New Benders' Decomposition Approaches for W-CDMA Telecommunication Network Design

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.
I understand that my thesis may be made electronically available to the public.

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Abstract

Network planning is an essential phase in successfully operating state-of-the-art telecommunication systems. It helps carriers increase revenues by deploying the right technologies in a cost effective manner. More importantly, through the network planning phase, carriers determine the capital needed to build the network as well as the competitive pricing for the offered services. Through this phase, radio tower locations are selected from a pool of candidate locations so as to maximize the net revenue acquired from servicing a number of subscribers. In the Universal Mobile Telecommunication System (UMTS) which is based on the Wideband Code Division Multiple Access scheme (W-CDMA), the coverage area of each tower, called a cell, is not only affected by the signal's attenuation but is also affected by the assignment of the users to the towers. As the number of users in the system increases, interference levels increase and cell sizes decrease. This complicates the network planning problem since the capacity and coverage problems cannot be solved separately.

To identify the optimal base station locations, traffic intensity and potential locations are determined in advance, then locations of base stations are chosen so as to satisfy minimum geographical coverage and minimum quality of service levels imposed by licensing agencies. This is implemented through two types of power control mechanisms. The power based power control mechanism, which is often discussed in literature, controls the power of the transmitted signal so that the power at the receiver exceeds a given threshold. On the other hand, the signal-to-interference ratio (SIR) based power control mechanism controls the power of the transmitted signal so that the ratio of the power of the received signal over the power of the interfering signals exceeds a given threshold. Solving the SIR based UMTS/W-CDMA network planning problem helps network providers in designing efficient and cost effective network infrastructure. In contrast to the power based UMTS/W-CDMA network planning problem, the solution of the SIR based model results in higher profits. In SIR based models, the power of the transmitted signals is decreased which lowers the interference and therefore increases the capacity of the overall network. Even though the

SIR based power control mechanism is more efficient than the power based power control mechanism, it has a more complex implementation which has gained less attention in the network planning literature.

In this thesis, a non-linear mixed integer problem that models the SIR based power control system is presented. The non-linear constraints are reformulated using linear expressions and the problem is exactly solved using a Benders decomposition approach. To overcome the computational difficulties faced by Benders decomposition, two novel extensions are presented. The first extension uses the analytic center cutting plane method for the Benders master problem, in an attempt to reduce the number of times the integer Benders master problem is solved. Additionally, we describe a heuristic that uses the analytic center properties to find feasible solutions for mixed integer problems. The second extension introduces a combinatorial Benders decomposition algorithm. This algorithm may be used for solving mixed integer problems with binary variables. In contrast to the classical Benders decomposition algorithm where the master problem is a mixed integer problem and the subproblem is a linear problem, this algorithm decomposes the problem into a mixed integer master problem and a mixed integer subproblem. The subproblem is then decomposed using classical Benders decomposition, leading to a nested Benders algorithm. Valid cuts are generated at the classical Benders subproblem and are added to the combinatorial Benders master problem to enhance the performance of the algorithm.

It was found that valid cuts generated using the analytic center cutting plane method reduce the number of times the integer Benders master problem is solved and therefore reduce the computational time. It was also found that the combinatorial Benders reduces the complexity of the integer master problem by reducing the number of integer variables in it. The valid cuts generated within the nested Benders algorithm proved to be beneficial in reducing the number of times the combinatorial Benders master problem is solved and in reducing the computational time that the overall algorithm takes. Over 110 instances of the UMTS/W-CDMA network planning problem ranging from 20 demand points and 10 base stations to 140 demand points and 30 base stations are solved to optimality.

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Chapter 1

Introduction

Universal Mobile Telecommunication Systems (UMTS) are the third generation of wireless cellular network standards. A major contribution of UMTS networks is the use of Wideband Code Division Multiple Access (W-CDMA) radio transmission technology, that represents a major evolution in offering a wide range of applications that require high data rates. For instance, mobiles can theoretically be allocated connections up to 2Mbps data rates. The shift towards the use of UMTS networks is motivated by its ability to offer end user applications at low costs. By the last quarter of 2006, UMTS subscriptions reached 100 million worldwide with a growth of 3 million new subscribers each month. Japan, the first country to deploy UMTS networks in 2001, offers the service for 30 million subscribers. Most European countries started to adopt UMTS technology in 2002. By the end of 2006, the number of UMTS subscribers in Europe reached 40 million. While currently UMTS is implemented in 130 networks in more than 50 countries, most cellular operators are expected to implement UMTS systems as part of their networks by 2007 (UMTS Forum White Paper, 2006).

A critical aspect of UMTS deployment is Radio Network Planning, which seeks to determine the location of transmitters for a certain area with specific traffic demand and propagation gains so as to minimize the cost of network deployment and operation. Similar problems have been discussed for the first and second generations of cellular networks. Typically, the service area is broken to several radio cells each served by a single transmitter known as a base station. Each base station communicates to mobiles by transmitting a radio signal, thus causing interference to other base stations.

In first generation cellular networks, mobiles differentiate between their own transmitters and interfering transmitters by assigning different frequencies for each cell. This is known as frequency division multiple access (FDMA). Frequencies may be reused by different cells since signal power depletes with distance, thus similar frequencies may be assigned to different cells separated by a large enough distance. Models and algorithms for FDMA networks are discussed in Koster (1999). An extensive survey for frequency assignment in FDMA networks is provided in Murphey et al. (1999).

In second generation cellular networks each frequency channel is shared among different users by dividing it into time slots. This is known as time division multiple access (TDMA). These networks are modeled and solved as coverage problems in Hao et al. (1997), Molina et al. (1999) and Mathar and Niessen (2000). Global System for Mobile Communication (GSM) combines TDMA and FDMA to divide a 25 MHZ bandwidth among 124 carrier frequencies each divided into eight time slots. As discussed in Naghshineh and Katzela (1996) and Berruto et al. (1998), the combination of TDMA and FDMA separates the network planning problem into two phases. The first phase identifies the locations of base stations taking into account signal propagation and system capacities, while the second phase distributes the frequencies to the base stations, as done in FDMA networks. Alternatively, instead of dividing a channel to frequencies as in FDMA or to time slots as in TDMA, code division multiple access (CDMA) encodes the data transmitted on each channel by a specific code thus allowing the receiver to differentiate between data signals and interfering signals. W-CDMA upgrades CDMA to include multiple features such as frequency and time division duplex modes (FDD and TDD), as well as adaptive power control based on signal to interference ratios (SIR). UMTS is built on second generation infrastructures and mainly GSM infrastructure to include W-CDMA features.

Even though UMTS and GSM are built on similar infrastructures, network planning techniques adopted in GSM are not appropriate to UMTS network planning. The literature discussing GSM network planning proposes two solution models. The first one is a coverage problem where base stations are identified, with the objective of maximizing signal levels in selected areas. Propagation models such as the Hata Model (Hata, 1980) are often used to identify the constraints. The second is a frequency assignment problem where for each base station, a frequency is selected from a pool of frequencies with the objective of minimizing interference levels. Techniques adopted by GSM network planning are not appropriate for UMTS network planning due to a number of fundamental differences between the two standards. In UMTS, base stations' coverage areas, i.e. cell sizes, are not static but rather dependent on the amount of served traffic. For instance a phenomenon, known as "Cell Breathing Effect" occurs as traffic load changes in a cell. Cell size decreases as traffic load increases whereas it increases as traffic load decreases. This dynamic behavior of cell areas does not allow the use of coverage problems similar to those used in GSM. Another fundamental difference between GSM and UMTS is that bandwidth in UMTS is shared among all base stations (frequency reuse factor equal to one), thus no frequency assignment problem exists for such networks. UMTS network planning should not be based on coverage but also on interference levels. Interference is dependent on the power emitted by each mobile station and controlled by power control mechanisms. Signal to Interference Ratio (SIR) is used as an interference level indicator. In contrast to GSM where different frequencies are used to eliminate interference, UMTS utilizes spread spectrum where each signal is spread using a pseudo random spreading code. Using spreading codes, the power of the interference is reduced by a spreading factor (SF). Spreading codes are mutually orthogonal hence decoding at receivers allows the distinction between the interfering signals.

1.1 Literature Review

Several papers discuss network planning for third generation networks. Tutschku (1998) models network planning as a supply chain where demand point are modeled as customers with stochastic independent demands. The problem is then reduced to finding the minimum number of base stations that can cover all demand points taking into account that each demand point is covered by at least a single base station. This approach may not accurately model UMTS systems since demands are not independent (i.e. SIR for each connection is dependent on the number of other connections.). In Lee and Kang (2000), a similar modeling scheme is adopted and the resulting problem is solved by a tabu search algorithm. A different objective is provided in Sherali et al. (1996) where base stations are located so as to minimize power loss (i.e minimize distance) between the base stations and the demand points.

Recent literature models SIR constraints explicitly. In particular in Galota et al. (2001), SIR is considered however intercell interference is neglected and only intracell interference is modeled. Amaldi et al. (2002) model intracell interference as a fraction of intercell interference and prove that the SIR constraint will then constitute a maximum cell capacity. Additionally, Amaldi et al. (2002) and Amaldi et al. (2003) distinguish between two UMTS network planning models. These models differ in the power control (PC) mechanism which is used to adjust the power of the transmitted signals. Two PC mechanisms are discussed: In the power based PC mechanism, signals are transmitted with a high enough power so that the power of the received signal exceeds a given threshold value. In the SIR based PC mechanism, signals are transmitted with a high enough power so that the SIR of the received signal exceeds a given threshold value. Power control minimizes power consumption and reduces interference by keeping the power of the transmitted signals high enough to guarantee signal quality requirements, and low enough so as to minimize interference with other signals. Due its complexity, the SIR based PC mechanism has gained less attention within the scope of network planning literature. In contrast, the power based PC mechanism provides less complex models, however resulting network plans are allocated more resources than required by networks operating under SIR based PC. Amaldi et al. (2003) show that the SIR based PC model provides more efficient plans compared to the power based PC model. Furthermore, the computed SIR levels are closer to the actual values of real systems. A tabu search algorithm is used in Amaldi et al. (2003) to find feasible solutions for the two models. An evaluation of the discussed power control mechanisms may be found in Yates (1995). Recent work of Kalvenes et al. (2006) builds on the work of Amaldi et al. (2003) and solves the power based PC model in Cplex using a priority branching algorithm. Even though this paper provides solutions with a gap of less than 10% from the optimal solution, no discussion is provided for the SIR based power control mechanism. Olinick and Rosenberger (2002) extend the work of Kalvenes et al. (2002) and solve a stochastic model for the power based power control problem. Based on the stochastic model, Olinick and Rosenberger (2002) state that the SIR based power control is non-linear and "an exact solution procedure appears to be beyond the capabilities of the current state-of-the art of mathematical programming techniques".

1.2 Contributions of this thesis

In this thesis, a profit maximization model for the SIR based power control system is presented and solved. In contrast to the power based UMTS/W-CDMA network planning problem which is often discussed in literature, solving the SIR based model results in more efficient network plans. In SIR based models, the power of the transmitted signals is decreased which lowers the interference and therefore increases the capacity of the overall network.

The presented SIR based model contains non-linear constraints. These constraints are reformulated using linear expressions and the model is exactly solved using a Benders Decomposition approach. To enhance Benders decomposition, two approaches that aim at accelerating the algorithm are presented and evaluated. The first extension uses the analytic center cutting plane method (ACCPM) to generate valid cuts that aim at reducing the number of times the integer Benders master problem is solved. This is done using a two-phase ACCPM algorithm. In the first phase, the linear (LP) relaxation of the problem is solved using the analytic center cutting plane algorithm in a Benders decomposition framework. This generates a set of valid cuts that are added to the original problem which is solved in phase II using the classical Benders approach. The valid cuts reduce the number of times the integer Benders master problem is solved, and therefore reduce the total computational time. Within the scope of the two-phase ACCPM algorithm, a heuristic that uses the analytic center properties to find feasible solutions of general mixed integer problems is discussed. This work is first to use ACCPM in this fashion and can be applied to general mixed integer problems.

As a second approach, a nested Benders decomposition algorithm is introduced where a classical Benders algorithm is used within a combinatorial Benders algorithm. Combinatorial Benders decomposition is used to solve mixed integer problems with binary variables. In contrast to the classical Benders decomposition algorithm where the problem is decomposed into a mixed integer master problem and a linear subproblem, this algorithm decomposes the problem into a mixed integer master problem and a mixed integer subproblem. This aims at reducing the complexity of the master problem by reducing the number of integer variables in it. This approach reduces the computational burden of solving hard integer Benders master problems. In addition, we propose a set of valid cuts that are generated at the classical Benders subproblem and are valid to the combinatorial Benders master problem.

This thesis contributes to the solvability of the profit maximization SIR based UMTS/W-CDMA network planning problem as well as of general mixed integer programs. Two novel algorithmic ideas are used. First, an ACCPM-based Benders decomposition is proposed. Second, a nested combinatorial/classical Benders decomposition is explored, enhanced using valid cuts, and tested. The UMTS/W-CDMA network planning problem is solved using the proposed algorithms. Problems of up to 140 demand points and 30 potential base station locations are solved to optimality within 10 minutes using the nested Benders algorithm. The ACCPM-based Benders algorithm was found to reduce the number of integer master problems solved. The proposed solution methodologies were efficient in solving the UMTS/W-CDMA network planning problem. Furthermore, the nested Benders decomposition and the two-phase ACCPM algorithms are general and can be applied to solve scheduling, location optimization and assignment problems.

1.3 Structure of this thesis

Following this introductory chapter, Chapter 2 presents the model formulation of the UMTS/W-CDMA network planning problem and describes the classical Benders decomposition algorithm. In Chapter 3, the analytic center cutting plane method (ACCPM) and its extension to Benders decomposition are discussed. Chapter 4 introduces the Combinatorial Benders decomposition and its extension to Nested Benders decomposition. Finally, Chapter 5 concludes this work.

Chapter 2

Problem Formulation and Solution Methodology

2.1 Problem Formulation

Given a set of I demand points (DP) $DP_1, \ldots, DP_I$ where at each DP_i, U_i stochastic and independent number of simultaneous connections are anticipated; and a set of J potential base station (BS) locations $BS_1, \ldots, BS_J$, the UMTS/W-CDMA network planning problem seeks to determine the location of the base stations and the assignment of the demand points to base stations so as to maximize the profit acquired from servicing a number of users. A fixed profit r_i is generated from each serviced user at location DP_i, $i = 1, \ldots, I$. Each base station location is associated with a cost c_j that includes the cost of building and operating a base station at location BS_j, $j = 1, \ldots, J$. A fixed cost λ is associated with power transmission. The power of the signal transmitted between DP_i and BS_j attenuates due to many factors such as distance, obstacles, and antenna setup. These factors are captured by a propagation gain factor which is dependent on BS location relative to each DP. For each link between DP_i and BS_j, a power gain factor g_{ij} is defined. The UMTS network setup is shown in Figure 2.1.

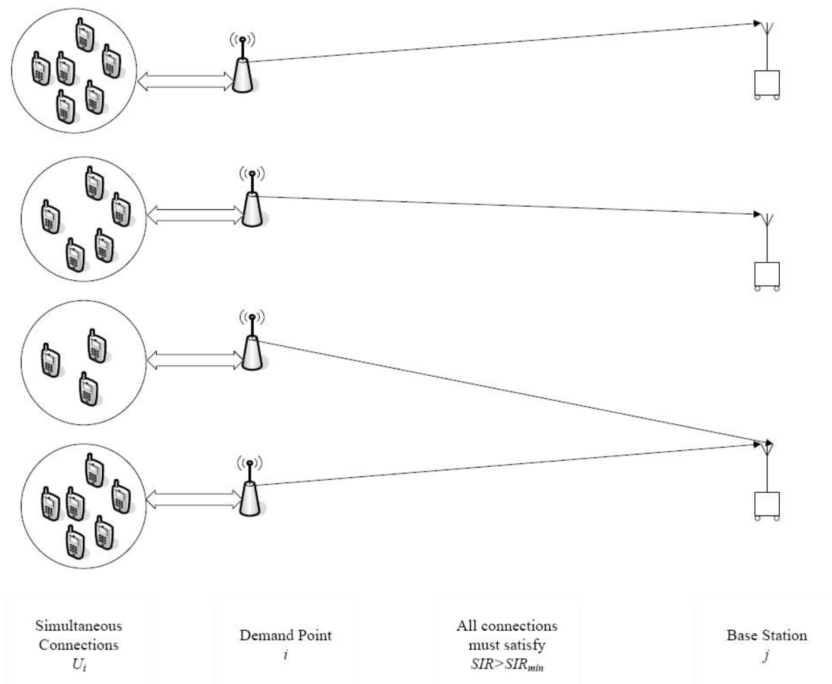

Figure 2.1: A UMTS Network.

We define the following decision variables:

$$y_j = \begin{cases} 1 & \text{if } BS_j \text{ location is selected} \\ 0 & \text{otherwise} \end{cases}$$

$$x_{ij} = \begin{cases} 1 & \text{if } DP_i \text{ is linked to } BS_j \\ 0 & \text{otherwise} \end{cases}$$

$$z_i = \begin{cases} 1 & \text{if } DP_i \text{ is linked to at least one base station} \\ 0 & \text{otherwise} \end{cases}$$

p_i: the power transmitted by each mobile station at DP_i

The presented model considers a SIR based PC mechanism. Parameter SIR_{min} specifies the target SIR level required for each signal. Therefore, given a connection between DP_i and BS_j, $SIR_{ij} \geq SIR_{min}$ should be satisfied. The SIR is given by the following equation:

$$SIR = SF \frac{P_{received}}{\alpha I_{in} + I_{out} + \eta},$$

where $P_{received}$ is the power of the received signal which is a factor of the transmitted power. Power loss is incurred due to environment and may be estimated using propagation models such as the Hata Model (Hata, 1980), and the 2 ray Model (Parsons, 1996). I_{in} is the power of interfering signals transmitted by the same BS (Intracell Interference), and I_{out} is the power of interfering signals transmitted by other BSs (Intercell Interference). η is the receiver thermal noise and $0 \leq \alpha \leq 1$ is the orthogonality loss factor. We assume that the number of spreading codes is higher than the number of available connections therefore we may consider $\alpha = 1$. We can safely take this assumption in the uplink direction (i.e. connection from DP to BS) since there would exist a very large number of orthogonal spreading codes. The power of the interference is reduced by a spreading factor (SF). Additional details are presented in Kim and Jeong (2000). Suppose a connection is established between a mobile station at location DP_i and a tower at location BS_j, a mobile station at DP_i transmits signals at a given power p_i. Due to signal attenuation, it is received by BS_j at a power $P_{received} = g_{ij}p_i$. At the same location DP_i, $U_i - 1$ other mobile stations are interfering at a power p_i each and received at BS_j with a power $(U_i - 1)g_{ij}p_i$. At each demand point location $DP_{k \neq i}$, U_k mobile stations are transmitting at a power p_k each, and received at a power $U_k g_{kj} p_k$. The total interference is $\sum_{k \neq i} U_k g_{kj} p_k$. Therefore, the SIR constraint can be formulated as:

$$SF \frac{p_i g_{ij}}{(U_i - 1)p_i g_{ij} + \sum_{k \neq i} U_k g_{kj} p_k + \eta} \geq SIR_{min} x_{ij} \quad \forall i \in I, \forall j \in J.$$

This constraint is redundant when $x_{ij} = 0$, otherwise when $x_{ij} = 1$ the SIR constraint is enforced. The UMTS/W-CDMA network planning problem is now formulated as follows:

$$[OP] : \max \sum_{i=1}^{I} r_i U_i z_i - \sum_{j=1}^{J} c_j y_j - \lambda \sum_{i=1}^{I} U_i p_i$$

$$\text{s.t. } x_{ij} - y_j \leq 0 \qquad \forall i \in I, \forall j \in J, \qquad (2.1)$$

$$SF \frac{p_i g_{ij}}{(U_i - 1)p_i g_{ij} + \sum_{k \neq i} U_k g_{kj} p_k + \eta} \geq SIR_{min} x_{ij} \qquad \forall i \in I, \forall j \in J, \qquad (2.2)$$

$$z_i - \sum_{j}^{J} x_{ij} \leq 0 \qquad \forall i \in I, \qquad (2.3)$$

$$\sum_{i}^{I} U_i z_i \geq \pi \sum_{i}^{I} U_i, \qquad (2.4)$$

$$0 \leq p_i \leq p_{max} \qquad \forall i \in I, \qquad (2.5)$$

$$z_i, y_j, x_{ij} \in \{0, 1\}.$$

Note that constraints (2.2) are non-linear. The objective maximizes the profit generated by the network. In contrast to the model provided in Kalvenes et al. (2006), we include the power cost in the objective function. Even though this complicates the model, it ensures that the optimal network plan guarantees the use of the lowest possible power levels. Constraint (2.1) ensures that DP_i cannot be linked to BS_j unless BS_j is selected. Constraint (2.2) ensures that a link between DP_i and BS_j is not valid unless it satisfies the minimum SIR condition $SIR_{ij} > SIR_{min}$. Constraints (2.3) and (2.4) ensure that at least π percent of the DP locations are covered. Parameter π is often imposed by agencies that regulate radio communication in concerned areas. Constraint (2.5) ensures that the power at which each DP is transmitting does not exceed a maximum of p_{max}.

2.2 Linearizing Constraints (2.2)

Constraints (2.2) are conditional non-linear constraints of the form

$$\text{if } x_{ij} = 1 \text{ then } SF\frac{p_i g_{ij}}{(U_i - 1)p_i g_{ij} + \sum_{k\neq i} U_k g_{kj} p_k + \eta} \geq SIR_{min}.$$

This constraint may be linearized through a big-M coefficient M_{ij} as follows.

$$SFp_i g_{ij} - [(U_i - 1)p_i g_{ij} + \sum_{k\neq i} U_k g_{kj} p_k + \eta]SIR_{min} \geq (x_{ij} - 1)M_{ij}. \tag{2.6}$$

The use of big-M is often not desirable. As discussed later, Benders decomposition eliminates the big-M coefficient from constraints (2.2) in the subproblem, however the resulting Benders cuts still depend on the big-M values. Choosing a sufficiently large big-M constant in a mixed integer problem affects the LP relaxation. In fact the strongest LP relaxation will result from choosing the smallest possible big-M value. In equation (2.6), a good value for the big-M may be selected as follows:

If $SFp_i g_{ij} > (U_i - 1)p_i g_{ij} SIR_{min}$ i.e. $SF > (U_i - 1)SIR_{min}$,
$SFp_i g_{ij} - [(U_i - 1)p_i g_{ij} + \sum_{k\neq i} U_k g_{kj} p_k + \eta]SIR_{min}$ attains its minimum at $p_i = 0$ and $p_k = p_{max}\, \forall k \neq i$. Thus, a good value for M_{ij} would be $(\sum_{k\neq i} U_k g_{kj} p_{max} + \eta)SIR_{min}$.

If $SFp_i g_{ij} < (U_i - 1)p_i g_{ij} SIR_{min}$ i.e. $SF < (U_i - 1)SIR_{min}$,

$SFp_ig_{ij} - [(U_i - 1)p_ig_{ij} + \sum_{k \neq i} U_k g_{kj} p_k + \eta]SIR_{min}$ attains its minimum at $p_i = p_{max}\, \forall i$. Thus, a good value for M_{ij} is $-SFp_{max}g_{ij} + [(U_i - 1)p_{max}g_{ij} + \sum_{k \neq i} U_k g_{kj} p_{max} + \eta]SIR_{min}$.

In the following section, we provide an exposition of Benders decomposition. Details on applying Benders decomposition to solve the UMTS/W-CDMA network planning problem are then provided.

2.3 Benders Decomposition

A number of optimization problems take the following general structure:

$$\min\ c^T x + d^T y$$

$$\text{s.t. } A_1 y \leq b_1, \tag{2.7}$$

$$A_2 x + My \geq b_2, \tag{2.8}$$

$$y \geq 0 \text{ and integer}, \tag{2.9}$$

$$x \geq 0, \tag{2.10}$$

which can be exploited through Benders decomposition (Benders, 1962; Geoffrion, 1972). By projecting problem (2.7)-(2.10) on the space defined by the y variables only, the resulting problem is:

$$\begin{aligned} &\min\ d^T y + \min\{c^T x | A_2 x \geq b_2 - My, x \geq 0\} \\ &\text{s.t. } A_1 y \leq b_1, \\ &\quad\ y \geq 0 \text{ and integer.} \end{aligned} \tag{2.11}$$

The inner minimization problem is rewritten as the dual maximization problem as follows:

$$\begin{aligned} &\max\ (b_2 - My)\lambda \\ &\text{s.t. } A_2^T \lambda \leq c, \\ &\quad\ \lambda \geq 0. \end{aligned}$$

Let H^P and H^R be the sets of extreme points and extreme rays of the set $\{\lambda | A_2^T \lambda \le c, \lambda \ge 0\}$. The problem formulated in (2.11) is equivalent to

$$\begin{aligned}
&\min\ d^T y + \theta \\
&\text{s.t. } A_1 y \le b_1, && (2.12) \\
&\quad (b_2 - My)\lambda \le \theta \quad \lambda \in H^P, && (2.13) \\
&\quad (b_2 - My)\mu \le 0 \quad \mu \in H^R, && (2.14) \\
&\quad y \ge 0 \text{ and integer.} && (2.15)
\end{aligned}$$

Starting with an initial empty set of extreme rays and extreme points, the cutting plane algorithm solves the relaxed IP master problem

$$\begin{aligned}
&\min\ d^T y \\
&\text{s.t. } A_1 y \le b_1, \\
&\quad y \ge 0 \text{ and integer.}
\end{aligned}$$

With fixed y values (obtained from the relaxed master problem), optimality cuts are generated from the extreme points of the subproblem. This can be done by solving the primal subproblem

$$\begin{aligned}
&\min\ c^T x \\
&\text{s.t. } A_2 x \ge b_2 - My, \\
&\quad x \ge 0,
\end{aligned}$$

or equivalently, its dual

$$\begin{aligned}
&\max\ (b_2 - My)\lambda \\
&\text{s.t. } A_2^T \lambda \le c, && (2.16) \\
&\quad \lambda \ge 0.
\end{aligned}$$

If the primal subproblem is infeasible or equivalently the dual subproblem is unbounded, then a feasibility cut is generated from the dual extreme ray of unboundedness. This can be found by solving the auxiliary subproblem (see Bazaraa and Jarvis, 1977)

$$
\begin{aligned}
& \max\ 0 \\
& \text{s.t.}\ \ (b_2 - My)\lambda = 1, && (2.17)\\
& \qquad A_2^T \lambda \leq 0, && (2.18)\\
& \qquad \lambda \geq 0.
\end{aligned}
$$

Generated cuts are appended to the master problem and the algorithm reiterates until an optimal solution is found.

2.4 Solving the UMTS/W-CDMA network planning problem using Benders Decomposition

The formulation of the UMTS base station location optimization problem presented in (2.1) - (2.5) falls within the generic structure of problem (2.7)-(2.10). As described earlier, this structure can be exploited using Benders decomposition. In particular, the fact that the problem presented in (2.1) - (2.5) in the x, y, and z variables alone is relatively easy to solve, motivates the use of Benders decomposition.

For fixed x_{ij}, y_j, and z_i, problem [OP] reduces to the primal subproblem:

$$
\begin{aligned}
&[\text{PSP}]\!: Z_{PSP} = \min\ \sum_{i}^{I} U_i p_i \\
&\quad \text{s.t.}\ \ p_i \leq p_{max} && \forall i, \quad (2.19)\\
&\qquad SF p_i g_{ij} - [(U_i - 1)p_i g_{ij} + \sum_{k \neq i} U_k g_{kj} p_k + \eta] SIR_{min} \geq (x_{ij} - 1)M_{ij} && \forall i, \forall j, \quad (2.20)\\
&\qquad p_i \geq 0 && \forall i, \quad (2.21)
\end{aligned}
$$

whose dual is:

$$
\begin{aligned}
\text{[DSP]: } & Z_{DSP} = \max\, p_{max} \sum_{i}^{I} \alpha_i + \sum_{i}^{I} \sum_{j}^{J} [(\overline{x}_{ij} - 1) M_{ij} + \eta SIR_{min}] \beta_{ij} \\
\text{s.t. } & \alpha_i + \sum_{j}^{J} \beta_{ij} (SFgij - (U_i - 1) g_{ij} SIR_{min}) - SIR_{min} \sum_{k \neq i} \sum_{j} U_k g_{kj} \beta_{kj} \leq U_i \qquad \forall i, \\
& \alpha_i \leq 0, \beta_{ij} \geq 0 \qquad \forall i, \forall j,
\end{aligned}
$$

which is equivalent to:

$$
Z_{DSP} = \max_{h \in H^P} \{ p_{max} \sum_{i}^{I} \alpha_i^h + \sum_{i}^{I} \sum_{j}^{J} [(\overline{x}_{ij} - 1) M_{ij} + \eta SIR_{min}] \beta_{ij}^h \}
$$

where $[\alpha_i^h, \beta_{ij}^h]$, $h \in H^P$, are the extreme points of the set:

$$
U = \left\{ \begin{array}{ll} (\alpha, \beta) : \alpha_i + \sum_j \beta_{ij} (SFgij - (U_i - 1) g_{ij} SIR_{min}) - SIR_{min} \sum_{k \neq i} \sum_j U_k g_{kj} \beta_{kj} \leq U_i & \forall i \\ \alpha_i \leq 0, \beta_{ij} \geq 0 & \forall i, \forall j \end{array} \right\}.
$$

If the set U is bounded, we generate an optimality cut of the form

$$
\theta - \sum_{i}^{I} \sum_{j}^{J} M_{ij} \beta_{ij}^h x_{ij} \geq p_{max} \sum_{i}^{I} \alpha_i^h + \sum_{i}^{I} \sum_{j}^{J} (\eta SIR_{min} - M_{ij}) \beta_{ij}^h, \quad h \in H^P \tag{2.22}
$$

where α_i^h and β_{ij}^h are the extreme points of [DSP].
If U is unbounded, we generate a feasibility cut of the form

$$
- \sum_{i}^{I} \sum_{j}^{J} M_{ij} \beta_{ij}^h x_{ij} \geq p_{max} \sum_{i}^{I} \alpha_i^h + \sum_{i}^{I} \sum_{j}^{J} (\eta SIR_{min} - M_{ij}) \beta_{ij}^h, \quad h \in H^R \tag{2.23}
$$

where α_i^h and β_{ij}^h are the extreme rays of [DSP].
The values of α_i^h and β_{ij}^h of the extreme ray are generated by solving the auxiliary sub-

problem

$$
\begin{aligned}
\text{[ASP]: } & \max\ 0 \\
\text{s.t. } & p_{max}\sum_{i}^{I}\alpha_i + \sum_{i}^{I}\sum_{j}^{J}[(\overline{x}_{ij}-1)M_{ij} + \eta SIR_{min}]\beta_{ij} = 1, \\
& \alpha_i + \sum_{j}^{J}\beta_{ij}(SFgij - (U_i-1)g_{ij}SIR_{min}) - SIR_{min}\sum_{k\neq i}\sum_{j} U_k g_{kj}\beta_{kj} \leq 0 \quad \forall i, \\
& \alpha_i \leq 0, \beta_{ij} \geq 0 \quad \forall i, \quad \forall j.
\end{aligned}
$$

Note that since x_{ij} is fixed in the subproblem, constraint (2.20) may be linearized as follows:

$$
p_i(SFg_{ij} - (U_i-1)g_{ij}SIR_{min}\overline{x}_{ij}) - SIR_{min}\overline{x}_{ij}\sum_{k\neq i} U_k g_{kj} p_k \geq \eta SIR_{min}\overline{x}_{ij} \quad \forall i, \forall j \quad (2.24)
$$

Replacing equation (2.20) by equation (2.24), the primal subproblem can be rewritten as:

$$
\begin{aligned}
\text{[PSP]: } & \min\ \sum_{i}^{I} U_i p_i \\
\text{s.t. } & p_i \leq p_{max} \quad \forall i, \\
& p_i(SFg_{ij} - (U_i-1)g_{ij}SIR_{min}\overline{x}_{ij}) - SIR_{min}\overline{x}_{ij}\sum_{k\neq i} U_k g_{kj} p_k \geq \eta SIR_{min}\overline{x}_{ij} \quad \forall i, \forall j, \\
& p_i \geq 0 \quad \forall i,
\end{aligned}
$$

whose dual is:

$$
\text{[DSP]: } \max\ p_{max}\sum_{i}^{I}\alpha_i + \sum_{i}^{I}\sum_{j}^{J}\eta SIR_{min}\overline{x}_{ij}\beta_{ij}
$$

$$
\text{s.t. } \alpha_i + \sum_{j}^{J}\beta_{ij}(SFgij - (U_i-1)g_{ij}SIR_{min}\overline{x}_{ij}) - SIR_{min}\sum_{k\neq i}\sum_{j} U_k g_{kj}\overline{x}_{ij}\beta_{kj} \leq U_i \ \forall i, \quad (2.25)
$$

$$
\alpha_i \leq 0, \beta_{ij} \geq 0 \quad \forall i, \forall j, \quad (2.26)
$$

and the auxiliary subproblem is:

$$
\begin{aligned}
&\text{[ASP]: } \max 0 \\
&\text{s.t. } p_{max}\sum_{i}^{I}\alpha_i + \sum_{i}^{I}\sum_{j}^{J}\eta SIR_{min}\overline{x}_{ij}\beta_{ij} = 1, \\
&\alpha_i + \sum_{j}^{J}\beta_{ij}(SFgij - (U_i - 1)g_{ij}SIR_{min}) - SIR_{min}\sum_{k\neq i}\sum_{j}U_k g_{kj}\beta_{kj} \leq 0 \quad \forall i, \\
&\alpha_i \leq 0, \beta_{ij} \geq 0 \quad \forall i, \quad \forall j.
\end{aligned}
$$

Note that the new [PSP], [DSP] and [ASP] formulations do not depend on the big-M. This eliminates any computational complexities resulting from the use of large big-M values at the level of the subproblem.

The Benders master problem is:

$$
\text{[MP]} := \max \sum_{i}^{I} r_i U_i z_i - \sum_{j}^{J} c_j y_j - \lambda\theta
$$

$$
\text{s.t. } \theta - \sum_{i}^{I}\sum_{j}^{J} M_{ij}\beta_{ij}^{h}x_{ij} \geq p_{max}\sum_{i}^{I}\alpha_i^h + \sum_{i}^{I}\sum_{j}^{J}(\eta SIR_{min} - M_{ij})\beta_{ij}^{h} \quad \forall h \in H^P \tag{2.27}
$$

$$
-\sum_{i}^{I}\sum_{j}^{J} M_{ij}\beta_{ij}^{h}x_{ij} \geq p_{max}\sum_{i}^{I}\alpha_i^h + \sum_{i}^{I}\sum_{j}^{J}(\eta SIR_{min} - M_{ij})\beta_{ij}^{h} \quad \forall h \in H^R \tag{2.28}
$$

$$
x_{ij} - y_j \leq 0 \quad \forall i, \forall j \tag{2.29}
$$

$$
z_i - \sum_{j}^{J} x_{ij} \leq 0 \quad \forall i \tag{2.30}
$$

$$
\sum_{i}^{I} U_i z_i \geq \pi \sum_{i}^{I} U_i \tag{2.31}
$$

$$
z_i, y_j, x_{ij} \in \{0, 1\} \tag{2.32}
$$

[MP] is a relaxation of [OP]. Therefore, solving [MP] yields an upper bound (UB) on the optimal solution of [OP]. Furthermore, starting with $\{\overline{x}_{ij}, \overline{y}_j, \overline{z}_i\}$ and solving [DSP] for

$\{\overline{p}_i\}$, forms a feasible solution to [OP]. Since [OP] is a maximization problem, the objective function evaluated at the point $\{\overline{x}_{ij}, \overline{y}_j, \overline{z}_i, \overline{p}_i\}$ gives a lower bound (LB). The upper and lower bounds act as a stopping criteria for the cutting plane algorithm. A sketch of the Benders algorithm applied to [OP] follows:

1. Start with UB $= \infty$; LB $=-\infty$. h is the iterations count, $h = 1$;

While LB $\neq$ UB

2. Solve [MP] and get solution $\{\overline{x}_{ij}, \overline{y}_j, \overline{z}_i\}$, and an upper bound UB_h.
3. Set $UB = UB_h$
4. Solve [DSP]
   - 4.1 If [DSP] is unbounded
     - 4.1.1 add feasibility cut (2.23)
   - 4.2 If [DSP] is bounded, get solution $\{\overline{p}_i\}$
     - 4.2.1 add optimality cut (2.22)
     - 4.2.2 Get LB_h from the objective function of [OP] evaluated at $\{\overline{x}_{ij}, \overline{y}_j, \overline{z}_i, \overline{p}_i\}$
     - 4.2.3 Update lower bound $LB = \max(LB, LB_h)$

End while

The classical Benders decomposition algorithm suffers from computational drawbacks due to solving an integer master problem at every iteration. Furthermore, the integer master problem gets harder since at each iteration new cuts are added. In the following chapter, a new method to accelerate Benders decomposition is introduced. This method is formed of two phases. In the first phase the LP relaxation of the problem is solved using the analytic center cutting plane method. This generates valid inequalities to be used in the second phase where the MIP problem is solved. These valid inequalities are expected to reduce the number of iterations therefore reducing the overall computational time.

Chapter 3

Accelerating Benders Decomposition using Central Cuts

3.1 Literature Review

The Benders decomposition algorithm suffers from a major computational bottleneck since the master problem, which is solved repeatedly, is an integer problem. Even when the master problem is a linear problem, the algorithm suffers from slow convergence. Nemhauser and Widhelm (1971) show that finding the geometric center of the linear master problem may be beneficial. However since finding the geometric center is a hard problem in itself, Marsten (1975) suggests adding box constraints to the master problem so that the solution of the master problem is restricted within a box centered near the previous solution. Geoffrion and Graves (1974) discuss the effect of the problem formulation on improving the computational efficiency due to the tighter linear relaxation (LP) within the branch and bound algorithm. Mevert (1977) illustrates the effect of adding an initial set of valid cuts to the master problem to tighten the feasible region of the master problem. In the same context, McDaniel and Devine (1977) propose the use of cuts generated from LP relaxation of the master problem. Magnanti and Wong (1981) exploit the structure of the subproblem in which multiple optimal solutions exist. In such a case, Magnanti and Wong (1981) use the optimal solution that generates the deepest cut. This cut is identified as a pareto-optimal cut. Cote and Laughton (1984) suggest that instead of solving the master problem to optimality, heuristics may be used to find an integer feasible solution in order to generate feasibility cuts. More recently, Rei et al. (2006) describe a local branching approach to be used within Benders decomposition to improve the lower and upper bounds at each iteration.

In this chapter, we propose a new approach that generates Benders' cuts from the analytic center of the master problem. Compared to Kelley's cutting plane method (Kelley, 1960) where the solution suggested by the master problem is its optimal solution, the analytic center cutting plane method proposes a central interior point of the feasible region of the master problem. This approach is known as the analytic center cutting plane method (ACCPM) (Goffin et al. (1992)). Accelerating Benders Decomposition via central cuts intersects with the various methods that were discussed in literature. Within the scope of the work of Nemhauser and Widhelm (1971), the analytic center often lies near the geometric center however is less computationally expensive to find. Since the master problem is an integer problem, the analytic center is generated from the LP relaxation of the master problem and valid cuts are generated similar to Mevert (1977) and McDaniel and Devine (1977). Additionally, heuristics may be used to find feasible integer points using the analytic center. These integer points generate valid cuts similar to Cote and Laughton (1984). The following section describes the analytic center cutting plane method.

3.2 ACCPM

Consider an optimization problem of the following general form:

$$
\begin{aligned}
[FMP] : \max\ & b^T y \\
\text{s.t. }\ & y \in Y = \{y : A^T y \leq c\}.
\end{aligned}
$$

A cutting plane method starts with an initial relaxation

$$
\begin{aligned}
[RMP] : \max\ & b^T y \\
\text{s.t. }\ & y \in \overline{Y},
\end{aligned}
$$

and chooses a feasible point $\overline{y}_i \in \overline{Y}$. A separation oracle either correctly asserts that $\overline{y}_i \in Y$ and generates an optimality cut of the form $a_i^T y \leq c_i$ or generates a feasibility cut that eliminates $\overline{y}$ from the set $\overline{Y}$, hence forming a tighter relaxation of the master problem.
If $\overline{y}_i$ is a feasible point of Y then $\theta_l = b^T \overline{y}$ is a lower bound on the optimal value of [FMP]. Furthermore by relaxation, any dual feasible point of [RMP] gives an upper bound θ_u to [FMP]. To see this, we consider the relaxed master problem $\{\max b^T y : A^T y \leq c\}$ and its corresponding dual $\{\min c^T x : Ax = b, x \geq 0\}$. Let $\overline{x}$ be a dual feasible point, x^* be the optimal dual solution and y^* be the optimal primal solution, then

$$\theta_u = c^T \overline{x} \geq c^T x^* = b^T y^*. \tag{3.1}$$

The upper and lower bounds are used as a stopping criteria for the cutting plane algorithm which is shown in Figure 3.1.

The analytic center is the point that maximizes the distance from the boundaries of the localization set or equivalently the logarithm of the product of the distances. The weighted analytic center adds a weight on a particular constraint to push the analytic center away from that constraint. In fact, Goffin and Vial (1993) showed that repeating a constraint is

Initialization Start with a polyhedron $Y_0 = \{A_0^T y \leq c_0 : l \leq y \leq u\}$. The constraints $l \leq y \leq u$ are added to ensure the boundedness of the polyhedron Y_0. The iteration count i is initialized to 0.

Iteration While $|\theta_u - \theta_l| > \varepsilon$

1. Compute the analytic center y_i^a of Y_i.
2. Generate cut $a_i^T y \leq c_i$ and a lower bound θ_l^i.
3. Update the lower bound $\theta_l = \max(\theta_l, \theta_l^i)$.
4. Update the polyhedron Y_i by adding the new cut $a_i^T y \leq c_i$ to form Y_{i+1}.
5. Update upper bound $\theta_u = \min(\theta_u, \theta_u^i)$.

End while

Figure 3.1: Analytic center cutting plane method.

equivalent to setting a weight on its corresponding slack in the potential function. Therefore, the weighted analytic center is the point that maximizes the weighted potential function given by:

$$\begin{aligned} \max \;\; & \ln v_0(b^T y - \theta_l) + \sum_{i=1}^{m} v_i \ln s_i \\ \text{s.t.} \;\; & A^T y + s = c, \\ & s > 0. \end{aligned} \tag{3.2}$$

Usually, a weight equal to the number of constraints is given to the bound constraint $b^T y \geq \theta_l$. This will force the analytic center away from the lower bound.

The first order optimality conditions for problem (3.2) are:

$$\widetilde{S}\widetilde{x} = \nu \tag{3.3}$$

$$\widetilde{A}\widetilde{x} = 0, \quad \widetilde{x} > 0 \tag{3.4}$$

$$\widetilde{A}^T y + \widetilde{s} = \widetilde{c}, \quad \widetilde{s} > 0 \tag{3.5}$$

where $\widetilde{x} = [x_0, x]^T, \widetilde{c} = [-\theta_l, c]^T, \widetilde{s} = [s_0, s]^T, s_0 = b^T y - \theta_l, \widetilde{A} = [-b, A]$ and $\nu = [m, 1, 1, \ldots, 1]^T$. If a primal feasible solution $\widetilde{x} > 0$ is available, then a primal Newton method is used. If a dual feasible solution $\widetilde{s} > 0$ is available, then a dual Newton method is used. If both a primal and a dual feasible points are available, then a primal-dual Newton method is used. Details of these methods are provided in Ye (1997) and Elhedhli and Goffin (2004).

3.2.1 Integer Analytic Centers

The analytic center presented in Section 3.2 assumes a continuous problem while the feasible region of the Benders master problem is defined over discrete points only. In this setting, the integer analytic center is defined as the closest discrete point to the continuous analytic center. Consider a relaxed Benders master problem of the following form:

$$\max \{b^T y : A^T y \leq c, y \text{ integer}\}.$$

To find the integer analytic center of the set $Y = \{A^T y \leq c, y \text{ integer}\}$, we first start by finding the continuous analytic center y^{ac} of the LP relaxation $Y_{LP} = \{A^T y \leq c\}$. To find the closest integer point to y^{ac}, a minimum distance problem of the following form is solved

$$\begin{aligned} \min \ & ||y - y^{ac}|| \\ \text{s.t. } & Ay \leq c, \\ & y \text{ integer.} \end{aligned} \tag{3.6}$$

Note that if the $\mathcal{L}_1$-norm is used to compute the distance $||y - y^{ac}||$ then problem (3.6) is a linear integer problem. This method was used in Atlason et al. (2004) in a simulation based ACCPM. Results showed that solving problem (3.6) is computationally expensive and it is advisable to use a heuristic to approximate the integer analytic center from the continuous one. This is detailed in Section 3.4.1.

In the following section we present a cutting plane algorithm formed of two phases. In the first phase, ACCPM is used to solve the LP relaxed problem. At each iteration, a heuristic is used to approximate the integer analytic center which is used to generate cuts that are valid to the original integer problem. In the second phase, Kelley's cutting plane method is used to solve the integer problem, warm started with the valid cuts generated in phase I.

3.3 A Two-Phase ACCPM Algorithm

This section describes the two-phase ACCPM algorithm for solving mixed integer problems taking the form (2.7)-(2.10). In the first phase of the algorithm, Benders decomposition is used to solve the LP relaxation:

$$\begin{aligned}
\min\ & c^T x + d^T y \\
\text{s.t.}\ & A_1 y \le b_1, \\
& A_2 x + M y \ge b_2, \\
& y \ge 0, \\
& x \ge 0.
\end{aligned}$$

The solution of the LP relaxed master problem, which may not necessarily be integer, is used to solve the subproblem and generate a new constraint. Since the IP region is contained in the LP region, then all the cuts that are valid to the LP problem are valid to the IP problem. In addition to the cuts that are generated from non-integer points, heuristics may be used to find integer points that are feasible to the IP master problem. These integer points are used to solve the subproblem and generate new constraints that are valid to the IP problem. Therefore by solving the LP relaxed problem, it is expected that a good number of cuts that are valid to the IP problem would be generated. These cuts are appended to the IP problem which is solved in phase II. Then by using Benders decomposition to solve the IP problem, the appended valid cuts are expected to reduce the number of iterations in which an integer master problem is solved.

ACCPM is used to solve the LP problem of phase I. The analytic center of the master problem is used to generate cuts from the subproblem. Let y^{ac}_{LP} be the continuous analytic center of the polyhedron $Y_1 = \{A_1 y \leq b_1\}$. The integer analytic center y^{ac}_{IP} is defined as the closest integer point to y^{ac}_{LP}. The integer analytic center can be either found by solving the minimum distance problem (3.6) or can be estimated using heuristics as detailed in section 3.4.1. The IP analytic center is used to generate cuts from the subproblem. These cuts are only valid to the IP master problem. We refer to LP central cuts as the feasibility and optimality cuts generated from the LP analytic center whereas the IP central cuts are the feasibility and optimality cuts generated from the IP analytic center. The two-phase ACCPM algorithm is described below:

Phase-I:

Initialization Initialize θ_l and θ_u to the initial lower and upper bounds, and choose a stopping parameter ϵ.

Iteration while $|\theta_u - \theta_l| > \epsilon$

1. Compute the LP analytic center of the LP relaxed master problem
2. Use the LP analytic center to solve the subproblem and generate LP cuts
3. Append the LP cuts to the LP relaxed master problem and to the IP master problem
4. Use the LP analytic center to compute the IP analytic center

5. Use the IP analytic center to solve the subproblem and generate IP cuts
6. Append the IP cuts to the IP master problem
7. Update upper and lower bounds

End while

Phase-II:

Initialization Initialize θ_l and θ_u to the initial lower and upper bounds, and choose a stopping parameter ϵ. Append all the central cuts generated from Phase-I to the IP master problem

Iteration while $|\theta_u - \theta_l| > \epsilon$

1. Solve the IP master problem
2. Use the solution of the master problem to solve the subproblem and generate a cut
3. Append the generated cut to the master problem
4. Update upper and lower bounds

End while

In the following section, the UMTS/W-CDMA network problem is solved using the two-phase ACCPM. Furthermore, we discuss how the upper and lower bounds are obtained in each phase.

3.4 Solving the UMTS/W-CDMA network planning problem via the two-phase ACCPM

As detailed in section 2.3, the UMTS/W-CDMA network planning problem can be solved iteratively through Benders decomposition where the master problem (2.27)-(2.32) and the dual subproblem (2.25)-(2.26) are solved iteratively. As detailed in section 3.3, Phase I of the algorithm solves the LP relaxation of the problem:

$$[\text{OP-LP}] : \max \sum_i^I r_i U_i z_i - \sum_j^J c_j y_j - \lambda \sum_i^I U_i p_i$$

$$\begin{aligned}
\text{s.t. } & x_{ij} - y_j \leq 0 && \forall i, \forall j, \\
& SF p_i g_{ij} - [(U_i - 1) p_i g_{ij} + \sum_{k \neq i} U_k g_{kj} p_k + \eta] SIR_{min} \geq (x_{ij} - 1) M_{ij} && \forall i, \forall j, \\
& z_i - \sum_j^J x_{ij} \leq 0 && \forall i, \\
& \sum_i^I U_i z_i \geq \pi \sum_i^I U_i, && \\
& 0 \leq p_i \leq p_{max} && \forall i, \\
& 0 \leq z_i \leq 1, 0 \leq y_j \leq 1, 0 \leq x_{ij} \leq 1 && \forall i, \forall j,
\end{aligned}$$

where the LP master problem is:

$$[\text{MP-LP}] : \max \sum_i^I r_i U_i z_i - \sum_j^J c_j y_j - \lambda \theta$$

$$\text{s.t. } \theta - \sum_i^I \sum_j^J M_{ij} \beta_{ij}^h x_{ij} \geq p_{max} \sum_i^I \alpha_i^h + \sum_i^I \sum_j^J (\eta SIR_{min} - M_{ij}) \beta_{ij}^h \quad \forall h \in H^P, \tag{3.7}$$

$$- \sum_i^I \sum_j^J M_{ij} \beta_{ij}^h x_{ij} \geq p_{max} \sum_i^I \alpha_i^h + \sum_i^I \sum_j^J (\eta SIR_{min} - M_{ij}) \beta_{ij}^h \quad \forall h \in H^R, \tag{3.8}$$

$$x_{ij} - y_j \leq 0 \quad \forall i, \forall j, \tag{3.9}$$

$$z_i - \sum_j^J x_{ij} \leq 0 \quad \forall i, \tag{3.10}$$

$$\sum_i^I U_i z_i \geq \pi \sum_i^I U_i, \tag{3.11}$$

$$0 \leq z_i \leq 1, 0 \leq y_j \leq 1, 0 \leq x_{ij} \leq 1. \tag{3.12}$$

The corresponding dual subproblem [DSP-LP] is identical to subproblem (2.25)-(2.26). In phase I, the LP and IP analytic centers are used to solve [DSP-LP] and hence generate a central cut. The LP analytic center is the analytic center of the polytope defined by constraints (3.7)-(3.12). The IP analytic center is generated from the LP analytic center by using the heuristics described in section 3.4.1. Computational results of the different heuristics is described in section 3.5. LP and IP central cuts are appended to [MP] which is solved in Phase II. Only LP cuts are appended to [MP-LP] which is solved in Phase I. A solution $\{\overline{x}_{ij}, \overline{y}_j, \overline{z}_i\}$ of [MP-LP] and its corresponding solution $\{\overline{p}_i\}$ of [DSP] yield a feasible solution $\{\overline{x}_{ij}, \overline{y}_j, \overline{z}_i, \overline{p}_i\}$ of [OP-LP]. The objective function evaluated at $\{\overline{x}_{ij}, \overline{y}_j, \overline{z}_i, \overline{p}_i\}$ gives a lower bound on the optimal solution of [OP-LP]. As described in section 3.2, the upper bound is found from a dual feasible point of [MP-LP]. The difference between the upper bound and the lower bound acts as a stopping criteria. Note that the optimal solution of the problem solved in Phase I is the optimal solution of the LP relaxation [OP-LP] of [OP] and may be used as an upper bound for the optimal solution [OP] which is solved in phase II. The optimal solution of [OP] is found by iteratively solving [MP] and [DSP], however instead of starting from an initial empty set of cuts, central cuts generated from the IP and LP analytic centers of [MP-LP] are added to [MP]. These cuts eliminate part of the feasible solutions of [MP] that are not optimal solutions of [OP], hence reducing the number of times an IP master problem is solved in phase II. In addition to the valid cuts that are generated from the IP and LP analytic center, the following valid cuts may be added to the master problem at every iteration:

Proposition 1 *For every solution $(\overline{x}, \overline{y}, \overline{z})$ of the relaxed master problem, [MP-LP] if in Phase-I or [MP] if in Phase-II, for which the subproblem is feasible, a valid cut of the following form*

$$\theta - \sum_{i}^{I} \sum_{j}^{J} \eta SIR_{min} \beta_{ij}^{h} x_{ij} \geq p_{max} \sum_{i}^{I} \alpha_i^h \tag{3.13}$$

may be added to the master problem.

Proof: Consider an optimal solution $(x^*, y^*, z^*, \theta^*)$ of the full master problem, and suppose that constraint (3.13) is violated, i.e.

$$\theta^* < p_{max} \sum_i^I \alpha_i^h + \sum_i^I \sum_j^J \eta SIR_{min} \beta_{ij}^h x_{ij}^*.$$

The optimal objective function value of the subproblem for fixed (x^*, y^*, z^*) is

$$\overline{\theta} = p_{max} \sum_i^I \alpha_i^h + \sum_i^I \sum_j^J \eta SIR_{min} \beta_{ij}^h x_{ij}^*.$$

Given an optimal solution (x^*, y^*, z^*) of the relaxed master problem and an optimal objective function value $\overline{\theta}$ of the subproblem, then $(x^*, y^*, z^*, \overline{\theta})$ is a feasible solution of the full master problem and has an objective function value $\overline{Z}_{[FMP]} = \sum_i^I r_i U_i z_i^* - \sum_j^J c_j y_j^* - \lambda\overline{\theta}$. Since the full master problem is a maximization problem, this value is then a lower bound on its optimal objective function value. The objective function evaluated at $(x^*, y^*, z^*, \theta^*)$ is equal to

$$Z_{[FMP]}^* = \sum_i^I r_i U_i z_i^* - \sum_j^J c_j y_j^* - \lambda\theta^*.$$

Since

$$\theta^* < p_{max} \sum_i^I \alpha_i^h + \sum_i^I \sum_j^J \eta SIR_{min} \beta_{ij}^h x_{ij}^* = \overline{\theta}.$$

then

$$Z_{[FMP]}^* < \overline{Z}_{[FMP]}$$

and $(x^*, y^*, z^*, \theta^*)$ is not an optimal solution. Furthermore, in order to have an optimal solution $(x^*, y^*, z^*, \theta^*)$, we should have

$$Z_{[FMP]}^* \geq \overline{Z}_{[FMP]}$$

and therefore

$$\theta^* \geq p_{max} \sum_i^I \alpha_i^h + \sum_i^I \sum_j^J \eta SIR_{min} \beta_{ij}^h x_{ij}^*.$$

$\square$

Proposition 2 *For every solution $(\overline{x}, \overline{y}, \overline{z})$ of the relaxed master problem, [MP-LP] if in Phase-I or [MP] if in Phase-II, for which the subproblem is infeasible, a valid cut of the following form*

$$-\sum_{i}^{I}\sum_{j}^{J} \eta SIR_{min}\beta_{ij}^{h}x_{ij} \geq p_{max}\sum_{i}^{I}\alpha_{i}^{h} \tag{3.14}$$

may be added to the master problem.

Proof: Consider a solution $(\overline{x}, \overline{y}, \overline{z})$ of the relaxed master problem for which the subproblem is infeasible, then the feasibility cut

$$-\sum_{i}^{I}\sum_{j}^{J} M_{ij}\beta_{ij}^{h}x_{ij} \geq p_{max}\sum_{i}^{I}\alpha_{i}^{h} + \sum_{i}^{I}\sum_{j}^{J}(\eta SIR_{min} - M_{ij})\beta_{ij}^{h}$$

is added to the master problem, where (α^h, β^h) are the components of the extreme ray generated by solving the alternative subproblem. Consider an optimal solution $(x^*, y^*, z^*, \theta^*)$ of the full master problem that violates constraint (3.14), that is,

$$p_{max}\sum_{i}^{I}\alpha_{i}^{h} + \sum_{i}^{I}\sum_{j}^{J} \eta SIR_{min}\beta_{ij}^{h}x_{ij}^{*} > 0. \tag{3.15}$$

Furthermore, since $(\alpha_i^h, \beta_{ij}^h)$ are the components of the extreme ray, then they satisfy

$$\alpha_{i}^{h} + \sum_{j}^{J}\beta_{ij}^{h}(SFgij - (U_i - 1)g_{ij}SIR_{min}) - SIR_{min}\sum_{k\neq i}\sum_{j}U_k g_{kj}\beta_{kj}^{h} \leq 0. \tag{3.16}$$

Having inequalities (3.15) and (3.16), implies that the auxiliary subproblem [ASP] with $x = x^*$ has a feasible solution $(\alpha_i^h, \beta_{ij}^h)$ and hence [DSP] is unbounded. Therefore, $(x^*, y^*, z^*, \theta^*)$ is not a feasible solution of the full master problem. $\square$

3.4.1 Heuristics for finding the IP analytic center

As described in section 3.2.1, solving problem (3.6) is computationally expensive and therefore heuristics are used to find feasible solutions. In this section we describe five heuristics that can be used to find integer solutions. Heuristic 1 is a generic heuristic that can be used to find feasible solutions of any MIP problem while the other four heuristics are specific to the UMTS/W-CDMA network planning problem.

Heuristic 1 - Central Rounding

Many heuristics address the problem of finding a feasible solution of the generic MIP problem

$$
\begin{aligned}
\min\ & b^T y \\
\text{s.t.}\ & A^T y \leq c, && && (3.17) \\
& y_j \text{ integer} && \forall j \in G, && (3.18) \\
& y_j \geq 0 && \forall j \in C. && (3.19)
\end{aligned}
$$

The feasibility pump, was recently introduced by Fischetti et al. (2005). This heuristic iteratively solves the LP relaxation of the MIP problem to generate a point y^* which is rounded to the nearest integer point $\tilde{y}$. The first point satisfies the LP constraints while the other satisfies the integrality constraints. Hopefully, the two points will converge to the same point within a finite number of iterations. Tests revealed that this algorithm suffers from stalling where the same points are generated repeatedly. Random perturbation that randomly shifts the values of $\tilde{y}$ up and down is effective in solving the stalling issue. The feasibility pump was proved to be successful for finding feasible solutions for 0-1 MIP problems, however it fails to solve problems with general integer variables. A rather more complicated extension of the feasibility pump is introduced in Bertacco et al. (2007). This extension improves the original feasibility pump so as to solve MIP problems with general integer variables. In addition to its complicated implementation, this method does not present any guarantees in terms of the quality of the feasible solution. This is mainly due to the fact that the objective function of the MIP is not used to find y^* and $\tilde{y}$.

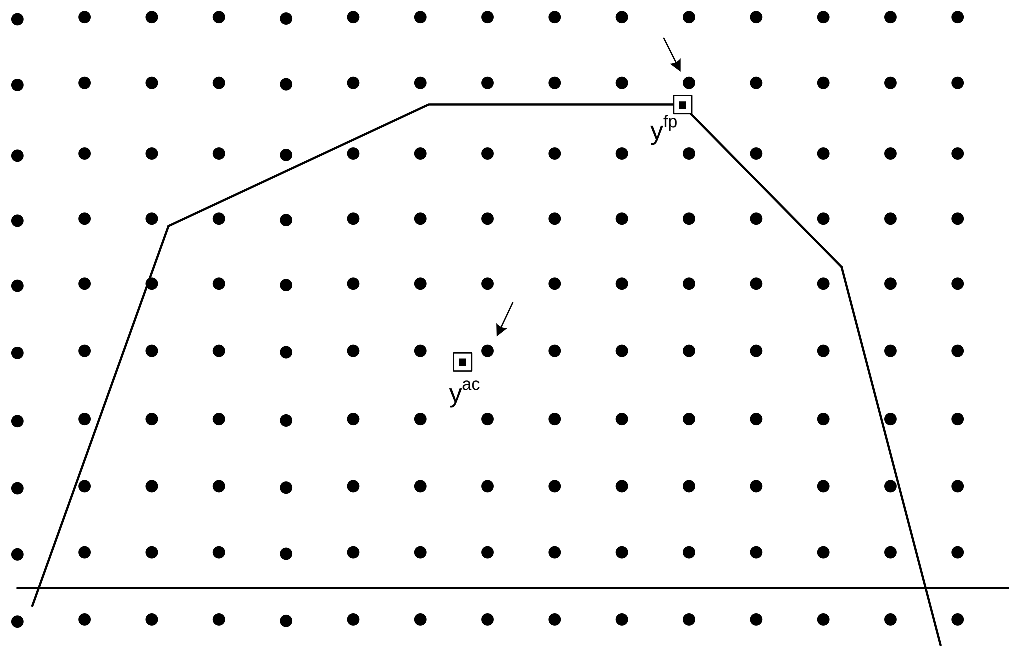

Figure 3.2: Rounding to the nearest integer. y^{ac}: Analytic Center, y^{fp}: Feasibility Pump solution, $\rightarrow$: Nearest integer point.

Central rounding is a new heuristic that takes advantage of analytic centers to generate feasible solutions of MIP problems. Compared to the feasibility pump, the efficiency of this algorithm stems from the fact that the analytic center lies near the center of the feasible region (Figure 3.2), so rounding to the nearest integer will most likely result in a feasible integer point. Considering problem (3.17)-(3.19), central rounding proceeds as follows: The continuous analytic center y^{ac} of the localization set

$$\mathcal{F} = \left\{ \begin{array}{ccc} b^T y & \leq & z_u \\ A^T y & \leq & c \\ y & \geq & 0 \end{array} \right\}$$

is calculated. An integer point y_I is found by rounding y^{ac} to the nearest integer point. If y_I is a feasible point, then the algorithm stops. On the other hand, if y_I is not feasible then the weight of the violated constraint is increased and the analytic center is recomputed. As detailed in section 3.2, the new weight will force the analytic center away from the violated

constraint and a new integer point is found by rounding to the nearest integer. This process is repeated until a feasible integer point is found. Unfortunately, the process of pushing the integer points towards feasibility is very slow and the weights might significantly increase so as to create computational problems. Having a weight on a constraint is identical to replicating the constraint. Therefore instead of incrementing the weight of the violated constraint, this constraint is replicated and appended to the set of constraints. Additionally in order to accelerate the process of finding a feasible integer solution, the right hand side of the added constraint is modified as follows:
Let $\overline{y}^{ac}$ be the continuous analytic center with $\overline{y}_I$ being the corresponding nearest integer point. Suppose that $\overline{y}_I$ is infeasible to constraint $A_1 y \leq c_1$. Knowing that replicating constraint $A_1 y \leq c_1$ will push the new analytic center $\hat{y}$ away from it such that $A_1 \hat{y} \leq A_1 \overline{y}^{ac} \leq c_1$ the right hand side of the replicated constraint is modified such that $A_1 y \leq A_1 \overline{y}^{ac}$. Note that $\overline{y}^{ac}$ is still a primal feasible point and therefore a new analytic center for the updated polyhedron can be computed efficiently. Note that in contrast to the feasibility pump, this algorithm does not suffer from stalling since at each iteration either a feasible integer solution is found or a cut that eliminates the current LP solution is added. Furthermore, the quality of the solution is insured by the bound z_u in the definition of the localization set $\mathcal{F}$. If the resulting feasible integer solution y_I does not satisfy the required quality, then the bound is set to $z_u = b^T y_I$ and the heuristic is rerun. This ensures a better quality solution in the next run. In the very first run z_u is set to $+\infty$.

In the two-phase ACCPM algorithm, central rounding is used to find a feasible integer solution of problem (3.6).

Heuristic 2

Similar to the central rounding heuristic, an integer point is found by rounding the LP analytic center to the nearest integer point. Moreover, whether the resulting integer point is feasible or not, a valid cut is generated from the subproblem. To show this, we consider the following

Lemma 1 *Every cut generated from the subproblem* (2.16) *is a valid cut.*

Proof: Consider a solution $\overline{y}$ not necessarily feasible to the full master problem (2.12)-(2.15). Using $\overline{y}$, let $\overline{\lambda}$ be the optimal solution of the subproblem (2.16) and a cut

$$(b_2 - My)\overline{\lambda} \leq \theta \tag{3.20}$$

is generated. Consider an optimal solution (y^*, θ^*) of the full master problem that violates constraint (3.20), i.e.

$$(b_2 - My^*)\overline{\lambda} = \overline{\theta} > \theta^*.$$

Note that $(y^*, \overline{\theta})$ is a feasible solution of the full master problem. The objective function of the full master problem evaluated at $(y^*, \overline{\theta})$ is equal to

$$\overline{Z}_{[FMP]} = d^T y^* + \overline{\theta}.$$

The objective function of the full master problem evaluated at (y^*, θ^*) is equal to

$$Z^*_{[FMP]} = d^T y^* + \theta^*.$$

Since

$$\overline{\theta} > \theta^*,$$

then

$$\overline{Z}_{[FMP]} > Z^*_{[FMP]}$$

and (y^*, θ^*) is not an optimal solution. $\square$

Heuristic 3

In this heuristic, the values of x_{ij} are rounded to the nearest integer. Therefore, if $x_{ij} < 0.5$ then set $x_{ij} = 0$ otherwise set $x_{ij} = 1$. Furthermore, if $\sum_i x_{ij} \geq 1$ then set $y_j = 1$ otherwise set $y_j = 0$. Additionally, if $\sum_j x_{ij} \geq 1$ then set $z_i = 1$ otherwise set $z_i = 0$.

Heuristic 4

Initialize $y_j = 0$, $\forall j$ and $z_i = 0$, $\forall i$. For each demand point i, find a base station j such that $x_{ij} = \max_j x_{ij}$ and then set $x_{ij} = 1$, $x_{ik} = 0$, $\forall k \neq j$, $y_j = 1$ and finally $z_i = 1$.

Heuristic 5

Initialize $y_j = 0, \forall j$ and $z_i = 0, \forall i$. Additionally, let $LP_{max} = \max x_{ij}$, $LP_{min} = \min x_{ij}$. If $x_{ij} \geq (LP_{max} - LP_{min})/2$ then set $x_{ij} = 1$, $z_i = 1$, and $y_j = 1$, otherwise set $x_{ij} = 0$. Note, that the resulting solution might violate constraint (2.31). In this case, find a demand point i such that $z_i = 0$, find a base station j such that $x_{ij} = \max_j x_{ij}$ then set $x_{ij} = 1$, $z_i = 1$ and $y_j = 1$. This process is repeated until constraint (2.31) is satisfied.

3.5 Computational Results

The classical Benders decomposition and the two-phase ACCPM were implemented in C and run on a Sunblade 2500 workstation with two 1.6 GHz processors and 2 Gb of RAM. The master problems and the subproblems were solved using cplex 10.1. Computational testing was done on a set of instances proposed by Amaldi et al. (2002). Potential base station and user locations were randomly selected from these instances. Propagation gains g_{ij} are calculated using Hata (1980) propagation model. The gain in dB is calculated as follows:

$$\begin{aligned} A_{ij} &= 69.55 + 26.16\log(f) - 13.82\log(H_j) \\ &\quad - [(1.1\log(f) - 0.7)H_i - (1.56\log(f) - 0.8)] \\ &\quad + [4.99 - 6.55\log(H_j)]\log(d_{ij}) \end{aligned}$$

where f is the center frequency measured in Mhz, H_i is the height of the transmitter at location i and H_j is the height of the base station at location j. H_i and H_j are measured

Data	Value	Description
SIR_{min}	0.01	Minimum SIR
f	2000	Operating frequency
H_j	10 m	Height of base station antenna
H_i	1 m	Height of mobile device antenna
SF	128	Spreading Factor
η	6.3e-14	Receiver thermal noise
p_{max}	0.15	Maximum power
r	145	Revenue from each serviced user
c	42	Base station cost
λ	0.42	Power cost

Table 3.1: Data for the test cases

in meters. A_{ij} is converted to propagation gain as follows

$$g_{ij} = 10^{-0.1A_{ij}}.$$

Parameters used for the test problems are shown in Table 3.1. We evaluate the effect of generating feasible cuts from the analytic center of the master problem through the two-phase ACCPM algorithm. The performance of the two-phase ACCPM algorithm is compared to the classical Benders decomposition algorithm. Results for 14 test cases solved using the classical Benders decomposition are shown in Table 3.2. For each test case, we ran 10 randomly generated problems and took the average. The first column of the table indicates the test case number. The second and third columns indicate the number of demand points (#DP) and the number of candidate base station locations (#BS) respectively. Column (4) indicates the number of iterations (Iter) and consequently the number of times the integer Benders master problem is solved. Finally, column (5) indicates the total CPU time in seconds spent on solving each test case.

Test Case (1)	#DP (2)	#BS (3)	Iter_{IP} (4)	CPU (s) (5)
1	8	2	28.1	1.7
2	9	2	39.2	2.9
3	10	2	43.4	5.0
4	5	3	33.1	2.4
5	6	3	55.9	7.1
6	7	3	95.5	25.4
7	8	3	131	59.1
8	9	3	203.4	206.6
9	10	3	225.7	637.3
10	5	4	65	12.5
11	6	4	126.2	49.5
12	7	4	221	212.8
13	8	4	433.6	2031.5
14	9	4	780	17722.3

Table 3.2: Computational Results - Classical Benders Decomposition

			Phase-I		Phase-II		
Test Case (1)	#DP (2)	#BS (3)	Iter_{LP} (4)	CPU (s) (5)	Iter_{IP} (6)	CPU (s) (7)	CPU (s) (8)
1	8	2	14.2	1.6	20.9	1.2	2.8
2	9	2	15.4	2	30.9	2.3	4.3
3	10	2	16.4	2.6	32	3.7	6.3
4	5	3	15.9	1.2	23.8	1.8	3
5	6	3	17.9	1.9	43.9	5.6	7.5
6	7	3	22.4	4.0	73.5	19.5	23.5
7	8	3	24.2	4.9	111.7	50.6	55.5
8	9	3	25.9	7	183.4	186.3	193.3
9	10	3	30.1	11.1	201.6	569.3	580.4
10	5	4	21.6	2.6	48.7	9.4	12.0
11	6	4	25.3	5.3	106.9	42.1	47.4
12	7	4	27.4	7.3	195.1	187.9	195.2
13	8	4	32.1	13.6	399.3	1871.1	1884.7
14	9	4	35.7	18	686.6	15598.3	15616.3

Table 3.3: Computational Results - Two-phase ACCPM Algorithm

The same 14 test cases are solved using the two-phase ACCPM algorithm and the results are shown in Table 3.3. Column (4) of the table indicates the number of iterations (Iter_{LP}) required to solve the LP relaxed problem (also indicates the number of valid cuts that are generated and added to the problem solved in Phase II). Column (5) indicates the CPU time required to solve the LP relaxed problem in Phase I. Column (6) indicates the number of iterations (Iter_{IP}) required to solve the problem in Phase II (also indicates the number of times the integer Master problem is solved). Column (7) indicates the CPU time required to solve the problem in Phase II. Finally Column (8) shows the total CPU time required by the two-phase ACCPM algorithm; that is the time spent on Phase I in addition to the time spent on Phase II.

We observe that the two-phase ACCPM algorithm becomes more efficient as the complexity of the problem increases. We notice that adding valid cuts from Phase-I always reduces the number of iterations and consequently the computational time of solving the integer problem in Phase-II. However, we notice that for relatively very small instances (test cases 1 through 5), even though the computational time required to solve the integer problem is decreased, the classical Benders decomposition achieves a better overall computational time. This is due to the overhead computational time required to generate the valid cuts using ACCPM in Phase-I, and indicates that a more efficient implementation of ACCPM will certainly enhance the two-phase ACCPM algorithm. For larger instances (test cases 6 through 14), we observe that the two-phase ACCPM algorithm takes less computational time than the classical Benders decomposition. Moreover, for test case 6, the computational time is reduced by 10%. On the other extreme, the computational time for test case 14 is reduced by 12%.

In an attempt to generate more cuts from Phase-I, we evaluate the performance of the two-phase ACCPM algorithm with the weight of the bound constraint set to 1. Recall from Section 3.2 that the weight of the bound constraint is set to be equal to the number

			Phase-I		Phase-II		
Test Case	#DP	#BS	Iter_{LP}	CPU (s)	Iter_{IP}	CPU (s)	CPU (s)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	8	2	25.9	2.5	13.4	0.74	3.2
2	9	2	28.7	3.6	22.8	1.7	5.3
3	10	2	30.5	4.8	26	3	7.8
4	5	3	26.0	1.9	18.9	1.4	3.3
5	6	3	27.8	2.9	39	5	7.9
6	7	3	32.6	4.6	73.3	19.5	24.1
7	8	3	35.0	6.8	104.4	47.1	53.9
8	9	3	39.6	9.9	172	175.1	185.0
9	10	3	40.6	13.3	192.7	544.5	557.8
10	5	4	31.9	3.7	43.6	8.4	12.1
11	6	4	38.1	6.8	102.1	40	46.8
12	7	4	41.4	10.2	188.8	181.7	191.9
13	8	4	45.6	15.3	394.3	1848.3	1863.6
14	9	4	51.3	23.5	629.2	14296.2	14319.7

Table 3.4: Computational Results - Two-phase ACCPM Algo. - Weight of bound constraint set to 1

of constraints. This reduces the number of iterations required by ACCPM to solve the LP problem. Since at each ACCPM iteration a valid cut is generated, reducing the number of iterations reduces the number of generated cuts. The 14 test cases are solved using the two-phase algorithm with the weight on the objective function set to 1. The results are shown in Table 3.5. We observe that the number of iterations of ACCPM in Phase-I and therefore the number of generated cuts is increased by approximately 50%. This is reflected in an increase in the computational time spent on solving Phase-I. However, this increase is compensated by a reduction in computational time spent on solving Phase-II. We observe that the additional cuts improve the computational performance of the two-phase algorithm.

To evaluate the effect of the central cuts, we compare the two-phase algorithm where valid cuts are generated from the analytic center of the master problem to a two-phase algorithm where valid cuts are generated from the extreme point (optimal solution) of the master problem (i.e., we compare the algorithm when in the first phase the master problem is solved using the analytic center cutting plane method and when it is solved using Kelley's cutting plane method). Results for the 14 test cases solved using the two-phase

			Phase-I		Phase-II		
Test Case (1)	#DP (2)	#BS (3)	Iter_{LP} (4)	CPU (s) (5)	Iter_{IP} (6)	CPU (s) (7)	CPU (s) (8)
1	8	2	9.3	0.0	22.6	1.3	1.3
2	9	2	11.0	0.1	31.8	2.4	2.5
3	10	2	11.3	0.1	35.8	4.2	4.3
4	5	3	12.0	0.0	24.3	1.8	1.8
5	6	3	15.2	0.1	45.5	5.8	5.9
6	7	3	18.3	0.1	76.0	20.2	20.3
7	8	3	19.0	0.1	115.6	52.2	52.3
8	9	3	23.3	0.2	184.4	187.4	187.6
9	10	3	25.0	0.2	207.3	585.4	585.6
10	5	4	19.3	0.1	51.3	9.86	9.9
11	6	4	25.0	0.1	109.9	43.2	43.3
12	7	4	26.6	0.2	200.0	192.6	192.8
13	8	4	31.4	0.3	408.8	1915.4	1915.7
14	9	4	36.0	0.4	706.7	16056.7	16057.1

Table 3.5: Computational Results - Two-phase Algorithm - Kelley Based

Kelley's cutting plane algorithm are shown in Table 3.5. We observe that the two-phase Kelley's based algorithm achieves better performance compared to the classical Benders algorithm. Compared to the ACCPM based algorithm, we observe that Kelley's cutting plane algorithm performs less iterations in Phase-I. This is also reflected in a significantly less computational time spent on solving Phase-I. This justifies our previous conclusion that a more efficient implementation of ACCPM will improve the overall performance of the two-phase algorithm. In Phase-II, ACCPM based algorithm consumes less iterations and consequently less time. Overall, Kelley's based algorithm achieves better performance for the relatively small instances (test cases 1 through 5) however for larger instances where the classical Benders decomposition suffers, we observe that the ACCPM based algorithm achieves better computational time compared to the Kelley's based algorithm.

Finally, we evaluate the effect of the rounding heuristics on the two-phase ACCPM algorithm. Since setting the weight on the bound constraint to 1 was shown to be beneficial, we use this same weight for the remaining tests. Results for the 14 test cases solved using the two-phase ACCPM with rounding Heuristic 1 are shown in Table 3.5. Eventhough Heuristic 1 (Central Rounding) consumes a significant time to find feasible integer solutions,

			Phase-I			Phase-II		
Test Case (1)	#DP (2)	#BS (3)	Iter_{LP} (4)	CPU (s) (5)	CPU_{H} (6)	Iter_{IP} (7)	CPU (s) (8)	CPU (s) (9)
1	8	2	25.9	2.5	6.0	9.5	0.5	9.0
2	9	2	28.7	3.6	11.7	17.7	1.3	16.6
3	10	2	30.5	4.8	1.9	15.1	1.9	8.5
4	5	3	26.0	1.9	5.9	13.3	1.0	8.8
5	6	3	27.8	2.9	10.1	29.5	3.8	16.8
6	7	3	32.6	4.6	16.9	51.9	13.8	35.3
7	8	3	35.0	6.8	28.7	77.5	41.3	76.7
8	9	3	39.6	9.9	44.3	138.3	158.5	212.7
9	10	3	40.6	13.3	68.5	154.5	436.3	518.2
10	5	4	31.9	3.7	18.6	35.1	6.8	29.1
11	6	4	38.1	6.8	35.8	77.5	30.4	72.9
12	7	4	41.4	10.2	61.1	173.6	167.2	238.4
13	8	4	45.6	15.3	101.0	371.3	1739.6	1855.9
14	9	4	51.3	23.5	108.5	602.1	13680.3	13812.3

Table 3.6: Computational Results - Two-phase ACCPM Algo. - Rounding Heuristic 1

generating cuts from integer solutions reduce the number of iterations in Phase-II of the algorithm. The computational overhead of finding feasible integer solutions makes the use of Heuristic-1 only beneficial in large instances where the computational time of Heuristic-1 is less significant and a better performance of the two-phase ACCPM algorithm is achieved. Heuristic-1 is computationally expensive since it is a generic heuristic that can be applied to any mixed integer problem. The remaining 4 heuristics are specific heuristics that apply to the UMTS/W-CDMA network planning problem. Results for the 14 test cases solved using the two-phase ACCPM algorithm with Heuristic-2, Heuristic-3, Heuristic-4 and Heuristic-5 are shown in Tables 3.7, 3.8, 3.9 and 3.10 respectively. The computational time of these heuristics is negligible so it is not explicitly shown in the tables. Results show that all these heuristics improve the computational performance of the two-phase ACCPM algorithm. Additionally, Heuristic-2 and Heuristic-3 achieve identical results since the cuts generated from the subproblem depend only on the x_{ij} variables, and these two heuristics behave similarly in terms of the x_{ij} variables. Heuristic-4 and Heuristic-5 achieve better results than Heuristic-2 and Heuristic-3. Finally, by comparing the computational results of Phase-II, we notice that Heuristic-1 achieves the best results compared to the other 4 heuristics. However due to the computational burden of Heuristic-1, only large instances achieve a better overall performance. A summary of the results is presented in Figures 3.3 and 3.4.

			Phase-I		Phase-II		
Test Case (1)	#DP (2)	#BS (3)	Iter_{LP} (4)	CPU (s) (5)	Iter_{IP} (6)	CPU (s) (7)	CPU (s) (8)
1	8	2	25.9	2.5	9.7	0.5	3.0
2	9	2	28.7	3.6	18.5	1.4	5.0
3	10	2	30.5	4.8	15.5	1.9	6.7
4	5	3	26.0	1.9	15.2	1.1	3.0
5	6	3	27.8	2.9	31.1	4.1	7.0
6	7	3	32.6	4.6	64.2	17.4	22.0
7	8	3	35.0	6.8	90.7	41.7	48.5
8	9	3	39.6	9.9	153.9	160.2	170.1
9	10	3	40.6	13.3	170.3	482.1	495.4
10	5	4	31.9	3.7	41.4	7.9	11.6
11	6	4	38.1	6.8	96.5	37.6	44.4
12	7	4	41.4	10.2	185.0	177.8	188.0
13	8	4	45.6	15.3	390.5	1827.6	1842.9
14	9	4	51.3	23.5	629.3	14291.9	14315.4

Table 3.7: Computational Results - Two-phase ACCPM Algo. - Rounding Heuristic 2

			Phase-I		Phase-II		
Test Case (1)	#DP (2)	#BS (3)	Iter_{LP} (4)	CPU (s) (5)	Iter_{IP} (6)	CPU (s) (7)	CPU (s) (8)
1	8	2	25.9	2.5	9.7	0.5	3.0
2	9	2	28.7	3.6	18.5	1.4	5.0
3	10	2	30.5	4.8	15.5	1.9	6.7
4	5	3	26.0	1.9	15.2	1.1	3.0
5	6	3	27.8	2.9	31.1	4.1	7.0
6	7	3	32.6	4.6	64.2	17.4	22.0
7	8	3	35.0	6.8	90.7	41.7	48.5
8	9	3	39.6	9.9	153.9	160.2	170.1
9	10	3	40.6	13.3	170.3	482.1	495.4
10	5	4	31.9	3.7	41.4	7.9	11.6
11	6	4	38.1	6.8	96.5	37.6	44.4
12	7	4	41.4	10.2	185.0	177.8	188.0
13	8	4	45.6	15.3	390.5	1827.6	1842.9
14	9	4	51.3	23.5	629.3	14291.9	14315.4

Table 3.8: Computational Results - Two-phase ACCPM Algo. - Rounding Heuristic 3

			Phase-I		Phase-II		
Test Case	#DP	#BS	Iter_{LP}	CPU (s)	Iter_{IP}	CPU (s)	CPU (s)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	8	2	25.9	2.5	9.3	0.5	3.0
2	9	2	28.7	3.6	15.3	1.2	4.8
3	10	2	30.5	4.8	15.3	1.8	6.6
4	5	3	26.0	1.9	15.1	1.1	3.0
5	6	3	27.8	2.9	30.7	4.0	6.9
6	7	3	32.6	4.6	63.1	17.1	21.7
7	8	3	35.0	6.8	87.1	41.1	47.9
8	9	3	39.6	9.9	149.1	152.1	162.0
9	10	3	40.6	13.3	161.3	456.1	469.4
10	5	4	31.9	3.7	41.1	7.8	11.5
11	6	4	38.1	6.8	96.2	37.6	44.4
12	7	4	41.4	10.2	184.7	177.4	187.6
13	8	4	45.6	15.3	382.5	1791.1	1806.4
14	9	4	51.3	23.5	617.5	14031.9	14055.4

Table 3.9: Computational Results - Two-phase ACCPM Algo. - Rounding Heuristic 4

			Phase-I		Phase-II		
Test Case	#DP	#BS	Iter_{LP}	CPU (s)	Iter_{IP}	CPU (s)	CPU (s)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	8	2	25.9	2.5	9.5	0.5	3.0
2	9	2	28.7	3.6	17.3	1.3	4.9
3	10	2	30.5	4.8	15.8	1.9	6.7
4	5	3	26.0	1.9	15.5	1.2	3.1
5	6	3	27.8	2.9	31.5	4.1	7.0
6	7	3	32.6	4.6	63.9	17.3	21.9
7	8	3	35.0	6.8	87.1	40.2	47.0
8	9	3	39.6	9.9	150.7	154.7	164.6
9	10	3	40.6	13.3	161.3	456.9	470.2
10	5	4	31.9	3.7	42.3	8.2	11.9
11	6	4	38.1	6.8	97.9	38.2	45.0
12	7	4	41.4	10.2	184.1	176.9	187.1
13	8	4	45.6	15.3	384.7	1801.1	1816.4
14	9	4	51.3	23.5	620.2	14091.2	14114.7

Table 3.10: Computational Results - Two-phase ACCPM Algo. - Rounding Heuristic 5

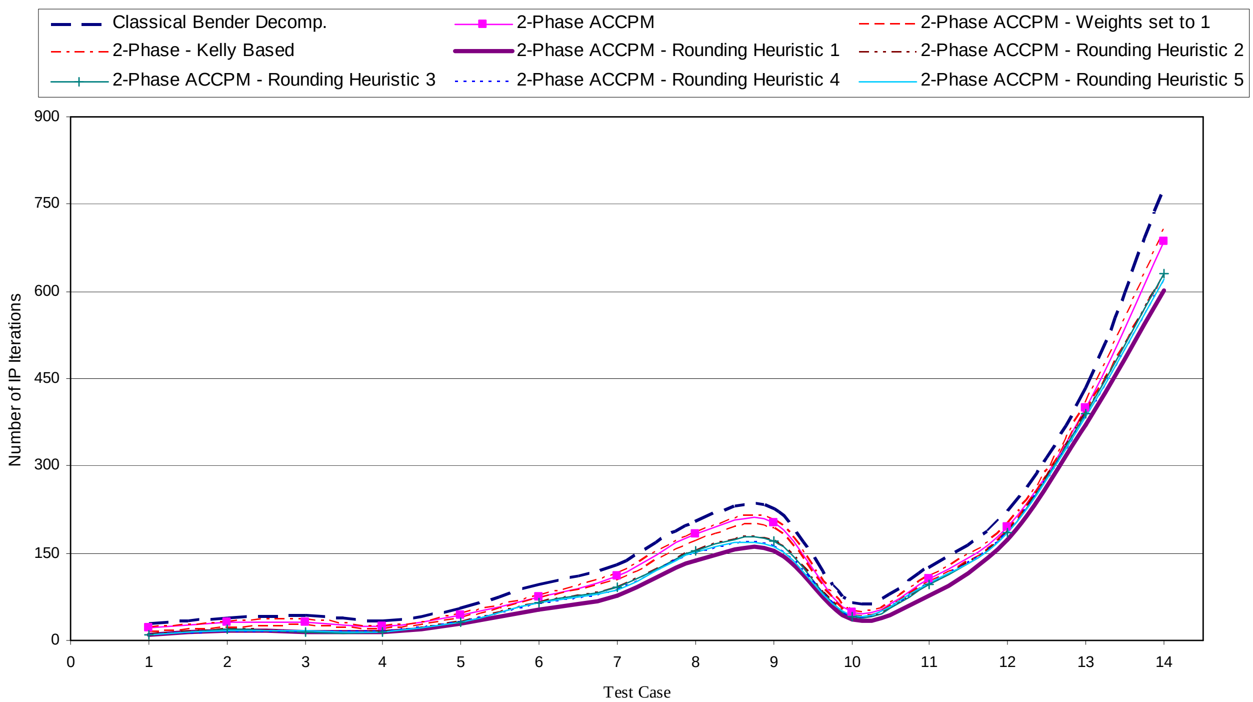

Figure 3.3: Number of times the IP master problem is solved.

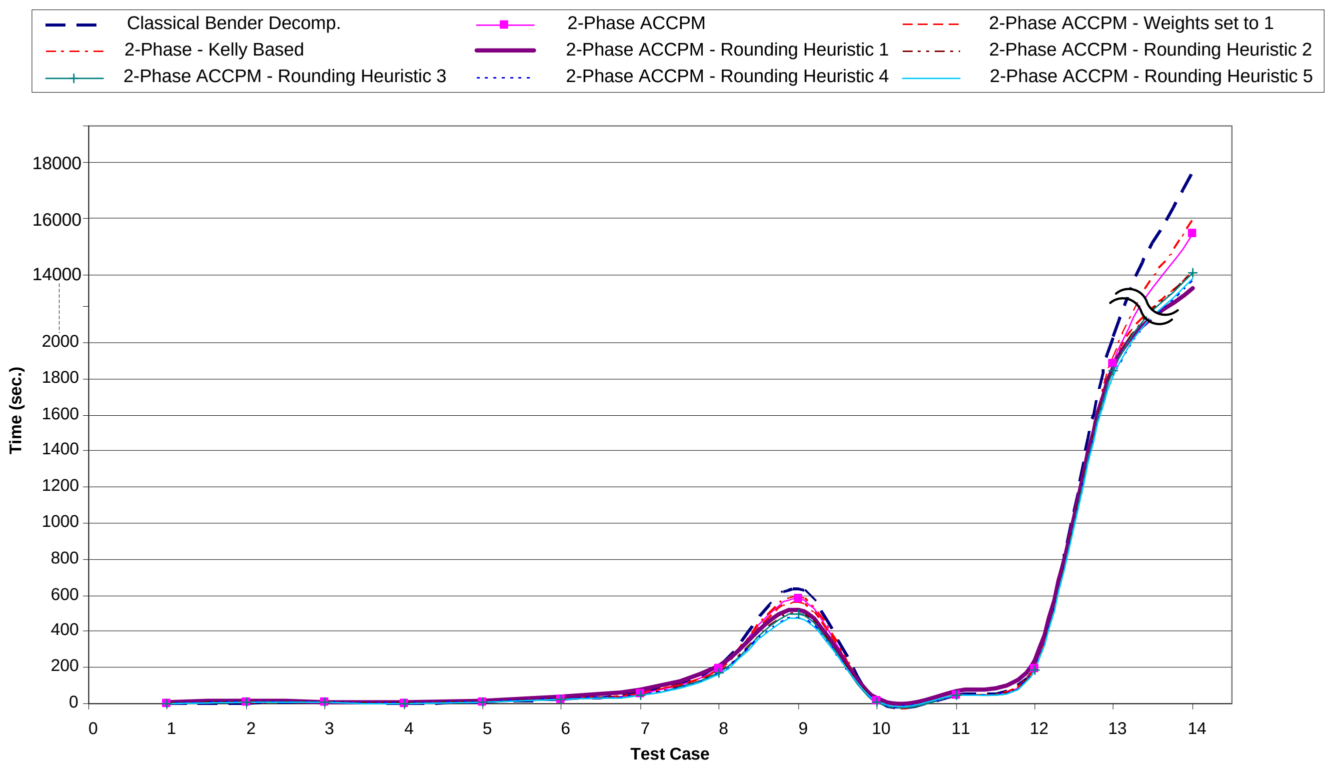

Figure 3.4: Total CPU time consumed by each algorithm.

In this chapter, the analytic center cutting plane method was used to generate valid cuts. The computational results show that these valid cuts reduce the number of iterations required to solve the MIP problem. In the following chapter we introduce a new decomposition approach that aims at reducing the complexity of the master problem by reducing the number of integer variables in it. Therefore rather than having a master problem defined over the full set of integer variables, the new decomposition approach formulates a master problem defined over a subset of the integer variables while the remaining integer variables are defined in the subproblem. This reduces the complexity of the master problem and aims at eliminating the bottleneck of solving a hard MIP problem at every iteration.

Chapter 4

Nested and Combinatorial Benders Decomposition Algorithms

Several techniques have been adopted to accelerate Benders decomposition. A new method that utilizes the analytic center cutting plane method to generate cuts was introduced in Chapter 3. Most research emphasizes accelerating Benders decomposition by reducing the number of iterations, therefore reducing the number of times the master problem is solved (see Chapter 3 for details). This chapter introduces a novel decomposition algorithm for solving binary integer problems. Rather than reducing the number of iterations, the new decomposition algorithm, Combinatorial Benders Decomposition, reduces the complexity of the integer master problem by reducing the number of integer variables in it. The resulting problem is formed by an integer master problem and an integer subproblem. The subproblem is then decomposed using the classical Benders decomposition leading to a nested Benders Decomposition. In the following section we present a general description of the combinatorial Benders decomposition. Then a detailed description of nested Benders decomposition applied to the UMTS/W-CDMA network planning problem follows.

4.1 Combinatorial Benders Decomposition Algorithm

Combinatorial Benders decomposition is applicable to MIP problems with the following structure:

$$
\begin{aligned}
\text{[P]: } & \min\ c^T x + d^T y \\
\text{s.t. } & Ax + My \geq b, \\
& x_i \in \{0,1\}, \\
& y_j \in \{0,1\} \quad \forall j \in B, \\
& y_j \text{ integer} \quad \forall j \in G, \\
& y_j \geq 0 \quad \forall j \in C,
\end{aligned}
$$

where x is a vector of binary variables and y is a vector of continuous, general-integer and binary variables. The case where $d = 0$ is considered in Codato and Fischetti (2006). In this work, we consider a general case where $d \neq 0$. In contrast to classical Benders decomposition where the subproblems are linear programs, combinatorial Benders decomposition leaves some of the integer variables for the subproblem, leading to a MIP subproblem. For fixed $\overline{x}$ variables, $[P]$ may be rewritten as:

$$
\begin{aligned}
\text{[MP]: } & \min\ c^T x + \theta(x) \\
\text{s.t. } & x_i \in \{0,1\},
\end{aligned}
$$

where

$$
\begin{aligned}
\text{[SP]: } \theta(x) = & \min\ d^T y \\
\text{s.t. } & My \geq b - A\overline{x}, \\
& y_j \in \{0,1\} \quad \forall j \in B, \\
& y_j \text{ integer} \quad \forall j \in G, \\
& y_j \geq 0 \quad \forall j \in C.
\end{aligned}
$$

For each x^h optimal to [MP], there exists a single value $\theta^h = d^T y^h$ optimal to [SP]. Furthermore, (x^h, y^h) is a feasible solution to [P]. Thus, the optimal solution of [P] is the best objective function value of [MP] evaluated at all possible (x^h, y^h). If x^h is infeasible for [SP], then x^h is infeasible for [P], thus at least one variable of x^h should be changed in order to make [SP] feasible. This is formulated as a feasibility cut of the form

$$\sum_{i:x_i^h=0} x_i + \sum_{i:x_i^h=1} (1 - x_i) \geq 1$$

and appended to $[MP]$. Otherwise, if there exists an optimal solution $\theta^h = d^T y^h$ of [SP] when x is fixed to x^h, a binary Benders cut of the form

$$M^h \sum_{i:x_i^h=0} x_i + M^h \sum_{i:x_i^h=1} (1 - x_i) + \theta \geq \theta^h \tag{4.1}$$

is generated and appended to $[MP]$. M^h is a large enough number that would make constraint (4.1) only active for the corresponding x^h. Hence, the master problem is:

$$\begin{aligned}
\text{[MP]: } & \min\ c^T x + \theta \\
\text{s.t. } & M^h \sum_{i:x_i^h=0} x_i^h + M^h \sum_{i:x_i^h=1} (1 - x_i^h) + \theta \geq \theta^h && \forall h \in H^P, && (4.2) \\
& \sum_{i:x_i^h=0} x_i^\hbar + \sum_{i:x_i^h=1} (1 - x_i^\hbar) \geq 1 && \forall \hbar \in H^R, && (4.3) \\
& x_i \in \{0, 1\}, \quad \theta \text{ is free}
\end{aligned}$$

where H^P and H^R are the sets of feasible and infeasible solutions to [SP]. If H^P and H^R enclose all solutions that are feasible and infeasible to P, then [MP] is equivalent to [P]. Otherwise, [MP(H^P, H^R)] is a relaxation of [P] and the objective function value of [MP] is a lower bound to the objective function value of [P]. Note that we assumed [MP] is always feasible. If solving [MP] to optimality is infeasible then [P] is also infeasible. Similar to the classical Benders decomposition algorithm, the upper and lower bounds are used as a stopping criteria for the cutting plane algorithm. A description follows:

1. Start with UB = ∞ ; LB =$-\infty$. h is the iterations count, h=1.

While LB $\neq$ UB

2. Solve [MP] and get solution $\overline{x}$, and a lower bound LB_h.

3. Set $LB = LB_h$.

4. Solve [SP]

4.1 If [SP] is infeasible

4.1.1 add feasibility cut of type (4.3)

4.2 If [SP] is feasible

4.2.1 Solve [SP] and get solution $\overline{y}$, get $UB_h = c^T\overline{x} + b^T\overline{y}_j$

4.2.2 add optimality cut of type (4.2)

4.2.3 Update upper bound $UB = \min(UB, UB_h)$

End while

Since $x \in \{0, 1\}$, there exists a finite number of constraints of the form (4.2) and (4.3) that may be appended to [MP]. Furthermore, none of the infeasible x variables may be generated more than once from [MP] since each infeasible x is eliminated by adding a constraint of type (4.3). If x is a feasible solution, then if x is generated twice, the algorithm terminates with an optimal solution x (UB=LB). As none of the x's are generated more than once, the combinatorial Benders decomposition will converge in a finite number of iterations.

4.1.1 Ensuring a bounded [MP] formulation

The master problem [MP] defined by (4.2)-(4.3) is bounded only if all possible combinations of the x variables are defined by the sets H^P and H^R. Since θ is a free variable, if there exists a feasible solution x^k where $k \notin \{H^P \cup H^R\}$ then [MP] is unbounded. Therefore, to ensure that [MP] is bounded at every iteration, the [MP] formulation should include a lower bound on $\theta(x)$. Since by construction [SP] is always bounded or otherwise [P] is unbounded, there always exists a lower bound $LB([SP]) = \min_x[SP(x)]$, where x is a feasible solution of [MP]. Therefore a constraint of the form

$$\theta \geq LB([SP]) \tag{4.4}$$

is added to the formulation of [MP] to ensure a bounded master problem. Alternatively if $LB([SP])$ is not known or cannot be easily found, a constraint of the form

$$\theta \geq \min(\theta^h) - \epsilon \qquad \forall \quad h \in H \tag{4.5}$$

may be added to [MP] to ensure boundedness, where ϵ is a positive number. In fact, at each iteration the optimal solution of [MP] is either (x^k, θ^k) with $k = h \in H^P$, thus $\theta^k = \theta^h$ otherwise $k \notin H$ and θ^k is less than all θ^h generated so far. Note that in the case where (4.5) is used instead of (4.4), if in an optimal solution constraint (4.5) is binding, then the objective function $c^T x^k + \theta^k$ is no longer a lower bound on the optimal objective function value of [P]. Alternatively, if (4.5) is not binding, therefore (4.5) has no effect on the [MP] formulations presented in (4.2)-(4.3) hence $c^T x^k + \theta^k$ is a lower bound on the optimal objective function value of [P]. We note that for the very first iteration, we may add any lower bound on the value of θ. This will not affect the solution of [MP] since none of the equations (4.2)-(4.3) is added and therefore would not affect the choice the x variables. Afterwards, the lower bound on θ is updated as described.

4.1.2 Avoiding Total Enumeration

In the case where $d \neq 0$ and if constraint (4.5) is used instead of (4.4), the algorithm will enumerate all possible combinations of the x variables before converging to an optimal solution. In fact, if there exists a combination x^k not visited (i.e. $k \notin \{H^P \cup H^R\}$), then there exists an optimal solution (x^k, θ^k) of [MP] where none of the constraints of type (4.2) is binding. To avoid a total enumeration case, it is advisable to exploit the structure of [SP] thus using a heuristic to generate multiple cuts of type (4.2)-(4.3) at each iteration.

This may be implemented in the sense that if a solution x^k is generated and if x^p is known to be worse than x^k then x^P cannot be part of an optimal solution. Hence, a constraint of the form (4.3) may be generated to eliminate x^p. In Section 4.3, a heuristic is implemented to generate multiple cuts when solving the UMTS/W-CDMA network planning problem. In the case where a similar heuristic cannot be implemented, it is advisable to apply the discussed decomposition approach when the number of x variables is relatively small.

In the following section the combinatorial Benders decomposition is used along with the classical Benders decomposition to form a nested Benders decomposition algorithm. This algorithm is then used to solve the UMTS/W-CDMA network planning problem.

4.2 Nested Benders Decomposition

For large instances of the problem formulated in Section 2.4, large number of cuts may be appended to the master problem before reaching an optimal solution. This increases the computational time required to solve the mixed integer master problem. Motivated by the fact that the number of base stations (y_j variables) is relatively smaller than the number of demand points (x_{ij} variables), fixing y_j variables will significantly reduce the computational requirements of finding the x_{ij} variables. The same motivation was exploited by Kalvenes et al. (2006) through priority branching over the base station locations in a branch and bound algorithm. However for large instances, the branch and bound tree will unreasonably grow to huge size. Fixing some binary variables through a Benders decomposition is explored next.
For fixed $\overline{y}_j$ and $\overline{z}_i$ values, the original problem (2.1)-(2.5) is decomposed via a Combinatorial Benders Decomposition into a master problem [CB-MP] and a subproblem [CB-SP].

For fixed $\overline{y}_j$ and $\overline{z}_i$, the original problem [OP] is rewritten as follows:

$$[\text{CB-SP}]: \theta_1 = \max -\sum_{i}^{I} U_i p_i$$

$$\begin{aligned}
\text{s.t. } & x_{ij} \leq \overline{y}_j && \forall i, \forall j \\
& SF\frac{p_i g_{ij}}{(U_i - 1)p_i g_{ij} + \sum_{k \neq i} U_k g_{kj} p_k + \eta} - SIR_{min} x_{ij} \geq 0 && \forall i, \forall j \\
& \sum_{j}^{J} x_{ij} \geq \overline{z}_i && \forall i \\
& 0 \leq p_i \leq_{max} && \forall i \\
& x_{ij} \in \{0, 1\} && \forall i, \forall j
\end{aligned}$$

The master problem is:

$$[\text{CB-MP}]: \max \sum_{i}^{I} r_i U_i z_i - \sum_{j}^{J} c_j y_j - \lambda\theta$$

$$\begin{aligned}
\text{s.t. } & \sum_{i:\overline{z}_i=0}^{I} z_i + \sum_{j:\overline{y}_j=0}^{J} y_j + \sum_{i:\overline{z}_i=1}^{I} (1 - z_i) + \sum_{j:\overline{y}_j=1}^{J} (1 - y_j) \geq 1 && \forall(\overline{z}_i, \overline{y}_j) \\
& M\sum_{i:\overline{z}_i=0}^{I} z_i + M\sum_{j:\overline{y}_j=0}^{J} y_j + M\sum_{i:\overline{z}_i=1}^{I} (1 - z_i) + M\sum_{j:\overline{y}_j=1}^{J} (1 - y_j) + \theta \geq \theta_1^h && \forall(\overline{z}_i, \overline{y}_j, \theta_1^h) \\
& \sum_{i}^{I} U_i z_i \geq \pi \sum_{i}^{I} U_i \\
& z_i, y_j \in \{0, 1\}
\end{aligned}$$

If [CB-SP] has an optimal solution $(\overline{x}_{ij}, \overline{p}_i)$, then $(\overline{x}_{ij}, \overline{y}_j, \overline{z}_i, \overline{p}_i)$ is a feasible solution of [OP]. If [SP] is infeasible, at least one value of the fixed variables $\overline{y}_j$ and $\overline{z}_i$ should be changed to break the infeasibility. Therefore if [CB-SP] is infeasible, a feasibility cut of the following form is generated and appended to [CB-MP].

$$\sum_{i:\overline{z}_i=0}^{I} z_i + \sum_{j:\overline{y}_j=0}^{J} y_j + \sum_{i:\overline{z}_i=1}^{I} (1 - z_i) + \sum_{j:\overline{y}_j=1}^{J} (1 - y_j) \geq 1 \tag{4.6}$$

If (x_{ij}^h, p_i^h) is an optimal solution of [CB-SP], an optimality cut of the following form is generated and appended to [CB-MP], where $\theta_1^h = -\sum_i^I U_i p_i^h$

$$M \sum_{i:\overline{z}_i=0}^{I} z_i + M \sum_{j:\overline{y}_j=0}^{J} y_j + M \sum_{i:\overline{z}_i=1}^{I} (1 - z_i) + M \sum_{j:\overline{y}_j=1}^{J} (1 - y_j) + \theta \geq \theta_1^h \tag{4.7}$$

M is a large enough number that makes the added cut redundant if any of the corresponding $(\overline{y}_j, \overline{z}_i)$ switch value.

Since [CB-SP] follows the same structure of problems described in section 2.3, and following the details presented in Section 2.4, [CB-SP], itself, may be solved via classical Benders Decomposition. For fixed $\overline{x}_{ij}$ values, [CB-SP] is decomposed into LP primal subproblem [CL-PSP] as follows:

$$\text{[CL-PSP]: min } \sum_i^I U_i p_i$$

$$\begin{aligned}
\text{s.t. } & p_i \leq p_{max} && \forall i \\
& SF \frac{p_i g_{ij}}{(U_i - 1) p_i g_{ij} + \sum_{k \neq i} U_k g_{kj} p_k + \eta} \geq SIR_{min} \overline{x}_{ij} && \forall i, \forall j \\
& p_i \geq 0.
\end{aligned}$$

Its dual problem is:

$$\text{[CL-DSP]: } \theta_2 = \max\, p_{max} \sum_i^I \alpha_i + \eta SIR_{min} \sum_i^I \sum_j^J \overline{x}_{ij} \beta_{ij}$$

$$\begin{aligned}
\text{s.t. } & \alpha_i + \sum_j^J \beta_{ij} (SF g_{ij} - (U_i - 1) g_{ij} SIR_{min} \overline{x}_{ij}) - SIR_{min} \sum_{k \neq i} \sum_j U_k g_{kj} \overline{x}_{ij} \beta_{kj} \leq U_i && \forall i \\
& \alpha_i \leq 0, \beta_{ij} \geq 0 && \forall i, \forall j
\end{aligned}$$

The master problem is then rewritten as

$$
\begin{aligned}
&[\text{CL-MP}] : \theta_1 = \min\ \theta_2 \\
&\text{s.t. } \theta_2 - \sum_i^I \sum_j^J M_{ij}\beta_{ij}^h x_{ij} \geq p_{max} \sum_i^I \alpha_i^h + \sum_i^I \sum_j^J (\eta SIR_{min} - M_{ij})\beta_{ij}^h && \forall h \quad (4.8)\\
&\quad - \sum_i^I \sum_j^J M_{ij}\beta_{ij}^h x_{ij} \geq p_{max} \sum_i^I \alpha_i^h + \sum_i^I \sum_j^J (\eta SIR_{min} - M_{ij})\beta_{ij}^h && \forall h \quad (4.9)\\
&\quad x_{ij} \leq \overline{y}_j && \forall i, \forall j \quad (4.10)\\
&\quad \sum_j^J x_{ij} \geq \overline{z}_i && \forall i \quad (4.11)\\
&\quad x_{ij} \in \{0, 1\}
\end{aligned}
$$

Further details about the [CB-SP] decomposition were presented in Section 2.3. As shown in Figure 4.1, the original problem [OP] is now solved via a Nested Benders decomposition. Since [CB-MP] is a relaxation of the original maximization problem [OP] then solving the [CB-MP] yields an upper bound (UB) for the optimal solution. On the other hand, if [CB-SP] (i.e. [CL-MP]-[CL-DSP]) is feasible, then solving [CL-MP]-[CL-DSP] iteratively to an optimal solution $\{\overline{x}_{ij}, \overline{p}_i\}$ along with the solution $\{\overline{y}_j, \overline{z}_i\}$ of [CB-MP] yield a feasible solution to the original problem. The objective function of the original problem evaluated at the point $\{\overline{x}_{ij}, \overline{y}_j, \overline{z}_i, \overline{p}_i\}$ gives a lower bound (LB). The upper and lower bounds are used as a stopping criterion. The algorithm is detailed below.

$$
\begin{aligned}
\text{[CB-MP]: } \max \sum_{i}^{I} r_i U_i z_i - \sum_{j}^{J} c_j y_j - \lambda\theta \\
\text{s.t. } \sum_{i:\overline{z}_i=0}^{I} z_i + \sum_{j:\overline{y}_j=0}^{J} y_j + \sum_{i:\overline{z}_i=1}^{I} (1-z_i) + \sum_{j:\overline{y}_j=1}^{J} (1-y_j) \geq 1 \qquad & \forall(\overline{z}_i, \overline{y}_j) \\
M \sum_{i:\overline{z}_i=0}^{I} z_i + M \sum_{j:\overline{y}_j=0}^{J} y_j + M \sum_{i:\overline{z}_i=1}^{I} (1-z_i) + M \sum_{j:\overline{y}_j=1}^{J} (1-y_j) + \theta \geq \theta_1^h \quad & \forall(\overline{z}_i, \overline{y}_j, \theta_1^h) \\
\sum_{i}^{I} U_i z_i \geq \pi \sum_{i}^{I} U_i \\
z_i, y_j \in \{0,1\}
\end{aligned}
$$

[CB-SP:] $\Downarrow$ Fix $(y_i, z_j)\forall i, \forall j$ $\Uparrow$ θ_1^h

$$
\begin{aligned}
\text{[CL-MP]: } \theta_1 = \min \theta_2 \\
\text{s.t. } \theta_2 - \sum_{i}^{I}\sum_{j}^{J} M_{ij}\beta_{ij}^h \overline{x}_{ij} \geq p_{max} \sum_{i}^{I} \alpha_i^h + \sum_{i}^{I}\sum_{j}^{J} (\eta SIR_{min} - M_{ij})\beta_{ij}^h \qquad & \forall h \\
-\sum_{i}^{I}\sum_{j}^{J} M_{ij}\beta_{ij}^h \overline{x}_{ij} \geq p_{max} \sum_{i}^{I} \alpha_i^h + \sum_{i}^{I}\sum_{j}^{J} (\eta SIR_{min} - M_{ij})\beta_{ij}^h \qquad & \forall h \\
x_{ij} \leq \overline{y}_j \qquad & \forall i, \forall j \\
\sum_{j}^{J} x_{ij} \geq \overline{z}_i \qquad & \forall i
\end{aligned}
$$

$\Downarrow$ Fix $x_{ij}\forall i, \forall j$ $\Uparrow$ α^h, β^h

$$
\begin{aligned}
\text{[CL-DSP]: } \max p_{max} \sum_{i}^{I} \alpha_i + \eta SIR_{min} \sum_{i}^{I}\sum_{j}^{J} \overline{x}_{ij}\beta_{ij} \\
\text{s.t. } \alpha_i + \sum_{j}^{J} \beta_{ij}(SFgij - (U_i - 1)g_{ij}SIR_{min}\overline{x}_{ij}) - SIR_{min}U_i \sum_{k \neq i}\sum_{j} g_{ij}\overline{x}_{kj}\beta_{kj} \leq U_i \quad & \forall i \\
\alpha_i \leq 0, \beta_{ij} \geq 0 \qquad \forall i, \forall j
\end{aligned}
$$

Figure 4.1: Nested Benders Decomposition.

Start with $UB_1 = \infty$; $LB_1 = -\infty$

While $LB_1 \neq UB_1$

Solve [CB-MP] and get solution $\{\overline{y}_j, \overline{z}_i\}$, and an upper bound UB_1.

While $LB_2 \neq UB_2$

Solve [CL-MP] and get solution $\{\overline{x}_{ij}\}$ and a lower bound LB_2.

IF [CL-MP] is feasible

Solve [CL-DSP]

If [CL-DSP] is unbounded

add feasibility cut to [CL-MP]

If [CL-DSP] is bounded

add optimality cut to [CL-MP]

Update upper bound $UB = \min(UB_2, UB_2^h)$

Else

[CB-SP] is infeasible

Break While

End IF

End while

IF [CB-SP] is feasible

Add optimality cut to [CB-MP]

Update LB_1 using current feasible solution $\{\overline{x}_{ij}, \overline{y}_j, \overline{z}_i, \overline{p}_i\}$

Else

Add feasibility cut to [CB-MP]

End IF

End while

The algorithm is summarized in Figure 4.1. A flowchart is presented in Figure 4.2. In an attempt to improve the performance of the nested Benders decomposition, the following section describes various valid inequalities that can be added to the problem.

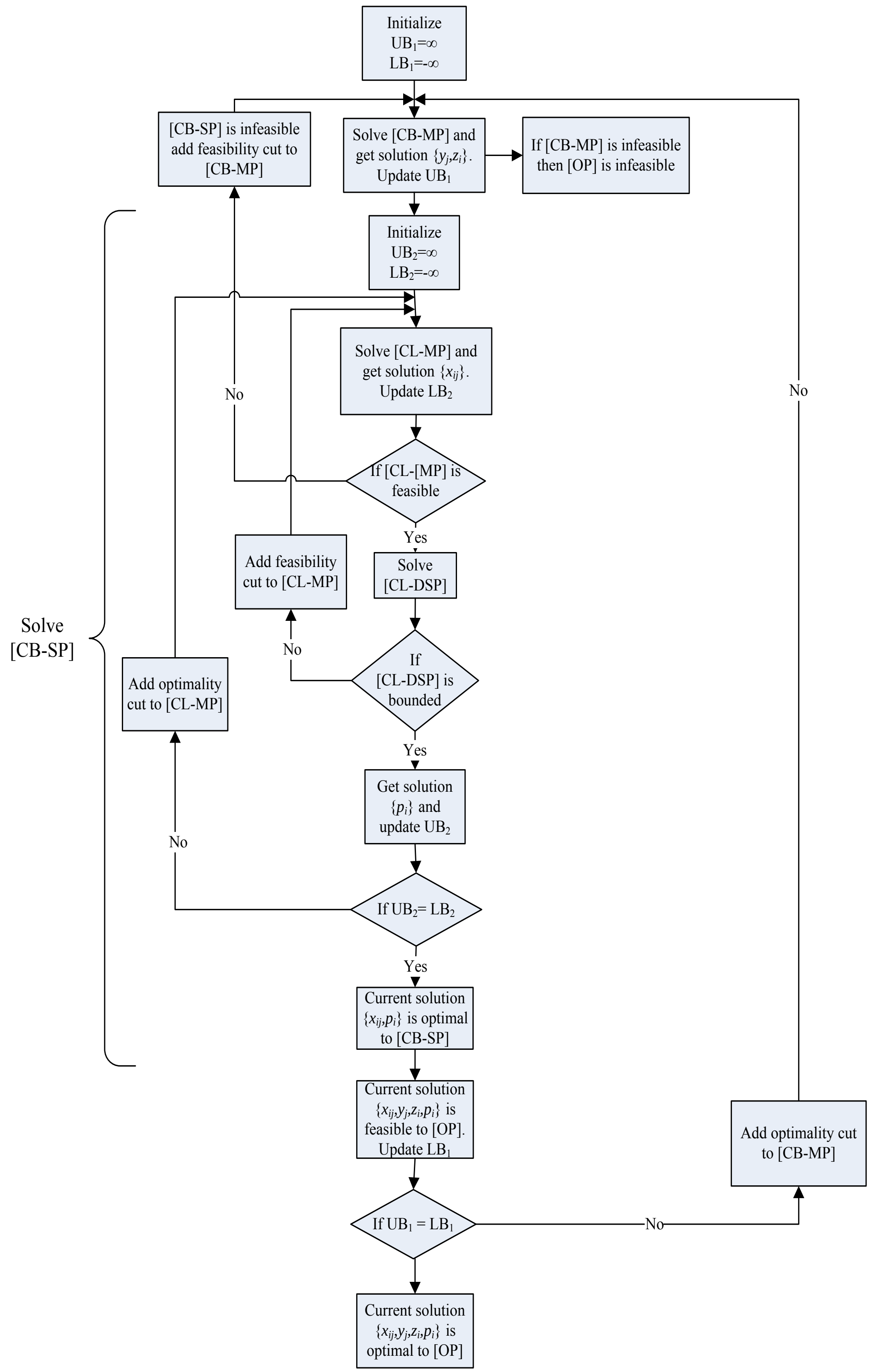

Figure 4.2: Nested Benders Decomposition Flowchart.

4.3 Valid Inequalities

Often, several iterations of the nested Benders algorithm are required before reaching an optimal solution. To reduce the number of iterations, valid inequalities may be added to the master problem. Additionally these inequalities will strengthen the LP bound thus accelerating the computational time required to solve the problem.

Proposition 3 *There is always an optimal solution with every demand point being assigned to exactly one base station.*

Proof: Consider an optimal solution for the problem in (2.1)-(2.5) where a demand point l is assigned to more than one base station indexed by set $\overline{J} \subseteq J$ such that $x^*_{lj} = 1$ for $j \in \overline{J}$.

Constraint (2.2) can be written as:

$$p_i(SFg_{ij} - (U_i - 1)g_{ij}SIR_{min}x_{ij}) \geq SIR_{min}x_{ij}(\sum_{k \neq i} U_k g_{kj} p_k + \eta) \quad \forall i, \forall j$$

Note that if $x_{ij} = 0$, this constraint is redundant. Otherwise if $x_{ij} = 1$, $SIR_{min}x_{ij}(\sum_{k \neq i} U_k g_{kj} p_k + \eta) > 0$ therefore $(SFg_{ij} - (U_i - 1)g_{ij}SIR_{min}x_{ij}) > 0$ and constraint (2.2) may be written as

$$p_i \geq \frac{SIR_{min}x_{ij}(\sum_{k \neq i} U_k g_{kj} p_k + \eta)}{SFg_{ij} - (U_i - 1)g_{ij}SIR_{min}x_{ij}} \quad \forall i, \forall j$$

This equation represents the power p_i required at DP i in order to meet the minimum SIR.

At optimality, the power transmitted by demand point l is:

$$p^*_l = \max_{j \in \overline{J}} \left\{ \frac{SIR_{min}x^*_{lj}(\sum_{k \neq l} U_k g_{kj} p^*_k + \eta)}{SFg_{lj} - (U_l - 1)g_{lj}SIR_{min}x^*_{lj}} \right\}$$

The objective function of [OP] evaluated at the optimal solution (x^*, y^*, z^*, p^*) is:

$$Z^*_{[OP]} = \sum_{i}^{I} r_i U_i z^*_i - \sum_{j}^{J} c_j y^*_j - \lambda(\sum_{i \neq l}^{I} U_i p^*_i + U_l p^*_l)$$

Consider a feasible solution $(\overline{x}, \overline{y}, \overline{z}, \overline{p})$ of [OP] where demand point l is assigned to a single base station $v \in \overline{J}$ such that $\overline{x}_{lv} = 1$ and $\overline{x}_{lj} = 0$ for $j \in \overline{J} \backslash \{v\}$. And let $\overline{y} = y^*, \overline{z} = z^*$, $\overline{p}_i = p_i^*$, $i \neq l$ and

$$\overline{p}_l = \frac{SIR_{min}\overline{x}_{lv}(\sum_{k\neq l} U_k g_{kv}\overline{p}_k + \eta)}{SFg_{lv} - (U_l - 1)g_{lv}SIR_{min}\overline{x}_{lv}} \leq \max_{j\in\overline{J}} \left\{ \frac{SIR_{min}x^*_{lj}(\sum_{k\neq l} U_k g_{kj}p^*_k + \eta)}{SFg_{lj} - (U_l - 1)g_{lj}SIR_{min}x^*_{lj}} \right\} = p^*_l.$$

By construction, $0 \leq \overline{p} \leq p^* \leq p_{max}$, and constraints (2.1)-(2.4) are satisfied. Therefore, $(\overline{x}, \overline{y}, \overline{z}, \overline{p})$ is a feasible solution. The objective function evaluated at $(\overline{x}, \overline{y}, \overline{z}, \overline{p})$ is:

$$\overline{Z}_{[OP]} = \sum_i^I r_i U_i \overline{z}_i - \sum_j^J c_j \overline{y}_j - \lambda(\sum_{i\neq l}^I U_i \overline{p}_i + U_l \overline{p}_l)$$

Since $\overline{p}_l \leq p^*_l$ then $\overline{Z}_{[OP]} \geq Z^*_{[OP]}$ therefore $(\overline{x}, \overline{y}, \overline{z}, \overline{p})$ is also an optimal solution. $\square$

The following valid inequality may be added to the problem in (2.1)-(2.5).

$$\sum_j x_{ij} \leq 1$$

Proposition 4 *If in an optimal solution $y^*_m = y^*_n = 1$, $g_{lm} < g_{ln}$ and $g_{km} > g_{kn}$ $\forall k \neq l$, then $x^*_{lm} = 0$*

Proof: Consider an optimal solution (x^*, y^*, z^*, p^*) of [OP] where $y^*_m = y^*_n = 1$. By proposition 3, there exists an optimal solution where demand point l is assigned to exactly one base station. Suppose demand point l is assigned to base station m such that $x^*_{lm} = 1$, $x^*_{ln} = 0$ and $x^*_{lj} = 0 \quad \forall j \neq n, m$, then

$$p^*_l = \max_j \left\{ \frac{SIR_{min}x^*_{lj}(\sum_{k\neq l} U_k g_{kj}p^*_k + \eta)}{SFg_{lj} - (U_l - 1)g_{lj}SIR_{min}x^*_{lj}} \right\} = \frac{SIR_{min}x^*_{lm}(\sum_{k\neq l} U_k g_{km}p^*_k + \eta)}{SFg_{lm} - (U_l - 1)g_{lm}SIR_{min}x^*_{lm}}.$$

Consider the alternative solution where demand point l is assigned to base station n such that $\overline{x}_{lm} = 0$, $\overline{x}_{ln} = 1$ and $\overline{x}_{lj} = 0 \quad \forall j \neq n, m$ then

$$\overline{p}_l = \max_j \left\{ \frac{SIR_{min}\overline{x}_{lj}(\sum_{k\neq l} U_k g_{kj}p^*_k + \eta)}{SFg_{lj} - (U_l - 1)g_{lj}SIR_{min}\overline{x}_{lj}} \right\} = \frac{SIR_{min}\overline{x}_{ln}(\sum_{k\neq l} U_k g_{kn}p^*_k + \eta)}{SFg_{ln} - (U_l - 1)g_{ln}SIR_{min}\overline{x}_{ln}}.$$

Since $g_{km} > g_{kn}$ and $g_{lm} < g_{ln}$ then it follows that

$$\overline{p}_l < p_l^*.$$

Note that since $\overline{p}_l < p_l^*$ therefore constraints (2.1)-(2.5) are satisfied, the solution $(\overline{x}, \overline{y}, \overline{z}, \overline{p})$ with $\overline{x}_{ln} = 1$ is a feasible solution and

$$\overline{Z}_{[OP]} < Z^*_{[OP]}$$

Therefore a feasible solution with $x_{lm} = 1$ can not be optimal. $\square$

To reduce the complexity of the problem, Amaldi et al. (2002), Amaldi et al. (2003) and Kalvenes et al. (2006) assume a power based power control and show that a DP is always connected to the BS with the highest gain. Proposition 4 shows that this assumption is not valid in UMTS/W-CDMA systems where power control is managed by SIR levels rather than power levels.

Proposition 5 *Given a feasibility cut (equation* (4.9)*): (that is generated at [CL-DSP] to be added to [CL-MP])*

$$-\sum_i^I \sum_j^J M_{ij}\beta_{ij}^h x_{ij} \geq p_{max} \sum_i^I \alpha_i^h + \sum_i^I \sum_j^J (\eta SIR_{min} - M_{ij})\beta_{ij}^h$$

corresponding to variables $y^*_{j_1} = y^*_{j_2} = ... = y^*_{j_s} = 1$*,* $z^*_{i_1} = z^*_{i_2} = ... = z^*_{i_s} = 1$ *and* $x^*_{i_1 j_1} = x^*_{i_2 j_2} = ... = x^*_{i_s j_s} = 1$ *for which* $\beta_{ij}^h > 0$*. If the following conditions are satisfied:*

$$(C): \begin{cases} g_{i_1 j} < g_{i_1 j_1} \, and \; g_{ij} > g_{ij_1}, & \forall i \neq i_1 \\ \vdots & \\ g_{i_s j} < g_{i_s j_s} \, and \; g_{ij} > g_{ij_s}, & \forall i \neq i_s, \end{cases}$$

then, the cut

$$\sum_{i \in \{i_1, i_2, \ldots, i_s\}} z_i - \sum_{j \notin \{j_1, j_2, \ldots, j_s\}} y_j \leq |s| - 1 \tag{4.12}$$

is valid to [CB-MP].

Proof: Given (y^*, z^*) values and condition (C), Proposition 2 implies that $x^*_{i_1j_1} = x^*_{i_2j_2} = ... = x^*_{i_sj_s} = 1$ in any optimal solution. First, note that this partial solution is infeasible to [CL-MP], and in particular to the feasibility cut:

$$-\sum_{i}^{I}\sum_{j}^{J} M_{ij}\beta^h_{ij}x_{ij} \;\;\geq\;\; p_{max}\sum_{i}^{I}\alpha^h_i + \sum_{i}^{I}\sum_{j}^{J}(\eta SIR_{min} - M_{ij})\beta^h_{ij}.$$

The latter inequality can be written as:

$$\sum_{p=1}^{s}\left(M_{i_pj_p}\beta^h_{i_pj_p}\right)x_{i_pj_p} \leq r \tag{4.13}$$

where $r = \sum_{p=1}^{s}(M_{i_pj_p} - \eta SIR_{min})\beta^h_{i_pj_p} - p_{max}\sum_{i}^{I}\alpha^h_i$, and is non-negative. Therefore, equation (4.13) implies that at least one of the $\left\{x^*_{i_1j_1}, x^*_{i_2j_2}, ..., x^*_{i_sj_s}\right\}$ has to be 0. So, at least one demand point among $\{i_1, i_2, ..., i_s\}$ should not be serviced, or at least one base station other than $\{j_1, j_2, ..., j_s\}$ should be selected (i.e. one or more of the demand points $\{i_1, i_2, ..., i_s\}$ is connected to a base station other than $\{j_1, j_2, ..., j_s\}$). This is captured by the cut $\sum\limits_{i\in\{i_1,i_2,...,i_s\}} z_i - \sum\limits_{j\notin\{j_1,j_2,...,j_s\}} y_j \leq |s| - 1$. □

Note that the valid cut (4.12) may be extended to lead to multiple cuts of the same form. The idea is that if any subset of $\left\{x^*_{i_1j_1}, x^*_{i_2j_2}, ..., x^*_{i_sj_s}\right\}$ violates (4.13), then by the previous Proposition, a cut of the form (4.12) is valid to [CB-MP]. This is detailed in the next proposition:

Proposition 6 *If a cut*

$$\sum_{i\in\{i_1,i_2,...,i_s\}} z_i - \sum_{j\notin\{j_1,j_2,...,j_s\}} y_j \leq |s| - 1$$

is valid to [CB-MP] and a subset of $\{x^*_{i_1j_1}, x^*_{i_2j_2}, ..., x^*_{i_sj_s}\}$ *indexed by* $\Psi^c \subseteq \{(i_1, j_1), (i_2, j_2), ..., (i_s, j_s)\}$, *denoted by* $\{x^*_{i_1j_1}, x^*_{i_2j_2}, ..., x^*_{i_qj_q}\}$ *violates* $\sum_{p=1}^{q}\left(M_{i_pj_p}\beta^h_{i_pj_p}\right)x_{i_pj_p} \leq r$, *then*

$$\sum_{i\in\{i_1,i_2,...,i_q\}} z_i - \sum_{j\notin\{j_1,j_2,...,j_q\}} y_j \leq |q| - 1 \tag{4.14}$$

is valid to [CB-MP]. □

Note that finding all the cuts that satisfy Proposition 4 involves enumerating all possible subsets Ψ^c, and selecting the ones that violate equation (4.13). To avoid total enumeration, we describe a heuristic that only enumerates the subsets that violate equation (4.13). The feasibility cut

$$\sum_{(i,j)} \left(M_{ij}\beta_{ij}^{h}\right) x_{ij} \leq r \tag{4.15}$$

that is generated from [CL-DSP] has the form of a knapsack constraint. Finding all possible x_{ij} combinations that violate the latter equation is a well known $\mathcal{NP}$-Hard problem. The algorithm that avoids trying all possible x_{ij} combinations proceeds as follows:

1. The coefficients of x_{ij} are sorted in a decreasing order.
2. Set m to be the number of x_{ij}.
3. Let k be the number of x_{ij} that take the value of 1. Start with $k = 1$.
4. If $k = m$ go to step 11.
5. Set the k left most x_{ij} variables to 1. If the resulting combination does not violate the constraint, increment k and go to step 4.
6. Set $n = k$ and select the 1 that is at position n.
7. If the value to the right of the selected 1 is 0, shift the selected 1 to the right, otherwise go to step 10.
8. If the resulting combination does not violate the constraint, shift selected 1 to the left and go to step 10.
9. Place all the 1s that are to the right of the selected 1 at the positions that are directly to its right. Go to step 7.
10. Starting from position $n-1$ and moving to the left, find the nearest 1. Decrement n and go to step 7. If none can be found, increment k and go to step 4.

11. End.

To illustrate the above algorithm, we consider the following example where a feasibility cut

$$5x_1 + 23x_2 + 16x_3 + 22x_4 \leq 28$$

is generated from [CL-DSP]. The detailed steps of the algorithm are shown in Table 4.1. Note that the algorithm enumerated ten combinations while the total enumeration has 16 combinations.

Step					k	n	Description
1	23	22	16	5	-	-	Sort Coefficients
5	1	0	0	0	1	1	Left most is set to 1. Constraint not violated
5	1	1	0	0	2	2	Left most are set to 1. Constraint is violated
6	1	**1**	0	0	2	2	1 at position 2 is selected
7	1	0	**1**	0	2	2	Shift selected 1 to the right
8	1	0	**1**	0	2	2	Constraint is violated
7	1	0	0	**1**	2	2	Shift selected 1 to the right
8	1	0	0	**1**	2	2	Constraint is not violated
8	1	0	**1**	0	2	2	Shift selected 1 to the left
10	**1**	0	1	0	2	1	1 at position 1 is selected
7	0	**1**	1	0	2	1	Shift selected 1 to the right
8	0	**1**	1	0	2	1	Constraint is violated
7	0	**1**	1	0	2	1	Go to step 10
5	1	1	1	0	3	3	Left most are set to 1. Constraint is violated
6	1	1	**1**	0	3	3	1 at position 3 is selected
7	1	1	0	**1**	3	3	Shift selected 1 to the right
8	1	1	0	**1**	3	3	Constraint is violated
10	1	**1**	0	1	3	2	1 at position 2 is selected
7	1	0	**1**	1	3	2	Shift selected 1 to the right
8	1	0	**1**	1	3	2	Constraint is violated
10	**1**	0	1	1	3	1	1 at position 1 is selected
7	0	**1**	1	1	3	1	Shift selected 1 to the right
8	0	**1**	1	1	3	1	Constraint is violated
10	0	**1**	1	1	4	1	No 1 is found at position $n-1$
4	1	1	1	1	4	1	$k = m$. Algorithm terminates

Table 4.1: Example of the search algorithm with $m = 4$

4.4 Computational Results

In this section, we evaluate the nested Benders decomposition algorithm in solving the UMTS/W-CDMA network planning problem. Potential base station and user locations were randomly selected from instances proposed by Amaldi et al. (2002). The parameters used for the test problems are shown in Table 3.1. For each test case, we ran 10 different problems and took the average. Tables 4.2, 4.3 and 4.4 show results for 111 test cases. The first column of the table indicates the test case number. The second column displays the uniform distribution from which the demand is drawn. The third and fourth columns indicate the number of demand point locations (DP) and the candidate base station locations (BS) respectively. Columns (5) and (6) indicate the number of times (Iter.) [CB-MP] is solved and the total CPU time spent solving [CB-MP]. Columns (7) and (8) indicate the number of times (Iter.) [CL-MP] is solved and the total CPU time spent solving [CL-MP]. Columns (9) and (10) indicate the number of times (Iter.) [CL-DSP] is solved and the total CPU time spent solving [CL-DSP]. Columns (11) and (12) indicate the number of optimality cuts (equation (4.7)) and the number of feasibility cuts (equation (4.6)) added to [CB-MP]. Columns (13) and (14) indicate the number of optimality cuts (equation (4.8)) and the number of feasibility cuts (equation (4.9)) added to [CL-MP]. Columns (15) and (16) indicate the number of valid cuts (equation (4.14)) added to [CB-MP] as well as the total CPU time taken by the heuristic to generate these cuts. Finally, column (17) indicates the total CPU time in seconds spent on solving each test case. All test cases are solved to optimality (0% gap).

First, we increase the number of demand points. Then, we increase the number of candidate base station locations. Finally, we increase the users' density on each demand point. We notice that the complexity of the problem increases with an increase in the number of users at each demand location. Moreover, for a low user density $U[1, 8]$, no valid cuts are generated using the heuristic described in Section 4.3. For a higher user density

$U[1, 32]$, valid cuts are generated. We observe that the heuristic consumes at most 2% of the total CPU time of each instance. Additionally, we evaluate the effect of adding valid cuts. For instances where valid cuts are generated (instances: 55, 66, 77, 88, 95, 96, 97, 98, 99, 106, 107, 108 and 109), the heuristic that generates the valid cuts is deactivated and the performance of the algorithm is evaluated. We observe that the addition of valid cuts significantly reduces the number of iterations and therefore the solution time. Moreover, test case 98 consumes 38% less computational time when valid cuts are added. Test case 106 consumes 7% less computational time when valid cuts are added. Furthermore, test case 99 could not be solved within a time limit of 96 hours whereas it was solved in 14572 seconds when valid cuts were added. Overall, a better performance is always achieved when valid cuts are added. Finally for small instances, we observe that increasing the number of demand points increases the time consumed on solving the subproblem. However, this does not affect the time consumed on solving the master problem. The reason is that for small instances all solutions suggested by the master problem are feasible to the subproblem. For large instances, some solutions suggested by the master problem are infeasible for the subproblem. This would require extra iterations to eliminate the infeasible solutions.

				[CB-MP]		[CL-MP]		[CL-DSP]						Valid Cuts		
Test Case (1)	d_m (2)	#DP (3)	#BS (4)	Iter (5)	CPU (s) (6)	Iter (7)	CPU (s) (8)	Iter (9)	CPU (s) (10)	Opt. Cut (4.7) (11)	Feas. Cut (4.6) (12)	Opt. Cut (4.8) (13)	Feas. Cut (4.9) (14)	Val. Cut (4.14) (15)	CPU (s) (16)	CPU (s) (17)
1	U[1,8]	20	10	11.00	0.07	21.00	0.15	21.00	0.15	11.00	0.00	21.00	0.00	0.00	0.00	0.79
2	U[1,8]	30	10	11.00	0.07	21.00	0.23	21.00	0.22	11.00	0.00	21.00	0.00	0.00	0.00	1.26
3	U[1,8]	40	10	11.00	0.07	21.00	0.35	21.00	0.32	11.00	0.00	21.00	0.00	0.00	0.00	2.00
4	U[1,8]	50	10	11.00	0.07	21.00	0.53	21.00	0.48	11.00	0.00	21.00	0.00	0.00	0.00	2.82
5	U[1,8]	60	10	11.00	0.07	21.00	0.69	21.00	0.68	11.00	0.00	21.00	0.00	0.00	0.00	3.84
6	U[1,8]	70	10	11.00	0.08	21.00	0.87	21.00	0.93	11.00	0.00	21.00	0.00	0.00	0.00	5.09
7	U[1,8]	80	10	11.00	0.09	21.00	1.09	21.00	1.25	11.00	0.00	21.00	0.00	0.00	0.00	6.62
8	U[1,8]	90	10	11.00	0.09	21.00	1.32	21.00	1.54	11.00	0.00	21.00	0.00	0.00	0.00	8.12
9	U[1,8]	100	10	11.00	0.09	21.00	1.61	21.00	1.93	11.00	0.00	21.00	0.00	0.00	0.00	9.97
10	U[1,8]	120	10	11.00	0.10	21.00	2.21	21.00	2.90	11.00	0.00	21.00	0.00	0.00	0.00	14.38
11	U[1,8]	140	10	11.00	0.10	21.00	2.88	21.00	4.25	11.00	0.00	21.00	0.00	0.00	0.00	20.00
12	U[1,8]	20	15	16.00	0.13	31.00	0.33	31.00	0.22	16.00	0.00	31.00	0.00	0.00	0.00	1.58
13	U[1,8]	30	15	16.00	0.13	31.00	0.62	31.00	0.35	16.00	0.00	31.00	0.00	0.00	0.00	2.77
14	U[1,8]	40	15	16.00	0.14	31.00	0.98	31.00	0.53	16.00	0.00	31.00	0.00	0.00	0.00	4.32
15	U[1,8]	50	15	16.00	0.14	31.00	1.38	31.00	0.79	16.00	0.00	31.00	0.00	0.00	0.00	6.26
16	U[1,8]	60	15	16.00	0.15	31.00	1.90	31.00	1.08	16.00	0.00	31.00	0.00	0.00	0.00	8.61
17	U[1,8]	70	15	16.00	0.15	31.00	2.46	31.00	1.51	16.00	0.00	31.00	0.00	0.00	0.00	11.50
18	U[1,8]	80	15	16.00	0.16	31.00	3.08	31.00	1.92	16.00	0.00	31.00	0.00	0.00	0.00	14.61
19	U[1,8]	90	15	16.00	0.16	31.00	3.81	31.00	2.43	16.00	0.00	31.00	0.00	0.00	0.00	18.31
20	U[1,8]	100	15	16.00	0.17	31.00	4.63	31.00	3.02	16.00	0.00	31.00	0.00	0.00	0.00	22.41
21	U[1,8]	120	15	16.00	0.19	31.00	6.38	31.00	4.57	16.00	0.00	31.00	0.00	0.00	0.00	31.98
22	U[1,8]	140	15	16.00	0.20	31.00	8.47	31.00	6.57	16.00	0.00	31.00	0.00	0.00	0.00	43.94
23	U[1,8]	20	25	26.00	0.31	51.00	1.19	51.00	0.43	26.00	0.00	51.00	0.00	0.00	0.00	4.60
24	U[1,8]	30	25	26.00	0.33	51.00	2.25	51.00	0.69	26.00	0.00	51.00	0.00	0.00	0.00	8.37
25	U[1,8]	40	25	26.00	0.32	51.00	3.62	51.00	1.06	26.00	0.00	51.00	0.00	0.00	0.00	13.42
26	U[1,8]	50	25	26.00	0.35	51.00	5.32	51.00	1.52	26.00	0.00	51.00	0.00	0.00	0.00	19.87
27	U[1,8]	60	25	26.00	0.35	51.00	7.33	51.00	2.09	26.00	0.00	51.00	0.00	0.00	0.00	27.57
28	U[1,8]	70	25	26.00	0.37	51.00	9.74	51.00	2.85	26.00	0.00	51.00	0.00	0.00	0.00	36.97
29	U[1,8]	80	25	26.00	0.39	51.00	12.34	51.00	3.65	26.00	0.00	51.00	0.00	0.00	0.00	47.25
30	U[1,8]	90	25	26.00	0.39	51.00	15.27	51.00	4.66	26.00	0.00	51.00	0.00	0.00	0.00	59.10
31	U[1,8]	100	25	26.00	0.40	51.00	18.50	51.00	5.68	26.00	0.00	51.00	0.00	0.00	0.00	72.11
32	U[1,8]	120	25	26.00	0.43	51.00	26.18	51.00	8.33	26.00	0.00	51.00	0.00	0.00	0.00	102.65
33	U[1,8]	140	25	26.00	0.46	51.00	34.90	51.00	11.89	26.00	0.00	51.00	0.00	0.00	0.00	139.70
34	U[1,8]	20	30	31.00	0.45	61.00	1.88	61.00	0.55	31.00	0.00	61.00	0.00	0.00	0.00	6.92
35	U[1,8]	30	30	31.00	0.45	61.00	3.61	61.00	0.90	31.00	0.00	61.00	0.00	0.00	0.00	12.95
36	U[1,8]	40	30	31.00	0.46	61.00	5.89	61.00	1.38	31.00	0.00	61.00	0.00	0.00	0.00	20.84
37	U[1,8]	50	30	31.00	0.49	61.00	8.73	61.00	1.94	31.00	0.00	61.00	0.00	0.00	0.00	31.06
38	U[1,8]	60	30	31.00	0.51	61.00	12.16	61.00	2.66	31.00	0.00	61.00	0.00	0.00	0.00	43.09
39	U[1,8]	70	30	31.00	0.52	61.00	15.98	61.00	3.65	31.00	0.00	61.00	0.00	0.00	0.00	57.83
40	U[1,8]	80	30	31.00	0.55	61.00	20.45	61.00	4.66	31.00	0.00	61.00	0.00	0.00	0.00	73.85
41	U[1,8]	90	30	31.00	0.56	61.00	25.52	61.00	5.91	31.00	0.00	61.00	0.00	0.00	0.00	92.75
42	U[1,8]	100	30	31.00	0.58	61.00	31.06	61.00	7.27	31.00	0.00	61.00	0.00	0.00	0.00	113.11
43	U[1,8]	120	30	31.00	0.60	61.00	43.83	61.00	10.52	31.00	0.00	61.00	0.00	0.00	0.00	161.51
44	U[1,8]	140	30	31.00	0.65	61.00	58.66	61.00	14.90	31.00	0.00	61.00	0.00	0.00	0.00	219.34
45	U[1,16]	20	10	11.00	0.07	21.00	0.13	21.00	0.12	11.00	0.00	21.00	0.00	0.00	0.00	0.75

Table 4.2: Computational Results - Nested Benders Decomposition

				[CB-MP]		[CL-MP]		[CL-DSP]						Valid Cuts		
Test Case (1)	d_m (2)	#DP (3)	#BS (4)	Iter (5)	CPU (s) (6)	Iter (7)	CPU (s) (8)	Iter (9)	CPU (s) (10)	Opt. Cut (4.7) (11)	Feas. Cut (4.6) (12)	Opt. Cut (4.8) (13)	Feas. Cut (4.9) (14)	Val. Cut (4.14) (15)	CPU (s) (16)	CPU (s) (17)
46	U[1,16]	30	10	11.00	0.07	21.00	0.24	21.00	0.21	11.00	0.00	21.00	0.00	0.00	0.00	1.27
47	U[1,16]	40	10	11.00	0.07	21.00	0.35	21.00	0.32	11.00	0.00	21.00	0.00	0.00	0.00	1.93
48	U[1,16]	50	10	11.00	0.07	21.00	0.50	21.00	0.47	11.00	0.00	21.00	0.00	0.00	0.00	2.79
49	U[1,16]	60	10	11.00	0.07	21.00	0.68	21.00	0.66	11.00	0.00	21.00	0.00	0.00	0.00	3.82
50	U[1,16]	70	10	11.00	0.08	21.00	0.85	21.00	0.92	11.00	0.00	21.00	0.00	0.00	0.00	5.11
51	U[1,16]	80	10	11.00	0.08	21.00	1.08	21.00	1.17	11.00	0.00	21.00	0.00	0.00	0.00	6.54
52	U[1,16]	90	10	11.00	0.09	21.00	1.32	21.00	1.52	11.00	0.00	21.00	0.00	0.00	0.00	8.10
53	U[1,16]	100	10	11.00	0.09	21.00	1.59	21.00	1.90	11.00	0.00	21.00	0.00	0.00	0.00	9.94
54	U[1,16]	120	10	11.00	0.10	21.00	2.18	21.00	2.83	11.00	0.00	21.00	0.00	0.00	0.00	14.30
55	U[1,16]	140	10	31.80	12.14	62.60	11.35	45.30	8.96	14.50	17.30	28.00	17.30	17.30	0.50	71.23
55*	U[1,16]	140	10	71.50	33.02	103.90	19.35	46.90	9.44	14.50	57.00	28.00	18.90	0.00	0.00	113.76
56	U[1,16]	20	15	16.00	0.11	31.00	0.33	31.00	0.21	16.00	0.00	31.00	0.00	0.00	0.00	1.57
57	U[1,16]	30	15	16.00	0.12	31.00	0.62	31.00	0.34	16.00	0.00	31.00	0.00	0.00	0.00	2.74
58	U[1,16]	40	15	16.00	0.14	31.00	0.99	31.00	0.55	16.00	0.00	31.00	0.00	0.00	0.00	4.33
59	U[1,16]	50	15	16.00	0.15	31.00	1.39	31.00	0.76	16.00	0.00	31.00	0.00	0.00	0.00	6.31
60	U[1,16]	60	15	16.00	0.15	31.00	1.89	31.00	1.07	16.00	0.00	31.00	0.00	0.00	0.00	8.61
61	U[1,16]	70	15	16.00	0.14	31.00	2.47	31.00	1.47	16.00	0.00	31.00	0.00	0.00	0.00	11.55
62	U[1,16]	80	15	16.00	0.16	31.00	3.11	31.00	1.91	16.00	0.00	31.00	0.00	0.00	0.00	14.70
63	U[1,16]	90	15	16.00	0.17	31.00	3.83	31.00	2.41	16.00	0.00	31.00	0.00	0.00	0.00	18.40
64	U[1,16]	100	15	16.00	0.17	31.00	4.64	31.00	3.02	16.00	0.00	31.00	0.00	0.00	0.00	22.50
65	U[1,16]	120	15	16.00	0.20	31.00	6.43	31.00	4.52	16.00	0.00	31.00	0.00	0.00	0.00	32.10
66	U[1,16]	140	15	28.50	2.90	56.00	17.07	43.10	9.01	15.60	12.90	30.20	12.90	12.90	0.40	80.09
66*	U[1,16]	140	15	41.50	3.58	70.60	22.23	44.70	9.36	15.60	25.90	30.20	14.50	0.00	0.00	96.46
67	U[1,16]	20	25	26.00	0.32	51.00	1.20	51.00	0.43	26.00	0.00	51.00	0.00	0.00	0.00	4.62
68	U[1,16]	30	25	26.00	0.32	51.00	2.27	51.00	0.70	26.00	0.00	51.00	0.00	0.00	0.00	8.37
69	U[1,16]	40	25	26.00	0.33	51.00	3.65	51.00	1.07	26.00	0.00	51.00	0.00	0.00	0.00	13.53
70	U[1,16]	50	25	26.00	0.35	51.00	5.31	51.00	1.51	26.00	0.00	51.00	0.00	0.00	0.00	19.91
71	U[1,16]	60	25	26.00	0.37	51.00	7.37	51.00	2.07	26.00	0.00	51.00	0.00	0.00	0.00	27.69
72	U[1,16]	70	25	26.00	0.38	51.00	9.70	51.00	2.83	26.00	0.00	51.00	0.00	0.00	0.00	36.94
73	U[1,16]	80	25	26.00	0.38	51.00	12.35	51.00	3.65	26.00	0.00	51.00	0.00	0.00	0.00	47.30
74	U[1,16]	90	25	26.00	0.40	51.00	15.25	51.00	4.59	26.00	0.00	51.00	0.00	0.00	0.00	58.93
75	U[1,16]	100	25	26.00	0.43	51.00	18.60	51.00	5.65	26.00	0.00	51.00	0.00	0.00	0.00	72.13
76	U[1,16]	120	25	26.00	0.42	51.00	26.08	51.00	8.21	26.00	0.00	51.00	0.00	0.00	0.00	102.40
77	U[1,16]	140	25	46.50	12.92	92.00	73.67	71.20	16.86	25.70	20.80	50.40	20.80	20.80	0.70	277.28
77*	U[1,16]	140	25	68.50	13.16	114.80	94.46	72.00	17.01	25.70	42.80	50.40	21.60	0.00	0.00	338.10
78	U[1,16]	20	30	31.00	0.45	61.00	1.90	61.00	0.56	31.00	0.00	61.00	0.00	0.00	0.00	6.93
79	U[1,16]	30	30	31.00	0.46	61.00	3.64	61.00	0.91	31.00	0.00	61.00	0.00	0.00	0.00	12.92
80	U[1,16]	40	30	31.00	0.47	61.00	5.90	61.00	1.37	31.00	0.00	61.00	0.00	0.00	0.00	20.90
81	U[1,16]	50	30	31.00	0.49	61.00	8.74	61.00	1.94	31.00	0.00	61.00	0.00	0.00	0.00	31.09
82	U[1,16]	60	30	31.00	0.52	61.00	12.11	61.00	2.67	31.00	0.00	61.00	0.00	0.00	0.00	43.15
83	U[1,16]	70	30	31.00	0.52	61.00	16.03	61.00	3.66	31.00	0.00	61.00	0.00	0.00	0.00	57.96
84	U[1,16]	80	30	31.00	0.55	61.00	20.49	61.00	4.68	31.00	0.00	61.00	0.00	0.00	0.00	74.00
85	U[1,16]	90	30	31.00	0.55	61.00	25.66	61.00	5.86	31.00	0.00	61.00	0.00	0.00	0.00	93.31
86	U[1,16]	100	30	31.00	0.58	61.00	30.99	61.00	7.22	31.00	0.00	61.00	0.00	0.00	0.00	113.35

*: Valid cuts are not added

Table 4.3: Computational Results - Nested Benders Decomposition

				[CB-MP]		[CL-MP]		[CL-DSP]						Valid Cuts		
Test Case (1)	d_m (2)	#DP (3)	#BS (4)	Iter (5)	CPU (s) (6)	Iter (7)	CPU (s) (8)	Iter (9)	CPU (s) (10)	Opt. Cut (4.7) (11)	Feas. Cut (4.6) (12)	Opt. Cut (4.8) (13)	Feas. Cut (4.9) (14)	Val. Cut (4.14) (15)	CPU (s) (16)	CPU (s) (17)
87	U[1,16]	120	30	31.00	0.63	61.00	43.86	61.00	10.38	31.00	0.00	61.00	0.00	0.00	0.00	161.76
88	U[1,16]	140	25	46.50	12.27	92.00	66.59	71.20	16.04	25.70	20.80	50.40	20.80	20.80	0.69	247.63
88*	U[1,16]	140	30	43.00	1.69	81.70	87.59	67.00	16.69	28.30	14.70	55.60	11.40	0.00	0.00	315.67
89	U[1,32]	20	10	11.00	0.07	21.00	0.13	21.00	0.12	11.00	0.00	21.00	0.00	0.00	0.00	0.74
90	U[1,32]	30	10	11.00	0.07	21.00	0.24	21.00	0.21	11.00	0.00	21.00	0.00	0.00	0.00	1.26
91	U[1,32]	40	10	11.00	0.07	21.00	0.36	21.00	0.31	11.00	0.00	21.00	0.00	0.00	0.00	1.94
92	U[1,32]	50	10	11.00	0.08	21.00	0.50	21.00	0.47	11.00	0.00	21.00	0.00	0.00	0.00	2.79
93	U[1,32]	60	10	11.00	0.07	21.00	0.68	21.00	0.64	11.00	0.00	21.00	0.00	0.00	0.00	3.82
94	U[1,32]	70	10	11.00	0.08	21.00	0.88	21.00	0.88	11.00	0.00	21.00	0.00	0.00	0.00	5.08
95	U[1,32]	80	10	77.90	9.97	155.20	10.92	109.00	5.97	31.70	46.20	62.80	46.20	46.20	0.69	56.86
95*	U[1,32]	80	10	113.00	11.66	190.70	14.01	109.40	6.07	31.70	81.30	62.80	46.60	0.00	0.00	67.87
96	U[1,32]	90	10	143.80	25.73	287.10	24.69	185.80	12.84	42.50	101.30	84.50	101.30	101.30	1.62	128.69
96*	U[1,32]	90	10	243.00	35.79	388.20	33.93	187.70	13.01	42.50	200.50	84.50	103.20	0.00	0.00	165.46
97	U[1,32]	100	10	145.70	28.27	291.10	28.64	181.80	15.10	36.40	109.30	72.40	109.40	109.40	1.81	150.35
97*	U[1,32]	100	10	274.00	57.68	419.30	43.22	181.70	15.47	36.40	237.60	72.40	109.30	0.00	0.00	222.95
98	U[1,32]	120	10	141.70	31.41	282.90	35.72	156.20	17.76	15.00	126.70	29.00	127.20	127.20	2.21	188.58
98*	U[1,32]	120	10	314.00	74.46	455.90	59.97	156.90	18.39	15.00	299.00	29.00	127.90	0.00	0.00	305.93
99	U[1,32]	140	10	874.40	12595.35	1749.90	586.22	933.70	149.97	58.20	816.20	116.40	817.30	817.30	17.87	14572.64
99*	U[1,32]	140	10	-	-	-	-	-	-	-	-	-	-	-	-	-
100	U[1,32]	20	15	16.00	0.14	31.00	0.34	31.00	0.23	16.00	0.00	31.00	0.00	0.00	0.00	1.66
101	U[1,32]	30	15	16.00	0.13	31.00	0.62	31.00	0.35	16.00	0.00	31.00	0.00	0.00	0.00	2.78
102	U[1,32]	40	15	16.00	0.13	31.00	0.98	31.00	0.54	16.00	0.00	31.00	0.00	0.00	0.00	4.37
103	U[1,32]	50	15	16.00	0.14	31.00	1.41	31.00	0.77	16.00	0.00	31.00	0.00	0.00	0.00	6.29
104	U[1,32]	60	15	16.00	0.14	31.00	1.87	31.00	1.05	16.00	0.00	31.00	0.00	0.00	0.00	8.61
105	U[1,32]	70	15	16.00	0.16	31.00	2.46	31.00	1.40	16.00	0.00	31.00	0.00	0.00	0.00	11.43
106	U[1,32]	80	15	139.30	45.67	279.40	38.90	209.40	12.60	69.30	70.00	139.40	70.00	70.00	1.09	184.90
106*	U[1,32]	80	15	191.50	42.34	331.20	45.12	209.00	12.51	69.30	122.20	139.40	69.60	0.00	0.00	198.18
107	U[1,32]	90	15	256.20	122.39	513.70	88.38	356.60	27.03	99.10	157.10	199.20	157.40	157.40	2.69	433.32
107*	U[1,32]	90	15	401.50	130.79	660.10	110.31	357.70	26.94	99.10	302.40	199.20	158.50	0.00	0.00	500.00
108	U[1,32]	100	15	255.60	137.56	512.50	101.29	340.50	31.21	83.60	172.00	167.90	172.60	172.60	2.99	503.84
108*	U[1,32]	100	15	448.00	205.34	706.00	134.96	341.60	31.46	83.60	364.40	167.90	173.70	0.00	0.00	665.17
109	U[1,32]	120	15	249.00	162.02	499.00	126.82	285.20	36.04	35.20	213.80	70.10	215.10	215.10	4.01	633.51
109*	U[1,32]	120	15	508.00	262.69	760.80	187.45	288.00	36.28	35.20	472.80	70.10	217.90	0.00	0.00	906.98
110	U[1,8]	140	20	1.00	0.01	1.00	0.47	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00	1.74
111	U[1,16]	140	20	11.70	0.50	11.70	5.45	0.00	0.00	1.00	10.70	0.00	0.00	0.00	0.00	20.39

*: Valid cuts are not added

-: Not solved within the time limit 96 hours

Table 4.4: Computational Results - Nested Benders Decomposition

Chapter 5

Conclusion

The thesis presents and solves an analytic model of the UMTS/W-CDMA network planning problem. It helps network operators achieve business objectives by determining the capital required to build a cost effective infrastructure, optimizing network assets, and improving performance. The solved analytic model is a profit maximization model for the UMTS/W-CDMA network planning problem with SIR based power control. The proposed non-linear mixed integer program is first linearized and then solved using Benders decomposition. Two novel algorithmic ideas were developed. The first is a two-phase ACCPM-based Benders decomposition, that uses ACCPM for the master problem. Five heuristics are described to find feasible integer solutions at which cuts are generated. More importantly, the first heuristic is generalizable to find feasible integer solutions for mixed integer problems. The computational results showed that generating cuts from integer points improves the efficiency of the two-phase ACCPM algorithm. It was shown that the two-phase ACCPM algorithm achieves better computational performance compared to the classical Benders decomposition.

Second, we propose and test a nested Benders decomposition algorithm. More than 110 instances of the UMTS/W-CDMA network planning problem with sizes up to 140 demand point locations and 30 candidate base station locations are solved.

Future work on the UMTS/W-CDMA network planning problem will incorporate demand uncertainties into the model. A simulation-optimization approach to solve the resulting model appears to be promising. Additional features related to spreading factors, pilot signal powers and soft handover are worth including into future models. Special problems encountered in network planning such as finding the maximum user capacity of a given network plan and upgrading an existing network plan will also be considered. Furthermore, future work will investigate the use of the two-phase ACCPM, the combinatorial Benders decomposition and the nested Benders decomposition algorithms to solve other problems.

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