding rational noncommutative power series σ whose Hankel matrix has minimal rank of say q, subject to $\phi(\sigma) = s$, is the power series for which q is the smallest value such that

$$\begin{aligned}
T_q &= \{\{k^w\} : \{k^w\} \in K^\Gamma, \operatorname{rank}\mathcal{H}_{\bar{P}\times\bar{P}}(\pi(\{k^w\})) \leq q\} \quad q \leq m && (2.69)\\
V_q &= \{\{k^w\} : \{k^w\} \in K^\Gamma, \operatorname{rank}\mathcal{H}_{\bar{P}\times(\bar{P}+1)}(\pi(\{k^w\})) \leq q\} \quad q \leq m && (2.70)\\
U_q &= \{\{k^w\} : \{k^w\} \in K^\Gamma, \operatorname{rank}\mathcal{H}_{(\bar{P}+1)\times\bar{P}}(\pi(\{k^w\})) \leq q\} \quad q \leq m && (2.71)\\
(T_q &- T_{q-1}) \cap (V_q - V_{q-1}) \cap (U_q - U_{q-1}) \neq \emptyset && (2.72)
\end{aligned}$$

and

$$\operatorname{rank} H_{\bar{P}\times\bar{P}}(\sigma) = \operatorname{rank} H_{\bar{P}\times\bar{P}+1}(\sigma) = \operatorname{rank} H_{\bar{P}+1\times\bar{P}}(\sigma) = q. \qquad (2.73)$$

Several possible limitations to this technique should be pointed out

1. *A priori* knowledge for an upper bound of rank of $\mathcal{H}(\sigma)$ is needed.

2. The minimal realization for a commutative power series is defined in terms of the minimality of the realization of an associated noncommutative power series.

3. Finding the smallest q such that (2.69)-(2.72) are satisfied, is equivalent to a number of conditions on the minors of the corresponding submatrices of the Hankel matrix which can be expressed as a system of nonlinear algebraic equations.

2.3 The Implicit Model

In this section, we will look at the *implicit* representations of linear systems, minimal order with respect to these representations, and necessary conditions for minimality suggested by Aplevich [2]. It will be shown by an example that these conditions are not sufficient for minimality.

Consider the n-D system described by the behavioural equations

$$[E_1 + F_1\lambda_1, \cdots, E_n + F_n\lambda_n, G] \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ w \end{bmatrix} = 0, \qquad (2.74)$$

where the λ_i are distinct linear operators, w is the external vector, the x_i are internal vectors of dimension $n_i \geq 0$, and F_i, E_i, and G are constant matrices of appropriate dimensions.

In [2], a detailed study of 1-D dynamical systems represented by these representations was given. The order of these systems was defined to be the dimension of x, equal to the number of operators needed. Using this definition of order, four rank conditions which proved to be necessary and sufficient for minimality were found. It was also shown that other definitions in the literature for proper systems and polynomial matrix fraction descriptions are special cases of this definition of minimality. The rank conditions established have been defined for the other first order representations by Kuijper [33].

It has been suggested ([2], chapter 10) that a natural generalization of the order of the models described by (2.74), is to consider the order with respect to each operator, say n_i, and let the order of the representation to be $n = \sum_i n_i$. The following four rank conditions have been proven to be necessary for minimality of systems described by (2.74):

Let $\bar{F}_i$ and $\bar{E}_i$ denote the matrices $[F_1, \cdots, F_n]$ and $[E_1, \cdots, E_n]$ respectively with blocks F_i and E_i deleted.

Condition 1 F_i has full column rank.

Condition 2 $[F_i, \bar{E}_i, \bar{F}_i, G]$ has full row rank.

Condition 3 $[E_i + \lambda F_i, \bar{E}_i, \bar{F}_i, G]$ has full row rank for all $\lambda \in \mathbb{C}$.

Condition 4 $[E_i + \lambda F_i]$ has full column rank for all $\lambda \in \mathbb{C}$.

One has to observe that different representations of a given dynamical system of the same order may have different numbers of the given operators, as the following example will show. This may be of importance if reduction of the size of one operator may be more costly than reduction with respect to the others.

Example 11 *Consider the dynamical system described by the transfer function $(1 - z_1 z_2)y = (z_1 + z_2)u$. In the following pages, it will be shown that any minimal implicit realization of this dynamical system over real numbers will be of order three. Two third order minimal implicit realizations of this system are given by:*

For $\Lambda_1 = [z_1, z_2, z_2]$

$$F_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}, \tag{2.75}$$

and for $\Lambda_2 = [z_1, z_1, z_2]$

$$F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}. \tag{2.76}$$

However, the first representation requires Λ *to have one* z_1 *and two* z_2, *whereas the second representation has two* z_1 *and one* z_2.

To show that smallest realization of form (2.74) over reals is of order 3. we will use a similar argument to the one used in example 6.

Assume that there exists an order 2 real realization of form (2.74) of $[1 - z_1 z_2, \; z_1 + z_2] \begin{bmatrix} y \\ u \end{bmatrix}$. *In chapter 5, it will be shown that a necessary condition of minimality is that* $[F \; G]$ *be of full rank. Using this and any of the following (if necessary)*

1. *add a constant multiple of one row to another,*
2. *scale* y *and* u *by a real constant* $\alpha \neq 0$. *(i.e.* $\frac{\alpha y}{\alpha u}$ *has the same transfer function as* $\frac{y}{u}$*),*
3. *a change of basis in the internal variable by multiplying by a real constant* $\alpha \neq 0$. *(i.e.*$x_{new} = \alpha x_{old}$*),*

In this fashion, it is always possible to make the given implicit realization look like one of the following cases without affecting the external behaviour:

$$\textbf{Case 1} \begin{bmatrix} 1 & 0 & E_{11} & E_{12} & 0 & -G_{12} \\ 0 & 1 & E_{21} & E_{22} & 0 & -G_{22} \\ 0 & 0 & E_{31} & E_{32} & -1 & G_{32} \end{bmatrix},$$

Case 2 $\begin{bmatrix} 1 & F_{12} & E_{11} & E_{12} & 0 & 0 \\ 0 & F_{22} & E_{21} & E_{22} & 1 & 0 \\ 0 & F_{32} & E_{31} & E_{32} & 0 & -1 \end{bmatrix}$,

Case 3 $\begin{bmatrix} F_{11} & 1 & E_{11} & E_{12} & 0 & 0 \\ F_{21} & 0 & E_{21} & E_{22} & -1 & 0 \\ F_{31} & 0 & E_{31} & E_{32} & 0 & 1 \end{bmatrix}$.

Case 4 $\begin{bmatrix} 1 & 0 & E_{11} & E_{12} & -G_{11} & 0 \\ 0 & 1 & E_{21} & E_{22} & -G_{21} & 0 \\ 0 & 0 & E_{31} & E_{32} & G_{31} & -1 \end{bmatrix}$.

It is easy to see that case 1 just reduces to example 6 and as was shown, it cannot happen if the constants are taken over reals.

Case 2 cannot occur since we must have

$$\det \begin{bmatrix} z_1 + E_{11} & F_{12}z_2 + E_{12} \\ E_{21} & F_{22}z_2 + E_{22} \end{bmatrix} = z_1 + z_2. \tag{2.77}$$

or

$$F_{22}z_1z_2 + E_{22}z_1 + E_{11}F_{22}z_2 + E_{11}E_{22} - E_{12}E_{21} = z_1 + z_2, \tag{2.78}$$

which implies that

$$F_{22} = 0,\ E_{22} = 1,\ F_{12} = \frac{-1}{E_{21}} \textit{ and } E_{11} = E_{12}E_{21}.$$

We also must have

$$[\, E_{31} \quad F_{32}z_2 + E_{32}\,] \begin{bmatrix} -(F_{12}z_2 + E_{12}) \\ z_1 + E_{11} \end{bmatrix} = z_1z_2 - 1, \tag{2.79}$$

which implies

$$E_{32} = 0,\ F_{32} = 1,\ \textit{and } E_{31} = \frac{-1}{E_{12}}.$$

Let $E_{11} = \alpha$*. The above equations imply that*

$$\alpha^2 = -1,$$

which clearly has no real solutions.

For case 3 to happen, we must have

$$\det \begin{bmatrix} F_{11}z_1 + E_{11} & z_2 + E_{12} \\ F_{21}z_1 + E_{21} & E_{22} \end{bmatrix} = z_1 + z_2, \tag{2.80}$$

or

$$-F_{21}z_1z_2 + F_{11}E_{22}z_1 + E_{21}z_2 + -E_{12}F_{21}z_1E_{11}E_{22} - E_{12}E_{21} = z_1 + z_2, \tag{2.81}$$

which implies that

$$F_{21} = 0,\ E_{21} = 1,\ F_{11} = \frac{-1}{E_{22}} \text{ and } E_{12} = E_{11}E_{22}.$$

We also must have

$$\begin{bmatrix} F_{31}z_1 + E_{31} & E_{32} \end{bmatrix} \begin{bmatrix} -(z_2 + E_{12}) \\ F_{11}z_1 + E_{11} \end{bmatrix} = z_1z_2 - 1, \tag{2.82}$$

which implies

$$F_{31} = 0,\ E_{31} = 0, \text{ and } E_{32} = \frac{-1}{E_{11}}.$$

Let $E_{12} = \alpha$*. The above equations imply that*

$$\alpha^2 = -1,$$

which clearly has no real solutions.

Case 4 cannot happen since we must have

$$\det \begin{bmatrix} z_1 + E_{11} & E_{12} \\ E_{21} & z_2 + E_{22} \end{bmatrix} = z_1 + z_2 \tag{2.83}$$

or

$$z_1z_2 + z_1E_{22} + z_2E_{11} + E_{11}E_{22} - E_{22}E_{21} = z_1 + z_2, \tag{2.84}$$

which is impossible, regardless of values of F_{ij} *or* E_{ij}*.*

The next example will illustrate that conditions (1-4) are not sufficient for minimality as defined above.

Example 12 *(example 7 revisited): The third order Roesser realization given for the transfer function describing*

$$[1 - z_1 z_2,\ z_1 - z_2] \begin{bmatrix} y \\ u \end{bmatrix}$$

can be rewritten in the form (2.74) for $\Lambda = [z_1, z_2, z_2]$ *and*

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}. \tag{2.85}$$

It is easy to see that conditions 1-4 hold, since

(a) $rank \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1$, *and* $rank \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = 2$.

(b)

$$rank \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix} = 4,$$

$$\text{and } rank \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix} = 4,$$

(c)

$$rank\begin{bmatrix} -1+\lambda & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix} = 4 \quad \forall \lambda \in \mathbb{C}.$$

$$\text{and } rank\begin{bmatrix} \lambda & -1 & 0 & 1 & 0 & -1 \\ 0 & -1+\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 \end{bmatrix} = 4 \quad \forall \lambda \in \mathbb{C}.$$

(d) $rank\begin{bmatrix} \lambda \\ 0 \\ -1 \\ -1 \end{bmatrix} = 1 \; \forall \lambda \in \mathbb{C}$, *and* $rank\begin{bmatrix} 0 & -1 \\ \lambda & -1 \\ \lambda & 0 \\ 0 & 1 \end{bmatrix} \; \forall \lambda \in \mathbb{C}$.

However, an implicit realization of order 2 can be written for $\Lambda' = [z_1, z_2]$ *and*

$$F' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \; E' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \; G' = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}. \tag{2.86}$$

2.4 Polynomial-Matrix Models

In chapter 5, I will show how an n-D version of the generalized theorem due to Forney can be used to classify all AR representations of a given dynamical system and will show how to obtain a minimal representation of a given class. In this section, this theorem and some of its related concepts will be given.

Many 1-D system analysis problems have been solved using polynomial matrices to represent system external behaviours. Popov, Rosenbrock, Wolovich and Forney are a few of many who successfully have used this approach (For an extensive list of references of these authors and others, see Kailath [29]).

The work in particular that we are interested in, is that of Forney [21]. Consider a $k \times n$ matrix G whose entries are rational functions in some indeterminate x with coefficients is some field F, that is $F(x)$. Forney showed how to find a *minimal* basis for the row space of G, that is the vector space V_G generated by the set of all linear combinations of the rows of G.

To accomplish this, definitions of *order* for G, as well as a minimal basis for V_G must be given. Forney defined the *degree* of a polynomial n-tuple $\mathbf{g} = (g_1, \cdots, g_n)$, deg $(\mathbf{g})$ to be the greatest degree of the components g_i, $1 \leq i \leq n$. Then the *order* of G was simply defined to be the sum of the degrees of rows of G. Furthermore, a minimal basis of V_G was defined to be any $k \times n$ polynomial matrix G^* such that G^* is a basis of V_G and it has the least order among all possible polynomial bases of V_G.

The main resulting theorem was:

Theorem 1 *Consider the polynomial matrix $G \in F^{k \times n}[x]$, for some field F and indeterminate x with order $m = m_1 + \cdots + m_k$ where the m_i are the degrees of rows g_i of G, $1 \leq i \leq k$. Then G is a minimal order basis for the row image V_G if and only if any of the following equivalent statements hold:*

1(a) *G is nonsingular modulo $p(x)$ for all irreducible polynomials $p(x) \in F[x]$, and*

1(b) *the high order coefficient matrix G_h has full rank.*

2(a) *The GCD of the $k \times k$ minors of G is 1, and*

2(b) *their greatest degree is m.*

3(a) *If* $\mathbf{y} = \mathbf{x}G$ *is a polynomial* n*-tuple, then* $\mathbf{x}$ *must be a polynomial* k*-tuple. and*

3(b) $deg\, \mathbf{y} = max_{1 \leq i \leq k}(deg\, x_i + deg\, \mathbf{g}_i)$.

4 *The row degree.* $deg\ \mathbf{g}_i$ *are such that for all integers* $d \geq 0$*. the dimension of vector space of all polynomial* n*-tuple in* V_G *with degree less than* d*. over the base field* F *is equal to* $\sum_{i.deg\,\mathbf{g}_i}(d - deg\,\mathbf{g}_i)$.

This theorem and its results are closely related to those of Popov. Rosenbrock and Wolovich. However, one of the interesting points about this theorem, besides its simplicity and elegance, is the fact that Forney realized that this theorem can be reformulated using the language of p-adic valuations (see appendix in pp. 516 of [21]).

Chapter 3

Models of n-D Dynamical Systems

Some of the results in this thesis are based on the behavioural model of n-D systems. first introduced in chapter 1. One of the purposes of this chapter is to give a formal introduction to the behavioural model. Different representations of behavioural models will be defined, with recasting techniques for obtaining representations of different forms.

Section 3.1, is a self-contained exposition of several possible behavioural models. specifically the AR, ARMA, and MA models.

In section 3.2, ARMA representations of first-degree in the internal variables and zeroth-degree in the external variables will be described. A realization method for obtaining an externally equivalent ARMA representation from an AR representation will be given. A systematic way to obtain an equivalent representation of a different first-degree form will be also given. A condensed survey of Gröbner bases is also included, and will be used to find externally equivalent AR representations from given first-degree ARMA representations.

3.1 Modeling

In chapter 1, we give a set-theoretic framework which characterizes a (n-D) dynamical system as an entity which is embedded in its environment and which interacts with it through external variables. The system is defined by the set of all trajectories which satisfy the system laws. This set is called the system behaviour. Formalizing the notion of behaviour yields the following definition of dynamical systems:

Definition 7 *A dynamical system* $\sum$ *is defined by a triplet* $(\mathcal{T}, \mathcal{W}, \mathcal{B})$, *where* $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2 \times \cdots \mathcal{T}_n$ *is an index set,* $\mathcal{W}$ *is a signal space where system variables take their values, and* $\mathcal{B} \subseteq \mathcal{W}^{\mathcal{T}}$ *is the behaviour of the system.*

The level of generality used above allows for easy introduction of some typical, general properties of dynamical systems:

linearity A dynamical system $\sum = (\mathcal{T}, \mathcal{W}, \mathcal{B})$ is *linear* if $\mathcal{W}$ and $\mathcal{B}$ are linear subspaces of $\mathcal{W}^{\mathcal{T}}$.

shift-invariance A dynamical system $\sum = (\mathcal{T}, \mathcal{W}, \mathcal{B})$ is *shift-invariant* if $\mathcal{T}$ is an additive semigroup and $\forall (T_1 = (t_{i_1}, t_{i_2}, \cdots, t_{i_n}), T_2 = (t_{j_1}, t_{j_2}, \cdots, t_{j_n})$ and $T_1 + T_2 \in \mathcal{T})$ it holds that $\{w(T_1) \in \sum\} \Rightarrow \{w(T_1 + T_2) \in \sum\}$.

completeness A dynamical system $\sum = (\mathcal{T}, \mathcal{W}, \mathcal{B})$ is *complete* if $\{w \in \mathcal{B}\} \Leftrightarrow \{w|_{\mathcal{K}} \in \mathcal{B}|_{\mathcal{K}}$ for all finite $\mathcal{K} \subseteq \mathcal{T}\}$.

We will mostly be assuming that the discrete time set is $\mathcal{T} = \mathbb{Z}^n$ and $\mathcal{W} = \mathbb{R}^q$ and will be concerned with the behaviours that are linear, shift-invariant, and complete.

Another notion which arises with respect to behavioural modeling is that of internal variables. Internal variables are normally introduced in order to facilitate writing the dynamical equations of a given system. A typical example of internal variables is given by the state variables. Definition(7) can be rewritten to include internal variables:

Definition 8 *A dynamical system, $\sum_x$, is defined by a quadruple $(\mathcal{T}.\mathcal{W}.\mathcal{X}.\mathcal{B}_e)$. where $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2 \times \cdots \mathcal{T}_n$ is an index set. $\mathcal{W}$ is a signal space where system variables take their values. $\mathcal{X}$ is the set of internal variables. and $\mathcal{B}_e \subseteq (\mathcal{W} \times \mathcal{X})^{\mathcal{T}}$ is the extended behaviour of the system.*

Let the projection P be defined such that $P_w(w.x) = w$ and $P_x(w.x) = x$. We will refer to $\mathcal{B}_x = P_x\mathcal{B}_e$ as 'internal' behaviour and $\mathcal{B}_w = P_w\mathcal{B}$ as the 'external' behaviour of the system.

3.1.1 Parameterization

The dynamical model defined in the last section is an abstract mathematical object, and the difference between such an object and its *representation* by means of equations should be emphasized. As an example of such equations let E be an abstract set, normally taken to be $E = \{0, 1\}$. Then $\mathcal{B}$ can be represented by the equations induced by a map $f : \mathcal{W}^{\mathcal{T}} \rightarrow E$ such that $\mathcal{B} = \{w \in \mathcal{W} | f(w) = 0\}$. It has to be noted that even though the behaviour of a dynamical model is *unique*, the same is not true of the equations which describe the model.

It is also possible to use concrete parameters, such as polynomials, matrices, and matrix polynomials to determine the equations that induce a mathematical model for a given linear, shift-invariant dynamical system.

Next we will look at some of the possible representations of these models: specifically ones which are known as AR, ARMA, and MA models.

AR-representation: External behaviour may be expressed by a linear relationship between external variables in Definition 7 at different points of the time set, yielding the so-called **Autoregressive**(AR) representation. For $\mathcal{T} = \mathbb{Z}^n$ and $\mathcal{W} = \mathbb{R}^q$, AR-representations are described by equations of the form:

$$\sum_{i_n=a_n}^{b_n} \sum_{i_{n-1}=a_{n-1}}^{b_{n-1}} \cdots \sum_{i_1=a_1}^{b_1} R_{i_1,i_2,\cdots,i_n} w(t_1+i_1, t_2+i_2, \cdots, t_n+i_n) = 0 \tag{3.1}$$

for all $T = (t_1, t_2, \cdots, t_n) \in \mathbb{Z}^n$ and $b_j \leq i_j \leq a_j \qquad \forall j \;\; 1 \leq j \leq n$
where $R_{i_1,i_2,\cdots,i_n} \in \mathbb{R}^{g \times q}$ for some positive integer g. In other words, $\sum$ is described by g scalar linear equations with entries of $R_{i_1,i_2,\cdots,i_n}$ as parameters and $w_1, w_2, \cdots, w_q$ as the variables. It is often convenient to write the above equation in the polynomial form:

$$R(\mathbf{z})w = 0 \tag{3.2}$$

where $\mathbf{z} = (z_1, z_2, \cdots, z_n)$, z_i is the shift operator in the i-th direction, and

$$R(\mathbf{z}) = \sum_{i_n=a_n}^{b_n} \sum_{i_{n-1}=a_{n-1}}^{b_{n-1}} \cdots \sum_{i_1=a_1}^{b_1} R_{i_1,i_2,\cdots,i_n} z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n}. \tag{3.3}$$

Note that this defines $\mathcal{B} = \ker R(\mathbf{z})$ where $R(\mathbf{z})$ is a linear map from $(\mathbb{R}^q)^{\mathbb{Z}^n}$ to $(\mathbb{R}^g)^{\mathbb{Z}^n}$.

It should also be noted that a n-D system which is linear, shift-invariant, and complete always has an AR-representation, and that all n-D systems parameterized by AR-representations are linear, shift-invariant, and complete. The proof of this statement in n-D is similar to the proofs given in 1-D by Willems (see [56], page 567) and in 2-D by Rocha (see [47], page 15), and hence will be omitted.

ARMA and MA-representations: Internal behaviour may be defined through a linear relationship between external and internal variables in Definition 8 at points of the time set. This will yield the so-called **Autoregressive-Moving-Average** (ARMA) representation. A dynamical system $(\mathbb{Z}^n, \mathbb{R}^q \times \mathbb{R}^n, \mathcal{B}_e)$ can be written in the polynomial form:

$$R(\mathbf{z})w = M(\mathbf{z})\xi \tag{3.4}$$

where $w : \mathbb{Z}^n \to \mathbb{R}^q$, $\xi : \mathbb{Z}^n \to \mathbb{R}^n$, $R(\mathbf{z}) \in \mathbb{R}^{g \cdot q}[\mathbf{z}]$ and $M(\mathbf{z}) \in \mathbb{R}^{g \cdot n}[\mathbf{z}]$. First-degree representations are often of interest since they provide an easy updating scheme for behavioural equations. In this thesis, we are interested in a special type of ARMA model which is first-degree in the internal variables and zeroth-degree in the external variables. In the next section, we will give some of the possible ARMA representations of this form.

A special class of ARMA-representation is the one where there are no constraints on the internal variables. These representations are called **Moving-Average**(MA) representations. In polynomial form MA-representation can be written as:

$$w = M(\mathbf{z})\xi \tag{3.5}$$

where $M(\mathbf{z}) \in \mathbb{R}^{g \cdot n}[\mathbf{z}]$.

3.2 Realization

In this section, we will give two different forms of ARMA representations which are first-degree in the internal variables and zeroth-degree in the external variables and we show their relationships to an AR representation of an n-D system. The first form is

$$F \Lambda Q \xi + E \xi + G w = 0 \tag{3.6}$$

where $F \in \mathbb{R}^{g\times h}$, $\Lambda = \text{diag}[z_{i_1}, z_{i_2}, \cdots, z_{i_h}]$,with i_j in $\{1, \cdots, n\}$ for $j = 1, \cdots, h$, possibly not all distinct, $Q \in \mathbb{R}^{h\times m}$, $E \in \mathbb{R}^{g\times m}$, $G \in \mathbb{R}^{g\times l}$.

It is easy to see that the implicit representation (2.74) in the previous chapter is a special case of (3.6) for $Q = I$ of appropriate dimension. In contrast with 1-D systems, the matrix Q has been included to allow change of basis in the space of x, while preserving the presentation as first-degree. The presence of Q can affect the number of internal variables needed, as the following example will show.

Example 13 *Consider the 2-D input-output system described by the transfer function $(z_1z_2-1)y = (z_1+z_2)u$. We have already shown that the smallest Roesser model realization for this transfer function requires a state vector of dimension three (see example 6 in page 12), and the smallest implicit model realization requires an internal vector of dimension three (see example 11 in page 36). However, a realization of the form (3.6) with a smaller number of internal variables is given by:*

$$\left(\begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1/2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & -1/2 \end{bmatrix}\right)\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & -1 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix} = 0. \tag{3.7}$$

In fact, I will show in chapter 4 that the minimum number of internal variables needed for a representation in from (3.6) is two.

The second form is

$$F\Lambda Q\xi = E\xi \tag{3.8}$$

$$N\xi = w \tag{3.9}$$

where $N \in \mathbb{R}^{l\times m}$. Using a terminology similar to 1-D [33], we refer to (3.6) as a *dual pencil* representation and (3.8)-(3.9) as a *pencil* representation. If decomposition of the external variable into inputs and outputs is also known (i.e. $w = \begin{bmatrix} y \\ u \end{bmatrix}$), we have the following first-degree *descriptor* representation

$$F\Lambda Q\xi = A\xi + Bu \tag{3.10}$$

$$y = C\xi + Du, \tag{3.11}$$

for some matrices A, B, C and D with entries in $\mathbb{R}$.

In the remainder of this section, systematic ways to obtain equivalent dual pencil, pencil and descriptor representations from an AR representation will be given, as well as the reverse. These methods will be illustrated by examples.

Using the definition of external behaviour given in page 46, define two representations to be *externally equivalent* if they have the same external behaviour. For example, two AR representations given by $R_1 w = 0$ and $R_2 w = 0$ are equivalent if $\ker R_1 = \ker R_2$. Similarly, the AR representation in (3.2) and pencil representation in (3.8)-(3.9) are equivalent if $\ker R = N(\ker(F\Lambda Q - E))$.

The matrix operations used in the following subsections to obtain equivalent representations of different forms are pre- and post-multiplication by a constant matrix, deletion of inactive internal variables, and deletion of rows corresponding to redundant constraints. These operations are the same as transformations used in [51] and preserve external equivalence.

It should also be easy to see then that the (implicit) Roesser state space model (2.1)-(2.2) is of descriptor form (3.10)-(3.11) for Q an identity matrix and F, A square constant matrices. The following recasting methods then allow for embedding of Roesser models in any of the above representations. Fornasini-Marchesini models are also included since I will show in the Appendix A how to obtain an equivalent Roesser representation for a given Fornasini-Marchesini model.

3.2.1 Recasting the AR representation as a dual pencil representation

A direct representation of an n-D system of form (3.6) will be obtained from the AR-representation (3.2). Without loss of generality, assume that the polynomials in $R(z_1, \cdots, z_n)$ are linear with respect to all their indeterminates, because any monomial $z_1^{k_1} \cdots z_n^{k_n}$ can be considered as $(\overbrace{z_1 \cdots z_1}^{k_1 \text{times}}) \cdots (\overbrace{z_n \cdots z_n}^{k_n \text{times}})$. For clarity, introduce the intermediary operators, $s_1 = \cdots = s_{k_1} = z_1, s_{k_1+1} = \cdots = s_{k_1+k_2} = z_2, \cdots$. These operators can always be replaced by their corresponding z_i's at the end of the algorithm. Rewrite (3.2) as

$$(A_{1,2,\cdots,m} s_1 s_2 \cdots s_m + A_{1,2,\cdots,m-1} s_1 s_2 \cdots s_{m-1} + \cdots A_1 s_1 + A_0) w = 0, \tag{3.12}$$

where the A_i's are constant matrices of appropriate dimension. Let I_n, and 0_n be $n \times n$ square identity and zero matrices. An equivalent representation of form (3.6) is given by inspection:

$$-F = \left[\begin{array}{c|c|c|c|c} \begin{matrix} I_l \\ 0_l \end{matrix} & I_{2l} & & & \\ \cline{1-2} & 0_{2l} & I_{4l} & \ddots & I_{2^{m-1}l} \\ \cline{1-3} & & 0_{4l} & \ddots & \\ & & & & 0_{2^{m-1}l} \end{array}\right], \quad E = \left[\begin{array}{c|c|c|c} \begin{matrix} 0 \\ I_l \end{matrix} & & & \\ & I_{2l} & & \\ & & \ddots & \\ & & & I_{2^{m-1}l} \end{array}\right], \tag{3.13}$$

and

$$G = [H_0, H_1, \cdots, H_{2^m - 1}], \tag{3.14}$$

where $H_j = A_{d_1, \cdots, d_k}$, where the d_i's are the locations of ones if j is written in binary form. For example, $H_6 = A_{23}$ since $6 = 110$ which has ones in locations 2 and 3, or $H_{13} = A_{134}$ since $13 = 1101$ which has ones in locations 1, 3, and 4. In the above, Q

is an appropriate identity matrix. The justification of the method is based on two facts. One is that for any given field K and indeterminate $s_1, \cdots, s_n$, a polynomial in $K[z_1, \cdots, s_n]$ can be considered to be a polynomial in $K[s_1][s_2]\cdots[s_n]$. The other is that any equation in the form:

$$(A_1 s_1 + A_0)w = 0 \tag{3.15}$$

can be rewritten as

$$\begin{bmatrix} -s_1 I & A_0 \\ I & A_1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ w \end{bmatrix} = 0. \tag{3.16}$$

For example, if polynomials in (3.2) are linear with respect to indeterminate s_1, s_2, and s_3, rewriting (3.2) in the form

$$(A_0 + A_1 s_1 + A_2 s_2 + A_3 s_3 + A_{12} s_1 s_2 + A_{13} s_1 s_3 + A_{23} s_2 s_3 + A_{123} s_1 s_2 s_3)w = 0, \tag{3.17}$$

and then applying (3.16) recursively twice yields

$$\begin{bmatrix} -s_1 I & -s_2 I & 0 & -s_3 I & 0 & 0 & 0 & A_0 \\ I & 0 & -s_2 I & 0 & -s_3 I & 0 & 0 & A_1 \\ 0 & I & 0 & 0 & 0 & -s_3 I & 0 & A_2 \\ 0 & 0 & I & 0 & 0 & 0 & -s_3 I & A_{12} \\ 0 & 0 & 0 & I & 0 & 0 & 0 & A_3 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & A_{13} \\ 0 & 0 & 0 & 0 & 0 & I & 0 & A_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & I & A_{123} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_{21} \\ \xi_{22} \\ \xi_{31} \\ \xi_{32} \\ \xi_{33} \\ \xi_{34} \\ w \end{bmatrix} = 0. \tag{3.18}$$

In the above example, if any rows of A_3, A_{13}, A_{23} or A_{123} are zero, the corresponding internal variables are zero by inspection and can be removed from the equations. This reduction of the dimension of internal variables required, as well as other reduction criteria will be formalized in the following chapter .

It is straightforward to prove by induction that F, Λ, Q, E, and G are given as described above, and to see that at each stage the behavioural polynomial matrix equations describing the external vector can be considered in two parts, for example as $P(s_1, \cdots, s_{n-1})w + s_n Q(s_1, \cdots, s_{n-1})w$. Now, by the induction hypothesis, (3.16) can be applied to both $P(s_1, \cdots, s_{n-1})$ and $Q(s_1, \cdots, sz_{n-1})$ with the addition of an n to the indices of corresponding matrices H of $Q(s_1, \cdots, s_{n-1})$ due to the factor z_n. Writing the indices in binary form, and using the fact that in each step, say the i-th, a new internal vector of dimension $2^{i-1}g$ is added, will complete the proof. This method will be illustrated by the following example.

Example 14 *Consider the 2-D AR representation:*

$$\begin{bmatrix} z_1 z_2 & z_1 & z_2^2 \\ z_1 + z_2 & z_2^2 + z_1 z_2 + 1 & z_1 z_2 \end{bmatrix} w = 0. \tag{3.19}$$

Introduce $s_1 = z_1$ and $s_2 = s_3 = z_2$, so (3.19) can be rewritten as

$$\begin{bmatrix} s_1 s_2 & s_1 & s_2 s_3 \\ s_1 + s_2 & s_2 s_3 + s_1 s_2 + 1 & s_1 s_2 \end{bmatrix} w = 0. \tag{3.20}$$

Rewriting this as (3.17), the only non-zero coefficient matrices are

$$\begin{aligned} A_0 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \\ A_{12} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \ A_{23} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \end{aligned}$$

Because A_3, A_{13} and A_{123} are zero in (3.18) x_{31}, x_{32}, and x_{34} are identically zero and can be removed to get

$$\begin{bmatrix} -z_1 I & -z_2 I & 0 & 0 & A_0 \\ I & 0 & -z_2 I & 0 & A_1 \\ 0 & I & 0 & -z_2 I & A_2 \\ 0 & 0 & I & 0 & A_{12} \\ 0 & 0 & 0 & I & A_{23} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_{21} \\ \xi_{22} \\ \xi_{33} \\ w \end{bmatrix} = 0. \tag{3.21}$$

Thus a representation of form (3.6) is given by $Q = I_8$, $\Lambda = diag[z_1 I_2, z_2 I_2, z_2 I_2, z_2 I_2]$,

$$F = \begin{bmatrix} -I_2 & -I_2 & 0 & 0 \\ 0 & 0 & -I_2 & 0 \\ 0 & 0 & 0 & -I_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_2 \end{bmatrix},$$

and

$$G = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}^T.$$

It should be noted that the above elementary approach can also be used to rewrite an equivalent first-degree representation of a general ARMA representation.

3.2.2 Recasting the dual pencil representation as a pencil representation

To obtain a pencil representation from a dual pencil, and hence from an AR representation, assume that $[F, G]$ has full row rank. Otherwise, the reductive methods of next chapter can be used to find an equivalent representation such that this assumption holds. Rewrite (3.6) in the following block matrix form:

$$[F \quad E \quad G] \begin{bmatrix} \Lambda Q \xi \\ \xi \\ w \end{bmatrix} = 0. \tag{3.22}$$

Pre-multiplying by a non-singular matrix $K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$, post-multiplying G by a permutation matrix $J = [J_1, J_2]$, and letting $J^T w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$, equation (3.22) can be rewritten as

$$\begin{bmatrix} K_1F & K_1E & 0 & L \\ 0 & K_2E & I & M \end{bmatrix} \begin{bmatrix} \Lambda Q\xi \\ \xi \\ w_1 \\ w_2 \end{bmatrix} = 0, \tag{3.23}$$

for some constant matrices L and M, and can be rewritten in pencil form:

$$[K_1F\Lambda Q, 0] \begin{bmatrix} \xi \\ w_2 \end{bmatrix} = [-K_1E, -L] \begin{bmatrix} \xi \\ w_2 \end{bmatrix} \tag{3.24}$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -K_2E & -M \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi \\ w_2 \end{bmatrix}. \tag{3.25}$$

Example 15 *Consider the dual pencil representation obtained in the previous example. Let* $\xi_{33} = \begin{bmatrix} \xi_{331} \\ \xi_{332} \end{bmatrix}$. *Since* $[F \quad G]$ *is not full row rank (i.e. rank = 9), pre-multiplying by a non-singular matrix which does the following row operations,* $R_8 \leftrightarrow R_{10}, -(R_9 + R_8) + R_{10} \rightarrow R_{10}$, *followed by a change of basis which is represented by the column operations* $K_8 + K_6 \rightarrow K_6, -K_8 + K_7 \rightarrow K_7$, *will imply that* ξ_{332} *can be set identically to zero, and hence can be eliminated from the equations. Pre-multiply the resulting equations by a non-singular matrix* P *which does the following sequence of row operations:* $-R_7 + R_6 \rightarrow R_6, -R_7 + R_4 \rightarrow R_4, -R_8 + R_3 \rightarrow R_3, -R_8 + R_2 \rightarrow R_2$ *to get a pencil representation of form (3.8) and (3.9) for*

$\Lambda = diag[z_1 I_2,\ z_2 I_2,\ z_2 I_2,\ z_2 I_2]$,

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_{21} \\ \xi_{22} \\ \xi_{331} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_{21} \\ \xi_{22} \\ \xi_{331} \end{bmatrix} \quad (3.26)$$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_{21} \\ \xi_{22} \\ \xi_{331} \end{bmatrix}. \quad (3.27)$$

3.2.3 Recasting the pencil representations as a descriptor representation

To convert (3.8) and (3.9) to form (3.10) and (3.11), assume that a partition of the external variable into inputs and outputs is known. Then equations (3.9) can be written as

$$\begin{bmatrix} P_1 N \\ P_2 N \end{bmatrix} \xi = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \xi = \begin{bmatrix} y \\ u \end{bmatrix} \quad (3.28)$$

for some permutation matrix $P = [P_1^T,\ P_2^T]^T$. An equivalent descriptor representation of (3.8) and (3.28) can be written trivially as

$$\begin{aligned} \begin{bmatrix} F\Lambda Q \\ 0 \end{bmatrix} \xi &= \begin{bmatrix} E \\ N_2 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ -I \end{bmatrix} u. & (3.29) \\ y &= N_1 \xi. & (3.30) \end{aligned}$$

It is possible to get a descriptor representation of (3.8) and (3.28) with fewer internal variables if in (3.8), column rank $\begin{bmatrix} Q \\ N_2 \end{bmatrix}$ > column rank$Q$. Taking the same approach as in [33], let $\xi = H\eta = [H_1,\ H_2,\ H_3]\eta$ be a change of basis and $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ be a non-singular matrix such that (3.8) and (3.28) can be rewritten as

$$\begin{aligned} F\Lambda\,[\,QH_1 \quad 0 \quad 0\,]\eta &= [\,EH_1 \quad EH_2 \quad EH_3\,]\eta & (3.31) \\ \begin{bmatrix} y \\ P_1 u \\ P_2 u \end{bmatrix} &= \begin{bmatrix} N_1H_1 & N_1H_2 & N_1H_3 \\ P_1N_2H_1 & I & 0 \\ P_2N_2H_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} & (3.32) \end{aligned}$$

where QH_1 is full column rank. From (3.32), it can be seen that η_2 can be solved explicitly in terms of η_1 and u. Eliminating η_2 from (3.31) and (3.32) yields the following descriptor representation:

$$\begin{bmatrix} F \\ 0 \end{bmatrix} \Lambda\,[\,QH_1 \quad 0\,] \begin{bmatrix} \eta_1 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} EH_1 - EH_2P_1N_2H_1 & EH_3 \\ P_2N_2H_1 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_3 \end{bmatrix} + \begin{bmatrix} EH_2P_1 \\ -P_2 \end{bmatrix} u, \tag{3.33}$$

$$y = [\,N_1H_1 - N_1H_2P_1N_2H_1 \quad N_1H_3\,]\eta + N_1H_2P_1u. \tag{3.34}$$

Example 16 *Consider the 2-D pencil representation given by:*

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \tag{3.35}$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} y \\ u_1 \\ u_2 \end{bmatrix} \tag{3.36}$$

Since Q is not full column rank, a change of basis $\xi = H\eta$ which is represented by the column operation $(-K_1 + K_2) + K_3 \to K_3$, and pre-multiplication by a non-singular matrix P which is represented by the row operation $2R_2 + R_1 \to R_1$ yields $\eta_3 = -\eta_2 + u_2$. An equivalent descriptor representation is given by

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{3.37}$$

$$y = [-3 \quad 2] \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + [0 \quad 3] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{3.38}$$

3.2.4 Recasting the descriptor representation as a pencil representation

A pencil representation equivalent to a descriptor representation (3.10)-(3.11) can be written trivially as

$$[F\Lambda Q \quad 0] \begin{bmatrix} \xi \\ \xi_2 \end{bmatrix} = [A \quad B] \begin{bmatrix} \xi \\ \xi_2 \end{bmatrix} \tag{3.39}$$

$$\begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi \\ \xi_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}. \tag{3.40}$$

It is possible to get a pencil representation with fewer internal variables if $[F, B]$ is not of full row rank. Pre-multiplying (3.10) by a non-singular matrix $P = [P_1^T, P_2^T, P_3^T]^T$, post-multiply u by a permutation matrix $J = [J_1, J_2]$, letting

$J^T u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and using a change of basis $\xi = [H_1, H_2]\eta$ allows equation (3.10) to be rewritten as

$$\begin{bmatrix} P_1F \\ 0 \\ 0 \end{bmatrix} \Lambda [QH_1 \quad QH_2] \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} P_1AH_1 & P_1AH_2 \\ P_2AH_1 & P_2AH_2 \\ 0 & P_3AH_3 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} P_1BJ_1 & P_1BJ_2 \\ I & P_2BJ_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{3.41}$$

where P_1F and P_3AH_3 are full row and column rank respectively. This implies η_2 can be set identically to zero, and an externally equivalent pencil representation of (3.10) and (3.11) is given by

$$P_1F\Lambda[QH_1 \quad 0] \begin{bmatrix} \eta_1 \\ u_2 \end{bmatrix} =$$

$$[P_1AH_1 - (P_1AH_2)P_2AH_1 \quad P_1BJ_2 - (P_1AH_2)P_2BJ_2] \begin{bmatrix} \eta_1 \\ u_2 \end{bmatrix} \tag{3.42}$$

$$\begin{bmatrix} y \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} CH_1 - (DJ_1)P_2AH_1 & DJ_2 - (DJ_1)P_2BJ_2 \\ -P_2AH_1 & -P_2BJ_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} \eta_1 \\ u_2 \end{bmatrix} \tag{3.43}$$

Example 17 *Consider the 2-D descriptor representation:*

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},$$

(3.44)

$$y = [1 \quad -1 \quad 0 \quad 1]\xi + [1 \quad 1 \quad -1]u. \tag{3.45}$$

Since $[F, B]$ is not full row rank (i.e. rank = 4), pre-multiply by the non-singular matrix $P = [P_1^T, P_2^T, P_3^T]^T$ which does the following row operations. $R_3 \leftrightarrow R_1$, $-(R_1 + R_2) + R_3 \rightarrow R_3$, $(R_1 - 2R_2) + R_4 \rightarrow R_4$, $R_3 \leftrightarrow R_4$. Then $P_3 A = [1, 0, 0, 0]$, so ξ_1 can be set identically to zero and an equivalent pencil representation is given by:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_2 \\ \xi_3 \\ \xi_4 \\ u_2 \\ u_3 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & -2 & -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} \xi_2 \\ \xi_3 \\ \xi_4 \\ u_2 \\ u_3 \end{bmatrix} \tag{3.46}$$

$$\begin{bmatrix} y \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_2 \\ \xi_3 \\ \xi_4 \\ u_2 \\ u_3 \end{bmatrix} \tag{3.47}$$

3.2.5 Recasting the pencil and dual Pencil representations as AR representations

Recasting the pencil into dual pencil representations is obvious, since a pencil representation can always be rewritten as:

$$\begin{bmatrix} F \\ 0 \end{bmatrix} \Lambda Q\xi + \begin{bmatrix} -E \\ -N \end{bmatrix} \xi + \begin{bmatrix} 0 \\ I \end{bmatrix} w = 0. \tag{3.48}$$

It is always possible to eliminate the internal variables in a dual pencil representation to produce an externally equivalent AR representation. The Gröbner basis algorithm can be used. First, we give a brief introduction of this algorithm.

Gröbner basis

Most of the discussion in this thesis depends heavily on computation with multivariate polynomials. In this section, I give a brief overview of the Gröbner basis algorithm which among many of its applications allows for the simplification and solution of equations represented by the given multivariate polynomials. For more details, the reader is referred to [1] [5], [8], and [9]. This algorithm has been successfully applied to some of the existing problems in multidimensional system theory such as obtaining a first-degree pencil representation for autonomous behaviours by Fornasini *et.al.*[19], solving the canonical Cauchy problem by Oberst [44], and finding a computational scheme generating the system behaviour by Rocha [48]. I will discuss in chapter 5 how some of the work presented in that chapter can be implemented and may be extended using a Gröbner basis.

Consider a system of equations of the form

$$\begin{cases} p_1(x_1, \cdots, x_n) = 0 \\ \vdots \\ p_1(x_1, \cdots, x_n) = 0 \end{cases},$$

where the p_i's are elements of the ring $\mathcal{R}$ of n-variate polynomials with coefficients over some field F. Denote the set $\{p_1, \cdots, p_k\}$ by P and the solution set of the system by Σ. Often, there is more than one set P for a solution set Σ. In fact, any polynomial in the ideal generated by P, that is

$$I = \{< P >\} = \{< p_1, \cdots, p_k >\} = \{\sum_{i=1}^{k} a_i p_i, \quad a_i \in F[x_1, \cdots, x_n]\},$$

also has the same solution set. The set P is said to be a *basis* for this ideal. The general idea of the algorithm is to compute a 'simpler' basis P', one for which the system P' is easier to solve. It should be noted that by the Hilbert basis theorem, any ideal $I \subset F[x_1, \cdots, x_n]$ has a finite generating basis.

First, the definition for the *degree* of a multivariate polynomial will be given. A key ingredient of the notion of degree is the ordering of the terms for polynomials in $F[x_1, \cdots, x_n]$.

Identify any monomial $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ by the n-tuple of exponents $\alpha = (\alpha_1, \cdots, \alpha_n) \in F_+^n$. Any order used must be a *total order*; that is, given any $\mathbf{x}^\alpha, \mathbf{x}^\beta$, exactly one of the following three relations must hold:

$$\mathbf{x}^\alpha < \mathbf{x}^\beta, \quad \mathbf{x}^\alpha = \mathbf{x}^\beta, \text{or} \quad \mathbf{x}^\alpha > \mathbf{x}^\beta.$$

The total orderings used in practice are based on some pre-defined but possibly arbitrary ordering of the first-degree monomials, the variables themselves. Suppose, without loss of generality, that $x_1 > x_2 > \cdots > x_n$. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n) \in F_+^n$. Then the following are three commonly used total orderings:

Lexicographic order $\mathbf{x}^\alpha >_{lex} \mathbf{x}^\beta \Longleftrightarrow$ {the first indices α_i and β_i from the left, which are different, satisfy $\alpha_i > \beta_i$.

Graded Lexicographic order $\mathbf{x}^\alpha >_{glex} \mathbf{x}^\beta \Longleftrightarrow (\sum_{i=1}^{n} \alpha_i > \sum_{i=1}^{n} \beta_i$ or $(\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i$ and $\mathbf{x}^\alpha >_{lex} \mathbf{x}^\beta)$.

Graded Reverse Lexicographic order $\mathbf{x}^\alpha >_{grlex} \mathbf{x}^\beta \Longleftrightarrow (\sum_{i=1}^{n} \alpha_i > \sum_{i=1}^{n} \beta_i$ or $(\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i$ and the first coordinates α_i and β_i from the right, which are different, satisfy $\alpha_i > \beta_i)$.

Two polynomials f and g are *equivalent* with respect to an ideal if their differences belongs to the ideal. The first question is for a given polynomial f and a system of polynomials P, how to reduce f to a polynomial g of smaller order that is equivalent to f with respect to the ideal generated by P. Assuming that an acceptable ordering is chosen, and given non-zero polynomials f, g, and h in $k[x_1, \cdots, x_n]$, f *reduces* to g modulo h ($f \rightarrow_h g$) if the leading monomial of f is divisible by the leading term of h and

$$g = f - \frac{lc(f)}{lc(h)} uh,$$

where $lc(f), lc(h)$ are leading coefficients of f and h respectively, and u is some monomial in $k[x_1, \cdots, x_n]$. A polynomial f is said to be *reduced* with respect to an ideal I if its leading monomial is not a multiple of the leading monomials of some element in I. Furthermore, f is said to be *completely reduced* with respect to I if none of its monomials is a multiple of the leading monomial of some element in I.

Definition 9 *A* Gröbner *(or* standard*) basis of an ideal I in $\mathcal{R}$ with respect to a given ordering, is a generating set of polynomials P for which the reduction of any element of I yields zero.*

A Gröbner basis is called *reduced* if each polynomial in the basis is completely reduced with respect to all the others.

Some useful theorems on Gröbner bases which are due to Buchberger are as follows [8].

Theorem 2 *Every ideal has at least one Gröbner basis with respect to each acceptable ordering.*

Theorem 3 *Two ideals are equal if and only if they have the same reduced Gröbner basis (with respect to the same order).*

Theorem 4 *A system of polynomial equations has a finite number of solutions (in complex space) if and only if each variable occurs in one of the leading monomials of the corresponding Gröbner basis for the lexicographical ordering.*

Modifications to the original Buchberger algorithm and additional results can be found in the earlier cited references and their bibliographies.

Next, use a lexicographical ordering $x_1 > \cdots > x_n > w_1 > \cdots w_p$. The ordering between $\mathbf{x}$ variables, and $\mathbf{w}$ variables can be chosen arbitrarily, as long as $\mathbf{x} > \mathbf{w}$. For monomial X_1 and X_2 in $\mathbf{x}$ variables only, and monomial Y_1 and Y_2 in $\mathbf{w}$ variables only, define an *elimination* order with $\mathbf{x}$ variables greater than $\mathbf{w}$ variables to be:

$$X_1Y_1 > X_2Y_2 \Longleftrightarrow \{X_1 >_{\mathbf{x}} X_2 \text{ or } X_1 = X_2 \text{ and } Y_1 >_{\mathbf{w}} Y_2\}.$$

Let I be the non-zero ideal in $\mathbb{R}[x_1, \cdots, x_n, w_1, \cdots, w_p]$ which corresponds to the polynomial equations given by the dual pencil representation of the system. Let G be a Gröbner basis of this ideal with respect to the elimination order defined above. Then $G \cap \mathbb{R}[w_1, \cdots, w_p]$ is a Gröbner basis for the ideal $I \cap \mathbb{R}[w_1, \cdots, w_p]$. (For more details, see discussion on 'Elimination' in [1] page 69, 'Elimination Theory' in [9] page 114, or 'Implicitization of Rational Parameterizations' in [5] page 328).

Example 18 *Consider the pencil representation obtained in the previous example. Its Gröbner basis (computed using Maple) is given by*

```
> gbasis([z_1*x_3, -z_2*x_2+x_3+x_4+u_2+u_3,
> -z_3*(x_3+x_4)+x_2-2*x_3-x_4-2*u_2-u_3,
>-u_1+x_3+x_4+2*u_2,
> -y-x_2+x_3+2*x_4+3*u_2-u_3],
> [x_2,x_3,x_4,y,u_1,u_2,u_3],plex);
[(-1 + z_2 z_3 + z_2) x_2 + (-z_3 + 1) u_2 - z_3 u_3, x_3,
    (-1 + 2 z_2) u_2 + (z_2 - 1) u_3 + (-1 + z_2 z_3 + z_2) x_4,
    (-1 + z_2 z_3 + z_2) y + (-3 z_2 z_3 + z_3 + z_2) u_2
     + (z_2 z_3 + z_3 + 3 z_2 - 3) u_3,
    (-1 + z_2 z_3 + z_2) u_1 + (1 - 2 z_2 z_3) u_2 + (z_2 - 1) u_3]
```

As can be seen, the last two equations contain of y, u_1, u_2, u_3 only, and correspond to an equivalent AR representation given by

$$\begin{bmatrix} -1+z_2z_3+z_2 & 0 & -3z_2z_3+z_3+z_2 & z_2z_3+z_3+3z_2-3 \\ 0 & -1+z_2z_3+z_2 & 1-2z_2z_3 & z_2-1 \end{bmatrix} \begin{bmatrix} y \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0. \tag{3.49}$$

Chapter 4

Minimality of ARMA Representations

In chapter 2, we have shown that the minimality of the first-degree representations considered is an open problem, and not all necessary and sufficient conditions for the ordering implied by the suggested definitions are known. This also implied that the existence of a minimal first-degree representation of the given form cannot be guaranteed. In this chapter definitions of minimal order for first-degree ARMA representations introduced in chapter 3, Dual Pencil representations and Pencil representations, are given. Some necessary conditions under which representations are minimal will be derived.

4.1 Dual Pencil Representations

In this section, I will give a definition of minimality for the dual pencil representation introduced in chapter 3.

As previously pointed out, implicit representations given by (2.74) can be thought of as a special case of dual pencil representations (3.6) for $Q = I$ of appropriate dimensions. A natural definition of minimality is to require that the system be described by as few equations, as few operators, and as few internal variables as possible. But there is no guarantee that all these quantities can be minimized simultaneously. It may be that more than one definition of minimality will be required, as in [50]. The following is another possible working definition of minimality:

Definition 10 *Associate with every dual pencil representation of a given system,*

$$F\Lambda Q\xi + E\xi + Gw = 0,$$

with $F \in K^{g\times h}$, $\Lambda = diag[z_{i_1}, z_{i_2}, \cdots, z_{i_h}]$, $Q \in K^{h\times m}$, $E \in K^{g\times m}$, $G \in K^{g\times l}$*: a triple* $(g, h, m) \in \mathbb{Z}_+^3$*. Define a partial order on these triples such that* $(g, h, m) \leq (g', h', m')$ *if and only if* $g \leq g'$, $h \leq h'$, *and* $m \leq m'$*. The dual pencil representation given by (3.6) is called* minimal *if there is no other dual pencil representation with an associated triple for which* $(g', h', m') < (g, h, m)$.

The existence of a minimal first-degree representation is guaranteed since a lower bound on the partial ordering on the triples, namely $(0, 0, 0)$ does exist. However, this minimal representation may not be unique, since it is possible to have two minimal representation with corresponding triples (g, h, m) and (g', h', m') such that say $g > g'$ but $h < h'$. Furthermore, with every given triple, there may be more than one corresponding representation, since in the suggested definition of minimality h and m deal with the total number operators and internal variables respectively. For example, it may be possible to have two minimal realization with the same associated triple but with different numbers of individual operators as the following example will illustrate.

Example 19 *In example 13 on page 49, we showed that one realization of the transfer function* $(1 - z_1 z_2)y = (z_1 + z_2)u$ *in the form (3.6) is given by* $F_1 = I_3$, $\Lambda_1 = [z_1, z_2, z_2]$, *and*

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1/2 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 0 & -1/2 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0 & 0 \\ -1 & -1 \\ 1/2 & -1/2 \end{bmatrix}. \tag{4.1}$$

We will show below that this representation is a minimal order Dual Pencil representation, and any other minimal Dual Pencil representation will have an associated triplet of $(3, 3, 2)$. *For example, another minimal Dual Pencil representation of this system is given by* $F_2 = I_3$, $\Lambda_2 = [z_1, z_1, z_2]$, *and*

$$Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -1 & 1 \\ -2 & 1 \\ 1 & -1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \\ 0 & 0 \end{bmatrix}. \tag{4.2}$$

However, Λ_1 *has one* z_1 *and two* z_2 *operators, whereas* Λ_2 *requires two* z_1 *and one* z_2 *operators.*

To show that minimal realizations of form (2.74) will have an associated triplet of $(3, 3, 2)$, *we will use the fact that* Λ *has to have at least one* z_1 *and one* z_2. *Then propositions 2 and 3 of to follow show that two conditions of minimality are that of* $[F, G]$ *has full row rank, and* Q *has full column rank. Finally, we will use similar arguments to in example 11 on page 36. These conditions will reduce associated triplets of possible Dual Pencil representations of smaller order to* $(3, 3, 1)$, $(3, 2, 2)$, $(3, 2, 1)$, $(2, 3, 2)$, $(2, 3, 1)$, $(2, 2, 2)$, $(2, 2, 1)$, $(1, 3, 2)$, $(1, 3, 1)$, $(1, 2, 2)$, *and* $(1, 2, 1)$. *We will show how Dual Pencil representations with associated triplet of* $(3, 2, 2)$ *and* $(3, 2, 1)$ *cannot describe a system with the transfer function given in this question. The arguments for the other triplets are similar, and will not be given here.*

Case 1: Associated triplet of $(3, 2, 2)$

Since Q *must be full column rank, it is alway possible to rewrite any Dual Pencil representation with an associated triplet of* $(3, 2, 2)$ *to look like*

$$\left(\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \\ F_{31} & F_{32} \end{bmatrix}\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \\ E_{31} & E_{32} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \\ G_{31} & G_{32} \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix}=0. \tag{4.3}$$

This is the same as the implicit model realization problem of example 11. and as shown on page 36. cannot happen.

Case 2:Associated triplet of $(3, 2, 1)$

Any Dual Pencil representation with this associated triplet can be rewritten in one of the three following forms:

$$(a)\ F=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\quad Q=\begin{bmatrix} 1 \\ Q_{21} \end{bmatrix},\ E=\begin{bmatrix} E_{11} \\ E_{21} \\ E_{31} \end{bmatrix},\quad G=\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \\ G_{31} & G_{32} \end{bmatrix},\ or$$

$$(b)\ F=\begin{bmatrix} 1 & F_{12} \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\quad Q=\begin{bmatrix} 1 \\ Q_{21} \end{bmatrix},\ E=\begin{bmatrix} E_{11} \\ E_{21} \\ E_{31} \end{bmatrix},\quad G=\begin{bmatrix} 0 & G_{12} \\ 1 & G_{22} \\ 0 & G_{32} \end{bmatrix},\ or$$

$$(c)\ F=\begin{bmatrix} 1 & F_{12} \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\quad Q=\begin{bmatrix} 1 \\ Q_{21} \end{bmatrix},\ E=\begin{bmatrix} E_{11} \\ E_{21} \\ E_{31} \end{bmatrix},\quad G=\begin{bmatrix} G_{11} & 0 \\ G_{21} & 1 \\ G_{31} & 0 \end{bmatrix}.$$

Case 2(a) yields the following relations between the internal and external variables:

$$(z_1 + E_{11})x = -G_{11}y - G_{12}u, \tag{4.4}$$

$$(Q_{21}z_2 + E_{21})x = -G_{21}y - G_{22}u, \tag{4.5}$$

$$E_{31}x = -G_{31}y - G_{32}u. \tag{4.6}$$

Similarly, for case 2(b) we have

$$((z_1 + F_{12}Q_{21}z_2) + E_{11})x = -G_{12}u, \tag{4.7}$$

$$E_{21}x = -y - G_{22}u, \tag{4.8}$$

$$E_{31}x = -G_{32}u, \tag{4.9}$$

and for case 2(c) we have

$$((z_1 + F_{12}Q_{21}z_2) + E_{11})x = -G_{11}u, \tag{4.10}$$

$$E_{21}x = -G_{12}y - u, \tag{4.11}$$

$$E_{31}x = -G_{31}y. \tag{4.12}$$

It is easy to see that none of the three cases described above can produce input-output relations described by $(z_1z_2 - 1)y = (z_1 + z_2)u$.

It will be shown that the following rank conditions are necessary for minimality of ARMA models of the form (3.6).

Proposition 1 *If (3.6) is minimal, then* $[F, G]$ *has full row rank.*

Proof: Rewrite (3.6) in the following block matrix form:

$$[F \quad E \quad G]\begin{bmatrix} \Lambda Q\xi \\ \xi \\ w \end{bmatrix} = 0. \tag{4.13}$$

Assume rank $[F, G] = g - r$, $r > 0$. Then there exists a non-singular matrix $P = [P_1^T, P_2^T]^T$ such that $P[F, G] = \begin{bmatrix} P_1F & P_1G \\ 0 & 0 \end{bmatrix}$, where $[P_1F, P_1G]$ is full row rank. There also exists a change of basis $\xi = [H_1, H_2]\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ such that $P_2EH =$

$[0, P_2EH_2]$ with P_2EH_2 of full column rank, and such that (4.13) can be rewritten as:

$$\begin{bmatrix} P_1F \\ 0 \end{bmatrix} \Lambda[QH_1 \quad QH_2] \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} P_1EH_1 & P_1EH_2 \\ 0 & P_2EH_2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} P_1G \\ 0 \end{bmatrix} w = 0. \quad (4.14)$$

Since P_2EH_2 is full column rank, this implies that η_2 can be set identically to zero. Eliminating η_2 from (4.13) will yield the externally equivalent model:

$$P_1F\Lambda QH_1\eta_1 + P_1EH_1\eta_1 + P_1Gw = 0, \quad (4.15)$$

with reduced internal vector, and fewer equations. □

Next, suppose that for a given representation of the form (3.6), $[F, G]$ has full row rank. Then, it is alway possible to rewrite the representation given by (3.6) in state space form. This provides a tool for generalizing existing and forthcoming results on minimality of state space models as shown below. There exists a non-singular $P = [P_1^T, P_2^T]^T$ such that:

$$P[F \quad G] = \begin{bmatrix} P_1F & P_1G \\ 0 & P_2G \end{bmatrix} \quad (4.16)$$

where P_1F and P_2G are both full row rank. There always exists a state-space representation equivalent to the model represented by (3.6), if there is a freedom to define the entries of w as inputs u and outputs y, since P_1 and P_2 can be chosen such that:

$$P[F \quad G] = \begin{bmatrix} P_1F & P_1G \\ 0 & P_2G \end{bmatrix} = \begin{bmatrix} P_1F & 0 & B \\ 0 & -I & D \end{bmatrix}, \quad (4.17)$$

for some matrices B and D. Letting $w = \begin{bmatrix} y \\ u \end{bmatrix}$ and $P_1E = A$, $P_2E = C$ we have:

$$-P_1F\Lambda Q\xi = A\xi + Bu \quad (4.18)$$

$$y = C\xi + Du. \quad (4.19)$$

Proposition 2 *If (3.6) is minimal, then Q has full column rank .*

Proof: Rewrite (3.6) in the following block matrix form:

$$[F \quad E \quad G] \begin{bmatrix} \Lambda Q\xi \\ \xi \\ w \end{bmatrix} = 0. \tag{4.20}$$

Assume rank $Q = k < m$. Then there exists a non-singular matrix $H = [H_1, H_2]$ and a change of basis $\xi = [H_1, H_2] \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$ such that $QH = [QH_1, 0]$ with rank QH_1 = rank Q, and non-singular $P = [P_1^T, P_2^T]^T$ such that $PEH_2 = \begin{bmatrix} 0 \\ P_2EH_2 \end{bmatrix}$ with P_2EH_2 of full row rank, such that (4.20) can be rewritten as:

$$\begin{bmatrix} P_1F \\ P_2F \end{bmatrix} \Lambda[QH_1 \quad 0] \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} P_1EH_1 & 0 \\ P_2EH_1 & P_2EH_2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} P_1G \\ P_2G \end{bmatrix} w = 0. \tag{4.21}$$

Then because P_2EH_2 has full row rank, η_2 and the corresponding equations can be dropped, leaving the externally equivalent model:

$$P_1F\Lambda QH_1\eta_1 + P_1EH_1\eta_1 + P_1Gw = 0, \tag{4.22}$$

with reduced internal vector, and fewer equations. □

Proposition 3 *Assume (within pre-multiplication by a permutation matrix) that in (3.6) $\Lambda = diag[z_{i_1}I_{h_1}, \cdots, z_{i_t}I_{h_t}]$, for i_j in $\{1, \cdots, n\}$, and some positive integers $h_1, \cdots, h_t$. Block partition F and Q into $[F_1, \cdots, F_t]$ and $[Q_1^T, \cdots, Q_t^T]^T$ respectively, where each $F_i, (Q_i)$ $i = 1, \cdots, t$ has full columns (rows). If (3.6) is minimal, then each $F_i, (Q_i)$ has full column (row) rank .*

Proof: Assume that F_1, (Q_1) is not full column (row) rank. Equation (3.6) in block matrix form can be rewritten as the following:

$$\left(F_1 z_{i_1} I_{h_1} Q_1 + \sum_{j=2}^{t} F_j z_j I_{m_{i_j}} Q_j \right) \xi + E\xi + Gw = 0. \tag{4.23}$$

Then there always exists a non-singular matrix $H = [H_1, H_2]$ ($P = [P_1^T, P_2^T]^T$) such that $F_1Q_1 = [F_1H_1, 0]H^{-1}Q_1$, (or $F_1Q_1 = F_1P^{-1}\begin{bmatrix} P_1Q_1 \\ 0 \end{bmatrix}$) where F_1H_1 is full column rank of say k (or P_1Q_1 is full row rank of say p). It is clear from (4.23) that $h_1 - k$ (or $h_1 - p$) z_{i_1}'s can be deleted to get

$$\left((F_1H_1)z_{i_1}I_k(H^{-1}Q_1) + \sum_{j=2}^{t} F_j z_{i_j} I_{h_j} Q_j\right)\xi + E\xi + Gw = 0. \tag{4.24}$$

(or

$$\left((F_1P^{-1})z_{i_1}I_p(P_1Q_1) + \sum_{j=2}^{t} F_j z_{i_j} I_{h_j} Q_j\right)\xi + E\xi + Gw = 0,) \tag{4.25}$$

where the number of the operators is reduced . □

Conjecture 2 *If (3.6) is minimal, then $[F\Lambda Q + E]$ must be full column rank for all $\Lambda = diag[\lambda_1, \lambda_2, \cdots, \lambda_m] \in K^{m\times m}$, where K is the field in which constant values are taken.*

I will show how this conjecture allows us to deal with cases such as example 7 and example 6 which were extensively used in chapter 2 to examine the minimality conditions for Roesser and implicit models.

Example 20 *(example 7 revisited): It is easy to check that $[F\Lambda Q + E]$ looses rank for $\Lambda = I$ and $-I$, that is $z_1 = z_2 = \pm 1$ (see page 40). For example, it is straightforward to show that the third column is equal to the negative of the sum of first and second columns, evaluated at $(z_1, z_2) = (1, 1)$. Let H^{-1} be the elementary matrix that adds the first and second columns to the third column. i.e.*

$$H^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Rewrite the original dual pencil equation as

$$(F\Lambda Q + E)H^{-1}H\xi + EH^{-1}H\xi + Gw = 0.$$

Let $H\xi = \eta$ *or*

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \xi_1 - \xi_3 \\ \xi_2 - \xi_3 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}.$$

This yields the following matrix equation:

$$\begin{bmatrix} z_1 & 0 & z_1 - 1 & 0 & -1 \\ 0 & z_2 & z_2 - 1 & 0 & -1 \\ -1 & 0 & z_2 - 1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ y \\ u \end{bmatrix} = 0. \tag{4.26}$$

I will show how η_3 *can be removed to obtain a dual pencil representation in* η_1, η_2, y, *and* u. *To do this, use row operations on matrix equation (4.26) over the field of rational real functions in* z_1 *and* z_2. *For the conjecture to be true, this should always yield to first-degree equations in the remaining* η *terms, and zeroth-degree equations in external variables. The row operations used for (4.26) are as follows:* $z_1R_3 + R_1 \rightarrow R_1$, $\frac{1}{z_1 z_2 - 1}R_1$, $-(z_2 - 1)R_1 + R_2 \rightarrow R_2$, $-(z_2 - 1)R_1 + R_3 \rightarrow R_3$, $\frac{1}{z_2}R_2$, $(z_1z_2 - 1)R_2$, $(z_1z_2 - 1)R_3$, $-z_1R_3 + R_2 \rightarrow R_2$, $-z_2R_2 + R_3 \rightarrow R_3$, $\frac{1}{z_1 z_2 - 1}R_2$, $\frac{1}{z_1 z_2 - 1}R_2$. *Removing* η_3 *form the matrix equation (4.26), yields the following dual pencil representation:*

$$\begin{bmatrix} z_1 & 1 & 0 & -1 \\ 1 & z_2 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ y \\ u \end{bmatrix} = 0. \tag{4.27}$$

Using $\Lambda = -I$*, the same approach as above can be taken for* $H^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ *to get another minimal representation of the dynamical system given by:*

$$\begin{bmatrix} z_1 & -1 & 0 & -1 \\ -1 & z_2 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \eta_1' \\ \eta_2' \\ y \\ u \end{bmatrix} = 0. \tag{4.28}$$

Example 21 *(example 6 revisited): It was shown that the smallest real gain realization of the dynamical system described by the AR representation* $[1 - z_1 z_2 \quad z_1 + z_2] \begin{bmatrix} y \\ u \end{bmatrix} = 0$ *is of order 2. (see page 36)*

It is easy to check that $[F\Lambda Q + E]$ *is full column rank for all* $\Lambda = diag[z_1, z_2] \in \mathbb{R}^{2\times 2}$*. However, if the constants are taken over the field of complex numbers, it is straightforward to check that* $[F\Lambda Q + E]$ *losses rank for* $\Lambda = diag[i, -i]$ *and* $\Lambda = diag[-i, i]$*. For example, it can be shown that the second column is equal to* $-i$ *times the first column plus* i *times the second column. Taking a similar approach as the previous example, let*

$$H^{-1} = \begin{bmatrix} 1 & -i & 0 \\ 0 & 1 & 0 \\ 0 & -i & 1 \end{bmatrix}, \quad and \quad H\xi = \eta.$$

Using the following row operations: $z_1 R_2 + R_1 \to R_1$, $\frac{-1}{z_1 z_2 - 1} R_1$, $-(z_2 + i)R_1 + R_2 \to R_2$, $(iz_2 - 1)R_1 + R_3 \to R_3$, $\frac{1}{z_2} R_3$, $(z_1 z_2 - 1)R_2$, $(z_1 z_2 - 1)R_3$, $-z_1 R_2 + iR_3 \to R_3$, $-z_2 R_3 + R_2 \to R_2$, $\frac{1}{z_1 z_2 - 1} R_2$, $\frac{1}{z_1 z_2 - 1} R_2$ *and removing* η_2 *yields the following dual* *pencil representation:*

$$\begin{bmatrix} -z_1 & i & 0 & 1 \\ i & -z_2 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ y \\ u \end{bmatrix} = 0. \tag{4.29}$$

4.2 Pencil Representations

In this section, I will give a definition of minimality for the pencil representations introduced in chapter 3, similar to the notion of minimality for dual pencil representations introduced in the previous section.

Definition 11 *Associate with every pencil representation of a given system given by (3.8)-(3.9):*

$$\begin{aligned} F\Lambda Q\xi &= E\xi \\ N\xi &= w, \end{aligned}$$

with $F \in K^{g\times h}$, $\Lambda = diag[z_{i_1}, z_{i_2}, \cdots, z_{i_h}]$, $Q \in K^{h\times m}$, $E \in K^{g\times m}$, $N \in K^{q\times m}$; *a triple* $(g, h, m) \in \mathbb{Z}_+^3$, *with a partial order similar to the one in the last definition. The representation given by (3.8)-(3.9) is called* minimal *if there is no other pencil representation with an associated triple for which* $(g', h', m') < (g, h, m)$.

All the rank conditions found for dual pencil representations in the previous section are also applicable to pencil representations, since a pencil representation can always be rewritten as:

$$\begin{bmatrix} F \\ 0 \end{bmatrix} \Lambda Q\xi + \begin{bmatrix} -E \\ -N \end{bmatrix} \xi + \begin{bmatrix} 0 \\ I \end{bmatrix} w = 0.$$

In particular, the two following propositions are immediate.

Proposition 4 *If the dual pencil representation given by (3.8)-(3.9) is minimal, then F has full row rank .*

Proposition 5 *If (3.6) is minimal, then* $\begin{bmatrix} F\Lambda Q - E \\ -N \end{bmatrix}$ *must be full column rank for all* $\Lambda = diag[\lambda_1, \lambda_2, \cdots, \lambda_m] \in K^{m\times m}$.

We conclude this chapter by pointing out again that not all necessary and sufficient conditions for the ordering implied by definitions given is this chapter are known. However these definitions allow sufficiency conditions to be investigated, and provide a framework for investigation.

Chapter 5

Minimal AR representations

In this chapter, a classification of externally equivalent AR representations will be given. A definition of *order* associated with any given AR representation and consequently a definition of *minimality* of polynomial bases describing an AR representation will be given. An n-D generalization of Forney's result [21] will be stated and proved, and an example illustrating the theorem will be given.

Sections 5.1 and 5.2 deal with the computational aspects of obtaining a minimal basis for an AR representation, namely factorization of multivariate polynomials over the given fields, and p-adic valuations with related concepts. These sections are a condensed survey of some of the classical and recent results dealing with these two topics. Readers familiar with these works can skip over this section to the following one which is the main result of this chapter.

Section 5.3 uses the concepts introduced in sections 5.1 and 5.2 to give an n-D generalization of a theorem due to Forney. The proof of this theorem, as well as an example illustrating its result, will be given.

5.1 Multivariate polynomial factorization

In this section we summarize some of the algorithms used in factorization of multivariate polynomials, basic algebraic definitions, and results for polynomial equations that will be needed in the following section. For more details the reader is referred to textbooks such as [7], [23], and papers such as [30], and [54]. The reader is assumed to be familiar with the axioms of a commutative ring R, as a set together with two binary operations $+$ and $\times$ such that

$(R, +)$ is an abelian group

$\times$ is commutative and associative

the distributive laws hold in R

The commutative ring of interest is the ring of multivariate polynomials in indeterminate $x_1, \cdots, x_n$ with coefficients in F, for some arbitrary field F, written as $F[\mathbf{x}] = F[x_1, \cdots, x_n]$. The a quotient field is $F(\mathbf{x})$. A commutative ring R with an identity and no zero divisors is called an *integral domain*. The elements in R which have an inverse are called *units*. Elements a and b of R such that $a = ub$ for some unit u in R are called *associates*. An element $0 \neq r \in R$ is called *irreducible* if whenever $r = ab$, either a or b is a unit. An element $0 \neq p \in R$ is called a *prime* if it is not a unit and whenever $p|ab$, either $p|a$ or $p|b$. An integral domain R is called a *Unique Factorization Domain (UFD)* if for all its non-unit elements $0 \neq r \in R$, the following properties hold:

- r can be written as a finite product of irreducibles p_i of R.
- This decomposition is unique up to associates.

It is easy to show that $F[\mathbf{x}]$ is a UFD. It is also easy to prove that in a UFD a non-zero element is a prime if and only if it is irreducible. henceforth we will use these two notions interchangeably. This is the property which we will utilize for factorization in $F(x_1, \cdots, x_n), n \geq 2$, since many of the existing definitions and results in 1-D rely on the fact that $F(x_1)$ is also a *principle ideal domain*. a property which does not hold for $n \geq 2$.

The remainder of this section deals with factorization of multivariate polynomials over $\mathbb{Q}$, $\mathbb{R}$. $\mathbb{C}$, and F where F is an arbitrary field. First a summary of discussions of factorization over rational numbers. which has long history. will be given. In the following subsections, extensions of these methods to reals and complex coefficients, as well as to polynomials with coefficients in an arbitrary field using algorithms suggested by Trager and Gianni will be shown.

5.1.1 Factorization over $\mathbb{Q}(x)$

The problem of testing a polynomial for irreducibility and factoring polynomials into irreducibles has a long history and can be traced back to Schubert. Eisenstien. and Kronecker [30]. However, most of the present work stems from the work of Berlekamp [6], Zassenhaus [59], and Wang [54] [53], (in the case of multivariate integral polynomials).

The factorization problem over $\mathbb{Q}$ can be converted to factorization over $\mathbb{Z}$ by multiplying the polynomial by the least common multiple of its coefficients. The main idea of factorizations over integers is to reduce the problem of factorization of multivariate polynomials to those of univariate polynomials by identifying $\mathbb{Z}[x_1, \cdots, x_n]$ as $\mathbb{Z}[x_2, \cdots, x_n][x_1]$, factorizing this univariate polynomial over $\mathbb{Z}_p$, where p is a prime number, then lifting the solution in $\mathbb{Z}_p[x_1]$ to a desired solution in $\mathbb{Z}[x_1]$ by solving certain nonlinear equations, and extending this solution to $\mathbb{Z}[x_1, \cdots, x_n]$ by an ideal-adic version of Newton's method, for solving nonlinear equations. Here are some basic definitions which will be needed.

Definition 12 *Let p be an n-variate polynomial in $\mathbb{Z}[x_1, \cdots, x_n]$. Express p as a univariate polynomial in x_1 of degree k_1 with coefficients in $\mathbb{Z}[x_2, \cdots, x_n]$, i.e.*

$$p(x_1, \cdots, x_n) = \sum_{i=1}^{k_1} p_i(x_2, \cdots, x_n)x_1^i \tag{5.1}$$

where $p_i(x_2, \cdots, x_n) \in \mathbb{Z}[x_2, \cdots, x_n]$. The content of p (cont(p)) is the g.c.d. of all the coefficients p_i. The primitive part of p (pp(p)) is defined as pp(p)= p/cont(p). p is called primitive if the p_i's are relatively prime.

It is a known classical result (Gauss' lemma) that the product of any two primitive polynomials is itself primitive. So, factorizing an n-variate polynomial reduces to the subproblems of factoring the content and the primitive part of p. Therefore, assume that the given polynomial p is primitive and monic. p is *square-free* if it has no repeated factors. One can test p for being square-free by checking for common roots of p and its partial derivative with respect to x_1. The g.c.d. algorithm can be used to write p as a product of square-free polynomials. Henceforth, assume that p is a monic, square-free primitive polynomial.

First, consider factorization of an univariate polynomial, say $p(x) \in \mathbb{Z}[x]$:

• Choose a prime q such that q does not divide p or dp/dx. This will ensure that p remains square-free modulo q.

• Factor p into irreducible polynomials in $\mathbb{Z}_q$ using Berlekamp's algorithm:

Berlekamp's factorization algorithm: Suppose that in $\mathbb{Z}_q$ $p(x)$ is a polynomial of degree n and factorizes into r irreducible polynomials $p_i(x)$, that is

$$p(x) = p_1(x) \cdots p_r(x) \mod q, \tag{5.2}$$

where the p_i are relatively prime (otherwise p is not square-free). Furthermore, suppose there exists a polynomial $v(x)$ such that

$$v(x) \equiv s_i \mod p_i \quad i = 1, \cdots, r, \tag{5.3}$$

for some integers s_i modulo q. By the Chinese Remainder Theorem (CRT), the degree of $v(x)$ is less than that of $p(x)$. For such a polynomial v if $s_i \neq s_j$ for some $i, j \in 1, \cdots, r$, then $\gcd(p, v - s_i)$ is divisible by p_i, but not by p_j. Hence, the calculation of the $\gcd(p, v - s_i)$ will lead to a decomposition of p.

Fermat's little theorem implies that for any polynomial $f(x)$,

$$f(x)^q \equiv f(x^p) \mod q.$$

Also in $\mathbb{Z}_q$, we have

$$v(x)^q \equiv s_i{}^q \equiv s_i \equiv v(x) \mod p_i,$$

and by CRT

$$v(x)^q \equiv v(x) \mod p(x).$$

Hence

$$v(x^q) \equiv v(x) \mod p(x). \tag{5.4}$$

We know that the degree of $v(x)$ is less than $p(x)$. Suppose $v(x)$ can be written as

$$v(x) = v_0 + v_1 x + \cdots + v_{n-1} x^{n-1}, \; v_i \in \mathbb{Z}_q \quad i = 1, \cdots, n-1.$$

Berlekamp's basic idea was to solve the problem of finding the coefficients v_i of all possible v, as a system of linear equations in the v_i using (5.4). Let $q_{j,i} \in \mathbb{Z}_q$ $0 \leq i, j \leq n-1$ be such that

$$x^{qj} \equiv q_{j,n-1} x^{n-1} + \cdots + q_{j,1} x + q_{j,0} \mod p(x).$$

Consider the polynomial $v(x)$ as a vector of its coefficients $\nu = (v_0, \cdots, v_{n-1})$. It is easy to see that ν can be found from the equation $\nu Q = \nu$ where

$$Q = \begin{bmatrix} q_{0,0} & \cdots & q_{0,n-1} \\ \vdots & & \vdots \\ q_{n-1,0} & \cdots & q_{n-1,n-1} \end{bmatrix}.$$

• Now assuming that using the algorithm above, we have a complete factorization of $p(x) \in \mathbb{Z}_q$ such that

$$p(x) = p_1(x) \cdots p_r(x) \mod q,$$

we will show how the factors p_i can be lifted to some factors $p_i'(x)$ in $\mathbb{Z}[x]$ using the two conditions which have to be met:

1. $p(x) = p_1'(x) \cdots p_r'(x)$,
2. $p_i(x) \equiv p_i'(x) \mod q \quad i = 1, \cdots, r$.

The existence of solutions subject to the two conditions above is guaranteed by a classical result known as Hensel's lemma. There are a variety of results in the literature which go under this name, however their common feature is that the existence of an approximate solution of an equation implies the existence of an exact solution, and subject to some given conditions an approximate solution is a "good enough" solution. The equation in question here is the nonlinear equation

$$F(p_1' \cdots p_r') = p(x) - p_1' \cdots p_r' = 0.$$

Hensel's lemma: Let $p(x)$ be a polynomial over the integers. Let q be a prime and $p_1(x), \cdots, p_r(x) \in \mathbb{Z}_q$ be relatively prime polynomials over $\mathbb{Z}_q$ such that

$$p(x) = p_1(x) \cdots p_r(x) \mod q.$$

Then for any integer $k \geq 1$, there exists polynomials $p_1^k \cdots p_r^k \in \mathbb{Z}_{q^k}$ such that

- $p(x) \equiv p_1^k(x) \cdots p_r^k(x) \mod q^k$,
- $p_i^k(x) \equiv p_i(x) \mod p(x) \quad i = 1, \cdots, r.$

Hensel's lemma allows us to lift factors to $\mathbb{Z}_{q^k}$ for $k \geq 1$. Calculate an integer bound N such that all the coefficients of all factors of $p(x)$ in $\mathbb{Z}[x]$ are absolutely bounded by N using one of the existing methods. Choose k such that $q^k > 2N$. Then the factors in $\mathbb{Z}_{q^k}$ or some products of them will give us all the irreducible factors of $p(x)$ in $\mathbb{Z}[x]$.

The factorization of polynomials in n variables ($n \geq 2$) is done in the same way as the one-variable case. The first $n-1$ variables are fixed to obtain a polynomial in one variable. A modified version of Hensel's lemma can be used to find a 1-D factorization. These factors then can be lifted to $\mathbb{Z}[x_1, \cdots, x_n]$ by an inverse reduction and trial divisions.

In the following subsections, two algorithms due to Trager [52], and Gianni and Trager [24] will be discussed. One uses the extension of the algorithms discussed above over extension fields of $\mathbb{Q}$ which includes all coefficients of a given polynomial to obtain a factorization. The other uses the Gröbner basis algorithm to give a factorization over any arbitrary field.

5.1.2 Factorization over $\mathbb{R}$(x) and $\mathbb{C}$(x)

The main idea of the following algorithm by Trager [52] is to obtain a factorization over $\mathbb{R}$ or $\mathbb{C}$ by using an extension field of $\mathbb{Q}$ which contains all the coefficients of the given polynomial. The factorization problem over the required extension field is then reduced to factorization over $\mathbb{Q}$ using a function called *norm*. Once the factorization is obtained over $\mathbb{Q}$, the factors can be lifted to those over $\mathbb{R}$ or $\mathbb{C}$ by using the relationship between the norms of the factors and the actual factors. In the following, definitions and the algorithm will be formally defined.

Definition 13 *Let K be an extension field of F (i. e. K is a field such that $F \subset K$). An element $\alpha \in K$ is called* algebraic *over F if α is a root of some non-zero polynomial $f \in F[\mathbf{x}]$. K is called an* algebraic extension *of F if every element in K is algebraic over F.*

For example, $i \in \mathbb{C}$ is algebraic over $\mathbb{Q}$, as is any element in $\mathbb{R}$ and $\mathbb{C}$ which can be obtained through addition, multiplication or exponentiation (including factors of powers) which includes almost all the coefficients necessary for describing dynamical systems.

First consider the 1-D factorization problem by fixing all but one of the variables, say x. The factors once obtained can be lifted to the n-D ones by trial and error similar to the one in the last subsection. It is a known fact that for any K an extension field of F and $\alpha \in K$ an algebraic element over F, $F(\alpha)$ is isomorphic to $F[x]/ < p(x) >$ where $p(\mathbf{x}) \in F[\mathbf{x}]$ is the unique monic irreducible polynomial of α of degree, say $h \geq 1$. Also any element $\beta \in F(\alpha)$ can be written uniquely in the form of

$$\beta = f_0 + f_1\alpha + \cdots + f_{h-1}\alpha^{h-1}, \quad f_i \in F.$$

$p(x)$ is referred to as the *irreducible* (or *minimal*) polynomial of α.

Definition 14 *Let $p(x)$ be the unique, monic irreducible polynomial of α over F. The* conjugates *of α over F are defined to be the remaining distinct roots of $p(x)$, say $\alpha_2, \cdots, \alpha_h$.*

Let $\beta \in F(\alpha)$ be represented as $\beta = f_0 + \cdots + f_{h-1}\alpha^{h-1}$, $f_i \in F$: then the conjugates of β, β_i are defined to be

$$\beta_i = f_0 + f_1\alpha_i + \cdots + f_{h-1}\alpha_i^{h-1}, \quad \text{for } i = 2, \cdots, h.$$

$Norm(\beta)$ is defined to be the product of all the conjugates of β in $F(\alpha)$.

It is easy to show that Norm(β) $\in F$, for any $\beta \in F(\alpha)$. As stated earlier, the Norm function can be used to convert the factorization problem in $F(\alpha)$ to a factorization in F. The following theorem will show the relationship between factors of the Norm of a given polynomial and its factors.

Theorem 5 *[52] If $\beta(x, \alpha)$ is an irreducible polynomial over $F(\alpha)$, then Norm(β) is a power of an irreducible polynomial over F.*

By allowing for a proper substitution for x [52], it is always possible to ensure the Norm(β) is square-free and hence obtain a complete factorization of β over $F(\alpha)[x]$ by calculating the gcd of β and all the factors of Norm(β).

5.1.3 Factorization over arbitrary fields

In this subsection the algorithm for factorization of polynomials with coefficients in an arbitrary field, given by Trager and Gianni, which uses the Gröbner basis algorithm will be given.

Theorem 6 *[24] Let $f, \in F[x_1, \cdots, x_n]$ be a given polynomial in all the n variables to be factorized. Consider a total degree ordering with x_1 greater than the other variables. Suppose J is a maximal ideal in $F[x_2, \cdots, x_n]$ and g and h two multivariate polynomials are in all the n variables such that*

1. $f \equiv gh$ mod J,

2. The ideal generated by g, h, and J is the whole ring $F[x_1, \cdots, x_n]$,

3. The ideal generated by the leading coefficient of f with respect to x_1 and J is $F[x_2, \cdots, x_n]$.

Then the reduced Gröbner basis of the ideal generated by f, g^k, and J^k for sufficiently large $k \in \mathbb{N}$ contains a unique polynomial g_k of least degree such that

(a) $g | g_k \mod J$,

(b) $g_k | f$.

The description of the methods in the above two subsections is only intended to give an overall sketch of factorization of polynomials in several variables and it should be noted that the factorization problem remains an active area of research.

5.2 p-adic valuations

In this subsection, I will give a definition of p-adic valuation for multivariate polynomials and polynomial vectors. In the following section, these notions will be used to define an order associated with an AR representation of a given dynamical system and to show the relationship between different AR representations of the given system. p-adic analysis is of particular interest and importance in many areas of mathematical research such as number theory and representation theory. This is equally true of the general notion of a valuation together with some of its related concepts.

I will give the definition of a valuation with respect to $\mathcal{R} = \mathbb{C}(x_1, \cdots, x_n)$. This and related concepts can be defined over more general fields. For more details on p-adic valuations the reader is referred to textbooks such as [3], [32] and [39].

Admissible definitions of order for a n-variate polynomial were given in chapter 3. Using any admissible ordering, a polynomial can be written in descending order of its monomial terms. The largest monomial is referred to as the *leading monomial* and the (multi-)degree of the given polynomial is taken to be $\sum_{i=1}^{n} \alpha_i$ if glex or grlex are used and α_1 if lex ordering is used.

The non-archmidean p-adic valuation to be used is an integer valued function $\nu : \mathbb{C}(x_1, \cdots, x_n) \to \mathbb{C} \cup \infty$ such that for all $k_1, k_2 \in \mathbb{C}(x_1, \cdots, x_n)$ the following axioms hold:

(i) $\nu(0) = \infty$,

(ii) $\nu(k_1 k_2) = \nu(k_1) + \nu(k_2)$,

(iii) $\nu(k_1 + k_2) \geq \min(\nu(k_1), \nu(k_2))$.

The following two valuations which satisfy the axioms above will be used.

Let p be any irreducible integer polynomial, and k any nonzero polynomial in $\mathbb{C}(x_1, \cdots, x_n)$. By the discussion in the earlier section, it is always possible to write k as

$$k = p^{\rho} \frac{u}{v},$$

where $\rho \in \mathbb{C}$, $u, v \in \mathbb{C}[x_1, \cdots, x_n]$, and $p \not| u$, $p \not| v$. Define $\nu_p(k)$ to be ρ. It is easy to check that axioms (i)-(iii) hold.

The other valuation which will be used is denoted by ν_∞ and for $k = \frac{u}{v}$, $u, v \in \mathbb{C}[x_1, \cdots, x_n]$ is equal to deg v - deg u. Again, it is straightforward to check that the required axioms hold.

Both valuations are defined to be *trivial* on the constant polynomials in the sense that for all $k \in \mathbb{Q}$, the valuation is equal to ∞ if $k = 0$ and 0 otherwise.

It should be noted that for any $0 \neq k \in \mathbb{C}(x_1, \cdots, x_n)$, and all irreducibles p $\sum_p \nu_p(k) = -\nu_\infty(k)$.

Next, we extend these notions to polynomial vectors. Consider a vector V over $\mathbb{C}(x_1, \cdots, x_n)$ of dimension n. An extension of a p-adic valuation ν on $\mathbb{C}(x_1, \cdots, x_n) \triangleq \mathcal{R}$ to a *norm* on $\mathcal{R}^n$ is defined as a integer value function $\Upsilon : \mathcal{R}^n \to \mathbb{C} \cup \infty$, such that $\Upsilon|_{\mathcal{R}} = \nu$ and for all $k \in \mathcal{R}$ and $a, b \in \mathcal{R}^n$ the following axioms hold:

(i) $\Upsilon(0) = \infty$,

(ii) $\Upsilon(ka) = \nu(k) + \Upsilon(a)$,

(iii) $\Upsilon(a + b) \geq \min(\Upsilon(a), \Upsilon(b))$.

Suppose $\{e_1, \cdots, e_n\}$ is a basis of V. Then every $\mathbf{a} \in V$ can be written uniquely as

$$\mathbf{a} = \sum_{i=1}^{n} a_i e_i,$$

for some $a_i \in \mathbb{C}(x_1, \cdots, x_n)$.

The norm we are going to use is defined as:

$$\Upsilon_p(\mathbf{a}) = \min_i \nu_p(a_i), \quad \Upsilon_\infty(\mathbf{a}) = \min_i \nu_\infty(a_i).$$

It is easy to check that the axioms (i)-(iii) hold.

We conclude this section by defining the p-adic *residue* for a given polynomial vector, and defining orthogonal vectors.

It is easy to show that for any $k \in \mathbb{C}(x_1, \cdots, x_n)$ and any irreducible polynomial p, k can be represented by a (possibly infinite) p-adic power series:

$$k = p^\rho k_1 + p^{\rho+1} k_2 + \cdots,$$

where $k_i \in \mathbb{C}(x_1, \cdots, x_n)$ and $\rho = \nu_p(k)$. Then the p-adic residue of a given polynomial vector $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{C}^n[x_1, \cdots, x_n]$ is defined to be the n-tuple of coefficients of $p^{\Upsilon_p(\mathbf{a})}$ in the power series expansions of the a_i.

Next, consider a vector $\mathbf{g}$ and let $\Sigma = \{\mathbf{g}_1, \cdots, \mathbf{g}_k\}$ be a k-vector set such that

$$\mathbf{g} = \sum_{j=1}^{k} h_j \mathbf{g}_j,$$

for some $h_j \in \mathbb{C}(x_1, \cdots, x_n)$. By property (iii) of norm definition in the previous page

$$\nu_p(\mathbf{g}) \geq \min_j \nu_p(h_j \mathbf{g}_j),$$

for all irreducibles p. The set Σ is called *p-orthogonal* if

$$\nu_p(\mathbf{g}) = \min_j \nu_p(h_j \mathbf{g}_j),$$

for a given p, and called *globally orthogonal* if it is p-orthogonal with respect to all the p's.

5.3 Classification of AR representations

In chapter 3, it was shown that a minimal-order Roesser realization, obtained from a proper 2-D single-input, single-output transfer function, with degrees n and m with respect to the operators z_1 and z_2, was defined to have an order equal to $n + m$. As has been pointed out earlier, contrary to the 1-D case [56], generically, the transfer function is not proper for $n \geq 2$. Furthermore, to the author's knowledge, a definition of order with respect to multi-input, multi-output systems has not been defined. An interesting question in its own right is the existence of more than one (possibly infinitely many) transfer functions describing the given dynamical system. In the behavioural setting, one can pose the question of how different AR representations of a given behaviour are related. In answering this question, I will define equivalence classes of AR representations. After giving a definition of *order* for AR representations, I will show how one can find a least-order representation for a given dynamical system.

Consider a behaviour of a system described by two AR representations $R_1 w = 0$ and $R_2 w = 0$, where w is the set of external variables with values in some K-valued vector space V, for some field K, and R_1 and R_2 two polynomial matrices with the number of columns equal to the dimension of V, and the number of the rows (possibly different) equal to the number of equations. From the algebra of vector spaces, the row spaces of R_1 and R_2 are identical. Hence, it is always possible to find polynomial matrices X and Y such that $XR_1 = R_2$, and $YR_2 = R_1$. Polynomial matrices X and Y can be computed using any mapping that takes a basis of X to a basis of Y and vice versa. In other words, all AR representations of a given behaviour form an equivalence class of the polynomial matrices R_i, where any two members of this class have the above property.

Next, for a given equivalence class and a one of its representative R_i we define *order* of R_i, and show how a least-order representative of this class can be found.

In [21] (appendix, pp. 516), Forney showed that p-adic valuations can be used to define a minimal basis for a rational vector space over a ring of polynomials in one variable, say $F[x]$ for some field F. In 1-D, this may not represent the most practical way of obtaining such minimal bases compare to the other existing methods, due to the fact that $F[x]$ is an Euclidean domain. However, as pointed out earlier in this thesis, $F[\mathbf{x}]$ is no longer an Euclidean domain in the n-D ($n \geq 2$) case. Hence, using the fact that $F[\mathbf{x}]$ is still a UFD, a generalization of Forney's result can provide a powerful method of classifying the vector spaces over $F(\mathbf{x})$.

Certain differences between the 1-D and n-D case should be pointed out:

- Definition of the degree of a polynomial is not unique.
- ν_p and ν_∞ are not the only possible valuations over $F(\mathbf{x})$.

First, we need the following definitions:

Let $\mathcal{P}$ be the set of all irreducibles in $F[\mathbf{x}]$ and $\mathcal{P}^*$ be the index set of p's of the valuations ν_p where either $p \in \mathcal{P}$, or $\nu_p = \nu_\infty$; that is $\mathcal{P}^* = \mathcal{P} \cup \{\infty\}$.

Definition 15 *For a rational function m-tuple* $\mathbf{a}$, *the* defect *of* $\mathbf{a}$ *is defined as*

$$\mathit{def}(\mathbf{a}) = -\sum_{p\in\mathcal{P}^*} \Upsilon_p(\mathbf{a}).$$

It is easy to see that def($\mathbf{a}$) is always non-negative. Furthermore, for any polynomial h, $0 \neq h \in F(\mathbf{x})$, we have

$$\text{def}(h\mathbf{a}) = -\sum_{p\in\mathcal{P}^*} \Upsilon_p(h\mathbf{a}) = -\sum_{p\in\mathcal{P}^*} \nu_p(h) - \sum_{p\in\mathcal{P}^*} \Upsilon_p(\mathbf{a}) = \text{def}(\mathbf{a}).$$

As in the 1-D case, this implies that for a m-tuple $\mathbf{a}$ of rational functions, def $\mathbf{a}$ = deg $\mathbf{a}$, since if h is chosen to be the ratio of the least common multiple of the denominators of all entries of $\mathbf{a}$ divided by the greatest common divisor of the numerators, $\sum_{p\in\mathcal{P}} \Upsilon_P(\mathbf{a}) = 0$ and $\Upsilon_\infty(\mathbf{a})$ = -deg $\mathbf{a}$.

Definition 16 *For a $k \times m$ matrix G with entries in $F(\mathbf{x})$ and rows $\mathbf{g}_i$, define the* order *of G to be $\sum$ def $\mathbf{g}_i$.*

Definition 17 *Let V be the k-dimensional vector space span of the rows of a $k \times m$ matrix G over $F(\mathbf{x})$. Define $k \times m$ matrix G' over $F(\mathbf{x})$ to be a* minimal basis *for V if G' is a basis for V and it has the least-order among all bases for V.*

The following theorem is a restatement of Forney's result for the n-D case. The proof is new.

Theorem 7 *Let $k \times m$ matrix G be a basis for a k-dimensional row-space V of polynomial n-tuple over $F(\mathbf{x})$, and let order of G be $\sum$ def $\mathbf{g}_i$, where $\mathbf{g}_i$ are the rows of G. The the following statements are equivalent:*

1. *G is a minimal basis for V.*
2. *$[G]_p$ has full rank over the residue class field $\bar{F}_p$ of polynomials modulo p, for all $p \in \mathcal{P}^*$.*
3. *Let $h \in \mathbb{Z}$ be the minimum p-valuation obtainable over all the $k \times k$ minors of G, then*
$$h = \sum_{i=1}^{k} \Upsilon_p(\mathbf{g}_i) \quad \forall p \in \mathcal{P}^*.$$
4. *The rows of G are globally orthogonal.*

Proof: The logical order of the proof is:

$$1 \Rightarrow \begin{matrix} 3 \\ \Updownarrow \\ 2 \end{matrix} \Rightarrow 4 \Rightarrow 1.$$

($1 \Rightarrow 2$) Assume that G is a minimal basis. Also, assume to contrary, that $[G]_p$ is not full rank over $\bar{F}_p$ for some irreducible polynomial p. That is,

$$\exists f_i \in \bar{F}_p \quad \text{such that} \quad \sum_{i=1}^{k} f_i[\mathbf{g}_i]_p \equiv 0 \bmod p.$$

Note that $\deg f_i < \deg p$. Let

$$\mathbf{g}' = \sum f_i \mathbf{g}_i. \tag{5.5}$$

It is easy to see that

$$[\mathbf{g}']_p = \sum f_i[\mathbf{g}_i]_p \equiv 0 \bmod p.$$

This implies that $\mathbf{g}'$ is divisible by p, and $\mathbf{g}'/p$ has degree

$$\begin{aligned} \deg(\mathbf{g}'/p) &= \deg \mathbf{g}' - \deg p \\ &\leq \max(\deg f_i + \deg \mathbf{g}_i) - \deg p \\ &< \max(\deg \mathbf{g}_i) \quad \text{since} \quad \deg f_i < \deg p, \end{aligned}$$

where the maximum is only taken over $\mathbf{g}_i$ for which $f_i \neq 0$. Without loss of generality, say $\mathbf{g}_1$ is the row for which this maximum was obtained. Using (5.5), $\mathbf{g}_1$ can be replaced by $\mathbf{g}'/p$ to get a basis of lower order, which is a contradiction.

($2 \Leftrightarrow 3$) Let $N = (n_{ij})$ be any $k \times k$ matrix, let σ be any permutation on the set $\{1, \cdots, k\}$, and $\epsilon(\sigma)$ the sign of the permutation σ. Then the determinant of N is given by [10]:

$$\det N = \sum \epsilon(\sigma) n_{\sigma(1),1} n_{\sigma(2),2} \cdots n_{\sigma(k),k}. \tag{5.6}$$

This implies that

$$\nu_p(\text{any } k \times k \text{ minor}) \geq \sum_{i=1}^{k} \Upsilon_p(\mathbf{g}_i).$$

Assume that the p-valuations of all $k \times k$ minors of G are always greater than the minimum value h. We are going to show that $[G]_p$ cannot have full rank over $\bar{F}_p$ for some irreducible polynomial $p \in \mathcal{P}^*$.

Let, for a given p, $v_p(\mathbf{g}_i) = a_i$, $i = 1, \cdots, k$. Consider the $k \times k$ minor for which each column has the entry with factor p of power a_i in the $\mathbf{g}_i$-th row. The term corresponding to the product of terms of power a_i cancel out (mod p) if and only if there exists a linear combination of the rows of $[G]_p$ which are congurant to zero mod p.

($2 \Rightarrow 4$) Assume $[G]_p$ has full rank over $\bar{F}_p$ for all irreducible polynomial $p \in \mathcal{P}^*$. Let $g' = \sum_{i=1}^{k} h_i \mathbf{g}_i$ for some $h_i \in F$. By the definition of p-adic norm,

$$\nu_p(g') \geq \min_i \nu_p(h_i \mathbf{g}_i) \quad \text{for all } \; p \in \mathcal{P}^*.$$

If $\nu_p(g') > \min_i \nu_p(h_i \mathbf{g}_i)$ for some $p \in \mathcal{P}^*$, and $\min_i \nu_p(h_i \mathbf{g}_i) = h_s g_s$ for some $s \in 1, \cdots, k$, then this implies that $\nu_p(h_s g_s)$ has been canceled out by $\sum_{\substack{i=1 \\ i \neq s}}^{k} \nu_p(h_i g_i)$. But this is contradicts $[G]_p$ having full rank over $\bar{F}_p$ for all irreducible polynomial $p \in \mathcal{P}^*$.

($4 \Rightarrow 1$) Assume the $\mathbf{g}_i$ are globally orthogonal. If G is not a minimal basis for V, then there exist a matrix G' which is also a basis for V, and has smaller order than order of G. Let $\mathbf{g}'_1, \cdots, \mathbf{g}'_k$ be the rows of G'. Since rows of G are a basis for V, we have for $j = 1, \cdots, k$:

$$\mathbf{g}'_j = \sum_{i=1}^{k} k_{ji} \mathbf{g}_i \quad \text{for } k_{ji} \in F(\mathbf{x}).$$

Since the $\mathbf{g}_i$ are globally orthogonal

$$\begin{aligned} \sum_{p \in \mathcal{P}^*} \Upsilon_p(\mathbf{g}'_j) &= \sum_{p \in \mathcal{P}^*} \Upsilon_p(k_{ji} \mathbf{g}_i) \\ &= \min_i \sum_{p \in \mathcal{P}^*} \Upsilon_p(k_{ji} \mathbf{g}_i), \end{aligned}$$

but order $G' <$ order G implies that

$$\sum_{j=1}^{k} \sum_{p \in \mathcal{P}^*} \Upsilon_p(\mathbf{g}'_j) > \sum_{i=1}^{k} \sum_{p \in \mathcal{P}^*} \Upsilon_p(\mathbf{g}_i). \tag{5.7}$$

Let $r \in 1, \cdots, k$ be such that

$$\sum_{p \in \mathcal{P}^*} \Upsilon_p(\mathbf{g}'_r) \geq \sum_{p \in \mathcal{P}^*} \Upsilon_p(\mathbf{g}'_j)$$

for $j \in 1, \cdots, k$ and $j \neq r$. Then (5.7) implies that

$$\begin{aligned}\sum_{p\in\mathcal{P}^*} \Upsilon_p(\mathbf{g}'_r) &> \min_i \sum_{p\in\mathcal{P}^*} \Upsilon_p(\mathbf{g}_i), \\ &> \min_i \sum_{p\in\mathcal{P}^*} \Upsilon_p(k_{ji}\mathbf{g}_i) \quad \forall k_{ji} \in F(\mathbf{x}),\end{aligned}$$

which is a contradiction. □

The following example will illustrate this theorem.

Example 22 *Consider the 2-D behaviour described by the AR representation $Gw = 0$, where G is given by:*

$$\begin{bmatrix} z_2 - 1 & z_2^2 - z_1 z_2 - z_2 + z_1 & z_1 z_2^2 - z_1^2 z_2 - z_2 + z_{12} \\ z_2^2 - z_1 z_2 - z_2 + z_1 & 0 & -z_1^2 z_2^2 + z_1^3 z_2 + z_1 z_2^2 - z_1^2 z_2 + z_1 z_2 - z_1^2 - z_2 + z_1 \end{bmatrix}. \tag{5.8}$$

Let the ordering of monomials be given by the total degree with $z_2 > z_1$. Factoring over $\mathbb{R}(z_1, z_2)$, G can be written as

$$\begin{bmatrix} p_1 & p_1 p_2 & p_2 p_3 \\ p_1 p_2 & 0 & p_2 p_3 p_4 \end{bmatrix}, \tag{5.9}$$

where $p_1 = z_2 - 1$, $p_2 = z_2 - z_1$, $p_3 = z_1 z_2 - 1$, and $p_4 = -z_1 + 1$. Then we have

$$[G]_{p_1} = \begin{bmatrix} 0 & 0 & -(z_1 - 1)^2 \\ 0 & 0 & (z_1 - 1)^3 \end{bmatrix}. \tag{5.10}$$

Label the rows of G as $\mathbf{g}_1$ and $\mathbf{g}_2$. Let

$$\begin{aligned}\mathbf{g}_{21} &= (-p_4)\mathbf{g}_1 + \mathbf{g}_2 \\ &\equiv 0 \quad mod p_1.\end{aligned}$$

Then

$$\mathbf{g}_{21}/p_1 = [p_1 \quad -p_1 p_4 \quad 0].$$

Note that $\deg \mathbf{g}_2 \geq \deg \mathbf{g}_1$. *Replace the second row of* G *by* $\mathbf{g}_{21}/p_1$ *to get*

$$G' = \begin{bmatrix} p_1 & p_1p_2 & p_2p_3 \\ p_1 & -p_1p_4 & 0 \end{bmatrix}. \tag{5.11}$$

It is easy to check that $[G']_{p_1}$ *is full rank. However,*

$$[G']_{p_1} = \begin{bmatrix} z_1 - 1 & 0 & 0 \\ z_1 - 1 & 0 & 0 \end{bmatrix},$$

which clearly is not full rank. Let $\mathbf{g}'_{12} = \mathbf{g}'_1 - \mathbf{g}'_2$ *where* $\mathbf{g}'_i, i = 1, 2$ *are rows of* G'. *Then* $\mathbf{g}'_{12}/p_2$ *is given by*

$$\mathbf{g}'_{12}/p_2 = [0 \quad p_2 \quad p_3].$$

Note that $\deg \mathbf{g}'_1 \geq \deg \mathbf{g}'_2$. *Replacing the first row of* G, *by* $\mathbf{g}'_{12}/p_2$ *yields*

$$G'' = \begin{bmatrix} 0 & p_2 & p_3 \\ p_1 & -p_1p_4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & z_2 - z_1 & z_1z_2 - 1 \\ z_2 - 1 & z_1z_2 + z_1 - z_2 + 1 & 0 \end{bmatrix}. \tag{5.12}$$

Similarly, $[G'']_{p_i}, i = 1, \cdots, 4$ *is full rank. Hence* G'' *represents a least-order AR representation of the behaviour in (5.8).*

It is elementary algebra to show that the above reduction is equivalent to pre-multiplying G *by a non-singular matrix* X *of rational functions such that* $XG = G''$, *where* X *is given by*

$$X = \begin{bmatrix} \frac{p_1+p_4}{p_1p_2} & \frac{-1}{p_1p_2} \\ \frac{-p_4}{p_1} & \frac{1}{p_1} \end{bmatrix}.$$

Similarly, $G = YG''$ *for*

$$Y = \begin{bmatrix} p_2 & 1 \\ p_2p_4 & p_2 \end{bmatrix}.$$

Chapter 6

Conclusions and Suggestions for Future Work

The work in this thesis covers two fundamental questions:

1. what is the relationship between different representations of the same behaviour?
2. under what conditions is a given representation minimal?

In modeling and classification of n-D dynamical systems, I have used behavioural models of two forms, namely AR and ARMA representations. For ARMA representation, I have considered Dual Pencil, Pencil, and Descriptor representations in a behavioural setting of linear, time-invariant dynamical systems.

With respect to the first question, in chapter 5 the relationship between all AR representations of a given behaviour which forms an equivalence class was given. In chapter 3, I presented a realization method for obtaining equivalent first-degree

ARMA representations from AR representations. In the same chapter. the relationship between equivalent Dual Pencil, Pencil, and Descriptor representations was also given. Some rank conditions were found which allow us to find an equivalent first-degree model of a different form with fewer auxiliary variables. The relationship between the two well known models of Roesser and Fornasini-Marchesini was given in the appendix of this thesis.

The second question was dealt with in chapters 4 and 5. In chapter 5. I have associated a definition of order to AR representations of a given dynamical system and have given necessary and sufficient conditions under which a given AR representation has the least possible order. In chapter 4, a definition of minimality for Dual Pencil and Pencil representations was given. Some necessary conditions under which these two representations are minimal were found. In chapter 2, I also gave a literature survey of minimality definitions and conditions for some known n-D models and examined the minimality conditions using illustrative examples.

Through the work presented in this thesis, I have shown how to classify all AR representations of a given behaviour and how to obtain a least order representation for a given class. This naturally includes the definition of order associated with input-output systems with improper transfer functions, as well as the multi-input multi-output case, and shows the relationship between different representations of such systems.

I have brought together a summary of existing models from a diverse background, which together with the work done in this thesis shows that the definition of minimality for first-degree ARMA representations is model dependent. Also, I have illustrated the complexity of minimality issues of n-D representations compared to their 1-D counterparts. This complexity is due partially to the presence of more than one operator in the representation of a given behaviour and may imply for instance that reducing the degree of one operator in the representation is more costly than reducing the degree needed for some other operator. The generality of the Dual Pencil and Pencil representations and the corresponding definitions of minimality provides a framework for investigating conditions that characterize minimal first-degree representations.

Future work which naturally follows from the results presented, some of which has been mentioned in the body of this thesis, includes the following:

- Youla and Gnavi [58] have shown that zero-, minor-, variety-, and factor-primeness do not imply equivalent characterizations of polynomial matrices for n-D systems with $n > 2$. In an upcoming paper, Fornasini and Valcher [20] have illustrated the differences between the 2-D and n-D case of these properties and have found conditions under which these properties of a given polynomial matrix can be equivalent. The theorem in chapter 5 used to classify the AR representations may be strengthened using this and related results.

- The links between Oberst's works 'On the minimal number of trajectories determining a multidimensional system' [43] and 'Canonical Cauchy's problem' for input-output systems [44], and the definition of minimal AR representation in chapter 5 should be investigated.

- It has already been shown that some of the known reduction algorithms for Roesser models may affect stability [46], [45]. Existing or new stability conditions should be used to ensure that reduction methods used preserve stability.

- Future work should focus on finding all necessary and sufficient conditions for minimality of first-degree ARMA representations. These conditions will be reformulated for the existing n-D models. Recasting methods should be modified to preserve minimality. This may also require using Gröbner basis algorithms similar to the one used by Fornasini *et al* [19] in state-space realization of autonomous AR systems. These conditions should also be extended to the known n-D models. It may also be possible to give definitions of controllability and observability for first-degree ARMA representations and establish a connection between these two concepts and minimality.

- Applications of results of this thesis to areas such as gain scheduling and signal processing should be thoroughly investigated.

Appendix A

In this appendix an equivalent Roesser model is obtained for an arbitrary Fornasini-Marchesini model in n-dimensions, and vice-versa. Both the regular and singular cases are covered.

Although for 2-D systems, this relationship has been extensively studied [14], [34], [38], the relationship between the general n-D Roesser and Fornasini-Marchesini models has not been made explicitly clear. Such a relationship is useful since often in the early stages of modeling of a given physical system a Fornasini-Marchesini model is obtained, whereas an equivalent first order Roesser model would simplify analysis. We are going to give a systematic way to translate between local-state models, via a simplified notation. The extension to n-D requires careful attention to notation and has not appeared in the literature. The development is explicit, and is illustrated by examples.

A.1 Equivalence of n-dimensional Roesser and Fornasini-Marchesini models

In the models considered, the notation $x(i_1, \cdots, i_n)$ will represent a variable dependent on n independent indices, written as a vector $\mathcal{I} = (i_1, \cdots, i_n)$. Then the general n-D Roesser model can be written as:

$$\overbrace{\begin{bmatrix} \hat{E}_{11} & \cdots & \hat{E}_{1n} \\ \vdots & & \vdots \\ \hat{E}_{n1} & \cdots & \hat{E}_{nn} \end{bmatrix}}^{\hat{E}} \overbrace{\begin{bmatrix} \hat{x}_1(\mathcal{I}+e_1) \\ \hat{x}_2(\mathcal{I}+e_2) \\ \vdots \\ \hat{x}_n(\mathcal{I}+e_n) \end{bmatrix}}^{\hat{x}'} = \overbrace{\begin{bmatrix} \hat{A}_{11} & \cdots & \hat{A}_{1n} \\ \vdots & & \vdots \\ \hat{A}_{n1} & \cdots & \hat{A}_{nn} \end{bmatrix}}^{\hat{A}} \overbrace{\begin{bmatrix} \hat{x}_1(\mathcal{I}) \\ \hat{x}_2(\mathcal{I}) \\ \vdots \\ \hat{x}_n(\mathcal{I}) \end{bmatrix}}^{\hat{x}} + \overbrace{\begin{bmatrix} \hat{B}_1 \\ \vdots \\ \hat{B}_n \end{bmatrix}}^{\hat{B}} u(\mathcal{I}) \tag{A.1}$$

$$y(\mathcal{I}) = \underbrace{\begin{bmatrix} \hat{C}_1, & \cdots, & \hat{C}_n \end{bmatrix}}_{\hat{C}} \begin{bmatrix} \hat{x}_1(\mathcal{I}) \\ \hat{x}_2(\mathcal{I}) \\ \cdots \\ \hat{x}_n(\mathcal{I}) \end{bmatrix} + \hat{D}u(\mathcal{I}). \tag{A.2}$$

where e_j is a vector which is zero except in the j-th entry, where it is 1.

Using a similar notation, with $\mathcal{V} = (1, 1, \cdots, 1)$, the general n-D Fornasini-Marchesini model can be written as:

$$\begin{aligned} Ex(\mathcal{I}+\mathcal{V}) = {} & A_0 x(\mathcal{I}) + \sum_{j=1}^{n} A_j x(\mathcal{I}+e_j) + \sum_{1 \le j < k \le n} A_{jk} x(\mathcal{I}+e_j+e_k) + \cdots \\ & + \sum_{j=1}^{n} A_{1,\cdots,j-1,j+1,\cdots,n} x(\mathcal{I}+\mathcal{V}-e_j) + B_0 u(\mathcal{I}) + \sum_{j=1}^{n} B_j u(\mathcal{I}+e_j) + \\ & + \sum_{1 \le j < k \le n} B_{jk} u(\mathcal{I}+e_j+e_k) + \cdots + \sum_{j=1}^{n} B_{1,\cdots,j-1,j+1,\cdots,n} u(\mathcal{I}+\mathcal{V}-e_j), \end{aligned} \tag{A.3}$$

$$y(\mathcal{I}) = Cx(\mathcal{I}) + Du(\mathcal{I}). \qquad \forall i_1, i_2, \cdots, i_n \geq 0 \tag{A.4}$$

The above equations are a direct n-D generalization of the model given by Kurek [35]. Generalizations of the two proposed Fornasini-Marchesini models [14], [17] can be obtained by setting appropriate A's and B's in (A.3) to zero. As a notational example, a 3-D Fornasini-Marchesini model has the form:

$$\begin{aligned} Ex(\mathcal{I}+\mathcal{V}) = {} & A_0x(\mathcal{I}) + A_1x(\mathcal{I}+e_1) + A_2x(\mathcal{I}+e_2) + A_3x(\mathcal{I}+e_3) + A_{12}x(\mathcal{I}+e_1+e_2)+ \\ & A_{13}x(\mathcal{I}+e_1+e_3) + A_{23}x(\mathcal{I}+e_2+e_3) + B_0u(\mathcal{I}) + B_1u(\mathcal{I}+e_1) + B_2u(\mathcal{I}+e_2) + \\ & B_3u(\mathcal{I}+e_3) + B_{12}u(\mathcal{I}+e_1+e_2) + B_{13}u(\mathcal{I}+e_1+e_3) + B_{23}u(\mathcal{I}+e_2+e_3), \qquad (A.5) \\ y(\mathcal{I}) = {} & Cx(\mathcal{I}) + Du(\mathcal{I}). \qquad (A.6) \end{aligned}$$

To recast the Roesser model (A.1), and (A.2) into the Fornasini-Marchesini form (A.3) and (A.4), consider one of the following:

Case 1: If the matrix $\hat{E}$ is non-singular, both sides of (A.1) can be pre-multiplied by $\hat{E}^{-1}$ to obtain a regular Roesser representation, and thus without loss of generality, assume that $\hat{E} = I$, the identity matrix. An equivalent Fornasini-Marchesini model can be obtained by the substitutions: $x(\mathcal{I}) = \hat{x}(\mathcal{I})$, $C = \hat{C}$, $D = \hat{D}$, $E = \hat{E} = I$, $A_{1,2,\cdots,i-1,i+1,\cdots,n} = [a_{hk}]_{m\times m}$, where

$$a_{hk} = \begin{cases} \hat{A}_{hk} & \text{for } h = i \\ 0 & \text{otherwise} \end{cases},$$

and $m = \sum_{j=1}^{n} n_j$, where n_j is the order of $x_j(x)$. Similarly, $B_{1,2,\cdots,i-1,i+1,\cdots,n} = [b_{hk}]_{m\times p}$, where

$$b_{hk} = \begin{cases} \hat{B}_h & \text{for } h = i \\ 0 & \text{otherwise} \end{cases},$$

where p is the order of the input vector, and all other A_i and B_j are equal to zero.

Case 2: If the matrix $\hat{E}$ is singular, an equivalent Fornasini-Marchesini model can be obtained by allowing:

$$x(\mathcal{I}) = \begin{bmatrix} \hat{x}(\mathcal{I}) \\ \hline \hat{x}_1(\mathcal{I}+e_1-e_2) \\ \hat{x}_1(\mathcal{I}+e_1-e_3) \\ \vdots \\ \hat{x}_1(\mathcal{I}+e_1-e_n) \\ \hline \vdots \\ \hline \hat{x}_n(\mathcal{I}+e_n-e_1) \\ \hat{x}_n(\mathcal{I}+e_n-e_2) \\ \vdots \\ \hat{x}_n(\mathcal{I}+e_n-e_{n-1}) \end{bmatrix}.$$

$D = \hat{D}$, $C_{l\times mn} = [\hat{C}, 0, \cdots, 0]$, where l is the order of the output vector. In addition, $E = [J_0, \cdots, J_n]$, where $J_0 = \mathrm{diag}[\hat{E}_{11}, \cdots, \hat{E}_{nn}]$, and $J_k = [w]_{m\times(n-1)n_j}$ where

$$w_{xy} = \begin{cases} \hat{E}_{xk} & \text{if } (y = x \text{ and } x < k) \text{ or } (y = x-1 \text{ and } x > k) \\ 0 & \text{otherwise} \end{cases}.$$

i.e.

$$J_0 = \begin{bmatrix} \hat{E}_{11} & 0 & 0 & 0 & 0 \\ 0 & \hat{E}_{22} & 0 & 0 & 0 \\ 0 & 0 & \hat{E}_{33} & 0 & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & 0 & \hat{E}_{nn} \end{bmatrix}, J_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \hat{E}_{21} & 0 & 0 & 0 \\ 0 & \hat{E}_{31} & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & \hat{E}_{n1} \end{bmatrix},$$

$$J_2 = \begin{bmatrix} \hat{E}_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \hat{E}_{32} & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & \hat{E}_{n2} \end{bmatrix}, \cdots, J_n = \begin{bmatrix} \hat{E}_{1n} & 0 & 0 & 0 & 0 \\ 0 & \hat{E}_{2n} & 0 & 0 & 0 \\ 0 & 0 & \hat{E}_{3n} & 0 & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & 0 & \hat{E}_{n-1,n} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, let $A_{1,2,\cdots,i-1,i+1,\cdots,n} = [a_{hk}]_{m\times mn}$, where

$$a_{hk} = \begin{cases} \hat{A}_{hk} & \text{for } h = i \text{ and } 1 \le k \le n \\ 0 & \text{otherwise} \end{cases},$$

and $B_{1,2,\cdots,i-1,i+1,\cdots,n} = [b_{hk}]_{m\times p}$, where

$$b_{hk} = \begin{cases} \hat{B}_h & \text{for } h = i \\ 0 & \text{otherwise} \end{cases},$$

and all other A_i and B_j are equal to zero.

Case 1 is the n-D generalization of the one given in [14]. The justification of both cases is based on the following: let z_i be an operator that has the effect of advancing the i-th subscript of the function upon which it is operating, i.e.

$$x(\mathcal{I} + e_i) = z_i x(\mathcal{I}). \tag{A.7}$$

Then assuming the necessary boundary conditions, the equation

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \hat{E}_{ij}\hat{x}_j(\mathcal{I}+e_j) + \hat{E}_{ii}\hat{x}_i(\mathcal{I}+e_i) = \sum_{k=1}^{n} \hat{x}_i\hat{A}_{ik}(\mathcal{I}) + \hat{B}_i u(\mathcal{I}), \quad i = 1,\cdots,n \tag{A.8}$$

can be rewritten as

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \hat{E}_{ij}\hat{x}_j(\mathcal{I}+e_j+\mathcal{V}-e_i) + \hat{E}_{ii}\hat{x}_i(\mathcal{I}+\mathcal{V}) = \sum_{k=1}^{n} \hat{A}_{ik}\hat{x}_k(\mathcal{I}+\mathcal{V}-e_i) + \hat{B}_i u(\mathcal{I}+\mathcal{V}-e_i) \tag{A.9}$$

The corresponding x, E, A, B,and D of an equivalent Fornasini-Marchesini model can be obtained by associating the vectors and matrices used in the Roesser model with the corresponding quantities in the Fornasini-Marchesini model.

To recast the Fornasini-Marchesini model (A.3) and (A.4) into the Roesser form (A.1) and (A.2), let z_i be defined as above, so for example, the 3-D Fornasini-Marchesini model given in (A.5) can be rewritten as:

$$
\begin{aligned}
&Ez_1z_2z_3x(\mathcal{I}) = \qquad\qquad (A.10)\\
&\quad A_0x(\mathcal{I})+A_1z_1x(\mathcal{I})+A_2z_2x(\mathcal{I})+A_3z_3x(\mathcal{I})+A_{12}z_1z_2x(\mathcal{I})+A_{13}z_1z_3x(\mathcal{I})+A_{23}z_2z_3x(\mathcal{I})+\\
&\quad B_0u(\mathcal{I}) + B_1z_1u(\mathcal{I})+B_2z_2u(\mathcal{I})+B_3z_3u(\mathcal{I})+B_{12}z_1z_2u(\mathcal{I})+B_{13}z_1z_3u(\mathcal{I})+B_{23}z_2z_3u(\mathcal{I}).
\end{aligned}
$$

Let μ =dim $x(\mathcal{I})$. Define the partial state x_{n-1} of dimension $2^{n-2}\mu$ to be:

$$
x_{n-1} = \begin{bmatrix} -M_{2^{n-2}} & N_{2^{n-2}} \\ \vdots & \vdots \\ -M_{2^{n-1}-2} & N_{2^{n-1}-2} \\ E & 0 \end{bmatrix} \begin{bmatrix} x(\mathcal{I}+e_n) \\ u(\mathcal{I}+e_n) \end{bmatrix} - \begin{bmatrix} K_{2^{n-2}} & H_{2^{n-2}} \\ \vdots & \vdots \\ K_{2^{n-1}-1} & N_{2^{n-1}-1} \end{bmatrix} \begin{bmatrix} x(\mathcal{I}) \\ u(\mathcal{I}) \end{bmatrix}. \qquad (A.11)
$$

where $K_j = A_{d_1,\cdots,d_k}$, where the d_i are the locations of ones if j is written in binary form. For example, $K_5 = A_{13}$ since $5 = 101$ which has ones in locations 1 and 3, or $K_{13} = A_{134}$ since $13 = 1101$ which has ones in locations 1, 3, and 4. Similarly $H_j = B_{d_1,\cdots,d_k}$, $M_j = A_{d_1,\cdots,d_k,n}$, and $N_j = B_{d_1,\cdots,d_k,n}$, for example $H_9 = B_{14}$, $M_7 = A_{1235}$, and $N_{14} = B_{2345}$.

Next, recursively define the partial states x_m of dimension $2^{m-1}\mu$ for $m = 2,\cdots,n-1$:

$$
x_{n-m} = \begin{bmatrix} -M_{2^{n-m-1}} & N_{2^{n-m-1}} \\ \vdots & \vdots \\ -M_{2^{n-m}-1} & N_{2^{n-2}-1} \end{bmatrix} \begin{bmatrix} x(\mathcal{I}+e_n) \\ u(\mathcal{I}+e_n) \end{bmatrix} - \begin{bmatrix} K_{2^{n-m-1}} & H_{2^{n-m-1}} \\ \vdots & \vdots \\ K_{2^{n-m}-1} & N_{2^{n-m}-1} \end{bmatrix} \begin{bmatrix} x(\mathcal{I}) \\ u(\mathcal{I}) \end{bmatrix} +
$$

$$
+I_{2^{n-m-1}}\begin{bmatrix} x_{n-1,2^{n-m-1}+1}(\mathcal{I}+e_{n-1}) \\ \vdots \\ x_{n-1,2^{n-m}}(\mathcal{I}+e_{n-1}) \end{bmatrix} + \cdots + I_{2^{n-m-1}}\begin{bmatrix} x_{n-m+1,2^{n-m-1}+1}(\mathcal{I}+e_{n-m+1}) \\ \vdots \\ x_{n-m+1,2^{n-m}}(\mathcal{I}+e_{n-m+1}) \end{bmatrix} \quad \text{(A.12)}
$$

The partial states defined above correspond to the partial states $\hat{x}_1(\mathcal{I})$ to $\hat{x}_{n-1}(\mathcal{I})$ of an equivalent Roesser model, and propagate only in the i-th direction for $i = 1, \cdots, n-1$. Then the equivalent Roesser model can be obtained by the substitutions:

$$
\hat{x}(\mathcal{I}) = \left[\begin{array}{c} x_1(\mathcal{I}) \\ \hline \vdots \\ \hline x_{n-1}(\mathcal{I}) \\ \hline x(\mathcal{I}) \\ u(\mathcal{I}) \end{array}\right] \quad \text{(A.13)}
$$

and

$$
\hat{E} = \left[\begin{array}{cc|c|c|c|c|cc}
I_\mu & & & & & & & \\
& I_{2\mu} & & & & & -A_n & B_n \\
0_{\mu\times\mu} & & I_{4\mu} & & & I_{2^{n-2}\mu} & -M_1 & N_1 \\
\cline{1-1} & 0_{2\mu\times 2\mu} & & \ddots & & & & \\
\cline{1-3} & & 0_{4\mu\times 4\mu} & & & & \vdots & \vdots \\
& & & \ddots & & & & \\
& & & & & 0_{2^{n-2}\mu\times 2^{n-2}\mu} & -M_{2^{n-1}-2} & N_{2^{n-1}-2} \\
& & & & & & E & 0
\end{array}\right], \quad \text{(A.14)}
$$

$$
\hat{A} = \left[\begin{array}{c|c|c|c|cc}
0 & & & & A_0 & 0 \\
I_\mu & & & & K_1 & H_1 \\
& I_{2\mu} & & & K_2 & H_2 \\
& & \ddots & & \vdots & \vdots \\
& & & I_{2^{n-2}\mu} & K_{2^{n-1}-1} & H_{2^{n-1}-1}
\end{array}\right] \quad \text{(A.15)}
$$

and

$$\hat{B} = [B_0{}^T, 0, \cdots, 0]^T \quad \hat{C} = [0, \cdots, C, 0], \quad \hat{D} = D$$

It should be noted that if the only non-zero B is B_0, the last column of $\hat{E}$ and $\hat{A}$ can be removed to reduce the order of the introduced state.

As an example, in the 2-D case, a Roesser equivalent of a second-order Fornasini-Marchesini model results in:

$$\begin{bmatrix} I & -A_2 \\ 0 & E \end{bmatrix} \begin{bmatrix} x_1(\mathcal{I}+e_1) \\ x(\mathcal{I}+e_2) \end{bmatrix} = \begin{bmatrix} 0 & A_0 \\ I & A_1 \end{bmatrix} \begin{bmatrix} x_1(\mathcal{I}) \\ x(\mathcal{I}) \end{bmatrix} + \begin{bmatrix} B_0 \\ 0 \end{bmatrix} u(\mathcal{I}), \quad \text{(A.16)}$$

$$y(\mathcal{I}) = [0 \quad C] \begin{bmatrix} x_1(\mathcal{I}) \\ x(\mathcal{I}) \end{bmatrix} + Du(\mathcal{I}), \quad \text{(A.17)}$$

which is the same as the one given in [38]. For a 4-th order general Fornasini-Marchesini model, the above method implies that:

$$\begin{bmatrix} I & I & 0 & I & 0 & 0 & 0 & -A_4 & B_4 \\ 0 & 0 & I & 0 & I & 0 & 0 & -A_{14} & B_{14} \\ 0 & 0 & 0 & 0 & 0 & I & 0 & -A_{24} & B_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 & I & -A_{124} & B_{124} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -A_{34} & B_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -A_{134} & B_{134} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -A_{234} & B_{234} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & E & 0 \end{bmatrix} \begin{bmatrix} x_1(\mathcal{I}+e_1) \\ x_{21}(\mathcal{I}+e_2) \\ x_{22}(\mathcal{I}+e_2) \\ x_{31}(\mathcal{I}+e_3) \\ x_{32}(\mathcal{I}+e_3) \\ x_{33}(\mathcal{I}+e_3) \\ x_{34}(\mathcal{I}+e_3) \\ x(\mathcal{I}+e_4) \\ u(\mathcal{I}+e_4) \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & A_1 & B_1 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & A_2 & B_2 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & A_{12} & B_{12} \\ 0 & 0 & 0 & I & 0 & 0 & 0 & A_3 & B_3 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & A_{13} & B_{13} \\ 0 & 0 & 0 & 0 & 0 & I & 0 & A_{23} & B_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & I & A_{123} & B_{123} \end{bmatrix} \overbrace{\begin{bmatrix} x_1(\mathcal{I}) \\ x_{21}(\mathcal{I}) \\ x_{22}(\mathcal{I}) \\ x_{31}(\mathcal{I}) \\ x_{32}(\mathcal{I}) \\ x_{33}(\mathcal{I}) \\ x_{34}(\mathcal{I}) \\ x(\mathcal{I}) \\ u(\mathcal{I}) \end{bmatrix}}^{x(\mathcal{I})} + \begin{bmatrix} B_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(\mathcal{I}) \quad \text{(A.18)}$$

$$y(\mathcal{I}) = [0 \quad 0 \quad 0 \quad C \quad 0]x(\mathcal{I}) + Du(\mathcal{I}). \qquad \text{(A.19)}$$

The justification of the method is based on two facts: one is that for any given field K with indeterminates $x_1, \cdots, x_n$, a polynomial in $K[x_1, \cdots, x_n]$ can be considered to be a polynomial in $K[x_1][x_2] \cdots [x_n]$. The other is that any equation in the form:

$$A_1 z_1 x + A_0 x + B_1 z_1 u + B_0 u = 0 \qquad \text{(A.20)}$$

can be rewritten as

$$\begin{bmatrix} -z_1 I & A_0 & B_0 \\ I & A_1 & B_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x \\ u \end{bmatrix} = 0. \qquad \text{(A.21)}$$

Hence, $n-1$ recursive applications of (A.21) to (A.3) will yield (A.13), (A.14), and (A.15). For example, (A.10) can be written as:

$$\begin{bmatrix} -z_1 I & A_0 + A_2 z_2 + A_3 z_3 + A_{23} z_{23} & B_0 + B_2 z_2 + B_3 z_3 + B_{23} z_{23} \\ I & A_1 + A_{12} z_2 + A_{13} z_3 - E z_2 z_3 & B_1 + B_{12} z_2 + B_{13} z_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x \\ u \end{bmatrix} = 0, \text{(A.22)}$$

$$\begin{bmatrix} -z_1 I & -z_2 I & 0 & A_0 + A_3 z_3 & B_0 + B_3 z_3 \\ I & 0 & -z_2 I & A_1 + A_{13} z_3 & B_1 + B_{13} z_3 \\ 0 & I & 0 & A_2 + A_{23} z_3 & B_2 + B_{23} z_3 \\ 0 & 0 & I & A_{12} - E z_3 & B_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x \\ u \end{bmatrix} = 0 \qquad \text{(A.23)}$$

which in the Roesser form, is:

$$\begin{bmatrix} I & I & 0 & -A_3 & B_3 \\ 0 & 0 & I & -A_{13} & B_{13} \\ 0 & 0 & 0 & -A_{23} & B_{23} \\ 0 & 0 & 0 & E & 0 \end{bmatrix} \begin{bmatrix} x_1(\mathcal{I}+e_1) \\ x_2(\mathcal{I}+e_2) \\ x_3(\mathcal{I}+e_2) \\ x(\mathcal{I}+e_3) \\ u(\mathcal{I}+e_3) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & A_0 & 0 \\ I & 0 & 0 & A_1 & B_1 \\ 0 & I & 0 & A_2 & B_2 \\ 0 & 0 & I & A_{12} & B_{12} \end{bmatrix} \begin{bmatrix} x_1(\mathcal{I}) \\ x_2(\mathcal{I}) \\ x_3(\mathcal{I}) \\ x(\mathcal{I}) \\ u(\mathcal{I}) \end{bmatrix} + \begin{bmatrix} B_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(\mathcal{I}). \text{(A.24)}$$

It is straightforward to prove by induction that M, N, K, and H are given as described above, and to see that at each stage the polynomial describing the relationship between the state vector and the input can be considered as two part, for example as $P(z_1, \cdots, z_{n-1}) + z_n Q(z_1, \cdots, z_{n-1})$. Now, by the induction hypothesis, (A.21) can be applied to both $P(z_1, \cdots, z_{n-1})$ and $Q(z_1, \cdots, z_{n-1})$ with the addition of an n to the indices of corresponding M, N, K, and H of $Q(z_1, \cdots, z_{n-1})$ due to the factor z_n. Writing the indices in binary form, and using the fact that in each step, say the i-th, a new state of dimension $2^{i-1}\mu$ is added, will complete the proof.

The algorithms above will be illustrated by the following further examples. Example 23 and 24 show respectively transformations of given regular and singular Roesser models into Fornasini-Marchesini models (i.e. cases 1 and 2). Example 25 shows how to obtain an equivalent Roesser model to a given (singular) Fornasini-Marchesini model.

Example 23 *Consider the 4-D regular Roesser model described by the following equations:*

$$\begin{bmatrix} \hat{x}_1^1(\mathcal{I}+e_1) \\ \hat{x}_2^1(\mathcal{I}+e_2) \\ \hat{x}_2^2(\mathcal{I}+e_2) \\ \hat{x}_3^1(\mathcal{I}+e_3) \\ \hat{x}_4^1(\mathcal{I}+e_4) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 & 4 \\ 0 & 2 & 0 & -3 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 1 & 0 \\ 1 & -1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{x}_1^1(\mathcal{I}) \\ \hat{x}_2^1(\mathcal{I}) \\ \hat{x}_2^2(\mathcal{I}) \\ \hat{x}_3^1(\mathcal{I}) \\ \hat{x}_4^1(\mathcal{I}) \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(\mathcal{I})$$

$$y(\mathcal{I}) = [1,0,1,0,-1] \begin{bmatrix} \hat{x}_1^1(\mathcal{I}) \\ \hat{x}_2^1(\mathcal{I}) \\ \hat{x}_2^2(\mathcal{I}) \\ \hat{x}_3^1(\mathcal{I}) \\ \hat{x}_4^1(\mathcal{I}) \end{bmatrix}. \tag{A.25}$$

A Fornasini-Marchesini representation of the model described by these equations can be rewritten by letting:

$$x(\mathcal{I}) = \begin{bmatrix} \hat{x}_1^1(\mathcal{I}) \\ \hat{x}_2^1(\mathcal{I}) \\ \hat{x}_2^2(\mathcal{I}) \\ \hat{x}_3^1(\mathcal{I}) \\ \hat{x}_4^1(\mathcal{I}) \end{bmatrix},$$

where the only non-zero A_i or B_j are

$$A_{123} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & -1 \end{bmatrix}, A_{124} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_{134} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -3 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{234} = \begin{bmatrix} -1 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B_{123} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_{124} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, B_{134} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, B_{234} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

or

$$\begin{aligned} x(\mathcal{I} + \mathcal{V}) = {} & A_{123}x(\mathcal{I}+e_1+e_2+e_3) + A_{124}x(\mathcal{I}+e_1+e_2+e_4) + A_{134}x(\mathcal{I}+e_1+e_3+e_4) + \\ & +A_{234}x(\mathcal{I}+e_2+e_3+e_4) + B_{123}u(\mathcal{I}+e_1+e_2+e_3) + B_{124}u(\mathcal{I}+e_1+e_2+e_4) + \\ & +B_{134}u(\mathcal{I}+e_1+e_3+e_4) + B_{234}u(\mathcal{I}+e_2+e_3+e_4). \end{aligned}$$

and

$$y(\mathcal{I}) = Cx(\mathcal{I}) \quad \textit{where} \quad C = [1, 0, 1, 0, -1].$$

Example 24 *Consider the 2-D singular Roesser model described by the following equations:*

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1^1(\mathcal{I}+e_1) \\ \hat{x}_2^1(\mathcal{I}+e_2) \\ \hat{x}_2^2(\mathcal{I}+e_2) \end{bmatrix} &= \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 1 \\ -2 & 0 & 5 \end{bmatrix} \begin{bmatrix} \hat{x}_1^1(\mathcal{I}) \\ \hat{x}_2^1(\mathcal{I}) \\ \hat{x}_2^2(\mathcal{I}) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(\mathcal{I}) \\ y(\mathcal{I}) &= [-1, 1, 0] \begin{bmatrix} \hat{x}_1^1(\mathcal{I}) \\ \hat{x}_2^1(\mathcal{I}) \\ \hat{x}_2^2(\mathcal{I}) \end{bmatrix} \end{aligned} \tag{A.26}$$

A Fornasini-Marchesini representation of the model described by these equations can be rewritten by letting:

$$x(\mathcal{I}) = \begin{bmatrix} \hat{x}_1^1(\mathcal{I}) \\ \hat{x}_2^1(\mathcal{I}) \\ \hat{x}_2^2(\mathcal{I}) \\ \hat{x}_1^1(\mathcal{I}+e_1-e_2) \\ \hat{x}_2^1(\mathcal{I}-e_1+e_2) \\ \hat{x}_2^2(\mathcal{I}-e_1+e_2) \end{bmatrix},$$

and

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 5 & 0 & 0 & 0 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

or

$$\begin{aligned} Ex(\mathcal{I}+\mathcal{V}) &= A_1x(\mathcal{I}+e_1) + A_2x(\mathcal{I}+e_2) + B_1u(\mathcal{I}+e_1) + B_2u(\mathcal{I}+e_2), \\ y(\mathcal{I}) &= Cx(\mathcal{I}) \;\; where \;\; C = [-1, 1, 0, 0, 0, 0]. \end{aligned} \tag{A.27}$$

Example 25 *Consider the 3-D Fornasini-Marchesini model described by the following equations:*

$$\overbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}^{E} \begin{bmatrix} x^1(\mathcal{I}+\mathcal{V}) \\ x^2(\mathcal{I}+\mathcal{V}) \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}}^{A_0} \begin{bmatrix} x^1(\mathcal{I}) \\ x^2(\mathcal{I}) \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}^{A_2} \begin{bmatrix} x^1(\mathcal{I}+e_2) \\ x^2(\mathcal{I}+e_2) \end{bmatrix} +$$
$$+ \overbrace{\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}}^{A_{12}} \begin{bmatrix} x^1(\mathcal{I}+e_1+e_2) \\ x^2(\mathcal{I}+e_1+e_2) \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & -2 \\ 6 & 1 \end{bmatrix}}^{A_{23}} \begin{bmatrix} x^1(\mathcal{I}+e_2+e_3) \\ x^2(\mathcal{I}+e_2+e_3) \end{bmatrix} + \overbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}^{B_0} u(\mathcal{I}) +$$

$$+\overbrace{\begin{bmatrix}-1\\1\end{bmatrix}}^{B_1}u(\mathcal{I}+e_1)+\overbrace{\begin{bmatrix}0\\2\end{bmatrix}}^{B_3}u(\mathcal{I}+e_3)+\overbrace{\begin{bmatrix}1\\1\end{bmatrix}}^{B_{13}}u(\mathcal{I}+e_1+e_3), \tag{A.28}$$

$$y(\mathcal{I}) = [-2 \quad 3]\begin{bmatrix}x^1(\mathcal{I})\\x^2(\mathcal{I})\end{bmatrix}+2u(\mathcal{I}). \tag{A.29}$$

Let

$$\hat{x}(\mathcal{I}) = \begin{bmatrix}x_{11}(\mathcal{I})\\x_{12}(\mathcal{I})\\x_{21}(\mathcal{I})\\x_{22}(\mathcal{I})\\x^1(\mathcal{I})\\x^2(\mathcal{I})\\u(\mathcal{I})\end{bmatrix}.$$

Then using the method described earlier, as in (A.24), an equivalent Roesser form for (A.28), and (A.29) results in:

$$\begin{bmatrix}1&0&1&0&0&0&0&0&0\\0&1&0&1&0&0&0&0&2\\0&0&0&0&1&0&0&0&1\\0&0&0&0&0&1&0&0&1\\0&0&0&0&0&0&0&2&0\\0&0&0&0&0&0&-6&1&0\\0&0&0&0&0&0&1&1&0\\0&0&0&0&0&0&0&0&0\end{bmatrix}\hat{x}'(\mathcal{I}) = \begin{bmatrix}0&0&0&0&0&0&1&1&0\\0&0&0&0&0&0&-1&0&0\\1&0&0&0&0&0&0&0&-1\\0&1&0&0&0&0&0&0&1\\0&0&1&0&0&0&0&1&0\\0&0&0&1&0&0&0&1&0\\0&0&0&0&1&0&2&-3&0\\0&0&0&0&0&1&0&1&0\end{bmatrix}\hat{x}(\mathcal{I})+\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}u(\mathcal{I}) \tag{A.30}$$

$$y(\mathcal{I}) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -2 \quad 3 \quad 0]\,\hat{x}(\mathcal{I})+2u(\mathcal{I}). \tag{A.31}$$

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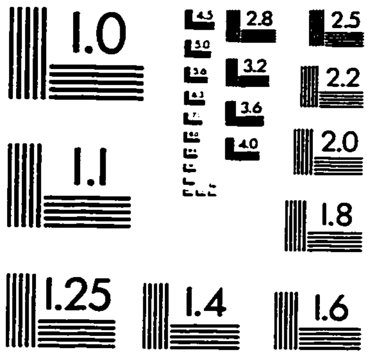
IMAGE EVALUATION TEST TARGET (QA-3)



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