Stability and Boundedness of Impulsive Systems with Time Delay

by

Qing Wang

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Abstract

The stability and boundedness theories are developed for impulsive differential equations with time delay. Definitions, notations and fundamental theory are presented for delay differential systems with both fixed and state-dependent impulses. It is usually more difficult to investigate the qualitative properties of systems with state-dependent impulses since different solutions have different moments of impulses. In this thesis, the stability problems of nontrivial solutions of systems with state-dependent impulses are "transferred" to those of the trivial solution of systems with fixed impulses by constructing the so-called "reduced system". Therefore, it is enough to investigate the stability problems of systems with fixed impulses. The exponential stability problem is then discussed for the system with fixed impulses. A variety of stability criteria are obtained and numerical examples are worked out to illustrate the results, which shows that impulses do contribute to the stabilization of some delay differential equations. To unify various stability concepts and to offer a general framework for the investigation of stability theory, the concept of stability in terms of two measures is introduced and then several stability criteria are developed for impulsive delay differential equations by both the single and multiple Lyapunov functions method. Furthermore, boundedness and periodicity results are discussed for impulsive differential systems with time delay. The Lyapunov-Razumikhin technique, the Lyapunov functional method, differential inequalities, the method of variation of parameters, and the partitioned matrix method are the main tools to obtain these results. Finally, the application of the stability theory to neural networks is presented. In applications, the impulses are considered as either means of impulsive control or perturbations. Sufficient conditions for stability and stabilization of neural networks are obtained.

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Chapter 1

Introduction

Impulsive delay differential systems arise naturally from a wide variety of applications such as orbital transfer of satellites, impact and constrained mechanics, sampled-data systems, spacecraft control, ecosystem management, and inspection processes in operations research [11, 74, 112]. For instance, impulsive phenomena was observed in Bautin's shock model of a clock mechanism ([13]), Kruger-Thiemer's study of drug distribution in the human body ([53]), Liu and Rohlf's control of Lotka-Volterra models ([78]), just to name a few. In fact, various physical processes undergo abrupt changes of state at certain moments of time between intervals of continuous evolution. The duration of these changes is often negligible in comparison with that of the entire evolution process and thus the abrupt changes can be well-approximated in terms of instantaneous changes of state, i.e. impulses. On the other hand, time delay occurs frequently in many applications as diverse as economics, feedback control, secure communication and population dynamics [20, 32, 35, 38, 40, 54, 55]. For example, model of population growth can be described by an impulsive delay differential equation when maturity and management are considered, where the time delay characterizes the retarded effect of reproduction or the interaction within or between species and the impulses describe some abrupt factors such as emigration, immigration, disease and the like ([34, 77]). While in the application to secure communication, impulsive delay differential equations are used to model the error dynamics, where time delay, which occurs in the differential system as well as in the impulses, describes the delay caused by transmission and sampling; and impulses are utilized to stabilize the error dynamics ([45, 46, 47]). When both time delay and impulses are involved, impulsive delay differential systems become a natural framework for mathematical modelling of many such physical phenomena.

An impulsive delay differential equation usually consists of three elements: namely, a continuous system of delay differential equations, which governs the motion of the dynamical system between impulsive or resetting events; a discrete system of difference equations, which controls the way the system states are instantaneously changed when a setting event occurs; and a criterion for determining when the states of the system are to be reset [56, 84, 85, 87]. Consequently, the solutions of an impulsive system with time delay are normally piecewise continuous, which causes a number of difficulties. For instance, if $x(t)$ is piecewise continuous, x_t may be discontinuous everywhere as a function of t; many simple functionals which are continuous on $\mathbb{R}_+ \times C([-\tau,0],\mathbb{R}^n)$ cannot be extended continuously to $\mathbb{R}_+ \times PC([-\tau,0],\mathbb{R}^n)$; and some properties of solutions, such as existence, uniqueness, stability, and boundedness, may be changed greatly by impulses [10, 75, 76, 77, 94].

Despite the wide applications, the study of impulsive delay differential equations is in its relative infancy [12]. An early article on this subject was published in 1986 by Anokhin [2]. In addition, the investigation history of impulsive ordinary differential equations is not long. Early work on impulsive ordinary differential equations was published in 1960 by Milman and Myshkis ([96]). Since then, quite a few classical results on ordinary differential equations have been extended to impulsive differential equations ([56]). Compared to impulsive ordinary differential equations, delay differential equations has been studied for a much longer time, as far back as the eighteenth century by many well-known mathematicians such as Euler, Lagrange and Laplace [102]. After many generations of mathematician's efforts, the theory of delay differential equations has matured a great deal and a number of monographs are dedicated to this subject, see [14, 20, 28, 35, 36, 50, 97, 134] for example.

The study of impulsive delay differential systems has been slow due to some technical difficulties. Recent research work has tended to focus on special classes of equations such as delay differential difference equations with impulses [8, 129], linear or scalar impulsive delay differential equations [4, 19, 34, 91, 133], first-order or second-order impulsive delay differential systems [22, 41, 65, 131]. However, there have appeared some papers that focus on more general impulsive delay differential systems and aim at revealing the essential difference caused by impulses ([75, 77, 109, 126, 128, 129]).

Existence and uniqueness are the most fundamental qualitative properties of impulsive systems with time delay. Early research results on existence and uniqueness have been obtained by Krishna and Anokhin [51], and Shen [103, 105] for some special class of equations. Results on these subjects for more general systems have been published by Ballinger and Liu [10, 76].

In recent years, stability and its applications to differential equations have been extensively studied [21, 26, 27, 29, 30, 37, 42, 48, 49, 57, 58, 59, 67, 89, 95, 99, 100, 101, 127, 132]. Furthermore, many results have been extended to impulsive differential equations ([1, 6, 7, 8, 56, 61, 104, 125]). Significant progress on stability of impulsive delay differential equations has been made during the past decade, see [18, 68, 74, 75, 81, 90, 93, 106, 113, 120, 121, 131, 133] and the references therein.

The study of stability of differential systems with delay is usually more challenging than that of systems without delay. Nonetheless, most of the tools such as the Lyapunov functional method, Razumikhin techniques, and the comparison method have been successfully applied to the study of impulsive delay differential systems ([75, 106, 108]). Quite recently, a number of interesting results on uniform asymptotic stability were obtained, where some restriction on the derivative of the Lyapunov function is relaxed. The non-positiveness requirement of the derivative of the Lyapunov function along solutions of equations has been regarded as necessary for uniformly asymptotic stability in the literature. But now it is allowed to be positive (see [75, 126]) even though this kind of assumption normally causes instability for delay differential systems without impulsive effects ([72, 108]). Nevertheless, to the best of our knowledge, these kinds of conditions had not been developed to obtain exponential stability until recently [87, 118].

Exponential stability is one of the most investigated problems in the stability analysis of impulsive systems since it has played an important role in many areas such as designs and applications of neural networks, population growth models and synchronization in secure communications ([23, 25, 44, 72, 88, 119]). However, results on exponential stability for impulsive delay differential system are very few compared to those on uniform stability and asymptotic stability. Most of the early works mainly focuse on specific classes of equations such as scalar equations and linear equations. Even less work is done on impulsive stabilization [17, 65, 71, 83, 94, 118, 126, 128]. In this thesis, assumptions allowing the derivatives of Lyapunov function or functional to be positive are used to impulsively stabilize delay differential equations.

Compared to the stability of trivial solution, there is little work done on the stability of nontrivial solutions of delay differential equations with state-dependent impulses due to some theoretical and technical difficulties [1, 61, 86]. In the classical stability theory, stability of nontrivial solutions can be converted to that of the trivial solution by change of variables. However, this method can not be extended to delay differential equations with state-dependent impulses because different solutions may have different moments of impulses. We have solved this problem recently by introducing the reduced system and utilizing the definition of quasi-stability. Because moments of impulsive effect of a nontrivial solution $\widetilde{x}(t)$ of a system with state-dependent impulses need not be the same as those of a neighboring solution $x(t)$, demanding that the difference of $x(t)$ and $\widetilde{x}(t)$ be small for all $t \geq t_0$ seems unreasonable. So it is natural to require that the difference be small for all $t \geq t_0$ except a small neighborhood of each impulse point. This leads to the concept of quasi-stability, see reference [61]. We will discuss this issue thoroughly in the later chapters.

To unify a variety of stability concepts and to offer a general framework for investigation of stability theory, introducing the concept of stability in terms of two measures has been proven to be very useful, see [62, 63, 85, 119] and references therein. This concept has generated renewed interest among many researchers recently and some interesting results have appeared in the literature [31, 73, 79, 85, 123, 130]. In this thesis, we obtain several stability criteria in terms of two measures by single and multiple Lyapunov functions method combined with Razumikhin technique and apply some of the results to the Lotka-Volterra system.

On the other hand, boundedness theory has played a significant role in the existence of periodic solutions and it has many applications in areas such as biological population management, secure communication and chaos control, [9, 45, 46, 47, 73, 77]. The theory has been greatly developed during the past decades (see [15, 16, 52, 69, 70, 80, 84, 92, 107] and the references therein). In this thesis, we have established several boundedness criteria for delay differential equations with fixed and state-dependent impulses. Those results are applicable to population growth dynamics and impulsive synchronization for secure communication.

One of the interesting applications of stability is to design neural networks with good stability properties, see [5, 24, 33, 39, 66, 83, 114, 115, 117, 122, 124, 135]. We have applied some of the exponential stability results and techniques to cellular neural networks (CNNs) and high order Hopfield type neural networks. We have discussed possible effects of impulsive perturbations on stability of neural networks and have obtained some stability criteria to keep the stability property of delayed neural networks under impulsive disturbance. We have also developed some results to impulsively stabilize neural networks.

Various methods, such as LMI tools, the method of variation of parameters, differential inequalities, Laplace transform, Lyapunov functional or function method (combined with Razumikhin technique) and so on, have been successfully utilized in the investigations of the stability and boundedness, see [3, 4, 16, 18, 43, 64, 82, 87, 98, 108, 109, 110, 116] for example. Most of our results in this thesis are established by using some of these methods.

There are many challenging and important problems still largely unexplored about impulsive delay differential systems. But I shall mainly focus on some qualitative properties and their applications in this thesis. In the next chapter, some definitions, notation and basic theory for impulsive delay differential systems will be presented. Then, in Chapter 3, the stability problems of nontrivial solutions of delay differential equations with state-dependent impulses are "transferred" to those of trivial solution of systems with fixed impulses by the construction of the "reduced system". Thereafter, in Chapter 4, we establish some exponential stability criteria for delay differential equations with fixed impulses. We also obtain conditions to impulsively stabilize delay differential equations. Meanwhile, numerical examples are presented to illustrate the results. Several theorems on the stability in terms of two measures are developed in Chapter 5 based on the single and multiple Lyapunov functions method together with Razumikhin technique. We also apply some results and techniques to obtain stability criteria for the Lotka-Volterra system. Several boundedness results are presented for delay differential equations with both fixed and state-dependent impulses in Chapter 6. Periodicity results are established for system with fixed impulses by use of the Horn's fixed point theorem. In Chapter 7, the applications of stability to neural networks are presented, where impulses are considered either as means of perturbations or control. Numerical examples illustrate that impulses do contribute to the stabilization of neural networks. Finally, in Chapter 8, conclusions and research plan are given.

Chapter 2

Preliminaries

This chapter summarizes some basic general information on impulsive delay differential equations and introduces concepts and fundamental theory.

2.1 Impulsive Delay Differential Equations

Impulsive delay differential equations differ greatly from ordinary differential equations in the sense that the state undergoes abrupt changes at certain moments and the derivative of the state depends not only on time and the present state, but also on the past states. Thus an impulsive delay differential equation is usually defined as a delay differential equation coupled with a difference equation. In order to introduce a general impulsive delay differential system, we need the following notation.

Denote $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, $\mathbb{R}^n$ the n-dimensional real space equipped with any vector norm $||\cdot||$, and $\mathbb{N}$ the set of positive integers, i.e., $\mathbb{N} = \{1, 2, \cdots\}$. Let $\lambda_{\max}(Q)$ (or $\lambda_{\min}(Q)$) denote the maximum (or minimum) eigenvalue of a symmetric matrix Q. For any matrix A, let A^T represent the transpose of A, and $\|A\|$ denote the norm of A induced by the Euclidean vector norm, i.e., $\|A\| = [\lambda_{\max}(A^TA)]^{\frac{1}{2}}$. Denote $\psi(t^+) = \lim_{s\to t^+}\psi(s)$ and $\psi(t^-) = \lim_{s\to t^-}\psi(s)$. For $a, b \in \mathbb{R}$ with $a < b$ and for $S \subset \mathbb{R}^n$, we define the following classes of functions.

$$PC([a,b],S) = \Big\{\psi : [a,b] \to S \;\Big|\; \psi(t) = \psi(t^+), \forall t \in [a,b); \psi(t^-) \text{ exists in } S, \forall t \in (a,b], \text{ and } \psi(t^-) = \psi(t) \text{ for all but at most a finite number of points } t \in (a,b]\Big\},$$

$$PC([a,b),S) = \Big\{\psi : [a,b) \to S \;\Big|\; \psi(t) = \psi(t^+), \forall t \in [a,b); \psi(t^-) \text{ exists in } S, \forall t \in (a,b), \text{ and } \psi(t^-) = \psi(t) \text{ for all but at most a finite number of points } t \in (a,b)\Big\},$$

and

$$PC([a,\infty),S) = \Big\{\psi : [a,\infty) \to S \;\Big|\; \forall c > a, \psi|_{[a,c]} \in PC([a,c],S)\Big\}.$$

Given a constant $\tau > 0$, we equip the linear space $PC([-\tau,0],\mathbb{R}^n)$ with the norm $\|\cdot\|_\tau$ defined by $\|\psi\|_\tau = \sup_{-\tau\le s\le 0}\|\psi(s)\|$. For the case $\tau = \infty$, $\|\psi\|_\tau = \|\psi\|_\infty = \sup_{-\infty<s\le 0}\|\psi(s)\|$ for any $\psi \in PC((-\infty,0],\mathbb{R}^n)$.

Consider the impulsive delay differential equation with state-dependent impulses

$$x'(t) = f(t,x_t), \qquad t \neq \tau_k(x(t^-)), \tag{2.1a}$$

$$\Delta x(t) = I_k(x(t^-)), \qquad t = \tau_k(x(t^-)), \tag{2.1b}$$

where $f : \mathbb{R}_+ \times PC([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n$, $t \in \mathbb{R}_+$, $\tau_k \in C(\mathbb{R}^n,\mathbb{R}_+)$, $\Delta x(t) = x(t^+) - x(t^-)$, $I_k \in C(\mathbb{R}^n,\mathbb{R}^n)$, $k \in \mathbb{N}$, and $x_t \in PC([-\tau,0],\mathbb{R}^n)$ is defined by $x_t(s) = x(t+s)$ for $-\tau \le s \le 0$. Here we assume $x(t^+) = x(t)$. In other words, we assume solutions of (2.1) are right-continuous.

The initial condition for system (2.1) is given by

$$x_{t_0} = \phi, \tag{2.2}$$

where $t_0 \in \mathbb{R}_+$ and $\phi \in PC([-\tau,0],\mathbb{R}^n)$.

We use the notation $A \backslash B$ to denote the difference of two sets A and B (i.e. $A \backslash B = \{t \mid t \in A$ and $t \notin B\}$). Let $J \subset \mathbb{R}_+$ be an interval of the form $[a, b)$ where $0 \leq a < b \leq \infty$ and let $D \subset \mathbb{R}^n$ be an open set. We first introduce the following definitions from [12].

Definition 2.1.1 *A function $x \in PC([t_0 - \tau, t_0 + \alpha], D)$ where $\alpha > 0$ and $[t_0, t_0 + \alpha] \subset J$ is said to be a* solution of (2.1) *if*

(i) the set $T = \{t \in (t_0, t_0 + \alpha] \mid t = \tau_k(x(t^-))$ for some $k\}$ of impulse times is finite (possibly empty);

(ii) x is continuous at each $t \in (t_0, t_0 + \alpha] \backslash T$;

(iii) the derivative of x exists and is continuous at all but at most a finite number of points t in $(t_0, t_0 + \alpha)$;

(iv) the right-hand derivative of x exists and satisfies the delay differential equation (2.1a) for all $t \in [t_0, t_0 + \alpha) \backslash T$; and

(V) x satisfies the delay difference equation (2.1b) for all $t \in T$.

If in addition, x satisfies the initial condition (2.2)*, then it is said to be a* solution of (the initial value problem) (2.1) & (2.2) *and we write $x(t) = x(t, t_0, \phi)$.*

Definition 2.1.2 *A function $x \in PC([t_0 - \tau, t_0 + \beta), D)$ where $0 < \beta \leq \infty$ and $[t_0, t_0 + \beta) \subset J$ is said to be a* solution of (2.1) (solution of (2.1) & (2.2)) *if for each $0 < \alpha < \beta$ the restriction of x to $[t_0 - \tau, t_0 + \alpha]$ is a solution of (2.1)* (solution of (2.1) & (2.2)) *and if $\beta < \infty$, then the derivative of x exists and is continuous at all but at most a finite number of points t in $(t_0, t_0 + \beta)$ and the set $T = \{t \in (t_0, t_0 + \beta) \mid t = \tau_k(x(t^-))$ for some $k\}$ is finite.*

Definition 2.1.3 *If x and y are solutions of (2.1) on the intervals J_1 and J_2, respectively, where J_2 properly contains J_1 and both intervals have the same closed left endpoint, and if $x(t) = y(t)$ for $t \in J_1$, then y is said to be a proper continuation of x to the right, or simply a continuation of x. A solution x of (2.1)defined on J_1 is said to be continuable if there exists some continuation y of x. Otherwise x is said to be noncontinuable and the interval J_1 is called a maximal interval of existence of x.*

Definition 2.1.4 *A solution $x(t) = x(t, t_0, \phi)$ of (2.1)-(2.2) is said to be* unique *if given any other solution $y(t) = y(t, t_0, \phi)$ of (2.1)-(2.2), $x(t) = y(t)$ on their common interval of existence.*

A special case of system (2.1)-(2.2) that we will mainly focus on in later chapters is the delay differential equation with fixed impulses

$$x'(t) = f(t, x_t), \qquad t \neq t_k, \tag{2.3a}$$

$$\Delta x(t) = I_k(x(t^-)), \qquad t = t_k,\ k \in \mathbb{N}, \tag{2.3b}$$

$$x_{t_0} = \phi, \tag{2.3c}$$

where the t_k are constants and satisfy $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, with $t_k \to \infty$ as $k \to \infty$.

Definition 2.1.5 *A function $x \in PC([t_0 - \tau, t_0 + \alpha], D)$ where $\alpha > 0$ and $[t_0, t_0 + \alpha] \subset J$ is said to be a* solution of (2.3) *if*

(i) x is continuous at each $t \neq t_k$ in $(t_0, t_0 + \alpha]$;

(ii) the derivative of x exists and is continuous at all but at most a finite number of points t in $(t_0, t_0 + \alpha)$;

(iii) the right-hand derivative of x exists and satisfies the delay differential equation (2.3a) for all $t \in [t_0, t_0 + \alpha)$;

(iv) x satisfies the delay difference equation (2.3b) at each $t_k \in (t_0, t_0 + \alpha]$;

(v) x satisfies the initial condition (2.3c).

Definition 2.1.6 *A function $x \in PC([t_0 - \tau, t_0 + \beta), D)$ where $0 < \beta \leq \infty$ and $[t_0, t_0 + \beta) \subset J$ is said to be a* solution of (2.3) *if for each $0 < \alpha < \beta$ the restriction of x to $[t_0 - \tau, t_0 + \alpha]$ is a solution of (2.3).*

The definitions of continuation and uniqueness of solutions of system (2.3) are the same as Definitions 2.1.3 and 2.1.4 for system (2.1)-(2.2), respectively (see [10]).

2.2 Fundamental Properties

Existence, continuation and uniqueness are the most important fundamental properties of a dynamical system. In this section, we introduce some results for system (2.1)-(2.2) and (2.3) from [10] and [76].

Definition 2.2.1 *A functional $f: J \times PC([-\tau, 0], D) \to \mathbb{R}^n$ is said to be composite-PC, if for each $t_0 \in J$ and $0 < \alpha \leq \infty$, where $[t_0, t_0 + \alpha) \subset J$, if $x \in PC([t_0 - \tau, t_0 + \alpha), D)$, then the composite function g defined by $g(t) = f(t, x_t)$ is an element of the function class $PC([t_0, t_0 + \alpha), \mathbb{R}^n)$.*

Definition 2.2.2 *A functional $f: J \times PC([-\tau, 0], D) \to \mathbb{R}^n$ is said to be quasi-bounded, if for each $t_0 \in J$ and $\alpha > 0$, where $[t_0, t_0 + \alpha] \subset J$, and for each compact set $F \subset D$, there exists some $M > 0$ such that $\|f(t, \psi)\| \leq M$ for all $(t, \psi) \in [t_0, t_0 + \alpha] \times PC([-\tau, 0], F)$.*

Definition 2.2.3 *A functional $f: J \times PC([-\tau, 0], D) \to \mathbb{R}^n$ is said to be continuous in ψ, if for each fixed $t \in J$, $f(t, \psi)$ is a continuous function of ψ on $PC([-\tau, 0], D)$.*

Definition 2.2.4 *A functional $f: J \times PC([-\tau, 0], D) \to \mathbb{R}^n$ is said to be locally Lipschitz in ψ, if for each $t_0 \in J$ and $\alpha > 0$, where $[t_0, t_0 + \alpha] \subset J$, and for each compact set $F \subset D$, there exists some $L > 0$ such that $\|f(t, \psi_1) - f(t, \psi_2)\| \leq L\|\psi_1 - \psi_2\|_\tau$ for all $t \in [t_0, t_0 + \alpha]$ and $\psi_1, \psi_2 \in PC([-\tau, 0], F)$.*

Theorem 2.2.1 (Local Existence [76]) *Assume that f is composite-PC, quasi-bounded and continuous in ψ and that $\tau_k \in C^1(D, \mathbb{R}_+)$ for $k = 1, 2, \ldots$. Furthermore, assume that whenever $t^* = \tau_k(x^*)$ for some $(t^*, x^*) \in J \times D$ and some k, then there exists a $\delta > 0$, where $[t^*, t^* + \delta] \subset J$, such that*

$$\nabla \tau_k(x(t)) \cdot f(t, x_t) \neq 1, \tag{2.4}$$

for all $t \in (t^*, t^* + \delta]$ and for all functions $x \in PC([t^* - \tau, t^* + \delta], D)$ that are continuous on $(t^*, t^* + \delta]$ and satisfy $x(t^*) = x^*$ and $\|x(s) - x^*\| < \delta$ for $s \in [t^*, t^* + \delta]$. Then for each $(t_0, \phi) \in J \times PC([-\tau, 0], D)$, there exists a solution $x = x(t_0, \phi)$ of (2.1) & (2.2) on $[t_0 - \tau, t_0 + \beta]$ for some $\beta > 0$.

Theorem 2.2.2 (Continuation [76]) *Assume that f is composite-PC, quasi-bounded and continuous in ψ and that $\tau_k \in C^1(D, \mathbb{R}_+)$ for $k = 1, 2, \ldots$ and the limit $\lim_{k\to\infty} \tau_k(x) = \infty$ is uniform in x. Furthermore, assume that*

$$\nabla \tau_k(\psi(0)) \cdot f(t, \psi) < 1, \tag{2.5}$$

for all $(t, \psi) \in J \times PC([-\tau, 0], D)$ and $k = 1, 2, \ldots$. Finally, assume that $\psi(0) + I_k(\psi(0)) \in D$ and

$$\tau_k(\psi(0) + I_k(\psi(0)) \leq \tau_k(\psi(0)), \tag{2.6}$$

for all $\psi \in PC([-\tau, 0], D)$ for which $\psi(0^-) = \psi(0)$ and for all $k = 1, 2, \ldots$. Then for every continuable solution x of (2.1), there exists a continuation y of x that is non-continuable. Moreover, any solution x of (2.1) can intersect each impulse hyper-surface $($in the sense that $t = \tau_k(x(t^-)))$ at most once.

Theorem 2.2.3 (Uniqueness [76]) *Assume that f is composite-PC and locally Lipschitz in ψ. Then there exists at most one solution of (2.1) & (2.2) on $[t_0 - \tau, t_0 + \beta)$ where $0 < \beta \leq \infty$ and $[t_0, t_0 + \beta) \subset J$.*

The following results are presented for system (2.3).

Theorem 2.2.4 (Local Existence [10]) *Assume that f is composite-PC, quasi-bounded and continuous in its second variable. Then for each $(t_0, \phi) \in J \times PC([-\tau, 0], D)$ there exists a solution $x = x(t_0, \phi)$ of (2.3) on $[t_0 - \tau, t_0 + \beta]$ for some $\beta > 0$.*

Theorem 2.2.5 (Continuation [10]) *Assume that f is composite-PC, quasi-bounded and continuous in its second variable. Let $(t_0, \phi) \in J \times PC([-\tau, 0], D)$ and let $x = x(t_0, \phi)$ be any solution of (2.3). If x is defined on a closed interval of the form $[t_0 - \tau, t_0 + \alpha]$, where $\alpha > 0$ and $[t_0, t_0 + \alpha] \subset J$, then x is continuable. If x is defined on an interval of the form $[t_0 - \tau, t_0 + \beta)$, where $0 < \beta < \infty$ and $[t_0, t_0 + \beta] \subset J$, and if x is noncontinuable then for every compact set $G \subset D$ there exists a sequence of numbers $\{t_k\}$ with $t_0 < t_k < t_{k+1} < t_0 + \beta$ for $k = 1, 2, \ldots$ and $\lim_{k\to\infty} t_k = t_0 + \beta$ such that $x(t_k) \notin G$.*

Theorem 2.2.6 (Uniqueness [10]) *Assume that f is composite-PC and locally Lipschitz in its second variable. Then there exists at most one solution of (2.3) on $[t_0 - \tau, t_0 + \beta)$ where $0 < \beta \leq \infty$ and $[t_0, t_0 + \beta) \subset J$.*

These theorems on existence, continuation and uniqueness represent the groundwork upon which further qualitative analysis can be performed on the wide class of impulsive delay differential equations considered in this thesis.

In Chapter 4-6, we assume that $f(t, \psi)$ is $composite$-PC, $quasi$-$bounded$ and $continuous\ in\ \psi$ so that the initial value problem (2.3) has a solution $x(t, t_0, \phi) \triangleq x(t)$ existing in a maximal interval I. In Chapter 4, we also assume $f(t, 0) = I_k(0) = 0$ for all $t \in \mathbb{R}_+$ and $k \in \mathbb{N}$ so that system (2.3) admits the trivial solution. In Chapter 6, we also suppose that f is $locally\ Lipschitz\ in\ \psi$ so that (2.3) has a unique solution.

2.3 Notation and Definitions

In this section, we introduce notation and definitions that will be useful in this thesis.

In order to make use of Lyapunov method in our theorems, we must first define the following properties [12], [75].

Definition 2.3.1 *A function $V(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ belongs to class ν_0 if*

(A1) V is continuous on each of the sets $[t_{k-1}, t_k) \times \mathbb{R}^n$ and for all $x,\ y \in \mathbb{R}^n$ and $k \in \mathbb{N}$, $\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$ exists;

(A2) $V(t, x)$ is locally Lipschitz in $x \in \mathbb{R}^n$, and for all $t \geq t_0$, $V(t, 0) \equiv 0$.

Definition 2.3.2 *A functional $V : \mathbb{R}_+ \times PC([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}_+$ is said to belong to the class $\nu_0(\cdot)$ (a set of Lyapunov like functionals) if*

(B1) V is continuous on $[t_{k-1}, t_k) \times PC([-\tau, 0], \mathbb{R}^n)$ and for all $\psi,\ \phi \in PC([-\tau, 0], \mathbb{R}^n)$, and $k \in \mathbb{N}$, $\lim_{(t,\psi)\to(t_k^-,\phi)} V(t, \psi) = V(t_k^-, \phi)$ exists;

(B2) $V(t, \psi)$ is locally Lipschitz in ψ in each compact set in $PC([-\tau, 0], \mathbb{R}^n)$, and for all $t \geq t_0$, $V(t, 0) \equiv 0$.

Definition 2.3.3 *A functional $V(t, \psi) : \mathbb{R}_+ \times PC([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}_+$ belongs to class $\nu_0^*(\cdot)$ if $V(t, \psi) \in \nu_0(\cdot)$ and for any $x \in PC([t_0 - \tau, \infty), \mathbb{R}^n)$, $V(t, x_t)$ is continuous for $t \geq t_0$.*

Definition 2.3.4 *Given a functional $V \in \nu_0^*(\cdot)$, the upper right-hand derivative of V with respect to system (2.3) is defined by*

$$D^+_{(2.3)}V(t,\psi) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x_{t+h}(t,\psi)) - V(t,\psi)],$$

for $(t,\psi) \in \mathbb{R}_+ \times PC([-\tau,0],\mathbb{R}^n)$.

Definition 2.3.5 *Given a function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, the upper right-hand derivative of V with respect to system (2.3) is defined by*

$$D^+_{(2.3)}V(t,\psi(0)) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, \psi(0) + hf(t,\psi)) - V(t,\psi(0))],$$

for $(t,\psi) \in \mathbb{R}_+ \times PC([-\tau,0],\mathbb{R}^n)$.

Note that in Definition 2.3.5, $D^+_{(2.3)}V(t,\psi(0))$ is a functional whereas V is a function. Moreover, Definition 2.3.5 is consistent with the earlier definition of the derivative of a functional in Definition 2.3.4.

In later chapters, we may drop the subscript and simply write D^+V or V' where it is understood which system the derivative of V is with respect to.

Next, we define stability for the impulsive system (2.3).

Definition 2.3.6 *The trivial solution of system* (2.3) *is said to be*

(S1) **stable** *if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists some $\delta = \delta(t_0,\epsilon) > 0$ such that if $\phi \in PC([-\tau,0],D)$ with $\|\phi\|_\tau \le \delta$ and $x = x(t_0,\phi)$ is any solution of* (2.3)*, then $x(t,t_0,\phi)$ is defined and $\|x(t,t_0,\phi)\| \le \epsilon$ for all $t \ge t_0$;*

(S2) **uniformly stable** *if δ in (S1) is independent of t_0;*

(S3) **asymptotically stable** *if (S1) holds and for every $t_0 \in \mathbb{R}_+$, there exists some $\eta = \eta(t_0) > 0$ such that if $\phi \in PC([-\tau,0],D)$ with $\|\phi\|_\tau \le \eta$, then $\lim_{t\to\infty} x(t,t_0,\phi) = 0$;*

(S4) **uniformly asymptotically stable** *if (S2) holds and there exists some $\eta > 0$ such that for every $\gamma > 0$, there exists some $T = T(\eta,\gamma) > 0$ such that if $\phi \in PC([-\tau,0],D)$ with $\|\phi\|_\tau \le \eta$, then $\|x(t,t_0,\phi)\| \le \gamma$ for $t \ge t_0 + T$;*

(S5) **unstable** *if (S1) fails to hold.*

Definition 2.3.7 *The trivial solution of system (2.3) is said to be exponentially stable, if for any initial data $x_{t_0} = \phi$, there exists an $\alpha > 0$, and for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\|x(t, t_0, \phi)\| < \varepsilon e^{-\alpha(t-t_0)}, \quad \textit{for all } t \geq t_0, \tag{2.7}$$

whenever $\|\phi\|_\tau < \delta$, $t_0 \in \mathbb{R}_+$ and $\phi \in PC([-\tau, 0], \mathbb{R}^n)$.

Definition 2.3.8 *The trivial solution of system (2.3) is said to be globally exponentially stable if, for any initial data $x_{t_0} = \phi$, there exist constants $\alpha > 0, M \geq 1$ such that*

$$\|x(t, t_0, \phi)\| \leq M\|\phi\|_\tau e^{-\alpha(t-t_0)}, \quad t \geq t_0,$$

where $t_0 \in \mathbb{R}_+$, $\phi \in PC([-\tau, 0], \mathbb{R}^n)$.

We define the following sets for later use.

$$\begin{array}{lcl}
S(\rho) & = & \{x \in \mathbb{R}^n \bigm| \|x\| < \rho, \text{ for } \rho > 0\}, \\
PC(\rho) & = & \{\psi \in PC([-\tau, 0], \mathbb{R}^n) \bigm| \|\psi\|_\tau < \rho\}, \\
G & = & \{(t, \psi) \bigm| t \in \mathbb{R}_+,\ \psi \in PC(\rho)\}, \\
K_0 & = & \big\{H \in C(\mathbb{R}_+, \mathbb{R}_+) \bigm| \ H(0) = 0, \text{ and } H(s) > 0 \text{ for } s > 0\ \big\}, \\
K & = & \big\{g \in K_0 \bigm| \ g \text{ is strictly increasing in } s\ \big\}, \\
K_1 & = & \{\psi \in K \ \bigm| \psi(s) < s \text{ for s} > 0\ \}, \\
K_2 & = & \{\phi \in K \ \bigm| \phi(u) \geq u \text{ for u} > 0\ \}, \\
K_3 & = & \{g \in K_0 \bigm| \text{g is nondecreasing in s}\}, \\
K_4 & = & \{g \in K \bigm| g(s) \to \infty \text{ as } s \to \infty\}, \\
\Omega & = & \big\{\omega(t, u) \bigm| \omega \in C([t_{k-1}, t_k) \times \mathbb{R}_+, \mathbb{R}_+)\ ,\ k \in \mathbb{N};\ \text{for } \textit{each } x \in \mathbb{R}_+ \text{ and} \\
 & & \quad k \in \mathbb{N},\ \lim_{(t,u)\to(t_k^-,x)} \omega(t, u) =\ \omega(t_k^-, x) \text{ exists}\ \big\},
\end{array}$$

$$\begin{array}{lcl}
\Gamma & = & \big\{h \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+) \bigm| \inf_{(t,x)} h(t, x) = 0\big\}, \\
\Gamma_0 & = & \big\{h_0 : \mathbb{R}_+ \times PC([-r, 0], \mathbb{R}^n) \to \mathbb{R}_+ \bigm| h_0(t, \phi) = \sup_{-r \leq s \leq 0} h^0(t + s, \phi(s)), \\
 & & \quad \text{where } h^0 \in \Gamma\big\}, \\
CK & = & \big\{\ a \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+) \bigm| a \text{ is nondecreasing with respect to the} \\
 & & \quad \text{second variable, and } a(\cdot, 0) = 0\big\}.
\end{array}$$

Chapter 3

Systems with State-dependent Impulses

This chapter discusses the stability problems of nontrivial solutions of delay differential equations with state-dependent impulses. It is well-known that the stability of a nontrivial solution of a delay differential equation with fixed impulses can be transferred to the stability of the trivial solution by a change of variable. However, this is invalid for a system with state-dependent impulses. The objective of this chapter is to solve this problem. We finally "transfer" the stability problems of systems with state-dependent impulses to those of systems with fixed impulses by introducing the concept of quasi-stability and constructing the so-called "reduced system".

The remainder of this chapter is organized as follows. In Section 3.1, we introduce the concept of quasi-stability and some other notation and definitions. In Section 3.2, we construct the reduced system, a medium to relate systems with state-dependent impulse effect and systems with fixed impulse effect. Finally in Section 3.3, we obtain criteria on quasi-stability by using some known results for systems with fixed impulses.

3.1 Quasi-stability

For systems with state-dependent impulses, impulse moments of a nontrivial solution $\widetilde{x}(t)$ need not be the same as those of a neighboring solution $x(t)$. Thus to demand that the difference of $x(t)$ and $\widetilde{x}(t)$ be small for all $t \geq t_0$ seems unreasonable. And hence it is natural to require that the difference be small for all $t \geq t_0$ except in a small neighborhood of each impulse point. This leads to the concept of quasi-stability [62, 63].

Let $T \subset \mathbb{R}$ be a fixed interval. Then $PC(T, \mathbb{R}^n)$ denotes the set of functions $U : T \to \mathbb{R}^n$, which are piecewise continuous with discontinuity of the first kind. Assume that a set of points of discontinuity of every function $u \in PC(T, \mathbb{R}^n)$ is no more than countable and does not have a finite limit point.

Definition 3.1.1 *A function $u_2 \in PC(T, \mathbb{R}^n)$ is said to belong to an ϵ-neighborhood of $u_1 \in PC(T, \mathbb{R}^n)$ if*

(i) every point t_k of discontinuity of $u_2(t)$ lies in an ϵ-neighborhood of some discontinuous point $\hat{t}_k$ of $u_1(t)$, i.e., $|t_k - \hat{t}_k| < \epsilon$;

(ii) for all $t \in T$ which are not in an ϵ-neighborhood of the point of discontinuity of $u_1(t)$, the inequality $\|u_1(t) - u_2(t)\| < \epsilon$ holds.

The following definitions on quasi-stability are in the spirit of [62, 63].

Definition 3.1.2 *The solution $\widetilde{x}(t)$ of system (2.1) is said to be*

(S1) quasi-stable if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that for every solution $x(t)$ of equation (2.1), $(x_{t_0} - \widetilde{x}_{t_0}) \in PC(\delta)$ implies that $x(t)$ belongs to an ϵ-neighborhood of $\widetilde{x}(t)$ when $t \geq t_0$;

(S2) quasi-uniformly stable if δ in (S1) is independent of t_0;

(S3) quasi-asymptotically stable if (S1) holds and for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exists some $\eta = \eta(t_0) > 0$ and $T(t_0, \epsilon) > 0$ such that for every solution $x(t)$ of equation (2.1), $(x_{t_0} - \widetilde{x}_{t_0}) \in PC(\eta)$ implies that $x(t)$ lies in an ϵ-neighborhood of $\widetilde{x}(t)$ when $t \geq t_0 + T$;

(S4) quasi-uniformly asymptotically stable, if (S2) holds and there exists some $\eta > 0$ such that for every $\epsilon > 0$, there exists some $T = T(\eta, \epsilon) > 0$ such that $(x_{t_0} - \widetilde{x}_{t_0}) \in PC(\eta)$ implies that $x(t)$ lies in an ϵ-neighborhood of $\widetilde{x}(t)$ when $t \geq t_0 + T$;

(S5) quasi-unstable, if (S1) fails to hold.

For convenience, we list the following assumptions to be satisfied in later sections.

(A1) $f(t, \psi)$ is composite-PC.

(A2) $f(t, \psi)$ is locally Lipschitz in ψ.

(A3) There exists some $0 < \rho_1 \leq \rho$ such that $x \in S(\rho_1)$ implies that $x + I_k(x) \in S(\rho)$ for all $k \in \mathbb{N}$.

(A4) There exist $\mu \in K$ and positive constants $L_k \in \mathbb{R}$ such that for any $x, y \in S(\rho)$ and $k \in \mathbb{N}$,

$$\|I_k(x) - I_k(y)\| \leq \mu(\|x - y\|), \quad \|\tau_k(x) - \tau_k(y)\| \leq L_k\|x - y\|.$$

(A5) $\tau_k(x + I_k(x)) \leq \tau_k(x)$ for all $k \in \mathbb{N}$ and $x \in S(\rho)$.

(A6) $\tau_{k+1}(x + I_k(x)) \geq \tau_k(x)$ for all $k \in \mathbb{N}$ and $x \in S(\rho)$.

(A7) For any $(t, \psi) \in G_k = \{(t, \psi) \mid t \in [\underline{t}_k, \bar{t}_k],\ \psi \in PC(\rho)$, where $\underline{t}_k = \inf_{x \in S(\rho)} \tau_k(x)$ and $\bar{t}_k = \sup_{x \in S(\rho)} \tau_k(x)\}$, there exists $M_k > 0$ such that

$$\sup_{G_k} \|f(t, \psi)\| = M_k < \infty. \tag{3.1}$$

(A8) $t_0 < \tau_1(x) < \tau_2(x) < \cdots,\ \lim_{k \to \infty} \tau_k(x) = \infty$ is uniform in x.

(A9) $\tau_k \in C^1(D, \mathbb{R}_+)$, and for any $k \in \mathbb{N}$, there exists $a > 0$ such that

$$L_k \cdot M_k \leq a < 1. \tag{3.2}$$

Remark 3.1.1 *If $(A2)$ holds, then clearly f is also continuous in ψ. If in addition, $(A1)$ holds, then f is also quasi-bounded [10, 76].*

Remark 3.1.2 *It was shown in the previous chapter that if conditions $(A1)$, $(A2)$ and $(A9)$ hold, the initial value problem (2.1)-(2.2) has a unique solution $x(t, t_0, \phi)$ existing on some interval I, which can be extended to a maximal interval if $(A5)$ holds. Furthermore, if $(A8)$ holds, then any solution of (2.1) intersects each impulse hyper-surface at most once.*

3.2 Reduced System

In this section, we shall convert, by constructing a reduced system, the stability problem of a nontrivial solution of the state-dependent impulsive system to that of the trivial solution of the system with fixed impulses.

Let $\widetilde{x}(t)$ be a given solution of equation (2.1) with points of discontinuity at t_k, i.e. $t_k = \tau_k(\widetilde{x}(t_k^-)),\ k \in \mathbb{N}$. Let $x(t)$ be any solution of equation (2.1) with points of discontinuity at s_k, i.e. $s_k = \tau_k(x(s_k^-)),\ k \in \mathbb{N}$.

For any fixed $k \in \mathbb{N}$, we construct a map $\Phi_k : G \to \mathbb{R}^n$ as follows. There are two cases to consider.

Case 1: $t_k \le s_k$.

Given any $x \in \mathbb{R}^n$, denote $a_k(t),\ t \in [t_k, s_k]$ the solution of

$$x'(t) = f(t, x_t), \tag{3.3}$$

which passes through the point (t_k, x); and $b_k(t),\ t \in [t_k, s_k]$ the solution of equation (3.3) which passes through the point $(s_k, a_k(s_k) + I_k(a_k(s_k)))$, see Figure 3.1. Then we have

$$\begin{aligned} a_k(t) &= x + \int_{t_k}^t f(s, a_{ks})ds, \quad t \in [t_k, s_k], \\ b_k(t) &= a_k(s_k) + I_k(a_k(s_k)) + \int_{s_k}^t f(s, b_{ks})ds, \quad t \in [t_k, s_k], \end{aligned}$$

where $a_{kt}(s) = a_k(t+s)$ for $-\tau \le s \le 0$ and $b_{kt}(s) = b_k(t+s)$ for $-\tau \le s \le 0$.

Define

$$\begin{aligned} \Phi_k(x) &= b_k(t_k) - a_k(t_k) = b_k(t_k) - x \\ &= \int_{t_k}^{s_k} f(t, a_{kt})dt + \int_{s_k}^{t_k} f(t, b_{kt})dt + I_k(x + \int_{t_k}^{s_k} f(t, a_{kt})dt). \end{aligned}$$

Case 2: $t_k > s_k$.

Given any $x \in \mathbb{R}^n$, denote $a_k(t)$, $t \in [s_k, t_k]$ the solution of $x'(t) = f(t, x_t)$ without impulse effect at $t = s_k$, which satisfies $a_k(s_k) = x$; and $b_k(t)$, $t \in [s_k, t_k]$ the solution of $x'(t) = f(t, x_t)$ with impulse effect at $t = s_k$, which starts at the point $(s_k, a_k(s_k) + I_k(a_k(s_k)))$, see Figure 3.2. Then

$$\begin{aligned} a_k(t) &= x + \int_{s_k}^t f(s, a_{ks})ds, \quad t \in [s_k, t_k], \\ b_k(t) &= a_k(s_k) + I_k(a_k(s_k)) + \int_{s_k}^t f(s, b_{ks})ds \\ &= x + I_k(x) + \int_{s_k}^t f(s, b_{ks})ds, \quad t \in [s_k, t_k]. \end{aligned}$$

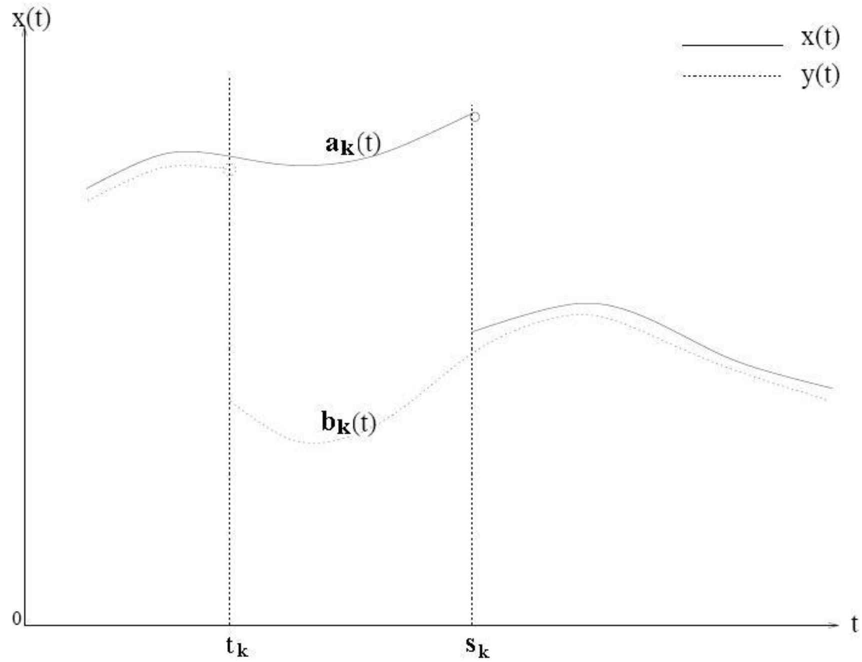

Figure 3.1: Reduced system, case 1: $t_k \leq s_k$.

We define

$$\begin{aligned}\Phi_k(y) &= b_k(t_k) - a_k(t_k) = b_k(t_k) - y \\ &= I_k(x) + \int_{s_k}^{t_k} f(s, b_{ks})ds + \int_{t_k}^{s_k} f(s, a_{ks})ds \\ &= \int_{t_k}^{s_k} f(s, a_{ks})ds + \int_{s_k}^{t_k} f(s, b_{ks})ds + I_k(y + \int_{t_k}^{s_k} f(s, a_{ks})ds),\end{aligned}$$

where $y = a_k(t_k) = x + \int_{s_k}^{t_k} f(s, a_{ks})ds$.

Now we consider the system with fixed impulses

$$\begin{aligned} y'(t) &= f(t, y_t), & t \neq t_k, \\ \Delta y|_{t=t_k} &= \Phi_k(y), & k \in \mathbb{N}, \end{aligned} \tag{3.4}$$

where

$$\Phi_k(y) = \begin{cases} \int_{t_k}^{s_k} f(t, a_{kt})dt + \int_{s_k}^{t_k} f(t, b_{kt})dt + I_k(y + \int_{t_k}^{s_k} f(t, a_{kt})dt), & t_k \neq s_k, \\ I_k(y), & t_k = s_k. \end{cases}$$

System (3.4) is called the reduced system of system (2.1).

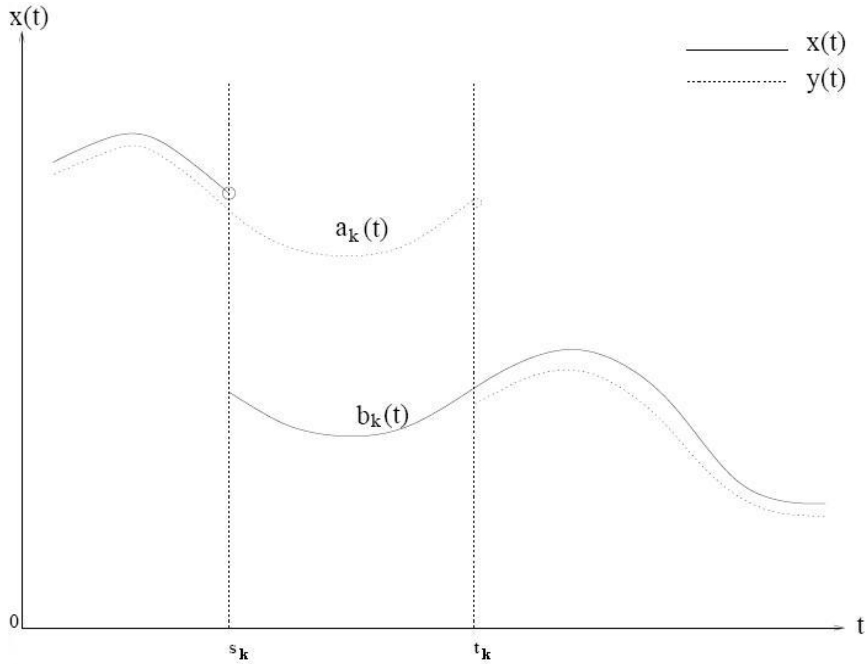

Figure 3.2: Reduced system, case 2: $t_k > s_k$.

Remark 3.2.1 *By the definition of $\Phi_k(y)$, we see that $\Phi_k(y)$ and its reduced system are well-defined and $\lim_{s_k \to t_k} \Phi_k(y) = I_k(y)$.*

Definition 3.2.1 *System (2.1) is said to be quasi-equivalent to system (3.4) on G, if for every solution $x(t) : [t_0, \beta) \to \mathbb{R}^n,\ \beta \in \mathbb{R}_+$ with $(t, x(t)) \in G$ of system (2.1), there exists a solution $y(t)$ of system (3.4) with $y_{t_0} = x_{t_0}$ such that*

$$x(t) = y(t), \quad for\ all\ \ t \in [t_0, \beta) \backslash \cup_{k \in \mathbb{N}} < t_k, s_k >, \tag{3.5}$$

where $< t_k, s_k >$ denotes $[t_k, s_k]$ if $t_k \leq s_k$; otherwise, it denotes $[s_k, t_k]$; and

$$\begin{aligned} x(t_k) &= y(t_k^-), \ \ x(s_k) &= y(s_k), \ \ if \ \ t_k \leq s_k, \\ x(t_k) &= y(t_k), \ \ x(s_k^-) &= y(s_k), \ \ if \ \ t_k > s_k. \end{aligned} \tag{3.6}$$

Conversely, for every solution $y(t) : [t_0, \beta) \to \mathbb{R}^n,\ \beta \in \mathbb{R}_+$ of system (3.4) with $(t, y(t)) \in G$, there exists a solution $x(t)$ of system (2.1) with $x_{t_0} = y_{t_0}$ which satisfies (3.5) and (3.6). $x(t)$ (or $y(t)$) is said to correspond to $y(t)$ (or $x(t)$) by quasi-equivalence.

Lemma 3.2.1 *For any* $x \in \mathbb{R}^n$ *and* $k \in \mathbb{N}$,

$$\|\Phi_k(x) - I_k(x)\| \le \frac{2L_kM_k}{1-L_kM_k}\|x-\widetilde{x}(t_k^-)\| + \mu(\frac{L_kM_k}{1-L_kM_k}\|x-\widetilde{x}(t_k^-)\|).$$

Proof. Let us assume $t_k \le s_k$. From previous statement (Case 1), for any fixed $x \in \mathbb{R}^n$,

$$\begin{aligned} a_k(t) &= x + \textstyle\int_{t_k}^t f(s, a_{ks})ds, \quad t \in [t_k, s_k], \\ b_k(t) &= a_k(s_k) + I_k(a_k(s_k)) + \textstyle\int_{s_k}^t f(s, b_{ks})ds, \quad t \in [t_k, s_k]. \end{aligned}$$

Then we have

$$\begin{aligned} \|a_k(t) - \widetilde{x}(t_k^-)\| &= \|x + \textstyle\int_{t_k}^t f(s, a_{ks})ds - \widetilde{x}(t_k^-)\| \\ &\le \|x - \widetilde{x}(t_k^-)\| + M_k \cdot |s_k - t_k|, \quad t \in [t_k, s_k]. \end{aligned} \tag{3.7}$$

Thus by (3.7) we obtain

$$\begin{aligned} |s_k - t_k| &= |\tau_k(a_k(s_k)) - \tau_k(\widetilde{x}(t_k^-))| \\ &\le L_k \cdot \|a_k(s_k) - \widetilde{x}(t_k^-)\| \\ &\le L_k \cdot \|x - \widetilde{x}(t_k^-)\| + L_kM_k \cdot |s_k - t_k|. \end{aligned}$$

Therefore,

$$|s_k - t_k| \le \frac{L_k \cdot \|x - \widetilde{x}(t_k^-)\|}{1 - L_kM_k}. \tag{3.8}$$

Similarly, we can prove that inequality (3.8) holds for the case $t_k > s_k$.

Then,

$$\begin{aligned} &\|\Phi_k(x) - I_k(x)\| \\ &= \|b_k(t_k) - a_k(t_k) - I_k(x)\| \\ &= \|\textstyle\int_{t_k}^{s_k} f(t, a_{kt})dt + \int_{s_k}^{t_k} f(t, b_{kt})dt + I_k(x + \int_{t_k}^{s_k} f(t, a_{kt})dt) - I_k(x)\| \\ &\le \|\textstyle\int_{t_k}^{s_k} f(t, a_{kt})dt\| + \|\int_{s_k}^{t_k} f(t, b_{kt})dt\| + \|I_k(x + \int_{t_k}^{s_k} f(t, a_{kt})dt) - I_k(x)\| \\ &\le 2M_k|s_k - t_k| + \mu(M_k|s_k - t_k|) \\ &\le \tfrac{2L_kM_k}{1-L_kM_k}\|x - \widetilde{x}(t_k^-)\| + \mu(\tfrac{L_kM_k}{1-L_kM_k}\|x - \widetilde{x}(t_k^-)\|). \end{aligned}$$

∎

3.3 Stability Criteria

In this section, we establish stability theorems by transferring the stability of nontrivial solutions of a state-dependent impulsive system to that of the trivial solution of a system with fixed impulse effect. As we will see, the reduced system is a bridge to connect these two impulsive functional differential equations.

It is obvious that $\widetilde{x}(t)$ is a solution of both system (2.1) and system (3.4). Let $x(t)$ be a solution of system (2.1)-(2.2), $y(t)$ be a solution of system (3.4) which corresponds to $x(t)$ by quasi-equivalence, and $u(t) = y(t) - \widetilde{x}(t)$. Then $u(t)$ satisfies the following system

$$\begin{aligned} u'(t) &= F(t, u_t), && t \neq t_k, \\ \Delta u &= J_k(u(t^-)), && t = t_k, \quad k \in \mathbb{N}, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} F(t, u_t) &= f(t, \widetilde{x}_t + u_t) - f(t, \widetilde{x}_t), \\ J_k(u) &= \Phi_k(\widetilde{x}(t_k^-) + u) - I_k(\widetilde{x}(t_k^-)). \end{aligned} \tag{3.10}$$

Obviously, we have $F(t, 0) = 0,\ J_k(0) = 0$. Thus system (3.9) possesses the trivial solution $u(t) \equiv 0$.

We denote $J_k(u)$ in equation (3.10) as $J_k(u) = P_k(u) + Q_k(u)$ for later use, where $P_k(u) = I_k(\widetilde{x}(t_k^-) + u) - I_k(\widetilde{x}(t_k^-))$ and $Q_k(u) = \Phi_k(\widetilde{x}(t_k^-) + u) - I_k(\widetilde{x}(t_k^-) + u)$.

Lemma 3.3.1 $\|Q_k(u)\| \leq h_k(\|u\|)$, *where* $h_k(s) = \frac{2}{1-L_kM_k}s + \mu(\frac{1}{1-L_kM_k}s)$.

Proof. From Lemma 3.2.1, equation (3.10) and the assumption $L_kM_k < 1$, we can easily obtain our result. ∎

Now we introduce some stability results of the trivial solution of the delay differential equation with fixed impulses (2.3) from [75] and [108].

Lemma 3.3.2 *([108]) Assume that there exist* $V_1(t, x) \in \nu_0,\ V_2(t, \psi) \in \nu_0^*(\cdot),\ w_1,\ w_2 \in K$ *and* $\psi_1 \in K_0$ *such that*

(i) $w_1(\|\psi(0)\|) \leq V(t, \psi) \leq w_2(\|\psi\|_\tau)$,
where $V(t, \psi) = V_1(t, \psi(0)) + V_2(t, \psi) \in \nu_0(\cdot)$;

(ii) $V_1(t_k, x + I_k(x)) - V_1(t_k^-, x) \leq -\lambda_k \psi_1(V_1(t_k^-, x))$, *for any* $x \in S(\rho_1)$ *and* $k \in \mathbb{N}$, *where* $\lambda_k \geq 0$ *with* $\sum_{k=1}^{\infty} \lambda_k = \infty$;

(iii) for any solution $x(t)$ *of equation (2.3), the upper right-hand derivative of* V *satisfies*

$$D^+V(t, x_t) \leq 0,$$

and for any $t \geq t_0$ *and* $\alpha > 0$, *there is some* $\beta > 0$ *such that* $V(t, x_t) \geq \alpha$ *implies* $V_1(t, x(t)) \geq \beta$.

Then the trivial solution of equation (2.3) is uniformly stable and asymptotically stable.

Theorem 3.3.1 *Assume that there exist* $V_1(t, x) \in \nu_0$, $V_2(t, \psi) \in \nu_0^*(\cdot)$, w_1, $w_2 \in K$ *and* $\psi_1 \in K_0$ *such that*

(i) $w_1(\|\psi(0)\|) \leq V(t, \psi) \leq w_2(\|\psi\|_\tau)$, *where* $V(t, \psi) = V_1(t, \psi(0)) + V_2(t, \psi) \in \nu_0(\cdot)$;

(ii) there exists $\alpha_1 \in K$ *such that, for any* $x \in S(\rho_1)$ *and* $k \in \mathbb{N}$,

$$V_1(t_k, x + P_k(x)) - V_1(t_k^-, x) \leq -\alpha_1(\|x\|),$$

and

$$\overline{L}_1 \cdot h_k(\|x\|) - \alpha_1(\|x\|) \leq -\lambda_k \psi_1(V_1(t_k^-, x)),$$

where $\lambda_k \geq 0$ *with* $\sum_{k=1}^{\infty} \lambda_k = \infty$ *and* $\overline{L}_1 > 0$ *is the Lipschitz constant of* V_1;

(iii) for any solution $x(t)$ *of equation (3.9), the upper right-hand derivative of* V *satisfies*

$$D^+V(t, x_t) \leq 0,$$

and for any $t \geq t_0$ *and* $\alpha > 0$, *there is some* $\beta > 0$ *such that* $V(t, x_t) \geq \alpha$ *implies* $V_1(t, x(t)) \geq \beta$.

Then the nontrivial solution $\widetilde{x}(t)$ *of equation (2.1) is quasi-uniformly stable and quasi-asymptotically stable.*

Proof. We shall first show uniform and asymptotic stability of the trivial solution of equation (3.9):

By condition (ii) and Lemma 3.3.1, we have

$$\begin{aligned} V_1(t_k, x + J_k(x)) - V_1(t_k^-, x) &= (V_1(t_k, x + J_k(x)) - V_1(t_k, x + P_k(x))) \\ &\quad +(V_1(t_k, x + P_k(x)) - V_1(t_k^-, x)) \\ &\le \overline{L}_1 \cdot \|Q_k(x)\| - \alpha_1(\|x\|) \le \overline{L}_1 \cdot h_k(\|x\|) - \alpha_1(\|x\|) \\ &\le -\lambda_k \psi_1(V_1(t_k^-, x)), \end{aligned}$$

which implies that condition (ii) of Lemma 3.3.2 holds. By Lemma 3.3.2, we know that the trivial solution of equation (3.9) is uniformly stable and asymptotically stable.

From the construction of system (3.9) and the definition of stability, we know the nontrivial solution $\widetilde{x}(t)$ of equation (3.4) is uniformly stable and asymptotically stable.

Now, we show quasi-uniform stability of solution $\widetilde{x}(t)$ of equation (2.1):

Because of the uniform stability of solution $\widetilde{x}(t)$ of equation (3.4), we have, for any $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, let $\epsilon_1 = \epsilon \cdot \inf_{k\in\mathbb{N}}\{1, \frac{1-L_kM_k}{L_k}\}$. There exists $\delta = \delta(\epsilon) > 0$ such that if $y(t) = y(t, t_0, \phi_1)$ is a solution of equation (3.4), then $\|\phi_1 - \phi\|_\tau \le \delta$ implies $\|y(t) - \widetilde{x}(t)\| \le \epsilon_1, \;\; t \ge t_0$.

Let $x(t) = x(t, t_0, \phi_1)$ be a solution of equation (2.1), which corresponds to $y(t)$ by quasi-equivalence. Then by quasi-equivalence of $y(t)$ and $x(t)$, we have

$$\|x(t) - \widetilde{x}(t)\| \le \epsilon_1, \quad t \notin [t_k, s_k),\; k \in \mathbb{N}, \tag{3.11}$$

where s_k is the impulse point of $x(t)$. Assume, without loss of generality, that $s_k \ge t_k$.

Then

$$\begin{aligned} s_k - t_k &= \tau_k(x(s_k^-)) - \tau_k(\widetilde{x}(t_k^-)) \le L_k \cdot \|x(s_k^-) - \widetilde{x}(t_k^-)\| \\ &\le L_k \cdot (\|x(t_k^-) - \widetilde{x}(t_k^-)\| + \|x(s_k^-) - x(t_k)\|) \\ &\le L_k \cdot (\|x(t_k^-) - \widetilde{x}(t_k^-)\| + M_k(s_k - t_k)), \end{aligned}$$

i.e.

$$s_k - t_k \le \frac{L_k\epsilon_1}{1 - L_kM_k} \le \epsilon, \tag{3.12}$$

which implies quasi-uniform stability of the nontrivial solution $\widetilde{x}(t)$ of equation (2.1).

Next, we shall show quasi-asymptotic stability of the nontrivial solution $\widetilde{x}(t)$ of equation (2.1):

From the quasi-uniform stability of $\widetilde{x}(t)$, we know $\widetilde{x}(t)$ is quasi-stable. From the asymptotic stability of the nontrivial solution $\widetilde{x}(t)$ of equation (3.3), for any $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, let $\epsilon_1 = \epsilon \cdot \inf_{k\in\mathbb{N}}\{1, \frac{1-L_kM_k}{L_k}\}$. There exists some $\eta = \eta(t_0) > 0$ and $T = T(\epsilon, t_0) > 0$ such that if $y(t) = y(t, t_0, \phi_1)$ is a solution of equation (3.3), then $\|\phi_1 - \phi\|_\tau \leq \eta$ implies

$$\|y(t) - \widetilde{x}(t)\| \leq \epsilon_1, \quad t \geq t_0 + T. \tag{3.13}$$

Let $x(t) = x(t, t_0, \phi_1)$ be a solution of equation (2.1) with impulse points s_k $(k \in \mathbb{N})$, which corresponds to $y(t)$ by quasi-equivalence. Then by quasi-equivalence of $y(t)$ and $x(t)$, we have

$$\|x(t) - \widetilde{x}(t)\| \leq \epsilon_1, \ \ t \geq t_0 + T \text{ and } t \notin [t_k, s_k), \ k \in \mathbb{N}. \tag{3.14}$$

Similarly proceeding as in the proof of quasi-uniform stability, we can obtain (3.12), which completes our proof. ∎

Lemma 3.3.3 *([108]) Assume that there exist* $V_1(t,x) \in \nu_0,\ V_2(t,\psi) \in \nu_0^*(\cdot),\ w_1,\ w_2 \in K$ *and* $c(s) \in K_0$ *such that*

(i) $w_1(\|\psi(0)\|) \leq V(t,\psi) \leq w_2(\|\psi\|_\tau)$,
where $V(t,\psi) = V_1(t,\psi(0)) + V_2(t,\psi) \in \nu_0(\cdot)$;

(ii) $|V_1(t_k, x + I_k(x)) - V_1(t_k^-, x)| \leq \beta_k \cdot V_1(t_k^-, x)$, *for any* $x \in S(\rho_1)$ *and* $k \in \mathbb{N}$, *where* $\beta_k \geq 0$ *with* $\sum_{k=1}^\infty \beta_k < \infty$;

(iii) for any solution $x(t)$ *of equation (2.3), the upper right-hand derivative of* V *satisfies*

$$D^+V(t,x_t) \leq -g(t)c(V(t,x_t)),$$

where $g \in C(J, \mathbb{R}_+)$ *and satisfies*

$$\int_{t_0}^\infty g(t)dt = \infty.$$

Then the trivial solution of equation (2.3) is uniformly stable and asymptotically stable.

Theorem 3.3.2 *Assume that there exist* $V_1(t,x) \in \nu_0,\ V_2(t,\psi) \in \nu_0^*(\cdot),\ w_1,\ w_2 \in K$ *and* $c(s) \in K_0$ *such that* *(i)* $w_1(\|\psi(0)\|) \leq V(t,\psi) \leq w_2(\|\psi\|_\tau)$, *where* $V(t,\psi) = V_1(t,\psi(0)) + V_2(t,\psi) \in \nu_0(\cdot)$;

(ii) *for each* $x \in S(\rho_1)$ *and* $k \in \mathbb{N}$,

$$\overline{L}_1(h_k(\|x\|) + \mu(\|x\|)) \leq \beta_k \cdot V_1(t_k^-, x),$$

where $\beta_k \geq 0$ *with* $\sum_{k=1}^{\infty} \beta_k < \infty$ *and* $\overline{L}_1$ *is the Lipschitz constant of* V_1*;*

(iii) *for any solution* $x(t)$ *of equation (3.9), the upper right-hand derivative of* V *satisfies*

$$D^+V(t,x_t) \leq -g(t)c(V(t,x_t)),$$

where $g \in C(J,\mathbb{R}_+)$ *and satisfies*

$$\int_{t_0}^{\infty} g(t)dt = \infty.$$

Then the nontrivial solution $\widetilde{x}(t)$ *of equation (2.1) is quasi-uniformly stable and quasi-asymptotically stable.*

Proof. By condition (ii), we have

$$\begin{aligned}
&|V_1(t_k, x + J_k(x)) - V_1(t_k^-, x)| \\
&\leq |V_1(t_k, x + J_k(x)) - V_1(t_k, x + P_k(x))| \\
&\quad + |V_1(t_k, x + P_k(x)) - V_1(t_k^-, x)| \\
&\leq \overline{L}_1 \cdot \|Q_k(x)\| + \overline{L}_1 \|P_k(x)\| \leq \overline{L}_1 \cdot (h_k(\|x\|) + \mu(\|x\|)) \\
&\leq \beta_k \cdot (V_1(t_k^-, x)),
\end{aligned}$$

which implies condition (ii) of Lemma 3.3.3 holds. By Lemma 3.3.3 we know the trivial solution of equation (3.9) is uniformly stable and asymptotically stable. This means the nontrivial solution $\widetilde{x}(t)$ of equation (3.4) is uniformly stable and asymptotically stable. We can use the same method as we have done in the proof of Theorem 3.3.1 to obtain our results. ∎

Lemma 3.3.4 *([75]) Assume that there exist functions* $V(t,x) \in \nu_0,\ a,b,c \in K,\ g \in K_3$ *and* $p \in PC(\mathbb{R}_+,\mathbb{R}_+)$ *such that*

(i) $b(\|x\|) \leq V(t,x) \leq a(\|x\|)$, *for all* $(t,x) \in [-\tau,\infty) \times S(\rho)$*;*

(ii) *the upper right-hand derivative of V with respect to system (2.3) satisfies*

$$D^+V(t,\psi(0)) \le p(t)c(V(t,\psi(0)), \textit{for all } t \ne t_k \textit{ in } \mathbb{R}_+ \textit{ and } \psi \in PC([-\tau,0],S(\rho)),$$

whenever $V(t,\psi(0) \ge g(V(t+s,\psi(s))$ *for* $s \in [-\tau,0]$*;*

(iii) $V(t_k,\psi(0)+I_k(\psi)) \le g(V(t_k^-,\psi(0)))$, *for all* $(t_k,\psi) \in \mathbb{R}_+ \times PC([-\tau,0],S(\rho_1))$ *for which* $\psi(0^-) = \psi(0)$*; and*

(iv) $\tau = \sup_{k\in Z}\{t_k - t_{k-1}\} < \infty,\ M_1 = \sup_{t\ge 0}\int_t^{t+\tau} p(s)ds < \infty$, *and* $M_2 = \inf_{q\ge 0}\int_{g(q)}^q \frac{(ds)}{(c(s))}$ $ds > M_1$.

Then the trivial solution of equation (2.3) is uniformly asymptotically stable.

Theorem 3.3.3 *Assume that there exist functions* $V(t,x) \in \nu_0,\ a,b,c \in K,\ p \in PC(\mathbb{R}_+,\mathbb{R}_+)$ *and* $g \in K_3$ *such that*

(i) $b(\|x\|) \le V(t,x) \le a(\|x\|)$, *for all* $(t,x) \in [-\tau,\infty) \times S(\rho)$*;*

(ii) *the upper right-hand derivative of V with respect to system (3.9) satisfies*

$$D^+V(t,\psi(0)) \le p(t)c(V(t,\psi(0)),\ \textit{for all } t \ne t_k \textit{ in } \mathbb{R}_+,\ \textit{and } \psi \in PC([-\tau,0],S(\rho)),$$

whenever $V(t,\psi(0) \ge g(V(t+s,\psi(s))$ *for* $s \in [-\tau,0]$*;*

(iii) $V(t_k,\psi(0)+J_k(\psi)) \le g(V(t_k^-,\psi(0)))$, *for all* $(t_k,\psi) \in \mathbb{R}_+ \times PC([-\tau,0],S(\rho_1))$ *for which* $\psi(0^-) = \psi(0)$*; and*

(iv) $\tau = \sup_{k\in Z}\{t_k - t_{k-1}\} < \infty,\ M_1 = \sup_{t\ge 0}\int_t^{t+\tau} p(s)ds < \infty$, *and* $M_2 = \inf_{q\ge 0}\int_{g(q)}^q \frac{(ds)}{(c(s))}$ $ds > M_1$.

Then the nontrivial solution $\widetilde{x}(t)$ *of equation (2.1) is quasi-uniformly asymptotically stable.*

Proof. By Lemma 3.3.4, we obtain the uniformly asymptotical stability of the trivial solution of equation (3.9), which implies the uniformly asymptotical stability of the nontrivial solution of equation (3.4). Then by a proof similar to that of Theorem 3.3.1, we can obtain our result. ■

Now we introduce an instability result for the trivial solution of system (2.3) from [108].

Lemma 3.3.5 *([108]) Assume that there exist $V_1(t,x) \in \nu_0,\ V_2(t,\psi) \in \nu_0^*(\cdot),\ w_1 \in K$ and $\psi_1 \in K_0$ such that ψ_1 is nondecreasing and the following assumptions hold:*

(i) $w_1(\|x\|) \le V_1(t,x)$;

(ii) for any solution $x(t)$ of equation (2.3), the upper right-hand derivative of V satisfies

$$D^+V(t,x_t) \ge 0, \quad \text{where } V = V_1 + V_2,$$

and for all $t \ge t_0$ and $\alpha > 0$, there is some $\beta > 0$ such that $V(t,x_t) \ge \alpha$ implies $\|x(t)\| \ge \beta$;

(iii) for each $k \in \mathbb{N}$ and $x \in S(\rho_1)$,

$$V_1(t_k, x + I_k(x)) - V_1(t_k^-, x) \ge \lambda_k \psi_1(V_1(t_k^-, x)),$$

where $\lambda_k \ge 0$ with $\sum_{k=1}^{\infty} \lambda_k = \infty$.

Then the trivial solution of equation (2.3) is unstable.

Theorem 3.3.4 *Assume that there exist $V_1(t,u) \in \nu_0,\ V_2(t,u_t) \in \nu_0^*(\cdot),\ w_1 \in K$ and $\psi_1 \in K_0$ such that ψ_1 is nondecreasing and the following assumptions hold:*

(i) $w_1(\|u\|) \le V_1(t,u)$;

(ii) for any solution $u(t)$ of equation (3.9), the upper right-hand derivative of V satisfies

$$D^+V(t,u_t) \ge 0, \quad \text{where } V = V_1 + V_2,$$

and for all $t \ge t_0$ and $\alpha > 0$, there is some $\beta > 0$ such that $V(t,u_t) \ge \alpha$ implies $\|u(t)\| \ge \beta$;

(iii) for each $k \in \mathbb{N}$ and $u \in S(\rho_1)$,

$$V_1(t_k, u + J_k(u)) - V_1(t_k^-, u) \ge \lambda_k \psi_1(V_1(t_k^-, u)),$$

where $\lambda_k \ge 0$ with $\sum_{k=1}^{\infty} \lambda_k = \infty$.

Then the nontrivial solution $\widetilde{x}(t)$ of equation (2.1) is quasi-unstable.

Proof. From Lemma 3.3.5 we obtain the instability of the trivial solution of equation (3.9), which implies that the nontrivial solution $\widetilde{x}(t)$ of equation (3.4) is unstable. That is, there exists $\epsilon_0 > 0$, and for any $\delta > 0$, there exists some $t^* \geq t_0$ such that $\|\phi_1 - \phi\|_\tau \leq \delta$ implies

$$\|y(t^*, t_0, \phi_1) - \widetilde{x}(t^*, t_0, \phi)\| > \epsilon_0. \tag{3.15}$$

If this $t^* \in T_1 = \{t \mid |t - t_k| \geq \epsilon_0,\ k \in \mathbb{N}\}$, then we obtain the quasi-instability of solution $\widetilde{x}(t)$ of equation (2.1); otherwise, we have, for any $\epsilon > 0$, all points t^* which make (3.15) hold are not in any ϵ-neighborhood of $t_k, k \in \mathbb{N}$, which implies the quasi-stability of $\widetilde{x}(t)$ of equation (2.1). Next, we will show that this case could not happen. Suppose not. Then $\|u(t)\| \leq \epsilon$ holds for all $t \geq t_0$ and $|t - t_k| \geq \epsilon$ whenever $\|\phi\|_\tau \leq \delta$, where $u(t) = u(t, t_0, \phi)$ is a solution of equation (3.9). Let $V_1(t) = V_1(t, u(t))$, $V_2(t) = V_1(t, u_t)$ and $V(t) = V_1(t) + V_2(t)$. By conditions (ii) and (iii), we have

$$V(t_k - \epsilon) - V(t_{k-1} + \epsilon) \geq 0.$$

Let $\epsilon \to 0$. By the continuity of $V(t)$ on the interval $[t_{k-1}, t_k)$, we have

$$V(t_k^-) \geq V(t_{k-1}).$$

Furthermore,

$$V(t_{k-1}) - V(t_{k-1}^-) = V_1(t_{k-1}) - V_1(t_{k-1}^-) \geq \lambda_{k-1}\psi_1(V_1(t_{k-1}^-)).$$

It is obvious that $V(t) \geq V(t_0)$ for all $t \geq t_0$. Then by condition (ii), there is a $\beta > 0$ such that $\|u(t)\| \geq \beta$ for $t \geq t_0$, and thus $V_1(t_{k-1}^-) \geq w_1(\|u(t_{k-1}^-)\|) \geq w_1(\beta)$. Then we have

$$V(t_k) - V(t_{k-1}) \geq \lambda_{k-1}\psi_1(w_1(\beta)),$$

which implies

$$V(t_k) \geq V(t_m) + \psi_1(w_1(\beta)) \sum_{j=m+1}^{k} \lambda_j \to \infty, \qquad as\ k \to \infty.$$

This contradiction shows that solution $\widetilde{x}(t)$ of equation (2.1) is quasi-unstable. ∎

Theorem 3.3.5 *Assume that there exist* $V_1(t,x) \in \nu_0,\ V_2(t,\psi) \in \nu_0^*(\cdot),\ w_1 \in K$ *and* $\psi_1 \in K_0$ *such that* ψ_1 *is nondecreasing and the following assumptions hold:*

(i) $w_1(\|x\|) \le V_1(t,x)$;

(ii) for any solution $x(t)$ *of equation (3.9), the upper right-hand derivative of* V *satisfies*

$$D^+V(t,x_t) \ge 0, \quad \text{where } V = V_1 + V_2,$$

and for all $t \ge t_0$ *and* $\alpha > 0$, *there is some* $\beta > 0$ *such that* $V(t,x_t) \ge \alpha$ *implies* $\|x(t)\| \ge \beta$;

(iii) for each $k \in \mathbb{N}$ *and* $x \in S(\rho_1)$,

$$V_1(t_k, x + P_k(x)) - V_1(t_k^-, x) \ge \lambda_k \psi_1(V_1(t_k^-, x)) + \overline{L}_1 \cdot h_k(\|x\|),$$

where $\lambda_k \ge 0$ *with* $\sum_{k=1}^{\infty} \lambda_k = \infty$, *and* $\overline{L}_1$ *is the Lipshitz constant of* V_1.

Then the nontrivial solution $\widetilde{x}(t)$ *of equation (2.1) is quasi-unstable.*

Proof. By condition (iii) we have

$$\begin{aligned}
&V_1(t_k, x + J_k(x)) - V_1(t_k^-, x)\\
&= (V_1(t_k, x + J_k(x)) - V_1(t_k, x + P_k(x)))\\
&\quad + (V_1(t_k, x + P_k(x)) - V_1(t_k^-, x))\\
&\ge -\overline{L}_1 \cdot \|Q_k(x)\| + (V_1(t_k, x + P_k(x)) - V_1(t_k^-, x))\\
&\ge -\overline{L}_1 \cdot h_k(\|x\|) + \lambda_k \psi_1(V_1(t_k^-, x)) + \overline{L}_1 \cdot h_k(\|x\|)\\
&\ge \lambda_k \psi_1(V_1(t_k^-, x)),
\end{aligned}$$

which implies condition (ii) of Theorem 3.3.4 holds. By Theorem 3.3.4, we know the nontrivial solution $\widetilde{x}(t)$ of equation (2.1) is quasi-unstable. ∎

Theorem 3.3.6 *If there exists some* $M > 0$ *such that* $\|f(t,\psi)\| \le M$ *holds for all* $t \in [t_0 - \tau, \infty)$ *and* $\psi \in PC([-\tau, 0], \mathbb{R}^n)$, *then the quasi-stability of the solution* $\widetilde{x}(t)$ *of system (2.1) implies the stability of solution* $\widetilde{x}(t)$ *of system (3.4).*

Proof. For any $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, let $\epsilon_1 = \epsilon \cdot (1+4M)^{-1}$ and $T_1 = \{t \in \mathbb{R}_+ \mid |t - t_k| \geq \epsilon_1, k \in \mathbb{N}\}$. The quasi-stability of $\widetilde{x}(t)$ implies that there exists some $\delta = \delta(\epsilon_1, t_0) > 0$ such that $\|\phi_1 - \phi\|_\tau \leq \delta$ implies

$$\|x(t) - \widetilde{x}(t)\| \leq \epsilon_1, \quad t \in T_1,$$

and

$$|t_k - s_k| < \epsilon_1, \quad k \in \mathbb{N},$$

where $x(t) = x(t, t_0, \phi_1)$ is a solution of system (2.1) with initial value $x_{t_0} = \phi_1$, and $s_k,\ k \in \mathbb{N}$ are impulse points of $x(t)$.

Let $y(t)$ be a solution of system (3.4) with initial value $y_{t_0} = x_{t_0} = \phi_1$, which corresponds to $x(t)$ by quasi-equivalence. Then we have

$$\|y(t) - \widetilde{x}(t)\| \leq \epsilon_1 \leq \epsilon, \quad for\ all\ t \in T_1.$$

Let $t \notin T_1$, which means $|t - t_k| < \epsilon_1$ for some $k \in \mathbb{N}$. Denote $\hat{t} = t_k + \epsilon_1$, then $\hat{t} \in T_1$ and $|t - \hat{t}| < 2\epsilon_1$, thus we have

$$y(\hat{t}) = x(\hat{t}) \;\; and \;\; \|y(\hat{t}) - \widetilde{x}(\hat{t})\| \leq \epsilon_1.$$

Thus for all $t \notin T_1$, we have

$$\begin{aligned} \|y(t) - \widetilde{x}(t)\| &= \|y(\hat{t}) + \textstyle\int_{\hat{t}}^{t} f(s, y_s) ds - \widetilde{x}(\hat{t}) - \int_{\hat{t}}^{t} f(s, \widetilde{x}_s)) ds\| \\ &\leq \|y(\hat{t}) - \widetilde{x}(\hat{t})\| + \| \textstyle\int_{\hat{t}}^{t} f(s, y_s) ds - \int_{\hat{t}}^{t} f(s, \widetilde{x}_s)) ds\| \\ &\leq \epsilon_1 + 4M\epsilon_1 = \epsilon, \end{aligned}$$

which completes our proof. ∎

Remark 3.3.1 *Theorem 3.3.6 is established to guarantee stability by avoiding solutions having drastic changes in each* ϵ*-neighborhood of any impulse point, which may cause instability though solutions are quasi-stable.*

Chapter 4

Systems with Fixed Impulses

In this chapter, we obtain exponential stability criteria for the trivial solution of a system with fixed impulses (2.3), since we have solved the stability problems of systems with fixed impulses in the preceding chapter. Based on the Lyapunov function and functional method, conditions to impulsively stabilize delay differential equations and to maintain the exponential stability under impulsive perturbations are obtained. Numerical examples are also worked out to illustrate our results.

4.1 Global Exponential Stability

In this section, we develop Lyapunov-Razumikhin methods and establish several exponential stability theorems which provide sufficient conditions for maintaining the exponential stability property of the trivial solution of a delay differential system without impulses.

Theorem 4.1.1 *Assume that there exist a function $V \in \nu_0$, and constants $p > 0,\ c_1 > 0,\ c_2 > 0,\ \lambda > 0,\ d_k \geq 0,\ k \in \mathbb{N}$, such that the following conditions hold:*

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$*;*

(ii) the upper right-hand derivative of V with respect to system (2.3) satisfies

$$D^+V(t,\varphi(0)) \leq -m(t)V(t,\varphi(0)),\ \textit{for all } t \neq t_k \textit{ in } \mathbb{R}_+,$$

whenever $V(t,\varphi(0)) \geq V(t+s,\varphi(s))e^{-\int_{t-\tau}^{t} m(s)ds}$ *for* $s \in [-\tau, 0]$*, where* $m(t) \in PC([t_0-\tau,\infty),\mathbb{R}_+)$ *and* $\inf_{t\geq t_0-\tau} m(t) \geq \lambda$*;*

(iii) $V(t_k,\varphi(0)+I_k(\varphi)) \leq (1+d_k)V(t_k^-,\varphi(0))$*, with* $\sum_{k=1}^{\infty} d_k < \infty$*, and* $\varphi(0^-) = \varphi(0)$.

Then the trivial solution of system (2.3) is globally exponentially stable.

Proof. Let $x(t) = x(t,t_0,\phi)$ be a solution of system (2.3) and $V(t) = V(t,x(t))$. We shall show

$$V(t) \leq c_2 \prod_{i=0}^{k-1}(1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t} m(s)ds}, \quad t \in [t_{k-1},t_k), \quad k \in \mathbb{N},$$

where $d_0 = 0$. Let

$$Q(t) = \begin{cases} V(t) - c_2\prod_{i=0}^{k-1}(1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t} m(s)ds}, & t \in [t_{k-1},t_k), \quad k \in \mathbb{N}, \\ V(t) - c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t} m(s)ds}, & t \in [t_0-\tau,t_0]. \end{cases}$$

We need to show $Q(t) \leq 0$ for all $t \geq t_0$. It is clear that $Q(t) \leq 0$ for $t \in [t_0-\tau,t_0]$, since $Q(t) \leq V(t) - c_2\|\phi\|_\tau^p \leq 0$ by condition (i).

Take $k = 1$. We shall show $Q(t) \leq 0$ for $t \in [t_0,t_1)$. In order to do this we let $\alpha > 0$ be arbitrary and show that $Q(t) \leq \alpha$ for $t \in [t_0,t_1)$. Suppose not, then there exists some $t \in [t_0,t_1)$ so that $Q(t) > \alpha$. Let $t^* = \inf\{t \in [t_0,t_1) : Q(t) > \alpha\}$. Since $Q(t) \leq 0 < \alpha$ for $t \in [t_0-\tau,t_0]$, we know $t^* \in (t_0,t_1)$. Note that $Q(t)$ is continuous on $[t_0,t_1)$, then $Q(t^*) = \alpha$ and $Q(t) \leq \alpha$ for $t \in [t_0-\tau,t^*]$.

Notice $V(t^*) = Q(t^*) + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)ds}$ and for $s \in [-\tau,0]$, we have

$$\begin{aligned} V(t^*+s) &= Q(t^*+s) + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*+s} m(s)ds} \\ &\leq \alpha + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*-\tau} m(s)ds} \\ &\leq (\alpha + c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)ds})e^{-\int_{t^*}^{t^*-\tau} m(s)ds} \\ &= V(t^*)e^{\int_{t^*-\tau}^{t^*} m(s)ds}. \end{aligned}$$

So by condition (ii), we have $D^+V(t^*) \leq -m(t^*)V(t^*)$. Thus we obtain

$$\begin{aligned} D^+Q(t^*) &= D^+V(t^*) + m(t^*)c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)ds} \\ &\leq -m(t^*)(V(t^*) - c_2\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)ds}) \\ &= -m(t^*)\alpha \\ &< 0, \end{aligned}$$

which contradicts the definition of t^*, and so we obtain $Q(t) \le \alpha$ for all $t \in [t_0, t_1)$. Let $\alpha \to 0^+$. We have $Q(t) \le 0$ for $t \in [t_0, t_1)$.

Now we assume that $Q(t) \le 0$ for $t \in [t_0, t_m),\ m \ge 1$. We shall show that $Q(t) \le 0$ for $t \in [t_0, t_{m+1})$.

By condition (iii), we have

$$\begin{aligned} Q(t_m) &= V(t_m) - c_2 \prod_{i=0}^m (1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t_m} m(s)ds} \\ &\le (1+d_m)V(t_m^-) - c_2 \prod_{i=0}^m (1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t_m} m(s)ds} \\ &= (1+d_m)Q(t_m^-) \\ &\le 0. \end{aligned}$$

Let $\alpha > 0$ be arbitrary. We need to show $Q(t) \le \alpha$ for $t \in (t_m, t_{m+1})$. Suppose not. Let $t^* = \inf\{t \in [t_m, t_{m+1}) : Q(t) > \alpha\}$. Since $Q(t_m) \le 0 < \alpha$, by the continuity of $Q(t)$, we obtain, $t^* > t_m$, $Q(t^*) = \alpha$ and $Q(t) \le \alpha$ for $t \in [t_0, t^*]$.

Since $V(t^*) = Q(t^*) + c_2 \prod_{i=0}^m (1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)ds}$, then for any $s \in [-\tau, 0]$, we have

$$\begin{aligned} V(t^*+s) &\le Q(t^*+s) + c_2 \prod_{i=0}^m (1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*+s} m(s)ds} \\ &\le \alpha + c_2 \prod_{i=0}^m (1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*-\tau} m(s)ds} \\ &\le (\alpha + c_2 \prod_{i=0}^m (1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)ds}) e^{-\int_{t^*}^{t^*-\tau} m(s)ds} \\ &= V(t^*) e^{\int_{t^*-\tau}^{t^*} m(s)ds}. \end{aligned}$$

Thus by condition (ii), we have $D^+V(t^*) \le -m(t^*)V(t^*)$. Thus we have

$$\begin{aligned} D^+Q(t^*) &= D^+V(t^*) + m(t^*)c_2 \prod_{i=0}^m (1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)ds} \\ &\le -m(t^*)(V(t^*) - c_2 \prod_{i=0}^m (1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t^*} m(s)ds}) \\ &= -m(t^*)\alpha \\ &< 0. \end{aligned}$$

Again this contradicts the definition of t^*, which implies $Q(t) \le \alpha$ for all $t \in [t_m, t_{m+1})$. Let $\alpha \to 0^+$. We have $Q(t) \le 0$ for all $t \in [t_m, t_{m+1})$. So $Q(t) \le 0$ for all $t \in [t_0, t_{m+1})$. Thus by the method of induction, we obtain

$$V(t) \le c_2 \prod_{i=0}^{k-1} (1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^{t} m(s)ds}, \quad t \in [t_{k-1}, t_k), \quad k \in \mathbb{N}.$$

By condition (i)-(iii), we have

$$c_1\|x\|^p \le V(t) \le c_2 \prod_{i=0}^{k-1}(1+d_i)\|\phi\|_\tau^p e^{-\int_{t_0}^t m(s)ds} \le c_2 M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \ge t_0,$$

which yields

$$\|x\| \le \big(\frac{c_2 M}{c_1}\big)^{\frac{1}{p}}\|\phi\|_\tau e^{-\frac{\lambda}{p}(t-t_0)}, \quad t \ge t_0,$$

where $M = \prod_{i=1}^{\infty}(1+d_i) < \infty$ since $\sum_{k=1}^{\infty} d_k < \infty$. Thus the proof is complete. ■

Example 4.1.1 *Consider the impulsive nonlinear delay differential equation*

$$\begin{cases} x'(t) &= -a(t)x(t) + \frac{b(t)}{1+x^2(t)}x(t-\tau), \quad t \ge t_0 = 0,\ t \ne t_k, \\ x(t_k) &= (1+c_k)x(t_k^-), \quad t_k = k,\ k \in \mathbb{N}, \\ x_{t_0} &= \phi, \end{cases} \tag{4.1}$$

where constants $\tau,\ c_k > 0$ *with* $\sum_{k=1}^{\infty} c_k < \infty$*, functions* $a \in C(\mathbb{R}, \mathbb{R}_+),\ b \in C(\mathbb{R}, \mathbb{R}),\ \phi \in PC([-\tau, 0], \mathbb{R}^n)$*. If* $a(t) \ge |b(t)|e^{-\lambda\tau} + \lambda,\ \lambda > 0$*, then the trivial solution of system (4.1) is globally exponentially stable.*

Proof. Set $V(x) = V(t,x) = |x|$, $m(t) = \lambda$ for all $t \ge t_0 - \tau$, where $\lambda > 0$ is a constant, then we have

$$\begin{aligned} D^+V(t,\varphi(0)) &\le \mathrm{sgn}(\varphi(0))[-a(t)\varphi(0) + \tfrac{b(t)}{1+\varphi^2(0)}\varphi(-\tau)] \\ &\le -a(t)|\varphi(0)| + |b(t)| \cdot |\varphi(-\tau)| \\ &\le -a(t)V(\varphi(0)) + |b(t)| \cdot V(\varphi(-\tau)). \end{aligned} \tag{4.2}$$

For any solution $x(t)$ of equation (4.1) such that

$$V(t,\psi(0)) \ge V(t+s,\varphi(s))e^{\int_{t-\tau}^t m(s)ds}, \quad \text{for } s \in [-\tau, 0],$$

we have $V(\varphi(-\tau)) \le e^{-\lambda\tau}V(\varphi(0))$. Therefore,

$$D^+V(t,\varphi(0)) \le [-a(t) + b(t)e^{-\lambda\tau}]V(\varphi(0)).$$

Since $a(t) \ge |b(t)|e^{-\lambda\tau} + \lambda$, it follows that

$$D^+V(t,\varphi(0)) \le -\lambda V(\varphi(0)) \le -m(t)V(\varphi(0)),$$

whenever $V(t, \varphi(0)) \geq V(t+s, \varphi(s))e^{\int_{t-\tau}^{t} m(s)ds}$ for $s \in [-\tau, 0]$, i.e., condition (ii) of Theorem 4.1.1 holds.

Moreover,

$$V(t_k, \varphi(0) + I_k(\varphi)) = (1 + c_k)V(t_k^-, \varphi(0)).$$

Thus by Theorem 4.1.1, the trivial solution of system (4.1) is globally exponentially stable.The numerical simulation of this example with initial function

$$\phi(t) = \begin{cases} 0, & t \in [-1, 0), \\ 1.7, & t = 0, \end{cases}$$

and $\lambda = \tau = 1,\ b(t) = t^2,\ a(t) = 2 + t^2$, $c_k = \frac{1}{2^k}$ is given in Figure 4.1.

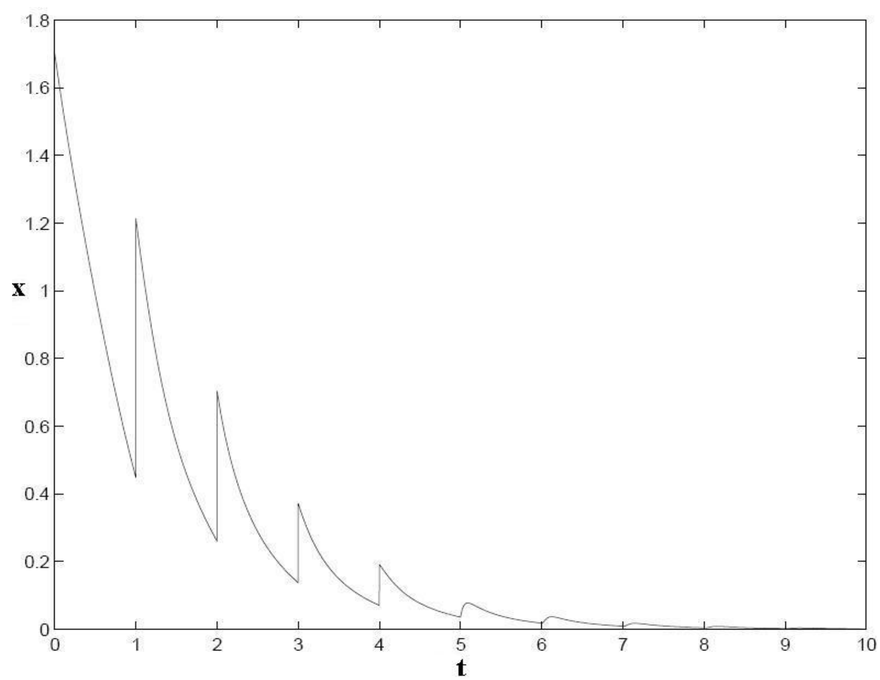

Figure 4.1: Numerical simulation of Example 4.1.1, impulsive system.

It should be noted that when $1 + x^2$ is omitted, system (4.1) becomes the well-known linear case which has been studied by several authors, see for example, [34, 126]. ■

Corollary 4.1.1 *Assume that there exist function* $V(t, x) \in \nu_0$ *and constants* $p > 0, q > 1, c_1 > 0, c_2 > 0, \delta > 1, \lambda > 0, d_k \geq 0, k \in \mathbb{N}$*, such that* *(i)* $c_1\|x\|^p \le V(t,x) \le c_2\|x\|^p$;

(ii) *the upper right-hand derivative of* V *with respect to system (2.3) satisfies*

$$D^+V(t,\varphi(0)) \le -\lambda V(t,\varphi(0)),\ \textit{for all } t \ne t_k \textit{ in } \mathbb{R}_+,$$

whenever $qV(t,\varphi(0)) \ge V(t+s,\varphi(s))$ *for* $s \in [-\tau, 0]$;

(iii) $V(t_k, \varphi(0) + I_k(\varphi)) \le (1+d_k)V(t_k^-, \varphi(0))$, *with* $\sum_{k=1}^{\infty} d_k < \infty$, *and* $\varphi(0^-) = \varphi(0)$.

Then system (2.3) is globally exponentially stable.

Proof. The conclusion follows by setting $m(t) \equiv \lambda$ and $q = e^{\lambda\tau}$ in Theorem 4.1.1. ■

Next, we shall apply Corollary 4.1.1 to some special cases of system (2.3).

Consider the impulsive delay system of the form

$$\begin{cases} \dot{x}(t) = g(t, x(t), x(t-h_1(t)), x(t-h_2(t)), \cdots, x(t-h_m(t))), & t \in [t_{k-1}, t_k), \\ \Delta x(t) = I_k(x(t^-)), \quad t = t_k, k \in \mathbb{N}, \\ x_{t_0} = \phi, \end{cases} \tag{4.3}$$

where $g \in C(\mathbb{R}_+ \times \mathbb{R}^{n\times(m+1)}, \mathbb{R}^n)$, and the function $h_k(t)$ is continuous, and $t - h_k(t)$ is strictly increasing on $\mathbb{R}_+$ and satisfying $0 \le h_k(t) \le \tau$ for $t \in \mathbb{R}_+$.

Corollary 4.1.2 *Assume that conditions* (i), (iii) *of Corollary 4.1.1 hold, while condition* (ii) *is replaced by*

*(ii)** *there exist positive constants* $\lambda > 0, \lambda_i > 0, i = 1, 2, \cdots, m$, *such that, for all* $(t, x, y_1, \cdots, y_m) \in [t_{k-1}, t_k) \times \mathbb{R}^{n\times(m+1)}, k \in \mathbb{N}$,

$$V_t(t,x) + V_x(t,x)g(t,x,y_1, \ \cdots, y_m) \le -\lambda V(t,x) + \sum_{i=1}^{m} \lambda_i V(t-h_i(t), y_i).$$

If λ *is chosen such that* $\lambda > \sum_{i=1}^{m}\lambda_i$, *then system (4.3) is globally exponentially stable.*

Proof. If $\lambda > \sum_{i=1}^{m}\lambda_i$, we know that the equation

$$\lambda - q\sum_{i=1}^{m}\lambda_i = \frac{\ln q}{\tau}$$

has a unique root satisfying

$$1 < q < \frac{\lambda}{\sum_{i=1}^{m} \lambda_i}.$$

Thus, for $\varphi \in C([-\tau, 0], \mathbb{R}^n)$, and $t \in [t_{k-1}, t_k), k \in \mathbb{N}$, if

$$V(t + \theta, \varphi(\theta)) \le qV(t, \varphi(0)), -\tau \le \theta \le 0,$$

then, by condition $(ii)^*$, we have

$$\begin{aligned} D^+V(t, \varphi(0)) &= V_t(t, \varphi(0)) + V_x(t, \varphi(0))g(t, \varphi(0), \varphi(-h_1(t)), \cdots, \varphi(-h_m(t)) \\ &\le -\lambda V(t, \varphi(0)) + \sum_{i=1}^{m} \lambda_i V(t - h_i(t), \varphi(-h_i(t))) \\ &\le -\left(\lambda - q\sum_{i=1}^{m} \lambda_i\right) V(t, \varphi(0)). \end{aligned}$$

This implies by Corollary 4.1.1 that the trivial solution of system (4.3) is globally exponentially stable. ■

Example 4.1.2 *Consider the following nonlinear impulsive delay system*

$$\begin{cases} \dot{x}(t) = Ax(t) + F(t, x(t), x(t - \tau)), \quad t \ge t_0 = 0, \ t \ne t_k, \\ \Delta x(t) = C_k x(t^-), \quad t = t_k, \quad k \in \mathbb{N}, \\ x_{t_0} = \phi, \end{cases} \tag{4.4}$$

where

$$A = \begin{bmatrix} -10 & 0 & 3 \\ 0 & -15 & 8 \\ 3 & 8 & -24 \end{bmatrix},$$

and $F(t, x(t), x(t - \tau)) = \frac{1}{11}\left(\frac{x_1(t-\tau)}{1+\sin^2 t + \|x(t)\|^2} \quad x_2(t - \tau)\sin(x_3(t)) \quad x_2(t - \tau)\cos(x_3(t))\right)^T$, τ *is a positive constant.*

Because A is Hurwitz, there exist positive definite symmetric matrices Q and P such that

$$A^T Q + QA = -P. \tag{4.5}$$

Let

$$Q = \begin{bmatrix} 7 & 2 & -3 \\ 2 & 5 & 0 \\ -3 & 0 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} 158 & 74 & -148 \\ 74 & 150 & -70 \\ -148 & -70 & 162 \end{bmatrix},$$

so that equation (4.5) holds.

Let $V(t,x) = x^T Q x$, *then*

$$\begin{aligned} D^+V(t,x) &= x'^T Qx + x^T Qx' \;=\; -x^T Px + 2x^T QF \\ &\le -11\|x\|^2 + \|Q\|(\|x\|^2 + \|F\|^2) \\ &\le -\|x\|^2 + 10\|F\|^2 \;\le\; -\|x\|^2 + \tfrac{1}{12}\|x(t-\tau)\|^2 \\ &\le -\tfrac{1}{\lambda_{\max}(Q)} V(t,x) + \tfrac{1}{12\lambda_{\min}(Q)} V(t-\tau, x(t-\tau)) \\ &\le -\tfrac{1}{10} V(t,x) + \tfrac{1}{12} V(t-\tau, x(t-\tau)), \end{aligned}$$

which implies that condition $(ii)^*$ *of Corollary 4.1.2 holds.*

Choose $d_k = \frac{1}{2^{k-4}}$ *and*

$$C_k = \begin{bmatrix} \frac{1}{2^k} & \frac{3}{2^{k+1}} & 0 \\ -\frac{3}{2^k} & -\frac{1}{2} + \frac{1}{2^{k+1}} & 0 \\ 0 & 0 & \frac{3}{2^k} \end{bmatrix}, \quad k \in \mathbb{N},$$

then condition (iii) *of Corollary 4.1.2 holds. Thus by Corollary 4.1.2, system (4.4) is globally exponentially stable. The numerical simulation of this example with* $\tau = \frac{1}{3}$, $t_k = k$ *is given in Figure 4.2.*

Next, we shall consider two special cases of g.

Case 1.

$$g(t,x,y_1,\cdots,y_m) = Ax + G(t,x,y_1,\cdots,y_m), \quad \text{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{4.6}$$

where $A \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix, and $G \in C(\mathbb{R}_+ \times \mathbb{R}^{n\times(m+1)}, \mathbb{R}^n)$.

Since A is a Hurwitz matrix, there exists a unique positive definite symmetric matrix Q such that

$$QA + A^T Q = -I, \tag{4.7}$$

where I is the identity matrix.

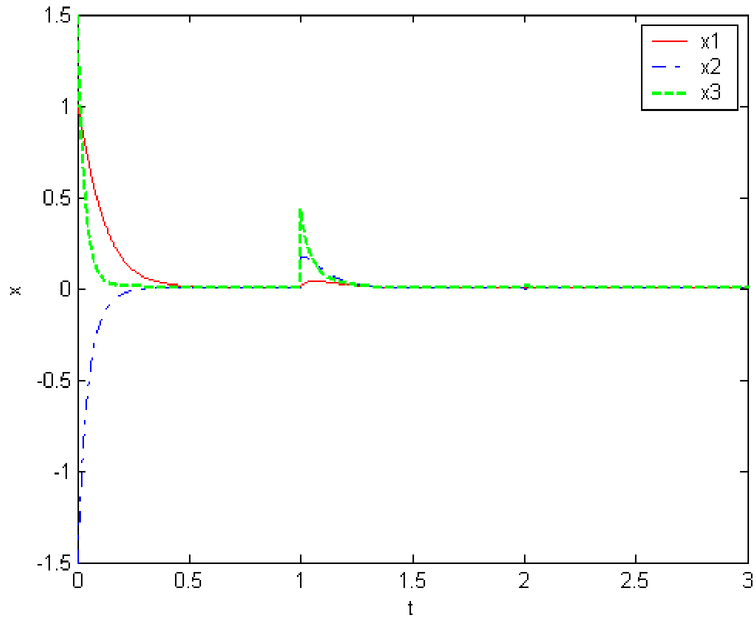

Figure 4.2: Numerical simulation of Example 4.1.2, impulsive system.

Corollary 4.1.3 *Let $p \geq 2$. Assume that conditions (4.6) and (4.7) hold and the matrix A is Hurwitz. If there exist nonnegative real numbers $\alpha_i, i = 0, 1, 2, \cdots, m$, such that for all $(t, x, y_1, \cdots, y_m) \in [t_{k-1}, t_k) \times \mathbb{R}^{n\times(m+1)}$, $k \in \mathbb{N}$, the following conditions hold:*

(i)

$$\|G(t, x, y_1, \cdots, y_m)\| \leq \alpha_0\|x\| + \sum_{i=1}^{m} \alpha_i\|y_i\|; \tag{4.8}$$

(ii)

$$\frac{p}{2\lambda_{\max}(Q)} - \frac{\alpha_0 p\lambda_{\max}(Q)}{\lambda_{\min}(Q)} - \frac{(p-1)\lambda_{\max}(Q)}{(\lambda_{\min}(Q))^{\frac{1}{2}}}\sum_{i=1}^{m}\alpha_i - \frac{\lambda_{\max}(Q)}{(\lambda_{\min}(Q))^{\frac{p+1}{2}}}\sum_{i=1}^{m}\alpha_i > 0; \tag{4.9}$$

(iii) there exists a sequence $\{d_k\}$ with $d_k \geq 0$, and $\sum_{k=0}^{\infty} d_k < \infty$, such that for all $\varphi \in PC([-\tau, 0], \mathbb{R}^n$, the following inequality holds:

$$\|\varphi(0) + I_k(\varphi(0))\| \leq (1 + d_k)^{\frac{1}{p}}\sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}}\|\varphi(0)\|. \tag{4.10}$$

Then system (4.3) is globally exponentially stable.

Proof. Let $V(x) = \left(x^TQx\right)^{\frac{p}{2}}$, where Q is the matrix given by equation (4.7). Then, for all $(t,x) \in [t_{k-1}, t_k) \times \mathbb{R}^n$, and $y_i = x(t-h_i(t)), k \in \mathbb{N}, i = 1, 2, \cdots, m$, we have

$$\begin{aligned}
D^+V(t,x) &= p\left(x^TQx\right)^{\frac{p}{2}-1} x^TQg(t,x,y_1,\cdots,y_m) \\
&= p\left(x^TQx\right)^{\frac{p}{2}-1} x^TQ(Ax + G(t,x,y_1,\cdots,y_m)) \\
&\le \frac{p}{2}\left(x^TQx\right)^{\frac{p}{2}-1} x^T(QA + A^TQ)x + p\left(x^TQx\right)^{\frac{p}{2}-1}\|x\|\|Q\|\left(\alpha_0\|x\| + \sum_{i=1}^m \alpha_i\|y_i\|\right) \\
&\le -\frac{p}{2\lambda_{\max}(Q)}\left(x^TQx\right)^{\frac{p}{2}} + \frac{\alpha_0 p\lambda_{\max}(Q)}{\lambda_{\min}(Q)}\left(x^TQx\right)^{\frac{p}{2}} \\
&\quad + p\lambda_{\max}(Q)\sum_{i=1}^m \alpha_i\left(x^TQx\right)^{\frac{p}{2}-1}\|x\|\|y_i\|. \qquad (4.11)
\end{aligned}$$

In order to estimate the last term of (4.11), we shall make use of the following well-known inequality

$$a^\mu b^{1-\mu} \le \mu a + (1-\mu)b, \quad \text{for all} \quad a, b \ge 0, 0 \le \mu < 1. \qquad (4.12)$$

From (4.12), we have

$$\begin{aligned}
\left(x^TQx\right)^{\frac{p}{2}-1}\|x\|\|y_i\| &\le \frac{1}{(\lambda_{\min}(Q))^{\frac{1}{2}}}(x^TQx)^{\frac{p-1}{2}}\|y_i\| \\
&= \frac{1}{(\lambda_{\min}(Q))^{\frac{1}{2}}}\left[(x^TQx)^{\frac{p}{2}}\right]^{\frac{p-1}{p}}(\|y_i\|^p)^{\frac{1}{p}} \\
&\le \frac{1}{(\lambda_{\min}(Q))^{\frac{1}{2}}}\left(\frac{p-1}{p}(x^TQx)^{\frac{p}{2}} + \frac{1}{p}\|y_i\|^p\right) \\
&\le \frac{1}{(\lambda_{\min}(Q))^{\frac{1}{2}}}\left(\frac{p-1}{p}(x^TQx)^{\frac{p}{2}} + \frac{1}{p(\lambda_{\min}(Q))^{\frac{p}{2}}}\left(y_i^TQy_i\right)^{\frac{p}{2}}\right). \quad (4.13)
\end{aligned}$$

Substituting (4.13) into (4.11), we obtain

$$\begin{aligned}
D^+V(t,x) &\le -\left(\frac{p}{2\lambda_{\max}(Q)} - \frac{\alpha_0 p\lambda_{\max}(Q)}{\lambda_{\min}(Q)} - \frac{(p-1)\lambda_{\max}(Q)}{(\lambda_{\min}(Q))^{\frac{1}{2}}}\sum_{i=1}^m \alpha_i\right)V(t,x) \\
&\quad + \frac{\lambda_{\max}(Q)}{(\lambda_{\min}(Q))^{\frac{p+1}{2}}}\sum_{i=1}^m \alpha_i V(t-h_i(t), x(t-h_i(t))). \qquad (4.14)
\end{aligned}$$

On the other hand, by condition (iii), we obtain

$$
\begin{aligned}
V(t_k, \varphi(0) + I_k(\varphi)) &= \Big\{ \big(\varphi(0) + I_k(\varphi)\big)^T Q \big(\varphi(0) + I_k(\varphi)\big) \Big\}^{\frac{p}{2}} \\
&\leq \Big\{ \lambda_{\max}(Q) \|\varphi(0) + I_k(\varphi)\|^2 \Big\}^{\frac{p}{2}} \leq \Big\{ (1 + d_k)^{\frac{2}{p}} \lambda_{\min}(Q) \|\varphi(0)\|^2 \Big\}^{\frac{p}{2}} \\
&\leq (1 + d_k) \Big\{ \big(\varphi(0)\big)^T Q \big(\varphi(0)\big) \Big\}^{\frac{p}{2}} \\
&= (1 + d_k) V(t_k^-, \varphi(0)). \qquad (4.15)
\end{aligned}
$$

Thus, the conclusion of the corollary follows readily from Corollary 4.1.2. ■

Case 2.

$$
g(t, x, y_1, \cdots, y_m) = g(t, x, x, \cdots, x) + G(t, x, y_1, \cdots, y_m), \quad \text{for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \quad (4.16)
$$

where $G(t, x, y_1, \cdots, y_m) = g(t, x, y_1, \cdots, y_m) - g(t, x, x, \cdots, x)$. In this case, the time delay helps to stabilize the system.

Corollary 4.1.4 *Let $p \geq 2$. Assume that (4.16) holds and there exist nonnegative real numbers $\lambda > 0$ and $\alpha_i, i = 1, 2, \cdots, m$, such that for all $(t, x, y_1, \cdots, y_m), (t, \bar{x}, \bar{y}_1, \cdots, \bar{y}_m) \in [t_{k-1}, t_k) \times \mathbb{R}^{n \times (m+1)}, k \in \mathbb{N}$, the following conditions hold:*

(i)

$$
x^T g(t, x, x, \cdots, x) \leq -\lambda \|x\|^2; \quad (4.17)
$$

(ii)

$$
\|g(t, \bar{x}, \bar{y}_1, \cdots, \bar{y}_m) - g(t, x, y_1, \cdots, y_m)\| \leq \alpha_0 \|\bar{x} - x\| + \sum_{i=1}^{m} \alpha_i \|\bar{y}_i - y_i\|; \quad (4.18)
$$

(iii)

$$
p\lambda - \big((p-1) + 2^p\big) \sum_{i=1}^{m} \alpha_i > 0; \quad (4.19)
$$

(iv) for all $\varphi \in PC([-\tau, 0], \mathbb{R}^n)$, and $d_k \geq 0$ with $\sum_{k=0}^{\infty} d_k < \infty$,

$$\|\varphi(0) + I_k(\varphi(0))\| \leq (1 + d_k)^{\frac{1}{p}} \|\varphi(0)\|. \tag{4.20}$$

Then system (4.3) is globally exponentially stable.

Proof. Let $V(t, x) = \|x\|^p$. Then, for all $(t, x) \in [t_{k-1}, t_k) \times \mathbb{R}^n$ and $y_i = x(t - h_i(t)), k \in \mathbb{N}, i = 1, 2, \cdots, m$, we have

$$\begin{aligned} D^+V(t, x) &= p\|x\|^{p-2} x^T g(t, x, y_1, \cdots, y_m) \\ &= p\|x\|^{p-2} x^T \big(g(t, x, x, \cdots, x) + G(t, x, y_1, \cdots, y_m)\big) \\ &\leq -p\lambda \|x\|^p + p \sum_{i=1}^{m} \alpha_i \big\|x\|^{p-1} \|x - y_i\|. \end{aligned} \tag{4.21}$$

From (4.12), we obtain

$$\begin{aligned} &\|x\|^{p-1}\|x - y_i\| = \big(\|x\|^p\big)^{\frac{p-1}{p}} \big(\|x - y_i\|^p\big)^{\frac{1}{p}} \leq \frac{p-1}{p}\|x\|^p + \frac{1}{p}\|x - y_i\|^p \\ &\leq \frac{p-1}{p}\|x\|^p + \frac{2^{p-1}}{p}\big(\|x\|^p + \|y_i\|^p\big) = \frac{(p-1) + 2^{p-1}}{p}\|x\|^p + \frac{2^{p-1}}{p}\|y_i\|^p. \end{aligned} \tag{4.22}$$

Substituting (4.22) into (4.21), it follows that

$$D^+V(t, x) \leq -\left(p\lambda - \big((p-1) + 2^{p-1}\big) \sum_{i=1}^{m} \alpha_i\right) V(t, x) + 2^{p-1} \sum_{i=1}^{m} \alpha_i V(t - h_i(t), x(t - h_i(t))).$$

Thus, the conclusion of the corollary follows readily from Corollary 4.1.2. ■

When the functions g, I_k, $k \in \mathbb{N}$ are linear, then for $m = 1$, system (4.3) reduces to the following linear impulsive delay system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t - h(t)), \quad t \in [t_{k-1}, t_k), \\ \Delta x(t) = C_k x(t^-), \quad t = t_k, k \in \mathbb{N}, \\ x_{t_0} = \phi, \end{cases} \tag{4.23}$$

where $t - h(t)$ is strictly increasing on $\mathbb{R}_+$ and $0 \leq h(t) \leq \tau$.

Corollary 4.1.5 *Assume that $A + A^T$ is negative definite and for some constant $q > 1$,*

$$-\frac{1}{2q^{\frac{1}{2}}}\lambda_{\max}(A + A^T) > \|B\|. \tag{4.24}$$

Furthermore, assume that $C_k, k \in \mathbb{N}$, and for some $d_k \geq 0$ with $\sum_{k=0}^{\infty} d_k < \infty$,

$$\|I + C_k\| \leq (1 + d_k)^{\frac{1}{2}}. \tag{4.25}$$

Then system (4.23) is globally exponentially stable.

Proof. It follows from Corollary 4.1.1 by choosing $V(x) = \|x\|^2$. ■

Example 4.1.3 *Consider the following linear impulsive delay system*

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t - \frac{1}{4}(1 + e^{-t})), & t \neq k, \quad k \in \mathbb{N}, \\ \Delta x(t) = C_k x(t^-), & t = k, \quad k \in \mathbb{N}, \\ x_{t_0} = \phi, \end{cases} \tag{4.26}$$

where

$$A = \begin{bmatrix} -13 & 20 & 0 \\ 7 & -35 & 15 \\ 0 & 14 & -20 \end{bmatrix}, \quad B = \begin{bmatrix} -1.5 & 1 & 0 \\ 1 & -0.3 & 0.5 \\ 0 & 1 & -0.1 \end{bmatrix},$$

and

$$C_k = \begin{bmatrix} \frac{1}{2^k} & \frac{3}{2^{k+1}} & 0 \\ -\frac{3}{2^k} & -\frac{1}{2} + \frac{1}{2^{k+1}} & 0 \\ 0 & 0 & \frac{3}{2^k} \end{bmatrix}.$$

Choose $q = 2,\ \delta = 2$ and $\tau = \frac{1}{2}$. Then we have

$$-\frac{1}{2q^{\frac{1}{2}}}\lambda_{\max}(A + A^T) = 2.39 \quad \text{and} \quad \|\mathrm{B}\| = [\lambda_{\max}(\mathrm{B^T B})]^{\frac{1}{2}} = 2.15,$$

so inequality (4.24) holds.

Furthermore, choose $d_k = \frac{1}{2^{k-4}}$. Then for all $k \in \mathbb{N}$, we have

$$\begin{aligned} \|I + C_k\| &= [\lambda_{\max}(I + C_k)^T(I + C_k)]^{\frac{1}{2}} \\ &= 1 + \tfrac{3}{2^k} \ \leq \ (1 + d_k)^{\frac{1}{2}}. \end{aligned}$$

Then we know from Corollary 4.1.5 that system (4.26) is globally exponentially stable. The numerical simulation of this example is given in Figure 4.3.

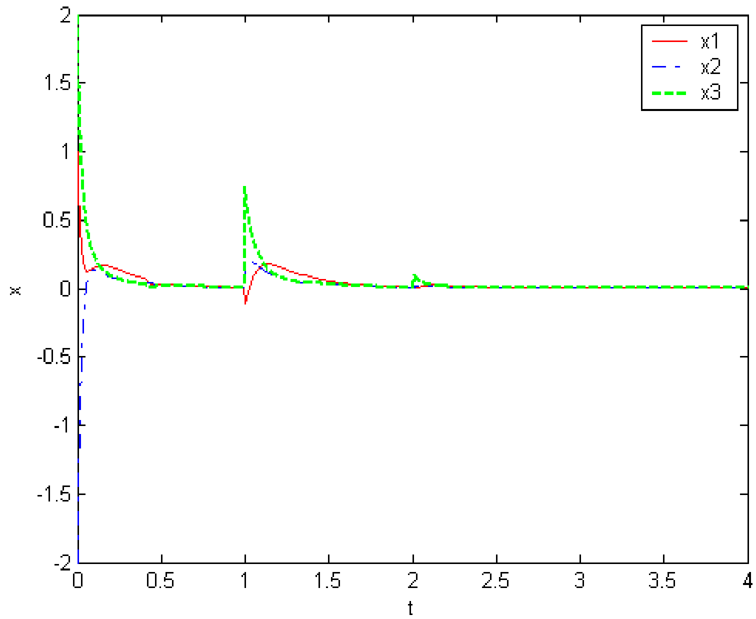

Figure 4.3: Numerical simulation of Example 4.1.3, impulsive system.

Theorem 4.1.2 *Assume that there exist a function $V \in \nu_0$, constants $p > 0, q > 1, c_1 > 0, c_2 > 0$ and $\eta \geq \frac{\ln q}{\tau}$ such that*

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$;

(ii) the upper right-hand derivative of V with respect to system (2.3) satisfies

$$D^+V(t,\varphi(0)) \leq -\eta V(t,\varphi(0)), \text{ for all } t \neq t_k \text{ in } \mathbb{R}_+,$$

whenever $qV(t,\varphi(0)) \geq V(t+s,\varphi(s))$ for $s \in [-\tau,0]$;

(iii) $V(t_k,\varphi(0)+I_k(\varphi)) \leq \psi_k(V(t_k^-,\varphi(0)))$, *where $\varphi(0^-) = \varphi(0)$, and $\psi_k(s)$ is continuous, $0 \leq \psi_k(as) \leq a\psi_k(s)$ and $\psi_k(s) \geq s$ hold for any $a \geq 0$ and $s \geq 0$, and there exists $H \geq 1$ such that*

$$\psi_k(\psi_{k-1}(\cdots(\psi_1(s))\cdots))/s \leq H, \quad s > 0, \quad k \in \mathbb{N}.$$

Then the trivial solution of system (2.3) is globally exponentially stable.

Proof. Choose $q = e^{\lambda\tau} > 1$ for some $\lambda > 0$. We shall show

$$V(t) \leq \psi_{k-1}(\psi_{k-2}(\cdots(\psi_1(\psi_0(V(t_0))))\cdots))e^{-\lambda(t-t_0)}, \qquad t \in [t_{k-1}, t_k), \ \ k \in \mathbb{N},$$

where $\psi_0(s) = s$ for any $s \in \mathbb{R}$. Let

$$Q(t) = \begin{cases} V(t) - \psi_{k-1}(\psi_{k-2}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t-t_0)}, & t \in [t_{k-1}, t_k), \ \ k \in \mathbb{N}, \\ Q(t_0), & t \in [t_0 - \tau, t_0]. \end{cases}$$

When $k = 1$, we shall show $Q(t) \leq 0$ for all $t \in [t_0, t_1)$. In order to do this, we shall show that $Q(t) \leq \alpha$ for any arbitrarily given $\alpha > 0$. Suppose that there exists some $t \in [t_0, t_1)$ so that $Q(t) > \alpha$. Let $t^* = \inf\{t \in [t_0, t_1) : Q(t) > \alpha\}$, since $Q(t_0) \leq V(t_0) - V(t_0) = 0 < \alpha$ and hence $Q(t) \leq \alpha$ for $t \in [t_0 - \tau, t_0]$, we know $t^* \in (t_0, t_1)$. Note that $Q(t)$ is continuous on $[t_0, t_1)$. Then $Q(t^*) = \alpha$ and $Q(t) \leq \alpha$ for $t \in [t_0 - \tau, t^*]$.

Since $V(t^*) = Q(t^*) + V(t_0)e^{-\lambda(t^*-t_0)}$, then for $s \in [-\tau, 0]$, we have

$$\begin{aligned} V(t^* + s) &= Q(t^* + s) + V(t_0)e^{-\lambda(t^*+s-t_0)} \\ &\leq \alpha + V(t_0)e^{-\lambda(t^*-t_0)}e^{\lambda\tau} \\ &\leq (\alpha + V(t_0)e^{-\lambda(t^*-t_0)})e^{\lambda\tau} \\ &= V(t^*)e^{\lambda\tau} \\ &\leq qV(t^*). \end{aligned}$$

So by condition (ii), we have $D^+V(t^*) \leq -\eta V(t^*)$. Then we have

$$\begin{aligned} D^+Q(t^*) &= D^+V(t^*) + \lambda V(t_0)e^{-\lambda(t^*-t_0)} \\ &\leq -\eta V(t^*) + \lambda V(t_0)e^{-\lambda(t^*-t_0)} \\ &\leq -\lambda(V(t^*) - V(t_0)e^{-\lambda(t^*-t_0)}) \\ &= -\lambda\alpha \\ &< 0, \end{aligned}$$

which contradicts the definition of t^*, so we obtain $Q(t) \leq \alpha$ for all $t \in [t_0, t_1)$. Let $\alpha \to 0^+$. We have $Q(t) \leq 0$ for $t \in [t_0, t_1)$.

Now we assume that $Q(t) \leq 0$ for $t \in [t_0, t_k), \ k \geq 1$. Then we shall show $Q(t) \leq 0$ for $t \in [t_0, t_{k+1})$. Let $\alpha > 0$ be arbitrary. We shall show $Q(t) \leq \alpha$ for $t \in (t_k, t_{k+1})$. Suppose not. Let $t^* = \inf\{t \in [t_k, t_{k+1}) : Q(t) > \alpha\}$.

By condition (iii), we have

$$\begin{aligned} Q(t_k) &= V(t_k) - \psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t_k-t_0)} \\ &\le \psi_k(V(t_k^-)) - \psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t_k-t_0)} \\ &\le \psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots)e^{-\lambda(t_k-t_0)}) - \psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t_k-t_0)} \\ &\le e^{-\lambda(t_k-t_0)}\psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots)) - \psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t_k-t_0)} \\ &\le 0. \end{aligned}$$

Since $Q(t_k) \le 0 < \alpha$, by the continuity of $Q(t)$, we have $t^* > t_k,\ Q(t^*) = \alpha$ and $Q(t) \le \alpha$ for $t \in [t_0 - \tau, t^*]$.

Since $V(t^*) = Q(t^*) + \psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t^*-t_0)}$; when $t^* + s \ge t_k$ for all $s \in [-\tau, 0]$, we have, for any $s \in [-\tau, 0]$,

$$\begin{aligned} V(t^*+s) &= Q(t^*+s) + \psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t^*+s-t_0)} \\ &\le \alpha + \psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t^*-\tau-t_0)} \\ &\le (\alpha + \psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t^*-t_0)})e^{\lambda\tau} \\ &\le V(t^*)e^{\lambda\tau} \\ &\le qV(t^*). \end{aligned}$$

When $t^* + s < t_k$ for some $s \in [-\tau, 0]$, note that $0 \le \psi_k(as) \le a\psi_k(s)$ and $\psi_k(s) \ge s$ hold for any $a \ge 0$ and $s \ge 0$. Then we have, for any $s \in [-\tau, 0]$ and $m < k$ with $m, k \in \mathbb{N}$,

$$\psi_m(\psi_{m-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t^*+s-t_0)} \le \psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t^*+s-t_0)}.$$

So in this case, we can also obtain that $V(t^*+s) \le qV(t^*)$ holds for all $s \in [-\tau, 0]$. Thus by condition (ii), we have $D^+V(t^*) \le -\eta V(t^*)$, and then we have

$$\begin{aligned} D^+Q(t^*) &= D^+V(t^*) + \lambda\psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t^*-t_0)} \\ &\le -\eta V(t^*) + \lambda\psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t^*-t_0)} \\ &\le -\lambda[V(t^*) - \psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t^*-t_0)}] \\ &\le -\lambda\alpha \\ &< 0. \end{aligned}$$

Again this contradicts the definition of t^*, which implies $Q(t) \le \alpha$ for all $t \in [t_k, t_{k+1})$. Let $\alpha \to 0^+$. We have $Q(t) \le 0$ for all $t \in [t_k, t_{k+1})$. So $Q(t) \le 0$ for all $t \in [t_0, t_{k+1})$ which proves, by the method of induction, $V(t) \leq \psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t-t_0)}$ for $t \in [t_{k-1}, t_k),\ k \in \mathbb{N}$.

By condition (iii), we obtain

$$\begin{aligned}\psi_k(\psi_{k-1}(\cdots(\psi_0(V(t_0)))\cdots))e^{-\lambda(t-t_0)} &= \psi_k(\psi_{k-1}(\cdots(\psi_1(V(t_0)))\cdots))e^{-\lambda(t-t_0)} \\ &\leq HV(t_0)e^{-\lambda(t-t_0)},\ \ t \geq t_0.\end{aligned}$$

Thus by condition (i), we have

$$c_1\|x\|^p \leq V(t) \leq c_2 H\|\phi\|_\tau^p e^{-\lambda(t-t_0)},\ \ t \geq t_0,$$

i.e.

$$\|x\| \leq (\frac{c_2 H}{c_1})^{\frac{1}{p}}\|\phi\|_\tau e^{-\frac{\lambda(t-t_0)}{p}},\quad t \geq t_0,$$

which completes our proof. ■

Example 4.1.4 *Consider the impulsive nonlinear delay differential equations*

$$\begin{cases} x'(t) &= -y(t)\sin(x(t-1)) - 4x(t) + y(t-1), \quad t \neq k,\ k \in \mathbb{N}, \\ y'(t) &= x(t)\sin(x(t-1)) - 3y(t), \quad t \neq k,\ k \in \mathbb{N}, \\ x(t_k) &= (1+\frac{2}{k^2})x(t_k^-), \quad t = k,\ k \in \mathbb{N}, \\ y(t_k) &= (1-\frac{3}{k^2})y(t_k^-), \quad t = k,\ k \in \mathbb{N}, \\ x_{t_0} &= \phi_1,\ \ y_{t_0} = \phi_2, \quad t_0 = 0, \end{cases} \tag{4.27}$$

where $\phi_i \in PC([-\tau, 0], \mathbb{R}^n)$ for $i = 1, 2$. Then the trivial solution of system (2.3) is globally exponentially stable.

Proof. Choose $V(x, y) = V(t, x, y) = x^2 + y^2$, then

$$\begin{aligned} D^+V(t, \varphi_1, \varphi_2) &= \varphi_1(0)(-\varphi_2(0)\sin(\varphi_1(-1)) - 4\varphi_1(0) \\ &\qquad +\varphi_2(-1)) + \varphi_2(0)(-\varphi_1(0)\sin(\varphi_1(-1)) - 3\varphi_2(0)) \\ &= -4\varphi_1^2(0) + \varphi_1(0)\varphi_2(-1) - 3\varphi_2^2(0) \\ &\leq -4\varphi_1^2(0) + \tfrac{1}{2}(\varphi_1^2(0) + \varphi_2^2(-1)) - 3\varphi_2^2(0) \\ &\leq -6V(\varphi_1(0), \varphi_2(0)) + V(\varphi_1(-1), \varphi_2(-1)). \end{aligned}$$

Let $q = 2,\ \eta = 4$, whenever $qV(\varphi_1(0), \varphi_2(0)) \geq V(\varphi_1(0), \varphi_2(0))$ for $s \in [-1, 0]$. We have

$$\begin{aligned} D^+V(t, \varphi_1, \varphi_2) &\leq -6V(\varphi_1(0), \varphi_2(0)) + V(\varphi_1(-1), \varphi_2(-1)) \\ &\leq -6V(\varphi_1(0), \varphi_2(0)) + 2V(\varphi_1(0), \varphi_2(0)) \\ &\leq -4V(\varphi_1(0), \varphi_2(0)), \end{aligned}$$

i.e., condition (ii) of Theorem 4.1.2 holds.

At last, to check condition (iii), let $\psi_k(s) = (1 + \frac{5}{k^2})(s)$, $k \in \mathbb{N},\ s \in \mathbb{R}$, then for any $k \in \mathbb{N}$,

$$\begin{aligned} V(\varphi_1(0) + \tfrac{2}{k^2}\varphi_1(0), \varphi_2(0) - \tfrac{3}{k^2}\varphi_2(0)) &= \tfrac{1}{2}((1 + \tfrac{2}{k^2})^2\varphi_1^2(0) + (1 - \tfrac{3}{k^2})^2\varphi_2^2(0)) \\ &\leq \psi_k(V(\varphi_1(0), \varphi_2(0))), \end{aligned}$$

i.e., condition (iii) is satisfied. Thus by Theorem 4.1.2, the trivial solution of system (4.27) is globally exponentially stable. The numerical simulation of this example with initial function

$$\phi_1(t) = \begin{cases} 0, & t \in [-1, 0), \\ 2.7, & t = 0, \end{cases} \qquad \phi_2(t) = \begin{cases} 0, & t \in [-1, 0), \\ -2.1, & t = 0, \end{cases}$$

is given in Figure 4.4.

■

Corollary 4.1.6 *Assume that conditions (i), (ii) of Theorem 4.1.2 hold and, condition (iii) is replaced by*

*(iii)** $V(t_k, \varphi(0) + I_k(\varphi)) \leq \psi_k(V(t_k^-, \varphi(0)))$, *where* $\varphi(0^-) = \varphi(0)$ *and* $\psi_k(s) = (1 + \frac{k}{k^3+s^2})s$ *for all* $k \in \mathbb{N}$.

Then the trivial solution of system (2.3) is globally exponentially stable.

Proof. Notice

$$\psi_k(s) = (1 + \frac{k}{k^3 + s^2})s \leq |s|(1 + \frac{1}{k^2}), \quad k \in \mathbb{N},$$

then by Theorem 4.1.2, the result holds. ■

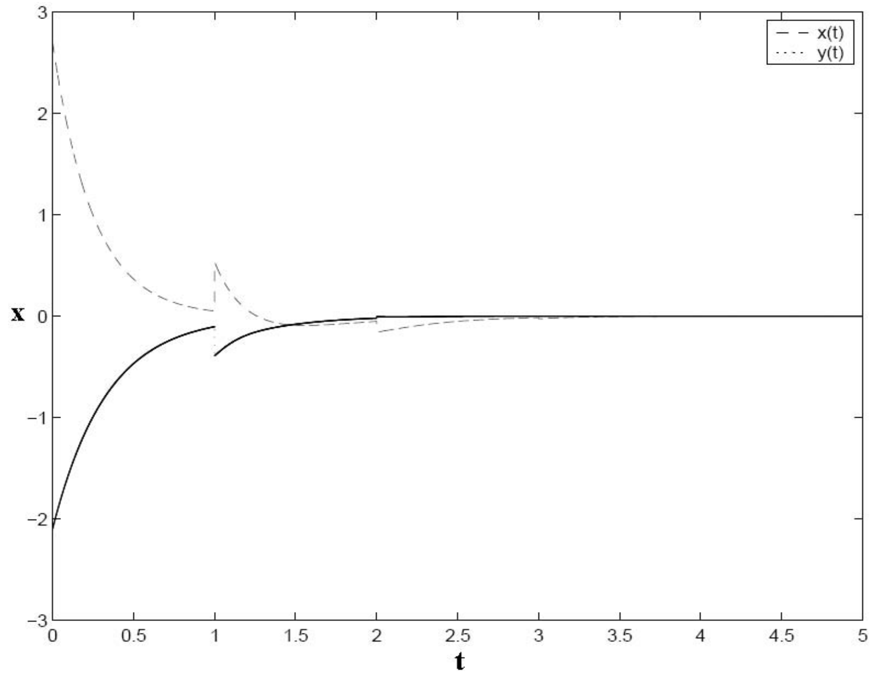

Figure 4.4: Numerical simulation of Example 4.1.4, impulsive system.

4.2 Stabilization via Lyapunov-Razumikhin Method

In this section, we establish several criteria for global exponential stability of impulsive delay differential equation (2.3), which are then used to impulsively stabilize delay differential equations.

Theorem 4.2.1 *Assume that there exist function $V \in \nu_0$, constants $p, c_1, c_2, \lambda > 0$ and $\alpha > \tau$ such that*

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$, for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$;

(ii) the upper right-hand derivative of V with respect to system (2.3) satisfies

$$D^+V(t,\varphi(0)) \leq 0, \text{for all } t \in [t_{k-1}, t_k),\ k \in \mathbb{N},$$

whenever $qV(t,\varphi(0)) \geq V(t+s,\varphi(s))$ for $s \in [-\tau, 0]$, where $q \geq e^{2\lambda\alpha}$ is a constant;

(iii) $V(t_k, \varphi(0) + I_k(\varphi)) \le d_k V(t_k^-, \varphi(0))$, *where* $d_k > 0$, $\forall k \in \mathbb{N}$ *are constants;*

(iv) $\tau \le t_k - t_{k-1} \le \alpha$ *and* $\ln(d_k) + \lambda\alpha < -\lambda(t_{k+1} - t_k)$.

Then the trivial solution of the impulsive system (2.3) is globally exponentially stable with convergence rate $\frac{\lambda}{p}$.

Proof. Choose $M \ge 1$ such that

$$c_2\|\phi\|_\tau^p < M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)} \le qc_2\|\phi\|_\tau^p. \tag{4.28}$$

Let $x(t) = x(t, t_0, \phi)$ be any solution of system (2.3) with $x_{t_0} = \phi$, and $v(t) = V(t, x)$. We shall show

$$v(t) \le M\|\phi\|_\tau^p e^{-\lambda(t_k - t_0)}, \ \ t \in [t_{k-1}, t_k), \ k \in \mathbb{N}. \tag{4.29}$$

We first show that

$$v(t) \le M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)}, \ \ t \in [t_0, t_1). \tag{4.30}$$

From condition (i) and (4.28), we have, for $t \in [t_0 - \tau, t_0]$,

$$v(t) \le c_2\|x\|^p \le c_2\|\phi\|_\tau^p < M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)}.$$

If (4.30) is not true, then there must exist some $\bar{t} \in (t_0, t_1)$ such that

$$v(\bar{t}) > M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)} > c_2\|\phi\|_\tau^p \ge v(t_0 + s), \ \ s \in [-\tau, 0], \tag{4.31}$$

which implies that there exists some $t^* \in (t_0, \bar{t})$ such that

$$v(t^*) = M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)}, \quad \text{and} \ \ v(t) \le M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)}, \quad \text{for } t_0 - \tau \le t \le t^*, \tag{4.32}$$

and there exists $t^{**} \in [t_0, t^*)$ such that

$$v(t^{**}) = c_2\|\phi\|_\tau^p, \ \ \text{and} \ \ v(t) \ge c_2\|\phi\|_\tau^p, \ \text{for } t^{**} \le t \le t^*. \tag{4.33}$$

Then we obtain, for any $t \in [t^{**}, t^*]$,

$$v(t + s) \le M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)} \le qc_2\|\phi\|_\tau^p \le qv(t), \ \ s \in [-\tau, 0]. \tag{4.34}$$

Thus by condition (ii), we have $D^+v(t) \leq 0$ for $t \in [t^{**}, t^*]$, and then we obtain $v(t^{**}) \geq v(t^*)$, i.e., $c_2\|\phi\|_\tau^p \geq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}$, which contradicts (4.28). Hence (4.30) holds and then (4.29) is true for $k = 1$.

Now we assume that (4.29) holds for $k = 1, 2, \cdots, m$, i.e.

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_k-t_0)}, \quad t \in [t_{k-1}, t_k),\ k = 1, 2, \cdots, m. \tag{4.35}$$

We shall show that (4.29) holds for $k = m + 1$, i.e.

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)}, \quad t \in [t_m, t_{m+1}). \tag{4.36}$$

For the sake of contradiction, suppose (4.36) is not true. Then we define

$$\bar{t} = \inf\{t \in [t_m, t_{m+1})|v(t) > M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)}\}.$$

By the continuity of $v(t)$ in the interval $[t_m, t_{m+1})$, we have

$$\begin{aligned} v(\bar{t}) &= M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \quad \text{and} \\ v(t) &\leq M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)}, \ \text{for } t \in [t_m, \bar{t}). \end{aligned} \tag{4.37}$$

Since

$$\begin{aligned} v(t_m) &\leq d_m v(t_m^-) \\ &< e^{-\lambda\alpha} e^{-\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(t_m-t_0)} \\ &< M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)}, \end{aligned}$$

i.e.

$$v(t_m) < e^{-\lambda\alpha} M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} < v(\bar{t}),$$

which implies that there exists some $t^* \in (t_m, \bar{t})$ such that

$$v(t^*) \geq e^{-\lambda\alpha} M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \quad \text{and} \quad D^+v(t^*) > 0. \tag{4.38}$$

Then we know $t^* + s \in [t_{m-1}, \bar{t})$ for $s \in [-\tau, 0]$ since $\tau \leq t_k - t_{k-1} \leq \alpha$. By (4.35) and (4.37), we obtain

$$\begin{aligned} v(t^* + s) &\leq M\|\phi\|_\tau^p e^{-\lambda(t_m-t_0)} \\ &= M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} e^{\lambda(t_{m+1}-t_m)} \\ &\leq e^{\lambda\alpha} M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \\ &\leq qv(t^*), \quad s \in [-\tau, 0]. \end{aligned}$$

Then from condition (ii), we obtain $D^+v(t^*) \leq 0$, contradicting (4.38). This implies that the assumption is not true, and hence (4.29) holds for $k = m+1$. Thus by mathematical induction, we obtain that (4.29) holds, so we have

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k).$$

Hence by condition (i), we have

$$\|x\| \leq M^*\|\phi\|_\tau e^{-\frac{\lambda}{p}(t-t_0)}, \quad t \in [t_{k-1}, t_k),\ k \in \mathbb{N},$$

where $M^* \geq \max\{1, [\frac{M}{c_1}]^{\frac{1}{p}}\}$. This implies that the trivial solution of system (2.3) is globally exponentially stable with convergence rate $\frac{\lambda}{p}$. ■

Remark 4.2.1 *It is well-known that, in the stability theory of functional differential equations, the condition $D^+V(t,x) \leq 0$ can not even guarantee the asymptotic stability of a functional differential system (see [38]). However, as we can see from Theorem 4.2.1, impulses have played an important role in exponentially stabilizing a functional differential system.*

Example 4.2.1 *Consider the following impulsive delay differential system*

$$\begin{cases} x_1'(t) &= x_2(t) - 0.001x_1(t),\ t \geq 0,\ t \neq k, \\ x_2'(t) &= -x_1(t) - 0.001x_2(t) + x_3^2(t),\ t \geq 0,\ t \neq k, \\ x_3'(t) &= -(0.005 + x_2 + t^2\sin^2(x_1(t)))x_3(t) \\ & \quad +0.001x_3(t-0.07),\ t \geq 0,\ t \neq k, \\ x(k) &= d_k^{\frac{1}{2}}x(k^-), \qquad k \in \mathbb{N}, \end{cases} \tag{4.39}$$

where $x = (x_1, x_2, x_3)^T,\ d_k, \tau \geq 0$. Assume that d_k satisfies

$$d_k \leq e^{-(\alpha+1)\lambda}, \tag{4.40}$$

where $\alpha,\ \lambda > 0$ are constants. Then the trivial solution of (4.39) is exponentially stable with convergence rate $\frac{\lambda}{2}$.

Proof. Choose $V(t,x) = \frac{1}{2}\|x\|^2 = \frac{1}{2}\sum_{i=1}^3 |x_i|^2$ so that condition (i) of Theorem 4.2.1 holds for $c_1 = c_2 = \frac{1}{2},\ p = 2$.

Calculate the upper right-hand derivative of V with respect to equation (4.39)

$$\begin{aligned} D^+V(t,x(t)) &= x_1(t)x_1'(t)+x_2(t)x_2'(t)+x_3(t)x_3'(t) \\ &= -0.001(|x_1(t)|^2+|x_2(t)|^2)-0.005|x_3(t)|^2 \\ &\quad -t^2\sin^2(x_1(t))x_3^2(t)+0.001x_3(t)x_3(t-0.07) \\ &\leq -0.001\|x(t)\|^2-t^2\sin^2(x_1(t))x_3^2(t)+0.0005x_3^2(t-0.07). \end{aligned}$$

Choose $\lambda=0.25,\ \alpha=1,\ q=2>e^{0.5}=1.6487$. Whenever $qV(t,\varphi(0))\geq V(t+s,\varphi(s))$ for $s\in[-0.5,0]$, i.e., $\|x(t+s)\|^2\leq 2\|x(t)\|^2$ for $s\in[-0.07,0]$, we have

$$\begin{aligned} D^+V(t,x(t)) &\leq -0.001\|x(t)\|^2-t^2\sin^2(x_1(t))x_3^2(t)+0.001\|x(t)\|^2 \\ &\leq -t^2\sin^2(x_1(t))x_3^2(t) \\ &\leq 0, \end{aligned}$$

which implies condition (ii) of Theorem 4.2.1 holds.

Furthermore, we have

$$V_1(k,x(k))=d_kx(k^-),$$

which, together with (4.40), yields that condition (iii) and (iv) of Theorem 4.2.1 hold. Then by Theorem 4.2.1, the trivial solution of (4.39) is globally exponentially stable, and its convergence rate is $\frac{\lambda}{2}$. The numerical simulation of this delay differential equation with $\tau=0.5$, $\alpha=1,\lambda=0.25$ and initial functions $\phi_1(t)=\phi_2(t)=\phi_3(t)=0$ for $t\in[-0.07,0)$ and $\phi_1(0)=0.8,\ \phi_2(0)=-0.5,\ \phi_3(0)=2.502$ is given in Figure 4.5; while the simulation of the impulsive system with $d_k=0.36,\ t_k=k$ for $k\in\mathbb{N}$ is given in Figure 4.6. ■

Remark 4.2.2 *The trivial solution of the corresponding delay differential equation (4.39) without impulses is stable but not asymptotically stable (see Figure 4.5), as we can see in Example 4.2.1, impulses do contribute to the exponentially impulsive stabilization of the system (see Figure 4.6).*

Theorem 4.2.2 *Assume that there exist function $V\in\nu_0$ and constants $p,c,c_1,c_2>0$ and $\alpha>\tau$, $\lambda>c$ such that*

(i) $c_1\|x\|^p\leq V(t,x)\leq c_2\|x\|^p$, for any $t\in\mathbb{R}_+$ and $x\in\mathbb{R}^n$;

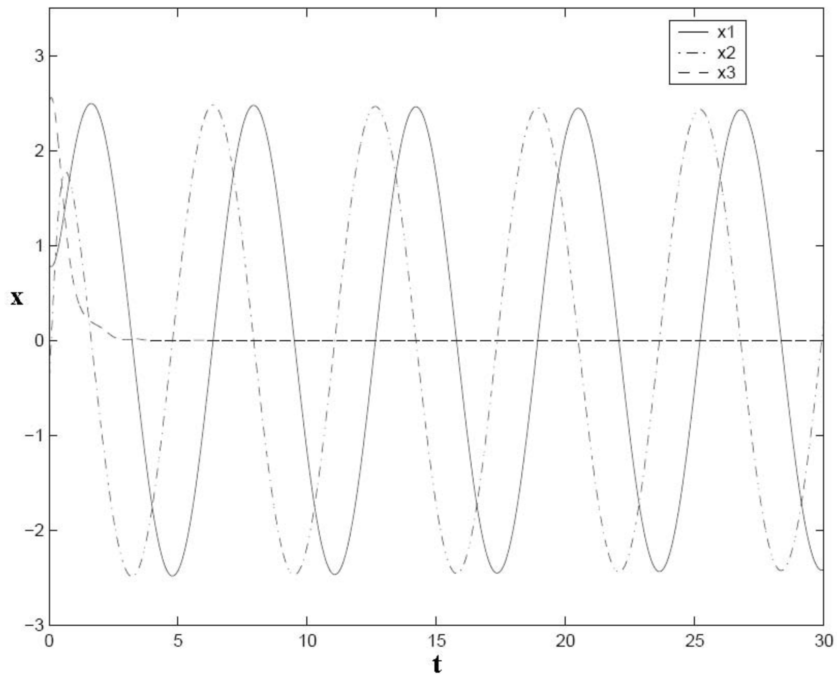

Figure 4.5: Numerical simulation of Example 4.2.1, system without impulses.

(ii) the upper right-hand derivative of V *with respect to system (2.3) satisfies*

$$D^+V(t,\varphi(0)) \le cV(t,\varphi(0)),\ \textit{for all } t \in [t_{k-1},t_k),\ \ k \in \mathbb{N},$$

whenever $qV(t,\varphi(0)) \ge V(t+s,\varphi(s))$ *for* $s \in [-\tau,0]$*, where* $q \ge e^{2\lambda\alpha}$ *is a constant;*

(iii) $V(t_k,\varphi(0)+I_k(\varphi)) \le d_kV(t_k^-,\varphi(0))$*, where* $d_k > 0$*,* $\forall k \in \mathbb{N}$ *are constants;*

(iv) $\tau \le t_k - t_{k-1} \le \alpha$ *and* $\ln(d_k) + \lambda\alpha < -\lambda(t_{k+1} - t_k)$*.*

Then the trivial solution of the impulsive system (2.3) is globally exponentially stable and the convergence rate is $\frac{\lambda}{p}$*.*

Proof. Choose $M \ge 1$ such that

$$c_2\|\phi\|_\tau^p < M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}e^{-\alpha c} < M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} \le qc_2\|\phi\|_\tau^p. \tag{4.41}$$

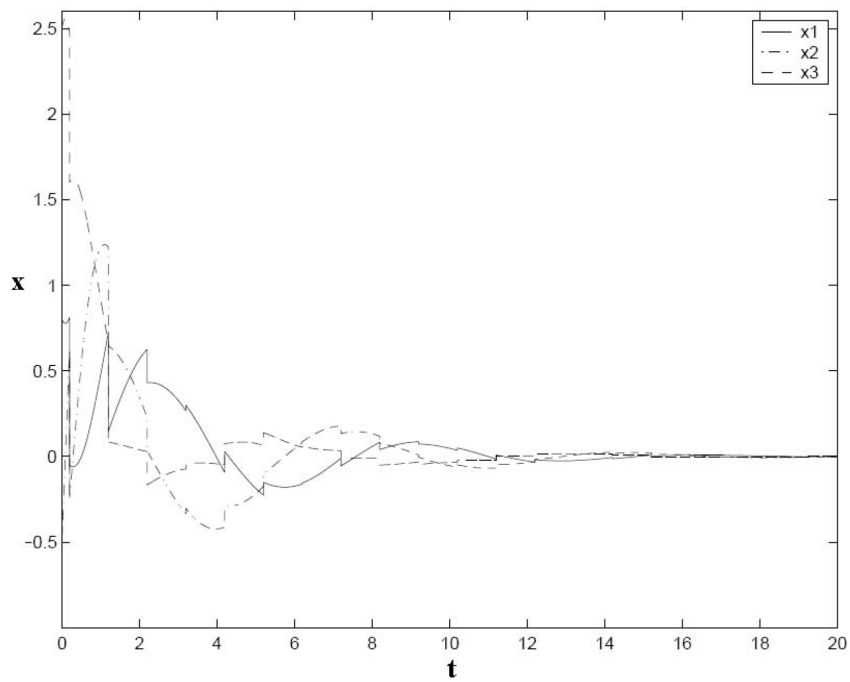

Figure 4.6: Numerical simulation of Example 4.2.1, impulse-stabilized system.

Let $x(t) = x(t, t_0, \phi)$ be any solution of system (2.3) with $x_{t_0} = \phi$, and $v(t) = V(t, x)$. We shall show

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_k - t_0)}, \;\; t \in [t_{k-1}, t_k), \; k \in \mathbb{N}. \tag{4.42}$$

We first show that

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)}, \;\; t \in [t_0, t_1). \tag{4.43}$$

From condition (i) and (4.41), we have, for $t \in [t_0 - \tau, t_0]$,

$$v(t) \leq c_2\|x\|^p \leq c_2\|\phi\|_\tau^p < M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)} e^{-\alpha c}.$$

If (4.43) is not true, then there must exist some $\bar{t} \in (t_0, t_1)$ such that

$$\begin{aligned} v(\bar{t}) &> M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)} > M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)} e^{-\alpha c} \\ &> c_2\|\phi\|_\tau^p \geq v(t_0 + s), \quad s \in [-\tau, 0], \end{aligned} \tag{4.44}$$

which implies that there exists some $t^* \in (t_0, \bar{t})$ such that

$$v(t^*) = M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)}, \;\text{ and }\; v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_1 - t_0)}, \quad t_0 - \tau \leq t \leq t^*, \tag{4.45}$$

and there exists $t^{**} \in [t_0, t^*)$ such that

$$v(t^{**}) = c_2\|\phi\|_\tau^p, \text{ and } v(t) \geq c_2\|\phi\|_\tau^p, \quad t^{**} \leq t \leq t^*. \tag{4.46}$$

Then we obtain, for any $t \in [t^{**}, t^*]$,

$$v(t+s) \leq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} \leq qc_2\|\phi\|_\tau^p \leq qv(t), \quad s \in [-\tau, 0], \tag{4.47}$$

thus by condition (ii), we obtain $D^+v(t) \leq cv(t)$ for $t \in [t^{**}, t^*]$, and then we have $v(t^{**}) \geq v(t^*)e^{-\alpha c}$, i.e., $c_2\|\phi\|_\tau^p \geq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}e^{-\alpha c}$, which contradicts (4.41). Hence (4.43) holds and then (4.42) is true for $k = 1$.

Now we assume that (4.42) holds for $k = 1, 2, \cdots, m (m \in \mathbb{N},\ m \geq 1)$, i.e.

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_k-t_0)}, \quad t \in [t_{k-1}, t_k),\ k = 1, 2, \cdots, m. \tag{4.48}$$

From condition (iii) and (4.48), we have

$$\begin{aligned} v(t_m) &\leq d_m v(t_m^-) \\ &< e^{-\lambda\alpha}e^{-\lambda(t_{m+1}-t_m)}M\|\phi\|_\tau^p e^{-\lambda(t_m-t_0)} \\ &< M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)}. \end{aligned} \tag{4.49}$$

Next, we shall show that (4.42) holds for $k = m + 1$, i.e.

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)}, \quad t \in [t_m, t_{m+1}). \tag{4.50}$$

For the sake of contradiction, suppose (4.50) is not true. Then we define

$$\bar{t} = \inf\{t \in [t_m, t_{m+1}) | v(t) > M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)}\}.$$

From (4.49), we know $\bar{t} \neq t_m$. By the continuity of $v(t)$ in the interval $[t_m, t_{m+1})$, we have

$$v(\bar{t}) = M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \quad \text{and} \quad v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)}, \quad t \in [t_m, \bar{t}]. \tag{4.51}$$

From (4.49), we have

$$v(t_m) < e^{-\lambda\alpha}M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} < v(\bar{t}),$$

which implies that there exists some $t^* \in (t_m, \bar{t})$ such that

$$v(t^*) = e^{-\lambda\alpha}M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \quad \text{and} \quad v(t^*) \leq v(t) \leq v(\bar{t}), \quad t \in [t^*, \bar{t}]. \tag{4.52}$$

Then we know $t+s \in [t_{m-1}, \bar{t}]$ for $t \in [t^*, \bar{t}]$ and $s \in [-\tau, 0]$ since $\tau \le t_k - t_{k-1} \le \alpha$. By (4.48) and (4.51), we have, for $t \in [t^*, \bar{t}]$,

$$\begin{aligned} v(t+s) &\le M\|\phi\|_\tau^p e^{-\lambda(t_m - t_0)} \\ &= M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} e^{\lambda(t_{m+1}-t_m)} \\ &\le e^{\lambda\alpha} M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \\ &= e^{2\lambda\alpha} v(t^*) \le q v(t), \quad s \in [-\tau, 0]. \end{aligned}$$

Then from condition (ii), we obtain $D^+v(t) \le cv(t)$; since $\lambda > c$, we have

$$v(\bar{t}) \le v(t^*)e^{\alpha c} = e^{-\lambda\alpha} M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} e^{\alpha c} < v(\bar{t}),$$

a contradiction with (4.52). This implies the assumption is not true, and hence (4.42) holds for $k = m+1$. Thus by mathematical induction, (4.42) holds. Hence,

$$v(t) \le M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k).$$

Then by condition (i), we obtain

$$\|x\| \le M^*\|\phi\|_\tau e^{-\frac{\lambda}{p}(t-t_0)}, \quad t \in [t_{k-1}, t_k),\ k \in \mathbb{N},$$

where $M^* \ge \max\{1, [\frac{M}{c_1}]^{\frac{1}{p}}\}$. This implies that the trivial solution of system (2.3) is globally exponentially stable with convergence rate $\frac{\lambda}{p}$. ■

Remark 4.2.3 *If the condition $\lambda > c$ is removed in Theorem 4.2.2, then we require $q \ge \max\{e^{\alpha c}, e^{2\lambda\alpha}\}$ in condition (ii) and condition (iv) to be strengthened. The details are stated in the following result whose proof is similar and thus omitted.*

Theorem 4.2.3 *Assume that there exist function $V \in \nu_0$ and constants $p, c, c_1, c_2, \lambda > 0$ and $\alpha > \tau$ such that*

(i) $c_1\|x\|^p \le V(t,x) \le c_2\|x\|^p$, for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$;

(ii) the upper right-hand derivative of V with respect to system (2.3) satisfies

$$D^+V(t, \varphi(0)) \le cV(t, \varphi(0)),\ \textit{for all } t \in [t_{k-1}, t_k),\ k \in \mathbb{N},$$

whenever $qV(t, \varphi(0)) \ge V(t+s, \varphi(s))$ for $s \in [-\tau, 0]$, where $q \ge \max\{e^{\alpha c}, e^{2\lambda\alpha}\}$ is a constant;

(iii) $V(t_k, \varphi(0) + I_k(\varphi)) \leq d_k V(t_k^-, \varphi(0))$*, where* $d_k > 0$*,* $\forall k \in \mathbb{N}$ *are constants;*

(iv) $\tau \leq t_k - t_{k-1} \leq \alpha$ *and* $\ln(d_k) + (\lambda + c)\alpha < -\lambda(t_{k+1} - t_k)$*.*

Then the trivial solution of the impulsive system (2.3) is globally exponentially stable and the convergence rate is $\frac{\lambda}{p}$*.*

Remark 4.2.4 *It is well-known that, in the stability theory of delay differential equations, the condition* $D^+V(t,x) \leq cV(t,x)$ *allows the derivative of the Lyapunov function to be positive which may not even guarantee the stability of a delay differential system (see [75, 94] and Example 4.2.2). However, as we can see from Theorem 4.2.2 and 4.2.3, impulses have played an important role in exponentially stabilizing a delay differential system.*

Next, we shall apply the previous theorems to the following linear impulsive delay system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t - \tau(t)), \quad t \in [t_{k-1}, t_k), \\ \Delta x(t) = C_k x(t^-), \quad t = t_k, k \in \mathbb{N}, \\ x_{t_0} = \phi, \end{cases} \tag{4.53}$$

where $t - \tau(t)$ is strictly increasing on $\mathbb{R}_+$ and $0 \leq \tau(t) \leq \tau$.

Corollary 4.2.1 *If there exist constants* $\alpha,\ \lambda > 0$ *such that*

(i) for some constant $q \geq e^{2\lambda\alpha}$*,* $\lambda_{\max}(A) + q^{\frac{1}{2}}\|B\| < \frac{\lambda}{2}$*;*

(ii) $\tau \leq t_k - t_{k-1} \leq \alpha$ *and*

$$\ln\|I + C_k\| + \frac{\lambda\alpha}{2} < -\frac{\lambda}{2}(t_{k+1} - t_k). \tag{4.54}$$

Then system (4.53) is globally exponentially stable and its convergence rate is $\frac{\lambda}{2}$*.*

Proof. It follows from Theorem 4.2.2 by choosing $V(x) = \|x\|^2$. ∎

Corollary 4.2.2 *If there exist constants* $\alpha,\ \lambda > 0$ *such that*

(i) for some constant $q > 0$*,* $q \geq \max\{e^{c\alpha}, e^{2\lambda\alpha}\}$*, where* $c = 2(\lambda_{\max}(A) + q^{\frac{1}{2}}\|B\|)$*;*

(ii) $\tau \leq t_k - t_{k-1} \leq \alpha$ *and*

$$\ln \|I + C_k\| + \frac{\alpha}{2}(\lambda + c) < -\frac{\lambda}{2}(t_{k+1} - t_k). \tag{4.55}$$

Then system (4.53) is globally exponentially stable and its convergence rate is $\frac{\lambda}{2}$.

Proof. It follows from Theorem 4.2.3 by choosing $V(x) = \|x\|^2$. ■

Now we give an example and its simulation to illustrate our results.

Example 4.2.2 *Consider the following linear impulsive delay system*

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t - \frac{1}{40}(1 + e^{-t})), \quad t \geq t_0 = 0, \ t \neq k, \\ \Delta x(t) = C_k x(t^-), \quad t = k, \ \ k \in \mathbb{N}, \\ x_{t_0} = \phi, \end{cases} \tag{4.56}$$

where

$$A = \begin{bmatrix} 0.1 & 0.2 & -0.1 \\ 0.2 & 0.15 & 0.3 \\ 0 & 0.24 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.12 & 0.03 & 0 \\ 0.12 & -0.2 & 0.05 \\ 0 & 0.14 & -0.1 \end{bmatrix},$$

and

$$C_k = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -0.8 & 0 \\ 0 & 0 & -0.4 \end{bmatrix},$$

then $\lambda_{\max}(A) = 0.4388$, $\|B\| = [\lambda_{\max}(BB^T)]^{\frac{1}{2}} = 0.2905$ and $\|I + C_k\| = 0.6$. Choose $q = 2, \ \lambda = 1.7, \ \tau = 0.05, \ \alpha = 0.2$. The conditions of Corollary 4.2.1 hold:

(i) $q = 2 \geq e^{2\lambda\alpha} = 1.9739, \ \lambda_{\max}(A) + q^{\frac{1}{2}}\|B\| = 0.8496 < \frac{\lambda}{2} = 0.85$;

(ii) $0.05 = \tau \leq t_k - t_{k-1} \leq \alpha = 0.2, \ \ln \|I + C_k\| + \frac{\lambda\alpha}{2} = -0.6808 < -\frac{\lambda}{2}(t_{k+1} - t_k) = -0.17$.

Thus by Corollary 4.2.1, the trivial solution of (4.56) is globally exponentially stable with convergence rate 0.85.

Furthermore, the conditions of Corollary 4.2.2 also hold:

(i) $c = 2(\lambda_{\max}(A) + q^{\frac{1}{2}}\|B\|) = 1.6992, \ q = 2 \geq \max\{e^{c\alpha}, e^{2\lambda\alpha}\} = 1.9739$;

(ii) $0.05 = \tau \leq t_k - t_{k-1} \leq \alpha = 0.2,\ \ln\|I + C_k\| + \frac{(\lambda+c)\alpha}{2} = -0.1709 < -\frac{\lambda}{2}(t_{k+1} - t_k) = -0.17.$

Thus from Corollary 4.2.2, it follows that the trivial solution of (4.56) is globally exponentially stable with convergence rate 0.85.

The numerical simulation of this impulsive delay differential equation with the initial function $(3.7H(t), -2.1H(t), 2.502H(t))^T$, where $H(t)$ is the Heaviside step function, is given in Figure 4.7, the graph of the solution of the corresponding system without impulse is given in Figure 4.8.

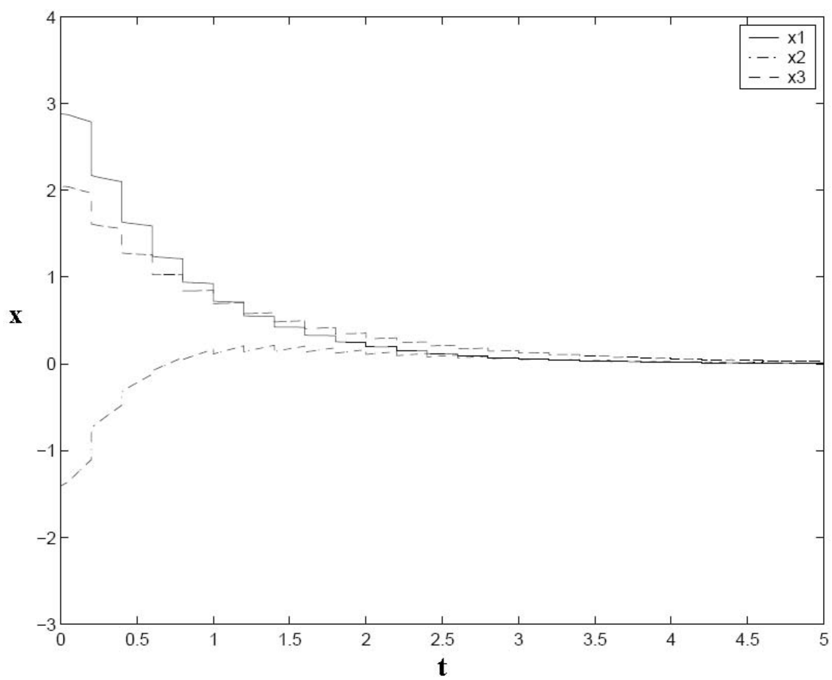

Figure 4.7: Numerical simulation of Example 4.2.2, impulse-stabilized system.

Remark 4.2.5 *As we see from Figures 4.7 and 4.8, the trivial solution of system (4.56) without impulse is unstable; however, after impulsive control, the trivial solution becomes globally exponentially stable. This implies that impulse may be used to exponentially stabilize some delay differential systems.*

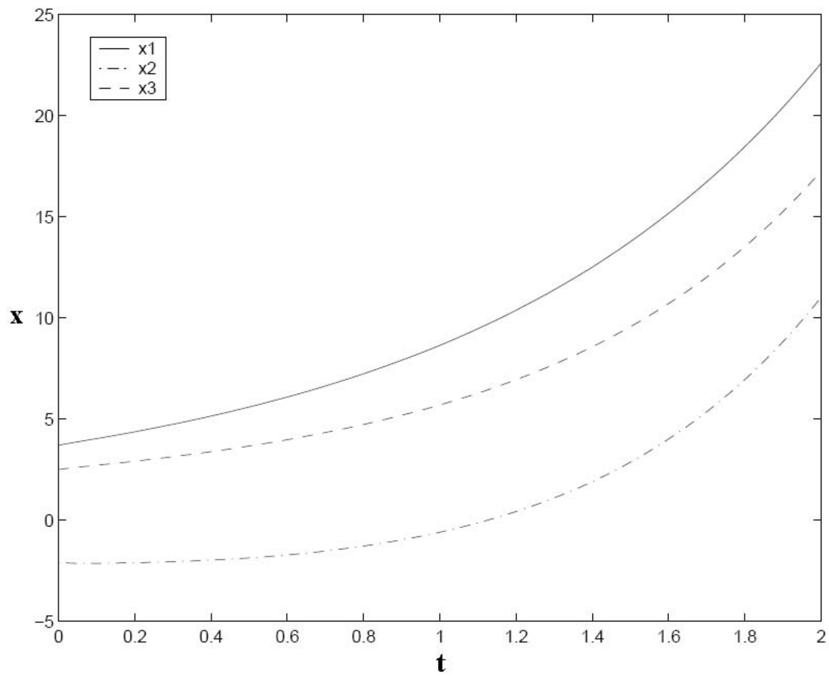

Figure 4.8: Numerical simulation of Example 4.2.2, system without impulses.

4.3 Stabilization via the Lyapunov Functional Method

In this section, the Lyapunov functional method is proposed in the context of impulsive stabilization problems of delay differential systems. We improve or generalize some known results in [3, 65].

Our first two results show that an unstable system can be made exponentially stable by appropriate sequence of impulses.

Theorem 4.3.1 *Assume that there exist* $V_1 \in \nu_0,\ V_2 \in \nu_0^*(\cdot)$, $0 < p_1 \le p_2$, *and constants* $\alpha, l, c, c_1, c_2, c_3 > 0,\ d_k \ge 0,\ k \in \mathbb{N}$, *such that*

(i) $c_1\|x\|^{p_1} \le V_1(t,x) \le c_2\|x\|^{p_1}$, $0 \le V_2(t,\psi) \le c_3\|\psi\|_\tau^{p_2}$, $t \in \mathbb{R}_+,\ x \in \mathbb{R}^n,\ \psi \in PC([-\tau,0],\mathbb{R}^n)$*;*

(ii) for each $k \in \mathbb{N}$ *and* $x \in \mathbb{R}^n$,

$$V_1(t_k, x + I_k(x)) \le d_k V_1(t_k^-, x);$$

(iii) *for* $V(t,\psi) = V_1(t,\psi(0)) + V_2(t,\psi)$, *the upper right-hand derivative of* V *with respect to system (2.3) satisfies*

$$D^+V(t,\psi) \le cV(t,\psi), \quad \textit{for all } t \in [t_{k-1}, t_k),\ \psi \in PC([-\tau,0],\mathbb{R}^n),\ k \in \mathbb{N};$$

(iv) *for any* $k \in \mathbb{N}$, $\tau \le t_k - t_{k-1} \le l$, *and* $\ln(d_k + \frac{c_3}{c_1} e^{(\frac{p_2}{p_1}-1)ckl}) \le -(\alpha + c)l$.

Then the trivial solution of system (2.3) is exponentially stable.

Proof. Let $x(t) = x(t, t_0, \phi)$ be any solution of system (2.3) with $\|\phi\|_\tau < \delta$. Let $v_1(t) = V_1(t, x(t))$ and $v_2(t) = V_2(t, x_t)$, $v(t) = v_1(t) + v_2(t)$. For any given $\varepsilon \in (0, 1]$, choose $\delta = \delta(\varepsilon) > 0$ such that

$$c_2\delta^{p_1} + c_3\delta^{p_2} < c_1\varepsilon^{p_1}e^{-(\alpha+c)l}.$$

From condition (iii), we have

$$v(t) \le v(t_{k-1})e^{c(t-t_{k-1})}, \ \ \text{for } t \in [t_{k-1}, t_k),\ k \in \mathbb{N}. \tag{4.57}$$

We shall prove

$$\begin{aligned} &v(t) < c_1\varepsilon^{p_1}e^{-(\alpha+c)kl}e^{c(t-t_0)}, \quad \text{and} \\ &\|x(t)\| < \varepsilon e^{-\frac{\alpha}{p_1}(t-t_0)}, \qquad t \in [t_{k-1}, t_k),\ k \in \mathbb{N}. \end{aligned} \tag{4.58}$$

For $k = 1$, we obtain, by conditions (i), (iv) and (4.57),

$$\begin{aligned} v(t) &\le v(t_0)e^{c(t-t_0)} \\ &\le (c_2\delta^{p_1} + c_3\delta^{p_2})e^{c(t-t_0)} \\ &< c_1\varepsilon^{p_1}e^{-(\alpha+c)l}e^{c(t-t_0)}, \end{aligned}$$

and thus

$$\begin{aligned} \|x(t)\|^{p_1} &\le \tfrac{1}{c_1}v(t) \\ &\le \varepsilon^{p_1}e^{-(\alpha+c)l}e^{c(t_1-t_0)} \\ &< \varepsilon^{p_1}e^{-\alpha l} \le \varepsilon^{p_1}e^{-\alpha(t_1-t_0)} \\ &\le \varepsilon^{p_1}e^{-\alpha(t-t_0)}, \qquad t \in [t_0, t_1). \end{aligned} \tag{4.59}$$

Hence

$$\|x(t)\| \le \varepsilon e^{-\frac{\alpha}{p_1}(t-t_0)}, \quad t \in [t_0, t_1).$$

Suppose (4.58) holds for $k = j$, i.e.

$$v(t) < c_1 \varepsilon^{p_1} e^{-(\alpha+c)jl} e^{c(t-t_0)}, \quad \text{and} \\ \|x(t)\| < \varepsilon e^{-\frac{\alpha}{p_1}(t-t_0)}, \quad t \in [t_{j-1}, t_j), \ j \geq 2. \tag{4.60}$$

We shall prove (4.58) holds for $k = j + 1$.

From condition (i) and (4.60), we have, for $t \in [t_{j-1}, t_j)$,

$$\|x(t)\|^{p_1} \leq \frac{1}{c_1} v_1(t) \leq \frac{1}{c_1} v(t) < \varepsilon^{p_1} e^{-(\alpha+c)jl} e^{c(t_j-t_0)}, \tag{4.61}$$

and thus

$$\|x_{t_j^-}\|_\tau = \sup_{-\tau \leq s < 0} \|x(t_j + s)\| < \varepsilon e^{-\frac{\alpha+c}{p_1}jl} e^{\frac{c}{p_1}(t_j-t_0)}.$$

By condition (ii) and the continuity of $v_2(t)$ at each t_j, we obtain

$$v_1(t_j) \leq d_j v_1(t_j^-) \leq d_j v(t_j^-) < d_j c_1 \varepsilon^{p_1} e^{-(\alpha+c)jl} e^{c(t_j-t_0)},$$

and

$$v_2(t_j) = v_2(t_j^-) \leq c_3 \|x_{t_j^-}\|_\tau^{p_2} < c_3 \varepsilon^{p_2} e^{-\frac{p_2}{p_1}(\alpha+c)jl} e^{\frac{p_2}{p_1}c(t_j-t_0)}.$$

Thus, in view of $p_1 \leq p_2$, we obtain

$$\begin{aligned} v(t_j) &= v_1(t_j) + v_2(t_j) \\ &< d_j c_1 \varepsilon^{p_1} e^{-(\alpha+c)jl} e^{c(t_j-t_0)} + c_3 \varepsilon^{p_2} e^{-\frac{p_2}{p_1}(\alpha+c)jl} e^{\frac{p_2}{p_1}c(t_j-t_0)} \\ &\leq c_1 \varepsilon^{p_1} e^{-(\alpha+c)jl} (d_j e^{c(t_j-t_0)} + \tfrac{c_3}{c_1} e^{\frac{p_2}{p_1}c(t_j-t_0)}) \\ &\leq c_1 \varepsilon^{p_1} e^{-(\alpha+c)jl} (d_j + \tfrac{c_3}{c_1} e^{(\frac{p_2}{p_1}-1)cjl}) e^{c(t_j-t_0)}. \end{aligned} \tag{4.62}$$

By condition (iv), we obtain

$$v(t_j) \leq c_1 \varepsilon^{p_1} e^{-(\alpha+c)jl} e^{-(\alpha+c)l} e^{c(t_j-t_0)} \leq c_1 \varepsilon^{p_1} e^{-(\alpha+c)(j+1)l} e^{c(t_j-t_0)}.$$

By (4.57) and (4.62), we have, for $t \in [t_j, t_{j+1})$,

$$v(t) \leq v(t_j) e^{c(t-t_j)} < c_1 \varepsilon^{p_1} e^{-(\alpha+c)(j+1)l} e^{c(t-t_0)}.$$

Thus

$$\begin{aligned} \|x(t)\|^{p_1} &< \varepsilon^{p_1} e^{-(j+1)(\alpha+c)l} e^{c(t-t_0)} \leq \varepsilon^{p_1} e^{-(j+1)(\alpha+c)l} e^{c(t_{j+1}-t_0)} \\ &\leq \varepsilon^{p_1} e^{-(j+1)(\alpha+c)l} e^{c(j+1)l} \leq \varepsilon^{p_1} e^{-(j+1)\alpha l} \\ &\leq \varepsilon^{p_1} e^{-\alpha(t_{j+1}-t_j+t_j-t_{j-1}+\cdots+t_1-t_0)} \\ &\leq \varepsilon^{p_1} e^{-\alpha(t_{j+1}-t_0)} \\ &\leq \varepsilon^{p_1} e^{-\alpha(t-t_0)}, \quad t \in [t_j, t_{j+1}), \end{aligned}$$

which implies (4.58) holds for $k = j + 1$. Thus we conclude by induction that (4.58) holds for all $k \in \mathbb{N}$. Hence, we obtain

$$\|x(t)\| < \varepsilon e^{-\frac{\alpha}{p_1}(t-t_0)},\ t \geq t_0,$$

i.e., the trivial solution of system (2.3) is exponentially stable. ∎

Remark 4.3.1 *It should be noted that condition (iii) allows $D^+V(t,\psi) > 0$ for all $t \in \mathbb{R}_+$ and $\psi(0) \neq 0$, which means that the underlying continuous system may be unstable. On the other hand, condition (iv) means that the impulses must be frequent and their amplitude must be suitably related to the growth rate of V. The constant c_3 in condition (i) is usually a function of the delay τ. In such a case, the result is delay-dependent.*

Corollary 4.3.1 *Assume that there exist $V_1 \in \nu_0,\ V_2 \in \nu_0^*(\cdot)$ and constants $p, \alpha, l, c, c_1, c_2, c_3 > 0,\ d_k \geq 0,\ k \in \mathbb{N}$, such that*

(i) $c_1\|x\|^p \leq V_1(t,x) \leq c_2\|x\|^p,\ 0 \leq V_2(t,\psi) \leq c_3\|\psi\|_\tau^p,\ t \in \mathbb{R}_+,\ x \in \mathbb{R}^n,\ \psi \in PC([-\tau,0],\mathbb{R}^n)$;

(ii) for each $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$, $V_1(t_k, x + I_k(x)) \leq d_k V_1(t_k^-, x)$;

(iii) for $V(t,\psi) = V_1(t,\psi(0)) + V_2(t,\psi)$, the upper right-hand derivative of V with respect to system (2.3) satisfies

$$D^+V(t,\psi) \leq cV(t,\psi), \quad \textit{for all } t \in [t_{k-1}, t_k),\ \psi \in PC([-\tau,0],\mathbb{R}^n),\ k \in \mathbb{N};$$

(iv) for any $k \in \mathbb{N}$, $\tau \leq t_k - t_{k-1} \leq l$, and $\ln(d_k + \frac{c_3}{c_1}) \leq -(\alpha + c)l$.

Then the trivial solution of system (2.3) is exponentially stable.

Proof. Let $p_1 = p_2 = p$ in Theorem 4.3.1. ∎

Theorem 4.3.2 *Assume that conditions (i)-(iii) in Theorem 4.3.1 hold, and condition (iv) in Theorem 4.3.1 is replaced by*

$(iv)'$ *for any $k \in \mathbb{N}$, $t_k - t_{k-1} \leq l$, $\ln(\frac{c_2 d_k + c_3 e^{\frac{p_2}{p_1}\alpha\tau}}{c_1}) \leq -(\alpha + c)l$.*

Then the trivial solution of system (2.3) is exponentially stable.

Proof. Let $x(t) = x(t, t_0, \phi)$ be any solution of system (2.3) with $\|\phi\|_\tau < \delta$. Let $v_1(t) = V_1(t, x(t))$ and $v_2(t) = V_2(t, x_t)$, $v(t) = v_1(t) + v_2(t)$. For any given $\varepsilon \in (0, 1]$, choose $\delta = \delta(\varepsilon) > 0$ such that $c_2\delta^{p_1} + c_3\delta^{p_2} < c_1\varepsilon^{p_1}e^{-(\alpha+c)l}$. From condition (iii), we have

$$v(t) \le v(t_{k-1})e^{c(t-t_{k-1})}, \text{ for } t \in [t_{k-1}, t_k),\ k \in \mathbb{N}. \tag{4.63}$$

We shall prove

$$\|x(t)\| < \varepsilon e^{-\frac{\alpha}{p_1}(t-t_0)}, \qquad t \in [t_{k-1}, t_k),\ k \in \mathbb{N}. \tag{4.64}$$

For $k = 1$, we obtain, by conditions (i), $(iv)'$ and (4.63),

$$\begin{aligned} v(t) &\le v(t_0)e^{c(t-t_0)} \\ &\le (c_2\delta^{p_1} + c_3\delta^{p_2})e^{c(t-t_0)} \\ &< c_1\varepsilon^{p_1}e^{-(\alpha+c)l}e^{c(t-t_0)}, \end{aligned}$$

and thus

$$\begin{aligned} \|x(t)\|^{p_1} &\le \tfrac{1}{c_1}v(t) \le \varepsilon^{p_1}e^{-(\alpha+c)l}e^{c(t_1-t_0)} \\ &< \varepsilon^{p_1}e^{-\alpha l} \le \varepsilon^{p_1}e^{-\alpha(t_1-t_0)} \\ &\le \varepsilon^{p_1}e^{-\alpha(t-t_0)}, \qquad t \in [t_0, t_1). \end{aligned} \tag{4.65}$$

Hence

$$\|x(t)\| \le \varepsilon e^{-\frac{\alpha}{p_1}(t-t_0)}, \quad t \in [t_0, t_1).$$

Suppose (4.64) holds for $k = j$, i.e.

$$\|x(t)\| < \varepsilon e^{-\frac{\alpha}{p_1}(t-t_0)}, \ t \in [t_{j-1}, t_j),\ j \ge 2. \tag{4.66}$$

We shall prove (4.64) holds for $k = j + 1$.

From condition (i) and (4.66), we have, for $t \in [t_{j-1}, t_j)$,

$$\begin{aligned} v_1(t_j^-) &\le c_2\|x(t_j)\|^{p_1} \\ &\le c_2\varepsilon^{p_1}e^{-\alpha(t_j-t_0)}, \end{aligned} \tag{4.67}$$

and

$$\begin{aligned} \|x_{t_j^-}\|_\tau &= \sup_{-\tau\le s<0}\|x(t_j + s)\| \\ &< \varepsilon e^{-\frac{\alpha}{p_1}(t_j-t_0-\tau)}. \end{aligned}$$

By condition (ii) and the continuity of $v_2(t)$ at each t_j, we obtain

$$v_1(t_j) \le d_jv_1(t_j^-) < d_jc_2\varepsilon^{p_1}e^{-\alpha(t_j-t_0)},$$

and

$$v_2(t_j) = v_2(t_j^-) \le c_3 \|x_{t_j^-}\|_\tau^{p_2} < c_3 \varepsilon^{p_2} e^{-\frac{p_2}{p_1}\alpha(t_j - t_0)} e^{\frac{p_2}{p_1}\alpha\tau}.$$

Thus, in view of condition $(iv)'$ and the fact $p_1 \le p_2$, we obtain

$$\begin{aligned} v(t_j) &= v_1(t_j) + v_2(t_j) \\ &< d_j c_2 \varepsilon^{p_1} e^{-\alpha(t_j - t_0)} + c_3 \varepsilon^{p_2} e^{-\frac{p_2}{p_1}\alpha(t_j - t_0)} e^{\frac{p_2}{p_1}\alpha\tau} \\ &\le (c_2 d_j + c_3 e^{\frac{p_2}{p_1}\alpha\tau}) \varepsilon^{p_1} e^{-\alpha(t_j - t_0)} \\ &\le c_1 \varepsilon^{p_1} e^{-(\alpha + c)l} e^{-\alpha(t_j - t_0)}. \end{aligned} \tag{4.68}$$

Then by (4.63) and (4.68), we have, for $t \in [t_j, t_{j+1})$,

$$\begin{aligned} v(t) &\le v(t_j) e^{c(t - t_j)} < c_1 \varepsilon^{p_1} e^{-(\alpha + c)l} e^{-\alpha(t_j - t_0)} e^{c(t - t_j)} \\ &\le c_1 \varepsilon^{p_1} e^{-\alpha l} e^{-\alpha(t_j - t_0)} \\ &\le c_1 \varepsilon^{p_1} e^{-\alpha(t - t_j)} e^{-\alpha(t_j - t_0)} \\ &\le c_1 \varepsilon^{p_1} e^{-\alpha(t - t_0)}. \end{aligned}$$

Thus

$$\|x(t)\| < \varepsilon e^{-\frac{\alpha}{p_1}(t - t_0)},\ t \in [t_j, t_{j+1}),$$

which shows that (4.64) holds for $k = j + 1$. Thus we conclude by induction that (4.64) holds for all $k \in \mathbb{N}$. Hence, we obtain

$$\|x(t)\| < \varepsilon e^{-\frac{\alpha}{p_1}(t - t_0)},\ t \ge t_0,$$

i.e., the trivial solution of system (2.3) is exponentially stable. ■

Remark 4.3.2 *It should be noted that in condition $(iv)'$ of Theorem 4.3.2, we removed the lower bound of the impulsive interval length, which means we may add impulses more frequently to relax the condition on d_k in the case $p_2 > p_1$ (see Example 4.3.2); but in the case $p_1 = p_2$, condition (iv) of Theorem 4.3.1 is sharper than condition $(iv)'$ of Theorem 4.3.2 (see Example 4.3.1).*

Next, we shall discuss the application of our results to impulsive stabilization of second-order linear delay differential equations. Some examples are also worked out to illustrate our results.

Consider the following second-order linear delay differential equation

$$\begin{cases} x''(t) + b(t)x'(t) + a(t)x(t-\tau) = 0, & t \geq t_0, \\ x(t) = \phi(t), \quad x'(t) = \psi(t), & t_0 - \tau \leq t \leq t_0, \end{cases} \tag{4.69}$$

and the corresponding equation with impulses

$$\begin{cases} x''(t) + b(t)x'(t) + a(t)x(t-\tau) = 0, & t \geq t_0,\ t \neq t_k, \\ x(t_k) = I_k(x(t_k^-)), & x'(t_k) = J_k(x'(t_k^-)), \\ x(t) = \phi(t), \quad x'(t) = \psi(t), & t_0 - \tau \leq t \leq t_0, \end{cases} \tag{4.70}$$

where $t_0 < t_1 < \cdots < t_k < \cdots$, $k \in \mathbb{N}$, $\lim_{k\to\infty} t_k = +\infty$ and $I_k,\ J_k,\ \phi,\ \psi \in C(\mathbb{R}, \mathbb{R})$ with $I_k(0) = J_k(0) = 0,\ k \in \mathbb{N}$.

Definition 4.3.1 *System (4.69) is said to be exponentially stabilized by impulses, if there exist a sequence $\{t_k\}$ and function sequences $\{I_k\}$ and $\{J_k\}$ such that the solutions of (4.70) are exponentially stable.*

Let $x_1(t) = x(t),\ x_2(t) = x'(t)$. We rewrite system (4.69) in vector form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 0 & -b(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -a(t) & 0 \end{bmatrix} \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix}. \tag{4.71}$$

Proposition 4.3.1 *If $a, b \in C([t_0, +\infty), \mathbb{R})$ and $|a(t)| \leq A$, $|b(t)| \leq B$ for all $t \in [t_0, +\infty)$ $(A,\ B \geq 0)$, and there exists some constant $\alpha > 0$ such that*

$$\ln(d_k + A\tau) < -(\alpha + 1 + A + 2B)l, \tag{4.72}$$

where $I_k(u) = J_k(u) = \sqrt{\frac{d_k}{2}}u$ for any $u \in \mathbb{R}$, $\tau \leq t_k - t_{k-1} \leq l$.
Then systems (4.69) and (4.71) can be exponentially stabilized by impulses.

Proof. Let $z(t) = (x_1(t), x_2(t))^T$, choose $V(t, z_t) = V_1(t, z) + V_2(t, z_t)$, where $V_1(t, z) = x_1^2(t) + x_2^2(t)$, $V_2(t, z_t) = \int_{t-\tau}^{t} |a(s+\tau)| x_1^2(s) ds$. Then, $\|z(t)\|^2 \leq V_1(t, z) \leq \|z(t)\|^2$ and $0 \leq V_2(t, z_t) \leq A\tau \|z(t)\|_\tau^2$, which implies condition (i) of Corollary 4.3.1 holds with $c_1 = c_2 = 1,\ c_3 = A\tau$, and $p = 2$.

We calculate D^+V along the solution of (4.69) and (4.71)

$$\begin{aligned}
D^+V(t,z_t) &= 2x_1(t)x_1'(t) + 2x_2(t)x_2'(t) + |a(t+\tau)|x_1^2(t) - |a(t)|x_1^2(t-\tau) \\
&= 2x(t)x'(t) - 2b(t)x'^2(t) - 2a(t)x'(t)x(t-\tau) + |a(t+\tau)|x^2(t) \\
&\qquad -|a(t)|x^2(t-\tau) \\
&\le x^2(t) + x'^2(t) + 2|b(t)|x'^2(t) + |a(t)|(x'^2(t) + x^2(t-\tau)) \\
&\qquad +|a(t+\tau)|x^2(t) - |a(t)|x^2(t-\tau) \\
&\le (1+A+2B)(x^2(t)+x'^2(t)) \\
&\le (1+A+2B)V(t,z_t),
\end{aligned}$$

which implies condition (iii) of Corollary 4.3.1 holds with $c = 1+A+2B$. Condition (ii) and (iv) of Corollary 4.3.1 come from the assumptions of our proposition. Thus by Corollary 4.3.1, the solutions of (4.70) are exponentially stable. ■

Remark 4.3.3 *Let $b(t) \equiv 0$ (in system (4.69)) for all $t \in [t_0, +\infty)$. Then $B = 0$ (in Proposition 4.3.1). Choose $d_k = \frac{1}{10}A\tau$. Then $A\tau < e^{-(1+A)\tau}$ from (4.72). This also shows that system (4.69) can be exponentially stabilized by impulses by Theorem 1 in [65].*

Remark 4.3.4 *If we apply condition $(iv)'$ of Theorem 4.3.2 instead of condition (iv) of Theorem 4.3.1 in Proposition 4.3.1, then d_k must satisfy*

$$\ln(d_k + A\tau e^{\alpha\tau}) < -(\alpha + 1 + A + 2B)l,$$

instead of (4.72) to exponentially stabilize the same system, which is more restrictive than (4.72). Thus in the case $p_1 = p_2$, condition (iv) in Theorem 4.3.1 is better than $(iv)'$ in Theorem 4.3.2.

Example 4.3.1 *Consider the following linear impulsive delay differential system*

$$\begin{cases} x''(t) + 0.012x'(t) - 0.12x(t-0.05) = 0, & t \ge 0,\ t \ne k, \\ x(k) = \sqrt{\frac{d}{2}}x(k^-), \quad x'(k) = \sqrt{\frac{d}{2}}x'(k^-), & k \in \mathbb{N}, \end{cases} \tag{4.73}$$

where $d = 0.28$.

Let $x_1(t) = x(t),\ x_2(t) = x'(t)$. We rewrite system (4.73) as

$$\begin{aligned}
&\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -0.012 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.12 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-0.05) \\ x_2(t-0.05) \end{bmatrix}, \\
&\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} 0.3742 & 0 \\ 0 & 0.3742 \end{bmatrix} \begin{bmatrix} x_1(k^-) \\ x_2(k^-) \end{bmatrix}.
\end{aligned} \tag{4.74}$$

It is known that the corresponding equation without impulses is unstable. The numerical simulation of this delay differential equation with initial function

$$\phi_1(t) = \begin{cases} 0, & t \in [-0.05, 0), \\ -2.1, & t = 0, \end{cases} \quad \phi_2(t) = \begin{cases} 0, & t \in [-0.05, 0), \\ 2.102, & t = 0, \end{cases}$$

is given in Figure 4.9.

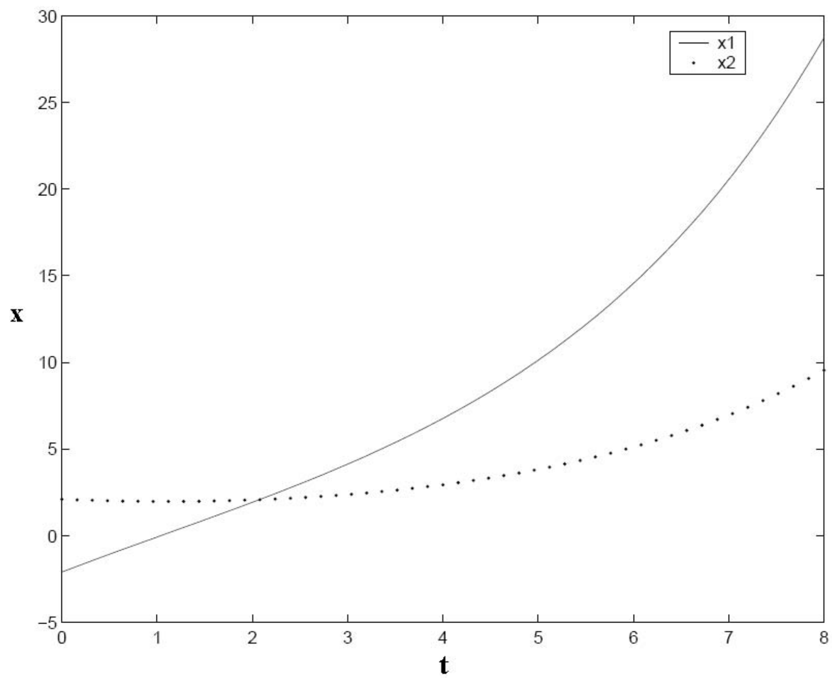

Figure 4.9: Numerical simulation of Example 4.3.1, unstable system without impulses.

However, if we choose $A = 0.12,\ B = 0.012,\ l = 1,\ \tau = 0.05,\ \alpha = 0.1$, then

$$d_k + A\tau = 0.286 < e^{-(\alpha+1+A+2B)l} = 0.2882,$$

which implies (4.72) holds. Hence by Proposition 4.3.1, the unstable equation $x''(t)+0.012x'(t)-0.12x(t-0.05) = 0$ can be exponentially stabilized by impulses, see Figure 4.10.

Remark 4.3.5 *Theorem 5.2 in [3] can not be applied effectively to system (4.74). Actually, we can find*

$$A_1(t) = \begin{bmatrix} 0 & 0 \\ -0.12 & 0 \end{bmatrix}, \quad A_2(t) = \begin{bmatrix} 0 & -1 \\ 0.012 & 0 \end{bmatrix},$$

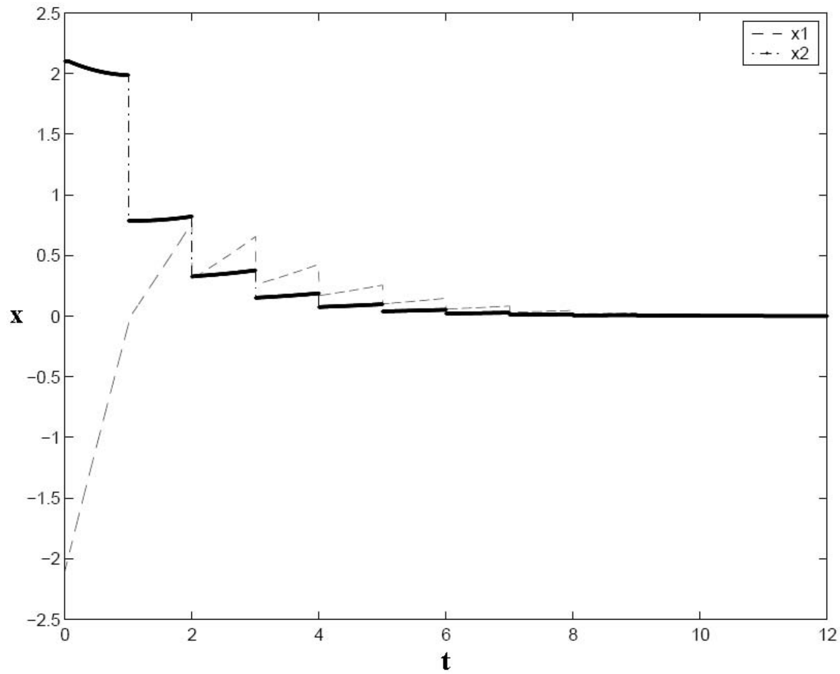

Figure 4.10: Numerical simulation of Example 4.3.1, impulse-stabilized system.

which gives us $\|A_1(t)\| = 0.12,\ \|A_2(t)\| = 1$. *Moreover, we have* $\delta = 0.05,\ \rho = \sigma = 1,\ B = 0.3742$, *which yields*

$$(0.12 + 1)(\frac{-0.3742^2}{\ln(0.3742)} + 2) = 2.3995 > 1,$$

i.e. one of the conditions in Theorem 5.2 ([3]) does not hold.

Example 4.3.2 *Consider the following impulsive nonlinear delay differential system*

$$\begin{cases} x'(t) = (4+\sin(t))x(t) + \frac{1+\cos(t)}{2}\frac{x^3(t-\tau)}{x(t)+0.2} - \frac{1}{2}x^3(t), & t \geq 0,\ t \neq 0.05k, \\ x(0.05k) = 2^{\frac{1}{2}} d_k^{\frac{1}{4}} x(0.05k^-), & k \in \mathbb{N}, \end{cases} \tag{4.75}$$

where $\tau > 0$ *is constant and* $d_k \geq 0$. *Assume that* d_k *satisfies*

$$d_k \leq e^{-0.05(\alpha+5.5)} - 2\tau e^{1.5\alpha\tau}. \tag{4.76}$$

Then the trivial solution of (4.75) is exponentially stable.

Proof. Choose $V_1(t,x) = \frac{x^4}{4}$ and $V_2(t,x_t) = \frac{1}{2}\int_{-\tau}^{0} x^6(t+s)ds$. Then condition (i) of Theorem 4.3.2 holds for $c_1 = c_2 = \frac{1}{4}$, $c_3 = \frac{1}{2}\tau$, $p_1 = 4$, and $p_2 = 6$.

Calculate the derivative of V along equation (4.75)

$$\begin{aligned} D^+V(t,x_t) &= D^+V_1(t,x) + D^+V_2(t,x_t) \\ &= x^3(t)x'(t) + \tfrac{1}{2}x^6(t) - \tfrac{1}{2}x^6(t-\tau) \\ &\leq 5x^4(t) + x^2(t)x^3(t-\tau) - \tfrac{1}{2}x^6(t-\tau) \\ &\leq 5x^4(t) + \tfrac{1}{2}x^4(t) \\ &\leq 5.5V(t,x_t), \end{aligned}$$

which implies condition (iii) of Theorem 4.3.2 holds with $c = 5.5$.

Moreover, we have

$$V_1(t_k, x + I_k(x)) = \frac{1}{4}x^4(k) = d_k x(t_k^-) \leq d_k V_1(t_k^-, x(t_k^-)),$$

which, together with (4.76), implies that conditions (ii) and $(iv)'$ of Theorem 4.3.2 hold. Then by Theorem 4.3.2, the trivial solution of (4.75) is exponentially stable. The numerical simulation of this delay differential equation with $\tau = 0.1$, $\alpha = 0.1$, $d_k = 0.5$, and initial function

$$\phi(t) = \begin{cases} 0, & t \in [-0.1, 0), \\ -1.45, & t = 0, \end{cases}$$

is given in Figure 4.12.

Remark 4.3.6 *From Example 4.3.2 we can see impulses are used to stabilize an unstable system (see Figure 4.11 and Figure 4.12). And the allowance of $p_1 \neq p_2$ in the conditions of our results makes the choices of Lyapunov functionals wider. Note Theorem 4.3.1 can not apply here since $p_2 > p_1$. By condition (iv), we need $d_k + 0.2e^{0.1375k} \leq e^{-0.28}$. It is impossible to find such a series $\{d_k\}$ to satisfy this relationship as k gets larger and larger.*

■

Our next result shows that impulses can contribute to make a stable system exponentially stable.

Theorem 4.3.3 *Assume that there exist $V_1 \in \nu_0,\ V_2 \in \nu_0^*(\cdot)$, constants $p_1, p_2 > 0$ with $p_1 \leq p_2$, and $\alpha, c_1, c_2, c_3 > 0,\ d_k \geq 0,\ k \in \mathbb{N}$ such that*

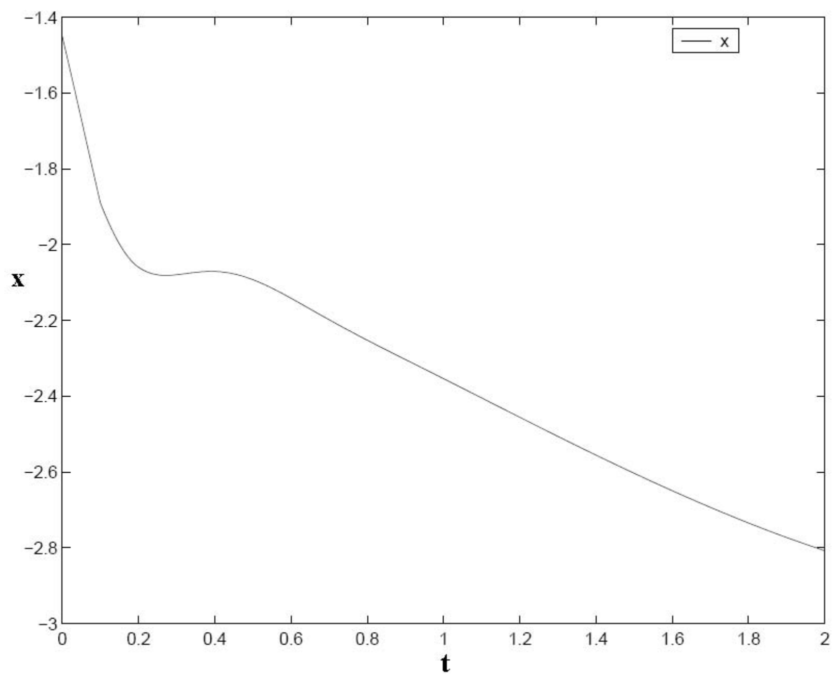

Figure 4.11: Numerical simulation of Example 4.3.2, unstable system without impulses.

(i) $c_1\|x\|^{p_1} \le V_1(t,x) \le c_2\|x\|^{p_1}$, $0 \le V_2(t,\psi) \le c_3\|\psi\|_\tau^{p_2}$, $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, $\psi \in PC([-\tau,0],\mathbb{R}^n)$*;*

(ii) for each $k \in \mathbb{N}$ *and* $x \in \mathbb{R}^n$*,*

$$V_1(t_k, x + I_k(x)) \le d_k V_1(t_k^-, x);$$

(iii) for $V(t,\psi) = V_1(t,\psi(0)) + V_2(t,\psi)$*, the upper right-hand derivative of* V *with respect to system (2.3) satisfies*

$$D^+V(t,\psi) \le 0, \text{ for all } t \in [t_{k-1}, t_k),\ \psi \in PC([-\tau,0],\mathbb{R}^n),\ k \in \mathbb{N};$$

(iv) for any $k \in \mathbb{N}$*,* $\ln(d_k + \frac{c_3}{c_1}e^{\frac{\alpha p_2}{p_1}\tau}) \le -\alpha(t_{k+1} - t_k)$.

Then the trivial solution of (2.3) is exponentially stable.

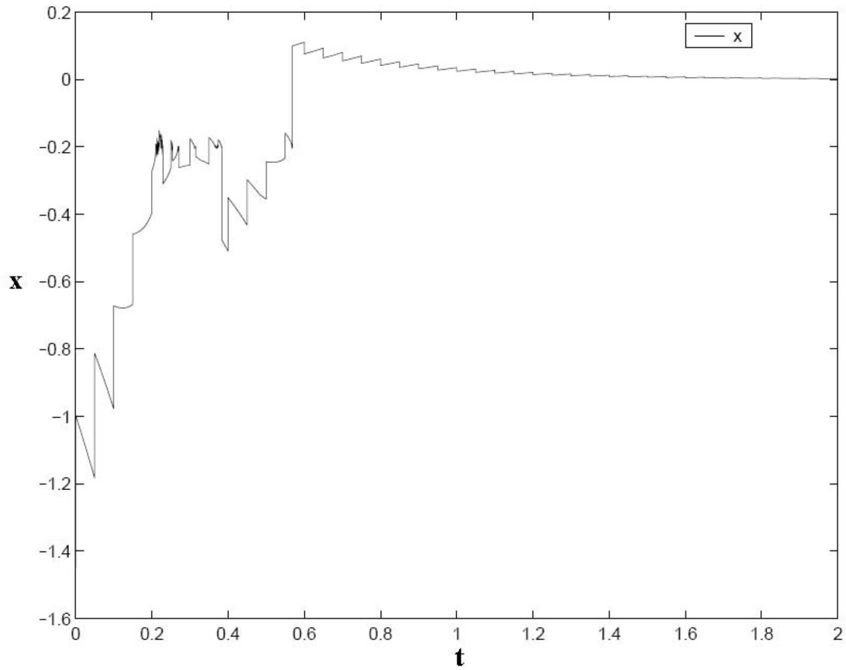

Figure 4.12: Numerical simulation of Example 4.3.2, impulse-stabilized system.

Proof. Let $x(t) = x(t, t_0, \phi)$ be any solution of system (2.3) with $\|\phi\|_\tau < \delta$. Let $v_1(t) = V_1(t, x(t))$ and $v_2(t) = V_2(t, x_t)$, $v(t) = v_1(t) + v_2(t)$. For any given $\varepsilon \in (0, 1]$, choose $\delta = \delta(\varepsilon) > 0$ such that

$$c_2\delta^{p_1} + c_3\delta^{p_2} < c_1\varepsilon^{p_1}e^{-\alpha(t_1-t_0)}.$$

From condition (iii), we have

$$v(t) \le v(t_{k-1}), \quad \text{for } t \in [t_{k-1}, t_k),\ k \in \mathbb{N}. \tag{4.77}$$

Now we shall prove

$$v(t) < c_1\varepsilon^{p_1}e^{-\alpha(t-t_0)}, \quad \text{for } t \in [t_{k-1}, t_k),\ k \in \mathbb{N}. \tag{4.78}$$

When $k = 1$, from (4.77) and the choice of δ, we obtain

$$\begin{aligned} v(t) &\le v(t_0) \le v_1(t_0) + v_2(t_0) \\ &\le c_2\delta^{p_1} + c_3\delta^{p_2} \\ &< c_1\varepsilon^{p_1}e^{-\alpha(t_1-t_0)} \\ &\le c_1\varepsilon^{p_1}e^{-\alpha(t-t_0)}, \quad \text{for } t \in [t_0, t_1). \end{aligned}$$

Suppose (4.78) holds for $k = j$, i.e.

$$v(t) < c_1\varepsilon^{p_1}e^{-\alpha(t-t_0)}, \quad \text{for } t \in [t_{j-1}, t_j),\ j \ge 2. \tag{4.79}$$

We shall prove (4.78) holds for $k = j + 1$.

By (4.79) and condition (ii), we obtain

$$v_1(t_j) \le d_j v_1(t_j^-) < d_j c_1\varepsilon^{p_1}e^{-\alpha(t_j-t_0)}. \tag{4.80}$$

From (4.79) and condition (i), we have

$$\|x(t)\|^{p_1} \le \frac{v_1(t)}{c_1} \le \frac{v(t)}{c_1} < \varepsilon^{p_1}e^{-\alpha(t-t_0)}, \quad t \in [t_{j-1}, t_j), \tag{4.81}$$

which yields

$$\begin{aligned} \|x_{t_j^-}\|_\tau^{p_2} &= \sup_{-\tau\le s<0} \|x(t_j + s)\|^{p_2} \\ &< \varepsilon^{p_2}e^{-\frac{\alpha p_2}{p_1}(t_j-t_0-\tau)} \\ &\le e^{\frac{\alpha p_2}{p_1}\tau}\varepsilon^{p_2}e^{-\frac{\alpha p_2}{p_1}(t_j-t_0)}. \end{aligned}$$

Then by condition (i) and the continuity of $v_2(t)$ at each t_j, we obtain

$$v_2(t_j) = v_2(t_j^-) \le c_3\|x_{t_j^-}\|_\tau^{p_2} < c_3 e^{\frac{\alpha p_2}{p_1}\tau}\varepsilon^{p_2}e^{-\frac{\alpha p_2}{p_1}(t_j-t_0)}. \tag{4.82}$$

Thus, in view of (4.80), (4.82) and the fact $p_1 \le p_2$, we obtain

$$\begin{aligned} v(t_j) &= v_1(t_j) + v_2(t_j) \\ &< c_1 d_j\varepsilon^{p_1}e^{-\alpha(t_j-t_0)} + c_3 e^{\frac{\alpha p_2}{p_1}\tau}\varepsilon^{p_2}e^{-\frac{\alpha p_2}{p_1}(t_j-t_0)} \\ &\le c_1\varepsilon^{p_1}(d_j + \varepsilon^{p_2-p_1}\tfrac{c_3}{c_1}e^{\frac{\alpha p_2}{p_1}\tau})e^{-\alpha(t_j-t_0)} \\ &\le c_1\varepsilon^{p_1}(d_j + \tfrac{c_3}{c_1}e^{\frac{\alpha p_2}{p_1}\tau})e^{-\alpha(t_j-t_0)}. \end{aligned}$$

Then by condition (iv) we have $v(t_j) < c_1\varepsilon^{p_1}e^{-\alpha(t_{j+1}-t_0)}$, which, together with (4.77), gives

$$v(t) \le v(t_j) < c_1\varepsilon^{p_1}e^{-\alpha(t_{j+1}-t_0)} \le c_1\varepsilon^{p_1}e^{-\alpha(t-t_0)}, \ \ t \in [t_j, t_{j+1}).$$

This implies (4.78) holds for $k = j+1$. Thus we conclude by induction that (4.78) is true. Then by condition (i), we obtain

$$\|x(t)\| < \varepsilon e^{-\frac{\alpha}{p_1}(t-t_0)}, \quad t \ge t_0,$$

which implies that the trivial solution of system (2.3) is exponentially stable and the proof is complete. ∎

Remark 4.3.7 *From condition (iv) of Theorem 4.3.3, we see that when the underlying continuous system is stable (but not asymptotically stable) impulses are not required to be very frequent. Thus the upper bound on the time interval between consecutive impulses is removed.*

Example 4.3.3 *Consider the following impulsive delay differential system*

$$\begin{cases} x_1'(t) &= x_2(t), \quad x_2'(t) = -x_1(t) + x_3^2(t), & t \ge 0, \ t \ne k, \\ x_3'(t) &= -(5 + x_2 + t^2\sin^2(x_1(t)))x_3(t) + 5x_3(t-\tau), & t \ge 0, \ t \ne k, \\ x(k) &= d_k^{\frac{1}{2}}x(k^-), & k \in \mathbb{N}, \end{cases} \tag{4.83}$$

where $x = (x_1, x_2, x_3)^T, \ d_k, \tau \ge 0$. Assume that d_k satisfies

$$d_k \le e^{-\alpha} - 5\tau^{\alpha\tau}, \tag{4.84}$$

where $\alpha > 0$ is a constant.
Then the trivial solution of (4.83) is exponentially stable.

Proof. Choose $V_1(t,x) = \|x\|^2$ and $V_2(t,x_t) = 5\int_{t-\tau}^{\tau} x_3^2(s)ds$ so that condition (i) of Theorem 4.3.3 holds for $c_1 = c_2 = 1$, $c_3 = 5\tau$ and $p_1 = p_2 = 2$.

Calculate the derivative of V with respect to equation (4.83)

$$\begin{aligned} D^+V(t,x_t) &= D^+V_1(t,x) + D^+V_2(t,x_t) \\ &= 2x_1(t)x_1'(t) + 2x_2(t)x_2'(t) + 2x_3(t)x_3'(t) + 5x_3^2(t) - 5x_3^2(t-\tau) \\ &= -10x_3^2(t) - 2t^2\sin^2(x_1(t))x_3^2(t) + 10x_3(t)x_3(t-\tau) \\ &\quad +5x_3^2(t) - 5x_3^2(t-\tau) \le 0, \end{aligned}$$

which implies condition (iii) of Theorem 4.3.3 holds.

Furthermore, we have

$$V_1(k, x(k)) = d_k x(k^-),$$

which, together with (4.84), yields that condition (ii) and (iv) of Theorem 4.3.3 hold. Then by Theorem 4.3.3, the trivial solution of (4.83) is exponentially stable. The numerical simulation of this delay differential equation with $\tau = 0.07$, $\alpha = 0.2$, $d_k = 0.36$, and initial functions $\phi_1(t) = \phi_2(t) = \phi_3(t) = 0$ for $t \in [-0.07, 0)$ and $\phi_1(0) = 0.8$, $\phi_2(0) = -0.5$, $\phi_3(0) = 2.502$ is given in Figure 4.13.

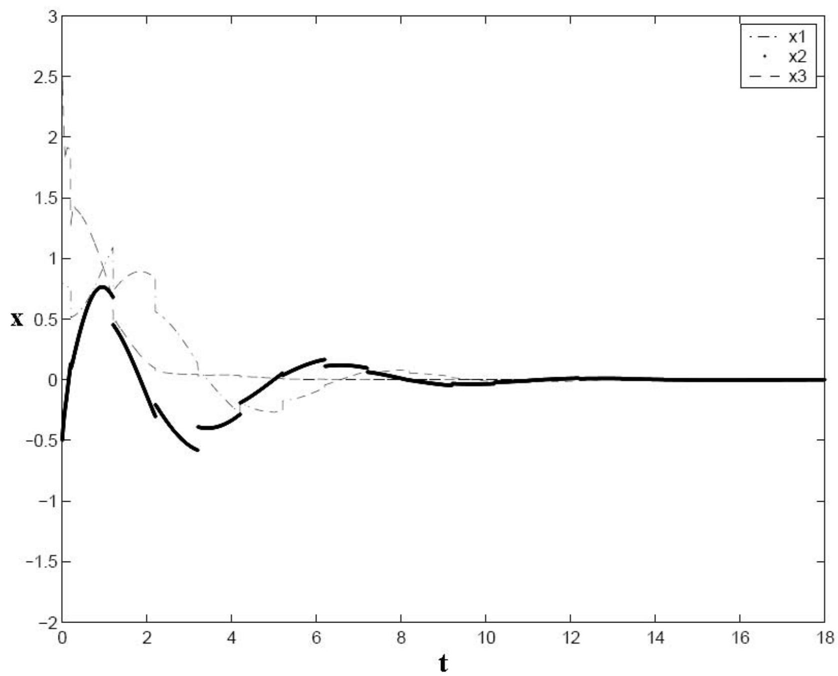

Figure 4.13: Numerical simulation of Example 4.3.3, impulse-stabilized system.

Remark 4.3.8 *The trivial solution of the corresponding delay differential equation (4.83) without impulses is stable but not exponentially stable (see Figure 4.14). As we can see in Example 4.3.3, impulses do contribute to the exponentially impulsive stabilization of the system.*

■

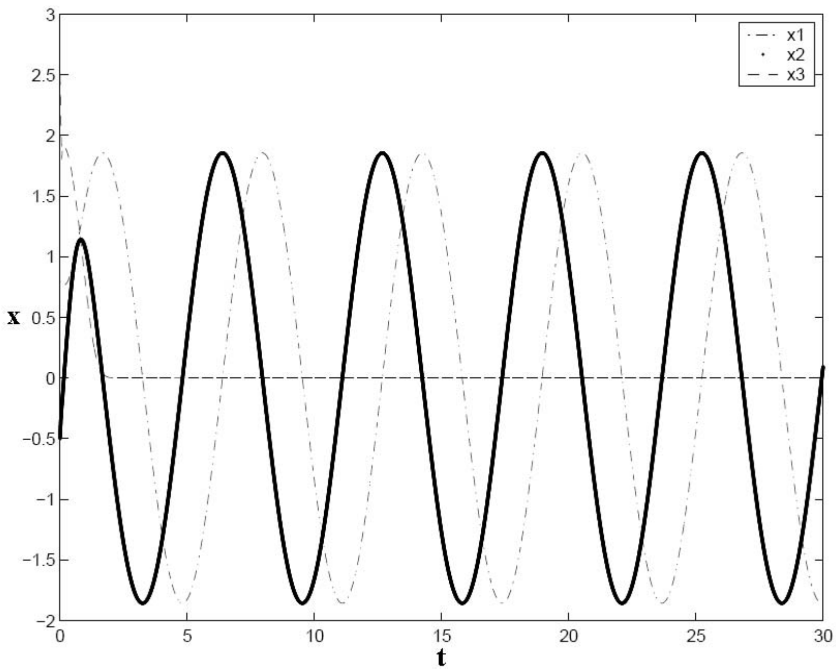

Figure 4.14: Numerical simulation of Example 4.3.3, system without impulses.

Example 4.3.4 *Consider the following impulsive delay differential system*

$$\begin{cases} x'(t) &= a(t)x^3(t) + b(t)x^3(t-\tau), \qquad t \geq 0,\ t \neq k, \\ x(k) &= d_k^{\frac{1}{4}} \sin(x(k^-)), \qquad k \in \mathbb{N}, \end{cases} \tag{4.85}$$

where $a(t) \leq -\delta,\ b(t) \leq \delta,\ \delta, \tau > 0$ *are constants and* $d_k \geq 0$*. If* d_k *satisfies*

$$d_k \leq 4e^{-\alpha} - 2\delta\tau e^{1.5\delta\tau}, \tag{4.86}$$

then the trivial solution of (4.85) is exponentially stable.

Proof. Choose $V_1(t,x) = \frac{x^4}{4}$ and $V_2(t,x_t) = \frac{\delta}{4}\int_{-\tau}^{0} x^6(t+s)ds$. Then condition (i) of Theorem 4.3.3 holds for $c_1 = c_2 = \frac{1}{4}$, $c_3 = \frac{1}{2}\delta\tau$, $p_1 = 4$, and $p_2 = 6$.

Calculate the derivative of V with respect to equation (4.85)

$$
\begin{aligned}
D^+V(t,x_t) &= D^+V_1(t,x) + D^+V_2(t,x_t)\\
&= x^3(t)x'(t) + \tfrac{\delta}{2}x^6(t) - \tfrac{\delta}{2}x^6(t-\tau)\\
&= x^3(t)[a(t)x^3(t) + b(t)x^3(t-\tau)] + \tfrac{\delta}{2}x^6(t) - \tfrac{\delta}{2}x^6(t-\tau)\\
&\le a(t)x^6(t) + \tfrac{b(t)}{2}x^6(t) + \tfrac{b(t)}{2}x^6(t-\tau) + \tfrac{\delta}{2}x^6(t) - \tfrac{\delta}{2}x^6(t-\tau)\\
&= (a(t) + \tfrac{b(t)}{2} + \tfrac{\delta}{2})x^6(t) - \tfrac{1}{2}(\delta - b(t))x^6(t-\tau)\\
&\le 0,
\end{aligned}
$$

which implies condition (iii) of Theorem 4.3.3 holds.

Moreover, we have

$$
\begin{aligned}
V_1(t_k, x + I_k(x)) &= \tfrac{1}{4}x^4(k) = \tfrac{d_k}{4}\sin^4 x(k^-)\\
&\le \tfrac{d_k}{4}x^4(k^-) \le \tfrac{d_k}{4}V_1(t_k^-, x(t_k^-)),
\end{aligned}
$$

which, together with (4.86), implies that conditions (ii) and (iv) of Theorem 4.3.3 hold. Then by Theorem 4.3.3, the trivial solution of (4.85) is exponentially stable. The numerical simulation of this delay differential equation with $a(t) = -2e^t,\ b(t) = 1.5\sin(t),\ \delta = 2,\ \tau = 0.25,\ \alpha = 0.01,\ d_k = 1.8$, and initial function

$$
\phi(t) = \begin{cases} 0, & t \in [-0.25, 0),\\ -1.45, & t = 0, \end{cases}
$$

is given in Figure 4.15.

■

Remark 4.3.9 *From Example 4.3.4 we can see that impulses are used to keep the stability properties of the system. And the allowance of $p_1 \neq p_2$ in the conditions of our results makes the choices of Lyapunov functionals wider.*

On the other hand, a well-behaved system may lose its (asymptotic) stability due to uncontrolled impulsive inputs. Thus it is necessary to have a criterion under which the stability properties of a system can be preserved under impulsive perturbations. The following theorem provides a set of sufficient conditions.

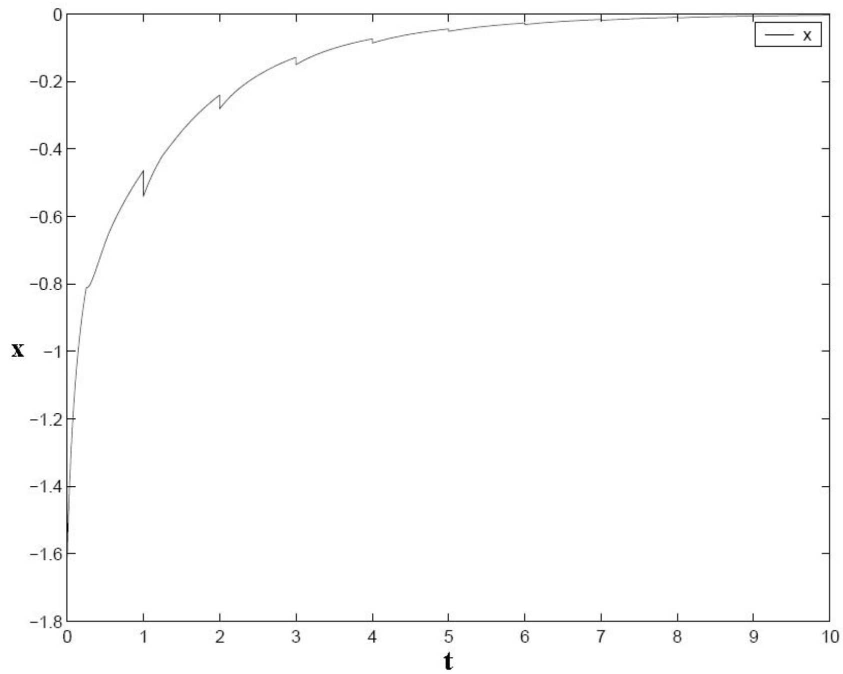

Figure 4.15: Numerical simulation of Example 4.3.4, impulsive system.

Theorem 4.3.4 *Assume that there exist* $V_1 \in \nu_0,\ V_2 \in \nu_0^*(\cdot)$, *and constants* $p_1, p_2, c, c_1, c_2, c_3 > 0,\ d_k \geq 1,\ k \in \mathbb{N}$ *such that*

(i) $c_1\|x\|^{p_1} \leq V_1(t,x) \leq c_2\|x\|^{p_1},\ 0 \leq V_2(t,\psi) \leq c_3\|\psi\|_\tau^{p_2},\ t \in \mathbb{R}_+,\ x \in \mathbb{R}^n,\ \psi \in PC([-\tau,0],\mathbb{R}^n)$*;*

(ii) for each $k \in \mathbb{N}$ *and* $x \in \mathbb{R}^n$*,*

$$V_1(t_k, x + I_k(x)) \leq d_k V_1(t_k^-, x);$$

(iii) for $V(t,\psi) = V_1(t,\psi(0)) + V_2(t,\psi)$*, the upper right-hand derivative of* V *with respect to system (2.3) satisfies*

$$D^+V(t,\psi) \leq -cV(t,\psi), \ \textit{for all}\ t \in [t_{k-1}, t_k),\ \psi \in PC([-\tau,0],\mathbb{R}^n),\ k \in \mathbb{N};$$

(iv) for any $k \in \mathbb{N}$*,* $\ln(d_k) \leq \frac{c}{2}(t_k - t_{k-1})$*.*

Then the trivial solution of (2.3) is exponentially stable.

Proof. Let $x(t) = x(t, t_0, \phi)$ be any solution of system (2.3) with $\|\phi\|_\tau < \delta$. Let $v_1(t) = V_1(t, x(t))$ and $v_2(t) = V_2(t, x_t)$, $v(t) = v_1(t) + v_2(t)$. For any given $\varepsilon > 0$, choose $\delta = \delta(\varepsilon) > 0$ such that

$$c_2\delta^{p_1} + c_3\delta^{p_2} < c_1\varepsilon^{p_1}.$$

From condition (iii), we have

$$v(t) \le v(t_{k-1})e^{-c(t-t_{k-1})}, \quad \text{for } t \in [t_{k-1}, t_k),\ k \in \mathbb{N}. \tag{4.87}$$

From condition (ii) and the fact $d_k \ge 1$, we have

$$\begin{aligned} v(t_k) &= v_1(t_k) + v_2(t_k) \\ &\le d_k v_1(t_k^-) + v_2(t_k^-) \\ &\le d_k v(t_k^-), \quad \forall\, k \in \mathbb{N}. \end{aligned} \tag{4.88}$$

From (4.87) and (4.88), we obtain

$$v(t) \le \prod_{i=0}^{k-1} d_i v(t_0) e^{-c(t-t_0)}, \quad t \in [t_{k-1}, t_k). \tag{4.89}$$

By condition (iv), we have

$$\prod_{i=0}^{k-1} d_i \le e^{\frac{c}{2}[(t_{k-1}-t_{k-2})+(t_{k-2}-t_{k-3})+\cdots+(t_1-t_0)]} \le e^{\frac{c}{2}(t_{k-1}-t_0)},$$

which, together with (4.89), gives

$$\begin{aligned} v(t) &\le e^{\frac{c}{2}(t_{k-1}-t_0)} v(t_0) e^{-c(t-t_0)} \le e^{\frac{c}{2}(t-t_0)} v(t_0) e^{-c(t-t_0)} \\ &\le v(t_0) e^{-\frac{c}{2}(t-t_0)} \le (v_1(t_0) + v_2(t_0)) e^{-\frac{c}{2}(t-t_0)} \\ &\le (c_2\delta^{p_1} + c_3\delta^{p_2}) e^{-\frac{c}{2}(t-t_0)} \\ &< c_1\varepsilon^{p_1} e^{-\frac{c}{2}(t-t_0)}, \qquad t \in [t_{k-1}, t_k). \end{aligned}$$

Hence by condition (i), we obtain

$$\|x(t)\| < \varepsilon e^{-\frac{c}{2p_1}(t-t_0)}, \quad t \ge t_0,$$

which implies that the trivial solution of system (2.3) is exponentially stable and the proof is complete. ∎

Remark 4.3.10 *Theorem 4.3.4 tells us to what extent we can relax the restriction on impulses to keep the exponential stability property of a system (see Example 4.3.5).*

Example 4.3.5 *Consider the impulsive functional differential equation*

$$\begin{cases} x'(t) &= -\frac{1+e^2}{50}x(t) + \frac{1}{50}\int_{t-\tau}^{t} x(s)\cos(x(s))e^{-(t-s)}ds, \quad t \geq 0,\ t \neq 2k, \\ x(t_k) &= I_k(x(t_k^-)), \quad t_k = 2k,\ k \in \mathbb{N}, \end{cases} \tag{4.90}$$

where $I_k(x) \in C(\mathbb{R}, \mathbb{R}),\ \tau \in (0, e^2]$.

Let $V(t, x_t) = V_1(t, x) + V_2(t, x_t)$, where $V_1(t, x) = |x|$ and

$$V_2(t, x_t) = 0.02 \int_{t-\tau}^{t} [e^2 - (t-s)]e^{-(t-s)}|x(s)|ds.$$

Since $\tau > 0$, we have

$$\int_{t-\tau}^{t} [e^2 - (t-s)]e^{-(t-s)}ds = e^2(1 - e^{-\tau}) + e^{-\tau}(1-\tau) - 1 > 0, \quad t \geq 0.$$

Thus condition (i) of Theorem 4.3.4 holds with $c_1 = c_2 = 1$, $p_1 = p_2 = 1$ and $c_3 = 0.02[e^2(1 - e^{-\tau}) + e^{-\tau}(1-\tau) - 1]$.

For any solution $x(t)$ of (4.90), we have

$$\begin{aligned} D^+V(t, x_t) &\leq 0.02[-(1+e^2)|x(t)| + \textstyle\int_{t-\tau}^{t} e^{-(t-s)}|x(s)|ds + e^2|x(t)|] \\ &\quad -\tfrac{e^2-\tau}{50e^\tau}|x(t-\tau)| + \tfrac{1}{50}\textstyle\int_{t-\tau}^{t}[-1 - e^2 + (t-s)]e^{-(t-s)}|x(s)|ds] \\ &\leq -\tfrac{1}{50}[|x(t)| + \textstyle\int_{t-\tau}^{t}[e^2 - (t-s)]e^{-(t-s)}|x(s)|ds] - \tfrac{e^2-\tau}{50e^\tau}|x(t-\tau)|] \\ &\leq -0.02V(t, x_t), \end{aligned}$$

which implies condition (iii) of Theorem 4.3.4 holds with $c = 0.02$.

Now if we suppose $|I_k(x)| \leq d_k|x|$, where $d_k \geq 0$ and satisfies $d_k \leq e^{0.02}$, then by Theorem 4.3.4 we know that the trivial solution of (4.90) is exponentially stable. The numerical simulation of this delay differential equation with $\tau = 1.6$, $d_k = 1.01$ and initial function

$$\phi(t) = \begin{cases} 0, & t \in [-1.6, 0), \\ 0.82, & t = 0, \end{cases}$$

is given in Figure 4.16.

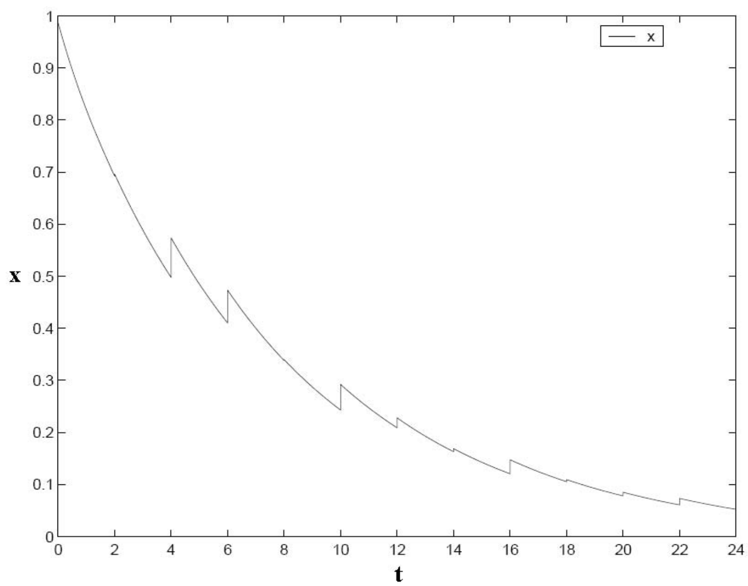

Figure 4.16: Numerical simulation of Example 4.3.5, impulsive system.

Chapter 5

Stability in Terms of Two Measures

In this chapter, we use the Lyapunov-Razumikhin technique to investigate the stability problem in terms of two measures for impulsive functional differential equations utilizing the ideas developed in [75, 89, 109]. We first obtain several Razumikhin-type stability criteria for impulsive functional differential equations via a single Lyapunov function. These criteria are applied to obtain partial stability, uniform stability and uniform asymptotical stability for a class of impulsive functional differential equations. Then the stability criteria via two Lyapunov functions are derived and the results are applied to obtain stability properties of the Lotka-Volterra system.

Definition 5.0.1 *Let $h \in \Gamma,\ h_0 \in \Gamma_0$. Then system (2.3) is said to be*

(S1). (h_0, h)*-equi-stable (equi-S for short), if for each $\varepsilon > 0$ and $t_0 \geq 0$, there exists some $\delta = \delta(\varepsilon, t_0) > 0$, such that $h_0(t_0, \phi) < \delta$ implies $h(t, x(t)) < \varepsilon$ for $t \geq t_0$, where $x(t) = x(t, t_0, \phi)$ is any solution of system (2.3);*

(S2). (h_0, h)*-uniformly stable (US for short), if the δ in $(S1)$ is independent of t_0;*

(S3). (h_0, h)*-equi-asymptotically stable (equi-AS for short), if $(S1)$ holds and for each $\varepsilon > 0$ and $t_0 \geq 0$, there exist some $\delta = \delta(t_0) > 0$ and $T = T(\varepsilon, t_0) > 0$, such that $h_0(t_0, \phi) < \delta$ implies $h(t, x(t)) < \varepsilon$ for $t \geq t_0 + T$;*

(S4). (h_0, h)*-uniformly asymptotically stable (UAS for short), if $(S2)$ holds and for each $\gamma > 0$ and $t_0 \geq 0$, there exist some $\eta = \eta(\gamma) > 0$ and $T = T(\gamma) > 0$ such that $h_0(t_0, \phi) < \eta$ implies $h(t, x(t)) < \gamma$ for any $t \geq t_0 + T$.*

Based on the Definition 5.0.1 and the usual stability concepts, it is easy to formulate other kinds of stability in terms of two measures (h_0, h) [60]. We shall give a few examples of the two measures (h_0, h) to demonstrate the generality of the Definition 5.0.1. It is obvious that Definition 5.0.1 reduces to

(1) the stability of the trivial solution of (2.3) if

$$h(t,x) = h_0(t,x) = \|x\|;$$

(2) the stability of the nontrivial solution $\tilde{x}(t)$ of (2.3) if

$$h(t,x) = h_0(t,x) = \|x - \tilde{x}(t)\|;$$

(3) the stability of an invariant set $A \subset \mathbb{R}^n$, if

$$h(t,x) = h_0(t,x) = d(x,A);$$

where d is the distance function;

(4) partial stability of the trivial solution of (2.3) if

$$h(t,x) = \|x^{(s)}\|, \quad 1 \le s < n, \quad \text{and} \quad h_0(t,x) = \|x\|;$$

where $x^{(s)} = (x_{m_1}, x_{m_2}, \cdots, x_{m_s})^T$ with $1 \le m_i < n$ and $1 \le i \le s$;

(5) the stability of conditionally invariant set B with respect to A, where $A \subset B \subset \mathbb{R}^n$, if

$$h(t,x) = d(x,B) \quad \text{and} \quad h_0(t,x) = d(x,A).$$

Several other combinations of choices are possible for (h, h_0) in addition to those given above.

5.1 Single Lyapunov Function Method

In this section, we state and prove our main results via a single Lyapunov function. The first result is on US.

Theorem 5.1.1 *Assume that there exist functions $V \in \nu_0,\ w,\ w_i \in K,\ i = 1, 2,$ and constant $\rho > 0$ such that*

(i) $V(t, \phi(0)) \leq w_2(h_0(t, \phi))$ *and* $w_1(h(t, \phi(0))) \leq V(t, \phi(0))$*, where* $\phi \in PC([-\tau, 0], \mathbb{R}^n)$*, if* $h(t, \phi(0)) < \rho$*; and* $h(t_0, \phi(0)) \leq w(h_0(t_0, \phi))$*;*

(ii) $V(t_k, x + I_k(x)) \leq (1 + b_k)V(t_k^-, x),$ *if* $h(t, x) < \rho$*; where* $b_k \geq 0$ *with* $\sum_{k=1}^{\infty} b_k < \infty$*;*

(iii) for any solution $x(t)$ *of (2.3),* $D^+V(t, x(t)) \leq 0,$ *whenever* $V(t, x(t)) \geq V(s, x(s))$ *for* $t \geq s$ *and* $h(t, x(t)) < \rho$*;*

(iv) there exists some $\rho_0 \in (0, \rho)$ *such that* $h(t_k, x) < \rho_0$ *implies* $h(t_k, x + I_k(x)) < \rho$*.*

Then (2.3) is (h_0, h)*-*US*.*

Proof. Since $\sum_{k=1}^{\infty} b_k < \infty$, it follows that $\prod_{k=1}^{\infty}(1 + b_k) = M$ and $1 \leq M < \infty$. For any given $\varepsilon \in (0, \rho_0)$, choose $\delta > 0$ such that $Mw_2(\delta) < w_1(\varepsilon)$ and $w(\delta) < \varepsilon$. Let $x(t) = x(t, t_0, \phi)$ be any solution of (2.3). We shall show $h_0(t_0, \phi) < \delta$ implies

$$h(t, x(t)) < \varepsilon, \qquad \forall t \geq t_0.$$

By the choice of δ, $h(t_0, x(t_0)) < \varepsilon$.

We claim that

$$h(t, x(t)) < \varepsilon, \qquad t \in [t_0, t_1). \tag{5.1}$$

Suppose (5.1) is not true. Then there is a $t^* \in (t_0, t_1)$ such that $h(t^*, x(t^*)) = \varepsilon$ and $h(t, x(t)) < \varepsilon$ for $t \in [t_0, t^*)$. Since $\varepsilon < \rho_0 < \rho$, by condition (i), we have

$$\begin{aligned} V(t^*, x(t^*)) &\geq w_1(h(t^*, x(t^*))) = w_1(\varepsilon) \\ &> Mw_2(\delta) \geq w_2(\delta) \\ &\geq V(s, x(s)), \quad \text{for } s \in [t_0 - \tau, t_0]. \end{aligned} \tag{5.2}$$

Let $U(t) = \sup_{s \in [t_0 - \tau, t]} V(s, x(s))$. Then by (5.2), we have

$$U(t^*) > w_2(\delta) \geq U(t_0), \tag{5.3}$$

which implies there exists some $\widehat{t} \in [t_0, t^*]$ such that $D^+U(\widehat{t}) > 0$, where $D^+U(\widehat{t}) = \limsup_{\alpha \to 0^+} \frac{1}{\alpha}[U(\widehat{t} + \alpha) - U(\widehat{t})]$.

We now show $D^+U(t) = 0$ for all $t \in [t_0, t^*]$.

For $t \in [t_0, t^*]$, we have $U(t) \geq V(t, x(t))$ by the definition of $U(t)$. If $U(t) > V(t, x(t))$, then by continuity of $V(t, x(t))$, there exists some $\sigma > 0$ such that $V(t + h, x(t + h)) \leq U(t)$ for $0 < h < \sigma$. Thus $U(t + h) = U(t)$ for $0 < h < \sigma$, which implies $D^+U(t) = 0$. If $U(t) = V(t, x(t))$, then $V(t, x(t)) \geq V(s, x(s))$ for $t \geq s$, so $D^+V(t, x(t)) \leq 0$ by condition (iii) and the fact that $h(t, x(t)) \leq \varepsilon < \rho_0 < \rho$ for all $t \in [t_0, t^*]$. This implies $V(t + h, x(t + h)) \leq V(t, x(t))$ and hence $U(t + h) \leq U(t)$ for $h > 0$ sufficiently small. Thus $D^+U(t) \leq 0$, which, together with $D^+U(t) \geq 0$, gives $D^+U(t) = 0$ for all $t \in [t_0, t^*]$. This contradicts $D^+U(\widehat{t}) > 0,\ \widehat{t} \in [t_0, t^*]$ and thus (5.1) is true.

Since $h(t, x(t)) < \varepsilon$ for $t \in [t_0, t_1)$, it follows $D^+U(t) = 0,\ t \in [t_0, t_1)$ in view of the argument that proves $D^+U(t) = 0$ for $t \in [t_0, t^*]$. Hence

$$V(t, x(t)) \leq U(t) = U(t_0) \leq w_2(\delta), \quad t \in [t_0, t_1), \tag{5.4}$$

and so

$$w_1(h(t_1, x(t_1))) \leq V(t_1, x(t_1)) \leq (1 + b_1)V(t_1^-, x(t_1^-)) \leq (1 + b_1)w_2(\delta) < w_1(\varepsilon),$$

which implies $h(t_1, x(t_1)) < \varepsilon$.

Next, we shall show

$$h(t, x(t)) < \varepsilon, \quad t \in [t_1, t_2).$$

If not, then there exists some $t^* \in (t_1, t_2)$ such that $h(t^*, x(t^*)) = \varepsilon$ and $h(t, x(t)) < \varepsilon$ for $t \in [t_1, t^*)$. Since $\varepsilon < \rho_0 < \rho$, by condition (i),

$$\begin{aligned} V(t^*, x(t^*)) &\geq w_1(h(t^*, x(t^*))) = w_1(\varepsilon) \\ &> Mw_2(\delta) \geq (1 + b_1)w_2(\delta) \\ &\geq V(s, x(s)), \quad \text{for } s \in [t_0 - \tau, t_1], \end{aligned}$$

so $U(t^*) > (1 + b_1)w_2(\delta) \geq U(t_1)$, which implies there exists some $\widehat{t} \in [t_1, t^*]$ such that $D^+U(\widehat{t}) > 0$. By the same argument that proves (5.1), we obtain $h(t, x(t)) < \varepsilon$ for $t \in [t_1, t_2)$. Further more, we have

$$V(t_2, x(t_2)) \leq (1 + b_2)V(t_2^-, x(t_2^-)) \leq (1 + b_2)(1 + b_1)w_2(\delta).$$

Thus it follows, by repeating the same argument,

$$h(t, x(t)) < \varepsilon, \text{ for } t \geq t_0,$$

which completes the proof. ■

The second result is on equi-AS, where the function V is assumed to diverge to infinity as t tends to infinity and $h(t, x) < \rho$.

Theorem 5.1.2 *Assume that there exist functions $V \in \nu_0,\ w \in K$ and constant $\rho > 0$ such that*

(i) $V(t, \phi(0)) \leq a(t, h_0(t, \phi))$, *if* $h(t, \phi(0)) < \rho$*;* $h(t_0, \phi(0)) \leq a_1(t_0, h_0(t_0, \phi))$*, where* $\phi \in PC([-\tau, 0], \mathbb{R}^n)$*,* $a,\ a_1 \in CK$*;*

(ii) $V(t, x) \geq \xi(t) w(h(t, x))$*, for* $t \geq t_0$ *and* $h(t, x) < \rho$;
where $\xi(t)$ *is a continuous and strictly increasing function such that* $\xi(0) = 1$, $\lim_{t\to+\infty} \xi(t) = +\infty$*;*

(iii) $V(t_k, x + I_k(x)) \leq (1 + b_k) V(t_k^-, x)$*, if* $h(t, x) < \rho$; *where* $b_k \geq 0$ *with* $\sum_{k=1}^{\infty} b_k < \infty$*;*

(iv) for any solution $x(t)$ *of (2.3),* $D^+V(t, x(t)) \leq 0$, *whenever* $V(t, x(t)) \geq V(s, x(s))$ *for* $t \geq s$ *and* $h(t, x(t)) < \rho$*;*

(v) there exists some $\rho_0 \in (0, \rho)$ *such that* $h(t_k, x) < \rho_0$ *implies* $h(t_k, x + I_k(x)) < \rho$*.*

Then (2.3) is (h_0, h)*-*$equi\text{-}AS$*.*

Proof. From condition (ii), we know, for $t \geq t_0$ and $h(t, x(t)) < \rho$,

$$V(t, x) \geq \xi(t) w(h(t, x)) \geq w(h(t, x)).$$

Then, for any given $\varepsilon > 0$ and $t_0 \geq 0$, by choosing $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$a(t_0, \delta) < M^{-1} w(\varepsilon) \quad \text{and} \quad a_1(t_0, \delta) < \varepsilon,$$

where $M = \prod_{k=1}^{\infty}(1 + b_k)$. Similarly to the proof of (h_0, h)-US in Theorem 5.1.1, the (h_0, h)-$equi$-S of (2.3) can be obtained, i.e., for $\rho_0 > 0$, there exists $\delta_0 = \delta_0(t_0, \rho_0) > 0$ such that $h_0(t_0, \phi) < \delta_0$ implies

$$h(t, x(t)) < \rho_0, \quad \forall t \geq t_0, \tag{5.5}$$

where $x(t) = x(t, t_0, \phi)$ is any solution of (2.3).

Moreover, we can obtain

$$V(t, x(t)) \leq Ma(t_0, \delta_0), \quad \forall t \geq t_0.$$

Since $\lim_{t\to+\infty} \xi(t) = +\infty$, then for any $\varepsilon \in (0, \rho_0)$, there exists $T = T(\varepsilon, t_0) > 0$ such that

$$\xi(t) \ > \ \frac{Ma(t_0, \delta_0)}{w(\varepsilon)}, \quad \text{for} \quad t \geq t_0 + T. \tag{5.6}$$

From condition (ii) and (5.5), (5.6), for $t \geq t_0 + T$,

$$w(h(t, x(t))) \leq \frac{V(t, x(t))}{\xi(t)} \leq \frac{Ma(t_0, \delta_0)}{\xi(t)} < w(\varepsilon),$$

which implies (2.3) is (h_0, h)-$equi$-AS. ■

Remark 5.1.1 *It should be noted that Theorem 5.1.1 and Theorem 5.1.2 are applicable to systems with infinite delay.*

Example 5.1.1 *Consider the following impulsive delay differential equations*

$$\begin{array}{rcl} x'(t) & = & -y(t)\cos(x(t)y(t-h)) - \frac{x(t)}{2(t+h+1)}, \ \ t \neq k, \\ y'(t) & = & 2(t+h+\frac{3}{2})x(t)\cos(x(t)y(t-h)), \ \ t \neq k, \\ x(k) & = & (1+\frac{2}{k^2})x(k^-), \ k \in \mathbb{N}, \\ y(k) & = & (1-\frac{3}{k^2})y(k^-), \ k \in \mathbb{N}, \\ x_{t_0} & = & \phi_1, \quad y_{t_0} \ = \ \phi_2, \end{array} \tag{5.7}$$

where $h > 0$, and $\phi_1,\ \phi_2 \in PC([-h, 0], \mathbb{R}^n)$. Then the trivial solution of (5.7) is equi-asymptotically x-stable.

Proof. Let $h_0(t, z(t)) = \|z(t)\|_h = \sup_{-h\leq s\leq 0} \|z(t+s)\|$ and $h(t, z(t)) = |x(t)|$, where $z(t) = (x(t), y(t))^T$, then (5.7) is (h_0, h)-$equi$-AS reduces to the trivial solution of (5.7) is equi-asymptotically x-stable (see [42] and [63]).

Let $V(t, x(t), y(t)) = \frac{1}{2}[x^2(t)+y^2(t)]+(t+h+1)x^2(t)$, $a(t, s) = (t+h+2)s$ and $a_1(t, s) = s$ for any $s,\ t \geq 0$ and $b_k = \frac{4}{k^2} + \frac{4}{k^4}$. Then, we have

$$\begin{array}{rcl} V(t, x(t), y(t)) & = & \dfrac{1}{2}(x^2(t) + y^2(t)) + (t+h+1)x^2(t) \\ & \leq & (t+h+2)\|z(t)\|_h^2, \end{array}$$

and

$$\begin{aligned} V(t_k, x(t_k), y(t_k)) &= \frac{1}{2}[x^2(k) + y^2(k)] + (k+h+1)x^2(k) \\ &= \frac{1}{2}[(1+\frac{4}{k^2}+\frac{4}{k^4})x^2(k^-) + (1-\frac{6}{k^2}+\frac{9}{k^4})y^2(k^-)] \\ &\quad +(k+h+1)(1+\frac{4}{k^2}+\frac{4}{k^4})x^2(k^-) \\ &\leq [1+\frac{4}{k^2}+\frac{4}{k^4}]V(t_k^-, x(t_k^-), y(t_k^-)). \end{aligned}$$

Thus conditions (i) and (iii) of Theorem 5.1.2 are satisfied.

The derivative of V along solutions of (5.7) is given by

$$\begin{aligned} D^+V(t, x(t), y(t)) &= x(t)x'(t) + y(t)y'(t) + x^2(t) + 2(t+h+1)x(t)x'(t) \\ &= -\frac{x^2(t)}{2(t+h+1)} \leq 0. \end{aligned}$$

Moreover, $V(t, x(t), y(t)) = \frac{1}{2}[x^2(t) + y^2(t)] + (t+h+1)x^2(t) \geq \xi(t)w(|x(t)|)$, where $\xi(t) = \frac{t+h+1}{h+1},\ w(|x(t)|) = (h+1)|x(t)|^2$. And for any $\rho > 0$, choose $\rho_0 = \frac{\rho}{4}$. If $h(t_k, x) = |x| < \rho_0$, then $h(t_k, x + I_k(x)) = (1+\frac{2}{k^2})|x| < 4\rho_0 = \rho$. So conditions (ii) and (v) of Theorem 5.1.2 hold.

Thus, by Theorem 5.1.2, the trivial solution of (5.7) is equi-asymptotically x-stable. ■

The next result is on UAS where the upper right-hand derivative of V is not assumed to be negative.

Theorem 5.1.3 *Suppose* (2.3) *is* (h_0, h)-US, *and there exist functions* $V(t, x) \in \nu_0,\ w_i \in K,\ i = 1, 2,$ *and constant* $\rho > 0$ *such that*

(i) $w_1(h(t, x)) \leq V(t, x) \leq w_2(h(t, x)),$ *if* $h(t, x) < \rho$;

(ii) $V(t_k, x + I_k(x)) \leq \psi_k(V(t_k^-, x)),$ *if* $h(t, x) < \rho$; *where* $\psi_k \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\psi_k(s) \geq s$ *and* $\frac{\psi_k(s)}{s}$ *is nondecreasing for* $s > 0$, *and for any* $a_1 > 0$, *there is a constant* M *so that*

$$\sum_{k=1}^{\infty} [\frac{\psi_k(a_1)}{a_1} - 1] = M < \infty;$$

(iii) there exist constants $T^* > 0$ *and* $g \in C(\mathbb{R}, \mathbb{R}_+)$ *such that for any solution* $x(t)$ *of (2.3)*

$$D^+V(t, x(t)) \leq -F(t, h(t, x(t))) + g(t), \qquad t \geq T^*,$$

whenever $h(t, x(t)) < \rho$*, and* $P(V(t, x(t))) > V(t+s, x(t+s)),\ -\tau \leq s \leq 0$*, where* $P \in C(\mathbb{R}_+, \mathbb{R}_+),\ P(s) > s$ *for* $s > 0$ *and* $F(t, h(t, x(t))) \geq \psi(t, \eta) \geq 0$ *for* $h(t, x(t)) \geq \eta > 0$*, where* $\psi(t, \eta)$ *is measurable;*

(iv) for any given $\eta > 0,\ \lim_{p \to \infty} \inf_{t \geq 0} \int_t^{t+p} \psi(s, \eta) ds = \infty$ *and* $\int_0^\infty g(t)dt = \Omega < \infty$*.*

Then (2.3) *is* (h_0, h)*-*UAS*.*

Proof. Since (2.3) is (h_0, h)-US, for any $\beta \in (0, \rho)$, there exists a $\delta > 0$ independent of t_0, such that $h_0(t_0, \phi) < \delta$ implies $h(t, x(t)) < \beta$ for all $t \geq t_0$; then by condition (i), if $h_0(t_0, \phi) < \delta$, we have

$$V(t, x(t)) \leq w_2(\beta), \quad \forall t \geq t_0. \tag{5.8}$$

Thus, for all $t \geq t_0$, $h(t, x(t)) < \beta < \rho$.

For any $\varepsilon \in (0, \beta)$, choose

$$0 < 2a < \min\Big\{w_1(\varepsilon), \inf_{\frac{w_1(\varepsilon)}{2} \leq s \leq w_2(\beta)} \{P(s) - s\}\Big\}. \tag{5.9}$$

Since $\sum_{k=1}^\infty [\frac{\psi_k(a_1)}{a_1} - 1] < \infty$, there exists $K^* \in \mathbb{N}$ such that

$$\sum_{k=K^*}^\infty [\frac{\psi_k(a_1)}{a_1} - 1] < \frac{a}{2w_2(\beta)}. \tag{5.10}$$

From $\int_0^\infty g(t)dt = \Omega$, there exists $\widehat{T} > 0$ such that

$$\int_{t_0 + \widehat{T}}^\infty g(s)ds < \frac{a}{2}. \tag{5.11}$$

By condition (iv), for $\eta = w_2^{-1}(\frac{w_1(\varepsilon)}{2})$, there exists $\widetilde{T} > 0$ such that

$$\int_t^{t+\widetilde{T}} \psi(t, \eta)dt > \Omega + w_2(\beta)(1 + M), \quad \forall t \geq t_0. \tag{5.12}$$

Let N be the first positive integer such that

$$w_2(\beta) \le w_1(\varepsilon) + Na. \tag{5.13}$$

We shall show that, for any $i = 0, 1, \cdots, N$,

$$V(t, x(t)) \le w_1(\varepsilon) + (N - i)a, \quad t \ge t_0 + t_{K^*} + \widehat{T} + i(\widetilde{T} + \tau). \tag{5.14}$$

It is clear that (5.14) holds for $i = 0$, since from (5.8) and (5.13),

$$V(t, x(t)) \le w_2(\beta) \le w_1(\varepsilon) + Na, \quad \forall t \ge t_0.$$

Suppose (5.14) holds for $i = k,\ k \in \{0, 1, \cdots, N-1\}$, i.e.

$$V(t, x(t)) \le w_1(\varepsilon) + (N - k)a, \quad t \ge \tau_k, \tag{5.15}$$

where $\tau_k = t_0 + t_{K^*} + \widehat{T} + k(\widetilde{T} + \tau),\ k \in \{0, 1, \cdots, N-1\}$.

We shall show (5.14) holds for $i = k+1,\ k \in \{0, 1, \cdots, N-1\}$, i.e.

$$V(t, x(t)) \le w_1(\varepsilon) + (N - k - 1)a, \quad t \ge \tau_{k+1}. \tag{5.16}$$

Let $\bar{I}_k = [\tau_k + \tau, \tau_{k+1}]$. We claim that there exists some $t^* \in \bar{I}_k$, such that

$$V(t^*, x(t^*)) < w_1(\varepsilon) + (N - k - 2)a. \tag{5.17}$$

Otherwise, for all $t \in \bar{I}_k$, we have

$$V(t, x(t)) \ge w_1(\varepsilon) + (N - k - 2)a. \tag{5.18}$$

From (5.9) we have $a < \frac{w_1(\varepsilon)}{2}$. By (5.8), (5.18) and $k \le N - 1$, we obtain

$$\frac{w_1(\varepsilon)}{2} \le V(t, x(t)) \le w_2(\beta), \quad \forall t \in \bar{I}_k. \tag{5.19}$$

Then by (5.9), (5.15), (5.18) and (5.19), we have, for any $t \in \bar{I}_k$,

$$\begin{aligned} P(V(t, x(t))) &> V(t, x(t)) + 2a \\ &\ge w_1(\varepsilon) + (N - k - 2)a + 2a \\ &\ge w_1(\varepsilon) + (N - k)a \\ &\ge V(t + s, x(t + s)), \qquad \forall s \in [-\tau, 0]. \end{aligned}$$

From condition (iii), we have, for any $t \in \bar{I}_k$,

$$D^+V(t, x(t)) \leq -F(t, h(t, x(t))) + g(t). \tag{5.20}$$

On the other hand, condition (i) and (5.19) imply that, for any $t \in \bar{I}_k$,

$$w_2(h(t, x(t))) \geq V(t, x(t)) \geq \frac{w_1(\varepsilon)}{2},$$

i.e.

$$h(t, x(t)) \geq w_2^{-1}(\frac{w_1(\varepsilon)}{2}) = \eta > 0. \tag{5.21}$$

From (5.21) and the assumption on F, we have

$$F(t, h(t, x(t))) \geq \psi(t, \eta) \geq 0,$$

which, together with (5.20), gives

$$D^+V(t, x(t)) \leq -\psi(t, \eta) + g(t), \quad \forall t \in \bar{I}_k. \tag{5.22}$$

Integrating (5.22) from $\tau_k + \tau$ to τ_{k+1}, and noticing $\tau_{k+1} = \tau_k + \tau + \widetilde{T}$, from (5.8), (5.18), conditions (ii) and (iv), we have

$$\begin{aligned} V(\tau_{k+1}, x(\tau_{k+1})) &\leq V(\tau_k + \tau, x(\tau_k + \tau)) - \int_{\tau_k+\tau}^{\tau_{k+1}} \psi(s, \eta) ds + \int_{\tau_k+\tau}^{\tau_{k+1}} g(s) ds \\ &\quad + \sum_{\tau_k+\tau < t_k \leq \tau_{k+1}} (V(t_k, x(t_k)) - V(t_k^-, x(t_k^-))) \\ &\leq w_2(\beta) - \int_{\tau_k+\tau}^{\tau_k+\tau+\widetilde{T}} \psi(s, \eta) ds + \int_0^\infty g(s) ds \\ &\quad + \sum_{\tau_k+\tau < t_k \leq \tau_{k+1}} V(t_k^-, x(t_k^-))[\frac{\psi_k(V(t_k^-, x(t_k^-)))}{V(t_k^-, x(t_k^-))} - 1] \\ &\leq w_2(\beta) - \int_{\tau_k+\tau}^{\tau_k+\tau+\widetilde{T}} \psi(s, \eta) ds + \Omega \\ &\quad + w_2(\beta) \sum_{k=1}^{\infty} [\frac{\psi_k(w_2(\beta))}{w_2(\beta)} - 1] \\ &\leq w_2(\beta)(1 + M) - \int_{\tau_k+\tau}^{\tau_k+\tau+\widetilde{T}} \psi(s, \eta) ds + \Omega \\ &< 0. \end{aligned}$$

This contradicts $V(t, x(t)) \geq 0$, so (5.17) holds.

Now we prove, for all $t \geq t^*$,

$$V(t, x(t)) \leq w_1(\varepsilon) + (N - k - 1)a. \tag{5.23}$$

Assume $t^* \in [t_q, t_{q+1})$ for some $q \geq K^*$. We first show that (5.23) holds for $t \in [t^*, t_{q+1})$.

Suppose not. Then there exists some $\widehat{t} = \{\inf_{t\in[t^*,t_{q+1})} : V(t, x(t)) > w_1(\varepsilon) + (N-k-1)a\}$, and then by the continuity of $V(t, x(t))$ on $[t^*, t_{q+1})$ we have

$$V(\widehat{t}, x(\widehat{t})) = w_1(\varepsilon) + (N - k - 1)a, \tag{5.24}$$

and $\widehat{t} > t^*$, since $V(t^*, x(t^*)) < w_1(\varepsilon) + (N - k - 2)a$. Thus we have

$$V(t^*, x(t^*)) < w_1(\varepsilon) + (N - k - 2)a < V(\widehat{t}, x(\widehat{t})),$$

which implies that there exists some $\underline{t} \in (t^*, \widehat{t})$ such that

$$\begin{aligned} &V(\underline{t}, x(\underline{t})) = w_1(\varepsilon) + (N - k - 2)a, \quad \text{and} \\ &V(\underline{t}, x(\underline{t})) \leq V(t, x(t)) \leq V(\widehat{t}, x(\widehat{t})), \quad \text{for all } t \in [\underline{t}, \widehat{t}]. \end{aligned}$$

Then as for (5.19), we can obtain

$$\frac{w_1(\varepsilon)}{2} \leq V(t, x(t)) \leq w_2(\beta), \quad \forall t \in [\underline{t}, \widehat{t}].$$

Thus for all $t \in [\underline{t}, \widehat{t}]$, we have

$$\begin{aligned} P(V(t, x(t))) &> V(t, x(t)) + 2a \geq V(\underline{t}, x(\underline{t})) + 2a \\ &= w_1(\varepsilon) + (N - k - 2)a + 2a \\ &= w_1(\varepsilon) + (N - k)a \\ &\geq V(t + s, x(t + s)), \qquad \forall s \in [-\tau, 0]. \end{aligned}$$

By the same argument to obtain (5.21), we know (5.21) holds for $t \in [\underline{t}, \widehat{t}]$. Then by condition (iii), we have

$$D^+V(t, x(t)) \leq -F(t, h(t, x(t))) + g(t) \leq g(t). \tag{5.25}$$

Integrating both sides of (5.25) from $\underline{t}$ to $\widehat{t}$ and using (5.11) and the fact $\bar{t} > t_0 + \widehat{T}$, we obtain

$$\begin{aligned} V(\widehat{t}, x(\widehat{t})) &\leq V(\underline{t}, x(\underline{t})) + \textstyle\int_{\underline{t}}^{\widehat{t}} g(s)ds \\ &\leq w_1(\varepsilon) + (N-k-2)a + \textstyle\int_{t_0+\widehat{T}}^{\infty} g(s)ds \\ &< w_1(\varepsilon) + (N-k-2)a + \tfrac{a}{2} \\ &< w_1(\varepsilon) + (N-k-1)a, \end{aligned}$$

which contradicts (5.24) and shows that (5.23) holds for any $t \in [t^*, t_{q+1})$.

Next, we prove that (5.23) holds for any $t \geq t_{q+1}$. Suppose not. Then there exists some $t^{**} = \{\inf_{t \geq t_{q+1}} : V(t, x(t)) > w_1(\varepsilon) + (N-k-1)a\}$, and we have

$$V(t^{**}, x(t^{**})) \geq w_1(\varepsilon) + (N-k-1)a. \tag{5.26}$$

So $t^{**} \geq t_{q+1} > t^*$. Let $\bar{t} = \{\sup_{t \in [t^*, t^{**}]} : V(t, x(t)) \leq w_1(\varepsilon) + (N-k-2)a\}$. Then $\bar{t} < t^{**}$, and either $V(\bar{t}, x(\bar{t})) = w_1(\varepsilon) + (N-k-2)a$, $\bar{t} \neq t_k$ for any $k \in \mathbb{N}$; or $V(\bar{t}, x(\bar{t})) < w_1(\varepsilon) + (N-k-2)a$, $\bar{t} = t_p^-$ with some $p \geq q$.

By the definition of $\bar{t}$, in both cases, we have

$$V(t, x(t)) \geq w_1(\varepsilon) + (N-k-2)a \geq \frac{w_1(\varepsilon)}{2},\ t \in [\bar{t}, t^{**}]. \tag{5.27}$$

By (5.8) and (5.27), we have

$$\frac{w_1(\varepsilon)}{2} \leq V(t, x(t)) \leq w_2(\beta), \quad \forall t \in [\bar{t}, t^{**}],$$

which, together with (5.9), (5.15) and (5.27), implies that for every $t \in [\bar{t}, t^{**}]$,

$$\begin{aligned} P(V(t, x(t))) &> V(t, x(t)) + 2a \geq w_1(\varepsilon) + (N-k-2)a + 2a \\ &= w_1(\varepsilon) + (N-k)a \geq V(t+s, x(t+s)), \qquad \forall s \in [-\tau, 0]. \end{aligned}$$

Then by condition (iii), we know that

$$D^+V(t, x(t)) \leq -F(t, h(t, x(t))) + g(t), \quad t \in [\bar{t}, t^{**}].$$

By condition (i) and (5.27), we know $h(t, x(t)) \geq w_2^{-1}(\frac{w_1(\varepsilon)}{2}) = \eta > 0$ holds for $t \in [\bar{t}, t^{**}]$. By condition (iii), we have

$$D^+V(t, x(t)) \leq -\psi(t, \eta) + g(t) \leq g(t), \quad t \in [\bar{t}, t^{**}]. \tag{5.28}$$

For the first case, i.e., $V(\bar{t}, x(\bar{t})) = w_1(\varepsilon) + (N-k-2)a$ and $\bar{t} \neq t_k$ for any $k \in \mathbb{N}$, by integrating both sides of (5.28) from $\bar{t}$ to t^{**} and using (5.10), (5.11),

$$\begin{aligned}
V(t^{**}, x(t^{**})) &\leq V(\bar{t}, x(\bar{t})) + \int_{\bar{t}}^{t^{**}} g(s)ds \\
&+ \sum_{\bar{t} \leq t_k \leq t^{**}} \left[V(t_k, x(t_k)) - V(t_k^-, x(t_k^-)) \right], \\
&< w_1(\varepsilon) + (N-k-2)a + \int_{t_0+\widehat{T}}^{\infty} g(s)ds + w_2(\beta) \sum_{k=K^*}^{\infty} [\frac{\psi_k(w_2(\beta))}{w_2(\beta)} - 1] \\
&< w_1(\varepsilon) + (N-k-2)a + \frac{a}{2} + w_2(\beta) \cdot \frac{a}{2w_2(\beta)} \\
&\leq w_1(\varepsilon) + (N-k-1)a,
\end{aligned}$$

which contradicts (5.26).

For the second case, i.e., $V(\bar{t}, x(\bar{t})) < w_1(\varepsilon) + (N-k-2)a$ and $\bar{t} = t_q^-$ with some $q \geq p$, integrating both sides of (5.28) from $\bar{t}$ to t^{**} gives

$$\int_{\bar{t}}^{t^{**}} D^+V(t, x(t))dt = \int_{t_q^-}^{t^{**}} D^+V(t, x(t))dt = \int_{t_q}^{t^{**}} D^+V(t, x(t))dt \leq \int_{\bar{t}}^{t^{**}} g(s)ds,$$

and by (5.10) and (5.11), we have

$$\begin{aligned}
V(t^{**}, x(t^{**})) &\leq V(t_q^-, x(t_q^-)) + \left[V(t_q, x(t_q)) - V(t_q^-, x(t_q^-)) \right] \\
&+ \int_{\bar{t}}^{t^{**}} g(s)ds + \sum_{k=q+1}^{\infty} [V(t_k, x(t_k)) - V(t_k^-, x(t_k^-))] \\
&< w_1(\varepsilon) + (N-k-2)a + \int_{t_0+\widehat{T}}^{\infty} g(s)ds + w_2(\beta) \sum_{k=q}^{\infty} [\frac{\psi_k(w_2(\beta))}{w_2(\beta)} - 1] \\
&< w_1(\varepsilon) + (N-k-2)a + \frac{a}{2} + w_2(\beta) \cdot \frac{a}{2w_2(\beta)} \\
&\leq w_1(\varepsilon) + (N-k-1)a,
\end{aligned}$$

which also contradicts (5.26).

For both cases, we obtain a contradiction, which shows that (5.23) holds for any $t \geq t^*$. And hence (5.16) is true since $t^* \leq \tau_{q+1}$.

So by induction, (5.14) holds for $i \in \{0, 1, \cdots, N\}$. Let $i = N$ in (5.14). We have

$$w_1(h(t, x(t))) \le V(t, x(t)) \le w_1(\varepsilon), \quad \forall t \ge \tau_N = t_0 + T^*,$$

i.e.

$$h(t, x(t)) \le \varepsilon, \quad \forall t \ge \tau_N = t_0 + T^*,$$

where $T^* = t_{K^*} + \widehat{T} + N(\widetilde{T} + \tau)$ is independent of t_0. The proof is complete. ■

Corollary 5.1.1 *Suppose* (2.3) *is* (h_0, h)*-*US*, and there exist functions* $V(t, x) \in \nu_0$, $w_i \in K$, $i = 1, 2, 3$, *and constant* $\rho > 0$ *such that*

(i) $w_1(h(t, x)) \le V(t, x) \le w_2(h(t, x))$, *if* $h(t, x) < \rho$*;*

(ii) $V(t_k, x + I_k(x)) \le (1 + b_k)V(t_k^-, x)$, *if* $h(t, x) < \rho$, *where* $b_k \ge 0$ *and* $\sum_{k=1}^{\infty} b_k < \infty$*;*

(iii) for any $\lambda \ge 0$*, there exist constants* $T^* > 0$ *such that for any solution* $x(t)$ *of (2.3),*

$$D^+V(t, x(t)) \le -b(t)\ [\ w_3(h(t, x(t))) - \lambda\ ], \qquad t \ge T^*,$$

whenever $h(t, x(t)) < \rho$*, and* $P(V(t, x(t))) > V(t + s, x(t + s))$, $-\tau \le s \le 0$, *where* $P \in C(\mathbb{R}_+, \mathbb{R}_+)$, $P(s) > s$ *for* $s > 0$, $b(t) \ge 0$*;*

(iv) $\lim_{T\to\infty} \inf_{t\ge 0} \int_t^{t+T} b(s)ds = \infty$.

Then (2.3) is (h_0, h)*-*UAS*.*

Proof. Let $\psi_k(s) = (1+b_k)s$ in condition (ii) of Theorem 5.1.3 with $b_k \ge 0$ and $\sum_{k=1}^{\infty} b_k < \infty$. Then condition (ii) of Theorem 5.1.3 holds. Let $\lambda = \frac{w_3(\sigma)}{2}$ for any $\sigma > 0$. We have

$$b(t)\ [\ w_3(\sigma) - \lambda\ ]\ = b(t)\frac{w_3(\sigma)}{2} \triangleq \widetilde{\psi}(t, \sigma), \ \ \forall t \ge T^*.$$

Then condition (iii) of Theorem 5.1.3 can be rewritten so that there exist constants $T^* > 0$ such that for any solution $x(t)$ of (2.3)

$$D^+V(t, x(t)) \le -b(t)\ [\ w_3(h(t, x(t))) - \lambda\ ]\ \triangleq\ -F(t, h(t, x(t))), \qquad t \ge T^*,$$

whenever $h(t,x(t)) < \rho$, and $P(V(t,x(t))) > V(t+s,x(t+s)),\ -\tau \le s \le 0$, where $P \in C(\mathbb{R}_+,\mathbb{R}_+),\ P(s) > s$ for $s > 0$ and $F(t,h(t,x(t))) \ge \widetilde{\psi}(t,\sigma) \ge 0$ for $h(t,x(t)) \ge \sigma > 0$. Together with condition (iv), we know condition (iv) of Theorem 5.1.3 holds. This completes the proof. ■

Let $b(t) = 1$ in Corollary 5.1.1. Then condition (iii) becomes the one used in [93] to obtain UAS, and we can obtain the following result.

Corollary 5.1.2 *Suppose* (2.3) *is* (h_0,h)-US, *and there exist functions* $V(t,x) \in \nu_0,\ w_i \in K,\ i = 1,2,3$, *and constant* $\rho > 0$ *such that*

(i) $w_1(h(t,x)) \le V(t,x) \le w_2(h(t,x)),$ *if* $h(t,x) < \rho$;

(ii) $V(t_k, x + I_k(x)) \le (1+b_k)V(t_k^-, x),$ *if* $h(t,x) < \rho$, *where* $b_k \ge 0$ *and* $\sum_{k=1}^{\infty} b_k < \infty$;

(iii) for any $\lambda \ge 0$, *there exist constants* $T^* > 0$ *such that for any solution* $x(t)$ *of (2.3),*

$$D^+V(t,x(t)) \le -w_3(h(t,x(t))) + \lambda, \qquad t \ge T^*,$$

whenever $h(t,x(t)) < \rho$, *and* $P(V(t,x(t))) > V(t+s,x(t+s)),\ -\tau \le s \le 0$, *where* $P \in C(\mathbb{R}_+,\mathbb{R}_+),\ P(s) > s$ *for* $s > 0$.

Then (2.3) is (h_0,h)-UAS.

Example 5.1.2 *Consider the scalar impulsive delay differential equation*

$$\begin{aligned} x'(t) &= -a(t)x(t) + b(t)x(t-r_0(t)) + \int_{-\infty}^{t} c(t,s,x(s))ds,\ t \ge t_0,\ t \ne t_k, \\ x(t_k) &= (1+b_k)x(t_k^-),\ k \in \mathbb{N}, \\ x_{t_0} &= \phi, \end{aligned} \tag{5.29}$$

where $a, b \in C(\mathbb{R}_+,\mathbb{R}),\ c(t,s,x) \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R},\mathbb{R})$, $r_0 \in C(\mathbb{R}_+,\mathbb{R}_+)$ *and* $r_0(t) \le \tau_0$, *and* $b_k > 0$ *and* $\sum_{k=1}^{\infty} b_k < \infty$, $\phi \in PC((-\infty,t_0],\mathbb{R}^n)$. *Suppose*

(i). there exists function $q \in L^1[0,\infty)$ *such that*

$$|c(t,s,x)| \le a(t)q(t-s)|x|, \tag{5.30}$$

(ii). there exists constant $\alpha \geq 0$ *such that*

$$|b(t)| \leq \alpha a(t), \quad \textit{and} \quad \alpha + \int_0^\infty q(s)ds < 1, \tag{5.31}$$

(iii). $\lim_{T\to\infty} \inf_{t\geq 0} \int_t^{t+T} a(s)ds = +\infty.$

Then (5.29) is US *and* UAS*.*

Proof. Let $h_0(t, x(t)) = \|x(t)\|_\tau = \sup_{-\tau\leq s\leq 0} |x(t+s)|$ (when $\tau = \infty$, $\|x(t)\|_\tau = \sup_{-\tau<s\leq 0}$ $|x(t+s)|$) and $h(t, z(t)) = |x(t)|$. Then (5.29) is (h_0, h)-US (or (h_0, h)-UAS) reduces to the trivial solution of (5.29) is uniformly stable (or uniformly asymptotically stability), see [63, 62].

Let $V(t, x) = |x(t)|$. Then we have

$$V(t_k, x(t_k)) = (1 + b_k)x(t_k^-) \leq (1 + b_k)|x(t_k^-)|,$$

i.e.

$$V(t_k, x(t_k)) \leq (1 + b_k)V(t_k^-, x(t_k^-)).$$

By (5.30) and (5.31), we have, whenever $V(s, x(s)) \leq V(t, x(t))$ for $s \leq t$,

$$\begin{aligned} D^+V(t, x(t)) &\leq -a(t)|x(t)| + |b(t)||x(t - \tau(t))| + \int_{-\infty}^t c(t, s, x(s))ds \\ &\leq -a(t)\Big\{1 - \alpha - \int_{-\infty}^t q(t - s)ds\Big\}V(t, x(t)) \\ &\leq 0. \end{aligned}$$

Thus from Theorem 5.1.1, we obtain (5.29) is US.

From condition (i) and (ii), we have $q(s) \in L^1[0, \infty)$ and there exists constant $p > 1$ such that

$$\beta \triangleq 1 - p(\alpha + \int_0^\infty q(s)ds) > 0. \tag{5.32}$$

Since (5.29) is US, let $|x(t)| \leq L$. Then for any $\sigma > 0$, there is some $\widetilde{T} > 0$ such that

$$\int_{\widetilde{T}}^\infty q(s)ds < \frac{\sigma}{L}. \tag{5.33}$$

In order to apply Corollary 5.1.1, choosing $\tau = \max\{\tau_0, \widetilde{T}\}$ and using (5.32) and (5.33), we calculate $D^+V(t, x(t))$ again

$$\begin{aligned} D^+V(t,x(t)) &\leq -a(t)\big\{1 - p\alpha - p\int_{t-\tau}^{t} q(t-s)ds\big\}|x(t)| \\ &\quad +La(t)\int_{-\infty}^{t-\tau} q(t-s)ds \\ &\leq -a(t)\big\{1 - p\alpha - p\int_{0}^{\infty} q(s)ds\big\}|x(t)| + La(t)\int_{\tau}^{\infty} q(s)ds \\ &\leq -a(t)\big\{\beta|x(t)| - \sigma\big\}, \end{aligned}$$

whenever $pV(t, x(t)) \geq V(t+s, x(t+s))$ for $s \in [-\tau, 0]$.

So all conditions of Corollary 5.1.1 are satisfied and then (5.29) is UAS. ■

5.2 Multiple Lyapunov Functions Method

We state and prove stability in terms of two measures via two Lyapunov functions in this section.

Theorem 5.2.1 *Suppose (2.3) is (h_0, h)-US, $w_i \in K,\ V,\ H \in \nu_0$ such that*

- *(i)* $0 \leq V(t,x) \leq w_1(h(t,x))$ *and* $w_2(h(t,x)) \leq H(t,x) \leq w_3(h(t,x))$, *if* $h(t,x) < \rho$;
- *(ii)* $V(t_k, x + I_k(x)) \leq \psi_k(V(t_k^-, x))$ *and* $H(t_k, x + I_k(x)) \leq \psi_k(H(t_k^-, x))$, *if* $h(t,x) < \rho$, *where* $\psi_k \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\psi_k(s) \geq s$ *and* $\frac{\psi_k(s)}{s}$ *is nondecreasing for* $s > 0$, *and for any* $a_1 > 0$, *there is a constant* M *so that*

$$\sum_{k=1}^{\infty}\Big[\frac{\psi_k(a_1)}{a_1} - 1\Big] = M < \infty;$$

- *(iii) there exist constants* $T^* > 0$ *and* $g \in C(\mathbb{R}, \mathbb{R}_+)$ *such that for any solution* $x(t)$ *of (2.3) and* $t \geq T^*$,

$$\begin{aligned} D^+V(t,x(t)) &\leq -F(t, h(t,x(t))) + g(t), \\ D^+H(t,x(t)) &\leq -F_1(t, h(t,x(t))), \end{aligned}$$

whenever $h(t,x(t)) < \rho$ *and* $P(H(t,x(t))) > H(t+s,x(t+s))$ *for* $-\tau \le s \le 0$, *where* $P \in C(\mathbb{R}_+,\mathbb{R}_+)$, $P(s) > s$ *for* $s > 0$ *and* $F(t,h(t,x(t))) \ge \psi(t,\eta) \ge 0$ *for* $h(t,x(t)) \ge \eta > 0$, *where* $\psi(t,\eta)$ *is measurable;* $F_1(t,h(t,x(t))) \ge 0$*;*

(iv) for any given $\eta > 0$, $\lim_{p\to\infty} \inf_{t\ge 0} \int_t^{t+p} \psi(s,\eta)ds = \infty$ *and* $\int_0^\infty g(t)dt = \Omega < \infty$*;*

(v) there exists some $0 < \rho_0 < \rho$ *such that* $h(t_k,x) < \rho_0$ *implies* $h(t_k,x+I_k(x)) < \rho$.

Then (2.3) is (h_0,h)*-*UAS.

Proof. Since (2.3) is (h_0,h)-US, then for $\rho_0 > 0$, there exists a $\delta > 0$ independent of t_0, such that $h_0(t_0,\phi) < \delta$ implies $h(t,x(t)) < \rho_0$ for all $t \ge t_0$. Choose a $\beta > 0$ so that $w_3(\beta) = w_2(\rho_0)$. Then if $h_0(t_0,\phi) < \delta$, we have

$$H(t,x(t)) \le w_3(\beta) \text{ and } V(t,x(t)) \le w_1(\rho_0), \quad \forall t \ge t_0. \tag{5.34}$$

Thus, for any $t \ge t_0$, we have $h(t,x(t)) \le \rho_0 < \rho$.

For any $\varepsilon \in (0, \min\{\rho_0,\beta\})$, choose

$$0 < 2a < \min\Big\{w_2(\varepsilon), \inf_{\frac{w_2(\varepsilon)}{2} \le s \le w_3(\beta)} \{P(s)-s\}\Big\}. \tag{5.35}$$

Since $\sum_{k=1}^{\infty} [\frac{\psi_k(a_1)}{a_1} - 1] < \infty$, there exists $K^* \in \mathbb{N}$ such that

$$\sum_{k=K^*}^{\infty} [\frac{\psi_k(a_1)}{a_1} - 1] < \frac{a}{w_3(\beta)}. \tag{5.36}$$

By condition (iv), for $\eta = w_3^{-1}[\frac{w_2(\varepsilon)}{2}]$, there exists $\widetilde{T} > 0$ such that

$$\int_t^{t+\widetilde{T}} \psi(t,\eta)dt > \Omega + w_1(\rho_0)(1+M), \quad \forall t \ge t_0. \tag{5.37}$$

Let N_0 be the first positive integer such that

$$w_3(\beta) \le w_2(\varepsilon) + N_0 a. \tag{5.38}$$

We shall show that, for any $i = 0, 1, \cdots, N_0$,

$$H(t,x(t)) \le w_2(\varepsilon) + (N_0 - i)a, \quad t \ge t_0 + t_{K^*} + i(\widetilde{T} + \tau). \tag{5.39}$$

It is clear that (5.39) holds for $i = 0$ since from (5.34) and (5.38),

$$H(t, x(t)) \leq w_3(\beta) \leq w_2(\varepsilon) + N_0 a, \quad \forall t \geq t_0.$$

Suppose (5.39) holds for $i = k$, i.e.

$$H(t, x(t)) \leq w_2(\varepsilon) + (N_0 - k)a, \quad t \geq \tau_k,\ k = 0, 1, \cdots, N_0 - 1, \tag{5.40}$$

where $\tau_k = t_0 + t_{K^*} + k(\widetilde{T} + \tau),\ k = 0, 1, \cdots, N_0 - 1.$

We shall show (5.39) holds for $i = k + 1$, i.e.

$$H(t, x(t)) \leq w_2(\varepsilon) + (N_0 - k - 1)a, \quad t \geq \tau_{k+1},\ k = 0, 1, \cdots, N_0 - 1. \tag{5.41}$$

Let $\bar{I}_k = [\tau_k + \tau, \tau_{k+1}]$, we claim that there exists some $t^* \in \bar{I}_k$, such that

$$H(t^*, x(t^*)) < w_2(\varepsilon) + (N_0 - k - 2)a. \tag{5.42}$$

Otherwise, for all $t \in \bar{I}_k$, we have

$$H(t, x(t)) \geq w_2(\varepsilon) + (N_0 - k - 2)a. \tag{5.43}$$

From (5.35) we have $a < \frac{w_2(\varepsilon)}{2}$, noticing $k \leq N_0 - 1$, (5.34) and (5.43), we obtain

$$\frac{w_2(\varepsilon)}{2} \leq H(t, x(t)) \leq w_3(\beta), \quad \forall t \in \bar{I}_k. \tag{5.44}$$

Then by (5.35), (5.40) and (5.44), we have, for any $t \in \bar{I}_k$,

$$\begin{aligned} P(H(t, x(t))) &> H(t, x(t)) + 2a \\ &\geq w_2(\varepsilon) + (N_0 - k - 2)a + 2a \\ &\geq w_2(\varepsilon) + (N_0 - k)a \\ &\geq H(t + s, x(t + s)), \qquad\qquad \forall s \in [-\tau, 0]. \end{aligned}$$

From condition (iii), we have, for any $t \in \bar{I}_k$,

$$D^+ V(t, x(t)) \leq -F(t, h(t, x(t))) + g(t). \tag{5.45}$$

On the other hand, condition (i) and (5.44) imply, for any $t \in \bar{I}_k$,

$$w_3(h(t, x(t))) \geq H(t, x(t)) \geq \frac{w_2(\varepsilon)}{2},$$

i.e.

$$h(t,x(t)) \geq w_3^{-1}(\frac{w_2(\varepsilon)}{2}) = \eta > 0. \tag{5.46}$$

From (5.46) and the assumption on F, we have

$$F(t,h(t,x(t))) \geq \psi(t,\eta) \geq 0,$$

together with (5.45), we obtain

$$D^+V(t,x(t)) \leq -\psi(t,\eta) + g(t), \quad \forall t \in \bar{I}_k. \tag{5.47}$$

Integrating (5.47) from $\tau_k + \tau$ to τ_{k+1}, and noticing $\tau_{k+1} = \tau_k + \tau + \widetilde{T}$, from (5.34), (5.37) and conditions (ii) and (iv), we have

$$\begin{aligned} V(\tau_{k+1}, x(\tau_{k+1})) &\leq V(\tau_k+\tau, x(\tau_k+\tau)) - \textstyle\int_{\tau_k+\tau}^{\tau_{k+1}} \psi(s,\eta)ds + \int_{\tau_k+\tau}^{\tau_{k+1}} g(s)ds \\ &\quad + \textstyle\sum_{\tau_k+\tau<t_k\leq\tau_{k\pm1}} (V(t_k,x(t_k)) - V(t_k^-,x(t_k^-))) \\ &\leq w_1(\rho_0) - \textstyle\int_{\tau_k+\tau}^{\tau_k+\tau+\widetilde{T}} \psi(s,\eta)ds + \int_0^\infty g(s)ds \\ &\quad + \textstyle\sum_{\tau_k+\tau<t_k\leq\tau_{k+1}} V(t_k^-,x(t_k^-))[\frac{\psi_k(V(t_k^-,x(t_k^-)))}{V(t_k^-,x(t_k^-))} - 1] \\ &\leq w_1(\rho_0) - \textstyle\int_{\tau_k+\tau}^{\tau_k+\tau+\widetilde{T}} \psi(s,\eta)ds + \Omega \\ &\quad + w_1(\rho_0)\textstyle\sum_{k=1}^{\infty}[\frac{\psi_k(w_1(\rho_0))}{w_1(\rho_0)} - 1] \\ &\leq w_1(\rho_0)(1+M) - \textstyle\int_{\tau_k+\tau}^{\tau_k+\tau+\widetilde{T}} \psi(s,\eta)ds + \Omega \\ &< 0. \end{aligned}$$

This contradicts $V(t,x(t)) \geq 0$, so (5.42) holds.

Now we prove, for all $t \geq t^*$,

$$H(t,x(t)) \leq w_2(\varepsilon) + (N_0 - k - 1)a. \tag{5.48}$$

Assume $t^* \in [t_q, t_{q+1})$ for some $q \geq K^*$. We first show that (5.48) holds for $t \in [t^*, t_{q+1})$.

Suppose not. Then there exists some $\widehat{t} = \{\inf_{t\in[t^*,t_{q+1})} :\ H(t,x(t)) > w_2(\varepsilon) + (N_0 - k - 1)a\}$, and then by the continuity of $H(t,x(t))$ on $[t^*, t_{q+1})$ we have

$$H(\widehat{t}, x(\widehat{t})) = w_2(\varepsilon) + (N_0 - k - 1)a, \tag{5.49}$$

and $\widehat{t} > t^*$, since $H(t^*,x(t^*)) < w_2(\varepsilon) + (N_0 - k - 2)a$. Thus we have

$$H(t^*,x(t^*)) < w_2(\varepsilon) + (N_0 - k - 2)a < H(\widehat{t}, x(\widehat{t})),$$

which implies that there exists some $\underline{t} \in (t^*, \widehat{t})$ such that

$$H(\underline{t}, x(\underline{t})) = w_2(\varepsilon) + (N_0 - k - 2)a, \quad \text{and}$$
$$H(\underline{t}, x(\underline{t})) \le H(t, x(t)) \le H(\widehat{t}, x(\widehat{t})), \quad \text{for all } t \in [\underline{t}, \widehat{t}].$$

Then as for (5.44), we can obtain

$$\frac{w_2(\varepsilon)}{2} \le H(t, x(t)) \le w_3(\beta), \quad \forall t \in [\underline{t}, \widehat{t}].$$

Thus for all $t \in [\underline{t}, \widehat{t}]$, we have

$$\begin{aligned} P(H(t, x(t))) &> H(t, x(t)) + 2a \ge H(\underline{t}, x(\underline{t})) + 2a \\ &= w_2(\varepsilon) + (N_0 - k - 2)a + 2a \\ &= w_2(\varepsilon) + (N_0 - k)a \\ &\ge H(t+s, x(t+s)), \qquad \forall s \in [-\tau, 0]. \end{aligned}$$

Then by condition (iii), we have

$$D^+H(t, x(t)) \le -F_1(t, h(t, x(t))) \le 0. \tag{5.50}$$

Then we have $H(\widehat{t}, x(\widehat{t})) \le H(\underline{t}, x(\underline{t}))$, which is a contradiction and shows that (5.48) holds for any $t \in [t^*, t_{q+1})$.

Next, we prove that (5.48) holds for any $t \ge t_{q+1}$. Suppose not. Then there exists some $t^{**} = \{\inf_{t \ge t_{q+1}} : H(t, x(t)) > w_2(\varepsilon) + (N_0 - k - 1)a\}$, and we have

$$H(t^{**}, x(t^{**})) \ge w_2(\varepsilon) + (N_0 - k - 1)a. \tag{5.51}$$

So $t^{**} \ge t_{q+1} > t^*$. Let $\bar{t} = \{\sup_{t \in [t^*, t^{**}]} : \ H(t, x(t)) \le w_2(\varepsilon) + (N_0 - k - 2)a\}$. Then $\bar{t} < t^{**}$, and either $H(\bar{t}, x(\bar{t})) = w_2(\varepsilon) + (N_0 - k - 2)a, \ \bar{t} \ne t_k$ for any $k \in \mathbb{N}$; or $H(\bar{t}, x(\bar{t})) < w_2(\varepsilon) + (N_0 - k - 2)a, \ \bar{t} = t_p^-$ with some $p \ge q$.

By the definition of $\bar{t}$, in both cases, we have

$$H(t, x(t)) \ge w_2(\varepsilon) + (N_0 - k - 2)a \ge \frac{w_2(\varepsilon)}{2}, \ t \in [\bar{t}, t^{**}]. \tag{5.52}$$

By (5.34) and (5.52), we have

$$\frac{w_2(\varepsilon)}{2} \leq H(t,x(t)) \leq w_3(\beta), \ \ \forall t \in [\bar{t}, t^{**}],$$

which implies that for every $t \in [\bar{t}, t^{**}]$,

$$\begin{aligned} P(H(t,x(t))) &> H(t,x(t)) + 2a \geq w_2(\varepsilon) + (N_0 - k - 2)a + 2a \\ &= w_2(\varepsilon) + (N_0 - k)a \geq H(t+s, x(t+s)), \qquad \forall s \in [-\tau, 0]. \end{aligned}$$

Then by condition (iii), we know that

$$D^+H(t,x(t)) \leq -F_1(t, h(t,x(t))) \leq 0, \quad t \in [\bar{t}, t^{**}]. \tag{5.53}$$

For the first case, i.e., $H(\bar{t}, x(\bar{t})) = w_2(\varepsilon) + (N_0 - k - 2)a$ and $\bar{t} \neq t_k$ for any $k \in \mathbb{N}$, by integrating both sides of (5.53) from $\bar{t}$ to t^{**} and using (5.36),

$$\begin{aligned} H(t^{**}, x(t^{**})) &\leq H(\bar{t}, x(\bar{t})) + \sum_{\bar{t} \leq t_k \leq t^{**}} \big[H(t_k, x(t_k)) - H(t_k^-, x(t_k^-)) \big], \\ &< w_2(\varepsilon) + (N_0 - k - 2)a + w_2(\beta) \sum_{k=K^*}^{\infty} [\frac{\psi_k(w_2(\beta))}{w_2(\beta)} - 1] \\ &< w_2(\varepsilon) + (N_0 - k - 2)a + w_2(\beta) \cdot \frac{a}{2w_2(\beta)} \\ &< w_2(\varepsilon) + (N_0 - k - 1)a, \end{aligned}$$

which contradicts (5.51).

For the second case, i.e., $H(\bar{t}, x(\bar{t})) < w_2(\varepsilon) + (N_0 - k - 2)a$ and $\bar{t} = t_q^-$ with some $q \geq p$, integrating both sides of (5.53) from $\bar{t}$ to t^{**} gives

$$\int_{\bar{t}}^{t^{**}} D^+H(t,x(t))dt = \int_{t_q^-}^{t^{**}} D^+H(t,x(t))dt = \int_{t_q}^{t^{**}} D^+H(t,x(t))dt \leq 0,$$

and by (5.36), we have

$$\begin{aligned} H(t^{**}, x(t^{**})) \;&\le\; H(t_q^-, x(t_q^-)) + \big[\, H(t_q, x(t_q)) - H(t_q^-, x(t_q^-)) \,\big] \\ &+ \sum_{k=q+1}^{\infty} [H(t_k, x(t_k)) - H(t_k^-, x(t_k^-))] \\ &<\; w_2(\varepsilon) + (N_0 - k - 2)a + w_2(\beta) \sum_{k=q}^{\infty} [\frac{\psi_k(w_2(\beta))}{w_2(\beta)} - 1] \\ &<\; w_2(\varepsilon) + (N_0 - k - 2)a + w_2(\beta) \cdot \frac{a}{2w_2(\beta)} \\ &<\; w_2(\varepsilon) + (N_0 - k - 1)a, \end{aligned}$$

which also contradicts (5.51).

Thus we know (5.48) holds in both cases, and hence (5.41) is true since $t^* \le \tau_{k+1}$.

So by induction, (5.39) holds for $i = 0, 1, \cdots, N_0$. Let $i = N_0$ in (5.39). We have

$$w_2(h(t, x(t))) \le H(t, x(t)) \le w_2(\varepsilon), \quad \forall t \ge \tau_{N_0} = t_0 + T^*,$$

i.e.

$$h(t, x(t)) \le \varepsilon, \quad \forall t \ge \tau_{N_0} = t_0 + T^*,$$

where $T^* = t_{K^*} + N_0(\widetilde{T} + \tau)$ is independent of t_0. The proof is complete. ∎

Corollary 5.2.1 *If condition (iii) and (iv) of Theorem 5.2.1 are replaced by (iii') and (iv') respectively,*

(iii') *for any $\lambda_i \ge 0 (i = 1, 2)$, there exist constants $T^* > 0$ such that for any solution $x(t)$ of (2.3) and $t \ge T^*$,*

$$\begin{aligned} D^+V(t, x(t)) &\le -b(t)\,[\, w_4(h(t, x(t))) - \lambda_1 \,] + g(t), \\ D^+H(t, x(t)) &\le -c(t)\,[\, w_5(h(t, x(t))) - \lambda_2 \,], \end{aligned}$$

whenever $h(t, x(t)) < \rho$, and $P(V(t, x(t))) > V(t + s, x(t + s)),\ -\tau \le s \le 0$, where $P \in C(\mathbb{R}_+, \mathbb{R}_+),\ P(s) > s$ for $s > 0$, $b(t),\ c(t) \ge 0$;

(iv') $\lim_{T\to\infty} \inf_{t\ge 0} \int_t^{t+T} b(s)ds = \infty.$

Then the result is still true.

Proof. Let $\lambda_1 = \frac{w_4(\sigma)}{2}$ for any $\sigma > 0$, we have

$$b(t)\ [\ w_4(\sigma) - \lambda_1\ ]\ = b(t)\frac{w_4(\sigma)}{2} \triangleq \widetilde{\psi}(t,\sigma),\ \ \forall t \geq T^*,$$

so condition (iii) of Theorem 5.2.1 can be rewritten so that there exist constants $T^* > 0$ such that for any solution $x(t)$ of (2.3),

$$\begin{aligned} D^+V(t,x(t)) &\leq -b(t)\ [\ w_4(h(t,x(t))) - \lambda_1\ ]\ + g(t) \\ &\triangleq\ -F_1(t,h(t,x(t)))\ + g(t),\quad t \geq T^*, \end{aligned}$$

whenever $h(t,x(t)) < \rho$ and $P(V(t,x(t))) > V(t+s,x(t+s)),\ -\tau \leq s \leq 0$, where $P \in C(\mathbb{R}_+,\mathbb{R}_+),\ P(s) > s$ for $s > 0$ and $F(t,h(t,x(t))) \geq \widetilde{\psi}(t,\sigma) \geq 0$ for $h(t,x(t)) \geq \sigma > 0$. Moreover, let $F_1(t,h(t,x(t))) = c(t)\ [\ w_5(h(t,x(t))) - \lambda_2\ ]$, then $F_1(t,h(t,x(t))) \geq 0$. Together with condition (iv'), we know condition (iv) of Theorem 5.2.1 holds. This completes the proof. ■

Remark 5.2.1 *In Corollary 5.2.1, if we let $h_0(t,x(t)) = \|x(t)\|_\tau,\ h(t,x(t)) = \|x(t)\|$, where $\|\cdot\|$ is any norm in $\mathbb{R}^n$, and let $x + I_k(x) \equiv x$, then we can obtain the same UAS result in Theorem 2.1 in reference [98].*

Corollary 5.2.2 *If condition (ii) of Theorem 5.2.1 is replaced by (ii'),*

(ii') $V(t_k, x + I_k(x)) \leq (1 + b_k)V(t_k^-, x)$ *and* $H(t_k, x + I_k(x)) \leq (1 + b_k)H(t_k^-, x)$, *if* $h(t,x) < \rho$, *where* $b_k > 0$ *and* $\sum_{k=1}^{\infty} b_k < \infty$.

Then the result is still true.

Proof. Let $\psi_k(s) = (1 + b_k)s$ in condition (ii) of Theorem 5.2.1, together with $b_k > 0$ and $\sum_{k=1}^{\infty} b_k < \infty$, then condition (ii) of Theorem 5.2.1 holds. ■

5.3 Application to the Lotka-Volterra System

Functional differential equations are frequently used to model population dynamics. The Lotka-Volterra equation for predator-pray problems or competing species has often been considered ([54, 31, 98]). When population levels repeatedly undergo changes of relatively short duration (due, for instance, to stocking or harvesting of species), these events may be more suitably modelled by an impulsive functional differential equation (see [77] and references therein). In this section, we apply the results in the previous section to the stability analysis of the time-delayed Lotka-Volterra equations.

Consider the following Lotka-Volterra system subject to impulsive effects

$$x_i'(t) = b_i(x_i(t))\Big\{r_i(t) - a_i(t)x_i(t) + \sum_{j=1}^{n}\int_{-\infty}^{t} x_j(s)d\mu_{ij}(t,s)\Big\},\ t \geq t_0 = 0,\ t \neq t_k, \quad (5.54a)$$

$$x_i(t_k) = c_{ik}x_i(t_k^-) + (1 - c_{ik})x_i^*,\ i = 1, \cdots, n, \quad (5.54b)$$

$$x(\theta) = \phi(\theta), \quad \theta \in (-\infty, 0], \quad (5.54c)$$

where $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is assumed to be a positive equilibrium of system (5.54a) and (5.54c), and the initial functions satisfy

$$\phi_i(\theta) \geq 0, \quad \phi_i(0) > 0, \quad \text{for } i = 1, 2, \cdots, n. \quad (5.55)$$

Suppose for $i,\ j = 1, 2, \cdots, n,\ k \in \mathbb{N}$, the following conditions hold

(A1). Constants $c_{ik} \in [0, 1]$ and functions $b_i \in K_0$ and for any $0 < \beta \ll 1,\ \int_0^\beta \frac{ds}{b_i(s)} = +\infty$;

(A2). $r_i(t) \geq 0$ and $a_i(t) \geq 0$ are continuous functions;

(A3). $\mu_{ij}(t, s)$ have bounded variation for any $t \in \mathbb{R}$ and $s \leq t$, and satisfy

$$\int_{-\infty}^{u} |d\mu_{ij}(t, s)| \leq a_i(t)\hat{\mu}_{ij}(t, u),$$

where $\hat{\mu}_{ij}(t, u)(u \leq t)$ are nondecreasing with respect to u, and there exist constants $\gamma_{ij} \geq 0$ with $\gamma_{ii} < 1$ such that $\hat{\mu}_{ij}(t, t) \leq \gamma_{ij}$, and for any $\varepsilon > 0$, there exists constant $h > 0$ such that $\hat{\mu}_{ij}(t, t - h) \leq \varepsilon,\ \forall t \geq 0$.

Remark 5.3.1 *From Lemma 3.1 ([98]), we know the solutions of (5.54a) and (5.54c) are positive in their maximal interval of existence. So the solutions of (5.54) are positive in their maximal interval of existence, since $c_{ik} \in [0,1]$ for any $i = 1, \cdots, n,\ k \in \mathbb{N}$.*

Theorem 5.3.1 *Assume conditions $(A1)$-$(A3)$ hold, and*

(i). $b_i(s)$ *are nondecreasing and for any* $i = 1, \cdots, n$, *and* $t \in \mathbb{R}$,

$$\lim_{p \to +\infty} \int_t^{t+p} a_i(s)ds = +\infty;$$

(ii). Γ_1 *is an M-matrix, where*

$$\Gamma_1 = \begin{bmatrix} 1-\gamma_{11} & -\gamma_{12} & \cdots & -\gamma_{1n} \\ -\gamma_{21} & 1-\gamma_{22} & \cdots & -\gamma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -\gamma_{n1} & -\gamma_{n2} & \cdots & 1-\gamma_{nn} \end{bmatrix}.$$

Then x^* is uniformly asymptotically stable.

Proof. Rewriting system (5.54a) and (5.54b), we obtain

$$y_i'(t) = b_i(y_i(t)+x_i^*)\big\{-a_i(t)y_i(t) + \sum_{j=1}^{n}\int_{-\infty}^{t} y_j(s)d\mu_{ij}(t,s)\big\},\ t \geq 0,\ t \neq t_k, \tag{5.56a}$$

$$y_i(t_k) = c_{ik}y_i(t_k^-),\ i = 1, \cdots, n, \tag{5.56b}$$

where $y_i(t) = x_i(t) - x_i^*, i = 1, \cdots, n$. Since Γ_1 is an M-matrix, there exist positive constants $d_i,\ i = 1, \cdots, n$, such that

$$d_i(1-\gamma_{ii}) > \sum_{i \neq j}^{n} d_j\gamma_{ij},\quad i = 1, \cdots, n. \tag{5.57}$$

Choose

$$\begin{aligned} H(y(t)) &= \max\{d_i^{-1}|y_i(t)| :\ 1 \leq i \leq n\}, \\ N_H &= \{i \in \{1, 2, \cdots, n\} :\ H(y(t)) = d_i^{-1}|y_i(t)|,\ t \geq 0\}. \end{aligned}$$

For any $i \in N_H$, using (5.57), we calculate $D^+H(y(t))$

$$\begin{aligned} D^+H(y(t)) &\le b_i(x_i(t))\big\{-a_i(t)|y_i(t)| + \textstyle\sum_{j=1}^n \int_{-\infty}^t |y_j(s)||d\mu_{ij}(t,s)|\big\} \\ &\le -b_i(x_i(t))d_i^{-1}\big\{d_i(1-\gamma_{ii}) - \textstyle\sum_{i\neq j}^n d_j\gamma_{ij}\big\}a_i(t)H(y(t)) \\ &\le 0, \end{aligned} \tag{5.58}$$

whenever $H(s, y(s)) \le H(t, y(t))$ for $s \le t$.

And

$$\begin{aligned} H(y(t_k)) &= \max\{d_i^{-1}|y_i(t_k)| : 1 \le i \le n\} \\ &= \max\{d_i^{-1}c_{ik}|y_i(t_k^-)| : 1 \le i \le n\} \\ &\le H(y(t_k^-)). \end{aligned}$$

Thus we know from Theorem 5.1.1 that the trivial solution of (5.56) is uniformly stable.

Now choose $h_0(t, y(t)) = \|y(t)\|_\infty = \sup_{-\infty<s\le 0}\left\{\max_{1\le i\le n}\{d_i^{-1}|y_i(t+s)|\}\right\}$, $h(t, y(t)) = \|y(t)\|_n = \max_{1\le i\le n}\{d_i^{-1}|y_i(t)|\}$, then for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\|_\infty \le \delta$ implies $|y_i(t)| \le \varepsilon$ for $t \ge t_0,\ i = 1, 2, \cdots, n$.

Then for $t \ge t_0$, let

$$V(y(t)) = \max_{1\le i\le n}\left\{d_i^{-1}V_i(t),\ \text{where } V_i(t) = \int_0^{|y_i|}\frac{du}{b_i(x_i^* + \text{sgn}(y_i)u)}\right\}, \tag{5.59}$$

and

$$N_V = \{i \in \{1, 2, \cdots, n\} :\ V(y(t)) = d_i^{-1}V_i(t),\ t \ge 0\}.$$

We have $\frac{1}{b_i(x_i^* + \text{sgn}(y_i)u)} > 0$ since

$$x_i^* + \text{sgn}(y_i)u = \begin{cases} x_i^* + u \ge x_i^* > 0, & \text{if } y_i \ge 0, \\ x_i^* - u \ge x_i^* + y_i = x_i > 0, & \text{if } y_i < 0, \end{cases}$$

so $V(y(t)) \ge 0$.

And we have $V(y(t_k)) \le V(y(t_k^-))$ since

$$V_i(y(t_k)) = \int_0^{c_{ik}|y_i(t_k^-)|}\frac{du}{b_i(x_i^* + \text{sgn}(y_i)u)} \le V_i(y(t_k^-)),$$

in view of $c_{ik} \in [0, 1]$.

Moreover, choose $\rho > 0$ such that $\rho < \min_{1\le i\le n}\{d_i^{-1}x_i^*\}$, then there exists $\eta > 0$ such that

$$\rho \le \min_{1\le i\le n}\{d_i^{-1}(x_i^* - \eta)\}.$$

Thus, together with (5.59), we have, for $h(t, y(t)) = \|y(t)\|_n < \rho$,

$$d_i^{-1}(x_i^* - \eta) \ge \rho > \|y(t)\|_n \ge d_i^{-1}|y_i(t)|,$$

i.e. $x_i^* - |y_i(t)| \ge \eta$. Since $b_i(s)$ are nondecreasing, we have

$$\frac{1}{b_i(x_i^* + \text{sgn}(y_i)u)} \le \frac{1}{b_i(x_i^* - |y_i(t)|)} \le \frac{1}{b_i(\eta)},\ \forall u \in (0, |y_i(t)|),$$

which implies $V(y(t)) \le \max_{1\le i\le n}\big\{\frac{d_i^{-1}}{b_i(\eta)}|y_i(t)|\big\}$ when $h(t, y(t)) < \rho$, thus condition (i) of Corollary 5.2.1 holds.

By assumption $(A3)$, for any given $\sigma_1 > 0$, there exists $h > 0$ such that

$$\varepsilon \sum_{j=1}^{n} \hat{\mu}_{ij}(t, t-h) \le \sigma_1, \quad \text{for } i = 1, 2, \cdots, n. \tag{5.60}$$

By (5.57), there exists $\rho_1 > 1$ such that

$$d_i > \rho_1 \sum_{j=1}^{n} d_j \gamma_{ij}, \quad i = 1, 2, \cdots, n. \tag{5.61}$$

By assumption $(A2)$ and inequalities (5.60) and (5.61), for any $i \in N_V$, we have

$$\begin{aligned} D^+V(y(t)) &= -d_i^{-1}a_i(t)|y_i(t)| + \textstyle\sum_{j=1}^{n} d_i^{-1}\big\{\int_{t-h}^{t} + \int_{-\infty}^{t-h}\big\}|y_j(s)d\mu_{ij}(t,s)| \\ &\le -d_i^{-1}a_i(t)\big\{H(y(t))[d_i - \rho_1 \textstyle\sum_{j=1}^{n} d_j\gamma_{ij}] - \sigma_1\big\}, \end{aligned}$$

whenever $H(y(s)) \le \rho_1 H(y(t))$ for $s \in [t-h, t]$. Similarly, we obtain

$$D^+H(y(t)) \le -b_i(x_i(t))a_i(t)d_i^{-1}\big\{H(y(t))[d_i - \rho_1 \sum_{j=1}^{n} d_j\gamma_{ij}] - \sigma_1\big\},$$

whenever $H(y(s)) \le \rho_1 H(y(t))$ for $s \in [t-h, t]$. Choose $P(s) = \rho_1 s$. Then all conditions of Corollary 5.2.1 are satisfied, and hence the equilibrium x^* of system (5.54) is uniformly asymptotically stable.

■

Chapter 6

Boundedness and Periodicity

This chapter discusses boundedness for system with fixed and state-dependent impulses. Some periodicity results are also obtained for system (2.3).

We first introduce some boundedness definitions.

Definition 6.0.1 *Solutions of system (2.3) (or (2.1)) are said to be*

(B1) uniformly bounded (UB for short) if for every $B_1 > 0$, there exists some $B_2 = B_2(B_1) > 0$ such that if $t_0 \in \mathbb{R}_+$ and $\phi \in PC([-\tau, 0], \mathbb{R}^n)$ with $\|\phi\|_\tau \leq B_1$, then any solution $x(t, t_0, \phi)$ is defined and $\|x(t, t_0, \phi)\| \leq B_2$ for all $t \geq t_0$;

(B2) uniformly ultimately bounded (UUB for short) with bound B if (B1) holds and for every $B_3 > 0$, there exists some $T = T(B_3) > 0$ such that if $\phi \in PC([-\tau, 0], \mathbb{R}^n)$ with $\|\phi\|_\tau \leq B_3$, then for any $t_0 \in \mathbb{R}_+$, $\|x(t, t_0, \phi)\| \leq B$ for $t \geq t_0 + T$.

6.1 Systems with Fixed Impulses

In order to investigate boundedness and periodicity for system (2.3), we make the following assumptions.

(1) $f(t + T, \psi) = f(t, \psi)$ for any $t \in \mathbb{R}_+$ and $\psi \in PC([-\tau, 0], \mathbb{R}^n)$;

(2) there exists a positive integer q such that $t_{k+q} = t_k + T$ and $I_{k+q}(x) = I_k(x)$ for $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

We now present the Horn's fixed point theorem for later use.

Lemma 6.1.1 **(Horn's Theorem [111])** *Let $S_0 \subset S_1 \subset S_2$ be convex subsets of the Banach space X, with S_0 and S_2 compact and S_1 open relative to S_2. Let $P: S_2 \to X$ be a continuous mapping such that, for some integer $m > 0$,*

(a) $P^j(S_1) \subset S_2$, $1 \leq j \leq m-1$, *and*

(b) $P^j(S_1) \subset S_0$, $m \leq j \leq 2m-1$.

Then P has a fixed point in S_0.

Theorem 6.1.1 *If solutions of system (2.3) are UUB with bound B at $t_0 = 0$, then (2.3) has a T-periodic solution.*

Proof. There is a $B_1 > B$ such that $\|\phi\|_\tau \leq B$, $\phi \in PC([-\tau, 0], \mathbb{R}^n)$, $t \geq 0$ imply that $\|x(t, 0, \phi)\| < B_1$ for all $t \geq 0$. There exists $B_2 > B_1 + 1$ such that $\|\phi\|_\tau \leq B_1 + 1$, $\phi \in PC([-\tau, 0], \mathbb{R}^n)$, $t \geq 0$ imply that $\|x(t, 0, \phi)\| \leq B_2$. Also, there is a positive integer m such that $\|\phi\|_\tau \leq B_1 + 1$, $\phi \in PC([-\tau, 0], \mathbb{R}^n)$, $t \geq mT - \tau$ imply that $\|x(t, 0, \phi)\| < B$. Finally, there exists $L > 0$ such that $\|\phi\|_\tau \leq B_2$, $\phi \in PC([-\tau, 0], \mathbb{R}^n)$, $0 \leq t \leq mT$ imply that $\|x'(t, 0, \phi)\| \leq L$. Let

$$S_0 = \big\{\phi \in PC([-\tau, 0], \mathbb{R}^n) : \|\phi\|_\tau \leq B, \ \|\phi(u) - \phi(v)\| \leq L|u - v|, \ u, v \in [t_{k-1}, t_k)\big\},$$
$$S_2 = \big\{\phi \in PC([-\tau, 0], \mathbb{R}^n) : \|\phi\|_\tau \leq B_2, \ \|\phi(u) - \phi(v)\| \leq L|u - v|, \ u, v \in [t_{k-1}, t_k)\big\},$$

and

$$S_1 = \big\{\phi \in PC([-\tau, 0], \mathbb{R}^n) : \|\phi\|_\tau < B_1 + 1\big\} \bigcap S_2.$$

Then one may easily prove that S_i, $(i = 0, 1, 2)$ are convex, and S_1 is open in S_2. By Lemma 2.4 in [7], we find that S_0 and S_2 are compact.

Define $P: S_2 \to PC([-\tau, 0], \mathbb{R}^n)$ by

$$P\phi = x(s + T, 0, \phi), \quad \text{for } \phi \in S_2, \ -\tau \leq s \leq 0.$$

Now $x(t + T, 0, \phi)$ is a solution for $t \geq 0$ and its initial function is $P\phi$. Hence

$$x(t + T, 0, \phi) = x(t, 0, P\phi), \quad t \geq -\tau, \tag{6.1}$$

by the uniqueness theorem.

Similarly, since

$$P^2\phi = x(s+T,0,P\phi), \quad -\tau \le s \le 0,$$

we obtain

$$x(t+T,0,P\phi) = x(t,0,P^2\phi), \quad t \ge -\tau. \tag{6.2}$$

Now in (6.1), let t be replaced by $t+T$, so that by (6.2),

$$x(t+2T,0,\phi) = x(t+T,0,P\phi) = x(t,0,P^2\phi).$$

In general, by induction we have

$$P^j\phi = x(s+jT,0,\phi), \quad -\tau \le s \le 0, \;\; j = 1,2,3,\cdots.$$

We claim that

$$P^j(S_1) \subset S_2, \;\; \text{for } j = 1,2,3,\cdots.$$

In fact, for each $\phi \in S_1$, we know that

$$\|x(s+jT,0,\phi)\| \le B_2, \quad \text{for all } j = 1,2,3,\cdots,$$

which implies $\|P^j\phi\| \le B_2$.

For $u,\, v \in [t_{k-1}, t_k)$, we have

$$\begin{aligned}\|(P^j\phi)(u) - (P^j\phi)(v)\| &= \|x(u+jT,0,\phi) - x(v+jT,0,\phi)\| \\ &\le \|x'(\xi,0,\phi)\|\,|u-v| \;\le\; L|u-v|,\end{aligned}$$

where ξ is between $u+jT$ and $v+jT$.

Hence $P^j\phi \in S_2$, for $\phi \in S_1$ and $j = 1,2,3,\cdots$, and we conclude that $P^j(S_1) \subset S_2$.

Next, by choice of m, using a similar argument to the above we conclude that

$$P^j(S_1) \subset S_0, \quad \text{for } j \ge m.$$

So all the conditions in Horn's theorem are satisfied, and thus P has a fixed point $\phi \in S_0$, i.e., $P\phi = \phi$. This shows that $x(t,0,\phi)$ and $x(t+T,0,\phi)$ are both solutions of (2.3) with the same initial function. By uniqueness we conclude that

$$x(t+T,0,\phi) = x(t,0,\phi), \quad \text{for all } t \ge -\tau.$$

■

Example 6.1.1 *Consider the impulsive delay differential equation*

$$\begin{array}{rcl} x'(t) & = & -a(t)x(t) + \int_{t-\tau}^{t} c(t-s)x(s)ds + f(t), \quad t \geq 0, \ t \neq t_k, \\ x(t_k) & = & bx(t_k^-), \quad k \in \mathbb{N}, \end{array} \tag{6.3}$$

where $-1 \leq b \leq 1$*,* $a,\ c,$ *and* f *are continuous functions in* $\mathbb{R}$*, and* $a(t+T) = a(t)$*,* $c(t+T) = c(t)$*, and* $f(t+T) = f(t)$ *for some* $T > 0$*, and there is a positive integer* q *such that* $t_{k+q} = t_k + T$*. Suppose*

$$-a(t) + \int_0^t |c(u)|du \leq -\alpha, \quad \alpha > 0.$$

Then by Example 4.2 in [93], the solution of (6.3) is uniformly ultimately bounded. Therefore by Theorem 6.1.1, system (6.3) has a T*-periodic solution.*

Next, we establish a new result on boundedness for system (2.3) using a Lyapunov like function with the Razumikhin method.

Theorem 6.1.2 *Assume that there exist functions* $V \in \nu_0$*,* $w_1,\ w_2 \in K$*,* $\psi \in K_1$*, and* $G \in K_0$ *with* G *nondecreasing such that*

(i) $w_1(\|x\|) \leq V(t,x) \leq w_2(\|x\|)$*;*

(ii) There exists a real number $H > 0$ *such that for any solution* $x(t) = x(t, t_0, \phi)$ *of (2.3),*

$$D^+V(t, x(t)) \leq g(t)G(V(t, x(t))),$$

if $V(t, x(t)) \geq H$ *and* $V(s, x(s)) \leq \psi^{-1}(V(t, x(t)))$ *for all* $t \geq t_0$ *and* $t - \tau \leq s \leq t$*, where* $g: \ [t_0, \infty) \to \mathbb{R}_+$ *is locally integrable;*

(iii) For all $k \in \mathbb{N}$ *and* $x \in \mathbb{R}^n$*,*

$$V(t_k, x + I_k(x)) \leq \psi(V(t_k^-, x));$$

(iv) There exist $\lambda_2 \geq \lambda_1 > 0$ *and* $A > 0$ *such that for all* $k \in \mathbb{N}$ *and* $\mu > 0$*,*

$$\lambda_1 \leq t_k - t_{k-1} \leq \lambda_2, \quad \int_{\psi(\mu)}^{\mu} \frac{du}{G(u)} - \int_{t_{k-1}}^{t_k} g(s)ds \geq A.$$

Then the solutions of (2.3) are UUB.

Proof. First we show UB. Let $B_1 \geq w_2^{-1}(H)$ be given, for any $t_0 \geq 0$ and $\|\phi\|_\tau \leq B_1$. Let $x(t) = x(t, t_0, \phi)$, $V(t) = V(t, x(t))$ and $w_1(B_2) = \psi^{-1}(w_2(B_1))$. Then

$$w_1(\|x(t)\|) \leq V(t) \leq w_2(\|x(t)\|) \leq w_2(B_1) < \psi^{-1}(w_2(B_1)) = w_1(B_2),\ t_0 - \tau \leq t \leq t_0.$$

We claim that

$$V(t) \leq \psi^{-1}(w_2(B_1)), \quad t_0 \leq t \leq t_1. \tag{6.4}$$

If (6.4) does not hold, then there exists a $\bar{t} \in (t_0, t_1)$ such that

$$V(\bar{t}) > \psi^{-1}(w_2(B_1)) > w_2(B_1) \geq V(t_0),$$

which implies that there is a $\hat{t} \in (t_0, \bar{t})$ such that

$$V(\hat{t}) = \psi^{-1}(w_2(B_1)), \quad V(t) \leq \psi^{-1}(w_2(B_1)), \quad t_0 - \tau \leq t \leq \hat{t},$$

and there exists $\breve{t} \in [t_0, \hat{t})$ such that

$$V(\breve{t}) = w_2(B_1), \quad V(t) \geq w_2(B_1), \quad \breve{t} \leq t \leq \hat{t}.$$

Therefore, for all $t \in [\breve{t}, \hat{t}]$, we have $V(t) \geq H$ and

$$V(s) \leq \psi^{-1}(w_2(B_1)) \leq \psi^{-1}(V(t)), \quad t - \tau \leq s \leq t.$$

By condition (ii) we have

$$D^+V(t) \leq g(t)G(V(t)), \quad \breve{t} \leq t \leq \hat{t},$$

and so

$$\int_{V(\breve{t})}^{V(\hat{t})} \frac{du}{G(u)} \leq \int_{\breve{t}}^{\hat{t}} g(s)ds \leq \int_{t_0}^{t_1} g(s)ds.$$

On the other hand, from condition (iv) we obtain

$$\int_{V(\breve{t})}^{V(\hat{t})} \frac{du}{g(u)} = \int_{w_2(B_1)}^{\psi^{-1}(w_2(B_1))} \frac{du}{G(u)} \geq \int_{t_0}^{t_1} g(s)ds + A > \int_{V(\breve{t})}^{V(\hat{t})} \frac{du}{G(u)},$$

which is a contradiction, and so (6.4) holds. From (6.4) and condition (iii) we have

$$V(t_1) \leq \psi(V(t_1^-)) \leq w_2(B_1). \tag{6.5}$$

In a similar way to the proof of (6.4) and (6.5) we obtain

$$V(t) \leq \psi^{-1}(w_2(B_1)), \;\; t_1 \leq t < t_2, \quad V(t_2) \leq w_2(B_1).$$

By simple induction, we can prove, in general, that

$$\begin{array}{lll} V(t) & \leq & \psi^{-1}(w_2(B_1)), \quad t_{i+1} \leq t < t_{i+2}, \\ V(t_{i+2}) & \leq & w_2(B_1), \quad i = 0, 1, 2, \cdots . \end{array}$$

Therefore we have

$$w_1(\|x(t)\|) \leq V(t) \leq \psi^{-1}(w_2(B_1)) = w_1(B_2), \quad t \geq t_0.$$

This proves UB.

Next we shall prove UUB. Let $B = w_1^{-1}(\psi^{-1}(\psi^{-1}(H)))$ and $B_3 \geq w_2^{-1}(H)$. In view of the proof of UB, we know that there exists $B_4 = w_1^{-1}(\psi^{-1}(w_2(B_3)))$ such that $\|\phi\|_\tau \leq B_3$ and $t_0 \in \mathbb{R}_+$ imply that

$$V(t) \leq \psi^{-1}(w_2(B_3)) = w_1(B_4), \quad t \geq t_0 - \tau.$$

Let N_0 be the smallest positive integer such that

$$\psi^{-1}(w_2(B_3)) \leq \psi(w_1(B)) + N_0 AG(\psi(w_1(B))). \tag{6.6}$$

Set $m_i = \min\left\{k \in \mathbb{N} : \, t_k - t_{m_{i-1}} \geq \tau\right\}$, $i = 1, 2, \cdots, N_0$. Here $m_0 = 0$. Since $V(t_k) \leq \psi(V(t_k^-)) \leq V(t_k^-)$ for all $k \in \mathbb{N}$, it is easy to see that for each interval $J_i = [t_{m_{i-1}}, t_{m_i}]$, $\sup_{t \in J_i} V(t) = L_i$ exists and either $L_i = V(t_{m_{i-1}})$ or $L_i = V(r_i^-)$ for some $r_i \in (t_{m_{i-1}}, t_{m_i}]$ (here $V(r_i^-) = V(r_i)$ when r_i is not an impulse point). We assume, without loss of generality, that $L_i = V(r_i^-)$, $i = 1, 2, \cdots, N_0$. When $L_i = V(t_{m_{i-1}})$ for some $i \in \{1, 2, \cdots, N_0\}$, the proof is similar and is omitted.

Let m^* be the smallest positive integer such that $m^*\lambda_1 \geq \tau$. It is clear that $\sum_{i=1}^{N_0}(t_{m_i} - t_{m_{i-1}}) \leq m^* N_0 \lambda_2$. Let $\gamma = m^* N_0 \lambda_2$. We will prove that

$$\|x(t)\| \leq B, \quad t \geq t_0 + \gamma. \tag{6.7}$$

To that end, we first show that if for some $i \in \{1, 2, \cdots, N_0\}$ we have

$$V(r_i^-) \le \psi(w_1(B)), \tag{6.8}$$

then

$$V(t) \le w_1(B), \quad t \ge t_{m_{N_0}}. \tag{6.9}$$

In fact, in view of (6.8), we obtain that

$$V(t) \le \psi(w_1(B)) < w_1(B), \quad t_{m_{i-1}} \le t \le t_{m_i}. \tag{6.10}$$

Next we prove that

$$V(t) \le w_1(B), \quad t_{m_i} \le t \le t_{m_{i+1}}. \tag{6.11}$$

If (6.11) is not valid, then there exists $\bar{t} \in (t_{m_i}, t_{m_{i+1}})$ such that

$$V(\bar{t}) > w_1(B) > \psi(w_1(B)) \ge V(t_{m_i}),$$

which implies that there is a $\hat{t} \in (t_{m_i}, \bar{t}]$ such that

$$V(\hat{t}) = w_1(B), \quad V(t) \le w_1(B), \quad t_{m_i} \le t \le \hat{t}, \tag{6.12}$$

and there exists $\breve{t} \in [t_{m_i}, \hat{t})$ such that

$$V(\breve{t}) = \psi(w_1(B)), \quad V(t) \ge \psi(w_1(B)), \quad \breve{t} \le t \le \hat{t}. \tag{6.13}$$

From (6.10), (6.12), and (6.13), we obtain that, for $\breve{t} \le t \le \hat{t}$,

$$V(s) \le w_1(B) \le \psi^{-1}(V(t)), \quad t - \tau \le s \le t,$$

and

$$V(t) \ge \psi(w_1(B)) = \psi^{-1}(H) > H.$$

Condition (ii) implies that

$$D^+V(t) \le g(t)G(V(t)), \quad \breve{t} \le t \le \hat{t},$$

and so

$$\begin{aligned} \int_{V(\breve{t})}^{V(\hat{t})} \frac{du}{G(u)} &\le \int_{\breve{t}}^{\hat{t}} g(s)ds < \int_{t_{m_i}}^{t_{m_{i+1}}} g(s)ds + A \\ &\le \int_{\psi(w_1(B))}^{w_1(B)} \frac{du}{G(u)} = \int_{V(\breve{t})}^{V(\hat{t})} \frac{du}{G(u)}. \end{aligned}$$

This is a contradiction, and so (6.11) holds. From (6.11) and condition (iii), we have

$$V(t_{m_{i+1}}) \le \psi(V(t^-_{m_{i+1}})) \le \psi(w_1(B)).$$

By induction, we obtain that

$$V(t) \le w_1(B), \quad t_{m_{i+k}} \le t \le t_{m_{i+k+1}},$$

and

$$V(t_{m_{i+k+1}}) \le \psi(w_1(B)), \quad k = 0, 1, 2, \cdots . \tag{6.14}$$

This shows that if (6.8) holds for some $i \in \{1, 2, \cdots, N_0\}$, then (6.9) holds.

Now we prove that (6.8) holds for some $i \in \{1, 2, \cdots, N_0\}$. If this is not true, then $V(r_i^-) > \psi(w_1(B))$ for all $i = 1, 2, \cdots, N_0$. We claim that

$$V(r_i^-) \le V(r_0^-) - iAG(\psi(w_1(B))), \quad i = 0, 1, 2, \cdots, N_0, \tag{6.15}$$

where $V(r_0^-) = \psi^{-1}(w_2(B_3))$. Obviously (6.15) holds for $i = 0$. Assume that (6.15) holds for some $0 \le j < N_0$. To prove that (6.15) holds for $j + 1$, we first show that

$$V(r_{j+1}^-) \le V(r_j^-). \tag{6.16}$$

In fact, since

$$V(t) \le V(r_j^-), \quad t_{m_{j-1}} \le t \le t_{m_j},$$

and

$$V(t_{m_j}) \le \psi(V(t^-_{m_j})) \le \psi(V(r_j^-)),$$

it follows, in a similar way to the proof of (6.14), that

$$V(t) \le V(r_j^-), \quad t_{m_{j+k}} \le t \le t_{m_{j+k+1}},$$

and

$$V(t_{m_{j+k+1}}) \le \psi(V(r_j^-)), \quad k = 0, 1, 2, \cdots .$$

By induction, (6.16) holds.

Next we consider two possible cases.

Case 1. $\psi(w_1(B)) < V(r_{j+1}^-) \le \psi(V(r_j^-))$.

It follows that, by condition (iv), we have

$$\int_{V(r_{j+1}^-)}^{\psi^{-1}(V(r_{j+1}^-))} \frac{du}{G(u)} \geq A,$$

and so

$$\begin{aligned} V(r_{j+1}^-) &\leq \psi^{-1}(V(r_{j+1}^-)) - AG(\psi(w_1(B))) \\ &\leq V(r_j^-) - AG(\psi(w_1(B))) \\ &\leq V(r_0^-) - (j+1)AG(\psi(w_1(B))). \end{aligned}$$

Case 2. $\psi(V(r_j^-)) < V(r_{j+1}^-) \leq V(r_j^-)$.

Let $r_{j+1} \in (t_{m_{j+k}}, t_{m_{j+k+1}}]$, $k \in \mathbb{N} \bigcup 0$. If $k = 0$, then

$$V(t_{m_{j+k}}) = V(t_{m_j}) \leq \psi(V(t_{m_j}^-)) \leq \psi(V(r_j^-)).$$

If $k > 0$, then we also have

$$V(t_{m_{j+k}}) \leq \psi(V(t_{m_{j+k}}^-)) \leq \psi(V(r_{j+1}^-)) \leq \psi(V(r_j^-)).$$

Therefore, there exists an $\bar{r} \in [t_{m_{j+k}}, r_{j+1})$ such that

$$V(\bar{r}) = \psi(V(r_j^-)), \tag{6.17}$$

and

$$\psi(V(r_j^-)) \leq V(t) \leq \psi^{-1}(w_2(B_3)), \quad \bar{r} \leq t < r_{j+1}, \tag{6.18}$$

which, together with (6.16), implies that for $\bar{t} \leq t < r_{j+1}$,

$$V(s) \leq V(r_j^-) \leq \psi^{-1}(V(t)), \quad t - \tau \leq s \leq t.$$

By condition (ii), we have

$$D^+V(t) \leq g(t)G(V(t)), \quad \bar{r} \leq t < r_{j+1},$$

thus,

$$\int_{V(\bar{r})}^{V(r_{j+1}^-)} \frac{du}{G(u)} \leq \int_{\bar{r}}^{r_{j+1}} g(s)ds \leq \int_{t_{m_{j+k}}}^{t_{m_{j+k+1}}} g(s)ds.$$

From (6.16) and condition (iv), we have

$$\int_{\psi(V(r_j^-))}^{V(r_{j+1}^-)} \frac{du}{G(u)} \leq \int_{\psi(V(r_j^-))}^{V(r_j^-)} \frac{du}{G(u)} - A,$$

and so

$$\int_{V(r_{j+1}^-)}^{V(r_j^-)} \frac{du}{G(u)} \geq A.$$

Therefore,

$$\begin{aligned} V(r_{j+1}^-) &\leq V(r_j^-) - AG(\psi(w_1(B))) \\ &\leq V(r_0^-) - (j+1)AG(\psi(w_1(B))). \end{aligned}$$

In view of cases 1 and 2, (6.15) holds for $i = j+1$. Then by induction, we have that (6.15) holds for all $i = 0, 1, 2 \cdots, N_0$. Therefore, by (6.6) and choosing $i = N_0$ in (6.15), we obtain that

$$\begin{aligned} V(r_{N_0}^-) &\leq V(r_0^-) - N_0 AG(\psi(w_1(B))) \\ &= \psi^{-1}(w_2(B_3)) - N_0 AG(\psi(w_1(B))) \\ &\leq \psi(w_1(B)). \end{aligned}$$

This contradicts $V(r_i^-) > \psi(w_1(B))$ for all $i = 1, 2, \cdots, N_0$. Therefore, there exists some $i \in \{1, 2, \cdots, N_0\}$ such that (6.8) holds, and thus (6.9) holds.

Since $t_0 + \gamma = t_0 + m^* N_0 \lambda_2 \geq t_{m_{N_0}}$, we have

$$w_1(\|x(t)\|) \leq V(t) \leq w_1(B), \quad t \geq t_0 + \gamma,$$

which implies that

$$\|x(t)\| \leq B, \quad t \geq t_0 + \gamma.$$

■

Example 6.1.2 *Consider the impulsive delay differential equation*

$$\begin{aligned} x'(t) &= a(t)x(t) + b(t)x(t-\tau), \quad t \geq 0,\ t \neq t_k, \\ x(t_k) &= cx(t_k^-), \quad k \in \mathbb{N}, \end{aligned} \tag{6.19}$$

where $\tau > 0$, $a(t),\ b(t) \in C(\mathbb{R}_+, \mathbb{R})$ *with* $a(t) \leq a$ *and* $b(t) \leq b$ *for some* $a, b \in \mathbb{R}$*, there exists* $T > 0$ *such that* $a(t+T) = a(t)$, $b(t+T) = b(t)$*, and* $t_{k+q} = t_k + T$ *for some* $q \in \mathbb{N}$, $0 = t_0 < t_1 < t_2 < \cdots < t_k$ *with* $t_k \to \infty$ *as* $k \to \infty$*. Assume that the following conditions are satisfied* *(i)* $0 < c < 1$ *and* $a + \sqrt{2}bc^{-1} > 0$*;*

(ii) $t_k - t_{k-1} < (-\frac{1}{2}\ln c) \setminus (a + \sqrt{2}bc^{-1})$ *for all* $k \in \mathbb{N}$.

Let $V(t,x) = V(x) = \frac{1}{2}x^2$, $\psi(s) = \frac{1}{2}c^2 s$, *and* $G(s) = s$. *Then*

$$V(x + I_k(x)) = V(cx) = \frac{1}{2}c^2x^2 = \psi(V(x)).$$

For any solution $x(t)$ *of system (6.19) that satisfies* $V(x(t+s)) \leq \psi^{-1}(V(x(t)))$ *for* $-\tau \leq s \leq 0$, *we have* $|x(t-\tau)| \leq \sqrt{2}c^{-1}|x(t)|$. *Therefore,*

$$\begin{aligned} D^+V(x(t)) &\leq a(t)x^2(t) + b(t)x(t)x(t-\tau) \\ &\leq ax^2(t) + \sqrt{2}bc^{-1}x^2(t) \\ &= g(t)G(V(x)), \end{aligned}$$

where $g(t) = 2(a + \sqrt{2}bc^{-1}) > 0$. *Let* $A = \ln 2$. *Then for any* $\mu > 0$ *and* $k \in \mathbb{N}$, *we have*

$$\begin{aligned} \int_{\psi(\mu)}^{\mu} \frac{du}{G(u)} - \int_{t_{k-1}}^{t_k} g(s)ds &> \int_{c^2\mu}^{\mu} \frac{du}{s} + \frac{2\ln c}{a+\sqrt{2}bc^{-1}}(a + \sqrt{2}bc^{-1}) \\ &= \ln 2 - 2\ln c + 2\ln c \ = \ A. \end{aligned}$$

Thus, by Theorem 6.1.2, we know that the solutions of system (6.19) are UUB, and thus by Theorem 6.1.1, system (6.19) has a T*-periodic solution.*

Remark 6.1.1 *When* $a(t) \equiv a > 0,\ b(t) \equiv b > 0$, *the solutions of the equation*

$$y'(t) = ay(t) + by(t-\tau), \tag{6.20}$$

are unbounded for any initial function $\phi \not\equiv 0$, *and so system (6.20) has no non-zero periodic solution [36]. Thus, the impulsive perturbations in system (6.19) are responsible for the periodic solution in the above example.*

6.2 Systems with State-dependent Impulses

In this section, we establish some boundedness criteria for delay differential equations with state-dependent impulses (2.1). Some examples are also discussed to illustrate the effectiveness of our results.

We assume the following hypotheses hold.

(A1) $f(t,\psi)$ is $composite$-PC, $continuous\ in\ \psi$ and $quasi$-$bounded$.

(A2) $\tau_k \in C^1(D,\mathbb{R}_+)$ for $k = 0,1,\cdots$, and for each $t^* \in J$, there exists some $\delta > 0$, where $[t^*, t^*+\delta] \subset J$ such that

$$\nabla\tau_k(\psi(0)) \cdot f(t,\psi) \neq 1, \tag{6.21}$$

for all $(t,\psi) \in (t^*, t^*+\delta] \times PC([-\tau,0],D)$ and $k = 0,1,\cdots$.

Remark 6.2.1 *From Chapter 2, we know that if conditions $(A1)$ and $(A2)$ hold, the initial value problem (2.1) has a solution $x(t,t_0,\phi)$ existing in a maximal interval I.*

Our first two results utilize the Lyapunov-Razumkhin technique and the last result employs the Lyapunov functional method.

Theorem 6.2.1 *Assume that there exist $V(t,x) \in \nu_0$, $W_1,\ W_2 \in K_4,\ W_3 \in K_0$ such that*

(i)

$$W_1(\|x\|) \leq V(t,x) \leq W_2(\|x\|);$$

(ii) for any $x \in \mathbb{R}^n$ and $\tau_k \in C^1(\mathbb{R}^n,\mathbb{R}_+)$,

$$V(\tau_k(x), x + I_k(x)) \leq (1+b_k)V(\tau_k^-(x),x),\ k = 0,1,\cdots,$$

where $b_k \geq 0$ with $\sum_{k=1}^{\infty} b_k < \infty$;

(iii) there exists some constant $\rho > 0$ such that for any solution $x(t)$ of system (2.1),

$$D^+V(t,x(t)) \leq -W_3(\|x(t)\|),$$

whenever $\|x(t)\| \geq \rho$ and $P(V(t,x(t))) > V(s,x(s))$ for $s \in [t-\tau,t]$ and $t \geq t_0$, where $P \in C(\mathbb{R}_+,\mathbb{R}_+)$ and $P(s) > Ms$ for $s > 0$, where $M = \prod_{k=1}^{\infty}(1+b_k)$.

Then the solutions of (2.1)-(2.2) are UUB.

Proof. We first show uniform boundedness. Let $B_1 > 0$ and assume, without loss of generality, that $B_1 \geq \rho$. Choose $B_2 = W_1^{-1}(MW_2(B_1))$. For any $t_0 \in \mathbb{R}_+$ and $\|\varphi\|_\tau \leq B_1$, let $x(t) = x(t,t_0,\varphi)$ be a solution of (2.1)-(2.2), which exists in a maximal interval $I = [t_0-\tau, t_0+\beta)$.

If $\beta < \infty$, then there exists some $t \in (t_0, t_0 + \beta)$ for which $\|x(t)\| > B_2$. We will prove that $\|x(t)\| \leq B_2$ which in turn will imply that $\beta = \infty$ and hence the solutions of (2.1)-(2.2) are uniformly bounded.

For simplicity, let $\tau_0 = t_0 \in \mathbb{R}_+$ be the initial time and denote impulse moments $\tau_k(x(\tau_k^-))$ for $k = 1, 2, \cdots$, by t_k when there is no confusion.

In order to prove uniform boundedness, we first show

$$\begin{aligned} V(t) &< \tfrac{1}{M}(1+b_0)\cdots(1+b_m)W_1(B_2), \quad t_m \leq t < t_{m+1}; \quad \text{and} \\ V(t_{m+1}) &\leq \tfrac{1}{M}(1+b_0)\cdots(1+b_m)(1+b_{m+1})W_1(B_2), \; m = 0, 1, \cdots, \end{aligned} \tag{6.22}$$

where $V(t) = V(t, x(t))$ and $b_0 = 0$.

Now we shall show (6.22) holds for $m = 0$, i.e.

$$V(t) < \frac{1}{M} W_1(B_2), \quad t_0 \leq t < t_1. \tag{6.23}$$

For $t_0 - \tau \leq t \leq t_0$, we have

$$W_1(\|x(t)\|) \leq V(t) \leq W_2(\|x(t)\|) < W_2(B_1) = \frac{1}{M} W_1(B_2). \tag{6.24}$$

If (6.23) does not hold, then there is some $\bar{t} \in (t_0, t_1)$ such that

$$V(\bar{t}) = \frac{1}{M} W_1(B_2), \quad \text{and} \quad V(t) \leq \frac{1}{M} W_1(B_2), \quad t_0 - \tau \leq t \leq \bar{t},$$

and

$$D^+ V(\bar{t}) \geq 0. \tag{6.25}$$

Thus

$$P(V(\bar{t})) > MV(\bar{t}) \geq V(s), \quad \bar{t} - \tau \leq s \leq \bar{t},$$

and from $W_2(\|x(\bar{t})\|) \geq V(\bar{t}) = \frac{1}{M} W_1(B_2) = W_2(B_1)$ we have

$$\|x(\bar{t})\| \geq B_1 \geq \rho,$$

then by assumption (iii), we obtain

$$D^+ V(\bar{t}) \leq -W_3(\|x(\bar{t})\|) < 0,$$

which contradicts (6.25), and hence (6.23) holds. By (6.23) and assumption (ii), we have

$$\begin{aligned} V(t_1) &= V(t_1, x(t_1^-) + I_m(t_1, x(t_1^-))) \\ &\le (1+b_1)V(t_1^-, x(t_1^-)) = (1+b_1)V(t_1^-) \\ &\le \frac{1}{M}(1+b_1)W_1(B_2), \end{aligned}$$

which implies that (6.22) holds for $m = 0$.

Now suppose that (6.22) holds for $m \le i-1$ and $i = 1, 2, \cdots$. We prove that (6.22) holds for $m = i$, i.e.

$$\begin{array}{c} V(t) < \frac{1}{M}(1+b_0)\cdots(1+b_i)W_1(B_2), \ \ t_i \le t < t_{i+1}; \ \ \text{and} \\ V(t_{i+1}) \le \frac{1}{M}(1+b_0)\cdots(1+b_i)(1+b_{i+1})W_1(B_2), \ i = 1, 2, \cdots . \end{array} \tag{6.26}$$

First we prove that

$$V(t) \le \frac{1}{M}(1+b_0)\cdots(1+b_i)W_1(B_2), \ \ t_i \le t < t_{i+1}. \tag{6.27}$$

If (6.27) does not hold, then there is some $\bar{t} \in (t_i, t_{i+1})$ such that

$$V(\bar{t}) > \frac{1}{M}(1+b_0)\cdots(1+b_i)W_1(B_2) \ge V(t_i),$$

and so there exists a $t^* \in (t_i, \bar{t}\,]$ such that

$$V(t^*) \ge \frac{1}{M}(1+b_0)\cdots(1+b_i)W_1(B_2), \quad \text{and} \quad V(t) \le V(t^*), \ \ t^* - \tau \le t \le t^*,$$

and

$$D^+V(t^*) \ge 0. \tag{6.28}$$

Then we have

$$P(V(t^*)) > MV(t^*) \ge V(s), \ \ t^* - \tau \le s \le t^*,$$

and

$$\|x(t^*)\| \ge B_1 \ge \rho,$$

since $W_2(\|x(t^*)\|) \ge V(t^*) \ge M^{-1}(1+b_0)\cdots(1+b_i)W_1(B_2) \ge M^{-1}W_1(B_2) = W_2(B_1)$. By assumption (iii),

$$D^+V(t^*) \le -W_3(\|x(t^*)\|) < 0,$$

which contradicts (6.28) and so (6.27) holds.

From (6.27) and assumption (ii), we have

$$V(t_{i+1}) \le V(t_{i+1}^-)(1+b_{i+1}) \le \frac{1}{M}(1+b_0)\cdots(1+b_{i+1})W_1(B_2),$$

which implies that (6.26) holds for $m = i$, and hence (6.22) holds for all $m = 0, 1, \cdots$.

Therefore, we have

$$W_1(\|x(t)\|) \le V(t) \le W_1(B_2), \quad t \ge t_0.$$

This proves uniform boundedness.

Now we will prove UUB.

Let $B = W_1^{-1}(MW_2(\rho))$. Then $W_1(B) = MW_2(\rho)$. Let $B_3 \ge \rho$ be given. By the preceding arguments, we can find a $B_4 > B$ such that $\|\varphi\|_\tau \le B_3$ implies

$$V(t) \le W_1(B_4), \quad t \ge t_0.$$

Let

$$0 < d < \inf\{P(s) - Ms : \frac{1}{M}W_1(B) \le s \le W_1(B_4)\},$$

and N be the first positive integer such that

$$W_1(B) + Nd \ge MW_1(B_4).$$

Set $\gamma = \inf_{\rho \le s \le B_4} W_3(s)$. Then $\gamma > 0$. We first show that

$$V(t) \le W_1(B) + (N-1)d, \quad t \ge t_0 + h, \tag{6.29}$$

where $h > \max\{(1+A)W_1(B_4)/\gamma, \tau\},\ A = \sum_{k=1}^\infty b_k$.

Suppose for all $t \in I_1 = [t_0, t_0 + h]$,

$$V(t) > \frac{1}{M}[W_1(B) + (N-1)d].$$

Then $M^{-1}W_1(B) < V(t) \le W_1(B_4)$ for $t \in I_1$. Thus, for $t \in I_1$, we have

$$\begin{aligned} P(V(t)) &> MV(t) + d > \frac{M}{M}[W_1(B) + (N-1)d] + d \\ &= W_1(B) + Nd \ge W_1(B_4) \ge V(s), \quad t - \tau \le s \le t, \end{aligned}$$

and

$$\|x(t)\| \geq \rho,$$

since $W_2(\|x(t)\|) \geq V(t) > M^{-1}W_1(B) = W_2(\rho)$. By assumption (iii), we have for $t \in I_1$,

$$D^+V(t) \leq -W_3(\|x(t)\|) \leq -\gamma,$$

and so

$$\begin{aligned} V(t) &\leq V(t_0) - \gamma(t-t_0) + \sum_{t_0<t_j\leq t} [V(t_j) - V(t_j^-)] \\ &\leq W_1(B_4) - \gamma(t-t_0) + \sum_{t_0<t_j\leq t} b_j V(t_j^-) \\ &\leq W_1(B_4) - \gamma(t-t_0) + AW_1(B_4). \end{aligned}$$

Let $t = t_0 + h$. We have

$$V(t_0+h) < (1+A)W_1(B_4) - \gamma \cdot \frac{(1+A)W_1(B_4)}{\gamma} = 0.$$

This is a contradiction, thus there is a $t^* \in I_1$ such that

$$V(t^*) \leq \frac{1}{M}[W_1(B) + (N-1)d].$$

Let $q = \inf\{k \in \mathbb{N} : t_k > t^*\}$. We claim that

$$V(t) \leq \frac{1}{M}[W_1(B) + (N-1)d], \quad t^* \leq t < t_q. \tag{6.30}$$

Otherwise, there is a $\bar{t} \in (t^*, t_q)$ such that

$$V(\bar{t}) > \frac{1}{M}[W_1(B) + (N-1)d] \geq V(t^*).$$

This implies that there is a $\hat{t} \in (t^*, \bar{t}\,]$ such that

$$V(\hat{t}) \geq \frac{1}{M}[W_1(B) + (N-1)d],$$

and

$$D^+V(\hat{t}) \geq 0.$$

Thus

$$\begin{aligned} P(V(\hat{t})) &> MV(\hat{t}) + d \geq W_1(B) + (N-1)d + d \\ &= W_1(B) + Nd \geq W_1(B_4) \geq V(s), \;\; \hat{t} - \tau \leq s \leq \hat{t}, \end{aligned}$$

and

$$\|x(\hat{t})\| \geq \rho,$$

since $W_2(\|x(\hat{t})\|) \geq V(\hat{t}) \geq M^{-1}W_1(B) = W_2(\rho)$. By assumption (iii),

$$D^+V(\hat{t}) \leq -W_3(\|x(\hat{t})\|) < 0.$$

This is a contradiction and so (6.30) holds. From (6.30) and assumption (ii), we have

$$V(t_q) \leq (1+b_q)V(t_q^-) \leq \frac{1}{M}(1+b_q)[W_1(B) + (N-1)d].$$

Similarly, we can prove that

$$V(t) \leq \frac{1}{M}(1+b_q)[W_1(B) + (N-1)d], \;\; t_q \leq t < t_{q+1},$$

and

$$V(t_{q+1}) \leq \frac{1}{M}(1+b_q)(1+b_{q+1})[W_1(B) + (N-1)d].$$

By induction, we can prove in general that

$$V(t) \leq \frac{1}{M}(1+b_q)\cdots(1+b_{q+i})[W_1(B) + (N-1)d], \;\; t_{q+i} \leq t < t_{q+i+1},$$

and

$$V(t_{q+i+1}) \leq \frac{1}{M}(1+b_q)\cdots(1+b_{q+i+1})[W_1(B) + (N-1)d], \;\; i = 0, 1, 2, \cdots .$$

Thus (6.29) holds. Similarly, we may prove that

$$V(t) \leq W_1(B) + (N-2)d, \;\; t \geq t_0 + 3h,$$

and by induction, we have

$$V(t) \leq W_1(B) + (N-j)d, \;\; t \geq t_0 + (2j-1)h, \;\; j = 1, 2, \cdots, N.$$

Thus we obtain

$$W_1(\|x(t)\|) \leq V(t) \leq W_1(B), \ \ t \geq t_0 + (2N-1)h.$$

Let $T = (2N-1)h$. Then

$$\|x(t)\| \leq B, \ \ t \geq t_0 + T.$$

This proves UUB. ■

Example 6.2.1 *Consider the scalar equation*

$$\begin{cases} x'(t) = A(t)x(t) + \int_{t-\tau}^{t} C(t-s)x(s)ds + f(t), \ \ t \neq x(t) + k, \\ x(t_k) - x(t_k^-) = b_k x(t_k^-), \ t = x(t) + k, \ \ k \in \mathbb{N}, \end{cases} \tag{6.31}$$

where A, C, and f are continuous functions, $|f(t)| \leq L$ for some $L > 0$. For the impulsive perturbations, we assume that $b_k \geq 0$ and $\sum_{k=1}^{\infty} b_k < \infty$. Suppose $A(t) < 0$ and

$$A(t) + M\int_0^{\tau} |C(u)|du \leq -\alpha,$$

where $\alpha > 0$ and $M = \prod_{k=1}^{\infty}(1+b_k)$. Let $V(t,x) = |x|$ and $q > 1$ such that

$$A(t) + Mq\int_0^{\tau} |C(u)|du \leq -\frac{\alpha}{2},$$

and let $P(s) = Mqs$. Then for any solution $x(t) = x(t, t_0, \varphi)$ such that

$$P(V(t, x(t))) > V(s, x(s)) \text{ for } t \geq \sigma, \ t - \tau \leq s \leq t,$$

we have

$$\begin{aligned} D^+V(t, x(t)) \ &\leq \ A(t)|x(t)| + q\int_0^{\tau} |C(u)||x(t-u)|du + |f(t)| \\ &\leq \ L - \frac{\alpha}{2}|x(t)| \leq -\frac{\alpha}{4}|x(t)|, \quad \text{if } |x(t)| \geq H = \frac{4L}{\alpha}, \end{aligned}$$

and

$$V(t_k, x + I_k(x)) = |x + b_k x| = (1+b_k)V(t_k^-, x).$$

By Theorem 6.2.1, we obtain UUB for (6.31).

Next we shall establish a Razumikhin-type theorem on boundedness by using Lyapunov functionals.

Theorem 6.2.2 *Assume that there exist* $V_1(t,x) \in \nu_0,\ V_2(t,\phi) \in \nu_0^*(\cdot)$ *and* $W_1,\ W_2,\ W_3 \in K_4$ *such that*

(i) $W_1(\|\phi(0)\|) \le V(t,\phi) \le W_2(\|\phi\|_\tau)$, *where* $V(t,\phi) = V_1(t,\phi(0)) + V_2(t,\phi) \in \nu_0(\cdot)$*;*

(ii) for each $x \in \mathbb{R}^n$ *and* $\tau_k \in C^1(\mathbb{R}^n, \mathbb{R}_+)$, $k \in \mathbb{N}$,

$$V_1(\tau_k(x), x + I_k(x)) \le (1 + b_k)V_1(\tau_k^-(x), x),$$

where $b_k \ge 0$ *with* $\sum_{k=1}^{\infty} b_k < \infty$*;*

(iii) for any solution $x(t) = x(t, t_0, \varphi)$ *with* $t_0 \in \mathbb{R}_+, \varphi \in PC([-\tau, 0], \mathbb{R}^n)$,

$$D^+V(t, x_t(t_0,\varphi)) \le A, \quad \textit{if}\ \ V(t, x_t(t_0,\varphi)) \ge W_2(\|\varphi\|_\tau)\ \ \textit{for}\ \ t_0 \le t \le t_0 + \tau;$$

and

$$D^+V(t, x_t(t_0,\varphi)) \le A - W_3(V(t, x_t(t_0,\varphi))), \quad \textit{if}$$

$$P(V(t, x_t(t_0,\varphi))) > V(s, x_s(t_0,\varphi)), \quad \textit{for} \quad t \ge t_0 + \tau,\ \ t - \tau \le s \le t,$$

where $A > 0$ *is a constant and* $P(s)$ *is defined as in Theorem 6.2.1.*

Then the solutions of (2.1)-(2.2) are UUB.

Proof. First, we prove the uniform boundedness.

Let $B_1 > 0$ and assume, without loss of generality, that $B_1 \ge W_2^{-1}(MW_3^{-1}(A))$. Choose $B_2 > 0$ such that $\frac{1}{M}W_1(B_2) = M(W_2(B_1) + A\tau)$. For any $t_0 \in R_+$ and $\|\varphi\|_\tau \le B_1$, let $x(t) = x(t, t_0, \varphi)$ be a solution of (2.1)-(2.2), which exists in a maximal interval $I = [t_0 - \tau, t_0 + \beta)$. We will prove that $\|x(t)\| \le B_2$ which implies that $\beta = \infty$ and hence the solutions of (2.1)-(2.2) are uniformly bounded.

For simplicity, let $\tau_0 = t_0 \in \mathbb{R}_+$ be the initial time and denote impulse moments $\tau_k(x(\tau_k^-))$ for $k = 1, 2, \cdots$, by t_k when there is no confusion.

Let $x(t) = x(t, t_0, \varphi), V_1(t) = V_1(t, x(t)), V_2(t) = V_2(t, x_t(t_0, \varphi))$ and $V(t) = V_1(t) + V_2(t)$. Obviously,

$$V(t_0) \le W_2(\|\varphi\|_\tau) < W_2(B_1).$$

For each $t \in [t_0, t_0 + \tau]$, if $V(t) < W_2(\|\varphi\|_\tau)$, then $V(t) < W_2(B_1)$; While if $V(t) \geq W_2(\|\varphi\|_\tau)$,then by assumption (iii), $D^+V(t) \leq A$. Since $V_2(t)$ is continuous, it then follows from assumption (ii) that

$$V(t_k) - V(t_k^-) = V_1(t_k) - V_1(t_k^-) \leq b_k V_1(t_k^-) \leq b_k V(t_k^-).$$

Suppose that $t_1, t_2, \cdots, t_m$ are all impulse points situated in $(t_0, t_0 + \tau]$ such that

$$t_0 < t_1 < t_2 < \cdots < t_m \leq t_0 + \tau.$$

Then we obtain for $t_0 \leq t \leq t_0 + \tau$,

$$\begin{cases} D^+V(t) \leq A, & t \neq t_i, \\ V(t_i) \leq (1 + b_i)V(t_i^-), & i = 0, 1, \cdots, m. \end{cases} \tag{6.32}$$

We claim that for $t_0 \leq t \leq t_0 + \tau$,

$$V(t) \leq V(t_0) \prod_{t_0 < t_i \leq t} (1 + b_i) + \int_{t_0}^{t} \prod_{s < t_i \leq t} (1 + b_i) A ds. \tag{6.33}$$

Let $t \in [t_0, t_1)$. Then from (6.32) we obtain,

$$V(t) \leq V(t_0) + A(t - t_0),$$

and

$$V(t_1) \leq (1 + b_1)V(t_1^-) \leq (1 + b_1)[V(t_0) + A(t_1 - t_0)].$$

Hence (6.33) is true for $t \in [t_0, t_1]$. For $t \in [t_1, t_2)$, we have from (6.32) that

$$\begin{aligned} V(t) &\leq V(t_1) + A(t - t_1) \\ &\leq (1 + b_1)[V(t_0) + A(t_1 - t_0)] + A(t - t_1). \end{aligned}$$

Now assume that (6.33) holds for $t \in [t_0, t_j),\ (l < j < m)$. Then for $t \in [t_j, t_{j+1})$, it follows from (6.32) and the induction hypothesis that

$$\begin{aligned} V(t) &\leq V(t_j) + A(t - t_j) \\ &\leq (1 + b_j)V(t_j^-) + A(t - t_j) \\ &\leq (1 + b_j)\left(V(t_0) \prod_{t_0 < t_i \leq t_{j-1}} (1 + b_i) + \int_{t_0}^{t_j} \prod_{s < t_i \leq t_{j-1}} (1 + b_i) A ds \right) + A(t - t_j) \\ &= V(t_0) \prod_{t_0 < t_i \leq t} (1 + b_i) + \int_{t_0}^{t_j} \prod_{s < t_i \leq t} (1 + b_i) A ds + \int_{t_j}^{t} \prod_{s < t_i \leq t} (1 + b_i) A ds. \end{aligned}$$

This implies that (6.33) holds for $t \in [t_0, t_{j+1})$. By induction, we see that (6.33) holds for $t_0 \le t \le t_0 + \tau$. Thus

$$W_1(|x(t)|) \le V(t) \le M(W_2(B_1) + A\tau) = M^{-1}W_1(B_2), \ \ t_0 \le t \le t_0 + \tau.$$

Next, we will prove that

$$V(t) \le \frac{1}{M}W_1(B_2), \ \ t_0 + \tau \le t < t_{m+1}. \tag{6.34}$$

If (6.34) does not hold, then there exists a $\bar{t} \in (t_0 + \tau, t_{m+1})$ such that

$$V(\bar{t}) > \frac{1}{M}W_1(B_2) \ge V(t_0 + \tau),$$
$$V(t) \le V(\bar{t}), \ \ t_0 \le t \le \bar{t},$$

and $D^+V(\bar{t}) \ge 0$. Then $P(V(\bar{t})) > MV(\bar{t}) \ge V(s)$ for $\bar{t} - \tau \le s \le \bar{t}$. Hence by assumption (iii),

$$D^+V(\bar{t}) \le A - W_3(V(\bar{t})).$$

Since $D^+V(\bar{t}) \ge 0$, it follows that

$$V(\bar{t}) \le W_3^{-1}(A) \le \frac{1}{M}W_2(B_1) < \frac{1}{M}W_1(B_2).$$

This is a contradiction and so (6.34) holds. From (6.34) and assumption (ii), we have

$$\begin{aligned} V(t_{m+1}) &= V_1(t_{m+1}) + V_2(t_{m+1}) \\ &\le (1 + b_{m+1})V_1(t_{m+1}^-) + (1 + b_{m+1})V_2(t_{m+1}) \\ &= (1 + b_{m+1})V(t_{m+1}^-) \le (1 + b_{m+1})\frac{1}{M}W_1(B_2). \end{aligned}$$

Similarly, we may prove

$$V(t) \le \frac{1 + b_{m+1}}{M}W_1(B_2), \ \ t_{m+1} \le t < t_{m+2},$$

$$V(t_{m+2}) \le \frac{1}{M}(1 + b_{m+1})(1 + b_{m+2})W_1(B_2).$$

By induction, one may prove in general that

$$V(t) \le \frac{1}{M}(1 + b_{m+1}) \cdots (1 + b_{m+i})W_1(B_2), \ \ t_{m+i} \le t < t_{m+i+1},$$

$$V(t_{m+i+1}) \leq \frac{1}{M}(1+b_{m+1})\cdots(1+b_{m+i+1})W_1(B_2), \quad i = 1, 2, \cdots .$$

Thus we have

$$W_1(\|x(t)\|) \leq V(t) \leq W_1(B_2), \quad \text{for} \quad t \geq t_0,$$

or

$$\|x(t)\| \leq B_2, \quad \text{for} \quad t \geq t_0.$$

This proves uniform boundedness as required.

Next we show UUB. Let $B = W_1^{-1}(MW_3^{-1}(A+1))$. Then $V(t) \geq \frac{1}{M}W_1(B) = W_3^{-1}(A+1)$ implies $D^+V(t) \leq -1$ by assumption (iii). Let $B_3 > 0$ be given with $B_3 > W_2^{-1}(W_3^{-1}(A))$, and choose $B_4 > \max\{B_3, B\}$ such that for any $t_0 \in \mathbb{R}_+$ and $\varphi \in PC(B_3)$,

$$V(t) \leq W_1(B_4) \quad \text{and} \quad \|x(t)\| \leq B_4, \quad t \geq t_0.$$

Let

$$0 < d < \inf\{P(s) - Ms : \frac{1}{M}W_1(B) \leq s \leq W_1(B_4)\},$$

and N be the first positive integer such that

$$\frac{1}{M}[W_1(B) + Nd] \geq W_1(B_4).$$

We first show that

$$V(t) \leq W_1(B) + (N-1)d, \quad t \geq t_0 + h, \tag{6.35}$$

where

$$h > \max\{W_1(B_4)(1+M^*), \tau\}, \quad M^* = \sum_{k=1}^{\infty} b_k.$$

Suppose that for all $t \in [t_0, t_0+h] = J_1$,

$$V(t) > \frac{1}{M}[W_1(B) + (N-1)d].$$

Then for $t \in J_1,\ M^{-1}W_1(B) < V(t) \leq W_1(B_4)$. Thus

$$\begin{aligned} P(V(t)) &> MV(t) + d \\ &> \frac{M}{M}[W_1(B) + (N-1)d] + d = W_1(B) + Nd \\ &\geq W_1(B_4) \geq V(s), \quad t - \tau \leq s \leq t. \end{aligned}$$

By assumption (iii),we have for $t \in J_1$,

$$D^+V(t) \leq A - W_3(V(t)) \leq -1,$$

and so

$$\begin{aligned} V(t) &\leq V(t_0) - (t - t_0) + \sum_{t_0 < t_j \leq t} [V(t_j) - V(t_j^-)] \\ &\leq W_1(B_4) - (t - t_0) + \sum_{t_0 < t_j \leq t} [V_1(t_j) - V_1(t_j^-)] \\ &\leq W_1(B_4) - (t - t_0) + \sum_{t_0 < t_j \leq t} b_j V(t_j^-) \\ &< W_1(B_4)(1 + M^*) - (t - t_0). \end{aligned}$$

Thus

$$V(t_0 + h) < W_1(B_4)(1 + M^*) - W_1(B_4)(1 + M^*) = 0.$$

This is a contradiction and so there exists some $t^* \in J_1$ such that

$$V(t^*) \leq \frac{1}{M}[W_1(B) + (N - 1)d].$$

Let $m = \inf\{k \in \mathbb{N} : t_k > t^*\}$. We claim that

$$V(t) \leq \frac{1}{M}[W_1(B) + (N - 1)d], \quad t^* \leq t < t_m. \tag{6.36}$$

If (6.36) does not hold, then there is a $\bar{t} \in (t^*, t_m)$ such that

$$V(\bar{t}) > \frac{1}{M}[W_1(B) + (N - 1)d] \geq V(t^*).$$

Thus, there must be a $\hat{t} \in (t^*, \bar{t}]$ such that $D^+V(\hat{t}) \geq 0$ and $V(\hat{t}) \geq M^{-1}[W_1(B) + (N - 1)d]$. Thus

$$\begin{aligned} P(V(\hat{t})) &> MV(\hat{t}) + d \geq W_1(B) + (N - 1)d + d \\ &= W_1(B) + Nd \geq W_1(B_4) \geq V(s), \quad \hat{t} - \tau \leq s \leq \hat{t}. \end{aligned}$$

By assumption (iii) we have

$$D^+V(\hat{t}) \leq A - W_3(V(\hat{t})) \leq -1.$$

This is a contradiction and so (6.36) holds. From (6.36) and assumption (ii), we obtain

$$\begin{aligned} V(t_m) &= V_1(t_m) + V_2(t_m) \\ &\leq (1+b_m)V_1(t_m^-) + (1+b_m)V_2(t_m) = (1+b_m)V(t_m^-) \\ &\leq (1+b_m)M^{-1}[W_1(B) + (N-1)d]. \end{aligned}$$

Now we prove

$$V(t) \leq \frac{1+b_m}{M}[W_1(B) + (N-1)d], \;\; t_m \leq t < t_{m+1}. \tag{6.37}$$

Suppose there is a $\bar{t} \in (t_m, t_{m+1})$ such that

$$V(\bar{t}) > \frac{1+b_m}{M}[W_1(B) + (N-1)d] \geq V(t_m).$$

Then there must be a $\hat{t} \in (t_m, \bar{t}]$ such that $D^+V(\hat{t}) \geq 0$ and

$$V(\hat{t}) \geq \frac{1+b_m}{M}[W_1(B) + (N-1)d].$$

Thus

$$\begin{aligned} P(V(\hat{t})) &> MV(\hat{t}) + d \geq (1+b_m)[W_1(B) + (N-1)d] + d \\ &\geq W_1(B) + Nd \geq W_1(B_4) \geq V(s), \;\; \hat{t} - \tau \leq s \leq \hat{t}. \end{aligned}$$

By assumption (iii) we have

$$D^+V(\hat{t}) \leq A - W_3(V(\hat{t})) \leq -1.$$

This contradiction shows that (6.37) holds. Thus

$$V(t_{m+1}) \leq (1+b_{m+1})V(t_{m+1}^-) \leq \frac{1}{M}(1+b_m)(1+b_{m+1})[W_1(B) + (N-1)d].$$

By induction, one may prove in general that

$$V(t) \leq \frac{1}{M}(1+b_m)\cdots(1+b_{m+i-1})[W_1(B) + (N-1)d], \;\; t_{m+i-1} \leq t < t_{m+i},$$

$$V(t_{m+i}) \leq \frac{1}{M}(1+b_m)\cdots(1+b_{m+i})[W_1(B) + (N-1)d], \;\; i = 1, 2, \cdots.$$

Thus (6.35) holds.

Next we prove that

$$V(t) \le W_1(B) + (N-2)d, \quad t \ge t_0 + 3h. \tag{6.38}$$

Suppose that for all $t \in [t_0 + 2h, t_0 + 3h] = J_2$,

$$V(t) > \frac{1}{M}[W_1(B) + (N-2)d].$$

Then for $t \in J_2$, we have, by (6.35),

$$\begin{aligned} P(V(t)) &> MV(t) + d > W_1(B) + (N-2)d + d \\ &= W_1(B) + (N-1)d \ge V(s), \quad t - \tau \le s \le t. \end{aligned}$$

By assumption (iii), we have, for $t \in J_2$,

$$D^+V(t) \le A - W_3(V(t)) \le -1,$$

and so

$$\begin{aligned} V(t) &\le V(t_0+2h) - (t - t_0 - 2h) + \sum_{t_0+2h<t_j\le t} [V(t_j) - V(t_j^-)] \\ &\le W_1(B_4) - (t - t_0 - 2h) + \sum_{t_0+2h<t_j\le t} [V_1(t_j) - V_1(t_j^-)] \\ &\le W_1(B_4) - (t - t_0 - 2h) + \sum_{t_0+2h<t_j\le t} b_j V(t_j^-) \\ &< W_1(B_4)(1 + M^*) - (t - t_0 - 2h). \end{aligned}$$

Thus

$$V(t_0 + 3h) < W_1(B_4)(1 + M^*) - W_1(B_4)(1 + M^*) = 0,$$

which is a contradiction and so there exists some $t^* \in J_2$ such that

$$V(t^*) \le \frac{1}{M}[W_1(B) + (N-2)d].$$

Let $m = \inf\{k \in \mathbb{N} : t_k > t^*\}$. We claim that

$$V(t) \le \frac{1}{M}[W_1(B) + (N-2)d], \quad t^* \le t < t_m. \tag{6.39}$$

If (6.39) does not hold, then there is a $\bar{t} \in (t^*, t_m)$ such that

$$V(\bar{t}) > \frac{1}{M}[W_1(B) + (N-2)d] \geq V(t^*).$$

Thus, there must be a $\hat{t} \in (t^*, \bar{t}]$ such that $D^+V(\hat{t}) \geq 0$ and $V(\hat{t}) \geq M^{-1}[W_1(B) + (N-2)d]$. Thus

$$P(V(\hat{t})) > MV(\hat{t}) + d \geq W_1(B) + (N-1)d \geq V(s), \;\; \hat{t} - \tau \leq s \leq \hat{t}.$$

By assumption (iii) we have

$$D^+V(\hat{t}) \leq A - W_3(V(\hat{t})) \leq -1.$$

This is a contradiction and so (6.39) holds. From (6.39) and assumption (ii), we have

$$\begin{aligned} V(t_m) &= V_1(t_m) + V_2(t_m) \\ &\leq (1+b_m)V_1(t_m^-) + (1+b_m)V_2(t_m) = (1+b_m)V(t_m^-) \\ &\leq (1+b_m)M^{-1}[W_1(B) + (N-2)d]. \end{aligned}$$

By induction, one may prove in general that

$$V(t) \leq \frac{1}{M}(1+b_m)\cdots(1+b_{m+i})[W_1(B) + (N-2)d], \;\; t_{m+i} \leq t < t_{m+i+1},$$

$$V(t_{m+i+1}) \leq \frac{1}{M}(1+b_m)\cdots(1+b_{m+i+1})[W_1(B) + (N-2)d], \;\; i = 0, 1, 2, \cdots.$$

Thus (6.38) holds. Similarly, one may prove that

$$V(t) \leq W_1(B) + (N-3)d, \;\; t \geq t_0 + 5h.$$

By induction, one may prove in general that

$$V(t) \leq W_1(B) + (N-i)d, \;\; t \geq t_0 + (2i-1)h, \;\; i = 1, 2, \cdots, N.$$

Thus we obtain that

$$W_1(\|x(t)\|) \leq V(t) \leq W_1(B), \;\; t \geq t_0 + (2N-1)h.$$

Let $T = (2N-1)h$. Then $\|x(t)\| \leq B$ for $t \geq t_0 + T$. This proves UUB and so the proof is complete. ■

Next we shall establish a theorem on boundedness by using Lyapunov functionals.

Theorem 6.2.3 *Assume that there exist $V(t,\phi) \in \nu_0^*(\cdot)$, $W_1, W_2 \in K_4,\ W_3 \in C(\mathbb{R}_+, \mathbb{R}_+)$ and constants $d_k, e_k \geq 0$ with $\sum_{k=1}^{\infty} d_k < \infty$ and $e = \sum_{k=1}^{\infty} e_k < \infty$ such that*

(i)

$$W_1(\|\phi(0)\|) \leq V(t,\phi) \leq W_2(\|\phi\|_\tau);$$

(ii) for each $x \in \mathbb{R}^n$ and $\tau_k \in C^1(\mathbb{R}^n, \mathbb{R}_+)$, $k \in \mathbb{N}$,

$$V(\tau_k(x), x + I_k(x)) \leq (1 + d_k)V(\tau_k^-(x), x) + e_k;$$

(iii) for any solution $x(t) = x(t, t_0, \varphi)$ with $t_0 \in \mathbb{R}_+, \varphi \in PC([-\tau, 0], \mathbb{R}^n)$,

$$D^+V(t, x_t) \leq -W_3(\|\phi(0)\|).$$

Then solutions of (2.1)-(2.2) are UB.
If, in addition, we have $W_3 \in K$, $\liminf_{s\to\infty} W_3(s) > 0$, and $\tau_k - \tau_{k-1} > L$ for some $L > 0$ and $k \in \mathbb{N}$, then solutions of (2.1)-(2.2) are UUB.

Proof. First, the uniform boundedness follows by the fact that $V(t,\phi)$ is non-increasing in each interval between impulse moments and mathematical induction, if we let $B_1 > 0$ and choose $B_2 > 0$ such that $W_1(B_2) = d(W_2(B_1) + e)$, where $d = \prod_{k=1}^{\infty}(1 + d_k)$.

For any $t_0 \in \mathbb{R}_+$ and $\|\varphi\|_\tau \leq B_1$, let $x(t) = x(t, t_0, \varphi)$ be a solution of (2.1)-(2.2), which exists in a maximal interval $I = [t_0 - \tau, t_0 + \beta)$. Again, we let $\tau_0 = t_0 \in \mathbb{R}_+$ be the initial time and denote impulse moments $\tau_k(x(\tau_k^-))$ for $k = 1, 2, \cdots$, by t_k when there is no confusion.

Next we shall show UUB.

Let $B_1^* = B_2(1)$ where B_2 is defined as in the first part by $B_2(B_1) = W_1^{-1}(d(W_2(B_1) + e))$. Next let $B = B_2(B_1^*)$ and note that $B \geq B_1^* \geq 1$.

Let $V(t) = V(t, x_t)$, $B_1 > 0$. By our assumption on f there exists a constant $M = M(B_1) \geq \|f(t,\phi)\|$ for all $(t,\phi) \in \mathbb{R}_+ \times PC([-\tau, 0], \mathbb{R}^n)$ with $\|\varphi\|_\tau \leq B_2(B_1)$. Without loss of generality let us assume that $M > \max\{1/\tau, 1/L\}$. Let $N = N(B_1)$ be some positive integer satisfying $N > 2M[2W_2(B_1) + d^*d(W_2(B_1) + e) + e]/W_3(1/2)$, where $d^* = \sum_{k=1}^{\infty} d_k$, and define $T = T(B_1) = 2(N + 1)\tau$. We know $\|x(t)\| \leq B_2(B_1)$ for $t \in [t_0 - \tau, \infty)$ from the first part. We will show that $\|x(t)\| \leq B$ for $t \geq t_0 + T$. We consider two cases:

Case 1: If $\|x_{t^*}\|_\tau \leq 1$ for some $t^* \in [t_0, t_0+T]$ then either $t^* \neq t_k$ for any k in which case $\|x(t^*)\| \leq \|x_{t^*}\|_\tau \leq 1 \leq B_1^*$ or else $t^* = t_k$ for some k in which case $W_1(\|x(t^*)\|) \leq V(t^*) \leq (1+d_k)V(t^{*-}) + e_k \leq d(V(t^{*-})+e) \leq d(W_2(\|x(t^{*-})\|)+e) \leq d(W_2(1)+e)$ implying $\|x(t^*)\| \leq W_1^{-1}(d(W_2(1)+e)) = B_1^*$. So either way $\|x(t^*)\| \leq B_1^*$. The restriction of x to $[t^*, \infty)$ is a solution of (2.1) with $\|x_{t^*}\|_\tau \leq B_1^*$ and initial time t^* and so by the uniform boundedness results we know $\|x(t)\| \leq B = B_2(B_1^*)$ for $t \geq t^* - \tau$ and in particular for $t \geq t_0 + T$.

Case 2: Suppose $\|x_t\|_\tau > 1$ for all $t \in [t_0, t_0+T]$. Then on every interval $[t, t+\tau] \subset [t_0, t_0+T]$ there exists some $\bar{t} \in [t, t+\tau]$ such that $\|x(\bar{t})\| > 1$ and moreover we may assume, without loss of generality, that $\bar{t} \neq t_k$ for any k. Thus for $j = 1, 2, \ldots, N$, there exists some $\hat{t}_j \in [t_0 + (2j-1)\tau, t_0 + 2j\tau]$ with $\hat{t}_j \neq t_k$ for any k and $\|x(\hat{t}_j)\| > 1$. Note that $t_{j+1} - \hat{t}_j \geq \tau$ for all j. For each j consider the interval $[\hat{t}_j - 1/(2M), \hat{t}_j + 1/(2M)]$. Since $M > 1/\tau$, these intervals are non-overlapping and each is contained in $[t_0, t_0+T]$. Each interval has length $1/M < L$ and so can contain at most one impulse time t_k. Suppose that there are no impulse times in $[\hat{t}_j, \hat{t}_j + 1/(2M)]$. Since $\|x(t)\| \leq B_2(B_1)$ for all $t \geq t_0 - \tau$ then $\|x_t\|_\tau \leq B_2(B_1)$ for $t \in [\hat{t}_j, \hat{t}_j + 1/(2M)]$ which implies $\|x'(t)\| = \|f(t, x_t)\| \leq M$ at almost all points in this interval. Thus $\|x(t)\| > 1/2$ on this interval. This in turn implies $D^+m(t) \leq -W_3(1/2)$ on this interval and so $V(t)$ decreases by at least $W_3(1/2)/(2M)$. A similar argument shows that $V(t)$ decreases by at least $W_3(1/2)/(2M)$ on $[\hat{t}_j - 1/(2M), \hat{t}_j]$ if this interval is free from impulses.

On $[t_0, t_0+T]$, $V(t)$ is of bounded variation since it is non-increasing except possibly at the discrete impulse times t_k where it may undergo a jump discontinuity. Since $V(t)$ decreases by at least $W_3(1/2)/(2M)$ on either $[\hat{t}_j, \hat{t}_j + 1/(2M)]$ or $[\hat{t}_j - 1/(2M), \hat{t}_j]$ for each j, the negative variation of $V(t)$ on $[t_0, t_0+T]$ must be no less than $NW_3(1/2)/(2M)$. Since $V(t_0) \leq W_2(B_1)$ and $V(t) \leq d(W_2(B_1)+e)$ on $[t_0, t_0+T]$, the positive variation of $V(t)$ on $[t_0, t_0+T]$ is bounded above by $V(t_0) + \sum_{k:\ t_k \in (t_0, t_0+T)} (d_k V(t_k) + e_k) \leq W_2(B_1) + \sum_{k:\ t_k \in (t_0, t_0+T)} (d_k(d(W_2(B_1) + e)) + e_k) \leq W_2(B_1) + d^* d(W_2(B_1)+e) + e$. Since the difference between the positive and negative variations is $V(t_0+T) - V(t_0)$, $V(t_0+T) \leq V(t_0) + W_2(B_1) + d^* d(W_2(B_1)+e) + e - NW_3(1/2)/(2M) \leq 2W_2(B_1) + d^* d(W_2(B_1)+e) + e - NW_3(1/2)/(2M) < 0$ by our choice of N, which is impossible.

Since case 2 cannot occur, then case 1 must be satisfied in which case we have already established that for our choice of $T(B_1)$ we have $\|x(t)\| \leq B$ for $t \geq t_0 + T$. This proves UUB.

■

Example 6.2.2 *Consider the scalar impulsive delay differential equation*

$$\begin{aligned} x' &= -p(t)x(t) + q(t)x(t-\tau) + w(t),\ t \neq x^3(t) + 2k, & \text{(6.40a)} \\ \Delta x(t) &= h_k x(t), \ \ t = x^3(t) + 2k, & \text{(6.40b)} \end{aligned}$$

where $\tau > 0$, $p,\ q \in PC(\mathbb{R}_+, \mathbb{R})$, w is a square integrable function on $\mathbb{R}_+$ (i.e. $\int_0^\infty w^2(t)dt < \infty$), $h_k > 0$ for $k \in \mathbb{N}$ and $\sum_{k=1}^\infty h_k < \infty$. Assume that for some $M_1 > 1/2$ and $0 < M_2 < M_1 - 1/2$, $p(t) \geq M_1$ and $|q(t)| \leq M_2$ for all $t \in \mathbb{R}_+$. We will show that the conditions of Theorem 6.2.3 are satisfied and thereby conclude that solutions of this impulsive delay differential equation are uniformly ultimately bounded.

To begin with we note that f satisfies assumption $(A1)$.

Define the Lyapunov functional V by

$$V(t,\psi) = \psi^2(0) + M_2 \int_{-\tau}^0 \psi^2(s)ds + \int_t^\infty w^2(s)ds. \tag{6.41}$$

Clearly V satisfies condition (i) of Theorem 6.2.3 with $W_1(s) = s^2$ and $W_2(s) = (1 + M_2\tau)s^2 + \int_0^\infty w^2(t)dt$. Differentiating V along solutions of (6.40) gives us

$$\begin{aligned} D^+_{(6.40)}V(t,\psi) &= 2\psi(0)\left[-p(t)\psi(0) + q(t)\psi(-\tau) + w(t)\right] \\ &\quad + M_2\left[\psi^2(0) - \psi^2(-\tau)\right] - w^2(t) \\ &\leq (-2M_1 + M_2)\psi^2(0) + 2M_2|\psi(0)\psi(-\tau)| \\ &\quad + 2\psi(0)w(t) - M_2\psi^2(-\tau) - w^2(t) \\ &\leq (-2M_1 + M_2 + 1)\psi^2(0) + 2M_2|\psi(0)\psi(-\tau)| \\ &\quad - M_2\psi^2(-\tau) \\ &\leq -K\psi^2(0), \end{aligned} \tag{6.42}$$

where $K = 2M_1 - 2M_2 - 1 > 0$. Thus condition (iii) of Theorem 6.2.3 is satisfied with $W_3(s) = Ls^2$.

Finally, let us check condition (ii). If $t_0 \in \mathbb{R}_+$ and $x \in PC([t_0 - \tau, \infty), \mathbb{R})$ with discontinuities occurring only at impulse times, then

$$V(t, x_t) = x^2(t) + M_2 \int_{t-\tau}^t x^2(s)ds + \int_t^\infty w^2(s)ds \tag{6.43}$$

is also continuous at all points except possibly impulse times. Moreover,

$$\begin{aligned} V(t_k, x_{t_k}) &= (1+h_k)^2 x^2(t_k) + M_2 \int_{t_k-\tau}^{t_k} x^2(s)ds + \int_{t_k}^{\infty} w^2(s)ds \\ &\le (1+h_k)^2 V(t_k, x_{t_k}) = (1+d_k) V(t_k, x_{t_k}), \end{aligned} \tag{6.44}$$

where $d_k = 2h_k + h_k^2 > 0$. *Since* $\sum_{k=1}^{\infty} h_k < \infty$, $\sum_{k=1}^{\infty} d_k < \infty$.

We can therefore conclude in light of Theorem 6.2.3 that solutions of system (6.40) are uniformly ultimately bounded. Note that in this example, the boundedness conclusion is independent of the delay term τ. *Also, what is interesting is that solutions are uniformly ultimately bounded despite the fact that the state* x *increases in magnitude at each impulse time.*

Chapter 7

Applications to Neural Networks

Neural networks have been successfully employed in various areas such as pattern recognition, associative memory, and combinatorial optimization. The stability analysis of neural networks has allowed them to become an important technical tool in recent years.

One of the most investigated problems in the study of neural networks is global exponential stability of the equilibrium point. If an equilibrium of a neural network is globally exponentially stable, it means that the domain of attraction of the equilibrium point is the whole space and the convergence is in real time. This is significant both theoretically and practically. Such neural networks are known to be well-suited for solving some class of optimization problems. In fact, a globally exponentially stable neural network is guaranteed to compute the global optimal solution independently of the initial condition, which in turn implies that the network is devoid of spurious suboptimal responses.

In this chapter, we investigate the exponential stability of neural networks by applying the stability criteria and techniques in the previous chapters to neural networks.

7.1 Impulsive Stabilization

We discuss the global exponential stability of cellular neural networks via Lyapunov functionals and functions in this section.

Denote the norm $||x|| = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$ with $x = (x_1, x_2, \cdots, x_n)^T$ and $p = 1$ or 2.

Consider the impulsive delayed cellular neural networks described by the following impulsive delay differential equations

$$\begin{cases} \frac{du_i(t)}{dt} & = -c_i u_i(t) + \sum_{j=1}^n a_{ij} f_j(u_j(t)) + \sum_{j=1}^n b_{ij} f_j(u_j(t - t_j)) \\ & \quad +J_i, \qquad t \in [t_{k-1}, t_k), \\ \Delta u_i(t_k) & = I_{ik}(u_i(t_k^-)), \quad k \in \mathbb{N}, \\ u_{it_0} & = \phi_i, \quad i = 1, 2, \cdots, n, \end{cases} \tag{7.1}$$

where $u_i(\cdot)$ is the state representing the membrane potential of the i^{th} unit; J_i is a constant denoting the external bias or input from outside the network to the i^{th} unit; $a_{ij},\ b_{ij}$ are constants; where τ_i, bounded by τ, are constants denoting the transmission delay; n corresponds to the number of units in a neural network; $f_i : PC([-\tau, 0], \mathbb{R}) \to \mathbb{R}$ is the activation function satisfying

$$|f_i(u_i)| \le N_i, \qquad \forall\, u_i \in \mathbb{R}, \tag{7.2}$$

$$0 \le \frac{f_i(u_i) - f_i(v_i)}{u_i - v_i} \le L_i, \qquad \forall u_i \ne v_i, u_i, v_i \in \mathbb{R},\ i = 1, 2, \cdots, n. \tag{7.3}$$

And $\phi_i \in PC([-\tau, 0], \mathbb{R})$ is the initial function; $I_{ik} \in PC([-\tau, 0], \mathbb{R})$ represents the effects of impulsive control or perturbation; t_k is impulse moment and $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, with $t_k \to \infty$ as $k \to \infty$; $\Delta u_i(t) = u_i(t^+) - u_i(t^-)$; and $u_{it}, u_{it^-} \in PC([-\tau, 0], \mathbb{R})$ are defined by $u_{it}(s) = u_i(t + s), u_{it^-}(s) = u_i(t^- + s)$ for $-\tau \le s \le 0$, respectively.

From [10], we know that system (7.1) without impulses (or $I_{ik}(s) = s$ for any $s \in \mathbb{R}$) has at least one equilibrium point if conditions (7.2) and (7.3) hold. Denote one of the equilibrium points by $u^* = [u_1^*, u_2^*, \cdots, u_n^*]^T$. We shall investigate the global exponential stability of this equilibrium point u^*.

Define $x_i(\cdot) = u_i(\cdot) - u_i^*$ and then system (7.1) can be simplified as

$$\begin{cases} \frac{dx_i(t)}{dt} & = -c_i x_i(t) + \sum_{j=1}^n a_{ij} \big(f_j(x_j(t) + u_j^*) - f_j(u_j^*) \big) \\ & \quad + \sum_{j=1}^n b_{ij} \big(f_j(x_j(t - \tau_j) + u_j^*) - f_j(u_j^*) \big), \quad t \ge t_0,\ t \ne t_k, \\ \Delta x_i(t_k) & = I_{ik}(x_i(t_k^-) + u_i^*), \quad k \in \mathbb{N}, \\ x_{it_0} & = \phi_i - u_i^*, \quad i = 1, 2, \cdots, n. \end{cases} \tag{7.4}$$

Assume $I_{ik}(u_i^*) = 0$ so that system (7.4) admits the trivial solution. Then the stability problem of the equilibrium point u^* of system (7.1) is equivalent to the stability problem of the trivial solution of system (7.4).

The following results focus on impulsive stabilization of neural networks via Lyapunov functionals.

Theorem 7.1.1 *Assume that there exist constants* $l, \alpha, d > 0$ *such that* $\tau \leq t_k - t_{k-1} \leq l$, $|I_i(y + u_i^*) + y| \leq d|y|$ *for any* $y \in \mathbb{R}$ *and* $i = 1, 2, \cdots, n$ *with*

$$\ln(d + \lambda\tau) \leq -(\alpha + c)l,$$

where $c = \max_{1\leq i\leq n}\{-c_i + \sum_{j=1}^{n}(|a_{ji}| + |b_{ji}|)L_i\} > 0$*, and* $\lambda = \max_{1\leq j\leq n}\{\sum_{i=1}^{n} |b_{ij}|L_j\}$*. Then the equilibrium point* $u = u^*$ *of system (7.1) is globally exponentially stable.*

Proof. Choose the Lyapunov functional

$$V(t, x_t) = \sum_{i=1}^{n}(|x_i(t)| + \sum_{j=1}^{n} |b_{ij}|L_j \int_{t-\tau_j}^{t} |x_j(s)|ds).$$

Then the upper right-hand derivative of V with respect to system (7.4) is

$$\begin{aligned} D^+V(t, x_t) \;\leq\; & \sum_{i=1}^{n}[-c_i|x_i(t)| + \sum_{j=1}^{n} |a_{ij}|L_j|x_j(t)| \\ & + \sum_{j=1}^{n} |b_{ij}|L_j|x_j(t - \tau_j)| + \sum_{j=1}^{n} |b_{ij}|L_j(|x_j(t)| - |x_j(t - \tau_j)|)] \\ = \; & \sum_{i=1}^{n}[-c_i + \sum_{j=1}^{n}(|a_{ji}| + |b_{ji}|)L_i]|x_i(t)| \\ \leq \; & cV(t, x_t), \qquad t \in [t_{k-1}, t_k),\ k \in \mathbb{N}, \end{aligned}$$

where $c = \max_{1\leq i\leq n}\{-c_i + \sum_{j=1}^{n}(|a_{ji}| + |b_{ji}|)L_i\} > 0$. Thus we have

$$V(t, x_t) \leq V(t_{k-1}, x_{t_{k-1}})e^{c(t-t_{k-1})}, \quad t \in [t_{k-1}, t_k),\ k \in \mathbb{N}. \tag{7.5}$$

Then, for $t \in [t_0, t_1)$,

$$\begin{aligned} V(t, x_t) &\leq V(t_0, x_{t_0})e^{c(t-t_0)} \\ &\leq [\|x(t_0)\| + \max_{1\leq j\leq n}\{\textstyle\sum_{i=1}^{n} |b_{ij}|L_j\}\tau\|\phi - u^*\|_\tau]e^{c(t-t_0)} \\ &\leq (1 + \lambda\tau)\|\phi - u^*\|_\tau e^{c(t_1-t_0)}, \end{aligned} \tag{7.6}$$

where $\lambda = \max_{1\leq j\leq n}\{\sum_{i=1}^{n} |b_{ij}|L_j\}$ and $\|x(t)\| = \sum_{i=1}^{n} |x_i|$ and so

$$\|x(t)\| \leq V(t, x_t) \leq M\|\phi - u^*\|_\tau e^{-\alpha(t-t_0)}, \quad t \in [t_0, t_1), \tag{7.7}$$

where $M = (1+\lambda\tau)e^{(c+\alpha)l}$. So from (7.6), we have

$$\begin{aligned} \|x(t_1^-)\| &\leq (1+\lambda\tau)\|\phi - u^*\|_\tau e^{c(t_1-t_0)}, \\ \|x_{t_1^-}\|_\tau &\leq (1+\lambda\tau)\|\phi - u^*\|_\tau e^{c(t_1-t_0)}. \end{aligned} \tag{7.8}$$

Therefore we obtain

$$\begin{aligned} V(t_1, x_{t_1}) &= \textstyle\sum_{i=1}^n (|x_i(t_1)| + \sum_{j=1}^n |b_{ij}| L_j \int_{t_1-\tau_j}^{t_1} |x_j(s)|ds) \\ &\leq d\|x(t_1^-)\| + \textstyle\sum_{i=1}^n \sum_{j=1}^n |b_{ij}| L_j |x_{j t_1^-}|_\tau \tau_j \\ &\leq d\|x(t_1^-)\| + \tau\lambda \|x_{t_1^-}\|_\tau \leq (d+\lambda\tau)\|x_{t_1^-}\|_\tau \\ &\leq (d+\lambda\tau)(1+\lambda\tau)\|\phi - u^*\|_\tau e^{c(t_1-t_0)} \\ &\leq e^{-(c+\alpha)l} M \|\phi - u^*\|_\tau e^{-\alpha l}. \end{aligned} \tag{7.9}$$

Thus, for $t \in [t_1, t_2)$,

$$\begin{aligned} V(t, x_t) &\leq V(t_1, x_{t_1}) e^{c(t-t_1)} \leq V(t_1, x_{t_1}) e^{cl} \\ &\leq e^{-2\alpha l} M \|\phi - u^*\|_\tau \\ &\leq M\|\phi - u^*\|_\tau e^{-\alpha(t-t_0)}, \end{aligned}$$

and hence

$$\|x(t)\| \leq M\|\phi - u^*\|_\tau e^{-\alpha(t-t_0)}, \quad t \in [t_1, t_2).$$

Next we shall show that

$$V(t_i, x_{t_i}) \leq e^{-(i+1)\alpha l} e^{-cl} M \|\phi - u^*\|_\tau, \quad i \in \mathbb{N}. \tag{7.10}$$

We know (7.10) holds for $i = 1$ in view of (7.9). If we assume that it holds for $i = k$, i.e.

$$V(t_k, x_{t_k}) \leq e^{-(k+1)\alpha l} e^{-cl} M \|\phi - u^*\|_\tau, \quad k \in \mathbb{N},$$

then we have, for $t \in [t_k, t_{k+1})$,

$$\begin{aligned} V(t, x_t) &\leq V(t_k, x_{t_k}) e^{c(t-t_k)} \leq V(t_k, x_{t_k}) e^{cl} \\ &\leq e^{-(k+1)\alpha l} M \|\phi - u^*\|_\tau, \end{aligned}$$

and

$$\|x(t)\| \leq V(t, x_t) \leq e^{-(k+1)\alpha l} M \|\phi - u^*\|_\tau,$$

$$\|x_{t_{k+1}^-}\|_\tau \le V(t, x_t) \le e^{-(k+1)\alpha l} M\|\phi - u^*\|_\tau.$$

Therefore

$$\begin{aligned} V(t_{k+1}, x(t_{k+1})) &= \textstyle\sum_{i=1}^n [|x_i(t_{k+1})| + \sum_{j=1}^n |b_{ij}|L_j| \int_{t_{k+1}-\tau_j}^{t_{k+1}} |x_j(s)|ds] \\ &\le d\|x(t_{k+1}^-)\| + \lambda\tau\|x_{t_{k+1}^-}\|_\tau \\ &\le (d + \lambda\tau)e^{-(k+1)\alpha l} M\|\phi - u^*\|_\tau \\ &\le e^{-(\alpha+c)l} e^{-(k+1)\alpha l} M\|\phi - u^*\|_\tau \\ &\le e^{-(k+2)\alpha l} e^{-cl} M\|\phi - u^*\|_\tau, \end{aligned}$$

which implies that (7.10) holds for $i = k + 1$, and hence (7.10) holds for any $i \in \mathbb{N}$. So we have

$$\begin{aligned} \|x(t)\| &\le V(t, x(t)) \le V(t_i, x(t_i))e^{c(t-t_i)} \\ &\le e^{-(i+1)\alpha l} M\|\phi - u^*\|_\tau \\ &\le M\|\phi - u^*\|_\tau e^{-\alpha(t-t_0)}, \qquad t \in [t_i, t_{i+1}),\ i \in \mathbb{N}, \end{aligned}$$

which, together with (7.7), yields the global exponential stability of u^*. ■

If we change the proof in Theorem 7.1.1 a little bit, we have the following result in which the lower bound of the length of the successive impulses is relaxed, but the restriction on the amplitude of impulses is stronger than in Theorem 7.1.1.

Theorem 7.1.2 *Assume that there exist constants $l, \alpha, d > 0$ such that $t_k - t_{k-1} \le l$, $|I_i(y + u_i^*) + y| \le d|y|$ for any $y \in \mathbb{R}$ with*

$$\ln(d + \lambda\tau e^{\alpha\tau}) \le -(\alpha + c)l,$$

where $c = \max_{1\le i\le n}\{-c_i + \sum_{j=1}^n (|a_{ji}| + |b_{ji}|)L_i\} > 0$, and $\lambda = \max_{1\le j\le n}\{\sum_{i=1}^n |b_{ij}|L_j\}$. Then the equilibrium point $u = u^*$ of system (7.1) is globally exponentially stable.

Proof. By choosing the same Lyapunov functional and using the same argument as in Theorem 7.1.1, we obtain that $D^+V(t, x_t) \le cV(t, x_t)$ for $t \in [t_{k-1}, t_k)$ and $k \in \mathbb{N}$, and $\|x(t)\| \le M\|\phi - u^*\|_\tau e^{-\alpha(t-t_0)}$ for $t \in [t_0, t_1)$. Next we shall show by mathematical induction that

$$\|x(t)\| \le M\|\phi - u^*\|_\tau e^{-\alpha(t-t_0)}, \quad t \in [t_{i-1}, t_i), \ \ i \in \mathbb{N}. \tag{7.11}$$

We have that (7.11) holds for $i = 1$ by the same argument in Theorem 7.1.1. Then assume it holds for $i = k$, i.e.

$$\|x(t)\| \leq M\|\phi - u^*\|_\tau e^{-\alpha(t-t_0)}, \quad t \in [t_{k-1}, t_k). \tag{7.12}$$

We show that (7.11) holds for $i = k + 1$:

$$\|x(t)\| \leq M\|\phi - u^*\|_\tau e^{-\alpha(t-t_0)}, \quad t \in [t_k, t_{k+1}).$$

From (7.12), we obtain

$$\|x(t_k^-)\| \leq M\|\phi - u^*\|_\tau e^{-\alpha(t_k-t_0)},$$

and

$$\|x_{t_k^-}\|_\tau \leq M\|\phi - u^*\|_\tau e^{\alpha\tau} e^{-\alpha(t_k-t_0)},$$

and hence

$$\begin{aligned} V(t_k, x_{t_k}) &= \textstyle\sum_{i=1}^n [|x_i(t_k)| + \sum_{j=1}^n |b_{ij}| L_j \int_{t_k-\tau_j}^{t_k} |x_j(s) ds|] \\ &\leq \textstyle\sum_{i=1}^n [d|x_i(t_k^-)| + \sum_{j=1}^n |b_{ij}| L_j \int_{t_k^- - \tau_j}^{t_k^-} |x_j(s) ds|] \\ &\leq d\|x(t_k^-)\| + \lambda\tau \|x_{t_k^-}\|_\tau \\ &\leq M(d + \lambda\tau e^{\alpha\tau})\|\phi - u^*\|_\tau e^{-\alpha(t_k-t_0)} \\ &\leq e^{-(\alpha+c)l} M\|\phi - u^*\|_\tau e^{-\alpha(t_k-t_0)}. \end{aligned}$$

Then for $t \in [t_k, t_{k+1})$, we have

$$\begin{aligned} V(t, x_t) &\leq V(t_k, x_{t_k}) e^{c(t-t_k)} \leq V(t_k, x_{t_k}) e^{cl} \\ &\leq e^{-\alpha l} M\|\phi - u^*\|_\tau e^{-\alpha(t_k-t_0)} \\ &\leq M\|\phi - u^*\|_\tau e^{-\alpha(t-t_0)}. \end{aligned}$$

This gives

$$\|x(t)\| \leq V(t, x_t) \leq M\|\phi - u^*\|_\tau e^{-\alpha(t-t_0)}, \quad t \in [t_k, t_{k+1}),$$

which implies (7.11) holds for all $t \geq t_0$ and completes the proof. ■

When c is non-positive, the method in Theorem 7.1.1 can not be applied. Instead, using a method similar to the one used to prove Theorem 7.1.2, we can obtain the following result.

Theorem 7.1.3 *Assume that there exist constants* $\alpha,\ d > 0$ *such that* $\max_{1\le i\le n}\{-c_i+\sum_{j=1}^{n}(|a_{ji}|+|b_{ji}|)L_i\} \triangleq c \le 0$ *and* $|I_i(y+u_i^*)+y| \le d|y|$ *for any* $y \in \mathbb{R}$ *with*

$$\ln(d+\lambda\tau e^{\alpha\tau}) \le -\alpha(t_k - t_{k-1}),\quad k \in \mathbb{N},$$

where $\lambda = \max_{1\le j\le n}\{\sum_{i=1}^{n}|b_{ij}|L_j\}$. *Then the equilibrium point* $u = u^*$ *of system (7.1) is globally exponentially stable.*

Proof. Choosing the same Lyapunov functional and using the same argument as in Theorem 7.1.1, we obtain that $D^+V(t,x_t) \le cV(t,x_t)$ for $t \in [t_{k-1},t_k)$. Since $c \ge 0$, we have $D^+V(t,x_t) \le 0$ for $t \ge t_0$. Then similarly to Theorem 7.1.1, we have

$$\begin{aligned}\|x(t)\| &\le V(t,x_t) \le V(t_0,x_{t_0})\\ &\le (1+\lambda\tau)\|\phi-u^*\|_\tau\\ &\le M\|\phi-u^*\|_\tau e^{-\alpha(t-t_0)},\quad t\in[t_0,t_1),\end{aligned}$$

where $M = (1+\lambda\tau)e^{cl}$. And hence we have

$$\|x(t_1^-)\| \le M\|\phi-u^*\|_\tau e^{-\alpha(t_1-t_0)},$$

and

$$\|x_{t_1^-}\|_\tau \le M\|\phi-u^*\|_\tau e^{\alpha\tau}e^{-\alpha(t_1-t_0)}.$$

Then for $t \in [t_1,t_2)$, we obtain

$$\begin{aligned}V(t,x_t) &\le V(t_1,x_{t_1}) \le d\|x(t_1^-)\| + \lambda\tau\|x_{t_1^-}\|_\tau\\ &\le (d+\lambda\tau e^{\alpha\tau})M\|\phi-u^*\|_\tau e^{-\alpha(t_1-t_0)}\\ &\le e^{-\alpha(t_2-t_1)}M\|\phi-u^*\|_\tau e^{-\alpha(t_1-t_0)}\\ &\le M\|\phi-u^*\|_\tau e^{-\alpha(t-t_0)},\end{aligned}$$

which gives

$$\|x(t)\| \le V(t,x_t) \le M\|\phi-u^*\|_\tau e^{-\alpha(t-t_0)},\quad t\in[t_1,t_2).$$

Similarly, we can prove that

$$\|x(t)\| \le V(t,x_t) \le M\|\phi-u^*\|_\tau e^{-\alpha(t-t_0)},\quad t\in[t_{k-1},t_k),\ k\in\mathbb{N},$$

and so the result follows. ∎

By using the same method in Theorem 7.1.1 and a different Lyapunov functional, we have the following result.

Theorem 7.1.4 *Assume that there exist constants* $l, \alpha, d > 0$ *such that* $\tau \leq t_k - t_{k-1} \leq l$, $|I_i(y + u_i^*) + y| \leq d|y|$ *for any* $y \in \mathbb{R}$ *with*

$$\ln(d^2 + \lambda\tau) \leq -(c + 2\alpha)l,$$

where $c = \max_{1\leq i\leq n}\{-2c_i + \sum_{j=1}^n(|a_{ji}|L_i + |a_{ij}|L_j + |b_{ji}|L_i + |b_{ij}|L_j)\} > 0$, *and* $\lambda = \max_{1\leq j\leq n}\{\sum_{i=1}^n |b_{ij}|L_j\}$. *Then the equilibrium point* $u = u^*$ *of system (7.1) is globally exponentially stable.*

Proof. Choose the Lyapunov functional

$$V(t, x_t) = \sum_{i=1}^n [x_i^2(t) + \sum_{j=1}^n |b_{ij}|L_j \int_{t-\tau_j}^t x_j^2(s)ds].$$

Then the upper right-hand derivative of V with respect to system (7.4) is

$$\begin{aligned}
D^+V(t, x_t) &= \sum_{i=1}^n [2x_i(t)\{-c_i x_i(t) + \sum_{j=1}^n a_{ij}(f_j(x_j(t) + u_j^*) - f_j(u_j^*)) + \sum_{j=1}^n b_{ij} \\
\times &\quad (f_j(x_j(t - \tau_j) + u_j^*) - f_j(u_j^*))\} + \sum_{j=1}^n |b_{ij}|L_j(x_j^2(t) - x_j^2(t - \tau_j))] \\
\leq &\quad \sum_{i=1}^n [-2c_i x_i^2(t) + \sum_{j=1}^n 2|x_i(t)||a_{ij}|L_j|x_j(t)| + \sum_{j=1}^n 2|x_i(t)||b_{ij}|L_j \\
\times &\quad |x_j(t - \tau_j)| + \sum_{j=1}^n |b_{ij}|L_j(|x_j(t)|^2 - |x_j(t - \tau_j)|^2)] \\
\leq &\quad -2\sum_{i=1}^n c_i x_i^2(t) + \sum_{i=1}^n\sum_{j=1}^n |a_{ij}|L_j(|x_i(t)|^2 + |x_j(t)|^2) + \sum_{i=1}^n\sum_{j=1}^n |b_{ij}|L_j \\
\times &\quad (|x_i(t)|^2 + |x_j(t - \tau_j)|^2) + \sum_{i=1}^n\sum_{j=1}^n |b_{ij}|L_j(|x_j(t)|^2 - |x_j(t - \tau_j)|^2) \\
\leq &\quad \sum_{i=1}^n [-2c_i + \sum_{j=1}^n (|a_{ji}|L_i + |a_{ij}|L_j + |b_{ji}|L_i + |b_{ij}|L_j)]|x_i(t)|^2 \\
\leq &\quad cV(t, x_t), \qquad t \in [t_{k-1}, t_k),\ k \in \mathbb{N},
\end{aligned}$$

where $c = \max_{1\le i\le n}\{-2c_i + \sum_{j=1}^{n}(|a_{ji}|L_i + |a_{ij}|L_j + |b_{ji}|L_i + |b_{ij}|L_j)\} > 0$. Thus we have

$$V(t,x_t) \le V(t_{k-1}, x_{t_{k-1}})e^{c(t-t_{k-1})}, \quad t \in [t_{k-1}, t_k),\ k \in \mathbb{N}. \tag{7.13}$$

Then, for $t \in [t_0, t_1)$,

$$\begin{aligned} V(t,x_t) &\le V(t_0, x_{t_0})e^{c(t-t_0)} \\ &\le [\|x(t_0)\|^2 + \max_{1\le j\le n}\{\textstyle\sum_{i=1}^{n}|b_{ij}|L_j\}\tau\|\phi - u^*\|_\tau)^2]e^{c(t-t_0)} \\ &\le (1+\lambda\tau)\|\phi-u^*\|_\tau^2 e^{c(t_1-t_0)}, \end{aligned} \tag{7.14}$$

where $\lambda = \max_{1\le j\le n}\{\sum_{i=1}^{n}|b_{ij}|L_j\}$, $\|x(t)\| = (\sum_{i=1}^{n}|x_i|^2)^{\frac{1}{2}}$. Then

$$\|x(t)\| \le \sqrt{V(t,x_t)} \le M\|\phi - u^*\|_\tau e^{-\alpha(t-t_0)}, \quad t \in [t_0, t_1), \tag{7.15}$$

where $M = \sqrt{1+\lambda\tau}e^{(\frac{c}{2}+\alpha)l}$. So from (7.14) and (7.15), we have

$$\begin{aligned} \|x(t_1^-)\|^2 &\le (1+\lambda\tau)\|\phi-u^*\|_\tau^2 e^{c(t_1-t_0)}, \\ \|x_{t_1^-}\|_\tau^2 &\le (1+\lambda\tau)\|\phi-u^*\|_\tau^2 e^{c(t_1-t_0)}. \end{aligned} \tag{7.16}$$

Therefore we obtain

$$\begin{aligned} V(t_1, x_{t_1}) &= \textstyle\sum_{i=1}^{n}(|x_i(t_1)|^2 + \sum_{j=1}^{n}|b_{ij}|L_j\int_{t_1-\tau_j}^{t_1}|x_j(s)|^2 ds) \\ &\le d^2\|x_1(t_1^-)\|^2 + \textstyle\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|L_j|x_{j_{t_1^-}}|_\tau^2\tau_j \\ &\le d^2\|x_1(t_1^-)\|^2 + \tau\lambda\|x_{t_1^-}\|_\tau^2 \le (d^2+\lambda\tau)\|x_{t_1^-}\|_\tau^2 \\ &\le (d^2+\lambda\tau)(1+\lambda\tau)\|\phi-u^*\|_\tau^2 e^{c(t_1-t_0)} \\ &\le e^{-(c+2\alpha)l}M^2\|\phi-u^*\|_\tau^2 e^{-2\alpha l}. \end{aligned}$$

Thus we have, for $t \in [t_1, t_2)$,

$$\begin{aligned} V(t,x_t) &\le V(t_1, x_{t_1})e^{c(t-t_1)} \le V(t_1, x_{t_1})e^{cl} \\ &\le e^{-4\alpha l}M^2\|\phi-u^*\|_\tau^2 \\ &= M^2\|\phi-u^*\|_\tau^2 e^{-2\alpha(t-t_0)}, \end{aligned}$$

and hence we obtain

$$\|x(t)\| \le M\|\phi-u^*\|_\tau e^{-\alpha(t-t_0)}, \ \ t \in [t_1, t_2).$$

Similarly to the proof that used in Theorem 7.1.1, we can derive

$$\|x(t)\| \le M\|\phi-u^*\|_\tau e^{-\alpha(t-t_0)}, \ \ t \ge t_0,$$

which completes the proof. ■

Remark 7.1.1 *Compared to Theorem 7.1.1, the result in Theorem 7.1.4, if both theorems are applicable (i.e. $c > 0$), seems less restrictive for small values of α and l. Hence we expect a larger bound on the impulse amplitude d from Theorem 7.1.4 in the stability analysis of the same problem; however, for big values of c, either the difference is small or we can obtain a larger bound on the impulse amplitude d from Theorem 7.1.1, see Example 7.1.1.*

Example 7.1.1 *Consider the following cellular neural networks with time delay*

$$\begin{cases} \frac{du_1(t)}{dt} & = -u_1(t) + \frac{1}{2}f(u_1(t)) + \frac{1}{2}f(u_2(t)) - \frac{3}{2}f(u_1(t-0.01)) \\ & \quad -\frac{3}{2}f(u_2(t-0.075)) + 2, \qquad t \geq 0; \\ \frac{du_2(t)}{dt} & = -0.5u_2(t) + \frac{1}{2}f(u_1(t)) + f(u_2(t)) - f(u_1(t-0.01)) \\ & \quad -\frac{1}{2}f(u_2(t-0.075)) + 2, \qquad t \geq 0; \\ x_0 & = \phi, \end{cases} \tag{7.17}$$

where $f(s) = \frac{1}{2}(|s+1| - |s-1|)$ for any $s \in \mathbb{R}$, $\phi \in PC([-0.01, 0], \mathbb{R}^2)$.

By direct computation, we know that $u^* = (0.5, 4.5)$ is the unique equilibrium point of the cellular neural networks (7.17).

It is easy to check that the condition in Corollary 3 ([24]) does not hold since

$$\begin{aligned} c_1 &< |a_{11}| + |a_{12}| + |b_{11}| + |b_{12}|; \\ c_2 &< |a_{21}| + |a_{22}| + |b_{21}| + |b_{22}|, \end{aligned}$$

and the condition in [5] is not satisfied since the matrix $-(A + A^T)$ is not positive definite, so the equilibrium point u^* might not be exponentially stable. Actually, the numerical simulation shows that u^* is not even asymptotically stable. See Figure 7.1 for graphs of the solutions with different initial functions $\phi = (-0.1H(t), 4H(t))^T$, $(2H(t), 6H(t))^T$, $(6H(t), -5H(t))^T$, where $H(t)$ is the Heaviside step function.

Let $c = 3 > 0$, $\lambda = \frac{5}{2}$, $\tau = 0.01$. Choose $\alpha = 0.1$ and $l = 0.5$. Then by Theorem 7.1.1, the estimate for the impulse amplitude which can stabilize this cellular neural network is

$$d \leq e^{-(\alpha+c)l} - \lambda\tau \leq 0.1872.$$

Thus we can choose the impulse control functions $I_i(s) = -0.85s + 0.85u_i^*$ for any $s \in \mathbb{R}$ to stabilize cellular neural network (7.17). See Figure 7.2 for the numerical simulations with the same initial functions as in Figure 7.1.

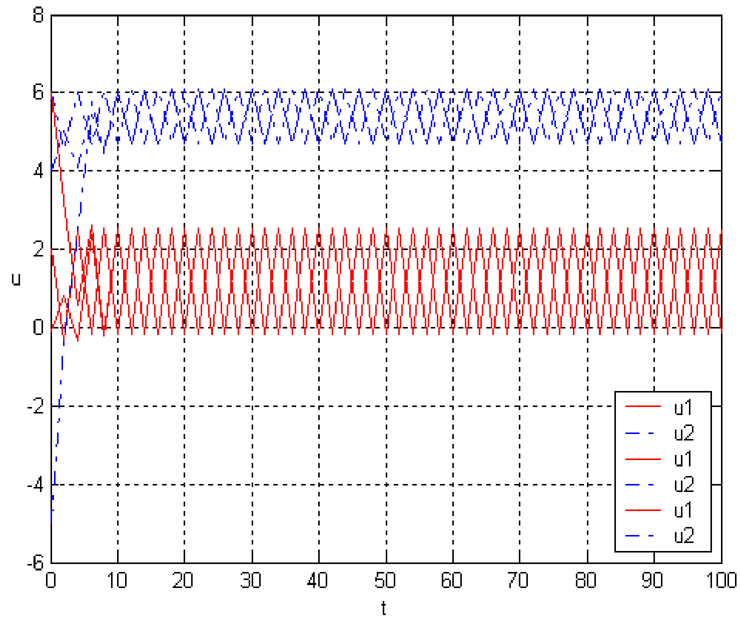

Figure 7.1: Numerical simulation of Example 7.1.1, system without impulses.

However, if we use Theorem 7.1.4 instead, we obtain $c = 5.5 > 0$ and an estimate for the impulse amplitude which can stabilize this cellular neural network is

$$d \leq \sqrt{e^{-(2\alpha+c)l} - \lambda\tau} \leq 0.1669,$$

which allows a smaller bound for the impulse amplitude. If we choose smaller l, for example, $l = 0.25$ in both theorems, then from Theorem 7.1.4 we obtain $d \leq 0.4588$, which is a larger bound for the amplitude of impulsive control than the result $d \leq 0.4357$ from Theorem 7.1.1. This verifies the prediction in Remark 7.1.1.

Using the same Lyapunov functional as in Theorem 7.1.4 and a similar method as in Theorem 7.1.2 and Theorem 7.1.4, we have the following result.

Theorem 7.1.5 *Assume that there exist constants $l, \alpha, d > 0$ such that $t_k - t_{k-1} \leq l$, $|I_i(y + u_i^*) + y| \leq d|y|$ for any $y \in \mathbb{R}$ with*

$$\ln(d^2 + \lambda\tau e^{2\alpha\tau}) \leq -(c + 2\alpha)l,$$

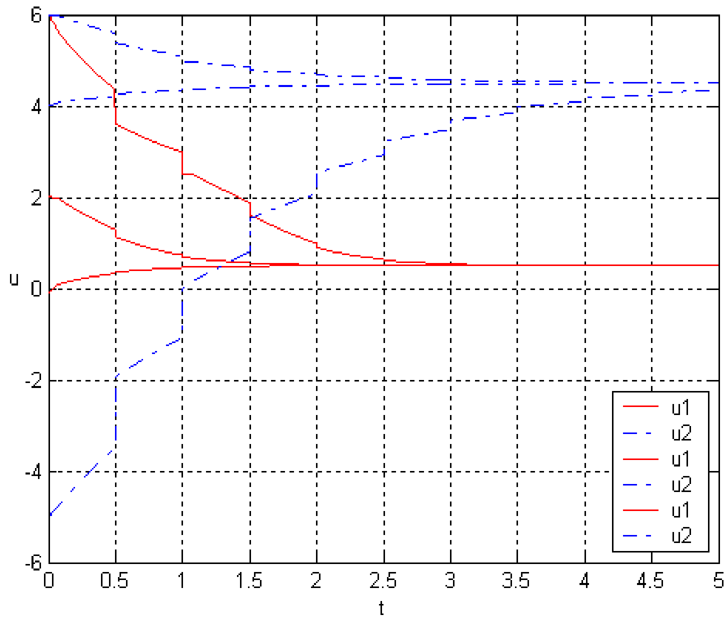

Figure 7.2: Numerical simulation of Example 7.1.1, system with impulsive control.

where $c = \max_{1\le i\le n}\{-2c_i+\sum_{j=1}^{n}(|a_{ji}|L_i+|a_{ij}|L_j+|b_{ji}|L_i+|b_{ij}|L_j)\} > 0$, $\lambda = \max_{1\le j\le n}\{\sum_{i=1}^{n} |b_{ij}|L_j\}$. *Then the equilibrium point* $u = u^*$ *of system (7.1) is globally exponentially stable.*

Using the same Lyapunov functional as in Theorem 7.1.4 and a similar method as was used to prove Theorem 7.1.3 and Theorem 7.1.4, we have the following result.

Theorem 7.1.6 *Assume that there exist constants* $\alpha,\ d > 0$ *such that* $\max_{1\le i\le n}\{-2c_i+\sum_{j=1}^{n}(L_i |a_{ji}| + |a_{ij}|L_j + |b_{ji}|L_i + |b_{ij}|L_j)\} \triangleq c \le 0$ *and* $|I_i(y+u_i^*)+y| \le d|y|$ *for any* $y \in \mathbb{R}$ *with*

$$\ln(d^2+\lambda\tau e^{2\alpha\tau}) \le -\alpha(t_k - t_{k-1}),\quad k \in \mathbb{N}^*,$$

where $\lambda = \max_{1\le j\le n}\{\sum_{i=1}^{n} |b_{ij}|L_j\}$. *Then the equilibrium point* $u = u^*$ *of system (7.1) is globally exponentially stable.*

Remark 7.1.2 *Notice that the cellular neural networks without impulses might be stable when* $c \le 0$ *in Theorem 7.1.6 and Theorem 7.1.3, see Example 7.1.2.*

In Example 7.1.1, we show how Theorem 7.1.1 and Theorem 7.1.4 can be applied to stabilize the unstable neural network (7.17). Next we shall try to use Theorem 7.1.3 to determine the stability of neural networks. In fact, when $c \leq 0$, the neural network without impulses might be stable, as the following example indicates.

Example 7.1.2 *Consider the following cellular neural networks with time delay*

$$\begin{cases} \frac{du_1(t)}{dt} &= -2u_1(t) + \frac{2}{3}f(u_1(t)) + \frac{1}{5}f(u_2(t)) + \frac{4}{5}f(u_1(t-0.01)) \\ & \quad +\frac{1}{3}f(u_2(t-0.075)) + 3, \qquad t \geq 0; \\ \frac{du_2(t)}{dt} &= -2u_2(t) + \frac{1}{3}f(u_1(t)) + \frac{4}{5}f(u_2(t)) + \frac{1}{5}f(u_1(t-0.01)) \\ & \quad +\frac{2}{3}f(u_2(t-0.075)) + 2.5, \qquad t \geq 0; \\ x_0 &= \phi, \end{cases} \tag{7.18}$$

where $f(s) = \frac{1}{2}(|s+1| - |s-1|)$ *for any* $s \in \mathbb{R}$, $\phi \in PC([-0.01, 0], \mathbb{R}^2)$.

By direct computation, we know that $u^* = (2.5, 2.25)$ is the unique equilibrium point of the cellular neural networks (7.18). Moreover, $c_i = 2 = \sum_{j=1}^n (|a_{ji}| + |b_{ji}|)L_i = \frac{1}{2}\sum_{j=1}^n (|a_{ji}|L_i + |a_{ij}|L_j + |b_{ji}|L_i + |b_{ij}|L_j)$, which yields $c = 0$ in Theorem 7.1.3 and Theorem 7.1.6. The simulations, with initial functions $\phi = (-5H(t), 5H(t))^T,\ (2H(t), 3H(t))^T,\ (3H(t), 2H(t))^T$ (see Figure 7.3), show that the equilibrium point u^* of this system might be globally exponentially stable.

7.2 Exponential Stability

This section investigates the problem of exponential stability for a class of impulsive cellular neural networks with time delay. By dividing the network state variables into subgroups according to the characters of the neural networks, some sufficient conditions for exponential stability are derived by Lyapunov functionals and the method of variation of parameters. These conditions are given in terms of some blocks of the interconnection matrix, which extend and improve some of the known results in the literature. Our results show that impulses may be used to stabilize the cellular neural networks with time delay if they are not stable (or exponentially stable). On the other hand, if impulses are input disturbances, criteria on the magnitude and frequency of the impulses are also established to maintain the stability property of the original system. Our

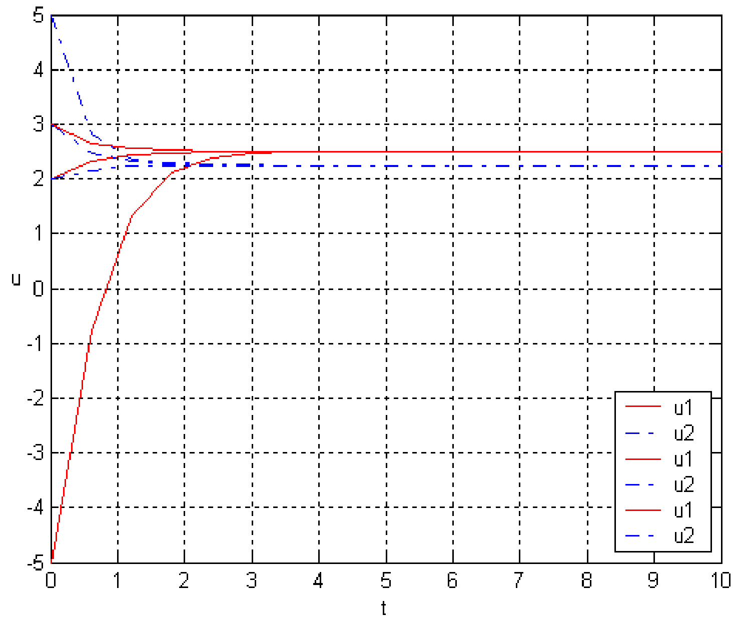

Figure 7.3: Numerical simulation of Example 7.1.2, stable system when $c = 0$.

results generalize and improve some of the known results. Two numerical examples are given to illustrate our results.

Given a constant $\tau^* > 0$, we equip the linear space $PC([-\tau^*, 0], \mathbb{R}^n)$ with the norm $\|\cdot\|_{\tau^*}$ defined by $\|\psi\|_{\tau^*} = \sup_{-\tau^* \leq s \leq 0} \|\psi(s)\|$.

Consider the impulsive delayed cellular neural networks described by the following impulsive delay differential equations

$$\begin{cases} \frac{dx(t)}{dt} = -x(t) + Af(x(t)) + Bf(x(t-\tau)) + u, & t \in [t_{k-1}, t_k), \\ x(t) = I_k(x(t^-)), & t = t_k,\ k \in \mathbb{N}, \\ x_{t_0} = \phi, \end{cases} \tag{7.19}$$

where $x(\cdot) = [x_1(\cdot), x_2(\cdot), \cdots, x_n(\cdot)]^T$ is the state vector representing the membrane potential of the units $i = 1, 2, \cdots, n$; $u = [u_1, u_2, \cdots, u_n]^T$ is a constant vector denoting the external bias or input from outside the network to the units; $A, B \in \mathbb{R}^{n\times n}$ are feedback matrices; $x(t-\tau) = [x_1(t-\tau_1), x_2(t-\tau_2), \cdots, x_n(t-\tau_n)]^T$, where τ_i, bounded by τ^*, are constants denoting the transmission delay; $f : PC([-\tau^*, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is the activation function given by $f_i(x_i(\cdot)) = \frac{1}{2}(|x_i(\cdot) + 1| - |x_i(\cdot) - 1|),\ \ i = 1, 2, \cdots, n$; $\phi \in PC([-\tau^*, 0], \mathbb{R}^n)$ is the initial function; $I_k \in PC([-\tau^*, 0], \mathbb{R}^n)$ represents the effects of impulsive control or perturbation; t_k is impulse moment and $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, with $t_k \to \infty$ as $k \to \infty$; $\Delta x(t) = x(t^+) - x(t^-)$; and $x_t, x_{t^-} \in PC([-\tau^*, 0], \mathbb{R}^n)$ are defined by $x_t(s) = x(t+s), x_{t^-}(s) = x(t^- + s)$ for $-\tau^* \le s \le 0$, respectively.

From [10, 135], we know that system (7.19) without impulses (or $I_k(x) = x$ for any $x \in \mathbb{R}^n$) has at least one equilibrium point. Denote one of the equilibrium points by $x^* = [x_1^*, x_2^*, \cdots, x_n^*]^T$. We shall investigate the exponential stability of this equilibrium point x^*. The concept of exponential stability is defined as follows.

Definition 7.2.1 *An equilibrium x^* of system (7.19) is said to be exponentially stable, if for any initial function $x_{t_0} = \phi \in PC([-\tau^*, 0], \mathbb{R}^n)$, there exists some $M,\ \alpha > 0$ such that*

$$\|x(t, t_0, \phi) - x^*\| \le M\|\phi - x^*\|_{\tau^*} e^{-\alpha(t-t_0)}, \quad \textit{for all } t \ge t_0, \tag{7.20}$$

where $t_0 \in \mathbb{R}_+$.

Define $y(\cdot) = x(\cdot) - x^*$ and then system (7.19) can be simplified as

$$\begin{cases} \frac{dy(t)}{dt} &= -y(t) + A(f(y(t) + x^*) - f(x^*)) + B(f(y(t-\tau) + x^*) - f(x^*)), \\ & \qquad\qquad\qquad\qquad\qquad\qquad t \ge t_0,\ t \ne t_k, \quad k \in \mathbb{N}, \\ y(t) &= I_k(y(t^-) + x^*) - x^*, \qquad t = t_k, \quad k \in \mathbb{N}, \\ y_{t_0} &= \phi - x^*. \end{cases} \tag{7.21}$$

Let us divide the set $\widetilde{I} = 1, 2, \cdots, n$ into subsets $\widetilde{I}_1,\ \widetilde{I}_2,\ \widetilde{I}_3$ such that $\widetilde{I} = \widetilde{I}_1 \bigcup \widetilde{I}_2 \bigcup \widetilde{I}_3$, where $\widetilde{I}_1 = \{i \in \widetilde{I}\,|\, x_i^* > 1\}$, $\widetilde{I}_2 = \{i \in \widetilde{I}\,|\, |x_i^*| \le 1\}$ and $\widetilde{I}_3 = \{i \in \widetilde{I}\,|\, x_i^* < -1\}$. Then we rearrange the order of $y_i,\ x_i^*$ and let $z = \{z_1, z_2, \cdots, z_n\}$ equate the rearranged $y = \{y_{i_1}, y_{i_2}, \cdots, y_{i_n}\}$, $\phi^* = \{\phi_1^*, \phi_2^*, \cdots, \phi_n^*\}$ equate the rearranged $\phi = \{\phi_{i_1}, \phi_{i_2}, \cdots, \phi_{i_n}\}$ and $z^* = \{z_1^*, z_2^*, \cdots, z_n^*\}$ equate the rearranged $x^* = \{x_{i_1}^*, x_{i_2}^*, \cdots, x_{i_n}^*\}$ such that $\widetilde{I}_1 = \{1, 2, \cdots, r\}$, $\widetilde{I}_2 = \{r+1, r+2, \cdots, r+m\}$, $\widetilde{I}_3 = \{r+m+1, r+m+2, \cdots, n\}$, where $r,\ m$ are nonnegative integers. In order to introduce the method, we assume that $\widetilde{I}_2 \ne \emptyset$. Let

$$z(t) = \begin{bmatrix} z^{(1)}(t) \\ z^{(2)}(t) \\ z^{(3)}(t) \end{bmatrix},$$

where

$$\begin{aligned} z^{(1)}(t) &= (z_1(t), z_2(t), \cdots, z_r(t))^T, \\ z^{(2)}(t) &= (z_{r+1}(t), z_{r+2}(t), \cdots, z_{r+m}(t))^T, \\ z^{(3)}(t) &= (z_{r+m+1}(t), z_{r+m+2}(t), \cdots, z_n(t))^T. \end{aligned}$$

Assume that $I_k(y(t^-) + x^*) - x^* = [I_{1k}^T(z^{(1)}(t^-)), I_{2k}^T(z^{(2)}(t^-)), I_{3k}^T(z^{(3)}(t^-))]^T$, where $I_{1k} \in C(\mathbb{R}^r, \mathbb{R}^r)$, $I_{2k} \in C(\mathbb{R}^m, \mathbb{R}^m)$, $I_{3k} \in C(\mathbb{R}^{n-r-m}, \mathbb{R}^{n-r-m})$ with $I_{ik}(0) \equiv 0$ for $i = 1, 2, 3$. Then system (7.21) is rewritten by

$$\begin{cases} \frac{dz^{(1)}(t)}{dt} = -z^{(1)}(t) + A_{11}g(z^{(1)}(t)) + A_{12}g(z^{(2)}(t)) + A_{13}g(z^{(3)}(t)) \\ +B_{11}g(z^{(1)}(t-\tau)) + B_{12}g(z^{(2)}(t-\tau)) + B_{13}g(z^{(3)}(t-\tau)),\ t \geq t_0, t \neq t_k, \\ \frac{dz^{(2)}(t)}{dt} = -z^{(2)}(t) + A_{21}g(z^{(1)}(t)) + A_{22}g(z^{(2)}(t)) + A_{23}g(z^{(3)}(t)) \\ +B_{21}g(z^{(1)}(t-\tau)) + B_{22}g(z^{(2)}(t-\tau)) + B_{23}g(z^{(3)}(t-\tau)),\ t \geq t_0, t \neq t_k, \\ \frac{dz^{(3)}(t)}{dt} = -z^{(3)}(t) + A_{31}g(z^{(1)}(t)) + A_{32}g(z^{(2)}(t)) + A_{33}g(z^{(3)}(t)) \\ +B_{31}g(z^{(1)}(t-\tau)) + B_{32}g(z^{(2)}(t-\tau)) + B_{33}g(z^{(3)}(t-\tau)),\ t \geq t_0, t \neq t_k, \\ z^{(1)}(t_k) = I_{1k}(z^{(1)}(t_k^-)), \quad z^{(2)}(t_k) = I_{2k}(z^{(2)}(t_k^-)), \\ z^{(3)}(t_k) = I_{3k}(z^{(3)}(t_k^-)), \quad z_{t_0} = \phi^* - z^*, \quad k \in \mathbb{N}, \end{cases} \tag{7.22}$$

where $[g^T(z^{(1)}(\cdot)), g^T(z^{(2)}(\cdot)), g^T(z^{(3)}(\cdot))]^T = f(z(\cdot) + z^*) - f(z^*)$.

Let $q = \min\{\min_{i \in \widetilde{I}_1}(x_i^* - 1), \min_{i \in \widetilde{I}_3}(-1 - x_i^*)\}$, then $q > 0$. Assume that the initial function ϕ satisfies $\sup_{t_0 - \tau_i \leq t \leq t_0} |\phi_i(t) - x_i^*| < q$ for $i = 1, 2, \cdots, n$. Then by continuity, there exists a $T > 0$, such that $|z_i(t)| < q$ for $t \in [-\tau_i, T)$. By the choice of q it is easy to verify that if $z_i^* \in \widetilde{I}_j$, then $z_i(t) + z_i^* \in \widetilde{I}_j$ for $j = 1, 3$ and $t \in [t_0 - \tau_i, T)$; thus for any $t \in [t_0, T)$, we have

$$f_i(z_i(t) + z_i^*) - f_i(z_i^*) = 0,\ f_i(z_i(t - \tau_i) + z_i^*) - f_i(z_i^*) = 0, \quad \forall i \in \widetilde{I}_1 \bigcup \widetilde{I}_3,$$

and then $g(z^{(1)}(\cdot)) \equiv g(z^{(3)}(\cdot)) \equiv 0$. Hence for any $t \in [0, T)$, system (7.22) is equivalent to

$$\begin{cases} \frac{dz^{(1)}(t)}{dt} &= -z^{(1)}(t) + A_{12}g(z^{(2)}(t)) + B_{12}g(z^{(2)}(t-\tau)),\ t \geq t_0,\ t \neq t_k, \\ \frac{dz^{(2)}(t)}{dt} &= -z^{(2)}(t) + A_{22}g(z^{(2)}(t)) + B_{22}g(z^{(2)}(t-\tau)),\ t \geq t_0,\ t \neq t_k, \\ \frac{dz^{(3)}(t)}{dt} &= -z^{(3)}(t) + A_{32}g(z^{(2)}(t)) + B_{32}g(z^{(2)}(t-\tau)),\ t \geq t_0,\ t \neq t_k, \\ z^{(1)}(t_k) &= I_{1k}(z^{(1)}(t_k^-)), \quad z^{(2)}(t_k) = I_{2k}(z^{(2)}(t_k^-)), \\ z^{(3)}(t_k) &= I_{3k}(z^{(3)}(t_k^-)), \quad z_{t_0} = \phi^* - z^*, \qquad k \in \mathbb{N}. \end{cases} \tag{7.23}$$

Next, we shall discuss the exponential stability of the equilibrium point x^* of system (7.19). In the theorems, we first obtain an exponential estimate of the second group of the neurons by employing Lyapunov functionals. Then we obtain exponential estimates of the other two groups of neurons by using the method of variation of parameters. For convenience, we shall use $\|\phi - x^*\|_{\tau^*}$ instead of $\|\phi^* - z^*\|_{\tau^*}$ in the following presentations, since they are equal to each other.

Theorem 7.2.1 *Assume that $I_{1k} = I_{3k} = E$ with E representing the identity map, and there exist constants $l, \alpha,\ d_{2k} > 0$ such that $\tau^* \le t_k - t_{k-1} \le l$ and $\|I_{2k}(x)\| \le d_{2k}\|x\|$ for any $x \in \mathbb{R}^m$ and $k \in \mathbb{N}$ with*

$$\ln(d_{2k}^2 + \tau^*\|B_{22}\|) \le -(\alpha + c)l,$$

where $c = 2(\|A_{22}\| + \|B_{22}\| - 1) > 0$. Then the equilibrium point $x = x^*$ of system (7.19) is exponentially stable.

Proof. Choose Lyapunov functional

$$V(t, z_t^{(2)}) = \|z^{(2)}(t)\|^2 + \|B_{22}\| \sum_{j=r+1}^{r+m} \int_{t-\tau_j}^{t} z_j^2(s)ds.$$

Then the upper right-hand derivative of V with respect to the second equation of system (7.23) is

$$\begin{aligned} D^+V(t, z_t^{(2)}) &= 2z^{(2)T}(t)\big[-z^{(2)}(t) + A_{22}g(z^{(2)}(t)) \\ &\quad + B_{22}g(z^{(2)}(t-\tau))\big] + \|B_{22}\|(\|z^{(2)}(t)\|^2 - \|z^{(2)}(t-\tau)\|^2) \\ &\le 2\big(-1 + \|A_{22}\| + \|B_{22}\|\big)\|z^{(2)}(t)\|^2 \\ &\le cV(t, z_t^{(2)}), \qquad t \in [t_{k-1}, t_k)\bigcap[t_0, T),\ k \in \mathbb{N}, \end{aligned}$$

where $c = 2(\|A_{22}\| + \|B_{22}\| - 1) > 0$. Thus we have

$$V(t, z_t^{(2)}) \le V(t_{k-1}, z_{t_{k-1}}^{(2)})e^{c(t-t_{k-1})}, \quad t \in [t_{k-1}, t_k)\bigcap[t_0, T),\ k \in \mathbb{N}. \tag{7.24}$$

Then, we obtain that, for $t \in [t_0, t_1)\bigcap[t_0, T)$,

$$\begin{aligned} V(t, z_t^{(2)}) &\le V(t_0, z_{t_0}^{(2)})e^{c(t-t_0)} \\ &\le (\|\phi - x^*\|_{\tau^*}^2 + \|B_{22}\|\tau^*\|\phi - x^*\|_{\tau^*}^2)e^{cl} \\ &\le (1 + \tau^*\|B_{22}\|)e^{cl}\|\phi - x^*\|_{\tau^*}^2 \\ &\le M^2\|\phi - x^*\|_{\tau^*}^2 e^{-\alpha(t-t_0)}, \end{aligned} \tag{7.25}$$

where $M = \sqrt{(1+\tau^*\|B_{22}\|)e^{(c+\alpha)l}}$, so we have

$$\|z^{(2)}(t)\|^2 \le V(t, z_t^{(2)}) \le M^2\|\phi - x^*\|_{\tau^*}^2 e^{-\alpha(t-t_0)}, \;\; t \in [t_0, t_1)\bigcap[t_0, T),$$

i.e.

$$\|z^{(2)}(t)\| \le M\|\phi - x^*\|_{\tau^*} e^{-\frac{\alpha}{2}(t-t_0)}, \;\; t \in [t_0, t_1)\bigcap[t_0, T). \tag{7.26}$$

And we have

$$\begin{aligned} V(t_1, z_{t_1}^{(2)}) &= \|z^{(2)}(t_1)\|^2 + \|B_{22}\| \textstyle\sum_{j=r+1}^{r+m} \int_{t_1-\tau_j}^{t_1} z_j^2(s)ds \\ &= d_{21}^2\|z^{(2)}(t_1^-)\|^2 + \|B_{22}\| \textstyle\sum_{j=r+1}^{r+m} \int_{t_1^- - \tau_j}^{t_1^-} z_j^2(s)ds \\ &\le d_{21}^2\|z^{(2)}(t_1^-)\|^2 + \|B_{22}\|\tau^*\|z^{(2)}(t_1^-)\|_{\tau^*}^2. \end{aligned} \tag{7.27}$$

By (7.25) we have

$$\begin{aligned} \|z_t^{(2)}\|_{\tau^*}^2 &\le V(t, z_t^{(2)}) \le (1+\tau^*\|B_{22}\|)e^{cl}\|\phi - x^*\|_{\tau^*}^2 \\ &\le M^2\|\phi - x^*\|_{\tau^*}^2 e^{-\alpha(t_1-t_0)}. \end{aligned}$$

Then by inequality (7.26) and the assumptions, we have

$$\begin{aligned} V(t_1, z_{t_1}^{(2)}) &\le (d_{21}^2 + \|B_{22}\|\tau^*)\|z_{t_1^-}^{(2)}\|_{\tau^*}^2 \\ &\le (d_{21}^2 + \|B_{22}\|\tau^*)M^2\|\phi - x^*\|_{\tau^*}^2 e^{-\alpha(t_1-t_0)} \\ &\le e^{-(\alpha+c)l}M^2\|\phi - x^*\|_{\tau^*}^2 e^{-\alpha(t_1-t_0)}. \end{aligned} \tag{7.28}$$

Thus by (7.24) and (7.28), we have

$$\begin{aligned} V(t, z_t^{(2)}) &\le V(t_1, z_{t_1}^{(2)})e^{c(t-t_1)} \\ &\le M^2\|\phi - x^*\|_{\tau^*}^2 e^{-\alpha(t_1-t_0)}e^{-\alpha l} \\ &\le M^2\|\phi - x^*\|_{\tau^*}^2 e^{-\alpha(t_2-t_0)}, \quad t \in [t_1, t_2)\bigcap[t_0, T). \end{aligned} \tag{7.29}$$

So we have

$$\|z^{(2)}(t)\|^2 \le V(t, z_t^{(2)}) \le M^2\|\phi - x^*\|_{\tau^*}^2 e^{-\alpha(t-t_0)}, \;\; t \in [t_1, t_2)\bigcap[t_0, T),$$

i.e.

$$\|z^{(2)}(t)\| \le M\|\phi - x^*\|_{\tau^*} e^{-\frac{\alpha}{2}(t-t_0)}, \;\; t \in [t_1, t_2)\bigcap[t_0, T). \tag{7.30}$$

Similarly, we can obtain

$$\|z^{(2)}(t)\| \leq M\|\phi - x^*\|_{\tau^*} e^{-\frac{\alpha}{2}(t-t_0)},\ t \in [t_{k-1}, t_k) \bigcap [t_0, T),\ \forall k \in \mathbb{N}, \tag{7.31}$$

which implies

$$\|z^{(2)}(t)\| \leq M\|\phi - x^*\|_{\tau^*} e^{-\frac{\alpha}{2}(t-t_0)},\ \ t \in [t_0, T). \tag{7.32}$$

By the assumption, $z^{(1)}(t)$ and $z^{(3)}(t)$ are continuous for $t \in [t_0, T)$. Then by equation (7.23) and the method of variation of parameters, we have

$$\begin{aligned} z^{(i)}(t) \ \ &= \ \ z^{(i)}(t_0)e^{-(t-t_0)} + \int_{t_0}^{t} e^{-(t-s)}\big(A_{i2}g(z^{(2)}(s)) + B_{i2}g(z^{(2)}(s-\tau))\big)ds, \\ &\qquad\qquad \text{for } t \in [t_0, T),\ \ i = 1, 3. \end{aligned} \tag{7.33}$$

Using the estimate of $z^{(2)}(t)$ (i.e. (7.32)), we have

$$\begin{aligned} \|z^{(i)}(t)\| \ &\leq \ \|z^{(i)}(t_0)\|e^{-(t-t_0)} + \int_{t_0}^{t} e^{-(t-s)}\big(\|A_{i2}\|\|z^{(2)}(s)\| \\ &\qquad + \|B_{i2}\|\|z^{(2)}(s-\tau)\|\big)ds \\ &\leq \ \|\phi - x^*\|_{\tau^*}\bigg(e^{-(t-t_0)} + \int_{t_0}^{t} e^{-(t-s)}M\big(\|A_{i2}\| \\ &\qquad + \|B_{i2}\|e^{\frac{\alpha}{2}\tau^*}\big)e^{-\frac{\alpha}{2}(s-t_0)}ds\bigg) \\ &\leq \ \|\phi - x^*\|_{\tau^*}\bigg(e^{-(t-t_0)} + e^{-t}M\big(\|A_{i2}\| + \|B_{i2}\|e^{\frac{\alpha}{2}\tau^*}\big) \\ &\qquad \times \int_{t_0}^{t} e^{s-\frac{\alpha}{2}(s-t_0)}ds\bigg), \qquad\qquad t \in [t_0, T),\ i = 1, 3. \end{aligned} \tag{7.34}$$

By direct calculation, we obtain from (7.34) that

(1) when $\alpha = 2$, for $t \in [t_0, T),\ i = 1, 3$, we have

$$\|z^{(i)}(t)\| \leq \|\phi - x^*\|_{\tau^*}\big[1 + MT(\|A_{i2}\| + \|B_{i2}\|e^{\tau^*})\big]e^{-(t-t_0)};$$

(2) when $0 < \alpha < 2$, i.e. $1 - \frac{\alpha}{2} > 0$, for $t \in [t_0, T),\ i = 1, 3$, we have

$$\|z^{(i)}(t)\| \leq \|\phi - x^*\|_{\tau^*}\left[1 + M\frac{\|A_{i2}\| + \|B_{i2}\|e^{\frac{\alpha}{2}\tau^*}}{1 - \frac{\alpha}{2}}\right]e^{-\frac{\alpha}{2}(t-t_0)};$$

(3) when $\alpha > 2$, i.e. $1 - \frac{\alpha}{2} < 0$, for $t \in [t_0, T),\ i = 1, 3$, we have

$$\|z^{(i)}(t)\| \leq \|\phi - x^*\|_{\tau^*} \left[1 + M \frac{\|A_{i2}\| + \|B_{i2}\| e^{\frac{\alpha}{2}\tau^*}}{\frac{\alpha}{2} - 1} \right] e^{-(t-t_0)}.$$

In summary, we have

$$\|z^{(i)}(t)\| \leq \widetilde{M} \|\phi - x^*\|_{\tau^*} e^{-\widetilde{\alpha}(t-t_0)}, \quad t \in [t_0, T), \quad i = 1, 3,$$

where $\widetilde{\alpha} = \min\{1, \frac{\alpha}{2}\}$ and

$$\begin{aligned} \widetilde{M} \quad = \quad & \max_{\{i=1,3\}} \{1 + MT(\|A_{i2}\| + \|B_{i2}\| e^{\tau^*}),\ 1 + M \tfrac{\|A_{i2}\| + \|B_{i2}\| e^{\frac{\alpha}{2}\tau^*}}{1 - \frac{\alpha}{2}}, \\ & 1 + M \tfrac{\|A_{i2}\| + \|B_{i2}\| e^{\frac{\alpha}{2}\tau^*}}{\frac{\alpha}{2} - 1}\}. \end{aligned}$$

This, together with (7.32), implies that

$$\|z^{(i)}(t)\| \leq \max\{M, \widetilde{M}\} \|\phi - x^*\|_{\tau^*} e^{-\widetilde{\alpha}(t-t_0)}, \quad t \in [t_0, T),\ i = 1, 2, 3.$$

If we choose $\|\phi - x^*\|_{\tau^*} < \frac{q}{\max\{M, \widetilde{M}\}}$, then $\|z^{(i)}(t)\| < q$ for any $t \in [t_0, T)$ and $i = 1, 2, 3$. Thus by repeating these procedures, we can show that the same result holds for any $t \in [T, T_1),\ [T_1, T_2),\ \cdots, [T_{n-1}, T_n)$ with $T_n \to \infty$ as $t \to \infty$. So under the assumptions of the theorem, the interval of existence of the solution of system (7.23) is $[t_0, \infty)$ and the trivial solution of system (7.23) is exponentially stable, and hence the equilibrium point x^* of system (7.19) is exponentially stable. ∎

Remark 7.2.1 *Notice that in Theorem 7.2.1, the neural networks (7.19) without impulses might be unstable under the condition $\|A_{22}\| + \|B_{22}\| > 1$, see Example 7.2.1 and [135]. Thus the impulses here play an important role to stabilize the neural networks. According to the method we used in this theorem, it is good enough to control the second group of neurons $z^{(2)}(t)$. Hence in Corollary 7.2.1, we obtain sufficient conditions on impulses of the other two groups of neurons such that the stability property is maintained for certain impulsive perturbations.*

Example 7.2.1 *Consider the following cellular neural networks with time delay*

$$\begin{cases} \frac{dx_1(t)}{dt} & = -x_1(t) - 2f_1(x_1(t)) + f_2(x_2(t)) + \frac{1}{3}f_3(x_3(t)) + 2f_1(x_1(t-0.25)) \\ & \quad +\frac{1}{2}f_2(x_2(t-0.5)) + \frac{2}{3}f_3(x_3(t-0.3)) + 3, \qquad t \geq 0, \\ \frac{dx_2(t)}{dt} & = -x_2(t) + 3f_1(x_1(t)) + \frac{3}{2}f_2(x_2(t)) + f_3(x_3(t)) + \frac{1}{2}f_1(x_1(t-0.25)) \\ & \quad -\frac{1}{4}f_2(x_2(t-0.5)) + 5f_3(x_3(t-0.3)) + \frac{5}{2}, \qquad t \geq 0, \\ \frac{dx_3(t)}{dt} & = -x_3(t) - f_1(x_1(t)) + \frac{1}{4}f_2(x_2(t)) + f_3(x_3(t)) - 2f_1(x_1(t-0.25)) \\ & \quad +\frac{1}{2}f_2(x_2(t-0.5)) + f_3(x_3(t-0.3)) + 3, \qquad t \geq 0, \\ x_0 & = \phi. \end{cases} \tag{7.35}$$

By direct computation, we know that $x^* = (2, 0, -2)$ is an isolated equilibrium point of the cellular neural networks (7.35). Since $x_i^* \in \widetilde{I}_i$ for $i = 1, 2, 3$, $z^{(i)}(\cdot) = y_i(\cdot)$. Let $y = x - x^*$, for $\|\phi_i - x_i^*\|_{0.5} < \delta = 1$, $\|y_i\|_{\tau_i} < 1$, and then system (7.35) is simplified as

$$\begin{cases} \frac{dy_1(t)}{dt} & = -y_1(t) + g(y_2(t)) + \frac{1}{2}g(y_2(t-0.5)), \qquad t \geq 0, \\ \frac{dy_2(t)}{dt} & = -y_2(t) + \frac{3}{2}g(y_2(t)) - \frac{1}{4}g(y_2(t-0.5)), \qquad t \geq 0, \\ \frac{dy_3(t)}{dt} & = -y_3(t) + \frac{1}{4}g(y_2(t)) + \frac{1}{2}f(x_2(t-0.5)), \qquad t \geq 0, \\ y_0 & = \phi - x^*, \end{cases} \tag{7.36}$$

where $g(y_2(\cdot)) = f_2(y_2(\cdot) + x_2^*) - f_2(x_2^*)$.

We notice that $c_i = 1 < \sum_{j=1}^{3}(|a_{ij}| + |b_{ij}|)$ for $i = 1, 2, 3$, so Corollary 3 in [24] cannot be used. And $\|A_{22}\| + \|B_{22}\| = 1.75 > 1$, Theorem 1 in [135] cannot be applied. In fact, the equilibrium x^* is not stable, see Figure 7.4 for the simulation of the non-impulsive system with the initial functions $\phi = (2.05H(t), 0.15H(t), -1.85H(t))^T$, $\phi = (1.9H(t), -0.2H(t), -2.2H(t))^T$ and $\phi = (2.2H(t), -0.4H(t), -2.4H(t))^T$, where $H(t)$ is the Heaviside step function. From the graph, we can see that the solutions starting from the neighborhood of x^* all go away from x^* and converge to another equilibrium point $(3.5, 1.25, -1.25)^T$.

By Theorem 7.2.1, applying the impulsive control $y_2(k) = 0.02y_2(k^-)$ will stabilize the delayed cellular neural networks with $d_{2k} = 0.02$, $\tau^* = 0.5$, $\alpha = 0.5$, $c = 1.5$. The simulations of the impulsive cellular neural networks with time delay and the same initial functions are given in Figure 7.5.

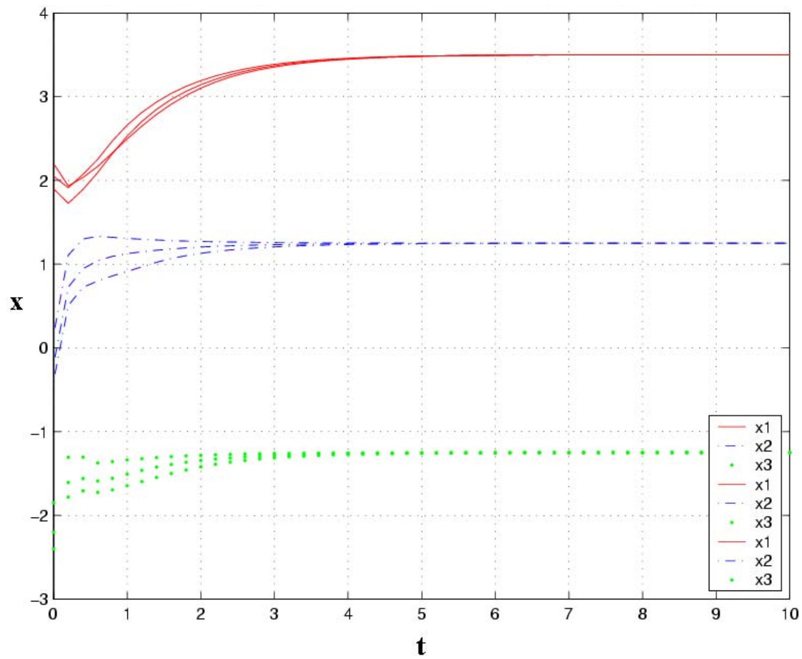

Figure 7.4: Numerical simulation of Example 7.2.1, system without impulses.

Theorem 7.2.2 *Assume that there exist constants* $b_{ik} \geq 0$ *with* $\sum_{k=1}^{\infty} b_{ik} < \infty$ *for* $i = 1, 2, 3$ *and* $k \in \mathbb{N}$ *such that* $\|A_{22}\| + \|B_{22}\| < 1$ *and*

$$\|I_{ik}(z^{(i)})\|^2 \leq (1 + b_{ik})\|z^{(i)}\|^2, \qquad i = 1, 2, 3, \quad k \in \mathbb{N};$$

where $z^{(1)} \in \mathbb{R}^r$, $z^{(2)} \in \mathbb{R}^m$ *and* $z^{(3)} \in \mathbb{R}^{n-r-m}$. *Then the equilibrium point* $x = x^*$ *of system (7.19) is exponentially stable.*

Proof. Since $\|A_{22}\| + \|B_{22}\| < 1$, there exists some $\alpha \in (0, 1)$ such that

$$2 - 2\|A_{22}\| - (1 + e^{\alpha\tau^*})\|B_{22}\| - \alpha \geq 0. \tag{7.37}$$

Choose the Lyapunov functional

$$V(t, z_t^{(2)}) = \|z^{(2)}(t)\|^2 e^{\alpha t} + \|B_{22}\| \sum_{j=r+1}^{r+m} \int_{t-\tau_j}^{t} z_j^2(s) e^{\alpha(s+\tau_j)} ds.$$

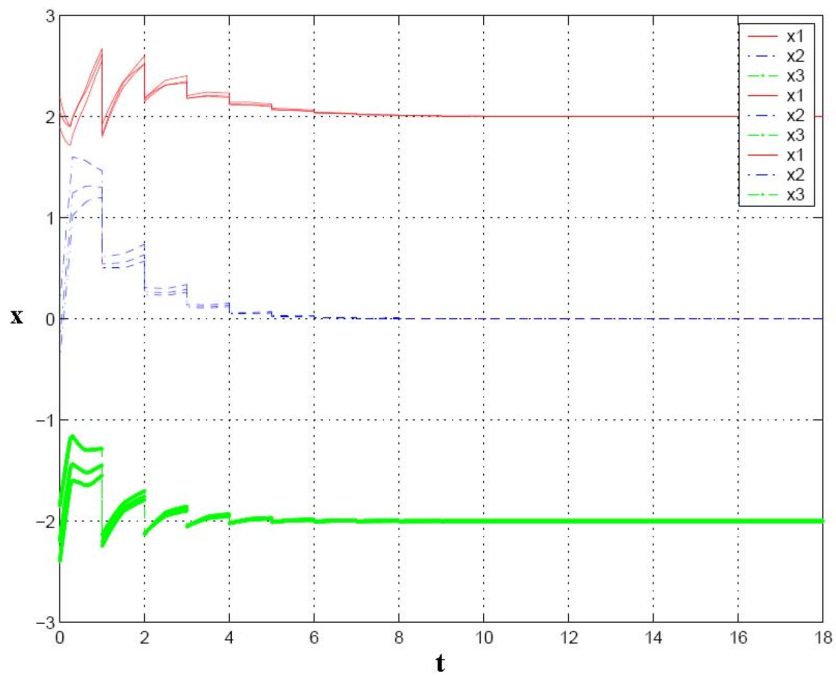

Figure 7.5: Numerical simulation of Example 7.2.1, system with impulsive control.

Then the upper right-hand derivative of V with respect to the second equation of system (7.23) is

$$\begin{aligned} D^+V(t, z_t^{(2)}) &= 2e^{\alpha t} z^{(2)T}(t)\big(- z^{(2)}(t) + A_{22}g(z^{(2)}(t)) + B_{22}g(z^{(2)}(t-\tau)\big) \\ &\quad +\alpha\|z^{(2)}(t)\|^2 e^{\alpha t} + \|B_{22}\| \sum_{j=r+1}^{r+m} \big(z_j^{(2)}(t)e^{\alpha(t+\tau_j)} - z_j^{(2)}(t-\tau_j)e^{\alpha t}\big) \\ &\le -\big(2-\alpha-2\|A_{22}\| - e^{\alpha\tau^*}\|B_{22}\|\big)\|z^{(2)}(t)\|^2 e^{\alpha t} \\ &\quad +2e^{\alpha t}\|B_{22}\|\|z^{(2)}(t)\|\|z^{(2)}(t-\tau)\| - \|B_{22}\|\|z^{(2)}(t-\tau)\|^2 e^{\alpha t} \\ &\le -(2-\alpha-2\|A_{22}\| - \|B_{22}\| - e^{\alpha\tau^*}\|B_{22}\|)\|z^{(2)}(t)\|^2 e^{\alpha t} \\ &\le 0, \qquad\qquad t\in[t_{k-1}, t_k)\bigcap[t_0, T),\ k\in\mathbb{N}. \end{aligned}$$

Hence we have

$$V(t, z_t^{(2)}) \le V(t_k, z_{t_k}^{(2)}), \qquad t\in[t_k, t_{k+1})\bigcap[t_0, T),\ k\in\{0\}\bigcup\mathbb{N}.$$

From the hypothesis, we have

$$\begin{aligned} V(t_k, z_{t_k}^{(2)}) &= \|z^{(2)}(t_k)\|^2 e^{\alpha t_k} + \|B_{22}\| \sum_{j=r+1}^{r+m} \int_{t_k-\tau_j}^{t_k} z_j^2(s) e^{\alpha(s+\tau_j)} ds \\ &\le (1+b_{2k}) \|z^{(2)}(t_k^-)\|^2 e^{\alpha t_k^-} + \|B_{22}\| \sum_{j=r+1}^{r+m} \int_{t_k^- -\tau_j}^{t_k^-} z_j^2(s) e^{\alpha(s+\tau_j)} ds \\ &\le (1+b_{2k}) V(t_k^-, z_{t_k^-}^{(2)}), \quad k \in \mathbb{N}, \end{aligned}$$

so

$$\begin{aligned} V(t, z_t^{(2)}) &\le V(t_k, z_{t_k}^{(2)}) \le (1+b_{2k}) V(t_k^-, z_{t_k^-}^{(2)}) \\ &\le \cdots \\ &\le \prod_{j=1}^{k} (1+b_{2j}) V(t_0, z_{t_0}^{(2)}) \\ &\le M_2 V(t_0, z_{t_0}^{(2)}), \quad t \in [t_k, t_{k+1}) \bigcap [t_0, T),\ k \in \{0\} \bigcup \mathbb{N}, \end{aligned}$$

where $M_2 = \prod_{j=1}^{\infty}(1+b_{2j}) < \infty$. Then we obtain

$$\begin{aligned} \|z^{(2)}(t)\|^2 e^{\alpha t} &\le V(t, z_t^{(2)}) \le M_2 V(t_0, z_{t_0}^{(2)}) \\ &\le M_2 e^{\alpha t_0} (1 + \tfrac{1}{\alpha} e^{\alpha \tau^*} \|B_{22}\|) \|\phi - x^*\|_{\tau^*}^2, \quad t \in [t_k, t_{k+1}) \bigcap [t_0, T), \end{aligned}$$

i.e.

$$\|z^{(2)}(t)\| \le M_2^* \|\phi - x^*\|_{\tau^*} e^{-\frac{\alpha}{2}(t-t_0)}, \quad t \in [t_k, t_{k+1}) \bigcap [t_0, T), \tag{7.38}$$

where $M_2^* = \sqrt{M_2(1 + \frac{1}{\alpha} e^{\alpha \tau^*} \|B_{22}\|)}$.

By equation (7.23) and the method of variation of parameters, we have

$$\begin{aligned} z^{(i)}(t) &= z^{(i)}(t_k) e^{-(t-t_k)} + \int_{t_k}^{t} e^{-(t-s)} \big(A_{i2} g(z^{(2)}(s)) + B_{i2} g(z^{(2)}(s-\tau))\big) ds, \\ &\qquad \text{for } t \in [t_k, t_{k+1}) \bigcap [t_0, T), \quad i = 1, 3. \end{aligned} \tag{7.39}$$

In view of (7.39) and the hypothesis, we have

$$\begin{aligned} \|z^{(i)}(t)\| &\le \|z^{(i)}(t_k)\| e^{-(t-t_k)} + \int_{t_k}^{t} e^{-(t-s)} \big(\|A_{i2}\| \|z^{(2)}(s)\| \\ &\quad + \|B_{i2}\| \|z^{(2)}(s-\tau)\|\big) ds \\ &\le \sqrt{1+b_{ik}} \|z^{(i)}(t_k^-)\| e^{-(t-t_k)} + \int_{t_k}^{t} e^{-(t-s)} \big(\|A_{i2}\| \|z^{(2)}(s)\| \\ &\quad + \|B_{i2}\| \|z^{(2)}(s-\tau)\|\big) ds \\ &\le (1+b_{ik}) \|z^{(i)}(t_k^-)\| e^{-(t-t_k)} + \int_{t_k}^{t} e^{-(t-s)} \big(\|A_{i2}\| \|z^{(2)}(s)\| \\ &\quad + \|B_{i2}\| \|z^{(2)}(s-\tau)\|\big) ds, \quad \text{for } t \in [t_k, t_{k+1}) \bigcap [t_0, T), \quad i = 1, 3, \end{aligned}$$

which, together with

$$\begin{aligned}\|z^{(i)}(t_k^-)\| &\leq \|z^{(i)}(t_{k-1})\|e^{-(t_k^- - t_{k-1})} + \int_{t_{k-1}}^{t_k^-} e^{-(t_k^- - s)}\big(\|A_{i2}\|\|z^{(2)}(s)\| \\ &\quad +\|B_{i2}\|\|z^{(2)}(s-\tau)\|\big)ds, \quad i=1,3,\end{aligned}$$

yields

$$\begin{aligned}\|z^{(i)}(t)\| &\leq (1+b_{ik})\Big[\|z^{(i)}(t_{k-1})\|e^{-(t_k^- - t_{k-1})} + \int_{t_{k-1}}^{t_k^-} e^{-(t_k^- - s)}\big(\|A_{i2}\|\|z^{(2)}(s)\| \\ &\quad +\|B_{i2}\|\|z^{(2)}(s-\tau)\|\big)ds\Big]e^{-(t-t_k)} + \int_{t_k}^{t} e^{-(t-s)}\big(\|A_{i2}\|\|z^{(2)}(s)\| \\ &\quad +\|B_{i2}\|\|z^{(2)}(s-\tau)\|\big)ds \\ &\leq (1+b_{ik})\|z^{(i)}(t_{k-1})\|e^{-(t-t_{k-1})} + (1+b_{ik})\int_{t_{k-1}}^{t} e^{-(t-s)}\big(\|A_{i2}\| \\ &\quad \times\|z^{(2)}(s)\| + \|B_{i2}\|\|z^{(2)}(s-\tau)\|\big)ds \\ &\leq (1+b_{ik})(1+b_{i\,k-1})\|z^{(i)}(t_{k-1}^-)\|e^{-(t-t_{k-1})} + (1+b_{ik})\int_{t_{k-1}}^{t} e^{-(t-s)} \\ &\quad \times\big(\|A_{i2}\|\|z^{(2)}(s)\| + \|B_{i2}\|\|z^{(2)}(s-\tau)\|\big)ds. \end{aligned} \tag{7.40}$$

Then by (7.39) again, we have

$$\begin{aligned}\|z^{(i)}(t)\| &\leq (1+b_{ik})(1+b_{i\,k-1})\big[\|z^{(i)}(t_{k-2})\|e^{-(t_{k-1}^- - t_{k-2})} + \int_{t_{k-2}}^{t_{k-1}^-} e^{-(t_{k-1}^- - s)} \\ &\quad \times\big(\|A_{i2}\|\|z^{(2)}(s)\| + \|B_{i2}\|\|z^{(2)}(s-\tau)\|\big)ds\big]e^{-(t-t_{k-1})} + (1+b_{ik}) \\ &\quad \times\int_{t_{k-1}}^{t} e^{-(t-s)}\big(\|A_{i2}\|\|z^{(2)}(s)\| + \|B_{i2}\|\|z^{(2)}(s-\tau)\|\big)ds \\ &\leq (1+b_{ik})(1+b_{i\,k-1})\big[\|z^{(i)}(t_{k-2})\|e^{-(t-t_{k-2})} + \int_{t_{k-2}}^{t} e^{-(t-s)} \\ &\quad \times\big(\|A_{i2}\|\|z^{(2)}(s)\| + \|B_{i2}\|\|z^{(2)}(s-\tau)\|\big)ds\big].\end{aligned}$$

It follows by induction

$$\begin{aligned}\|z^{(i)}(t)\| &\le \prod_{j=1}^{k}(1+b_{ij})\Big[\|z^{(i)}(t_0)\|e^{-(t-t_0)}+\int_{t_0}^{t}e^{-(t-s)}\big(\|A_{i2}\|\|z^{(2)}(s)\| \\ &\quad +\|B_{i2}\|\|z^{(2)}(s-\tau)\|\big)ds\Big] \\ &\le M_i\Big[\|z^{(i)}(t_0)\|e^{-(t-t_0)}+\int_{t_0}^{t}e^{-(t-s)}\big(\|A_{i2}\|\|z^{(2)}(s)\| \\ &\quad +\|B_{i2}\|\|z^{(2)}(s-\tau)\|\big)ds\Big], \qquad \text{for } t\in[t_k,t_{k+1})\bigcap[t_0,T),\end{aligned}$$

where $M_i=\prod_{j=1}^{\infty}(1+b_{ij}),\ i=1,3$. By (7.40) and the estimate of $z^{(2)}(t)$ (i.e. (7.38)), we have

$$\begin{aligned}\|z^{(i)}(t)\| &\le M_i\Big[\|z^{(i)}(t_0)\|e^{-(t-t_0)}+\int_{t_0}^{t}e^{-(t-s)}\big(\|A_{i2}\|\|z^{(2)}(s)\| \\ &\quad +\|B_{i2}\|\|z^{(2)}(s-\tau)\|\big)ds\Big] \\ &\le M_i\|\phi-x^*\|_{\tau^*}\Big(e^{-(t-t_0)}+\int_{t_0}^{t}e^{-(t-s)}M_2^*\big(\|A_{i2}\| \\ &\quad +\|B_{22}\|e^{\frac{\alpha}{2}\tau^*}\big)e^{-\frac{\alpha}{2}(s-t_0)}ds\Big) \\ &\le M_i\|\phi-x^*\|_{\tau^*}\Big(e^{-(t-t_0)}+e^{-t}M_2^*\big(\|A_{i2}\|+\|B_{22}\|e^{\frac{\alpha}{2}\tau^*}\big) \\ &\quad \times\int_{t_0}^{t}e^{s-\frac{\alpha}{2}(s-t_0)}ds\Big), \qquad t\in[t_0,T),\ i=1,3. \qquad (7.41)\end{aligned}$$

By direct calculation, we obtain from (7.41) that

(1) when $\alpha=2$, for $t\in[t_0,T),\ i=1,3$, we have

$$\|z^{(i)}(t)\|\le M_i\|\phi-x^*\|_{\tau^*}\big[1+M_2^*T(\|A_{i2}\|+\|B_{i2}\|e^{\tau^*})\big]e^{-(t-t_0)};$$

(2) when $0<\alpha<2$, i.e. $1-\frac{\alpha}{2}>0$, for $t\in[t_0,T),\ i=1,3$, we have

$$\|z^{(i)}(t)\|\le M_i\|\phi-x^*\|_{\tau^*}\left[1+M_2^*\frac{\|A_{i2}\|+\|B_{i2}\|e^{\frac{\alpha}{2}\tau^*}}{1-\frac{\alpha}{2}}\right]e^{-\frac{\alpha}{2}(t-t_0)};$$

(3) when $\alpha > 2$, i.e. $1 - \frac{\alpha}{2} < 0$, for $t \in [t_0, T),\ i = 1, 3$, we have

$$\|z^{(i)}(t)\| \le M_i\|\phi - x^*\|_{\tau^*}\left[1 + M_2^* \frac{\|A_{i2}\| + \|B_{i2}\|e^{\frac{\alpha}{2}\tau^*}}{\frac{\alpha}{2} - 1}\right] e^{-(t-t_0)}.$$

In summary, we have

$$\|z^{(i)}(t)\| \le \widetilde{M}\|\phi - x^*\|_{\tau^*} e^{-\widetilde{\alpha}(t-t_0)}, \quad t \in [t_0, T), \quad i = 1, 3,$$

where $\widetilde{\alpha} = \min\{1, \frac{\alpha}{2}\}$ and

$$\begin{aligned}\widetilde{M} \;\; = \;\; & \max\nolimits_{\{i=1,3\}}\{M_i[1 + M_2^* T(\|A_{i2}\| + \|B_{i2}\|e^{\tau^*})], \\ & M_i[1 + M_2^* \tfrac{\|A_{i2}\| + \|B_{i2}\|e^{\frac{\alpha}{2}\tau^*}}{1-\frac{\alpha}{2}}], \\ & M_i[1 + M_2^* \tfrac{\|A_{i2}\| + \|B_{i2}\|e^{\frac{\alpha}{2}\tau^*}}{\frac{\alpha}{2}-1}]\}.\end{aligned}$$

This, together with (7.38), implies that

$$\|z^{(i)}(t)\| \le \max\{M_2^*, \widetilde{M}\}\|\phi - x^*\|_{\tau^*} e^{-\widetilde{\alpha}(t-t_0)}, \quad t \in [t_0, T).$$

If we choose $\|\phi - x^*\|_{\tau^*} < \frac{q}{\max\{M_2^*, \widetilde{M}\}}$, then $\|z^{(i)}(t)\| < q$ for any $t \in [t_0, T)$ and $i = 1, 2, 3$. Thus by repeating these procedures, we can obtain that the same result holds for any $t \in [T, T_1),\ [T_1, T_2), \cdots, [T_{n-1}, T_n)$ with $T_n \to \infty$ as $t \to \infty$. So under the assumptions of the theorem, the existing interval of the solution of system (7.23) is $[t_0, \infty)$ and the trivial solution of system (7.23) is exponentially stable, and hence the equilibrium point x^* of system (7.19) is exponentially stable. ∎

Remark 7.2.2 *Theorem 7.2.2 is a generalization of Theorem 1 in [135], when $I_{ik}(x) \equiv x$, Theorem 7.2.2 is reduced to Theorem 1 in [135]. This result gives some sufficient conditions under which the stability property will not be destroyed by impulsive perturbations, see Example 7.2.2.*

Similarly to Theorem 7.2.1 and 7.2.2, we can prove the following result.

Corollary 7.2.1 *Assume that all the conditions of Theorem 7.2.1 hold except the assumption: $I_{1k} = I_{3k} = E$ with E representing the identity map, is changed to*
(i') there exists $b_{ik} > 0$ with $\sum_{k=1}^{\infty} b_{ik} < \infty$ such that

$$\|I_{ik}(z^{(i)})\|^2 \le (1 + b_{ik})\|z^{(i)}\|^2, \qquad i = 1, 3, \quad k \in \mathbb{N};$$

where $z^{(1)} \in \mathbb{R}^r$ and $z^{(3)} \in \mathbb{R}^{n-r-m}$, then the same result holds.

Next, we discuss an example to illustrate our previous result.

Example 7.2.2 *Consider the following cellular neural networks with time delay described by*

$$\begin{cases} \frac{dx_1(t)}{dt} &= -x_1(t) + 0.6f_1(x_1(t)) - f_2(x_2(t)) + 0.8f_3(x_3(t)) \\ & \quad +0.6f_1(x_1(t-0.1)) - 0.5f_2(x_2(t-0.2)) + 0.4f_3(x_3(t-0.1)) + 2, \quad t \geq 0, \\ \frac{dx_2(t)}{dt} &= -x_2(t) + 2.4f_1(x_1(t)) - f_2(x_2(t)) + 0.5f_3(x_3(t)) \\ & \quad -0.1f_2(x_2(t-0.2)) + 0.5f_3(x_3(t-0.1)) + 3, \qquad t \geq 0, \\ \frac{dx_3(t)}{dt} &= -x_3(t) - 1.2f_1(x_1(t)) + 0.7f_2(x_2(t)) + 0.3f_3(x_3(t)) \\ & \quad -0.6f_1(x_1(t-0.1)) + 0.2f_3(x_3(t-0.1)) + 1, \qquad t \geq 0, \\ x_0 &= \phi. \end{cases} \tag{7.42}$$

By simple computation, we know that $x^* = (1.46, 4.1, -0.2)$ is a unique equilibrium point of the cellular neural networks (7.42). Since $x_1^*, x_2^* \in \widetilde{I}_1$ and $x_3^* \in \widetilde{I}_2$, we have $z^{(1)}(\cdot) = (y_1(\cdot), y_2(\cdot))^T = (x_1(\cdot) - x_1^*, x_2(\cdot) - x_2^*)^T$ and $z^{(2)}(\cdot) = y_3(\cdot) = x_3(\cdot) - x_3^*$. So we can rewrite system (7.42) as

$$\begin{cases} \frac{dz^{(1)}(t)}{dt} &= -z^{(1)}(t) + A_{11}g(z^{(1)}(t)) + A_{12}g(z^{(2)}(t)) + B_{11}g(z^{(1)}(t-\tau)) \\ & \quad +B_{12}g(z^{(2)}(t-0.1)), \quad t \geq 0, \\ \frac{dz^{(2)}(t)}{dt} &= -z^{(2)}(t) + A_{21}g(z^{(1)}(t)) + A_{22}g(z^{(2)}(t)) + B_{21}g(z^{(1)}(t-\tau)) \\ & \quad +B_{22}g(z^{(2)}(t-0.1)), \quad t \geq 0, \\ z_0 &= \phi - z^*, \end{cases} \tag{7.43}$$

where

$$\begin{aligned} A_{11} &= \begin{bmatrix} 0.6 & -1 \\ 2.4 & -1 \end{bmatrix}, & A_{12} &= \begin{bmatrix} 0.8 \\ 0.5 \end{bmatrix}, \\ B_{11} &= \begin{bmatrix} 0.6 & -0.5 \\ 0 & -0.1 \end{bmatrix}, & B_{12} &= \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}, \\ A_{21} &= \begin{bmatrix} -1.2 & 0.7 \end{bmatrix}, & A_{22} &= \begin{bmatrix} 0.3 \end{bmatrix}, \\ B_{21} &= \begin{bmatrix} -0.6 & 0 \end{bmatrix}, & B_{22} &= \begin{bmatrix} 0.2 \end{bmatrix}, \end{aligned}$$

and $g(z(\cdot)) = f(z(\cdot) + z^*) - f(z^*)$.

Then for $\|\phi_i - z_i^*\|_{0.2} < q = 0.46$, $\|z_i\|_{\tau_i} < 0.46$, system (7.43) is simplified as

$$\begin{cases} \frac{dz^{(1)}(t)}{dt} & = -z^{(1)}(t) + A_{12}g(z^{(2)}(t)) + B_{12}g(z^{(2)}(t-0.1)), \quad t \geq 0, \\ \frac{dz^{(2)}(t)}{dt} & = -z^{(2)}(t) + A_{22}g(z^{(2)}(t)) + B_{22}g(z^{(2)}(t-0.1)), \quad t \geq 0, \\ z_0 & = \phi - z^*. \end{cases} \tag{7.44}$$

Notice that $c_i < \sum_{j=1}^{3}(|a_{ij}| + |b_{ij}|)$ for $i = 1, 2, 3$, so Corollary 3 in [24] can not determine the stability of system (7.42). However, the condition of Theorem 1 in [135] holds since $\|A_{22}\| + \|B_{22}\| = 0.5 < 1$, which implies that the equilibrium x^* of system (7.42) is exponentially stable, see Figure 7.6 for the simulation of the non-impulsive system with the initial functions $\phi = (1.6H(t), 3.7H(t), 0.2H(t))^T$, $\phi = (1.2H(t), 4.2H(t), 0)^T$ and $\phi = (2H(t), 4.5H(t), -0.5H(t))^T$, where $H(t)$ is the Heaviside step function. From the graph, we can see that the solutions starting from the neighborhood of x^* converge to x^*.

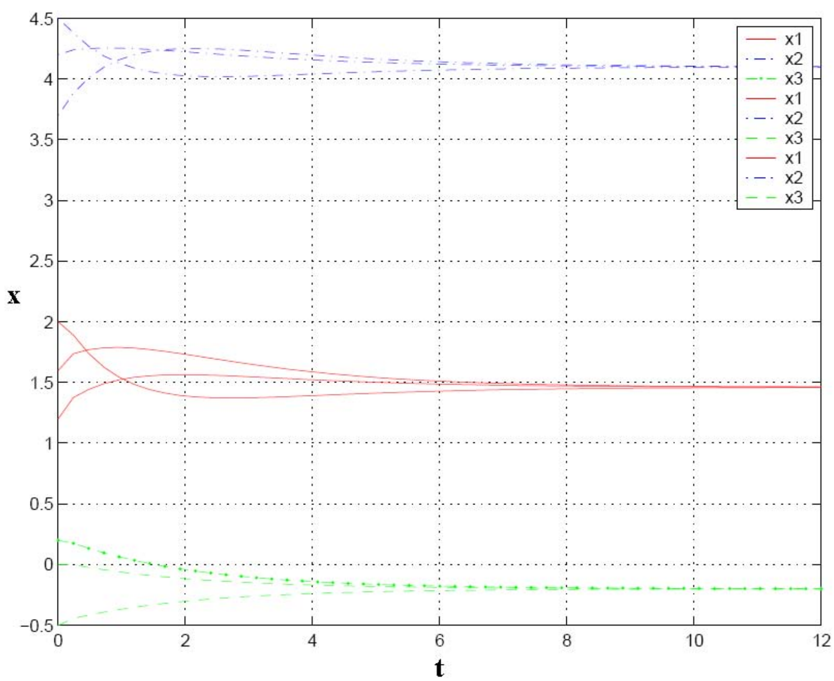

Figure 7.6: Numerical simulation of Example 7.2.2, system without impulses.

By Theorem 7.2.2, the impulsive perturbation $z_i(k) = (1+(-1)^i \frac{i}{k^2})z_i(k^-)$ will not affect the stability property of delayed cellular neural networks. The simulations of the impulsive cellular neural networks with time delay and the same initial functions are given in Figure 7.7.

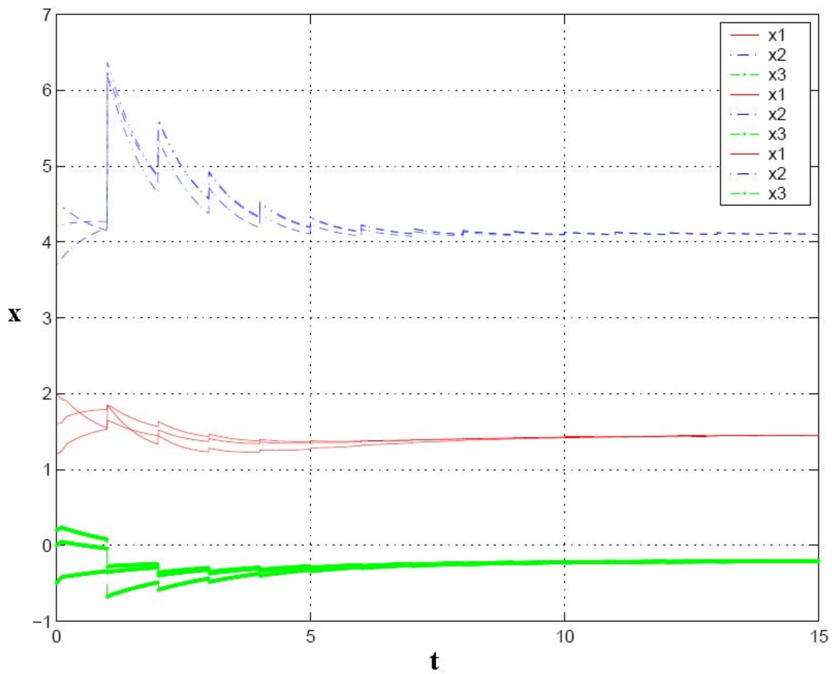

Figure 7.7: Numerical simulation of Example 7.2.2, impulse-disturbed system.

Remark 7.2.3 *Theorem 1 in [5] can not determine the stability of the equilibrium point x^* of system (7.42) because condition (i) : $-(A + A^T)$ is positive definite does not hold. In fact, the eigenvalues of $-(A + A^T)$ are -1.729, -1.0442, and 2.9732, which implies that $-(A + A^T)$ is neither positive definite nor negative definite. Moreover, Corollary 3 in [24] can not apply to Example 7.2.2 since $c_i < \sum_{j=1}^{3}(|a_{ij}| + |b_{ij}|)u_j$ for $i = 1, 2, 3$. While by Theorem 7.2.2, the equilibrium point x^* of system (7.42) is exponential stable with $I_{ik}(s) \equiv s$ for $i = 1, 2, 3$ and $k \in \mathbb{N}$.*

7.3 High Order Hopfield Type Neural Networks

It is known that high-order neural networks have stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order neural networks.

In this section, we discuss the exponential stability of high-order neural networks with time-varying delays.

We consider the impulsive high order Hopfield type neural networks with time-varying delays described by

$$\begin{cases} C_i u_i'(t) = -\frac{u_i(t)}{R_i} + \sum_{j=1}^n P_{ij} g_j(u_j(t-\tau_j(t)))+ \\ \quad \sum_{j=1}^n \sum_{l=1}^n P_{ijl} g_j(u_j(t-\tau_j(t))) g_l(u_l(t-\tau_l(t))) + I_i,\ t \in [t_{k-1}, t_k), \\ \Delta u_i(t_k) = d_{ik} u_i(t_k^-) + \sum_{j=1}^n W_{ij}^{(k)} h_j(u_j(t_k^-)) \\ \quad + \sum_{j=1}^n \sum_{l=1}^n W_{ijl}^{(k)} h_j(u_j(t_k^-)) h_l(u_l(t_k^-)), \quad k \in \mathbb{N}, \end{cases} \tag{7.45}$$

where $i = 1, 2, \cdots, n$,

$$\Delta u_i(t_k) = u_i(t_k) - u_i(t_k^-),\ u_i(t_k^-) = \lim_{t \to t_k^-} u_i(t),\ k \in \mathbb{N},$$

the time sequence $\{t_k\}$ satisfies $0 < t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$, and $\lim_{k\to\infty} t_k = \infty$; $C_i > 0$, $R_i > 0$, and I_i are, respectively, the capacitance, the resistance, and the external input of the ith neuron; P_{ij}, W_{ij} and P_{ijl}, W_{ijl} are, respectively, the first and second order synaptic weights of the neural networks; and $\tau_i(t)$, $(i = 1, 2, \cdots, n)$, is the transmission delay of the ith neuron such that $0 \le \tau_i(t) \le \tau$, where τ is a constant.

The initial condition for system (7.45) is given by

$$u_i(s) = \psi_i(s), \quad s \in [t_0 - \tau, t_0], \tag{7.46}$$

where $\psi_i : [t_0 - \tau, t_0] \to \mathbb{R}$, $(i = 1, 2, \cdots, n)$, is a piecewise continuous function.

We assume throughout that the neuron activation functions $g_i(u)$, $h_i(u)$, $i = 1, 2, \cdots, n$, are continuously differentiable and satisfy the following conditions:

$$|g_i(u_i)| \le M_i \text{ and } |h_i(u_i)| \le N_i, \quad \forall\, u_i \in \mathbb{R}, \tag{7.47}$$

$$0 \le \frac{g_i(u_i)-g_i(v_i)}{u_i-v_i} \le K_i \quad \text{and} \quad 0 \le \frac{h(u_i)-h(v_i)}{u_i-v_i} \le L_i, \quad \forall\, u_i \ne v_i, u_i, v_i \in \mathbb{R},\ i = 1, 2, \cdots, n. \tag{7.48}$$

Remark 7.3.1 *It follows from Theorem 1 in [122] that system (7.45) has an equilibrium point if conditions (7.47) and (7.48) hold.*

Define

$$M = [M_1, M_2, \cdots, M_n]^T, \quad N = [N_1, N_2, \cdots, N_n]^T, \tag{7.49}$$

and

$$L = \text{diag}(L_1, L_2, \cdots, L_n), \quad K = \text{diag}(K_1, K_2, \cdots, K_n). \tag{7.50}$$

Let $u^* = [u_1^*, u_2^*, \cdots, u_n^*,]^T$ be an equilibrium point of system (7.45), and set

$$x_i(t) = u_i(t) - u_i^*, \quad i = 1, 2, \cdots, n,$$

$$d_{ik}u_i^* + \sum_{j=1}^{n} W_{ij}^{(k)} h_j(u_j^*) + \sum_{j=1}^{n}\sum_{l=1}^{n} W_{ijl}^{(k)} h_j(u_j^*) h_l(u_l^*) = 0, \quad \forall k \in \mathbb{N},$$

$$f_i(x_i(t - \tau_i(t))) = g_i(u_i(t - \tau_i(t))) - g_i(u_i^*),$$

and

$$\varphi_i(x_i(t)) = h_i(u_i(t)) - h_i(u_i^*), \ i = 1, 2, \cdots, n.$$

Then, for each $i = 1, 2, \cdots, n$,

$$|f_i(z)| \leq K_i\, |z|, \ \text{ and } \ zf_i(z) \geq 0, \ \ \forall\, z \in \mathbb{R}, \tag{7.51}$$

$$|\varphi_i(z)| \leq L_i\, |z|, \ \text{ and } \ z\varphi_i(z) \geq 0, \ \forall\, z \in \mathbb{R}. \tag{7.52}$$

System (7.45) may be rewritten as follows.

$$\begin{aligned} C_i x_i'(t) &= -\tfrac{x_i(t)}{R_i} + \sum_{j=1}^{n} \left(P_{ij} + \sum_{l=1}^{n} (P_{ijl} + P_{ilj})\zeta_l \right) f_j(x_j(t - \tau_j(t))), \ t \in [t_{k-1}, t_k), \\ \Delta x_i(t_k) &= d_{ik} x_i(t_k^-) + \sum_{j=1}^{n} \left(W_{ij}^{(k)} + \sum_{l=1}^{n} (W_{ijl}^{(k)} + W_{ilj}^{(k)})\xi_l \right) \varphi_j(x_j(t_k^-)), \ k \in \mathbb{N}, \end{aligned} \tag{7.53}$$

where $i = 1, 2, \cdots, n$; ζ_l is between $g_l(u_l(t - \tau_l(t)))$ and $g_l(u_l^*)$, and ξ_l is between $h_l(u_l(t_k^-))$ and $h_l(u_l^*)$.

Define

$$C = \text{diag}(C_1, C_2, \cdots, C_n), \quad R = \text{diag}(R_1, R_2, \cdots, R_n),$$

$$D_k = \text{diag}(d_{1k}, d_{2k}, \cdots, d_{nk}), \ W^{(k)} = (W_{ij}^{(k)})_{n\times n}, \ P = (P_{ij})_{n\times n},$$

$$W_i^{(k)} = (W_{ijl}^{(k)})_{n\times n},\, P_i = (P_{ijl})_{n\times n},\ i = 1, 2, \cdots, n,$$

$$P_H = (P_1 + P_1^T,\ P_2 + P_2^T, \cdots, P_n + P_n^T)^T,$$

$$\Xi^{(k)} = (W_1^{(k)} + [W_1^{(k)}]^T, \cdots, W_n^{(k)} + [W_n^{(k)}]^T)^T,$$

$$\varphi(x(t^-)) = [\varphi_1(x_1(t^-)), \cdots, \varphi_n(x_n(t^-))]^T,$$

$$f(x(t-\tau(t))) = [f_1(x_1(t-\tau_1(t))), \cdots, f_n(x_n(t-\tau_n(t)))]^T,$$

$$\zeta = [\zeta_1, \zeta_2, \cdots, \zeta_n]^T,\ \ \Gamma = \text{diag}(\zeta, \zeta, \cdots, \zeta),\ \ \xi = [\xi_1, \xi_2, \cdots, \xi_n]^T,$$

$$\Lambda = \text{diag}(\xi, \xi, \cdots, \xi),\ \ \Delta x = [\Delta x_1, \Delta x_2, \cdots, \Delta x_n]^T,$$

$$x(t-\tau(t)) = [x_1(t-\tau_1(t)), x_2(t-\tau_2(t)), \cdots, x_n(t-\tau_n(t))]^T.$$

Then, system (7.53) may be written in the following equivalent form.

$$\begin{cases} x'(t) & = & -C^{-1}R^{-1}x(t) + C^{-1}(P + \Gamma^T P_H)f(x(t-\tau(t))),\ t \in [t_{k-1}, t_k), \\ \Delta x(t_k) & = & D_k x(t_k^-) + (W^{(k)} + \Lambda^T \Xi_k)\varphi(x(t_k^-)), \qquad k \in \mathbb{N}. \end{cases} \tag{7.54}$$

The initial condition for system (7.54) is given by

$$x(t) = \phi(t),\ t \in [t_0 - \tau, t_0], \tag{7.55}$$

where

$$\phi(t) = [\phi_1(t), \phi_2(t), \cdots, \phi_n(t)]^T,\ t \in [t_0 - \tau, t_0],$$

and

$$\phi_i(t) = \psi_i(t) - u_i^*,\ t \in [t_0 - \tau, t_0],\ i = 1, 2, \cdots, n.$$

In this section, we denote with $\|\cdot\|$ either any vector norm or the induced matrix norm. In addition, let $\|x(t)\|_{(p)} = \big(\sum_{i=1}^{n} |x_i(t)|^p\big)^{\frac{1}{p}}$ for any $x(t) \in \mathbb{R}^n$ and $p = 1, 2$.

Next, we shall apply the results in the previous chapters to obtain some sufficient conditions for the globally exponential stability of Hopfield type neural networks with time-varying delays. The theorems and Example 7.3.1 show that some Hopfield type neural networks may be exponentially stabilized by impulses.

Let $u^* = [u_1^*, u_2^*, \cdots, u_n^*]^T$ be an equilibrium point of system (7.45). Then $x = [0, 0, \cdots, 0]^T$ is an equilibrium point of system (7.53) or (7.54). To obtain the global exponential stability of the equilibrium point u^* of system (7.45), it is sufficient to establish the global exponential stability of the trivial solution of system (7.53) or (7.54).

Theorem 7.3.1 *Let* $a = \lambda_{\min}(R^{-1}C^{-1})$, $A = (a_{ij})_{n\times n}$, $a_{ij} = |P_{ij}| + \sum_{k=1}^{n} |P_{ijk} + P_{ikj}|M_k$ *for any* $i, j = 1, 2, \cdots, n$ *and* $k \in \mathbb{N}$, *and* $b_k \geq [\|I + D_k\|_{(2)} + \max_{1\leq i\leq n}\{L_i\}(\|W^{(k)}\|_{(2)} + \|\Xi_k\|_{(2)}\|N\|_{(2)})]^2,\ k \in \mathbb{N}$, *if*

(i) $a \geq \frac{1+q}{2}\mu$, *for any* $q \geq e^{\lambda\alpha}$, *where* $\lambda,\ \alpha > 0$ *are constants,* $\mu = \lambda_{\max}(B)$ *and*

$$B = \begin{bmatrix} 0 & C^{-1}AK \\ KA^TC^{-1} & 0 \end{bmatrix};$$

(ii) $\tau \leq t_k - t_{k-1} \leq \alpha$ *and* $\ln(b_k) + \lambda\alpha < -\lambda(t_{k+1} - t_k)$ *for any* $k \in \mathbb{N}$.

Then the equilibrium point u^* *of system (7.45) is globally exponentially stable with convergence rate* $\frac{\lambda}{2}$.

Proof. Choose the Lyapunov function $V(t, x(t))$ to be

$$V(t, x(t)) = \frac{1}{2}\|x(t)\|_{(2)}^2 = \frac{1}{2}\sum_{i=1}^{n} x_i^2(t).$$

Then condition (i) of Theorem 4.2.1 holds with $c_1 = c_2 = \frac{1}{2}$ and $p = 2$. For $t \neq t_k$, by computing the upper right-hand derivative of $V(t)$ along the trajectories of system (7.53), we obtain

$$\begin{aligned} D^+V(t,x(t)) &= -\sum_{i=1}^{n}\frac{x_i^2(t)}{R_iC_i}+\sum_{i=1}^{n}\sum_{j=1}^{n}\big(P_{ij}+\sum_{k=1}^{n}(P_{ijk}+P_{ikj})\zeta_k\big) \\ &\quad \times\frac{x_i(t)}{C_i}f_j(x_j(t-\tau_j(t))) \\ &\le -\min_{1\le i\le n}\big(\frac{1}{R_iC_i}\big)\sum_{i=1}^{n}x_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}\big(|P_{ij}| \\ &\quad +\sum_{k=1}^{n}|P_{ijk}+P_{ikj}|M_k\big)\frac{K_j}{C_i}|x_i(t)||x_j(t-\tau_j(t))|. \end{aligned}$$

Denote $a=\lambda_{\min}(R^{-1}C^{-1})$, $|x(t)| = \begin{bmatrix} |x_1(t)|, & |x_2(t)|, & \cdots, & |x_n(t)| \end{bmatrix}^T$ and

$$|x(t-\tau(t))| = \begin{bmatrix} |x_1(t-\tau_1(t))|, & \cdots, & |x_n(t-\tau_n(t))| \end{bmatrix}^T.$$

We rewrite the above expression in the following matrix form

$$D^+V(t,x(t))\le -a\sum_{i=1}^{n}x_i^2(t)+\frac{1}{2}\begin{bmatrix} |x(t)| \\ |x(t-\tau(t))| \end{bmatrix}^T B\begin{bmatrix} |x(t)| \\ |x(t-\tau(t))| \end{bmatrix},$$

where $B=\begin{bmatrix} 0 & C^{-1}AK \\ KA^TC^{-1} & 0 \end{bmatrix}$. Since B is symmetric, let $\mu=\lambda_{\max}(B)$. We have

$$D^+V(t,x(t))\le -2aV(t,x(t))+\mu V(t,x(t))+\mu V(t-\tau(t),x(t-\tau(t))).$$

Whenever $V(t+s,x(t+s))\le qV(t,x(t))$ for $s\in[-\tau,0]$, we obtain from condition (i) that

$$D^+V(t,x(t))\le -[2a-\mu(1+q)]V(t,x(t))\le 0,$$

which implies that condition (ii) of Theorem 4.2.1 holds.

From (7.54), we have

$$\begin{aligned} V(t_k,x) &= \tfrac{1}{2}\|x(t_k)\|_{(2)}^2=\tfrac{1}{2}\|x(t_k^-)+\Delta x(t_k^-)\|_{(2)}^2 \\ &= \tfrac{1}{2}\|(I+D_k)x(t_k^-)+(W^{(k)}+\Lambda^T\Xi_k)\varphi(x(t_k^-))\|_{(2)}^2 \\ &\le \tfrac{1}{2}(\|I+D_k\|_{(2)}\|x(t_k^-)\|_{(2)}+(\|W^{(k)}\|_{(2)}+\|\Lambda^T\|_{(2)}\|\Xi_k\|_{(2)}) \\ &\quad \times\|\varphi(x(t_k^-))\|_{(2)})^2 \\ &\le \tfrac{1}{2}[\|I+D_k\|_{(2)}+\max_{1\le i\le n}\{L_i\}(\|W^{(k)}\|_{(2)}+\|\Xi_k\|_{(2)}\|N\|_{(2)})]^2 \\ &\quad \times\|x(t_k^-)\|_{(2)}^2. \end{aligned}$$

By condition (ii) and the assumptions $\|\varphi(x(t))\|_{(2)} \leq \max_{1\leq i\leq n}\{L_i\}\|x(t)\|_{(2)}$, $\Lambda^T\Lambda = \|\xi\|_{(2)}^2 I$ and $\|\xi\|_{(2)} \leq \|N\|_{(2)}$, we obtain

$$\begin{aligned} V(t_k, x) &\leq [\|I + D_k\|_{(2)} + \max_{1\leq i\leq n}\{L_i\}(\|W^{(k)}\|_{(2)} + \|\Xi_k\|_{(2)}\|N\|_{(2)})]^2 \\ &\quad \times V(t_k^-, x) \\ &\leq b_k V(t_k^-, x), \end{aligned}$$

which, together with the condition (ii), implies that the conditions (iii) and (iv) of Theorem 4.2.1 hold, so we obtain that the trivial solution of system (7.53) or (7.54) is globally exponentially stable with convergence rate $\frac{\lambda}{2}$, i.e., the equilibrium point u^* of system (7.45) is globally exponentially stable with convergence rate $\frac{\lambda}{2}$. ■

Theorem 7.3.2 *Let* $a = \min_{1\leq i\leq n}\{\dfrac{1}{R_iC_i}\}$, $b = \max_{1\leq i\leq n}\{\sum_{j=1}^{n}(|P_{ij}| + \sum_{k=1}^{n}|P_{ijk} + P_{ikj}|M_k\frac{K_j}{C_i})\}$, *and for any* $k \in \mathbb{N}$, $b_k \geq \max_{1\leq i\leq n}\{|1 + d_{ik}|\} + \max_{1\leq i\leq n}\{\sum_{j=1}^{n}(|W_{ji}^{(k)}| + \sum_{l=1}^{n}|W_{jil}^{(k)} + W_{jli}^{(k)}|N_l)L_i\}$. *If*

(i) $a \geq qb$ *with* $q \geq e^{2\lambda\alpha}$, *where* $\lambda,\ \alpha > 0$ *are constants;*

(ii) $\tau \leq t_k - t_{k-1} \leq \alpha$ *and* $\ln(b_k) + \lambda\alpha < -\lambda(t_{k+1} - t_k)$ *for any* $k \in \mathbb{N}$.

Then the equilibrium point u^* *of system (7.45) is globally exponentially stable with convergence rate* λ.

Proof. Choose the Lyapunov function $V(t, x(t))$ to be

$$V(t, x(t)) = \|x(t)\|_{(1)} = \sum_{i=1}^{n}|x_i(t)|.$$

Then condition (i) of Theorem 4.2.1 holds with $c_1 = c_2 = p = 1$. For $t \neq t_k$, the upper right-hand derivative of $V(t)$ along the trajectories of system (7.53) is

$$\begin{aligned} D^+_{(7.53)}V(t, x(t)) &= -\sum_{i=1}^{n}\tfrac{dx_i(t)}{dt}\mathrm{sgn}(x_i(t)) \\ &= -\sum_{i=1}^{n}\tfrac{1}{R_iC_i}|x_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n}\big(P_{ij} + \sum_{k=1}^{n}(P_{ijk} + P_{ikj})\zeta_k\big) \\ &\quad \times\tfrac{f_j(x_j(t-\tau_j(t)))}{C_i}\mathrm{sgn}(x_i(t)). \end{aligned}$$

By (7.51) we have

$$\begin{aligned} D^+_{(7.53)}V(t,x(t)) &\le -\sum_{i=1}^{n}\tfrac{1}{R_iC_i}|x_i(t)| + \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(|P_{ij}| + \sum_{k=1}^{n}(|P_{ijk}+P_{ikj}|)\\ &\quad \times M_k\Big)\tfrac{K_j}{C_i}|x_j(t-\tau_j(t)|\\ &\le -\min_{1\le i\le n}\left\{\frac{1}{R_iC_i}\right\}\sum_{i=1}^{n}|x_i(t)| + \max\nolimits_{1\le i\le n}\sum_{j=1}^{n}\Big(|P_{ij}|\\ &\quad + \sum_{k=1}^{n}|P_{ijk}+P_{ikj}|M_k\Big)\tfrac{K_j}{C_i}\textstyle\sum_{j=1}^{n}|x_j(t-\tau_j(t))|. \end{aligned}$$

By condition (i), we have

$$D^+_{(7.53)}V(t,x(t)) \le -aV(t) + bV(t+s) \ \le\ -(a-bq)V(t) \le 0,$$

whenever $V(t+s,x(t+s)) \le qV(t,x(t))$ for $s \in [-\tau, 0]$. This implies that the condition (ii) of Theorem 4.2.1 holds.

By condition (ii), we have, from (7.53)

$$\begin{aligned} V(t_k,x) &= \textstyle\sum_{i=1}^{n}|x_i(t_k)| \ =\ \sum_{i=1}^{n}\big|(1+d_{ik})x_i(t_k^-) + \sum_{j=1}^{n}\big(W_{ij}^{(k)}\\ &\quad + \textstyle\sum_{l=1}^{n}(W_{ijl}^{(k)} + W_{ilj}^{(k)})\xi_l\big)\varphi_j(x_j(t_k^-))\big|\\ &\le \max\nolimits_{1\le i\le n}\{|1+d_{ik}|\}\textstyle\sum_{i=1}^{n}|x_i(t_k^-)| + \max_{1\le i\le n}\big[\sum_{j=1}^{n}\big(|W_{ji}^{(k)}|\\ &\quad + \textstyle\sum_{l=1}^{n}|W_{jil}^{(k)} + W_{jli}^{(k)}|N_l\big)L_i\big]\sum_{i=1}^{n}|x_i(t_k^-)|\\ &\le b_kV(t_k^-,x), \end{aligned}$$

which, together with condition (ii), implies that conditions (iii) and (iv) of Theorem 4.2.1 hold, so we obtain that the trivial solution of system (7.53) or (7.54) is globally exponentially stable with convergence rate λ, i.e., the equilibrium point u^* of system (7.45) is globally exponentially stable with convergence rate λ. ∎

Remark 7.3.2 *Theorems 7.3.1 and 7.3.2 also give the relation between the measure of strength of the impulse b_k and the upper bound of the impulsive interval α:*

$$b_k \ <\ e^{-2\lambda\alpha}, \quad \lambda > 0. \tag{7.56}$$

Note that 7.56 is sufficient but not necessary for the impulsive stabilization of the neural network (7.45).

Now, we discuss an example to illustrate our results.

Example 7.3.1 *Consider the following impulsive high order Hopfield type neural networks*

$$\begin{cases} u_1'(t) &= -0.2u_1(t) - \tanh(0.1u_1(t-0.5e^{-t})) \\ & \quad +0.6\tanh(0.1u_2(t-0.5)) + 0.6, \quad t \in [t_{k-1}, t_k), \\ u_2'(t) &= -0.1852u_2(t) + 0.9\tanh(0.1u_1(t-0.5)) \\ & \quad -0.9\tanh(0.1u_2(t-0.5)) + 0.067, \quad t \in [t_{k-1}, t_k), \\ \Delta u_i(k) &= d_{ik}(u_i(k^-) - u_i^*), \quad k \in \mathbb{N}. \end{cases} \tag{7.57}$$

By computation, system (7.57) has a unique equilibrium point $(u_1^*, u_2^*)^T = (2.2017, 0.9530)^T$. *In the notation of Theorem 7.3.1, we have* $C = I,\ R^{-1} = diag(0.2, 0.1852),\ g_1(u) = g_2(u) = \tanh(0.1u),\ I_1 = 0.6,\ I_2 = 0.067,\ \tau = 0.5,\ M_1 = M_2 = 1,\ K_1 = K_2 = 0.1,\ P_1 = (P_{1ij})_{2\times 2} = P_2 = (P_{2ij})_{2\times 2} = 0$, *and*

$$P = (P_{ij})_{2\times 2} = \begin{bmatrix} -1 & 0.6 \\ 0.9 & -0.9 \end{bmatrix}.$$

By simple calculation, we have $a = \lambda_{\min}(R^{-1}C^{-1}) = 0.1852$ *and*

$$B = \begin{bmatrix} 0 & 0 & 0.1 & 0.06 \\ 0 & 0 & 0.09 & 0.09 \\ 0.1 & 0.06 & 0 & 0 \\ 0.09 & 0.09 & 0 & 0 \end{bmatrix},$$

so $\mu = \lambda_{\max}\{B\} = 0.0213$. *Then*

$$a = 0.1852 > \frac{1+q}{2}\mu = 0.0215,$$

where $q = e^{0.02} = 1.0202$.

Choose $h_1(u) = -\frac{u_1^*+u_2^*}{2}$ *and* $h_2(u) = \frac{u_1^*-u_2^*}{2}$, $W_{ijl}^{(k)} = 0$, $D_k = diag(d_{1k}, d_{2k}) = diag(-0.5, -0.5)$ *and*

$$W^{(k)} = (W_{ij}^{(k)})_{2\times 2} = \begin{bmatrix} d_{1k} & -d_{1k} \\ d_{2k} & d_{2k} \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & -0.5 \end{bmatrix},$$

then $N_1 = 1.6,\ N_2 = 0.6$ *and* $L_1 = L_2 = 0$. *Let* $\alpha = 1\ \lambda = 0.01$, $t_k = k$ *for any* $k \in \mathbb{N}$. *Choose* $b_k = 0.65$, *then* $b_k \leq e^{-\lambda\tau - \lambda(t_{k+1}-t_k)} = 0.9802$ *and*

$$\begin{aligned}&[\|I+D_k\|_{(2)} + \max_{1\leq i\leq n}\{L_i\}(\|W^{(k)}\|_{(2)} + \|\Xi_k\|_{(2)}\|N\|_{(2)})]^2 \\ &\quad = 0.25 \leq b_k = 0.65.\end{aligned}$$

Then the conditions of Theorem 7.3.1 hold, and the equilibrium point u^* of system (7.57) is globally exponentially stable with convergence rate 0.005.

Moreover, the conditions of Theorem 7.3.2 also hold:

(1) $a = \min_{1\le i\le 2} \frac{1}{R_iC_i} = 5 \ \ge \ qb = 1.0202 \times 1.8$,

(2) $\max_{1\le i\le n}\{|1+d_{ik}|\} + \max_{1\le i\le n}\{\sum_{j=1}^{n}(|W_{ji}^{(k)}| + \sum_{l=1}^{n}|W_{jil}^{(k)} + W_{jli}^{(k)}|N_l)L_i\} = 0.5 \le b_k = 0.65$, $\tau = 0.5 \le t_k - t_{k-1} = 1 \le \alpha = 1$, *and* $b_k \le e^{-\lambda\tau-\lambda(t_{k+1}-t_k)} = 0.9802$.

Thus by Theorem 7.3.2, the equilibrium point u^* of system (7.57) is globally exponentially stable with convergence rate 0.01. As is apparent from the conditions laid out above, on the basis of both theorems in this section, global exponential stability is obtained for system (7.57). As predicted by the forerunning analysis, the faster convergent speed is achieved in the case of Theorem 7.3.2.

The numerical simulation of this impulsive delay differential equation with initial functions

$$\begin{aligned} \phi_1(t) &= \begin{cases} 0, & t \in [-0.5, 0), \\ -2.1, & t = 0, \end{cases} \\ \phi_2(t) &= \begin{cases} 0, & t \in [-0.5, 0), \\ 1.1, & t = 0, \end{cases} \end{aligned}$$

is given in Figure 7.8, the graph of solution of the corresponding system without impulse is given in Figure 7.9.

Remark 7.3.3 *As is shown from the above pictures, the equilibrium point $u^* = (2.2017, 0.9530)$ of system (7.57) without impulse is stable but not asymptotically stable, however, after impulsive control, the equilibrium point of this system becomes globally exponentially stable, which implies that impulse can be used to exponentially stabilize some high order Hopfield type neural networks with time-varying delays.*

The next theorem shows that to what extent impulsive perturbation Hopfield type neural networks (7.45) or (7.54) can endure without destroying the globally exponential stability property.

Theorem 7.3.3 *Assume that there exist constant $q > 1$ and matrices $\bar{P}$, D_k, $W^{(k)}$, $\Xi_k \in \mathbb{R}^{n\times n}$, $\forall k \in \mathbb{N}$ with $\bar{P} > 0$ such that*

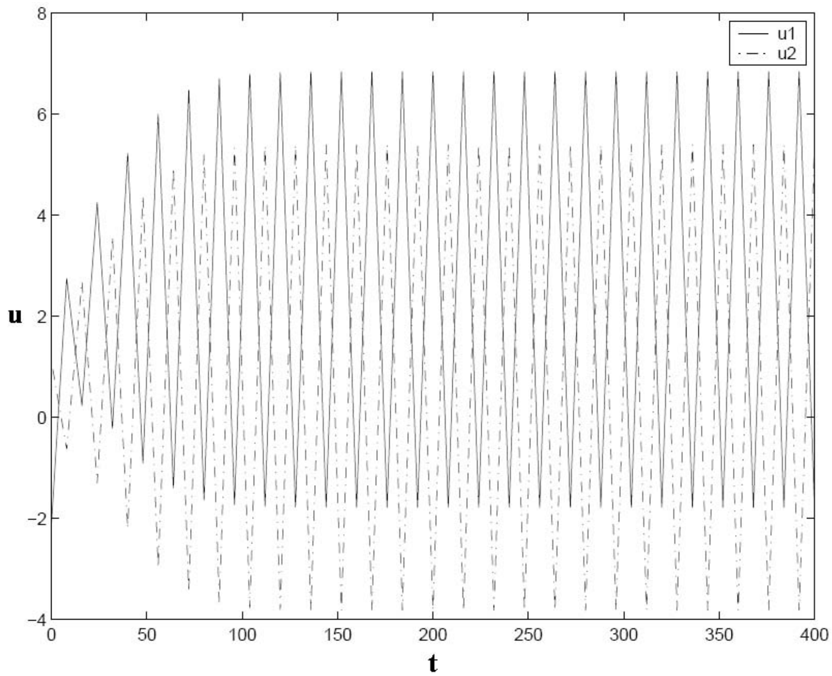

Figure 7.8: Numerical simulation of Example 7.3.1, system without impulses.

(i) $\frac{1}{\lambda_{\max}(\bar{P})} - (1+q)\frac{\max_{1\le i\le n}\{K_i\}}{\lambda_{\min}(\bar{P})}(\|\bar{P}C^{-1}P\|_{(2)} + \|\bar{P}C^{-1}\|_{(2)}\|M\|_{(2)}\|P_H\|_{(2)}) \ge \frac{\ln q}{\tau}$*;*

(ii) $\frac{\lambda_{\max}(\bar{P})}{\lambda_{\min}(\bar{P})}[\|I+D_k\|_{(2)}+\max_{1\le i\le n}\{L_i\}(\|W^{(k)}\|_{(2)}+\|\Xi_k\|_{(2)}\|N\|_{(2)})]^2 \le 1+b_k$*, where* $b_k > 0$ *for all* $k \in \mathbb{N}$ *and* $\sum_{k=1}^{\infty} b_k < \infty$*.*

Then the equilibrium point u^* *of system (7.54) is globally exponentially stable.*

Proof. Let $A = -C^{-1}R^{-1}$, A is Hurwitz, then there exists a unique positive definite symmetric matrix $\bar{P}$ such that

$$\bar{P}A + A^T\bar{P} = -I. \tag{7.58}$$

Choose the Lyapunov function $V(t,x(t)) = x^T(t)\bar{P}x(t)$, $\bar{P}$ is given by equation (7.58). For any $(t,x) \in [t_{k-1},t_k) \times \mathbb{R}^n$, we obtain that

$$\begin{aligned} D^+V(t,x) &= 2x^T(t)\bar{P}x'(t) \\ &\le -\frac{x^T(t)\bar{P}x(t)}{\lambda_{\max}(\bar{P})} + \frac{\max_{1\le i\le n}\{K_i\}}{\lambda_{\min}(\bar{P})}(\|\bar{P}C^{-1}P\|_{(2)} \\ &\quad +\|\bar{P}C^{-1}\|_{(2)}\|M\|_{(2)}\|P_H\|_{(2)})[x^T(t)\bar{P}x(t) + x^T(t-\tau)\bar{P}x(t-\tau)], \end{aligned} \tag{7.59}$$

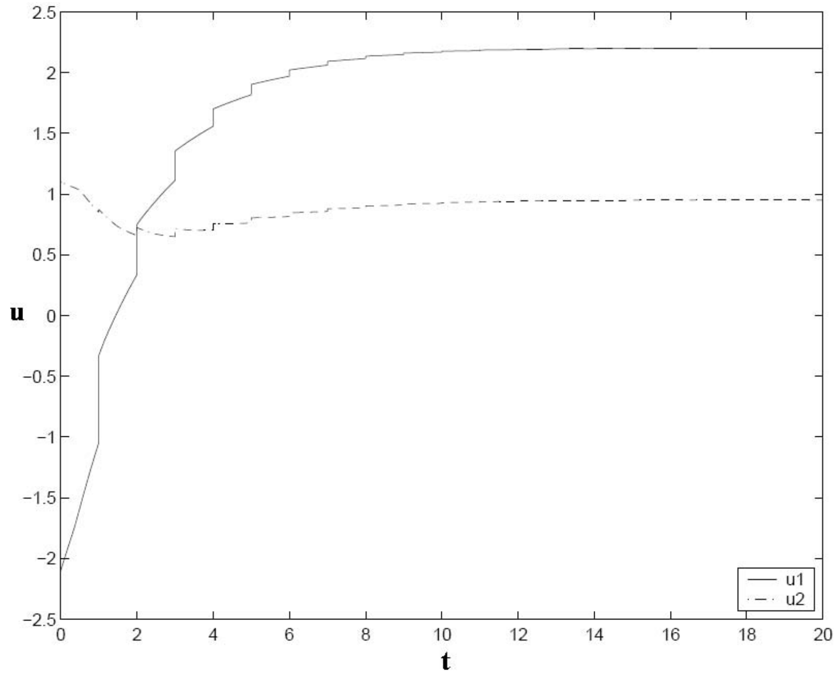

Figure 7.9: Numerical simulation of Example 7.3.1, impulsive system.

whenever $qV(t,x(t)) \geq V(t+s,x(t+s))$ for any $s \in [-\tau, 0]$, i.e., $x^T(t+s)\bar{P}x(t+s) \leq qx^T(t)\bar{P}x(t)$, we get

$$\begin{aligned} D^+V(t,x) \leq & -[\tfrac{1}{\lambda_{\max}(\bar{P})} - \tfrac{\max_{1\leq i\leq n}\{K_i\}}{\lambda_{\min}(\bar{P})}(\|\bar{P}C^{-1}P\|_{(2)} \\ & +\|\bar{P}C^{-1}\|_{(2)}\|M\|_{(2)}\|P_H\|_{(2)})(1+q)]x^T(t)\bar{P}x(t) \ \leq \ -\eta V(t,x(t)), \end{aligned} \tag{7.60}$$

where $\eta = \frac{1}{\lambda_{\max}(\bar{P})} - \frac{\max_{1\leq i\leq n}\{K_i\}}{\lambda_{\min}(\bar{P})}(\|\bar{P}C^{-1}P\|_{(2)} + \|\bar{P}C^{-1}\|_{(2)}\|M\|_{(2)}\|P_H\|_{(2)})(1+q)$. And by condition (i), $\eta \geq \frac{\ln q}{\tau}$, so the condition (ii) of Theorem 4.1.2 holds.

By condition (ii) and $\|\varphi(x(t))\|_{(2)} \leq \max_{1\leq i\leq n}\{L_i\}\|x(t)\|_{(2)}$, $\Lambda^T\Lambda = \|\xi\|^2_{(2)}I$, and $\|\xi\|_{(2)} \leq \|N\|_{(2)}$, we have

$$\begin{aligned} & V(t_k,x) = x^T(t_k)\bar{P}x(t_k) \\ & = (x(t_k^-) + \Delta x(t_k^-))^T\bar{P}(x(t_k^-) + \Delta x(t_k^-)) \\ & \leq \tfrac{\lambda_{\max}(\bar{P})}{\lambda_{\min}(\bar{P})}[\|I + D_k\|_{(2)} + \max_{1\leq i\leq n}\{L_i\} \\ & \times(\|W^{(k)}\|_{(2)} + \|\Xi_k\|_{(2)}\|N\|_{(2)})]^2V(t_k^-,x) \\ & \leq (1+b_k)V(t_k^-,x), \end{aligned}$$

i.e., condition (iii) of Theorem 4.1.2 holds with $\psi_k(s)=(1+b_k)s$ and $H=\prod_{i=1}^{\infty}(1+b_k)$. Hence from Theorem 4.1.2, we obtain that the trivial solution of system (7.54) is globally exponentially stable, i.e., the equilibrium point u^* of system (7.45) is globally exponentially stable. ■

Example 7.3.2 *Consider the following impulsive high order Hopfield type neural networks*

$$\begin{cases} C_i u_i'(t) &= -\frac{u_i(t)}{R_i} + \sum_{j=1}^n P_{ij} g_j(u_j(t-\tau_j(t))) + \sum_{j=1}^n \sum_{l=1}^n P_{ijl} \\ & \times g_j(u_j(t-\tau_j(t))) g_l(u_l(t-\tau_l(t))) + I_i, \quad t\in[t_{k-1},t_k),\ i=1,2, \\ & \Delta u_i(k) = d_{ik}u_i(k^-) + \sum_{j=1}^n W_{ij}^{(k)} h_j(u_j(k^- - \tau_j(k))) \\ & \ + \sum_{j=1}^n \sum_{l=1}^n W_{ijl}^{(k)} h_j(u_j(k^- - \tau_j(k))) h_l(u_l(t_k^- - \tau_l(t_k))),\ i=1,2,\ k\in\mathbb{N}, \end{cases} \tag{7.61}$$

where $C=$ *diag*$(2,3)$, $R=$ *diag*$(2.5,1.8)$, $g_1(u)=tanh(0.12u)$, $g_2(u)=tanh(0.4u)$, $I_1=1.2$, $I_2=0.2$ *and*

$$\begin{aligned} P &= (P_{ij})_{2\times 2} = \begin{bmatrix} -0.1 & 0.03 \\ 0.12 & -0.05 \end{bmatrix}, \\ P_1 &= (P_{1ij})_{2\times 2} = \begin{bmatrix} 0.08 & -0.1 \\ 0.1 & 0.06 \end{bmatrix}, \\ P_2 &= (P_{2ij})_{2\times 2} = \begin{bmatrix} -0.06 & 0.4 \\ -0.4 & -0.05 \end{bmatrix}, \end{aligned}$$

then $M_1=M_2=1$, $K_1=0.12$, $K_2=0.4$, *and system (7.61) has an equilibrium point* $(u_1^*,u_2^*)^T=(2.954,0.404)^T$.

By simple calculation, we have $\bar{P}=\text{diag}\{2.5,2.6998\}$, $A=-C^{-1}R^{-1}=\text{diag}\{-0.2,-0.1852\}$, and

$$P_H=(P_1+P_1^T,P_2+P_2^T)^T=\begin{bmatrix} 0.16 & 0 \\ 0 & 0.12 \\ -0.12 & 0 \\ 0 & -0.1 \end{bmatrix},$$

and $\bar{P}C^{-1}=\text{diag}\{1.25,0.8999\}$,

$$\bar{P}C^{-1}P=\begin{bmatrix} -0.125 & 0.0375 \\ 0.108 & -0.045 \end{bmatrix},$$

so $\lambda_{\max}\{\bar{P}\} = 2.6998$ and $\lambda_{\min}\{\bar{P}\} = 2.5$, $\|P_H\|_{(2)} = (\lambda_{\max}(P_H^T P_H))^{\frac{1}{2}}$, $\|\bar{P}C^{-1}\|_{(2)} = 1.25$, $\|\bar{P}C^{-1}P\|_{(2)} = 0.175$, and then

$$\frac{1}{\lambda_{\max}(\bar{P})} - \frac{\max_{1\le i\le n}\{K_i\}}{\lambda_{\min}(\bar{P})}[\|\bar{P}C^{-1}P\|_{(2)} + \|\bar{P}C^{-1}\|_{(2)}\|M\|_{(2)} \\ \times\|P_H\|_{(2)}](1+q) = 0.231 > 0.1 > \frac{\ln(1.05)}{0.5},$$

where $q = 1.05 > 1$.

Choose $h_1(u) = -\frac{u_1^*+u_2^*}{2}$ and $h_2(u) = \frac{u_1^*-u_2^*}{2}$, $W_{ijl}^{(k)} = 0$, $D_k = \text{diag}(d_{1k}, d_{2k})$ and

$$\begin{aligned} W^{(k)} &= (W_{ij}^{(k)})_{2\times 2} = \begin{bmatrix} d_{1k} & -d_{1k} \\ d_{2k} & d_{2k} \end{bmatrix} \\ &= \begin{bmatrix} -0.2 + \frac{1}{2k^2} & 0.2 - \frac{1}{2k^2} \\ -0.2 + \frac{1}{2k(k+1)} & -0.2 + \frac{1}{2k(k+1)} \end{bmatrix}, \end{aligned}$$

then $N_1 = 1.8,\ N_2 = 1.3$ and $L_1 = L_2 = 0$, and we have

$$\frac{\lambda_{\max}(\bar{P})}{\lambda_{\min}(\bar{P})}[\|I + D_k\|_{(2)} + \max_{1\le i\le n}\{L_i\}(\|W^{(k)}\|_{(2)} + \\ \|\Xi_k\|_{(2)}\|N\|_{(2)})]^2 = [\frac{2.6998}{2.5}(0.8 + \frac{1}{2k^2})]^2 \le 1 + \frac{2}{k^2}.$$

Then the conditions of Theorem 7.3.3 hold, and the equilibrium point u^* of system (7.61) is globally exponentially stable.

The numerical simulation of this impulsive delay differential equation with initial functions $\phi_1(t) = \phi_2(t) = 0$ for $t \in [-0.5, 0)$, and $\phi_1(0) = -2.1$, $\phi_2(0) = 1.1$ is given in Figure 7.10, the graph of solution of the corresponding system without impulse is given in Figure 7.11.

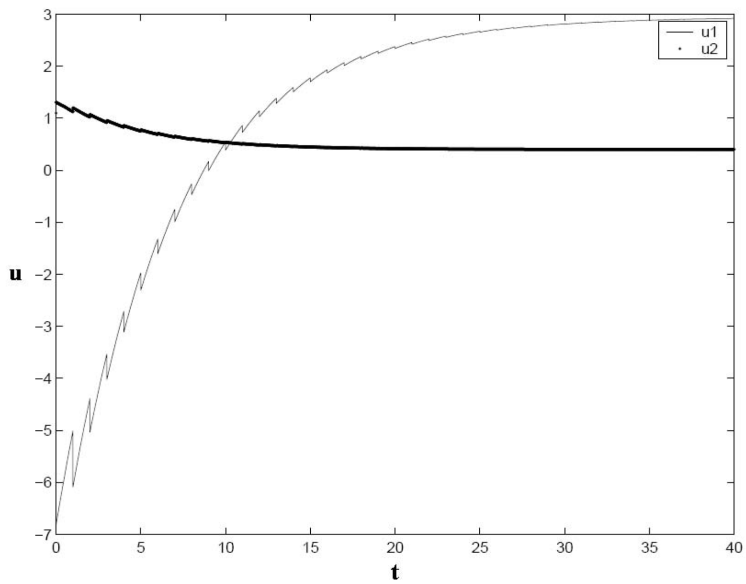

Figure 7.10: Numerical simulation of Example 7.3.2, system with impulsive disturbances.

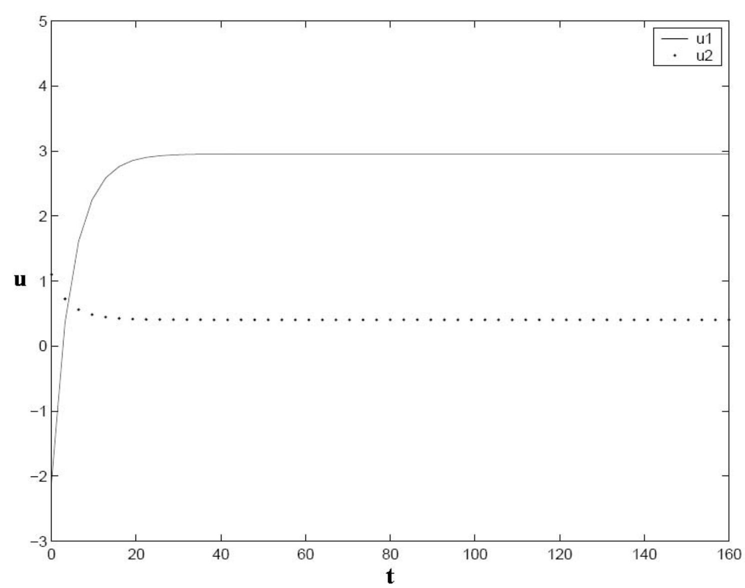

Figure 7.11: Numerical simulation of Example 7.3.2, system without impulses.

Chapter 8

Conclusions and Future Research

The stability and boundedness theories of impulsive systems with time delay have been developed in this thesis. Impulses have been treated as either perturbations or a means of controls. Conditions to maintain or to obtain the desirable stability or boundedness properties of delay differential systems have been given. Various methods, such as Lyapunov-Razumikhin technique, Lyapunov functional method, variation of parameters, and differential inequalities, have been utilized.

After introducing some important definitions, notation and fundamental results for impulsive delay differential equations in Chapter 2, the stability problem of nontrivial solutions of delay differential systems with state-dependent impulses has been solved in Chapter 3. By constructing the “reduced system”, the stability of systems with state-dependent impulses has been “transferred” to that of systems with fixed impulses.

In Chapter 4, many (global) exponential stability criteria have been established. Our results have revealed the essential role that impulses may play in stabilizing delay differential equations. Several numerical examples have been also worked out to illustrate the theorems. Stability criteria in terms of two measures have been established for impulsive delay differential equations in Chapter 5. Some results have been applied to Lotka-Volterra systems with time delay and impulsive effects.

Boundedness results have been obtained for impulsive delay differential equations with both fixed and state-dependent impulses in Chapter 6. Those results are applicable to population growth dynamics and impulsive synchronization for secure communication.

The application of stability theory to neural networks has been discussed in Chapter 7. We have applied the results and techniques in Chapter 4 to obtain (global) exponential stability of cellular neural networks and high order Hopfield type neural networks with time delays and impulsive effects. We have discussed possible effects of impulsive perturbation on stability of neural networks and have obtained some stability criteria to keep the stability property of delayed neural networks under impulsive disturbance. We have also developed some results to impulsively stabilize neural networks with time delay.

In addition to the work dedicated to the stability and boundedness theory in this thesis, there are still many interesting problems unexplored in the theory of impulsive delay differential equations and its applications.

Based on the Lyapunov method and some known results on stability and robust stability, I plan to study necessary and sufficient conditions of optimality for optimal impulsive control problems as well as the design of robust controls that will guarantee the stability and robust stability of the closed-loop system. There are basically two ways of using the Lyapunov method for impulsive control design. The first technique involves hypothesizing one form of control law and then finding a Lypapunov function to justify the choice. The second technique requires hypothesizing a Lyapunov function candidate and then finding a control law to make this candidate a real Lyapunov function. In design, one often has the freedom to deliberately modify the dynamics through designing an appropriate controller in such a way that a chosen function becomes a Lyapunov function for the "closed-loop system", so the application of the Lyapunov theory to the design of stabilizing controllers for impulsive control systems can be very rewarding.

While the Lyapunov method has been used to investigate stability in terms of two measures for impulsive delay differential equations, boundedness criteria in terms of two measures for impulsive (abstract) delay differential equations could be similarly established. The techniques from the stability investigations of impulsive systems can also be applied to switching systems, since both systems have some common properties.

Boundedness is an important property of a dynamical system. It has played a crucial role in the existence of periodic solutions. It would be interesting to work on boundedness for impulsive systems with finite and infinite time delay, and then explore further applications to population growth models. Moreover, boundedness, together with attractivity, gives rise to the concept of Lagrange stability. This concept has been used in chaos synchronization for secure communication. It would be very useful to obtain some results on Lagrange stability for impulsive delay systems and explore the applications of impulsive synchronization for secure communication based on the work that we have done on boundedness and stability of impulsive delay differential equations.

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