Material Modelling for Structural Analysis of Polyethylene

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The purpose of this work was to develop a practical method for constitutive modelling of polyethylene, based on a phenomenological approach, which can be applied for structural analysis. Polyethylene (PE) is increasingly used as a structural material, for example in pipes installed by trenchless methods where relatively low stiffness of PE reduces the required installation forces, chemical inertness makes it applicable for corrosive environments, and adequate strength allows to use it for sewer, gas and water lines. Polyethylene exhibits time-dependent constitutive behaviour, which is also dependent on the applied stress level resulting in nonlinear stress-strain relationships. Nonlinear viscoelastic theory has been well established and a variety of modelling approaches have been derived from it. In order to be able to realistically utilize the nonlinear modelling approaches in design, a simple method is needed for finding the constitutive formulation for a specific polyethylene type.

In this study, time-dependent constitutive relationships for polymers are investigated for polyethylene materials. Creep tests on seven polyethylene materials were conducted and the experimental results indicate strong nonlinear viscoelasticity in the material responses. Creep tests on seven materials were conducted for 24 hours for modelling purposes. However, creep tests up to fourteen days were performed on one material to study long-term creep behaviour. Multiple-stepped creep tests were also investigated. Constant rate (load and strain rate) tensile tests were conducted on two of the seven polyethylene materials.

A practical approach to nonlinear viscoelastic modelling utilizing both multi-Kelvin element theory and power law functions to model creep compliance is presented. Creep tests are used to determine material parameters and models are generated for four different polyethylene materials. The corroboration of the models is achieved by comparisons with the results of different tensile creep tests, with one dimensional step loading test results and with test results from load and displacement rate loading.

Acknowledgements

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This work benefits very much from the discussions of "creep" group, an inter-disciplinary research team on polymer engineering at University of Waterloo. The team is supervised by Dr. Alexander Penlidis and Dr. Maria Anna Polak and the current students are Joy Cheng, Jose Alvarado, and the author.

The material tests for this research were conducted at the laboratories in Civil Engineering and Mechanical Engineering at University of Waterloo. Some of the specimens were prepared in the Department of Chemical Engineering. The author would like to take the opportunity to thank the lab technicians who have given assistance and advice. Special thanks to Richard Morrison and Doug Hirst from Civil Engineering Laboratory, who have assisted on the tests closely.

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Thanks to the author's family and friends for their constant support and help, among them, his wife Jin Zhao Liu, his sister Peilin Liu, and friends, Ying An, Hui Liu, Xianxun Yuan, Yuxin Liu and many others.

The author wants to express his appreciation especially to his parents, Changtie Liu and Suxin Yao, whose love always accompanies him and makes everything possible.

Table of Contents

List of Tables

List of Illustrations

Chapter 1

Introduction

Polyethylenes are increasingly used in many aspects of our lives, including infrastructure and building construction. The rationale for the research presented herein is the need for structural analysis and design tools for polymeric structures and structural components. Such analytical tools should be capable of capturing essential features in the material behaviour, being at the same time relatively simple, versatile and computationally efficient.

1.1 Constitutive Modelling of Polymers

A constitutive model defines the structural material property in terms of the relationships between stresses and strains and plays a critical role in the structural analysis procedure. Once a constitutive model is defined, the equilibrium equations can be set up with appropriate loading and structural boundary conditions (constraints). Such equilibrium equations in general are differential equations with structural displacements or internal forces as variables and can be solved by recognized techniques (e.g. Finite Element method). The structural response in terms of reaction (stress) and deformation (strain) can be solved in a post process of the solved equation variables and compared with certain structural failure criteria. A constitutive relation can be linear or nonlinear, elastic or plastic, time dependent or not. It mathematically describes the mechanical response of a real material.

One of the common construction applications of polyethylene are pipes for gas, water and sewer systems. Specifically, immediate need for research presented herein comes from related work on modelling polyethylene pipes installed or rehabilitated using trenchless methods (Cheng and Polak 2007). Polyethylene (PE) pipes have been used for their combination of strength, flexibility, corrosion resistance and lightweight. In modern

pipeline installation practices, pipes are often pulled underground through horizontal boreholes (horizontal drilling) which, in order to minimize pulling loads, require rather low stiffness of a pipe. Such installations impose tensile and bending deformations on the pipe and often result in certain amount of scratching of the pipe's outer surface. Subsequent utilization subjects the pipe to internal and external hoop stresses, often variable over time. All these loads result in complex 3-D stress states within pipe wall. Thus, in short term, the pipe can be subject to longitudinal stresses due to pulling load (which if too high would render the pipe useless, however in proper installations these stresses remain low), and then, in long term, the pipe is subjected to rather low but variable hoop stresses which combined with scratching can results in stress cracking and premature failure. The stress and strains levels in a pipe are complex 3-dimensional but usually low; however, the pipe is subjected to complex load history making hereditary effects very important in effective modelling. Also, adequate long term strength and performance of the PE pipes is probably the major concern in the design and construction of such pipeline systems, however, no adequate predictive models exist that can link stress cracking (environmental stress crack resistance, ESCR) determined using standard laboratory tests with the real conditions to which pipes are subjected.

Creep behaviour of polymers is related to their long term strength and to ESCR. It also best represents hereditary nature of the behaviour of a polymeric structure (e.g. pipe) which is subject to complex load history. Based on creep test results, hereditary type constitutive models can be developed. Constitutive models are part of structural analysis formulations; structural analysis of pipeline systems allows determining strains and stresses under loads and thus is essential in the design process. Many constitutive models for PE have been proposed. Modelling often addresses either viscoelastic (defined rather arbitrarily as low strain, before yield behaviour) or post yield, before necking or deformations at very large strains. The last two problems are not really applicable for structural design; structures are never allowed to carry loads which cause excessive deformations. In fact, from the structural analysis point of view yield or stress cracking is considered a failure. Most of the approaches published in literature utilize rather complex mathematical expressions with several material parameters, and although many

capture the behaviour well they are difficult to apply in practical cases. Additional issue arises due to the variety of available polyethylenes, each with a specific material behaviour which should be described by material specific parameters. Also, the type and time of loading used for material parameters calibration will influence the performance of the model. In other words, a material model developed and calibrated for one polyethylene will work well only for this particular material. Therefore, design engineers have a considerable difficulty finding an easy and yet rational way for performing structural analysis which captures both time and nonlinear effects of polymeric structures (see also comments by Drozdov 1997a).

Polyethylene is a semicrystalline polymer with large hydrocarbon chain molecules which are capable of adopting two distinct arrangements; the crystalline and the amorphous phases. The material behaviour of semicrystalline polymers is often categorized as viscoelastic or viscoplastic which suggests a combination of viscous flow typical for fluids, with either elastic or plastic characteristics typical for solids. In reality, these idealized types of behaviour exist simultaneously in the behaviour of polyethylene in any relative proportions. The proportion of the viscous, elastic and plastic characteristics depends on the rate of loading, time, loading history, stress level and temperature. It also depends strongly on the molecular structure including crystallinity, molecular weight, molecular weight distributions, and short or long chain branching. This is why different polyethylenes respond differently to loads and should be modelled using different material parameters specific for a given material.

Modelling of the polyethylene, a semicrystalline polymer, can be categorized as either micromechanical or macromechanical. Micromechanical approaches start with the analysis of microstructure of the material on the molecular level; the crystalline and amorphous phases are modelled considering their specific characteristics. The macroscopic behaviour can be simulated from the physically based micromechanical modelling. These models (e.g. Nikolov *et al.*, 2000, Alvarado *et al.*, 2006) are very important for studies on the behaviour of polyethylenes under simple tension, compression or shear loads since such simple behaviour defines the behaviour of the

material under complex loads which are present in structures made from these materials. Micromechanical studies can also serve as an excellent research tool for linking chemical composition of the polymer with its mechanical response to applied loads (Li et al. 2002). However, for practical analysis of real structures, macromechanical, phenomenological models are needed, which consist of mathematical equations relating strains to stresses at the macro, structural level.

The macromechanical modelling generally uses the experimental behaviour under the simple loads to define material factors to be used in mathematical equations describing the relations between stresses and strains. Within the macro-modelling approaches, a vast amount of papers have been written on the behaviour before yield which in case of polymers is time dependent. These papers addressed this class of material models. Several researchers derived their relationships in the form of viscoelastic or viscoplastic equations, dependent on time, loading history, loading rate, calibrated based on uniaxial tensile or compressive tests. The very well known, and often followed, nonlinear time-dependent formulation was proposed by Schapery (1969). Krishnashwamy *et al*. (1992) and Zhang and Moore (1997) developed phenomenological integral time-dependent models applicable for finite element structural analysis. They addressed the nonlinearity of PE behaviour by formulating material coefficients as functions of stress. Lu *et al.* (2000) proposed phenomenological creep modelling for ABS used in pipes, using different mathematical formulae depending on stress and temperature levels. An adaptive link theory was proposed for modelling polymers by Drozdov (1997, 1999) and Drozdov and Kalamkarov (1996) and the integral form viscoelastic formulation is obtained from the model. The model proposed by Popelar *et al*. (1990) is based on relaxation tests. It predicts stress-strain response in uniaxial constant strain rate test well at small strain rate ($<10^{-3}$ s^{-1}). Beijer and Spoormaker (2000) investigated the performance of integral model formulations under small strain (less than 5%) and under strains closer to yield and including yield. They pointed out the difference in the character of time-dependent behaviour of HDPE with increasing stress approaching yield. Lai and Bakker (1995) investigated creep of polyethylene and their formulation includes aging through the time shift function. A theoretical method for predicting hoop stresses in polyethylene pipes,

using a visco-hyperelastic constitutive model, was presented by Guan and Boot (2001). A model proposed by Duan *et al*. (2001) follows the phenomenological unified approach to predict the stress-strain relationship for semicrystalline and glassy polymers for large spectrum of deformations - before yield, during viscoplastic phase after yield, and strain hardening. It was fit on test data on polymethyl-methacrylate (PMMA).

Although substantial work on nonlinear time-dependent modelling has been done, an engineer facing a problem of modelling behaviour of a structure made form a given creeping material will have a considerable difficulty using the nonlinear models and therefore linear viscoelastic, or even simple linear elastic analysis would normally be adopted. Such analysis procedures are readily available in many commercial finite element packages. The objective of the presented work was to develop a practical method for constitutive modelling for polyethylene that would include time and nonlinear effects with the accuracy acceptable for analysis of structures and structural components. Thus the method should be easily calibrated for a given material based on tests and then developed into a format that can be applied for structural analysis (e.g. finite element analysis).

The thesis presents the proposed formulation and the relevant material testing conducted on samples from four different polyethylenes. Applicability of the formulation for modelling variety of loading scenarios is shown. The theoretical background used in constitutive modelling is discussed first. Classical formulations based on integral equations are adopted for linear viscoelastic modelling based on either multiple Kelvin elements or viscoplastic modelling based on power law functions. Creep testing is used for experimental determination of material parameters which are calculated using a numerical procedure. The method for including nonlinearity and the procedure for modelling load rate effects are discussed. The constitutive models are presented for four different polyethylenes tested as part of this research. The models are verified by comparison with creep responses at different stress levels, with step loading test results and load rate testing. The numerical procedure for including load rate effects is described.

1.2 Outline of the Thesis

In this study, our research focuses on the time dependent mechanical properties of polyethylene. Phenomenological models capturing the viscoelastic and viscoplastic material nature of PE have been investigated. A group of seven polyethylene materials have been tested. Procedures of generating viscoelastic and viscoplastic constitutive models that have standardized forms have been developed. Time (rate) behaviours of the materials were simulated by the models and compared with laboratory tests. Satisfactory agreement between numerical simulations and laboratory tests has been documented.

The models are generated for PE pipe material; however, the modelling procedure and finite element routine are not limited to PE and can be used for other polymers with similar macro-mechanical properties.

This thesis is organized as follows:

- Chapter 1: A brief introduction of the research background.

- Chapter 2: Laboratory tests for seven polyethylene materials, including tensile creep tests, tensile rate tests, and their combinations.

- Chapter 3: Nonlinear viscoelastic/viscoplastic modelling of creep behaviour of polymers.

- Chapter 4: Modelling rate behaviour of polymers.

- Chapter 5: Summary and further discussion.

Chapter 2

Material Tests on Polyethylenes

Polyethylene, like other polymers, exhibits time-dependent material properties. Creep, time relaxation, and rate effects are the typical loading responses for such materials. One dimensional creep, time-relaxation, and rate(strain/stress) tests can be used for defining time-dependent material properties for polymers.

In a creep test, the load is applied to the material so that the stress is kept constant and the strain grows with time; the strain history is recorded as a material response. In a time relaxation test, the material specimen deformation is maintained constant and the stress in the specimen decreases with time; the stress history is recorded. In a constant strain rate test, the specimen is loaded in such a way that the strain rate of the specimen remains constant and the corresponding stress is recorded; the stress-strain relationship can be obtained for different testing rates. In a constant stress rate test, the material is loaded with a stress of constant growth and the strain history is recorded; the stress-strain relationship can thus be obtained for the testing conditions.

Due to the viscoelastic/viscoplastic properties of polymers, the stress/strain relationships vary for the same material depending on the testing conditions. Unlike the case of a linear elastic material, the mechanical properties of a viscoelastic material can not be determined from testing at one specific loading condition. A group of tests under systematically changing conditions (e.g. creep testing with a group of stress levels, relaxation testing with a group of strain levels, or constant rate testing with a group of rate values) have to be performed for each material in order to capture the mechanical properties of each material.

Tensile creep tests at different stress levels are conducted on polyethylenes in this study. Tensile creep testing is commonly used because it is easy to control and the strain-time data obtained are easy to reconcile with classical viscoelastic model formulations.

2.1 Material Samples and Specimen Preparation

Samples of seven polyethylene materials were tested. Six of them are high density polythylenes (HDPE) and one is medium density polyethylene (MDPE). A list is shown in Table 2.1.

Table 2.1 List of Tested Polyethylenes.

No.	Material	Type	Code
1	HDPE	pipe	HDPE-pipe
2	HDPE	resin	HDPE-resin1
3	HDPE	resin	HDPE-resin2
4	HDPE	resin	HDPE-resin3
5	HDPE	resin	HDPE-resin4
6	PE80	resin	PE80-resin
7	MDPE	pipe	MDPE-pipe

HDPE-pipe and MDPE-pipe are two pipe materials. Test specimens were cut directly from pipe samples. For HDPE-pipe, the specimens are cut off from the pipe sample with length along the pipe axis and width along the radius of the pipe. For MDPE-pipe, the specimen axis is along the pipe axis and width along the circumference of the pipe wall. The specimens are machined down to the required dimensions, Figure 2.1.

HDPE-resins are resin samples. The test specimens were made from these resins. The resins were heated up to 160°C (320°F) in the mould and then cooled down to room temperature on the hot plates with the heater off. The resins were first hot pressed into about 7-inch square plates with the designed specimen thickness and then machined down to the required dimensions, Figure 2.1.

The tensile specimens have dimensions that conform to ASTM D638-02a (ASTM 2002), shown in Figure 2.1.

The specimen's thickness and width are measured at three locations along the length of the sample to account for the dimension variation along the specimen length. The averages of the measurements are taken in calculating the initial cross-sectional area for the specimen.

2.2 Test Set-up

A tensile creep test is performed by applying a load through a lever using dead weights. The test set-up is shown in Figure 2.2. The test equipment is a four-pole frame with a tensile rod-clipper device going vertically along the centre axis. A lever on the top converts gravitational loading to tensile force pulling upwards. The test specimen is clipped vertically along the tensile rods.

A clip-on strain gauge is used to measure displacement in the sample between the gauge knife edges, which are 25.4 mm apart initially (gauge length). The measured displacement is recorded by a data acquisition system connected to the strain gauge (data are saved on the computer hard drive).

The static load is provided by applying dead weights on the loading arm of the lever. The ratio of the lever arms is 10:1; therefore, the tensile force applied on to the specimen is 10 times the total weights applied.

Small deformation is assumed for the tensile creep tests. Engineering stress and strain are used in calculating the material responses. The stress is calculated as the ratio of applied tension to the initial cross-sectional area at all testing stages so that a constant tensile stress is assumed for constant loads. The strain at each measured time is calculated as the ratio of the recorded displacement to the initial gauge length. The displacements are measured and recorded at 30 second intervals.

2.3 Tensile Creep Tests (24hr)

Creep tests were conducted on the seven polyethylene materials for 24 hours at different stress levels, listed in Table 2.1. The results at 4 or 5 stress levels were used to generate the constitutive models and the other results were used to evaluate the performance of the models.

Tables 2.2 summarizes the creep tests on the seven materials. Figures 2.3 (a) to (g) show the strain-time curves for the tested materials; and Figures 2.4 (a) to (g) show the corresponding compliance-time curves. For convenience of comparison, the maximum strain in the figures is limited to 10%. Creep tests on HDPE-pipe at high stresses are shown in Figures 2.5 and 2.6.

Table 2.2 24-hour Creep Test Summary.

	Stress [MPa]						
No.	**HDPE-pipe**	**HDPE-Resin1**	**HDPE-Resin2**	**HDPE-Resin3**	**HDPE-Resin4**	**PE-80**	**MDPE-pipe**
1	2.97	2.67	2.68	2.53	2.59	3.05	3.12
2	5.47	4.55	4.73	5.14	6.73	6.14	5.08
3	5.97	5.15	5.58	6.97	9.71	8.37	5.10
4	6.67	6.23	5.76	10.51	13.01	10.18	6.23
5	7.71	7.14	6.53	12.84		12.46	6.71
6	10.31	7.58	7.28	16.77			7.67
7	11.55	10.58	8.23	17.38			8.40
8	12.19		10.61	19.75			9.32
9	12.50		11.67				
10	13.52		13.72				
11	14.62						

Compliance is defined as the ratio of strain and stress at specific time. A comparison of compliance-time curves at different stresses indicates stress influence on the creep behaviour. It can be seen in Figure 2.4 that the compliance-time curves at different testing stresses diverge from one another for all tested materials. Creep behaviour cannot be

described by a single compliance curve, which means that the materials exhibit nonlinear viscous behaviour; creep compliance for the tested materials depends on both time and stress.

At high stress levels, creep strain grows very fast from the start of the testing and large deformation occurs within the first two hours (Figure 2.5). Excessive deformation far beyond practically acceptable levels was observed. There is a hardening plateau observed near the end of the tests. Testing with larger deformation is restrained in this study by instrumental limitations (lever arm travel range and strain gauge capacity). Engineering strain is used in Figure 2.5 assuming small deformations. Note that at such large deformation level engineering stress and strain are not valid any more. Figure 2.6 shows the corresponding compliance-time curves.

2.4 Long-Term Creep Tests

For practical applications, e.g. pipelines, the plastic materials are subjected to loading for longer time. Creep behaviour for time longer than 24 hours is therefore needed. Results from creep tests on HDPE-pipe for long test durations are shown in this section.

The creep tests on HDPE-pipe are listed in Table 2.3. The corresponding strain-time curves are drawn in Figure 2.7. The time grid in each figure is in terms of days. The compliance-time curves for tests in Table 2.3 are shown in Figure 2.8.

Table 2.3 Long-Term Creep Test Summary for HDPE-pipe.

No.	Stress [MPa]	Duration [day]	Figure
1	6.89	7	2.7 (a)
2	4.42	14	2.7 (b)
3	6.08	14	2.7 (c)
4	8.15	14	2.7 (d)
5	10.62	14	2.7 (e)

It can be seen that, comparing 14-day results, Figure 2.7, with the 24-hour test results, Figure 2.3 (a), the creep strain keeps growing with time after tested for 24 hours. At low

stress (e.g. 4.42 MPa), strain growth is not significant after the first 24 hours; the major part of the deformation occurs within the first two days. At higher stress (e.g. 8.15 MPa), large strain growth can still be observed at the end of 14- days.

2.5 Short-Term Creep Behaviour

To study creep behaviour of polyethylenes within different time windows, strain-time curves for 24-hour creep test results are reproduced for the first 3 hours of the tests and shown in Figure 2.9 (a) to (g).

2.6 Creep Test with Step Loading

Creep tests have also been performed for stepped loading. The specimen first creeps for a period of time under a constant stress and then creeps for a period of time at another (increased or decreased) constant stress level. For increased stress test, progressive strain growth can be observed; for decreased stress test, strain reduction under sustained loading can be observed.

A summary of the tests are listed in Table 2.4.

Table 2.4 Stepped Loading Creep Test Summary for HDPE-pipe.

No.	Stress [MPa]	Figure	Duration [hr]
1	5.25 - 8.31	2.10	24
2	5.33 - 10.55	2.11	6
3	10.07 - 13.20	2.12	24
4	10.59 - 5.35	2.13	24
5	8.36 - 5.29	2.14	24
6	13.31 - 10.16	2.15	24
7	2.90 - 8.68	2.16	6
8	2.91 - 8.72	2.17	6
9	2.90 - 8.70	2.18	6
10	2.89 - 5.78 - 8.66 - 5.78	2.19	12
11	1.46 - 2.92 - 4.38 - 5.84 - 7.30	2.20	12

2.7 Tensile Rate Tests

Tensile rate tests were also conducted on two polyethylene materials on MTS tester: HDPE-pipe and HDPE-resin2. The specimens have the same dimensions as in creep tests. Two groups of tests were done on each material: constant load rate and constant strain rate. For constant load rate tests, the tension loads are monitored to be constant. The engineering stress in the specimens can be approximated to be constant for each test when the deformation is small. For constant strain rate tests, the loads are monitored so that the strain rate reading on the clip-on strain gauge is constant for each test. Engineering stress can be calculated from the loading record. The specimens are tensioned to failure: either necking occurs or the specimen breaks.

Table 2.5 (a) and (b) list the load rate tests and strain rate tests, respectively, on HDPE-pipe. Table 2.6 (a) and (b) list the tests on HDPE-resin2. The stress values in the tables were estimated based on nominal cross-sectional dimensions. The actual engineering stress is evaluated based on the actual measurements of each specimen tested.

Table 2.5 (a) Load Rate Test on HDPE-pipe.

No.	Load Rate (N/s)	Estimated Stress Rate (MPa/s)
1	65.0	1.0
2	6.5	0.1
3	3.25	0.05
4	0.65	0.01

Table 2.5 (b) Strain Rate Test on HDPE-pipe.

No.	Strain Rate (/s)
1	0.05
2	0.01
3	0.005

For HDPE-pipe, the load-time curves for constant load rate tests are drawn in Figure 2.21 (a) and the corresponding stress-strain curves are drawn in Figure 2.21 (b). The failure stress, corresponding to the maximum load (~1600N), is about 24.6 MPa.

Table 2.6 (a) Load Rate Test on HDPE-Resin2.

No.	Load Rate (N/s)	Estimated Stress Rate (MPa/s)
1	78.0	1.0
2	7.8	0.1
3	3.9	0.05

Table 2.6 (b) Strain Rate Test on HDPE-Resin2.

No.	Strain Rate (/s)
1	0.01
2	0.001
3	0.0005

For HDPE-Resin2, the load-time curves for constant load rate tests are shown in Figure 2.23 (a) and the corresponding stress-strain curves are shown in Figure 2.23 (b). The failure stress, corresponding to the maximum load (~300N), is about 4 MPa.

For the constant strain rate tests, strain measurements were recorded by the clip-on strain gauge; the tester adjusts the tension according to the strain reading to monitor strain control tests. It can be seen from the strain-time curves in Figure 2.22 (a) for HDPE-pipe (and Figure 2.24 (a) for HDPE-Resin2) that the constant strain rates are not monitored as well as load rates in the case of constant load rate tests. This is understood as the strain is a feedback from the loading and there is a lag between the time strain being read by the system and the response on loading being made accordingly. The stress-strain curves for constant strain rate tests are shown in Figure 2.22 (b) for HDPE-pipe and Figure 2.24(b) for HDPE-Resin2, respectively.

2.8 Complex Tests

Two additional complex tests combining rate control were conducted on HDPE-pipe samples using MTS tester. In the first test, the specimen is loaded to a stress level at a constant load rate, and the load is kept constant allowing the material to creep. Then, the specimen is loaded to a higher stress at a different constant load rate, and allowed to creep for a certain time. In the second test, the specimen is loaded at constant load rate, creeps, unloaded at the same load rate to a lower stress, and finally creeps for a period of

time. The stress-time curve for the first test is drawn in Figure 2.25 (a) and the corresponding strain-time curve in Figure 2.25 (b). Figure 2.26 (a) and (b) show the results of the second test. The experimental results of the complex tests are used later in confirming the performance of the models generated for HDPE-pipe.

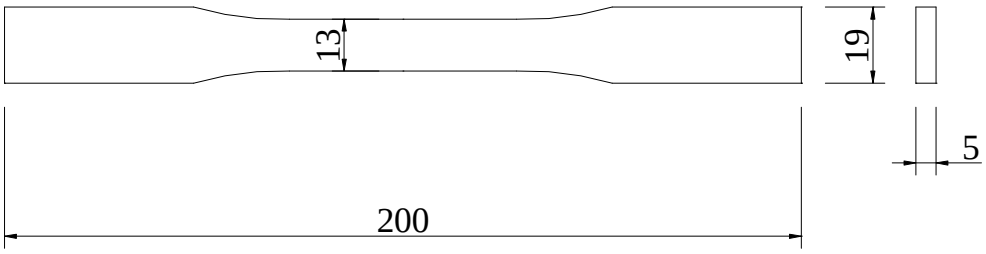


Figure 2.1 Plastic Tensile Test Specimen (units in mm).

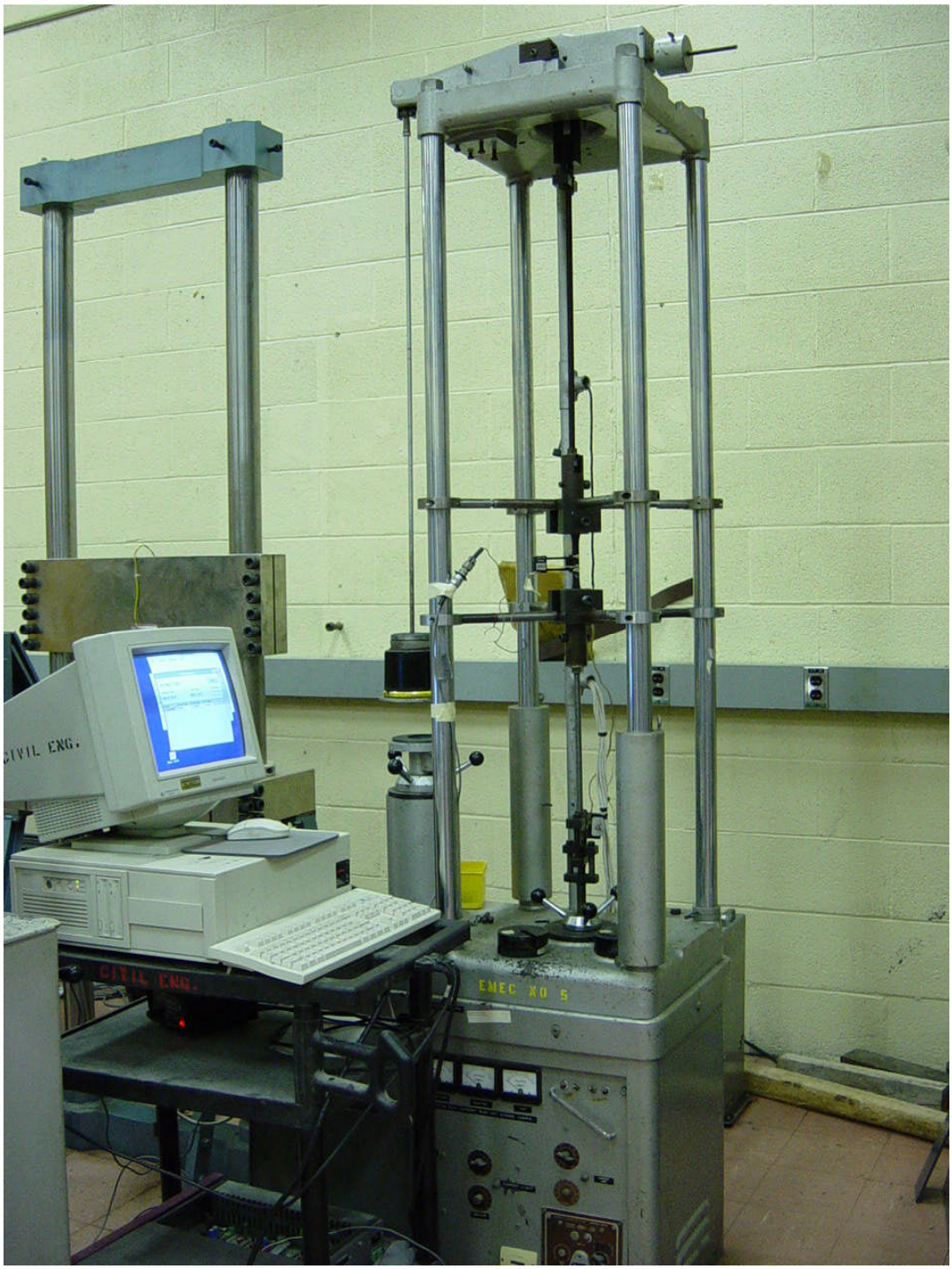


Figure 2.2 Creep test set-up.

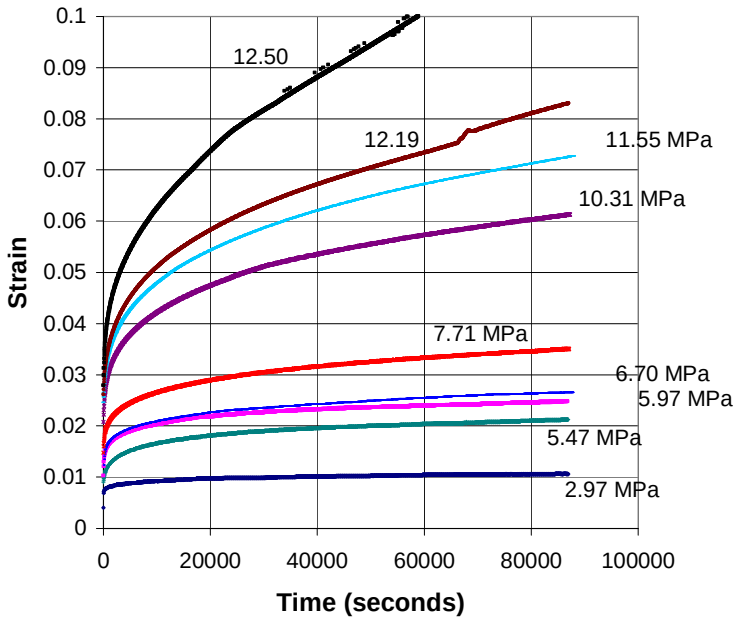


Figure 2.3 (a) Strain-time curves for 24-hour creep tests on HDPE-pipe.

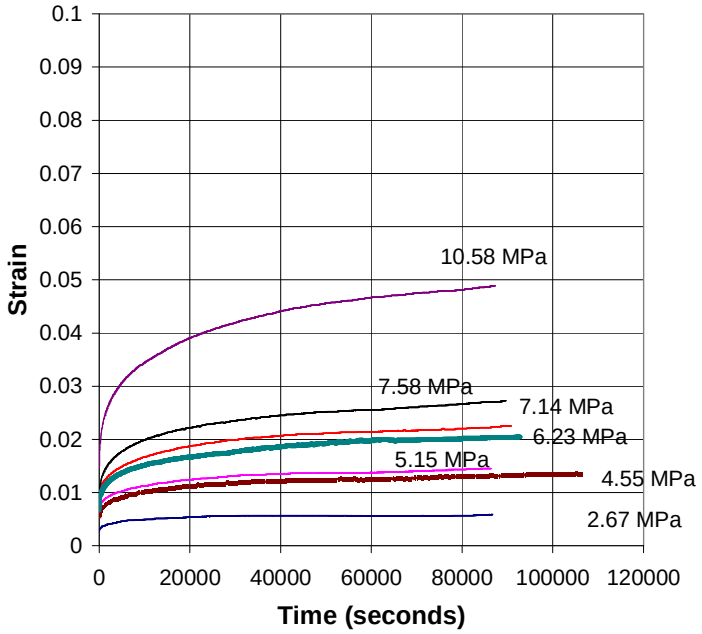


Figure 2.3 (b) Strain-time curves for 24-hour creep tests on HDPE-resin1.

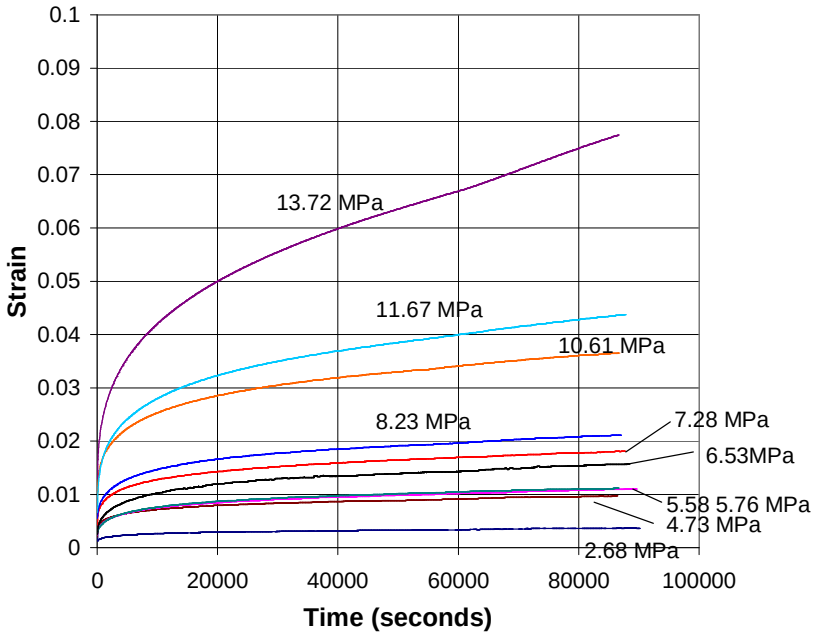


Figure 2.3 (c) Strain-time curves for 24-hour creep tests on HDPE-resin2.

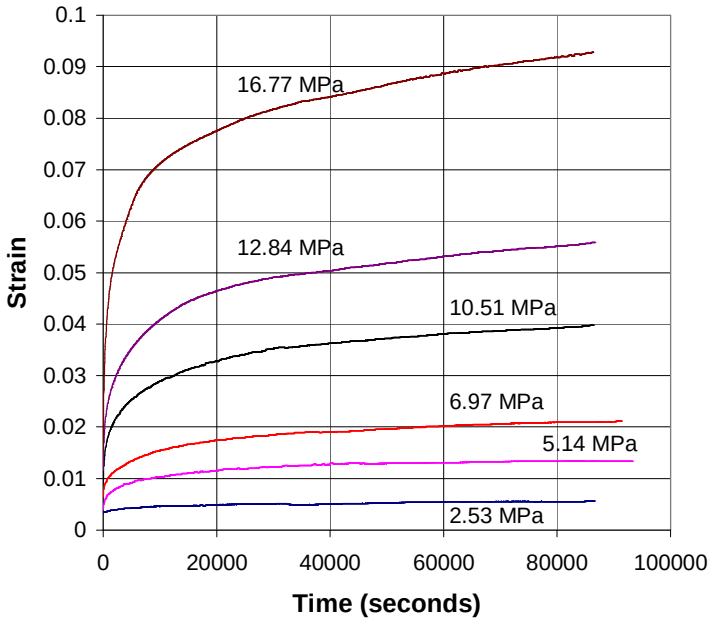


Figure 2.3 (d) Strain-time curves for 24-hour creep tests on HDPE-resin3.

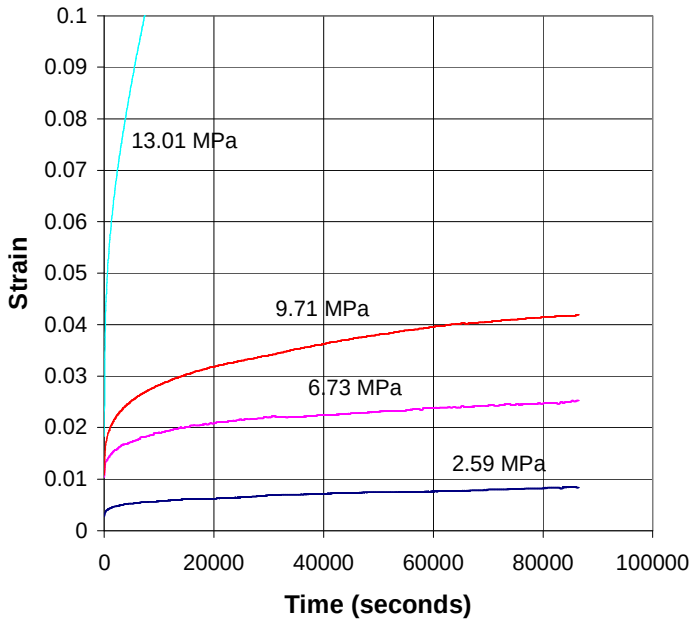


Figure 2.3 (e) Strain-time curves for 24-hour creep tests on HDPE-resin4.

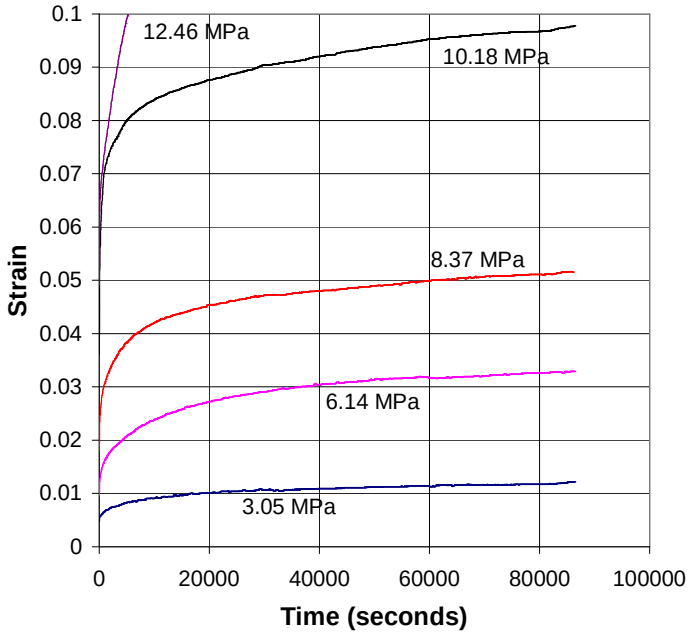


Figure 2.3 (f) Strain-time curves for 24-hour creep tests on PE80-resin.

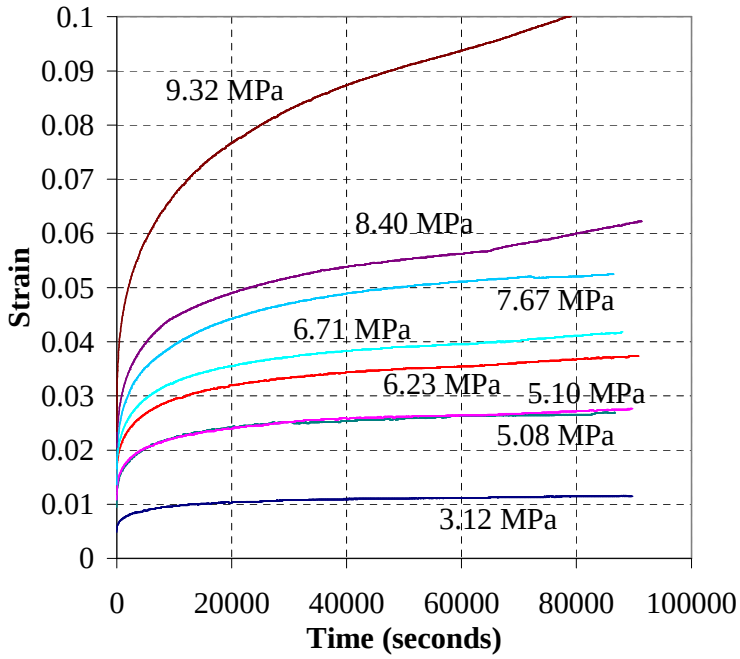


Figure 2.3 (g) Strain-time curves for 24-hour creep tests on MDPE-pipe.

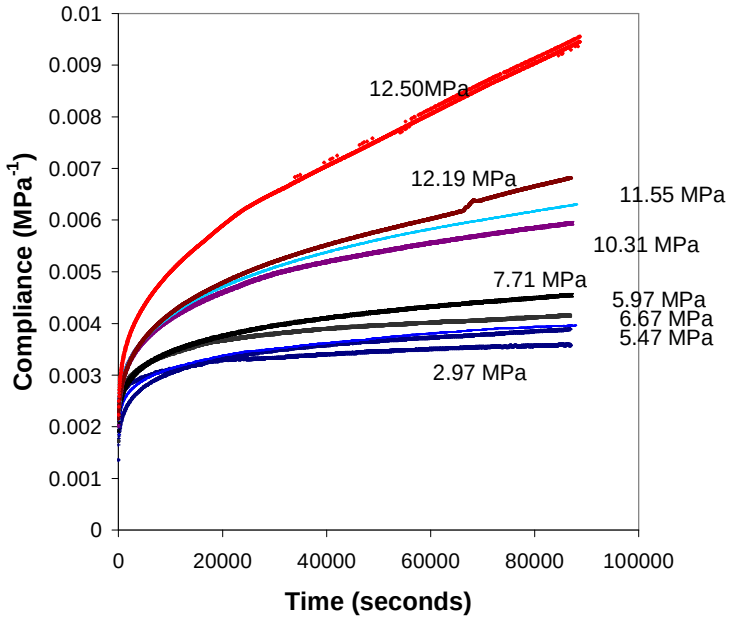


Figure 2.4 (a) Compliance-time curves for 24-hour creep tests on HDPE-pipe.

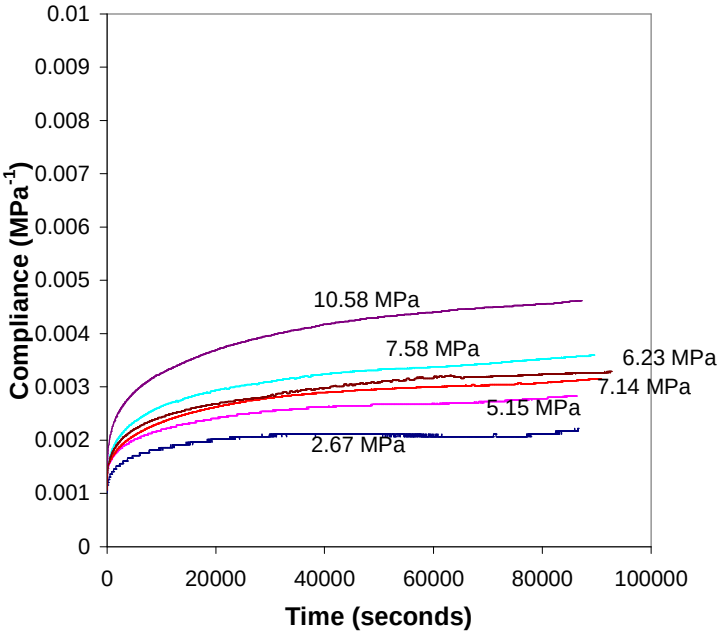


Figure 2.4 (b) Compliance-time curves for 24-hour creep tests on HDPE-resin1.

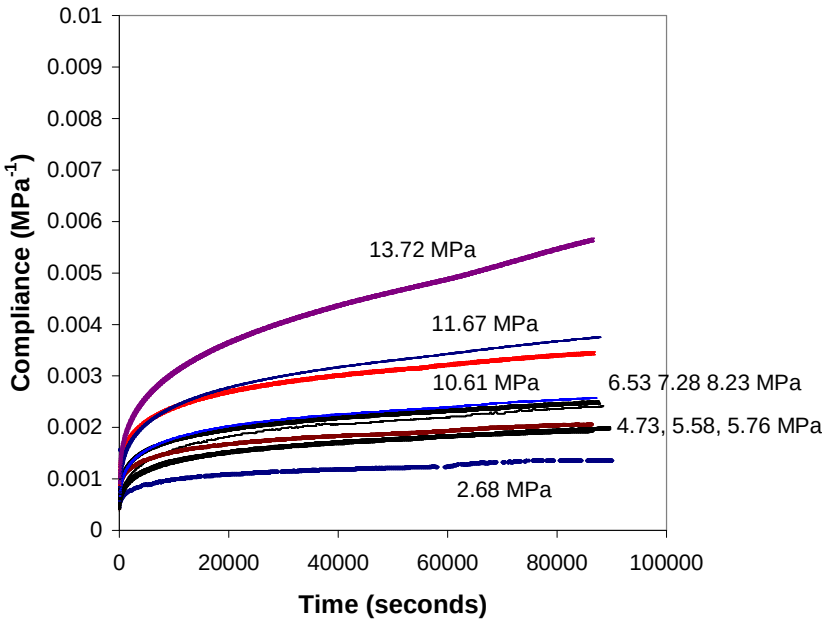


Figure 2.4 (c) Compliance-time curves for 24-hour creep tests on HDPE-resin2.

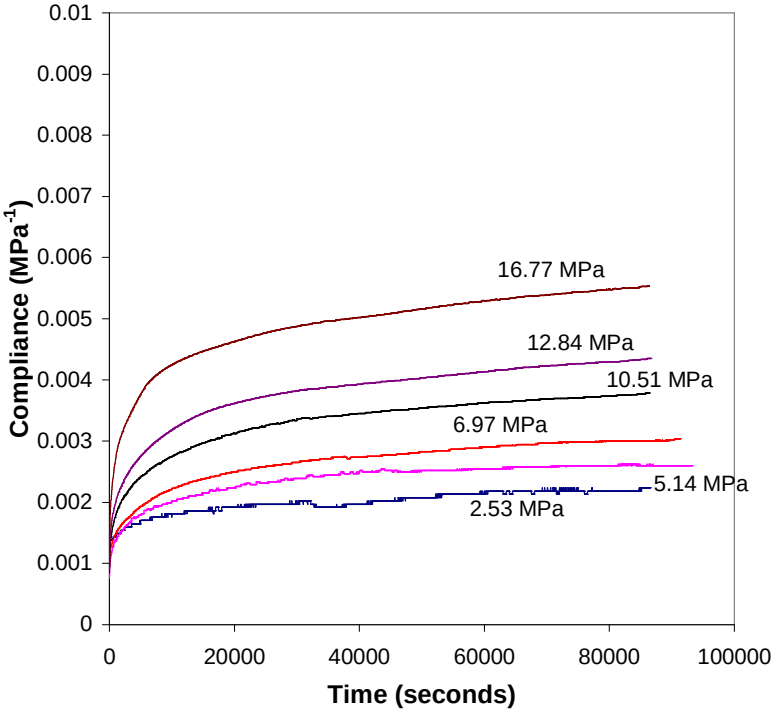


Figure 2.4 (d) Compliance-time curves for 24-hour creep tests on HDPE-resin3.

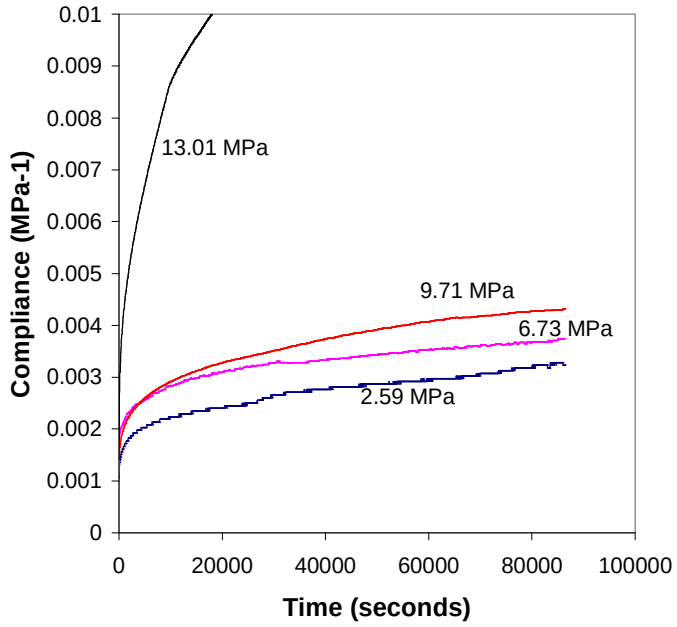


Figure 2.4 (e) Compliance-time curves for 24-hour creep tests on HDPE-resin4.

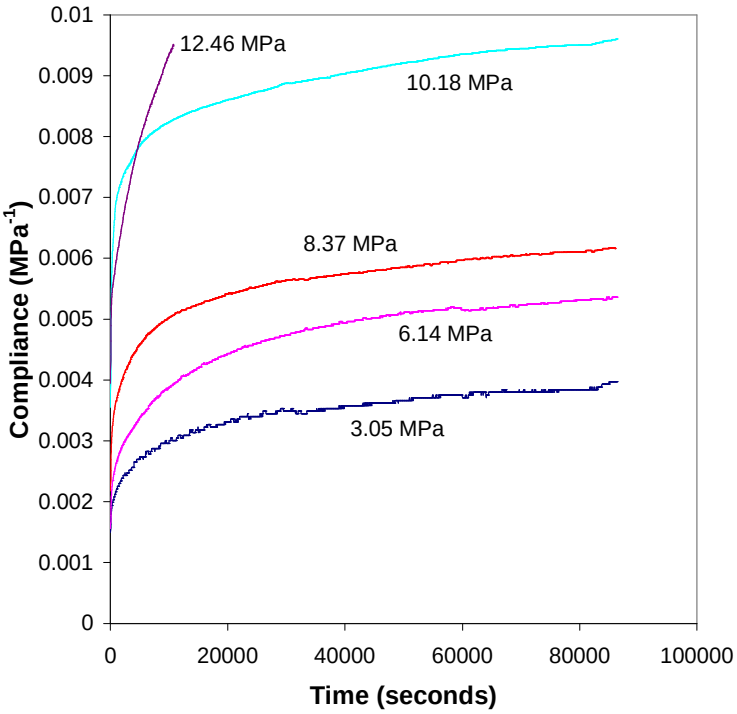


Figure 2.4 (f) Compliance-time curves for 24-hour creep tests on PE80-resin.

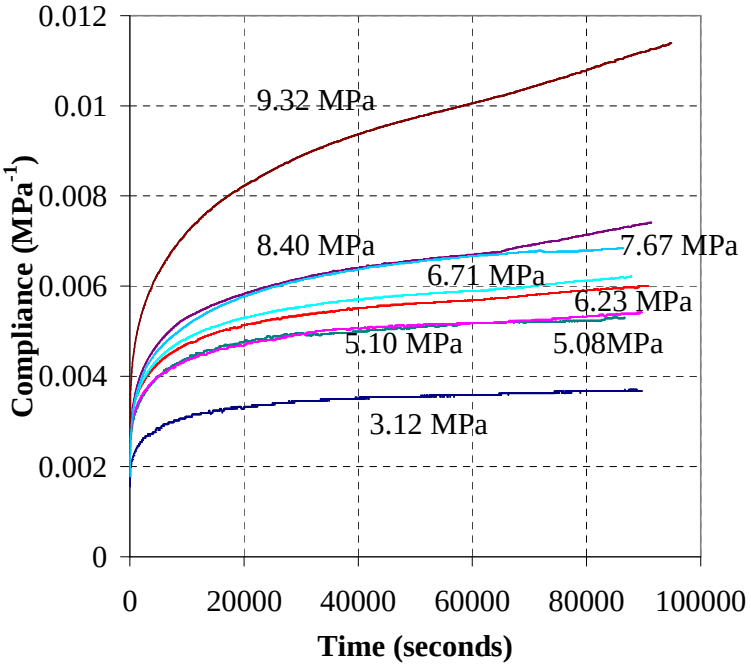


Figure 2.4 (g) Compliance-time curves for 24-hour creep tests on MDPE-pipe.

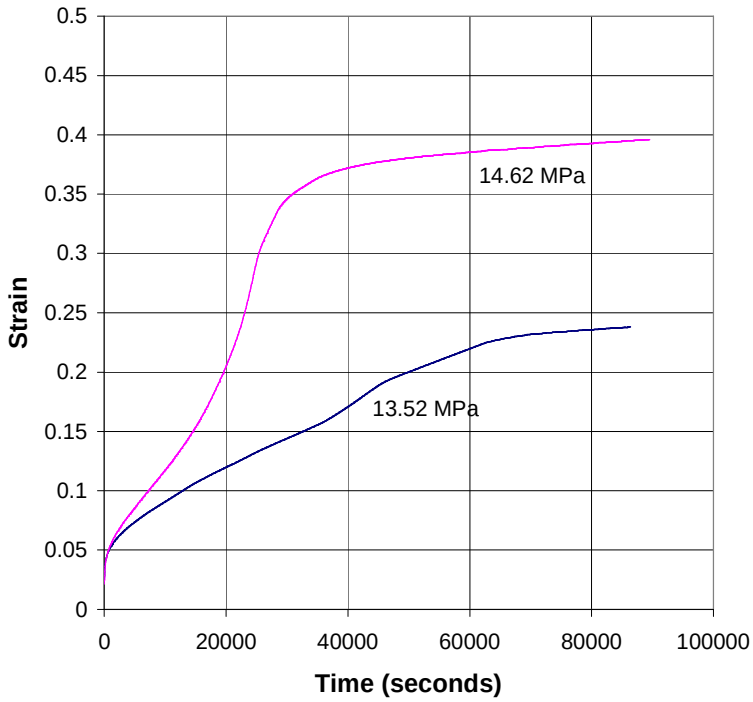


Figure 2.5 Strain-time curves for 24-hour creep tests at high stress on HDPE-pipe.

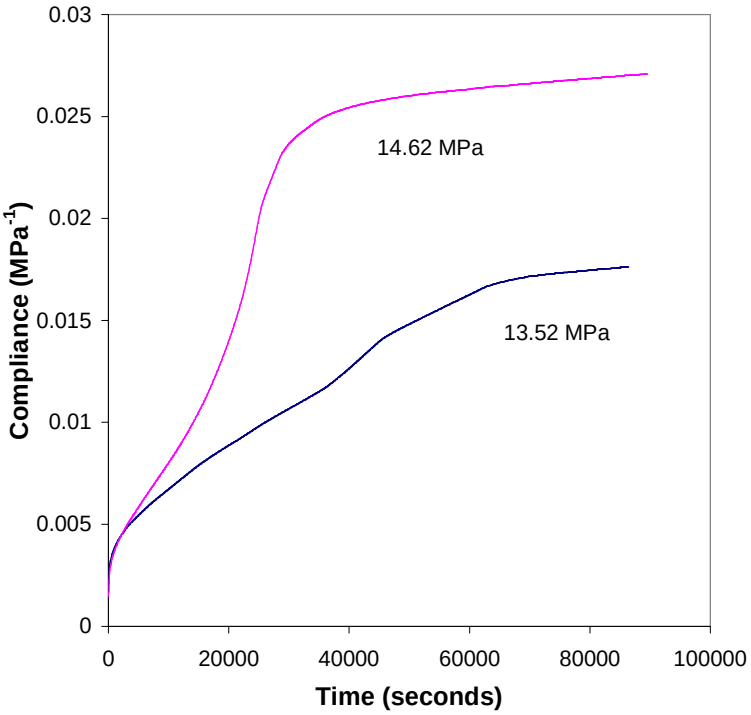


Figure 2.6 Compliance-time curves for 24-hour creep tests at high stress on HDPE-pipe.

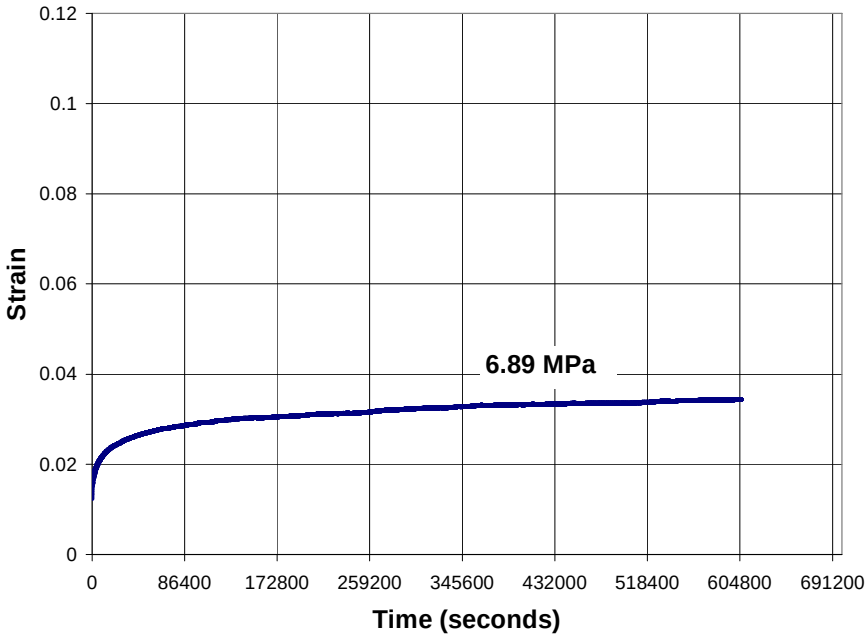


Figure 2.7 (a) Strain-time curves for 7-day creep tests at 6.89 MPa on HDPE-pipe.

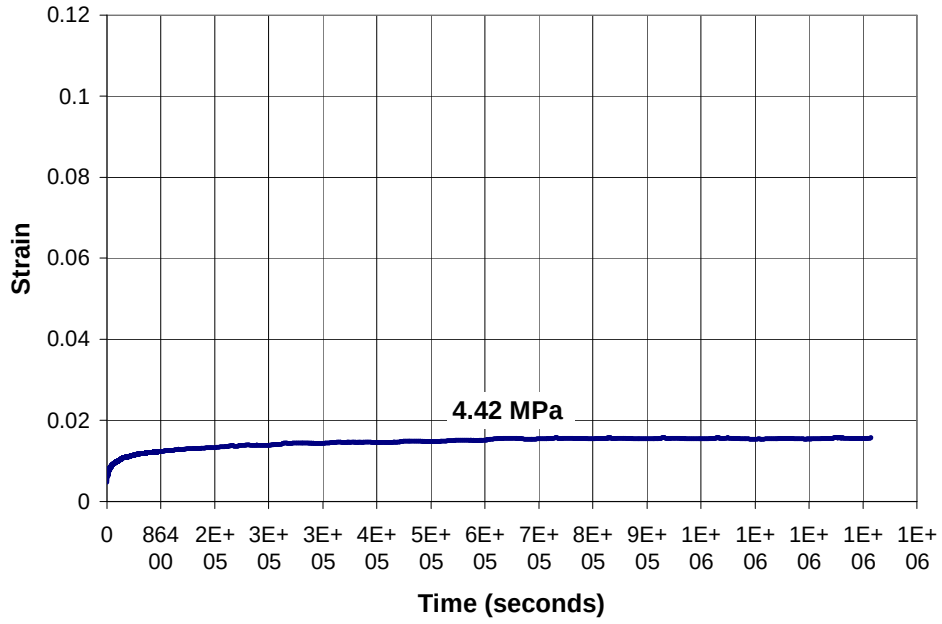


Figure 2.7 (b) Strain-time curves for 14-day creep tests at 4.42 MPa on HDPE-pipe.

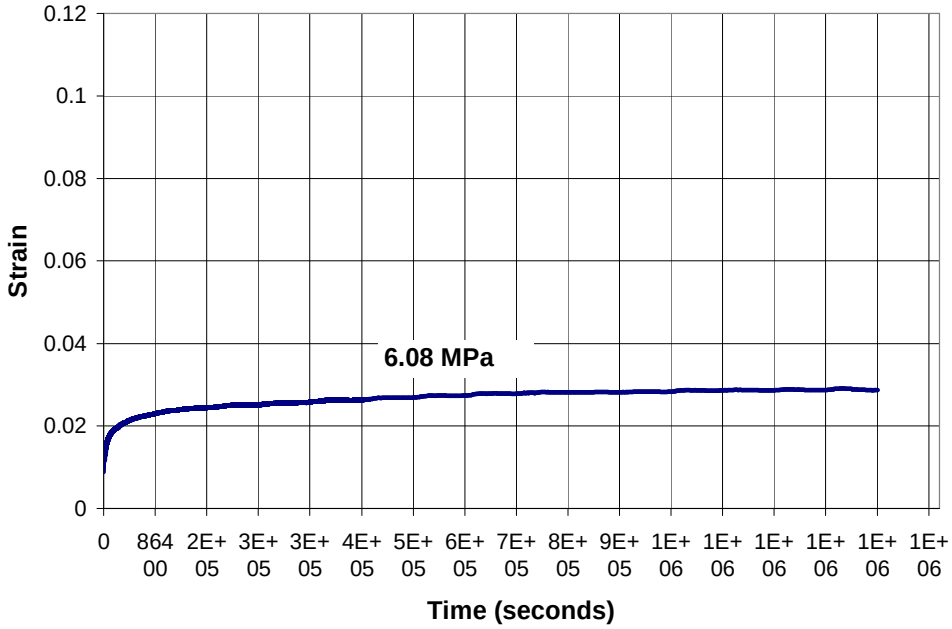


Figure 2.7 (c) Strain-time curves for 14-day creep tests at 6.08 MPa on HDPE-pipe.

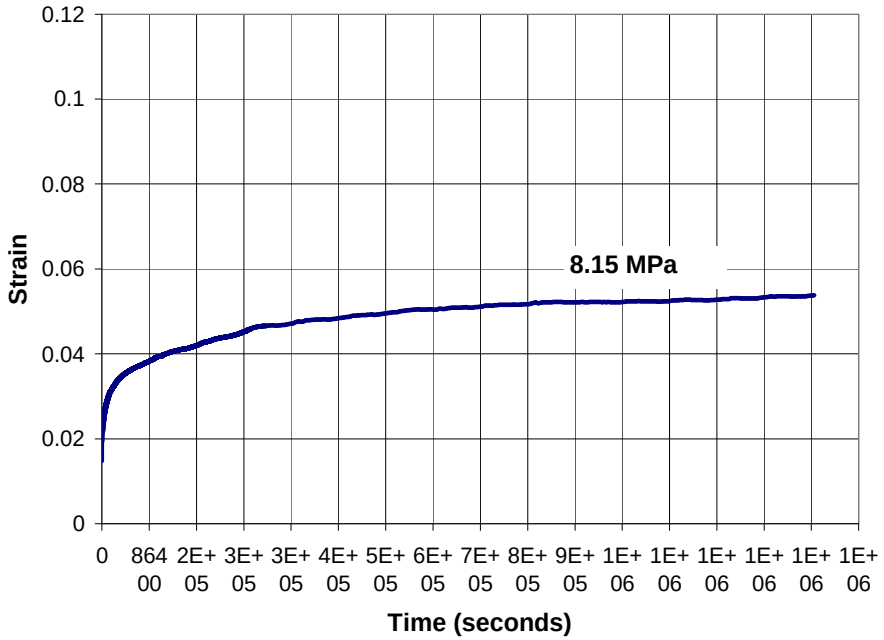


Figure 2.7 (d) Strain-time curves for 14-day creep tests at 8.15 MPa on HDPE-pipe.

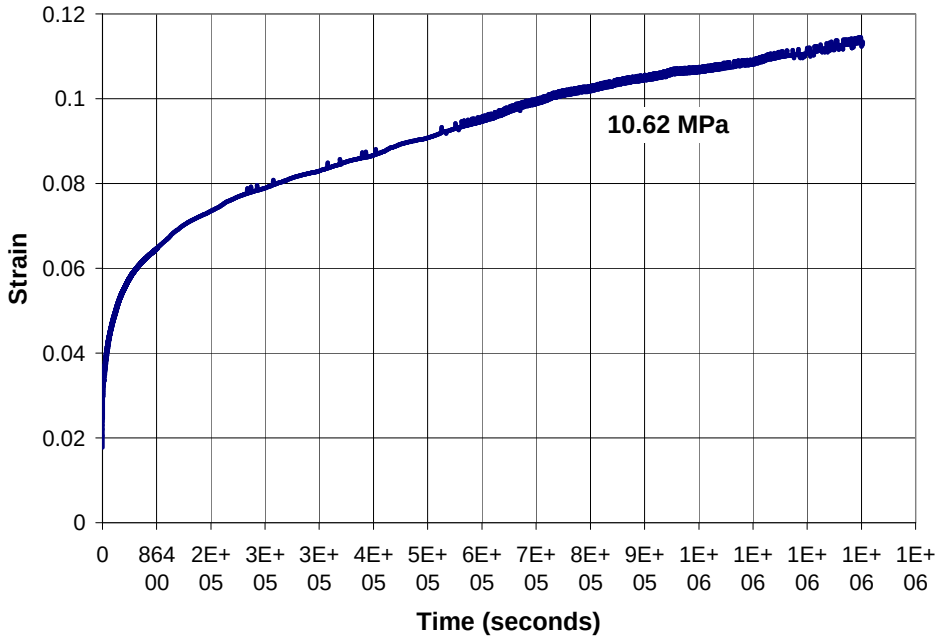


Figure 2.7 (e) Strain-time curves for 14-day creep tests at 10.62 MPa on HDPE-pipe.

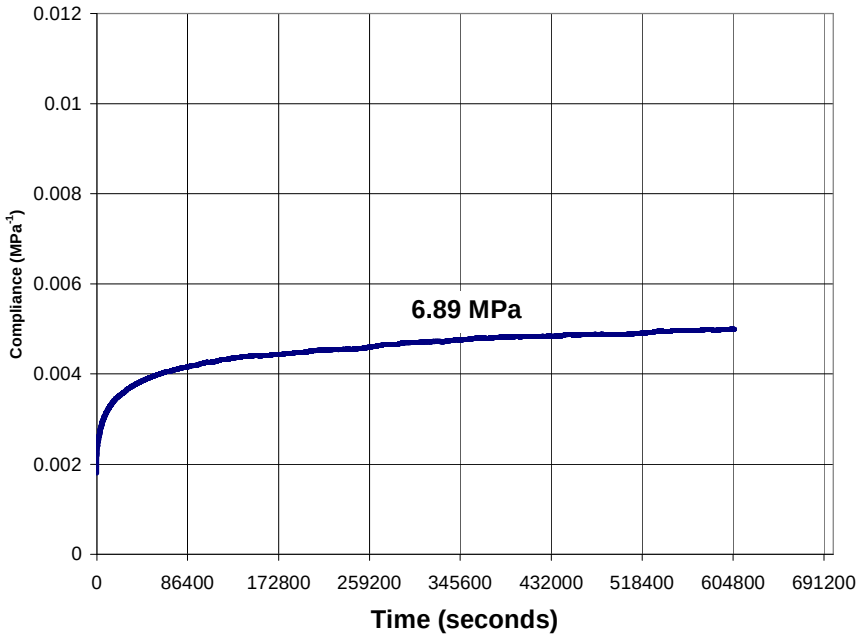


Figure 2.8 (a) Compliance-time curves for 7-day creep tests at 6.89 MPa on HDPE-pipe.

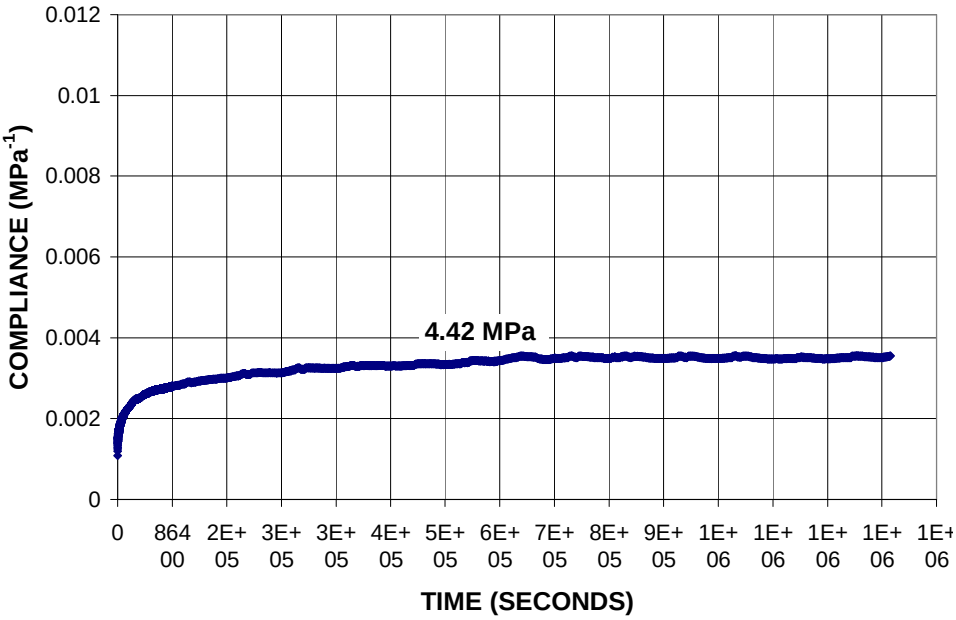


Figure 2.8 (b) Compliance-time curves for 14-day creep tests at 4.42 MPa on HDPE-pipe.

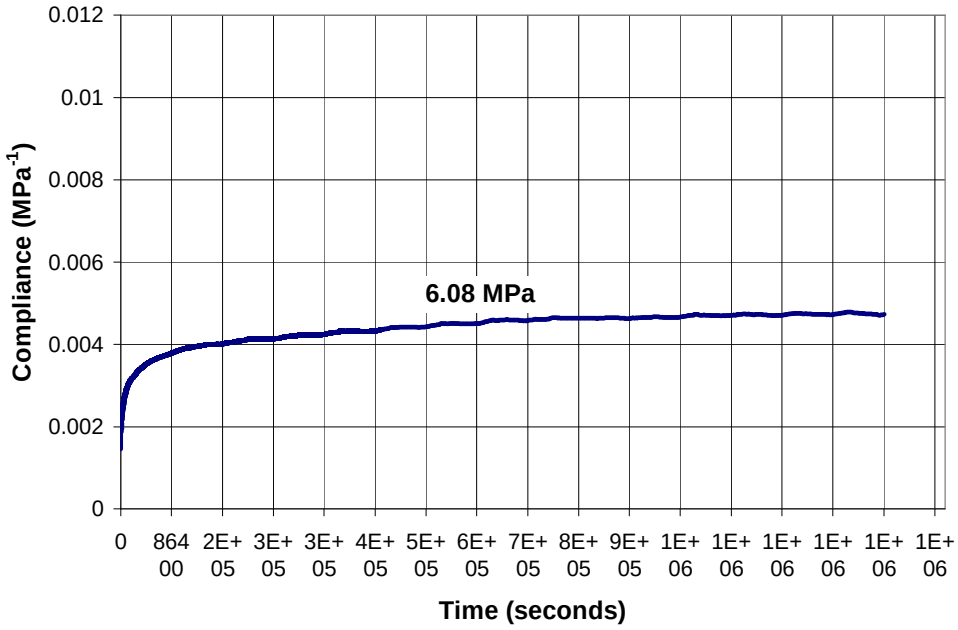


Figure 2.8 (c) Compliance-time curves for 14-day creep tests at 6.08 MPa on HDPE-pipe.

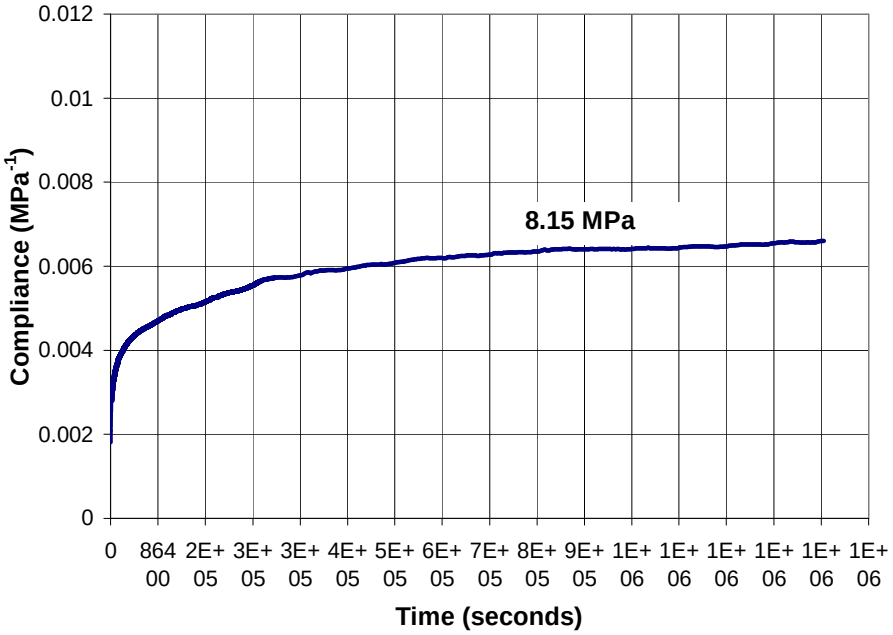


Figure 2.8 (d) Compliance-time curves for 14-day creep tests at 8.15 MPa on HDPE-pipe.

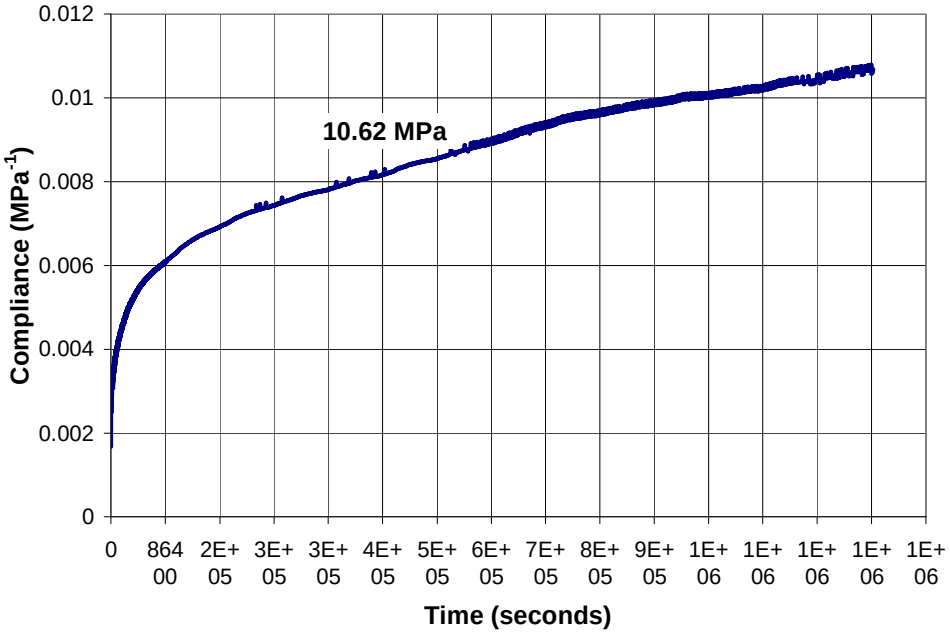


Figure 2.8 (e) Compliance-time curves for 14-day creep tests at 10.62 MPa on HDPE-pipe.

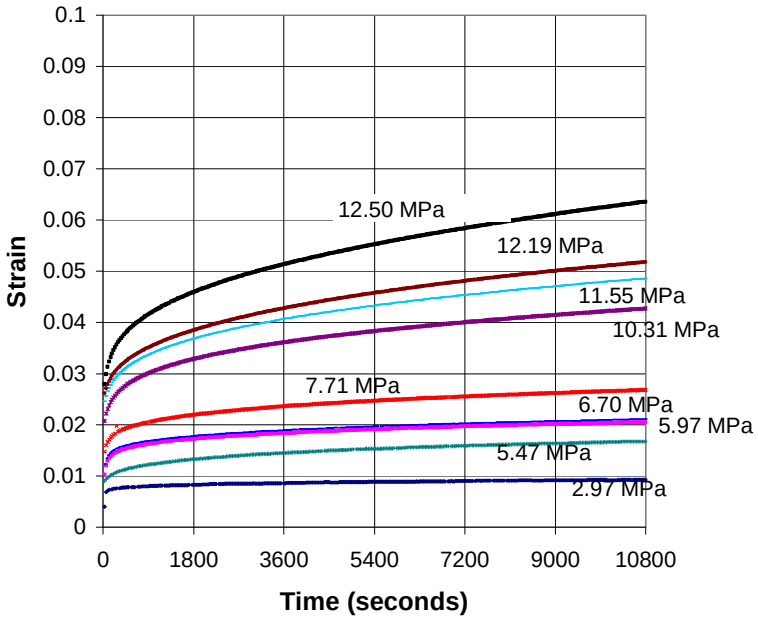


Figure 2.9 (a) Strain-time curves for 3-hour creep tests on HDPE-pipe.

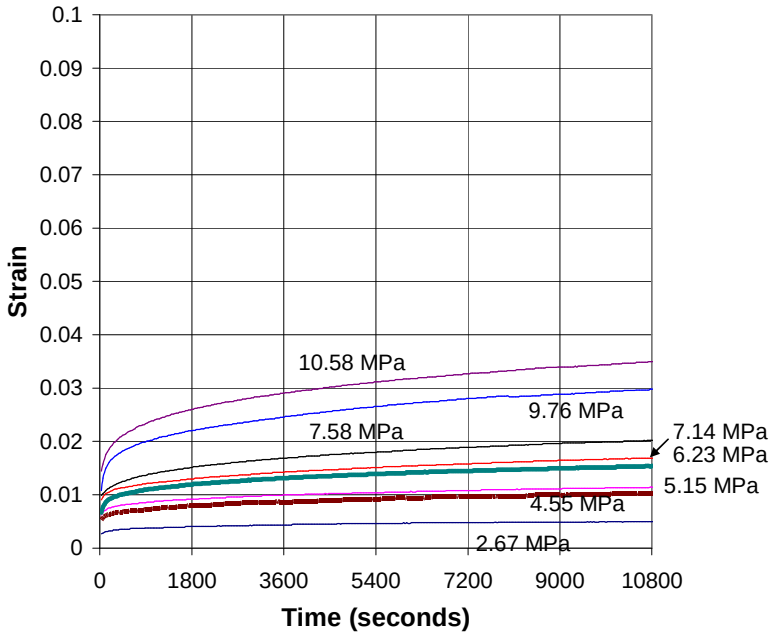


Figure 2.9 (b) Strain-time curves for 3-hour creep tests on HDPE-resin1.

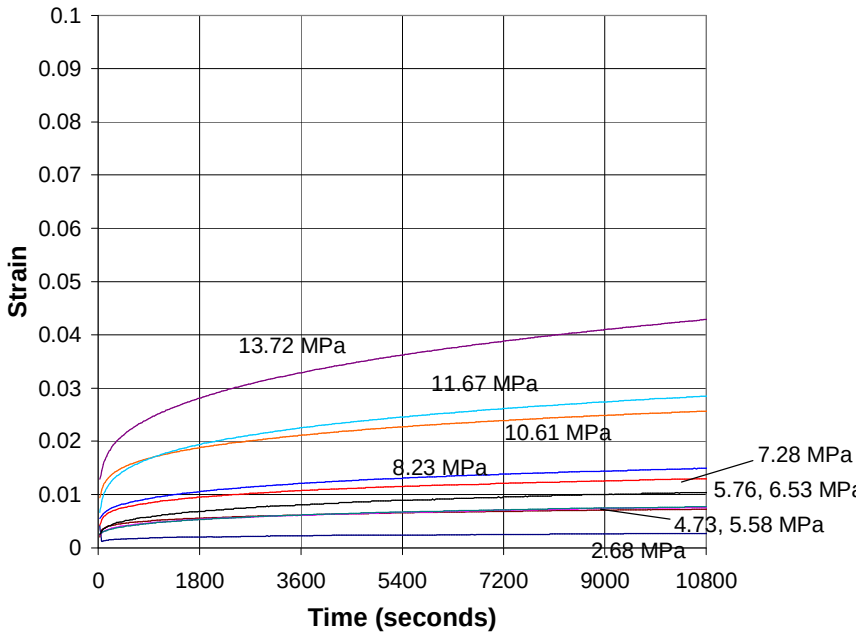


Figure 2.9 (c) Strain-time curves for 3-hour creep tests on HDPE-resin2.

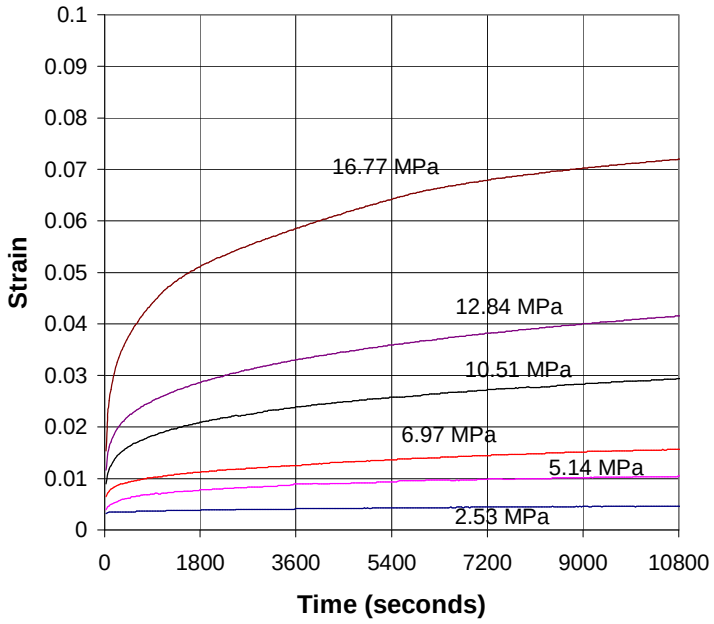


Figure 2.9 (d) Strain-time curves for 3-hour creep tests on HDPE-resin3.

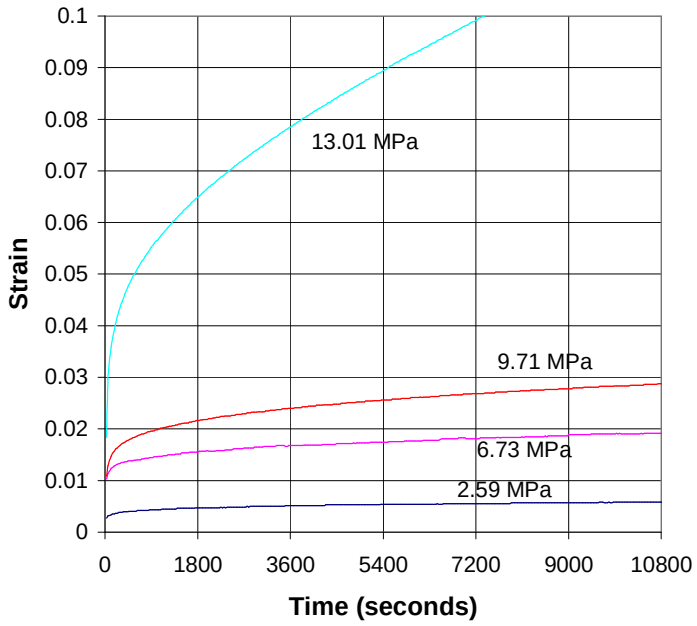


Figure 2.9 (e) Strain-time curves for 3-hour creep tests on HDPE-resin4.

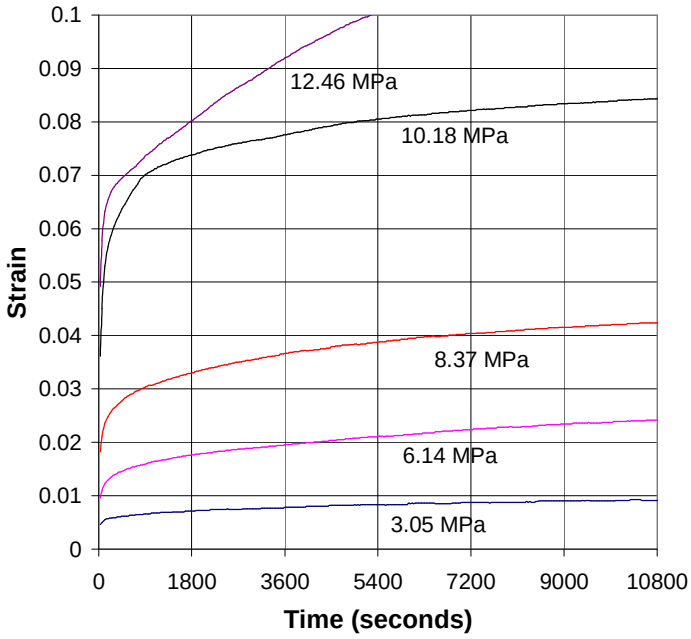


Figure 2.9 (f) Strain-time curves for 3-hour creep tests on PE80-resin.

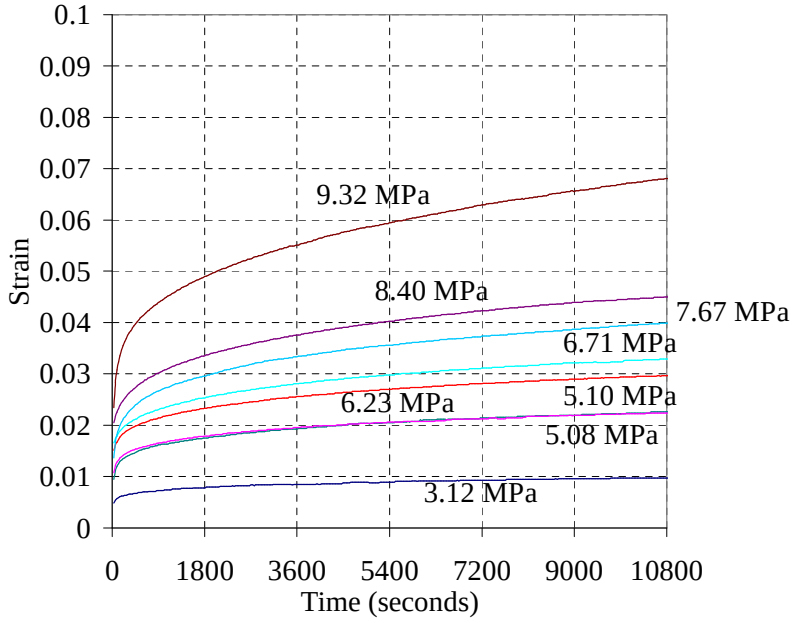


Figure 2.9 (g) Strain-time curves for 3-hour creep tests on MDPE-pipe.

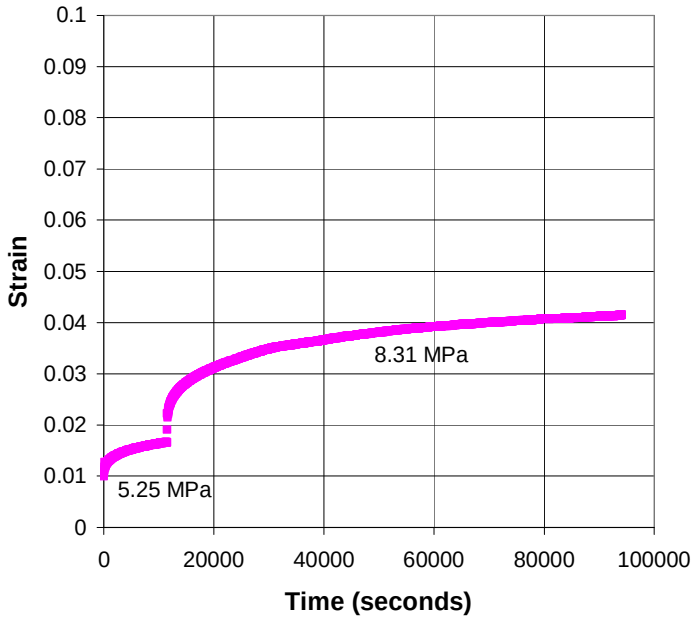


Figure 2.10 Strain-time curves for stepped loading creep tests on HDPE-pipe (5.25-8.31 MPa).

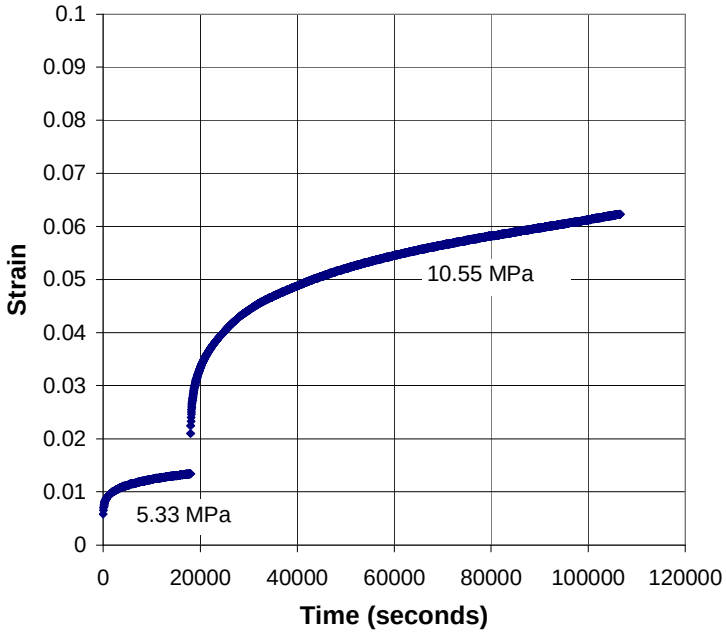


Figure 2.11 Strain-time curves for stepped loading creep tests on HDPE-pipe (5.33-10.55 MPa).

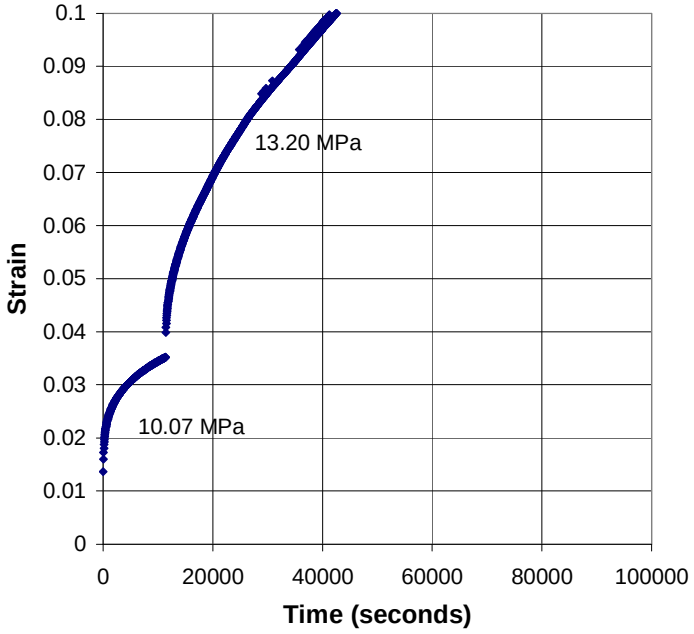


Figure 2.12 Strain-time curves for stepped loading creep tests on HDPE-pipe (10.07-13.20 MPa).

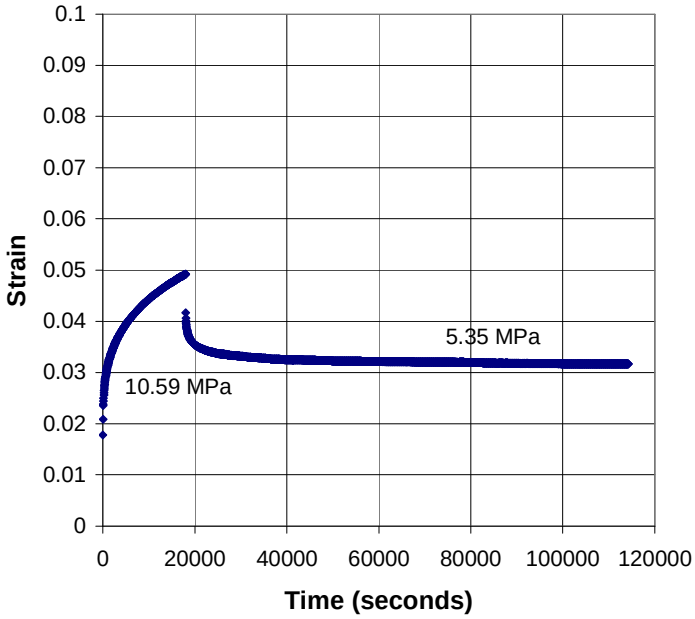


Figure 2.13 Strain-time curves for stepped loading creep tests on HDPE-pipe (10.59-5.35 MPa).

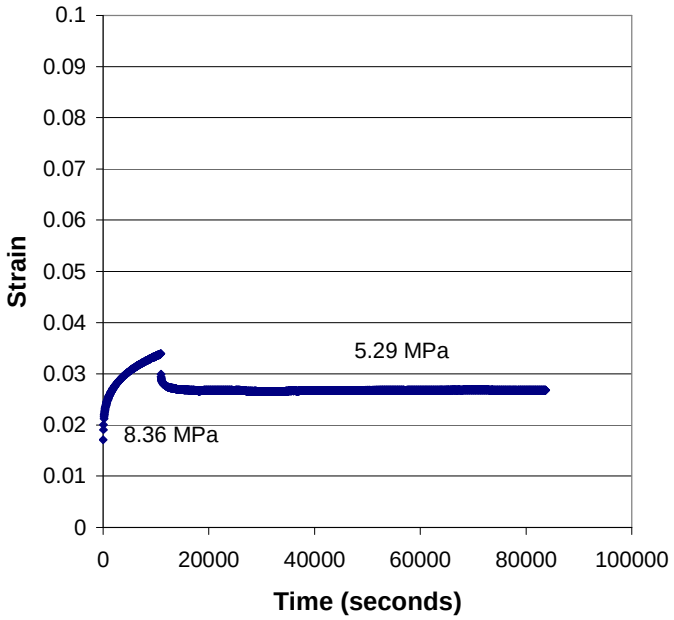


Figure 2.14 Strain-time curves for stepped loading creep tests on HDPE-pipe (8.36-5.29 MPa).

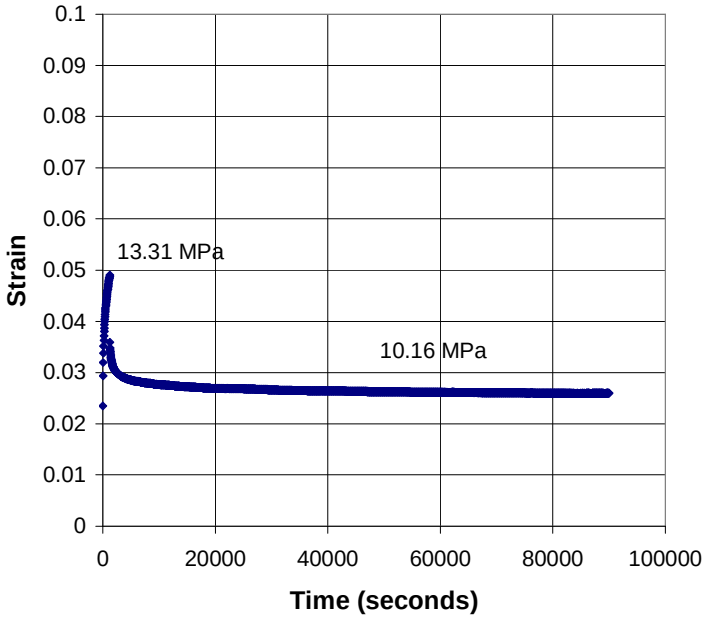


Figure 2.15 Strain-time curves for stepped loading creep tests on HDPE-pipe (13.31-10.16 MPa).

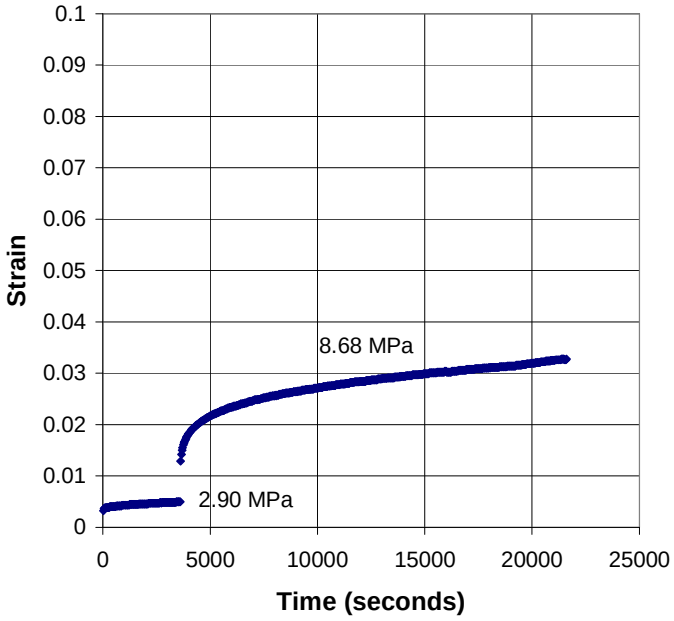


Figure 2.16 Strain-time curves for stepped loading creep tests on HDPE-pipe (2.90-8.68 MPa).

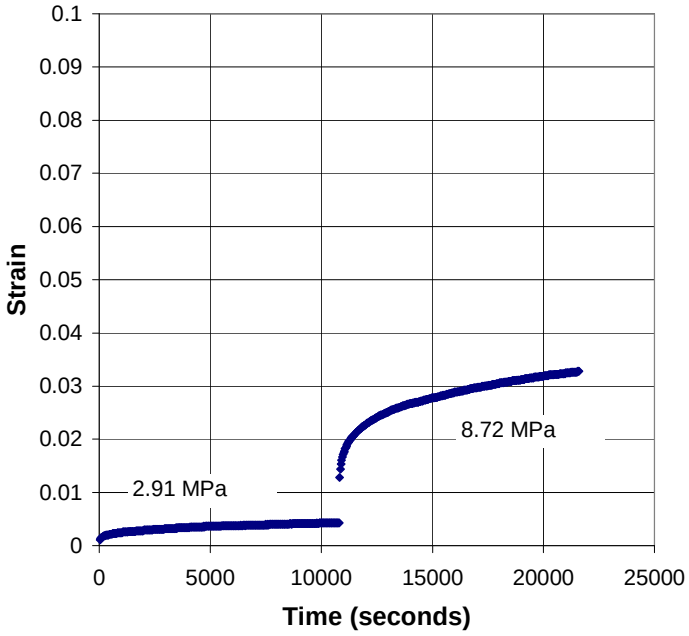


Figure 2.17 Strain-time curves for stepped loading creep tests on HDPE-pipe (2.91-8.72 MPa).

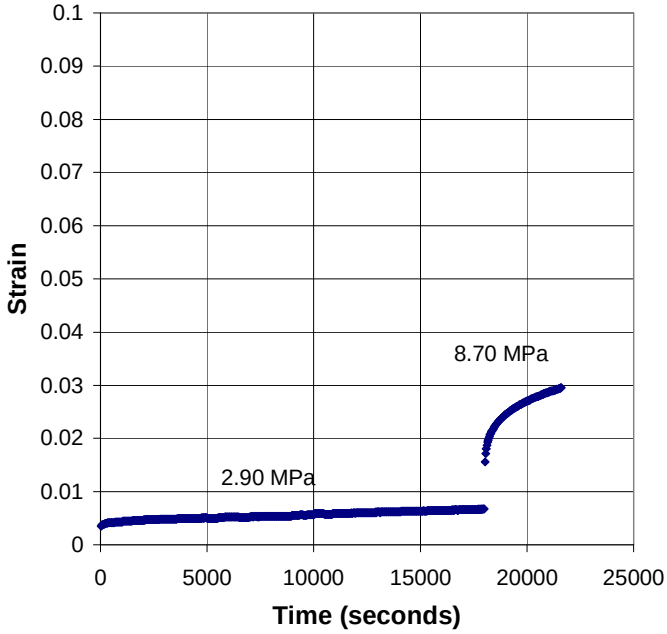


Figure 2.18 Strain-time curves for stepped loading creep tests on HDPE-pipe (2.90-8.70 MPa).

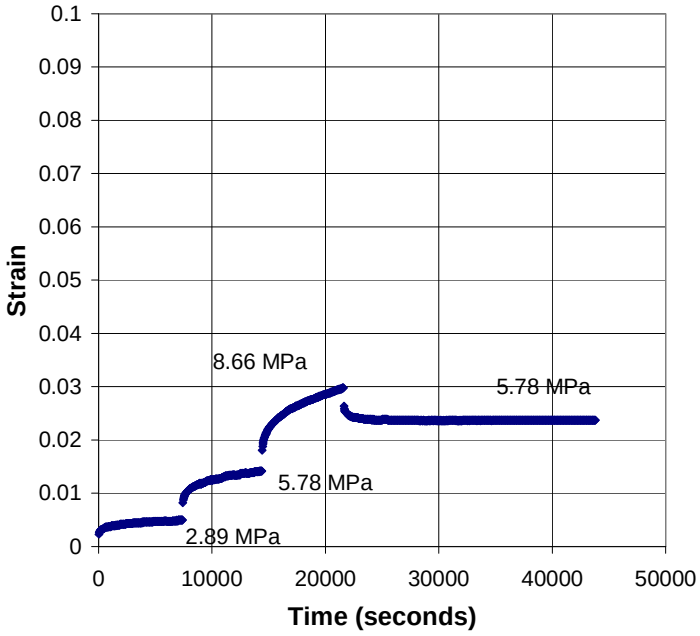


Figure 2.19 Strain-time curves for stepped loading creep tests on HDPE-pipe (2.89-5.78-8.66-5.78 MPa).

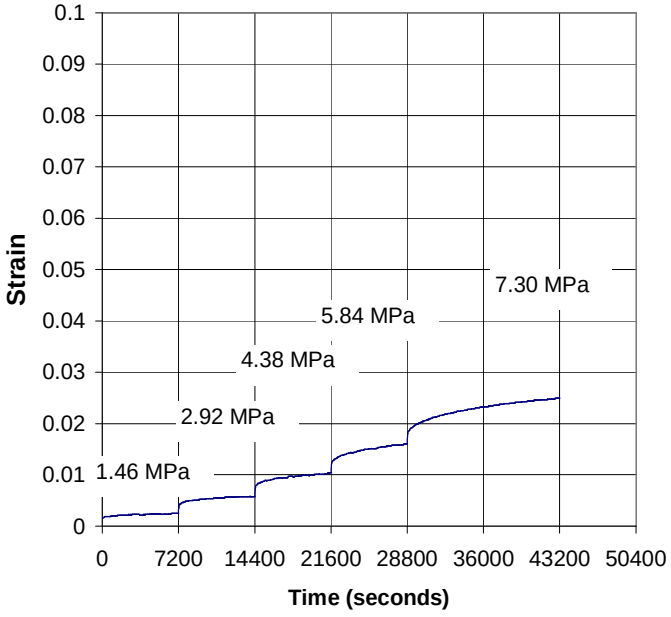


Figure 2.20 Strain-time curves for stepped loading creep tests on HDPE-pipe (1.46-2.92-4.38-5.84-7.30 MPa).

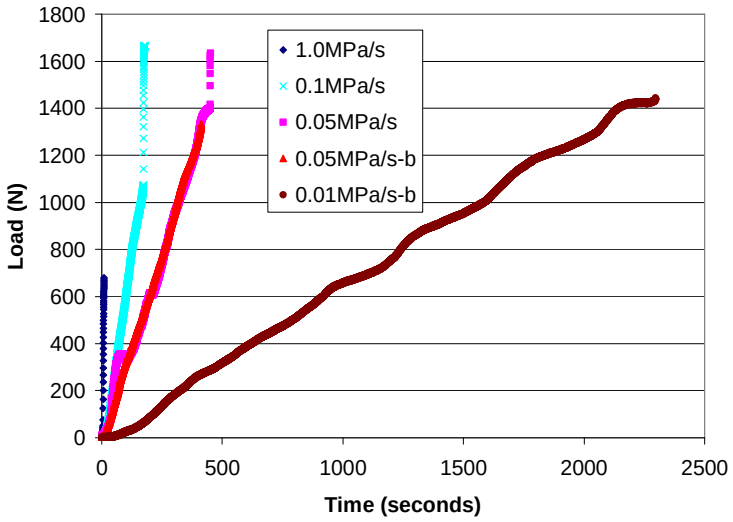


Figure 2.21 (a) Constant load rate tests on HDPE-pipe: load vs. time.

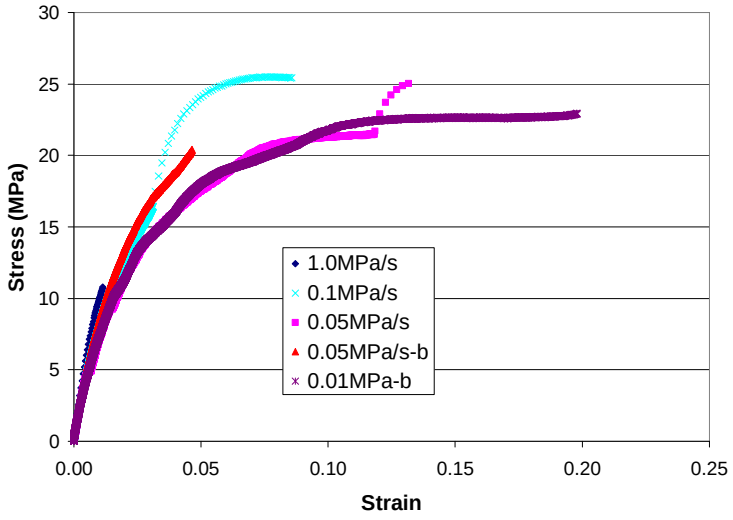


Figure 2.21 (b) Constant load rate tests on HDPE-pipe: stress vs. strain.

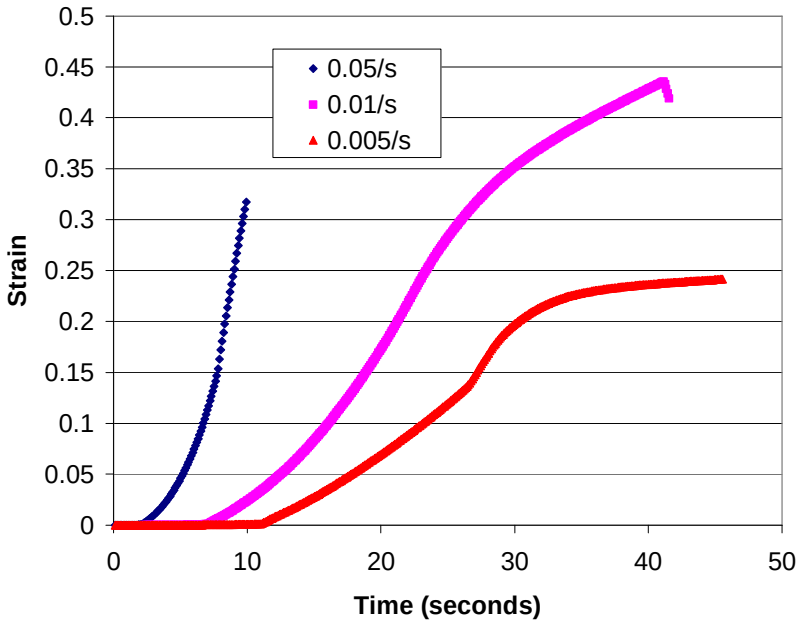


Figure 2.22 (a) Constant Strain rate tests on HDPE-pipe: Strain vs. time

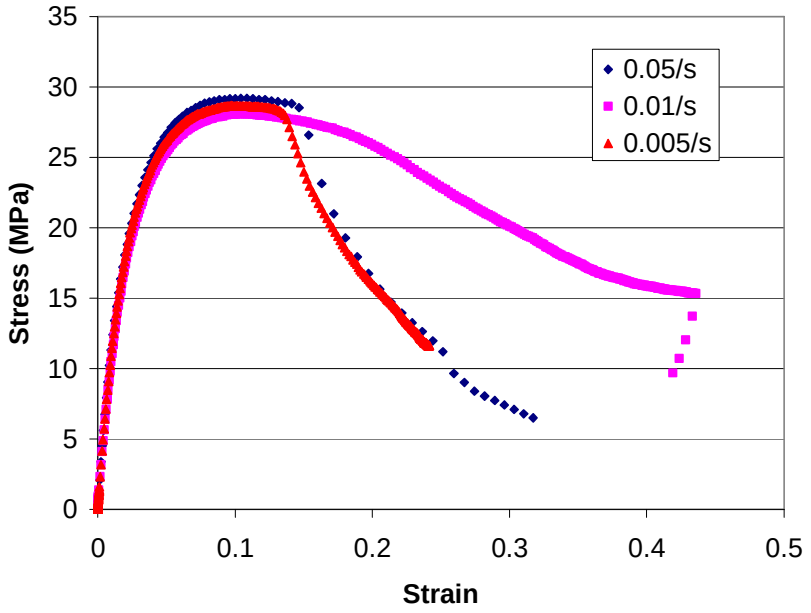


Figure 2.22 (b) Constant Strain rate tests on HDPE-pipe: stress vs. strain

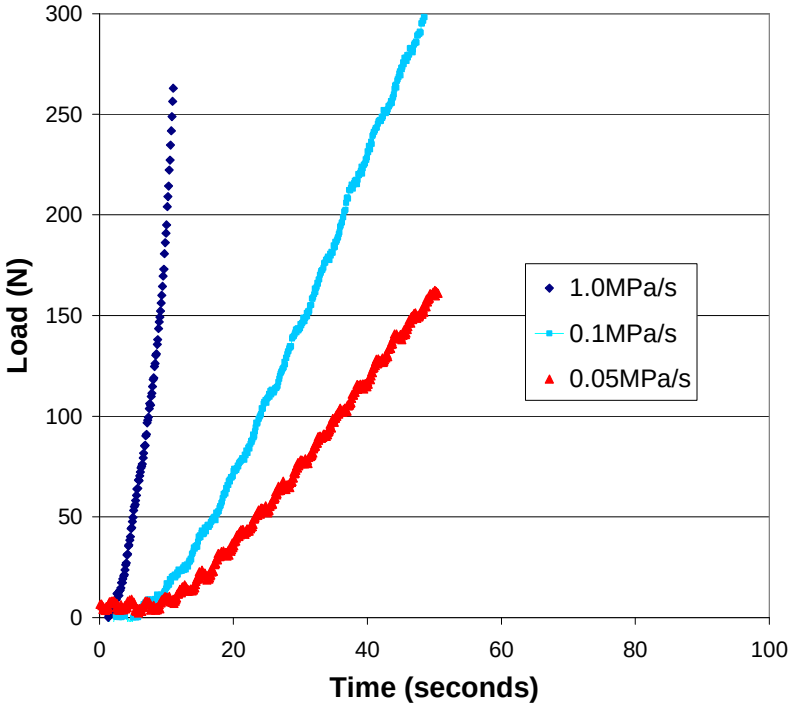


Figure 2.23 (a) Constant load rate tests on HDPE-Resin2: load vs. time.

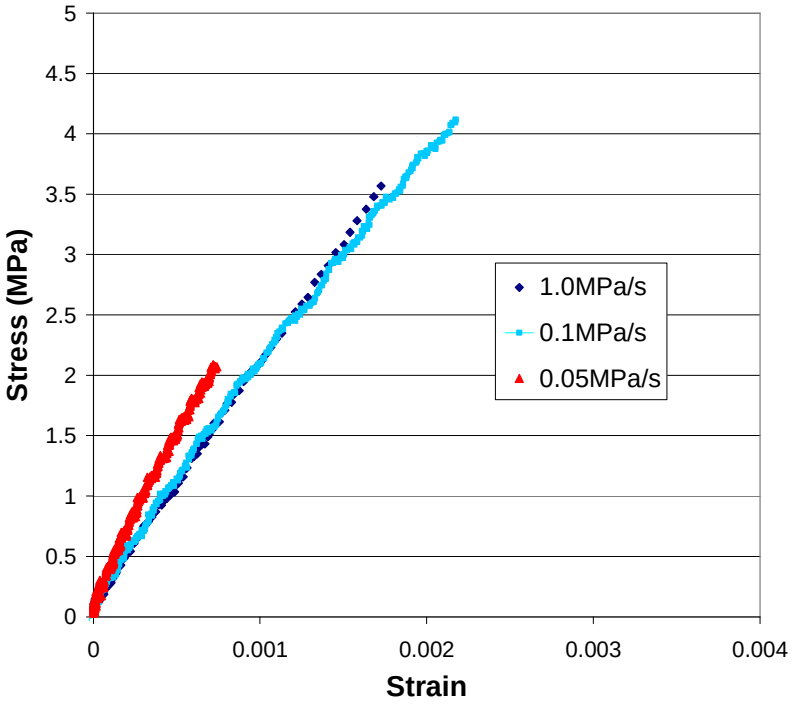


Figure 2.23 (b) Constant load rate tests on HDPE-Resin2: stress vs. strain.

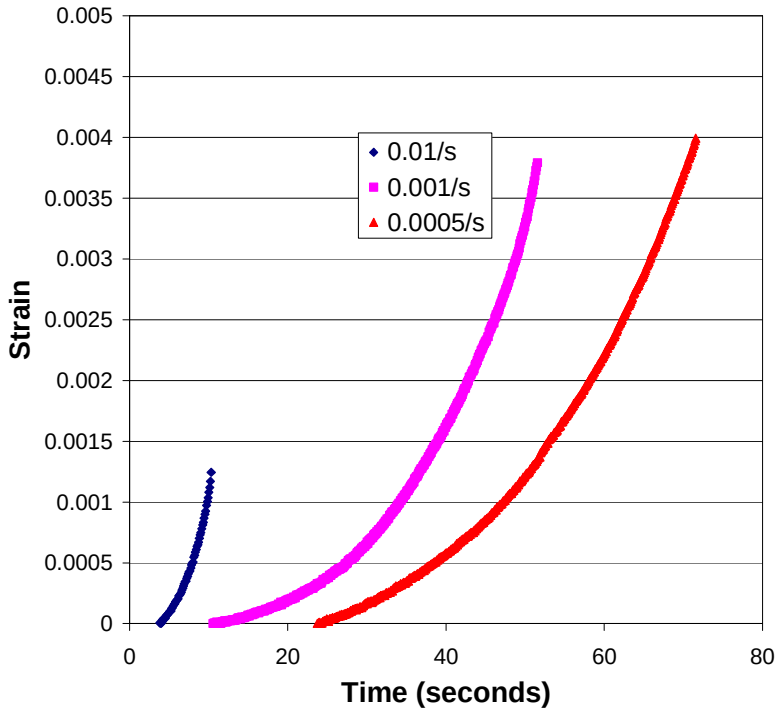


Figure 2.24 (a) Constant Strain rate tests on HDPE-Resin2: Strain vs. time

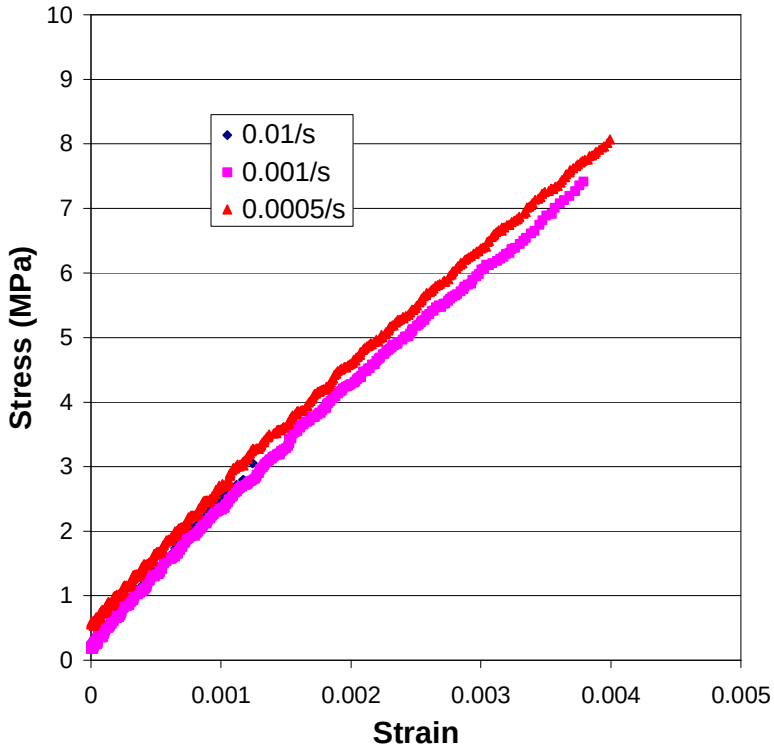


Figure 2.24 (b) Constant Strain rate tests on HDPE-Resin2: stress vs. strain

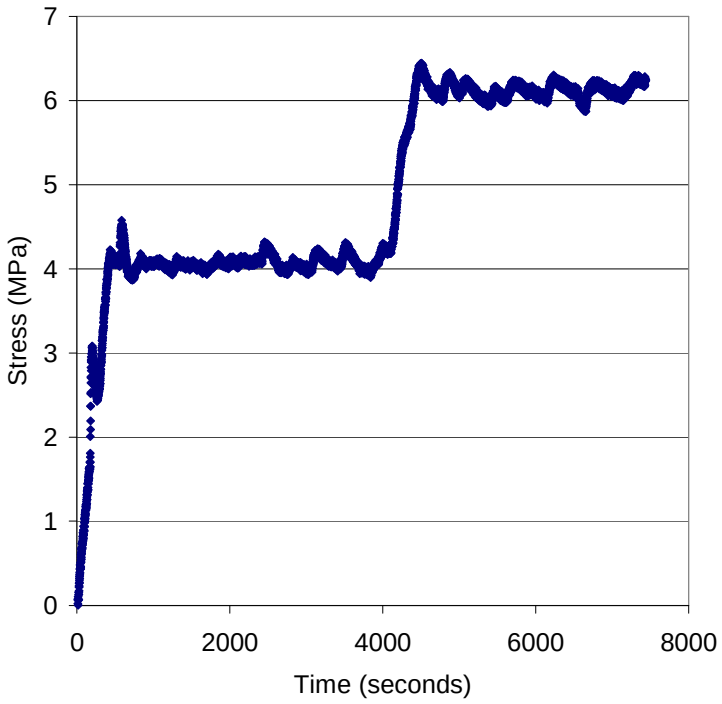


Figure 2.25 (a) Stress vs. time for complex-test-1.

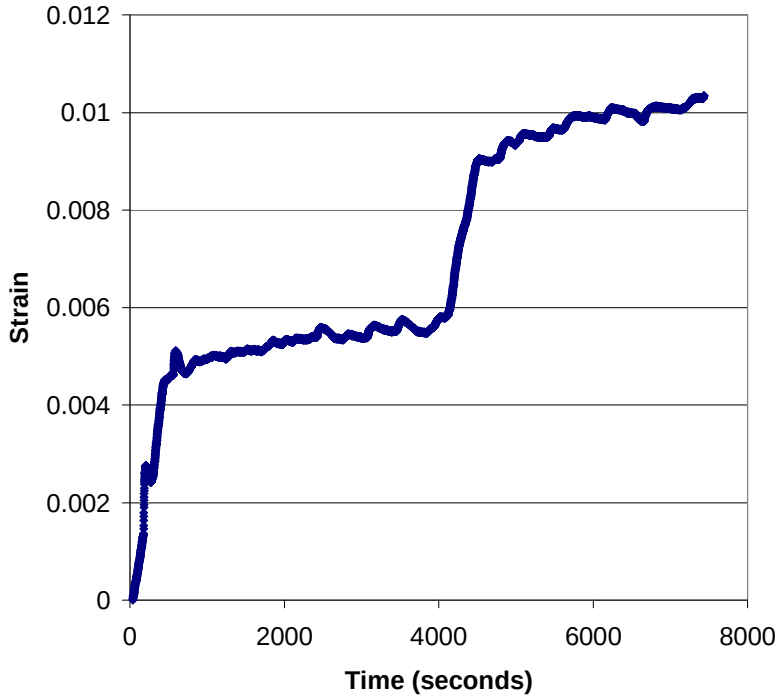


Figure 2.25 (b) Strain vs. time for complex-test-1.

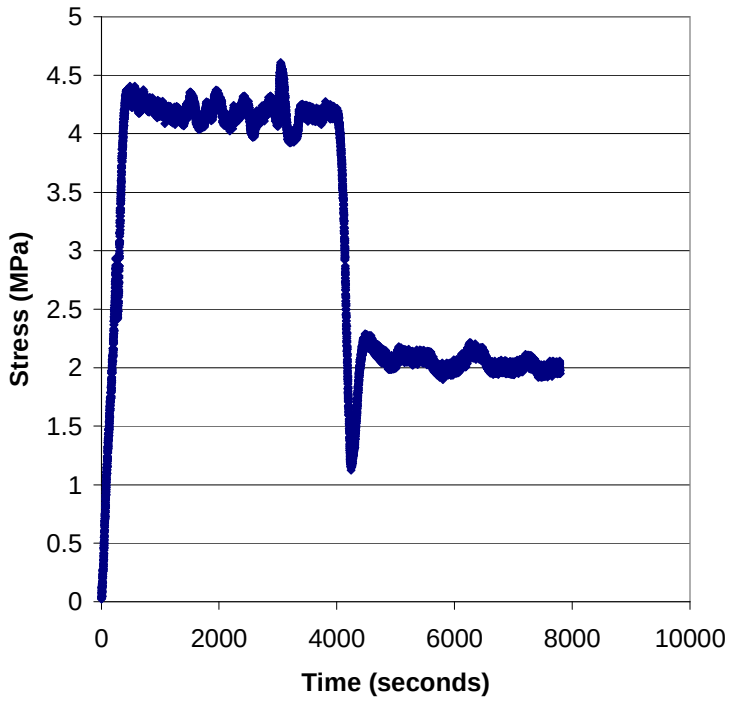


Figure 2.26 (a) Stress vs. time for complex-test-2.

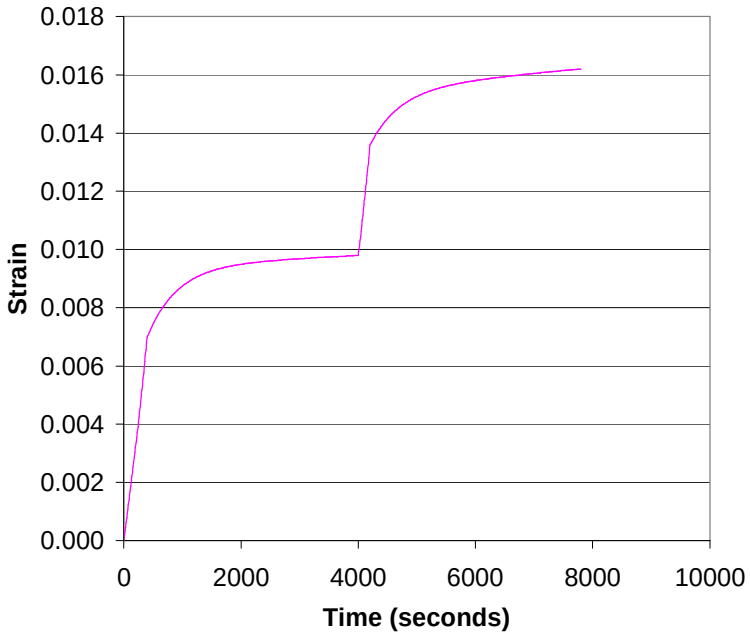


Figure 2.26 (b) Strain vs. time for complex-test-2.

Chapter 3

Modelling Creep Behaviour Using Nonlinear Viscoelastic/Viscoplastic Models

3.1 Introduction

The material behaviour of semicrystalline polymers, like polyethylene, is often categorized as viscoelastic or viscoplastic, which suggests a combination of viscous flow typical for fluids, with either elastic or plastic characteristics typical for solids. In reality, the challenge in studying and modelling these materials comes from the fact that all of these three idealized types of behaviour exist simultaneously, in any relative proportion. The proportion of the viscous, elastic and plastic characteristics in the behaviour of polyethylene depends on the rate of loading, time, loading history, stress level and temperature. It also depends strongly on the micro-structure including crystallinity, molecular weight, molecular weight distribution and branching. Thus, different polyethylenes will exhibit different material responses and should be modelled using different values and relationships of the material parameters; a macromechanical model created for one type of polyethylene will not necessarily work well for another one. Polyethylene also exhibits a nonlinear behaviour meaning that its load-deformation relationship depends on the level of applied stress. The creep responses at different stress levels are described by different coefficients of the creep compliance functions.

Linear viscoelasticity theories have long been established and viscoelastic models are formed mathematically in either integral or differential equation forms (Flügge 1967; Lockett 1972). From a practical point of view, integral form models can be discretized and the resulting formulation implemented in a numerical fashion. Differential form equations require finding the solution to the equations. Depending on the formulation of the model, material parameters for these constitutive models are found from creep, relaxation or constant strain rate tests. Several linear and nonlinear viscoelastic and

viscoplastic models (Krishnaswamy *et al.* 1992; Zhang *et al.* 1997; Lu *et al.* 2000) have been proposed for practical applications in finite element analysis. These models are material and application specific, which means that the derived equations and material parameters obtained from experiments work well for the given material and for the time period used in testing (for material calibration). The mathematical forms of these models are complex and thus complicate the calibration of material parameters. Due to wide variations in polymer types, material parameters should in fact be generated for every specific material. A simple and effective constitutive modelling approach is important for the analysis of polymeric structures.

This chapter presents a simple approach to the development of nonlinear, time dependent macromechanical modelling for polyethylene. The modelling is done based on creep responses at different stress levels, which are used as an input for the polymeric material modelling routine (PMMR). The PMMR routine is used to create a nonlinear material model for the given polyethylene, which can then be utilized by a structural analysis program, e.g. finite element analysis program. The aim of the presented work is to create a rational modelling approach which would be practical for use in real engineering design analyses. Thus it must be simple and easy to calibrate based on testing.

The chapter presents briefly the theoretical background related to viscoelastic and viscoplastic modelling of polyethylene. The importance of modelling creep response is emphasized. A modelling procedure that provides a simple form of constitutive equations, upon which finite element formulation can be established, is proposed based on short term axial tension creep experiments. The performance of the model with respect to the material parameter selection in the modelling process is discussed. To simplify the approach at this stage of the research, physical aging effects are not considered.

3.2 Creep and Time-Dependent Modelling

Creep, time-relaxation, and rate effects are the typical phenomena of time-dependent material behaviours. This means that for viscoelastic/viscoplastic materials, material

responses under loading are not only dependent on stress and strain conditions, but also on time. Thus, the mechanical properties are functions of time. Time effects must be included in the experiments used to determine the material parameters in the constitutive equations.

Creep is the phenomenon whereby the material deformation grows with time while the stress in the material remains constant. In a creep test, the stress in the tested sample is kept constant and the strain is recorded with time. A strain-time curve is obtained for a given stress. The stress-strain relationship can be expressed by Eq. (3.1)

$$\varepsilon(t) = \sigma_c \psi(t) \tag{3.1}$$

where σ_c is the constant stress, and $\psi(t)$ the creep compliance. The constitutive relationship is uniquely defined by creep compliance, which is a function of time. The task in material modelling is to find a function $\psi(t)$ that is the best fit for the test results.

Tensile creep tests are easy to control and are commonly used to study the time-dependent effects in materials. Creep test results for several polyethylene materials were presented in the previous chapter.

3.3 Linear Viscoelastic/Viscoplastic Modelling

In Eq. (3.1), the creep function, $\psi(t)$, is a function of time only. Material property so defined is independent of stress/strain and in such a case the stress-strain relationship of the material is considered linear viscoelastic (or viscoplastic). The creep beahviour at different stresses can be described by one compliance-time curve for linear viscoelastic/viscoplastic material. When the material property is not only a function of time but also of stress or strain, the stress-strain relationship is nonlinear viscoelastic (or viscoplastic). A couple of compliance-time curves must be used to describe creep behaviour at different stresses. Material parameter dependence of stress or strain must be found from experiments with the material in question.

Boltzmann superposition principle of linear systems can be applied to study linear materials. By linear superposition, the material response at a point of time can be expressed as the result of the addition of the loading responses of small time intervals in the material history. The standard modelling approach for viscoelastic modelling stress strain relationship for time dependent materials can take an integral form (Flügge, 1967; Lockett, 1972):

$$\sigma(t) = \int_{-\infty}^{t} \phi(t-\tau)\dot{\varepsilon}(\tau)d\tau \tag{3.2}$$

Alternatively, the strain of the material at time t can also be expressed via an integral form:

$$\varepsilon(t) = \int_{-\infty}^{t} \psi(t-\tau)\dot{\sigma}(\tau)d\tau \tag{3.3}$$

where functions ϕ and ψ are known as the stress-relaxation function and the creep compliance function, respectively.

When material aging is not significant and is not considered, the stress or strain depend on material loading history only. Thus, Equations (3.2) and (3.3) can be written as (3.4) and (3.5), correspondingly:

$$\sigma(t) = \int_{0}^{t} \phi(t-\tau)\dot{\varepsilon}(\tau)d\tau \tag{3.4}$$

$$\varepsilon(t) = \int_{0}^{t} \psi(t-\tau)\dot{\sigma}(\tau)d\tau \tag{3.5}$$

Since material property is independent of the testing procedure, it should be uniquely defined by one of (3.4) and (3.5). Either one of the functions ϕ or ψ should be adequate in defining material behaviour.

Linear viscoelastic behaviour is also very often expressed by differential formulations, which can be established by equilibrium conditions for rhelogical models.

The transition from the integral to the differential formulation can be shown by adopting first an appropriate function from the creep or relaxation functions (Lockett 1972). In the following derivation, the creep function is used.

Based on test results we know that the creep function, ψ, is monotonically increasing (see Figure 2.4), and thus can be written as:

$$\psi(t) = \psi_0 + \int_0^{\infty} Y(\tau)\left[1 - \exp(-t/\tau)\right]d\tau \tag{3.6}$$

If $Y(\tau)$ is simplified by taking discrete values (time discretization):

$$Y(\tau) = \sum_{i=1}^{N} C_i \delta(\tau - \tau_i) \tag{3.7}$$

where δ is the Dirac delta function. Then by substitution:

$$\psi(t) = \psi_0 + \sum_{i=1}^{N} C_i\left[1 - \exp(-t/\tau_i)\right] \tag{3.8}$$

By substituting the relaxation function (3.8) into equation (3.5) and by integration we obtain:

$$\varepsilon(t) = \psi_0\sigma(t) + \sigma(t)\sum_{i=1}^{N} C_i - \sum_{i=1}^{N} C_i e^{-t/\tau_i}\int_0^{t} \exp(\tau/\tau_i)\dot{\sigma}(\tau)d\tau. \tag{3.9}$$

In the simple case of only one relaxation time τ_1 ($N = 1$), Equation (3.9) becomes:

$$\varepsilon(t) = \psi_0\sigma(t) + C_1\sigma(t) - C_1 e^{-t/\tau_1}\int_0^{t} \exp(\tau/\tau_1)\dot{\sigma}(\tau)d\tau \tag{3.10}$$

and, upon differentiation with respect to t and some manipulation, it can be shown that

$$(\psi_0 + C_1)\sigma(t) + \tau_1\psi_0\dot{\sigma}(t) = \varepsilon(t) + \tau_1\dot{\varepsilon}(t). \tag{3.11}$$

This represents the case if one spring and one Kelvin element is used in a rheological model (Lockett 1972).

In general, when more relaxation times are used ($N > 1$), the differentiation formulation of linear viscoelastic modelling takes the form:

$$[\,p_0 + p_1\frac{\partial}{\partial t} + p_2\frac{\partial^2}{\partial t^2} + \ldots]\sigma(t) = [\,q_0 + q_1\frac{\partial}{\partial t} + q_2\frac{\partial^2}{\partial t^2} + \ldots]\varepsilon(t) \tag{3.12}$$

where p_α and q_α are functions of ψ_0, C_i and τ_i.

Determination of Creep Functions

Phenomenological modelling of the behaviour of viscoelastic solids requires determining either creep or relaxation functions. Creep functions are often used and will be discussed herein.

Consider the case of a material subject to constant stress:

$$\sigma(t) = \sigma_c H(t) \tag{3.13}$$

where *H(*t) is a Heaviside step function. Considering that $\dot{H}(t) = \delta(t)$, where δ is the Dirac delta function, upon substitution of Equation (3.13) into Equation (3.5), Eq. (3.1) is obtained:

$$\varepsilon(t) = \sigma_c \psi(t) \tag{3.1}$$

Simple creep tests under different constant values of stress can be used to experimentally determine the shape of the creep (compliance) function, $\psi(t)$.

Linear viscoelastic modelling

For a linear viscoelastic material subject to constant stress applied at time 0 (Figure 3.1), an elastic strain occurs instantaneously followed by the viscous strain growth. The strain growth rate slows with time. After a certain time, the strain becomes constant.

When the loading is removed (the stress in the sample becomes zero), there is an instantaneous strain recovery. The remaining deformation recovers gradually with time and the strain becomes zero if the recovery time is long enough. Figure 3.1 illustrates creep and strain recovery of a viscoelastic material.

Viscoelasticity can be described by rheological models that consist of springs and dashpots. Springs are linear elastic elements and dashpots are linear viscous elements. Various combinations of springs and dashpots simulate different material behaviour

under loading (Flügge, 1967; Christensen, 1971). Viscoelastic solids with creep response shown in Figure 3.1 are often modelled by a one-dimension model of one spring and a series of Kelvin (Voigt) elements, Figure 3.2. A Kelvin element consists of one spring and one dashpot parallel to each other, while the deformations of the spring and the dashpot under loading are assumed equal at any time.

Creep function, $\psi(t)$, for the viscoelastic solid model in Figure 3.2 can be written as:

$$\psi(t) = \psi_e + \psi_v(t) = \frac{1}{E_0} + \sum_{i=1}^{N} \frac{1}{E_i} \left\{ 1 - \exp\left(-\frac{t}{\tau_i} \right) \right\} \tag{3.14}$$

where, $\psi_e = \frac{1}{E_0}$, represents the instantaneous elastic component; and $\psi_v(t) = \sum_{i=1}^{n} \frac{1}{E_i} \left\{ 1 - \exp\left(-\frac{t}{\tau_i} \right) \right\}$ represents the viscoelastic time effects. Material constants are the elastic moduli of the springs E_0, E_i, and the viscous moduli of the dashpots, η_i, with $\tau_i = \frac{\eta_i}{E_i}$; N is the number of Kelvin elements. Since the creep function is independent of stress or strain, the model is linear. The constitutive equation for the viscoelastic model can be obtained by substituting Eq. (3.14) into Eq. (3.5),

$$\varepsilon(t) = \int_0^t \left\{ \frac{1}{E_0} + \sum_{i=1}^{N} \frac{1}{E_i} \left\{ 1 - \exp\left(-\frac{t-\tau}{\tau_i} \right) \right\} \right\} \dot{\sigma}(\tau) d\tau \tag{3.15}$$

Linear viscoplastic modelling

When plastic (permanent) deformation occurs under excessive loading, the material behaves more like a liquid. Upon loading, instantaneous elastic deformation occurs. Then the deformation (strain) keeps growing with a decreasing rate but does not approach an asymptotic value. When the load is removed, there is an instantaneous elastic strain recovery and elastic recovery over time; however, there is a residual plastic strain that remains in a solid (plastic deformation).

For viscoplastic behaviour, the creep compliance, $\psi(t)$, can be expressed by a power function:

$$\psi(t) = \psi_e + \psi_p(t) = \frac{1}{E_0} + C_0 t^{C_1} \tag{3.16}$$

where, $\psi_e = \frac{1}{E_0}$, is the instantaneous elastic component as in a viscoelastic model; $\psi_p(t) = C_0 t^{C_1}$ represents the time-dependent viscoplastic components. E_0, C_0 and C_1 are the material constants. $\psi_p(t)$, as a power law function, will not approach an asymptotic value in contrast to $\psi_v(t)$ defined by the exponential terms. The constitutive equation for the viscoplastic model can be obtained by substituting Eq. (3.16) into Eq. (3.5),

$$\varepsilon(t) = \int_0^t \left\{ \frac{1}{E_0} + C_0 (t-\tau)^{C_1} \right\} \dot{\sigma}(\tau) d\tau \tag{3.17}$$

Material constants in the creep compliance of Eq. (3.14) and Eq. (3.16) can be calibrated from creep test results by using Eq. (3.1).

3.4 Nonlinear Viscoelastic/Viscoplastic Modelling

Most polymeric materials exhibit creep behaviour that is dependent on both time and stress level. Linear viscoelastic or viscoplastic behaviour discussed in the previous section is only an idealized approximation which can be used when the stress level is low. As shown in Figure 2.4, creep behaviour can not be described by a single compliance-time curve for the tested materials. All polytheylenes tested in this study show nonlinear behaviour. Nonlinear models must be used in this general case.

The creep compliance function, ψ, is not only dependent on time but also on stress or strain. Thus the material properties depend on the stress/strain state as well. To incorporate stress/strain into the material function, ψ, a common method is to include the stress influence on the material function, ψ, utilizing the model formulations obtained

for the linear cases. Such modelling results have been presented by Krishnaswamy *et al*. (1992), Zhang and Moore (1997), and Lu *et al*. (2000).

3.4.1 Nonlinear Viscoelastic Modelling

For a nonlinear viscoelastic model, the creep compliance, $\psi(\sigma,t)$, can be written as:

$$\psi(\sigma,t)=\psi_e(\sigma)+\psi_v(\sigma,t)=\frac{1}{E_0(\sigma)}+\sum_{i=1}^{n}\frac{1}{E_i(\sigma)}\left\{1-\exp\left(-\frac{t}{\tau_i(\sigma)}\right)\right\} \quad (3.18)$$

where, in this case, $E_0(\sigma)$, $E_i(\sigma)$, and $\tau_i(\sigma)$ are all functions of stress.

The constitutive equation for the viscoelastic model can be obtained by substituting Eq. (3.18) into Eq. (3.5),

$$\varepsilon(t)=\int_0^t\left\{\frac{1}{E_0(\sigma)}+\sum_{i=1}^{n}\frac{1}{E_i(\sigma)}\left\{1-\exp\left(-\frac{t-\tau}{\tau_i(\sigma)}\right)\right\}\right\}\dot{\sigma}(\tau)d\tau \quad (3.19)$$

3.4.2 Nonlinear Viscoplastic Modelling

For a nonlinear viscoplastic model, the creep compliance, $\psi(\sigma,t)$, can be written as:

$$\psi(\sigma,t)=\psi_e(\sigma)+\psi_p(\sigma,t)=\frac{1}{E_0(\sigma)}+C_0(\sigma)t^{C_1(\sigma)} \quad (3.20)$$

where, in this case, E_0, C_0 and C_1 are all functions of stress.

The constitutive equation for the viscoplastic model can be written as,

$$\varepsilon(t)=\int_0^t\left\{\frac{1}{E_0(\sigma)}+C_0(\sigma)t^{C_1(\sigma)}\right\}\dot{\sigma}(\tau)d\tau \quad (3.21)$$

In both cases, the models have the same format as in the linear cases, respectively, but the material functions are modified to include stress influence and need to be calibrated according to stress as well as time.

3.4.3 Development of Nonlinear Modelling Procedure

In order to obtain an effective modelling solution for different materials, a simplified modelling method is used in this study. Creep tests are done for a few selected stress levels, σ_1, …, σ_n. For each stress level a separate set of material parameters is obtained. The sets of constants for all creep tests create an array of material constants. The material parameters for stresses other than the tested stresses are obtained by linear interpolation. Thus, piece-wise linear functions are assumed for the material functions. The modelling is written into a numerical procedure PMMR and can be easily repeated for any material.

The linear interpolation of material parameters can be expressed by Eq. (3.22),

$$E_0(\sigma) = E_0(\sigma_m) + \frac{\sigma - \sigma_m}{\sigma_n - \sigma_m}\left[E_0(\sigma_n) - E_0(\sigma_m)\right] \tag{3.22}$$

$$x_i(\sigma) = \frac{1}{E_i(\sigma)} = x_i(\sigma_m) + \frac{\sigma - \sigma_m}{\sigma_n - \sigma_m}\left[x_i(\sigma_n) - x_i(\sigma_m)\right] \tag{3.23}$$

where, $x_i = \frac{1}{E_i}$; σ_m and σ_n are the stresses used for model development. For σ, $\sigma_m < \sigma < \sigma_n$.

Eq. (3.24) and (3.25) describe the linear interpolation for a viscoplastic model,

$$C_0(\sigma) = C_0(\sigma_m) + \frac{\sigma - \sigma_m}{\sigma_n - \sigma_m}\left[C_0(\sigma_n) - C_0(\sigma_m)\right] \tag{3.24}$$

$$C_1(\sigma) = C_1(\sigma_m) + \frac{\sigma - \sigma_m}{\sigma_n - \sigma_m}\left[C_1(\sigma_n) - C_1(\sigma_m)\right] \tag{3.25}$$

The interpolation of the instantaneous elastic parameter, $E_0(\sigma)$, is the same as in the viscoelastic case, Eq. (3.22).

The presented two-step curve-fitting approach works well at a given stress for a material for which the model is developed. However, since each polyethylene behaves differently under a creep test, the compliance-stress relationship for models should theoretically be

redefined for each specific material. This means that for each specific material new compliance functions (of stress) representing material parameters should be created.

3.5 Nonlinear Least-Squares-Fitting

The material properties (creep compliance) can be determined from strain data by Equation (3.1).

In the case of constant stress, for viscoelastic modelling, Eq. (3.1) becomes:

$$\varepsilon(t) = \sigma_c \left\{ \frac{1}{E_0} + \sum_{i=1}^{N} \frac{1}{E_i} \left\{ 1 - \exp\left(-\frac{t}{\tau_i} \right) \right\} \right\} \tag{3.26}$$

where σ_i denotes the constant engineering stress. E_0, E_i, and τ_i are constants defining the strain-time curve at stress σ_i.

A linear least squares fitting is used by assuming values for the relaxation times τ_i in Equation (3.26), which then becomes a linear equation with variables $x_0 = \frac{1}{E_0}$, and $x_i = \frac{1}{E_i}$:

$$\varepsilon(t) = \sigma_c \left\{ x_0 + \sum_{i=1}^{N} \left\{ 1 - \exp\left(-\frac{t}{\tau_i} \right) \right\} x_i \right\} \tag{3.27}$$

We define the squared error of the fitting

$$f(x_i) = \sum_{j=1}^{M_k} \left(\varepsilon_j - \hat{\varepsilon}_j \right)^2 \tag{3.28}$$

where M_k is the number of strain measurements, $\hat{\varepsilon}_j$ is a strain measurement at time t_i, and ε_j is the corresponding theoretical value.

By setting $\frac{df}{dx_l} = 0 \ (l = 1,..., N)$, to minimize $f(x_i)$, we obtain a set of linear simultaneous equations with respect to the material parameters x_i:

$$[A]_{N\times N}\{x\}_{N\times 1} = \{F\}_{N\times 1} \tag{3.29}$$

where,

$$A_{ij} = \sum_{p=1}^{M_k}\left\{\sigma_c\left[1-\exp\left(-\frac{t_p}{\tau_j}\right)\right]\left[1-\exp\left(-\frac{t_p}{\tau_i}\right)\right]\right\},\ i, j = 1,\ldots, N \tag{3.30}$$

$$F_i = \sum_{p=1}^{M_k}\left\{\left(\hat{\varepsilon}_p - \sigma_c x_0\right)\left[1-\exp\left(-\frac{t_p}{\tau_i}\right)\right]\right\},\ i = 1,\ldots, N \tag{3.31}$$

In Equation (3.28) and (3.31), $\hat{\varepsilon}_p$ denotes strain measurement at time $t = t_p$; M_k is the number of strain measurements.

x_0 is evaluated independently directly from the instantaneous material response:

$$x_0 = \frac{1}{E_0} = \frac{\hat{\varepsilon}_0}{\sigma_c} \tag{3.32}$$

where $\hat{\varepsilon}_0$ is the instantaneous strain measured at $t = t_0 \approx 0$. The number of Kelvin elements, N, for the model is defined in the following way. First, $N = 1$ is assumed and Equation (3.30) and (3.31) are applied to define the model and a creep test is simulated by the model. If the simulated strains cause a fitting error that is within the convergence criterion, the model is acceptable. Otherwise N is increased to 2 and a model is generated and evaluated again. The procedure is repeated until a model is accepted according to the strain convergence criterion or, until N is too large, which means the model search failed and different relaxation times, τ_i, need to be assumed.

The creep test strain convergence criterion is chosen as:

$$\frac{\left(\sum_{j=1}^{M_k}\left(\frac{\varepsilon_j - \hat{\varepsilon}_j}{\hat{\varepsilon}_j}\right)^2\right)^{1/2}}{M_k} < e \tag{3.33}$$

where $\hat{\varepsilon}_j$ are the experimental measurements, ε_j are the model simulations, and e is a pre-set value to limit curve fitting error. For the polyethylene materials under study, e was selected to be between 0.01 and 0.02. A smaller value will result in a larger N. In this thesis, the models for the tested polyethylenes were obtained for e selected to be around 0.01.

For viscoplastic modelling, the creep test is defined by:

$$\varepsilon(t) = \frac{\sigma_c}{E_0} + \sigma_c C_0 t^{C_1} \tag{3.34}$$

or,

$$\ln\left(\frac{\varepsilon(t)}{\sigma_c} - \frac{1}{E_0}\right) = \ln C_0 + C_1 \ln t \tag{3.35}$$

where, σ_c denotes the constant engineering stress.

Equation (3.35) is a linear equation with $\ln C_0$ and C_1 being the variables and linear least-squares fitting can be applied.

3.6 Modelling results

Nonlinear viscoelastic and viscoplastic models are generated, respectively, for four tested materials listed in Table 2.1. The materials are: HDPE-pipe, HDPE-resin1, HDPE-resin2, and medium density polyethylene MDPE-pipe.

Creep test data used were 24 hour test results.

3.6.1 Viscoelastic Models

Five creep tests are considered for each polyethylene material for model generation. The nonlinear viscoelastic models are given in Table 3.1 (a~d). The curve-fitting converge criterion, e , was selected to be around 0.01 in calibrating the tested polyethylenes. The

number of relaxation times $N = 3$ for all the five materials at all stress levels. Satisfactory modelling results have been found at such accuracy level, as shown in Figures 3.3 and 3.5.

Simulations of creep tests using the generated models are shown in Figures 3.3, and 3.5. Excellent agreements between simulation and experiments can be observed comparing Figures 3.3 or Figures 3.5 with Figures 2.3. A two-step creep test is simulated and compared with test data for HDPE-pipe in Figure 3.7 a. Excellent agreement between simulation and experiment can be observed.

3.6.2 Viscoplastic Models

Five creep tests are considered for each polyethylene material for model generation. The nonlinear viscoplastic models are given in Table 3.2 (a~d). Simulations of creep tests using the generated models are shown in Figures 3.4, and 3.6. A two-step creep test is simulated and compared with test data for HDPE-pipe in Figure 3.7 b. The simulated curves should be considered together with the corresponding test curves shown in Chapter 2 to compare simulations with experiments. The material parameters in Eq. (3.34) are calibrated at each stress level. The numerical calculation is simpler than in the case of calibrating viscoelastic models since there are only two unknown to be decided in this case and the parameters are defined in one step while no iteration is needed.

Comparing the test results and model simulations shows that, in both viscoelastic and viscoplastic modelling cases, the simulated creep strain curves are almost identical to the test results, indicating good curve-fitting results; the simulated strain curves at stresses not used for model generation are close to the test results, indicating that the nonlinear modelling method works well for creep simulations.

3.7 Discussion

Experimental results in Chapter 2 have shown time-effects on polyethylene under loading. Deformation of polyethylene does not depend on the instantaneous loading but

on the duration (time) of the loading. In case of rate loading, the loading rate affects the material response. Polyethylene, like other polymers, exhibits viscoelastic behaviour. Creep results also indicate nonlinear viscoelastic properties for polyethylene even at low stress levels (stresses below half of tensile strength). The creep behaviour at different stress levels can not be described by a single compliance curve. The material properties are also stress dependent.

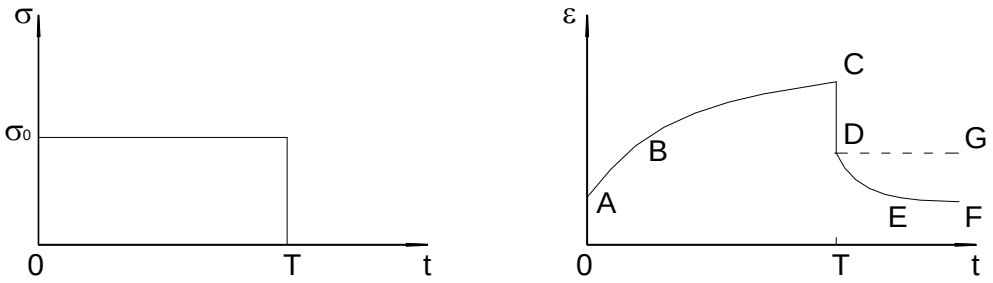
Mathematical formulation for linear viscoelasticity has been established based on superposition theory. A Kelvin element (spring-dashpot) model with exponential functions of time is one kind of such models for a viscoelastic solid. Material parameters for the model can be estimated via calibration from creep tests. Such a model can be used to model linear viscoelastic behaviour. In case of nonlinear viscoelasticity, material parameters in the constitutive equations are functions of both time and stress. In previous research, prescribed functions of time and stress had be defined for curve fitting to the test data. Such procedure can be mathematically cumbersome. In this study, a two-step curve-fitting modelling approach was proposed. In the first step, linear viscoelastic behaviour is assumed for an individual creep curve. Material parameters for a multi-Kelvin element model can be obtained from linear least-squares fitting. In the second step, piece-wise linear functions of stress were assumed for the material parameters obtained in the first step. The final nonlinear model can be applied by simply linearly interpolating the material parameters according to stress.

An alternative to the multi-Kelvin element model is the power law function model (viscoplastic model). A power law function does not asymptotically converge to a constant value as exponential functions and thus describes slightly different creep behaviour from a multi-Kelvin element model at the tail of a creep curve. The two-step curve-fitting can apply to a power law function model in the same manner.

Models for both viscoelastic (multi-Kelvin element) and viscoplastic (power law function) cases have been generated for four polyethylenes tested, shown in Chapter 2. Material parameters were calibrated using creep data. Creep simulation of creep tests and

stepped creep tests demonstrated good performance of the models at both stresses at model calibration and other stresses.

The models can be applied to other polymers and have simple forms which are convenient for numerical applications.



(a) (b)

Figure 3.1 Creep and recovery for a viscoelastic material: (a) Stress with time; (b) resulting strain variation with time

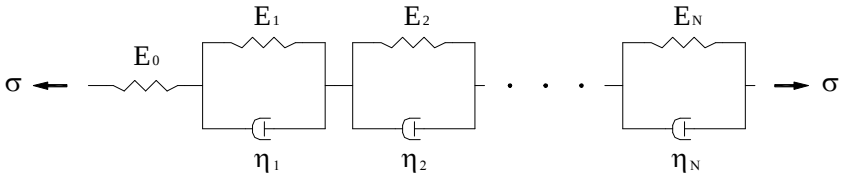


Figure 3.2. Multiple Kelvin element viscoelastic solid.

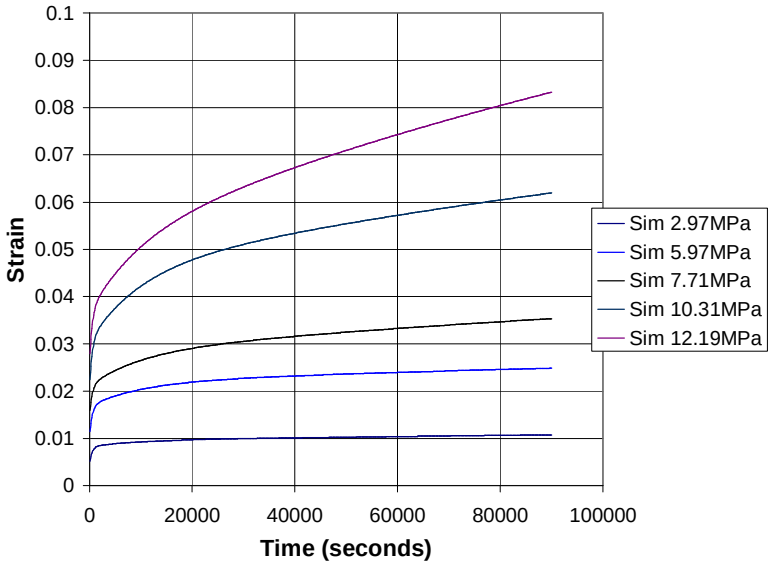


Figure 3.3 a. Simulated creep tests using viscoelastic model at modelling stresses for HDPE-pipe.

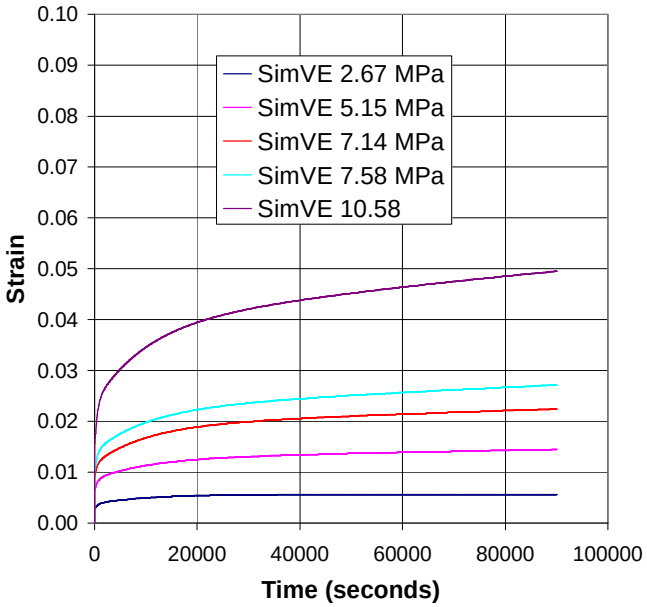


Figure 3.3 b. Simulated creep tests using viscoelastic model for HDPE-resin 1.

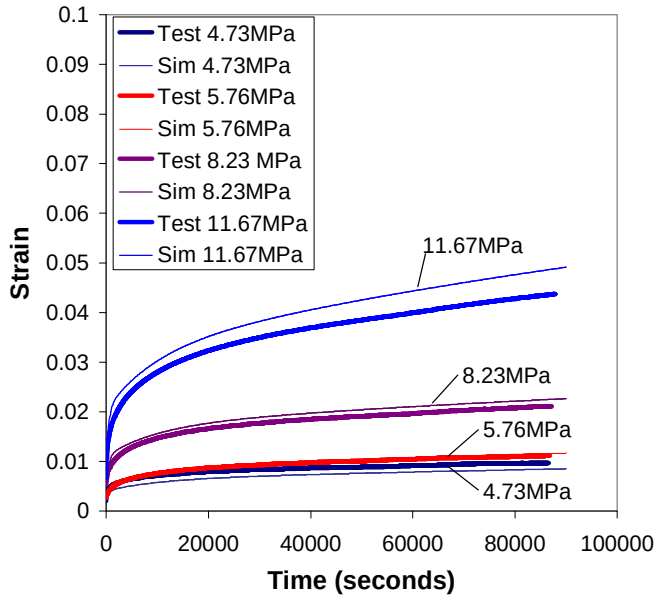


Figure 3.3 c. Simulated creep tests using viscoelastic model for HDPE-resin 2.

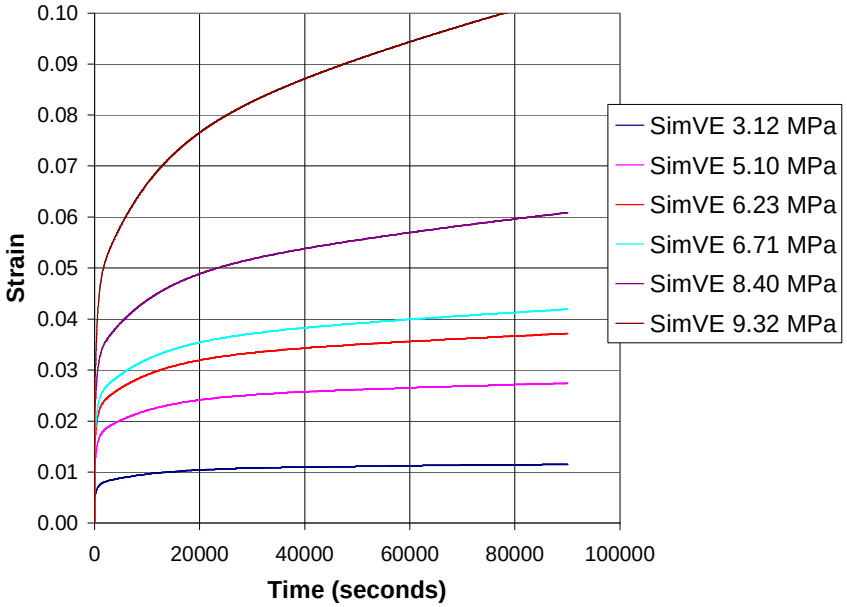


Figure 3.3 d. Simulated creep tests using viscoelastic model for MDPE-pipe.

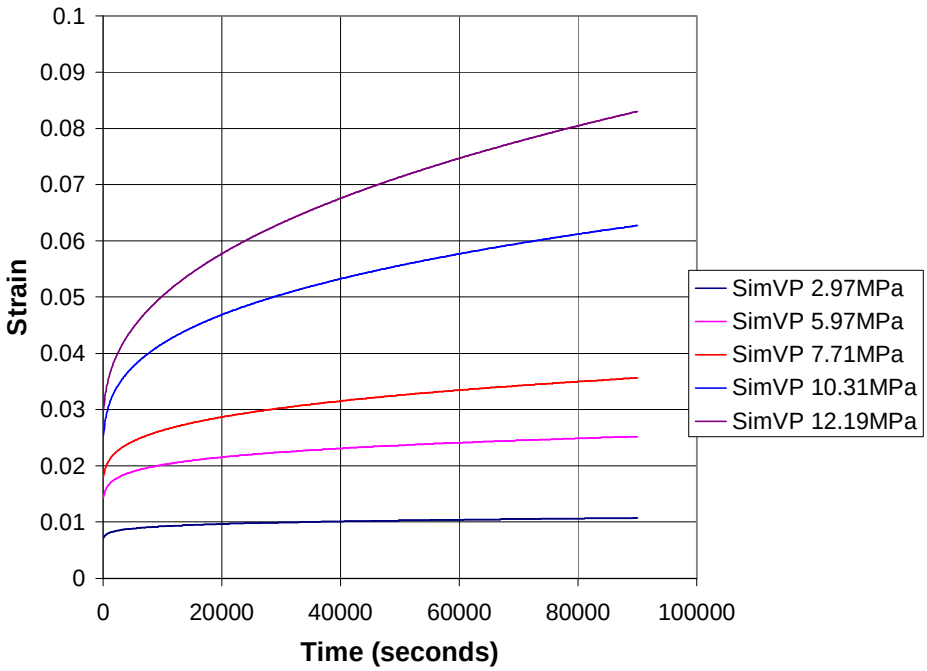


Figure 3.4 a. Simulated creep tests using viscoplastic model at modelling stresses for HDPE-pipe.

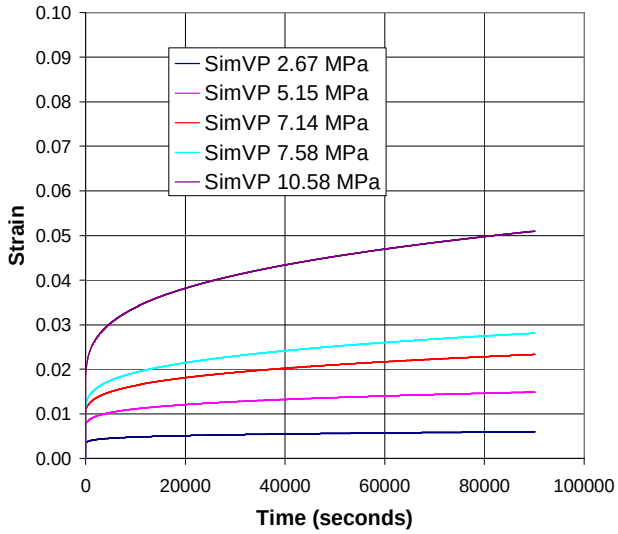


Figure 3.4 b. Simulated creep tests using viscoplastic model for HDPE-resin 1.

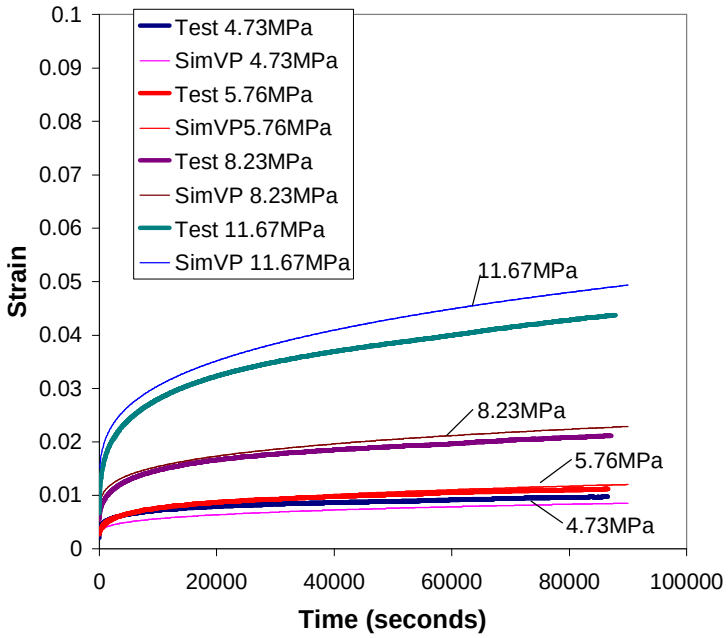


Figure 3.4 c. Simulated creep tests using viscoplastic model for HDPE-resin 2.

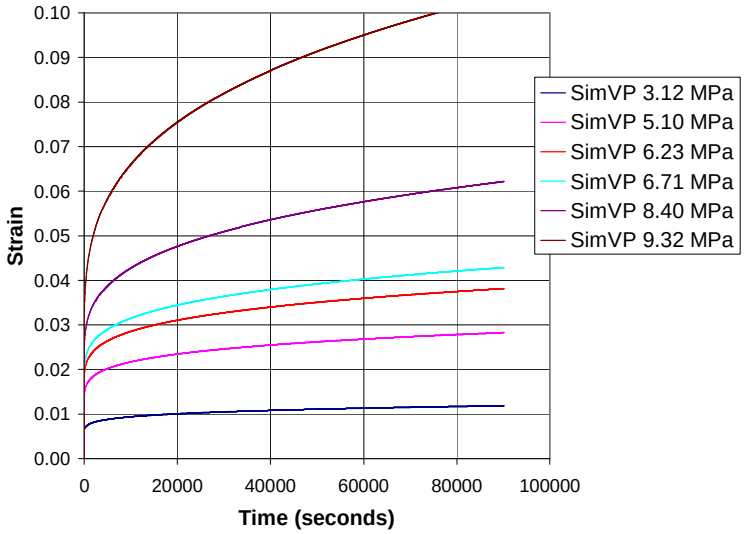


Figure 3.4 d. Simulated creep tests using viscoplastic model for MDPE-pipe.

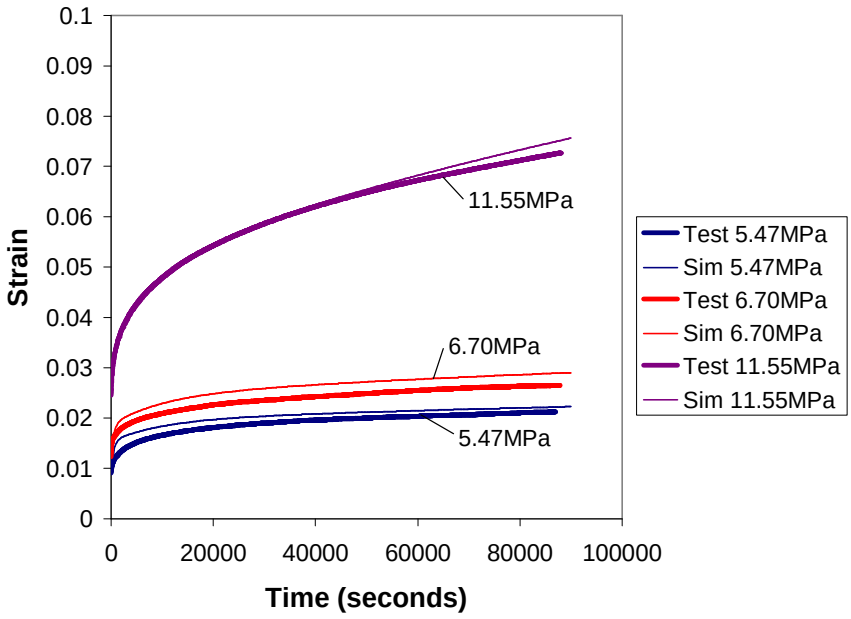


Figure 3.5 Simulated creep tests using viscoelastic model at other stresses for HDPE-pipe.

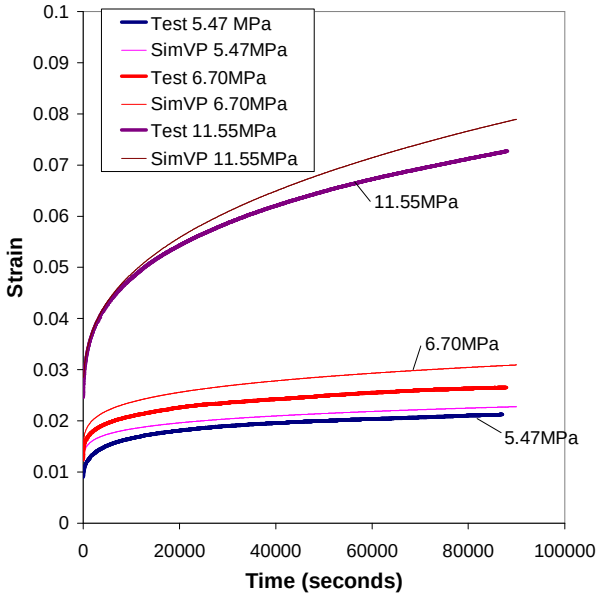


Figure 3.6 Simulated creep tests using viscoplastic model at other stresses for HDPE-pipe.

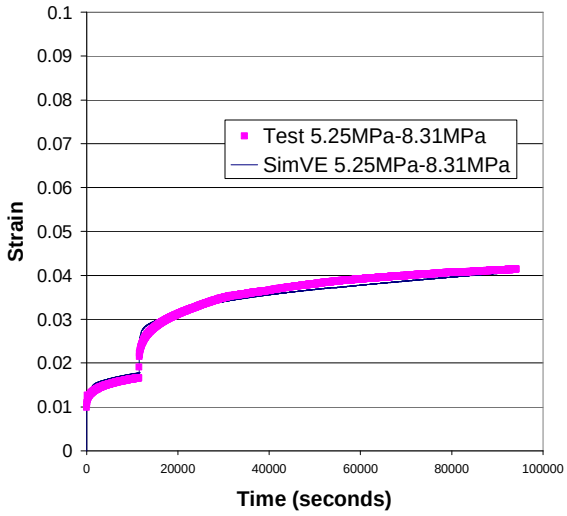


Figure 3.7 a. Two step loading creep test simulated using viscoelastic model for HDPE-pipe.

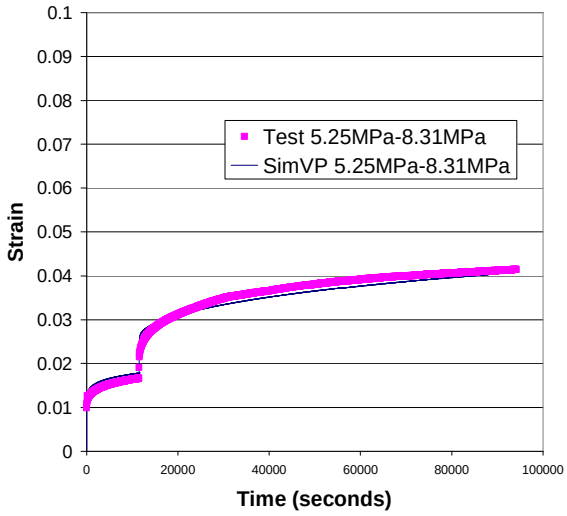


Figure 3.7 b. Two step loading creep test simulated using viscoplastic model for HDPE-pipe.

Table 3.1a Nonlinear viscoelastic model. HDPE-pipe.

Number of Kelvin elements		3		
		τ_1	τ_2	τ_3
stress	E_0	500	10000	200000
		E_1	E_2	E_3
2.97	650	797.3889	2320.3566	925.0882
5.97	580	913.5936	1212.2605	695.0461
7.71	520	1224.7911	1104.9922	385.8572
10.31	500	1034.2045	694.1084	226.4555
12.19	470	1128.4448	806.0972	140.6875

Table 3.1b Nonlinear viscoelastic model. HDPE-resin 1.

Number of Kelvin elements		3		
		τ_1	τ_2	τ_3
stress	E_0	500	10000	200000
		E_1	E_2	E_3
2.67	990	2473.5339	1434.3650	1.0e8
5.15	830	2153.6304	1319.8418	949.4745
7.14	790	2614.5305	993.8024	747.7686
7.58	770	1771.7237	959.6445	537.9008
10.58	730	1153.4563	706.9109	352.5731

Table 3.1c Nonlinear viscoelastic model. HDPE-resin 2.

Number of Kelvin elements		3		
stress	E_0	τ_1	τ_2	τ_3
		500	10000	200000
		E_1	E_2	E_3
2.68	2500	2848.6134	3650.6457	1053.8829
5.58	2300	2125.6411	1811.4240	696.3469
7.28	1700	1515.4295	1537.4866	603.9634
10.60	1200	1180.3846	1111.9421	405.5838
13.72	1100	999.9933	810.1940	145.0453

Table 3.1d Nonlinear viscoelastic model. MDPE-pipe.

Number of Kelvin elements		3		
stress	E_0	τ_1	τ_2	τ_3
		500	10000	200000
		E_1	E_2	E_3
3.12	640	1137.4169	1067.2127	1168.6089
5.10	470	804.3798	718.0750	588.7810
6.23	420	813.3631	668.5170	422.0754
8.40	410	690.8382	572.2448	224.6822
9.32	390	419.6169	363.3388	106.4053

Table 3.2a Nonlinear viscoplastic model. HDPE-pipe.

stress	E_0	C_0	C_1
2.97	650	0.4960e-3	0.1254
5.97	580	0.2956e-3	0.1872
7.71	520	0.1232e-3	0.2706
10.31	500	0.1130e-3	0.3145
12.19	470	0.5517e-4	0.3893

Table 3.2b Nonlinear viscoplastic model. HDPE-resin 1.

stress	E_0	C_0	C_1
2.67	990	0.1242e-3	0.2012
5.15	830	0.8856e-4	0.2587
7.14	790	0.6390e-4	0.3019
7.58	770	0.7807e-4	0.3007
10.58	730	0.1298e-3	0.2876

Table 3.2c Nonlinear viscoplastic model. HDPE-resin 2.

stress	E_0	C_0	C_1
2.68	2500	0.5515e-4	0.2517
5.58	2300	0.6842e-4	0.2753
7.28	1700	0.1143e-3	0.2485
10.60	1200	0.1346e-3	0.2623
13.72	1100	0.7354e-4	0.3642

Table 3.2d Nonlinear viscoplastic model. MDPE-pipe.

stress	E_0	C_0	C_1
3.12	640	0.2341e-3	0.1983
5.10	470	0.2927e-3	0.2153
6.23	420	0.2385e-3	0.2414
8.40	410	0.1946e-3	0.2839
9.32	390	0.3080e-3	0.2918

Chapter 4

Model Application – Rate Effects

4.1 One-Dimensional Constitutive Equations

In this chapter, applications of one-dimensional models are discussed. Unlike conventional linear elastic or plastic constitutive models, which describe only strain and stress relationships either in full stress/strain forms or in incremental forms, the models developed for a viscoelastic/viscoplastic case have time integral forms that relate stress, strain, and their time rates. This is how time effects are included in the models. The constitutive equations for materials with time-dependent properties are not simple functions. The nonlinear viscoelastic/viscoplastic models generated in Chapter 3 have integral forms as Eq. (3.4) and (3.5). Eq. (3.4) expresses stress response to strain rate effects over time; whereas Eq. (3.5) expresses strain response to stress rate effects over time. The models are defined in this study by giving specific creep compliance functions for the material under investigation. This chapter deals with the numerical application of constitutive equations of integral form models and the time/rate effects defined by the models are studied.

The constitutive equations for the models that have defined by creep compliance are re-written here,

$$\varepsilon(t) = \int_0^t \psi(t-\tau)\dot{\sigma}(\tau)d\tau \tag{4.1}$$

where strain $\varepsilon(t)$ is expressed as accumulated responses to stress rate, $\dot{\sigma}(\tau)$, over time through a creep compliance, $\psi(t-\tau)$. In Eq. (4.1), a linear viscoelastic/viscoplastic formulation is used as an example, where the creep compliance is only function of time. In general, creep compliance is function of time and stress for the models generated in Chapter 3.

As revealed by Eq. (3.14) and (3.16), the creep functions for the models under study can be separated into two parts: an instantaneous elastic part and a viscous (either viscoelastic or viscoplastic) part. Then Eq. (4.1) can be written as,

$$\varepsilon(t) = \int_0^t \left[\psi^e(\tau) + \psi^v(t-\tau)\right]\dot{\sigma}(\tau)d\tau \tag{4.2}$$

Or, for viscoplastic models,

$$\varepsilon(t) = \int_0^t \left[\psi^e(\tau) + \psi^p(t-\tau)\right]\dot{\sigma}(\tau)d\tau \tag{4.3}$$

For the application of Eq. (4.2) or (4.3), stress rates over loading time are the input and strain history is the output. The application of constitutive equations, Eq. (4.1) to (4.3), is discussed for different loading cases.

In the case of creep, stress, as the loading result, is given by a Heaviside step function:

$$\sigma(t) = \sigma_c H(t) \tag{4.4}$$

Then stress rate for creep is,

$$\dot{\sigma}(t) = \sigma_c \dot{H}(t) = \sigma_c \delta(t) \tag{4.5}$$

where δ is the Dirac delta function. If a constant stress is applied to the material at t=0, then,

$$\dot{\sigma}(t) = \sigma_c \delta(0) \tag{4.6}$$

When stress rate is defined by Eq. (4.6), Eq. (4.1) becomes,

$$\varepsilon(t) = \int_0^t \psi(t-\tau)\sigma_c \delta(0)d\tau = \sigma_c \psi(t) \tag{4.7}$$

Eq. (4.7) is the creep equation. For viscoelastic models Eq. (4.7) is written as,

$$\varepsilon(t) = \sigma_c \left\{ \frac{1}{E_0} + \sum_{i=1}^{n} \frac{1}{E_i}\left[1 - \exp\left(-\frac{t}{\tau_i}\right)\right]\right\} \tag{4.8}$$

For viscoplastic models Eq. (4.7) is written as,

$$\varepsilon(t) = \sigma_c \left\{ \frac{1}{E_0} + C_0 (t)^{C_1} \right\} \tag{4.9}$$

In Eqs. (4.8) and (4.9), stress and strain are related by simple functions and no integration has to be conducted.

Viscoelastic/viscoplastic models that have the form of Eq. (4.8) or Eq. (4.9) have been used as material models for general loading cases (Krishnashwamy, *et al.*, 1992; Zhang *et al.*, 1997), even though the condition to obtain the equations is that the stress is constant. The advantage of doing so is that the stress-strain relationship is largely simplified. In such a case, the constitutive equation becomes,

$$\varepsilon(t) = \sigma\psi(t) \tag{4.10}$$

By differentiation with respect to time on both sides of Eq. (4.10), assuming that creep compliance is independent of stress for simplification, we obtain,

$$\dot{\varepsilon}(t) = \dot{\sigma}\psi(t) \tag{4.11}$$

Obviously, Eqs. (4.10) and (4.11) do not remain true for a general load case. The stress-strain relationship has to be defined by the integral in Eqs. (4.2) and (4.3) so that the time (stress) effect is included for the entire loading history.

Numerical integration can be used on computing Eqs. (4.2) or (4.3) for viscoelastic and viscoplastic models, respectively. Eq. (4.2) is used below to demonstrate the procedure.

When the load history includes stress changing with time, it can be approximated as a series of small time steps in which constant values of stress are applied . The stress σ_1 is applied abruptly at time t_0 and held constant till time $t_1 = t_0 + \Delta t_1$. At time t_1 , the strain is equal to:

$$\varepsilon_1 = \left[\psi^e(\sigma_1) + \psi^v(\sigma_1, t_1 - t_0)\right]\sigma_1 \tag{4.12}$$

At time t_1 a new stress σ_2 is applied in a jump and held constant till time t_2 . At time t_1 the creep behaviour is calculated as if at this instant stress σ_1 is removed and at the same time stress σ_2 is applied. Both stresses are considered as independent and calculation of strains at time t_2 includes strain recovery from time t_1 to t_2:

$$\varepsilon_2 = \psi^e(\sigma_2)\sigma_2 + \psi^v(\sigma_1, t_2 - t_0)\sigma_1 + \left[\psi^v(\sigma_2, t_2 - t_1)\sigma_2 - \psi^v(\sigma_1, t_2 - t_1)\sigma_1\right] \tag{4.13}$$

At the end of the next step, $t_3 = t_2 + \Delta t_3$, σ_2 is removed while σ_3 is applied and held constant, and strain is calculated as:

$$\varepsilon_3 = \psi^e(\sigma_3)\sigma_3 + \psi^v(\sigma_1, t_3 - t_0) + \left[\psi^v(\sigma_2, t_3 - t_1)\sigma_2 - \psi^v(\sigma_1, t_3 - t_1)\sigma_1\right] + \left[\psi^v(\sigma_3, t_3 - t_2)\sigma_3 - \psi^v(\sigma_2, t_3 - t_2)\sigma_2\right] \tag{4.14}$$

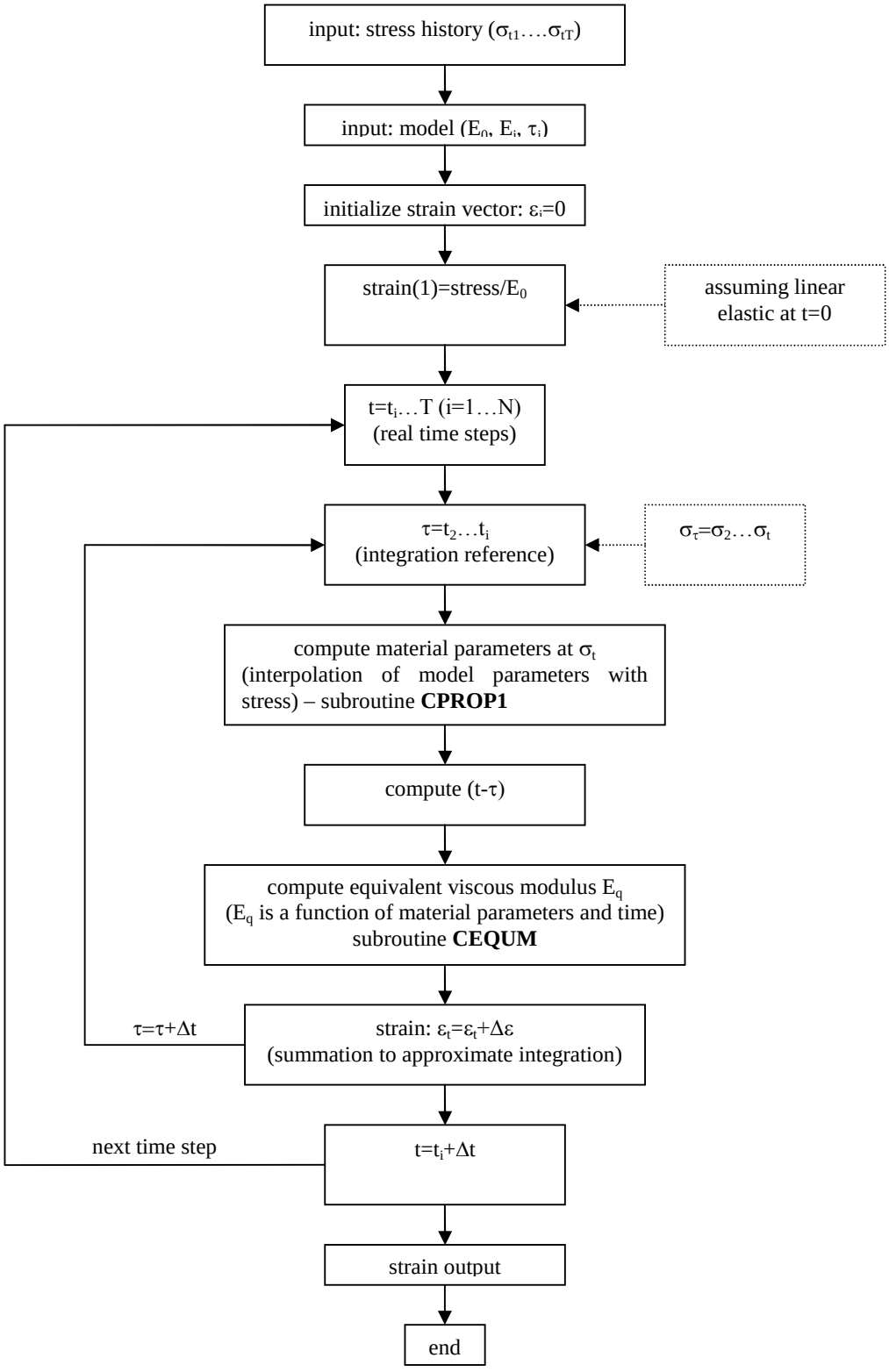
The time integration can be written for the N time steps in a summation form assuming small time steps:

$$\varepsilon_k = \psi^e(\sigma_k) + \sum_{j=1}^{k}\left\{\psi^v(\sigma_j, t_k - t_{j-1})\sigma_j - \psi^v(\sigma_{j-1}, t_k - t_{j-1})\sigma_{j-1})\right\} \tag{4.15}$$

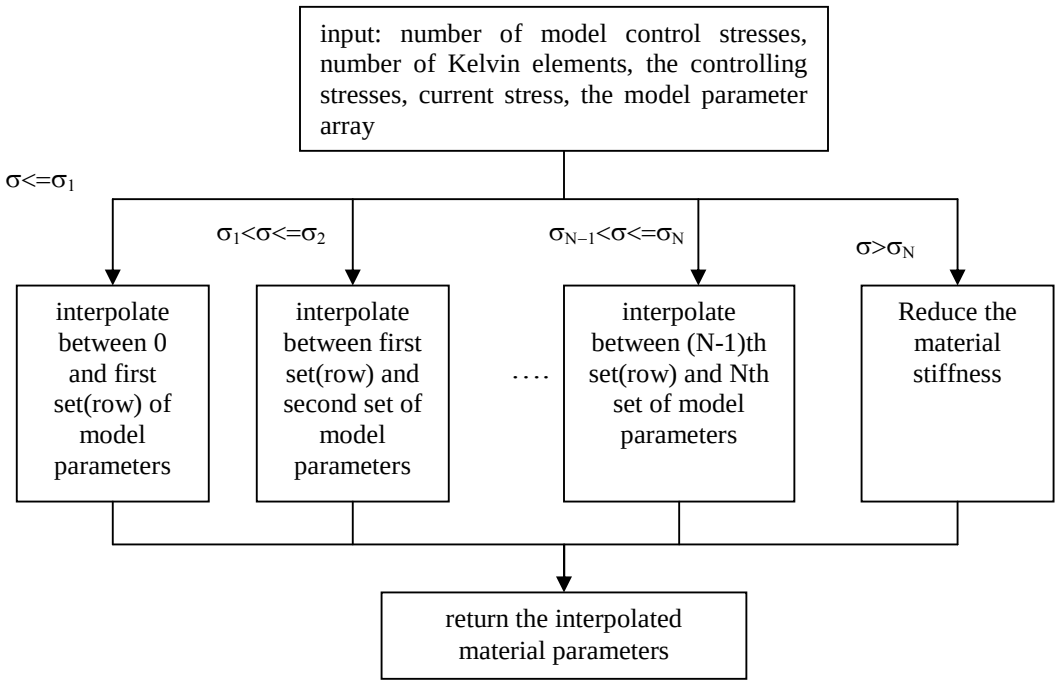
in which, time t is divided into k discrete intervals. The stress values at the start and end of the j^{th} time interval, (t_{j-1}, t_j), are σ_{j-1} and σ_j, respectively. The strain at the end of the k^{th} time interval is ε_k.

Given a stress history of a loading response, the strain response of the tested material can be calculated from the summation in equation (4.15). The flow chart is illustrated on the following page.

Flow chart of calculating one dimensional loading response.



Subroutine CPROP1: compute material parameters at given stress for integration



For viscoelastic models, by substitution of the model, equation (4.12) is written as:

$$\varepsilon_k = \frac{1}{E_0[\sigma_k]}\sigma_k + \sum_{j=1}^{k}\left\{\sum_{i=1}^{n}\frac{1}{E_i[\sigma_j]}\left[1-\exp\left(-\frac{t_k - t_{j-1}}{\tau_i}\right)\right]\sigma_j - \sum_{i=1}^{n}\frac{1}{E_i[\sigma_{j-1}]}\left[1-\exp\left(-\frac{t_k - t_{j-1}}{\tau_i}\right)\right]\sigma_{j-1}\right\} \quad (4.16)$$

and, for the first three time steps,

$$\varepsilon_1 = \frac{1}{E_0[\sigma_1]}\sigma_1 + \left\{\sum_{i=1}^{n}\frac{1}{E_i[\sigma_1]}\left[1-\exp\left(-\frac{t_1 - t_0}{\tau_i}\right)\right]\sigma_1\right\}$$

$$\varepsilon_2 = \frac{1}{E_0[\sigma_1]}\sigma_2 + \left\{\sum_{i=1}^{n}\frac{1}{E_i[\sigma_1]}\left[1-\exp\left(-\frac{t_2 - t_0}{\tau_i}\right)\right]\sigma_1\right\} + \left\{\sum_{i=1}^{n}\frac{1}{E_i[\sigma_2]}\left[1-\exp\left(-\frac{t_2 - t_1}{\tau_i}\right)\right]\sigma_2 - \sum_{i=1}^{n}\frac{1}{E_i[\sigma_1]}\left[1-\exp\left(-\frac{t_2 - t_1}{\tau_i}\right)\right]\sigma_1\right\}$$

$$\varepsilon_3 = \frac{1}{E_0[\sigma_1]}\sigma_3 + \left\{\sum_{i=1}^{n}\frac{1}{E_i[\sigma_1]}\left[1-\exp\left(-\frac{t_3 - t_0}{\tau_i}\right)\right]\sigma_1\right\} + \left\{\sum_{i=1}^{n}\frac{1}{E_i[\sigma_2]}\left[1-\exp\left(-\frac{t_3 - t_1}{\tau_i}\right)\right]\sigma_2 - \sum_{i=1}^{n}\frac{1}{E_i[\sigma_1]}\left[1-\exp\left(-\frac{t_3 - t_1}{\tau_i}\right)\right]\sigma_1\right\} + \left\{\sum_{i=1}^{n}\frac{1}{E_i[\sigma_3]}\left[1-\exp\left(-\frac{t_3 - t_2}{\tau_i}\right)\right]\sigma_3 - \sum_{i=1}^{n}\frac{1}{E_i[\sigma_2]}\left[1-\exp\left(-\frac{t_3 - t_2}{\tau_i}\right)\right]\sigma_2\right\}$$

For viscoplastic models, Eq. (4.12) is written as:

$$\varepsilon_k = \frac{1}{E_0[\sigma_k]}\sigma_k + \sum_{j=1}^{k}\left\{\left\{C_0[\sigma_j](t_k - t_{j-1})^{C_1[\sigma_j]}\right\}\sigma_j - \left\{C_0[\sigma_{j-11}](t_k - t_{j-1})^{C_1[\sigma_{j-1}]}\right\}\sigma_{j-1}\right\} \quad (4.17)$$

and, for the first three time steps,

$$\varepsilon_1 = \left\{\frac{1}{E_0[\sigma_1]} + C_0[\sigma_1](t)^{C_1[\sigma_1]}\right\}\sigma_1$$

$$\varepsilon_2 = \frac{1}{E_0[\sigma_1]}\sigma_1 + \left\{C_0[\sigma_1](t_2 - t_0)^{C_1[\sigma_1]}\right\}\sigma_1 + \left\{\left\{C_0[\sigma_2](t_2 - t_1)^{C_1[\sigma_2]}\right\}\sigma_2 - \left\{C_0[\sigma_1](t_2 - t_1)^{C_1[\sigma_2]}\right\}\sigma_1\right\}$$

$$\varepsilon_3 = \frac{1}{E_0[\sigma_1]}\sigma_1 + \left\{C_0[\sigma_1](t_2 - t_0)^{C_1[\sigma_1]}\right\}\sigma_1 + \left\{\left\{C_0[\sigma_2](t_3 - t_1)^{C_1[\sigma_2]}\right\}\sigma_2 - \left\{C_0[\sigma_1](t_3 - t_1)^{C_1[\sigma_2]}\right\}\sigma_1\right\} + \left\{\left\{C_0[\sigma_3](t_3 - t_2)^{C_1[\sigma_2]}\right\}\sigma_3 - \left\{C_0[\sigma_2](t_3 - t_2)^{C_1[\sigma_2]}\right\}\sigma_2\right\}$$

Note, in Eqs. (4.13), (4.16), or (4.17), the stress and strain relationship can not be inversed directly; strain is calculated from the defined stress history. When the strain history is given, the corresponding integral formulation for stress (relaxation function) has to be used in calculating the stress responses.

4.2 Numerical results

The HDPE-pipe material samples are loaded to 25.0 MPa at constant stress rates, using the models generated for HDPE-pipe in Chapter 3. The stress rates are: 1.0 MPa/s, 0.1 MPa/s, 0.05 MPa/s, and 0.01 MPa/s. The strain response is calculated for each case. The simulation results, using the viscoelastic model generated in Chapter 3, and the corresponding experiment results are drawn in Figures 4.1 to 4.4.

Complex tests on HDPE-pipe shown in Chapter 2 were also simulated using the nonlinear viscoelastic model. The simulated curves for two complex tests are shown in Figure 4.5 and Figure 4.6, respectively, together with experiment results.

Same simulations results can be obtained using the viscoplastic model generated in Chapter 3 for HDPE-pipe. The simulations for constant load (stress) rate tests are shown in Figures 4.7 to 4.10 for stress rates 1.0 MPa/s, 0.1 MPa/s, 0.05 MPa/s, and 0.01 MPa/s, respectively. The results for the complex tests are shown in Figures 4.10 and 4.11, respectively.

4.3 Discussion

In this chapter, material models generated for HDPE-pipe are used for one-dimensional load applications. Constant stress rate tension tests and combination of constant stress rate and creep tests are simulated.

Rate tension tests have been conducted using an MTS tester. The tester is load monitored: tension forces are programmed and are inputs to the control system while strain in the tested specimen is measured by a clip-on strain gauge and recorded with time as strain response history.

For constant stress rate tests, constant tension rates are used. The stress is approximated to be constant when the material deformation is small and can be calculated as engineering stress. For constant strain rate tests, an initial tension is picked up by the tester and is adjusted later by the strain reading to achieve a constant strain rate. The strain gauge measures displacement between the knife edges and is calibrated as (engineering) strain.

Polyethylene has much lower stiffness than steel and has larger deformation than steel under similar levels of loading. The required load levels are low for polyethylene compared to the loading accuracy of the tester. For high stress (load) rate (e.g. 1.0 MPa/s), the loading rate was too fast resulting in excessive tensions within short time so that the loading rate was not properly monitored (as shown in Figures 4.1 (a) and 4.7 (a)). For low stress rate, the force was too small and the tester did not respond properly. For the same reason, during constant strain rate tests, very often the initial tension was too large and the resulted strain would jump over the specified rate. The adjustment on the force was so large that the force oscillated. The right deformation could not be found and this causes errors.

For each rate test shown, the strain responses from simulations are close to those from experiments, even though there is a slight discrepancy between the simulation and test

results in the stress-strain relationship, Figures 4.1 to 4.4. The discrepancy may have resulted from the error in stress monitoring in the experiments, which can be observed in the stress-time graphs. A more accurate tension tester or monitor systems should be used to get more reliable experimental data.

For the same reason, discrepancy can be observed in the strain simulations for the complex tests. However, the simulations still gave acceptable results because the constant loading (creep) procedures are easier to be monitored than the loading rate procedures and the creep responses in between the short time rate loading dominate the overall material responses. The complex test simulation attests good performance of the material model.

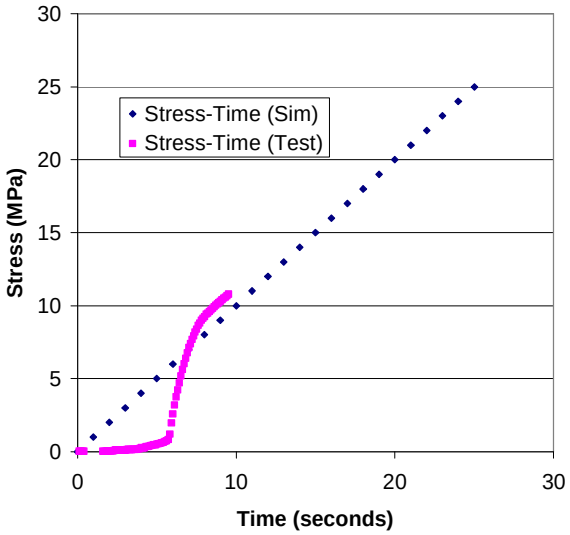


Figure 4.1 (a) Constant load rate (1.0MPa/s) simulations on HDPE-pipe using nonlinear viscoelastic model: stress vs. time.

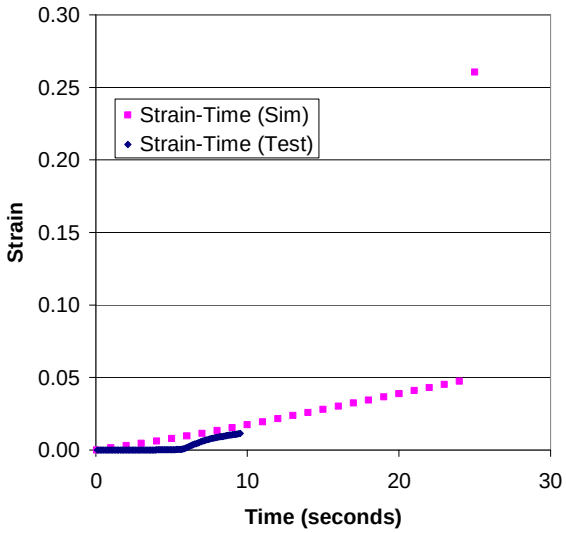


Figure 4.1 (b) Constant load rate (1.0 MPa/s) simulations on HDPE-pipe using nonlinear viscoelastic model: strain vs. time.

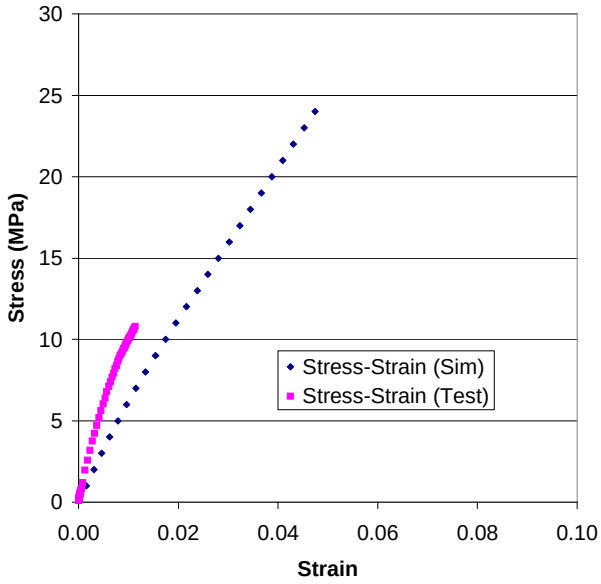


Figure 4.1 (c) Constant load rate (1.0 MPa/s) simulations on HDPE-pipe using nonlinear viscoelastic model: stress vs. strain.

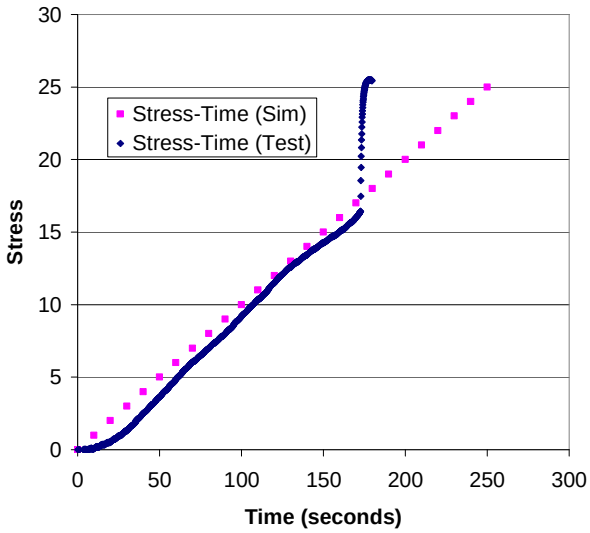


Figure 4.2 (a) Constant load rate (0.1MPa/s) simulations on HDPE-pipe using nonlinear viscoelastic model: stress vs. time.

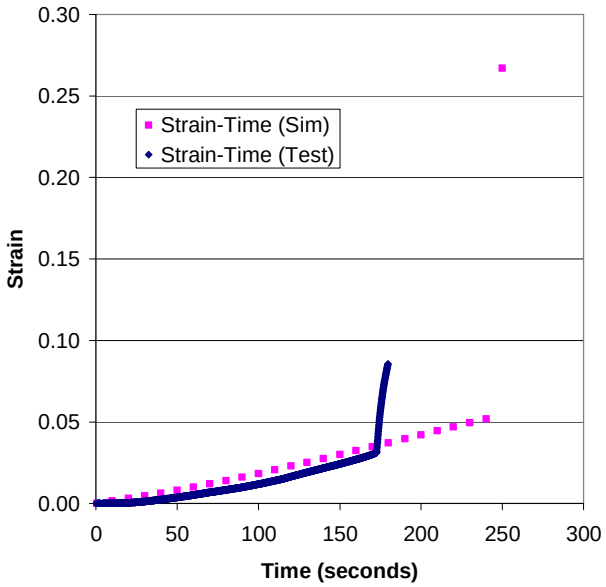


Figure 4.2 (b) Constant load rate (0.1 MPa/s) simulations on HDPE-pipe using nonlinear viscoelastic model: strain vs. time.

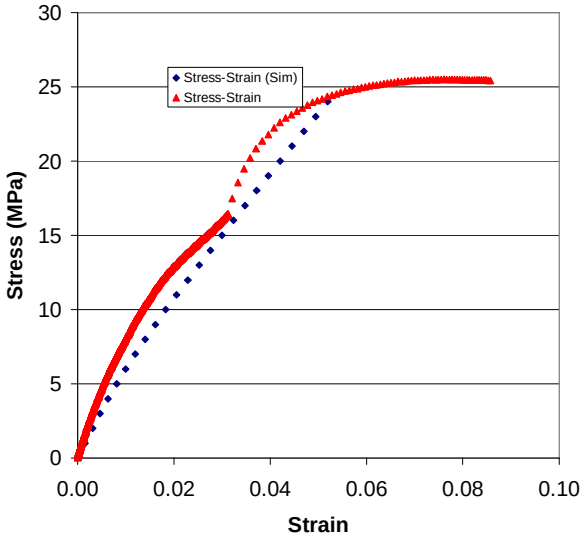


Figure 4.2 (c) Constant load rate (0.1MPa/s) simulations on HDPE-pipe using nonlinear viscoelastic model: stress vs. strain.

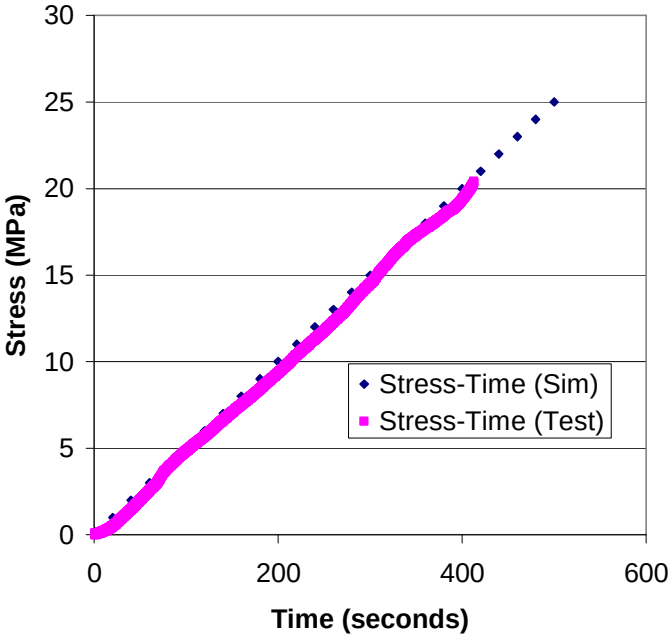


Figure 4.3 (a) Constant load rate (0.05MPa/s) simulations on HDPE-pipe using nonlinear viscoelastic model: stress vs. time.

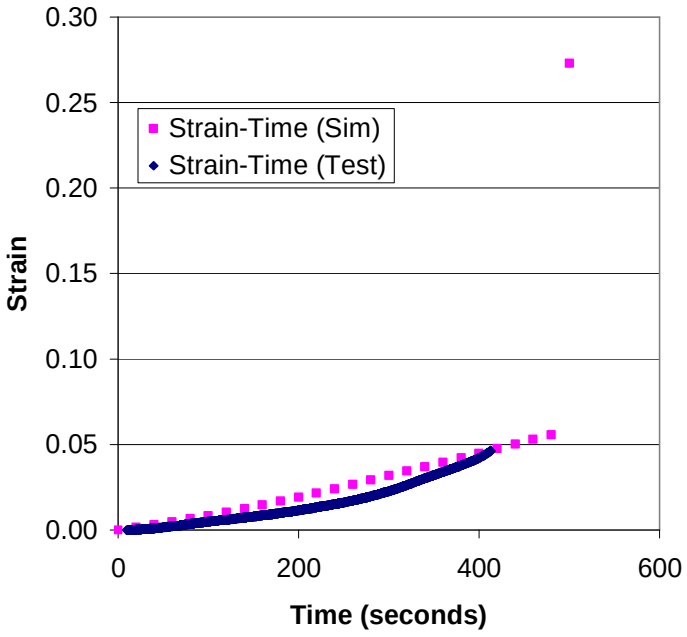


Figure 4.3 (b) Constant load rate (0.05MPa/s) simulations on HDPE-pipe using nonlinear viscoelastic model: strain vs. time.

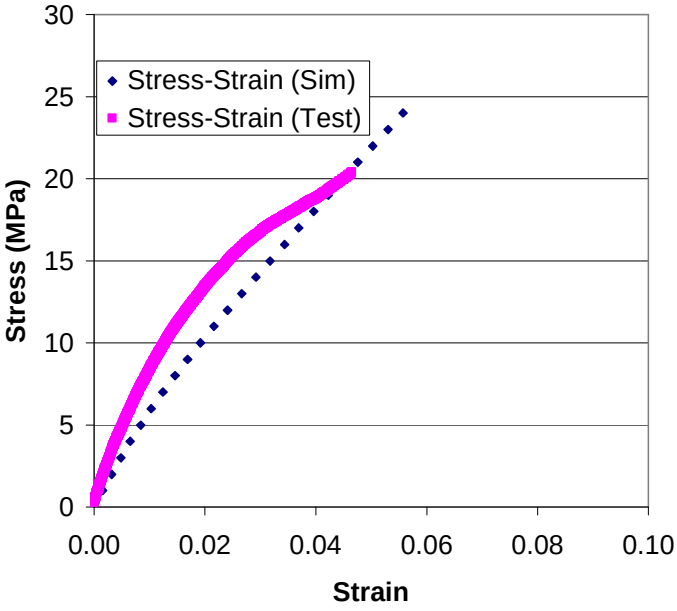


Figure 4.3 (c) Constant load rate (0.05MPa/s) simulations on HDPE-pipe using nonlinear viscoelastic model: stress vs. strain.

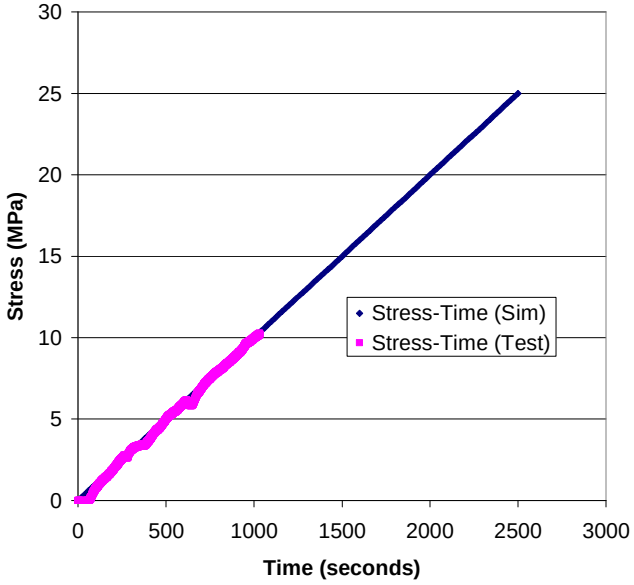


Figure 4.4 (a) Constant load rate (0.01MPa/s) simulations on HDPE-pipe using nonlinear viscoelastic model: stress vs. time.

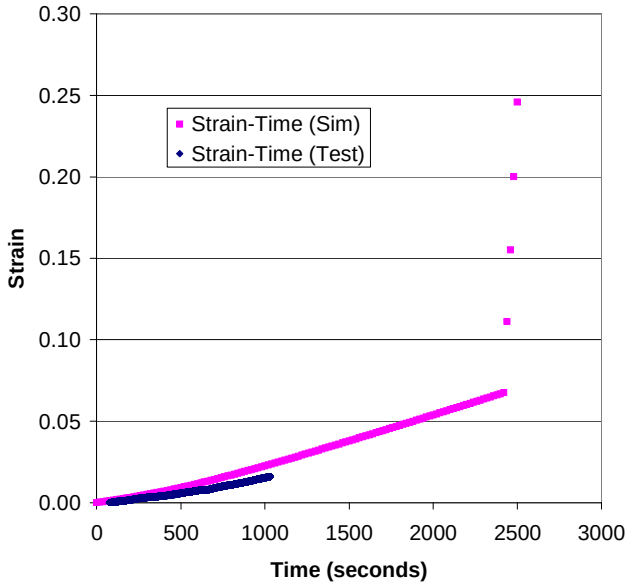


Figure 4.4 (b) Constant load rate (0.01MPa/s) simulations on HDPE-pipe using nonlinear viscoelastic model: strain vs. time.

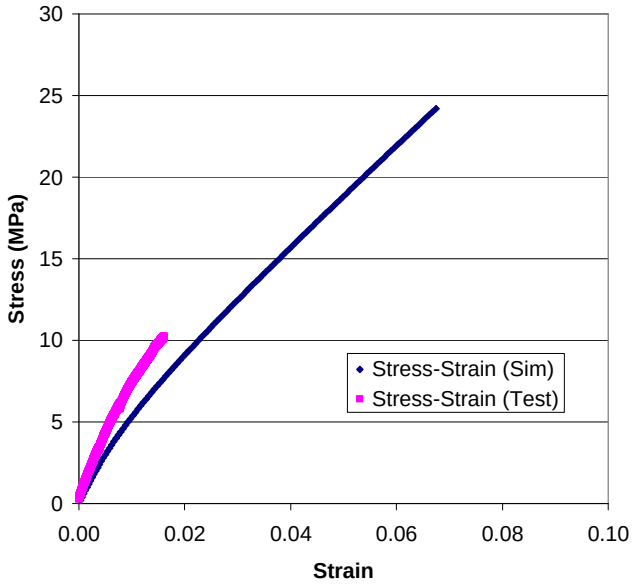


Figure 4.4 (c) Constant load rate (0.01MPa/s) simulations on HDPE-pipe using nonlinear viscoelastic model: stress vs. strain.

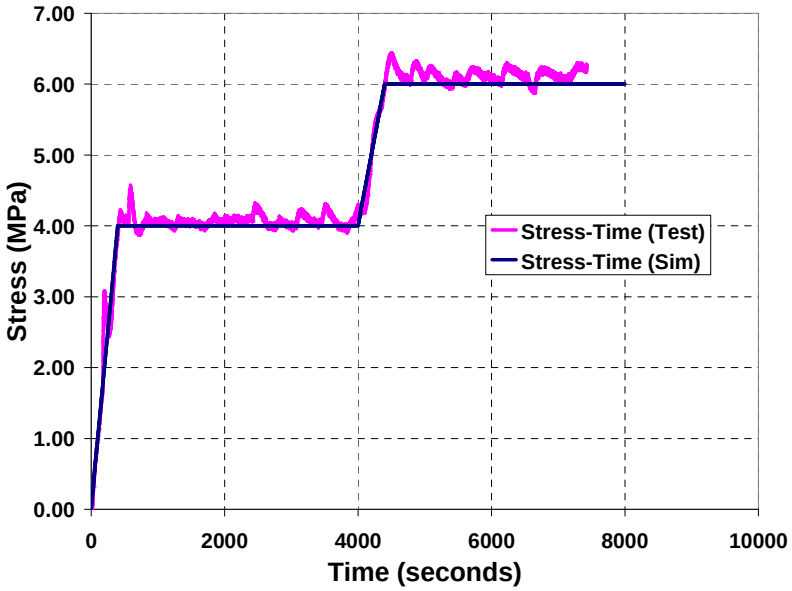


Figure 4.5 (a) Simulations of complex-test-1 on HDPE-pipe using nonlinear viscoelastic model: stress vs. time.

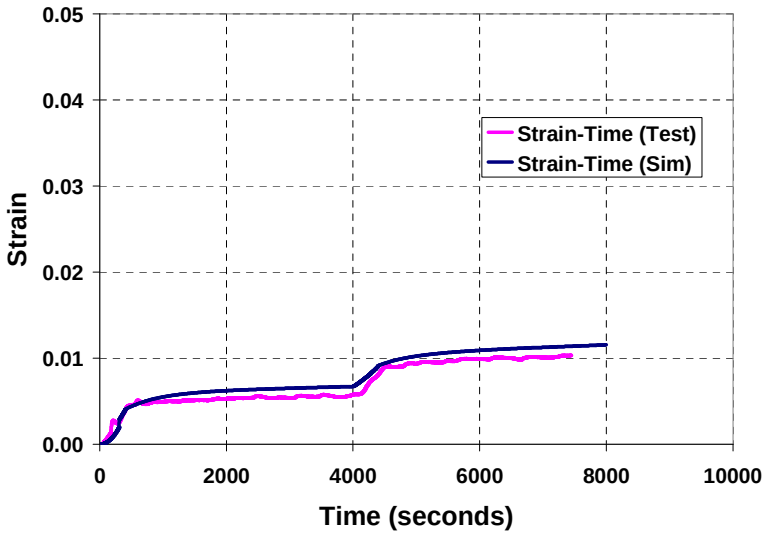


Figure 4.5 (b) Simulations of complex-test-1 on HDPE-pipe using nonlinear viscoelastic model: strain vs. time.

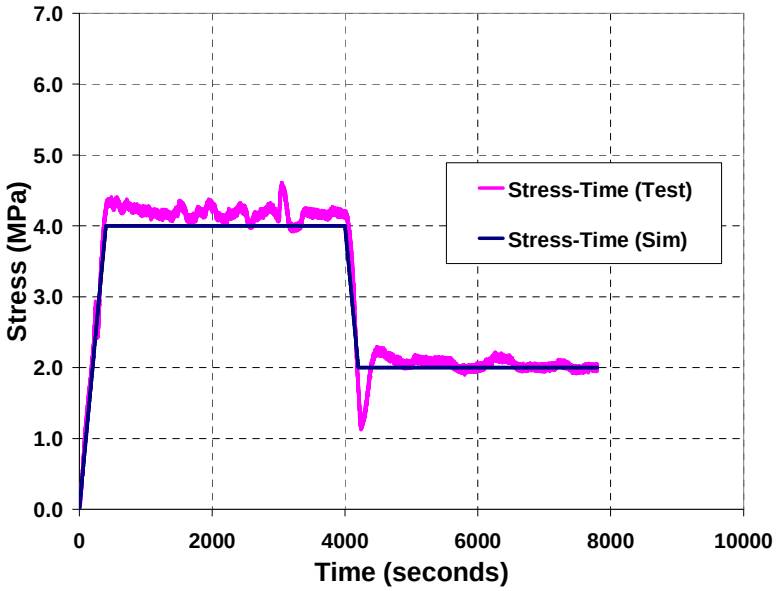


Figure 4.6 (a) Simulations of complex-test-2 on HDPE-pipe using nonlinear viscoelastic model: stress vs. time.

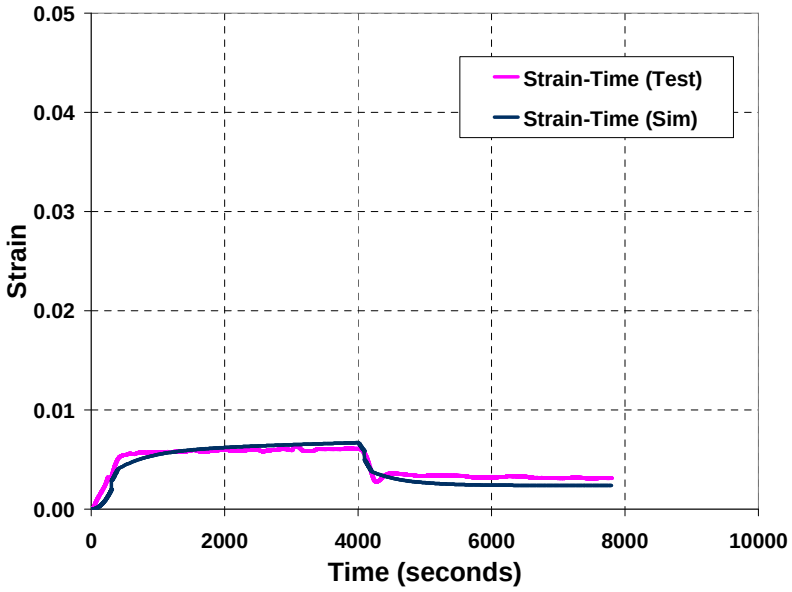


Figure 4.6 (b) Simulations of complex-test-2 on HDPE-pipe using nonlinear viscoelastic model: strain vs. time.

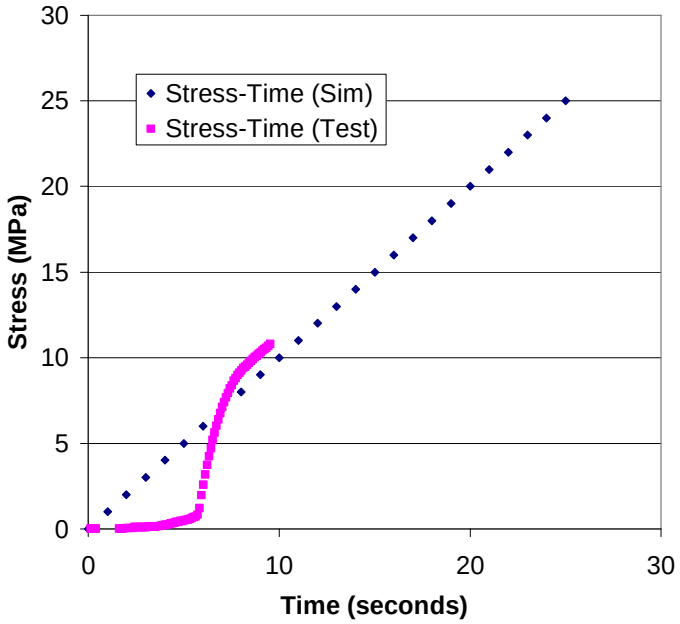


Figure 4.7 (a) Constant load rate (1.0MPa/s) simulations on HDPE-pipe using nonlinear viscoplastic model: stress vs. time.

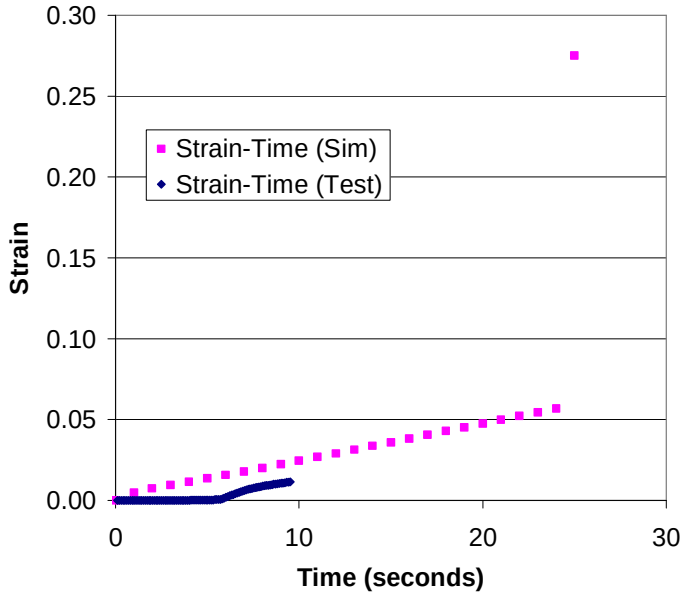


Figure 4.7 (b) Constant load rate (1.0MPa/s) simulations on HDPE-pipe using nonlinear viscoplastic model: strain vs. time.

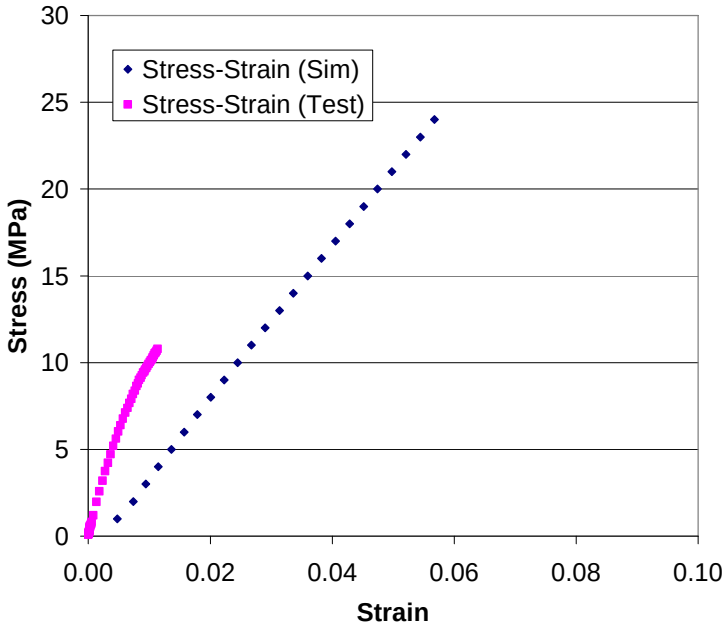


Figure 4.7 (c) Constant load rate (1.0MPa/s) simulations on HDPE-pipe using nonlinear viscoplastic model: stress vs. strain.

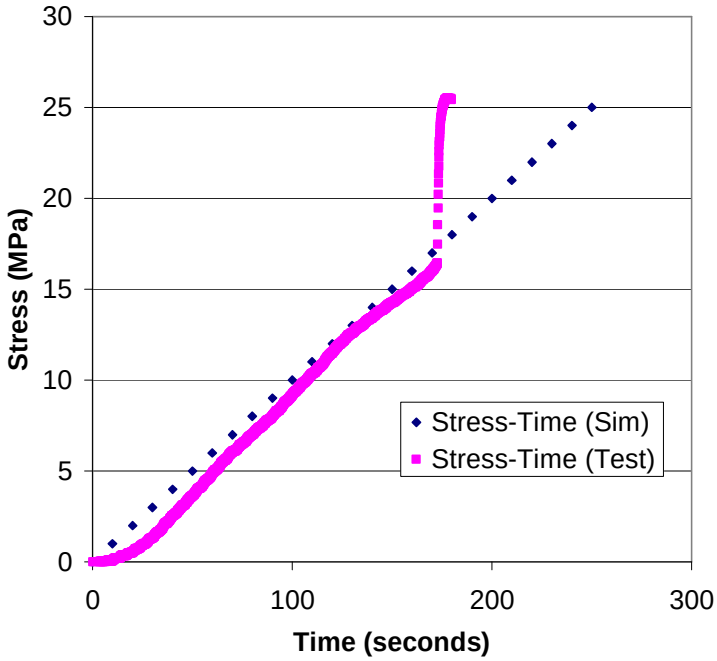


Figure 4.8 (a) Constant load rate (0.1 MPa/s) simulations on HDPE-pipe using nonlinear viscoplastic model: stress vs. time.

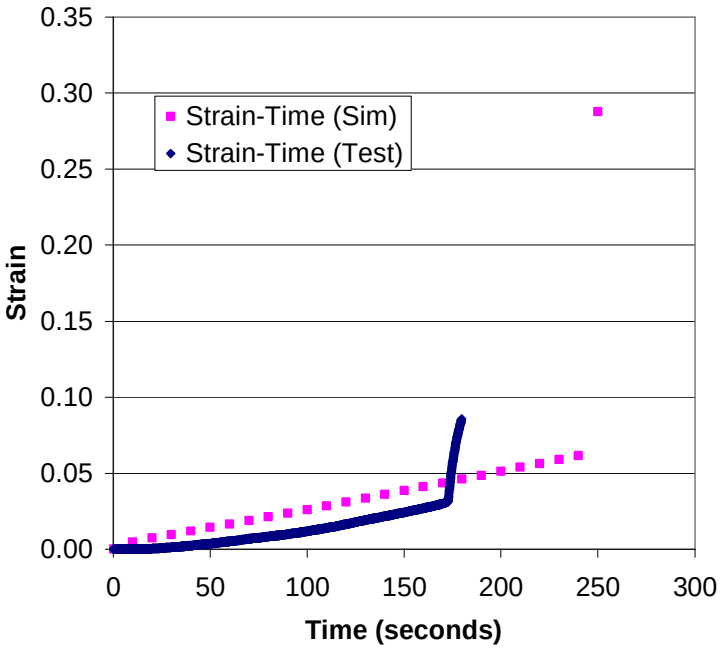


Figure 4.8 (b) Constant load rate (0.1 MPa/s) simulations on HDPE-pipe using nonlinear viscoplastic model: strain vs. time.

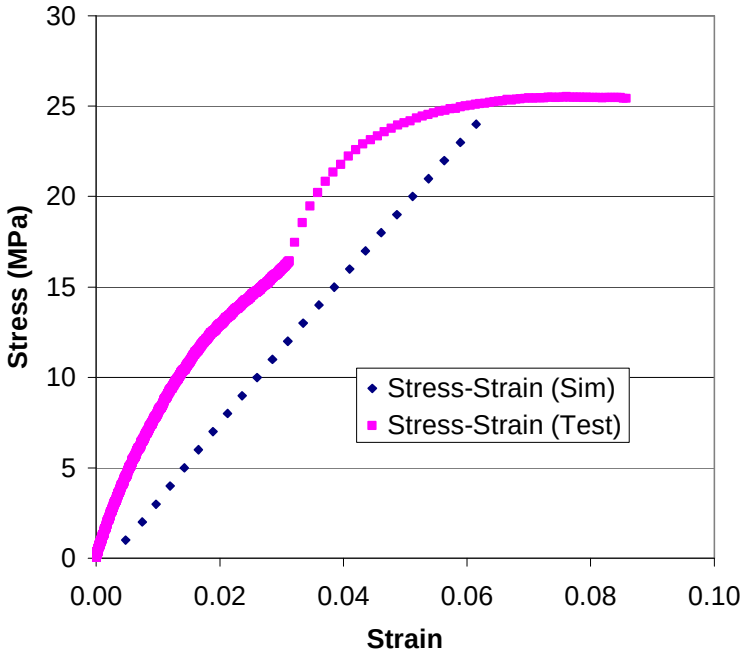


Figure 4.8 (c) Constant load rate (0.1 MPa/s) simulations on HDPE-pipe using nonlinear viscoplastic model: stress vs. strain.

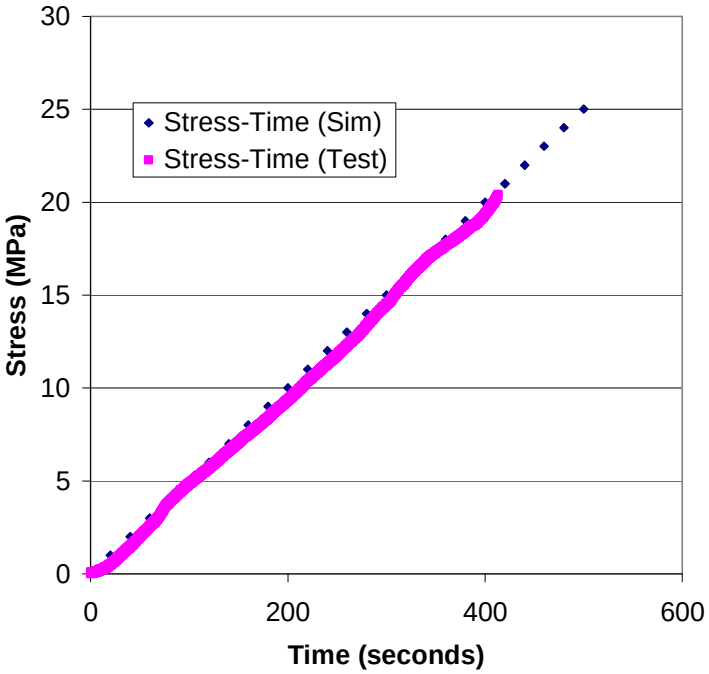


Figure 4.9 (a) Constant load rate (0.05 MPa/s) simulations on HDPE-pipe using nonlinear viscoplastic model: stress vs. time.

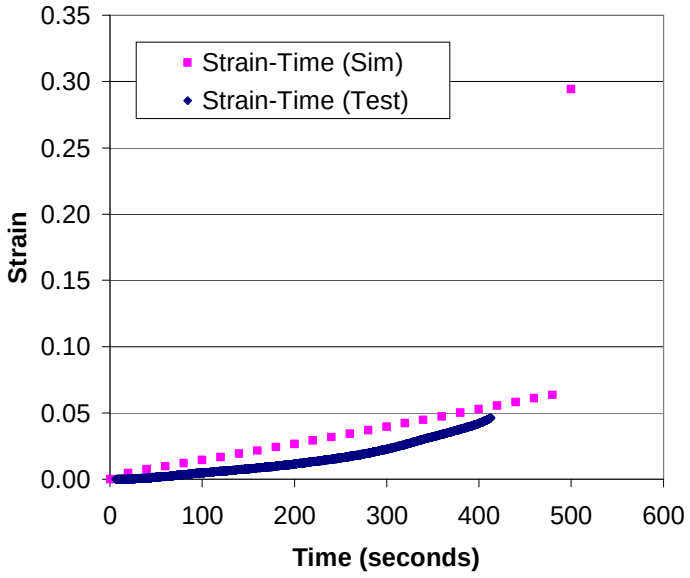


Figure 4.9 (b) Constant load rate (0.05 MPa/s) simulations on HDPE-pipe using nonlinear viscoplastic model: strain vs. time.

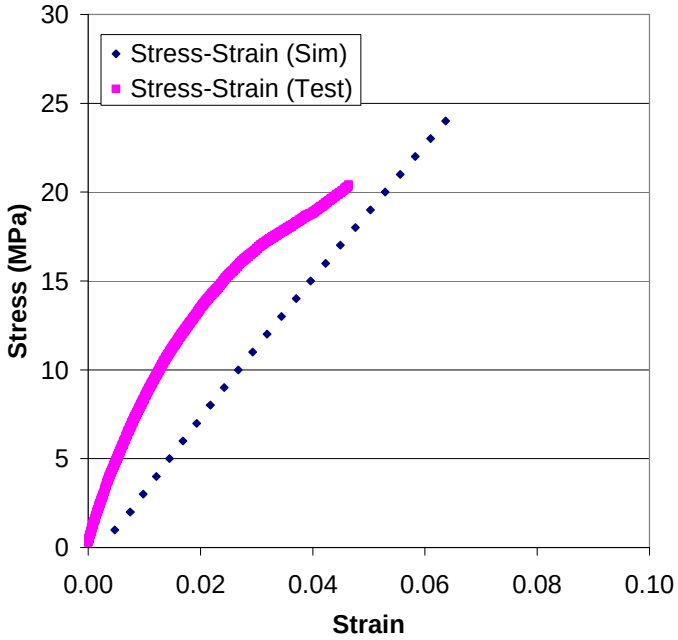


Figure 4.9 (c) Constant load rate (0.05 MPa/s) simulations on HDPE-pipe using nonlinear viscoplastic model: stress vs. strain.

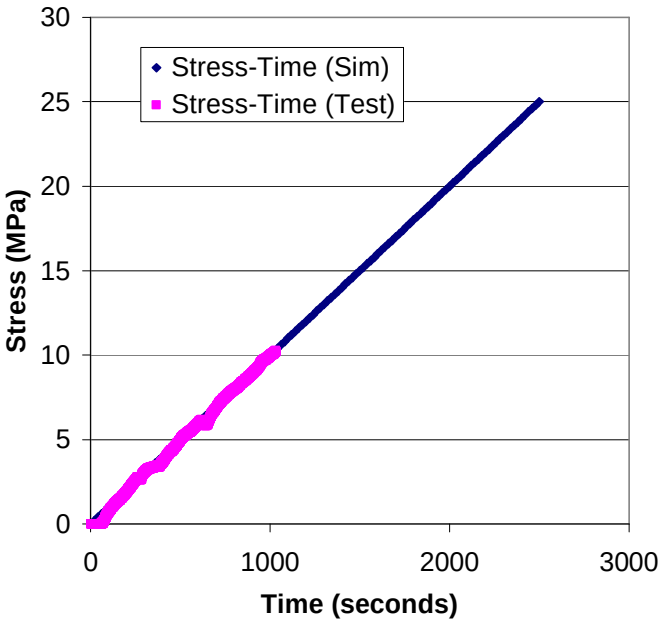


Figure 4.10 (a) Constant load rate (0.01 MPa/s) simulations on HDPE-pipe using nonlinear viscoplastic model: stress vs. time.

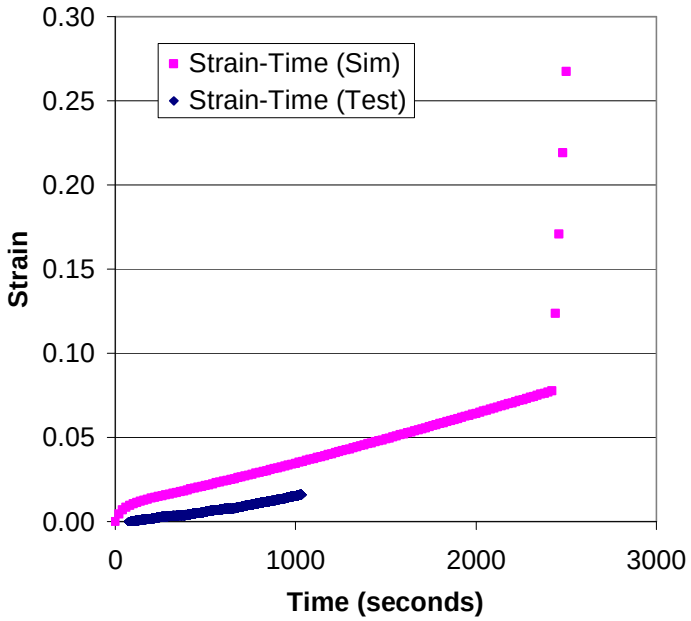


Figure 4.10 (b) Constant load rate (0.01 MPa/s) simulations on HDPE-pipe using nonlinear viscoplastic model: strain vs. time.

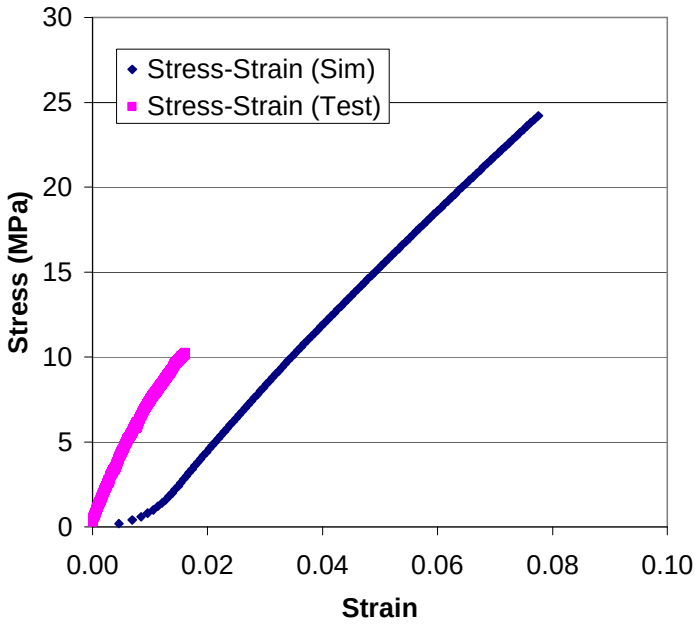


Figure 4.10 (c) Constant load rate (0.01 MPa/s) simulations on HDPE-pipe using nonlinear viscoplastic model: stress vs. strain.

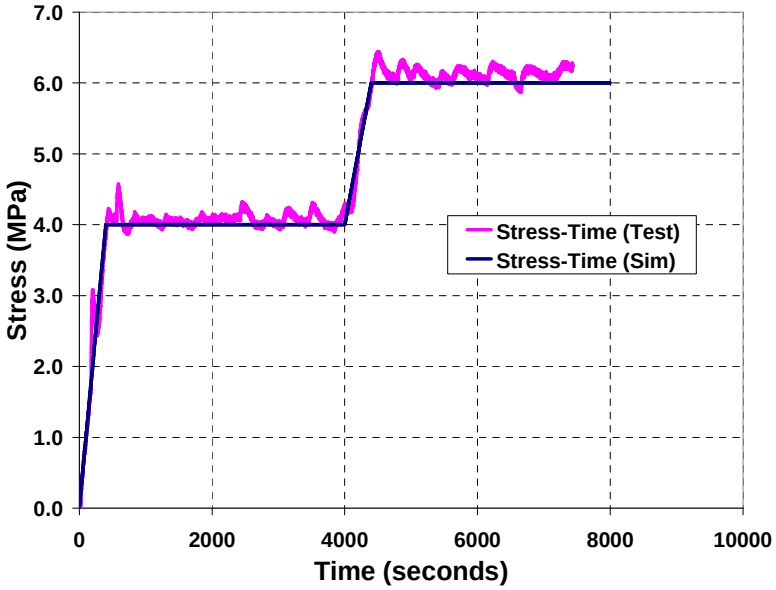


Figure 4.11 (a) Simulations of complex-test-1 on HDPE-pipe using nonlinear viscoplastic model: stress vs. time.

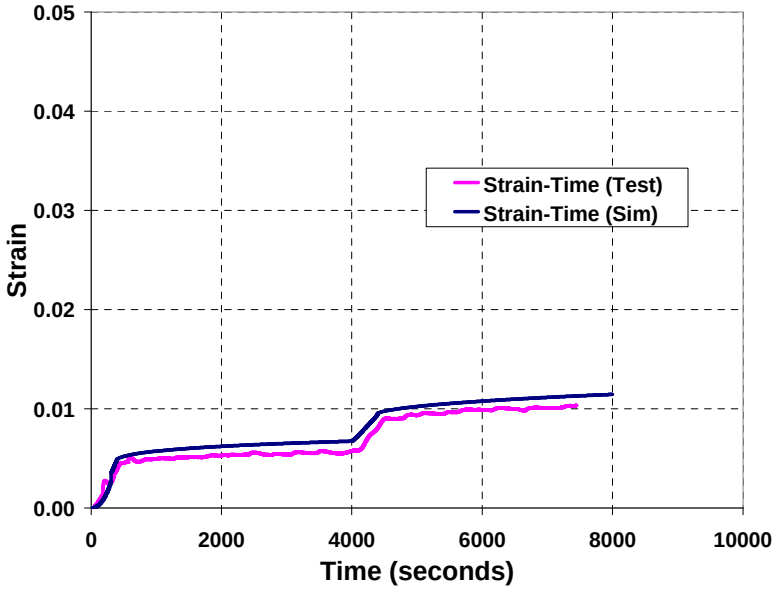


Figure 4.11 (b) Simulations of complex-test-1 on HDPE-pipe using nonlinear viscoplastic model: strain vs. time.

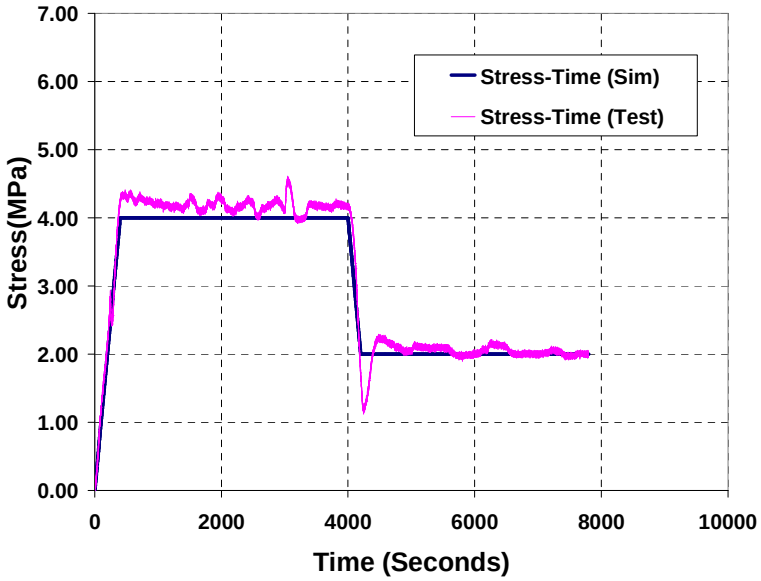


Figure 4.12 (a) Simulations of complex-test-2 on HDPE-pipe using nonlinear viscoplastic model: stress vs. time.

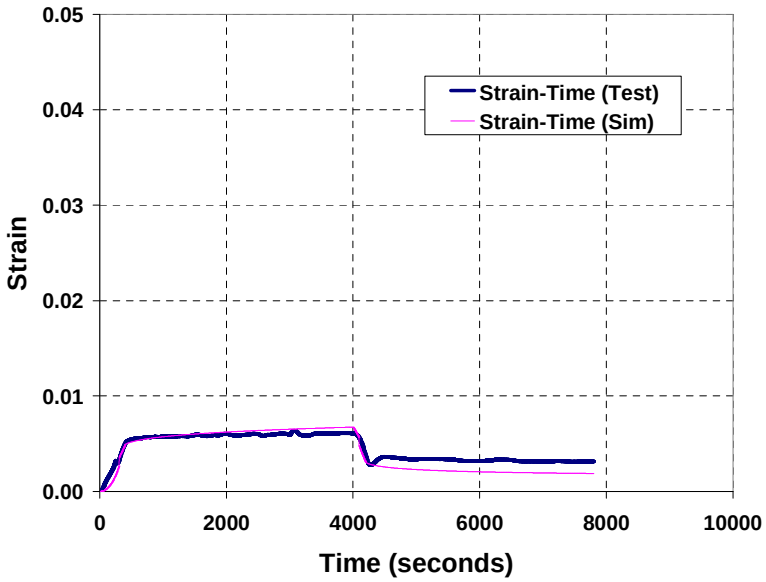


Figure 4.6 (b) Simulations of complex-test-2 on HDPE-pipe using nonlinear viscoplastic model: strain vs. time.

Chapter 5

Summary and Discussion

5.1 Tests on Polymers

Seven polyethylene materials have been tested for 24-hour creep. High density polyethylene pipe material, HDPE-pipe, was also tested for a longer duration (14-day) creep. The results indicate strong nonlinear viscoelasticity of the materials. The creep data were used for generating material models for four of the materials. Stepped creep tests, constant load rate tests, and constant strain rate tests were conducted on HDPE-pipe.

Creep test is easy to handle using the static load creep tester. 24-hour is feasible for practical modelling purposes. An MTS tester was used for rate tension tests. During a constant load rate test, the load applied is monitored and input into the control systems, and the displacement and strain are recorded. The test control in this case was simple even though there was some load fluctuation observed. During a constant strain rate test, the strain is recorded and used as feedback to monitor the applied load. In such case, the load is difficult to control due to the delayed response of the tester. Some of the tests failed when the tester jumped over the targeted load. Another reason causing such problem is that the tested polyethylene has low stiffness compared to conventional structural materials, e.g. steel and concrete, and the testing load required is much smaller. The tester was not properly set up for the designed tests.

Creep tests results on the selected polyethylenes indicate nonlinear viscoelastic behaviours of the materials. Under constant loading, the strain in the tested sample keeps growing with time. At different stress levels the creep compliance-time diverge shortly after loading started. Such creep response can not be modelled by a linear viscoelastic model accurately. At low stresses (blow half of the yield strength, 25 MPa for HDPE-

pipe), the major deformation in a creep test occurs in the first a few days. At higher stresses the strain growth is still large after 14 days.

The rate tests also reveal rate-dependent viscoelastic behaviour of the tested materials, HDPE-pipe and HDPE-resin2.

5.2 Material modelling

A modelling approach was introduced for nonlinear viscoelastic and viscoplastic modelling. The models have an integral form based on classical linear viscoelastic cases and nonlinearity was incorporated by linear interpolation of the material parameters with respect to stress. The viscoelastic model generated corresponds to the traditional rheological linear viscoelastic one-spring-plus-a-series-of-Kelvin(Voigt)-elements modelling approach. The models were calibrated from a group of creep tests at increasing stresses by linear-least-squares-fitting. The mathematical formulation is simple and the calibration procedure is standard and was incorporated into FORTRAN subroutines. The models have tabular representations instead of complex functions, which are easy for numerical applications.

Viscoelastic and viscoplastic models for four tested polyethylene materials were generated. Model simulations were compared with corresponding experimental results. Good model performance were achieved demonstrated by creep simulations, stepped creep simulations, constant stress rate simulations and combined constant stress rate and creep simulations.

In the model calibration, relaxation times are assumed constant to linearize the curve fitting. These relaxation times define the shape of the creep compliance curves. Selection of the values for the relaxation times is arbitrary and the study shows that a wide range of values can be chosen to achieve good modelling results. However, the accuracy of the models depends on the time window selected for curve fitting. Since the relaxation times define the direction of the tails of the compliance curves, the performance at longer times,

outside of the time window used for model development, can be different for different selection of these relaxation times, even though the models all perform well within the time window. Selection of relaxation times and time extrapolation of the models are important topics for research on further practical applications of the models.

5.3 Model Applications

The viscoelastic and viscoplastic models generated have integral forms that relate stress, strain, and their rates. The constitutive equations are not simple functions as in linear elastic or incremental plastic cases. The application of the models requires numerical procedures, normally finite element analysis. The models generated in this study are presented as tabulated arrays and can be easily used as an input for numerical procedures.

In this study, only one-dimensional applications were discussed. The models are applied to one-dimensional loading using a numerical integration procedure. Stepped creep tests and constant load rate tests were simulated using both viscoelastic and viscoplastic models for HDPE-pipe and compared with experiments. The results for stepped creep tests, constant stress rate tests, and combined stress rate and creep tests were good. The one-dimensional numerical procedure adopted in the study is valid can be expend to three dimensional cases for more complicated numerical applications.

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