Genus one partitions

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

We obtain a tight upper bound for the genus of a partition, and calculate the number of partitions of maximal genus. The generating series for genus zero and genus one rooted hypermonopoles is obtained in closed form by specializing the genus series for hypermaps. We discuss the connection between partitions and rooted hypermonopoles, and suggest how a generating series for genus one partitions may be obtained via the generating series for genus one rooted hypermonopoles. An involution on the poset of genus one partitions is constructed from the associated hypermonopole diagrams, showing that the poset is rank-symmetric. Also, a symmetric chain decomposition is constructed for the poset of genus one partitions, which consequently shows that it is strongly Sperner.

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Chapter 1

Introduction

A brief overview

Noncrossing partitions were first considered by Kreweras [23] in 1972. He investigated them as a subposet of the lattice of partitions under the refinement order, and derived numerous enumerative and order theoretic results.

Kreweras demonstrated that noncrossing partitions are a member of the Catalan family of combinatorial objects. To date there are over 100 known members in this family, some of which include plane rooted binary trees, Dyck paths, and triangulations of convex polygons. A comprehensive list compiled by Stanley can be found in [29].

The lattice of partitions Π_n is one of the classical posets which have been extensively studied. It is well-known that the partition lattice is semimodular but not rank-symmetric. In comparison, Kreweras showed that the lattice of noncrossing partitions Π_n^0 is not semimodular, but is self-dual, and therefore rank-symmetric. Later, Simion and Ullman [27] showed that Π_n^0 admits a symmetric chain decomposition, and hence is strongly Sperner. Edelman's analysis of multichain enumeration in Π_n^0 [12], and Björner's observation that Π_n^0 is EL-shellable [6], are other examples of work on the noncrossing partition lattice. Also, various identities involving Catalan numbers can be proved combinatorially by using the lattice of noncrossing partitions.

Aside from enumerative combinatorics, noncrossing partitions play an important role in the study of Birkhoff-Lewis equations and the Four Colour Theorem. Birkhoff and Lewis [5] studied two kinds of chromials of planar cubic maps; the free and the constrained. Relations between these polynomials came to be known as the Birkhoff-Lewis equations. It has been proposed that a solution to these equations would lead to an algebraic solution to the Four Colour Theorem. Tutte [31] showed that these equations are recoverable from the matrix of chromatic joins. As Cautis and Jackson showed in [9], the determinant of this matrix can be found by using the Temperley-Lieb algebra, which is a subalgebra of the partition algebra that corresponds to noncrossing partitions.

There are many connections between noncrossing partitions and other areas of mathematics that are not mentioned here. For an excellent survey on the topic, see Simion [26].

One way in which the study of noncrossing partitions may be generalized is to study partitions where certain kinds of crossings are allowed to occur. For example, Chen et al [10] studied k-crossings and k-nestings of matchings and partitions, and showed that crossing numbers and nesting numbers are distributed symmetrically over all partitions of n.

A different way to approach crossings of partitions is to classify them according to the surface that their diagram embeds on. Jackson [20] observed that every partition can be associated with a number called its genus, which is the genus of the corresponding hypermap, so that the set of noncrossing partitions is the set of genus zero partitions. A special case was considered by Cori and Marcus [11], who studied chord diagrams (equivalent to partitions where each block has size 2) with respect to their genus. They derived formulas for counting the number of nonisomorphic chord diagrams of genus one, and of maximal genus. Such diagrams are pertinent to the study of Vassiliev invariants for knots.

The purpose of this thesis is to study enumerative and order-theoretic aspects of partitions with respect to genus, with a focus on partitions of genus one. Jackson obtained computational evidence that the poset of genus one partitions is rank-symmetric. A major result of this thesis (Theorem 4.9) is the construction of a symmetric chain decomposition for the poset of genus one partitions, which shows that the poset is rank-symmetric. We also present a conjecture relating genus one partitions and genus one rooted hypermonopoles, which leads to a generating series for the number of genus one partitions with respect to the number of blocks, and hence would give another proof that the genus one partition poset is rank-symmetric.

Outline of the thesis

The thesis is organized as follows: Chapter 2 introduces relevant background material on maps and hypermaps that enable us to define genus for partitions. We present three different kinds of diagrams associated with partitions, and examine the relationships between partitions, permutations, and rooted hypermonopoles; these diagrams and relations are used extensively in later chapters. We derive a new result (Propositions 2.6 and 2.7) on the upper bound for the genus of a partition of n, enumerate partitions of maximal genus, and give an application of the upper bound result.

The focus of Chapter 3 is to enumerate genus zero and genus one partitions via the enumeration of rooted hypermonopoles. We use the genus series for rooted hypermaps developed by Goulden and Jackson [15] as a starting point and first specialize to a generating series for rooted hypermonopoles. From this, the generating series for genus zero and genus one rooted hypermonopoles are then derived. Necessary background information on hypergeometric functions, representations of the symmetric group, symmetric functions, and integer partitions is reviewed. We conclude the chapter by proposing a conjecture relating the set of genus one partitions to the set of genus one rooted hypermonopoles.

Chapter 4 is based on work by Simion and Ullman [27]. They used partition diagrams to show that the lattice Π_n^0 of genus zero partitions has nice structural properties. We follow a similar approach to show that the poset Π_n^1 of genus one partitions also has similarly desirable structural properties; the major new result (Theorem 4.9) of this chapter is the construction of a symmetric chain decomposition for Π_n^1 which mimics the construction of symmetric chains for Boolean lattices through 'parenthetization'. We begin by giving an overview of poset theory, and compare known results regarding the structure of the partition lattice and the genus zero partition lattice. Instead of using diagrams to define the order-reversing involution on Π_n^0 given by Simion and Ullman, we reformulate their involution in terms of a product of permutations by exploring the relationship between partitions, permutations, and hypermonopoles (Proposition 4.4). We also construct a new involution on Π_n^1 (Proposition 4.5).

Chapter 2

Partitions and their genus

2.1 Definitions and preliminaries

2.1.1 Set partitions

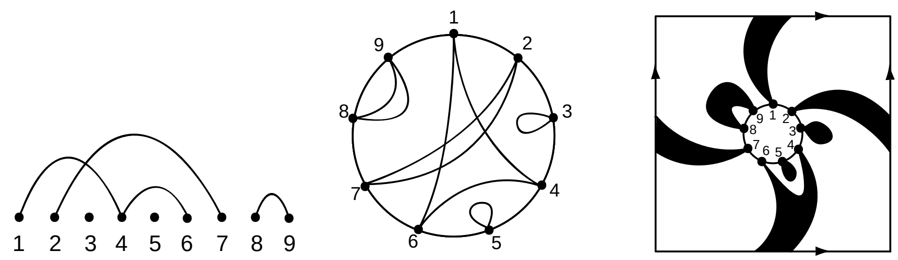
Definition 2.1. *A set partition π of $[n] = \{1, \ldots, n\}$ is a set of nonempty, pairwise disjoint subsets $\pi_1, \cdots, \pi_k$ of $[n]$, such that $\pi_1 \cup \cdots \cup \pi_k = [n]$. The subsets π_i are called the blocks or parts of π.*

For simplicity, we denote a set partition by $\pi = \pi_1/\pi_2/\cdots/\pi_k$, and say that π is a partition of n. Although the blocks of a partition are unordered, it is customary to present a partition with its blocks in increasing order of their minimum element, and with the elements of each block also written in increasing order. For example, $\pi = 1\ 4\ 6/2\ 7/3/5/8\ 9$ is a partition of 9 into five blocks.

Two elements in a block are said to be *cyclically adjacent* to each other if they are adjacent to each other in the increasing order, or if the two elements are the smallest and largest elements in that block. For example, the elements 1 and 6 are cyclically adjacent in the partition 1 4 6/2 7/3/5/8 9.

A partition is *noncrossing*, or *planar*, if there do not exist elements $a < b < c < d$ with a, c in one block, and b, d in a different block.

It is often useful to represent partitions by diagrams, and there are various standard ways to do this. Given a partition π of n, the *circular diagram* of π consists of n points on a circle, with the points labelled 1 through n in the clockwise direction. A chord joins the points i and j across the interior of the



$\pi = 1\ 4\ 6/2\ 7/3/5/8\ 9$

Figure 2.1: Linear, circular, and hypermonopole diagrams.

circle if and only if i and j are cyclically adjacent to each other in the same block of π.

Notice that the circular diagram of a noncrossing partition may be drawn in the plane without any chords crossing each other.

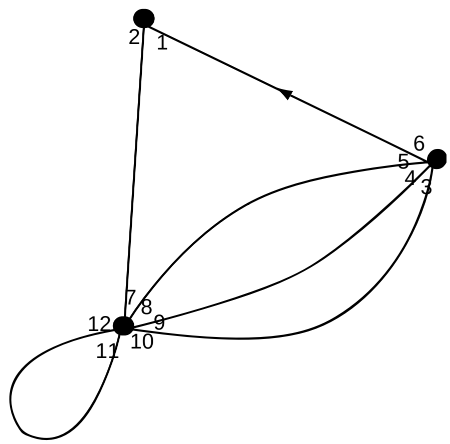
The *linear diagram* of π consists of n points on a line labelled 1 through n from left to right. An arc above the line joins i and j if and only if i and j are adjacent to each other in the same block of π. In other words, we linearize the circular diagram by cutting the arc on the circle between 1 and n, and omitting the chords that join the smallest element with the largest element in each block.

A less standard representation of a partition is its hypermonopole diagram. It is particularly useful in the discussion of partitions with respect to genus, and it shall be examined in the following sections.

2.1.2 Maps

In this thesis, a *surface* is taken to be a Hausdorff space in which every point has a neighbourhood homeomorphic to the open unit disc in $\mathbb{R}^2$. All surfaces encountered in this thesis are assumed to be compact, orientable, and without boundary. The fixed orientation of each surface is the clockwise direction.

A *map* is a 2-cell embedding of a connected graph into a surface. That is, the deletion of the embedded graph from the surface decomposes the surface into a disjoint union of regions that are homeomorphic to open discs. A *rooted map* is a map with a distinguished vertex, edge, and face, all of which are mu-



$$\sigma = (1\ 2)(3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12)$$

$$\alpha = (1\ 6)(2\ 7)(3\ 10)(4\ 9)(5\ 8)(11\ 12)$$

Figure 2.2: Rotation system for a rooted map.

tually incident. In this thesis, we only study maps in connected and orientable surfaces, hence a rooted map on such surfaces may be regarded as a map with a distinguished directed edge. By convention, the head of the distinguished edge is the root vertex, and the face on the left of the distinguished edge is the root face.

Two rooted maps are equivalent if there exists a diffeomorphism between the surfaces that sends one map to the other, taking the root edge of one map to the root edge of the other. The only automorphism for rooted maps is the trivial automorphism. The task of distinguishing between equivalent rooted maps is greatly simplified by a combinatorial axiomatization of maps. We describe one way that this can be done.

Suppose $\mathfrak{m}$ is a map whose underlying graph has n edges. Label each vertex-edge incidence of the graph with the integers 1 through $2n$, with the restriction that the head of the root edge is labelled 1. The *vertex permutation* of $\mathfrak{m}$ is the permutation $\sigma \in S_{2n}$, whose disjoint cycles are the lists of labels at each vertex, read in the fixed orientation. The *edge permutation* of $\mathfrak{m}$ is the permutation $\alpha \in S_{2n}$, whose disjoint cycles are the labels at the ends of each edge. Note that α consists of n cycles of length two, and hence is a fixed point free involution.

For example, Figure 2.2 shows a graph embedded in the sphere. The graph has three vertices and six edges, with the head of the root edge labelled 1. The corresponding vertex and edge permutations are $\sigma = (1\,2)(3\,4\,5\,6)(7\,8\,9\,10\,11\,12)$ and $\alpha = (1\ 6)(2\ 7)(3\ 10)(4\ 9)(5\ 8)(11\ 12)$. Deleting the graph from the surface leaves five open discs. Consider the permutation

$$\varphi = \alpha^{-1}\sigma = (1\ 7\ 5)(2\ 6\ 10\ 12)(3\ 9)(4\ 8)(11).$$

Notice that each cycle of φ describes a face of the map; the labels in each cycle of φ are the labels encountered along the boundary of each face in the opposite orientation. The permutation $\varphi = \alpha^{-1}\sigma$ is called the *face permutation* of the map. This example illustrates the fact that the face permutation of an embedded graph is recoverable from its vertex and edge permutations.

A *pure rotation system* on the graph is the assignment of an ordered cyclic list of edge incidences at each vertex. Thus, if the edge permutation is fixed to be $(1\ 2)(3\ 4)\cdots(2n-1\ 2n)$, then the vertex permutation is a pure rotation system on the graph.

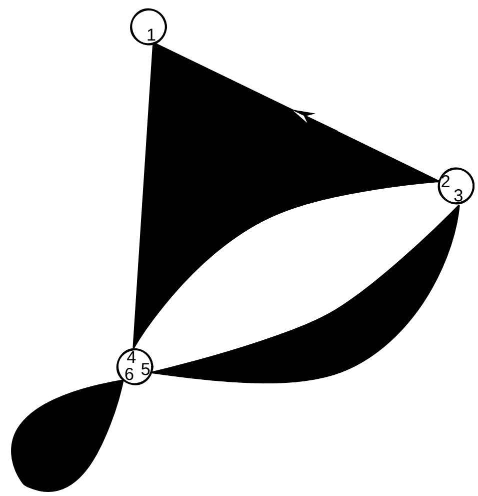
The following theorem, as it appears in [18], is sometimes referred to as the Embedding Theorem.

Theorem 2.2. *Every pure rotation system for a graph induces (up to orientation-preserving equivalence of embeddings) a unique embedding of the graph into an oriented surface. Conversely, every embedding of a graph into an oriented surface induces a unique pure rotation system.* □

In other words, a pair of vertex and edge permutations (σ, α) encode a map. Notice that since the underlying graph of a map is connected, then the subgroup $\langle\sigma, \alpha\rangle$ of S_{2n} generated by σ and α must be transitive on the set $\{1, \ldots, 2n\}$.

Conversely, a natural question that one may ask is: does every pair of permutations $(\sigma, \alpha) \in S_{2n} \times S_{2n}$ encode a map? From the physical interpretation of the permutation α, we know that it must have n cycles of length two. Also, if $\langle\sigma, \alpha\rangle$ is not transitive on $\{1, \ldots, 2n\}$, then (σ, α) encodes the vertex and edge incidences of a disconnected graph, but it does not encode an embedding of the graph into a surface. These are the only restrictions on α and σ.

The Embedding Theorem is an invaluable tool for solving the problem of enumeration of maps. For more details about the combinatorial axiomatization of maps, see [21].



$\sigma = (1)(2\ 3)(4\ 5\ 6) \quad \alpha = (1\ 2\ 4)(3\ 5)(6)$

Figure 2.3: Rotation system for a rooted hypermap.

2.1.3 Hypermaps

A *hypermap* is a two-face colourable map. By convention, the faces of a hypermap are coloured black or white; a black face is called a *hyperedge*, while a white face is called a *hyperface*. A *rooted hypermap* on an orientable surface is a hypermap with a directed edge such that the head of the edge is the root vertex, the left side of the edge is a hyperedge, and the right side of the edge is a hyperface. Alternately, a hypermap may be viewed as a map whose edges are allowed to have more than two ends.

Like regular maps, rooted hypermaps on orientable surfaces are also encoded by pairs of permutations. If instead of labelling each vertex-edge incidence, we label each vertex-hyperedge incidence from 1 through n, then a hypermap may be encoded by a pair of permutations in S_n, rather than S_{2n}. As before, let $\sigma \in S_n$ be the vertex permutation whose disjoint cycles are the lists of incidence labels at each vertex of the graph, encountered in the fixed orientation. Let $\alpha \in S_n$ be the *hyperedge permutation* whose disjoint cycles are the lists of incidence labels of the hyperedges, encountered in the opposite orientation. Then $\varphi = \alpha^{-1}\sigma$ is the *hyperface permutation* whose disjoint cycles are the lists of incidence labels of the hyperfaces, also encountered in the opposite orientation.

The faces of the map in Figure 2.2 are in fact two-colourable. Figure 2.3 shows a rotation system for a two-face colouring of the map. The hypermap has three vertices, three hyperedges, and two hyperfaces. The corresponding vertex and hyperedge permutations are $\sigma = (1)(2\ 3)(4\ 5\ 6)$ and $\alpha = (1\ 4\ 2)(3\ 5)(6)$. The hyperface permutation is $\varphi = \alpha^{-1}\sigma = (1\ 2\ 5\ 6)(3\ 4)$.

2.2 Partitions and rooted hypermonopoles

Definition 2.3. *A hypermonopole is a hypermap with one vertex.*

Circular diagrams of noncrossing partitions are closely related to diagrams of planar rooted hypermonopoles: the circular diagram may be embedded into the sphere, with the regions corresponding to the blocks of the partition coloured black and the rest of the regions within the circle coloured white. By continuously shrinking the circle to a point across the outer face of the sphere, a diagram of a rooted hypermonopole is obtained, with the shrunken circle representing the vertex, and the black regions representing the hyperedges of the hypermonopole. The root edge of the hypermonopole is chosen so that its head is labelled 1, and the face lying on the left side of the edge is a hyperedge.

In this way, each noncrossing partition is associated to a unique planar rooted hypermonopole, and vice versa. This idea can be extended to all partitions as follows.

Let Π_n denote the set of all partitions of n, and let $S_n^<$ denote the subset of permutations of n whose disjoint cycles are *increasing*. That is, each cycle of the permutation may be written in the form $(i_1\ i_2\ \cdots\ i_r)$ with $i_1 < i_2 < \cdots < i_r$.

Define the function

$$\Phi : \Pi_n \longrightarrow S_n^<,$$

by sending $\pi \in \Pi_n$ to the permutation in $S_n^<$ whose disjoint cycles are formed from ordering the elements in the blocks of π in increasing numerical order. For example,

$$\Phi(1\ 4\ 6/2\ 7/3/5/8\ 9) = (1\ 4\ 6)(2\ 7)(3)(5)(8\ 9).$$

It is clear that Φ is a bijection.

Observe that if the underlying graph of a rooted hypermonopole has n edges, then without loss of generality, we can assume that its vertex is encoded by the permutation $\sigma = (1\ 2\ \cdots\ n)$. Thus each rooted hypermonopole is uniquely determined by its hyperedge permutation. Conversely, any permutation in S_n uniquely encodes a rooted hypermonopole with n edges, since σ is an n-cycle and any subgroup of S_n with an n-cycle as a generator is transitive on the set $\{1, \ldots, n\}$.

Let H_n denote the set of rooted hypermonopoles with n edges, and let $H_n^<$ denote the subset of rooted hypermonopoles whose hyperedge permutation consists of increasing disjoint cycles only.

Define the function

$$\Psi : S_n^< \longrightarrow H_n^<$$

by sending $\alpha \in S_n^<$ to the rooted hypermonopole encoded by α. Clearly, Ψ is also a bijection.

Thus, a partition $\pi \in \Pi_n$ with k blocks is uniquely associated to the permutation

$$\alpha_\pi = \Phi(\pi)$$

in S_n with k increasing cycles. In turn, π is also uniquely associated to the rooted hypermonopole

$$\mathfrak{h}_\pi = \Psi(\Phi(\pi))$$

with n edges and k hyperedges, whose hyperedge permutation has k increasing cycles. Throughout this thesis, α_π is called the *permutation associated to the partition* π, and $\mathfrak{h}_\pi$ is called the *rooted hypermonopole associated to the partition* π.

The following diagram may be a useful reference:

$$\Pi_n \longleftrightarrow S_n^< \longleftrightarrow H_n^<$$

$$\pi \leftrightarrow \alpha_\pi \leftrightarrow \mathfrak{h}_\pi$$

$$\text{blocks of } \pi \;\leftrightarrow\; \text{cycles of } \alpha_\pi \;\leftrightarrow\; \text{hyperedges of } \mathfrak{h}_\pi.$$

Remark. Traditionally, a noncrossing partition π is associated to the planar rooted hypermonopole whose hyperedge permutation consists of disjoint increasing cycles that correspond to the blocks of π. Here, we are extending the convention to higher genus partitions. The condition of increasing cycles is necessary; for example, 123 and 132 represent the same set partition, but the rooted hypermonopole encoded by the permutation $(1\ 2\ 3)$ is embeddable in the sphere, while the rooted hypermonopole encoded by $(1\ 3\ 2)$ can only be embedded in the torus. By our convention, the permutation $(1\ 2\ 3)$ corresponds to the partition 123, while $(1\ 3\ 2)$ does not correspond to any partition.

2.3 Genus of partitions

For a permutation $\alpha \in S_n$, let $l(\alpha)$ denote the number of disjoint cycles in α. If a rooted hypermap $\mathfrak{m}$ with n edges is encoded by the permutations (σ, α), then $l(\sigma)$ is the number of vertices in $\mathfrak{m}$, and $l(\alpha) + l(\alpha^{-1}\sigma)$ is the number of faces in $\mathfrak{m}$. The *genus* $g(\mathfrak{m})$ of the hypermap is given by the Euler-Poincaré formula

$$l(\sigma) - n + l(\alpha) + l(\alpha^{-1}\sigma) = 2 - 2g(\mathfrak{m}). \tag{2.1}$$

Definition 2.4. *The* genus *of a partition $\pi \in \Pi_n$ is the genus of its associated rooted hypermonopole $\mathfrak{h}_\pi$. That is,*

$$g(\pi) = \frac{1}{2}(1 + n - l(\alpha_\pi) - l(\alpha_\pi^{-1}\sigma)),$$

where $\sigma = (1\ 2 \cdots n)$.

For example, if $\pi = 1\ 4\ 6/2\ 7/3/5/8\ 9$, then $\alpha_\pi = (1\ 4\ 6)(2\ 7)(3)(5)(8\ 9)$, $\alpha_\pi^{-1}\sigma = (1\ 7\ 9\ 6\ 2\ 3)(4\ 5)(8)$, so that $l(\alpha_\pi) = 5$, $l(\alpha^{-1}\sigma) = 3$, and $g(\alpha_\pi) = \frac{1}{2}(1 + 9 - 3 - 5) = 1$. Indeed, Figure 2.1 shows that the rooted hypermonopole $\mathfrak{h}_\pi$ has a 2-cell embedding on the torus.

This definition makes it apparent that the set of noncrossing partitions is exactly the set of genus zero partitions. Table 2.1 shows the number of partitions of n with genus g. The numbers in the column corresponding to $g = 0$ are the Catalan numbers, as expected.

2.4 The hypermonopole diagram

Genus zero partitions have nice pictorial representations in the plane, since their associated rooted hypermonopoles are embeddable in the sphere. The structure of their hyperedges and hyperfaces can be easily visualized this way. Although higher genus hypermonopoles do not have a 2-cell embedding in the sphere, it is still possible to have nice pictorial representations of them in the plane which clearly show the structure of their faces, since every surface can be represented as a polygon in the plane.

2.4.1 Classification of orientable surfaces

For $g \in \mathbb{N}$, let $\mathcal{S}_g$ denote the standard g-holed torus, so that $\mathcal{S}_0$ denotes the sphere, $\mathcal{S}_1$ denotes the torus, etc. The following theorem is a standard one that

	g=0	1	2	3	4	5
n=1	1					
2	2					
3	5					
4	14	1				
5	42	10				
6	132	70	1			
7	429	420	28			
8	1440	2310	399	1		
9	4862	12012	4179	94		
10	13796	60060	36498	2620	1	
11	58786	291720	282282	45430	352	
12	208012	1385670	1999998	600655	19261	1

Table 2.1: Number of partitions of n with respect to genus g

may be found in most books on classical topology.

Theorem 2.5. *(The classification theorem for orientable surfaces) Every closed, connected, orientable surface is homeomorphic to one of the standard surfaces* $\mathcal{S}_g$ *for* $g \in \mathbb{N}$.

Previously in Equation (2.1), we defined the genus of a hypermap as an integer that is dependent on the number of vertices, edges and faces of the hypermap without an explicit mention of the surface that the hypermap is embedded in. It can be shown that any map $\mathfrak{m}$ with a 2-cell embedding in $\mathcal{S}_g$ satisfies the equation

$$V(\mathfrak{m}) - E(\mathfrak{m}) + F(\mathfrak{m}) = 2 - 2g,$$

where $V(\mathfrak{m})$, $E(\mathfrak{m})$, and $F(\mathfrak{m})$ respectively denote the number of vertices, edges, and faces of the map. This equation is also known as the Euler-Poincaré equation, and is consistent with Equation (2.1). The integer $2 - 2g$ is the *Euler characteristic* of the surface $\mathcal{S}_g$, and the *genus* of $\mathcal{S}_g$ is g. It can be shown that both the Euler characteristic and the genus are total invariants of surfaces, so the genus of a map may also be defined as the genus of the surface that it is embedded in.

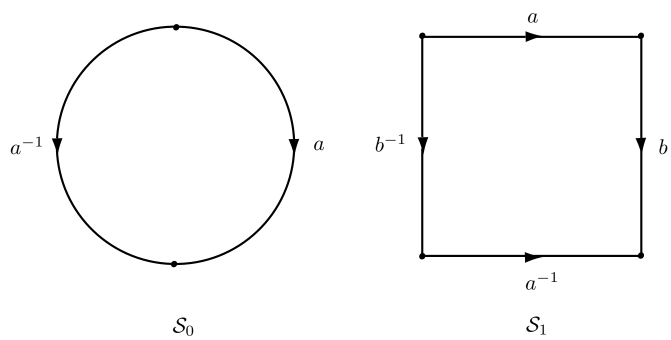


Figure 2.4: Polygonal representation of the sphere and the torus.

2.4.2 Polygonal representation of orientable surfaces

The sphere $\mathcal{S}_0$ may be represented in the plane by a circle. See Figure 2.4. The two edges labelled a and a^{-1} are identified in the direction of the arrows. This is the standard polygonal representation of $\mathcal{S}_0$.

The surfaces $\mathcal{S}_g$ for $g \geq 1$ may be obtained from the sphere by attaching g handles. A *handle* can represented in the plane by a square whose pairs of opposite sides are identified, with an open disc removed from the square. Handle attachment corresponds to removing an open disc from the sphere, and joining the handle and the sphere together along the boundaries of the open discs. Thus, for $g \geq 1$, each g-holed surface $\mathcal{S}_g$ can be represented by a polygon with $4g$ sides. The edges of the polygon are labelled $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ in the clockwise direction, with the edges a_i, a_i^{-1} identified in pairs along the direction of their arrows, and the edges b_i, b_i^{-1} identified in pairs along the direction of their arrows. Under these identifications, the $4g$ corners of the polygon represent the same point on the surface. These are the *standard polygonal representations* of the surfaces $\mathcal{S}_g$ for $g \geq 1$.

Remark. The result of handle attachment to the sphere is the connected sum of $\mathcal{S}_0$ and the torus $\mathcal{S}_1$. More formally, given two surfaces $\mathcal{X}$ and $\mathcal{Y}$, let D be an open disc on $\mathcal{X}$ and let E be an open disc on $\mathcal{Y}$, with ∂D denoting the boundary of D. Let $f : \partial D \to \partial E$ be an orientation-reversing homeomorphism, and let $\sim_f$ be an equivalence relation on $(\mathcal{X} - D) \coprod (\mathcal{Y} - E)$ induced by f. The *connected sum* of $\mathcal{X}$ and $\mathcal{Y}$ is the quotient space $\mathcal{X} \# \mathcal{Y} = ((\mathcal{X} - D) \coprod (\mathcal{Y} - E)) / \sim_f$.

The references [30] and [18] offer a more detailed account of the classification of surfaces from two different points of view. See [30] for a more detailed account of polygonal representations.

The hypermonopole diagram

The *hypermonopole diagram* of a genus g partition $\pi \in \Pi_n$ is a diagram of the 2-cell embedding of its associated rooted hypermonopole $\mathfrak{h}_\pi$ in the standard polygonal representation of the standard genus g surface. See Figure 2.1 for an example.

2.5 An upper bound for the genus of a partition

Using the Euler-Poincaré formula and the previous observation that a rooted hypermonopole corresponds to a partition exactly when each disjoint cycle of its hyperedge permutation is increasing, a tight upper bound for the genus of a partition of n can be obtained.

Proposition 2.6. *For $m \geq 1$, the partitions of $n = 2m$ are of genus at most $m-1$. In particular for $m \geq 2$, $1\ 3\ 5\ \cdots\ 2m-1/2\ 4\ 6\ \cdots\ 2m$ is the only genus $m-1$ partition of $2m$.*

Proof. Let π be a partition of $n = 2m$ whose associated rooted hypermonopole is encoded by the permutations (σ, α). Since $l(\alpha)$ and $l(\alpha^{-1}\sigma)$ are always greater than or equal to 1, then

$$2g(\pi) = 2m + 1 - l(\alpha) - l(\alpha^{-1}\sigma) \leq 2m - 1.$$

Thus $g(\pi) \leq m - 1$, since $g(\pi)$ is a nonnegative integer.

When $m = 1$, the two partitions $1\ 2$ and $1/2$ of $n = 2$ both have genus zero.

Now suppose $m \geq 2$ and π is a genus $m-1$ partition of $2m$. Euler's formula implies

$$l(\alpha) + l(\alpha^{-1}\sigma) = 3. \tag{2.2}$$

Since $l(\alpha)$, $l(\alpha^{-1}\sigma) \geq 1$ there are two ways of achieving this equality. Recall $l(\alpha)$ is the number of blocks in π. If π has one block only, then π must be the genus zero partition $1\ 2\ \cdots\ n$. This implies $m - 1 = g(\pi) = 0$, which is not an admissible case since we are only dealing with $m \geq 2$. Therefore, we are left with the case $l(\alpha) = 2$ and $l(\alpha^{-1}\sigma) = 1$.

So suppose $\alpha = c_1 c_2 \in S_n$ where c_1 and c_2 are disjoint cycles. Furthermore, suppose for contradiction that one of the cycles contains cyclically adjacent integers. (Recall that 1 and n are considered to be cyclically adjacent.) Without loss of generality, let $c_1 = (\cdots j\ j+1 \cdots)$. Then

$$\alpha^{-1}\sigma = (\cdots j+1\ j \cdots) c_2^{-1} (1 \cdots j\ j+1 \cdots n) = (j)(j+1 \cdots) \cdots ,$$

so that $l(\alpha^{-1}\sigma) \geq 2$, which does not satisfy Equation (2.2).

Therefore, the two disjoint cycles c_1 and c_2 of α do not contain cyclically adjacent integers. Since the cycles must be increasing, the only way this can occur is if $\alpha = (1\ 3\ 5\ \cdots\ 2m-1)(2\ 4\ 6\ \cdots\ 2m)$. It is straightforward to verify that $1\ 3\ 5\ \cdots\ 2m-1/2\ 4\ 6\ \cdots\ 2m$ is a genus $m-1$ partition. $\square$

This proof shows that whenever π contains a pair of cyclically adjacent integers j, $j+1$ in one of its blocks, then the permutation $\alpha^{-1}\sigma$ will contain the singleton cycle (j).

When n is an odd integer, the situation is more complicated.

Proposition 2.7. *For $m \geq 2$, the partitions of $n = 2m-1$ are of genus at most $m-2$.*

Proof. When $m = 2$, it is easy to check that every partition of $n = 3$ has genus zero.

Now let $m \geq 3$ and let π be a partition of $n = 2m-1$ whose associated rooted hypermonopole is encoded by the permutations (σ, α). As before, if $l(\alpha) = 1$, then $g(\pi) = 0$, so we can assume $l(\alpha) \geq 2$. Then

$$2g(\pi) = 2m - l(\alpha) - l(\alpha^{-1}\sigma) \leq 2m - 3.$$

Since $g(\pi)$ is a nonnegative integer, then $g(\pi) \leq m-2$. $\square$

We show that this bound is tight by counting the number of genus $m-2$ partitions of $2m-1$.

Proposition 2.8. *For $m \geq 3$, there are $\frac{1}{3}(4^{m-1}-1) + 2m - 1$ genus $m-2$ partitions of $2m-1$.*

Proof. Let π be a genus $m-2$ partition of $n = 2m-1$ whose associated rooted hypermonopole is encoded by the permutations (σ, α). By the Euler-Poincaré formula,

$$l(\alpha) + l(\alpha^{-1}\sigma) = 2m - 1 + 2 - 2(m-2) - 1 = 4.$$

Since $g(\pi) = m - 2 \geq 1$, then $l(\alpha)$ is at least 2, and there are two cases to consider.

Case 1. Suppose $l(\alpha) = 2$ and $l(\alpha^{-1}\sigma) = 2$. Then one cycle has at least m elements and that cycle must contain a pair of cyclically adjacent integers. On the other hand, recall the earlier observation that each occurrence of a pair of cyclically adjacent integers in a cycle of α gives rise to a singleton cycle in the permutation $\alpha^{-1}\sigma$. Then (assuming $\alpha \neq (1\ 2 \cdots n)$, since such an α has genus zero)

$$2 = l(\alpha^{-1}\sigma) \geq 1 + \#\text{ of cyclically adjacent integers in } \alpha,$$

so the number of cyclically adjacent integers in α is at most 1.

Conversely, suppose $\alpha \in S_{2m-1}$ has two increasing cycles in which only one cycle contains a pair of cyclically adjacent integers, say j and $j+1$ (where $j+1 = 1$ if $j = n$). Then it is easy to check that $\alpha^{-1}\sigma$ has one singleton cycle (j) and one other cycle, so that $l(\alpha^{-1}\sigma) = 2$.

If j and $j+1$ is the pair of cyclically adjacent integers in α, then there is only one way to arrange the rest of the elements in two increasing cycles, namely,

$$\alpha = (\cdots\ j-4\ j-2\ j\ j+1\ j+3\ j+5\ \cdots)(\cdots\ j-3\ j-1\ j+2\ j+4\ \cdots).$$

Since there are n possible pairs of cyclically adjacent integers, then there are n permutations of this form.

Case 2. Suppose $l(\alpha) = 3$ and $l(\alpha^{-1}\sigma) = 1$. By similar reasoning as in Proposition 2.6, no cycle of α can have cyclically adjacent integers.

A cycle $(k_1\ k_2 \cdots k_s)$ has an *ascent* at k_i if $k_i < k_{i+1}$ for some $1 \leq i \leq s-1$, or $k_i < k_1$ if $i = s$. For example, an increasing cycle of length s has $s-1$ ascents.

Suppose $\alpha = (a_1\ a_2 \cdots a_p)(b_1\ b_2 \cdots b_q)(c_1\ c_2 \cdots c_r) \in S_{2m-1}$ is a permutation with three increasing disjoint cycles, none of which contain cyclically adjacent integers. Without loss of generality, we may assume that a_1, b_1, and c_1 are the smallest integers in their respective cycles, and that $a_1 = 1$. We shall see that the permutation $\alpha^{-1}\sigma$ has exactly two ascents.

By definition, an ascent occurs at x in $\alpha^{-1}\sigma$ if and only if $x < \alpha^{-1}(\sigma(x))$. Note that $x \neq n$. Suppose $\sigma(x)$ is not the smallest integer in its cycle in α, so that in particular $\sigma(x) \neq 1$ or 2. Then $\alpha^{-1}(\sigma(x)) = \alpha^{-1}(x+1) \leq x - 1$, since α^{-1} has decreasing cycles with no cyclically consecutive integers. Thus an ascent does not occur in this case.

Now suppose $\sigma(x)$ is the smallest integer in its cycle. If $\sigma(x) = a_1 = 1$ so that $x = n$, then an ascent does not occur at x. Thus ascents in $\alpha^{-1}\sigma$ can occur at x only when $\sigma(x) = b_1$ or c_1. On the other hand, if $\sigma(x) = b_1 \leq b_q$, then

$$1 \leq x = b_1 - 1 < b_q = \alpha^{-1}(b_1) = \alpha^{-1}(\sigma(x)).$$

The case when $\sigma(x) = c_1$ is similar, and thus there are two ascents in the cycles of $\alpha^{-1}\sigma$. Since every cycle contains at least one ascent, this implies that $\alpha^{-1}\sigma$ has at most two disjoint cycles. By the Euler-Poincaré formula,

$$l(\alpha^{-1}\sigma) + 2g = 2m - 3.$$

It follows that $l(\alpha^{-1}\sigma)$ is odd, and therefore must equal one.

Thus we have reduced this case to the problem of counting permutations with three increasing cycles such that no cycle contains cyclically adjacent integers.

Claim 1: The number of permutations in S_{2m-1} with three increasing cycles such that no cycle contains cyclically adjacent integers is equal to the number of walks of length $2m - 2$ from one particular to vertex to a particular one of its neighbours in the triangle graph C_3.

Proof. We shall explicitly construct a bijection. Let $n = 2m - 1$. Suppose the vertices of C_3 are labelled A, B and C. Given a permutation $\alpha \in S_n$ with three disjoint increasing cycles, let λ_A be the cycle that contains 1, let λ_C be the cycle that contains n, and let λ_B be the remaining cycle. (Since α does not have cyclically adjacent integers, λ_A and λ_C are distinct cycles.) Construct the sequence $\Upsilon(\alpha) = u_1 u_2 \cdots u_n$ where u_i is the name of the cycle that contains i. For example, if $\alpha = (1\ 3\ 6)(2\ 4)(5\ 7) \in S_7$, then $\lambda_A = (1\ 3\ 6)$, $\lambda_B = (2\ 4)$, and $\lambda_C = (5\ 7)$, and the sequence corresponding to α is $\Upsilon(\alpha) = ABABCAC$.

For each $\alpha \in S_n$ with the above stated properties, $\Upsilon(\alpha)$ defines a walk on C_3 from vertex A to vertex C of length $2m - 2$.

Conversely, given any walk γ from A to C of length $2m - 2$ in C_3, the construction $\Upsilon^{-1}(\gamma)$ can be obtained in the obvious way. Notice that walks of even length between distinct vertices in C_3 must visit every vertex of C_3 at least once, thus $\Upsilon^{-1}(\gamma)$ is indeed a permutation with three disjoint increasing cycles not containing adjacent integers. It is easy to see that Υ is a bijection. $\square$

The incidence matrix of the triangle graph C_3 is

$$C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

That is, $C = J - I$ where I is the identity matrix and J is the matrix of all ones. The number we seek the coefficient of x^{2m-2} in any off-diagonal entry of the matrix $(I - xC)^{-1}$.

We have

$$(I - xC)^{-1} = (I - x(J - I))^{-1} = (1 + x)^{-1}(I - x/(1 + x)J)^{-1}.$$

Notice that $x/(1 + x)J$ is a matrix of rank 1.

Claim 2: If A is a matrix of rank 1, then $(I - A)^{-1} = I + (1 - \mathrm{tr}(A))^{-1}A$.

Proof. Since A has rank 1, we can write $A = uv^T$ where u, v are column vectors. Since $v^T u$ is a scalar, then

$$A^2 = (uv^T)(uv^T) = u(v^T u)v^T = (v^T u)(uv^T) = (v^T u)A.$$

Observe that $v^T u = \mathrm{tr}(v^T u) = \mathrm{tr}(uv^T) = \mathrm{tr}(A)$, so $A^2 = \mathrm{tr}(A)A$. By induction,

$$A^n = \mathrm{tr}(A)^{n-1}A$$

for all integers $n \geq 1$. Thus

$$(I - A)^{-1} = I + \sum_{i \geq 1} A^i = I + \sum_{i \geq 1} \mathrm{tr}(A)^{i-1}A = I + (1 - \mathrm{tr}(A))^{-1}A,$$

as required. □

Since $x/(1 + x)J$ is a rank 1 matrix with trace $3x/(1 + x)$, so it follows from the Claim that

$$(I - x/(1 + x)J)^{-1} = I + x/(1 - 2x)J.$$

Therefore, the coefficient of x^{2m-2} in the $(1, 2)$th entry of $(1 + x)^{-1}(I - x/(1 + x)J)^{-1}$ is

$$\begin{aligned}
[x^{2m-2}]\frac{x}{(1-2x)(1+x)} &= [x^{2m-3}]\frac{1}{(1-2x)(1+x)} \\
&= [x^{2m-3}]\sum_{i,j \geq 0} (-1)^i 2^j x^{i+j} \\
&= (-1)^{2m-3}\sum_{j=0}^{2m-3} (-2)^j.
\end{aligned}$$

The last expression is a geometric series whose sum is $\frac{1}{3}(4^{m-1} - 1)$. Combining cases one and two, the result follows. □

Remark. A substring ij of a sequence in $\{1,\dots,r\}$ is a *level* if $i=j$. A *Smirnov sequence* is a sequence in $\{1,\dots,r\}$ with no levels. We can alternately count the number of Smirnov sequences in $\{1,2,3\}$ of length n that start with 1 and end with 3. From Section 2.4.16 in [14], the number of Smirnov sequences of length n is $[x^n](1-3x/(1+x))^{-1} = 3\cdot 2^{n-1}$. Using an alternating sum to remove the number of sequences whose last element equals the first, and dividing by 6 to ensure the sequences start with 1 and end with 3, the desired number is $\frac{1}{6}(3\cdot 2^{n-1}-3\cdot 2^{n-2}+\cdots-3\cdot 2)=\frac{1}{3}(2^{n-1}-1)$.

The numbers obtained from Propositions 2.6 and 2.8 agree with the numbers appearing in Table 2.1.

2.5.1 An application

The *matrix of chromatic joins* $M_n(t)$, is defined by

$$M_n(t) = [t^{l(\rho_1\vee\rho_2)}],$$

where ρ_1 and ρ_2 are noncrossing partitions of n, and $l(\rho_1\vee\rho_2)$ denotes the number of blocks of the join of the two partitions. Note that $M_n(t)$ is a C_n by C_n matrix, where C_n is the nth Catalan number defined in Equation (3.2). An important result is that the determinant of this matrix is nonzero, and factorizes nicely into a product of generalized Chebyshev polynomials. A first proof was given by Tutte in [32]. Another proof of this that makes use of the Temperley-Lieb algebra can be found in [8].

Let $M_n^g(t)$ denote the higher genus analogue of the matrix of chromatic joins. That is, $M_n^g(t)=[t^{l(\rho_1\vee\rho_2)}]$, where ρ_1, ρ_2 ranges over all partitions of genus $\leq g$. It is of interest to determine if the determinant of $M_n^g(t)$ also factorizes nicely, but currently this is still an open question.

For fixed n, the value of $M_n^g(t)$ is eventually constant, as $g\to\infty$. Proposition 4.5 in [8] deals with this case. Lindström's theorem applies to give

$$\det M_n^\infty(t) = \prod_{i=1}^{n}(t-i)^{b(n,i)},$$

where $b(n,i)=B_n-\sum_{j=0}^{i} S(n,j)$, B_n is the nth Bell number, and $S(n,j)$ is a Stirling number of the second kind. Given n, it was previously unknown for what values of g this formula holds. The upper bound results in this chapter now provides an answer.

Chapter 3

Counting partitions with respect to genus

It is well-known that the number of genus zero partitions of n with k blocks is the *Narayana number*

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}. \tag{3.1}$$

The total number of genus zero partitions of n can be obtained by summing over the Narayana numbers. Rewriting $\binom{n}{k-1}$ as $\binom{n}{n+1-k}$ and applying Vandermonde's convolution (Lemma 3.1), the number of genus zero partitions of n is the *Catalan number*

$$C_n = \sum_{k=1}^{n} N(n,k) = \frac{1}{n+1}\binom{2n}{n}. \tag{3.2}$$

A large number of interpretations for Catalan numbers may be found in [29].

Equation (3.1) was first proved in 1972 by Kreweras [23]. A purely combinatorial proof using well-parenthesized strings was given in 1980 by Edelman [12]. The generating series for Catalan numbers is $\sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$.

Since partitions may be viewed as a subset of rooted hypermonopoles, we can try to modify the character theoretic approach for enumerating rooted hypermaps in an attempt to enumerate partitions with respect to genus. But partitions only correspond to rooted hypermonopoles whose hyperedge permutations have increasing cycles, and as a consequence, this cuts into the conjugacy classes of permutations in S_n so that it is unlikely the character approach will work in general. However, we already know that there is a bijection between genus zero partitions and genus zero rooted hypermonopoles (Section 2.2), so the rooted hypermap generating series for orientable surfaces can be used to give another proof of Equations (3.1) and (3.2).

In the case where genus equals one, there is numerical evidence (see Table 3.1 for example) that the relationship between sets of genus one partitions and sets of genus one rooted hypermonopoles is nice enough so that the generating series for rooted hypermaps is useful in the enumeration of genus one partitions.

Previously, Goupil and Schaeffer [16] computed the connexion coefficients of the center of the group algebra of S_n and consequently obtained an expression for the number of genus g rooted hypermonopoles with n edges and hyperedge partition λ. By summing over all integer partitions $\lambda \vdash n$ with k parts, they obtained an expression for the number of genus g rooted hypermonopoles with n edges and k hyperedges. Their results are implicit in an earlier paper by Goulden and Jackson [15], and the hypermonopole series of prescribed genus can be obtained from Goulden and Jackson's genus series for rooted hypermaps.

We demonstrate how this can be done, for the genus zero and genus one rooted hypermonopole series, giving full details. We give a conjecture regarding genus one partitions at the end of the chapter, and prove some related results.

3.1 Background and notation

We begin this chapter by introducing some necessary background and notation. Results from representation theory that are used in Section 3.2 are developed.

For any positive integer j, the *falling factorial* is denoted by

$$x_{(j)} = x(x-1)\cdots(x-j+1),$$

and the *rising factorial* is denoted by

$$x^{(j)} = x(x+1)\cdots(x+j-1).$$

The falling and rising factorials are related by the identities $x_{(j)} = (x-j+1)^{(j)}$ and $x^{(j)} = (x+j-1)_{(j)}$.

Remark. In literature regarding hypergeometric series, the rising factorial is also called a *Pochhammer symbol*, and is usually denoted by $(x)_j$. To avoid confusion with the falling factorial, we use the above notation instead.

3.1.1 Hypergeometric series

A *hypergeometric series* is a series $f(z) = \sum_{j\geq 0} f_j z^j$ whose *term ratio* f_{j+1}/f_j is a rational function in j, with $f_0 = 1$.

For nonnegative integers p and q, the normal form of a hypergeometric series is

$$ {}_pF_q \left[\begin{array}{ccc|c} a_1 & \cdots & a_p & \multirow{2}{*}{z} \\ b_1 & \cdots & b_q & \end{array} \right] = \sum_{j\geq 0} \frac{(a_1)^{(j)} \cdots (a_p)^{(j)}}{(b_1)^{(j)} \cdots (b_q)^{(j)}} \frac{z^j}{j!}, $$

where the term ratio is given by the rational function

$$ \frac{1}{(1+j)} \frac{(a_1+j)\cdots(a_p+j)}{(b_1+j)\cdots(b_q+j)}. \tag{3.3} $$

The factor of $1+j$ in the bottom appears because of historical reasons.

If the indeterminate z is omitted from the notation, it is assumed that the series is evaluated at $z = 1$.

For example, the geometric series is a hypergeometric series whose term ratio is a constant:

$$ \frac{1}{1-z} = \sum_{j\geq 0} z^j = {}_1F_0 \left[\begin{array}{c|c} 1 & \multirow{2}{*}{z} \\ - & \end{array} \right]. $$

Another example is the exponential series whose term ratio is the rational function $1/(j+1)$:

$$ e^z = \sum_{j\geq 0} \frac{z^j}{j!} = {}_0F_0 \left[\begin{array}{c|c} - & \multirow{2}{*}{z} \\ - & \end{array} \right]. $$

Observe that if one of the upper parameters a_i is a negative integer, then the hypergeometric series terminates naturally as a polynomial. If one of the lower parameters b_i is a nonpositive integer, then the series is undefined. Also, if an upper parameter is equal to a lower parameter, the pair may be cancelled. If none of the upper parameters are negative integers, then the radius of convergence of the series depends on the values of p and q:

p, q	radius of convergence
$p < q+1$	∞
$p = q+1$	1
$p > q+1$	0

A power series $f(z) = \sum_{j\geq 0} f_j z^j$ may be put into hypergeometric form if $f_0 \neq 0$, and the factored form of its term ratio is given by Equation (3.3). In this case,

$$f(z) = f_0 \cdot {}_pF_q \left[\begin{array}{ccc|c} a_1 & \cdots & a_p & \multirow{2}{*}{z} \\ b_1 & \cdots & b_q & \end{array} \right].$$

The following is a standard identity which we shall make extensive use of. The binomial form can be found in [17] (Equation 5.22) and the hypergeometric form can be found in [28](Equation 1.7.7).

Proposition 3.1. *(Vandermonde's convolution)*

$$\sum_k \binom{r}{u+k}\binom{s}{v-k} = \binom{r+s}{u+v} \qquad \textit{(binomial form)}$$

$${}_2F_1 \left[\begin{array}{cc} a & -n \\ - & c \end{array} \right] = \frac{(c-a)^{(n)}}{(c)^{(n)}} \qquad \textit{(hypergeometric form),}$$

for integers u, v, and nonnegative integers n. □

3.1.2 Group representations

Let V be a vector space over a field F, and let $GL(V)$ denote the group of automorphisms of V, called the general linear group of V. A *representation* of a group G is a group homomorphism

$$\rho : G \to GL(V).$$

Alternately, a representation of G can be thought of as a G-module V with group action given by

$$g \cdot v = \rho(g)(v)$$

for all $v \in V$.

For example, if $V = \mathbb{C}[G]$ is the group algebra of G with basis $\{e_g : g \in G\}$, and $\rho : g \mapsto \rho_g$ where $\rho_g(h) = gh$ for all $h \in G$, then ρ is called the left regular representation of G.

A subspace $W \leq V$ is *G-invariant* if $g \cdot w \in W$ for all $w \in W$ and $g \in G$. In this case, W is a *subrepresentation* of V. A representation is *irreducible* if its only subrepresentations are 0 and itself.

Let $GL_n(F)$ denote the general linear group of an n-dimensional vector space over F. The *character* of a representation $\rho : G \to GL_n(F)$ is the function

$$\chi : G \to F : g \mapsto \mathrm{tr}\rho(g),$$

where $\mathrm{tr}\rho(g)$ is the trace of the matrix of $\rho(g)$ with respect to some fixed basis of V. A character is irreducible if its corresponding representation is irreducible.

Since $\mathrm{tr}(BAB^{-1}) = \mathrm{tr}(A)$ for an invertible matrix B, it follows that χ is constant on the conjugacy classes of G. That is, χ is a *class function*.

3.1.3 Integer partitions

For a positive integer n, the decomposition $n = \theta_1 + \theta_2 + \cdots + \theta_k$ of n into positive integers is called an *integer partition* of n. The order of the summands does not matter, so by convention, we list them in decreasing order and write $\theta = (\theta_1, \theta_2, \ldots) \vdash n$, where $\theta_i = 0$ for $i \geq k+1$. Each nonzero θ_i is called a *part* of the integer partition, and the number of parts of θ, denoted by $l(\theta)$, is the *length* of θ. An integer partition is also denoted as $[1^{a_1}, 2^{a_2}, \cdots]$, where a_i is the number of i's in the sequence θ.

The generating series for integer partitions is given by

$$\sum_{n\geq 0} \mathrm{ptn}(n) x^n = \prod_{i\geq 1} (1-x^i)^{-1},$$

where $\mathrm{ptn}(n)$ is the number of integer partitions of n.

The *Ferrers diagram* $\mathcal{F}_\theta$ of $\theta \vdash n$ is a diagram consisting of $l(\theta)$ rows of squares, where the ith row from the top contains θ_i squares and the rows are justified to the left. The *conjugate* of θ is the integer partition corresponding to the Ferrers diagram consisting of $l(\theta)$ columns where the ith column from the left contains θ_i squares. The conjugate is denoted by θ'.

A *hook* is an integer partition of the form $(n-j, 1, \ldots, 1) \vdash n$ where $0 \leq j < n$. Let $q_{i,j}$ be the square in the ith row and jth column of $\mathcal{F}_\theta$. The *hooklength* $\mathrm{hook}(q_{i,j})$ of $q_{i,j}$ is the sum of the number of squares in row i that appear to the right of $q_{i,j}$, the number of squares in column j that appear below $q_{i,j}$, and 1. That is,

$$\mathrm{hook}(q_{i,j}) = \theta_i - i + \theta'_j - j + 1.$$

Let

$$H_\theta = \prod_{q_{i,j} \in \mathcal{F}_\theta} \mathrm{hook}(q_{i,j}) \tag{3.4}$$

denote the product of the hooklengths of $\mathcal{F}_\theta$.

For example, $\theta = (5,5,3,2,2,2,1) \vdash 20$ may also be denoted as $[1, 2^3, 3, 5^2]$. Its conjugate is $\theta' = (7,6,3,2,2)$, and the product of the hooklengths of $\mathcal{F}_\theta$ is $H_\theta = 1^4 2^3 3^3 4^2 5^3 7^1 8^1 9^1 10^1 11^1 = 23950080000$.

Remark. H_θ is related to the degree f^θ of the irreducible character χ^θ (see page 28) of S_n by the Hooklength formula $f^\theta = n!/H_\theta$.

3.1.4 Symmetric functions

Let $\mathcal{P}$ denote the set of all integer partitions, with the empty partition $\emptyset$ adjoined. Let $\mathbf{x} = (x_1, x_2, \ldots)$ be an infinite set of indeterminates.

For $\theta = (\theta_1, \theta_2, \ldots) \in \mathcal{P}$, the *monomial symmetric function* indexed by θ is defined by

$$m_\theta(\mathbf{x}) = \sum_\lambda \mathbf{x}^\lambda,$$

summed over all distinct permutations $\lambda = (\lambda_1, \lambda_2, \ldots)$ of the entries of the vector $(\theta_1, \theta_2, \ldots)$, where $\mathbf{x}^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots$. Note $m_\emptyset = 1$.

The *elementary symmetric function* indexed by θ is defined by

$$e_\theta(\mathbf{x}) = e_{\theta_1}(\mathbf{x}) e_{\theta_2}(\mathbf{x}) \cdots,$$

where $e_n(\mathbf{x}) = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n} = m_{[1^n]}(\mathbf{x})$ for positive integers n, and $e_0(\mathbf{x}) = 1$. The generating series for the e_n is

$$E(t) = \sum_{n \geq 0} e_n t^n = \prod_{i \geq 1} (1 + x_i t).$$

The *complete symmetric function* indexed by θ is defined by

$$h_\theta(\mathbf{x}) = h_{\theta_1}(\mathbf{x}) h_{\theta_2}(\mathbf{x}) \cdots,$$

where $h_n(\mathbf{x}) = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n} = \sum_{\lambda \vdash n} m_\lambda$ for positive integers n, and $h_0(\mathbf{x}) = 1$. The generating series for the h_n is

$$H(t) = \sum_{n \geq 0} h_n t^n = \prod_{i \geq 1} (1 - x_i t)^{-1}.$$

The *power sum symmetric function* indexed by θ is defined by

$$p_\theta(\mathbf{x}) = p_{\theta_1}(\mathbf{x}) p_{\theta_2}(\mathbf{x}) \cdots,$$

where $p_n(\mathbf{x}) = x_1^n + x_2^n + \cdots = m_{[n]}(\mathbf{x})$ for positive integers n, and $p_0(\mathbf{x}) = 1$.

There are different ways to define the Schur symmetric functions. The definition given below follows a combinatorial approach. For $\theta \in \mathcal{P}$, a *semistandard Young tableau* of shape θ is a Ferrers diagram $\mathcal{F}_\theta$ whose squares are filled with positive integers that are weakly increasing along rows from left to right, and strictly increasing along columns from top to bottom. The *weight* of a semistandard Young tableau T is

$$\mathbf{x}^T = x_1^{m_1(T)} x_2^{m_2(T)} \cdots ,$$

where $m_i(T)$ is the number of occurrences of i in T.

The *Schur symmetric function* indexed by θ is defined by

$$s_\theta(\mathbf{x}) = \sum_T \mathbf{x}^T,$$

summed over all semistandard Young tableaux of shape θ. Note $s_\emptyset = 1$.

Let Λ^n denote the vector space of symmetric functions of homogeneous degree n over $\mathbb{Q}$, and let $\Lambda = \bigoplus_{n\geq 0} \Lambda^n$ denote the (graded) algebra of symmetric functions over $\mathbb{Q}$. Each of $\{m_\theta : \theta \vdash n\}$, $\{e_\theta : \theta \vdash n\}$, $\{h_\theta : \theta \vdash n\}$, $\{p_\theta : \theta \vdash n\}$ and $\{s_\theta : \theta \vdash n\}$ is a basis for Λ^n. Equivalently, $\{m_\theta : \theta \in \mathcal{P}\}$, $\{e_i : i \geq 1\}$, $\{h_i : i \geq 1\}$, $\{p_i : i \geq 1\}$ and $\{s_\theta : \theta \in \mathcal{P}\}$ each generate Λ as a $\mathbb{Q}$-algebra.

Change of basis for Λ^n

For an integer partition $\alpha = [1^{a_1} 2^{a_2} \cdots] \vdash n$, let ϖ_α denote the size of the centralizer of a permutation in S_n with cycle type α, so that

$$\varpi_\alpha = 1^{a_1} a_1! 2^{a_2} a_2! \cdots .$$

Let $\mathcal{C}_\alpha$ denote the conjugacy class of S_n indexed by α. Note $|\mathcal{C}_\alpha| = n!/\varpi_\alpha$.

Also, let

$$\varepsilon_\alpha = (-1)^{a_2 + a_4 + \cdots} = (-1)^{n - l(\alpha)}.$$

The following four Propositions are standard formulas for calculating change of basis matrices for Λ^n. More formulas can be found in [24].

Proposition 3.2. *For $n \in \mathbb{N}$,*

$$e_n = \sum_{\alpha \vdash n} \varepsilon_\alpha \varpi_\alpha^{-1} p_\alpha.$$

Proof. The generating series for the elementary symmetric functions is $E(t) =$

$\prod_{i\geq 1}(1+x_it)$. Thus

$$\begin{aligned} E(t) &= \exp\left(\sum_{i\geq 1}\log(1+x_it)\right) \\ &= \exp\left(-\sum_{i\geq 1}\sum_{j\geq 1}(-1)^j\frac{x_i^jt^j}{j}\right) \\ &= \exp\left(\sum_{j\geq 1}(-1)^{j-1}\frac{p_j}{j}t^j\right), \end{aligned}$$

and

$$e_n = [t^n]\exp\left(\sum_{j\geq 1}(-1)^{j-1}\frac{p_j}{j}t^j\right) = \sum_{\alpha\vdash n}\varepsilon_\alpha\varpi_\alpha^{-1}p_\alpha.$$

□

Proposition 3.3. *For $n\in\mathbb{N}$,*

$$h_n = \sum_{\alpha\vdash n}\varpi_\alpha^{-1}p_\alpha.$$

Proof. The generating series for the complete symmetric functions is $H(t) = \prod_{i\geq 1}(1-x_it)^{-1}$. Thus

$$H(t) = \exp\left(\sum_{i\geq 1}\log(1-x_it)^{-1}\right) = \exp\left(\sum_{i\geq 1}\sum_{j\geq 1}\frac{x_i^jt^j}{j}\right) = \exp\left(\sum_{j\geq 1}\frac{p_j}{j}t^j\right),$$

and

$$h_n = [t^n]\exp\left(\sum_{j\geq 1}\frac{p_j}{j}t^j\right) = \sum_{\alpha\vdash n}\varpi_\alpha^{-1}p_\alpha.$$

□

The next Proposition can be taken as an alternate definition of Schur functions. It is a standard formula which can be found in [24] (Equation I.3.4).

Proposition 3.4.

$$s_\theta = \det\left(h_{\theta_i-i+j}\right)_{d\times d}$$

for $\theta\in\mathcal{P}$, and $d\geq l(\theta)$. □

Let $\chi^\theta(\lambda)$ denote the irreducible character of S_n indexed by θ, evaluated on the conjugacy class indexed by λ.

Proposition 3.5. *For any $\theta \vdash n$,*

$$s_\theta = \sum_{\lambda \vdash n} \varpi_\lambda^{-1} \chi^\theta(\lambda) p_\lambda.$$

Proof. The proof involves character theory of the symmetric groups, so it is deferred to Section 3.1.5. ☐

A bilinear form

Define a bilinear form $\langle \cdot, \cdot \rangle$ on Λ by requiring

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu},$$

where $\delta_{\lambda\mu}$ denotes the Kronecker delta. It can be shown that under this bilinear form,

$$\langle p_\lambda, p_\mu \rangle = \varpi_\lambda \delta_{\lambda\mu}.$$

In other words, the power sum symmetric functions $\{p_\theta : \theta \vdash n\}$ form an orthogonal basis for Λ^n. It can also be shown that

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}, \tag{3.5}$$

so that the Schur functions $\{s_\theta : \theta \vdash n\}$ form an orthonormal basis for Λ^n. Refer to [24] (Chapter I.4) for details.

This bilinear form enables us to evaluate irreducible characters of the symmetric group in the next section.

3.1.5 Characters of the symmetric groups

We shall describe the irreducible characters of S_n.

For functions f and g on S_n with values in a commutative $\mathbb{Q}$-algebra, define the scalar product

$$\langle f, g \rangle_{S_n} = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) g(\sigma^{-1}).$$

Let R^n denote the $\mathbb{Z}$-module generated by the irreducible characters of S_n, and let

$$R = \bigoplus_{n \geq 0} R^n.$$

It can be shown that R is a commutative graded ring with unity. Since irreducible characters are class functions, then every element in R^n is a class function. For $f = \sum_{n\geq 0} f_n$ and $g = \sum_{n\geq 0} g_n$ where $f_n,\, g_n \in R^n$, R carries an inner product

$$\langle f, g\rangle = \sum_{n\geq 0} \langle f_n, g_n\rangle_{S_n}.$$

Given a permutation $\sigma \in S_n$, let $\tau(\sigma)$ denote the cycle type of σ. Define the function $\psi : S_n \to \Lambda^n$ by

$$\psi(\sigma) = p_{\tau(\sigma)}.$$

Note that $\psi(\sigma) = \psi(\sigma^{-1})$.

Define a $\mathbb{Z}$-linear mapping $\mathrm{ch} : R \to \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ by

$$\mathrm{ch}(f) = \langle f, \psi\rangle_{S_n} = \frac{1}{n!}\sum_{\sigma\in S_n} f(\sigma)\psi(\sigma),$$

for $f \in R^n$. This is called the *characteristic map* and it can be shown that ch is an isometric ring isomorphism of R onto Λ. Refer to Macdonald [24] (Chapter I.7) for full details.

Recall that $|\mathcal{C}_\lambda| = n!/\varpi_\lambda$, so for any class function f,

$$\mathrm{ch}(f) = \frac{1}{n!}\sum_{\lambda\vdash n} |\mathcal{C}_\lambda| f(\lambda) p_\lambda = \sum_{\lambda\vdash n} \varpi_\lambda^{-1} f(\lambda) p_\lambda.$$

Let η_n denote the trivial character of S_n, so that $\eta_n(\lambda) = 1$ for all $\lambda \vdash n$. Then $\mathrm{ch}(\eta_n) = \sum_{\lambda\vdash n} \varpi_\lambda^{-1} p_\lambda = h_n$. For $\lambda \vdash n$, let $\eta_\lambda = \eta_{\lambda_1}\eta_{\lambda_2}\cdots$, so that η_λ is the character induced by the trivial representation of $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots$. It follows that

$$\mathrm{ch}(\eta_\lambda) = h_\lambda.$$

For $\theta \vdash n$, define

$$\chi^\theta = \det(\eta_{\theta_i - i + j})_{1\leq i,j\leq n} \in R^n,$$

so that χ^θ is also a character of S_n. Then by Proposition 3.4,

$$\begin{aligned} s_\theta &= \det\left(h_{\theta_i - i + j}\right)_{n\times n} \\ &= \det\left(\mathrm{ch}(\eta_{\theta_i - i + j})\right)_{n\times n} \\ &= \mathrm{ch}(\det\left(\eta_{\theta_i - i + j}\right)_{n\times n}) \\ &= \mathrm{ch}(\chi^\theta). \end{aligned}$$

On the other hand, $\mathrm{ch}(\chi^\theta) = \sum_{\lambda\vdash n} \varpi_\lambda^{-1}\chi^\theta(\lambda)p_\lambda$. It remains to show that $\{\chi^\theta : \theta \vdash n\}$ is the set of irreducible characters of S_n, and Proposition 3.5 is proved.

Since ch is an isometry, by Equation (3.5) we have

$$\left\langle \chi^\lambda, \chi^\mu \right\rangle_{S_n} = \left\langle \mathrm{ch}(\chi^\lambda), \mathrm{ch}(\chi^\mu) \right\rangle = \langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu},$$

which shows that up to a difference of signs, χ^λ is an irreducible character of S_n for each $\lambda \vdash n$. Since the set of conjugacy classes of S_n is indexed by the integer partitions of n, therefore $\{\chi^\lambda : \lambda \vdash n\}$ must be the entire set of irreducible characters of S_n.

Remark. This correspondence between conjugacy classes and irreducible representations does not generally hold for arbitrary groups.

The following theorem is a standard result which gives the value of an irreducible character of S_n evaluated on a conjugacy class. Its proof can be found in [13] (Chapter 4). Let $\triangle = \triangle(x_1, \ldots, x_d)$ denote the *Vandermonde determinant*, which has several well-known formulas:

$$\begin{aligned} \triangle(x_1, \ldots, x_d) &= \det [x_i^{d-j}]_{d\times d} \\ &= \prod_{1\le i<j\le d} (x_i - x_j) \\ &= \sum_{\mu\in S_d} \mathrm{sgn}(\mu)\ x_1^{\mu(d)-1} \cdots x_d^{\mu(1)-1}. \end{aligned}$$

Theorem 3.6. *(Frobenius' formula) Let $\theta = (\theta_1, \ldots, \theta_n) \vdash n$. The irreducible character χ^θ of S_n indexed by θ, evaluated on the conjugacy class indexed by $\alpha = [1^{a_1}, \ldots, n^{a_n}]$ is given by*

$$\chi^\theta(\alpha) = [x_1^{t_1} \cdots x_d^{t_d}]\ \triangle(x_1, \ldots, x_d) \prod_{i=1}^{n} p_i(x_1, \ldots, x_d)^{a_i},$$

where $d \ge l(\theta)$, and $t_i = \theta_i + d - i$ for $1 \le i \le d$. $\square$

We can use Frobenius' formula to calculate the special case when α is a single cycle of length n.

Corollary 3.7. *([13] Exercise 4.16)*

$$\chi^\theta(n) = \begin{cases} (-1)^j & \text{if } \theta = [1^j, n-j] \text{ and } 0 \le j \le n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose $l(\theta) = d$. By Frobenius' formula,

$$\chi^{\theta}(n) = [x_1^{t_1} \cdots x_d^{t_d}] \triangle \cdot (x_1^n + \cdots + x_d^n).$$

Each monomial in $\triangle \cdot (x_1^n + \cdots + x_d^n)$ is of the form

$$x_1^{\mu(d)-1} x_2^{\mu(d-1)-1} \cdots x_i^{\mu(d-i+1)-1+n} \cdots x_d^{\mu(1)-1}$$

for some $\mu \in S_d$.

Note that $\{\mu(i) - 1 : 1 \leq i \leq d\} = \{0, 1, \ldots, d-1\}$. Also note that by definition, $t_1 > \cdots > t_d$. Since θ_d is assumed to be nonzero, then $t_d = \theta_d \geq 1$. Together with the previous observation, it follows that the coefficient of $x_1^{t_1} \cdots x_d^{t_d}$ is nonzero only when $t_1 = n$, $t_2 = d-1$, $t_3 = d-2, \ldots, t_d = 1$. In other words, $\theta = (n-d+1, 1, \ldots, 1)$, where $1 \leq d \leq n$. So assume $\theta = (n-j, 1, \ldots, 1) \vdash n$ for some $0 \leq j \leq n-1$. Then

$$\begin{aligned}
\chi^{\theta}(n) &= [x_1^n x_2^j x_3^{j-1} \cdots x_{j+1}] \triangle \cdot (x_1^n + \cdots + x_{j+1}^n) \\
&= [x_1^0 x_2^j x_3^{j-1} \cdots x_{j+1}] \sum_{\mu \in S_{j+1}} \operatorname{sgn}(\mu)\; x_1^{\mu(j+1)-1} \cdots x_{j+1}^{\mu(1)-1} \\
&= \operatorname{sgn}((1\ 2\ \cdots j+1)) \\
&= (-1)^j,
\end{aligned}$$

as required. $\square$

Lemma 3.8. *([8] Corollary 5.3) For $\alpha = [1^{a_1}, 2^{a_2}, \ldots] \vdash n$,*

$$\sum_{j=0}^{n-1} \chi^{[1^j, n-j]}(\alpha) u^j = \frac{1}{1+u} \prod_{i=1}^{n} (1 - (-u)^i)^{a_i}.$$

Proof. By Proposition 3.5, for α, $\theta \vdash n$,

$$\begin{aligned}
\langle p_\alpha, s_\theta \rangle &= \left\langle p_\alpha, \sum_{\lambda \vdash n} \varpi_\lambda^{-1} \chi^{\theta}(\lambda) p_\lambda \right\rangle \\
&= \sum_{\lambda \vdash n} \varpi_\lambda^{-1} \chi^{\theta}(\lambda) \langle p_\alpha, p_\lambda \rangle \\
&= \chi^{\theta}(\alpha).
\end{aligned}$$

Thus $\chi^{[1^j, n-j]}(\alpha) = \langle p_\alpha, s_{[1^j, n-j]} \rangle$.

Because power sum symmetric functions form an orthogonal basis for Λ^n with respect to this bilinear form, we wish to express the Schur function in terms of power sums. Begin by expressing the Schur function as a polynomial in complete symmetric functions. By Proposition 3.4,

$$s_{[1^j,n-j]} = \det \begin{bmatrix} h_{n-j} & h_{n-j+1} & h_{n-j+2} & \cdots & h_n \\ h_0 & h_1 & h_2 & \cdots & h_j \\ 0 & h_0 & h_1 & \cdots & h_{j-1} \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & h_0 & h_1 \end{bmatrix}.$$

Since $h_0 = 1$, and $s_{[1^j]} = \det (h_{1-i+j})_{j\times j}$, then by cofactor expansion along the first row,

$$\begin{aligned} s_{[1^j,n-j]} &= h_{n-j}s_{[1^j]} - h_{n-j+1}s_{[1^{j-1}]} + \cdots + (-1)^j h_n s_\emptyset \\ &= h_{n-j}e_j - h_{n-j+1}e_{j-1} + \cdots + (-1)^j h_n e_0 \\ &= [t^n u^j] \sum_{i\geq 0} h_i t^i \sum_{i\geq 0} e_i t^i u^i \sum_{i\geq 0} (-1)^i u^i \\ &= [t^n u^j] H(t)E(tu)\frac{1}{1+u}. \end{aligned}$$

From Propositions 3.2 and 3.3, we have $H(t) = \exp\left(\sum_{i\geq 1} \frac{p_i}{i} t^i\right)$, and $E(tu) = \exp\left(\sum_{i\geq 1} (-1)^{i-1} \frac{p_i}{i} t^i u^i\right)$. So

$$\begin{aligned} [t^n]H(t)E(tu) &= [t^n]\exp\left(\sum_{i\geq 1} \frac{p_i}{i} t^i\right) \exp\left(\sum_{i\geq 1} (-1)^{i-1} \frac{p_i}{i} t^i u^i\right) \\ &= [t^n]\exp\left(\sum_{i\geq 1} (1-(-u)^i)\frac{p_i}{i} t^i\right) \\ &= \sum_{\substack{(a_1,\ldots,a_n):\\ \sum i a_i = n}} \prod_{i=1}^{n} \frac{1}{a_i!}\left(\frac{p_i}{i}(1-(-u)^i)\right)^{a_i} \\ &= \sum_{\theta\vdash n} \varpi_\theta^{-1} p_\theta \prod_{i=1}^{n} (1-(-u)^i)^{m_i(\theta)}, \end{aligned}$$

where $m_i(\theta)$ is the number of occurrences of i in the sequence θ. Thus

$$\begin{aligned}
\chi^{[1^j,n-j]}(\alpha) &= \langle p_\alpha, s_{[1^j,n-j]}\rangle \\
&= [u^n]\frac{1}{1+u}\left\langle p_\alpha, \sum_{\theta\vdash n}\varpi_\theta^{-1}p_\theta\prod_{i=1}^{n}(1-(-u)^i)^{m_i(\theta)}\right\rangle \\
&= [u^n]\frac{1}{1+u}\sum_{\theta\vdash n}\varpi_\theta^{-1}\prod_{i=1}^{n}(1-(-u)^i)^{m_i(\theta)}\langle p_\alpha, p_\theta\rangle \\
&= [u^n]\frac{1}{1+u}\prod_{i=1}^{n}(1-(-u)^i)^{a_i}.
\end{aligned}$$

The result follows. □

Remark. When $\alpha = (n)$, this reduces to $\sum_{j=0}^{n-1}\chi^{[1^j,n-j]}(n)u^j = \frac{1}{1+u} - \frac{(-u)^n}{1+u}$, so $[u^j]\left(\frac{1}{1+u} - \frac{(-u)^n}{1+u}\right) = [u^j]\frac{1}{1+u} = (-1)^j$ for $0 \le j \le n-1$, which agrees with Corollary 3.7.

An alternate formula can be obtained for $\chi^{[1^j,n-j]}(\alpha)$ through Frobenius' formula. The result is $\chi^{[1^j,n-j]}(\alpha) = (-1)^j\sum_{r=0}^{j}\sum_{\lambda\vdash r}(-1)^{l(\lambda)}\binom{\alpha}{\lambda}$, where $\binom{\alpha}{\lambda} = \prod_i\binom{a_i}{l_i}$. Details are given in Appendix A.

3.2 Hypermonopole series

Given a hypermap $\mathfrak{h}$, its *hyperedge partition* is the sequence $\eta = (\eta_1, \eta_2, \ldots) = [1^{a_1}, 2^{a_2}, \ldots]$ where a_i is the number of hyperedges of $\mathfrak{h}$ with degree i. Hyperface partitions and vertex partitions of hypermaps are similarly defined.

Let $h(\eta, \phi, \nu)$ denote the number of hypermaps in orientable surfaces with hyperedge partition η, hyperface partition ϕ, and vertex partition ν. Let $H(x, y, z)$ be the generating series for rooted hypermaps in orientable surfaces, so that

$$H(x, y, z) = \sum_{\eta,\phi,\nu\in\mathcal{P}} h(\eta, \phi, \nu)\mathbf{x}_\eta\mathbf{y}_\phi\mathbf{z}_\nu,$$

where $\mathbf{x}_\eta$ denotes the monomial $x_{\eta_1}x_{\eta_2}\cdots$, etc.

The following theorem can be found in [15] whose proof follows from Theorem 3.2 and Lemma 3.3 in [22].

Theorem 3.9. *(Genus series for rooted hypermaps)*

$$H(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z})) = t\frac{\partial}{\partial t}\log\sum_{\theta\in\mathcal{P}} t^{|\theta|}H_\theta s_\theta(\mathbf{x})s_\theta(\mathbf{y})s_\theta(\mathbf{z})\Bigg|_{t=1},$$

where $p(\mathbf{x})$ *denotes* $p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots$. ☐

Let $a_n(p,q)$ be the number of rooted hypermonopoles with n edges, p hyperedges, and q hyperfaces. Let $A_n(x,y)$ be the generating series for rooted hypermonopoles with n edges in orientable surfaces, so that

$$A_n(x,y) = \sum_{p,q\geq 1} a_n(p,q) x^p y^q.$$

From Theorem 3.9 we obtain the following corollary for the generating series of rooted hypermonopoles, which was first proved in [8]. We offer a different proof below, but first, we need a summation lemma:

Lemma 3.10. *For an indeterminate* x *and any positive integer* n*,*

$$\sum_{i=0}^{j} \frac{\binom{x}{i}\binom{-x}{n-i}}{\binom{x}{j}\binom{-x}{n-j}} = \frac{(n-j)(x-j)}{nx}.$$

Proof. Proceed by induction on j. When $j=0$, $\frac{\binom{x}{0}\binom{-x}{n}}{\binom{x}{0}\binom{-x}{n}} = 1 = \frac{nx}{nx}$, so the base case holds. Now $\binom{x}{j} = \binom{x}{j-1}\frac{x-j+1}{j}$ and $\binom{-x}{n-j} = \binom{-x}{n-(j-1)}\frac{n-j+1}{-x-n+j}$, so

$$\begin{aligned}
\sum_{i=0}^{j} \frac{\binom{x}{i}\binom{-x}{n-i}}{\binom{x}{j}\binom{-x}{n-j}} &= 1 + \sum_{i=0}^{j-1} \frac{\binom{x}{i}\binom{-x}{n-i}}{\binom{x}{j-1}\binom{-x}{n-(j-1)}} \frac{j(-x-n+j)}{(n-j+1)(x-j+1)} \\
&= 1 + \frac{(n-(j-1))(x-(j-1))}{nx} \frac{j(-x-n+j)}{(n-j+1)(x-j+1)} \\
&= \frac{nx - jx - jn + j^2}{nx}.
\end{aligned}$$

☐

Corollary 3.11. *For positive integers* n*,*

$$A_n(x,y) = (-1)^{n-1} n! \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j}\binom{x+j}{n}\binom{y+j}{n}.$$

Proof. In order to obtain $A_n(x,y)$ from $H(p(\mathbf{x}), p(\mathbf{y}), p(\mathbf{z}))$, recall that the power sum symmetric functions $p_i(\mathbf{x})$ are algebraically independent, hence each $p_i(\mathbf{x})$ may be replaced by x_i. Furthermore, since we are not interested in keeping track of the degree of the hyperedges and hyperfaces in this corollary, each x_i may then be replaced by x, and similarly, each y_i may be replaced by y. The main idea in this proof is to make use of the homomorphisms $L_x : p_i(x) \mapsto x$ and $L_y : p_i(y) \mapsto y$.

The degree of a vertex of a hypermap is defined to be the number of vertex-hyperedge incidences. So, a rooted hypermonopole has n edges if and only if its vertex has degree n. Thus

$$\begin{aligned} A_n(p(\mathbf{x}), p(\mathbf{y})) &= [p_n(\mathbf{z})] t \frac{\partial}{\partial t} \log \sum_{\theta \in \mathcal{P}} t^{|\theta|} H_\theta s_\theta(\mathbf{x}) s_\theta(\mathbf{y}) s_\theta(\mathbf{z}) \Bigg|_{t=1} \\ &= [p_n(\mathbf{z})] n \sum_{\theta \vdash n} H_\theta s_\theta(\mathbf{x}) s_\theta(\mathbf{y}) s_\theta(\mathbf{z}). \end{aligned}$$

By Proposition 3.5 and Corollary 3.7,

$$\begin{aligned} [p_n(\mathbf{z})] s_\theta(\mathbf{z}) &= \varpi_{(n)}^{-1} \chi^\theta(n) \\ &= \begin{cases} (-1)^j/n & \text{if } \theta = [1^j, n-j], 0 \le j \le n-1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

And from the hooklength formula (Equation (3.4)), we have

$$H_{[1^j, n-j]} = n(n-j-1)! j! = \frac{n!}{\binom{n-1}{j}},$$

therefore

$$A_n(p(\mathbf{x}), p(\mathbf{y})) = n! \sum_{j=0}^{n-1} \frac{(-1)^j}{\binom{n-1}{j}} s_{[1^j, n-j]}(\mathbf{x}) s_{[1^j, n-j]}(\mathbf{y}).$$

In order to apply the homomorphisms L_x and L_y, it is necessary to express the Schur functions explicitly in terms of the p_i's. By Proposition 3.5 and Lemma 3.8,

$$\begin{aligned} s_{[1^j, n-j]}(\mathbf{x}) &= \sum_{\alpha \vdash n} \varpi_\alpha^{-1} \chi^{[1^j, n-j]}(\alpha) p_\alpha(\mathbf{x}) \\ &= [u^j] \frac{1}{1+u} \sum_{\substack{(a_1, \ldots, a_n): \\ \sum i a_i = n}} \prod_{i=1}^{n} \frac{1}{a_i!} \left(\frac{p_i(\mathbf{x})}{i} (1 - (-u)^i) \right)^{a_i}. \end{aligned}$$

For $n \ge 1$, we have

$$[w^n] \exp\left(\sum_{i \ge 1} c_i w^i \right) = \sum_{\substack{(a_1, \ldots, a_n): \\ \sum i a_i = n}} \frac{c_1^{a_1} \cdots c_n^{a_n}}{a_1! \cdots a_n!},$$

so

$$s_{[1^j, n-j]}(\mathbf{x}) = [u^j w^n] \frac{1}{1+u} \exp\left(\sum_{i \ge 1} \frac{p_i(\mathbf{x})}{i} (1 - (-u)^i) w^i \right).$$

Apply the homomorphism $L_x : p_i(\mathbf{x}) \mapsto x$ to get

$$\begin{aligned} L_x s_{[1^j,n-j]}(\mathbf{x}) &= [u^j w^n] \frac{1}{1+u} \exp\left(\sum_{i\geq 1} \frac{x}{i}(1-(-u)^i)w^i \right) \\ &= [u^j w^n] \frac{1}{1+u} \exp\left(x \sum_{i\geq 1} \frac{w^i}{i} - \frac{(-uw)^i}{i} \right) \\ &= [u^j w^n] \frac{1}{1+u} \exp(x(\log(1-w)^{-1} + \log(1+uw))) \\ &= [u^j w^n] \frac{1}{1+u} \left(\frac{1+uw}{1-w} \right)^x . \end{aligned}$$

We have

$$[u^j] \frac{(1+uw)^x}{1+u} = [u^j] \sum_{i\geq 0} (-u)^i \cdot \sum_{k\geq 0} \binom{x}{k} u^k w^k = \sum_{i=0}^{j} (-1)^i \binom{x}{j-i} w^{j-i}.$$

Changing variables $i \mapsto j-i$ yields

$$[u^j] \frac{(1+uw)^x}{1+u} = (-1)^j \sum_{i=0}^{j} (-1)^i \binom{x}{i} w^i,$$

so

$$\begin{aligned} [u^j w^n] \frac{(1+uw)^x}{1+u} (1-w)^{-x} &= [w^n](-1)^j \sum_{i=0}^{j} \binom{x}{i} (-w)^i \cdot \sum_{m\geq 0} \binom{-x}{m} (-w)^m \\ &= (-1)^{n+j} \sum_{i=0}^{j} \binom{x}{i} \binom{-x}{n-i}. \end{aligned}$$

By Lemma 3.10,

$$\begin{aligned} (-1)^{n+j} \sum_{i=0}^{j} \binom{x}{i} \binom{-x}{n-i} &= (-1)^{n+j} \binom{x}{j} \binom{-x}{n-j} \frac{(n-j)(x-j)}{nx} \\ &= (-1)^{n+j} \frac{x_{(j)}}{j!} \frac{(-x)_{(n-j)}}{(n-j)!} \frac{(n-j)(x-j)}{nx} \\ &= \frac{(-1)^n (n-1)!}{j!(n-1-j)!} \frac{(-1)^j (x-1)_{(j-1)} (x-j)(-x)_{(n-j)}}{n!} \\ &= (-1)^n \binom{n-1}{j} \frac{(-x+j)_{(n)}}{n!} \\ &= (-1)^n \binom{n-1}{j} \binom{-x+j}{n}. \end{aligned}$$

Therefore,

$$L_x s^{[1^j, n-j]}(\mathbf{x}) = (-1)^n \binom{n-1}{j}\binom{-x+j}{n} = \binom{n-1}{j}\binom{x-j+n-1}{n},$$

and it follows that

$$\begin{aligned} A_n(x,y) &= L_x L_y A_n(p(\mathbf{x}), p(\mathbf{y})) \\ &= n! \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j}\binom{x-j+n-1}{n}\binom{y-j+n-1}{n}. \end{aligned}$$

Switching the order of summation (changing variables $j \mapsto n-1-j$) yields the desired series. □

Remark. $A_n(x,y)$ is symmetric in x and y as expected, since hypermaps are face two-colourable.

In its present form, the series $A_n(x,y)$ appears to have degree $2n$. However it really has degree $n+1$, so we wish to find an expression for this series in which the highest degree term in the expression has degree $n+1$. With the help of hypergeometric functions, such a degree-respecting form can be obtained for the series. The following proof is due to Andrews [3].

Corollary 3.12.

$$A_n(x,y) = (n-1)! \sum_{j=1}^{\lfloor (n+1)/2 \rfloor} j \binom{x+y+n-2j}{n-2j+1}\binom{x}{j}\binom{y}{j}.$$

Proof. The first term in $A_n(x,y)$ is $(-1)^{n-1} n! \binom{x}{n}\binom{y}{n}$, and the term ratio for $A_n(x,y)$ is

$$\frac{(j+x+1)(j+y+1)(j+1-n)}{(j+1)(j+x-n+1)(j+y-n+1)}.$$

Therefore,

$$A_n(x,y) = (-1)^{n-1} n! \binom{x}{n}\binom{y}{n} \, {}_3F_2 \left[\begin{matrix} 1-n & x+1 & y+1 \\ - & x-n+1 & y-n+1 \end{matrix} \right].$$

At this point, we need the following

Claim:

$${}_3F_2 \left[\begin{matrix} -m & b & c \\ - & 1+a-b & 1+a-c \end{matrix} \right] = \frac{(-1)^m}{a^{(m)}} \cdot F,$$

where

$$F = \sum_{\frac{m}{2} \le j \le m} \frac{a^{(2j)}(1+a-b-c)^{(j)}(-m)^{(j)}}{(1+a-b)^{(j)}(1+a-c)^{(j)}(2j-m)!}.$$

Proof. The following formula, due to Whipple, can be found in [4] §4.5 (1):

$${}_4F_3\left[\begin{matrix} a & b & c & -m \\ - & 1+a-b & 1+a-c & w \end{matrix}\right] = \frac{(w-a)^{(m)}}{w^{(m)}} \cdot G,$$

where $G =$

$${}_5F_4\left[\begin{matrix} 1+a-w & \frac{1}{2}a & \frac{1}{2}(1+a) & 1+a-b-c & -m \\ - & 1+a-b & 1+a-c & \frac{1}{2}(1+a-w-m) & 1+\frac{1}{2}(a-w-m) \end{matrix}\right].$$

Note that $(\frac{1}{2}a)^{(j)}(\frac{1}{2}(a+1))^{(j)} = (\frac{1}{2})^{2j}a^{(2j)}$, so the right hand side of Whipple's formula is

$$\frac{(w-a)^{(m)}}{w^{(m)}} \sum_{j \ge 0} \frac{(1+a-w)^{(j)}a^{(2j)}(1+a-b-c)^{(j)}(-m)^{(j)}}{j!(1+a-b)^{(j)}(1+a-c)^{(j)}(1-(w-a)-m)^{(2j)}}.$$

Observe that as $w \to a$, the left hand side of Whipple's formula tends to the left hand side of our Claim, and at the same time,

$$(1+a-w)^{(j)} \to 1^{(j)} = j!,$$

and

$$\frac{(w-a)^{(m)}}{(1-(w-a)-m)^{(2j)}} = \frac{(-1)^m}{(1-(w-a))^{(2j-m)}} \to \frac{(-1)^m}{(2j-m)!}.$$

The result follows. □

Set $m = n-1$, $b = x+1$, $c = y+1$, $a = x+y-n+1$ in the Claim, and note that $\frac{z^{(m)}}{z^{(i)}} = (z+i)^{(m-i)}$ to get

$$A_n(x,y) = n!\binom{x}{n}\binom{y}{n} \sum_{\frac{n-1}{2} \le j \le n-1} \frac{(x+y)^{(2j-n+1)}(-n)^{(j)}(1-n)^{(j)}}{(x-n+1)^{(j)}(y-n+1)^{(j)}(2j-n+1)!}.$$

After routine simplification,

$$A_n(x,y) = (n-1)! \sum_{\frac{n-1}{2} \le j \le n-1} \binom{x+y-n+2j}{2j-n+1}\binom{x}{n-j}\binom{y}{n-j}(n-j).$$

Changing variables $j \mapsto n-j$ gives the result. □

Recently, Schaeffer and Vassilieva [25] constructed a bijection between unicellular partitioned bicoloured maps and 2-tuples of ordered bicoloured trees and partial permutations to prove that the numbers $B(m,n,N)$ of unicellular bicoloured maps with m white vertices, n black vertices and N edges satisfy

$$\sum_{m,n\geq 1} B(m,n,N)y^m z^n = N! \sum_{p,q\geq 1} \binom{N-1}{p-1,q-1}\binom{y}{p}\binom{z}{q}.$$

They observe that Adrianov [1] independently came up with the formula

$$\sum_{m,n\geq 1} B(m,n,N)y^m z^n = (N-1)! \sum_{k\geq 0} \frac{yz}{k+1}\binom{y+k}{k}\binom{z+k}{k}\binom{y+z}{N-1-2k},$$

and present a derivation of their formula from Adrianov's. Both these expressions are equivalent to the formulas in Corollary 3.11 and Corollary 3.12.

3.3 Genus zero rooted hypermonopoles and partitions

By the Euler-Poincaré formula, an indeterminate z can be introduced to the hypermonopole series to mark genus. This is

$$A(x,y,z) = z^{(n+1)/2} A_n(xz^{-1/2}, yz^{-1/2}),$$

so that $[z^g]A(x,y,z)$ gives the generating series for rooted hypermonopoles of genus g.

The genus zero rooted hypermonopole series, which we denote by $A_n^0(x,y)$, is

$$[z^{-(n+1)/2}]A_n(x/\sqrt{z}, y/\sqrt{z}),$$

which consists of the terms of highest degree in $A_n(x,y)$. Using the degree-respecting form for $A_n(x,y)$ from Corollary 3.12, the terms of highest degree are

$$(n-1)! \sum_{j=1}^{\lfloor (n+1)/2 \rfloor} \frac{j}{(n-2j+1)!j!j!}(x+y)^{n-2j+1}x^j y^j.$$

Expanding using the binomial theorem and simplifying, this becomes

$$\frac{1}{n}\sum_{j\geq 1}\sum_{i\geq 0} \frac{n!}{j!(j-1)!i!(n-2j-i+1)!} x^{i+j} y^{n+1-i-j}.$$

Make the change of variables $k = i + j$ to get

$$\frac{1}{n}\sum_{k\geq 1}\binom{n}{k}\sum_{i\geq 0}\binom{k}{i}\binom{n-k}{k-1-i}x^k y^{n+1-k}.$$

By Vandermonde's convolution, we get the final expression

$$\frac{1}{n}\sum_{k\geq 1}\binom{n}{k}\binom{n}{k-1}x^k y^{n+1-k}.$$

This is the generating series for genus zero rooted hypermonopoles with the number of hyperedges marked by x, and the number of hyperfaces marked by y.

Because of the bijection between genus zero rooted hypermonopoles and genus zero partitions, the coefficients $[x^k y^{n+1-k}]A_n(x, y)$ are the Narayana numbers $N(n, k)$ defined in Equation (3.1), as expected. By Vandermonde's convolution, setting $x = y = 1$ in $A_n^0(x, y)$ gives the Catalan number C_n.

We remark that the indeterminate y is redundant in this generating series, since given the degree of x, the degree of y is completely determined by the Euler-Poincaré formula.

3.4 Genus one rooted hypermonopoles

The genus one rooted hypermonopole series, which we denote by $A_n^1(x, y)$, is

$$[z^{-(n-1)/2}]A_n(x/\sqrt{z}, y/\sqrt{z}),$$

and consists of terms of degree $n-1$ in $A_n(x, y)$.

Before stating the result regarding this series, we first develop some facts about Stirling numbers of the first kind. Most are standard results which can be found in Chapter 6 of [17].

3.4.1 Stirling numbers of the first kind

For $n \geq 0$ and $0 \leq i \leq n$, the *unsigned Stirling number of the first kind* $|s_n^{(i)}|$, is the number of ways to arrange n objects into i cycles, and the *signed Stirling numbers of the first kind* is defined by $s_n^{(i)} = (-1)^{n-i}|s_n^{(i)}|$.

Stirling numbers of the first kind satisfy the recurrence ([17] Equation (6.8))

$$s_n^{(i)} = s_{n-1}^{(i-1)} - (n-1)s_{n-1}^{(i)}.$$

The following are generating series for the $s_n^{(i)}$, whose proofs can be obtained by induction and the use of the recurrence equation:

$$x_{(n)} = \sum_{i=0}^{n} s_n^{(i)} x^i,$$

and

$$\frac{1}{i!}(\log(1+x))^i = \sum_{n \geq i} s_n^{(i)} \frac{x^n}{n!}.$$

Also, $(x+j)_{(n)} = n! \sum_{i \geq 0} \binom{j}{i} x_{(n-j+i)}/(n-j+i)!$, which implies

$$[x^k](x+j)_{(n)} = n! \sum_{i=0}^{j} \binom{j}{i} \frac{s_{n-j+i}^{(k)}}{(n-j+i)!}. \tag{3.6}$$

From the combinatorial interpretation of (unsigned) Stirling numbers of the first kind, some special values can be deduced, including $s_n^{(n)} = 1$, and $s_{n+1}^{(n)} = -\binom{n+1}{2}$, for all $n \geq 0$. The next result is also needed later:

Lemma 3.13. *For all $n \geq 0$,*

$$s_{n+2}^{(n)} = \sum_{i=1}^{n+1} i \binom{i}{2} = \binom{n+2}{3} \frac{3n+5}{4}.$$

Proof. By the recursion formula,

$$\begin{aligned}
s_{n+2}^{(n)} &= s_{n+1}^{(n-1)} - (n+1) s_{n+1}^{(n)} \\
&= s_{n+1}^{(n-1)} + (n+1)\binom{n+1}{2} \\
&= \qquad \vdots \\
&= 1\binom{1}{2} + 2\binom{2}{2} + \cdots + (n+1)\binom{n+1}{2}.
\end{aligned}$$

For the second equality, proceed by induction. First we have $s_2^{(0)} = 0 = \binom{1}{2}$. Now

$$\begin{aligned}
\sum_{i=1}^{n+1} i \binom{i}{2} &= \sum_{i=1}^{n} i \binom{i}{2} + (n+1)\binom{n+1}{2} \\
&= \binom{n+1}{3} \frac{3n+2}{4} + \frac{(n+1)(n+1)n}{2} \\
&= \binom{n+2}{3} \frac{3n+5}{4}.
\end{aligned}$$

$\square$

We are now ready to proceed with the main result of this section. The first proof, given below, extracts terms of degree $n-1$ from the series $A_n(x,y)$ in Corollary 3.11, and uses properties of Stirling numbers of the first kind. The second proof, given in Appendix A, is due to Andrews [3] and utilizes the degree-respecting form of $A_n(x,y)$ in Corollary 3.12.

Theorem 3.14. *The generating series for genus one rooted hypermonopoles with the number of hyperedges marked by x, and the number of hyperfaces marked by y is*

$$A_n^1(x,y) = \frac{1}{6}\sum_{k=1}^{n-2}\binom{n+1}{2}\binom{n-1}{k-1}\binom{n-1}{k+1}x^k y^{n-1-k}.$$

Proof. By Corollary 3.11 and Equation (3.6), for fixed k,

$$[x^k y^{n-1-k}]A_n(x,y) = (-1)^{n-1}n!\sum_{i\geq 1}\sum_{t\geq 1}\frac{s_i^{(k)}s_t^{(n-1-k)}}{i!t!}J(i,t),$$

where

$$J(i,t) = \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\binom{j}{n-i}\binom{j}{n-t}.$$

Clearly, $J(i,t)$ is zero unless $1 \leq i,t \leq n$. Note that $J(i,t) = J(t,i)$.

In order to simplify this sum, we express it in terms of hypergeometric series. For $t \leq n$ and $t \leq i \leq n$, we have

$$J(i,t) = (-1)^m\binom{n-1}{m}\binom{m}{n-i}\binom{m}{n-t}\,{}_3F_2\left[\begin{array}{ccc} 1 & m+1 & m+1-n \\ - & m+1-n+i & m+1-n+t\end{array}\right],$$

where $m = n-t$. Similarly, this expression for $J(i,t)$ also holds for $i \leq n$ and $i \leq t \leq n$ where $m = n-i$. Thus instead of summing over all $0 \leq i,t \leq n$, we split the series $A_n^1(x,y)$ into two sums: let Z_1 be the sum over $1 \leq t \leq n$ and $t \leq i \leq n$, and let Z_2 be the sum over $1 \leq i \leq n$ and $i+1 \leq t \leq n$.

First consider Z_1, so that $m = n-t$. Now,

$${}_3F_2\left[\begin{array}{ccc} 1 & -t+1+n & -t+1 \\ - & -t+1+i & 1\end{array}\right] = {}_2F_1\left[\begin{array}{cc} -t+1+n & -t+1 \\ - & -t+1+i\end{array}\right].$$

By Vandermonde's convolution,

$${}_2F_1\left[\begin{array}{cc} -t+1+n & -t+1 \\ - & -t+1+i\end{array}\right] = \frac{(i-n)^{(t-1)}}{(i-t+1)^{(t-1)}} = \frac{\binom{i-n+t-2}{t-1}}{\binom{i-1}{t-1}}.$$

Thus

$$
\begin{aligned}
Z_1 &= n! \sum_{t=1}^{n} \sum_{i=t}^{n} \frac{s_i^{(k)} s_t^{(n-1-k)}}{i!t!} (-1)^{t-1} \binom{n-1}{n-t} \binom{n-t}{n-i} \frac{\binom{i-n+t-2}{t-1}}{\binom{i-1}{t-1}} \\
&= n! \sum_{t=1}^{n} \sum_{i=t}^{n} \frac{s_i^{(k)} s_t^{(n-1-k)}}{i!t!} \binom{n-1}{n-i} \binom{n-i}{t-1}.
\end{aligned}
$$

By the symmetry of $J(i,t)$, we similarly have

$$
Z_2 = n! \sum_{i=1}^{n} \sum_{t=i+1}^{n} \frac{s_i^{(k)} s_t^{(n-1-k)}}{i!t!} \binom{n-1}{n-t} \binom{n-t}{i-1}.
$$

Note that

$$
\binom{n-1}{n-i} \binom{n-i}{t-1} = \frac{(n-1)!}{(i-1)!(t-1)!(n-i-t+1)!} = \binom{n-1}{n-t} \binom{n-t}{i-1},
$$

so combining Z_1 and Z_2 yields

$$
[x^k y^{n-1-k}] A_n^1(x,y) = n! \sum_{t=1}^{n} \sum_{i=1}^{n} \frac{s_i^{(k)} s_t^{(n-1-k)}}{i!t!} \binom{n-1}{n-i} \binom{n-i}{t-1}.
$$

Make a change of variables by letting $u = i + t$ and this becomes

$$
n! \sum_{u \geq 2} \sum_{t \geq 1} \frac{s_{u-t}^{(k)} s_t^{(n-1-k)}}{(u-t)!t!} \binom{n-1}{n-u+t} \binom{n-u+t}{t-1}.
$$

Observe that $s_{u-t}^{(k)} s_t^{(n-1-k)}$ is nonzero if and only if $u-t \geq k$ and $t \geq n-1-k$; that is, if $u \geq n-1$. Also, $\binom{n-u+t}{t-1} = \binom{n-u+t}{n+1-u}$ is nonzero if and only if $n+1 \geq u$. Thus

$$
[x^k y^{n-1-k}] A_n^1(x,y) = \sum_{u=n-1}^{n+1} U(u),
$$

where

$$
U(u) = n! \sum_{t=1}^{u-1} \frac{s_{u-t}^{(k)} s_t^{(n-1-k)}}{(u-t)!t!} \binom{n-1}{n-u+t} \binom{n-u+t}{t-1}.
$$

When $u = n-1$, this corresponds to

$$
U(n-1) = n! \sum_{t=1}^{n-2} \frac{s_{n-1-t}^{(k)} s_t^{(n-k-1)}}{(n-1-t)!t!} \binom{n-1}{t+1} \binom{t+1}{t-1}.
$$

Notice that $s_{n-1-t}^{(k)} s_t^{(n-k-1)}$ is nonzero if and only if $n-1-t \geq k$ and $t \geq n-k-1$; that is, $t = n-k-1$. Thus,

$$\begin{aligned} U(n-1) &= \frac{n!}{k!(n-k-1)!} \binom{n-1}{n-k} \binom{n-k}{2} s_k^{(k)} s_{n-k-1}^{(n-k-1)} \\ &= \frac{n!(n-1)!}{2 \cdot k!(k-1)!(n-k-1)!(n-k-2)!}. \end{aligned}$$

When $u = n$, we have

$$U(n) = n! \sum_{t=1}^{n-1} \frac{s_{n-t}^{(k)} s_t^{(n-k-1)}}{(n-t)!t!} \binom{n-1}{t} \binom{t}{t-1}.$$

Similar to before, $s_{n-t}^{(k)} s_t^{(n-k-1)}$ is nonzero if and only if $n-k \geq t \geq n-k-1$. Thus

$$\begin{aligned} U(n) &= n! \left(\frac{s_{k+1}^{(k)} s_{n-k-1}^{(n-k-1)}}{(k+1)!(n-k-2)!} \binom{n-1}{k} + \frac{s_k^{(k)} s_{n-k}^{(n-k-1)}}{k!(n-k-1)!} \binom{n-1}{k-1} \right) \\ &= -(n-k-1) \binom{k+1}{2} \binom{n}{k+1} \binom{n-1}{k} - (n-k) \binom{n-k}{2} \binom{n}{k} \binom{n-1}{k-1} \\ &= -\frac{n!(n-1)!}{k!(k-1)!(n-k-1)!(n-k-2)!}. \end{aligned}$$

Lastly, when $u = n+1$,

$$U(n+1) = n! \sum_{t=1}^{n} \frac{s_{n+1-t}^{(k)} s_t^{(n-k-1)}}{(n+1-t)!t!} \binom{n-1}{t-1}.$$

The terms in this sum are nonzero when $n-k+1 \geq t \geq n-k-1$. Thus

$$\begin{aligned}
U(n+1) &= \frac{n!s_{k+2}^{(k)}s_{n-k-1}^{(n-k-1)}}{(k+2)!(n-k-1)!}\binom{n-1}{k+1} + \frac{n!s_{k+1}^{(k)}s_{n-k}^{(n-k-1)}}{(k+1)!(n-k)!}\binom{n-1}{k} \\
&\qquad + \frac{n!s_{k}^{(k)}s_{n-k+1}^{(n-k-1)}}{k!(n-k-1)!}\binom{n-1}{k-1} \\
&= \frac{1}{k+2}\binom{n}{k+1}\binom{n-1}{k+1}s_{k+2}^{(k)} + \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}\binom{k+1}{2}\binom{n-k}{2} \\
&\qquad + \frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1}s_{n-k+1}^{(n-k-1)} \\
&= \frac{1}{k+2}\binom{n}{k+1}\binom{n-1}{k+1}\binom{k+2}{3}\frac{3k+5}{4} \\
&\qquad + \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k}\binom{k+1}{2}\binom{n-k}{2} \\
&\qquad + \frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1}\binom{n-k+1}{3}\frac{3(n-k)+2}{4} \\
&= \frac{n!(n-1)!(6(n-k)(k+1)+(n+1))}{12\cdot(k+1)!(k-1)!(n-k)!(n-k-2)!}.
\end{aligned}$$

Gathering results, for $1 \leq k \leq n-2$, and $n \geq 3$,

$$[x^k y^{n-1-k}]A_n^1(x,y) = \sum_{u=n-1}^{n+1} U(u) = \frac{1}{6}\binom{n+1}{2}\binom{n-1}{k-1}\binom{n-1}{k+1}.$$

The result follows. □

3.5 Genus one partitions

Table 3.1 shows the number $T(n,k)$ of genus one partitions of n with k blocks. The numbers were computed by a Maple program. It points to the following conjecture:

Conjecture 3.15. *The number of genus one partitions of n with k blocks is equal to the number of genus one rooted hypermonopoles with $n-1$ edges and $k-1$ hyperedges. That is, $T(n,k) = \frac{1}{6}\binom{n}{2}\binom{n-2}{k}\binom{n-2}{k-2}$.*

In the next chapter, we give a related conjecture (Conjecture 4.10) concerning the structure of the genus one partition poset, and show that this would imply Conjecture 3.15.

T(n,k)	k=2	3	4	5	6	7	8	9
n=4	1							
5	5	5						
6	15	40	15					
7	35	175	175	35				
8	70	560	1050	560	70			
9	126	1470	4410	4410	1470	126		
10	210	3360	14700	23520	14700	3360	210	
11	330	6930	41580	97020	97020	41580	6930	330

Table 3.1: Numbers $T(n,k)$ of genus one partitions of n with k blocks.

A bijective proof can possibly be obtained by analyzing the associated rooted hypermonopole diagrams of genus one partitions. In terms of the diagrams, this conjecture states that the number of genus one rooted hypermonopoles with n edges and k hyperedges whose hyperedge permutation consists only of increasing cycles, is equal to the number of genus one rooted hypermonopoles with $n-1$ edges and $k-1$ hyperedges. In a sense, the cycles in the hyperedge edge permutation that are non-increasing should be 'broken' into two increasing cycles.

Corollary 3.16. *If Conjecture 3.15 holds, then for $n \geq 4$, the generating series for genus one partitions with respect to the number of blocks is*

$$P_n^1(x) = \frac{1}{6}\sum_{k=2}^{n-2}\binom{n}{2}\binom{n-2}{k}\binom{n-2}{k-2}x^k. \tag{3.7}$$

□

Corollary 3.17. *If Conjecture 3.15 holds, then for $n \geq 4$, the number of genus one partitions of n is*

$$T_n = \frac{1}{6}\binom{n-2}{2}\binom{2n-4}{n-2}. \tag{3.8}$$

Proof. Expanding the binomials and shifting the index of summation in Equation (3.7),

$$T_n = P_n^1(1) = \frac{1}{6}\sum_{k=0}^{n-4}\frac{n!(n-2)!}{2(k+2)!k!(n-k-2)!(n-k-4)!}.$$

The first term in this series is $\binom{n}{4}$, which is nonzero for $n \geq 4$.

The term ratio for this series is $\frac{(k+2-n)(k+4-n)}{(k+1)(k+3)}$, which implies

$$T_n = \binom{n}{4} {}_2F_1 \left[\begin{matrix} 4-n & 2-n \\ - & 3 \end{matrix} \right] = \binom{n}{4} \frac{(n-1)^{(n-2)}}{3^{(n-2)}},$$

by Vandermonde's convolution. Simplifying this gives the desired result. □

The sequence of numbers T_n satisfy a number of recurrence equations involving Catalan numbers.

Corollary 3.18. *For integers $n \geq 4$,*

$$T_n = \frac{4n-10}{n-4} T_{n-1}$$

$$T_n = \frac{1}{2} \binom{n-1}{3} C_{n-2}$$

$$T_n = 4T_{n-1} + \binom{n-2}{2} C_{n-3} \tag{3.9}$$

$$T_n = 2T_{n-1} + \binom{n-1}{3} C_{n-3}. \tag{3.10}$$

□

Remark that the numbers T_n satisfy a one term recurrence, as do the Catalan numbers C_n.

In particular, Equations (3.9) and (3.10) hint at the existence of a decomposition of the poset of genus one partitions into symmetric chains. This is explored in Section 4.5.

3.6 Higher genus partitions

Notice that the Narayana numbers are symmetric with respect to k:

$$N(n, k) = N(n, n+1-k)$$

for all $1 \leq k \leq n$. Similarly, the numbers $T(n, k)$ of genus one partitions of n with k blocks are conjectured to satisfy

$$T(n, k) = T(n, n-k)$$

for all $2 \leq k \leq n-2$.

W(n,k)	k=2	3	4	5	6	7
n=6	1					
7	7	21				
8	28	210	161			
9	84	1134	2184	777		
10	210	4410	15330	13713	2835	
11	462	13860	75675	110880	63063	8547

Table 3.2: Numbers $W(n, k)$ of genus two partitions of n with k blocks.

Table 3.2 shows the number $W(n, k)$ of genus two partitions of n with k blocks. These numbers were computed by a Maple program, and it appears that they are not symmetric with respect to the number of blocks. This suggests that the study of higher genus rooted hypermonopoles may not be directly related to the study of higher genus partitions, as the numbers of genus g rooted hypermonopoles of n are always symmetric with respect to the number of hyperedges.

The apparent symmetry of the numbers $T(n, k)$ suggest that the poset of genus one partitions satisfies nice structural properties. We shall explore this in Chapter 4.

Chapter 4

Posets of genus g partitions

The lattice of partitions of n is one of the classical lattices which have been studied extensively . The subposet of noncrossing partitions of n was first examined by Kreweras [23] in 1972, but it has since been studied extensively as well. In this section, we focus on studying the structural properties of the poset of genus one partitions of n.

We previously observed that the Narayana numbers $N(n, k)$ are symmetric with respect to k. This implies that the lattice of noncrossing partitions is rank-symmetric. Simion and Ullman show in [27] that the lattice of noncrossing partitions exhibits structural properties stronger than rank-symmetry; it is self-dual, and has a symmetric chain decomposition. Since the numbers $T(n, k)$ are conjectured to exhibit symmetry, it is of interest to find out whether the poset of genus one partitions possesses the same structural properties as the lattice of genus zero partitions.

In this chapter, we show that while the poset of genus one partitions is not self-dual, it does admit a symmetric chain decomposition. We construct this by mimicking the 'parenthesization' method that shows the Boolean lattices are a symmetric chain order. The structure of the hypermonopole diagram of a genus one partition is analyzed, which will later aid in the construction of an involution for the poset of genus one partitions. Finally, Conjecture 3.15 and Equation (3.10) from the previous Chapter suggest a particular decomposition for the poset of genus one partitions into posets of genus one and zero partitions. We give a conjecture regarding this decomposition, and show that this conjecture would imply the result of Conjecture 3.15.

4.1 Background and notation

4.1.1 Posets

A *partially ordered set*, or *poset*, for short, is a set P together with an *order relation* $\leq$; that is, for all a, b, $c \in P$, $\leq$ is

- reflexive: $a \leq a$,
- transitive: if $a \leq b$ and $b \leq c$ then $a \leq c$,
- antisymmetric: if $a \leq b$ and $b \leq a$ then $a = b$.

An element b is said to *cover* a, if $a \leq b$ and for $c \in P$, $a \leq c \leq b$ implies $a = c$ or $c = b$. Denote the covering relation by $a \lessdot b$.

The *dual* of a poset P, denoted by P^*, has underlying set with the reversed order relation of P. A poset is *self-dual* if $P \cong P^*$.

Let $P^{\min}$ and $P^{\max}$ respectively denote the set of minimal elements of P and the set of maximal elements of P. If P contains a unique minimal element, it is denoted by $\widehat{0}$. If P contains a unique maximal element, it is denoted by $\widehat{1}$.

A *chain* is a poset in which any two elements are comparable, and an *antichain* is a poset in which any two elements are incomparable. The *length* $l(P)$ of a poset P is the supremum of the sizes of the chains in P, and the *width* $w(P)$ of a poset P is the supremum of the sizes of the antichains in P. A chain in P is *saturated* or *unrefinable* if every covering relation in the chain is a covering relation in P.

A poset P is *ranked* if there is a rank function $\rho : P \to \mathbb{Z}$ such that $\rho(a) = 0$ for all $a \in P^{\min}$, $\rho(b) = \rho(a) + 1$ if $a \lessdot b$, and every maximal chain in P has the same length N. In this case, the *rank* of P is $N - 1$.

Note that every finite lattice has a unique minimal element and a unique maximal element. The elements in the lattice that cover $\widehat{0}$ are called *atoms*. A lattice is *atomic* if every element in L can be written as a join of a finite number of atoms.

A lattice is *(upper) semimodular* if for all a, $b \in L$,

$$a \wedge b \lessdot a \Rightarrow b \lessdot a \vee b.$$

It can be shown that all semimodular lattices are ranked; a proof can be found in [2] (Chapter 2). An equivalent definition of semimodularity is if for all a,

$b \in L$,

$$\rho(a \wedge b) + \rho(a \vee b) \leq \rho(a) + \rho(b),$$

where ρ is the rank function on L.

Given a ranked poset P and rank function ρ, let

$$P_j = \{a \in P : \rho(a) = j\}$$

denote the elements in the jth rank of P. The number of elements in P_j is the jth *Whitney number of the second kind*, $W_j(P)$.

Suppose P has rank $N-1$. If for all $0 \leq j \leq \lfloor (N-1)/2 \rfloor$,

$$W_j = W_{N-1-j} \qquad \text{and}$$

$$W_0 \leq W_1 \leq \cdots \leq W_{\lfloor (N-1)/2 \rfloor},$$

then P is *rank-symmetric* and *unimodal*. P has a *symmetric chain decomposition* if its underlying set can be decomposed into a disjoint union of unrefinable chains $\{C_i\}$ such that

$$\min\{\rho(a) : a \in C_i\} + \max\{\rho(a) : a \in C_i\} = N - 1$$

for each chain C_i. If P admits such a decomposition, then it is called a *symmetric chain order*. A symmetric chain order is necessarily rank-symmetric and unimodal.

Boolean lattices

Some examples that we shall later make extensive use of are Boolean lattices. For a nonnegative integer n, a finite *Boolean lattice* $\mathcal{B}_n$ is the power set of a set of size n, ordered by inclusion.

Boolean lattices are atomic, semimodular (in fact, distributive, which implies semimodularity), rank-symmetric, unimodal, self-dual, a symmetric chain order, and hence are strongly Sperner. See [2] (Chapter 8) for details.

4.1.2 Stirling numbers of the second kind

For $n \geq 0$ and $0 \leq k \leq n$, the *Stirling number of the second kind* $S(n,k)$, is the number of partitions of n with k blocks. Stirling numbers of the second kind satisfy the recurrence ([17] Equation (6.3))

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \tag{4.1}$$

and have the generating series

$$\frac{1}{k!}(e^x - 1)^k = \sum_{n \geq k} S(n,k) \frac{x^n}{n!}.$$

Also,

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n. \tag{4.2}$$

4.2 Posets of set partitions

The set Π_n of partitions of n can be made into a lattice under the *refinement order*: for partitions π_1, $\pi_2 \in \Pi_n$, $\pi_1 \leq \pi_2$ if and only if each block of π_1 is contained in a block of π_2. In this case, π_1 is said to be a refinement of π_2. The unique minimal element of Π_n is $\widehat{0} = 1/2/\cdots/n$ and the unique maximal element is $\widehat{1} = 1\ 2 \cdots\ n$.

Π_n is a geometric lattice, meaning it is atomic, upper semimodular, and has finite length. Its rank function is

$$\mathrm{rk}_n(\pi) = n - \mathrm{bk}(\pi),$$

where $\mathrm{bk}(\pi)$ is the number of blocks of π. So Π_n has rank $n-1$.

Lemma 4.1. *For $g \geq 1$, a genus g partition of n has at least two blocks and at most $n - 2g$ blocks.*

Proof. Let $\pi_1 = 1\ 3\ 5\ \cdots\ 2g+1/2\ 4\ 6\ \cdots\ 2g+2\ 2g+3 \cdots n-1\ n$ be a partition of n with two blocks. The associated hyperface permutation is $\alpha_{\pi_1}^{-1}\sigma = (n\ 2g+1\ 2g\ \cdots\ 2\ 1)(2g+2)(2g+3)\cdots(n-2)(n-1)$ and $l(\alpha_{\pi_1}^{-1}\sigma) = n - 2g - 1$. By the Euler-Poincaré equation, π_1 is a genus g partition.

Let $\pi_2 = 1\ 2g+1/2\ 2g+2/\cdots/2g\ 4g/4g+1/4g+2/\cdots/n-1/n$ be a partition of n with $n - 2g$ blocks. The associated hyperface permutation is $\alpha_{\pi_2}^{-1}\sigma = (2g+1\ 2\ 2g+3\ 4\ \cdots\ 2g-1\ 4g\ 4g+1\ 4g+2\ \cdots\ n-1\ n)$ and $l(\alpha_{\pi_2}^{-1}\sigma) = 1$. By the Euler-Poincaré equation, π_2 is a genus g partition. $\square$

For $g \in \{0, 1, 2, \ldots\}$, let Π_n^g denote the poset of genus g partitions of n under the refinement order. Since $1/2/\cdots/n$ and $1\ 2 \cdots\ n$ are genus zero partitions, Π_n^0 has rank function $\mathrm{rk}_n^0(\pi) = n - \mathrm{bk}(\pi)$ and rank $n-1$. For $g \geq 1$, by

Lemma 4.1, each Π_n^g has rank function $\mathrm{rk}_n^g(\pi) = n - 2g - \mathrm{bk}(\pi)$ and rank $n - 2g - 2$.

In particular, the lattice Π_n^0 is a subposet of Π_n; it is not a sublattice because the join of two noncrossing partitions is not necessarily noncrossing. For general $g \geq 1$, Π_n^g is also a subposet of Π_n, but not a sublattice. In particular, Π_n^g does not have unique minimal and maximal elements since $\widehat{0}$ and $\widehat{1}$ are genus zero partitions. The only exceptions occur when n is even and $g = \frac{n}{2} - 1$. In these cases, Π_n^g is the trivial lattice since its underlying set contains one element, as shown by Proposition 2.6. Observe that Π_n^g can be made into a lattice by adjoining $\widehat{0}$ and $\widehat{1}$.

4.2.1 Π_n versus Π_n^0

It is clear that both Π_n and Π_n^0 are atomic and have finite length, but while Π_n is upper semimodular for all positive integers n, Π_n^0 is not upper semimodular when $n \geq 3$. To see this, consider the partitions $\alpha_1 = 1\ 3/2/4/5/6/\cdots/n$ and $\alpha_2 = 1/2\ 4/3/5/6/\cdots/n$ in Π_n^0. They each have rank 1, so both partitions cover $\widehat{0}$, but their join in Π_n^0 is $\alpha_1 \vee \alpha_2 = 1\ 2\ 3\ 4/5/6/\cdots/n$, which has rank 3, and so it does not cover either α_1 or α_2.

A sequence $\{x_k\}_{k=0}^n$ of nonnegative real numbers is *logarithmically concave* if

$$x_{k-1}x_{k+1} \leq x_k^2$$

for all $1 \leq k \leq n-1$. A log concave sequence is clearly unimodal, since if it is not, then there exists k such that $x_{k-1} > x_k < x_{k+1}$, which contradicts the condition of log concavity.

For example, by a straightforward calculation, the binomial coefficients $\binom{n}{k}$ form a log concave sequence for $0 \leq k \leq n$. It follows that $\{N(n,k)\}_{k=1}^n = \{\frac{1}{n}\binom{n}{k}\binom{n}{k-1}\}_{k=1}^n$ is a log concave sequence. The Stirling numbers $\{S(n,k)\}_{k=1}^n$ can also be shown to be log concave, by induction on n and use of the recurrence equation (4.1). Therefore, both Π_n and Π_n^0 are unimodal.

However, Π_n^0 enjoys some nice structural properties that are not shared by Π_n. As we have shown in Chapter 3, the kth Whitney number of Π_n^0 is the Narayana number $N(n, n-k)$, which satisfies $N(n,k) = N(n, n+1-k)$. On the other hand, the kth rank of Π_n consists of partitions with $n-k$ blocks, so the kth Whitney number of Π_n is the Stirling number of the second kind $S(n, n-k)$. By the formula in Equation (4.2) for the Stirling numbers, it is not

Properties	Π_n	Π_n^0	Π_n^1
unimodal	✓	✓	✓
rank-symmetric	×	✓	✓
self-dual	×	✓	×
symmetric chain order	×	✓	✓
atomic	✓	✓	-
semimodular	✓	×	-

Table 4.1: Structural properties of Π_n, Π_n^0, and Π_n^1.

hard to see that $S(n,k) \neq S(n, n+1-k)$ in general. Thus Π_n^0 is rank-symmetric, while Π_n is not.

Table 4.1 is a summary of the structural properties of the lattices Π_n, Π_n^0, and the poset Π_n^1. The assertions made in the table that have not yet been discussed will be proved in the sections following.

4.3 The structure of hypermonopole diagrams of genus one partitions

In this section, we examine the hypermonopole diagram of genus one partitions in depth. Although genus one partitions are nonplanar, their blocks cross in a simple way so that the poset Π_n^1 has nice structural properties. Later we shall make extensive use of these diagrams to prove that Π_n^1 is rank-symmetric for all $n \geq 4$.

Given k points $x_1, x_2, \ldots, x_k$ on a circle, let $x_1 \prec x_2 \prec \cdots \prec x_k$ denote that the points are arranged in that order in the clockwise direction. For example, $x_1 \prec x_2 \prec \cdots \prec x_k$ and $x_k \prec x_1 \prec x_2 \prec \cdots \prec x_{k-1}$ describe the same circular order.

A block λ of a partition π is called a *crossing block* if there exists another block μ of π and integers $a\ c \in \lambda$ and $b\ d \in \mu$ such that either $a \prec b \prec c \prec d$. Otherwise, λ is said to be a *noncrossing block*.

Clearly, a genus zero partition does not have any crossing blocks, while a genus g partition has at least two crossing blocks, for $g \geq 1$.

Let v be a fixed point on a torus $\mathcal{T}$. Note that the choice of v is not important, as $\mathcal{T}$ is a connected surface. If $\mathfrak{h}$ is a genus one hypermonopole embedded in $\mathcal{T}$, then we can assume, without loss of generality, that the vertex of the hypermonopole coincides with the point v. The edges of the hypermonopole may be thought of as loops in $\mathcal{T}$ based at v. Thus, the boundary of a hyperedge of the hypermonopole consists of these loops based at v.

The *fundamental group* $\pi_1(\mathcal{T}, v)$ of $\mathcal{T}$ with base point v is the group of equivalence classes of loops in $\mathcal{T}$ based at v. It has the presentation

$$\pi_1(\mathcal{T}, v) = \langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z} \times \mathbb{Z},$$

where a is a meridional loop based at v, and b is a longitudinal loop based at v. More information on fundamental groups of surfaces can be found in [19].

Since $\pi_1(\mathcal{T}, v)$ is abelian, every loop in $\mathcal{T}$ based at v is homotopic to a loop of the form $a^i b^j$ for i, $j \in \mathbb{Z}$. The parametrization of these loops is not important for our present purpose, so we are effectively working with the (commutative) *positive monoid* of the presentation $\langle a, b \mid aba^{-1}b^{-1} \rangle$ of the fundamental group of $\mathcal{T}$. (An abelian monoid M is positive if the subset $\{x \in M : \exists y \in M \text{ such that } xy = 1\} = \{1\}$). Thus, every loop in $\mathcal{T}$ based at v is homotopic to a loop of the form $a^i b^j$ for nonnegative integers i and j.

Observe that since the deletion of the edges of a genus one hypermonopole decomposes the torus into a union of discs, then an edge of a hypermonopole must be homotopic to either the null loop, the meridional loop a, the longitudinal loop b, or the product ab of the meridional and longitudinal loops.

An edge of a hypermonopole is *trivial* if it is homotopic to the null loop based at v. Otherwise, the edge is *nontrivial*.

Lemma 4.2. *Given a hypermonopole diagram of a genus one partition, the boundary of the hyperedge that encodes a crossing part of the partition contains exactly two nontrivial edges. Moreover, these nontrivial edges are homotopic.*

Proof. Let π be a genus one partition and let $\mathfrak{h}_\pi$ be the corresponding hypermonopole embedded in $\mathcal{T}$ whose vertex coincides with the fixed base point v. Let λ be a crossing block of π. Then there exists another block μ and integers i, $j \in \lambda$, x, $y \in \mu$ such that they are arranged in the order $i \prec x \prec j \prec y$ around the vertex.

Recall that the vertex of $\mathfrak{h}_\pi$ is labelled $1, \ldots, n$ in the clockwise orientation. Let p be the integer in λ that is counterclockwise nearest to x. Similarly let

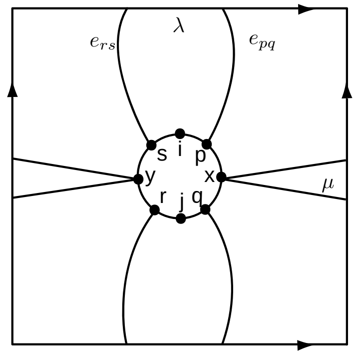


Figure 4.1: Edge e_{pq} is homotopic to edge e_{rs}.

$q \in \lambda$ be clockwise nearest to x, let $r \in \lambda$ be counterclockwise nearest to y, and let $s \in \lambda$ be clockwise nearest to y.

With this choice of p, q, r, and s, then p is cyclically adjacent to q in λ, and r is cyclically adjacent to s in λ, so there must be an edge e_{pq} joining p and q, and an edge e_{rs} joining r and s in the hypermonopole representation of α. Moreover, these two edges must both be nontrivial, otherwise, it contradicts the assumption that $p \prec x \prec q$ and $r \prec y \prec s$.

Without loss of generality, suppose e_{pq} is homotopic to the meridional loop a. Regarding the vertex of $\mathfrak{h}_\pi$ as a small circle, the edge e_{rs} does not intersect the circle at any other points other than at r and s, and it also does not intersect the edge e_{pq}. It follows that e_{rs} cannot be homotopic to the loops b or ab. See Figure 4.1. Therefore, e_{rs} is homotopic to a, and hence to e_{pq}.

To see that the boundary of the hyperedge that encodes λ has exactly two nontrivial edges, suppose t and u are cyclically adjacent integers in λ and e_{tu} is the edge joining them in the hypermonopole representation. By the choice of p, q, r and s, either $s \prec t \prec u \prec p$, or $q \prec t \prec u \prec r$.

The edges e_{pq} and e_{rs} bound a space that is homeomorphic to a disc, so in both cases, the edge e_{tu} is a loop in a space that is homeomorphic to a disc. From this it follows that e_{tu} is homotopic to the null path, and hence is a trivial edge. □

Remark. The above result does not hold for surfaces of higher genus because in those cases, boundaries of hyperedges may contain more than two nontrivial

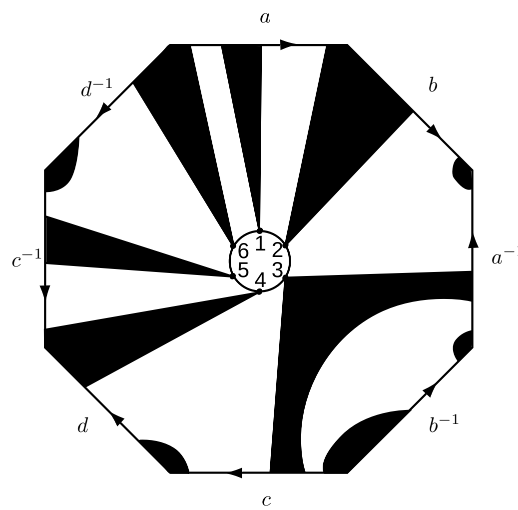


Figure 4.2: A genus two hypermonopole representing the partition 1 3 5/2 4 6 whose hyperedges have boundaries which contain more than two nontrivial edges

edges. See Figure 4.2 for a genus two example. This Lemma is key in understanding the relative simplicity of genus one partitions.

The above Lemma also shows that the hypermonopole diagram of any genus one partition has at least two, and at most three types of crossing blocks, in which the type of the crossing block is determined by the homotopy type of the nontrivial edges in the block. A crossing block is *type I crossing block* if its nontrivial edges are homotopic to the meridional loop a, it is *type II* if its nontrivial edges are homotopic to the longitudinal loop b, and it is *type III* if its nontrivial edges are homotopic to the loop ab.

It is convenient to classify noncrossing blocks as well. A noncrossing block has the same type as the crossing block that is closest to it in the counterclockwise direction.

4.4 Self-duality and rank-symmetry

4.4.1 Genus zero

In [27], Simion and Ullman construct an order-reversing involution on the lattice of genus zero partitions to show that it is self-dual. The involution was defined purely diagrammatically using the circular diagram of partitions. We show that this involution can be expressed explicitly as a homomorphism of S_n by exploring the relationship between partitions, permutations, and hypermonopoles. First, we briefly describe their construction.

Theorem 4.3. *(Simion, Ullman [27]) For each $n \geq 1$, Π_n^0 is self-dual.*

Proof. Given a partition $\pi \in \Pi_n^0$, consider its circular diagram. Recall that this consists of a circle with n points on it labelled 1 through n in the clockwise direction, and a chord joining labels i and j is drawn inside the circle if and only if i and j are cyclically adjacent to each other in a block of π. Construct a function $\omega_0 : \Pi_n^0 \to \Pi_n^0$ as follows: subdivide each arc of the circle by adding a new point in the middle of each arc, and label the new points 1 through n in the counter-clockwise direction, with the new point on the arc between the old labels $n-1$ and n labelled 1. Define $\omega_0(\pi)$ to be the coarsest partition obtainable from the circular diagram of π by joining the new labels with new chords that do not cross any existing chords.

For example, if $\pi = 1\,8/2\,3\,6/4\,5/7/9/10$, then $\omega_0(\pi) = 1\,2\,10/3\,4\,9/5\,7/6/8$. See Figure 4.3.

From this construction, it is clear that ω_0 is a well-defined involution on Π_n^0. To see that it is an order-reversing involution, suppose $\pi \leq \rho$ in Π_n^0. Then the chords of the circular diagram of $\omega_0(\rho)$ do not cross the chords of the circular diagram of π, so it follows that $\omega_0(\rho) \leq \omega_0(\pi)$. $\square$

As mentioned in Section 2.2, circular diagrams and hypermonopole diagrams of genus zero partitions may be regarded as the same. Under this identification, the new subdivision points in Simion and Ullman's construction correspond to the vertex-hyperface incidences of the associated hypermonopole of the partition. In other words, the action of ω_0 is equivalent to re-labelling the hyperfaces of $\mathfrak{h}_\pi$ in a clever way. Thus, instead of defining ω_0 purely diagrammatically, we can use the relations between partitions, hypermonopoles, and permutations, to

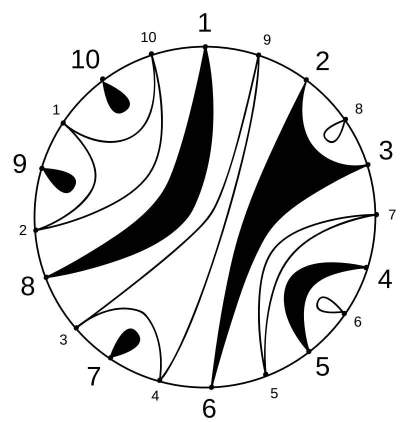


Figure 4.3: The partition $\pi = 1\ 8/2\ 3\ 6/4\ 5/7/9/10$ is shown in black. Its image under ω_0, $1\ 2\ 10/3\ 4\ 9/5\ 7/6/8$, is shown in white.

translate Simion and Ullman's construction into algebraic terms.

Let $\pi \in \Pi_n^0$, and let α_π denote the corresponding permutation in S_n. As before, $\sigma = (1\ 2\ \cdots n) \in S_n$ is the canonical vertex permutation.

The first step in Simion and Ullman's construction of $\omega_0(\pi)$ was to label the n new subdivision points. We need to re-label the vertex-hyperface incidences of the associated hypermonopole $\mathfrak{h}_\pi$, and this is equivalent to re-labelling the vertex-hyperedge incidences of $\mathfrak{h}_\pi$, since the hyperface permutation is obtainable from the hyperedge permutation. Considering the vertex of $\mathfrak{h}_\pi$ as a circle and assuming that the n points on the circle are evenly spaced, it is easy to see that the required re-labelling is accomplished by shifting the labels of the n points (vertex-hyperedge incidences) in the counter-clockwise direction by an angle of $4\pi/n$ radians, then reflecting the labels with respect to the axis through the center of the circle, and the vertex labelled 1.

Let $\iota \in S_n$ such that

$$\iota = \begin{cases} (1\ n-1)(2\ n-2)\cdots(\frac{n}{2}-1\ \frac{n}{2}+1)(\frac{n}{2})(n) & \text{if } n \text{ is even,} \\ (1\ n-1)(2\ n-2)\cdots(\frac{n-1}{2}\ \frac{n+1}{2})(n) & \text{if } n \text{ is odd.} \end{cases}$$

Note that ι is a permutation of order two.

Also,

$$\sigma\iota = \begin{cases} (1\ n)(2\ n-1)\cdots(\frac{n}{2}\ \frac{n}{2}+1) & \text{if } n \text{ is even,} \\ (1\ n)(2\ n-1)\cdots(\frac{n-1}{2}\ \frac{n+3}{2})(\frac{n+1}{2}) & \text{if } n \text{ is odd,} \end{cases}$$

so that $\sigma\iota$ is also a permutation of order two.

The re-labelling of points is equivalent to the mapping

$$\alpha_\pi \mapsto \iota^{-1}\alpha_\pi\iota = \iota\alpha_\pi\iota.$$

Remark that this is the conjugation of α_π by ι, so the cycle type of α_π is preserved, as is necessary. Moreover, the circular diagram of $\iota\alpha_\pi\iota$ remains planar. Also notice that the new labelling of the vertex-hyperedge incidences is in the counter-clockwise orientation, so the disjoint cycles of $\iota\alpha_\pi\iota$ are decreasing.

The next step is to find the hyperface permutation of $\mathfrak{h}_\pi$ with respect to the new labelling. Since the cycles of $\iota\alpha_\pi\iota$ are decreasing, the desired operation is

$$\iota\alpha_\pi\iota \mapsto (\iota\alpha_\pi\iota)^{-1}\sigma^{-1} = \iota\alpha_\pi^{-1}\iota\sigma^{-1}.$$

Note that the cycles of the resulting permutation remain decreasing. Reflecting the diagram puts the labels in the desired clockwise orientation. This corresponds to taking the inverse of $\iota\alpha_\pi^{-1}\iota\sigma^{-1}$. Thus we have the following theorem:

Proposition 4.4. *For $\pi \in \Pi_n^0$, with associated permutation α_π, let*

$$\widetilde{\omega_0} : S_n \to S_n$$

be the function defined by $\widetilde{\omega_0}(\alpha_\pi) = \sigma\iota\alpha_\pi\iota$, with ι and σ as given previously. Then $\omega_0(\pi)$ is the partition that corresponds to the permutation $\widetilde{\omega_0}(\alpha_\pi)$. □

We verify the previous example for $\pi = 1\ 8/2\ 3\ 6/4\ 5/7/9/10$. In this case, $\iota = (1\ 9)(2\ 8)(3\ 5)(4\ 6)$, $\alpha_\pi = (1\ 8)(2\ 3\ 6)(4\ 5)$, and $\sigma\iota\alpha_\pi\iota = (1\ 2\ 10)(3\ 4\ 9)(5\ 7)$.

One advantage of this theorem is instead of having to resort to drawing diagrams to find $\omega_0(\pi)$, we can compute it explicitly as a product of permutations through $\widetilde{\omega_0}(\alpha_\pi)$. It is easy to check that $\widetilde{\omega_0}$ is an involution:

$$\widetilde{\omega_0}(\widetilde{\omega_0}(\alpha_\pi)) = \widetilde{\omega_0}(\sigma\iota\alpha_\pi\iota) = \sigma\iota(\sigma\iota\alpha_\pi\iota)\iota = \alpha_\pi,$$

since $(\sigma\iota)^2 = \mathrm{id}$, and $\iota^2 = \mathrm{id}$.

We remark that we cannot explicitly show that $\widetilde{\omega_0}$ induces an order-reversing ω_0, since permutations do not inherently carry the refinement order of partitions. However, it is possible to show explicitly that ω_0 is a mapping that sends elements in rank k to elements in rank $n-1-k$.

First observe that a transposition $\tau = (u\ v)$ acts on the disjoint cycles of a permutation α in one of two ways. If u and v are in distinct cycles of α, then τ

'joins' the two cycles. That is, $l(\tau\alpha) = l(\alpha) - 1$. If u and v are in the same cycle of α, then τ 'cuts' that cycle. That is, $l(\tau\alpha) = l(\alpha)+1$. For example, if $\tau = (1\,3)$, then $\tau(1\,5\,4)(3\,7)(2\,6) = (1\,5\,4\,3\,7)(2\,6)$, and $\tau(1\,6\,3\,4\,7)(2\,5) = (1\,6)(3\,4\,7)(2\,5)$.

Suppose π has $n-k$ blocks so that it is an element in rank k of Π_n^0. Then α_π has $n-k$ disjoint cycles. Since $\iota\alpha_\pi\iota$ is in the same conjugacy class as α_π, then it also has $n-k$ disjoint cycles, say $\iota\alpha_\pi\iota = \lambda_1\lambda_2\cdots\lambda_{n-k}$. As observed before, each disjoint cycle λ_i is decreasing. Suppose λ_1 is the disjoint cycle containing 1. Let m_i and M_i respectively be the minimum and maximum element in the cycle λ_i, for $1 \leq i \leq n-k$. Let $\tau_i = (1\ i) \in S_n$, so that

$$\sigma = (1\ 2 \cdots n) = \tau_n\tau_{n-1}\cdots\tau_2.$$

Then the number of disjoint cycles in $\sigma\iota\alpha_\pi\iota$ depends on how each τ_i acts on $\tau_{i-1}\tau_{i-2}\cdots\tau_2\iota\alpha_\pi\iota$, for $2 \leq i \leq n$.

Claim: Each τ_{m_i} acts as a 'join' on $\tau_{m_i-1}\tau_{m_i-2}\cdots\tau_2\iota\alpha_\pi\iota$, and the remaining transpositions act as 'cuts'.

Proof. Without loss of generality, suppose $m_2 < m_3 < \cdots < m_{n-k}$. The integers $2, 3, \ldots, m_2 - 1$ must be in λ_1, or else it contradicts the minimality of m_2. Moreover, since λ_1 is a decreasing cycle, each of $\tau_2, \ldots, \tau_{m_2-1}$ acts as a cut on λ_1, and does not affect the cycles $\lambda_2, \ldots, \lambda_{n-k}$. Since the cycle λ_2 is unchanged under the actions of these transpositions, the elements 1 and m_2 remain in distinct cycles in the permutation $\tau_{m_2-1}\cdots\tau_2\iota\alpha_\pi\iota$, so τ_{m_2} acts on it as a join.

By the planarity of the diagram of $\iota\alpha_\pi\iota$, and the minimality of m_3, the elements $m_2+1, m_2+2, \cdots m_3-1$ are all contained in the cycle of $\tau_{m_2}\cdots\tau_2\iota\alpha_\pi\iota$ that contains 1. A similar argument applies as before; each of $\tau_{m_2+1}, \ldots, \tau_{m_3-1}$ acts as a cut on the cycle containing 1, and the elements 1 and m_3 are in distinct cycles in $\tau_{m_3-1}\cdots\tau_2\iota\alpha_\pi\iota$, so τ_{m_3} acts on it as a join. Repeating this argument, the Claim follows. $\square$

The Claim shows that $n-k-1$ of the τ_i's act as joins, and k of the τ_i's act as cuts. It follows that

$$l(\sigma\iota\alpha_\pi\iota) = (n-k) - (n-k-1) + k = k+1.$$

Therefore, $\widetilde{\omega_0}(\alpha_\pi)$ has $k+1$ cycles, meaning $\omega_0(\pi)$ is an element in rank $n-1-k$ of Π_n^0.

4.4.2 Genus one

There is an analogue of the construction of ω_0 in the case of genus one partitions.

Recall that a hypermonopole is a face two-colourable map, where by convention, the hyperedges are coloured black and the hyperfaces are coloured white. Reversing the colouring of a genus g hypermonopole with n edges and k hyperedges results in another genus g hypermonopole with n edges and $n-k-2g+1$ hyperedges, by the Euler-Poincaré formula. In other words, reversing the colouring amounts to taking the hyperedges as hyperfaces and vice versa. Clearly, the operation of reversing the colouring of a hypermap is an involution.

The most important property of the involution ω_0 is that it takes the hypermonopole diagram of a planar partition π with k blocks, and reverses its colouring to get another hypermonopole diagram that corresponds to a planar partition with $n-1-k$ blocks, so that ω_0 is a mapping between symmetric ranks of Π_n^0. This method of taking the reverse-coloured hypermonopole works in the case of genus zero partitions because the Euler-Poincaré formula satisfies

$$\#\text{ of hyperedges of }\pi + \#\text{ of hyperfaces of }\pi = n+1,$$

and the number of blocks of partitions in symmetric ranks sum up to $n+1$ in the lattice of genus zero partitions.

In the case of genus one partitions, the Euler-Poincaré formula gives

$$\#\text{ of hyperedges of }\pi + \#\text{ of hyperfaces of }\pi = n-1,$$

but the number of blocks of partitions in symmetric ranks add to n in the poset of genus one partitions, so we cannot simply use the reverse colouring of a genus one hypermonopole to obtain an involution between symmetric ranks of Π_n^1.

Another reason that this method does not generalize to the genus one case is due to the fact that reversing the colouring of a genus one hypermonopole does not necessarily yield a genus one hypermonopole that corresponds to a genus one partition! There is an algebraic way to handle ω_0 because the hyperface permutation of a genus zero hypermonopole is an increasing permutation, and hence is itself a permutation that corresponds to a genus zero partition. This is not true for higher genus partitions and their associated permutations. For example, for $\pi = 1\ 3/2\ 4 \in \Pi_4^1$, the associated hyperedge permutation is $\alpha_\pi = (1\ 3)(2\ 4)$, while the associated hyperface permutation is $\varphi_\pi = \alpha_\pi^{-1}\sigma = (1\ 4\ 3\ 2)$; the disjoint cycles of φ_π are not increasing, and hence φ_π does not correspond to any partition. In general, if the genus one partition π has three types of crossing blocks, then its hyperface permutation contains two cycles each with two circular descents, while the remaining cycles contain one circular descent. And if π has only two types of crossing blocks, then its hyperface permutation contains one cycle with three circular descents, while the remaining cycles contain one circular descent.

However, an involution ω_1 that takes partitions between symmetric ranks on Π_n^1 can be constructed by using the hypermonopole diagrams and breaking and gluing half-hyperfaces.

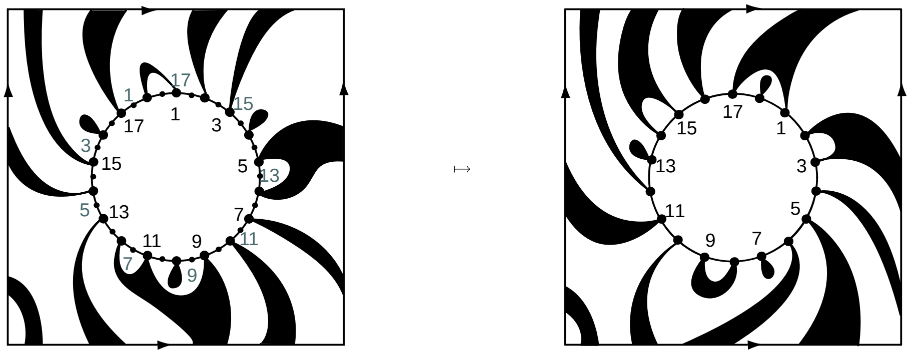
Suppose λ is a crossing block in the partition π, and η is the hyperedge that corresponds to λ in the hypermonopole diagram of π. The boundary walk of η contains exactly two nontrivial edges, thus the boundary of the polygon representing the torus separates the hyperedge η into two halves. Each half is called a *half-hyperedge* of $\mathfrak{h}_\pi$. More generally, if η is a hyperedge that contains j nontrivial edges in its boundary walk, then the boundary of the polygon representing the surface separates η into j 'halves'. *Half-hyperfaces* are defined similarly. The type of a half-hyperedge is the same as the type of the hyperedge that it is contained in. The type of a half-hyperface is the type of the hyperedge that lies immediately to its counterclockwise side.

To fix ideas, consider the example in Figure 4.4. The partition π has three type I crossing blocks 13 17/2 9 11 12/3 8, giving rise to six type I half-hyperedges, namely $h_1 = 3$, $h_2 = 2$, $h_3 = 17$, $h_4 = 13$, $h_5 = 9, 11, 12$, and $h_6 = 8$. And π has six type I half-hyperfaces, namely $f_1 = 3, 4$, $f_2 = 2$, $f_3 = 1, 17$, $f_4 = 13$, $f_5 = 12$, and $f_6 = 8$.

Proposition 4.5. *For $n \geq 4$, Π_n^1 is rank-symmetric.*

Proof. Let π be a genus one partition of n with k blocks. Consider its hypermonopole diagram, regarding the vertex as a small circle with n points on it, labelled 1 through n in the clockwise direction. As in the genus zero case, subdivide each arc of the circle by adding a new point in the middle of each arc of the circle, and label the new points 1 through n in the counter-clockwise direction, starting by assigning the label 1 to the new point on the arc that is between the old points $n-1$ and n.

Suppose π has j type I crossing blocks. Then there are $2j$ type I half-hyperedges, and $2j$ type I half-hyperfaces in the hypermonopole diagram of π. Of the $2j$ half-hyperfaces, exactly two are such that the hyperedge to its immedi-



$\pi = 1\ 18/2\ 9\ 11\ 12/3\ 8/4/5\ 6\ 14/7\ 15/10/13\ 17/16$ $\qquad$ $\omega_1(\pi) = 1\ 5\ 17/2\ 3\ 11/4\ 12/6\ 16/7/8\ 9/10\ 14\ 15/13/18$

Figure 4.4: The action of ω_1. The new labels of the subdivision are shown on the outside of the vertex of α.

ate clockwise side is of a different type. Starting at one of these half-hyperfaces, label the half-hyperfaces $f_1, \ldots, f_{2j}$ in the counterclockwise direction, and make j new faces by combining the half-hyperface f_i with f_{2j-i+1} for $1 \leq i \leq j$, using the new vertex-hyperface labels and arranging them in increasing order.

Repeat this procedure for type II and type III crossing half-hyperfaces to get a collection of new faces made from gluing half-hyperfaces in pairs, and also append the noncrossing hyperfaces of $\mathfrak{h}_\pi$ to this collection $\mathcal{K}$.

Define the function $\omega_1 : \Pi_n^1 \to \Pi_n^1$ as follows. For $\pi \in \Pi_n^1$, its image is the partition whose blocks correspond to the faces in $\mathcal{K}$.

It is easy to verify from the diagram that $\omega_1(\pi)$ has genus one; The type I, II, and III half-hyperfaces of π glue together to become the type I, II, and III crossing blocks of $\omega_1(\pi)$. It is also clear from the diagram that $\omega_1(\pi)$ is an involution on Π_n^1.

Lastly, to see that Π_n^1 is rank-symmetric, we need to show that ω_1 maps partitions with k blocks to partitions with $n-k$ blocks. If π has k blocks then $\mathfrak{h}_\pi$ has $n-k-1$ hyperfaces by the Euler-Poincaré formula. There are two cases to consider.

Case 1. π has three types of crossing blocks. As observed earlier in this section, π has $n-k-3$ increasing cycles in its hyperface permutation, so that each of these cycles correspond to a hyperface of π with two nontrivial edges in its boundary.

The remaining two non-increasing cycles each have two circular descents, and correspond to a hyperface with three nontrivial edges in its boundary. In the construction, the $n-k-3$ increasing hyperface cycles of π give rise to $n-k-3$ new blocks for $\omega_1(\pi)$, while the two non-increasing cycles correspond to six half-hyperfaces that were glued together to create three new blocks for $\omega_1(\alpha)$.

CASE 2. π has two types of crossing blocks. Similar to the first case, there are $n-k-2$ increasing cycles in its hyperface permutation, and one non-increasing cycle with three circular descents. The non-increasing cycle corresponds to four half-hyperfaces that were glued together to create two new blocks for $\omega_1(\alpha)$.

In either case, the resulting partition $\omega_1(\alpha)$ has $n-k$ blocks. This proves that Π_n^1 is rank-symmetric. $\square$

For example, recall that the partition π in Figure 4.4 has six type I crossing half-hyperfaces $f_1 = 3,4$, $f_2 = 2$, $f_3 = 1,17$, $f_4 = 13$, $f_5 = 12$, and $f_6 = 8$. Under the new labelling, these become $f_1 = 14,15$, $f_2 = 16$, $f_3 = 1,17$, $f_4 = 5$, $f_5 = 6$, and $f_6 = 10$. These glue together to form the new blocks 1 5 17/6 16/10 14 15. Similarly, the type II crossing half-hyperfaces give rise to the block 4 12, and the type III crossing half-hyperfaces give rise to the block 2 3 11. The noncrossing hyperfaces form the blocks 7/8 9/13/18. Thus $\omega_1(\alpha) = $ 1 5 17/2 3 11/4 12/6 16/7/8 9/10 14 15/13/18.

The involution ω_1 is not order-reversing. Unlike Π_n^0, Π_n^1 is in general not self-dual. To see this, let U_x denote the set of elements in P that cover x, and let D_x denote the set of elements in P that are covered by x. That is,

$$U_x = \{y \in P : x \lessdot y\} \quad \text{and} \quad D_x = \{y \in P : x \gtrdot y\}.$$

Proposition 4.6. Π_n^1 *is not self-dual, for* $n \geq 6$.

Proof. Let π be an element of rank 0 in Π_n^1, so that π has $n-2$ blocks; it has two crossing blocks each of size 2, and $n-4$ singleton blocks. Thus

$$|U_\pi| = \binom{n-2}{2} - 1,$$

since taking the union of any two blocks, as long as they are not both crossing blocks, will give rise to a genus 1 partition with $n-3$ blocks. In other words, the "up-degree" of any rank 0 element in the poset Π_n^1 is $\binom{n-2}{2} - 1$.

If Π_n^1 is self-dual, then every rank $n-4$ element of Π_n^1 should have "down-degree" equal to $\binom{n-2}{2} - 1$. We shall show that this is not the case.

Let $\beta = 1\ 3/2\ 4\ 5\ 6 \cdots n \in \Pi_n^1$. Then β has two blocks, so it is an element of rank $n-4$. Let λ denote the block $2\ 4\ 5\ 6 \cdots n$. Splitting off a contiguous block of integers of size k from λ, for any $k \in \{1, 2, \cdots, n-4\}$, results in a genus 1 partition in D_β. For example, splitting off the contiguous block $\{5, 6, 7\}$ of size 3 from λ yields the partition $\gamma = 1\ 3/2\ 4\ 8\ 9 \cdots n/5\ 6\ 7 \in D_\beta$.

There are $n-k-2$ ways of splitting a size k contiguous block of integers from λ. So

$$|D_\beta| \geq n-3+\cdots+2 = \frac{(n-3)(n-2)}{2} - 1 = \binom{n-2}{2} - 1.$$

But $1\ 3/2\ 5\ 6 \cdots (n-1)/4\ n$ is also a partition in D_β that is not obtained by splitting off any contiguous blocks of integers.

Therefore $|D_\beta| \geq \binom{n-2}{2}$, and the result follows. $\square$

Remark. Π_4^1 is the trivial poset so it is self-dual. Π_5^1 is a poset with two ranks, where every rank 0 element has up-degree 2 and every rank 1 element has down-degree 2, so Π_5^1 is also self-dual.

4.5 Symmetric chain decomposition

The proof that the poset of genus one partitions admits a symmetric chain decomposition borrows ideas from the proof for the genus zero case given in [27], so the genus zero case is included for comparison.

4.5.1 Genus zero

Two proofs are provided by Simion and Ullman in [27] to show that Π_n^0 is a symmetric chain order. The first is an existence proof, which relies on the following standard result, which can be found in [2].

Theorem 4.7. *The product of two symmetric chain orders is a symmetric chain order.* $\square$

Theorem 4.8. *(Simion, Ullman [27])* Π_n^0 *is a symmetric chain order for each* $n \geq 1$.

Existence proof. Proceed by induction on n. When $n = 1$, Π_1^0 is the trivial lattice, so it is a symmetric chain order. Assume the result is true for Π_k^0 for all

$k \leq n$. Let

$$R_1 = \{\pi \in \Pi_n^0 : 1 \text{ is a singleton block in } \pi\},$$

and for $2 \leq i \leq n$, let

$$R_i = \{\pi \in \Pi_n^0 : 1 \sim i \text{ and } i \text{ is the next smallest integer in that block}\},$$

where $a \sim b$ means a and b are in the same block.

Then the each of the induced sublattices R_1 and R_2 is isomorphic to Π_{n-1}^0, and moreover, $R_1 \cup R_2$ is isomorphic to the product of Π_{n-1}^0 and a two-element chain. And for $3 \leq i \leq n$, R_i is isomorphic to the lattice $\Pi_{i-2}^0 \times \Pi_{n-i+1}^0$. Note that $R_1 \cup R_2$ and each R_i for $3 \leq i \leq n$ is an interval in Π_n^0. By Theorem 4.7, $R_1 \cup R_2$ and each R_i for $3 \leq i \leq n$ is a symmetric chain order. The minimum and maximum elements in $R_1 \cup R_2$ are $\widehat{0} = 1/2/\cdots/n$ and $\widehat{1} = 1\ 2\ \cdots\ n$, respectively, so that $R_1 \cup R_2$ is embedded rank-symmetrically in Π_n^0. As for each R_i, the minimum element is $1\ i/2/3/\cdots/i-1/i+1/\cdots/n$, which has rank 1, while the maximum element is $1\ i\ i+1 \cdots n/2\ 3\ \cdots\ i-1$, which has rank $n-2$, so that the ranks of the minimum and maximum elements in each R_i sum to $n-1$, and each R_i is also symmetrically embedded in Π_n^0. This is a decomposition of Π_n^0 into symmetrically embedded symmetric chain orders, from which it follows that Π_n^0 is a symmetric chain order. $\square$

The second proof is a constructive proof, so that given any $\pi \in \Pi_n^0$, we are able to construct the symmetric chain that π lies on. The proof maintains the spirit of the classical 'parenthesization' method of showing that Boolean lattices are symmetric chain orders, so we first give a proof of this.

Recall that the elements of the Boolean lattice $\mathcal{B}_n$ are subsets of the power set of $[n] = \{1, \ldots, n\}$. Each element of $\mathcal{B}_n$ can alternately be represented by a string of n parentheses as follows. Let $Y \subset [n]$ be an element in $\mathcal{B}_n$. For $1 \leq i \leq n$, let the ith parenthesis in the string be a right parenthesis if $i \in Y$, and let it be a left parenthesis otherwise. Denote this string of parentheses by $w(Y)$. For example, $w(\emptyset)$ is a string of n left parentheses, and $w([n])$ is a string of n right parentheses.

These strings are used to construct symmetric chains in $\mathcal{B}_n$. Notice that if we take $w(\emptyset)$ and change the first left parenthesis to a right parenthesis, this new string corresponds to the element $\{1\} \in \mathcal{B}_n$, and $\{1\}$ covers $\emptyset$ in the lattice $\mathcal{B}_n$. If we successively change each left parenthesis to a right parenthesis starting from the left hand side of $w(\emptyset)$, then we obtain $n+1$ strings corresponding to

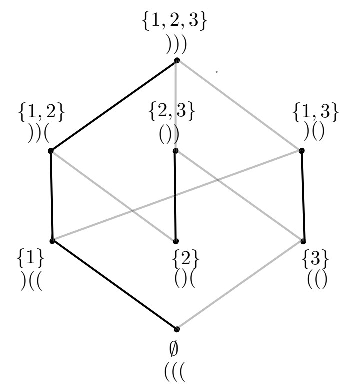


Figure 4.5: The Hasse diagram of the Boolean lattice $\mathcal{B}_3$. A symmetric chain decomposition of $\mathcal{B}_3$ making use of parentheses is depicted by darkened lines.

the elements $\emptyset, \{1\}, \{1, 2\}, \ldots, [n]$, which is an unrefinable chain in the lattice $\mathcal{B}_n$.

In general, given an element $Y \in \mathcal{B}_n$, the subset of parentheses in $w(Y)$ which match is the *core* of Y. Notice that the unmatched right parentheses all lie to the left of the unmatched left parentheses. Elements lying above Y in the chain are obtained by successively changing each unmatched left parenthesis to a right parenthesis in order from left to right, and elements lying below Y in the chain are similarly obtained by changing each unmatched right parenthesis to a left parenthesis in order from right to left. Notice that this process does not introduce additional matched parentheses, so the core of the chain is not affected and every element in the chain has the same core.

The chain created in this way is clearly unrefinable. To see that it is a symmetric chain, first notice that every unmatched parenthesis of the minimum element in the chain is a left parenthesis, while every unmatched parenthesis of the maximum element in the chain is a right parenthesis. Suppose the core of the chain has size $2m$. Then the minimum element on the chain has m right parentheses (from the core) while the rest are left parentheses, so the minimum element has rank m in $\mathcal{B}_n$. The maximum element on the chain has m left parentheses from the core, with the rest being right parentheses, so it is an element of rank $n - m$. Thus the chain is symmetric.

Figure 4.5 shows the symmetric chain decomposition of $\mathcal{B}_3$ constructed this way.

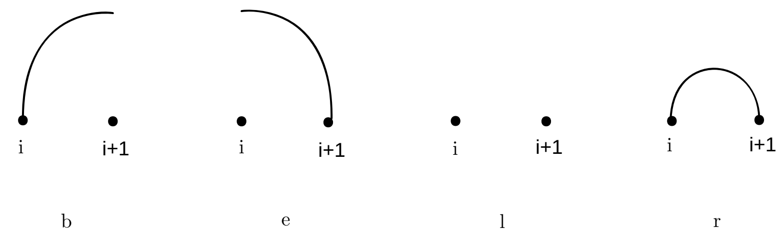


Figure 4.6: Associating a word to the linear diagram of a partition.

Now we are ready to construct symmetric chains for Π_n^0.

Constructive proof. To each $\pi \in \Pi_n^0$, we associate a word $w(\pi) = w_1 w_2 \cdots w_{n-1}$ of length $n-1$ from the alphabet $\{b, e, l, r\}$ as follows:

$$w_i = \begin{cases} b, & \text{if } i \nsim i+1 \text{ and } i \text{ is not the largest element in its block,} \\ e, & \text{if } i \nsim i+1 \text{ and } i+1 \text{ is not the smallest element in its block,} \\ l, & \text{if } i \nsim i+1 \text{ and } i \text{ is the largest element in its block,} \\ & \quad \text{and } i+1 \text{ is the smallest element in its block,} \\ r, & \text{if } i \sim i+1. \end{cases}$$

The alphabet corresponds to the four possible configurations of arcs on adjacent points in the linear diagram of a partition. See Figure 4.6.

Let $B = \{i : w_i = b\}$, and define the sets E, L, and R similarly. Note that $|B| + |E| + |L| + |R| = n - 1$. Regarding l and r as left and right parentheses respectively, let ML be the subset of L such that $i \in ML$ if and only if the left parenthesis corresponding to w_i is matched with a right parenthesis. Similarly define MR. Note that $|ML| = |MR|$. The *core* of the noncrossing partition π is the quadruple of sets $c(\pi) = (B, E, ML, MR)$.

For example, if $\pi = 1\,2\,10\,11\,12/3\,6\,9/4/5/7\,8/13\,16\,17/14/15$ is a genus zero partition of 17, then $w(\pi) = rbblebleerrlbler$, $B = \{2, 3, 6, 13\}$, $E = \{5, 8, 9, 15\}$, $ML = \{4, 7, 14\}$, $MR = \{10, 11, 17\}$. See Figure 4.7.

Now we are ready to construct the symmetric chain γ that a given π lies on. Similar to the construction for Boolean lattices, this is determined by the core of π, with the unmatched l's and r's taking the place of the unmatched left and right parentheses. Suppose there are s unmatched r's and t unmatched l's. As before, each unmatched r lies to the left of unmatched ls. By changing every unmatched r to an l, we obtain the word that corresponds to the minimum element $\widehat{0}_\gamma$ in the chain γ. Successive elements in γ are obtained by changing each unmatched l to an r in order from left to right, so that the maximum element $\widehat{1}_\gamma$ if the chain γ does not contain any unmatched l's. From this we see that γ is a chain of length $s+t+1$, and π is the $(r+1)$th smallest element on the chain.

Indeed, γ is an unrefined chain, since changing an l to an r corresponds to merging two blocks in a partition. Also, every element in a chain has the same core.

To see that γ is symmetrically embedded in Π_n^0, first observe that

$$\mathrm{bk}(\pi) = |B| + |L| + 1 = |E| + |L| + 1. \tag{4.3}$$

This comes from the fact that the number of blocks in π is the number of disjoint strings of arcs in the linear diagram of π, and the occurrence of $w_i = b$ or l in $w(\pi)$ signifies that a new string of arcs start at $i+1$. Similarly, $w_i = e$ or l signify that a string of arcs end at i.

Since $\widehat{0}_\gamma$ does not contain any unmatched r's, and $\widehat{1}_\gamma$ does not contain any unmatched l's, then

$$\mathrm{bk}(\widehat{0}_\gamma) = |B| + |ML| + (s+t) + 1,$$

and

$$\mathrm{bk}(\widehat{1}_\gamma) = |E| + |ML| + 1 = |E| + |MR| + 1.$$

Adding these two equations together yields

$$\begin{aligned}
\mathrm{bk}(\widehat{0}_\gamma) + \mathrm{bk}(\widehat{1}_\gamma) &= |B| + |E| + (|ML| + s) + (|MR| + t) + 2 \\
&= (|B| + |E| + |L| + |R|) + 2 \\
&= (n-1) + 2 \\
&= n+1,
\end{aligned}$$

which proves that γ is symmetrically embedded. Thus every partition in Π_n^0 lies on a unique symmetric chain, and Π_n^0 is a symmetric chain order. □

Continuing with the last example, π lies on the 3-element chain shown in Figure 4.7. This chain has the core $\{\{2,3,6,13\},\{5,8,9,15\},\{4,7,14\},\{10,11,17\}\}$.

Observation 1. From equation (4.3) we can conclude that $|B| = |E|$. But more is true. If the b's and e's are regarded as left and right parentheses, then they are completely matched. This can be seen from the interpretation that

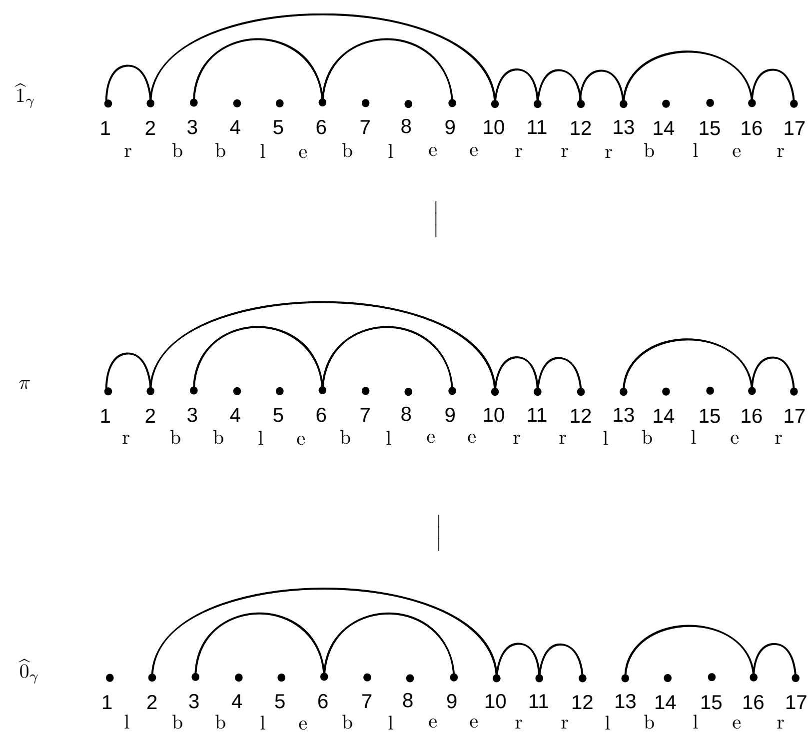


Figure 4.7: The symmetric chain γ in Π_{17}^0 containing the partition $\pi =$ 1 2 10 11 12/3 6 9/4/5/7/8/13 16 17/14/15.

$w_i = b$ means that an arc in the linear diagram begins at $i+1$, while $w_j = e$ means that an arc ends at j.

Observation 2. The choice to begin the linear diagram of a partition at 1 is an arbitrary one. By choosing to begin the linear diagram at any $i \in \{1, \ldots, n\}$, different symmetric chain decompositions can be obtained.

4.5.2 Genus one

The following proof for the case of genus one partitions is a constructive proof. As in the proof for the case of genus zero partitions, we shall make use of the linear diagrams of partitions and appeal to the parenthesization method.

Theorem 4.9. Π_n^1 *is a symmetric chain order for each* $n \geq 4$.

Proof. Let $\pi \in \Pi_n^1$. Then π has $t = 2$ or 3 types of blocks. Consider the set of crossing half-hyperedges. Exactly $2t$ of the half-hyperedges are such that the hyperedge to its counter-clockwise side is of a different type. Denote these $2t$ half-hyperedges by $A_1, \ldots, A_{2t}$, where A_i is the ordered list whose elements are the labels in the half-hyperedge read in clockwise order. Let

$$d = \min_{1 \leq i \leq 2t} \{\text{first label in the list } A_i\}.$$

The canonical hypermonopole diagram of π is the drawing in which the non-trivial edges of the crossing block containing d are homotopic to the meridional loop a; that is, the crossing block containing d is designated to be type I. If π has three types of crossing blocks, its canonical hypermonopole diagram is unique. If π has only two types of crossing blocks, then by convention, all blocks of π are either type I or type II.

For example if $\pi = 1/2\ 3\ 5\ 13/4/6/7\ 9/8\ 14/10\ 11/12$ as in Figure 4.8, then $2\ 3\ 5\ 13/4/6/7\ 9/10\ 11/12$ are the type I blocks, and $1/8\ 14$ are the type II blocks. In this example, $d = 2$.

To each $\pi \in \Pi_n^1$, we associate a word $w(\pi) = w_d w_{d+1} \cdots w_n w_1 \cdots w_{d-2}$ of length $n-1$ from the alphabet $\{b, e, l, r, g\}$ as follows: if i and $i+1$ are in blocks of different types, then

$$w_i = g.$$

Otherwise, if i and $i+1$ are in blocks of the same type, then as in the genus zero case, the letter to w_i is determined by the configuration of the arcs at the points i and $i+1$ as shown in Figure 4.6.

Let $B = \{i : w_i = b\}$, and similarly define the sets E, L, R, and G. Note that $|B| + |E| + |L| + |R| + |G| = n - 1$ and $|G| = 2t - 1$.

The next step in the construction is to define the core of π.

First recall $w_i = b$ signifies that an arc in the linear diagram of π begins at i, and $w_j = e$ signifies that an arc ends at $j + 1$, so one might expect that $|B| = |E|$ as in the genus zero case. However this is not necessarily true in the genus one case. Suppose an arc in the linear diagram begins at i and ends at j. One scenario that may occur is if $w_i = b$, but $w_j = g$ because an arc of a different type begins at j. A second scenario if is $w_j = e$ but $w_i = g$. The next claim shows that the second scenario cannot occur, so that $|B| \geq |E|$.

Claim 1. If an arc begins at i and ends at $j + 1$ with $w_j = e$, then $w_i = b$.

Suppose the claim is not true and $w_i = g$, so that the blocks containing i and $i + 1$ are of different types. By construction, the blocks that contain the elements $i + 2, i + 3, \dots, j$ cannot be the same type as the block containing i (they must be the same type as the block containing $i+1$, or are the third type). But $w_j = e$ means that the blocks containing j and $j+1$ are the same type, and since $j+1$ and i are in the same block by assumption, then the block containing j and the block containing i are the same type. This is a contradiction, so the claim holds. □

If an arc begins at i and ends at $j + 1$ with $w_j = e$, we match this e with either a b or l as follows. Let $i \prec j$ denote that i appears before j with respect to the order $d \prec d + 1 \prec \cdots \prec n \prec 1 \prec 2 \prec \cdots \prec d - 2$. If $w_k \neq g$ for any $i \prec k \prec j$, then match $w_j = e$ with $w_i = b$. Otherwise, match $w_j = e$ with the b or l that is counter-clockwise closest to the first g. By Claim 1, every e is matched. Let mB denote the set of labels which correspond to the letter b and match with an e, and let mL denote the set of labels which correspond to the letter l and match with an e, so that $|mB| + |mL| = |E|$.

Assign left parentheses to the remaining b's and l's, and assign right parentheses to the r's. Let MB denote the set of labels which correspond to the letter b and match with a right parenthesis, let ML denote the set of labels which correspond to the letter l and match with a right parenthesis, and let MR denote the set of labels which correspond to the letter r and match with a left parenthesis. Then we have $|MB| + |ML| = |MR|$. Let UB denote the set of labels which correspond to the letter b and are unmatched. Similarly define the sets UL and UR so that $B = mB \cup MB \cup UB$, $L = mL \cup ML \cup UL$ and

$R = MR \cup UR$.

The core of the genus one partition π is

$$c(\pi) = \{d, mB, mL, E, MB, ML, MR, G\}.$$

The next step is to construct the chain γ that π lies on, and this depends on the set of unmatched parentheses of π. As before, every unmatched right parenthesis lies to the left of unmatched left parentheses.

Elements that are above π on the chain are successively obtained by changing each unmatched left parenthesis to a right parenthesis in order from left to right, but since a left parenthesis may correspond to the letter l or b, additional care must be taken to ensure that this block-merging operation is consistent when it is reversed.

If the left parenthesis corresponds to $w_i = l$, then it is replaced by $w_i = r$ as before. If the left parenthesis corresponds to $w_i = b$, then there must be an arc in the linear diagram that starts at i and end at some label j, with $w_k = g$ for some $i \prec k \prec j$. Consider the planar area under the arc. If the label $w_p = l$ appears in the planar area, then replace $w_i = b$ with $w_i = r$, replace $w_p = l$ with $w_p = b$, and modify the linear diagram accordingly. If such a label l does not occur in the planar area, then the only other label that appears in the planar area is a g. In this case replace $w_i = b$ by $w_i = r$ and the arc is shifted to begin at that g.

The elements that are below π on the chain are successively obtained by changing each unmatched right parenthesis to a left parenthesis in order from right to left. If $w_i = r$ is a part of a crossing arc (signified by the presence of g's under the arc), then $w_i = r$ is replaced by $w_i = b$ and the construction described above is reversed. Otherwise, $w_i = r$ is simply replaced by $w_i = l$.

Indeed, the chain constructed this way is unrefinable. Also, every element on a chain has the same core because switching unmatched parentheses does not affect the core. The presence of the letters g prevent crossing blocks from merging as we move up the chain.

Figure 4.8 shows the chain that $\pi = 1/2\ 3\ 5\ 13/4/6/7\ 9/8\ 14/10\ 11/12$ lies on. The core of this chain consists of $d = 2$, $mB = \{3\}$, $mL = \{6\}$, $E = \{4, 12\}$, $MB = \emptyset$, $ML = \{9\}$, $MR = \{10\}$, $G = \{7, 8, 13\}$. The unmatched letters are $w_2 = r$ and $w_5 = w_{11} = w_{14} = l$, so that π is the second element on the chain, and the chain has length five.

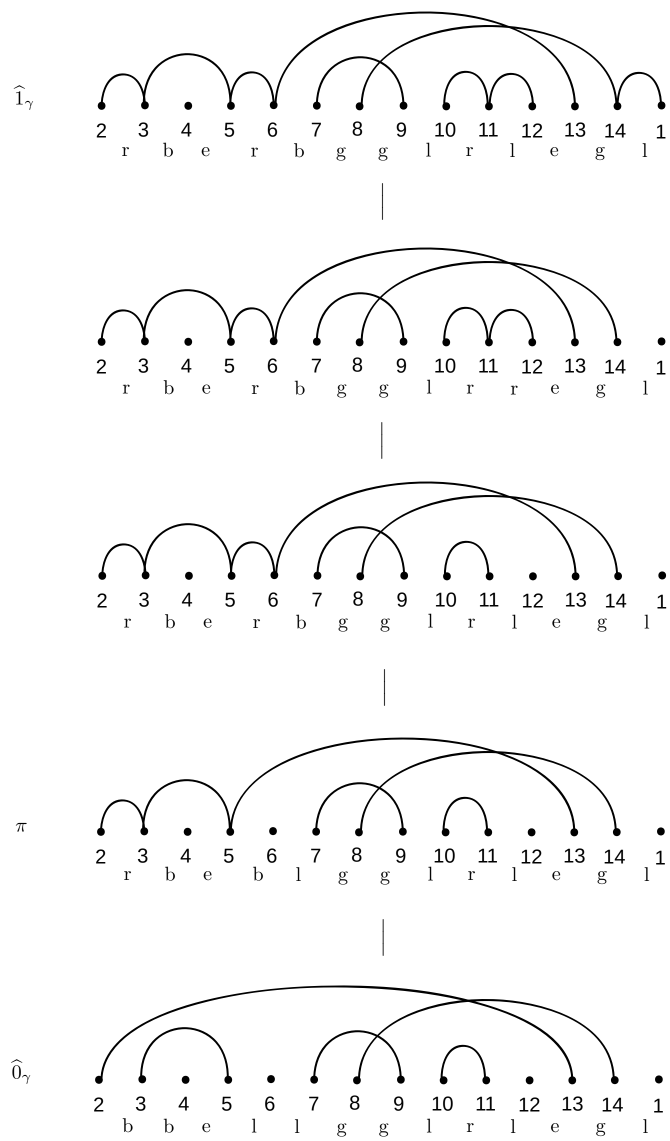


Figure 4.8: The symmetric chain γ in Π_{14}^1 containing the partition π = 1/2 3 5 13/4/6/7 9/8 14/10 11/12.

To see that the constructed chain γ is symmetrically embedded in Π_n^1, first observe that

$$\mathrm{bk}(\pi) = |B| + |L| + 1 + \frac{1}{2}(|G| - 1),$$

by similar reasons that Equation (4.3) holds for the genus zero case.

Let $\widehat{0}_\gamma$ and $\widehat{1}_\gamma$ respectively denote the minimum and maximum element of γ. $\widehat{0}_\gamma$ is obtained from π by changing every unmatched r to a b or l. Thus

$$\mathrm{bk}(\widehat{0}_\gamma) = (|mB|+|mL|)+(|MB|+|ML|)+|UB|+|UL|+|UR|+1+\frac{1}{2}(|G|-1).$$

Similarly

$$\begin{aligned}\mathrm{bk}(\widehat{1}_\gamma) &= (|mB| + |mL|) + (|MB| + |ML|) + 1 + \frac{1}{2}(|G| - 1) \\ &= |E| + |MR| + 1 + \frac{1}{2}(|G| - 1).\end{aligned}$$

Summing these two equations yield

$$\begin{aligned}\mathrm{bk}(\widehat{0}_\gamma) + \mathrm{bk}(\widehat{1}_\gamma) &= (|mB| + |MB| + |UB|) + (|mL| + |ML| + |UL|) \\ &\quad +(|MR| + |UR|) + |E| + 2 + (|G| - 1) \\ &= |B| + |L| + |R| + |E| + |G| + 1 \\ &= n,\end{aligned}$$

which shows that γ is symmetrically embedded. Thus every partition in Π_n^1 lies on a symmetric chain, and Π_n^1 is a symmetric chain order. $\square$

Conjecture 4.10. Π_n^1 *decomposes into a union of two* Π_{n-1}^1 *and* $\binom{n-1}{3}$ Π_{n-3}^0.

The proof of this Conjecture would inductively show that Π_n^1 is a symmetric chain order, and would prove Equation (3.10). We remark that this Conjecture holds for the cases $n = 4$ and $n = 5$.

Corollary 4.11. *If Conjecture 4.10 holds, then the* j*th rank of* Π_n^1 *has size*

$$T(n, n-2-j) = \frac{1}{6}\binom{n}{2}\binom{n-2}{j}\binom{n-2}{j+2}.$$

Proof. Recall that $T(n, k)$ denotes the number of genus one partitions of n with k blocks, so $T(n, n-2-j)$ is the cardinality of the jth rank of Π_n^1. Also recall that the Narayana number $N(n, n-j) = \frac{1}{n}\binom{n}{j}\binom{n}{j+1}$ is the cardinality of the jth rank of Π_n^0. Conjecture 4.10 implies

$$T(n, n-2-j) = T(n-1, n-3-j) + T(n-1, n-2-j) + \binom{n-1}{3} N(n-3, n-3-j).$$

Recall that the 0th rank of Π_n^1 has $\binom{n}{4}$ elements, therefore,

$$T(n, n-2) = \binom{n}{4} = \frac{1}{6}\binom{n}{2}\binom{n-2}{0}\binom{n-2}{2}.$$

By induction,

$$\begin{aligned} T(n, n-2-j) &= \frac{1}{6}\binom{n-1}{2}\binom{n-3}{j}\binom{n-3}{j+2} + \frac{1}{6}\binom{n-1}{2}\binom{n-3}{j-1}\binom{n-3}{j+1} \\ &\quad + \binom{n-1}{3}\frac{1}{n-3}\binom{n-3}{j}\binom{n-3}{j+1}. \end{aligned}$$

Routine simplification gives

$$T(n, n-2-j) = \frac{1}{6}\binom{n}{2}\binom{n-2}{j}\binom{n-2}{j+2}.$$

$\square$

In other words, Conjecture 4.10 predicts that the number of genus one partitions of n with k blocks is $T(n, k) = \frac{1}{6}\binom{n}{2}\binom{n-2}{k}\binom{n-2}{k-2}$.

4.5.3 Sperner property

An important max-min result on posets is Dilworth's Theorem, whose proof can be found in [2] (Theorem 8.14):

Theorem 4.12. *For a finite poset P, the minimum number of disjoint chains into which P can be decomposed is $w(P)$, the maximum size of an antichain in P.* $\square$

Sperner theorems deal with the study of maximal antichains in posets.

Let P be a finite ranked poset, let P_j denote the jth rank of P and let $W_j(P) = |P_j|$. Since each rank of P is an antichain, then $w(P) \geq \max_j W_j(P)$.

A finite ranked poset P is *Sperner* if

$$w(P) = \max_j W_j(P).$$

A *k-family* of P is a union of k antichains in P. A finite ranked poset is *k-Sperner* if the maximum cardinality of a k-family is equal to the sum of the k largest ranks of P. A poset is *strongly Sperner* if it is k-Sperner for all k. The following is a standard result.

Proposition 4.13. *A symmetric chain order is strongly Sperner.*

Proof. Let P be a finite symmetric chain order, and let $\mathcal{C}$ be a symmetric chain decomposition for P. For $k \geq 1$, let $P_{i_1}, \ldots, P_{i_k}$ be the k largest ranks of P, and let f_k denote the maximum cardinality of a k-family. Since $P_{i_1} \cup \ldots \cup P_{i_k}$ is itself a k-family, then $f_k \geq |P_{i_1} \cup \ldots \cup P_{i_k}|$.

Let A be a k-family. Then for any $C \in \mathcal{C}$, $|A \cap C| \leq k$. If the length of C is at least k, then $|C \cap (P_{i_1} \cup \ldots \cup P_{i_k})| = k$, so $|A \cap C| \leq |C \cap (P_{i_1} \cup \ldots \cup P_{i_k})|$. And if the length of C is less than k, then $|C \cap (P_{i_1} \cup \ldots \cup P_{i_k})| = |C|$, so $|A \cap C| \leq |C| = |C \cap (P_{i_1} \cup \ldots \cup P_{i_k})|$ in this case as well.

Therefore,

$$|A| = \sum_{C \in \mathcal{C}} |A \cap C| \leq \sum_{C \in \mathcal{C}} |C \cap (P_{i_1} \cup \ldots \cup P_{i_k})| = |P_{i_1} \cup \ldots \cup P_{i_k}|.$$

Since this holds for any k-family A, then $f_k \leq |P_{i_1} \cup \ldots \cup P_{i_k}|$. Result follows. $\square$

The above proposition shows that the posets Π_n^0 and Π_n^1 are strongly Sperner. It is interesting to compare this with the following result for Π_n:

Theorem 4.14. *(Canfield [7]). For sufficiently large n, the partition lattice Π_n is not Sperner.* $\square$

Remark. The symmetric chain decomposition property of a general poset can be proved by other means. For example, Corollary 8.66 in Aigner [2] states that any finite modular geometric lattice is a symmetric chain order. However, neither Π_n^0 nor Π_n^1 (when completed to a lattice by adjoining $\widehat{0}$ and $\widehat{1}$) are semimodular lattices, so Corollary 8.66 does not apply.

4.6 Further work

In Section 3.6, we gave evidence that the poset of genus two partitions is not rank-symmetric. Here we state two related conjectures.

Conjecture 4.15. *For $g \geq 2$, $n \geq 6$, and $0 \leq k \leq n - 2g - 2$, the Whitney numbers $W_k(\Pi_n^g)$ form a log concave sequence, and Π_n^g is unimodal.*

Conjecture 4.16. *For $g \geq 2$ and $n > 2g + 2$, Π_n^g is not rank-symmetric.*

It is of interest to find the generating series for partitions of genus g for each $g \in \mathbb{N}$.

A possible approach to proving Conjecture 3.15 is to find a direct bijection between the set of genus one rooted hypermonopoles with $n-1$ edges and $k-1$ hyperedges and the set of genus one partitions of n with k blocks.

Many of the known results regarding noncrossing partitions may have interesting analogues for higher genus partitions. A possible topic is Kreweras' proof [23] by induction that the Möbius function of Π_n^0 is $(-1)^{n-1}C_{n-1}$. What is the Möbius function of Π_n^g when is completed to a lattice by adjoining $\widehat{0}$ and $\widehat{1}$? Another result is that both Π_n and Π_n^0 are EL-shellable lattices. Is $\Pi_n^g \cup \widehat{0} \cup \widehat{1}$ also EL-shellable? Also mentioned in the Introduction is the fact that the determinant of the matrix of chromatic joins associated to noncrossing partitions has a nice factorization into generalized Chebyshev polynomials. It is of interest to investigate higher genus analogues of this result.

Finally, it may be worthwhile to mention that rooted hypermonopoles are in bijection with unicellular bicoloured maps; that is, one-faced maps whose vertices are two-coloured. The bijection is realized by taking map duals. This may provide a useful way of approaching the study of genus one partitions.

Appendix A

A.1 A second formula for the character $\chi^{[1^j,n-j]}(\alpha)$

This is a special case of a proof in [24] (I.7 Example 14).

By Frobenius' formula (Theorem 3.6),

$$\chi^{[1^j,n-j]}(\alpha) = [x_1^n x_2^j x_3^{j-1} \cdots x_{j+1}] \triangle \cdot \prod_{i \geq 1} p_i(x_1, \ldots, x_{j+1})^{a_i},$$

where $\triangle = \triangle(x_1, \ldots, x_{j+1})$ is the Vandermonde determinant. Since $\triangle \cdot \prod_{i \geq 1} p_i^{a_i}$ is a polynomial of homogeneous degree, then nothing is lost by setting $x_1 = 1$. Thus, shifting indices,

$$\begin{aligned}
\chi^{[1^j,n-j]}(\alpha) &= [x_1^j \cdots x_j] \prod_{s=1}^{j} (1 - x_s) \prod_{1 \leq r < s \leq j} (x_r - x_s) \prod_{i \geq 1} (1 + p_i(x_1, \ldots, x_j))^{a_i} \\
&= [x_1^j \cdots x_j] \triangle(x_1, \ldots, x_j) \prod_{s=1}^{j} (1 - x_s) \prod_{i \geq 1} (1 + p_i(x_1, \ldots, x_j))^{a_i} \\
&= [s_{[1^j]}] \prod_{i \geq 1} (1 - x_i) \prod_{i \geq 1} (1 + p_i)^{a_i} \\
&= \left\langle \prod_{i \geq 1} (1 - x_i) \prod_{i \geq 1} (1 + p_i)^{a_i}, s_{[1^j]} \right\rangle.
\end{aligned}$$

Observe that

$$\prod_{i \geq 1} (1 - x_i) = \sum_{i \geq 0} (-1)^i e_i,$$

and for $\alpha = [1^{a_1}, 2^{a_2}, \ldots]$, $\rho = [1^{r_1}, 2^{r_2}, \ldots]$,

$$\prod_{i \geq 1} (1 + p_i)^{a_i} = \prod_{i \geq 1} \sum_{r_i \geq 0} \binom{a_i}{r_i} p_i^{r_i} = \sum_{\rho \in \mathcal{P}} \binom{\alpha}{\rho} p_\rho.$$

Since $e_i = s_{[1^i]}$, then

$$\begin{aligned}\chi^{[1^j,n-j]}(\alpha) &= \sum_{i\geq 1}(-1)^i \left\langle e_i \sum_{\rho\in\mathcal{P}} \binom{\alpha}{\rho} p_\rho, s_{[1^j]}\right\rangle \\ &= \sum_{i\geq 1}(-1)^i \left\langle s_{[1^i]} \sum_{\rho\in\mathcal{P}} \binom{\alpha}{\rho} p_\rho, s_{[1^j]}\right\rangle \\ &= \sum_{i\geq 1}(-1)^i \left\langle \sum_{\rho\in\mathcal{P}} \binom{\alpha}{\rho} p_\rho, s_{[1^j]/[1^i]}\right\rangle,\end{aligned}$$

where $s_{[1^j]/[1^i]}$ is a skew Schur function. In this special case,

$$s_{[1^j]/[1^i]} = s_{[1^{j-i}]} = e_{j-i} = \sum_{\lambda\vdash j-i} \varepsilon_\lambda \varpi_\lambda^{-1} p_\lambda,$$

so

$$\begin{aligned}\chi^{[1^j,n-j]}(\alpha) &= \sum_{i\geq 1}(-1)^i \left\langle \sum_{\rho\in\mathcal{P}} \binom{\alpha}{\rho} p_\rho, \sum_{\lambda\vdash j-i} \varepsilon_\lambda \varpi_\lambda^{-1} p_\lambda \right\rangle \\ &= \sum_{i\geq 1}(-1)^i \sum_{\rho\in\mathcal{P}} \sum_{\lambda\vdash j-i} \binom{\alpha}{\rho} \varepsilon_\lambda \varpi_\lambda^{-1} \langle p_\rho, p_\lambda\rangle \\ &= \sum_{i\geq 1}(-1)^i \sum_{\lambda\vdash j-i} \binom{\alpha}{\lambda} \varepsilon_\lambda.\end{aligned}$$

It follows that

$$\chi^{[1^j,n-j]}(\alpha) = (-1)^j \sum_{i=0}^{j} \sum_{\lambda\vdash i} (-1)^{l(\lambda)} \binom{\alpha}{\lambda}.$$

A.2 A second proof of the genus one rooted hypermonopole series

This proof is due to Andrews [3].

Lemma A.1.

$$\sum_{j\geq 0} f(j) \binom{M}{j} \binom{N}{j+s} = \sum_{i=0}^{d} a_i \binom{M}{i} \binom{M+N-i}{M+s},$$

where $f(j) = a_0 + a_1 j + a_2 \binom{j}{2} + \cdots + a_d \binom{j}{d}$ is a general dth degree polynomial in j.

Proof.

$$\begin{aligned}\sum_{j\geq 0}\binom{j}{i}\binom{M}{j}\binom{N}{j+s} &= \sum_{j\geq i}\frac{M!}{i!(j-i)!(M-j)!}\binom{N}{j+s}\\ &= \sum_{j\geq 0}\frac{M!}{i!j!(M-i-j)!}\binom{N}{j+s+i}\\ &= \binom{M}{i}\sum_{j\geq 0}\binom{M-i}{j}\binom{N}{N-s-i-j}\\ &= \binom{M}{i}\binom{M+N-i}{N-s-i}\\ &= \binom{M}{i}\binom{M+N-i}{M+s},\end{aligned}$$

where the summation to closed form is by Vandermonde's convolution. □

In the proof of Corollary 3.12, we have

$$A_n(x,y) = (-1)^{n-1}n!\binom{x}{n}\binom{y}{n}\ {}_3F_2\left[\begin{matrix}1-n & x+1 & y+1\\ - & x-n+1 & y-n+1\end{matrix}\right].$$

Setting $a = k+m$ in Equation 1 on page 32 of [4],

$${}_3F_2\left[\begin{matrix}-m & b & c\\ - & k-b & k-c\end{matrix}\right] = \frac{k^{(m)}(k-b-c)^{(m)}}{(k-b)^{(m)}(k-c)^{(m)}}\cdot F,$$

where

$$F = \ {}_4F_3\left[\begin{matrix}-\frac{m}{2} & -\frac{m-1}{2} & b & c\\ - & \frac{k}{2} & \frac{k+1}{2} & b+c-k+1-m\end{matrix}\right].$$

Take $b = x+1$, $c = y+1$, $m = n-1$, $k = x+y+2-n$, so that

$${}_3F_2\left[\begin{matrix}1-n & x+1 & y+1\\ - & x-n+1 & y-n+1\end{matrix}\right] = \frac{(x+y+2-n)^{(n-1)}(-n)^{(n-1)}}{(y+1-n)^{(n-1)}(x+1-n)^{(n-1)}}\cdot G,$$

where

$$G = \ {}_4F_3\left[\begin{matrix}-\frac{(n-1)}{2} & -\frac{(n-1)}{2}+\frac{1}{2} & x+1 & y+1\\ - & \frac{x+y+2-n}{2} & \frac{x+y+2-n}{2}+\frac{1}{2} & 2\end{matrix}\right].$$

Now

$$\frac{(\frac{A}{2})^{(j)}(\frac{A}{2}+\frac{1}{2})^{(j)}}{(\frac{B}{2})^{(j)}(\frac{B}{2}+\frac{1}{2})^{(j)}} = \frac{A^{(2j)}}{B^{(2j)}},$$

so

$$
{}_3F_2\left[\begin{matrix} 1-n & x+1 & y+1 \\ - & x-n+1 & y-n+1 \end{matrix}\right]
$$

$$
= \frac{(x+y+2-n)^{(n-1)}(-1)^{n-1}n!}{(y+1-n)^{(n-1)}(x+1-n)^{(n-1)}} \sum_{j\geq 0} \frac{(-n+1)^{(2j)}(x+1)^{(j)}(y+1)^{(j)}}{(x+y+2-n)^{(2j)}j!(j+1)!}.
$$

Therefore,

$$
A_n(x,y) = \sum_{0\leq j\leq \frac{n-1}{2}} \frac{(1-n)^{(2j)}x^{(j+1)}y^{(j+1)}(x+y+2-n+2j)^{(n-1-2j)}}{j!(j+1)!}.
$$

Observe that

$$
x^{(j+1)} = x^{j+1} + \binom{j+1}{2}x^j + \left(3\binom{j+1}{4} + 2\binom{j+1}{3}\right) + \cdots,
$$

and

$$
\begin{aligned}
(x+y+2-n+2j)^{(n-1-2j)} &= (x+y)^{n-1-2j} - \binom{n-2j-1}{2}(x+y)^{n-2-2j} \\
&\quad + \left(3\binom{n-2j-1}{4} + 2\binom{n-2j-1}{3}\right) - \cdots.
\end{aligned}
$$

We want the coefficient of $x^k y^{n-1-k}$ in $A_n(x,y)$. And since $x^k y^{n-1-k}$ has total degree $n-1$, then it is given by

$$
\begin{aligned}
\sum_{0\leq j\leq \frac{n-1}{2}} \frac{(1-n)^{(2j)}}{j!(j+1)!} \Bigg\{ &\left(3\binom{n-2j-1}{4} + 2\binom{n-2j-1}{3}\right)\binom{n-3-2j}{k-j-1} \\
&-\binom{j+1}{2}\binom{n-2j-1}{2}\binom{n-2-2j}{k-j-1} \\
&-\binom{j+1}{2}\binom{n-2j-1}{2}\binom{n-2-2j}{k-j} \\
&+\left(3\binom{j+1}{4} + 2\binom{j+1}{3}\right)\binom{n-1-2j}{k-j-1} \\
&+\left(3\binom{j+1}{4} + 2\binom{j+1}{3}\right)\binom{n-1-2j}{k-j+1} \\
&+\binom{j+1}{2}^2\binom{n-1-2j}{k-j}\Bigg\}.
\end{aligned}
$$

Note that $(1-n)^{(2j)} = (n-1)!/(n-1-2j)!$. The previous sum can be rewritten so that Lemma A.1 applies to every term:

$$
\begin{aligned}
(n-1)! \sum_{0\le j\le \frac{n-1}{2}} \Bigg\{ & \frac{1}{8}\frac{(n-2j-3)(n-2j-4)}{(k-1)!(n-k-1)!}\binom{k-1}{j}\binom{n-k-1}{j+1} \\
& +\frac{1}{3}\frac{(n-2j-3)}{(k-1)!(n-k-1)!}\binom{k-1}{j}\binom{n-k-1}{j+1} \\
& -\frac{1}{2}\binom{j+1}{2}\frac{(n-2j-1)(n-2j-2)}{k!(n-k)!}\binom{k}{j}\binom{n-k}{j+1} \\
& +\frac{1}{4}\binom{j+1}{3}\frac{3j+2}{k!(n-k)!}\binom{n-k}{j}\binom{k}{j+1} \\
& +\frac{1}{4}\binom{j+1}{3}\frac{3j+2}{(k+1)!(n-k-1)!}\binom{k+1}{j}\binom{n-k-1}{j+1} \\
& +\binom{j+1}{2}^2\frac{1}{k!(n-k)!}\binom{k}{j}\binom{n-k}{j+1}\Bigg\}.
\end{aligned}
$$

This expression consists of six terms. The change of basis matrix

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 2 & 6 & 14 \\
0 & 0 & 0 & 6 & 36 \\
0 & 0 & 0 & 0 & 24
\end{bmatrix}
$$

converts any fourth degree polynomial with basis $\{1, j, j^2, j^3, j^4\}$ to a fourth degree polynomial with basis $\{1, \binom{j}{1}, \binom{j}{2}, \binom{j}{3}, \binom{j}{4}\}$. Hence each of the six terms is summable by Lemma A.1. The result follows after routine algebraic simplification. $\square$

Glossary of symbols

$A_n(x, y)$	generating series for rooted hypermonopoles with n edges, 34
$A_n^1(x, y)$	generating series for genus one rooted hypermonopoles, 42
$\mathcal{B}_n$	Boolean lattice of $2^{[n]}$, 52
C_n	nth Catalan number, 20
$\mathcal{C}_\alpha$	conjugacy class of S_n indexed by α, 26
e_θ	elementary symmetric function indexed by θ, 25
H_n	set of rooted hypermonopoles with n edges, 10
$H_n^<$	set of rooted hypermonopoles with n edges
	whose edge permutations are increasing, 10
H_θ	product of hooklengths of integer partition θ, 24
h_θ	complete symmetric function indexed by θ, 25
$\mathcal{F}_\theta$	Ferrers diagram of integer partition θ, 24
$\mathfrak{h}_\pi$	hypermonopole associated to π, 10
$g(\pi)$	genus of partition π, 11
m_θ	monomial symmetric function indexed by θ, 24
$m_i(\theta)$	multiplicity of i in integer partition θ, 33
$N(n, k)$	Narayana number, 20
$[n]$	set $\{1, 2 \ldots, n\}$, 4
$\mathcal{P}$	set of integer partitions, 24
P_j	jth rank of a ranked poset P, 51
p_θ	power sum symmetric function indexed by θ, 25
$\mathcal{S}_g$	standard g-holed torus, 11
S_n	symmetric group on n symbols, 6
$S_n^<$	set of permutations of n whose disjoint cycles are increasing, 9
s_θ	Schur symmetric function indexed by θ, 26
$s_n^{(k)}$	Stirling numbers of the first kind, 40
$S(n, k)$	Stirling numbers of the second kind, 52

T_n	number of genus one partitions of n, 46
$T(n,k)$	number of genus one partitions of n with k blocks, 45
$\mathbf{x}$	vector $(x_1, x_2, \ldots)$, 25
$W(n,k)$	number of genus two partitions of n with k blocks, 48
$W_j(P)$	size of jth rank of a ranked poset P, 51
α_π	permutation associated to π, 10
Λ^n	symmetric functions of homogeneous degree n over $\mathbb{Q}$, 26
Π_n	set of partitions of n, 9
Π_n^g	set of genus g partitions of n, 53
ϖ_α	size of centralizer of permutation in S_n with cycle type α, 26
σ	permutation $(1\ 2\ \cdots\ n)$, 9
$\theta \vdash n$	integer partition θ of n, 24
$\chi^\theta(\lambda)$	irreducible character of S_n indexed by θ evaluated on the conjugacy class $\mathcal{C}_\lambda$, 28
ω_0	an involution on Π_n^0, 58
ω_1	an involution on Π_n^1, 64
$\triangle(x_1, \ldots, x_n)$	Vandermonde determinant, 30
$[x^n]f$	coefficient of x^n in formal power series f, 30
$x_{(j)}$	falling factorial, 21
$x^{(j)}$	rising factorial, 21
$\widehat{0}$	unique minimum element in a poset, 50
$\widehat{1}$	unique maximum element in a poset, 50

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