Higher dimensional Taub-NUT spaces and applications

by

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Abstract

In the first part of this thesis we discuss classes of new exact NUT-charged solutions in four dimensions and higher, while in the remainder of the thesis we make a study of their properties and their possible applications.

Specifically, in four dimensions we construct new families of axisymmetric vacuum solutions using a solution-generating technique based on the hidden $SL(2,R)$ symmetry of the effective action. In particular, using the Schwarzschild solution as a seed we obtain the Zipoy-Voorhees generalisation of the Taub-NUT solution and of the Eguchi-Hanson soliton. Using the C-metric as a seed, we obtain and study the accelerating versions of all the above solutions. In higher dimensions we present new classes of NUT-charged spaces, generalising the previously known even-dimensional solutions to odd and even dimensions, as well as to spaces with multiple NUT-parameters. We also find the most general form of the odd-dimensional Eguchi-Hanson solitons. We use such solutions to investigate the thermodynamic properties of NUT-charged spaces in (A)dS backgrounds. These have been shown to yield counter-examples to some of the conjectures advanced in the still elusive dS/CFT paradigm (such as the maximal mass conjecture and Bousso's entropic N-bound). One important application of NUT-charged spaces is to construct higher dimensional generalisations of Kaluza-Klein magnetic monopoles, generalising the known 5-dimensional Kaluza-Klein soliton. Another interesting application involves a study of time-dependent higher-dimensional bubbles-of-nothing generated from NUT-charged solutions. We use them to test the AdS/CFT conjecture as well as to generate, by using stringy Hopf-dualities, new interesting time-dependent solutions in string theory. Finally, we construct and study new NUT-charged solutions in higher-dimensional Einstein-Maxwell theories, generalising the known Reissner-Nordström solutions.

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To My Parents And To My Brother

In the memory of Mişeta

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Chapter 1

Introduction

Among the major scientific achievements of twentieth century research in theoretical physics have been the remarkable advances in relativity theory and in quantum mechanics.

According to the theory of relativity the laws of physics should be the same for all the observers in the universe. This means that they must be formulated in a covariant (*i.e.* observer independent) way. The theory of relativity was created in two steps: the special theory of relativity, which reformulates and modifies the Newtonian equations of motion at so-called relativistic speeds (near the speed of light) was created in 1905, while later, in 1915, Einstein proposed the general theory of relativity, which is a consistent relativistic theory of gravity. To do so Einstein had to introduce the concept of curved spacetime and interpret gravity as an effect of geometric spacetime distortions, *i.e.* deviations from the flat geometry of special relativity. Furthermore, the curvature of spacetime is related to the stress-energy-momentum tensor of the matter in spacetime via the Einstein equation. In this way, the spacetime geometry is essentially tied up with the matter content in the spacetime.

On the other hand, quantum mechanics is the theory that describes the behaviour of matter (particles and light) at atomic and sub-atomic scales. According to this theory physical quantities do not have the continuous classical behaviour that we are used to, but turn out to be quantised, *i.e.* they can take only discrete values. Quantum mechanics contains also the famous Heisenberg uncertainty relations, according to which physical quantities (observables) cannot be determined with the same accuracy as in classical mechanics, an effect due to quantum fluctuations.

However relativity theory and quantum mechanics are not compatible with each other. In other words, relativity theory does not incorporate quantum effects in describing elementary particles at relativistic speeds, while quantum mechanics lacks a covariant (relativistically invariant) formulation. Ideally, if there exists a unified picture of Nature, there must exist then some underlying theory that unifies both theories and of which both relativity theory and quantum theory are special limits. Finding the unified picture has become the main aim of current research in theoretical physics.

The first successful attempt in achieving this goal was the unification of special relativity theory with quantum mechanics. The result was quantum field theory, the relativistic quantum theory of elementary particles, which uses ingredients from both relativity and quantum mechanics. From relativity we have the equivalence of mass and energy, while in quantum mechanics we have the possibility of creating energy “out of nothing” - even if for a very short interval of time. As a result, the vacuum is no more an empty place, particles being created and destroyed continuously. This fact is one of the most surprising consequences of quantum field theory. This synthesis entails a shift of the viewpoint from the non-relativistic quantum mechanic framework, (as described by the Schrödinger equation, where one quantises a single particle in a classical potential) to a quantum field theory, (where one identifies the particles with the modes of a field and quantises the field itself).This procedure goes under the name of ‘second quantisation’.

At present, incorporating the concept of gauge invariance, quantum field theory - in the form of the so-called Standard Model - is one of the most empirically accurate theories of Nature that we have. However, it is now believed that the Standard Model is not the end of the story. Despite the fact that it offers both a successful quantum and relativistic-invariant description of the strong and electro-weak interactions, it does not take the gravitational interaction into account. General relativity is incompatible with quantum field theory and at present we still do not have a good theory of quantum gravity. Even if from experimental point of view this is not really a concern at the present energy scales that we can probe in our experiments, one still hopes that one can find a unified conceptual frame in which quantum field theory and general relativity combine in a consistent manner. The search for this has become the most important challenge in theoretical physics.

The main problem in dealing with the quantum theory of the gravitational field is that gravity is not renormalizable. This means that usually, when we want to calculate the amplitudes for different processes to take place, we get infinite results (diverging integrals). Now, in a renormalizable theory, for instance in case of the gauge theories in particle physics, the number of these diverging amplitudes is finite and there is a consistent procedure to remove these infinities. However, in a nonrenormalizable theory, the number of divergent amplitudes is infinite and this makes the renormalization procedures useless. Besides the technical problem of renormalization, quantum gravity suffers also from conceptual problems. The most important one is related to the fact that gravity is a consequence of the curved geometry of space-time. If we want to quantise gravity, we must be prepared then to interpret this in terms of the quantisation of space-time.

In recent years we have witnessed remarkable developments in string theory and nowadays there is a growing consensus that string theory provides us with a consistent framework in which both a quantum theory of gravity and also the unification of all fundamental forces can be achieved. In light of these advances, we shall present in the next section a short introduction to this theory.

1.1 An introduction to String theory

The basic idea of string theory is that all matter is made up of very tiny strings. In other words, the elementary particles, instead of being point-like, are different vibrational modes of a string. As the string propagates through space-time, it will sweep out a two-dimensional surface, called the world-sheet of the string. In analogy with a point-like particle, we can write down an action to describe the string dynamics, that is proportional to the area of the world-sheet of the string. The constant of proportionality is called the string tension and it has units of $(mass)^2$. Usually one uses another parameter related to the string tension by the formula $T = \frac{1}{2\pi\alpha'}$, where α' is a quantity with the dimension $(length)^2$. This parameter introduces a fundamental length scale $\sqrt{\alpha'}$, which is the string scale, where the stringy effects become important. This parameter will also fix the typical size of a string to be extremely small (being of the order of the so-called Planck length, $10^{-33} cm$) so that at "low" energies all particles look like points and ordinary point particle theories (quantum field theories) provide an adequate description. Only when we magnify to scales resolving $10^{-33}cm$ do the particles look like strings; this fact explains why in all present day experiments the particles appear to be point-like objects.

String theory is a candidate for the unification of all fundamental forces and elementary particles. Unification is achieved in that all particles are actually vibrational modes of a single kind of string. Now, one mode of oscillation of the string is a massless, spin two state that has all the properties that can be identified as the graviton - the mediator of the gravitational interaction. Another important point to note is that a consistent quantum theory of strings is divergence-free. This means that string theory automatically gives us a finite quantum theory of gravity. However, unlike the usual point particle theories, we cannot construct a consistent string theory in any arbitrary space-time dimension [70, 71]. There exists a maximum allowable space-time dimension above which it is impossible to make the theory consistent. The reason is that a consistent string theory is conformally invariant on the string world-sheet and there is no known procedure to cancel or eliminate the conformal anomaly above a critical dimension. This critical dimension is essentially determined by the number of supersymmetries on the string world-sheet. The requirement of conformal invariance limits the number of allowed supersymmetries on the string world-sheet: one can have only theories with $N = 0, 1, 2, 4$ local supersymmetries. For the bosonic string ($N = 0$) the critical dimension is found to be $D = 26$. However the bosonic string theory suffers from the presence of a tachyon state in its spectrum. Another problem with the bosonic string is the fact that it does not contain fermions. Hence, the natural generalisation is to consider theories with supersymmetry, by introducing fermions on the world-sheet of the string. For a supersymmetric string theory with $N = 1$ the requirement of conformal invariance will fix the critical dimension to be $D = 10$. In case of a theory with $N = 2$ local supersymmetries on the string world-sheet, conformal invariance of the theory will fix the critical dimension to be $D = 4$, however the signature of the space-time metric is wrong: two of the four dimensions should be timelike! Furthermore, the theory with $N = 4$ local supersymmetries is completely unphysical, since the critical dimension in this case should be negative!

In conclusion, one should consider only theories with $N = 1$. The critical dimension of the space-time is then $D = 10$ and we are dealing with a supersymmetric string theory in ten dimensions. However, it turns out that instead of only one consistent superstring theory in ten dimensions, we actually have five different consistent superstring theories. They are named Type IIA, Type IIB, Type I, $SO(32)$ Heterotic and $E_8 \times E_8$ Heterotic. In the limit $\alpha' \to 0$ (the so-called "zero slope limit") the string tension T becomes infinite, so the size of the string shrinks to zero and we approximate the string by a point particle. After we impose the condition that the theory is to be free of any anomalies, the constraints on the theory can be interpreted as field equations derived from an effective action that will be the low-energy effective action of the respective superstring theory. The low-energy effective Lagrangians of these theories correspond to ten-dimensional supergravity theories. For instance, Type IIA corresponds to $N = 2$ nonchiral supergravity, type IIB reduces to $N = 2$ chiral supergravity, $E_8 \times E_8$ heterotic string theory reduces to $N = 1$ supergravity coupled to an $E_8 \times E_8$ Yang-Mills multiplet, $SO(32)$ heterotic string theory reduces to $N = 1$ supergravity coupled to an $SO(32)$ Yang-Mills multiplet, while type I superstring theory, which contains both open and closed strings, reduces in ten dimensions to $N = 1$ supergravity theory coupled to an $SO(32)$ Yang-Mills multiplet. We will describe in more detail the effective actions of these five superstring theories in Appendix A.

1.1.1 M-theory

As we have seen above, the requirement of quantum consistency fixes the dimension of the space-time to be ten. On the other hand, supersymmetry places an upper limit on the dimensionality of space-time [118]. If one requires gravity to be unique in four dimensions, imposing the condition that the helicities of the particles to be at most two, then the number N of supersymmetries allowed in the theory cannot exceed $N = 8$. This implies that the dimension of space-time is at most eleven. Indeed, there exists in 11 dimensions a unique $N = 1$ supergravity theory. Its action was known for a long time, but it was dismissed because the theory is nonrenormalizable and cannot accommodate chirality. This eleven dimensional supergravity is apparently not related to the superstring theories in ten dimensions. However, it has been known for a long time that the Kaluza-Klein reduction of the eleven-dimensional supergravity on a circle is Type IIA supergravity theory, which is the low-energy effective action of Type IIA superstring theory. In the limit where the string coupling constant becomes large (*i.e.* in the strong coupling regime of the Type

IIA supergravity theory), the radius of the eleventh dimension enlarges such that the description of the Type IIA supergravity theory, in this regime, effectively becomes eleven dimensional. On the other hand, when the string coupling constant of the theory is small enough, the radius of the eleventh dimension becomes very small. It is this regime defined by the perturbation theory (with the string coupling constant small) in which the Type IIA supergravity theory senses only ten dimensions.

The significance of this result was singled out in a series of observations made by Witten [139]. He argued that the differences among the five superstring theories in ten dimensions are just artifacts of the perturbative regime in which they are defined and that actually there exists one unique theory, M-theory, which unifies all the superstring theories in ten dimensions. This unified theory is essentially an eleven dimensional theory, which should correspond to the strong coupling regime of the Type IIA superstring theory. The low-energy effective action of M-theory would correspond then precisely with the eleven dimensional supergravity theory. One should note that for the present conjecture one has only a low-energy description of the theory, since until now one does not know yet the quantum theory that reduces to eleven dimensional supergravity at low energies. The unification cannot occur at the perturbative level (and it is exactly at this level where the five superstring theories appear to be different). If we go beyond the perturbative level, by considering non-perturbative effects, we find connections among the various superstring theories and these connections are expressed as dualities among the various superstring theories. In general, a duality between two different theories indicates that the physics described by those theories is actually the same. However, explicitly proving the existence of a duality is usually an impossible task to carry out in practice since the duality map often connects the strong respectively the weak coupling regimes of the theories in cause. One could then prove the existence of the duality only if one knew the non-perturbative behaviour of the respective theories. There are not so many theories whose non-perturbative behaviour can be discerned, and certainly this is not the case for the five superstring string theories in ten dimensions. This is why the dualities among string theories still remain at the stage of duality conjectures. However, if one has enough evidence supporting the duality conjectures, one can assume the duality between theories as a working hypothesis and, by studying the perturbative regime of one theory, one can get some hints about non-perturbative effects in the dual theory. It is only by the exclusive use of this duality concept that it has been possible to shed some light on the non-perturbative aspects of string theory. That is why the discovery of the various duality relations among the string theories has received the name of the 'Second Superstring Revolution'.

1.1.2 String dualities

String dualities can be classified as T-dualities, S-dualities and U-dualities. The first string duality to be discovered was T-duality in the context of heterotic string theory. It has been noticed by Narain [119] and Ginsparg [69] that the two heterotic superstring theories, heterotic $SO(32)$ and heterotic $E_8 \times E_8$, are equivalent when compactified on a circle. T-duality of string theory is a perturbative duality, in the sense that it relates string theories with the same string coupling. It holds order by order in the perturbative string theory and it has been extended to hold in the non-perturbative formulation of the string theory as well. Generally speaking, T-duality will map a theory with a large target space volume to a theory with a small target space volume. Another interesting example is the T-duality of the Type II superstring theories: Type IIA superstring theory compactified on a circle of radius R is dual to the Type IIB superstring theory compactified on a circle with radius proportional to $1/R$.

S-duality is a non-perturbative duality of string theory that transforms the string coupling to its inverse, while the moduli fields of the theory remain fixed. Generally speaking S-duality will relate the weak coupling regime of one theory with the strong coupling regime of the same (or different) string theory. An example of S-duality is the duality between the Type I superstring theory and the $SO(32)$ heterotic superstring theory in ten dimensions. Another example of weak-strong coupling duality is the $SL(2,Z)$ self-duality of the Type II superstring theory.

Finally, U-duality is a combination of T-duality and S-duality into a larger group of transformations. For Type II theories there is a T-duality symmetry group $SO(10-D, 10-D, Z)$ when we compactify the theory on a T^{10-D} torus and an $SL(2,Z)$ non-perturbative symmetry group that corresponds to the $SL(2,Z)$ symmetry of the Type IIB theory, which survives in the dimensional reduction process. Then the conjectured U-duality group will be generated by these two non-commuting groups. The exact form of the U-duality groups for the Type II string theory was conjectured in [94] to correspond to a discrete subgroup of the exceptional Cremmer-Julia symmetry groups $E_{11-D(11-D)}$ of maximal supergravities. The Cremmer-Julia symmetry groups appear in the dimensional reduction of eleven-dimensional supergravity to D dimensions, after we Poincaré dualise all fields with degree greater than $D/2$.

1.1.3 The AdS/CFT conjecture

It is now widely believed that a quantum theory of gravity should incorporate in some way the concept of holography. Roughly speaking, the idea of holography is to relate the quantum physics on a spacetime boundary to the classical geometrical properties of the spacetime. Originally the aim of this approach was to understand the quantum properties of black holes: since Bekenstein and Hawking found that the entropy of a black hole is proportional to its area this seemed to suggest that the effective QFT that would describe the microscopic degrees of freedom of a quantum black hole should live on its horizon, *i.e.* the local degrees of freedom are described by a theory in one lower dimension. In an extended version, the holographic principle states that for any Lorentzian manifold it should be possible to find a submanifold (a holographic screen) where all the quantum degrees of freedom are present.

In particular, as a candidate for a theory of quantum gravity string theory should involve holography. The AdS/CFT correspondence, proposed by Maldacena in 1997 [107], is a remarkable realization of this idea: here the screen is identified as the AdS boundary and duality asserts that conformal quantum field theories (CFT) living on this boundary provide a holographic description of the string theory in the AdS bulk. In fact AdS/CFT provides us with an explicit dictionary relating a theory of gravity in AdS background with a quantum gauge theory in lower dimensions. The entropy of a black hole in this background would be related then to the thermodynamic entropy of the boundary gauge theory at a finite temperature - the same with the Hawking temperature of the black hole.

A quite general phenomenon of the gravity/gauge theory correspondence is the UV/IR connection, *i.e.* the UV divergences in the field theory are related to IR divergences on the gravitational side. Regularizing these divergences in the field theory side explicitly provides us with a set of boundary counterterms in the gravity side. In recent years such counterterms proved to be an invaluable tool in analyzing the asymptotically AdS geometries and defining their conserved charges. We will consider these counterterms more closely in Chapter 4.

Recently, much attention has been devoted to a study of the de Sitter space (dS) and asymptotically dS spaces. This was partly motivated by recent results that seem to indicate that the Universe is currently undergoing a period of accelerated expansion and therefore it might approach a dS space in the far-future. Motivated by the analogy with the AdS/CFT correspondence there has also been advanced a proposal of a similar correspondence in the dS case [131]. This would entail a duality between quantum gravity on a dS background and a Euclidean conformal field theory on the boundary of the dS space. However, the status of the dS/CFT conjecture is still uncertain. The principal difficulty resides in the fact that no non-singular compactification of M/String-theory give rise to de Sitter spacetime. It is however possible to embed dS into a rather peculiar string theory, IIB* that is obtained by a timelike T-duality from the conventional Type IIA theory [93]. We will actually encounter the six-dimensional versions of Type II* theories in Chapters 6 and 7; for this purpose we gathered some of the details of these theories in Appendix C.

1.2 Overview of the thesis

Ever since the formulation of General Relativity, exact solutions have played an integral part in our understanding of the nature of spacetime. For example, much of our understanding of black hole thermodynamics and inflation were possible only with the discovery of the Kerr-Newman and FRW solutions respectively. Given the importance of such exact solutions, there is a corresponding impetus to derive new solutions which upon analysis would yield further insight into our universe. Since Einstein's equations in their unadulterated form consist of a series of coupled non-linear differential equations, obtaining solutions by hand is intractable unless some kind of simplifying symmetry is imposed in the ansatz. This motivated the development of many ingenious and powerful strategies to derive solutions to Einstein's equations. Four-dimensional solutions of the Einstein's equations have been studied extensively for many decades in the last century and there actually exists an encyclopedia of all the known four-dimensional solutions as well as an overview of the various solution-generating techniques [77]. In recent times, in view of the discovery of the supergravity theories and the remarkable advances of superstring theory, it has become of utmost importance to find and study solutions of Einstein's equations or the coupled Einstein-matter systems in higher dimensions. Various classes of solutions have been found, including black holes that generalise the four-dimensional Schwarzschild, Reissner-Nordström or Kerr solutions [117, 67].

The primary purpose of this thesis is to present and study the properties of a class of higher-dimensional generalisations of the so-called NUT-charged solutions in various backgrounds. Intuitively the NUT charge corresponds to a magnetic type of mass. The first solution in four dimensions describing such an object was presented in ref. [132, 120]. Although the Taub-NUT solution is not asymptotically flat (AF), it can be regarded as asymptotically locally flat (ALF). The difference appears in the topology of the boundary at infinity. If we consider as an example of an AF space the Euclidean version of the Schwarzschild solution then the boundary at infinity is simply the product $S^2 \times S^1$. By contrast, in the presence of a NUT charge, the spacetime has as boundary at infinity a twisted S^1 bundle over S^2. Only locally we can untwist the bundle structure to obtain the form of an AF spacetime. The bundles at infinity are labelled by the first Chern number, which is in fact proportional to the NUT charge [84]. The presence of a NUT charge induces a so-called Misner singularity in the metric, analogous to a 'Dirac string' in electromagnetism [115]. This singularity is only a coordinate singularity and can be removed by choosing appropriate coordinate patches. However, expunging this singularity comes at a price: in general we must make coordinate identifications in the spacetime that yield closed timelike curves in certain regions. We will later see that the higher-dimensional NUT-charged solutions share many of the remarkable properties of their four-dimensional versions.

Another motivation for our work comes from a more general study of gravitational entropy in four and higher dimensions [83]. Ever since the seminal papers of Bekenstein and Hawking, it has been known that the entropy of a black-hole is proportional to the area of the horizon. This relationship can be generalised to a wider class of spacetimes, namely those whose Euclidean sections cannot be everywhere foliated by surfaces of constant (Euclidean) time. These situations can occur if the Euclidean spacetime has non-trivial topology: the inability to foliate the spacetime leads to a breakdown of the concept of unitary Hamiltonian evolution, and mixed states with entropy will arise [84, 83]. Spacetimes that carry a NUT charge are in this broader class and in this regard obtaining new exact NUT-charged solutions and studying their thermodynamic properties is worthwhile.

The structure of this thesis is as follows: in the next two chapters (and also in the last one) we present new classes of NUT-charged spaces in four dimensions and higher, while in the remainder of the thesis we make a study of their properties and their possible applications.

More specifically, we construct in Chapter 2 new families of axisymmetric vacuum solutions in four dimensions using a solution-generating technique based on the hidden $SL(2,R)$ symmetry of the effective Lagrangian. Our method is based on the simple observation that a static axisymmetric metric as written in Weyl-Papapetrou form exhibits a simple 'scaling' symmetry that allows one to generate a family of new static vacuum axisymmetric solutions, indexed by a real parameter. We also make use of a charging method for static vacuum metrics, which dates back to Weyl [137]. We demonstrate a simpler alternative derivation of this transformation by using a $SL(2,R)$ symmetry of the reduced Lagrangian in three dimensions. However, unlike previous applications of this transformation, we show that with our simplified mapping and by combining this charging method with the scaling property, one is able to generate new solutions starting with the Schwarzschild solution as our seed metric. In particular, using the Scharzschild solution as a seed we are able to obtain the Zipoy-Voorhees generalisation of both the Taub-NUT solution and the Eguchi-Hanson soliton. Such families of solutions are parameterized by the value of a real parameter γ. The $\gamma = 1$ member of these families reduces to the Taub-NUT/Eguchi-Hanson solution. Finally, in the second part of Chapter 2, using the C-metric as a seed, we will be able to obtain an accelerating version of all the above solutions. Again, such families of solutions are parameterized by the value of a real parameter γ. The $\gamma = 1$ member of this family yields a new solution, which we interpreted as the accelerating version of the Taub-NUT solution.

We also discuss in more detail the interesting features of the Taub-NUT geometry. We will see that this geometry can be understood as a radial extension of a circle fibration over the sphere S^2. This observation will prove to be essential in the later chapters, where we will see that the higher dimensional NUT-charged spaces can be thought of in a very similar way as radial extensions of circle fibrations over products of Einsten-Kähler manifolds. In particular, we will provide the direct generalisation of such metrics to even and odd dimensions. The novelty of our solutions is that by associating a NUT charge N with every such Einsten-Kähler factor of the base space we obtain higher dimensional generalisations of Taub-NUT spaces that can have quite generally multiple NUT parameters. In our work we give the Lorentzian form of the solutions however, in order to understand the singularity structure of these spaces we have to concentrate mainly on a study of their Euclidean sections. In most of the cases the Euclidean section is simply obtained using the analytic continuations $t \rightarrow it$ and $N_j \rightarrow in_j$. To render such Euclidean metrics regular one follows a procedure as in Ref.[122] in which the basic idea is to turn all the singularities appearing in the metric into removable coordinate singularities. For generic values of the parameters the metrics are singular – it is only for careful choices of the parameters that they become regular. Having presented in Chapter 3 the most general forms of the non-rotating Taub-NUT spaces in higher dimensions in odd and also even dimensions, in the rest of this thesis we address some of their physical properties and possible applications.

In Chapter 4 we briefly review the path-integral approach to quantum gravity and its relationship to gravitational thermodynamics for asymptotically flat or asymptotically (A)dS spacetimes. In this approach, the partition function for the gravitational field is defined by a sum over all smooth Euclidean geometries which are periodic with a period β in imaginary time. The path-integral is computed by using the saddle point approximation in which one considers that the dominant contributions to the path-integral will come from metrics near the classical solutions of Euclidean Einstein's equations with the given boundary conditions. In the semiclassical limit this yields a relationship between gravitational entropy and other relevant thermodynamic quantities, such as mass, angular momentum, and other conserved charges. In particular, the gravitational entropy can then be regarded as arising from the the quantum statistical relation (or the generalised Gibbs-Duhem relation) applied to the path-integral formulation of quantum gravity. In general, for spaces that are asymptotically AdS or flat, we can compute the partition function using an analytic continuation of the action by rotating the time axis so that $t \rightarrow -iT$ in order to obtain a Euclidean signature metric. The positivity of the Euclidean action ensures a convergent path integral with which one can carry out any calculations (of action, entropy, etc.). The presumed physical interpretation of the results is then obtained by rotation back to a Lorentzian signature at the end of the calculation. However, for spaces that are asymptotically (A)dS, we describe two approaches toward doing thermodynamics. In one approach (referred to as the $\mathbb{R}$-approach), the analysis is carried out using the unmodified metric with Lorentzian signature; no analytic continuation is performed on the coordinates and/or the parameters that appear in the metric. In the alternative $\mathbb{C}$-approach one deals with an analytically continued version of the metric and at the end of the computation all the final results are analytically continued back to the Lorentzian sector. The $\mathbb{C}$-approach is closest to the more traditional method of obtaining Euclidean sections for asymptotically flat and AdS spacetime. The $\mathbb{R}$-approach refers to the Lorentzian section, and makes use of the path integral formalism only insofar as the generalised Gibbs-Duhem relation is employed.

The main result in Chapter 5 is the demonstration that the $\mathbb{R}$ and $\mathbb{C}$-approaches are equivalent, in the sense that we can start from the $\mathbb{C}$-approach results and derive by consistent analytic continuations (*i.e.* using a well-defined prescription for performing the analytic continuations) all the results from the $\mathbb{R}$-approach. There are no a-priori obstacles in taking the opposite view, in which the $\mathbb{C}$-approach results are derived from the respective $\mathbb{R}$-approach results. However, one could still argue that the $\mathbb{C}$-approach is the more basic one, as in it the periodicity conditions appear more naturally than in the $\mathbb{R}$-approach. As specific examples, we consider some of the NUT-charged spaces presented Chapter 3 and study their thermodynamic properties with some very interesting results.

In Chapter 6 we describe another application of the Taub-NUT-Eguchi-Hanson solitons in the construction of Kaluza-Klein (KK) magnetic monopoles. We begin by reviewing how the flat KK monopole can be obtained from the four dimensional Taub-Nut solution. We also briefly discuss the features of the monopole solution obtained by using the Euclidean Taub-Bolt solution. At this point we consider the solution obtained by dimensionally reducing an Eguchi-Hanson-like monopole and we prove that even if the four-dimensional metric is non-asymptotically flat, its geometry is nonetheless U-dual to that of a Taub-Bolt monopole. We next present a new metric ansatz which is a solution of vacuum Einstein's equations with cosmological constant in five dimensions and we perform a Kaluza-Klein reduction to obtain a new four-dimensional monopole solution. In the remaining sections in this chapter we consider similar monopole solutions in higher dimensions and we also perform Kaluza-Klein sphere reductions to four dimensions. In six dimensions we apply spatial and timelike Hopf-dualities to generate new solutions.

In Chapter 7 we describe new time-dependent bubble solutions that can be obtained from the higher-dimensional NUT-charged spaces by analytical continuation. This generalises previous studies of four-dimensional nutty bubbles. One five-dimensional locally asymptotically AdS solution in particular has a special conformal boundary structure of $AdS_3 \times S^1$. We compute its boundary stress tensor and related it to the properties of the dual field theory. Interestingly enough, we also find consistent six-dimensional bubble solutions that have only one timelike direction. The existence of such spacetimes with non-trivial topology is closely related to the existence of the Taub-NUT(-AdS) solutions with more than one NUT charge. Finally, we begin an investigation of generating new solutions from Taub-NUT spacetimes and 'nuttier' bubbles. Using the so-called Hopf duality, we provided new explicit time-dependent backgrounds in six dimensions.

Finally, in Chapter 8 we briefly present another generalisation of the NUT-charged spaces as solutions in Einstein-Maxwell theory. However, for space reasons, we confined ourselves to perform a simple singularity analysis of such metrics, leaving a full thermodynamical description for further work.

The thesis ends with a concluding chapter, highlighting the significance of the work carried out and highlighting possible directions for further research.

Notation

Our conventions generally are the ones from Wald's textbook [136]. We use $(-, +, ..., +)$ for the (Lorentzian) signature of the metric; in even D dimensions our metrics will be solutions of the vacuum Einstein field equations with cosmological constant $\Lambda = \pm\frac{(D-1)(D-2)}{2l^2}$, which can be expressed in the form $G_{ij} + \Lambda g_{ij} = 0$ or in the equivalent form $R_{ij} = \lambda g_{ij}$, where $\lambda = \frac{2\Lambda}{D-2} = \pm\frac{D-1}{l^2}$. By an abuse of terminology we will still call λ cosmological constant.

Chapter 2

Accelerating Taub-NUT and Eguchi-Hanson solitons in four dimensions

In this chapter we construct new solutions of the vacuum Einstein field equations in four dimensions via a solution generating method utilizing the $\mathrm{SL}(2, R)$ symmetry of the dimensionally reduced Lagrangian. In particular, using the Schwarzschild solution as the initial seed, we obtain on the way a new family of solutions, describing the generalisation *à la* Zipoy-Voorhees of the four dimensional Taub-NUT solution and also non-trivial generalisation of the four-dimensional Eguchi-Hanson solitons. Much like the original Zipoy-Voorhees solution, such metrics are parameterized by a real number γ. For $\gamma = 1$ we recover the usual Taub-NUT/Eguchi-Hanson solitons and, for higher positive integer values of γ, they can be interpreted as the 'superposition' of γ NUT-charged objects/solitons. This will be our first encounter with a NUT charged space and we will pause for a moment to discuss its properties in more detail.

In the second part of this chapter, we apply the same solution-generating method this time to the so-called C-metric. This metric is known to describe two black holes uniformly accelerated in opposite directions where the source of acceleration is a strut in between pushing apart the black holes or alternatively two strings pulling on the black holes from infinity (for a recent discussion of its properties see for instance [90, 91, 73, 75, 74, 72] and the references therein). For our purpose, it will prove more convenient to use the form of the C-metric given in [90] which has been cast into a nice factorized form. Using a similar procedure we finally obtain new vacuum solutions that we interpret as describing the accelerating Zipoy-Vorhees-like family of Taub-NUT solutions, respectively Eguchi-Hanson instantons. Finally, we focus on a particular member of this family and show that it represents an accelerating version of the Taub-NUT solution.

The structure of this chapter is as follows. In section 2.1 we describe our solution generating technique, which maps a static axisymmetric solution in vacuum to a new stationary vacuum solution of Einstein's gravity in four dimensions. We then apply this transformation on the Schwarzschild solution in section 2.2 and analyse the properties of the generated solutions. Next, we apply the same transformation technique on another seed, namely on the C-metric in section 2.4 and we consider more closely the properties of the generated accelerating Taub-NUT solution in section 2.5.

2.1 The solution-generating method

Of particular importance for the present work is a special type of simplifying ansatz, the *static* axisymmetric Weyl-Papapetrou metric, which was first proposed by Weyl in [137]

$$ds_4^2 = -e^{-\psi}dt^2 + e^{\psi}\left[e^{2\mu}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right]. \tag{2.1}$$

The metric is specified by the values of two functions ψ and μ, which are functions of the canonical Weyl variables ρ and z. Starting with a static vacuum axisymmetric solution as in (2.1), let us consider its dimensional reduction along the timelike direction down to three dimensions. The reduced Lagrangian can be written as:

$$\mathcal{L}_3 = eR - \frac{1}{2}e(\partial\psi)^2, \tag{2.2}$$

where we denote $e = \sqrt{g}$[1]. Then the equation of motion for ψ is readily seen to be $\Delta\psi = 0$, where Δ is the Laplacian constructed using the three-dimensional metric:

$$ds^2 = e^{2\mu}(d\rho^2 + dz^2) + \rho^2 d\varphi^2. \tag{2.3}$$

[1]More generally, in the remaining of this thesis we shall use $e = \sqrt{|g|}$, R for the Ricci scalar when writing down a Lagrangian, being understood that these quantities are defined using the metric in the dimension in which the respective Lagrangian is defined.

Now the key observation is that $\Delta\psi = e^{-2\mu}\Delta_{\mu=0}\psi$, where $\Delta_{\mu=0}$ is the Laplacian computed for a flat three-dimensional Euclidean metric, which corresponds to setting $\mu = 0$ in (2.3). Therefore any arbitrary solution of Laplace's equation in flat three dimensional space is automatically a valid solution of Laplace's equation in the curved background (2.3). Once we know ψ, the remaining function $\mu(\rho, z)$ is found by performing a simple line-integral using the relations:

$$\partial_z\mu = \frac{\rho}{2}\partial_\rho\psi\partial_z\psi, \qquad \partial_\rho\mu = \frac{\rho}{4}\left[(\partial_\rho\psi)^2 - (\partial_z\psi)^2\right]. \tag{2.4}$$

and the static axisymmetric Einstein's equations are now reduced to Laplace's equation on flat space.

Due to the linearity of the equation for ψ, construction of multi-black hole versions is easily carried out. The Weyl formalism has been recently extended to higher dimensions by Emparan and Reall [55] and the same line of thought can be used for the corresponding higher dimensional axisymmetric metrics.

Given the simplifications introduced by the above axisymmetric ansatz, one now has two choices. One may either try to solve the differential equations directly, or, in more general cases, try to further exploit the hidden symmetries of the dimensionally reduced Lagrangians and generate solutions using pre-existing solutions as seeds.

2.1.1 The 'scaling' symmetry

Consider for instance the following 'scaling' symmetry of the field equations: given a vacuum static solution described by the pair of functions (ψ, μ) then it is easily seen from (2.4) that the pair $(\gamma\psi, \gamma^2\mu)$ will describe new vacuum static axisymmetric solution of the field equations, where γ is any real parameter.

As an example of this scaling symmetry, let us apply it to a particular simple solution. Our seed metric will be the Schwarzschild solution:

$$ds^2 \;\; = \;\; -\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{2.5}$$

For our purposes, we have to convert it first to the Weyl form as:

$$\begin{aligned} ds^2 &= -e^{-\psi}dt^2 + e^{\psi}\left[e^{2\mu}(d\rho^2+dz^2)+\rho^2 d\varphi^2\right], \\ e^{-\psi} &= 1-\frac{2m}{r}, \qquad e^{2\mu} = \frac{r(r-2m)}{(r-m)^2-m^2\cos^2\theta}, \\ d\rho^2+dz^2 &= \frac{(r-m)^2-m^2\cos^2\theta}{r(r-2m)}\left(dr^2+r(r-2m)d\theta^2\right), \end{aligned} \tag{2.6}$$

where $\rho = \sqrt{r(r-2m)}\sin\theta$ and $z=(r-m)\cos\theta$ are the canonical Weyl coordinates.

Using now the scaling symmetry described above, we can write down a new vacuum solution of Einstein's field equations by taking $\psi \to \gamma\psi$ and $\mu \to \gamma^2\mu$. We obtain:

$$\begin{aligned} ds^2 &= e^{-\psi}dt^2 + e^{\psi}\left[e^{2\mu}(d\rho^2+dz^2)+\rho^2 d\varphi^2\right], \\ e^{-\psi} &= \left(1-\frac{2m}{r}\right)^{\gamma}, \qquad e^{2\mu} = \left(\frac{r(r-2m)}{(r-m)^2-m^2\cos^2\theta}\right)^{\gamma^2}, \end{aligned} \tag{2.7}$$

which is easily seen to be the Zipoy-Voorhees solution [142, 135]. For integer values of γ this metric describes the superposition of γ black holes.

2.1.2 Weyl's charging method: the $SL(2,R)$ approach

There also exist transformations similar to the Ehlers-Harrison transformation for the Ernst formalism [78, 57, 77] which map static vacuum solutions into stationary Einstein-Maxwell solutions. For the Weyl-Papapetrou ansatz, it has been long known that a transformation already exists that brings a static, axisymmetric vacuum solution to a non-trivial class of static solutions in Einstein Maxwell theory [59]. In particular, the Schwarzschild solution can be transformed into the Reissner-Nordström solution. In this section, we will demonstrate a simpler alternative derivation of this transformation using a $SL(2,R)$ symmetry of the reduced Lagrangian in three dimensions. However, unlike previous applications of this transformation, we show that with our simplified mapping, we are able to generate new solutions starting with the Schwarzschild solution as our seed metric.

We start with Einstein-Maxwell theory in four dimensions described by the Lagrangian:

$$\mathcal{L}_4 = eR - \frac{1}{4}eF_{(2)}^2, \tag{2.8}$$

where R is the Ricci scalar, $F_{(2)} = dA_{(1)}$ is the electromagnetic field strength which only has an electric component $A_{(1)} = \chi dt$ and we denote $e = \sqrt{-g}$. Let us first consider the dimensional reduction of the four-dimensional Lagrangian (2.8) to three dimensions on a timelike coordinate using the static Kaluza-Klein ansatz:

$$ds_4^2 = -e^{-\phi}dt^2 + e^{\phi}ds_3^2. \tag{2.9}$$

The reduced Lagrangian in three dimensions is then

$$\mathcal{L}_3 \;\; = \;\; eR - \frac{1}{2}e(\partial\phi)^2 + \frac{1}{2}ee^{\phi}(\partial\chi)^2. \tag{2.10}$$

Let us notice now that if we define the matrix:

$$\mathcal{M} = \begin{pmatrix} e^{\frac{\phi}{2}} & \frac{\chi}{2}e^{\frac{\phi}{2}} \\ \frac{\chi}{2}e^{\frac{\phi}{2}} & -e^{-\frac{\phi}{2}} + \frac{\chi^2}{4}e^{\frac{\phi}{2}} \end{pmatrix} \tag{2.11}$$

then the three-dimensional Lagrangian can be cast into the following form:

$$\mathcal{L}_3 \;\; = \;\; eR + e\,\mathrm{tr}\big[\partial\mathcal{M}^{-1}\partial\mathcal{M}\big], \tag{2.12}$$

The reduced Lagrangian is then manifestly invariant under general $SL(2,R)$ transformations if we consider the following transformation laws for the three-dimensional fields:

$$g_{\mu\nu} \to g_{\mu\nu}, \qquad \mathcal{M} \to \Omega^T\mathcal{M}\Omega, \qquad \Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1. \tag{2.13}$$

Starting now with a static axisymmetric vacuum solution described by the metric:

$$ds^2 \;\; = \;\; -e^{-\psi}dt^2 + e^{\psi}ds_3^2 \tag{2.14}$$

then performing the dimensional reduction on the timelike direction down to three dimensions and applying a general $SL(2,R)$ transformation parameterized as above, we obtain a static axisymmetric electrically charged solution of Einstein-Maxwell field equations, described by the fields:

$$\begin{aligned} ds^2 &= -e^{-\phi}dt^2 + e^{\phi}ds_3^2, \qquad A_{(1)} = \chi dt, \\ e^{\phi} &= e^{\psi}\frac{\left(1-\delta e^{-\psi}\right)^2}{4C^2\delta}, \qquad \chi = \frac{4C\delta}{e^{\psi}-\delta}, \end{aligned} \tag{2.15}$$

where, in terms of the parameters appearing in Ω, the new constants δ and C can be expressed as $\delta = c^2/a^2$ and $C = 1/(2ac)$. Note that in the limit in which $\Omega = I_2$, *i.e.* $c \to 0$ and $a = 1$, we have $\delta \to 0$ simultaneously with $C \to \pm\infty$ such that the product $C^2\delta \to 1/4$ remains constant.

As an example of this charging technique, let us generate the Reissner-Nordström solution starting from the Schwarzschild metric (2.5). The final solution can be written in the form:

$$\begin{aligned} ds^2 &= -\frac{4C^2\delta r(r-2m)}{((1-\delta)r+2\delta m)^2}dt^2 + \frac{((1-\delta)r+2\delta m)^2}{4C^2\delta r(r-2m)}dr^2 + \frac{((1-\delta)r+2\delta m)^2}{4C^2\delta}(d\theta^2+\sin^2\theta d\varphi^2), \\ A_{(1)} &= \frac{2C\left((1+\delta)r-2\delta m\right)}{(1-\delta)r+2\delta m}dt. \end{aligned} \tag{2.16}$$

For generic values $\delta \neq 1$ we can easily perform a redefinition of the radial coordinate, together with an appropriate constant scaling of the timelike coordinate and cast the solution into the usual Reissner-Nordström form. The electric charge is $Q = m/C$, while the mass of the solution is $M = (1+\delta)Q/(2\sqrt{\delta})$. Note that $\delta = 1$ is a special case as it leads to the Bertotti-Robinson metric [19, 127] and it describes therefore the extremally charged Reissner-Nordström solution for which $M = Q$.

At this point let us mention two important points regarding the above charging technique. First, this method is not restricted only to metrics with axisymmetric symmetry; it can be extended to any general static vacuum solution of Einstein's field equations. Second, we can easily consider a similar method to generate magnetically charged static solutions out of axial vacuum metrics. Consider for instance a magnetically charged solution described by the metric:

$$ds_4^2 = e^{-\phi}d\varphi^2 + e^{\phi}ds_3^2. \tag{2.17}$$

with a magnetic 1-form potential $A_{(1)} = \zeta d\varphi$. Performing a dimensional reduction down to three dimensions we obtain the Lagrangian:

$$\mathcal{L}_3 = eR - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{2}ee^{\phi}(\partial\zeta)^2. \tag{2.18}$$

Upon defining the matrix:

$$\mathcal{M} = \begin{pmatrix} e^{\frac{\phi}{2}} & \frac{\zeta}{2}e^{\frac{\phi}{2}} \\ \frac{\zeta}{2}e^{\frac{\phi}{2}} & e^{-\frac{\phi}{2}} + \frac{\zeta^2}{4}e^{\frac{\phi}{2}} \end{pmatrix} \tag{2.19}$$

one can express the three-dimensional Lagrangian into the following manifestly $SL(2,R)$-invariant form:

$$\mathcal{L}_3 \;=\; eR + 4e\mathrm{tr}\big[\partial\mathcal{M}^{-1}\partial\mathcal{M}\big]. \tag{2.20}$$

In analogy with the electrically charged case we can then generate new magnetically charged solutions out of uncharged axial symmetric spaces. For instance starting with a vacuum solution of the form:

$$ds^2 \;=\; e^{-\psi}d\varphi^2 + e^{\psi}ds_3^2 \tag{2.21}$$

after performing the general $SL(2,R)$ transformation we obtain a magnetically charged solution:

$$\begin{aligned} ds^2 &= e^{-\phi}d\varphi^2 + e^{\phi}ds_3^2, \qquad && A_{(1)} = \zeta d\varphi, \\ e^{\phi} &= e^{\psi}\frac{\left(1+\delta e^{-\psi}\right)^2}{4C^2\delta}, \qquad && \zeta = \frac{4C}{1+\delta e^{-\psi}}, \end{aligned} \tag{2.22}$$

where $\delta = c^2/a^2$ and $C = 1/(2ac)$. As an example of this second method of adding magnetic charge to an axial metric, let us apply it to the usual Schwarzschild solution (2.5). This yields the Melvin generalisation of the Schwarzschild solution [77]:

$$\begin{aligned} ds^2 &= \frac{(1+\delta r^2\sin^2\theta)^2}{4C^2\delta}\left[-\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2d\theta^2\right] + \frac{4C^2\delta}{(1+\delta r^2\sin^2\theta)^2}r^2\sin^2\theta d\varphi^2, \\ A_{(1)} &= \frac{4Cd\varphi}{1+\delta r^2\sin^2\theta}. \end{aligned} \tag{2.23}$$

Physically this metric describes a black hole immersed into a magnetic universe. This solution is not asymptotically flat. It has a curvature singularity at $r = 0$, which is hidden behind an event horizon at $r = 2m$. In the limit $\delta \to 0$ and $C \to \infty$, while keeping the product $C^2\delta$ constant, we recover the uncharged Schwarzschild solution. In the remainder of this chapter we shall not further pursue this second method and focus only on the electrically charged cases.

Once we obtained a static electrically charged solution in Einstein-Maxwell theory, the next step in our solution-generating technique will be to perform a dualisation of the electromagnetic potential and find the corresponding magnetically charged solutions. In our case it turns out that it is easier to compute the dual electromagnetic potential in the reduced three-dimensional theory. In this case we start with the three-dimensional Lagrangian (2.10) and dualise the scalar field to obtain a magnetic 2-form field strength $F_{(2)}$. Following the usual procedure, we add a term $d\chi \wedge F_{(2)}$ to the action and solve the equations of motion for the scalar field. Replacing the result in the action we finally express the Lagrangian in terms of the dual field as:[2]

$$\mathcal{L}_3 \;=\; eR - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{4}ee^{-\phi}F_{(2)}^2, \tag{2.24}$$

where the components of the two-form field strength are computed using the formula:

$$F_{\alpha\beta} \;=\; ee^{\phi}\epsilon_{\alpha\beta\mu}\partial^{\mu}\chi, \tag{2.25}$$

where $\epsilon_{\alpha\beta\mu}$ is the Levi-Civita symbol. After lifting the solution back to four-dimensions we obtain a magnetically charged static solution of the Einstein-Maxwell field equations.

2.1.3 Final mapping to a vacuum stationary solution

We are now ready to perform the last step in our solution-generating method, namely to map the magnetic solution to a vacuum axisymmetric stationary solution of Einstein's field equations in four dimensions. This actually involves two steps: we first map the magnetic solution of the Einstein-Maxwell theory to a solution of the Einstein-Maxwell-Dilaton (EMD) theory with a specific value of the dilaton coupling, namely the one corresponding to the Kaluza-Klein theory, *i.e.* $a = -\sqrt{3}$. To do this we shall employ the general results derived in [56] (see also [36] for a geometrical derivation of the respective mapping). Starting with a magnetostatic solution:

$$ds_4^2 \;=\; -e^{-\phi}dt^2 + e^{\phi}\big[e^{2\mu}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\big], \qquad A_{(1)} = A_{\varphi}d\varphi \tag{2.26}$$

the corresponding solution of the EMD system is:

$$\begin{aligned} ds_4^2 &= -e^{-\frac{\phi}{4}}dt^2 + e^{\frac{\phi}{4}}\left[\left(e^{2\mu}\right)^{\frac{1}{4}}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right], \\ A_{(1)} &= \frac{A_{\varphi}}{2}d\varphi, \qquad e^{\frac{\tilde{\phi}}{\sqrt{3}}} = e^{\frac{\phi}{4}}. \end{aligned} \tag{2.27}$$

[2]Note that we perform the dualisation using a three dimensional *Euclidean* metric.

This is none other than the dimensional reduction of a vacuum five-dimensional metric using the ansatz:

$$ds_5^2 \;\; = \;\; e^{-\frac{2\tilde{\phi}}{\sqrt{3}}}(d\tau + \frac{A_\varphi}{2}d\varphi)^2 + e^{\frac{\tilde{\phi}}{\sqrt{3}}}ds_4^2 \tag{2.28}$$

In our case it turns out that the five-dimensional metric is simply the trivial product of a four-dimensional Euclidean metric with a time direction. Since the five dimensional metric solves the vacuum Einstein equations it is manifest that the four-dimensional Euclidean metric will be Ricci flat, *i.e.* it solves the vacuum Einstein equations in four dimensions. Therefore, our final result is expressible in the form:

$$ds_4^2 \;\; = \;\; e^{-\frac{\phi}{2}}(d\tau + A_\varphi/2d\varphi)^2 + e^{\frac{\phi}{2}}\Bigg[\left(e^{2\mu}\right)^{\frac{1}{4}}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\Bigg]. \tag{2.29}$$

Let us also mention at this point that we will also interpret the generated five dimensional metric in terms of Kaluza-Klein magnetic monopoles. In particular, as we shall prove that the Taub-NUT solution is a particular case of the stationary four-dimensional Ricci flat metric (2.29), it turns out that our solution-generating method touches upon another very interesting subject, namely that of Kaluza-Klein magnetic monopoles, of which we will have more to say in Chapter 5.

We note that even if the charging method does not require any other symmetry beyond the static condition, this second step in our solution-generating technique can be applied *only* for stationary axisymmetric metrics that can be cast into the Weyl-Papapetrou form. However, this is not really a very stringent constraint as most of the physically interesting solutions can be cast in the Weyl-Papapetrou form. To understand the effects of this last step in our solution-generating method one could take for instance the magnetically charged four-dimensional Reissner-Nordström and map it to an Euclidean vacuum metric as in (2.29). However, given the presence of the square roots appearing in the factors $e^{\frac{\phi}{2}}$ it is easily seen that we obtain an axisymmetric NUT-charged like solution with naked singularities, whose physical interpretation is obscure at this point. On the other hand, since we restrict ourselves to axisymmetric metrics that can be cast in Weyl form, it turns out that before we apply the charging procedure we can use the scaling symmetry discussed in the previous section to scale the dilaton in the initial seed in such a way to cancel the awkward effect of the square-roots in the final expression of the metric.

In the following sections we will generate new solutions using this method, starting first with the Schwarzschild metric as our initial seed, then again with the C-metric. We will generate in this way the Zipoy-Voorhees extensions of the Taub-NUT solution and of the Eguchi-Hanson instanton, respectively, using the C-metric as seed, we will generate the accelerating version of these solutions.

2.2 Zipoy-Voorhees families of Taub-Nut and Eguchi-Hanson solitons

We are now ready to apply the solution-generating technique described in the previous section. We have already showed as an example how to generate the Zipoy-Voorhees family of solutions out of the Schwarzschild metric in section (2.1.1). The next step in our method is to charge it using the $SL(2,R)$-symmetry. We obtain:

$$\begin{aligned} ds^2 &= -e^{-\phi}dt^2 + e^{\phi}ds_3^2, \qquad A_{(1)} = \chi dt, \\ e^{\phi} &= \frac{\left(1-\delta\left(1-\frac{2m}{r}\right)^{\gamma}\right)^2}{4C^2\delta\left(1-\frac{2m}{r}\right)^{\gamma}}, \qquad \chi = \frac{4C\delta}{\left(1-\frac{2m}{r}\right)^{-\gamma}-\delta}. \end{aligned} \tag{2.30}$$

We can now easily find out its magnetic dual solution as being given by:

$$A_{\varphi}d\varphi = \frac{2m\gamma}{C}\cos\theta d\varphi. \tag{2.31}$$

while the expression for the metric remains unchanged. The electrically charged version of the Zipoy-Voorhees metric has been obtained previously in [124, 126, 125]. Finally, before we map this solution to a vacuum Euclidean solution in four-dimensions we will make a final rescaling of the parameter $\gamma \to 2\gamma$ and we also perform a global rescaling of the metric[3] by $2C\sqrt{\delta}$ to obtain:

$$\begin{aligned} ds^2 &= \frac{1-\delta\left(1-\frac{2m}{r}\right)^{2\gamma}}{\left(1-\frac{2m}{r}\right)^{\gamma}}\left[\left(\frac{r(r-2m)}{(r-m)^2-m^2\cos^2\theta}\right)^{\gamma^2-1}(dr^2+r(r-2m)d\theta^2)\right. \\ &\quad \left.+r(r-2m)\sin^2\theta d\varphi^2\right] + \frac{4C^2\delta\left(1-\frac{2m}{r}\right)^{\gamma}}{1-\delta\left(1-\frac{2m}{r}\right)^{2\gamma}}(d\tau+\frac{2m\gamma}{C}\cos\theta d\varphi)^2. \end{aligned} \tag{2.32}$$

[3]We are allowed to do so since the metric is a vacuum solution.

This general family of vacuum Euclidean solutions, indexed by a real parameter γ, is the main result of this section.

Let us consider some limiting cases of this metric. Taking $\delta \to 0$ and $C \to \infty$ while keeping the product $C^2\delta$ constant, the metric reduces to the Euclidean version of the Zipoy-Voorhees static metric. If we take $\gamma = 1$ and properly rescale the z coordinate to absorb some constant factor we obtain a spherically-symmetric metric:

$$ds^2 = \delta\frac{r(r-2m)}{r^2-\delta(r-2m)^2}(d\tau + 4m\cos\theta d\varphi)^2 + \left(r^2 - \delta(r-2m)^2\right)\left[\frac{dr^2}{r(r-2m)} + d\Omega^2\right].$$

We distinguish now two possibilities. If we set directly $\delta = 1$ we can cast the metric in the following form:

$$ds^2 = \frac{R^2}{4}\left(1 - \frac{\alpha^4}{R^4}\right)(d\tau + \cos\theta d\varphi)^2 + \frac{dR^2}{\left(1 - \frac{\alpha^4}{R^4}\right)} + \frac{R^2}{4}d\Omega^2, \tag{2.33}$$

after redefining $R^2 = 4m(r-m)$ and $\alpha = 4m$. This is the well-known Eguchi-Hanson soliton [53].

On the other hand, if $\delta \neq 1$ then by redefining the radial coordinate such that $R^2 - n^2 = r^2 - \delta(r-2m)^2$ with $(1-\delta)n^2 = 4m^2\delta$ and rescaling z we obtain:

$$\begin{aligned} ds^2 = {} & \frac{\left(R + \frac{n}{\sqrt{\delta}}\right)\left(R + n\sqrt{\delta}\right)}{R^2 - n^2}(d\tau + 2n\cos\theta d\varphi)^2 + \frac{R^2 - n^2}{\left(R + \frac{n}{\sqrt{\delta}}\right)\left(R + n\sqrt{\delta}\right)}dR^2 \\ & + (R^2 - n^2)d\Omega^2, \end{aligned}$$

which we recognise as the Euclidean version of the Taub-NUT metric. It is possible now to set again $\delta = 1$ and recover the Taub-Nut instanton.

In summary, setting $\gamma = 1$ in our general metric yields the Euclidean version of the usual Taub-NUT solution, whereas setting $\delta = 1$ yields the Eguchi-Hanson soliton. Therefore for general values of γ our solution (2.32) describes the Zipoy-Voorhees-like generalisation of such objects.

Another case of interest is the one that corresponds to negative values for δ. Setting $\delta \to -\delta$, from the general form of the metric (2.32), we obtain a metric with Lorentzian signature:[4]

$$ds^2 = \frac{1+\delta\left(1-\frac{2m}{r}\right)^{2\gamma}}{\left(1-\frac{2m}{r}\right)^{\gamma}}\left[\left(\frac{r(r-2m)}{(r-m)^2-m^2\cos^2\theta}\right)^{\gamma^2-1}(dr^2+r(r-2m)d\theta^2) + r(r-2m)\sin^2\theta d\varphi^2\right] - \frac{4C^2\delta\left(1-\frac{2m}{r}\right)^{\gamma}}{1+\delta\left(1-\frac{2m}{r}\right)^{2\gamma}}(dt+\frac{2m\gamma}{C}\cos\theta d\varphi)^2. \tag{2.34}$$

where now δ takes positive values only. Consider now the $\gamma = 1$ member of this family of solutions. After redefining the radial coordinate such that $R^2 + N^2 = r^2 + \delta(r-2m)^2$, where $(1+\delta)N^2 = 4\delta m^2$, we obtain:

$$ds^2 = -\frac{\left(R+\frac{N}{\sqrt{\delta}}\right)\left(R-N\sqrt{\delta}\right)}{R^2+N^2}(dt+2N\cos\theta d\varphi)^2 + \frac{R^2+N^2}{\left(R+\frac{N}{\sqrt{\delta}}\right)\left(R-N\sqrt{\delta}\right)}dR^2 + (R^2+N^2)d\Omega^2, \tag{2.35}$$

i.e. the Taub-NUT solution. On the other hand, setting $\delta \to 0$ and taking the limit $C \to \infty$ while keeping the product $C^2\delta$ constant, the metric is readily seen to reduce to the Zipoy-Voorhees metric. Therefore, the general metric (2.34) describes the general Zipoy-Voorhees-Taub-NUT family of vacuum solutions, indexed by a real number γ and having two independent parameters δ and N.

2.3 Taub-NUT solutions: The first encounter

In this section we will consider more closely the properties of the Taub-NUT solution in flat backgrounds, both in Lorentzian and Euclidean signature. This solution has been found initially by Taub in 1951 [132] and later extended by Newman, Unti and Tamburino in 1963 [120]. The properties of this solution have been later clarified by Misner in 1967 [115, 116].

[4]We changed the notation $\tau \to t$.

2.3.1 The Taub-NUT solution in four dimensions

The metric (2.35) is in fact the general form of the Lorentzian Taub-NUT space. Indeed, the four dimensional vacuum Taub-NUT solution is usually written as:

$$\begin{aligned} ds^2 &= -f(r)(dt + 2N\cos\theta d\varphi)^2 + \frac{dr^2}{f(r)} + (r^2+N^2)(d\theta^2 + \sin^2\theta d\varphi^2), \\ f(r) &= \frac{r^2 - 2mr - N^2}{r^2+N^2} \end{aligned} \tag{2.36}$$

and it is then clear that (2.35) can be cast into this form if we take $m = \frac{\delta-1}{2\sqrt{\delta}}N$. Here the angles θ and φ are the standard angles parameterizing S^2 and they have the following ranges: $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$.

There are two different regions of this spacetime, which appear as we vary r. They are distinguished by the sign of $g_{tt} = f(r)$. Notice that $f(r) = 0$ at $r_\pm = m \pm \sqrt{m^2+N^2}$ and they correspond in a sense to horizons for this geometry. More precisely, they are chronology horizons since across these horizons $f(r)$ will change sign and the coordinate r changes from spacelike to timelike and vice-versa.

For $r_- < r < r_+$ we have $f(r) < 0$ and it is clear that, in analogy with what happens when one crosses the horizon for a Schwarzschild black hole, r will be a timelike coordinate in this region while t is spacelike. This corresponds to a time-dependent geometry and in fact this was the original solution discovered by Taub [132]. As we shall prove later on, the coordinate t has the periodicity $8\pi N$ and in this case the spatial slices will have the topology S^3. However, the interesting feature of this geometry is that it naturally describes a Big Bang (the volume of this universe is zero at the 'initial' time $r = r_-$) followed by a Big Crunch (the volume becomes zero again at the 'final' time $r = r_+$) with the total volume of the universe reaching a maximum value in between.

On the other sides of this cosmological region, *i.e.* for $r < r_-$ and $r > r_+$, one has $f(r) > 0$. Now the coordinate t plays the role of time and the geometry is stationary. However, since for consistency t must have the periodicity $8\pi N$, this region has pathological features as it contains closed timelike curves (CTCs) through every point. The constant radial slices have the topology of a Lorentzian version of S^3, since t is fibered over S^2.

Notice that these two regions are naturally connected through the horizons. The metric is singular at the location of these horizons, however, let us notice that the Taub-NUT solution has no curvature singularities and this suggests that one can extend the metric to cover such regions (see for instance the discussion in [82]). We can focus on the regions close to these horizons. Considering for instance the horizon located at $r = r_-$, if we define $\tau = r - r_-$ then $f(r) \simeq -\tau/r_-$ and the metric on a section with constant (θ, φ) becomes:

$$ds^2 \simeq \frac{\tau}{r_-}dt^2 - \frac{d\tau^2}{\tau/r_-}. \tag{2.37}$$

This is a two-dimensional version of the Misner space [115]. In the four-dimensional geometry, this space is fibered over S^2. Therefore, the full Taub-NUT geometry describes a cosmological Taub region connected through Misner-like spaces with regions containing CTCs.

Moreover, it turns out that there are even geodesics that can pass from one region to another and this is one very interesting feature of this geometry: one starts in the cosmological region and after waiting a finite time one find oneself in a region containing CTCs. There also exist the so-called quasi-regular singularities [99, 100], which correspond to the end-points of incomplete and inextensible geodesics that spiral indefinitely around a topologically closed spatial dimension. However, since the Riemann tensor and all its derivatives remain finite in all parallelly propagated orthonormal frames we take the point of view that these represent some of mildest of types of singularities and we shall ignore them when discussing the singularity structure of the Taub-NUT solutions.

It is because of these very interesting features that the Taub-NUT solution has become renowned in the GR community as being 'a counterexample to almost anything' [115]. This solution is usually interpreted as describing a so-called gravitational dyon with both ordinary and magnetic mass. Here the NUT charge N plays the role dual to that of an ordinary mass m, in the same way in which the electric and magnetic charges are dual in Maxwell's theory of electromagnetism. In order to understand this analogy let us first recall the way Dirac-string singularities appear in the presence of a magnetic charge.

Dirac strings. The Hopf fibration

All experimental evidence so far tells us that all particles have electric charges that are integer multiples of a fundamental unit of electric charge. In 1931 Dirac showed that in order to have a consistent quantum mechanical description of the charged particles in the presence of magnetic monopoles, one has to impose a constraint on the values of the allowed electric and magnetic charges: the so-called *Dirac quantisation formula* [50]. Dirac considered the quantum mechanics of an electron (of charge e) in the magnetic field of a magnetic monopole of charge g. As it is well-known, on R^3 this field cannot be described by a smooth and single-valued vector potential. Indeed, assuming that one could write $\vec{B} = \nabla \times \vec{A}$ then, if one computes the magnetic flux through some closed surface that contains the monopole, one finds that the flux should be zero. On the other hand, this contradicts the fact that the respective flux should be $\Phi = 4\pi g$. However, Dirac found that $\vec{B} = \nabla \times \vec{A}^{\pm}$, with vector potentials $\vec{A}^{\pm} = A^{\pm}_{\varphi}\vec{e}_{\varphi}$. Now, the potentials A^{+}_{φ} and A^{-}_{φ} are singular at $\theta = \pi$ (*i.e.* the negative z-direction), respectively $\theta = 0$ (*i.e.* the positive z-direction). These singularities are known as *Dirac's string singularities.* The union of the regions in which $A^{\pm}_{\varphi}$ are defined covers the whole of R^3 and moreover, in the intersection of these regions the vector potentials are related through a gauge transformation $A^{+}_{\varphi} = A^{-}_{\varphi} + \partial_{\varphi}(2g\varphi)$. Notice, however, that the function $2g\varphi$ is multiple valued.

If one considers now an electron (of charge e) moving in the magnetic field of the monopole, then its wave functions corresponding to the different vector potentials should be related as $\Psi^{+} = e^{2ieg\varphi/\hbar c}\Psi^{-}$ and the wave function will be single-valued if and only if $eg = n\hbar c/2$, where n is a natural number. This is the celebrated *Dirac's quantisation condition*: it implies that the mere existence of the magnetic monopoles requires the quantisation of the electric charge in units of $\hbar c/(2g)n$.

In 1975 Wu and Yang reformulated Dirac's magnetic monopole theory in the modern language of fibre bundles, avoiding in this way the use of Dirac string singularities [140, 141]. They noticed that Dirac's magnetic monopole of charge $g = n\hbar c/(2e)$ has a very natural interpretation as a connection in a circle fibration over the two-sphere S^2. The basic idea was to consider a cover of the manifold with patches, each patch providing a local coordinate system. When two such patches overlap, the coordinate systems in the intersection must be related by diffeomorphisms. If one considers now the electromagnetic potential, it will be defined as a 1-form on the manifold. Therefore, its definitions on different patches must be related by the standard transformation rules of 1-forms under changes of coordinates. However, the new freedom introduced by the notion of fibre bundles is that these different definitions can also be related by gauge transformations.

To better understand this situation let us consider again the example of the Dirac monopole. We have two potentials (written here in natural units with $\hbar = c = e = 1$):

$$A^{\pm}_{\varphi} = \frac{n}{2}(\pm 1 - \cos\theta\varphi)d\varphi \tag{2.38}$$

and they form a connection 1-form defined over S^2. Notice that $A^{\pm}_{\varphi}$ are defined in two charts covering the whole of S^2, namely: U^+, covering the northern hemisphere including the equator and, respectively U^-, covering the southern hemisphere (again including the equator). The intersection $U^+ \cap U^-$ corresponds to a small band around the equator and it is parameterized by φ. Notice that when extended to $U^{\mp}$ the gauge fields $A^{\pm}$ are singular. In the overlap the two gauge fields are related by a gauge transformation and the discussion proceeds as above to find the quantisation condition. However, for consistency, the circle coordinate (let us denote it by τ) must be defined in the overlap such that $\tau^+ = \tau^- - n\varphi$, with integer n. Then the Dirac monopole corresponds to a principal $U(1)$-bundle over S^2. The monopole of charge $1/2$ is the connection on the celebrated Hopf fibration $S^3 \to S^2$, while the monopole of charge $n/2$ corresponds to the $U(1)$-bundle over S^2 with n points identified on the fibre S^1, *i.e.* the lens space. We can in fact write down directly the metric on S^3 as a fibration over S^2 as:

$$ds^2 = \frac{1}{4}(d\tau - \cos\theta d\varphi)^2 + \frac{1}{4}d\Omega^2. \tag{2.39}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric on the unit-sphere S^2. Such fibre bundles over spheres are classified by the values of certain topological invariants. In our case we consider the so-called first Chern class:

$$c_1 = -\frac{1}{2\pi}\int_{S^2} F \tag{2.40}$$

where $F = dA$. This is nothing but the magnetic charge up to a factor of 2π and it should be an integer, according to general topological arguments. This corresponds again to Dirac's quantisation formula.

The Misner string

Let us return now to the discussion of the Taub-NUT solution. Notice that the off-diagonal component in the metric is $A_{\varphi} = g_{t\varphi} = 2N\cos\theta$, which is sometimes called the gravito-magnetic potential. this is none other the electromagnetic field of a magnetic monopole of charge proportional to N. In this sense N can be considered as a sort of 'magnetic mass' and that is why the Taub-NUT solution is sometimes interpreted to describe a gravitational dyon.

We can understand now the four dimensional Taub-NUT geometry as a radial extension of a circle fibration over S^2. Indeed, for constant radius, we find that the geometry describes a principal $U(1)$-fibration over S^2, the circle coordinate being t. As we have seen in the previous section, A_φ cannot be globally defined without singularities – here they will be called *Misner string singularities* – over the whole of S^2 and we have to use two coordinate patches that cover the S^2. In analogy with the discussion of the Dirac strings, for consistency we must require that the time coordinate t in the intersection of such patches satisfies a relation of the form: $t^+ = t^- + 4N\varphi$. Since φ is periodic with periodicity 2π we deduce that t must be periodically identified with period $8\pi N$. Therefore, in order to eliminate the Misner string singularities we have to periodically identify the circle coordinate t with period $8\pi N$.

2.3.2 Euclidean Taub-NUT solutions

The Euclidean version of the Taub-NUT solution is easily found by performing the following analytical continuations $t \to i\tau$, $N \to in$. The metric becomes:

$$ds^2 = F_E(r)(d\tau - 2n\cos\theta d\varphi)^2 + F_E^{-1}(r)dr^2 + (r^2 - n^2)d\Omega^2,$$

where

$$F_E(r) = \frac{r^2 - 2mr + n^2}{r^2 - n^2}. \qquad (2.41)$$

The regularity of the Euclidean Taub-NUT solution requires that the period of τ be $\beta = 8\pi n$ to ensure removal of the Dirac-Misner string singularity. It is also necessary to eliminate the singularities in the metric that appear as r is varied over the manifold. Attention must be paid to the so-called endpoint values of r: these are the values for which the metric components become zero or infinite. For a complete manifold r must range between two adjacent endpoints – if any conical singularities occur at these points they must be eliminated. In the above metric, finite endpoints occur at $r = \pm n$ or at the simple zeros of $F_E(r)$. In general $r = \pm n$ are curvature singularities unless $F_E = 0$ there as well.

To eliminate a conical singularity at a zero r_0 of $F_E(r)$ we must restrict the periodicity of τ by:

$$\beta \;=\; \frac{4\pi}{|F_E'(r_0)|}, \tag{2.42}$$

and this will generally impose a restriction on the values of the parameters once we match it with $8\pi n$.

The Euclidean sections of the Taub-NUT space can be classified as follows [66]: in general, the $U(1)$ isometry generated by the Killing vector $\frac{\partial}{\partial\tau}$ (that corresponds to the coordinate τ that parameterizes the fibre S^1) can have a zero-dimensional fixed point set (referred to as a 'Nut' solution) or a two-dimensional fixed point set (correspondingly referred to as a 'Bolt' solution).

For the Nut solution we impose $F_E(r = n) = 0$ to ensure that the fixed point of the Killing vector $\frac{\partial}{\partial\tau}$ is zero-dimensional and also $\beta F_E'(r = n) = 4\pi$ in order to avoid the presence of the conical singularities at $r = n$. With these conditions we obtain $m = n$, yielding

$$F_E(r) = \frac{r-n}{r+n}. \tag{2.43}$$

For the bolt solution the Killing vector $\frac{\partial}{\partial\tau}$ has a two-dimensional fixed point set in the 4-dimensional Euclidean sector. The regularity of the solution is then ensured by the following conditions [122, 34]: $F(r = r_b) = 0$ and $\frac{4\pi}{F'(r_b)} = \frac{8\pi n}{k}$ where k is an integer while the period of τ is now given by $\beta = \frac{8\pi n}{k}$, i.e. we identify k points on the circle described by χ. It is easy to see that these conditions are satisfied for $r_b = \frac{2n}{k}$ and $m = m_p = \frac{n(4+k^2)}{4k}$. However, we must demand that $r \geq r_b > n$, so that the fixed point set of $\frac{\partial}{\partial\tau}$ is not zero-dimensional; this in turn avoids the curvature singularity at $r = n$ and forces $k = 1$. Then the period of the coordinate χ is $8\pi n$ and for the bolt solution we obtain [122]:

$$F_E(r) = \frac{(r-2n)\left(r-\frac{1}{2}n\right)}{r^2-n^2}. \tag{2.44}$$

2.3.3 Other extensions in four dimensions

In four dimensions, a particularly interesting class of solutions that generalise the Schwarzschild black-hole is the so-called C-metric. The static part of this metric was found by Levi-Civita almost one century ago (see for instance [77]), however, its physical interpretation was clarified only after Kinnersley and Walker's work decades later [97]. By performing appropriate coordinate definitions, they found that this metric describes a pair of causally disconnected black holes uniformly accelerating in opposite directions. The cause of the acceleration is understood in terms of nodal/conical singularities along the axis that connects the two black holes and these singularities are interpreted as strings/struts pulling or pushing the black holes apart. A more general class of electrovacuum spacetimes that includes and considerably generalises the C-metric was found by Plebański and Demiański [123]. Recent analyses of this class of solutions have been performed in [90, 91, 73, 75, 74, 72], where a new exact solution describing a pair of accelerating and rotating charged black holes having also a NUT-charge has been presented. However, an accelerating NUT solution *without rotation* has not been identified yet within that class. It is the goal of the next sections to construct and analyse such an accelerated NUT-charged solution.

2.4 Accelerated Zipoy-Voorhees-like family of solutions

Since in the previous sections starting from the Schwarzschild metric we have been able to generate the Taub-NUT metric and its Zipoy-Voorhees generalisation, we shall simply consider next the uncharged C-metric as the seed in our solution-generating procedure.

Expressed in the form given in [90] the C-metric takes the simple form:

$$ds^2 = \frac{1}{A^2(x-y)^2}\left[-(y^2-1)F(y)dt^2 + \frac{dy^2}{(y^2-1)F(y)} + \frac{dx^2}{(1-x^2)F(x)} + (1-x^2)F(x)d\varphi^2\right],$$

where $F(\xi) = 1 + 2mA\xi$. We restrict our attention to case in which $0 \leq 2mA < 1$ and, in order to preserve the signature of the metric we restrict the values of the coordinates such that:

$$-\frac{1}{2mA} \leq y \leq -1, \qquad -1 \leq x \leq 1. \tag{2.45}$$

In terms of these coordinates, spatial infinity corresponds to $x = y = -1$, the black hole horizon is located at $y = -\frac{1}{2mA}$, while acceleration horizon corresponds to $y = -1$. The part of the symmetry axis joining the black hole horizon with the acceleration horizon is $x = 1$, while the one joining the black hole horizon to infinity is $x = -1$.

In order to apply our solution generating technique we need to write the C-metric in Weyl form. Using the results from [90] we obtain:

$$\begin{aligned} ds^2 &= -e^{-\psi}dt^2 + e^{\psi}\left[e^{2\mu}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right], \\ e^{-\psi} &= \frac{(y^2-1)F(y)}{A^2(x-y)^2}, \qquad e^{2\mu} = \frac{(y^2-1)F(y)}{f(x,y)G(x,y)}, \end{aligned} \tag{2.46}$$

where

$$\begin{aligned} f(x,y) &= (y^2-1)F(x) + (1-x^2)F(y), \\ G(x,y) &= \left[1 + mA(x+y)^2\right]^2 - m^2A^2(1-xy)^2, \end{aligned} \tag{2.47}$$

while the canonical Weyl coordinates ρ and z are defined as:

$$\begin{aligned} \rho^2 &= \frac{(y^2-1)(1-x^2)F(x)F(y)}{A^4(x-y)^4}, \qquad z = \frac{(1-xy)[1+mA(x+y)]}{A^2(x-y)^2}, \\ d\rho^2 + dz^2 &= \frac{f(x,y)G(x,y)}{A^4(x-y)^4}\left(\frac{dy^2}{(y^2-1)F(y)} + \frac{dx^2}{(1-x^2)F(x)}\right). \end{aligned} \tag{2.48}$$

Consider now the scaling transformation $(\psi, \mu) \to (\gamma\psi, \gamma^2\mu)$, where γ is a real parameter. Applying it to the C-metric we obtain a new vacuum solution of the form:

$$\begin{aligned} ds^2 = &-\left[\frac{(y^2-1)F(y)}{A^2(x-y)^2}\right]^{\gamma} dt^2 + \left[A^2(x-y)^2\right]^{\gamma-2}\left[\frac{(1-x^2)F(x)}{[(y^2-1)F(y)]^{\gamma-1}}d\varphi^2\right. \\ &\left.+\frac{((y^2-1)F(y))^{\gamma^2-\gamma}}{\left[f(x,y)G(x,y)\right]^{\gamma^2-1}}\left(\frac{dy^2}{(y^2-1)F(y)} + \frac{dx^2}{(1-x^2)F(x)}\right)\right]. \end{aligned} \tag{2.49}$$

It is clear that by taking $\gamma = 1$ we recover the initial C-metric. On the other hand, let us consider the zero-acceleration limit of this metric. Performing the coordinate transformations:

$$x = \cos\theta, \quad y = -\frac{1}{Ar}, \quad t \to A^{2\gamma-1}t, \tag{2.50}$$

while taking the limit $A \to 0$ *and* rescaling the metric by a constant factor $A^{2\gamma-2}$ it is readily seen that we recover the Zipoy-Voorhees solution (2.7). Therefore, we could naively interpret the metric (2.49) as describing an accelerating version of the Zipoy-Voorhees solution. However, the fact that the above metric is not the 'proper' accelerating Zipoy-Voorhees solution can also be seen from the fact that in fact the $\gamma = 2$ of this family should reduce to the so-called accelerating Darmois solution. This is the coincident limit of the accelerating Bonnor dihole solution that was recently found by Teo in [134]. In fact, a different metric describing the proper accelerated Zipoy-Voorhees solution, however, written in a very symmetric form has been presented by Teo in the same work. The proper accelerating Zipoy-Voorhees solution reads:

$$
\begin{aligned}
ds^2 &= -e^{-\psi}dt^2 + e^{\psi}\left[e^{2\mu}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right], \qquad (2.51)\\
e^{-\psi} &= \frac{(y^2-1)F(y)}{A^2(x-y)^2}\left(\frac{F(y)}{F(x)}\right)^{\alpha-1}, \quad e^{2\mu} = \frac{(y^2-1)F(y)}{f(x,y)G(x,y)}\frac{F(y)^{\alpha^2-1}F(x)^{(\alpha-1)^2}}{G(x,y)^{\alpha^2-1}},
\end{aligned}
$$

where the canonical Weyl coordinates are again defined in (2.48). Indeed, we see that the $\alpha = 2$ member of this family is clearly different from the member $\gamma = 2$ of (2.49) and therefore we cannot actually interpret (2.49) as being an accelerating version of the Zipoy-Voorhees family. Nonetheless, since the metric (2.49) is used only as an intermediate step in our solution-generating technique, we shall not further discuss its properties at this point, but limit ourselves to notice that one can apply the scaling transformation on Teo's solution and generate a new family of vacuum metrics indexed by two distinct real parameters:

$$
\begin{aligned}
ds^2 &= -e^{-\psi}dt^2 + e^{\psi}\left[e^{2\mu}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right], \qquad (2.52)\\
e^{-\psi} &= \left[\frac{(y^2-1)F(y)}{A^2(x-y)^2}\left(\frac{F(y)}{F(x)}\right)^{\alpha-1}\right]^{\gamma}, \quad e^{2\mu} = \left(\frac{(y^2-1)F(y)}{f(x,y)G(x,y)}\frac{F(y)^{\alpha^2-1}F(x)^{(\alpha-1)^2}}{G(x,y)^{\alpha^2-1}}\right)^{\gamma^2},
\end{aligned}
$$

The next step is to charge the solution (2.52) using a general $SL(2,R)$ transformation. Using the formulae (2.15) we obtain:

$$
\begin{aligned}
ds^2 &= -e^{-\psi}\frac{1}{H_\gamma(x,y)}dt^2 + e^{\psi}H_\gamma(x,y)\left[e^{2\mu}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right],\\
A_{(1)} &= \frac{4C\delta}{\left(\frac{A^2(x-y)^2}{(y^2-1)F(y)}\left(\frac{F(x)}{F(y)}\right)^{\alpha-1}\right)^{\gamma} - \delta}dt, \qquad H_\gamma(x,y) = \frac{\left(1-\delta\left(\frac{(y^2-1)F(y)}{A^2(x-y)^2}\left(\frac{F(y)}{F(x)}\right)^{\alpha-1}\right)^{\gamma}\right)^2}{4C^2\delta}.
\end{aligned}
$$

Let us consider a few limiting cases of the above metric. Taking $\delta \to 0$ and $C \to \infty$ while keeping the product $C^2\delta$ constant we recover the uncharged metric (2.52). On the other hand, the $\gamma = 1$ member of this family should correspond to the charged accelerating Zipoy-Voorhees solution. In particular, for $\alpha = 1$ this should reduce to a charged version of the C-metric. However, unlike the known form of the electrically charged C-metric, in general, the above solution has a ring-like curvature singularity located at the roots of $H_\gamma(x, y) = 0$. Therefore its interpretation as a new form of the charged C-metric is dubious. Nonetheless, in absence of a true charged generalisation of the accelerating Zipoy-Voorhees family, we will make use of the above metric in our solution-generating method.

By dimensionally reducing this solution down to three dimensions and dualising the scalar field χ to an electromagnetic field as described in section 2.1, we find that the magnetic potential is given by:

$$A_\varphi = \frac{\gamma}{C}\frac{(1-x^2)\,(\alpha F(x) + (1-\alpha)F(y))}{A^2(x-y)^2} + \frac{2m\gamma\alpha x}{AC}, \tag{2.53}$$

while the metric remains unchanged in this process. Finally, taking $\gamma = 2$ and using (2.29) we find:

$$\begin{aligned} ds^4 = {} & \frac{(y^2-1)F(x)}{A^2(x-y)^2}\left(\frac{F(y)}{F(x)}\right)^\alpha \frac{C^2\delta}{H(x,y)}(dt + A_\varphi d\varphi)^2 + \frac{H(x,y)}{A^2(x-y)^2}\left[\left(\frac{F(x)}{F(y)}\right)^\alpha (1-x^2)F(y)d\varphi^2 \right. \\ & \left. + \frac{(F(x)F(y))^{\alpha(\alpha-1)}}{G(x,y)^{\alpha^2-1}}\left(\frac{dy^2}{(y^2-1)F(y)} + \frac{dx^2}{(1-x^2)F(x)}\right)\right], \end{aligned} \tag{2.54}$$

where we defined:

$$H(x,y) = \frac{1 - \delta\left(\frac{(y^2-1)F(x)}{A^2(x-y)^2}\left(\frac{F(y)}{F(x)}\right)^\alpha\right)^2}{2}. \tag{2.55}$$

This is the main result of this section. In the limit $\delta \to 0$, $C \to \infty$ (with $C^2\delta$ constant and rescaling the t coordinate by a constant factor) we recover the Euclidean form of the accelerating Zipoy-Voorhees solution (2.52). It is manifest that the C-metric corresponds to the $\alpha = 1$ member of this family. Another interesting limit to consider is the zero-acceleration limit. In this case it turns out that performing the coordinate transformations:

$$x = \cos\theta, \qquad y = -\frac{1}{Ar}, \qquad \delta \to \delta A^4, \qquad t \to \frac{t}{A}, \tag{2.56}$$

we recover the general Zipoy-Voorhees-Taub-NUT family of solutions (2.32) described in the previous section. Therefore, we expect that this metric describes the proper accelerating version of the family (2.32). Computing some of the curvature invariants for this metric one finds that generically there is a curvature singularity located at the roots of $H(x,y)=0$ as long as $x \neq y$. However if we consider negative values of δ (*i.e.* replace $\delta \to -\delta$) in the above metric we obtain a vacuum solution with Lorentzian signature and furthermore, we find that $H(x,y) > 0$ always (for $x \neq y$).

2.5 Properties of the accelerating Taub-NUT solution

In what follows we will concentrate our attention on the $\alpha = 1$ member of this Lorentzian family. The metric becomes:

$$\begin{aligned} ds^4 &= -\frac{(y^2-1)F(y)}{A^2(x-y)^2}\frac{C^2\delta}{H(x,y)}\left(dt+\frac{1}{C}\left(\frac{(1-x^2)F(x)}{A^2(x-y)^2}+\frac{2mx}{A}\right)d\varphi\right)^2 \\ &\quad +\frac{H(x,y)}{A^2(x-y)^2}\left[(1-x^2)F(x)d\varphi^2+\frac{dy^2}{(y^2-1)F(y)}+\frac{dx^2}{(1-x^2)F(x)}\right], \end{aligned} \tag{2.57}$$

where we denote:

$$H(x,y) = \frac{1+\delta\left(\frac{(y^2-1)F(y)}{A^2(x-y)^2}\right)^2}{2}.$$

Let us first notice that the C-metric, respectively the Taub-NUT metric are included as limiting cases in the above solution. Indeed, taking $\delta \to 0$ and $C \to \infty$ while keeping the product $C^2\delta$ constant, after rescaling the time coordinate with a constant factor we obtain the uncharged C-metric solution. On the other hand, the zero-acceleration limit is taken by performing the coordinate redefinitions and scalings of the parameters:

$$x = \cos\theta, \quad y = -\frac{1}{Ar}, \quad \delta \to A^4\delta, \quad t \to \frac{t}{A}, \tag{2.58}$$

in the limit $A \to 0$. It is readily seen that in this limit we obtain the Taub-NUT metric.

To understand the properties of this solution it turns out to be more convenient to consider the above metric in Weyl form:

$$\begin{aligned} ds^2 &= -e^{-\phi}(dt + A_\varphi d\varphi)^2 + e^{\phi}\left[e^{2\mu}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right], \\ e^{-\phi} &= e^{-\psi}\frac{2C^2\delta}{1+\delta e^{-2\psi}}, \qquad e^{2\mu} = \frac{(y^2-1)F(y)}{f(x,y)G(x,y)}, \end{aligned} \tag{2.59}$$

where the canonical Weyl coordinates ρ and z are given by (2.48) and the expressions of $e^{-\psi}$ and $e^{2\mu}$ in terms of the canonical Weyl coordinates are given in Appendix B. The general analysis of the above metric can now be done in parallel with the one corresponding to the uncharged C-metric. In particular, we see that if we restrict the values of m and A such that $0 \leq 2mA < 1$, then, in order to preserve the correct signature of the metric, the coordinates (x, y) have to take the range:

$$-1 \leq x \leq 1, \qquad -\frac{1}{2mA} \leq y \leq -1. \tag{2.60}$$

Now, it is well known that, in Weyl cylindrical coordinates, black hole horizons correspond to rods on the symmetry axis [56]. Our interpretation of the above solution as describing an accelerating NUT-charged black hole will rely on the identification of such rods on the symmetry axis.

We will define the symmetry axis to correspond to $\rho = 0$ *i.e.* it is the z-axis. Using (2.48) one sees that it corresponds to the four intervals: $x = -1$, $x = 1$, $y = -\frac{1}{2mA}$ and $y = -1$. As we shall prove bellow, $y = -\frac{1}{2mA}$ corresponds to the event horizon of the black hole, $y = -1$ is the acceleration horizon, the line $x = 1$ is the part of the symmetry axis between the event horizon and the acceleration horizon, while $x = -1$ is the part of the symmetry axis joining up the event horizon with asymptotic infinity.

To this end, notice that the asymptotic region $x = y = -1$ corresponds to $z = \pm\infty$, while the end-points of the range of the coordinates (x, y) are mapped into $z(x, y)$ as follows:

$$z_1 = z\left(-1, -\frac{1}{2mA}\right) = -\frac{m}{A}, \qquad z_2 = z\left(1, -\frac{1}{2mA}\right) = \frac{m}{A}, \qquad z_3 = z(1,-1) = \frac{1}{2A^2}.$$

Note now that $e^{-\phi}|_{\rho=0}$ vanishes at all the above points z_i, $i = 1..3$, it is positive for $z < z_1$ and $z_2 < z < z_3$ whereas both $e^{-\phi}|_{\rho=0}$ and $e^{2\mu}|_{\rho=0}$ are *zero* for $z_1 < z < z_2$ and $z > z_3$. We may then follow a similar analysis with the one performed in [49] to conclude that the regions $z_1 < z < z_2$ and $z > z_3$ are the Killing horizons of our accelerating solution. One can also see this by noting that the location of the horizons is given by the equation $g^{yy} = 0$, which in our case corresponds to the equation $(y^2 - 1)F(y) = 0$. Furthermore, by computing the area of each of the above horizons one can check that $z_1 < z < z_2$ has finite area and it corresponds then to a black hole horizon, while $z > z_3$ has infinite area and it corresponds to an accelerating horizon. Indeed, using the C-metric coordinates the area of the black hole horizon is readily found to be:

$$A_H = \int_0^{2\pi} \int_{-1}^{1} \sqrt{g_{\varphi\varphi} g_{xx}} dx d\varphi = \frac{8\pi m^2}{C\sqrt{\delta}(1 - 4m^2A^2)}. \tag{2.61}$$

while the area corresponding to $y = -1$ diverges.

Having determined that the above solution describes an accelerating object, let us turn now to a consideration of the 'cause' of the acceleration. The analysis of the conical singularities proceeds exactly as in the case of the uncharged C-metric. In particular, if we denote the periodicity of φ as $\Delta\varphi$, then along a portion of the axis where the metric function $e^{-\phi}$ is positive, $e^{-\phi} > 0$, the deficit of conical angle will be given by[5]:

$$\Delta = 2\pi - \Delta\varphi e^{-\mu}|_{\rho=0}. \tag{2.62}$$

Recall that if $\Delta < 0$ one has an excess of conical angle and this corresponds to a strut, if $\Delta > 0$ one has a deficit of conical angle that corresponds to a string, whereas if $\Delta = 0$ there is no conical singularity on that part of the symmetry axis. Since in our case the function $e^{2\mu}$ is precisely the same as the one corresponding to the uncharged C-metric, we deduce that in general there is a conical singularity residing in this solution and that, for appropriate values of $\Delta\varphi$, it can be chosen to lie along $z_2 < z < z_3$ (*i.e.* $x = 1$) *or* $z < z_1$ (*i.e.* $x = -1$). In particular we find:

$$\Delta_{x=\pm 1} = 2\pi - (1 \pm 2mA)\Delta\varphi. \tag{2.63}$$

One can remove the conical singularity on the segment $z_2 < z < z_3$ (*i.e.* $x = 1$) if one chooses $\Delta\varphi = 2\pi/(1 + 2mA)$ but then there will be a positive deficit angle for $z < z_1$

[5]The measurement of the proper circumference and proper radius must be performed in a frame for which the proper time $d\tau = dt + A_\varphi d\varphi = 0$.

(*i.e.* $x=-1$) and this can be interpreted as an semi-infinite cosmic string pulling on the black hole. Alternatively, for $\Delta\varphi = 2\pi/(1-2mA)$ one can eliminate the conical angle for $z<z_1$ (*i.e.* $x=-1$) but then there will be a excess of conical angle for $z_2<z<z_3$ (*i.e.* $x=1$). This is interpreted as a strut pushing on the black hole. The strut continues past the acceleration horizon and connects with the mirror black hole on the other side of it.[6]

Following the discussion in [91] let us consider next the presence of torsion singularities. In general these appear when the conical singularities possess a non-zero angular velocity, signified by a non-vanishing $\omega = g_{t\varphi}/g_{tt}$ along the symmetry axis. As is apparent from the metric written in Weyl-Papapetrou form, near the symmetry axis $\rho \to 0$ for a non-zero value of ω the coordinate φ will become a timelike coordinate and this will lead to the apparition of CTCs sufficiently close to the axis. In general these CTCs can be eliminated only when ω takes the *same* constant value along the entire axis of symmetry as in that case it is possible to perform a global coordinate transformation $t \to t - \omega|_{\rho=0}\varphi$ to give a metric without such pathologies. For our accelerating NUT solution we find that on the symmetry axis $\rho = 0$:

$$\omega|_{x=\pm 1} \;=\; \pm\frac{2m}{AC}, \tag{2.64}$$

and therefore at the first sight there are unavoidable torsion singularities associated with this metric. However, one can still perform a coordinate definition $t_N = t + \frac{2m}{AC}\varphi$ on the line $x=1$ respectively $t_S = t - \frac{2m}{AC}\varphi$ near $x=-1$. Since the coordinate φ is periodic, this will introduce a periodicity for the time coordinate. As we have seen in section 2.3 this is precisely what is expected in the case of a NUT-charged solution.

2.6 Summary

We have constructed new families of axisymmetric vacuum solutions in 4-dimensions. Using the Schwarzschild solution as seed we obtained the Zipoy-Voorhees generalisation of the Taub-NUT solution, respectively of the Eguchi-Hanson solitons. We then discussed in detail the interesting features of the Taub-NUT geometry. We have seen that this geometry

[6]The existence of the second black hole on the other side of the acceleration horizon is obscured by the use of the Weyl coordinates.

can be understood as a radial extension of a circle fibration over the sphere S^2. We will use this observation in Chapter 3, where we will see that the higher dimensional NUT-charged spaces can be thought of in a very similar way as radial extensions of circle fibrations over products of Einsten-Kähler manifolds. Finally, using the C-metric as seed, we obtained an accelerating version of the above solutions. These solutions are parameterized by a real parameter γ. The $\gamma = 1$ member of this family reduces to a new solution, which we interpreted as the accelerating version of the Taub-NUT solution.

Chapter 3

Higher-dimensional Taub-NUT solutions in cosmological backgrounds

3.1 Overview of the higher dimensional NUT-charged solutions

In this chapter we construct new solutions of the vacuum Einstein field equations with multiple NUT parameters, with and without cosmological constant. These solutions describe spacetimes with non-trivial topology that are asymptotically dS, AdS or flat and represent new generalisations of the spacetimes studied in refs. [11, 133, 9, 37].

As we have seen in Chapter 2, there are known extensions of the Taub-NUT solutions to the case when a cosmological constant is present and also in the presence of rotation [47, 68, 4, 98]. In these cosmological settings, the asymptotic structure is only locally de Sitter (for a positive cosmological constant) or anti-de Sitter (for a negative cosmological constant) and we speak about Taub-NUT-(A)dS solutions.

generalisations to higher dimensions follow closely the four-dimensional case [11, 122, 3, 133, 9, 37, 111, 105, 35, 60]. In constructing these metrics the idea is to regard Taub-NUT spacetimes as radial extensions of $U(1)$ fibrations over a $2k$-dimensional base space M endowed with an Einstein-Kähler metric g_M. Then the $(2k+2)$-dimensional Taub-NUT spacetime has the metric:

$$ds^2 = F^{-1}(r)dr^2 + (r^2 + N^2)g_M - F(r)(dt + NA)^2 \tag{3.1}$$

where t is the coordinate on the fibre S^1 and the one-form A has curvature $J = dA$, which is proportional to some covariantly constant 2-form. Here N is the NUT charge and $F(r)$ is a function of r.

Another class of solutions introduced in [111] used a generalised ansatz:

$$ds^2 = F^{-1}(r)dr^2 + (r^2 + N^2)g_M + \alpha r^2 g_Y - F(r)(dt + 2NA)^2 \tag{3.2}$$

in which one constructs the higher dimensional Taub-NUT space as a generalised fibration over an Einstein-Kähler manifold M endowed with the metric g_M. The non-trivial feature of this ansatz is that now the fibre contains besides the (r, t)-sector a general Einstein space Y, endowed with an Einstein metric g_Y, while α is a constant. This type of solution was later generalised to arbitrary dimensions by Lü, Page and Pope in [105].

In this chapter we generalise both of these types of solutions to include multiple NUT parameters in arbitrary dimensions. We first describe the generalisation of the ansatz (3.1) for an arbitrary number of factors M_i in the factored form of the base space B. To analyse the possible singularities of these metrics we switch over to their Euclidean sections by performing analytic continuations of the time coordinate t and of the NUT parameters. We go on to analyse the regularity constraints to be imposed on these Euclidean sections in order to obtain regular metrics that can be extended globally to cover the whole manifold. We find that for generic values of the parameters these metrics are singular: it is only for a astute choice of the parameters that they become regular. As an example of this general analysis we focus on the six-dimensional case and we explicitly consider the cases of a Taub-NUT-like fibration over the base spaces $S^2 \times S^2$ and the complex projective space CP^2.

In Section 3 we present a more general form of the solution (3.1) in which we replace the 2-dimensional factors M_i by arbitrary even-dimensional Einstein-Kähler manifolds. We use here the normalisation that the Ricci tensor for each manifold M_i can be written as $Ricci(M_i) = \delta_i g(M_i)$. For each factor M_i we associate a NUT parameter N_i. Consistent with what was conjectured in [111], we find that generically there are constraints to be imposed on the possible values of the cosmological constant λ, the NUT parameters N_i and the values of the various δ's. These solutions represent the multiple NUT parameter generalisation of the inhomogeneous Einstein metrics on complex line-bundles described in [122]. We find that we can cast these solutions in another form that explicitly encodes the constraint conditions into the metric.

Following the construction in [111], we discuss in section 4 the properties of the five dimensional metrics in some detail. In Section 5 we present the multiple NUT parameter extension of the metrics (3.2) constructed by Lü, Page and Pope [105]. In this case we replace the Einstein-Kähler manifold M by a product of Einstein-Kähler manifolds M_i with arbitrary even-dimensions and to each such factor we associate a NUT parameter N_i. As is apparent from the five dimensional examples, the case in which Y is one-dimensional is particularly interesting to us since it will provide us with the general form of the odd-dimensional Eguchi-Hanson-type solitons [42]. This case will be analysed in section 6.

3.2 A more general form of the Bais-Batenburg solution

We assume that the $(d-2)$-dimensional base space in our construction can be factored as a product of p factors, $B = M_1 \times \cdots \times M_p$ where M_i are 2-dimensional spaces of constant curvature with metrics g_{M_i}, normalised such that $\text{Ricci}(M_i) = \delta_i g(M_i)$, where δ_i are constants. The metric ansatz that we use is then:

$$ds_d^2 = -F(r)(dt + \sum_{i=1}^{p} 2N_i A_i)^2 + F^{-1}(r)dr^2 + \sum_{i=1}^{p}(r^2 + N_i^2)g_{M_i} \tag{3.3}$$

where

$$A_i = \begin{cases} \cos\theta_i d\phi_i, & \text{for } \delta = 1 \text{ (sphere)} \\ \theta_i d\phi_i, & \text{for } \delta = 0 \text{ (torus)} \\ \cosh\theta_i d\phi_i, & \text{for } \delta = -1 \text{ (hyperboloid)} \end{cases}$$

By solving the vacuum Einstein equations with cosmological constant we obtain:

$$F(r) = \frac{r}{\prod_{i=1}^{p}(r^2+N_i^2)}\left[\int^{r}\left(\delta_1 - \frac{d-1}{l^2}(s^2+N_1^2)\right)\frac{\prod_{i=1}^{p}(s^2+N_i^2)}{s^2}ds - 2m\right], \tag{3.4}$$

where m is a constant of integration, while the constraints on the values of the NUT parameters N_i and thecosmological constant λ can be expressed in the very simple form for every $i, j = 1 \ldots p$:

$$\lambda(N_j^2 - N_i^2) = \delta_j - \delta_i \tag{3.5}$$

If all the factors δ_i coincide then we can satisfy this constraint in two ways: either we can take all the NUT parameters to be equal $N_i = N$ and keep the cosmological constant non-vanishing or else we can take $\lambda = 0$ and keep the NUT parameters independent. However, if at least two factors δ_i are different, it is inconsistent to set $\lambda = 0$. In this case all the NUT parameters corresponding to identical δ_i factors must remain equal, while those corresponding to different δ_i factors must remain distinct such that the above constraints are still satisfied.

3.2.1 Singularity Analysis

Note that while the factors multiplying g_{M_i} are never zero, this is not so for the Euclidean section of the metric. Therefore, when we shall address the possible singularities of the above metrics we shall focus mainly on their Euclidean sections, recognising that the Lorentzian versions are singularity-free – apart from quasi-regular singularities [99, 100], which correspond to the end-points of incomplete and inextensible geodesics that spiral infinitely around a topologically closed spatial dimension. However, since the Riemann tensor and all its derivatives remain finite in all parallelly propagated orthonormal frames we take the point of view that these represent some of mildest of types of singularities and we shall ignore them when discussing the singularity structure of the Taub-NUT solutions. We also note that for asymptotically dS spacetimes that have no bolts quasi-regular singularities are absent [5].

Scalar curvature singularities have the possibility of manifesting themselves only in the Euclidean sections. These are simply obtained by the analytic continuations $t \to i\tau$

and $N_j \to in_j$, and can be classified by the dimensionality of the fixed point sets of the Killing vector $\xi = \partial/\partial\tau$ that generates a $U(1)$ isometry group. As we have seen in the previous chapter, in four dimensions the Killing vector that corresponds to the coordinate that parameterizes the fibre S^1 can have a zero-dimensional fixed point set (we speak about a 'Nut' solution in this case) or a two-dimensional fixed point set (referred to as a 'Bolt' solution). The classification in higher dimensions can be done in a similar manner. If this fixed point set dimension is $(d-1)$ the solution is called a Bolt solution; if the dimensionality is less than this then the solution is called a Nut solution.[1] If $d=3$, Bolts have dimension 2 and Nuts have dimension 0. However if $d>3$ then Nuts with larger dimensionality can exist [111, 105]. Note that fixed point sets need not exist; indeed there are parameter ranges of NUT-charged asymptotically dS spacetimes that have no Bolts [5].

The singularity analysis of these metrics is a direct application of the one given in [122]. In order to extend the local metrics presented above to global metrics on non-singular manifolds the idea is to turn all the singularities appearing in the metric into removable coordinate singularities. For generic values of the parameters in the solution the singularities are not removable, corresponding to conical singularities in the manifold. We are mainly interested in the case of compact Einstein-Kähler manifolds M_i. Generically the Kähler forms J_i on M_i can be equal to dA_i only locally. Hence we need to use a number of overlapping coordinate patches to cover the whole manifold. In order to render the 1-form $d\tau + \sum 2n_i A_i$ well-defined we need to identify τ periodically. In general this can be done if the ratios of all the parameters n_i are rational numbers. If we choose them to be positive integers we can define $q = \gcd\{n_1, \ldots, n_p\}$ and require the period of τ to be given by:

$$\beta = \frac{8\pi q}{k} \tag{3.6}$$

where k is a positive integer. It is also necessary to eliminate the singularities in the metric that appear as r is varied over M. Attention must be paid to the so-called endpoint values of r: these are the values for which the metric components become zero or infinite. For a complete manifold r must range between two adjacent endpoints – if any conical

[1] In this thesis we will keep up with this terminology. By Nut we will mean the Taub-NUT-Nut solution, while Bolt will refer to the Taub-NUT-bolt solution.

singularities occur at these points they must be eliminated. The finite endpoints occur at $r = \pm n_i$ or at the simple zeros of $F_E(r)$. In general $r = \pm n_i$ are curvature singularities unless $F_E = 0$ there as well. To eliminate a conical singularity at a zero r_0 of $F_E(r)$ we must restrict the periodicity of τ to be given by:

$$\beta = \frac{4\pi}{|F_E'(r_0)|} \tag{3.7}$$

and this will generally impose a restriction on the values of the parameters once we match it with (3.6). For compact manifolds the radial coordinate takes values between two finite endpoints and the regularity constraint must be imposed at both endpoints. If the manifold is noncompact then the cosmological constant is non-positive and the radial coordinate takes values between one finite endpoint r_0 and one infinite endpoint $r_1 = \infty$. For our asymptotically locally flat or $(A)dS$ solutions the infinite endpoints are not within a finite distance from any points $r \neq r_1$ so there is no regularity condition to be imposed at r_1. In this case the regularity conditions to be satisfied are that $F_E(r) > 0$ for $r \geq r_0$ and $\beta = \frac{4\pi}{|F_E'(r_0)|}$.

3.2.2 The six dimensional Taub-NUT metrics

Consider for example the six-dimensional case with a fibration over the base space $S^2 \times S^2$. If the cosmological constant is non-zero, $\lambda = -5/l^2$, then we must have $n_1 = n_2 = n$. Regularity of the 1-form $d\tau - 2nA$ forces the periodicity of τ to be given by $8\pi n/k$, where k is an integer. We must match this periodicity with the one emerging by requiring absence of conical singularities at the root r_0 of $F_E(r)$, which is

$$F_E(r) = \frac{3r^6 + (l^2 - 15n^2)r^4 - 3n^2(2l^2 - 15n^2)r^2 - 6mrl^2 - 3n^4(l^2 - 5n^2)}{3l^2(r^2 - n^2)^2} \tag{3.8}$$

from the Einstein equations using (3.4). The Nut solution corresponds to $r_0 = n$ in which case we obtain $\frac{4\pi}{|F_E'(r)|} = 12\pi n$. As there is no integer k for which the periodicities can be matched, we conclude that this solution is singular. Indeed it is easy to check that $r_0 = n$ is the location of a curvature singularity. To define a bolt solution it is sufficient to require $r_0 > n$ and the regularity condition in this case is given by $\frac{4\pi}{|F_E'(r)|} = \frac{8\pi n}{k}$, with k an integer.

Solving this constraint we find

$$r_0 = \frac{kl^2 \pm \sqrt{k^2l^4 - 80n^2l^2 + 400n^4}}{20n}.$$

If the cosmological constant vanishes then we can have different values for the NUT parameters. Without loss of generality, assume that $n_1 > n_2$ and that they are rationally related. Then it is easy to see that, in order to keep the metric positive definite, we have to restrict the range of the radial coordinate such that $r > n_1$. As above, the periodicity of the τ coordinate is found to be $8\pi n_2/k$, where k is an integer. We have to match this with the periodicity imposed on τ by eliminating the conical singularities at a root r_0 of $F_E(r)$. We distinguish two types of solutions: a Nut and a bolt. The Nut solution corresponds to $r_0 = n_1$ and in this case the periodicity $\frac{4\pi}{|F_E'(n_1)|} = 8\pi n_1$ cannot be matched with $8\pi n_2/k$ for any integer value of k. However note that $r_0 = n_1$ is not the location of a curvature singularity! On the other hand, the bolt solution corresponds to $r \geq r_0 > n_1$ and the periodicity is found to be $\frac{4\pi}{|F_E'(n_1)|} = \frac{8\pi n_1}{p}$, where p is some integer. It is now possible to match it with $8\pi n_2/k$ with k an integer such that $p/k = n_1/n_2$. The bolt solution is then non-singular.

The situation changes considerably if we take $B = CP^2$ as the base space [11, 122, 9]. In this case $p = 1$ in eq. (3.2) and the submanifold g_{M_1} has the metric

$$d\Sigma_2{}^2 = \frac{du^2}{\left(1+\frac{\delta u^2}{6}\right)^2} + \frac{u^2}{4\left(1+\frac{\delta u^2}{6}\right)^2}\left(d\psi + \cos(\theta)d\phi\right)^2 + \frac{u^2}{4\left(1+\frac{\delta u^2}{6}\right)}(d\theta^2 + \sin^2\theta d\phi^2) \quad (3.9)$$

with $F_E(r)$ still given by (3.8). However the one-form $2nA$ is now given by

$$A = \frac{u^2 n}{2\left(1+\frac{\delta u^2}{6}\right)}\left(d\psi + \cos\theta d\phi\right) \quad (3.10)$$

We need to find the smallest value of $\int 2ndA$ over a closed 2-chain. Changing coordinates so that $u = \sqrt{\frac{6}{\lambda}}\tan\chi$ the CP^2 metric can be written as [2]

$$\begin{aligned} ds^2 &= \frac{6}{\delta}\left(d\chi^2 + \frac{\sin^2\chi}{4}\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \sin^2\chi\cos^2\chi\left(d\psi + \cos\theta d\phi\right)^2\right) \quad (3.11) \\ A &= \frac{3n}{2\delta}\sin^2\chi\left(d\psi + \cos\theta d\phi\right) \end{aligned}$$

where $0 \leq \chi \leq \frac{\pi}{2}$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, and $0 \leq \psi \leq 4\pi$. We see from (3.11) that $\chi = 0$ is a 'nut' in this subspace, and so there is no closed 2-chain on which to integrate $2ndA$. However at $\chi = \frac{\pi}{2}$ the (θ, ϕ) sector is a 2-dimensional bolt. Hence at $\chi = \frac{\pi}{2}$ we obtain

$$\int 2ndA = 2\frac{3n}{2\delta}4\pi = \frac{12\pi n}{\delta}$$

implying that the periodicity of τ can be $12\pi n/k$, where we use the normalisation $\delta = 1$. Equating this to $\frac{4\pi}{|F'_E(r=n)|} = 12\pi n$ yields[2] $k = 1$, and the geometry at $r_0 = n$ is smooth. Thus we can obtain regular Nut and bolt solutions if the base space is CP^2. More generally, for CP^q the periodicity is $\frac{4\pi n(q+1)}{k\delta}$, with k an integer [133, 122].

3.3 A more general class of solutions: Even dimensions

In this section we present a more general class of Taub-NUT metrics in even dimension. These spaces are constructed as complex line bundles over a product of Einstein-Kähler spaces M_i, with dimensions $2q_i$ and metrics g_{M_i}. Then the total dimension is $d = 2(1 + \sum_i^p q_i)$. The metric ansatz that we use is the following:

$$ds_d^2 = -F(r)(dt + \sum_{i=1}^{p} 2N_iA_i)^2 + F^{-1}(r)dr^2 + \sum_{i=1}^{p}(r^2 + N_i^2)g_{M_i}$$

Here $J_i = dA_i$ is the Kähler form for the i-th Einstein-Kahler space M_i and we use the normalisation such that the Ricci tensor of the i-th manifold is $R_{ab} = \delta_i g_{ab}$. Then the general solution to Einstein's field equations with cosmological constant $\lambda = \pm(d-1)/l^2$ is given by:

$$F(r) = \frac{r}{\prod_{i=1}^{p}(r^2 + N_i^2)^{q_i}}\left[\int^r \left(\delta_1 \mp \frac{d-1}{l^2}(s^2 + N_1^2)\right)\frac{\prod_{i=1}^{p}(s^2 + N_i^2)^{q_i}}{s^2}ds - 2m\right] \quad (3.12)$$

[2]The parameter $m = \frac{4n^3(6n^2 - l^2)}{3l^2}$ is fixed by requiring that $F_E(n) = 0$.

while the constraints on the values of the NUT parameters N_i and the cosmological constant λ can be expressed in the very simple form for every $i, j = \overline{1,p}$:

$$\lambda(N_j^2 - N_i^2) = \delta_j - \delta_i \tag{3.13}$$

It is easy to see that if the Einstein-Kähler spaces are two-dimensional, *i.e.* $q_i = 1$, we recover the solution from the previous section.

The singularity analysis of these metrics proceeds as described in the previous section. The Euclidean section of these metrics is obtained by analytical continuation of the time coordinate and of the NUT parameters. From the general expression of the function $F_E(r)$[3] it is an easy matter to see that if the root $r_0 = n_j$ where n_j is the NUT parameter associated with an Einstein-Kähler manifold M_j of dimension $2q_j$ then:

$$\frac{4\pi}{|F_E'(n_j)|} = \frac{4\pi n_j(q_j+1)}{\delta_j} \tag{3.14}$$

otherwise, for generic roots r_0 we deduce that:

$$\beta = \frac{4\pi}{|F_E'(r_0)|} = \frac{4\pi r_0}{\delta_1 - \lambda(r_0^2 - n_1^2)} \tag{3.15}$$

These formulae are very useful in the singularity analysis of these metrics.

It is also possible to incorporate the above constraints directly in the metric. However, this would require the manifolds involved to be non-canonically normalised. Take for instance the 6-dimensional Taub-NUT fibration constructed over $S^2 \times S^2$. If we normalise the spheres such that their Einstein constant is $\delta = 1$ then the constraint equation on the parameters takes the form $\lambda(n_1^2 - n_2^2) = 1 - 1 = 0$ and we can have a solution with non-vanishing cosmological constant only if the NUT parameters are equal. Suppose now that we normalise the spheres such that their Einstein constants are δ_1, respectively δ_2. Then the constraint should read $\lambda(n_1^2 - n_2^2) = \delta_1 - \delta_2$. One way to change the Einstein constant in a general equation of the form $\text{Ricci}(M_i) = \lambda_i g(M_i)$ is to multiply the metric $g(M_i)$ by a constant factor $1/\delta_i$. This yields $\lambda_i\delta_i$ as the normalised Einstein constant for the new rescaled metric[4]. On the other hand, recall that for a $2q$-dimensional Einstein-Kähler with

[3] This is deduced from (3.12) by analytical continuation of all the NUT charges $N_i \to in_i$.

[4] The Ricci tensor is invariant under an overall rescaling of the metric by a constant.

Kähler form A_i, the product $(dA_i)^q$ is proportional to its volume form. When rescaling the metric by $1/\delta_i$ the volume form gets rescaled by a factor $1/\delta_i^q$ – hence we must rescale A_i by a factor of $1/\delta_i$ to obtain the Kähler form for the rescaled metric. For spheres we should then multiply A_i by $1/\delta_i$ and the metric elements $d\Omega_i^2$ by $1/\delta_i$, for each $i = 1, 2$. The expression for $F(r)$ remains unchanged in this process[5].

Applying this to the six-dimensional case, we obtain

$$\begin{aligned} ds^2 &= -F(r)\left(dt - \frac{2n_1}{\delta_1}\cos\theta_1 d\phi_1 - \frac{2n_2}{\delta_2}\cos\theta_2 d\phi_2\right)^2 + \frac{dr^2}{F(r)} \\ &\quad + \frac{r^2+n_1^2}{\delta_1}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{r^2+n_2^2}{\delta_2}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \end{aligned} \tag{3.16}$$

while the constraint equation takes the form

$$\lambda(n_1^2 - n_2^2) = \delta_1 - \delta_2$$

Solving this equation for δ_2 and replacing its value in the metric we obtain:

$$\begin{aligned} ds^2 &= -F(r)\left(dt - 2n_1\cos\theta_1 d\phi_1 - \frac{2n_2}{\delta_1 - \lambda(n_1^2 - n_2^2)}\cos\theta_2 d\phi_2\right)^2 + \frac{dr^2}{F(r)} \\ &\quad + \frac{r^2+n_1^2}{\delta_1}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{r^2+n_2^2}{\delta_1 - \lambda(n_1^2 - n_2^2)}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \end{aligned} \tag{3.17}$$

which is a solution of the Einstein field equations with cosmological constant λ for every value of the NUT parameters n_1 and n_2. Notice that now the constraint equation is already encoded in the metric, and we can for convenience scale $\delta_1 = 1$. When $n_2 = 0$ it reduces to the cosmological 6-dimensional Taub-NUT solution obtained previously in [111, 105].

It is interesting to note that the above form of the metric allows non-singular Nuts of intermediate dimensionality constructed over the base $S^2 \times S^2$. To see this let us notice that the absence of Misner string singularities can be accomplished if n_1/δ_1 and n_2/δ_2 are rationally related. Specifically, we can choose for example $n_1 = 2n_2$ and $\delta_1 = 1$ while $\delta_2 = 1/2$. To satisfy these relations it is enough to take $\lambda n_2^2 = -1/6$, where $\lambda = -5/l^2$ in 6-dimensions. Then regularity of the 1-form $(d\tau - 2n_1/\delta_1 A_1 - 2n_2/\delta_2 A_2)$ requires the periodicity of τ to be given by $\frac{8\pi n_1}{k\delta_1} = \frac{8\pi n_2}{k\delta_2}$ where k is some integer. It is easy to see that

[5] It can be read from the general form (3.12) for general values of δ's.

we can match this periodicity with $\frac{4\pi}{|F'_E(n_1)|} = \frac{8\pi n_1}{\delta_1}$ if we take $k = 1$. Then there exists a nut at $r = n_1$ which is completely regular – it can be easily checked that there are no curvature singularities in this case! We conclude that the Nut solution of intermediate dimensionality constructed over the base space $S^2 \times S^2$ is regular.

3.4 Warped-type Fibrations and Odd Dimensions

We can find a very general class of solutions of Einstein's field equations if we use the generalised ansatz [111, 105]

$$ds_d^2 = -F(r)(dt + \sum_{i=1}^{p} 2N_i A_i)^2 + F^{-1}(r)dr^2 + \sum_{i=1}^{p}(r^2 + N_i^2)g_{M_i} + \alpha r^2 g_Y \tag{3.18}$$

As before $J_i = dA_i$ is the Kähler form for the i-th Einstein-Kahler space M_i, Y is a q-dimensional Einstein space with metric g_Y and we use the normalisation such that the Ricci tensor of the i-th manifold is $R_{ab} = \delta_i g_{ab}$ and $R^Y_{ab} = \delta_Y g^Y_{ab}$.

Then the general solution of Einstein's field equations is given by

$$\begin{aligned} F(r) &= \frac{r^{1-q}}{\prod_{i=1}^{p}(r^2 + N_i^2)^{q_i}} \left[\int^r \left(\delta_1 \mp \frac{d-1}{l^2}(s^2 + N_1^2) \right) s^{q-2} \prod_{i=1}^{p}(s^2 + N_i^2)^{q_i} ds - 2m \right], \\ \alpha &= \frac{\delta_Y}{\delta_1 - \lambda N_1^2}, \end{aligned} \tag{3.19}$$

where the constraints on the values of cosmological constant $\lambda = \mp\frac{d-1}{l^2}$ and the NUT parameters N_i can be expressed in the following simple form:

$$\lambda(N_j^2 - N_i^2) = \delta_j - \delta_i \tag{3.20}$$

for every i, j. For $p = 1$ we recover the general solution found by Lü, Page and Pope in ref. [105].

We can treat the case $\delta_Y = 0$ (or $q = 1$) if we take the limit in which $\lambda N_1^2 = \delta_1$ in order to keep α finite in the above expressions. In general it is not necessary to have all the NUT parameters identical, though NUT parameters N_j corresponding to Einstein-Kähler spaces that have the same Einstein constants δ_j have to be equal.

3.5 Taub-NUT-(A)dS spacetimes in five dimensions

In even-dimensions the usual Taub-NUT construction corresponds to a $U(1)$-fibration over an even-dimensional Einstein space used as the base space. Since obviously this cannot be done in odd-dimensions, we must modify our metric ansatz in such a way that we can realize the $U(1)$-fibration as a fibration over an even dimensional subspace of the odd dimensional base space. In five dimensions our base space is three dimensional and we shall construct the NUT space as a partial fibration over a two-dimensional space of constant curvature. The spacetimes that we obtain are not trivial in the sense that we cannot naively set the NUT charge and/or the cosmological constant (now $\lambda = \frac{6}{l^2}$) to vanish. In particular, it turns out that both these limits are possible, however we will have more to say about this in the next section.

Consider first a fibration over S^2. The ansatz that we shall use in the construction of these spaces is the following:

$$ds^2 = -F(r)(dt - 2n\cos\theta d\phi)^2 + F^{-1}(r)dr^2 + (r^2+n^2)(d\theta^2 + \sin^2\theta d\phi^2) + r^2 dy^2 \quad (3.21)$$

The above metric is a solution of the Einstein field equations with cosmological constant λ provided

$$F(r) = \frac{4ml^2 - r^4 - 2n^2r^2}{l^2(r^2+n^2)} \quad (3.22)$$

and

$$n^2 = \frac{l^2}{4} \quad (3.23)$$

Let us consider next the Euclidean section of the above solution (obtained by making the analytical continuations $t \to i\chi$ and $n \to in$):

$$ds^2 = F_E(r)(d\chi - 2n\cos\theta d\phi)^2 + F_E^{-1}(r)dr^2 + (r^2-n^2)(d\theta^2 + \sin^2\theta d\phi^2) + r^2 dz^2 \quad (3.24)$$

where

$$F_E(r) = \frac{r^4 - 2n^2r^2 + 4ml^2}{l^2(r^2-n^2)} \quad (3.25)$$

and the constraint $\lambda n^2 = -\frac{3}{2}$ holds. Since we analytically continue n we must also analytically continue $l \to il$ for consistency with the initial constraint on λ and n^2.

In order to get rid of the usual Misner type singularity in the metric we have to assume that the coordinate χ is periodic with period β. Notice that for $r = n$ the fixed point of the Killing vector $\frac{\partial}{\partial\chi}$ is one dimensional and we shall refer to such a solution as being a NUT solution. However, for $r = r_b$, where $r_b > n$ is the largest root of $F_E(r)$, the fixed point set is three-dimensional and we shall refer to such solutions as bolt solutions. Note that for either situation the the period of χ must be $\beta = 8\pi n$ to ensure the absence of the Dirac-Misner string singularity.

In order to have a regular Nut solution we have to ensure the following additional conditions:

- $F_E(r = n) = 0$ in order to ensure that the fixed point of the Killing vector $\frac{\partial}{\partial\chi}$ is one-dimensional.
- $\beta F_E'(r = n) = 4\pi k$ (where k is an integer) in order to avoid the presence of conical singularities at $r = n$ (in other words, the periodicity of χ must be an integer multiple of the periodicity required for regularity in the (χ, r) section; we identify k points on the circle described by χ).

It is easy to see that the above conditions lead to $k = 1$ and $m_n = \frac{n^4}{4l^2} = \frac{l^2}{64}$. It is precisely for this value of the parameter m that the above solution becomes the Euclidean AdS spacetime in five-dimensions.

Let us now turn to the regularity conditions that we have to impose in order to obtain the bolt solutions. In order to have a regular bolt at $r = r_b$ we have to satisfy similar conditions as before, with $r_b > n$:

- $F_E(r = r_b) = 0$
- $\beta F_E'(r = r_b) = 4\pi k$ where k is an integer.

The above conditions lead to $r_b = \frac{kn}{2}$ and

$$m = m_b = -\frac{k^2 l^2 (k^2 - 8)}{1024} \tag{3.26}$$

To ensure that $r_b > n$ we have to take $k \geq 3$; as a consequence the curvature singularity at $r = n$ is avoided. We obtain the following family of bolt solutions, indexed by the integer k:

$$ds^2 = F_E(r,k)(d\chi - 2n\cos\theta d\phi)^2 + F_E^{-1}(r,k)dr^2 + (r^2 - n^2)(d\theta^2 + \sin^2\theta d\phi^2) + r^2 dz^2 \quad (3.27)$$

where

$$F_E(r,k) = \frac{256r^4 - 128l^2r^2 - k^2l^4(k^2-8)}{256l^2(r^2-n^2)} \quad (3.28)$$

and $n = \frac{l}{2}$. One can check directly that the bolt solution is not simply the AdS space in disguise by computing the curvature tensor of the bolt metric and comparing it with that of the Euclidean AdS space.

We can obtain NUT spaces with non-trivial topology if we make partial base fibrations over a two-dimensional torus T^2 or over the hyperboloid H^2. We obtain

$$ds^2 = -F(r)(dt - 2n\theta d\phi)^2 + F^{-1}(r)dr^2 + (r^2+n^2)(d\theta^2 + d\phi^2) + r^2dy^2 \quad (3.29)$$

for the torus, where

$$F(r) = \frac{4ml^2 + r^4 + 2n^2r^2}{l^2(r^2+n^2)} \quad (3.30)$$

where now the constraint equation takes the form $\lambda n = 0$ where $\lambda = -\frac{6}{l^2}$; we can have consistent Taub-NUT spaces with toroidal topology if and only if the cosmological constant vanishes. The Euclidean version of this solution, obtained by analytic continuation of the coordinate $t \to it$ and of the parameter $n \to in$ has a curvature singularity at $r = n$. Note that if we consider $n = 0$ in the above constraint we obtain the AdS/dS black hole solution in five dimensions with toroidal topology.

If the cosmological constant vanishes then we can have $n \neq 0$ and we obtain the following form of the metric

$$ds^2 = -F(r)(dt - 2n\theta d\phi)^2 + F^{-1}(r)dr^2 + (r^2+n^2)(d\theta^2 + d\phi^2) + r^2dy^2 \quad (3.31)$$

where

$$F(r) = \frac{4m}{r^2+n^2} \quad (3.32)$$

The asymptotic structure of the above metric is given by

$$ds^2 = \frac{4m}{r^2}(dt - 2n\theta d\phi)^2 + \frac{r^2}{4m}dr^2 + r^2(d\theta^2 + d\phi^2 + dy^2) \tag{3.33}$$

If y is an angular coordinate then the angular part of the metric parameterizes a three torus. The Euclidean section of the solution described by (3.31) is not asymptotically flat and has a curvature singularity localized at $r = 0$. However, let us notice that for $r \leq n$ the signature of the space becomes completely unphysical. Hence, for the Euclidean section, we should restrict the values of the radial coordinate such that $r \geq n$.

In the case of a fibration over the hyperboloid H^2 we obtain:

$$ds^2 = -F(r)(dt - 2n\cosh\theta d\phi)^2 + F^{-1}(r)dr^2 + (r^2 + n^2)(d\theta^2 + \sinh^2\theta d\phi^2) + r^2 dy^2 \tag{3.34}$$

where now $\lambda = -\frac{6}{l^2}$,

$$F(r) = \frac{r^4 + 2n^2r^2 - 4ml^2}{l^2(r^2 + n^2)} \tag{3.35}$$

and the constraint $n^2 = \frac{l^2}{4}$ holds.

The 'Euclidean' section of these spaces is described by the metric

$$ds^2 = F_E(r)(dt - 2n\cosh\theta d\phi)^2 + F_E^{-1}(r)dr^2 + (r^2 - n^2)(d\theta^2 + \sinh^2\theta d\phi^2) + r^2 dy^2 \tag{3.36}$$

where $n^2 = \frac{l^2}{4}$ and

$$F_E(r) = -\frac{r^4 - 2n^2r^2 + 4ml^2}{l^2(r^2 - n^2)} \tag{3.37}$$

and it is a Euclidean solution of the vacuum Einstein field equations with positive cosmological constant. The coordinates θ and ϕ parameterize a hyperboloid, which after performing appropriate identifications becomes a surface of any genus higher than 1. In general the metric has a curvature singularity located at $r = n = \frac{l}{2}$ with the exception of the case in which $m_n = -\frac{l^2}{64}$ when the space is actually the five-dimensional Euclidean dS space in disguise.

In order to discuss the possible singularities in the metric first let us notice the absence of Misner strings, the fibration over the hyperbolic space being trivial in this case. Moreover, if we impose the condition that there are no conical singularities at $r = r_p$, where r_p is the biggest root of $F_E(r)$, then we must set the periodicity β of the coordinate χ to be

$\frac{4\pi}{|F_E'(r=r_p)|}$. If we take $r_p = n$ we obtain $\beta = 8\pi n$ and $m = -\frac{l^2}{64}$, which means that the Nut solution is the dS space in disguise.

In order to determine the bolt solution one has to satisfy the following conditions:

- $F_E(r = r_b) = 0$
- $\frac{4\pi}{|F_E'(r_b)|} = \frac{8\pi n}{k}$ where k is an integer and the period of χ is now given by $\beta = \frac{8\pi n}{k}$; again we identify k points on the circle described by χ.

The above conditions lead to $r_b = \frac{kn}{2}$ and

$$m = m_b = \frac{k^2 l^2 (k^2 - 8)}{1024} \tag{3.38}$$

We must take $k \geq 3$ to ensure that $r_b > n$, which again avoids the curvature singularity at $r = n$. We obtain the following family of bolt solutions, indexed by the integer k:

$$ds^2 = F_E(r)(d\chi - 2n\cosh\theta d\phi)^2 + F_E^{-1}(r)dr^2 + (r^2 - n^2)(d\theta^2 + \sinh^2\theta d\phi^2) + r^2 dz^2 \tag{3.39}$$

where

$$F_E(r) = \frac{-256r^4 + 512n^2r^2 + k^2l^4(k^2 - 8)}{256l^2(r^2 - n^2)}$$

and $n = \frac{l}{2}$.

3.6 Eguchi-Hanson solitons in five dimensions

As advertised in the previous section, we would like to comment here on the limit when the NUT charge and/or the cosmological constant are zero. Central to our construction is the five dimensional metric (3.24) and recall that there is a constraint between the NUT parameter and the cosmological constant, which can be expressed as $4n^2 = l^2$. It should be clear that the limit in which the cosmological constant is vanishing, *i.e.* $l \to \infty$, the NUT parameter n should also diverge. On the other hand, the limit in which the NUT charge is vanishing is equivalent to the limit in which the cosmological constant diverges. However, as noted there, there is a way to evade this situation: after performing a change of coordinates it turns out that that the metric can be cast in such a form that allows us to take the limit of a vanishing cosmological constant.[6]

More precisely, start with the metric (3.21) in which we make use of the constraint (3.23):

$$ds^2 = -F(r)(dt - 2n\cos\theta d\phi)^2 + F^{-1}(r)dr^2 + (r^2+n^2)(d\theta^2 + \sin^2\theta d\phi^2) + r^2 dy^2, \quad (3.40)$$

where:

$$F(r) = \frac{4ml^2 - r^4 - 2n^2r^2}{l^2(r^2+n^2)} \quad (3.41)$$

Make now the coordinate change $\rho^2 = 4r^2 + l^2$ and define $a = 64ml^2 - l^4$. Then the metric becomes:

$$\begin{aligned} ds^2 &= \frac{\rho^2}{4}\left(1 - \frac{a}{\rho^4}\right)(dt + \cos\theta d\phi)^2 - \frac{d\rho^2}{\left(\frac{\rho^2}{l^2} - 1\right)\left(1 - \frac{a}{\rho^4}\right)} \\ &\quad + \frac{\rho^2}{4}(d\theta^2 + \sin^2\theta d\phi^2) + \left(\frac{\rho^2}{l^2} - 1\right) d\tilde{y}^2, \end{aligned} \quad (3.42)$$

where we have absorbed an $l/2$ factor into $\tilde{y} = yl/2$. This is a solution of the 5-dimensional Einstein equations with positive cosmological constant $\lambda = 6/l^2$, referred to as the Eguchi-Hanson soliton [42, 41, 113]. The limit in which the cosmological constant vanishes is now a smooth limit and the metric becomes the product of the four-dimensional Eguchi-Hanson metric with a trivial flat direction. We have now two possibilities depending on the relative magnitude of the parameters a and l. Consider first the case in which $l < a^{1/4} \leq \rho$. Then it is easy to see that $\rho = a^{1/4}$ corresponds to a cosmological horizon: inside it the coordinate t is timelike and there are closed timelike curves after one eliminates the Misner string singularity in the metric. For $a = 0$ the metric reduces to the usual de Sitter metric, with a cosmological horizon at $\rho = l$. Notice that the values $\rho < l$ are not allowed if $a \neq 0$. On the other hand, if $l > a^{1/4}$ then we must restrict $\rho \geq a^{1/4}$ and there is now a cosmological horizon at $\rho = l$. There are no closed timelike curves in this case and inside the cosmological horizon $\tilde{y}$ is the timelike coordinate.

[6]The metric obtained this way is a generalisation of the four-dimensional Eguchi-Hanson metric to higher dimensions [42, 41, 113].

There also exists a limit in which we can set the NUT parameter to zero. Namely, our 5-dimensional metric can be written quite generally in the form [113]

$$\begin{aligned} ds^2 &= -F(r)(dt + 2\frac{n}{\delta}\cos\theta d\phi)^2 + \frac{dr^2}{F(r)} + \frac{r^2+n^2}{\delta}(d\theta^2 + \sin^2\theta d\phi^2) + r^2 dy^2, \\ F(r) &= \frac{r^4 + 2n^2r^2 - 2ml^2}{r^2+n^2}, \end{aligned} \tag{3.43}$$

and the constraint equation is now simply $\delta = -\frac{4n^2}{l^2}$. This metric is then a solution of Einstein field equations with cosmological constant $\Lambda = -\frac{6}{l^2}$. Once we fix δ as above, there is no constraint on the values of Λ and n other than the requirement of a metric of Lorentzian signature — this can be easily accommodated by analytically continuing the coordinate $\theta \to i\theta$. Defining now a new NUT parameter $N = \frac{n}{\delta}$ and $\lambda = -\frac{4}{l^2}$, the above solution can be written in the following form:

$$\begin{aligned} ds^2 &= -F(r)(dt - 2N\cosh\theta d\phi)^2 + \frac{dr^2}{F(r)} + \frac{\lambda^2N^2r^2+1}{(-\lambda)}(d\theta^2 + \sinh^2\theta d\phi^2) + r^2 dy^2, \\ F(r) &= \frac{16N^2r^4 + 2r^2l^2 - ml^2}{l^2(16N^2r^2 + l^4)}. \end{aligned} \tag{3.44}$$

When $N \neq 0$, a change of coordinates will bring the metric into a form similar to the one discussed in Section 2. Notice however that, in the form written above, it is possible to take a smooth limit of the metric in which the NUT charge $N \to 0$. Then, we obtain a metric that is the trivial product of a 3-dimensional Schwarzschild AdS (described by the coordinates (t, r, y)) with a 2-dimensional hyperboloid (described by (θ, ϕ)).

3.7 Generalised higher dimensional Eguchi-Hanson solitons

The case $q = 1$ of the general solution given in (3.18) is particularly interesting to us, as it will provide a generalisation of Eguchi-Hanson metrics to arbitrary odd-dimensions [42]. For simplicity we shall work in the Euclidean sector. In this case the metric can be written as

$$ds_d^2 = F(r)(d\chi + \sum_{i=1}^{p} 2n_iA_i)^2 + F^{-1}(r)dr^2 + \sum_{i=1}^{p}(r^2 - n_i^2)g_{M_i} + r^2dy^2 \tag{3.45}$$

and we use $\delta_1 + \lambda n_1^2 = 0$ such that:

$$F(r) = \frac{1}{\prod_{i=1}^{p}(r^2 - n_i^2)^{q_i}}\Big[-\lambda \int^r \prod_{i=1}^{p}(s^2 - n_i^2)^{q_i} s ds - 2m\Big] \tag{3.46}$$

while the constraints on the values of the NUT parameters n_i and the cosmological constant λ take the form $\lambda n_i^2 = -\delta_i$. A positive value for the cosmological constant can still be accommodated if we take $\delta_i < 0$ (for instance a product of hyperboloids, for which $\delta_i = -1$). Let us take for simplicity a negative cosmological constant $\lambda = -\frac{d-1}{l^2}$ and let us suppose that all the δ_i's are the same, *i.e.* $\delta_i = \delta$ (for instance we can have a product of spheres or more generally products CP^{a_i} factors, for various values of a_i, normalised such that their cosmological constant is δ). Assume then that the base space contains a product of b_i CP^{a_i} factors. Then the dimension of the total space is $d = \sum_i 2a_i b_i + 3$, $n_i^2 = \frac{\delta l^2}{d-1} \equiv n^2$ and we have:

$$\begin{aligned} F(r) &= \frac{1}{(r^2 - n^2)^{\sum_i a_i b_i}}\Big[-\lambda \int^r (s^2 - n^2)^{\sum_i a_i b_i} s ds - 2m\Big] \\ &= \frac{1}{(r^2 - n^2)^{\sum_i a_i b_i}}\Big[\frac{(r^2 - n^2)^{\sum_i a_i b_i + 1}}{l^2} - 2m\Big] \end{aligned} \tag{3.47}$$

It is convenient at this time to make the change of variables such that $\rho^2 = r^2 - n^2$:

$$\begin{aligned} F(\rho) &= \frac{\rho^2}{l^2} - \frac{2m}{\rho^{d-3}} \\ &= \frac{\rho^2}{l^2}\Big[1 - \frac{2ml^2}{\rho^{d-1}}\Big] \equiv \frac{\rho^2}{l^2} g(\rho) \end{aligned} \tag{3.48}$$

It is now easy to see that the metric (3.45) with $p = \sum_i b_i$ can be written in the following form:

$$ds_d^2 = \frac{4\delta\rho^2}{d-1} g(\rho)(d\chi + \sum_{i=1}^{p} A_i)^2 + \frac{(d-1)d\rho^2}{\left(\delta + \frac{(d-1)\rho^2}{l^2}\right) g(\rho)} + \sum_{i=1}^{p} \rho^2 g_{M_i} + \frac{l^2}{d-1}\left(\delta + \frac{(d-1)\rho^2}{l^2}\right) dy^2$$

Making now the change of variables $(d-1)\rho^2 \to \rho^2$, defining $a^{d-1} \equiv 2ml^2(d-1)^{\frac{d-1}{2}}$

and rescaling y to absorb the constant factor $\frac{l^2}{d-1}$ we eventually obtain

$$\begin{aligned} ds_d^2 &= \frac{4\delta\rho^2}{(d-1)^2}\left(1-\frac{a^{d-1}}{\rho^{d-1}}\right)(d\chi+\sum_{i=1}^{p}A_i)^2+\frac{d\rho^2}{\left(\frac{\rho^2}{l^2}+\delta\right)\left(1-\frac{a^{d-1}}{\rho^{d-1}}\right)}+\sum_{i=1}^{p}\frac{\rho^2}{d-1}g_{M_i} \\ &+\left(\frac{\rho^2}{l^2}+\delta\right)dy^2 \end{aligned} \tag{3.49}$$

which is the most general (Euclidean) form of the odd-dimensional Eguchi-Hanson solitons [42], whose base space contains b_i factors CP^{a_i}. The general solution whose base space contains a number of unit curvature spheres $CP^1 = S^2$ has been analysed in [42, 41]. More generally we can replace the CP^a factors by arbitrary Einstein-Kähler manifolds M_i normalised such that their Einstein constants are equal $\delta_i = \delta$ for all $i = 1..p$. The parameter δ is not essential and it can be absorbed by an appropriate rescaling of the radial coordinate and redefinition of the parameter a. Without losing generality we can then set $\delta = 1$.

It is interesting to note that while the Eguchi-Hanson solitons constructed over Einstein-Kähler spaces are in general nonsingular there are also Lorentzian sections in odd-dimensions for which the curvature singularities at the origin can be easily avoided. Take for instance the five-dimensional metric:[7]

$$ds_5^2 = -\frac{\rho^2}{4}\left(1-\frac{a^4}{\rho^4}\right)(dt-\cosh\theta d\phi)^2+\frac{d\rho^2}{\left(\frac{\rho^2}{l^2}-1\right)\left(1-\frac{a^4}{\rho^4}\right)}+\frac{\rho^2}{4}(d\theta^2+\sinh^2\theta d\phi^2)+\left(\frac{\rho^2}{l^2}-1\right)dy^2$$

which is a solution of vacuum Einstein field equations with negative cosmological constant $\lambda = -\frac{4}{l^2}$. In order to keep the signature of the metric Lorentzian we must restrict the values of the radial coordinate such that $\rho > l$. Depending on the sign of the parameter a^4 we can have a horizon located at $\rho = a$ and in both situations the curvature singularity located at origin is avoided. In the limit in which the cosmological constant vanishes, *i.e.* $l \to \infty$, the metric describes the product of a four-dimensional Eguchi-Hanson-like metric with a flat direction and there is no way to avoid the curvature singularity at $r = 0$ while keeping the signature of the metric Lorentzian.[8]

[7]This metric can be formally obtained from (3.49) by setting $p = 1$ and $\delta = -1$ and replacing CP^1 by H^2 (see also [8]).

[8]Or at least allow it to be Riemannian.

3.8 Summary and overview of the thesis

In this chapter we constructed new solutions of the vacuum Einstein field equations with multiple NUT parameters, with and without cosmological constant. These solutions describe spacetimes with non-trivial topology that are asymptotically dS, AdS or flat and represent new generalisations of the spacetimes studied previously in literature.

We first described the generalisation of the Taub-NUT ansatz that corresponds to an arbitrary number of factors Einstein-Kähler spaces M_i in the factored form of the base space B. We have also provided a non-trivial generalisation of the Taub-NUT to odd dimensions. In particular, we found that the five-dimensional Taub-NUT spaces correspond to a generalisation of Eguchi-Hanson solitons and we provided the most general form of such generalised solitons in general odd-dimensions.

To analyse the possible singularities of these metrics we switched over to their Euclidean sections by performing analytic continuations of the time coordinate t and of the NUT parameters. We analysed the regularity constraints to be imposed on these Euclidean sections in order to obtain regular metrics that can be extended globally to cover the whole manifold. We found that for generic values of the parameters these metrics are singular: it is only for a astute choice of the parameters that they become regular. We considered as particular examples the NUT-charged spaces in five and six dimensions.

Having presented the most general forms of the non-rotating Taub-NUT spaces in higher dimensions, in the remainder of this thesis we shall address some of their physical properties and possible applications. For instance, in the next two chapters we shall consider their thermodynamical properties with some very interesting results. In Chapter 6 we describe another application of the Taub-NUT solitons in the construction of Kaluza-Klein magnetic monopoles. In Chapter 7 we describe new time-dependent bubble solutions that can be obtained from the higher-dimensional NUT-charged spaces by analytical continuation.

Finally, in Chapter 8 we briefly present another generalisation of the NUT-charged spaces as solutions in Einstein-Maxwell theory. However, for space reasons, we confined ourselves to perform a simple singularity analysis of such metrics, leaving a full thermodynamical description for further work.

Chapter 4

Gravitational thermodynamics

One of the most remarkable developments in theoretical physics was the discovery of the close relationship between the laws of thermodynamics and certain laws of black hole physics. In Einstein's theory of general relativity, black holes are classical solutions that represent matter that has collapsed down to a point of infinite density, forming a singularity. This singularity indicates that the classical GR description in no longer appropriate and that it must break down at the singularity. However, while classically the black holes are perfectly stable objects, when treated quantum mechanically it turns out that black holes radiate energy and they do 'evaporate' [80]. Since they emit thermal radiation with a well-defined temperature – the so-called Hawking temperature – they also have thermodynamical temperature and one can associate a gravitational entropy with them. While classically one can understand the need to associate an entropy with a black hole – to account for instance for the entropy of matter that falls into the black hole, a microscopic description of such entropy is puzzling. Indeed, if one regards the black hole as essentially a point mass singularity, then it is very hard to understand what could be the microscopic states whose counting would give rise to that entropy. To properly understand black hole thermodynamics from a microscopic point of view one should have recourse to a true quantum gravity theory.

For detailed thermodynamical computations relevant to black hole physics one should be able to compute the partition function of the system. It is well known that the partition function for quantum fields in the canonical ensemble can be related in general to a path-integral by analytic continuation. In this approach, the partition function for the gravitational field is defined by a sum over all smooth Euclidean geometries which are periodic with a period β in imaginary time. The path-integral is computed by using the saddle point approximation in which one considers that the dominant contributions will come from metrics near the classical solutions of the Euclidean Einstein's equations with the given boundary conditions. In the semiclassical limit this yields a relationship between gravitational entropy and other relevant thermodynamic quantities, such as mass, angular momentum, and other conserved charges. This relationship was first explored in the context of black holes by Gibbons and Hawking [65], who argued that the free energy is equal to the Euclidean gravitational action multiplied by the temperature. The gravitational entropy can then be regarded as arising from the quantum statistical relation applied to the path-integral formulation of quantum gravity [81].

In this chapter we briefly review the path-integral approach to quantum gravity and its relationship to gravitational thermodynamics for asymptotically flat or asymptotically (A)dS spacetimes.

4.1 The path-integral approach to quantum gravity for flat/AdS backgrounds

According to Feynman's idea, for a quantum field ϕ the amplitude for going from a state $|t_1, \phi_1\rangle$ to a state $|t_2, \phi_2\rangle$ can be expressed as a path-integral:

$$\langle t_2, \phi_2 | t_1, \phi_1 \rangle = \int_1^2 d[\phi] e^{\mathrm{i}I[\phi]} \tag{4.1}$$

over all possible intermediate field configurations between the initial and final states. However, using the Schrödinger picture, this amplitude can also be expressed as:

$$\langle t_2, \phi_2 | t_1, \phi_1 \rangle = \langle \phi_2 | e^{-\mathrm{i}H(t_2 - t_1)} | \phi_1 \rangle \tag{4.2}$$

where H is the Hamiltonian. By imposing the periodicity condition $\phi_1 = \phi_2$ for $t_2 - t_1 = -i\beta$, we sum over ϕ_1 to obtain:

$$\mathrm{Tr}\left[\exp(-\beta H)\right] = \int d[\phi] e^{-\hat{I}[\phi]} \tag{4.3}$$

The right-hand side is now a Euclidean path integral over all field configurations that are real on the Euclidean section and periodic in the imaginary time coordinate with period β, while $\hat{I}$ is the Euclidean action.

Inclusion of gravitational effects can be carried out by considering the initial state to include a metric on a surface S_1 at time t_1 evolving to another metric on a surface S_2 at time t_2, yielding the relation:

$$\langle g_2, \Phi_2, S_2 | g_1, \Phi_1, S_1 \rangle = \int D\left[g, \Phi\right] \exp\left(iI\left[g, \Phi\right]\right) \tag{4.4}$$

This then represents the amplitude to go from a state with metric and matter fields $[g_1, \Phi_1]$ on a surface S_1 to a state with metric and matter fields $[g_2, \Phi_2]$ on a surface S_2. The quantity $D\left[g, \Phi\right]$ is a measure on the space of all field configurations and $I\left[g, \Phi\right]$ is the action taken over all fields having the given values on the surfaces S_1 and S_2.

The left-hand side of (4.3) is simply the partition function Z for the canonical ensemble for a field at temperature β^{-1} . This allows us to make the connection with thermodynamics via $\ln Z = -\beta W$, where $W = \mathfrak{M} - TS$ is the Helmholtz free energy and $\mathfrak{M}$ is the total energy. Here Z can be interpreted as describing the partition function of a gravitational system at temperature β^{-1} contained in a (spherical) box of finite radius.

We can compute Z using an analytic continuation of the action in (4.4) so that the axis normal to the surfaces S_1, S_2 is rotated clockwise by $\frac{\pi}{2}$ radians into the complex plane [65] (*i.e.* by rotating the time axis so that $t \to -iT$) in order to obtain a Euclidean signature. The path-integral is computed by using the saddle point approximation in which one considers that the dominant contributions will come from metrics near the classical solutions of Euclidean Einstein's equations with the given boundary conditions. The positivity of the Euclidean action ensures a convergent path integral in which one can carry out any calculations (of action, entropy, etc.). The presumed physical interpretation of the results is then obtained by rotation back to a Lorentzian signature at the end of the calculation.

The counterterm action for AdS backgrounds. Conserved charges

For asymptotically AdS spacetimes, the action can be generally decomposed into two distinct parts:

$$I = I_B + I_{\partial B} \tag{4.5}$$

where the bulk (I_B) and boundary ($I_{\partial B}$) terms are the usual ones, given by

$$I_B = \frac{1}{16\pi G}\int_{\mathcal{M}} d^{d+1}x \sqrt{-g}\,(R - 2\Lambda) + \int_{\mathcal{M}} d^{d+1}x \sqrt{-g}\mathcal{L}_M(\Phi) \tag{4.6}$$

$$I_{\partial B} = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^{d}x \sqrt{h}K \tag{4.7}$$

where $\partial\mathcal{M}$ represents the spatial infinity, and $\int_{\partial\mathcal{M}}$ represents an integral over the boundary with the metric h_{ab} and extrinsic curvature K. The quantity $\mathcal{L}_M(\Phi)$ in (4.6) is the Lagrangian for the matter fields, which we will not be considering here. The bulk action is over the D-dimensional manifold $\mathcal{M}$, and the boundary action is the surface term necessary to ensure well-defined Euler-Lagrange equations.

There is however one major difficulty when applying the above procedure to spacetimes of interest. This is related to the fact that when computing the action (4.5) one generally gets infinite results for non-compact spaces. There is a standard procedure to cure this problem: one regularises the action by performing the so-called 'boundary substraction' prescription. For this, one restricts the spacetime to the interior of some bounded region and then one subtracts the action of some reference spacetime with the same boundary geometry [65]. One then takes the limit in which the boundary is pushed to infinity to obtain finite results for the final action. The idea is that, for an appropriate choice of the reference spacetime, the infinities in the initial action will be cancelled out by the infinities occurring in the action of the reference spacetime and the final result is finite. In fact one can further consider the variation of this regularised action with respect to the boundary metric and define an energy-momentum tensor that can be used to define conserved charges [31].

Unfortunately, such background subtraction procedures are marred with difficulties: even if some choices of such reference background spaces present themselves as 'natural', in general these choices are by no means unique. Moreover, it is not always possible to embed a boundary with a given induced metric into the reference background and, for different boundary geometries, one needs different reference backgrounds. A good example where these problems are encountered is provided by the NUT-charged spacetimes [84, 83, 37, 34, 54].

Recently, an alternative procedure has been proposed [13, 101]. This technique was inspired by the AdS/CFT correspondence and consists of adding to the action suitable boundary counterterms $I_{\rm ct}$, which are functionals only of curvature invariants of the induced metric on the boundary. Such terms will not interfere with the equations of motion because they are intrinsic invariants of the boundary metric. By choosing appropriate counterterms, which cancel the divergences, one can then obtain well-defined expressions for the action and the energy momentum of the spacetime. Unlike the background subtraction methods, this procedure is intrinsic to the spacetime of interest and it is unambiguous once the counterterm action is specified.

Thus we have to supplement the action (4.7) with [13, 46]:[1]

$$\begin{aligned} I_{\rm ct} &= \frac{1}{8\pi G}\int d^dx\sqrt{-h}\Bigg\{-\frac{d-1}{\ell}-\frac{\ell\Theta\left(d-4\right)}{2(d-2)}\mathsf{R}-\frac{\ell^3\Theta\left(d-6\right)}{2(d-2)^2(d-4)}\left(\mathsf{R}_{ab}\mathsf{R}^{ab}-\frac{d}{4(d-1)}\mathsf{R}^2\right) \\ &\quad +\frac{\ell^5\Theta\left(d-8\right)}{(d-2)^3(d-4)(d-6)}\left(\frac{3d+2}{4(d-1)}\mathsf{R}\mathsf{R}^{ab}\mathsf{R}_{ab}-\frac{d(d+2)}{16(d-1)^2}\mathsf{R}^3\right. \\ &\quad \left.-2\mathsf{R}^{ab}\mathsf{R}^{cd}\mathsf{R}_{acbd}-\frac{d}{4(d-1)}\nabla_a\mathsf{R}\nabla^a\mathsf{R}+\nabla^c\mathsf{R}^{ab}\nabla_c\mathsf{R}_{ab}\right)+...\Bigg\}, \end{aligned} \tag{4.8}$$

where R and R^{ab} are the curvature and the Ricci tensor associated with the induced metric h. The series truncates for any fixed dimension, with new terms entering at every new even value of d, as denoted by the step-function ($\Theta\left(x\right)=1$ provided $x\geq 0$, and vanishes otherwise).

Using these counterterms in odd and even dimensions, one can construct a divergence-free boundary stress tensor from the total action $I=I_B+I_{\partial B}+I_{\rm ct}$ by defining a boundary stress-tensor:

$$T_{ab}=\frac{2}{\sqrt{-\gamma}}\frac{\delta I}{\delta h^{ab}}. \tag{4.9}$$

[1]In odd dimensions, the action can also have logarithmic divergences that are not generally cancelled by these counterterms. However, these logarithmic terms appear as effect of a trace anomaly and we have to supplement the counterterm action with further terms. See for instance [128, 86, 112].

Thus a conserved charge

$$\mathfrak{Q}_\xi = \oint_\Sigma d^{d-1} S^a \; \xi^b T_{ab}, \tag{4.10}$$

can be associated with a closed surface Σ (with normal n^a), provided the boundary geometry has an isometry generated by a Killing vector ξ^a [26]. If $\xi = \partial/\partial t$ then $\mathfrak{Q}$ is the conserved mass/energy M.

The counterterm action for asymptotically flat spaces. Conserved charges

For asymptotically flat spacetimes, the gravitational action consists of the bulk Einstein-Hilbert term and it must be supplemented by the boundary Gibbons-Hawking term in order to have a well-defined variational principle [65]. In general $(d+1)$-dimensions, the gravitational action for an asymptotically flat spacetime is then taken to be:

$$I_B + I_{\partial B} = -\frac{1}{16\pi G}\int_M d^{d+1}x\sqrt{-g}R - \frac{1}{8\pi G}\int_{\partial M} d^d x\sqrt{-h}K \tag{4.11}$$

Here M is a $(d+1)$-dimensional manifold with metric $g_{\mu\nu}$, K is the trace of the extrinsic curvature $K_{ij} = \frac{1}{2}h_i^k \nabla_k n_j$ of the boundary ∂M with unit normal n^i and induced metric h_{ij}. When evaluated on non-compact solutions of the field equations it turns out that the action (4.11) diverges. The general remedy for this situation is again to consider the values of these quantities relative to those associated with some background reference spacetime, whose boundary at infinity has the same induced metric as that of the original spacetime. The background is chosen to have a topological structure that is compatible with that of the original spacetime and also one requires that the spacetimes approaches it sufficiently rapidly at infinity. This ‘background subtraction’ prescription suffers from all the problems already mentioned in the AdS case.

While there is a general algorithm for generating the counterterms for asymptotically AdS spacetimes [101], the asymptotically flat case is considerably less-explored (see however [109] for some new results in this direction). Early proposals [102, 108, 87] engendered study of proposed counterterm expressions for a class of $(d+1)$-dimensional asymptotically flat solutions whose boundary topology is $S^n \times R^{d-n}$ [101].

For asymptotically flat 4-dimensional spacetimes, the counterterm

$$I_{ct} = \frac{1}{8\pi G}\int d^3x\sqrt{-h}\sqrt{2\mathcal{R}} \tag{4.12}$$

was proposed [102, 108] to eliminate divergences that occur in (4.11). An analysis of the higher dimensional case [101] suggested in $(d+1)$-dimensions the counterterm:

$$I_{ct} = \frac{1}{8\pi G}\int d^d x\sqrt{-h}\frac{\mathcal{R}^{\frac{3}{2}}}{\sqrt{\mathcal{R}^2 - \mathcal{R}_{ij}\mathcal{R}^{ij}}}, \tag{4.13}$$

where $\mathcal{R}_{ij}$ is the Ricci tensor of the induced metric h_{ij} and $\mathcal{R}$ is the corresponding Ricci scalar. This counterterm removes divergences in the action for a general class of asymptotically flat spacetimes with boundary topologies $S^n \times R^{d-n}$.

There also exists another simpler counterterm that removes the divergences in the action for the general class of asymptotically flat spacetimes with boundary topologies $S^n \times R^{d-n}$ [101, 109, 110]:

$$I_{ct} = \frac{1}{8\pi G}\int d^d x\sqrt{-h}\sqrt{\frac{n\mathcal{R}}{n-1}}. \tag{4.14}$$

With an eye on a later application in Chapter 6 in the computation of the Kaluza-Klein monopole energy, we will focus in the remaining of this section to the $d = 4$ case.

By taking the variation of the action (4.13) with respect to the boundary metric h_{ij} we obtain the following boundary stress-energy tensor:

$$\begin{aligned} 8\pi G(T_{ct})^{ij} \;&=\; \frac{\mathcal{R}^{\frac{1}{2}}}{(\mathcal{R}^2 - \mathcal{R}_{kl}\mathcal{R}^{kl})^{\frac{3}{2}}}\left[3\mathcal{R}^{ij}\mathcal{R}_{kl}\mathcal{R}^{kl} - \mathcal{R}^{ij}\mathcal{R}^2 + 2\mathcal{R}\mathcal{R}^{ik}\mathcal{R}^{j}{}_{k} + \mathcal{R}^3 h^{ij} - \mathcal{R}\mathcal{R}_{kl}\mathcal{R}^{kl}h^{ij}\right] \\ &\quad +\Phi^{(i}{}_{k}{}^{;j)k} - \frac{1}{2}\Box\Phi^{ij} - \frac{1}{2}h^{ij}\Phi^{kl}{}_{;kl}, \end{aligned}$$

where:

$$\Phi^{ij} \;=\; \frac{\mathcal{R}^{\frac{1}{2}}}{(\mathcal{R}^2 - \mathcal{R}_{kl}\mathcal{R}^{kl})^{\frac{3}{2}}}\left[2\mathcal{R}\mathcal{R}^{ij} + \left(\mathcal{R}^2 - 3\mathcal{R}_{kl}\mathcal{R}^{kl}\right)h^{ij}\right],$$

so that the final boundary stress energy tensor is given by:

$$T_{ij} \;=\; \frac{1}{8\pi G}\left(K_{ij} - Kh_{ij} + (T_{ct})_{ij}\right) \tag{4.15}$$

For a five-dimensional asymptotically flat solution with a fibred boundary topology $R^2 \hookrightarrow S^2$,[2] we find that the action (4.11) can also be regularised using the following

[2] By $R^2 \hookrightarrow S^2$ we understand a that the boundary has the structure of a fibre bundle constructed over the base space S^2 whose fibres have a R^2 topology.

equivalent counterterm:

$$I_{ct} = \frac{1}{8\pi G} \int d^4x \sqrt{-h}\sqrt{2\mathcal{R}} \tag{4.16}$$

where $\mathcal{R}$ is the Ricci scalar of the induced metric on the boundary, h_{ij}. By taking the variation of this total action with respect to the boundary metric h_{ij}, it is straightforward to compute the boundary stress-tensor, including (4.16):

$$T_{ij} = \frac{1}{8\pi G} \left(K_{ij} - K h_{ij} - \Psi(\mathcal{R}_{ij} - \mathcal{R} h_{ij}) - h_{ij} \Box \Psi + \Psi_{;ij} \right) \tag{4.17}$$

where we denote $\Psi = \sqrt{\frac{2}{\mathcal{R}}}$. If the boundary geometry has an isometry generated by a Killing vector ξ^i, then $T_{ij}\xi^j$ is divergence free, from which it follows that the quantity

$$\mathcal{Q} = \oint_{\Sigma} d^3 S^i T_{ij} \xi^j,$$

associated with a closed surface Σ, is conserved. Physically, this means that a collection of observers on the boundary with the induced metric h_{ij} measure the same value of $\mathcal{Q}$, provided the boundary has an isometry generated by ξ. In particular, if $\xi^i = \partial/\partial t$ then $\mathcal{Q}$ is the conserved mass $\mathcal{M}$.

The counterterm (4.13) was proposed in [101] for five-dimensional spacetimes with boundary $S^2 \times R^2$, or $S^3 \times R$. On the other hand, the counterterm (4.13) is essentially equivalent to (4.16) for $S^2 \times R^2$ boundaries. We find that when the boundary is taken to infinity both expressions cancel the divergences in the action. Our choice of using (4.16) can be motivated by the fact that the expression for the boundary stress-tensor is considerably simpler. However, different counterterms can lead to different results when computing the energy, seriously constraining the various choices of the boundary counterterms (see for instance [88, 89] for a general study of the counterterm charges and a comparison with charges computed by other means in AdS context). As we shall see in Chapter 6, when applied to the Kaluza-Klein monopole both expressions lead to a background-independent Kaluza-Klein mass that agrees with other answers previously known in the literature; however, we do find slight discrepancies in the diagonal components of the boundary stress-tensor.

4.2 The path integral approach for asymptotically dS spacetimes

The extension of the previous path-integral considerations to the asymptotically dS spacestimes is not straightforward and in this thesis we follow the proposal made by Clarkson, Ghezelbash and Mann in [38, 40, 39].

In the case of $D = (d+1)$-dimensional asymptotically de Sitter spacetimes we replace the surfaces S_1, S_2 with histories H_1, H_2 that have spacelike unit normals and are surfaces that form the timelike boundaries of a given spatial region and so they will describe particular d-dimensional histories of $(d-1)$-dimensional subspaces of the full spacetime. The amplitude (4.4) becomes:

$$\langle g_2, \Phi_2, H_2 | g_1, \Phi_1, H_1 \rangle = \int D\left[g, \Phi\right] \exp\left(iI\left[g, \Phi\right]\right) \tag{4.18}$$

and describes quantum correlations between differing histories $[g_1, \Phi_1]$ and $[g_2, \Phi_2]$ of metrics and matter fields. The correlation between a history $[g_2, \Phi_2, H_2]$ with a history $[g_1, \Phi_1, H_1]$ is obtained from the square of the modulus of this amplitude.

The surfaces H_1, H_2 are joined by spacelike tubes at some initial and final times, so that the boundary and interior region are compact. In the limit where these times approach past and future infinity one obtains the correlation between the complete histories $[g_1, \Phi_1, H_1]$ and $[g_2, \Phi_2, H_2]$. This correlation is given by summing over all metric and matter field configurations that interpolate between these two histories. The quantity $\langle g_2, \Phi_2, H_2 | g_1, \Phi_1, H_1 \rangle$ depends only on the hypersurfaces H_1 and H_2 and the metrics and matter fields over these hypersurfaces. It does not depend on any special hypersurface between the hypersurfaces H_1 and H_2.

Since the action in (4.18) is real for Lorentzian metrics and real matter fields, the path integral will not converge as its argument will be oscillatory. Its treatment therefore requires some care.

In the asymptotically de Sitter case the action is in general negative definite near past and future infinity (outside of a cosmological horizon). The natural strategy would appear to be to analytically continue the coordinate orthogonal to the histories $[g_1, \Phi_1, H_1]$ and $[g_2, \Phi_2, H_2]$ to complex values by rotating the axis normal to the histories H_1, H_2

anticlockwise by $\frac{\pi}{2}$ radians into the complex plane. The action becomes pure imaginary, yielding a convergent path integral:

$$Z' = \int e^{+\hat{I}}, \tag{4.19}$$

since $\hat{I} = iI < 0$. In this case we must take

$$+\beta W = \ln Z' \tag{4.20}$$

In the semi-classical approximation this will lead to $\ln Z' = +I_{cl}$, where I_{cl} is the action evaluated on the solution of the Einstein equations.

Substituting this into (4.20) allows us to compute the entropy of the system:

$$S = \beta\mathfrak{M} - I_{cl} \tag{4.21}$$

As before, the presumed physical interpretation of the results is obtained by rotation back to a Lorentzian signature at the end of the calculation.

4.2.1 The counterterm action in de Sitter backgrounds

For a general asymptotically dS spacetime, the action can be decomposed into three distinct parts:

$$I = I_B + I_{\partial B} + I_{ct} \tag{4.22}$$

where the bulk (I_B) and boundary ($I_{\partial B}$) terms are the usual ones, given by

$$I_B = \frac{1}{16\pi G}\int_{\mathcal{M}} d^{d+1}x\, \sqrt{-g}\,(R - 2\Lambda + \mathcal{L}_M(\Phi)) \tag{4.23}$$

$$I_{\partial B} = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}^{\pm}} d^d x\, \sqrt{h^{\pm}}K^{\pm} \tag{4.24}$$

where $\partial\mathcal{M}^{\pm}$ represents future/past infinity, and $\int_{\partial\mathcal{M}^{\pm}} = \int_{\partial\mathcal{M}^{-}}^{\partial\mathcal{M}^{+}}$ represents an integral over a future boundary minus an integral over a past boundary, with the respective metrics $h^{\pm}$ and extrinsic curvatures $K^{\pm}$. The quantity $\mathcal{L}_M(\Phi)$ in (4.23) is the Lagrangian for the matter fields, which we shall not be considering here. The bulk action is over the $(d+1)$-dimensional manifold $\mathcal{M}$, and the boundary action is the surface term necessary to ensure well-defined Euler-Lagrange equations. For an asymptotically dS spacetime, the boundary $\partial\mathcal{M}$ will be a union of Euclidean spatial boundaries at early and late times.

The counter-term action I_{ct} in (4.22) appears in the context of the dS/CFT correspondence conjecture due to the counterterm contributions from the boundary quantum CFT [13, 85]. It has a universal form for both the AdS and dS cases and it can be generated by an algorithmic procedure, without reference to a background metric, with the result [61]

$$\begin{aligned} I_{ct} &= -\int d^d x\sqrt{h}\Bigg[-\frac{d-1}{l}+\frac{l\Theta(d-3)}{2(d-2)}\mathsf{R}-\frac{l^3\Theta(d-5)}{2(d-2)^2(d-4)}\left(\mathsf{R}_{ab}\mathsf{R}^{ab}-\frac{d}{4(d-1)}\mathsf{R}^2\right) \\ &\quad -\frac{l^5\Theta(d-7)}{(d-2)^3(d-4)(d-6)}\left(\frac{3d+2}{4(d-1)}\mathsf{R}\mathsf{R}^{ab}\mathsf{R}_{ab}-\frac{d(d+2)}{16(d-1)^2}\mathsf{R}^3\right. \\ &\quad \left. -2\mathsf{R}^{ab}\mathsf{R}^{cd}\mathsf{R}_{acbd}-\frac{d}{4(d-1)}\nabla_a\mathsf{R}\nabla^a\mathsf{R}+\nabla^c\mathsf{R}^{ab}\nabla_c\mathsf{R}_{ab}\right)+\ldots\Bigg] \end{aligned} \tag{4.25}$$

with R the curvature of the induced metric h at the future/past infinity and $\Lambda = \frac{d(d-1)}{2l^2}$. The step-function $\Theta(x)$ is unity provided $x \geq 0$ and vanishes otherwise. For example, in four ($d = 3$) dimensions, only the first two terms appear, and only these are needed to cancel divergent behavior in $I_B + I_{\partial B}$ near past and future infinity.

Conserved charges in dS backgrounds

Varying the action with respect to the boundary metric h_{ij} gives us the boundary stress-energy tensor:

$$T^{\pm ab} = \frac{2}{\sqrt{h^{\pm}}}\frac{\delta I}{\delta h^{\pm ab}} \tag{4.26}$$

If the boundary geometries have an isometry generated by a Killing vector $\xi^{\pm\mu}$, then $T^{\pm}_{ab}\xi^{\pm b}$ is divergence free, from which it follows that the quantity

$$\mathfrak{Q}^{\pm} = \oint_{\Sigma^{\pm}} d^{d-1}\varphi^{\pm}\sqrt{\sigma^{\pm}}n^{\pm a}T^{\pm}_{ab}\xi^{\pm b} \tag{4.27}$$

is conserved between histories of constant t, whose unit normal is given by $n^{\pm a}$. The φ^a are coordinates describing closed surfaces Σ, where we write the boundary metric(s) of the spacelike tube(s) as

$$h^{\pm}_{ab}d\hat{x}^{\pm a}d\hat{x}^{\pm b} = d\hat{s}^{\pm 2} = N_T^{\pm 2}dT^2 + \sigma^{\pm}_{ab}\left(d\varphi^{\pm a}+N^{\pm a}dT\right)\left(d\varphi^{\pm b}+N^{\pm b}dT\right) \tag{4.28}$$

where $\nabla_\mu T$ is a spacelike vector field that is the analytic continuation of a timelike vector field. Physically this means that a collection of observers on the hypersurface all observe the same value of $\mathfrak{Q}$ provided this surface has an isometry generated by ξ^b.

If $\partial/\partial T$ is itself a Killing vector, then we define the conserved quantity associated with it as the conserved mass[3] associated with the future/past surface $\Sigma^\pm$ at any given point T on the spacelike future/past boundary. Since all asymptotically de Sitter spacetimes must have an asymptotic isometry generated by $\partial/\partial T$, there is at least the notion of a conserved total mass $\mathfrak{M}^\pm$ for the spacetime in the limit that $\Sigma^\pm$ are future/past infinity.

4.2.2 The maximal mass conjecture

Following the above prescription to compute the conserved charges in dS backgrounds, Balasubramanian, de Boer and Minic put forward in Ref. [17] the following conjecture (referred to as the maximal mass conjecture or the BBM conjecture)[4]:

Any asymptotically de Sitter space whose mass exceeds that of de Sitter contains a cosmological singularity.

This conjecture is supported by explicit calculations of the masses of higher dimensional (topological and dilatonic) Schwarzschild-dS solutions: it was found that these masses are always less than those of dS spaces in the corresponding dimensions. In other words, the de Sitter space is more massive than the black hole spacetimes. Furthermore they argued that this result is consistent with the putative dS/CFT conjecture and also with the so-called Bousso bound [27, 28]:

The entropy of dS space is an upper bound for the entropy of any asymptotically dS space.

Balasubramanian et. al. argued that if dS is dual to a Euclidean CFT then defining the conserved charges in a manner purely analogous with the one in AdS/CFT they should correspond naturally to the energies and conserved quantities characterising states of the dual theory. Since generic field theories have entropies that increase with energy, then the large Schwarzschild-dS black holes should be mapped into states with lower energy than

[3] Technically, since $\partial/\partial T$ is a spacelike Killing vector, the conserved quantity associated with it should be a momentum; however, here we are following the mass definition proposed in [17].

[4] By cosmological singularity we shall understand here a curvature singularity.

that of dS. If Bousso's entropy bound is valid and there are no asymptotically dS spaces with entropy larger than that of dS, this would mean that a space with mass greater than that of dS should be pathological and contain cosmological singularities.

As we shall see in Chapter 5, the NUT-charged spacetimes provide us with an explicit counter-example to this conjecture.[5]

4.2.3 The $\mathbb{R}$ and $\mathbb{C}$-approaches to thermodynamics

In order to compute the thermodynamic relationships between conserved quantities in asymptotically de Sitter spacetimes, we must deal with the analytic continuation of the metric into a Euclidean section. Analytic continuation in the rotating case requires special care [25]. More recently it was noted that although the computation of conserved quantities does not depend upon such analytic continuation, the path-integral foundations of thermodynamics at asymptotic past/future infinity does, and that there are two apparently distinct ways of expressing the metric, depending on which set of Wick rotations is chosen [40, 38].

In the first approach, one deals with an analytically continued version of the metric that involves not only a complex rotation of the (spacelike) t coordinate $(t \to iT)$, but also an analytic continuation of the rotation and NUT charge parameters (if any) in the metric, yielding a metric of signature $(-,-,+,+,\ldots)$. Upon calculation, this will give rise to a negative action, and hence a negative definite energy. One must also periodically identify T with period β in order to eliminate possible conical singularities in the $(-,-)$ section of the metric. This is the so-called $\mathbb{C}$-approach, since it involves a rotation into the complex plane. One advantage of using this approach is that T is a 'time' coordinate and the conserved quantity associated with the Killing vector $\partial/\partial T$ can be identified unambiguously with a total mass of the system.

In the $\mathbb{R}$-approach, the analysis is carried out using the unmodified metric with Lorentzian signature; no analytic continuation is performed on the coordinates and/or the parameters that appear in the metric. This option appears because the t coordinate is spacelike outside the cosmological horizon, and so (a semi-classical path-integral) evaluation of thermody-

[5] As long as we do not interpret the quasiregular singularities as cosmological singularities.

namic quantities at past/future infinity does not necessarily require its analytical continuation [40]. Instead, one evaluates the action at past/future infinity, imposing periodicity in t, consistent with regularity at the cosmological horizon (given by the surface gravity of the cosmological horizon of the $(+,-)$ section). There is no need to analytically continue either the rotation parameters or NUT charges to complex values, and consequently there is no need to analytically continue any results to extract a physical interpretation. Note that we do not actually compute path-integral quantities using this approach; rather in this context the preceding path-integral methods are employed primarily as a device to justify eq. (4.21).

4.2.4 A simple example: The Schwarzschild-dS solution

As a simple application of this formalism consider the Scharwzschild-dS solution in 4-dimensions, outside the cosmological horizon

$$ds^2 = -\frac{d\tau^2}{F(\tau)} + F(\tau)dt^2 + \tau^2 d\Omega_2^2$$

where

$$F(\tau) = \left(\frac{\tau^2}{l^2} + \frac{2m}{\tau} - 1\right)$$

Working in the Lorentzian signature (what we called the $\mathbb{R}$-approach above) we obtain the following results for the action and the conserved mass [61]:

$$I_{SdS} = -\frac{\left(m + \frac{\tau_+^3}{l^2}\right)\beta_r}{2}, \quad \mathfrak{M} = -m,$$

Here τ_+ is the radius of the cosmological horizon and $\beta_r = \int dt$.

If the range of the t-coordinate is infinite the action will in general diverge. It is however tempting to impose a periodicity of the time coordinate even in the Lorentzian sector that is consistent with the periodicity of the analytically continued time $t \rightarrow iT$. We therefore turn to the $\mathbb{C}$-approach, in which the new metric has signature $(-,-,+,+)$. The sector (τ, T) will have a conical singularity unless the T coordinate is periodically identified with period

$$\beta_c = \frac{4\pi}{|F'(\tau_+)|} \tag{4.29}$$

Since under the analytic continuation $T \rightarrow it$ the periodicity β_c remains unaffected, there is no obstruction in considering a similar condition in the Lorentzian sector as well; by continuity we must require that $\beta_r = \beta_c$. This will render finite all the physical quantities of interest and allow a definition of the entropy in the Lorentzian sector by means of the extended Gibbs-Duhem relation (4.21). The result is $S = \pi\tau_+^2$, equal to one quarter of the area of the cosmological horizon.

While the Schwarzschild-dS case is somehow trivial, in the sense that the equivalence between the $\mathbb{C}$- and $\mathbb{R}$- approaches fixes an otherwise arbitrary periodicity in the space-like t coordinate, this method has been recently extended to the non-trivial case of four-dimensional Kerr-dS spacetimes [63]. In NUT-charged spacetimes the situation is considerably less trivial, since there are independent geometric reasons for fixing the periodicity of t in the $\mathbb{R}$-approach, *i.e.* in the Lorentzian sector.

4.3 Summary

In this chapter we briefly reviewed the path-integral approach to quantum gravity and its relationship to gravitational thermodynamics for asymptotically flat or asymptotically (A)dS spacetimes. In this approach, the partition function for the gravitational field is defined by a sum over all smooth Euclidean geometries which are periodic with a period β in imaginary time. The path-integral is computed by using the saddle point approximation in which one considers that the dominant contributions to the path-integral will come from metrics near the classical solutions of Euclidean Einstein's equations with the given boundary conditions. In the semiclassical limit this yields a relationship between gravitational entropy and other relevant thermodynamic quantities, such as mass, angular momentum, and other conserved charges. In particular, we have seen that the gravitational entropy can then be regarded as arising from the the quantum statistical relation or the generalised Gibbs-Duhem applied to the path-integral formulation of quantum gravity.

In general, for spaces that are asymptotically AdS or flat, we can compute the partition function using an analytic continuation of the action so that the axis normal to the surfaces S_1, S_2 is rotated clockwise by $\frac{\pi}{2}$ radians into the complex plane (*i.e.* by rotating the time axis so that $t \rightarrow -iT$) in order to obtain a Euclidean signature. The positivity of the Euclidean action ensures a convergent path integral in which one can carry out any calculations (of action, entropy, etc.). The presumed physical interpretation of the results is then obtained by rotation back to a Lorentzian signature at the end of the calculation.

However, for spaces that are asymptotically de Sitter, we have described two approaches to do thermodynamics. In one approach (referred to as the $\mathbb{R}$-approach), the analysis is carried out using the unmodified metric with Lorentzian signature; no analytic continuation is performed on the coordinates and/or the parameters that appear in the metric. In the alternative $\mathbb{C}$-approach one deals with an analytically continued version of the metric and at the end of the computation all the final results are analytically continued back to the Lorentzian sector.

We have also presented a set of counterterms that will cancel-out the divergences that appear in the gravitational action for spaces that are asymptotically (A)dS. These counterterms are motivated by the AdS/CFT, respectively by the still unknown dS/CFT conjecture. Since holography for asymptotically flat spaces is an even-less explored subject, there is no known[6] simple universal choice of the counterterm terms to cancel all the divergences in the action of such spaces. However, we have presented two counterterms that cancel out the divergences in a large class of asymptotically flat spacetimes with general boundary topology $S^n \times R^{d-n}$.

The results from this chapter will provide us with the main tools to use in the next chapters when we will discuss and analyse in detail the thermodynamics of various NUT-charged spaces.

[6]To our present knowledge.

Chapter 5

On the Thermodynamics of NUT Charged Spaces

As we have seen in the previous chapter, in asymptotically flat or (A)dS settings, the thermodynamic relationships between the various conserved charges may be established using the path-integral formalism of semi-classical quantum gravity. For spaces with rotation or NUT charge, such relationships depend upon how one analytically continues these parameters into a Euclidean section. We have described two main approaches to discuss the thermodynamics of these spaces.

In one approach (referred to as the $\mathbb{R}$-approach), the analysis is carried out using the unmodified metric with Lorentzian signature; no analytic continuation is performed on the coordinates and/or the parameters that appear in the metric. In the alternative $\mathbb{C}$-approach one deals with an analytically continued version of the metric and at the end of the computation all the final results are analytically continued back to the Lorentzian sector.

However, the simple analytic continuation of Euclidean time into the Lorentzian time $T \rightarrow it$ fails in many physically interesting cases (for example, if the spacetimes are manifestly not static, or even in the stationary cases). While this is a known issue in rotating spacetimes [25], it is shown most strikingly in the case of NUT-charged spacetimes. Consider for example Ricci flat Taub-NUT spacetime. As we have seen in Chapter 2,[1] this

[1] In this chapter we choose to denote by t, n the 'time', respectively the NUT charge in the *Lorentzian*

spacetime is non-singular only if we make the time coordinate t periodic with period $8\pi n$, in order to eliminate the Misner string singularity. To obtain the Euclidean sector we perform the analytic continuation of the time coordinate $t \to iT$ and of the NUT parameter $n \to iN$. To keep the Euclidean section non-singular, that is in order to eliminate a possible conical singularity that would appear in the (r, T) section, a constraint relating the mass to the NUT charge must be imposed, consistent with the preceding periodicity requirements (which imply $\beta = 8\pi n$). For the Taub-Nut solution this is $m = n$ whereas for the Taub-Bolt solution $m = \frac{5}{4}n$. However, the physical interpretation of the results in terms of the parameters appearing in the Lorentzian sector is somewhat problematic since a naive analytic continuation would send $T \to it$ and $N \to iN$ and render imaginary the physical quantities of interest.

Our main goal in this chapter is to clarify the relationship between the $\mathbb{R}$ and $\mathbb{C}$ approaches, with an eye toward understanding how to physically interpret the results obtained in each case. We find that the results of both these approaches are completely equivalent modulo analytic continuation. Furthermore, we provide an exact prescription that relates the results in both methods. Extending our methods to asympotically AdS/flat cases yields a physical interpretation of the thermodynamics of NUT-charged spacetimes in the Lorentzian sector. We discuss the constraints that appear by imposing the first law of thermodynamics. We find that the first law will hold precisely for the (asymptotically AdS) Bolt and Nut solutions that we obtain in the $\mathbb{C}$-and $\mathbb{R}$-approaches, which we take as a sign of the validity of our results regarding the thermodynamics of the Lorentzian Taub-NUT-(A)dS solutions. We also briefly discuss the case of higher dimensional NUT-charged spacetimes.

5.1 Thermodynamics of Taub-NUT-dS spaces

In asymptotically (A)dS/flat spacetimes with NUT charge there is an additional periodicity constraint for t that arises from demanding the absence of Misner-string singularities. When matched with the periodicity β, this yields an additional consistency criterion that relates the mass and NUT parameters, the solutions of which produce generalisations of

section, while T, N will be the 'time' and the NUT charges in the *Euclidean* section.

asymptotically flat Taub-Bolt/Nut space to the asymptotically (A)dS case.

For simplicity we shall concentrate mainly on the four-dimensional case. However, as we shall see in the last section, our results can be easily generalised to higher dimensional situations.

Consider the spherical Taub-NUT-dS solution, which is constructed as a circle fibration over the sphere in de Sitter background:

$$ds^2 = V(\tau)(dt + 2n\cos\theta d\phi)^2 - \frac{d\tau^2}{V(\tau)} + (\tau^2 + n^2)(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.1}$$

where

$$V(\tau) = \frac{\tau^4 + (6n^2 - l^2)\tau^2 + 2ml^2\tau - n^2(3n^2 - l^2)}{(\tau^2 + n^2)l^2} \tag{5.2}$$

As noted in the previous section, there are two different approaches to describe the thermodynamics of such solutions, namely the $\mathbb{C}$- and the $\mathbb{R}$-approach depending on the various analytic continuations that can be done.

5.1.1 The $\mathbb{C}$-approach results

In the $\mathbb{C}$-approach one analytically continues the coordinates in the (t, τ) sector such that the signature in this section becomes $(++)$ or $(--)$. One way to accomplish this is to analytically continue the coordinate $t \to iT$ and the nut charge parameter $n \to iN$. One obtains the metric:

$$ds^2 = -F(\tau)(dT - 2N\cos\theta d\phi)^2 - \frac{d\tau^2}{F(\tau)} + (\tau^2 - N^2)(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.3}$$

where now

$$F(\tau) = \frac{\tau^4 - (6N^2 + l^2)\tau^2 + 2ml^2\tau - N^2(3N^2 + l^2)}{(\tau^2 - N^2)l^2} \tag{5.4}$$

The conserved mass associated with this solution is [40, 38]

$$\mathcal{M}_c = -m \tag{5.5}$$

independently of whether the function $F(\tau)$ has any roots. Note that if we take $m < 0$, this will violate the maximal mass conjecture. When $F(\tau)$ has roots the parameters (m, N) are constrained relative to one another by additional periodicity requirements. Even in this case the maximal mass conjecture can be violated [40, 38].

Notice that indeed the signature of the metric becomes in this case $(--++)$, while the action becomes complex and the path-integral in (4.19) converges. When analysing the singularity structure of such spaces we have to take into account the presence of Misner string singularities as well as the possible conical singularities in the (T,τ) sector. To eliminate the Misner string singularity we impose the condition that the coordinate T has in general the periodicity $\frac{8\pi N}{q}$, where q is a positive integer which will also determine the topological structure of these solutions (see also [95]). To see this, notice that regularity of the 1-form $(dT - 2N\cos\theta d\phi)$ is achieved once we set the periodicity of t to be given compatible with the integrals of $2N\sin\theta d\theta\wedge d\phi$ over all 2-cycles in the base manifold. In 4-dimensions the base is S^2 thence the value of the integral is $8\pi N$. While one could simply consider the Bolt solution corresponding to $q=1$, if $q>1$ then the topology of the Bolt solution is in general that of an R^2/Z_q fibration over S^2. In order to get rid of the conical singularities in the (T,τ) sector we require the coordinate T be periodically identified with periodicity given by $\beta_c = \frac{4\pi}{|F'(\tau_c)|}$, where τ_c is a root of $F(\tau)$, i.e. $F(\tau_c)=0$, provided that such a root exists. For consistency, we have to match the values of the two obtained periodicities and this yields the condition:

$$\beta_c = \frac{4\pi}{|F'(\tau_c)|} = \frac{8\pi|N|}{q} \tag{5.6}$$

After some algebra, it can be readily checked that in this case we obtain

$$m = m_c = -\frac{\tau_c^4 - (l^2+6N^2)\tau_c^2 - N^2(l^2+3N^2)}{2l^2\tau_c}$$

where

$$\tau_c^{\pm} = \frac{ql^2 \pm \sqrt{q^2l^4 + 48N^2l^2 + 144N^4}}{12|N|}$$

In order to satisfy the condition $|\tau_c| > |N|$ we must consider the positive sign in the preceding expression for $N>0$, and the negative sign for $N<0$ and in the last case we should also require $q=1$.

Working at future infinity it can be shown that if function $F(\tau)$ has roots then we obtain the action, respectively the entropy:

$$I_c = -\frac{\beta_c(\tau_c^3 - 3N^2\tau_c + m_c l^2)}{2l^2}, \qquad S_c = \frac{\beta_c(\tau_c^3 - 3N^2\tau_c - m_c l^2)}{2l^2} \tag{5.7}$$

A more detailed account of the thermodynamics of these solutions can be found in [40, 38]. An interesting path-integral study of these solutions aimed at their cosmological interpretations appeared in [95]. However, while $\tau = \tau_c^+$ is the largest root of the upper branch solutions, it can easily be checked that for all lower branch solutions the function $F(\tau, \tau_c^-)$ has always two roots τ_1 and τ_2 such that $\tau_1 < \tau_c^- < \tau_2$. Hence these solutions, though they *apparently* respect the first law of thermodynamics, are not valid dS-bolt solutions. Rather they are the analytic continuation of lower-branch AdS-bolt solutions, as we shall see below. Furthermore, they have no counterpart in the $\mathbb{R}$-approach, as we shall also see.

5.1.2 The $\mathbb{R}$-approach results

In the $\mathbb{R}$-approach one does not analytically continue either the coordinates or the parameters in the metric. Instead one directly uses the metric in the Lorentzian signature

$$V(\tau) = \frac{\tau^4 + (6n^2 - l^2)\tau^2 + 2ml^2\tau - n^2(3n^2 - l^2)}{(\tau^2 + n^2)l^2} \tag{5.8}$$

where n is the nut charge. In [40] the conserved mass was found to be

$$\mathcal{M}_r = -m \tag{5.9}$$

for arbitrary values of the parameters (m, n). Again, setting $m < 0$ will violate the maximal mass conjecture.

Notice that in this case the coordinate t parameterizes a circle fibered over the 2-sphere with coordinates (θ, ϕ). In the $\mathbb{R}$ -approach one imposes directly the periodicity condition on the spacelike coordinate t:

$$\beta_r = \frac{4\pi}{|V'(\tau_r)|} = \frac{8\pi n}{k} \tag{5.10}$$

for points where $V(\tau_r) = 0$ (provided that τ_r exists) with k a positive integer. Since these surfaces are two-dimensional (they are the usual Lorentzian horizons) we shall still refer to them as 'bolts'. From the above condition one obtains:

$$m = m_r = -\frac{\tau_r^4 + (6n^2 - l^2)\tau_r^2 + n^2(l^2 - 3n^2)}{2l^2\tau_r} \tag{5.11}$$

where now

$$\tau_r^{\pm} = \frac{kl^2 \pm \sqrt{k^2l^4 - 144n^4 + 48n^2l^2}}{12n} \tag{5.12}$$

In order to have real roots we must impose the condition that the discriminant above be positive. This restricts the possible values of n and l such that:

$$|n| \leq \left(\frac{2 + \sqrt{4 + k^2}}{12}\right)^{\frac{1}{2}} l$$

Provided that such a τ_r exists we obtain

$$I_r = -\frac{\beta_r(m_r l^2 + \tau_r^3 + 3n^2\tau_r)}{2l^2}, \qquad S_r = -\frac{\beta_r(m_r l^2 - 3n^2\tau_r - \tau_r^3)}{2l^2} \tag{5.13}$$

for the action and entropy respectively. Although the additional periodicity constraint (5.6) imposes further restrictions, the maximal mass conjecture can again be violated for certain values of the parameters [40, 38]. It can be readily checked the first law of thermodynamics is satisfied for both the upper (τ_r^+)and lower (τ_r^-) branch solutions. However, as in the previous situation using the $\mathbb{C}$-approach, the function $V(\tau)$ will always have two roots τ_1 and τ_2 such that $\tau_1 < \tau_r^- < \tau_2$. That the first law holds for the lower branch is a direct consequence of the fact that this solution can be regarded as the analytic continuation $l \to il$ of one of the Bolt solutions in the Taub-NUT AdS case (see equation (5.39) bellow), for which the first law holds.

In what follows we shall restrict our analysis only to the upper branch solutions given by τ_r^+.

5.1.3 From the $\mathbb{C}$-approach to the $\mathbb{R}$-approach

As we have seen above we have two apparently distinct approaches for describing thermodynamics of Taub-NUT-dS spaces. In the $\mathbb{C}$-approach we consider the analytic continuation of the spacelike t coordinate and of the nut parameter. While this procedure will generally lead to a 'wrong signature' metric, this is simply a consequence of the fact that we work in the region outside the cosmological horizon; the metric inside the cosmological horizon has Euclidean signature. The signature in the (T, τ) sector is in this case $(-, -)$ and we impose a periodicity of the coordinate T in order to get rid of the possible conical singularities in this sector. When matched with the periodicity required by the absence of the Misner string singularity, this will fix, in general, the form of the mass parameter and the location of the nuts and bolts. We can now use the counterterm method to compute conserved quantities and study the thermodynamics of these solutions.

On the other hand, in the $\mathbb{R}$-approach we work directly with the fields defined in the Lorentzian signature section. There is a periodicity of the spacelike coordinate t that appears from the requirement that there are no Misner string singularities; however since the signature in the (t, τ) sector is now $(+, -)$ there are no conical singularities to be eliminated, so that there is no apparent reason to impose an extra condition as in (5.10). Indeed, in the absence of the Hopf-type fibration the coordinate t is not periodic.

However, there is no a-priori obstruction in formally satisfying eq. (5.10), and then using[2] the counterterm method to compute the conserved quantities and study the thermodynamics of these solutions as it was done in refs. [40, 38]. We shall show in what follows that in general the $\mathbb{R}$-approach results are just the analytic continuation of the $\mathbb{C}$-approach results and vice-versa. We shall later show that this affords a physical interpretation of the thermodynamics of Lorentzian nut-charged spacetimes.

To motivate this claim we consider the following. In order to obtain a Euclidean signature (positive or negative definite) in the (t, τ) sector we must perform the analytic continuations $t \to iT$ and $n \to iN$. However, since the function $V(\tau)$ depends only on n and not on t its analytic continuation will be given by:

$$F(\tau) = \frac{\tau^4 - (6N^2 + l^2)\tau^2 + 2ml^2\tau - N^2(3N^2 + l^2)}{(\tau^2 - N^2)l^2} \qquad (5.14)$$

It is readily seen that using these analytical continuations we obtain the metric used in the $\mathbb{C}$-approach. The key point to notice here is that we can go back to the Lorentzian section by again employing the analytic continuations $T \to it$ and $N \to in$. Since only

[2] Notice that we use the metric in the Lorentzian signature.

even powers of n appear we are guaranteed that the above continuations will take us from the Lorentzian section to the 'Euclidean' one and back.

In the $\mathbb{C}$-approach it makes sense to consider the removal of conical singularities in the (T, τ) sector (since the signature of the metric in that sector is $(--)$), as well as to match this periodicity condition with the one arising by requiring the absence of Misner string singularities. Hence we are fully entitled to impose the condition:

$$\beta_c = \frac{4\pi}{|F'(\tau_c)|} = \frac{8\pi N}{q} \tag{5.15}$$

where q is a positive integer and τ_c is such that $F(\tau_c) = 0$.

Let us consider now the effect of the analytic continuations $T \to it$ and $N \to in$ on the above condition. Since nothing depends on T explicitly, all that matters is the effect of the analytic continuation of the NUT charge. If we continue $N \to in$ we obtain $V(\tau) = F(\tau)|_{N=in}$. Thus in the above periodicity condition we obtain

$$\left(\frac{4\pi}{|F'(\tau_c)|}\right)_{N=in} = \frac{4\pi}{|V'(\tau_r)|} \tag{5.16}$$

where now τ_r is such that $V(\tau_r) = F(\tau_c) = 0$. Hence $\beta_r = (\beta_c)_{N=in}$. However, as we can see from the second equality in (5.15) we can consistently analytically continue $N \to in$ only if we also continue $q \to ik$. Again this assures us that $\beta_r = (\beta_c)_{N=in}$ and that it is real.

Thus the prescription to get the $\mathbb{R}$-results from the $\mathbb{C}$-results is as follows: using the $\mathbb{C}$-results perform the analytic continuations $T \to it$, $N \to in$ *and* $q \to ik$. A naive analytic continuation only of T and N but without continuing q is simply inconsistent[3]: from eq. (5.15) the left hand side remains real while the right hand side becomes complex!

Let us check this prescription by obtaining the $\mathbb{R}$-results starting from the $\mathbb{C}$-results for the thermodynamic quantities given above. Consider first the location of the bolts in the $\mathbb{C}$-approach:

$$\tau_c^+ = \frac{ql^2 + \sqrt{q^2 l^4 + 48N^2 l^2 + 144N^4}}{12N} \tag{5.17}$$

[3]If we rewrite eq. (5.15) by taking the absolute value of both sides then the continuation $q \to ik$ while no longer necessary, is still permitted.

If we perform $N \to in$ and $q \to ik$ we obtain $\tau_c \to \tau_r$ where:

$$\tau_r^+ = \frac{kl^2 + \sqrt{k^2l^4 - 144n^4 + 48n^2l^2}}{12n} \tag{5.18}$$

This is indeed the location of the 'bolt' in the $\mathbb{R}$-approach as we can see from (5.12).

Performing the analytic continuations $N \to in$ and $\tau_c^+ \to \tau_r^+$ in the expression for the mass parameter in the $\mathbb{C}$-approach we obtain $m_c \to m_r$ where:

$$m_r = -\frac{(\tau_r^+)^4 + (6n^2 - l^2)(\tau_r^+)^2 + n^2(l^2 - 3n^2)}{2l^2\tau_r^+} \tag{5.19}$$

which again is the mass parameter from the $\mathbb{R}$-approach. Notice that if we naively analytically continue $N \to in$ and ignore the condition $q \to ik$ we obtain imaginary values for the corresponding results in the $\mathbb{R}$-approach. However both the above analytic continuations conspire to always produce real quantities in the final results.

A closer look at the expressions for the action, conserved mass and the entropy in the $\mathbb{C}$-approach shows that if we perform the continuations $N \to in$, $\tau_c^+ \to \tau_r^+$ and $m_c \to m_r$ we obtain the respective expressions from the $\mathbb{R}$-approach.

5.2 No dS Nuts

It is known that in the asymptotically AdS/flat case, besides the usual Taub-Bolt solutions, one can also obtain the so-called Taub-Nut solutions. For these solutions the fixed-point set of the Killing vector $\frac{\partial}{\partial T}$ is zero-dimensional.

Superficially, a similar situation appears to hold in the $\mathbb{C}$-approach in dS backgrounds [40]. This can happen only if $\tau_c = N$ in the above equations, that is if $F(\tau_c = N) = 0$ and also

$$\frac{4\pi}{|F'(\tau_c = N)|} = \frac{8\pi N}{q} \tag{5.20}$$

are satisfied. Although such an equation has solutions, we find that the situation is somewhat more complicated than previously described in ref. [40].

Solving (5.20) we find $q = 1$, *i.e.* the periodicity of the T coordinate is $8\pi N$, while the mass parameter becomes:

$$m_c = \frac{N(l^2 + 4N^2)}{l^2} \tag{5.21}$$

and indeed $\tau_c = N$ is a fixed point set of zero dimensionality. However it is not the largest root of the function $F(\tau)$ as given in (5.14). Instead this nut is contained within a larger cosmological 'bolt' horizon located at $\tau = \tau_{ch} = \sqrt{4N^2 + l^2} - N$. In this sense there are no dS Nuts, *i.e.* no outermost cosmological horizons that are dimension zero fixed point sets of the Killing vector $\frac{\partial}{\partial T}$.

Note, however that if we insert this value for the mass parameter into eqs. (5.5) and subsequently (4.21) we obtain the action and respectively the entropy:

$$I_c = -\frac{4\pi N^2(l^2 + 2N^2)}{l^2}, \qquad S_c = -\frac{4\pi N^2(l^2 + 6N^2)}{l^2} \tag{5.22}$$

These values correspond to those derived for the Taub-Nut-dS solution studied in [40], and it is straightforward to show that the first law of thermodynamics is obeyed.

However the physical interpretation of this solution is not as a Taub-Nut in dS background, since the use of such formulae is predicated on $\tau_c = N$ being the largest root of F. Rather this solution is the AdS-NUT under the analytic continuation $l \to il$ (see (5.28) below). We shall discuss the corresponding solution when we address the AdS case.

It is straightforward to show that the putative 'bolt' solution, with $\tau_{ch} = \sqrt{4N^2 + l^2} - N$, yields an entropy that does not respect the first law of thermodynamics. This presumably is a consequence of the fact that we eliminated the conical singularity at the root $\tau_c = N$, by fixing the periodicity of the Euclidean time T to be $8\pi N$, while leaving a conical singularity that can not be eliminated at the outer root τ_{ch}! However, upon further inspection we find that if we choose to eliminate the conical singularity at the outer root and fix the periodicity of the Euclidean time coordinate to be $\beta_c = \frac{4\pi}{|F'(\tau_{ch})|}$, we still obtain a singular space. This is because the Misner string singularity cannot be simultaneously eliminated unless we impose a relationship between the NUT charge and the cosmological constant.[4] Furthermore, the entropy (as computed via the counterterm method) does not satisfy the first law. The difficulties in ascribing a consistent thermodynamic interpretation to this solution make its physical relevance a dubious prospect.

[4] In this case we might be able to recover a regular Euclidean instanton, having a topology similar with that of CP^2: with a nut at $\tau_c = N$ and a bolt at τ_{ch}.

5.3 Taub-NUT solutions in AdS/flat backgrounds

Motivated by the results of the previous sections we shall now extend our prescription to describe the thermodynamics of the Taub-NUT solutions in AdS or flat backgrounds. To our knowledge, the thermodynamics of such solutions have been discussed only in the $\mathbb{C}$-approach (*i.e.* in Euclidean regime) in [108, 84, 34, 37, 54]. To begin with, let us recall the metric of the Taub-NUT AdS solution in four dimensions:

$$ds^2 = -V(r)(dt - 2n\cos\theta d\phi)^2 + V^{-1}(r)dr^2 + (r^2+n^2)d\Omega^2 \tag{5.23}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the sphere S^2 and

$$V(r) = \frac{r^4 + (l^2+6n^2)r^2 - 2mrl^2 - n^2(l^2+3n^2)}{l^2(n^2+r^2)} \tag{5.24}$$

This metric is a solution of the vacuum Einstein field equations with negative cosmological constant $\lambda = -\frac{3}{l^2}$. In the limit $l \to \infty$ it reduces to the usual asymptotically (locally) flat Taub-NUT solution.

In the $\mathbb{C}$-approach we analytically continue the time coordinate $t \to iT$ and the NUT charge $n \to iN$. We obtain a Euclidean signature metric of the form:

$$ds^2 = F(r)(dT - 2N\cos\theta d\phi)^2 + F^{-1}(r)dr^2 + (r^2 - N^2)d\Omega^2 \tag{5.25}$$

where

$$F(r) = \frac{r^4 + (l^2-6N^2)r^2 - 2mrl^2 + N^2(l^2-3N^2)}{l^2(r^2-N^2)} \tag{5.26}$$

When discussing the singularity structure of these spaces we must impose two regularity conditions. First, removal of the Misner string singularities leads us to periodically identify the coordinate T with period $\frac{8\pi N}{q}$, where q is a non-negative integer. Now, if we match this value with the periodicity obtained by removing the conical singularities at the roots r_c of the function $F(r)$ we obtain in general

$$\beta_c = \frac{4\pi}{F'(r_c)} = \frac{8\pi N}{q} \tag{5.27}$$

Again we have two distinct cases to consider: in the Taub-Nut solution we impose $r_c = N$ (which makes the fixed-point set of the isometry ∂_T zero-dimensional), whereas for the bolt solutions $r_c = r_{b\pm} > N$ (for which the fixed-point set of ∂_T is two-dimensional).

For the Taub-Nut solution $r_c = N$, the periodicity of the coordinate T is found to be $8\pi N$, *i.e.* $q = 1$, and the value of the mass parameter is:

$$m_c = \frac{N(l^2 - 4N^2)}{l^2}$$

The action and entropy are:

$$I_c = \frac{4\pi N^2(l^2 - 2N^2)}{l^2}, \qquad S_c = \frac{4\pi N^2(l^2 - 6N^2)}{l^2} \tag{5.28}$$

while the specific heat $C = -\beta_c \partial_{\beta_c} S$ is given by:

$$C_c = \frac{8\pi N^2(12N^2 - l^2)}{l^2}$$

Notice that the energy $\mathfrak{M} = m$ becomes negative if $N > \frac{l}{2}$ while the action becomes negative for $N > \frac{l}{\sqrt{2}}$. When this latter inequality is saturated we recover the Euclidean AdS spacetime. The entropy is negative if $N > \frac{l}{\sqrt{6}}$ while for $N < \frac{l}{\sqrt{12}}$ the specific heat becomes negative, which signals thermodynamic instabilities. Therefore, as it has been argued in ref. [54], in order to obtain physically relevant solutions with both positive entropy and specific heat, one should restrict the values of the NUT charge such that:

$$\frac{l}{\sqrt{12}} \leq N \leq \frac{l}{\sqrt{6}} \tag{5.29}$$

The other possibility corresponds to the Bolt solutions, for which $r = r_c > N$. In this case the periodicity of the coordinate T is $\frac{8\pi N}{q}$, with q a non-negative integer. While value $q = 1$ is somehow singled out as it leads to identical periodicity with the one from the Nut solution, other values $q > 1$ are allowed as well. For a general q the topology on the boundary is that of a lens space S^3/Z_q, while the topology of the Bolt is in general that of an R^2/Z_q-fibration over S^2. The value of the mass parameter is

$$m_c = \frac{r_c^4 + (l^2 - 6N^2)r_c^2 + N^2(l^2 - 3N^2)}{2l^2 r_c} \tag{5.30}$$

while the location of the bolts is given by:

$$r_c = \frac{ql^2 \pm \sqrt{q^2l^4 - 48N^2l^2 + 144N^4}}{12N} \tag{5.31}$$

Notice that the condition $r_c > N$ restricts the values of the NUT charge parameter N such that (for $q = 1$):

$$N \leq \left(\frac{1}{6} - \frac{\sqrt{3}}{12} \right)^{\frac{1}{2}} l \tag{5.32}$$

For the bolt solutions the action is given by [37]

$$I_c = -\frac{\pi(r_c^4 - l^2 r_c^2 + N^2(3N^2 - l^2))}{3r_c^2 - 3N^2 + l^2} \tag{5.33}$$

and the entropy is

$$S_c = \frac{\pi(3r_c^4 + (l^2 - 12N^2)r_c^2 + N^2(l^2 - 3N^2))}{3r_c^2 + l^2 - 3N^2} \tag{5.34}$$

Note that the properties of the bolt solution with $r > r_{b+}$ are very different from those of the bolt solution with $r > r_{b-}$. It can be shown that the upper branch solution $r > r_{b+}$ is thermally stable whereas the lower branch $r > r_{b-}$ is thermally unstable [37, 54].

We shall now apply our prescription to convert all $\mathbb{C}$-results to the corresponding results in the $\mathbb{R}$-approach, for which the metric used has the Lorentzian signature.

Since we do not perform any analytic continuations in the $\mathbb{R}$-approach, the metric that we use is given by (5.23). The periodicity condition for the coordinate t is then given by:

$$\frac{4\pi}{|V'(r_r)|} = \frac{8\pi n}{k} \tag{5.35}$$

where $V(r_r) = 0$ and k is a positive integer.

5.4 Euclidean to Lorentzian

Before we plunge into the details of the Euclidean to Lorentzian transition by analytical continuation it is necessary first to discuss what is to become of the first law of the thermodynamics in this process.

5.4.1 When is the first law of thermodynamics satisfied?

It has been recently argued in [6] that there is a breakdown of the entropy/area relationship for NUT-charged AdS-spacetimes and that this result does not depend on the removal of Misner string singularities (if present) but rather is entirely a consequence of the first law of thermodynamics.

In the Euclidean sector (or the $\mathbb{C}$-approach) the argument goes as follows: using the counterterm method for a general bolt located at $r = r_c$ we compute

$$I_c = \frac{\beta_c}{2l^2}(l^2 m_c + 3N^2 r_c - r_c^3)$$

for the action, where $\beta_c = \frac{4\pi}{|F'(r_c)|}$ is the periodicity of the Euclidean time coordinate and r_c is the biggest root of $F(r)$ given in (5.25). This fixes the value of the mass parameter to be that given by (5.30). Using the boundary stress-energy tensor we can compute the conserved mass for this solution as being given by $\mathfrak{M} = m_c$ [54, 6]. We define now the entropy $S_c = \beta_c m_c - I_c$ by using the Gibbs-Duhem relation. It is easy to see that in order for the first law of thermodynamics $dS_c = \beta_c dm_c$ to hold in this case we must have:

$$m_c = \partial_{\beta_c} I_c \tag{5.36}$$

For generic values of r_c we find that the above relation is not satisfied in general. However, if we assume a functional dependence $r_c = r_c(N)$ then the first law is satisfied if and only if r_c is given by (5.31) where now q is a constant of integration or $r_c = \pm n$.

We can see now that we are guaranteed to have satisfied the first law of thermodynamics for the Nut and Bolt solutions in AdS backgrounds, even though no Misner-string singularities have been explicitly removed. We also find that using the expressions from (5.31) we obtain $\beta = \frac{8\pi|N|}{q}$. Now, removal of Misner-string singularities forces the parameter q to be an integer, but this is not required in order to satisfy the first law.

Let us consider next the restrictions imposed by the first law of thermodynamics in the $\mathbb{R}$-approach, *i.e.* in the Lorentzian solutions. Using the counterterm method for the Lorentzian solution given in (5.23) with a 'bolt' located at r_r, which is a root of (5.24), we obtain the action:

$$I_r = \frac{\beta_r}{2l^2}(l^2 m_r - 3n^2 r_r - r_r^3)$$

Here we set $\beta_r = \frac{4\pi}{|V'(r_r)|}$ to be the periodicity of the time coordinate, though there is no direct justification for this[5]. We find the value of the mass parameter m_r to be

$$m_r = \frac{r_r^4 + (l^2 + 6n^2)r_r^2 - n^2(l^2 + 3n^2)}{2l^2 r_r} \tag{5.37}$$

Using the boundary stress-energy tensor we can compute the conserved mass for this solution as being given by $\mathfrak{M} = m_r$. We can now define the entropy $S_r = \beta_r m_r - I_r$ by using the Gibbs-Duhem relation. It is easy to see that in order for the first law of thermodynamics $dS_r = \beta_r dm_r$ to hold in this case we must have:

$$m_r = \partial_{\beta_r} I_r$$

Again, for generic values of r_r we find that the above relation is not satisfied in general. However if we assume a functional dependence $r_r = r_r(n)$ then the first law is satisfied if and only if:

$$r_r = \frac{kl^2 \pm \sqrt{k^2 l^4 - 48n^2 l^2 - 144n^4}}{12n} \tag{5.38}$$

where k is a constant of integration. As we shall see in the next section, this is precisely the location of the Lorentzian bolt solutions, when k is an integer. The first law of thermodynamics will be automatically satisfied for these solutions. It is interesting to note that using the expressions from (5.37) and (5.38) we obtain $\beta = \frac{8\pi|n|}{k} = \frac{4\pi}{|V'(r_r)|}$. If we impose the further requirement that Misner string singularities be removed then k must be an integer.

5.4.2 The Bolt case

Let us consider now the analytic continuation of the bolt solutions from the $\mathbb{C}$-approach. In this case we perform the analytic continuations $T \to it$, $N \to in$ together with $q \to ik$. From (5.31) we obtain the location of the Lorentzian 'bolts' at:

$$r_r = \frac{kl^2 \pm \sqrt{k^2 l^4 - 48n^2 l^2 - 144n^4}}{12n} \tag{5.39}$$

[5] Note however that the removal of the Misner string singularity in the Lorentzian metric forces the time coordinate to be periodic.

the value of the mass parameter is:

$$m_r = \frac{r_r^4 + (l^2 + 6n^2)r_r^2 - n^2(l^2 + 3n^2)}{2l^2 r_r} \tag{5.40}$$

while the periodicity of the time coordinate t is given by $\beta_r = \frac{8\pi n}{k}$.

In order to obtain real values for r_r we must require that the discriminant is non-negative. This leads to the condition:

$$n \leq n_{max} = \left(\frac{\sqrt{4 + k^2} - 2}{12} \right)^{\frac{1}{2}} l \tag{5.41}$$

Then there is a maximum value n_{max} of the NUT charge for which the bolt solutions are physically acceptable. This means that below a certain temperature $T_{min} = \frac{k}{8\pi n_{max}}$ the bolt solutions do not exist.

If we analytically continue the action and the entropy of the bolt solutions we obtain:

$$I_r = -\frac{\pi(r_r^4 - l^2 r_r^2 + n^2(3n^2 + l^2))}{3r_r^2 + 3n^2 + l^2} \tag{5.42}$$

respectively

$$S_r = \frac{\pi(3r_r^4 + (l^2 + 12n^2)r_r^2 - n^2(l^2 + 3n^2))}{3r_r^2 + l^2 + 3n^2} \tag{5.43}$$

The specific heats can be computed using $C = -\beta\partial_\beta S = -n\partial_n S$; for brevity we shall not list here their explicit expressions.

In figure 5.1 we plot the masses of the upper branch ($r > r_{b+}$) and the lower branch ($r > r_{b-}$) solutions as a function of the NUT parameter n. We can see that there is a range for the NUT charge for which the mass of the lower branch solution becomes negative, while the mass of the upper branch solution is always positive.

We plot the entropy as a function of the NUT charge in figure 5.2, including the lower branch solutions. As is obvious from this figure, the entropy for the lower branch does become negative if $n < 1.19355$. The entropy for the upper branch solutions is always positive.

In figure 5.3 we plot the entropy and the specific heat versus the NUT charge for the upper branch solutions. In this case the entropy and the specific heat are always positive.

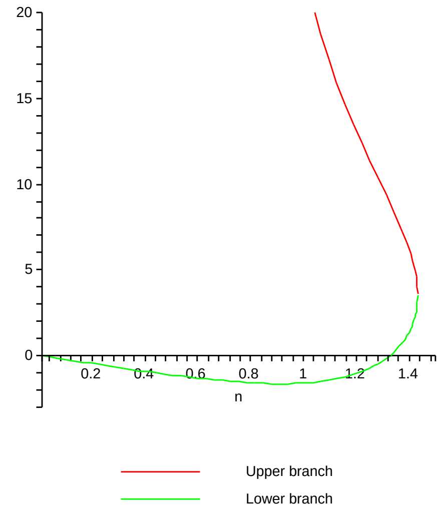

Figure 5.1: Plot of the upper ($r_b = r_{b+}$) and lower ($r_b = r_{b-}$) TB masses (for $k = 10$, $l = \sqrt{3}$).

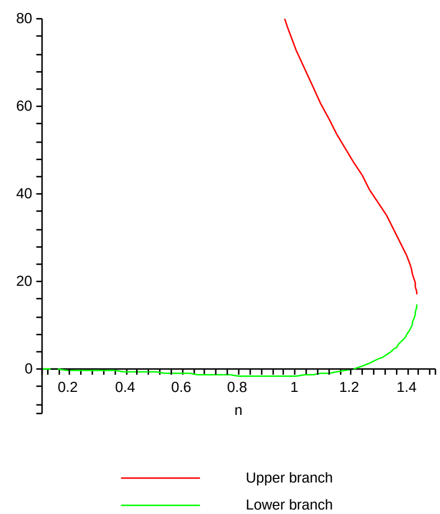

Figure 5.2: Plot of the upper ($r_b = r_{b+}$) and lower ($r_b = r_{b-}$) TB entropies (for $k = 10$, $l = \sqrt{3}$).

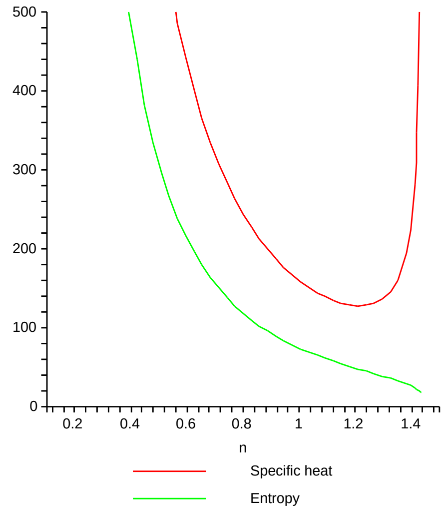

Figure 5.3: Plot of the upper branch bolt entropy and specific heat (for $k = 10$, $l = \sqrt{3}$).

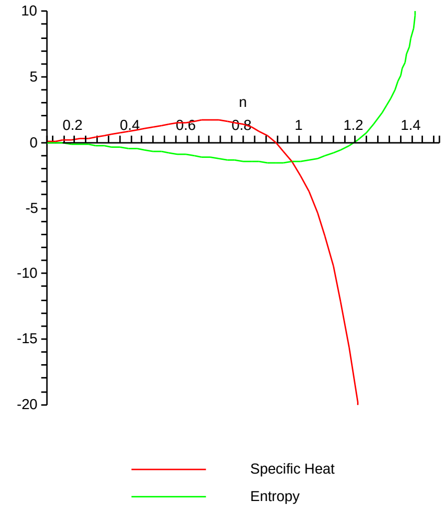

Figure 5.4: Plot of the lower branch bolt entropy and specific heat (for $k = 10$, $l = \sqrt{3}$).

In figure 5.4 we plot the entropy and the specific heat as a function of the NUT charge for the lower branch bolt solutions. We can see again that the entropy is negative if $n < 1.19355$ while the specific heat is positive if $n < 0.91338$ and negative otherwise, implying that the Lorentzian version of the lower branch solutions is thermally unstable. In both cases the specific heat diverges near $T = T_{min}$ (or $n = n_{max}$).

5.4.3 The Nut case

As we have seen above in the $\mathbb{C}$-approach, besides the usual Bolt solutions, one can also obtain the so-called Nut-solutions. For these solutions the fixed-point set of the Killing vector $\frac{\partial}{\partial T}$ is zero-dimensional. This can happen only if $\tau_c = N$ in the above equations, that is if $F(\tau_c = N) = 0$ and also

$$\frac{4\pi}{|F'(\tau_c = N)|} = \frac{8\pi N}{q} \tag{5.44}$$

are satisfied. Recall that $q = 1$ for the Nut solution in the $\mathbb{C}$-approach.

However, since we are interested in the analytic continuations that could take the $\mathbb{C}$-results to the $\mathbb{R}$-results we shall slightly modify our ansatz using the lesson learned in dealing with the Bolt cases. Namely, instead of focussing on $\tau_c = N$ (which clearly becomes imaginary when we analytically continue $N \to in$) we shall look for a solution of the form $\tau_c = pN$ where p is a positive real number. Then the usual Taub-Nut solution in the $\mathbb{C}$-approach corresponds to $p = 1$, while other values of $p > 1$ correspond to Bolt-type solutions.

The limit $p = 1$ must be treated with special care since in the Nut solution $\tau_c = N$ is a double root of the numerator of $F(r)$, while in the Bolt case we assume that τ_c is a single root. The difference arises when computing the periodicity $\beta_c = \frac{4\pi}{|F'(pN)|}$; accounting for the double root, it turns out that one should multiply by 2 the result from the Bolt case in order to recover the correct periodicity of the Nut.

It is easy to check now that the above conditions will fix the periodicity of the coordinate T to be $\beta_c = \frac{8\pi N}{q}$ where now q is a complicated function of p, l and N while the value of the mass parameter is given by

$$m_c = \frac{N[(1+p^2)l^2 + (p^4 - 6p^2 - 3)N^2]}{2pl^2} \tag{5.45}$$

Using the counterterm method, for a bolt located at $r_c = pN$ we obtain:

$$\begin{aligned} I_c &= \frac{\pi N^2[(p^2+1)l^2 - (p^4+3)N^2]}{l^2+3N^2(p^2-1)} \\ S_c &= \frac{\pi N^2[(p^2+1)l^2 - 3(p^4+3)N^2]}{l^2-3N^2(p^2-1)} \end{aligned} \tag{5.46}$$

Notice that in the limit $p \to 1$ one recovers the previous expressions for the action and respectively the entropy of the Taub-Nut-AdS solution (5.28) up to the factor of 2, as explained above.

Let us apply now our prescription for going from the $\mathbb{C}$-approach to the $\mathbb{R}$-approach. In this case we shall analytically continue $N \to in$ and also $q \to ik$ (which in the Nut case corresponds in fact to $p \to -ip$). Then the location of the nut becomes $\tau_r = pn$ in the $\mathbb{R}$-approach, yielding a real value for τ_r, while the value of the mass parameter is also real

$$m_r = \frac{n[(p^4+6p^2-3)n^2 - l^2(1-p^2)]}{2pl^2} \tag{5.47}$$

Now the periodicity of the coordinate t is given by $\frac{8\pi n}{|k|}$, with $k = (q)_{p=-ip}$. For $p = 1$ we obtain:

$$k = 2\left(\frac{6n^2}{l^2} + 1\right) \tag{5.48}$$

while value of the mass parameter is $m_r = \frac{2n^3}{l^2}$, the action is

$$I_r = -\frac{4\pi n^4}{l^2+6n^2} \tag{5.49}$$

However, since the location of the 'bolt' $r = n$ is not of the form (5.38) unless we assume a relationship between n and l we conclude that the first law of thermodynamics is not satisfied for the Lorentzian Taub-Nut-AdS solution in the $\mathbb{R}$-approach.

5.4.4 The flat-space limit

Leaving the more detailed study of the thermodynamics of the above solutions for future work, let us now briefly discuss the case in which the cosmological constant vanishes. Notice that this condition corresponds to $l \to \infty$ and in this limit we recover the Taub-NUT solutions in a flat background. Special care must be taken when discussing the analytic continuation from the Euclidean sector to the Lorentzian one. Let us consider first the Lorentzian Taub-Nut-AdS solution. In the limit $l \to \infty$ we obtain the action $I_r = 0$, the conserved mass is also zero in this limit. These results are in agreement with the expectation that the only way to have $r = n$ as a root of the Lorentzian function

$$F(r) = \frac{r^2 - 2mr - n^2}{r^2 + n^2}$$

is to take $m = 0$.

In the bolt case we obtain by analytic continuation $r_r = \frac{2n}{k}$ and the mass parameter is

$$m_r = n\frac{4 - k^2}{4k}$$

while the action and the entropy are given by:

$$I_r = \pi n^2 \frac{4 - k^2}{k^2}, \qquad S_r = \pi n^2 \frac{4 - k^2}{k^2}, \qquad C_r = 2\pi n^2 \frac{k^2 - 4}{k^2} \tag{5.50}$$

As in the Euclidean sector we have $q = 1$ (since if $q > 1$ then r_b is less than n) this will fix $k = 1$ in the above relations. Further, note that although these expressions satisfy the first law of thermodynamics, the entropy and specific heat for the Lorentzian bolt in the flat spacetime have opposite signs, which means that the solution is thermally unstable. This is not unexpected if we recall that the Taub-NUT solutions in flat background correspond to the lower-branch Taub-NUT-AdS solutions, which are thermodynamically unstable.

5.5 Higher dimensional Taub-NUT-dS spaces

The above results can be easily extended to higher dimensional Taub-NUT spacetimes. As an example, we shall focus only on the asymptotically de Sitter spacetimes.

First, let us notice the absence of higher dimensional Taub-Nut-dS solutions, which is a result analogous with that stating the absence of hyperbolic nuts in AdS-backgrounds [34, 6]. Quite generally we can see this by observing the behaviour of the function $F(\tau)$ near the root $\tau_c = N$. Since $F(\tau)$ takes negative values for points $\tau > \tau_c$ we deduce that there always exists a larger root of $F(\tau)$ that will contain the nut. Therefore, in our discussion we shall refer only to the higher-dimensional Bolt solutions. An analysis similar to the one performed in Section (5.4.1) assures us that the first law is satisfied in both approaches.

An analysis of the thermodynamics of the higher-dimensional Taub-NUT-dS spaces has been presented in [40]. It has been shown there that the thermodynamic behaviour of both the $\mathbb{R}$-approach and $\mathbb{C}$-approach quantities are qualitatively the same in $4s$-dimensions, a behaviour that is distinct from the common behaviour in $4s+2$ dimensions[6]. This means that spaces of dimensionality $8, 12, 16, ...$ have the same qualitative thermodynamic behaviour as the four-dimensional case, while spaces of dimensionality $10, 14, 18, ...$ have the same behaviour as the six-dimensional case. We shall now illustrate equivalence of the two approaches in six-dimensions; the other $4s+2$ higher-dimensional cases are analogous.

The Taub-NUT-dS metric in six dimensions, constructed over an $S^2 \times S^2$ base is given by:

$$ds^2 = V(\tau)(dt + 2n\cos\theta_1 d\varphi_1 + 2n\cos\theta_2 d\varphi_2)^2 - \frac{d\tau^2}{V(\tau)} + (\tau^2 + n^2)(d\Omega_1^2 + d\Omega_2^2)$$

where

$$\begin{aligned} V(\tau) &= \frac{3\tau^6 - (l^2 - 15n^2)\tau^4 - 3n^2(2l^2 - 15n^2)\tau^2 + 3n^4(l^2 - 5n^2) + 6ml^2\tau}{3(\tau^2 + n^2)^2 l^2} \\ d\Omega_i^2 &= d\theta_i^2 + \sin^2\theta_i d\varphi_i^2 \end{aligned} \tag{5.51}$$

Removal of the Misner string singularities in the metric forces us to take $12\pi|n|$ as the periodicity of the time coordinate. Similar with the 4-dimensional case, we shall impose an extra periodicity $\frac{4\pi}{|V'(\tau_r)|}$. Matching the two values leads to the condition:

$$\beta_r = \frac{4\pi}{|V'(\tau_r)|} = \frac{12\pi|n|}{k}$$

where $V(\tau_r) = 0$ and k is a positive integer.[7] This fixes the bolt location to be given by the formula:

$$\tau_r = \frac{kl^2 + \sqrt{k^2l^4 - 900n^4 + 180n^2l^2}}{30n}$$

[6] Here $s = 1, 2, ...$

[7] The parameter k will determine the topology on the boundary $\tau \to \infty$. For $k = 1$ the boundary is $Q(1,1)$, which is the 5-dimensional circle fibration over $S^2 \times S^2$, while for $k > 1$ we obtain $Q(1,1)/Z_k$.

and by requiring real values for τ_r we must restrict the allowed range of the NUT charge to be:

$$|n| \leq l\frac{\sqrt{90+30\sqrt{k^2+9}}}{30}$$

In the $\mathbb{C}$-approach we make the analytic continuations $t \to iT$ and $n \to iN$. Then the function $V(\tau)$ is continued to $F(\tau)$. Imposing the periodicity condition

$$\beta_c = \frac{4\pi}{|F'(\tau_c)|} = \frac{12\pi|N|}{q}$$

where τ_c is a root of $F(\tau)$ we find

$$\tau_c = \frac{ql^2 + \sqrt{q^2l^4 + 900N^4 + 180N^2l^2}}{30N}$$

It is easy to see now that starting, for instance, with the $\mathbb{C}$-quantities and using the analytic continuations $N \to in$ and $q \to ik$ we recover the corresponding $\mathbb{R}$-quantities and vice-versa. This equivalence extends to all thermodynamic quantities computed using the counterterm approach.

5.6 Summary

The work of this chapter was motivated by the observation made in Chapter 4 that the path-integral formalism can be extended to asymptotically de Sitter spacetimes to describe quantum correlations between timelike histories, providing a foundation for gravitational thermodynamics at past/future infinity [40]. The key result is the generalisation of the quantum statistical relation or the generalised Gibbs-Duhem relation (4.21) to asymptotically dS spacetimes.

In order to employ this relation it is generally necessary to analytically continue the spacetime near past/future infinity. There are two apparently distinct ways of doing this – the $\mathbb{R}$-approach and the $\mathbb{C}$-approach. The $\mathbb{C}$-approach is closest to the more traditional method of obtaining Euclidean sections for asymptotically flat and AdS spacetime. The $\mathbb{R}$-approach refers to the Lorentzian section, and makes use of the path integral formalism only insofar as the generalised Gibbs-Duhem relation is employed.

The main result of this chapter is the demonstration that the $\mathbb{R}$ and $\mathbb{C}$-approaches are equivalent, in the sense that we can start from the $\mathbb{C}$-approach results and derive by consistent analytic continuations (*i.e.*, using a well-defined prescription for performing the analytic continuations) all the results from the $\mathbb{R}$-approach. There are no a-priori obstacles in taking the opposite view, in which the $\mathbb{C}$-approach results are derived from the respective $\mathbb{R}$-approach results. However, one could still argue that the $\mathbb{C}$-approach is the more basic one, as in it the periodicity conditions appear more naturally than in the $\mathbb{R}$-approach.

On the other hand, the $\mathbb{R}$-approach, when used without the justification that comes from the $\mathbb{C}$-approach, raises some interesting questions. Even applied to simple cases such as the Schwarzschild-dS solution, one may take the view that in the absence of the nut charge one could still consider a periodicity on the time coordinate in the Lorentzian sector given by $\beta_r = 8\pi m$. A more orthodox interpretation would be that β_r in the Lorentzian sector is simply the inverse temperature (as related by the surface gravity of the black hole horizon) and is not related to a real periodicity of the time coordinate. Whether or not this is indeed a necessary condition remains to be seen.

Using this equivalence we then proposed an interpretation of the thermodynamical behaviour of nut-charged spacetimes. In the asymptotically dS case, we showed that while a subset of the Bolt solutions can have a sensible physical interpretation, the same does not hold for the Taub-Nut-dS solutions. Indeed, in the putative Taub-Nut-dS solution the nut is always enclosed in a larger cosmological 'bolt' and moreover it does not have a Lorentzian counterpart (*i.e.* it has no equivalent solution in the $\mathbb{R}$-approach). From these facts we conclude that there are no Taub-Nut-dS solutions. This situation holds despite the fact that a naive application of (4.21) to this case yields thermodynamic quantities that respect the first law of thermodynamics. Rather these quantities are the analytic continuations of their AdS counterparts under $l \to il$. Similar remarks apply to the lower-branch dS bolt cases. We have also found that this situation holds in higher dimensions: there are no Taub-Nut-dS solutions, in analogy with the non-existence of the hyperbolic nuts in AdS backgrounds [54].

Moreover, the $\mathbb{C}$-approach has been previously applied with success to more general cases - it has been proven to be very useful when treating for instance asymptotically AdS or flat Taub-NUT spaces. In particular, we have shown in section 5.4 that starting from the well-known results regarding the thermodynamics of the Nut and Bolt solutions in the Euclidean Taub-NUT-AdS case (which corresponds to our $\mathbb{C}$-approach) we can consistently make analytic continuations back to the Lorentzian sections, yielding a physical interpretation of the thermodynamics of such spacetimes. However, this holds only for the bolt solutions; we found that the Lorentzian AdS-Nut solution did not respect the first law of thermodynamics, rendering the physical interpretation of the Nut solution dubious at best.

Chapter 6

Higher dimensional Kaluza-Klein monopoles

6.1 Overview

Although magnetic monopole solutions were found in vacuum Kaluza-Klein theories in the 1980s [129, 76], an outstanding problem has been generalisation of these soliton solutions (here termed Kaluza-Klein monopoles, or KK monopoles) to include a cosmological constant. Given current theoretical interest in asymptotically (anti)-de Sitter spacetimes and recent experimental results that indicate that the universe does indeed possess a small positive cosmological constant, it is reasonable to pursue such an objective.

One such attempt was recently made by Onemli and Tekin [121]. They concluded that there is no five-dimensional static Kaluza-Klein monopole with cosmological constant. The metric ansatz they employed was tailored to describe a static 'Kaluza-Klein' monopole in an $AdS_2 \times AdS_3$ background. In the limit in which the cosmological constant λ tends to zero, the 'KK-AdS monopole' should reduce to the 'KK monopole' in flat space. Moreover, if the monopole charge tends to zero then the 'KK-AdS monopole' should reduce smoothly to the $AdS_2 \times AdS_3$ background. If we relax the requirement that the monopole solution be static it is easy to construct a time-dependent five-dimensional soliton that has all the desired properties. Two such solutions in five dimensions have been obtained in [121].

In the present chapter we consider other possible extensions of Kaluza-Klein monopole solutions that admit a cosmological constant. The essential ingredient in the original Kaluza-Klein monopole construction is a Euclidean section of the four dimensional Taub-NUT space; the 'trick' employed in [129, 76] to obtain the monopole solution was to lift this Euclidean section up to five-dimensions by adding a flat time coordinate and then to dimensionally reduce along the 'Euclidean time' direction from the Euclidean Taub-NUT section. However, in a presence of the cosmological constant it is not possible to use the above technique without introducing an explicit time dependence in the metric. Therefore, in order to obtain cosmological four-dimensional magnetic monopole solutions our strategy is to consider directly in five-dimensions the new cosmological Taub-NUT-like solutions discussed in sections 3.5, 3.6 and perform a Kaluza-Klein compactification along the fifth dimension. The new feature of these solutions is that the four-dimensional dilaton acquires a potential term as an effect of the cosmological constant. However their asymptotics are not very appealing physically since they are not asymptotically flat or $(A)dS$ in the Einstein frame. Their metric description simplifies when considered in the string frame: for our explicit examples the four-dimensional metric in the string frame is very similar to the AdS form in the (r, t) sector, except for a deficit of solid angle in the angular sector.

In higher than five dimensions we have more choices: we can consider solutions that are Ricci flat with different NUT parameters or we can consider Taub-NUT like spaces that are constructed as circle fibrations over base spaces that have non-trivial topology. We also perform Kaluza-Klein (KK) reductions of the above solutions down to four dimensions, obtaining new magnetic monopole solutions. More specifically, in six and seven dimensions we have considered non-singular Ricci-flat solutions for which one can use the KK trick to obtain similar KK magnetic brane solutions for which the background spaces are Ricci flat Bohm spaces of the form $S^p \times S^q$ and generically have conical singularities. We considered their further reduction down to four dimensions on Riemannian spaces of constant curvature and specifically considered such reductions on spheres. In contrast with the KK procedure to untwist the $U(1)$-fibration, we have considered in six dimensions another method that is known to untwist the circle fibration, namely Hopf duality in string theory. We extended these duality rules to the case of a timelike Hopf-duality of the truncated six-dimensional Type II theories and applied them to generate charged string solutions in six-dimensions. By performing sphere reductions we obtained the corresponding four-dimensional solutions. In general, the presence of the cosmological constant in the higher dimensional theory induces a scalar potential for the Kaluza-Klein scalar fields. If the isometry generated by the Killing vector $\frac{\partial}{\partial z}$, which is associated with the circle direction on which we perform the reduction has fixed points, then the dilaton, which describes the radius of that extra-dimension, will diverge at the fixed point sets and the D-dimensional metric will be singular at those points. In certain cases we find that the dilaton field also diverges at infinity. Respectively this means that, physically, the space-time decompactifies near the KK-brane and at infinity; the higher-dimensional theory should be used when describing such objects in those regions.

The organization of this chapter is as follows. We begin in section 6.2 by reviewing how the flat KK monopole can be obtained from the four dimensional Taub-Nut solution. We also briefly discuss the features of the monopole solution obtained by using the Euclidean Taub-Bolt solution. At this point we consider the solution obtained by dimensionally reducing an Eguchi-Hanson-like monopole and we prove that even if the four-dimensional metric is non-asymptotically flat, its geometry is nonetheless U-dual to that of a Taub-Bolt monopole.

We next present in section 6.3 the new metric ansatz which is a solution of vacuum Einstein's equations with cosmological constant in five dimensions and we perform a Kaluza-Klein reduction to obtain a new four-dimensional monopole solution. In the following sections we consider similar monopole solutions in higher dimensions and we also perform Kaluza-Klein sphere reductions to four dimensions. In six dimensions we apply spatial and timelike Hopf-dualities to generate new solutions.

6.2 Kaluza-Klein magnetic monopoles

We begin by reviewing the original Kaluza-Klein magnetic monopole solution in 4 dimensions that arises as a Kaluza-Klein compactification of a 5 dimensional vacuum metric [129, 76]. The essential ingredient used in the monopole construction is a four dimensional Euclidean version of the Taub-NUT solution:

$$ds^2 = F_E(r)(d\chi - 2n\cos\theta d\varphi)^2 + F_E^{-1}(r)dr^2 + (r^2 - n^2)d\Omega^2$$

where

$$F_E(r) = \frac{r^2 - 2mr + n^2}{r^2 - n^2} \tag{6.1}$$

As we have seen in Chapter 2, in general, the $U(1)$ isometry generated by the Killing vector $\frac{\partial}{\partial\chi}$ (that corresponds to the coordinate χ that parameterizes the fibre S^1) can have a zero-dimensional fixed point set (referred to as a 'Nut' solution) or a two-dimensional fixed point set (correspondingly referred to as a 'Bolt' solution). The regularity of the Nut solution forces $m = n$, yielding

$$F_E(r) = \frac{r - n}{r + n} \tag{6.2}$$

In the Bolt solution the Killing vector $\frac{\partial}{\partial\chi}$ has a two-dimensional fixed point set in the 4-dimensional Euclidean sector. The regularity of the Bolt solution is then ensured by taking $r \geq 2n$ and $m = 5n/4$. Then we obtain:

$$F_E(r) = \frac{(r - 2n)\left(r - \frac{1}{2}n\right)}{r^2 - n^2} \tag{6.3}$$

In both cases the periodicity of the χ-coordinate is $8\pi n$.

6.2.1 The Gross-Perry-Sorkin magnetic monopole

The GPS monopole solution is constructed as follows. Taking the product of the Euclidean Taub-Nut space-time with the real line, we obtain the following 5-dimensional Ricci flat metric:

$$ds^2 = -dt^2 + F_E(r)(d\chi - 2n\cos\theta d\varphi)^2 + F_E^{-1}(r)dr^2 + (r^2 - n^2)d\Omega^2$$

which solves the 5-dimensional vacuum Einstein equations.

If we perform now a Kaluza-Klein reduction along the coordinate χ (which is periodic with period $8\pi n$) we obtain the following 4-dimensional fields (with $\alpha = \frac{1}{2\sqrt{3}}$) [129, 76]

$$\begin{aligned} ds^2 &= -F_E^{\frac{1}{2}} dt^2 + F_E^{-\frac{1}{2}}(r)dr^2 + F_E^{\frac{1}{2}}(r^2 - n^2)d\Omega^2 \\ \mathcal{A} &= -2n\cos\theta d\varphi, \qquad e^{\frac{\phi}{\sqrt{3}}} = F_E^{-\frac{1}{2}}, \end{aligned} \tag{6.4}$$

which is a solution of the equations of motion derived from the action:

$$\mathcal{L} = eR - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{4}ee^{-\sqrt{3}\phi}\mathcal{F}^2, \tag{6.5}$$

where $\mathcal{F} = d\mathcal{A}$. It is clear now that the metric is asymptotically flat. The above solution describes a magnetic monopole and its properties have been discussed in detail in [129, 76].

6.2.2 Taub-Bolt monopoles

There are a few extensions of the above construction that we can consider. The obvious one to explore is the Taub-Bolt solution in four-dimensions instead of the Nut solution. As in the case of the Nut solution, we take the product with the real line and obtain a metric in five-dimensions that is a solution of the vacuum Einstein field equations. Performing the Kaluza-Klein compactification along the χ direction yields (6.4), where now $F_E(r)$ is given by (6.3). The five-dimensional metric is regular everywhere for $r \geq 2n$. However, the four-dimensional solution obtained by Kaluza-Klein reduction, while asymptotically flat, is now singular at the location of the bolt $r = 2n$ where the dilaton field diverges as expected.

The physical interpretation of this solution was recently clarified by Liang and Teo [103] (see also [33]). It corresponds to a pair of coincident extremal dilatonic black holes with opposite magnetic charges. To see this we can use as a seed in the KK procedure the Euclidean rotating version of the Bolt solution [47, 68]. We add a timelike flat direction in order to lift the solution to five dimensions, after which we reduce down to four dimensions. When $n \neq 0$ it has been shown in [103] that the above solution describes a pair of extremal dilatonic black holes carrying opposite but unbalanced magnetic charges and separated by a distance $2a$, a being the rotation parameter, which in this case serves as a measure of the proper distance between the black holes. In the limit $a \to 0$ we obtain the the solution (6.4) and (6.3) which corresponds then to a pair of coincident monopoles that carry opposite unbalanced magnetic charges. The total magnetic charge of the system is n.

Since the above dihole has unbalanced charges it is not possible to introduce a background magnetic field to stabilize the system [103]. Consequently the solution is unstable and is expected to decay (possibly to a pure Nut solution (the KK soliton in this case) with total charge n).

6.2.3 The black version of the Kaluza-Klein monopole

We will digress here for a moment to notice that the action (6.5) is the particular case $a = -\sqrt{3}$ of a more general Einstein-Maxwell-Dilaton theory, with arbitrary coupling constant a. In particular, the four-dimensional fields (6.4) describe the extremal version of the charged dilatonic black hole solution found by Garfinkle, Horowitz and Strominger [58]. The charged dilatonic black hole with coupling constant $a = -\sqrt{3}$ takes the form:

$$\begin{aligned} ds^2 &= -\frac{1-\frac{2m}{r}}{\left(1+\frac{2p}{r}\right)^{\frac{1}{2}}}dt^2 + \left(1+\frac{2p}{r}\right)^{\frac{1}{2}}\left[\frac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega^2\right], \\ \mathcal{A} &= -2\sqrt{p(p+m)}\cos\theta d\varphi, \qquad e^{\frac{2\phi}{\sqrt{3}}} = 1+\frac{2p}{r}. \end{aligned} \tag{6.6}$$

If we oxidise this black hole solution to five dimensions we obtain the following Ricci flat metric:

$$\begin{aligned} ds^2 &= -\left(1-\frac{2m}{r}\right)dt^2 + \left(1+\frac{2p}{r}\right)\left[\frac{dr^2}{1-\frac{2m}{r}} + r^2 d\Omega^2\right] \\ &\quad + \frac{1}{\left(1+\frac{2p}{r}\right)}(d\chi - 2\sqrt{p(p+m)}\cos\theta d\varphi)^2. \end{aligned} \tag{6.7}$$

This is the black version of the Kaluza-Klein monopole and it describes a five-dimensional black hole whose horizon is a squashed 3-sphere. The Kaluza-Klein monopole is recovered by setting $m = 0$, while if we set $p = 0$ we obtain the uniform black string solution.

6.2.4 The mass of the Kaluza-Klein Monopole

In General Relativity there are many known expressions for computing the energy in asymptotically flat spacetimes. The general idea is to study the asymptotic values of the gravitational field, far away from an isolated object, and compare them with those corresponding to a gravitational field in the absence of the respective object. However, most of these proposals will provide results that are relative to the choice of a reference background (be it a spacetime metric or merely a connection). The background must be chosen such that its topological properties match the solution whose action and conserved charges we want to compute. However, this does not fix the choice of the background and moreover, there might be cases in which the topological properties of the solution rule out any natural choice of the background.

As we have seen in Chapter 4, most of these difficulties are simply avoided once we resort to the counterterm-method. The main advantage of this approach is that it gives results that are intrinsic to the solutions considered, that is, the results are not 'relative' to some reference background. In this section we investigate the various local counterterm prescriptions, considered in Chapter 4, to compute the action and the conserved charges in the five-dimensional Kaluza-Klein theory and, more specifically, for the Kaluza-Klein monopole solution.

The computation of the mass

Before we apply the counterterm prescription to compute the components of the boundary stress-tensor, let us notice that the boundary topology of the KK monopole for constant, finite values of the radial coordinate r is that of a squashed 3-sphere times a real line. Therefore, one might expect that the proper counterterm action to use should be the one corresponding to an $S^3 \times R$ topology. However using that counterterm we find that it is impossible to cancel out the divergences as $r \to \infty$. Rather we note that, as $r \to \infty$, the boundary topology is that of a fibre bundle $R \times S^1 \hookrightarrow S^2$ as the radius of S^2 grows with r, while the radius of S^1 reaches a constant value. Thence, asymptotically, the choice of the counterterm (4.16) is natural and indeed, we find that using this counterterm we can eliminate the divergences in the action and obtain finite values for the total mass.

Using the metric with the general expression (6.1) for the function $F_E(r)$ we find

$$\begin{aligned}
8\pi G T^t{}_t &= \frac{m}{r^2} + O(r^{-3}), \\
8\pi G T^\chi{}_\chi &= \frac{2m}{r^2} + O(r^{-3}), \\
8\pi G T^\chi{}_\phi &= \frac{4mn\cos\theta}{r^2} + O(r^{-3}), \\
8\pi G T^\theta_\theta &= -\frac{m^2 - 2n^2}{2r^3} + O(r^{-4}), \\
8\pi G T^\phi_\phi &= -\frac{m^2 - 2n^2}{2r^3} + O(r^{-4}),
\end{aligned} \tag{6.8}$$

the rest of the terms being of order $O(r^{-3})$ or higher. Then the conserved mass associated with the Killing vector $\xi = \partial/\partial t$ is found to be:

$$\mathcal{M} = \frac{4\pi mn}{G}.$$

However using the counterterm (4.13) and the boundary stress-tensor (4.15) we obtain

$$\begin{aligned}
8\pi G T^{t}_{\ t} &= \frac{m}{r^2} + O(r^{-3}), \\
8\pi G T^{\chi}_{\ \chi} &= \frac{2m}{r^2} + O(r^{-3}), \\
8\pi G T^{\chi}_{\ \phi} &= \frac{4mn\cos\theta}{r^2} + O(r^{-3}), \\
8\pi G T^{\theta}_{\theta} &= -\frac{m^2 - 4n^2}{2r^3} + O(r^{-4}), \\
8\pi G T^{\phi}_{\phi} &= -\frac{m^2 - 4n^2}{2r^3} + O(r^{-4}).
\end{aligned} \tag{6.9}$$

It is easy to see that this boundary stress-energy tensor leads to the same mass as above. Notice however that some of the components of the stress-energy tensor (6.9) are different from the ones obtained in (6.8).

For Kaluza-Klein monopole we have $m = n$ and we obtain $\mathcal{M} = \frac{4\pi n^2}{G}$, which is easily seen to be the same with the one derived in [24, 48] by using a background subtraction procedure.[1] For the Bolt monopole we have $m = 5n/2$ and using either prescription (4.13) or (4.16) we obtain $\mathcal{M} = \frac{10\pi n^2}{G}$. In both cases the regularized action takes the form $I = \beta\mathcal{M}$, where β is the periodicity of the Euclidean time $\tau = it$. Upon application of the Gibbs-Duhem relation $S = \beta\mathcal{M} - I$ we find that the entropy is zero, as expected since there are no horizons.

The GPS monopole mass from the four dimensional perspective

It is instructive to compute the conserved mass after we perform the dimensional reduction along the χ direction down to four-dimensions. While both the metric and the fields in general have singularities at the origin, this is not necessarily an obstruction since the conserved charges are in general computed as surface integrals at infinity.

[1] The parameter λ_∞ used in [24] corresponds in our case to $4n$, while $k = 8\pi G$.

In four-dimensions we can use the counterterm (4.12), whose only difference from (4.16) is that we are integrating now over a three-dimensional boundary instead of a four-dimensional one. A similar computation with the one performed in five-dimensions yields

$$\begin{aligned} 8\pi G_4 T_t^t &= \frac{m}{r^2} + O(r^{-3}), \\ 8\pi G_4 T_\theta^\theta &= \frac{n^2 - m^2}{2r^3} + O(r^{-4}), \\ 8\pi G_4 T_\phi^\phi &= \frac{n^2 - m^2}{2r^3} + O(r^{-4}), \end{aligned} \tag{6.10}$$

for boundary stress-energy tensor, where G_4 is Newton's constant in four-dimensions. Then the conserved mass associated with the Killing vector $\xi = \partial/\partial t$ is found to be:

$$\mathcal{M} = \frac{m}{2G_4}.$$

Noting that we have the relation $G = 8\pi n G_4$ we find that the mass computed using the four-dimensional geometry agrees precisely with the one computed in the five-dimensional theory.

Finally, we shall compute the mass using the methods from [109]. In that work, Mann and Marolf put forward a new counterterm that is also given by a local function of the boundary metric and its curvature tensor. The new counterterm is taken to be the trace $\hat{K}$ of a symmetric tensor $\hat{K}_{ij}$ that is defined implicitly in terms of the Ricci tensor $\mathcal{R}_{ij}$ of the induced metric on the boundary via the relation

$$\mathcal{R}_{ik} = \hat{K}_{ik}\hat{K} - \hat{K}_i^m \hat{K}_{mk}. \tag{6.11}$$

In contrast to previous counterterm proposals (such as (4.13)) this new counterterm assigns an identically zero action to the flat background in any coordinate systems while giving finite values for asymptotically flat backgrounds. The renormalized action leads to the usual conserved quantities that can also be expressed in terms of a boundary stress-tensor whose expression involves the electric part of the Weyl tensor:[2]

$$T_{ij}^0 u^j = \frac{1}{8\pi G_4} r E_{ij} u^j.$$

[2]Even if the four-dimensional solution is not a vacuum metric, the net effect of the matter fields is to give only sub-leading order corrections and to leading order we can still replace the bulk Riemann tensor with the Weyl tensor.

Here E_{ij} is the pull-back to the boundary of the contraction of the bulk Weyl tensor $C_{\mu\nu\rho\tau}$ with the induced metric $h^{\mu\nu}$ while u^i is the normal to the spacelike surface Σ on the boundary. More precisely, introducing the unit normal vector to the boundary n^μ then the electric part of the bulk Weyl tensor is defined by:

$$E_{\mu\nu} = C_{\mu\rho\nu\tau} n^\rho n^\tau = -C_{\mu\rho\nu\tau} h^{\rho\tau}. \tag{6.12}$$

Now E_{ij} is simply the pull-back to the boundary of the above tensor. Computing this expression in the $r \to \infty$ limit and contracting with the Killing vector $\xi = \partial/\partial t$ we obtain:

$$T^0_{ij}\xi^i u^j = \frac{1}{8\pi G_4}\frac{m}{r^2} + O(r^{-3}),$$

while the conserved mass is found by simple integration to be:

$$\mathcal{M} = \frac{m}{2G_4},$$

in agreement with previous computations.

6.2.5 The mass of the black version of the KK soliton

For completeness we also present here a computation of the thermodynamic quantities of the black version of the Kaluza-Klein monopole. By employing a counterterm prescription using the counterterm (4.17) we arrive at the following components of the boundary stress-tensor:

$$\begin{aligned}
8\pi G T^t{}_t &= \frac{2m+p}{r^2} + O(r^{-3}) \\
8\pi G T^\chi_\chi &= \frac{2p+m}{r^2} + O(r^{-3}), \\
8\pi G T^\chi_\phi &= \frac{2\sqrt{p(p+m)}(2p+m)\cos\theta}{r^2} + O(r^{-3}), \\
8\pi G T^\theta_\theta &= \frac{p^2+pm-m^2}{2r^3} + O(r^{-4}), \\
8\pi G T^\phi_\phi &= \frac{p^2+pm-m^2}{2r^3} + O(r^{-4}).
\end{aligned} \tag{6.13}$$

By performing an analytical continuation of the time coordinate and eliminating the conical singularities we find the periodicity of the Euclidean time to be:

$$\beta = 8\pi\sqrt{m(m+p)}, \tag{6.14}$$

while the periodicity of the χ coordinate is easily seen to be $L = 8\pi\sqrt{p(m+p)}$. We find then the mass of the black soliton:

$$\mathcal{M} = \frac{4\pi(2m+p)\sqrt{p(p+m)}}{G}. \tag{6.15}$$

We can also evaluate the total action of this solution, by noting that the only non-zero contribution appears from the surface terms only. In particular we find the total action to be:

$$I = \frac{32\pi^2(m+p)^2\sqrt{mp}}{G}. \tag{6.16}$$

Finally, by means of the Gibbs-Duhem relation we evaluate the total entropy of this solution to be:

$$S = \beta\mathcal{M} - I = \frac{32\pi^2(p+m)m\sqrt{pm}}{G}. \tag{6.17}$$

Computing now the horizon area of the black soliton it is easily seen that the entropy is in fact equal to one quarter of the area, as expected.

6.3 Eguchi-Hanson monopoles

One could also use the Eguchi-Hanson soliton in the above Kaluza-Klein monopole construction, in place of the Taub-NUT solution. Indeed, both solutions present a similar structure involving a squashed 3-sphere so that it is natural to expect that the four-dimensional solution will describe a magnetic monopole object as well. However, there is one important difference between the two solutions. Recall that in the usual Taub-NUT solitons the radius of the χ coordinate remains finite in the asymptotic regions, while the radius of the corresponding coordinate in the Eguchi-Hanson soliton becomes infinite in the same limit. In particular, since the dilaton in four dimensions is directly connected to the radius of the extra dimension it turns out that the dilaton has to diverge as well at infinity and the four-dimensional solution is manifestly non-asymptotically flat. By contrast, in the Kaluza-Klein monopole the four-dimensional dilaton becomes constant in the asymptotic regions and the metric is asymptotically flat.

However, as it will become apparent in this section, it is possible to employ a particular U-duality transformation [44] and regularise in this way the asymptotic behaviour of the KK Eguchi-Hanson monopole. In particular, we will show that the Eguchi-Hanson soliton is U-dual to the usual Kaluza-Klein Taub-Nut/Bolt magnetic monopoles.

6.3.1 Eguchi-Hanson Kaluza-Klein Monopole

Start with the four dimensional Eguchi-Hanson metric to which we add a flat time direction[3]:

$$ds^2 = -dt^2 + \frac{r^2 f(r)}{4}(dz - \cos\theta d\phi)^2 + \frac{dr^2}{f(r)} + \frac{r^2}{4}(d\theta^2 + \sin^2\theta d\phi^2), \tag{6.18}$$

where

$$f(r) = 1 - \frac{\delta^4}{r^4}. \tag{6.19}$$

We follow the method described in [44] to dimensionally reduce from five to three dimensions, perform a $SL(2,R)$ transformation and oxidize back to four dimensions. This is done to scale and shift the harmonic functions so as to obtain asymptotically Minkowskian limits.

The ansatz for the dimensional reduction from five dimensions to four dimensions in Kaluza-Klein theory is given by:

$$ds_5^2 = e^{\frac{\psi}{\sqrt{3}}} ds_4^2 + e^{-\frac{2}{\sqrt{3}}\psi}(dz + \mathcal{A}^1_{(1)})^2. \tag{6.20}$$

Then the four dimensional theory will be described by the action:

$$\mathcal{L}_4 = eR - \frac{1}{2}e(\partial\psi)^2 - \frac{1}{4}e \cdot e^{-\sqrt{3}\psi}(\mathcal{F}^1_{(2)})^2. \tag{6.21}$$

Here $\mathcal{F}^1_{(2)} = d\mathcal{A}^1_{(1)}$ and $\mathcal{A}^1_{(1)} = \mathcal{A}^1_\mu dx^\mu$ is the one-form potential that appears in the Kaluza-Klein form of the metric. The superscript 1 is used to denote the one-form which appears

[3] We changed the notation for the extra coordinate since we reserve χ to denote a scalar field later on.

from the dimensional reduction from five to four dimensions. The subscript in the brackets is used to denote the degree of the respective differential form.

Performing dimensional reduction along the z-direction we obtain the four dimensional solution:

$$\begin{aligned} ds_4 &= -\frac{r\sqrt{f(r)}}{2}dt^2 + \frac{r}{2\sqrt{f(r)}}dr^2 + \frac{r^3\sqrt{f(r)}}{8}(d\theta^2 + \sin^2\theta d\phi^2) \\ e^{-\frac{2}{\sqrt{3}}\psi} &= \frac{r^2 f(r)}{4}, \qquad \mathcal{A}^1_{(1)} = -\cos\theta d\phi. \end{aligned} \tag{6.22}$$

Notice that as $r \to \infty$, $\psi \to -\infty$; near infinity the spacetime decompactifies and the physical description is essentially five-dimensional.

Let us now perform a further dimensional reduction down to three-dimensions. The ansatz for the dimensional reduction along the time direction t from four dimensions to three dimensions in Kaluza-Klein theory is given by:

$$ds_4^2 = e^{\varphi} ds_3^2 - e^{-\varphi}(dt + \mathcal{A}^2_{(1)})^2. \tag{6.23}$$

The four dimensional Lagrangian (6.21) will reduce in three dimensions to the Lagrangian given by:

$$\begin{aligned} \mathcal{L}_3 &= eR - \frac{1}{2}e(\partial\psi)^2 - \frac{1}{2}e(\partial\varphi)^2 + \frac{1}{4}e\cdot e^{-2\varphi}(\mathcal{F}^2_{(2)})^2 - \frac{1}{4}e\cdot e^{-\varphi-\sqrt{3}\psi}(\mathcal{F}'^1_{(2)})^2 \\ &\quad + \frac{1}{2}e\cdot e^{\varphi-\sqrt{3}\psi}(\mathcal{F}^1_{(1)2})^2. \end{aligned} \tag{6.24}$$

Here $\mathcal{F}^1_{(1)2} = d\mathcal{A}^1_{(0)2}$ and $\mathcal{F}^2_{(2)} = d\mathcal{A}^2_{(1)}$ where $\mathcal{A}^2_{(1)} = \mathcal{A}^2_\mu dx^\mu$ is the one-form potential that appears in the Kaluza-Klein form of the metric. We have defined $\mathcal{F}'^1_{(2)} = \mathcal{F}^1_{(2)} - \mathcal{F}^1_{(1)2} \wedge \mathcal{A}^2_{(1)}$. The superscript 2 is used to denote the dimensional reduction from four to three dimensions along the timelike direction. Note that the kinetic terms of the field strengths that have the value 2 of the internal index have the opposite sign to the one that corresponds to a usual dimensional reduction on a spacelike direction. Performing the dimensional reduction we get the following three-dimensional fields:

$$\begin{aligned} ds_3^2 &= \frac{r^2 dr^2}{4} + \frac{(r^4 - \delta^4)}{16}(d\theta^2 + \sin^2\theta d\phi^2), \\ e^{-\frac{2}{\sqrt{3}}\psi} &= \frac{r^2 f(r)}{4}, \quad e^{-2\varphi} = \frac{r^2 f(r)}{4}, \quad \mathcal{A}^1_{(1)} = -\cos\theta d\phi. \end{aligned} \tag{6.25}$$

6.3.2 $SL(2,R)$ invariance of the reduced action

Defining new scalar fields ϕ_1 and ϕ_2 by:

$$\phi_1 = \frac{\sqrt{3}}{2}\psi + \frac{\varphi}{2}, \qquad \phi_2 = \frac{\sqrt{3}}{2}\varphi - \frac{\psi}{2}. \tag{6.26}$$

we can see that the three dimensional Lagrangian can be cast in the following form:

$$\begin{aligned} \mathcal{L}_3 &= eR - \frac{1}{2}e(\partial\phi_1)^2 - \frac{1}{2}e(\partial\phi_2)^2 + \frac{1}{4}e \cdot e^{-\phi_1 - \sqrt{3}\phi_2}(\mathcal{F}^2_{(2)})^2 - \frac{1}{4}e \cdot e^{-2\phi_1}(\mathcal{F'}^1_{(2)})^2 \\ &\quad + \frac{1}{2}e \cdot e^{-\phi_1 + \sqrt{3}\phi_2}(\mathcal{F}^1_{(1)2})^2. \end{aligned} \tag{6.27}$$

We shall define a new radial coordinate $R = \frac{r^2}{4}$ and define $\alpha = \frac{\delta^2}{4}$. In the new variables our solution corresponds to:

$$\begin{aligned} ds_3^2 &= dR^2 + (R^2 - \alpha^2)(d\theta^2 + \sin^2\theta d\phi^2), \\ e^{\phi_1} &= \frac{R}{R^2 - \alpha^2}, \qquad e^{\phi_2} = 0, \qquad \mathcal{A}^1_{(1)} = -\cos\theta d\phi. \end{aligned} \tag{6.28}$$

Notice that it is consistent to truncate the fields $\mathcal{A}^2_{(1)}$ and $\mathcal{A}^1_{(0)}$ (which also vanish in our solution) in the above three dimensional Lagrangian. We next dualize the 1-form potential field $\mathcal{A}^1_{(1)}$ to a scalar field χ.

Details on the dualization

To dualize the 2-form $\mathcal{F}^1_{(2)}$ one adds the Lagrange multiplier of the form $d\chi \wedge \mathcal{F}^1_{(2)}$. Then the Lagrangian that dictates the dynamics of the $\mathcal{F}^1_{(2)}$ field is given by[4]:

$$\begin{aligned} \mathcal{L}_{\mathcal{F}^1_{(2)}} &= -\frac{1}{4}ee^{-2\phi_1}(\mathcal{F}^1_{(2)})^2 + d\chi \wedge \mathcal{F}^1_{(2)} \\ &= -\frac{1}{4}ee^{-2\phi_1}(\mathcal{F}^1)_{\mu_1\mu_2}(\mathcal{F}^1)^{\mu_1\mu_2} + \frac{1}{2}\epsilon^{\mu_1\mu_2\mu_3}(\mathcal{F}^1)_{\mu_1\mu_2}\partial_{\mu_3}\chi. \end{aligned} \tag{6.29}$$

Treating $\mathcal{F}^1_{(2)}$ as an auxiliary field we can trivially solve its equation of motion to obtain:

$$e(\mathcal{F}^1)^{\mu_1\mu_2} = e^{2\phi_1}\epsilon^{\mu_1\mu_2\mu_3}\partial_{\mu_3}.\chi. \tag{6.30}$$

[4]Here ϵ is the Levi-Civita symbol.

Replacing this expression in the above Lagrangian and noticing that $\epsilon^{ijk}\epsilon_{ijl} = 2\delta_l^k$ (since the three-dimensional space has Euclidean signature) we obtain in the end:

$$\begin{aligned}\mathcal{L}_{\mathcal{F}^1_{(2)}} &= -\frac{1}{4}ee^{-2\phi_1}(\mathcal{F}^1)_{\mu_1\mu_2}(\mathcal{F}^1)^{\mu_1\mu_2} + \frac{1}{2}\epsilon^{\mu_1\mu_2\mu_3}(\mathcal{F}^1)_{\mu_1\mu_2}\partial_{\mu_3}\chi \\ &= +\frac{1}{2}ee^{2\phi_1}\partial_\mu\chi\partial^\mu\chi. \end{aligned} \tag{6.31}$$

For the our solution, we obtain the scalar field χ to be given by

$$\chi = \frac{R^2 + \beta R + \alpha^2}{R}, \tag{6.32}$$

where β is an arbitrary integration constant.

In terms of the new fields the three dimensional Lagrangian becomes:

$$\mathcal{L}_3 = eR - \frac{1}{2}e(\partial\phi_1)^2 - \frac{1}{2}e(\partial\phi_2)^2 + \frac{1}{2}e\cdot e^{2\phi_1}(d\chi)^2. \tag{6.33}$$

The unusual sign for the kinetic term of the scalar field χ appears because we have performed the dualization in the three-dimensional space, which has Euclidean signature.

We shall next show that the above Lagrangian has a global $SL(2,R)$ symmetry, with the scalars ϕ_1 and χ parameterizing the coset $SL(2,R)/O(1,1)$. To see this we shall define the following 2×2 scalar matrix:

$$\mathcal{M} = \begin{pmatrix} e^{\phi_1} & \chi e^{\phi_1} \\ \chi e^{\phi_1} & -e^{-\phi_1} + \chi^2 e^{\phi_1} \end{pmatrix}. \tag{6.34}$$

Note that $\det\mathcal{M} = -1$ hence $\mathcal{M}$ is not an $SL(2,R)$ matrix.

Then it is easy to see that the truncated three dimensional Lagrangian can be written in the following compact form:

$$\mathcal{L} = eR - \frac{1}{2}e(\partial\phi_2)^2 + \frac{1}{4}e\mathrm{tr}[\partial\mathcal{M}^{-1}\partial\mathcal{M}]. \tag{6.35}$$

The Lagrangian is manifestly invariant under $SL(2,R)$ transformations if we consider the following transformation law of the potentials:

$$\mathcal{M} \to U^T\mathcal{M}U, \quad \phi_2 \to \phi_2, \tag{6.36}$$

where $U \in SL(2,R)/O(1,1)$. We can parameterize the matrix U in the form

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1. \tag{6.37}$$

Then the two scalar fields ϕ_1 and χ transform under (6.36) as:

$$\begin{aligned} e^{\phi_1'} &= a^2 e^{\phi_1} + 2ac\chi e^{\phi_1} - c^2 e^{-\phi_1} + c^2\chi^2 e^{\phi_1}, \\ \chi' e^{\phi_1'} &= abe^{\phi_1} + (ad+bc)\chi e^{\phi_1} - dce^{-\phi_1} + dc\chi^2 e^{\phi_1}. \end{aligned} \tag{6.38}$$

Then new primed solution after the $SL(2,R)$ transformation is:

$$\begin{aligned} ds_3^2 &= dR^2 + (R^2-\alpha^2)(d\theta^2 + \sin^2\theta d\phi^2), \\ \chi' &= \frac{(bc+da+2dc\beta)R^2 + (4ba+4da\beta+4dc\beta^2+dc\alpha^2+4bc\beta)R + da\alpha^2 + bc\alpha^2 + 2dc\beta\alpha^2}{(cR+2(a+c\beta))\,(2(a+c\beta)R+c\alpha^2)}, \\ e^{\phi_1'} &= \frac{(cR+2(a+c\beta))\,(2(a+c\beta)R+c\alpha^2)}{R^2-\alpha^2}, \qquad \phi_2 = 0. \end{aligned} \tag{6.39}$$

6.3.3 Oxidation from three to four dimensions

We are now ready to lift the solution from three to four dimensions. For this we have to dualise the scalar field strength $d\chi$ to an electromagnetic field strength.[5] Using the formula (6.30) we obtain in a straightforward manner:

$$\mathcal{F}^1_{(2)} = [(a+c\beta)^2 - 4\alpha^2c^2]\sin\theta d\theta\wedge d\phi. \tag{6.40}$$

Next we rotate back the scalar fields by inverting the relations (6.26) and we obtain:

$$\psi = \frac{\sqrt{3}}{2}\phi_1 - \frac{\phi_2}{2}, \qquad \varphi = \frac{\sqrt{3}}{2}\phi_2 + \frac{\phi_1}{2}. \tag{6.41}$$

Notice that for our solutions $\phi_2 = 0$ therefore $e^\psi = e^{\frac{\sqrt{3}}{2}\phi_1}$ and $e^\varphi = e^{\frac{\phi_1}{2}}$. Using (6.23) we obtain the following field configuration in four dimensions:

$$\begin{aligned} ds_4^2 &= F(R)^{\frac{1}{2}}dt^2 + F(R)^{-\frac{1}{2}}dR^2 + F(R)^{-\frac{1}{2}}(R^2-\alpha^2)(d\theta^2+\sin^2\theta d\phi), \\ e^{\frac{\psi}{\sqrt{3}}} &= F(R)^{-\frac{1}{2}}, \qquad \mathcal{A}^1_{(1)} = -[(a+c\beta)^2 - 4\alpha^2c^2]\cos\theta d\phi, \\ F(R) &= \frac{(2cR+p)(pR+2\alpha^2c)}{R^2-\alpha^2}, \end{aligned} \tag{6.42}$$

[5] We will drop from here on the prime superscript.

which is a solution of the equations of motion derived from the Lagrangian (6.21).

Let us notice that the asymptotic form of the metric is given by:

$$ds_4^2 \sim -dT^2 + d\tilde{R}^2 + \tilde{R}^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{6.43}$$

after rescaling $t = [2c(a+c\beta)]^{\frac{1}{4}}T$ and $\tilde{R} = [2c(a+c\beta)]^{\frac{1}{4}}R$. Moreover, since the dilaton field is constant at infinity the solution is asymptotically flat. It describes a magnetic monopole located at $R = \alpha$, with magnetic charge given by:

$$\frac{1}{4\pi}\int_{S_2} \mathcal{F}^1_{(2)} \;=\; (a + c\beta)^2 - 4\alpha^2 c^2. \tag{6.44}$$

We can further oxidize this solution to five dimensions and we obtain the following Ricci flat metric:

$$ds_5^2 \;=\; -dt^2 + \frac{R^2-\alpha^2}{g(R)}(dz - \mathcal{A}^1_{(1)})^2 + \frac{g(R)}{R^2-\alpha^2}dR^2 + g(R)(d\theta^2 + \sin^2\theta d\phi^2), \tag{6.45}$$

where[6]

$$g(R) \;=\; (2cR + p)(pR + 2\alpha^2 c), \qquad \mathcal{A}^1_{(1)} = (p^2 - 4\alpha^2c^2)\cos\theta d\phi. \tag{6.46}$$

Since this metric is Ricci flat and it is the trivial product of a four-dimensional metric with a time direction, we expect that:

$$ds_4^2 \;=\; \frac{R^2-\alpha^2}{g(R)}(dz - \mathcal{A}^1_{(1)})^2 + \frac{g(R)}{R^2-\alpha^2}dR^2 + g(R)(d\theta^2 + \sin^2\theta d\phi^2) \tag{6.47}$$

is a Ricci flat metric and it can be easily checked that this is indeed the case.

Let us discuss next the regularity conditions on this metric. Elimination of Misner string singularities requires the coordinate z be periodic with periodicity $8\pi n$, where $n = p^2 - 4\alpha^2c^2$. By requiring the absence of conical singularities in the (z, R) sector we obtain $2\pi(p + 2\alpha c)^2$ as the periodicity of the z coordinate. By equating this value with the periodicity required by the absence of Misner string singularities we obtain the following restriction: $\alpha c = \frac{3n}{8}$ on the values of the parameters n, α and c. However, it can be easily checked that after imposing this constraint the final solution depends on only one

[6] We denote $a + c\beta$ by p.

parameter (the NUT charge n), as the parameter α leads only to a global rescaling of the metric. As $g(R)$ is not zero at $R = \alpha$ the above four-dimensional solution corresponds to a Taub-Bolt solution and therefore the four-dimensional monopole is the Taub-Bolt magnetic monopole.

On the other hand, if we start with the flat space in five dimensions, *i.e.*if we consider $\alpha = 0$, then it has been proven in [44] that the final solution is the Kaluza-Klein soliton constructed using the Taub-Nut.

6.3.4 Monopole solutions in 4 dimensions

We are now ready to generate new magnetic monopole solutions in four dimensions using a similar procedure as in the GPS case. In order to obtain cosmological four-dimensional magnetic monopole solutions our strategy is to consider directly in five-dimensions the new cosmological Taub-NUT-like solutions discussed in Chapter 3 and perform a Kaluza-Klein compactification along the fifth dimension. As we have seen in section 3.6 the Euclidean version of these solutions can be cast into the form:

$$\begin{aligned} ds^2 &= \frac{r^2}{4}\left(1-\frac{a^4}{r^4}\right)(d\chi+\cos\theta d\phi)^2+\frac{dr^2}{\left(\pm\frac{r^2}{l^2}+1\right)\left(1-\frac{a^4}{r^4}\right)} \\ &+\frac{r^2}{4}(d\theta^2+\sin^2\theta d\phi^2)+\left(\pm\frac{r^2}{l^2}+1\right)dy^2, \end{aligned} \tag{6.48}$$

after performing a coordinate transformation and redefining the parameter m. This is a solution of the vacuum Einstein equations with cosmological constant $\Lambda = \mp 6/l^2$.

The Eguchi-Hanson soliton is non-singular if we periodically identify the coordinate χ with period $\beta = \frac{4\pi}{k}$ (to remove the Misner string singularities), where k is an integer. However, we have to match this periodicity with the one that appears by eliminating the conical singularities at $r = a$ in the (χ, r) sector. Considering for instance the AdS case, this condition leads to:

$$a^2 = l^2\left(\frac{k^2}{4}-1\right). \tag{6.49}$$

One has to consider then $k > 3$ and this yields $a > l$.

Notice that these regularity conditions remain unchanged if we we can further consider the analytic continuation $y \to it$. We can perform now a Kaluza-Klein reduction along the χ direction.

Since our initial space-time is a solution of the Einstein field equations with non-zero cosmological constant, the field content in four dimensions will be given by the metric tensor $g_{\mu\nu}$, a magnetic one-form potential $\mathcal{A}$ and a scalar field ϕ with a non-trivial scalar potential $V(\phi)$. We obtain in general:

$$\begin{aligned} ds^2 &= -\left(1+\frac{r^2}{l^2}\right)\frac{r}{2}f(r)^{\frac{1}{2}}dt^2 + \frac{rdr^2}{2\left(1+\frac{r^2}{l^2}\right)f(r)^{\frac{1}{2}}} + \frac{r^3}{8}f(r)^{\frac{1}{2}}d\Omega^2, \\ \mathcal{A} &= -\cos\theta d\varphi, \qquad e^{\frac{\phi}{\sqrt{3}}} = \frac{2}{r}f(r)^{-\frac{1}{2}}, \end{aligned} \tag{6.50}$$

where

$$f(r) = 1 - \frac{a^4}{r^4}.$$

This solution differs from the Kaluza-Klein GPS solution in terms of both the metric coefficients and by the fact that the scalar field ϕ has a potential of the exponential type

$$V(\phi) = -\frac{8}{l^2}e^{-\frac{\phi}{\sqrt{3}}},$$

indicating that our Kaluza-Klein dimensional reduction yields a massive scalar field.

To study the properties of the above 4-dimensional spaces, let us consider first the Taub-Nut solution, which we have seen in Chapter 3 that corresponds to the pure AdS solution in five-dimensions. For $a = 0$ we obtain the five-dimensional metric:

$$ds^2 = -(1+\frac{r^2}{l^2})dt^2 + \frac{1}{1+\frac{r^2}{l^2}}dr^2 + \frac{r^2}{4}\left[(d\chi - \cos\theta d\varphi)^2 + d\theta^2 + \sin^2\theta d\varphi^2\right].$$

After Kaluza-Klein compactification we obtain the following four-dimensional fields:

$$\begin{aligned} ds^2 &= -\frac{r}{2}(1+\frac{r^2}{l^2})dt^2 + \frac{rdr^2}{2\left(1+\frac{r^2}{l^2}\right)} + \frac{r^3}{8}d\Omega^2, \\ \mathcal{A} &= -\cos\theta d\varphi, \qquad e^{-\frac{\phi}{\sqrt{3}}} = \frac{r}{2}. \end{aligned} \tag{6.51}$$

The four-dimensional metric has a curvature singularity at $r = 0$, where the scalar field also diverges. Its asymptotic structure is given by

$$ds^2 = -\frac{2\tilde{r}^{4/3}}{l^2}dt^2 + \frac{l^2}{2\tilde{r}^{4/3}}d\tilde{r}^2 + \tilde{r}^2 d\Omega^2,$$

where we have rescaled t and defined $\tilde{r}$ by $\tilde{r}^2 = \frac{r^3}{8}$ and we can see that it is not asymptotically flat. However, one can check by computing some of the curvature scalars (like $R_{abcd}R^{abcd}$) that the above asymptotic metric is well-behaved at infinity and that it has a curvature singularity at $\tilde{r} = 0$. Notice however that at both $r = 0$ and $r \to \infty$ the dilaton is blowing up and the extra dimension opens up, which means that the physical description is effectively five-dimensional.

For the Bolt solution we must take $k \geq 3$ and $a \neq 0$. The compactified four-dimensional solution is given again by (6.50) and its asymptotic structure is the same as the one obtained from the five-dimensional AdS spacetime. Again the four-dimensional metric will have a curvature singularity at the bolt while the dilaton diverges at $r = a$ and at infinity.

Next, let us notice that in the string frame the situation changes as follows. The five-dimensional Nut solution reduces to the following metric:

$$ds^2 = -(1 + \frac{R^2}{l^2})dt^2 + \frac{1}{1 + \frac{R^2}{l^2}}dR^2 + \frac{R^2}{4}\left[d\theta^2 + \sin^2\theta d\varphi^2\right].$$

While this metric resembles a four-dimensional AdS metric with cosmological constant $\lambda = -\frac{3}{l^2}$ in fact there is a deficit of solid angle as the area of the 2-sphere is πR^2 instead of $4\pi R^2$. This behavior is characteristic of a global monopole [18]. The above metric has a curvature singularity at the origin (the location of the monopole) and the dilaton field diverges both at origin and at infinity. The magnetic charge is computed using the formula [130]:

$$\frac{1}{4\pi}\int_{S^2}\mathcal{F} = 1.$$

Note that if we reduce directly the five-dimensional metric (6.48) for $\Lambda > 0$, which is time-dependent (outside the cosmological horizon the coordinate r is timelike) we obtain a time-dependent four-dimensional magnetic monopole solution.

On the other hand, if we set the cosmological constant to zero, *i.e.* we take the limit $l \to \infty$, the five dimensional metric reduces to the Eguchi-Hanson times a trivial time direction. Therefore, upon a Kaluza-Klein compactification to four dimensions we obtain the Eguchi-Hanson monopole.

6.4 Higher dimensional magnetic monopoles

We now consider some of the higher dimensional Taub-NUT spaces constructed in Chapter 3. We will discuss in some detail the six-dimensional metrics and we will also use the Hopf-duality to generate new solutions.

6.4.1 Six dimensional metrics

In six dimensions the base space is four-dimensional and we can use products of the form $M_1 \times M_2$ of two-dimensional Einstein spaces or we can use CP^2 as a four-dimensional base space over which to construct the circle fibrations. If we use products of two dimensional Einstein spaces then we can consider all the cases in which M_i, $i = 1, 2$ can be a sphere S^2, a torus T^2 or a hyperboloid H^2. The circle fibration can be constructed in these cases over the whole base space $M_1 \times M_2$ or just over one factor space M_i.

We shall consider first the case in which $M_1 = M_2 = S^2$ and assume that the $U(1)$ fibration is constructed over the whole base space $S^2 \times S^2$. In what follows we shall look at the case of two different NUT charges, that is we set the cosmological constant to zero.

Let us consider the Euclidean section, obtained by the following analytic continuations $t \to i\chi$ and $n_j \to in_j$ where $j = 1, 2$:

$$\begin{aligned} ds^2 &= F_E(r)(d\chi - 2n_1 \cos\theta_1 d\varphi_1 - 2n_2 \cos\theta_2 d\varphi_2)^2 + F_E^{-1}(r)dr^2 \\ &\quad +(r^2 - n_1^2)(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2) + (r^2 - n_2^2)(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2), \end{aligned} \tag{6.52}$$

where

$$F_E(r) = \frac{r^4 - 3(n_1^2 + n_2^2)r^2 + 6mr - 3n_1^2 n_2^2}{3(r^2 - n_1^2)(r^2 - n_2^2)}. \tag{6.53}$$

This metric is a solution of the vacuum Einstein field equations without cosmological constant, for any values of the parameters n_1 and n_2. In order to analyze the possible singularities we have to consider two cases, depending on the values of the NUT charges n_1 and n_2. We will assume that the two NUT charges n_1 and n_2 differ, and we can set $n_1 > n_2$ without loss of generality. In this case in the Euclidean section the radius r cannot be smaller than n_1 or the signature of the spacetime will change. The Taub-Nut solution in this case corresponds to a two-dimensional fixed-point set located at $r = n_1$. There is still a curvature singularity located at $r = n_1$, removed by setting the periodicity of the coordinate χ to be $8\pi n_1$, while the value of the mass parameter must be $m = m_p = \frac{n_1^3+3n_1n_2^2}{3}$.

The Bolt solution corresponds to a four-dimensional fixed-point set located at $r = r_b = \frac{2n_1}{k}$, for which the periodicity of the coordinate χ is given by $\frac{8\pi n_1}{k}$ and the value of the mass parameter is $m = m_p = \frac{n_1(12n_2^2-4n_1^2)}{12}$. In order to avoid the curvature singularity at $r = n_1$ we must choose $k = 1$ so that $r > r_b = 2n_1$.

In the following we consider the Taub-Nut solution for which $n_1 > n_2$. The periodicity of the coordinate χ is taken to be $8\pi n_1$ while the value of the mass parameter is fixed to be $m = m_p = \frac{n_1^3+3n_1n_2^2}{3}$. For these values the six-dimensional metric is nonsingular at $r = n_1$.

Employing the usual Kaluza-Klein procedure we obtain a six-dimensional magnetic monopole: we add a flat time direction to obtain a seven-dimensional solution of the vacuum Einstein field equations and after that perform a Kaluza-Klein compactification along the coordinate χ using the metric ansatz:

$$ds_7^2 = e^{\frac{\phi}{\sqrt{10}}}ds_6^2 + e^{-\frac{4\phi}{\sqrt{10}}}(d\chi - A_{(1)})^2.$$

It is easy to check that we obtain the following six-dimensional fields

$$\begin{aligned} ds_6^2 &= -F_E^{\frac{1}{4}}dt^2 + F_E^{-\frac{3}{4}}dr^2 + F_E^{\frac{1}{4}}\left[(r^2-n_1^2)(d\theta_1^2+\sin^2\theta_1 d\varphi_1^2) + (r^2-n_2^2)(d\theta_2^2+\sin^2\theta_2 d\varphi_2^2)\right], \\ A_{(1)} &= -2n_1\cos\theta_1 d\varphi_1 - 2n_2\cos\theta_2 d\varphi_2, \qquad e^{-\frac{\phi}{\sqrt{10}}} = F_E^{\frac{1}{4}}, \end{aligned} \tag{6.54}$$

where we now restrict $r \geq n_1 > n_2$ and

$$F_E(r) = \frac{r^3 + n_1r^2 - (2n_1^2+3n_2^2)r + 3n_1n_2^2}{3(r+n_1)(r^2-n_2^2)}.$$

One can check that the above six-dimensional monopole solution has a curvature singularity located at $r = n_1$. It is interesting to note that the asymptotic structure of this solution, after rescaling the coordinates $t \to 3^{1/4}T$ and $r \to 3^{-3/8}R$ is given by

$$ds_{asymp}^2 = -dT^2 + dR^2 + \frac{1}{3}R^2(d\theta_1^2+\sin^2\theta_1 d\varphi_1^2) + \frac{1}{3}R^2(d\theta_2^2+\sin^2\theta_2 d\varphi_2^2).$$

The area of each 2-sphere is not $4\pi R^2$ but instead $\frac{4\pi R^2}{3}$: each has a deficit solid angle of $\frac{8\pi}{3}$ steradians. Furthermore, the above asymptotic form of the metric is Ricci flat, and can be obtained from our solution by setting $n = 0$. Therefore we conclude that the background for our monopole is a Ricci flat Bohm metric constructed as a cone over $S^2 \times S^2$ [23, 64].

The corresponding six-dimensional Lagrangian obtained after Kaluza-Klein reduction is given by

$$\mathcal{L}_6 = eR - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{4}ee^{-\frac{5}{\sqrt{10}}\phi}F_{(2)}^2,$$

where $F_{(2)} = dA_{(1)} = 2n_1\Omega_1 + 2n_2\Omega_2$ and we have denoted by Ω_i the volume form $\sin\theta_i d\theta_i \wedge d\varphi_i$ of the sphere M_i, $i = 1, 2$.

In the following we shall perform a Kaluza-Klein reduction on M_2. The general sphere reduction formulae have been presented in [29]. The metric ansatz that we have to use in the dimensional reduction from six to four dimensions is given by:

$$ds_6^2 = e^{\frac{\varphi}{\sqrt{2}}}ds_4^2 + e^{-\frac{\varphi}{\sqrt{2}}}(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2).$$

The dimensionally-reduced Lagrangian will take now the form

$$\mathcal{L}_4 = eR - \frac{1}{2}e(\partial\varphi)^2 - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{4}ee^{-\frac{\varphi}{\sqrt{2}}-\frac{5}{\sqrt{10}}\phi}F^2 + ee^{\sqrt{2}\varphi}R_2 - e2n_2^2 e^{\frac{3}{\sqrt{2}}\varphi - \frac{5}{\sqrt{10}}\phi},$$

where $F = dA = d(-2n_1\cos\theta_1 d\phi_1)$ and $R_2 = 4$ is the Ricci scalar of the sphere M_2. The full solution in four dimensions will be given by:

$$\begin{aligned} ds_4^2 &= -F_E^{\frac{1}{2}}(r^2 - n_2^2)dt^2 + F_E^{-\frac{1}{2}}(r^2 - n_2^2)dr^2 + F_E^{\frac{1}{2}}(r^2 - n_1^2)(r^2 - n_2^2)(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2), \\ F &= dA = 2n_1\sin\theta_1 d\theta_1 \wedge d\varphi_1, \qquad e^{-\frac{\phi}{\sqrt{10}}} = F_E^{\frac{1}{4}}, \qquad e^{-\frac{\varphi}{\sqrt{2}}} = F_E^{\frac{1}{4}}(r^2 - n_2^2). \end{aligned} \tag{6.55}$$

The asymptotic form of the above four-dimensional monopole metric is given by:

$$ds_{asymp}^2 \sim -Rdt^2 + dR^2 + \frac{4R^2}{3}(d\theta_1^2 + \sin^2\theta d\varphi_1^2), \tag{6.56}$$

(after defining $r^2 = \frac{2R}{3^{\frac{1}{4}}}$ and rescaling the time coordinate t) and we can see that the spacetime is not asymptotically flat. Moreover the metric has infinite redshift at the origin[7], which is also the location of a curvature singularity. It takes an infinite time for a photon to reach infinity and, indeed, the (r, t)-sector is asymptotically flat; however, while

[7] A similar BPS monopole solution with an infinite redshift at the origin but with a deficit of solid angle has been obtained in [79].

the metric it is singularity-free at infinity the scalar field φ diverges there. It is interesting to note that the asymptotic form has a surfeit of solid angle, as the area of a sphere of radius R is not $4\pi R^2$ but $\frac{16\pi r^2}{3}$. Asymptotically conical metrics are reminiscent of global monopoles [18]. The magnetic charge is computed to be $2n_1$.

The second case to discuss in six dimensions is a generalisation of the five-dimensional solution presented in the previous section. The metric ansatz is as follows:

$$\begin{aligned} ds^2 &= -F(r)(dt - 2n\cos\theta_1 d\phi_1)^2 + F^{-1}(r)dr^2 + (r^2+n^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \alpha r^2(d\theta_2^2 + d\phi_2^2), \\ F(r) &= \frac{-3r^5 + (l^2 - 10n^2)r^3 + 3n^2(l^2 - 5n^2)r + 6ml^2}{3rl^2(r^2+n^2)}, \end{aligned} \tag{6.57}$$

and the vacuum Einstein field equations with cosmological constant are satisfied if and only if $\alpha(2 - \lambda n^2) = 0$. Since α cannot be zero we must restrict the values of n and $\lambda = \frac{10}{l^2}$ such that $\lambda n^2 = 2$, forcing a positive cosmological constant. The Euclidean section is obtained by taking the analytic continuation $t \to i\chi$ and $n \to in$ and $l \to il$ with $l = \sqrt{5}n$. Notice that in this case the Taub-Nut solution corresponds to a two-dimensional fixed-point set of the vector field $\frac{\partial}{\partial\chi}$ located at $r = n$. The periodicity of the χ coordinate is in this case equal to $8\pi n$ and the value of the mass parameter is fixed to $m_b = \frac{n^3}{15}$. For this value of the mass parameter the solution is regular at the nut location. The Bolt spacetime has a four-dimensional fixed-point set of $\frac{\partial}{\partial\chi}$ located at $r_b = \frac{kn}{2}$ and the value of the mass parameter is $m_b = \frac{k^3n^3(20-3k^2)}{960}$ where the periodicity of the coordinate χ is $\frac{8\pi n}{k}$, where k is an integer. To ensure that $r_b > n$ we have to take $k > 3$; in this way the curvature singularity at $r = n$ is avoided as well. We next perform another analytic continuation of one of the coordinates on T^2 (say $\theta_2 \to it$) and then two Kaluza-Klein reductions along the χ and ϕ_2 directions down to four dimensions. We obtain the final solution:

$$\begin{aligned} ds^2 &= -r^3 F_E^{\frac{1}{2}} dt^2 + F_E^{-\frac{1}{2}}(r) r dr^2 + F_E^{\frac{1}{2}} r\left(r^2 - \frac{l^2}{5}\right) d\Omega^2, \\ \mathcal{A} &= -2n\cos\theta_1 d\phi_1, \qquad e^{\frac{-3\varphi_1}{\sqrt{6}}} = r^2, \qquad e^{\frac{-\varphi_2}{\sqrt{3}}} = r^{\frac{1}{3}} F_E^{-\frac{1}{2}}, \end{aligned} \tag{6.58}$$

where

$$F_E(r) = \frac{15r^5 - 5l^2r^3 + 30ml^2}{3l^2r(5r^2 - l^2)},$$

which is a solution of the equations of motion derived from the following Lagrangean:

$$\mathcal{L}_4 = eR - \frac{1}{2}e(\partial\varphi_1)^2 - \frac{1}{2}e(\partial\varphi_2)^2 - \frac{1}{4}ee^{-\sqrt{3}\varphi_2}\mathcal{F}^2 + 2ee^{\frac{\varphi_1}{\sqrt{6}} + \frac{\varphi_2}{\sqrt{3}}}\lambda,$$

with $\lambda = -\frac{10}{l^2}$ and $\mathcal{F} = d\mathcal{A}$. The asymptotic form of the metric is:

$$ds_4^2 = -R^2dt^2 + \frac{\sqrt{l}}{4R}dR^2 + R^2d\Omega^2,$$

after rescaling $R^2 = \frac{r^4}{l}$. It is clearly not asymptotically flat, it has infinite redshift at origin, which is also the location of a curvature singularity and as expected the dilaton fields diverge at infinity. However, notice that while φ_2 diverges at the root of F_E, φ_1 is finite there. The magnetic charge is found to be $2n$.

Hopf reductions in six dimensions

It is well-known that odd-dimensional spheres S^{2n+1} may be regarded as circle bundles over CP^n and one can use the so-called Hopf duality (a T-duality along the $U(1)$-fibre) to generate new solutions [51, 52, 45] by untwisting S^{2n+1} to $CP^n \times S^1$. The six-dimensional case is particularly interesting for us since it has been shown in [52] that it is possible to make consistent truncations of the maximal Type II supergravity theories to a bosonic sector which exhibits an $O(2,2)$ global symmetry with the T-duality transformation taking a simple form. The theories at hand are the toroidal reductions of Type IIA, respectively Type IIB ten-dimensional supergravities while the reduction ansatz for the fields is that the six-dimensional fields that are retained are precisely the ten-dimensional ones, with the spacetime indices restricted to run over the six-dimensional range only. The two truncated theories in $D = 6$ are then related by a T-duality transformation upon reduction to $D = 5$. The explicit mappings of the fields have been given in [52] and we follow their notational conventions. For convenience we also provide the derivation of the T-duality rules in Appendix C.

Let us start with the solution given in (7.25) in which we set $\lambda = 0$. We shall perform first the analytic continuations $t \to iz$, $n_1 \to in_1$ and subsequently $\varphi_2 \to it$:

$$\begin{aligned} ds^2 &= \tilde{F}(r)(dz - 2n_1\cos\theta_1 d\varphi_1 - 2n_2\cos\theta_2 dt)^2 + \tilde{F}^{-1}(r)dr^2 \\ &\quad +(r^2 - n_1^2)(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2) + (r^2 + n_2^2)(d\theta_2^2 - \sin^2\theta_2 dt^2), \end{aligned} \tag{6.59}$$

where

$$\tilde{F}(r) = \frac{r^4 - 3(n_1^2 - n_2^2)r^2 + 6mr + 3n_1^2n_2^2}{3(r^2 - n_1^2)(r^2 + n_2^2)}.$$

Considering the above metric as a solution of the pure gravity sector of the truncated Type IIA theory we can now perform a Hopf-duality along the spacelike z-direction to obtain a solution of six-dimensional Type IIB theory. Following the procedure given in Appendix C, we first perform a Kaluza-Klein dimensional reduction down to five dimensions to obtain:

$$\begin{aligned} ds^2_{5A} &= \tilde{F}(r)^{-\frac{2}{3}}dr^2 + \tilde{F}(r)^{-\frac{1}{2}}\left[(r^2-n_1^2)(d\theta_1^2+\sin^2\theta_1 d\varphi_1^2)+(r^2+n_2^2)(d\theta_2^2-\sin^2\theta_2 dt^2)\right], \\ e^{-\frac{\varphi_A}{\sqrt{6}}} &= \tilde{F}(r)^{\frac{1}{3}}, \quad \mathcal{A}_{(1)} = -2n_1\cos\theta_1 d\varphi_1 - 2n_2\cos\theta_2 dt. \end{aligned} \tag{6.60}$$

We can now rotate the scalars using (C.10) and map the field $\mathcal{A}_{(1)} \to A^{NS}_{(1)1}$ as in (C.9). Finally, lifting the solution back to six-dimensions we obtain the following Type IIB solution:

$$\begin{aligned} ds_{6B} &= \tilde{F}(r)^{-\frac{1}{2}}dz^2 + \tilde{F}(r)^{-\frac{1}{2}}dr^2 + \tilde{F}(r)^{\frac{1}{2}}(r^2-n_1^2)d\Omega_1^2 + \tilde{F}(r)^{\frac{1}{2}}(r^2+n_2^2)(d\theta_2^2-\sin^2\theta_2 dt^2), \\ e^{2\phi_1} &= e^{2\phi_2} = \tilde{F}(r), \qquad A^{NS}_{(2)} = -2n_1\cos\theta_1 d\varphi_2\wedge dz + 2n_2\cos\theta_2 dt\wedge dz. \end{aligned} \tag{6.61}$$

We can also make the analytic continuations $t \to i\varphi_2$ and $n_1 \to in_1$ to obtain the solution:

$$\begin{aligned} ds_{6B} &= -F(r)^{-\frac{1}{2}}dt^2 + F(r)^{-\frac{1}{2}}dr^2 + F(r)^{\frac{1}{2}}(r^2+n_1^2)d\Omega_1^2 + F(r)^{\frac{1}{2}}(r^2+n_2^2)d\Omega_2^2, \\ e^{2\phi_1} &= e^{2\phi_2} = F(r), \qquad A^{NS}_{(2)} = 2n_1\cos\theta_1 d\varphi_2\wedge dt - 2n_2\cos\theta_2 d\varphi_2\wedge dt. \end{aligned} \tag{6.62}$$

Were we to consider (6.59) as a solution of the pure gravity sector of Type IIB theory, then after performing the spacelike Hopf dualisation we would obtain as an intermediate step the five-dimensional solution (6.60) where now the fields belong to the Type IIB theory. Applying now the formulae (C.9), respectively rotating the scalars as in (C.10) and oxidizing the solution back to six dimensions we obtain:

$$\begin{aligned} ds_{6A} &= \tilde{F}(r)^{-\frac{1}{2}}dz^2 + \tilde{F}(r)^{-\frac{1}{2}}dr^2 + \bar{F}(r)^{\frac{1}{2}}(r^2-n_1^2)d\Omega_1^2 + \bar{F}(r)^{\frac{1}{2}}(r^2+n_2^2)(d\theta_2^2-\sin^2\theta_2 dt^2), \\ e^{2\phi_1} &= e^{2\phi_2} = \bar{F}(r), \qquad A_{(2)} = -2n_1\cos\theta_1 d\varphi_2\wedge dz + 2n_2\cos\theta_2 dt\wedge dz. \end{aligned} \tag{6.63}$$

Performing the analytic continuations we recover (6.62) except that we have now to replace $A^{NS}_{(2)}$ with $A_{(2)}$.

It is interesting to note that we can perform directly a timelike Hopf reduction in six-dimensions[8]. In this case if we start with a solution of Type IIA theory by performing the

[8] In which case we do not need to perform any analytical continuations.

timelike T-duality we obtain a solution of the appropriate truncation of Type IIB* theory [93]. If we start instead with a solution of Type IIB theory and perform a timelike Hopf duality we end up with a solution of an appropriate truncation of Type IIA* theory. The details of these reductions are gathered in the Appendix A.

As an example we shall perform a Hopf duality starting from Type IIA theory. Consider (7.25) as a solution of the pure gravity sector of the truncated six-dimensional Type IIA theory. Then the final solution of Type IIB* will be given by:

$$\begin{aligned} ds_{6B^*} &= -F(r)^{-\frac{1}{2}}dt^2 + F(r)^{-\frac{1}{2}}dr^2 + F(r)^{\frac{1}{2}}(r^2+n_1^2)d\Omega_1^2 + F(r)^{\frac{1}{2}}(r^2+n_2^2)d\Omega_2^2, \\ e^{2\phi_1} &= e^{2\phi_2} = F(r), \qquad A_{(2)}^{NS} = 2n_1\cos\theta_1 d\varphi_1\wedge dt - 2n_2\cos\theta_2 d\varphi_2\wedge dt, \end{aligned} \tag{6.64}$$

where now

$$F(r) = \frac{r^4 - 3(n_1^2+n_2^2)r^2 + 6mr - 3n_1^2n_2^2}{3(r^2+n_1^2)(r^2+n_2^2)}.$$

If we start with (7.25) as a solution of Type IIB then peforming a timelike Hopf dualisation we obtain a similar solution of Type IIA* for which:

$$A_{(2)} = 2n_1\cos\theta_1 d\varphi_1\wedge dt - 2n_2\cos\theta_2 d\varphi_2\wedge dt. \tag{6.65}$$

As we can see, some of the solutions obtained for Type IIA (respectively IIB) and Type IIA* (respectively IIB*) are identical after we perform appropriate analytic continuations[9]. This is to be expected once we notice that they are solutions of the $NSNS$-sector only, which is the same for both theories (their actions would differ only by the sign of the kinetic terms of the RR-fields).

As an application of these solutions let us set for convenience $n_2 = 0$ in (6.62) and perform a sphere reduction using the ansatz (6.4.1) down to a four-dimensional solution:

$$\begin{aligned} ds_{4B} &= -r^2dt^2 + r^2dr^2 + F(r)r^2(r^2+n_1^2)d\Omega_1^2, \\ e^{-\sqrt{2}\phi} &= r^4F(r), \qquad e^{2\phi_1} = e^{2\phi_2} = F(r), \qquad A_{(2)}^{NS} = 2n_1\cos\theta_1 d\varphi_2\wedge dt, \end{aligned} \tag{6.66}$$

which is a solution of the equations of motion derived from the following Lagrangian:

$$\mathcal{L}_{4B} = eR - \frac{1}{2}e(\partial\varphi)^2 - \frac{1}{2}e(\partial\phi_1)^2 - \frac{1}{2}e(\partial\phi_2)^2 - \frac{1}{4}ee^{-\frac{\varphi}{\sqrt{2}}-\varphi_1-\varphi_2}(F_{(3)}^{NS})^2 + ee^{\sqrt{2}\varphi}R_2.$$

[9] Which keep the metric and the fields real.

The asymptotic form of the metric (6.66) is (after defining $R = \frac{r^2}{2}$ and rescaling t)

$$ds_{4B} \sim -Rdt^2 + dR^2 + \frac{4}{3}R^2 d\Omega_1^2.$$

Amusingly, the asymptotic form of the metric is the same with (6.56). The magnetic charge is found to be $2n$ and notice that there is an excess of solid angle as the area of the asymptotic sphere is $\frac{16\pi R^2}{3}$ instead of the expected $4\pi R^2$.

6.4.2 Monopoles in $D \geq 7$ dimensions

Similarly, we can construct Kaluza-Klein monopoles in seven and higher dimensions. For example, in seven dimensions the base space is five-dimensional and can be factorized in the form $B = M \times Y$, where M is an even dimensional space endowed with an Einstein-Kähler metric and Y is a Riemannian Einstein space.

Let us consider the case in which $M = S^2$ while Y can be a sphere S^3, a torus T^3 or a hyperboloid H^3. The solution is [111]

$$ds^2 = -F(r)(dt^2 + 2n\cos\theta d\varphi)^2 + F^{-1}(r)dr^2 + (r^2+n^2)(d\theta^2 + \sin^2\theta d\varphi^2) + \beta r^2 g_Y,$$

where g_Y is the metric on the unit-sphere S^3, torus T^3 or hyperboloid H^3:

$$F(r) = \frac{4r^6 + (l^2+12n^2)r^4 + 2n^2(l^2+6n^2)r^2 + 4ml^2 + n^4(l^2+6n^2)}{4l^2r^2(r^2+n^2)}. \tag{6.67}$$

The cosmological constant is $\lambda = -\frac{15}{l^2}$ and the parameters β, n and λ are constrained via the relation $\beta(5 - 2\lambda n^2) = 10k$, where $k = 1, 0, -1$ for S^3, T^3 and H^3 respectively. We must have $\beta > 0$, which in turn imposes a joint constraint on λn^2 that can be satisfied in various ways depending on the value of k. Since we are interested in a Ricci flat solution we shall consider $\lambda = 0$ and also $k = 1$, in which case $Y = S^3$ and $\beta = 2$. The Euclidean section of this solution, which is obtained by analytic continuation of the coordinate $t \to i\chi$ and of the parameter $n \to in$ is given by:

$$ds^2 = F_E(r)(d\chi^2 + 2n\cos\theta d\varphi)^2 + F_E^{-1}(r)dr^2 + (r^2-n^2)(d\theta^2 + \sin^2\theta d\varphi^2) + 2r^2 g_Y,$$

where

$$F(r) = \frac{r^4 - 2n^2r^2 + n^4 + 4m}{4r^2(r^2-n^2)}. \tag{6.68}$$

The metric has a scalar curvature singularity located at $r = n$ as can be checked by computing for instance the Kretschman scalar. However, if we take $m = 0$ then the metric is well-behaved at $r = n$. Furthermore, if $r < n$ the signature of the metric is unphysical; therefore we can restrict ourselves to the interval $r \geq n$ for which the solution is non-singular. In order to obtain the magnetic brane in seven-dimensions we employ the usual procedure: add a flat timelike direction and compactify the Ricci-flat eight-dimensional metric using the ansatz

$$ds_8^2 = e^{\frac{\phi}{\sqrt{15}}} ds_7^2 + e^{-\frac{5\phi}{\sqrt{15}}}(d\chi - A_{(1)})^2.$$

We obtain the following seven-dimensional fields:

$$\begin{aligned} ds_7^2 &= -F_E^{\frac{1}{5}} dt^2 + F_E^{-\frac{4}{5}} dr^2 + F_E^{\frac{1}{5}} \left((r^2 - n^2)(d\theta^2 + \sin^2\theta d\varphi^2) + 2r^2 d\Omega_3^2 \right), \\ A_{(1)} &= -2n\cos\theta d\varphi, \qquad e^{-\frac{\phi}{\sqrt{15}}} = F_E^{\frac{1}{5}}, \end{aligned} \tag{6.69}$$

where now $r \geq n$ and

$$F_E(r) = \frac{r^2 - n^2}{4r^2},$$

which are a solution of the equations of motion derived from the following Lagrangian:

$$\mathcal{L}_7 = eR - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{4}ee^{-\frac{6}{\sqrt{15}}\phi} F_{(2)}^2.$$

The above seven-dimensional metric has a curvature singularity at $r = n$. Its asymptotic structure, after we rescale the coordinates $t \to 4^{1/10}T$ and $r \to 4^{-2/5}R$, is given by

$$ds_{asymp}^2 = -dT^2 + dR^2 + \frac{R^2}{4}(d\theta^2 + \sin^2\theta d\varphi^2) + 2^{3/5}R^2 d\Omega_3^2.$$

This space has a deficit of solid angle corresponding to the sphere S^2 while the factor S^3 has a surfeit of solid angle.

Let us perform now a further dimensional reduction of the above seven-dimensional solution on the three-sphere S^3. The metric ansatz that we can use in the dimensional reduction from 7 to 4 dimensions is given by:

$$ds_7^2 = e^{\frac{3\varphi}{2\sqrt{15}}} ds_4^2 + e^{-\frac{\varphi}{\sqrt{15}}} d\Omega_3^2.$$

The four-dimensional fields will be given by

$$\begin{aligned} ds_4^2 &= 2^{\frac{3}{2}} r^3 \left(-F_E^{\frac{1}{2}} dt^2 + F_E^{\frac{1}{2}} dr^2 + F_E^{\frac{1}{2}} (r^2 - n^2)(d\theta^2 + \sin^2\theta d\varphi^2) \right), \\ A_{(1)} &= -2n\cos\theta d\varphi, \qquad e^{-\frac{\phi}{\sqrt{15}}} = F_E^{\frac{1}{5}}, \qquad e^{-\frac{\varphi}{\sqrt{15}}} = 2r^2 F_E^{\frac{1}{5}}, \end{aligned} \tag{6.70}$$

and they are a solution of the equations of motion derived from the following dimensionally reduced Lagrangian:

$$\mathcal{L}_4 = eR - \frac{1}{2}e(\partial\varphi)^2 - \frac{1}{2}e(\partial\phi)^2 - \frac{1}{4}ee^{-\frac{3\varphi}{2\sqrt{15}} - \frac{6}{\sqrt{15}}\phi} F_{(2)}^2 + ee^{\frac{5}{\sqrt{15}}\varphi} R_3,$$

where $R_3 = 6$ is the curvature scalar of the unit sphere S^3.

One can check that the above four-dimensional solution has a scalar curvature singularity at $r = n$. Its asymptotics are given by:

$$ds^2 = R^3 \left[-dT^2 + dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2) \right],$$

after we make the rescaling $R = \sqrt{2}r$ and $T = t/2$. Consequently the spacetime that we obtain is conformally flat and singularity free at infinity. The magnetic charge is found to be $2n$.

generalisation to more than seven dimensions is straightforward. Another interesting solution can be found in eleven dimensions by using the ansatz:

$$ds^2 = -F(r)(dt + 2n\cos\theta d\phi)^2 + \frac{dr^2}{F(r)} + (r^2 + n^2)(d\theta^2 + \sin^2\theta d\phi^2) + 6r^2 d\Omega_7^2,$$

By solving the vacuum Einstein field equations we find:

$$F(r) = \frac{3r^8 + 4n^2r^6 + 24m}{24r^6(r^2 + n^2)}.$$

Here $d\Omega_7^2$ is the metric on the 7-sphere, normalized such that its Ricci tensor is $R_{ij} = 6g_{ij}$. The Euclidean section is obtained by analytic continuations $t \to i\chi$ and $n \to in$. We can also replace the sphere element by any other Einstein space of positive curvature. For example, if we embed the seven dimensional de Sitter solution we obtain the metric:

$$\begin{aligned} ds^2 &= \tilde{F}(r)(d\chi + 2n\cos\theta d\phi)^2 + \frac{dr^2}{\tilde{F}(r)} + (r^2 - n^2)(d\theta^2 + \sin^2\theta d\phi^2) \\ &\quad + r^2\left[-\left(1 - \frac{R^2}{6}\right) dt^2 + \frac{dR^2}{1 - \frac{R^2}{6}} + R^2 d\Omega_5^2 \right], \\ \tilde{F}(r) &= \frac{3r^8 - 4n^2 r^6 + 24m}{24r^6(r^2 - n^2)}. \end{aligned} \tag{6.71}$$

It can be easily checked there is a curvature singularity located at $r = 0$. Misner string singularities can be removed by requiring the χ coordinate to have period $8\pi n$ and $m = \frac{n^8}{24}$. The values of the coordinate r are then restricted to $r > n$ avoiding the curvature singularity at $r = 0$. This solution corresponds to a seven dimensional fixed-point set of the isometry ∂_χ; since this is not the maximal possible co-dimension, it is a Taub-Nut solution.

The other possibility is that of a nine-dimensional fixed-point set of the isometry ∂_χ, located at $r_b = 2n$. The periodicity of the χ coordinate is still $8\pi n$ but now the values of the r coordinate are such that $r \geq r_b = 2n$. This in turn avoids the curvature singularities located at $r = 0$ and $r = n$, provided the value of the mass parameter is $m = -\frac{64n^8}{3}$.

Let us perform now a Kaluza-Klein compactification along the coordinate χ. The reduction ansatz is:

$$ds_{11}^2 = e^{\frac{\varphi}{6}} ds_{10}^2 + e^{-\frac{4\varphi}{3}}(d\chi + \mathcal{A})^2,$$

and we obtain the following ten-dimensional fields:

$$\begin{aligned} ds_{10}^2 &= \tilde{F}^{\frac{1}{8}}(r) r^2 \left[-\left(1 - \frac{R^2}{6}\right) dt^2 + \frac{dR^2}{1 - \frac{R^2}{6}} + R^2 d\Omega_5^2 \right] \\ &\quad + \tilde{F}^{-\frac{7}{8}} dr^2 + \tilde{F}^{\frac{1}{8}}(r)(r^2 - n^2)(d\theta^2 + \sin^2\theta d\phi^2), \\ \mathcal{A} &= 2n\cos\theta d\phi, \qquad e^{-\frac{4\varphi}{3}} = \tilde{F}(r). \end{aligned} \tag{6.72}$$

Now let us perform a sphere reduction of this solution down to five-dimensions using the metric ansatz:

$$ds_{10}^2 = e^{\sqrt{\frac{5}{12}}\phi} ds_5^2 + e^{-\sqrt{\frac{3}{20}}\phi} d\Omega_5^2. \tag{6.73}$$

We obtain the following fields:

$$\begin{aligned} ds_{5A} &= -\tilde{F}^{\frac{1}{3}} r^{\frac{16}{3}} R^{\frac{10}{3}} \left(1 - \frac{R^2}{6}\right) dt^2 + \tilde{F}^{\frac{1}{3}} r^{\frac{16}{3}} R^{\frac{10}{3}} \frac{dR^2}{1 - \frac{R^2}{6}} + \tilde{F}^{-\frac{2}{3}} r^{\frac{10}{3}} R^{\frac{10}{3}} dr^2 + \tilde{F}^{\frac{1}{3}} R^{\frac{10}{3}} r^{\frac{10}{3}} (r^2 - n^2) d\Omega_2^2, \\ \mathcal{A} &= 2n \cos\theta d\phi, \qquad e^{-\frac{4\varphi}{3}} = \tilde{F}(r), \quad e^{-\sqrt{\frac{3}{20}}\phi} = \tilde{F}^{\frac{5}{24}} r^{\frac{10}{3}} R^{\frac{10}{3}}, \end{aligned} \tag{6.74}$$

which give a solution of the equations of motion derived from the Lagrangian:

$$\mathcal{L}_5 = eR - \frac{1}{2} e (\partial\varphi)^2 - \frac{1}{2} e (\partial\phi)^2 - \frac{1}{4} e e^{-\frac{3\varphi}{2} - \sqrt{\frac{5}{12}}\phi} (\mathcal{F}_{(2)})^2 + e e^{\frac{4}{\sqrt{15}}\phi} R_5,$$

where R_5 is the curvature scalar of the unit 5-sphere. We can dualize $\mathcal{F}_{(2)}$ to a 3-form field strength and we find that the above solution would describe a non-uniform electric string in five dimensions as our solution depends explicitly on the fifth dimension R. This solution is very likely to be unstable as in eleven dimensions the 'de Sitter horizon' is delocalised along the noncompact direction r.

Chapter 7

Nutty Bubbles

Many important problems in physics, such as cosmological evolution or black hole evaporation, involve time in an essential way. Therefore, a key problem in string theory is understanding its behaviour in time-dependent backgrounds. In order to carry out this investigation one needs to construct simple enough time dependent-solutions that would provide consistent test-beds on which one could try to address these problems.

'Bubbles of nothing' are smooth, time-dependent[1] vacuum solutions of Einstein's equations and so are consistent backgrounds for string theory, at least at leading order in α'. The characteristic feature of such a solution is that it has a (minimal) area with *no* space inside. For example, a bubble solution can be obtained from a four-dimensional static, spherically symmetric black hole by a double analytic continuation in the time coordinate and some other combination of coordinates on the S^2-section. This way, the sphere is effectively changed into a two-dimensional de Sitter spacetime.

The first example was provided by Witten [138] as the endstate of the decay of the Kaluza-Klein (KK) vacuum. Balasubramanian *et at.* [15] found a similar process in Anti-de Sitter (AdS) spacetime. That is, a certain orbifold in AdS (analogue of the flat space KK vacuum) decays via a bubble of nothing. This opens the possibility that highly non-perturbative processes in gravity might be described (via AdS/CFT correspondence [106]) as barrier penetration in a dual field theory effective potential.

[1] There are also time-independent bubbles. Properties of bubbles of nothing in different situations are studied in [138, 1, 20, 16, 62, 22, 32].

More recently there has been renewed interest in bubbles of nothing since it was pointed out that they provide new endpoints for Hawking evaporation [92]. Closed string tachyon condensation is at the basis of a topology changing transition from black strings to bubbles of nothing.

In a recent work Ghezelbash and Mann studied the so-called 'nutty bubbles' — time-dependent backgrounds obtained by double analytic continuations of the coordinates/parameters of (locally asymptotically flat AdS) NUT-charged solutions in four dimensions [62]. At that time no five (or higher)-dimensional NUT-charged solutions with more than one NUT charge were known and based on the properties of the NUT-charged spaces with only one NUT parameter, they conjectured that one cannot construct consistent nutty bubble solutions (with only one timelike direction) in higher dimensions. However, using the new higher dimensional Taub-NUT solutions described in Chapter 3 we are able to provide interesting time-dependent nutty bubble solutions in higher-dimensions.

This chapter is organized as follows: in the next section we construct bubble solutions starting from the 5-dimensional Taub-NUT solutions. Recall that in five dimensions these solutions have only one NUT parameter. Moreover there is in general a restriction that connects the value of the NUT parameter to the cosmological constant. The bubbles are obtained by performing appropriate double analytic continuations of the coordinates. While our focus is primarily on the asymptotically AdS solutions, we will also provide non-trivial bubble solutions that are asymptotically dS as well as a five-dimensional non-asymptotically flat solution. Remarkably, we find a locally asymptotically AdS solution with a boundary geometry of $AdS_3 \times S^1$. In the Discussion section at the end of this chapter we will calculate its boundary stress tensor and show that it has two pieces: one that depends on the parameters of the bubble, and the other one which is universal and is reproduced by the universal anomaly contribution to the stress tensor of Yang-Mills theory on $AdS_3 \times S^1$.

In the third section we construct interesting higher dimensional nutty bubbles from some of the six-dimensional Taub-NUT solutions. However we focus on describing in detail only a couple of representative six-dimensional spaces. In contrast to the lower-dimensional spaces, in higher than six-dimensions there can be at least two independent NUT parameters and, quite generically, we have seen that there exits a set of constraints that relate the values of these NUT parameters to the cosmological constant. We can analytically continue the NUT parameters independently, as long as we can still satisfy the constraints (or their analytically continued avatars). We consider first the case when the base space of the circle fibration characteristic of the Taub-NUT solution is $S^2 \times S^2$. In this case there are two independent NUT parameters only if the cosmological constant vanishes. In six dimensions there exists a different class of cosmological Taub-NUT solutions, which are characterized by one NUT parameter only. The fibration is constructed over the first 2-dimensional factor M_1 and we consider the warped product with M_2. The novelty of this type of solution is that the warp factor depends non-trivially on the cosmological constant and the NUT charge. The time-dependent bubble solutions are obtained by double analytic continuations of the coordinates *and* the nut parameter. Finally, we also present a method to generate new time-dependent solutions by using Hopf-dualities using the Hopf-duality rules deduced in Appendix C. We apply this method to some of our 6-dimensional bubble solutions to generate new interesting time-dependent backgrounds.

7.1 Five dimensional 'nutty' bubbles

As we have seen in Chapter 3, the five-dimensional Taub-NUT space is built as a 'partial' fibration over a two-dimensional Einstein-Kähler space. However, in five dimensions there is a constraint to be satisfied on the possible values of the NUT charge and the cosmological constant. The effect of this constraint is such that for a circle fibration over the sphere, S^2, the cosmological constant can take only positive values, for a fibration over the torus, T^2, the cosmological constant must vanish, while in the case of a fibration over the hyperboloid, H^2, the cosmological constant can have only negative values.[2] It is worth mentioning that one cannot simultaneously set the NUT charge and/or the cosmological constant to zero — *i.e.* there is no smooth limit in which one can obtain five dimensional Minkowski space in this way. However, we have also seen in Chapter 3 that there are some ways to evade this situation.

[2] We are considering here the Lorentzian sections of the metric.

7.1.1 Nutty bubbles in AdS

As noted above, the cosmological constant can be negative ($\Lambda = -\frac{6}{l^2}$) only in the case of a fibration over the hyperboloid H^2. The metric is

$$ds^2 = -F(r)(dt - 2n\cosh\theta d\phi)^2 + F^{-1}(r)dr^2 + (r^2+n^2)(d\theta^2 + \sinh^2\theta d\phi^2) + r^2dy^2, \quad (7.1)$$

where

$$F(r) = \frac{4r^4 + 2l^2r^2 - 16ml^2}{l^2(4r^2+l^2)}. \quad (7.2)$$

Moreover, there is a constraint on the NUT parameter $n^2 = \frac{l^2}{4}$, which we already used to simplify the expression of $F(r)$. If we analytically continue the coordinate $t \to i\chi$ and then perform further analytic continuations in the H^2 sector, the following distinct metrics are obtained:

$$\begin{aligned}
ds^2 &= F(r)(d\chi + l\cos t d\phi)^2 + F^{-1}(r)dr^2 + (r^2+l^2/4)(-dt^2 + \sin^2 t d\phi^2) + r^2dy^2\\
ds^2 &= F(r)(d\chi + l\sinh\theta dt)^2 + F^{-1}(r)dr^2 + (r^2+l^2/4)(d\theta^2 - \cosh^2\theta dt^2) + r^2dy^2,\\
ds^2 &= F(r)(d\chi + l\cosh\theta dt)^2 + F^{-1}(r)dr^2 + (r^2+l^2/4)(d\theta^2 - \sinh^2\theta dt^2) + r^2dy^2,\\
ds^2 &= F(r)(d\chi + le^{\theta} dt)^2 + F^{-1}(r)dr^2 + (r^2+l^2/4)(d\theta^2 - e^{2\theta}dt^2) + r^2dy^2. \quad (7.3)
\end{aligned}$$

They are solutions of the vacuum Einstein field equations with negative cosmological constant $\Lambda = -6/l^2$.

For the last three geometries the coordinate θ is no longer periodic and can take any real value. The geometry in the second bracket is described by a two-dimensional AdS space. As is well known, this space can have non-trivial identifications and so the 2-dimensional sector can describe a 2-dimensional black hole (as in the second and third metrics above), while the first metric describes pure AdS in standard coordinates. Notice however that in this case, the geometry of a fixed (χ, r, y)-slice is AdS_2 modified by the term $F(r)l^2\sinh^2\theta dt^2$ as an effect of the non-trivial fibration over the (θ, t)-sector. This extra term will vanish only at points where $F(r) = 0$ (thence, on the bubble).

After a coordinate transformation the last three metrics can be written in a compact form

$$\begin{aligned}
ds^2 &= F(r)(d\chi + lxdt)^2 + F^{-1}(r)dr^2 + (r^2+l^2/4)\left(\frac{dx^2}{x^2+k} - (x^2+k)dt^2\right)\\
&\quad + r^2dy^2, \quad (7.4)
\end{aligned}$$

with $k = -1, 1, 0$. They are all locally equivalent under changes of coordinates. However, depending on the identifications made, the global structure can be quite different.

The quartic function in the numerator of $F(r)$ can have only two real roots — for $m > 0$, one is positive (denoted by r_+) and the other one is negative (denoted by r_-):

$$r_\pm = \pm\frac{l}{2}\sqrt{\sqrt{1+64m/l^2}-1}. \tag{7.5}$$

The conical singularities at either root of $F(r)$ in the (χ, r)-sector can be eliminated if the periodicity of the χ-coordinate is

$$\beta = \frac{4\pi}{|F'(r_\pm)|} = \frac{2\pi l}{\sqrt{\sqrt{1+64m/l^2}-1}}. \tag{7.6}$$

Now, for $r > r_+$ (or $r < r_-$) the first three metrics will describe stationary backgrounds. Note that the metric (7.4) is stationary and it possesses the Killing vector $\xi = \frac{\partial}{\partial t}$. The norm of this Killing vector is

$$\xi \cdot \xi = l^2 x^2 F(r) - (r^2 + l^2/4)(x^2 + k),$$

and we find that in general there is an ergoregion iff

$$\left(\frac{4l^2 F(r)}{4r^2 + l^2} - 1\right) x^2 > k.$$

However, since the expression in the bracket is always negative we find that there exists an ergoregion only if $k = -1$ in which case the following constraint is obtained:

$$|x| < \frac{4r^2 + l^2}{l\sqrt{64m + l^2}}.$$

The ergoregion corresponds to a strip in the (r, x) plane bounded by the horizons located at $|x| = 1$ and two curves that asymptote to 1 for $r \to r_\pm$, while for large values of r the strip will largely broaden (see figure 7.1) and the curves asymptote to $\frac{4x^2}{l\sqrt{64m+l^2}}$. Also, as it is apparent from the above formula, the strip broadens as m decreases.

In the remaining cases, for $k = 0, 1$ there is no ergoregion.

The asymptotic structure of the above metrics is

$$ds^2 = r^2/l^2(d\chi + lx dt)^2 + l^2/r^2 dr^2 + r^2\left(\frac{dx^2}{x^2+k} - (x^2+k)dt^2\right) + r^2 dy^2. \tag{7.7}$$

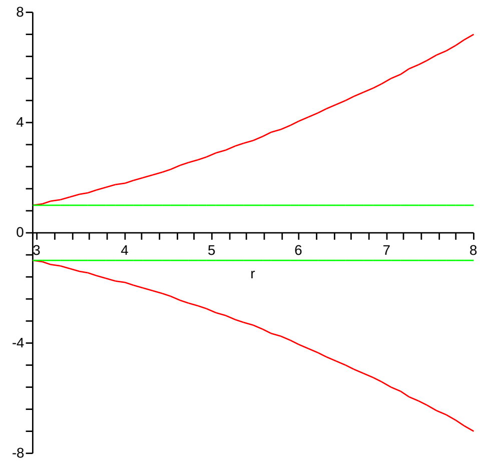

Figure 7.1: Ergoregion of the topological metric $k = -1$ (for $m = 20$ and $l = 1$).

Now, it is easy to read the boundary geometry — up to a conformal rescaling factor r^2/l^2, the boundary metric is

$$ds^2 \;=\; l^2(d\tilde{\chi} + x dt)^2 + l^2\left(\frac{dx^2}{x^2+k} - (x^2+k)dt^2\right) + l^2 dy^2. \tag{7.8}$$

Here, we use a rescaled coordinate $\tilde{\chi} = \chi/l$. From (7.6) it is easy to see that if $m = \frac{l^2 s^2(s^2+8)}{1024}$ then $\tilde{\chi}$ has periodicity $4\pi/s$, with s an integer. Remarkably, for $s = 1$ the boundary geometry is conformally flat. This can be easily seen from the fact that the boundary metric is the product of a 3-dimensional space of constant curvature (i.e. pure AdS_3) with a line (or a circle if we also compactify the y coordinate). Furthermore, one can also make nontrivial identifications in the AdS_3 sector which turn it into the BTZ black hole. We will have to say more about these solutions in the Discussion section.

Finally, let us consider the first metric from (7.3). Even if formally it can be transformed into the $k = -1$ metric by a coordinate transformation[3], if ϕ is periodic then the global structure of these spaces is completely different. The geometry in the (χ, t, ϕ)-sector

[3]Notice however that the range of the x coordinate is different: for the $k = -1$ metric $x \geq 1$, while for the first metric in (7.3) we must take $|x| \leq 1$. Another difference is that the periodicity in χ would be related to a periodicity in ϕ for the first metric, while for the second one, with $k = -1$, it would require us

resembles the usual Hopf-type fibration. The χ-circle is now fibred over the circle described by ϕ. However, the fibration is twisted as a function of time. At $t = 0$ we have a pair of orthogonal circles provided we define χ appropriately. As time increases we have the χ-circle twisting around relative to the ϕ-circle, while the ϕ-circle is getting bigger. The latter reaches a maximum, and then begins to shrink. However the χ-circle is still twisting, and by the time the ϕ-circle has shrunk back to zero, the χ-circle has twisted only 'halfway' round. Over this cycle the integral $\int d(l \cos t d\phi)$ is well-defined, and it equals $4\pi\ell$ since we are integrating t from 0 to π. This will set the periodicity of χ to be $4\pi l/s$, where s is an integer. Recall now that the quartic function $F(r)$ can have only two real roots, one positive (r_+) and one negative (r_-) for $m > 0$. If ϕ is an angular coordinate with period 2π, then in order to eliminate the Misner string singularity we require that the period $\beta = 4\pi/|F'(r_\pm)|$ be equal to $4\pi l/s$, where s is some integer. This further restricts the value of the mass parameter such that $m = \frac{l^2 s^2 (s^2+8)}{1024}$.

For $r > r_+$ (or $r < r_-$), this metric describes then a bubble located at $r = r_+$, which expands from zero size to a finite size and then contracts to zero size again. All the spacetime events are causally connected with each other. Near the initial expansion (or the final contraction) the scale factor is linear in time and the spacetime expands or contracts like a Milne universe. The boundary geometry for this bubble spacetime is given by

$$ds^2 = (d\chi + l \cos t d\phi)^2 + l^2(-dt^2 + \sin^2 t d\phi^2) + l^2 dy^2, \tag{7.9}$$

where χ is periodic with period $4\pi l/s$.

7.1.2 Nutty bubbles in dS

The Taub-NUT ansatz that we shall use in the construction of these spaces is the following:

$$ds^2 = -F(r)(dt - 2n \cos\theta d\varphi)^2 + F^{-1}(r)dr^2 + (r^2 + n^2)(d\theta^2 + \sin^2\theta d\varphi^2) + r^2 dz^2. \tag{7.10}$$

The above metric will be a solution of the Einstein field equations with positive cosmological constant $\Lambda = \frac{6}{l^2}$ provided

$$F(r) = \frac{4ml^2 - r^4 - 2n^2 r^2}{l^2(r^2 + n^2)}, \tag{7.11}$$

to make t periodic. Therefore we find that while there are no hyperbolic Misner strings for $k = -1$, the fourth metric could have Misner strings.

where the field equations impose the constraint[4] $4n^2 = l^2$. Notice that, for large values of r, the function $F(r)$ takes negative values and r becomes effectively a timelike coordinate as one should expect in a region outside the cosmological horizon. In order to remove the usual Misner string singularity in the metric, we have to assume that the coordinate t is periodic with period $4\pi l$. If we analytically continue the coordinate $t \to i\chi$ and one of the coordinates in the S^2 sector we obtain the following metrics:

$$\begin{aligned}
ds^2 &= F(r)(d\chi + l\cos\theta dt)^2 + F^{-1}(r)dr^2 + (r^2 + l^2/4)(d\theta^2 - \sin^2\theta dt^2) + r^2 dy^2, \\
ds^2 &= F(r)(d\chi + l\cosh t d\phi)^2 + F^{-1}(r)dr^2 + (r^2 + l^2/4)(-dt^2 + \sinh^2 t d\phi^2) + r^2 dy^2, \\
ds^2 &= F(r)(d\chi + l\sinh t d\phi)^2 + F^{-1}(r)dr^2 + (r^2 + l^2/4)(-dt^2 + \cosh^2 t d\phi^2) + r^2 dy^2, \\
ds^2 &= F(r)(d\chi + le^t d\phi)^2 + F^{-1}(r)dr^2 + (r^2 + l^2/4)(-dt^2 + e^{2t} d\phi^2) + r^2 dy^2. \qquad (7.12)
\end{aligned}$$

These metrics satisfy the vacuum Einstein field equations with positive cosmological constant $\Lambda = 6/l^2$, where $F(r)$ is given by (7.11). However, for large values of r the function $F(r)$ becomes negative and the signature of the spacetime will change accordingly. To avoid this situation one possibility is to consider two roots of the function $F(r)$ and to restrict the values of the r coordinate such that $F(r)$ is always positive. Namely we restrict the range of the r coordinate such that $r_- < r < r_+$, where $r_\pm$ are two roots of $F(r)$ and in this way we avoid the change in the metric signature. It is easy to see that if $m > 0$ then $F(r)$ has two real roots only if

$$r_\pm = \pm\frac{l}{2}\sqrt{\sqrt{1 + 64m/l^2} - 1}\,.$$

Fortunately, the conical singularities at the roots of $F(r)$ in the (χ, r)-sector can both be eliminated at the same time if we choose the periodicity of the χ-coordinate to be given by

$$\beta = \frac{4\pi}{|F'(r_\pm)|} = \frac{2\pi l}{\sqrt{\sqrt{1 + 64m/l^2} - 1}}\,.$$

To eliminate the Misner string singularity in the first metric in (7.12), we require that the period β be equal to $4\pi l/s$, where s is some integer. This further restricts the value of the mass parameter such that $m = \frac{l^2 s^2 (s^2+8)}{1024}$. Notice that, locally, all these metrics

[4] In the following we shall use this constraint to eliminate n from the metric.

are equivalent, being related by coordinate transformations. However, these spaces will be equivalent globally only if the coordinate ϕ is unwrapped. At every fixed (χ, r, y) the geometry is that of a perturbed two-dimensional de Sitter spacetime as an effect of the non-trivial fibration.

To better understand the geometry in the (χ, r, y) sector let us focus on a section with t, ϕ held fixed. Then the metric in the (χ, r, y)-sector becomes

$$ds^2 = F(r)d\chi^2 + F^{-1}(r)dr^2 + r^2dy^2.$$

We restrict the values of the r-coordinate between the two roots of $F(r)$ and since they have the same magnitude, we shall take $r_+^2 = r_-^2 = r_0^2$. Then, it is easy to see that we can write

$$F(r) = (r_0^2 - r^2)\frac{4r^2 + 2l^2 + 4r_0^2}{l^2(4r^2 + l^2)} = (r_0^2 - r^2)f(r),$$

where $f(r)$ is strictly positive everywhere. Now if we make the following change of coordinates

$$r^2 = r_0^2(1 - x^2),$$

the metric in the (χ, r, y) sector becomes

$$ds^2 = \frac{dx^2}{(1 - x^2)f(r_0\sqrt{1 - x^2})} + r_0^2(1 - x^2)dy^2 + r_0^2x^2f(r_0\sqrt{1 - x^2})d\chi^2.$$

A further change of coordinates $x = \sin\psi$ will bring it in the form:

$$ds^2 = \frac{d\psi^2}{f(r_0\cos\psi)} + r_0^2\cos^2\psi dy^2 + r_0^2\sin^2\psi f(r_0\cos\psi)d\chi^2,$$

where

$$f(r_0\cos\psi) = \frac{1}{l^2}\left[1 + \frac{4r_0^2}{4r_0^2\cos^2\psi + l^2}\right].$$

It can be easily seen that the geometry in this sector is one of a deformed 3-sphere. We conclude that our bubble metrics describe non-trivial fibrations of a 3-sphere over a 2-dimensional dS space.

If the coordinate ϕ is periodic, the circle geometry that it describes will evolve differently for each of the above geometries. For instance, for the second metric in (7.12) the evolution is that of a circle that begins with zero radius at $t = 0$ and then expands exponentially as $t \to \infty$, while for the third metric we obtain a de Sitter evolution of a circle with exponentially large radius at $t \to -\infty$ that exponentially shrinks to a minimal value and then expands again. Finally the fourth geometry describes the evolution of a circle which begins with zero radius at $t \to -\infty$ and then expands exponentially as $t \to \infty$. Similar to the four-dimensional situation considered in [62], a null curve in a geometry for which the bubbles are expanding has $|\dot{\phi}| \leq e^{-t}|\dot{t}|$ at late times, where the overdot refers to a derivative with respect to proper time. Hence, observers at different values of ϕ will eventually lose causal contact. On the other hand null rays at fixed ϕ and y obey the relation

$$\dot{r}^2 + V(r) = 0, \tag{7.13}$$

where $V(r) = p_\chi^2 + 4p_y^2 F(r)/r^2 - 4E^2F(r)/(4r^2 + l^2)$ is an effective potential, $p_\chi = \dot{\chi}F$ is the conserved momentum along the χ direction, $p_y = r^2\dot{y}$ is the conserved momentum along the y direction and $E = (r^2 + l^2/4)\dot{t}$ is the conserved energy. Generically, if $p_y = 0$ then the null geodesics oscillate between some minimal and maximal values of r, which can be chosen to be within the admitted range of r. Hence observers at any two differing values of r can be causally connected. However, if the observers are at different values of y then the effective potential diverges at $r = 0$, which means that there will be two regions that can be causally disconnected. It is easy to check that it is possible for observers at different values of χ, respectively y to be causally connected at fixed values of r.

Let us notice that the first metric from (7.12) is stationary since it possesses the Killing vector $\xi = \frac{\partial}{\partial t}$. The norm of this Killing vector is

$$\xi \cdot \xi \;\; = \;\; l^2 F(r) - (r^2 + l^2/4 - l^2 F(r)) \sin^2\theta \tag{7.14}$$

and so it becomes spacelike unless $\hat{\theta}(r) \leq |\theta| \leq \pi - \hat{\theta}(r)$. Here, we use the notation

$$\hat{\theta}(r) \;\; = \;\; \tan^{-1}\left(\frac{2l\sqrt{F(r)}}{\sqrt{4r^2 + l^2}}\right). \tag{7.15}$$

As in the four-dimensional case [62] these limits will describe an ‘ergocone’ for the spacetime. The above angle vanishes at $r = r_\pm$ while it attains a maximum value at $r = 0$.

7.1.3 Nutty Rindler bubbles in flat backgrounds

We can also obtain NUT spaces with non-trivial topology if we construct the circle fibration over a two-dimensional torus T^2,

$$ds^2 = -F(r)(dt - 2n\theta d\phi)^2 + F^{-1}(r)dr^2 + (r^2+n^2)(d\theta^2 + d\phi^2) + r^2dy^2 \tag{7.16}$$

where

$$F(r) = \frac{4ml^2 + r^4 + 2n^2r^2}{l^2(r^2+n^2)} \tag{7.17}$$

and the constraint equation takes now the form $\Lambda n^2 = 0$. Consistent Taub-NUT spaces with toroidal topology exist if and only if the cosmological constant vanishes. The Euclidean version of this solution, obtained by analytic continuation of the coordinate $t \to it$ and of the parameter $n \to in$, has a curvature singularity at $r = n$. Note that if we consider $n = 0$ in the above constraint we obtain the AdS/dS black hole solution in five dimensions with toroidal topology.

If the cosmological constant vanishes, then we can have $n \neq 0$ and the metric becomes

$$ds^2 = -F(r)(dt - 2n\theta d\phi)^2 + F^{-1}(r)dr^2 + (r^2+n^2)(d\theta^2 + d\phi^2) + r^2dy^2, \tag{7.18}$$

where

$$F(r) = \frac{4m}{r^2+n^2}. \tag{7.19}$$

The asymptotic structure of the above metric is given by

$$ds^2 = \frac{4m}{r^2}(dt - 2n\theta d\phi)^2 + \frac{r^2}{4m}dr^2 + r^2(d\theta^2 + d\phi^2 + dy^2). \tag{7.20}$$

If y is an angular coordinate then the angular part of the metric parameterizes a three torus. The Euclidean section of the solution described by (7.18) is not asymptotically flat and has a curvature singularity localized at $r = 0$. However, let us notice that for $r \leq n$ the signature of the space becomes completely unphysical. Hence, for the Euclidean section, we should restrict the values of the radial coordinate such that $r \geq n$.

Consider now the analytic continuations of the coordinates $t \to i\chi$ and $\theta \to -it$ (respectively $\phi \to -it$) in the case of fibration over a torus. We obtain the spacetimes

$$\begin{aligned} ds^2 &= F(r)(d\chi + 2ntd\phi)^2 + F^{-1}(r)dr^2 + (r^2+n^2)(-dt^2 + d\phi^2) + r^2dy^2, \\ ds^2 &= F(r)(d\chi + 2n\theta dt)^2 + F^{-1}(r)dr^2 + (r^2+n^2)(d\theta^2 - dt^2) + r^2dy^2 \end{aligned} \tag{7.21}$$

whose metrics are locally equivalent under coordinate transformations. However, if one of the coordinates ϕ or θ is periodic then they represent globally different spaces. Another metric — related to the above by coordinate transformations — is a generalisation to five-dimensions of the four dimensional nutty Rindler spacetime. Since $F(r)$ has no roots these spaces are not really bubbles. However they still represent interesting time-dependent backgrounds, with metrics given by

$$\begin{aligned} ds^2 &= F(r)(d\chi + nt^2 d\phi)^2 + F^{-1}(r)dr^2 + (r^2+n^2)(-dt^2 + t^2 d\phi^2) + r^2 dy^2, \\ ds^2 &= F(r)(d\chi + n\theta^2 dt)^2 + F^{-1}(r)dr^2 + (r^2+n^2)(d\theta^2 - \theta^2 dt^2) + r^2 dy^2. \end{aligned} \quad (7.22)$$

Let us notice that, by analogy with the four-dimensional case, in the first case, the geometry of a slice (r, θ, y) is that of a twisted torus which has a Milne-type evolution. The second geometry describes a static spacetime (with the Killing vector $\xi = \frac{\partial}{\partial t}$) with an ergoregion described by

$$\theta^2 > \frac{r^2+n^2}{n^2 F(r)}.$$

For large values of r, the ergoregion includes almost the entire (θ, r) plane except for a strip bounded by two curves, opposite the r-axis, which asymptote to parabolas. For small values of r the strip narrows and the boundary curves asymptote to $\pm n/\sqrt{2m}$. For the first geometry, as $t^2\dot{\phi}^2 \leq \dot{t}^2$ we have $\phi \sim \ln t$ and observers with different values of ϕ can communicate with each other for arbitrarily large t. In the second geometry we obtain $\dot{\theta}^2 \leq \theta^2 \dot{t}^2$, *i.e.* $\theta \sim e^{|t|}$ and we see that there is no restriction as to the maximum change of coordinate θ for points on the null curve as $t \to \pm\infty$ and observers at points with different θ can communicate with each other.

7.2 Higher dimensional Nutty Bubbles

We now consider some of the higher dimensional Taub-NUT spaces described in Chapter 3. As we have seen, the most general ansatz for the higher dimensional Taub-NUT spaces corresponds to factorizations of the base space of the form $B = \prod_i M_i \times Y$, where each factor M_i is endowed with an Einstein-Kähler metric g_{M_i} while Y is a general Einstein space with metric g_Y. In these cases one can consider the $U(1)$-fibration only over the factored space $M = \prod_i M_i$ of the base B and take then a warped product with the manifold Y. Quite generically, we can associate a NUT parameter N_i with every such factor M_i and the general ansatz is then given by

$$F^{-1}(r)dr^2 + \sum_i (r^2 + N_i^2)g_{M_i} + r^2 g_Y - F(r)(dt + \sum_i 2N_i\mathcal{A}_i)^2. \tag{7.23}$$

We now consider particular cases of these ansätze. To be more specific we shall focus on a couple of six-dimensional metrics.

7.2.1 Bubbles in flat backgrounds

In six dimensions the base space is four-dimensional and we can use products of the form $M_1 \times M_2$ of two-dimensional Einstein-Kähler spaces or we can use CP^2 as a four-dimensional base space over which to construct the circle fibrations. If we use products of two dimensional Einstein-Kähler spaces then we can consider all the cases in which M_i, $i = 1, 2$ can be a sphere S^2, a torus T^2 or a hyperboloid H^2. The circle fibration can be constructed over the whole base space $M_1 \times M_2$, in which case we can have two distinct NUT parameters associated with each factor M_i or, in the case of metrics with only one NUT parameter, just over one factor space M_1, in which case we also take the warped product with M_2 as in (7.23).

We shall consider first the case in which $M_1 = M_2 = S^2$ and assume that the $U(1)$ fibration is constructed over the whole base space $S^2 \times S^2$. Then the corresponding six-dimensional Taub-NUT solution is given by [111]

$$\begin{aligned} ds^2 &= -F(r)(dt - 2n_1\cos\theta_1 d\varphi_1 - 2n_2\cos\theta_2 d\varphi_2)^2 + F^{-1}(r)dr^2 \\ &\quad +(r^2+n_1^2)(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2) + (r^2+n_2^2)(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2), \end{aligned} \tag{7.24}$$

where

$$\begin{aligned} F(r) &= \frac{3r^6 + (l^2 + 5n_2^2 + 10n_1^2)r^4 + 3(n_2^2 l^2 + 10n_1^2 n_2^2 + n_1^2 l^2 + 5n_1^4)r^2}{3(r^2+n_1^2)(r^2+n_2^2)l^2} \\ &\quad - \frac{6ml^2 r + 3n_1^2 n_2^2(l^2 + 5n_1^2)}{3(r^2+n_1^2)(r^2+n_2^2)l^2}. \end{aligned} \tag{7.25}$$

Here the above metric is a solution of vacuum Einstein field equations with cosmological constant ($\lambda = -\frac{10}{l^2}$) if and only if $(n_1^2 - n_2^2)\lambda = 0$. Consequently, we see that differing values for n_1 and n_2 are possible only if the cosmological constant vanishes. For $n_1 = n_2 = n$ the above solution reduces to the six-dimensional solution found and studied in [11, 122, 9, 3, 133, 37]. In the case of only one NUT charge n there are no consistent analytic continuations of the coordinates that lead to acceptable time-dependent metrics with Lorentzian signature [62]. Basically, the reason for this is that if we perform analytic continuations of the coordinates on one factor space S^2 we also have to send $n \to in$, which will force us to analytically continue the coordinates in the second sphere S^2 yielding spaces with two timelike directions. However, if the NUT parameters are independent then we can analytically continue the coordinates in one factor M_i only and analytically continue the NUT parameter associated with the second factor M_j. This enables us to construct nutty bubble spacetimes in virtually any dimension. For this reason, in what follows we shall look at the case of two different NUT charges, that is we set the cosmological constant to zero. Let us consider the Euclidean section, obtained by the following analytic continuations $t \to i\chi$ and $n_j \to in_j$ where $j = 1, 2$:

$$\begin{aligned} ds^2 &= F_E(r)(d\chi - 2n_1 \cos\theta_1 d\varphi_1 - 2n_2 \cos\theta_2 d\varphi_2)^2 + F_E^{-1}(r)dr^2 \\ &\quad +(r^2 - n_1^2)(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2) + (r^2 - n_2^2)(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2), \\ F_E(r) &= \frac{r^4 - 3(n_1^2 + n_2^2)r^2 - 6mr - 3n_1^2 n_2^2}{3(r^2 - n_1^2)(r^2 - n_2^2)}. \end{aligned} \tag{7.26}$$

This metric is a solution of the vacuum Einstein field equations without cosmological constant, for any values of the parameters n_1 and n_2. We set $n_1 > n_2$ without loss of generality. In this case in the Euclidean section the radius r cannot be smaller than n_1 or the signature of the spacetime will change. The Taub-Nut solution in this case corresponds to a two-dimensional fixed-point set located at $r = n_1$. There is still a curvature singularity located at $r = n_1$. While superficially it would seem that this could removed by setting the periodicity of the coordinate χ to be $8\pi n_1$ (thereby setting $m = m_p = -\frac{n_1^3 + 3n_1 n_2^2}{3}$), a more careful analysis reveals that this Nut solution is actually still singular. This is because the nut parameters $n_{1,2}$ must be rationally related, in which case the periodicity of the χ coordinate is $8\pi n_2/k$, where k is an integer. As $n_2 < n_1$ it is not possible to match this periodicity with $8\pi n_1$ for any integer k.

On the other hand, the Bolt solution corresponds to $r \geq r_0 > n_1$ and the periodicity is found to be $\frac{4\pi}{|F'_E(n_1)|} = 4\pi r_0$. It is now possible to match it with $8\pi n_2/k$ with k the integer and we obtain $r_0 = \frac{2n_2}{k}$. The Bolt solution is then non-singular as long as $r_0 > n_1$, that is for $k = 1$ and $n_1 < 2n_2$.

We are now ready to perform analytic continuations on the sphere factors in order to generate new time-dependent backgrounds. For instance, we can consider $\theta_1 \to it + \frac{\pi}{2}$, which will force us to take $n_1 \to in_1$. We obtain the following time dependent solution:

$$\begin{aligned} ds^2 &= \tilde{F}_E(r)(d\chi + 2n_1 \sinh t d\phi_1 + 2n_2 \cos\theta_2 d\phi_2)^2 + \tilde{F}_E^{-1}(r)dr^2 \\ &\quad +(r^2+n_1^2)(-dt^2 + \cosh^2 t d\phi_1^2) + (r^2 - n_2^2)(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \\ \tilde{F}_E &= \frac{r^4 + 3(n_1^2 - n_2^2)r^2 - 6rm + 3n_1^2 n_2^2}{3(r^2+n_1^2)(r^2-n_2^2)}. \end{aligned} \tag{7.27}$$

More generally, as with the four-dimensional case, after performing appropriate analytic continuations we end up with the following metrics:

$$\begin{aligned} ds^2 &= \tilde{F}_E(r)(d\chi + 2n_1 \cosh t d\phi_1 + 2n_2 \cos\theta_2 d\phi_2)^2 + \tilde{F}_E^{-1}(r)dr^2, \\ &\quad +(r^2+n_1^2)(-dt^2 + \sinh^2 t d\phi_1^2) + (r^2 - n_2^2)(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \\ ds^2 &= \tilde{F}_E(r)(d\chi + 2n_1 e^t d\phi_1 + 2n_2 \cos\theta_2 d\phi_2)^2 + \tilde{F}_E^{-1}(r)dr^2, \\ &\quad +(r^2+n_1^2)(-dt^2 + e^{2t} d\phi_1^2) + (r^2 - n_2^2)(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \\ ds^2 &= \tilde{F}_E(r)(d\chi + 2n_1 \cos\theta dt + 2n_2 \cos\theta_2 d\phi_2)^2 + \tilde{F}_E^{-1}(r)dr^2, \\ &\quad +(r^2+n_1^2)(d\theta_1^2 - \sin^2\theta_1 dt^2) + (r^2 - n_2^2)(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2). \end{aligned} \tag{7.28}$$

They are also solutions of vacuum Einstein field equations with the same function $\tilde{F}_E$ as in (7.27).

While locally all these spaces are equivalent under coordinate transformations, if we compactify the coordinate ϕ_1(respectively θ_1 for the last metric in (7.28)) the global structure and in particular the evolution of the bubble will be different. The bubble will be located at the biggest root r_0 of $\tilde{F}_E(r)$ such that $r_0 > n_2$ and we also restrict the range of the r coordinate such that $r \geq n_2$. Elimination of the Misner string sets the periodicity of the χ coordinate to be $8\pi n_2/k$, which in turn must be matched with the periodicity $4\pi/|\tilde{F}'_E(r_0)|$, introduced after we eliminate any possible conical singularities in the (χ, r) sector. Again we have two solutions: a Nut and a Bolt.

The Nut solution corresponds to $r_0 = n_2$ and in this case the mass parameter is $n_2(3n_1^2 - n_2^2)/3$. Notice that the mass parameter can have either sign. The coordinate χ has periodicity $8\pi n_2$. It is very interesting to note that the fixed-point set of the isometry generated by $\partial/\partial\chi$ is effectively two dimensional. The induced geometry on the 'bubble-nut' is a two-dimensional de Sitter space, whose metrics are one of the following

$$\begin{aligned} ds_2^2 &= (n_1^2 + n_2^2)(-dt^2 + \cosh^2 t d\phi_1^2), \\ ds_2^2 &= (n_1^2 + n_2^2)(-dt^2 + \sinh^2 t d\phi_1^2), \\ ds_2^2 &= (n_1^2 + n_2^2)(-dt^2 + e^{2t} d\phi_1^2), \\ ds_2^2 &= (n_1^2 + n_2^2)(d\theta_1^2 - \sin^2\theta_1 dt^2). \end{aligned} \tag{7.29}$$

which differ globally but not locally. If the coordinate ϕ_1 is periodically identified then at any fixed time $r = n_2$ is our 'bubble-nut': a circle with minimal circumference that expands or contracts. The first three de Sitter geometries above correspond to three different evolutions of this circle: the first geometry describes the evolution of a circle with exponentially large radius at $t \to -\infty$, which shrinks to a minimal value and expands exponentially again for $t \to \infty$; the second geometry describes the evolution of a circle which begins with zero radius at $t = 0$ and expands exponentially, while the third geometry describes a circle that begins with exponentially small radius at $t \to -\infty$ and then expands exponentially. The last geometry is stationary as in these coordinates the metric has a Killing vector $\xi = \partial/\partial t$. The norm of this Killing vector is

$$\xi \cdot \xi = 4n_1^2 \tilde{F}_E(r) - (r^2 + n_1^2 - 4n_1^2 \tilde{F}_E(r)) \sin^2\theta_1$$

so that it will become spacelike unless $\hat{\theta}_1(r) \leq |\theta_1| \leq \pi - \hat{\theta}_1(r)$; here, we used the notation:

$$\hat{\theta}_1(r) = \tan^{-1}\left(\frac{2n_1\sqrt{\tilde{F}_E(r)}}{\sqrt{r^2 + n_1^2}}\right).$$

As in the four-dimensional case [62] these limits will describe an 'ergocone' for the spacetime. The above angle vanishes at $r = r_0$ and at infinity, while it attains a maximum value in between.

The Bolt solution corresponds to a four-dimensional fixed-point set of the isometry generated by $\partial/\partial\chi$. By solving the above constraint on the possible periodicities of the

χ coordinate we obtain the location of the bolt $r_0 = 2n_2/k$. Requiring that $r \geq r_0 > n_2$ implies $k = 1$, in which case the periodicity of the χ coordinate is $8\pi n_2$ and the mass parameter is $m = n_2(15n_1^2 + 4n_2^2)/12$. Notice that in this case the mass parameter is positive. The induced geometry on the 'bubble-bolt' is a two-dimensional de Sitter space times a sphere S^2:

$$\begin{aligned}
ds_2^2 &= (n_1^2 + 4n_2^2)(-dt^2 + \cosh^2 t d\phi_1^2) + 3n_2^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \\
ds_2^2 &= (n_1^2 + 4n_2^2)(-dt^2 + \sinh^2 t d\phi_1^2) + 3n_2^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \\
ds_2^2 &= (n_1^2 + 4n_2^2)(-dt^2 + e^{2t} d\phi_1^2) + 3n_2^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \\
ds_2^2 &= (n_1^2 + 4n_2^2)(d\theta_1^2 - \sin^2\theta_1 dt^2) + 3n_2^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2).
\end{aligned} \tag{7.30}$$

At any fixed time, $r = 2n_2$ is the 'bubble-bolt', which is topologically $S^1 \times S^2$. The S^2 factor is described by the (θ_2, ϕ_2) coordinates and it has constant size in time. On the other hand, the circle S^1 described by the ϕ_1 coordinate expands or contracts in time. Again, the first three geometries describe three different evolutions of this circle. The last geometry is static and it is easy to see that it possesses an ergocone with qualitatively the same features as described above for the static bubble-nut ergocone.

We can also consider Taub-NUT spaces for which both the 2-dimensional factors M_i are taken to be both a torus T^2 or a hyperboloid H^2. Such geometries and the nutty bubbles obtained from them are presented in Appendix A.

Finally, let us notice that all the nutty bubbles geometries exhibited so far have no curvature singularities. Generically, from the form of the metrics one would expect that $r = n_2$ be a curvature singularity. However, the bubble-nut solution described above is completely regular at $r = n_2$ as one can check by looking at some of the curvature invariants (for example $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$). For the bubble-bolt, this curvature singularity is simply avoided by requiring that $r \geq 2n_2$.

7.2.2 Bubbles in cosmological backgrounds

Another class of solutions is given for base spaces that are products of 2-dimensional Einstein manifolds $M_1 \times M_2$. In this case, the metric ansatz that we use to construct the Taub-NUT solution is the one given in (7.23), where now $M = M_1$ while $Y = M_2$.

As an example we shall consider again the case in which $M_1 = M_2 = S^2$. The metric is written in the form [111]:

$$\begin{aligned} ds^2 &= -F(r)(dt - 2n\cos\theta_1 d\phi_1)^2 + F^{-1}(r)dr^2 \\ &\quad +(r^2+n^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \alpha r^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2). \end{aligned} \tag{7.31}$$

In order to satisfy the field equations we must take

$$\alpha = \frac{2}{2-\lambda n^2}, \qquad F(r) = \frac{3r^5 + (l^2+10n^2)r^3 + 3n^2(l^2+5n^2)r - 6ml^2}{3rl^2(r^2+n^2)}.$$

The metric (7.31) is a solution of the vacuum Einstein field equations with cosmological constant $\lambda = -\frac{10}{l^2}$, for any values of n or λ. However, in order retain a metric of Lorentzian signature we must ensure that $\alpha > 0$, which translates in our case to $\lambda n^2 < 2$. For convenience, we have given above the form of $F(r)$ using a negative cosmological constant and in this case the constraint on n and λ is superfluous. We can also use a positive cosmological constant (we have to analytically continue $l \to il$ in $F(r)$) and as long as the above condition on α is satisfied the final metric has Lorentzian signature. The Euclidean section is

$$\begin{aligned} ds^2 &= F_E(r)(d\chi + 2n\cos\theta_1 d\phi_1)^2 + F_E^{-1}(r)dr^2 \\ &\quad +(r^2-n^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \alpha_E r^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \\ \alpha_E &= \frac{l^2}{l^2-5n^2}, \qquad F_E(r) = \frac{3r^5 + (l^2-10n^2)r^3 - 3n^2(l^2-5n^2)r - 6ml^2}{3rl^2(r^2-n^2)} \end{aligned} \tag{7.32}$$

obtained by continuing $t \to i\chi$ and $n \to in$.

We are now ready to construct the nutty bubbles. First, let us notice that we can analytically continue the coordinates independently in the two S^2 sectors. Let us perform the analytic continuation of one of the coordinates in the second S^2 factor, in which case we obtain the metrics:

$$
\begin{aligned}
ds^2 &= F_E(r)(d\chi + 2n\cos\theta_1 d\phi_1)^2 + F_E^{-1}(r)dr^2, \\
&\quad +(r^2-n^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \alpha_E r^2(-dt^2 + \cosh^2 t d\phi_2^2), \\
ds^2 &= F_E(r)(d\chi + 2n\cos\theta_1 d\phi_1)^2 + F_E^{-1}(r)dr^2, \\
&\quad +(r^2-n^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \alpha_E r^2(-dt^2 + \sinh^2 t d\phi_2^2), \\
ds^2 &= F_E(r)(d\chi + 2n\cos\theta_1 d\phi_1)^2 + F_E^{-1}(r)dr^2, \\
&\quad +(r^2-n^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \alpha_E r^2(-dt^2 + e^{2t} d\phi_2^2), \\
ds^2 &= F_E(r)(d\chi + 2n\cos\theta_1 d\phi_1)^2 + F_E^{-1}(r)dr^2, \\
&\quad +(r^2-n^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \alpha_E r^2(d\theta_2^2 - \sin^2\theta_2 dt^2).
\end{aligned} \tag{7.33}
$$

The above metrics are solutions of Einstein field equations with cosmological constant for any values of $\lambda = -10/l^2$ and n. In the case considered here, for a negative cosmological constant, α_E can have negative values if $5n^2 > l^2$. However, while in the Euclidean sector negative values of α_E are not permitted, for our nutty bubbles a negative value for α_E amounts to an overall sign change of the metric in the (t, ϕ_2) (respectively (t, θ_2)) sectors. We shall see that this can have a dramatic influence on the dynamical evolution of the bubble.

The bubble will be located at the highest root r_0 of $F_E(r)$ chosen such that $r_0 > n$ and in general we restrict the range of the r coordinate $r \geq r_0$. Elimination of the Misner string sets the periodicity of the χ coordinate to be $8\pi n/k$ and we also have to match it with the periodicity $4\pi/|F_E'(r_0)|$ introduced after we eliminate any possible conical singularities in the (χ, r) sector. Again we have two solutions: a Nut and a Bolt.

The Nut solution corresponds to a two-dimensional fixed-point set of the vector $\frac{\partial}{\partial\chi}$ located at $r = n$. The periodicity of the χ coordinate is in this case equal to $8\pi n$ and the value of the mass parameter is fixed to $m_n = \frac{n^3(4n^2-l^2)}{3l^2}$. Notice that the mass parameter can take any values: positive, negative or zero. There is a curvature singularity at the bubble location! Furthermore, if $n = \frac{l}{2}$ then curvature singularity present at $r = n$ disappears and the spacetime has constant curvature. The fixed-points set of the isometry generated by $\partial/\partial\chi$ is effectively two dimensional. The induced geometry on the 'bubble-nut' is a two-dimensional de Sitter space:

$$\begin{aligned} ds_2^2 &= \alpha_E n^2(-dt^2 + \cosh^2 t d\phi_2^2), \\ ds_2^2 &= \alpha_E n^2(-dt^2 + \sinh^2 t d\phi_2^2), \\ ds_2^2 &= \alpha_E n^2(-dt^2 + e^{2t} d\phi_2^2), \\ ds_2^2 &= \alpha_E n^2(d\theta_2^2 - \sin^2\theta_2 dt^2). \end{aligned} \tag{7.34}$$

If the coordinate ϕ_2 is periodically identified then at any fixed time $r = n$ is our 'bubble-nut': a circle with minimal circumference which expands or contracts. The first three de Sitter geometries above correspond to three different evolutions of this circle as with (7.12). The last geometry is static as in these coordinates the metric has a hypersurface-orthogonal Killing vector $\xi = \partial/\partial t$.

Now let us consider the effect of changing the sign of α_E. This can be easily accommodated by taking $l^2 < 5n^2$. As we can easily see from the metric induced on the bubble, a negative sign of α_E amounts to changing the induced de Sitter geometry of the bubble-nut into a two-dimensional anti-de Sitter geometry. As it is well known, this space can have non-trivial identifications and so it can describe for instance a two-dimensional black hole (as in the second and third metrics above), while the first metric describes pure AdS in standard coordinates. After a coordinate transformation these metrics can be written in the form:

$$\begin{aligned} ds^2 &= (-\alpha_E)n^2\left(\frac{dx^2}{x^2+k} - (x^2+k)dt^2\right), \\ ds_2^2 &= (-\alpha_E)n^2(-dt^2 + \sin^2 t d\phi_2^2), \end{aligned} \tag{7.35}$$

where $k = -1, 1, 0$ for the respective first three metrics in (7.34). They are all locally equivalent under changes of coordinates. However, depending on the identifications made, the global structure can be quite different. For example, the second metric from (7.35) can be locally transformed into the $k = -1$ metric by a coordinate transformation. However, if ϕ_2 is periodic then the global structure of these spaces is completely different. For $r \geq n$ this metric describes a bubble located at $r = n$, which expands from zero size to a finite size $(-\alpha_E n^2)$ and then contracts to zero size again. Near the initial expansion (or the final contraction) the scale factor is linear in time and the spacetime expands or contracts like a Milne universe.

The bubble-bolt geometry has a four-dimensional fixed-point set of $\frac{\partial}{\partial\chi}$ located at $r = r_b$ with:

$$r_b = \frac{kl^2 \pm \sqrt{k^2l^4 - 80n^2l^2 + 400n^4}}{20n}, \tag{7.36}$$

while the value of the mass parameter is:

$$m_b = \frac{3r_b^5 + (l^2 - 10n^2)r_b^3 - 3n^2(l^2 - 5n^2)r_b}{6l^2}. \tag{7.37}$$

The periodicity of the coordinate χ is $\frac{8\pi n}{k}$, where k is an integer. The induced geometry on the 'bubble-bolt' is a two-dimensional de Sitter space times a sphere S^2:

$$\begin{aligned}
ds_2^2 &= (r_b^2 - n^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \alpha_E r_b^2(-dt^2 + \cosh^2 t d\phi_2^2), \\
ds_2^2 &= (r_b^2 - n^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \alpha_E r_b^2(-dt^2 + \sinh^2 t d\phi_2^2), \\
ds_2^2 &= (r_b^2 - n^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \alpha_E r_b^2(-dt^2 + e^{2t} d\phi_2^2), \\
ds_2^2 &= (r_b^2 - n^2)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \alpha_E r_b^2(d\theta_2^2 - \sin^2\theta_2 dt^2).
\end{aligned} \tag{7.38}$$

At any fixed time, $r = r_b$ is the 'bubble-bolt', which is topologically $S^1 \times S^2$. The S^2 factor is described by the (θ_1, ϕ_1) coordinates and it has constant size in time. On the other hand, the circle S^1 described by ϕ_2, expands or contracts in time. Again, the first three geometries describe three different evolutions of this circle. The last geometry in (7.38) is static. As for the bubble-nut, changing the sign of α_E has dramatic consequences as it effectively turns the two-dimensional dS geometry into AdS.

The boundary geometry for these bubble spacetimes is given by

$$\begin{aligned}
ds^2 &= 4n^2/l^2(d\tilde{\chi} + \cos\theta_1 d\phi_1)^2 + (d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \alpha_E dS_2, \\
ds^2 &= 4n^2/l^2(d\tilde{\chi} + \cos\theta_1 d\phi_1)^2 + (d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + (-\alpha_E) d\Sigma_2,
\end{aligned} \tag{7.39}$$

where $\tilde{\chi} = \chi/2n$ is periodic with period 4π. Here dS_2 (respectively $d\Sigma_2$) describes the metric of a two-dimensional de Sitter space (respectively AdS). The (χ, θ_1, ϕ_1)-sector describes a squashed three-sphere, the squashing parameter being controlled by $4n^2/l^2$. It is interesting to note that, for negative α_E, one can perform identifications on the AdS part of the metric which turn it into a black-hole.

Finally, the other possibility to obtain bubble spacetimes is to analytically continue $t \to i\chi$ and one of the coordinates in the first S^2 factor in (7.31):

$$
\begin{aligned}
ds^2 &= F(r)(d\chi + 2n \sinh t d\phi_1)^2 + F^{-1}(r)dr^2 \\
&\quad +(r^2+n^2)(-dt^2 + \cosh^2 t d\phi_1^2) + \alpha r^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \\
ds^2 &= F(r)(d\chi + 2n \cosh t d\phi_1)^2 + F^{-1}(r)dr^2 \\
&\quad +(r^2+n^2)(-dt^2 + \sinh^2 t d\phi_1^2) + \alpha r^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \\
ds^2 &= F(r)(d\chi + 2n e^t d\phi_1)^2 + F^{-1}(r)dr^2, \\
&\quad +(r^2+n^2)(-dt^2 + e^{2t} d\phi_1^2) + \alpha r^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2), \\
ds^2 &= F(r)(d\chi + 2n \cos\theta_1 dt)^2 + F^{-1}(r)dr^2, \\
&\quad +(r^2+n^2)(d\theta_1^2 - \sin^2\theta_1 dt^2) + \alpha r^2(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2).
\end{aligned}
\tag{7.40}
$$

The above metrics are solutions of vacuum Einstein field equations with cosmological constant for any values of n or $\lambda = -10/l^2$. However, in order to keep the signature of the metric Lorentzian we have to ensure that $\alpha > 0$ i.e. $\lambda n^2 < 2$. We can have a positive[5] or negative cosmological constant as long as this relation is satisfied. Notice that for a negative cosmological constant α is always positive. The bubble is located at the biggest root r_0 of $F(r)$ and in order to eliminate a conical singularity in the (χ, r)-sector, we have to periodically identify χ with period given by $4\pi/|F'(r_0)| = 4\pi l^2 r_0/[l^2 + 5(r_0^2 + n^2)]$. At any fixed time, $r = r_0$ is the bubble, which is topologically $S^1 \times S^2$. The S^2 factor is described by the (θ_2, ϕ_2) coordinates and it has constant size. On the other hand, the circle S^1 described by ϕ_1 expands or contracts in time. Again, the first three geometries describe three different evolutions of this circle. The last geometry is stationary and it is easy to see that it possesses an ergocone with qualitatively the same features as described above for the static bubble-nut ergocones encountered in the previous sections.

Similar nutty bubbles can be obtained by considering Taub-NUT metrics for which $M_1 \neq M_2$. Such metrics have been studied in [111, 105, 113]. In six dimensions, such metrics can have two independent NUT parameters and there exists a constraint on the values of these NUT parameters and the cosmological constant. Quite generically this constraint makes it impossible to set the cosmological constant to zero. Having two NUT

[5] To write the solution for a positive cosmological constant we have to send $l \to il$ in $F(r)$ in (7.2.2).

parameters at our disposal it is very easy to construct various nutty bubble solutions by analytically continuing the coordinates in only one of the factors M_i.

7.2.3 Nutty bubbles and Hopf dualities

In this section we apply Hopf-duality rules from Appendix C to some of our nutty bubbles to generate new time dependent backgrounds. However, since these rules work only for solutions that do not have cosmological constant, we shall focus mainly on the cases in which $M_1 = M_2$. For simplicity, and just to illustrate the method we shall use just a few nutty bubble solutions as seeds.

Let us start with the solution given in (7.27). Considering this metric as a solution of the pure gravity sector of the truncated Type IIA theory we can now perform a Hopf-duality along the spacelike χ-direction to obtain a solution of six-dimensional Type IIB theory:

$$\begin{aligned} ds_{6B} &= \tilde{F}_E(r)^{-\frac{1}{2}}d\chi^2 + \tilde{F}_E(r)^{-\frac{1}{2}}dr^2 + \tilde{F}_E(r)^{\frac{1}{2}}(r^2+n_1^2)(-dt^2+\cosh^2 t d\phi_1^2) \\ &\quad + \tilde{F}_E(r)^{\frac{1}{2}}(r^2-n_2^2)(d\theta_2^2+\sin^2\theta_2 d\phi_2^2)^2 \\ e^{2\varphi_1} &= e^{2\varphi_2} = \tilde{F}_E(r), \qquad A_{(2)}^{NS} = 2n_1\sinh t d\phi_1\wedge d\chi + 2n_2\cos\theta_2 d\phi_2\wedge d\chi. \end{aligned} \tag{7.41}$$

Were we to consider (7.27) as a solution of the pure gravity sector of Type IIB theory, then after performing the spacelike Hopf dualisation we would obtain a solution of Type IIA theory:

$$\begin{aligned} ds_{6A} &= \tilde{F}(r)^{-\frac{1}{2}}d\chi^2 + \tilde{F}(r)^{-\frac{1}{2}}dr^2 + \bar{F}(r)^{\frac{1}{2}}(r^2+n_1^2)(-dt^2+\cosh^2 t d\phi_1^2) \\ &\quad + \bar{F}(r)^{\frac{1}{2}}(r^2-n_2^2)(d\theta_2^2+\sin^2\theta_2 d\phi_2^2) \\ e^{2\varphi_1} &= e^{2\varphi_2} = \bar{F}(r), \qquad A_{(2)} = 2n_1\sinh t d\phi_1\wedge d\chi + 2n_2\cos\theta_2 d\phi_2\wedge d\chi. \end{aligned} \tag{7.42}$$

The analysis of these charged bubbles proceeds as in the previous sections. The bubble will be located at the largest root of $\tilde{F}_E(r)$. Generically there exists a curvature singularity at the bubble location, which cannot be cured by any appropriate choices of the parameters. Another difference with the previous bubble solutions is that in the (χ, r)-sector there is no conical singularity to be eliminated and χ need not be compactified.

As another example of this method, let us consider a bubble solution, which corresponds to a six-dimensional Taub-NUT constructed as a circle fibration over $T^2 \times T^2$. The Euclidean version of such spaces is [111]:

$$\begin{aligned} ds^2 &= F_E(r)(d\chi + 2n_1\theta_1 d\phi_1 + 2n_2\theta_2 d\phi_2)^2 + F_E^{-1}(r)dr^2 \\ &\quad +(r^2 - n_1^2)(d\theta_1^2 + d\phi_1^2) + (r^2 - n_2^2)(d\theta_2^2 + d\phi_2^2) \\ F_E(r) &= \frac{3r^6 - 5(n_2^2 + 2n_1^2)r^4 + 15n_1^2(n_1^2 + 2n_2^2)r^2 + 6ml^2 r + 15n_1^4 n_2^2}{3(r^2 - n_1^2)(r^2 - n_2^2)l^2} \end{aligned} \tag{7.43}$$

The above metric is a solution of vacuum Einstein field equations with cosmological constant if and only if $(n_2^2 - n_1^2)\lambda = 0$. Hence in the case of a vanishing cosmological constant we can have two independent NUT charges in the metric. We can similarly analytically continue the coordinates from one factor space T^2 only:

$$\begin{aligned} ds^2 &= \tilde{F}_E(r)(d\chi + 2n_1 t d\phi_1 + 2n_2\theta_2 d\phi_2)^2 + \tilde{F}_E^{-1}(r)dr^2 \\ &\quad +(r^2 + n_1^2)(-dt^2 + d\phi_1^2) + (r^2 - n_2^2)(d\theta_2^2 + d\phi_2^2) \\ ds^2 &= \tilde{F}_E(r)(d\chi + 2n_1\theta_1 dt + 2n_2\theta_2 d\phi_2)^2 + \tilde{F}_E^{-1}(r)dr^2 \\ &\quad +(r^2 + n_1^2)(d\theta_1^2 - dt^2) + (r^2 - n_2^2)(d\theta_2^2 + d\phi_2^2) \\ \tilde{F}_E(r) &= \frac{2mr}{(r^2 + n_1^2)(r^2 - n_2^2)} \end{aligned} \tag{7.44}$$

However, if $\lambda \neq 0$ we are forced to have $n_1 = n_2$ and it is impossible to analytically continue the coordinates of the T^2 factors separately.

Taking the first bubble solution given in (7.44) as a solution of Type IIA theory, then after performing a Hopf duality along the χ direction we obtain:

$$\begin{aligned} ds_{6B} &= \tilde{F}_E(r)^{-\frac{1}{2}}d\chi^2 + \tilde{F}_E(r)^{-\frac{1}{2}}dr^2 + \tilde{F}_E(r)^{\frac{1}{2}}(r^2 + n_1^2)(-dt^2 + d\phi_1^2) + \tilde{F}_E(r)^{\frac{1}{2}}(r^2 - n_2^2)(d\theta_2^2 + d\phi_2^2) \\ e^{2\varphi_1} &= e^{2\varphi_2} = \tilde{F}_E(r), \qquad A_{(2)}^{NS} = 2n_1 t d\phi_1 \wedge d\chi + 2n_2\theta_2 d\phi_2 \wedge d\chi. \end{aligned} \tag{7.45}$$

which is a solution of the Type IIB theory. Notice that $F(r) = 0$ only if $r = 0$. On the other hand, we have to restrict the values of the radial coordinates such that $r \geq n_2$, or else the signature of this metric will change. There is however a curvature singularity at $r = n_2$, which cannot be eliminated by any appropriate choices of the parameter m.

7.3 Discussion

In this chapter, we have constructed a wide variety of time-dependent backgrounds using the standard techniques of analytic continuation. Since many of the presented solutions are locally asymptotically (A)dS, they are relevant in the context of gauge/gravity dualities.

For example, let us discuss one of our solutions (7.12) in the context of the AdS/CFT correspondence. The bulk-boundary correspondence in the Lorentzian section demands the inclusion of both normalizable and non-normalizable modes of the bulk fields [14]. The former propagate in the bulk and correspond to physical states while the latter serve as classical, non-fluctuating backgrounds and encode the choice of operator insertions in the boundary theory.

Since the bulk theory is a theory of gravity, one of the bulk fields will always be the graviton (metric perturbations). The AdS/CFT dictionary tells us that its dual operator is the stress-energy tensor of the CFT. We will compare the dual CFT stress tensor to the *rescaled* boundary stress tensor calculated from the bulk spacetime using the counterterm subtraction procedure of [13, 128]. Typically, the boundary of a locally asymptotically spacetime will be an asymptotic surface at some large radius r. However, the metric restricted to the boundary γ_{ab} diverges due to an infinite conformal factor r^2/ℓ^2, and so the metric upon which the dual field theory resides is usually defined using the rescaling

$$h_{ab} = \lim_{r\to\infty} \frac{\ell^2}{r^2}\gamma_{ab}. \tag{7.46}$$

Corresponding to the boundary metric h_{ab}, the stress-energy tensor $< \tau_{ab} >$ for the dual theory can be calculated using the following relation

$$\sqrt{-h}h^{ab} < \tau_{bc} >= \lim_{r\to\infty} \sqrt{-\gamma}\gamma^{ab}T_{bc}. \tag{7.47}$$

In our case, the boundary metric is

$$ds^2 = h_{ab}dx^a dx^b \;\; = \;\; (d\chi + lxdt)^2 + l^2\left(\frac{dx^2}{x^2+k} - (x^2+k)dt^2\right) + l^2 dy^2, \tag{7.48}$$

and so the conformal boundary, where the $\mathcal{N} = 4$ SYM lives, is $AdS_3 \times S^1$. The *rescaled* boundary stress tensor is

$$\begin{aligned}
\tau_t^t &= \frac{256m+5l^2}{1024\pi G l^3}, \\
\tau_\chi^t &= 0, \\
\tau_t^\chi &= -\frac{(l^2+64m)x}{64\pi G l^3} \\
\tau_\chi^\chi &= -\frac{11l^2+768m}{1024\pi G l^3}, \\
\tau_x^x &= \frac{5l^2+256m}{1024\pi G l^3}, \\
\tau_y^y &= \frac{l^2+256m}{1024\pi G l^3}.
\end{aligned} \tag{7.49}$$

Since the boundary metric is the product of a circle and a three-dimensional Einstein space, the trace anomaly vanishes. Indeed, as we expected, the stress tensor (7.49) is finite, covariantly conserved, and manifestly traceless.

For four dimensions, it was shown in [62] that in the special case when the NUT charge vanishes ($n = 0$), the metric (stress tensor) reduces to the 4-dimensional Schwarzschild-AdS metric (stress tensor). In five dimensions, the constraint between the NUT charge and the cosmological constant changes dramatically the situation. However, the limit we are interested here is $m = -64/l^2$. Then, the bulk geometry has constant curvature and it is the static bubble obtained from AdS_5 by analytic continuation. Indeed, in this case $F(r) = \frac{r^2}{l^2} + \frac{1}{4}$ and by redefining the coordinate $r^2 \to r^2 - \frac{l^2}{4}$ and rescaling y to absorb an l^2 factor, the metric can be cast in the form:

$$ds^2 = \left(\frac{r^2}{l^2} - \frac{1}{4}\right) dy^2 + \frac{dr^2}{\left(\frac{r^2}{l^2} - \frac{1}{4}\right)} + r^2\left[(d\tilde{\chi} + \sinh\theta dt)^2 + d\theta^2 - \cosh^2\theta dt^2\right].$$

One can recognize it as being the analytic continuation of AdS_5 with a non-canonically normalized H^3 factor. For this particular value of the parameter m, the stress tensor (7.49) becomes

$$\tau_b^a = \frac{N^2}{512\pi^2 l^4}\ \mathrm{diag}(1,1,1,-3), \tag{7.50}$$

where we have used the standard relation $l^3/G = 2N^2/\pi$ to rewrite the stress tensor in terms of field theory quantities.[6]

[6] The convention for the coordinates is $1,2,3,4 = t,\chi,x,y$.

Let us move now to the dual theory that is in terms of $\mathcal{N}=4$ SYM on the $AdS_3\times S^1$ spacetime. This is a conformally flat spacetime and, fortunately, there is a standard result for the stress tensor [21]:

$$\langle\tau^a_b\rangle = -\frac{1}{16\pi^2}\left(A^{(1)}H^a_b + B^{(3)}H^a_b\right) + \tilde{\tau}^a_b. \tag{7.51}$$

Here, ${}^{(1)}H^a_b$ and ${}^{(3)}H^a_b$ are conserved quantities constructed from the curvature (see [21] for their definitions), and $\tilde{\tau}^a_b$ is a traceless state-dependent part. In our case they are given by

$$\begin{aligned} {}^{(1)}H^a_b &= \frac{3}{8l^4}\text{diag}[1,1,1,-3], \\ {}^{(3)}H^a_b &= -\frac{1}{16l^4}\text{diag}[1,1,1,-3]. \end{aligned} \tag{7.52}$$

The coefficients A and B are calculated as in [16]. The trace of (7.51) is compared with the conformal anomaly for $\mathcal{N}=4$ SYM [128]:

$$\langle\tau^a_a\rangle = -\frac{1}{16\pi^2}\left(-6A\Box R - B(R_{ab}R^{ab} - 1/3R^2)\right) = \frac{(N^2-1)}{64\pi^2}(2R_{ab}R^{ab} - 2/3R^2). \tag{7.53}$$

This fixes $A=0$ and $B=(N^2-1)/2$ and so the field theory stress tensor becomes

$$\langle\tau^a_b\rangle = \frac{1}{2}\frac{(N^2-1)}{256\pi^2 l^4}\ \text{diag}(1,1,1,-3) + \tilde{\tau}^a_b. \tag{7.54}$$

In the large N limit, the geometrical part of the stress tensor precisely reproduces (7.50). The fact that the geometrical part is non zero is a direct consequence of analytic continuation — the quantum field theory on the AdS boundary can have a nonvanishing vacuum (Casimir) energy. Consequently, the above comparison of the stress tensor (7.50) to (7.54) does result in a non-trivial connection between them.

Chapter 8

New Taub-NUT-Reissner-Nordström spaces in higher dimensions

8.1 Overview

As we have seen in Chapter 2, in four dimensions there also exists NUT-charged generalisations of the Reissner-Nordström solution [4, 30] and they are included in the general family described recently by Griffiths and Podolsky [73, 75, 72]. However, until recently [114, 10], the higher dimensional generalisations of NUT-charged solutions in Einstein-Maxwell theory has not been discussed previously in the literature.

In this chapter, our main purpose is to provide the generalisation of these spaces to higher dimensions. These solutions will represent the electromagnetic generalisation of the NUT-charged spacetimes studied in refs. [105, 11, 133, 9, 37] as well as the NUT-charged generalisation of the higher dimensional Reissner-Nordström solutions [34].

Einstein-Maxwell theory in D-dimensions is described by the following action:

$$I = -\frac{1}{16\pi G}\int d^D x\sqrt{-g}\left[R-2\Lambda-F^2\right] \tag{8.1}$$

The equations of motion derived from this action can be written in the following form[1]:

$$
\begin{aligned}
R_{\mu\nu} &= 2\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{2(D-2)}F^2 g_{\mu\nu}\right) + \lambda g_{\mu\nu} \\
\nabla_\mu F^{\mu\nu} &= 0
\end{aligned}
\tag{8.2}
$$

where $F = dA$ is the electromagnetic 2-form field strength corresponding to the gauge potential A.

8.2 The general solution

Let us recall first the form of the 4-dimensional Taub-NUT-Reissner-Nordström metric. The metric is given by [30]:

$$
ds^2 = -f(r)(dt - 2N\mathcal{A})^2 + F^{-1}(r)dr^2 + (r^2+N^2)g_M \tag{8.3}
$$

where M is a 2-dimensional Einstein-Kähler manifold, which can be taken to be the unit sphere S^2, torus T^2 or the hyperboloid H^2. In each case we have:

$$
\mathcal{A} = \begin{cases} \cos\theta d\phi, & \text{for } \delta = 1 \text{ (sphere)} \\ \theta d\phi, & \text{for } \delta = 0 \text{ (torus)} \\ \cosh\theta d\phi, & \text{for } \delta = -1 \text{ (hyperboloid),} \end{cases}
$$

while the function $f(r)$ and the gauge field potential A have the following expressions:

$$
\begin{aligned}
f(r) &= \frac{r^4 + (l^2+6N^2)r^2 - 2ml^2r - 3N^4 + l^2(q^2-N^2)}{l^2(r^2+N^2)} \\
A &= -\frac{qr}{r^2+N^2}(dt - 2N\mathcal{A})
\end{aligned}
\tag{8.4}
$$

Here m, q and N are respectively the mass, charge and the NUT parameter. As one can see directly from the expression of the 1-form gauge potential, one noteworthy feature of this solution is that the electromagnetic field strength carries both electric and magnetic components. Moreover, if we try to compute the electric and magnetic charges the results will depend on the radius of the 2-sphere on which we integrate (see also [96]). However,

[1] We use here $\lambda = \pm\frac{D-1}{l^2}$.

if we take the limit in which the 2-sphere is pushed to infinity we find that the magnetic charge vanishes and the solution is purely electric with charge q.

We are now ready to present the general class of electrically-charged Taub-NUT metrics in even dimensions $D = 2d+2$. These spaces are constructed as complex line bundles over an Einstein-Kähler space M, with dimension $2d$ and metric g_M. The metric ansatz that we use is the following:

$$ds_D^2 = -f(r)(dt - 2N\mathcal{A})^2 + F^{-1}(r)dr^2 + (r^2 + N^2)g_M \tag{8.5}$$

Here $\mathcal{J} = d\mathcal{A}$ is the Kähler form for the Einstein-Kahler space M and we use the normalisation such that the Ricci tensor of the Einstein-Kähler space M is $R_{ab} = \delta g_{ab}$.

Motivated by the known four-dimensional solution we shall make the following ansatz for the electromagnetic gauge potential:

$$A \;=\; -\sqrt{\frac{D-2}{2}}\frac{qr}{(r^2+N^2)^{\frac{D-2}{2}}}(dt - 2N\mathcal{A}) \tag{8.6}$$

Then the general solution to Einstein's field equations with cosmological constant $\lambda = \pm(D-1)/l^2$ is given by:

$$\begin{aligned} f(r) \;=\; & \frac{r}{(r^2+N^2)^{\frac{D-2}{2}}}\left[\int^r \left(\delta \mp \frac{D-1}{l^2}(s^2+N^2)\right)\frac{(s^2+N^2)^{\frac{D-2}{2}}}{s^2}ds - 2m\right] \\ & + q^2\frac{(D-3)r^2+N^2}{(r^2+N^2)^{D-2}}. \end{aligned} \tag{8.7}$$

As in the 4-dimensional case the electromagnetic field strength has both electric and magnetic components. If we try to compute the electric and magnetic charges we obtain again results that depend on the radial coordinate r. However, if we push the integration surfaces to infinity the magnetic charge will vanish leaving us only with an effective electric charge.

As an example of this general solution, let us assume that the $(D-2)$-dimensional base space in our construction is a product of d factors, $M = M_1 \times \cdots \times M_d$ where M_i are two dimensional Einstein-Kähler spaces or more generally CP^n factors. In particular, we can use the sphere S^2, the torus T^2 or the hyperboloid H^2 as factor spaces. It is then easy to see that if $q = 0$ we recover the previously known higher dimensional Taub-NUT solutions with only one NUT parameter, studied in Chapter 3. On the other hand, if $N = 0$ then we recover the topological Reissner-Nordström-AdS solutions given in [34].

8.3 Regularity conditions

The singularity analysis performed here is a direct application of the one given in Chapter 3. In order to extend the local metrics presented above to global metrics on non-singular manifolds the idea is to turn all the singularities appearing in the metric into removable coordinate singularities. For generic values of the parameters the singularities are not removable, corresponding to conical singularities in the manifold. We are mainly interested in the case of a compact Einstein-Kähler manifold M. Generically the Kähler form $\mathcal{J}$ on M can be equal to $d\mathcal{A}$ only locally and we need to use a number of overlapping coordinate patches to cover the whole manifold. Now, in order to render the 1-form $d\tau - 2n\mathcal{A}$ well-defined we need to identify τ periodically. We will require the period of τ to be given by:

$$\beta = \frac{4\pi np}{k\delta} \tag{8.8}$$

where k is a positive integer, while p is a non-negative integer, defined as the integer such that the first Chern class, c_1, evaluated on $H_2(M)$ is $\mathbf{Z} \cdot p$, *i.e.* the integers divisible by p. Among all the Einstein-Kähler manifolds the integer p is maximised in CP^q, for which $p = q + 1$ [122]. It is also necessary to eliminate the singularities in the metric that appear as r is varied over M. The critical points are to the so-called endpoint values of r: these are the values for which the metric components become zero or infinite. For a complete manifold r must range between two adjacent endpoints and we must eliminate the conical singularities (if any) which occur at these points. The finite endpoins occur at $r = \pm n$ or at the simple zeros of $f_E(r)$. Quite generally, when the electrical charge q is zero, $r = \pm n$ give the location of curvature singularities unless $f_E = 0$ there as well. By contrast with the uncharged case, it turns out that if $q \neq 0$ then the curvature singularities at $r = \pm n$ cannot be eliminated for any choices of the parameters. This can be seen from the fact that $f_E(r)$ diverges badly when $r \to \pm n$ for any values of m and the components of the curvature tensor will diverge there as well. Therefore, in order to obtain non-singular

Euclidean sections we have to restrict the range of the radial coordinate such that the values $r = \pm n$ are avoided. We should then restrict our attention to simple roots r_0 of $f_E(r)$ different from $\pm n$. In general, to eliminate a conical singularity at a root r_0 of $f_E(r)$ we must restrict the periodicity of τ to be given by:

$$\beta = \frac{4\pi}{|f_E'(r_0)|}$$

and this will generally impose a restriction on the values of the parameters once we match it with (8.8). This condition will also fix the location of the bolt, which will be given by a solution of the equation:

$$npq^2\left[(D-3)^2r_0^4 - 2n^2r_0^2 + n^4\right] + \left[np\left(\delta - \lambda(r_0^2 - n^2)\right) - k\delta r_0\right](r_0^2 - n^2)^{D-1} = 0 \quad (8.9)$$

For compact manifolds the radial coordinate takes values between two finite endpoints and we have to impose this constraint at both endpoints. If the manifold is noncompact then the cosmological constant must be non-positive and the radial coordinate takes values between one finite endpoint r_0 and one infinite endpoint $r_1 = \infty$. Since for our asymptotically locally $(A)dS$ or flat solutions the infinite endpoints are not within a finite distance from any points $r \neq r_1$ there is no regularity condition to be imposed at r_1. In this case the only regularity conditions are that $f_E(r) > 0$ for $r \geq r_0$ and $\beta = \frac{4\pi}{|f_E'(r_0)|}$ to be satisfied. The only way to avoid a curvature singularity at $r = n$ is to restrict the values of the radial coordinate such that $r \geq r_0 > n$, *i.e.* the only non-singular Taub-NUT-RN spaces are the TNRN-bolt solutions[2].

8.4 The Taub-NUT-RN solution in six dimensions

As an illustration of the general analysis performed in the previous section, in this section we shall look more closely at a six-dimensional Taub-NUT-RN solution constructed over the four-dimensional base $S^2 \times S^2$. Performing the analytical continuations $t \to i\tau$, $N \to in$

[2]We are focussing here on non-compact manifolds. For compact manifolds we could restrict the range of the radial coordinate r to the interval between two adjacent roots $r_1 \leq r \leq r_2$ of $f_E(r)$ such that the values $\pm n$ are avoided.

and $q \to iq$, the metric in the Euclidean sector can be written in the form:

$$\begin{aligned} ds^2 &= f_E(r)\left(d\tau - 2n\cos\theta_1 d\phi_1 - 2n\cos\theta_2 d\phi_2\right)^2 + \frac{dr^2}{f_E(r)} \\ &\quad +(r^2-n^2)(d\theta_1^2+\sin^2\theta_1 d\phi_1^2) + (r^2-n^2)(d\theta_2^2+\sin^2\theta_2 d\phi_2^2) \end{aligned} \tag{8.10}$$

where the function $f_E(r)$ and the 1-form potential A are given by:

$$\begin{aligned} f_E(r) &= \frac{3r^6+(l^2-15n^2)r^4+3n^2(15n^2-2l^2)r^2-6ml^2r+3n^4(5n^2-l^2)}{3(r^2-n^2)^2l^2} \\ &\quad -\frac{q^2\,(3r^2-n^2)}{(r^2-n^2)^4} \\ A &= -\frac{\sqrt{2}qr}{(r^2-n^2)^2}\,(d\tau - 2n\cos\theta_1 d\phi_1 - 2n\cos\theta_2 d\phi_2) \end{aligned} \tag{8.11}$$

Regularity of the 1-form $d\tau - 2n\mathcal{A}$ forces the periodicity of the Euclidean time to be $\frac{8\pi n}{k}$, for some integer k. As mentioned in the previous section, the Nut solution is singular thence we restrict our attention directly to the Bolt solutions. These solutions correspond to a four-dimensional fixed-point set located at a simple root r_b of $f_E(r)$ and we restrict the values of the radial coordinate such that $r \geq r_b > n$. The periodicity of τ is given by $\frac{8\pi n}{k}$ and we have to match it with the periodicity obtained by eliminating the conical singularities at r_b. This will fix the location of the bolt as given by a root of

$$2nq^2(9r_b^4-2n^2r_b^2+n^4)+\left[2n\left(1+\frac{5}{l^2}(r_b^2-n^2)\right)-kr_b\right](r_b^2-n^2)^5=0$$

As this is a polynomial equation of rank 12, an analytical solution for r_b seems out of the question.

Finally, the value of the mass parameter is given by:

$$\begin{aligned} m_b &= \frac{3r_b^{10}+(l^2-21)r_b^8+n^2(78n^2-8l^2)r_b^6+10n^4(l^2-9n^2)r_b^4+(15n^8-9q^2l^2)r_b^2}{6(r_b^2-n^2)^2l^2r_b} \\ &\quad -\frac{3n^2(n^6l^2-q^2l^2-5n^8)}{6(r_b^2-n^2)^2l^2r_b}. \end{aligned}$$

Generically there is a curvature singularity at $r=n$, which is simply avoided if we restrict the range of the radial coordinate such that $r \geq r_b > n$. If $q=0$ we recover the six-dimensional cosmological Taub-NUT solution over the base space $S^2 \times S^2$, which was discussed in Chapter 3. If $n = 0$ the solution reduces to the topological Reissner-Nordström solution whose horizon topology is $S^2 \times S^2$.

Similar results are obtained for a fibration over CP^2. The only difference appears in the periodicity of τ, which has to be now $12\pi n/k$ and this will also modify the equation for r_b (as can be read from the general expression (8.9) with $p = 3$ and $\delta = 1$). Unlike the uncharged case, the NUT solution is singular as there will be a curvature singularity at $r = n$.

8.5 Summary

In this chapter we presented new families of higher dimensional solutions of the sourceless Einstein-Maxwell field equations with or without cosmological constant. Following the general structure of the NUT-charged spaces, these solutions are constructed as radial extensions of circle fibrations over even dimensional spaces that can be in general products of Einstein-Kähler spaces.

We have given the Lorentzian form of the solutions. However in order to understand the singularity structure of these spaces, we have concentrated mainly on their Euclidean sections – recognising that the Lorentzian versions are singularity-free – apart from quasi-regular singularities.

As previously discussed in Chapter 2, in general the Euclidean section is simply obtained using the analytic continuations $t \to i\tau$, $N \to in$ and $q \to iq$. Generically the Taub-NUT solutions present themselves in two classes: 'Nuts' and 'Bolts'. While in the uncharged case there can exist Nuts with intermediate dimensionality, for our Nut solutions the fix-point set has always dimension 0 and it is singular. Indeed, we found that at the Nut location there always exists a curvature singularity that cannot be eliminated for any choice of the parameters. This is clearly in contrast with the uncharged case: in absence of the electrical charge, there are Nut solutions that are non-singular – for example, if we use $M = CP^q$ then for an appropriate choice of the mass parameter we find that there is no curvature singularity at $r = n$. The only regular charged solutions are then the Bolt metrics. The regularity conditions require us to fix the periodicity of the Euclidean time τ. This periodicity is determined in two ways and by matching the two obtained values we get restrictions on the values of the parameters in our solutions. These restrictions will also fix the location of the bolt. However, as in D-dimensions the equation that we have to solve is a polynomial equation of rank $2D$, there is no chance to obtain the bolt location in a closed analytical form.

Chapter 9

Summary

It is appropriate at the conclusion of this thesis to summarise the main results that we have obtained and to point out some areas that need further clarification.

9.1 New exact solutions

In Chapter 2 we constructed new solutions of the vacuum Einstein field equations in four dimensions via a solution generating method utilizing the SL(2, R) symmetry of the dimensionally reduced action in three dimensions. Our method was based on the simple observation that a static axisymmetric metric as written in Weyl-Papapetrou coordinates exhibits a simple 'scaling' symmetry that allows one to generate a family of new static vacuum axisymmetric solutions, indexed by a real parameter. In particular, using this scaling symmetry one can easily generate the Zipoy-Voorhees solution from the Schwarzschild solution.

We also made use of a charging method for static vacuum metrics, which dates back to Weyl [137]. We demonstrated a simpler alternative derivation of this transformation by using a $SL(2, R)$ symmetry of the reduced Lagrangian in three dimensions. However, unlike previous applications of this transformation, we showed that with our simplified mapping and by combining this charging method with the scaling property, one is able to generate new solutions starting with the Schwarzschild solution as a seed. In particular, we obtained new vacuum stationary axisymmetric metrics. The Lorentzian version of the generated solutions gives a Zipoy-Voorhees like generalisation of the Taub-NUT solutions, while the Euclidean version gives a non-trivial generalisation of the Eguchi-Hanson solitons in four dimensions. Much like the original Zipoy-Voorhees solution [142, 135], such metrics are parameterized by a real number γ. For $\gamma = 1$ we recover the usual Taub-NUT/Eguchi-Hanson solitons and, for higher positive integer values of γ, they can be interpreted as the 'superposition' of γ NUT-charged objects/solitons.

Inspired by the success in obtaining the Zipoy-Voorhees Taub-NUT and Eguchi-Hanson solitons, we attempted the same procedure on a different seed metric, this time using another well-known Weyl solution, the C-metric. This metric is known to describe two black holes uniformly accelerated in opposite directions where the source of acceleration is a strut in between the black holes pushing them apart, or alternatively two strings pulling on the black holes from infinity. For our purpose, it was more convenient to use the form of the C-metric given in [90], which has been cast into a nice factorized form. Using a similar procedure to the one that generated the Zipoy-Voorhees Taub-NUT family, we finally obtained new vacuum solutions that we interpreted as describing the accelerating Zipoy-Vorhees-like family of Taub-NUT solutions, respectively Eguchi-Hanson instantons. We focused our attention on a particular member of this family and we showed that it describes the accelerated version of Taub-NUT space.

As avenues for further research, it would be interesting to study in more detail the connection of the singular charged C-metric that we obtained with the usual form of the C-metric. In particular, it would be interesting to find a proper dilatonic generalisation of the accelerated Zipoy-Voorhees metric, one that would reduce to the proper charged C-metric in the appropriate limit.

In Chapter 3 we considered higher dimensional solutions of the vacuum Einstein field equations with and without cosmological constant. These solutions are constructed as radial extensions of circle fibrations over even dimensional spaces that can be factored in general as products of Einstein-Kähler spaces. The novelty of our solutions is that, by associating a NUT charge with every such factor of the base space, we have obtained higher dimensional generalisations of Taub-NUT spaces that can have quite generally multiple NUT parameters. In our work we have given the Lorentzian form of the solutions, however, in order to understand the singularity structure of these spaces we have concentrated mainly on the their Euclidean sections. In most of the cases the Euclidean section is simply obtained using the analytic continuations $t \to it$ and $N_j \to in_j$. When continuing back the solutions to Lorentzian signature the roots of the function $F(r)$ will give the location of the chronology horizons since across these horizons $F(r)$ will change the sign and the coordinate r changes from spacelike to timelike and vice-versa.

To render such metrics regular one follows a procedure described in Ref. [122] in which the basic idea is to turn all the singularities appearing in the metric into removable coordinate singularities. For generic values of the parameters the metrics are singular – it is only for careful choices of the parameters that they become regular. Specifically, in order to globally define the 1-form $d\tau + \sum_{i=1}^{p} 2n_i A_i$ we use various coordinate patches to cover the manifold, defining the 1-form on each patch. This can be done consistently only if we identify τ periodically, while the NUT parameters n_i must be rationally related. On the other hand, $r = \pm n_i$ correspond to curvature singularities, unless we also require that $F_E = 0$ there as well. Now by removing the possible conical singularities at the roots of $F_E(r)$ (be they at $r = \pm n_i$ or elsewhere) we get another periodicity for the Euclidean time τ. By matching the two periodicities we obtained for τ we get another restriction on the value of the parameters appearing in the metric. As an example of this analysis we have considered the 6-dimensional Taub-NUT spaces constructed over both $S^2 \times S^2$ and CP^2. While the fibration over CP^2 is in general non-singular, we found that only the Bolt solution was non-singular for the fibration over $S^2 \times S^2$, with distinct NUT parameters.

While one could think that, more generally, there are no regular fibrations with distinct NUT parameters over base spaces that are products of identical factors, it turns out that this is not the case for fibrations over products of distinct manifolds. Take for instance the 8-dimensional metric constructed as a fibration over $CP^2 \times S^2$. If the cosmological constant is zero, we can have in general two distinct NUT parameters. There then exists a Nut solution of intermediate dimensionality: assuming that the NUT parameter corresponding to the CP^2 factor is n_1, while the one corresponding to S^2 is n_2, then the periodicity of the Euclidean time can be set to $8\pi n_2 = 12\pi n_1$. There exists a regular 4-dimensional nut located at $r = n_2 = \frac{3}{2} n_1$. As discussed in Chapter 3, there also exists another way to obtain Nuts of intermediate dimensionality constructed over spaces of the same nature. However the price to pay is the use of non-canonically normalised Einstein-Kähler manifolds.

We would also like to take the opportunity and comment at this point on the existence of Misner string singularities in cases where the base space contains 2-dimensional hyperboloids H^2 (respectively planar geometries T^2). In literature it is often stated that in these cases there are no hyperbolic (respectively planar) Misner strings [34, 6, 7]. From the general discussion in Chapter 3 we can see that this statement is true only if the Einstein-Kähler geometries H^2 (respectively T^2) are not compact. Otherwise, we find that the integral of the 2-form $2ndA$ over closed 2-cycles in H^2(respectively T^2) can have a finite value. This implies that the Euclidean time τ must have (under appropriate normalization of the compact space) periodicity $8\pi n/k$, for an integer k; we can therefore speak about hyperbolic (planar) Misner strings.

Our construction applies more generally, yielding multiple NUT-charged generalisations of inhomogeneous Einstein metrics on complex line bundles found in Ref. [111, 105]. In this case we replace the Einstein-Kähler manifold M by a product of Einstein-Kähler manifolds M_i with arbitrary even-dimensions. To each such factor we associate a NUT parameter N_i. As conjectured in [111], we find that, quite generally, in higher dimensions there are various constraints to be imposed on the possible values of the cosmological constant λ, the NUT parameters N_i and the values of the various δ's. These solutions represent the multiple NUT parameter extension of the inhomogeneous Einstein metrics on complex line-bundles described in [122]. It is also possible to cast these solutions into a different form, by explicitly encoding the constraint conditions into the metric. However this requires us to resort to non-canonically normalised Einstein-Kähler manifolds.

Finally, in Chapter 8 we briefly presented another generalisation of the NUT-charged spaces as solutions in Einstein-Maxwell theory. However, for space reasons, we confined ourselves to performing a simple singularity analysis of such metrics, leaving a full thermodynamic description for further work.

9.2 Properties and applications

In Chapter 4 we briefly reviewed the path-integral approach to quantum gravity and its relationship to gravitational thermodynamics for asymptotically flat or asymptotically AdS spacetimes. For detailed thermodynamic computations that include effects from the gravitational fields one should be able to compute the partition function of the respective systems. It is well known that the partition function for quantum fields in the canonical ensemble can be related in general to a path-integral by analytic continuation. In this approach, the partition function for the gravitational field is defined by a sum over all smooth Euclidean geometries which are periodic with a period β in imaginary time. The path-integral is computed by using the saddle point approximation in which one considers that the dominant contributions will come from metrics near the classical solutions of Euclidean Einstein's equations with the given boundary conditions. In the semiclassical limit this yields a relationship between gravitational entropy and other relevant thermodynamic quantities, such as mass, angular momentum, and other conserved charges. This relationship was first explored in the context of black holes by Gibbons and Hawking [65], who argued that the free energy is equal to the Euclidean gravitational action multiplied by the temperature. The gravitational entropy can then be regarded as arising from the the quantum statistical relation applied to the path-integral formulation of quantum gravity [81]. It has been recently noted that the path-integral formalism can be extended to asymptotically de Sitter spacetimes to describe quantum correlations between timelike histories, providing a foundation for gravitational thermodynamics at past/future infinity [40]. The key result is the generalisation of the Gibbs-Duhem relation (or the generalised quantum statistical relation) to asymptotically dS spacetimes. In order to employ this relation it is necessary to analytically continue the spacetime near past/future infinity. There are two apparently distinct ways of doing this – the $\mathbb{R}$-approach and the $\mathbb{C}$-approach. The $\mathbb{C}$-approach is closest to the more traditional method of obtaining Euclidean sections for asymptotically flat and AdS spacetime. The $\mathbb{R}$-approach refers to the Lorentzian section, and makes use of the path integral formalism only insofar as the generalised Gibbs-Duhem relation is employed.

The main result in Chapter 5 is the demonstration that the $\mathbb{R}$ and $\mathbb{C}$-approaches are equivalent, in the sense that we can start from the $\mathbb{C}$-approach results and derive by consistent analytic continuations (*i.e.*, using a well-defined prescription for performing the analytic continuations) all the results from the $\mathbb{R}$-approach. There are no *a-priori* obstacles in taking the opposite view, in which the $\mathbb{C}$-approach results are derived from the respective $\mathbb{R}$-approach results. However, one could still argue that the $\mathbb{C}$-approach is the more basic one, as in it the periodicity conditions appear more naturally than in the $\mathbb{R}$-approach. On the other hand, the $\mathbb{R}$-approach, when used without the justification that comes from the $\mathbb{C}$-approach, raises some interesting questions. Even applied to simple cases such as the Schwarzschild-dS solution, one may take the view that in the absence of the NUT charge one could still consider a periodicity on the time coordinate in the Lorentzian sector given by $\beta_r = 8\pi m$, which can be used when computing the action and the conserved charges. A more orthodox interpretation would be that β_r in the Lorentzian sector is simply the inverse temperature (as related to the surface gravity of the black hole horizon) and is not related to a real periodicity of the time coordinate. Whether or not this is indeed a necessary condition remains to be seen.

Using this equivalence we then proposed an interpretation of the thermodynamic behaviour of NUT-charged spacetimes. In the asymptotically dS case, we showed that while a subset of the Bolt solutions can have a sensible physical interpretation, the same does not hold for the Taub-Nut-dS solutions. Indeed, in the putative Taub-Nut-dS solution the Nut is always enclosed in a larger cosmological 'Bolt'. Moreover it does not have a Lorentzian counterpart (*i.e.* it has no equivalent solution in the $\mathbb{R}$-approach). From these facts we conclude that there are no Taub-Nut-dS solutions. This situation holds despite the fact that a naive application of (4.21) to this case yields thermodynamic quantities that respect the first law of thermodynamics. Rather these quantities are the analytic continuations of their AdS counterparts under $l \to il$. Similar remarks apply to the lower-branch dS bolt cases. We have also found that this situation holds in higher dimensions: there are no Taub-Nut-dS solutions, in analogy with the non-existence of the hyperbolic Nuts in AdS backgrounds [54, 7].

Moreover, the $\mathbb{C}$-approach has been previously applied with success to more general cases - it has been proven to be very useful when treating for instance asymptotically AdS or flat Taub-NUT spaces. In particular, we have shown here that starting from the well-known results regarding the thermodynamics of the Nut and Bolt solutions in the Euclidean Taub-NUT-AdS case (which corresponds to our $\mathbb{C}$-approach) we can consistently make analytic continuations back to the Lorentzian sections, yielding a physical interpretation of the thermodynamics of such spacetimes. However, this holds only for the Bolt solutions; we found that the Lorentzian AdS-Nut solution did not respect the first law of thermodynamics, rendering the physical interpretation of the Nut solution dubious at best.

In de Sitter backgrounds, it has been shown that there exist broad ranges of parameter space for which NUT-charged spacetimes violate both the maximal mass conjecture and the N-bound, in both four dimensions and in higher dimensions. However it was subsequently argued [12] that even if the Taub-NUT-dS spaces do violate the maximal mass conjecture they also suffer causal pathologies. However, this is not necessarily the case. As noted above in both the $\mathbb{R}$ and $\mathbb{C}$-approaches the maximal mass conjecture can be violated by choosing the parameter $m < 0$, independent of whether or not the metric function has any roots [38, 40]. A detailed discussion of this situation has appeared recently [5], where it was emphasized that globally hyperbolic asymptotically dS spacetimes exist that violate the maximal mass conjecture. In the present context this will take place whenever the parameters m and N are such that the function (5.2) has no roots. If horizons are present, the maximal mass conjecture and N-bound can both be violated – we have shown that this holds consistently for both approaches. Although these cases have regions containing closed timelike curves (CTC's), our computations pertain to regions where CTCs are absent, namely outside the cosmological horizon. However, one could argue that such spacetimes are causally unstable. Whether any spacetimes containing horizons exist that violate one or both conjectures and that satisfy rigid constraints of causal stability remains an open question.

In AdS backgrounds it would be interesting to understand how the AdS-CFT correspondence works in this case, since the Lorentzian sections of Taub-NUT spaces contain closed timelike curves, and so are causally pathological. In fact it was recently noted [6, 7] that the boundary metric for the four-dimensional Lorentzian Taub-NUT-AdS spacetime is in fact the three-dimensional Gödel metric. This metric also has a bad reputation as being causally ill-behaved since for generic values of the parameters, the Gödel spacetime admits CTC's through every point. The meaning of a quantum field theory in this background is still an open problem.

Recently, the authors of [63] used successfully the $\mathbb{R}$-approach methods to study the Kerr-dS space-times. As the metric is stationary in this case and the continuation to the complex section involves the analytical continuation of the rotation parameter $a \to ia$, it would be interesting to see if there is a similar prescription for the analytical continuation of the results from the $\mathbb{R}$-approach and the $\mathbb{C}$-approach and vice-versa.

In Chapter 6 we describe another application of the Taub-NUT-Eguchi-Hanson solitons in the construction of Kaluza-Klein magnetic monopoles. We began by reviewing how the flat Kaluza-Klein monopole can be obtained from the four dimensional Taub-Nut solution. We also briefly discussed the features of the monopole solution obtained by using the Euclidean Taub-Bolt solution. The physical interpretation of this 'monopole' solution was recently clarified by Liang and Teo [103]: it corresponds to a pair of coincident extremal dilatonic black holes with opposite magnetic charges. Motivated by this result, at this point we considered the solution obtained by dimensionally reducing an Eguchi-Hanson-like monopole and we have proven that even if the four-dimensional metric is non-asymptotically flat, its geometry is nonetheless U-dual to that of a Taub-Bolt monopole.

Next, using two distinct proposals for the boundary counterterm action we computed the mass of the Kaluza-Klein magnetic monopole and found agreement in both cases with previous results derived by other means [24, 48]. We also extended our results to the case of the Kaluza-Klein Bolt-monopole solution. In the general context of Kaluza-Klein theory it is also tempting to examine the energy from the point of view of the dimensionally-reduced theory. While the metric and also the fields do have in general singularities at origin, this is not necessarily an obstruction since the conserved charges are in general computed as surface integrals at infinity. In the four-dimensional theory, using the counterterm (4.12) proposed by Lau [102] and Mann [108] as well as the new counterterm proposed in [109] we computed the mass of the monopole and found it to be equal to the five-dimensional mass. A similar result was proved in [24] using background subtraction methods.

However, as we have seen in Chapter 3, the five-dimensional Eguchi-Hanson soliton in the limit of a vanishing cosmological constant reduces to this flat Eguchi-Hanson monopole solution. In light of this fact, we described in the remaining sections in Chapter 6 another application of the Taub-NUT-Eguchi-Hanson solitons in the construction of Kaluza-Klein magnetic monopoles. More specifically, we attempted to construct possible extensions of Kaluza-Klein monopole solutions that admit a cosmological constant. The essential ingredient in the original Kaluza-Klein monopole construction is a Euclidean section of the four dimensional Taub-NUT space; the 'trick' employed in [129, 76] to obtain the monopole solution was to lift this Euclidean section up to five-dimensions by adding a flat time coordinate and then to dimensionally reduce along the 'Euclidean time' direction from the Euclidean Taub-NUT section. However, in the presence of the cosmological constant it is not possible to use the above technique without introducing an explicit time dependence in the metric. Therefore, in order to obtain cosmological four-dimensional magnetic monopole solutions our strategy was to consider directly in five-dimensions the new cosmological Taub-NUT-like solutions discussed in sections 3.5, 3.6 in Chapter 3 and perform a Kaluza-Klein compactification along the fifth dimension. The new feature of these solutions is that the four-dimensional dilaton acquires a potential term as an effect of the cosmological constant. However their asymptotics are not very appealing physically since they are not asymptotically flat or $(A)dS$ in the Einstein frame. Their metric description simplifies when considered in the string frame: for our explicit examples the four-dimensional metric in the string frame is very similar to the AdS form in the (r, t) sector, except for a deficit of solid angle in the angular sector.

In higher than five dimensions we have more choices: we can consider solutions that are Ricci flat with different NUT parameters or we can consider Taub-NUT like spaces that are constructed as circle fibrations over base spaces that have non-trivial topology. We also performed Kaluza-Klein (KK) reductions of the above solutions down to four dimensions, obtaining new magnetic monopole solutions. More specifically, in six and seven dimensions we have considered non-singular Ricci-flat solutions for which one can use the KK trick to obtain similar KK magnetic brane solutions for which the background spaces are Ricci flat Bohm spaces of the form $S^p \times S^q$ and generically have conical singularities. We considered their further reduction down to four dimensions on Riemannian spaces of constant curvature and specifically considered such reductions on spheres. In contrast with the KK procedure to untwist the $U(1)$-fibration, we have considered in six dimensions another method that is known to untwist the circle fibration, namely Hopf duality in string theory. We extended these duality rules to the case of a timelike Hopf-duality of the truncated six-dimensional Type II theories and applied them to generate charged string solutions in six-dimensions. By performing sphere reductions we obtained the corresponding four-dimensional solutions. In general, the presence of the cosmological constant in the higher dimensional theory induces a scalar potential for the Kaluza-Klein scalar fields. If the isometry generated by the Killing vector that is associated with the circle direction on which we perform the reduction has fixed points, then the dilaton, which describes the radius of that extra-dimension, will diverge at the respective fixed point sets and the D-dimensional metric will be singular there. In certain cases we find that the dilaton field also diverges at infinity. Respectively this means that, physically, the space-time decompactifies near the KK-brane and at infinity; the higher-dimensional theory should be used when describing such objects in those regions.

In Chapter 7 we constructed new explicit solutions of general relativity from double analytic continuations of Taub-NUT spacetimes. This generalises previous studies of the four-dimensional nutty bubbles. One five-dimensional locally asymptotically AdS solution in particular has a special conformal boundary structure of $AdS_3 \times S^1$. We computed its boundary stress tensor and related it to the properties of the dual field theory. Interestingly enough, we also found consistent six-dimensional bubble solutions that have only one timelike direction. The existence of such spacetimes with non-trivial topology is closely related to the existence of the Taub-NUT(-AdS) solutions with more than one NUT charge. Finally, we began an investigation of generating new solutions from Taub-NUT spacetimes and nuttier bubbles. Using the so-called Hopf duality, we provided new explicit time-dependent backgrounds in six dimensions.

Appendix A

The effective actions of the Superstring theories in 10-dimensions

In the following, we present briefly the bosonic effective actions of the five superstring theories in ten dimensions [104].

- Type IIA superstring - This theory is a theory of closed strings. The $N = 1$ supersymmetry on the world - sheet of the string induces $N = 2$ supersymmetries between the bosonic and fermionic fields in space-time. However, in this case the two space-time supersymmetries appear with opposite chirality. Hence one is dealing with a $N = 2$, $D = 10$ non-chiral supergravity theory, whose bosonic action is given by:

$$S_{IIA} = \frac{1}{16\pi\alpha'^4}\Big[\int d^{10}x\sqrt{-g_{10}}[e^{-2\phi}(R_{10} + 4(\partial\phi)^2 - \frac{1}{12}H^2_{(3)}) - \frac{1}{4}F^2_{(2)} - \frac{1}{48}(F'_{(4)})^2] + \frac{1}{2}\int B_{(2)} \wedge F_{(4)} \wedge F_{(4)}\Big],$$

  where R_{10} is the Ricci scalar curvature of the space-time with metric $g_{\mu\nu}$ and $g_{10} = \det g_{\mu\nu}$. The dilaton field ϕ determines the value of the string coupling parameter $g_s = e^{-\phi}$. The antisymmetric tensor field strengths that appear in the action are defined in terms of the potentials by the formulae:

$$H_{(3)} = dB_{(2)}, \quad F_{(2)} = dA_{(1)}, \quad F_{(4)} = dA_{(3)}, \quad F'_{(4)} = F_{(4)} + A_{(1)} \wedge H_3.$$

The NS-NS sector of the action contains the graviton, the antisymmetric two form potential $B_{(2)}$, and the dilaton field ϕ. The RR sector contains a 1-form potential $A_{(1)}$ and a 3-form potential $A_{(3)}$. As one can see from the action, the NS-NS sector couples directly to the dilaton field, while the RR fields do not. Finally, the last term in the action is the so-called Chern-Simons term and it is a necessary consequence of the supersymmetry requirement.

- Type IIB superstring - This is a theory of closed strings with $N = 2$ supersymmetries. However in this case the two space-time supersymmetries appear with the same chirality, so the theory is chiral. The field content of this theory is the following: the $NS - NS$ sector contains the graviton $g_{\mu\nu}$, the dilaton field ϕ and an antisymmetric two-form potential $B_2^{(1)}$. The RR sector contains a scalar axion field χ, a two-form potential $B_2^{(2)}$, and a four-form potential $A_{(4)}$. The field equations for the four-form imply that its five-form field strength is self-dual. Because of this fact we cannot write down a covariant ten-dimensional low-energy action for the Type IIB superstring theory. However, we can drop the self-duality constraint by introducing new degrees of freedom at the level of the action. We can then use the following action, and impose the self-duality condition as an extra equation of motion for the four-form:

$$
\begin{aligned}
S_{IIB} &= \frac{1}{16\pi\alpha'^4}\Bigg[\int d^{10}x\sqrt{-g_{10}}[e^{-2\phi}(R_{10} + 2(\partial\phi)^2 - \frac{1}{12}(H_3^{(1)})^2) \\
&\qquad - \frac{1}{2}(\partial\chi)^2 - \frac{1}{12}(H_3^{(2)} + \chi H_3^{(1)})^2 - \frac{1}{240}(F_5)^2] + \int A_{(4)} \wedge H_3^{(2)} \wedge H_3^{(1)}\Bigg],
\end{aligned}
$$

where the RR field strengths are defined by

$$
H_3^{(2)} = dB_2^{(2)}, \quad F_{(5)} = dA_{(4)} + B_2^{(2)} \wedge H_3^{(1)}
$$

and we impose the self-duality condition $F_{(5)} = \star F_5$.

- Type I superstring - This is a theory of open strings. Notice that closed strings are also included in the theory since two interacting open strings can interact and join to form a closed string. The boundary conditions on the open strings halve the number of the supersymmetries in the theory from $N = 2$ to $N = 1$ supersymmetry. One can attach charges at the endpoints of the strings introducing in this way a Yang-Mills gauge group in the theory. However, the theory is free from anomalies and quantum mechanically consistent only if the gauge group is $SO(32)$. The low energy bosonic action for this $N = 1$, $D = 10$ supergravity theory is given by:

$$S_I = \frac{1}{16\pi\alpha'^4}\int d^{10}x\sqrt{-g_{10}}[e^{-2\phi}(R_{10} + 4(\partial\phi)^2) - \frac{1}{12}H^2_{(3)} - \frac{1}{4}e^{-\frac{2\phi}{2}}F^2_{(2)}] \quad (A.1)$$

where $F_{(2)}$ is the Yang-Mills field strength of the vector field corresponding to the $SO(32)$ group and $H_{(3)} = dB_{(2)}$ is the field strength of a two-form potential $B_{(2)}$. Notice that this field strength is not coupled to the dilaton field.

- Heterotic superstring - The heterotic superstring theories have only one supersymmetry $N = 1$. A Yang-Mills group arises from the compactification of the bosonic string theory on a 16-dimensional compact space. Quantum consistency restricts the gauge group to be $SO(32)$ or $E_8 \times E_8$. The bosonic part of the low-energy action of the heterotic superstring theory is given by:

$$S_H = \frac{1}{16\pi\alpha'^4}\int d^{10}x\sqrt{-g_{10}}e^{-2\phi}[R_{10} + 4(\partial\phi)^2 - \frac{1}{12}H^2_{(3)} - \frac{1}{4}F^{(2)}_2] \quad (A.2)$$

where $F_{(2)}$ is the Yang-Mills field strength corresponding to the gauge groups $SO(32)$ or $E_8 \times E_8$. Notice that Type I superstring theory and heterotic string theory have the same particle content. However, their actions differ because all the bosonic degrees of freedom couples directly to the dilaton field in heterotic theory, whereas the two-form potential is a RR degree of freedom in Type I theory.

- Eleven dimensional supergravity - This is the unique $N = 1$ supergravity theory in eleven dimensions. The bosonic content of the low energy effective action is given by the graviton $g_{\mu\nu}$ and a three-form potential $A_{(3)}$. The action is given by:

$$S_{11D} = \frac{1}{16\pi G_{11}}\left[\int d^{11}x\sqrt{-g_{11}}[R_{11} - \frac{1}{48}F^2_{(4)}] + \frac{1}{6}\int A_{(3)} \wedge F_{(4)} \wedge F_{(4)}\right] \quad (A.3)$$

where $F_{(4)} = dA_{(3)}$ is the four-form field strength of the three-form potential $A_{(3)}$. The eleven dimensional Newton constant G_{11} is the only parameter in the theory. The last term, the Chern-Simons term, arises as a direct consequence of supersymmetry.

Appendix B

The Weyl form of the accelerating Taub-NUT metric

Following Emparan and Reall [55] we introduce the notation:

$$\zeta_i = z - z_i, \qquad R_i = \sqrt{\rho^2 + \zeta_i^2}, \qquad Y_{ij} = \rho^2 + R_i R_j + \zeta_i \zeta_j. \tag{B.1}$$

It can then be shown [55] that:

$$\begin{aligned}
R_1 - \zeta_1 &= \frac{(y^2-1)F(x)}{A^2(x-y)^2}, \\
R_1 + \zeta_1 &= \frac{(1-x^2)F(y)}{A^2(x-y)^2}, \\
R_2 - \zeta_2 &= \frac{(x-1)(y+1)F(x)}{A^2(x-y)^2}, \\
R_2 + \zeta_2 &= -\frac{(x+1)(y-1)F(y)}{A^2(x-y)^2}, \\
R_3 - \zeta_3 &= \frac{(x-1)(y+1)F(y)}{A^2(x-y)^2}, \\
R_3 + \zeta_3 &= -\frac{(x+1)(y-1)F(x)}{A^2(x-y)^2},
\end{aligned} \tag{B.2}$$

while:

$$\begin{aligned}
Y_{12} &= \frac{(x-1)(y-1)(2mA-1)^2}{2A^4(x-y)^2} \\
Y_{13} &= \frac{(x-1)(y-1)F(x)F(y)}{2A^4(x-y)^2} \\
Y_{23} &= \frac{2F(x)F(y)}{A^4(x-y)^2}
\end{aligned} \tag{B.3}$$

Then the Weyl form of the uncharged C-metric corresponds to the following expressions:

$$\begin{aligned}
e^{-\psi} &= \frac{(R_1-\zeta_1)(R_3-\zeta_3)}{R_2-\zeta_2}, \\
e^{2\mu} &= \frac{1}{4(2mA-1)^2 R_1 R_2 R_3} \frac{Y_{12}Y_{23}}{Y_{13}} \frac{(R_1-\zeta_1)(R_3-\zeta_3)}{R_2-\zeta_2},
\end{aligned} \tag{B.4}$$

from which we can readily find $e^{-\phi}$ in (2.59). Finally, expressing x in terms of ρ and z we find [90]:

$$A_\varphi = \frac{1}{C}\left(\frac{(R_1+\zeta_1)(R_2-\zeta_2)}{R_3-\zeta_3} + \frac{2m}{A}\frac{F_1+F_2}{2F_0}\right), \tag{B.5}$$

where:

$$\begin{aligned}
F_0 &= 4m^2AR_1 + m(1+2mA)R_2 + m(1-2mA)R_3, \\
F_1 &= -4mR_1 - 2m(1+2mA)R_2 + 2m(1-2mA)R_3, \\
F_2 &= \frac{2m}{A^2}(1-2m^2A^2).
\end{aligned} \tag{B.6}$$

This completes the derivation of the Weyl-Papapetrou form of the accelerated Taub-NUT solution.

Appendix C

T-duality in six-dimensional Type II Superstring theories

The Lagrangian in $D = 6$ obtained by dimensional reduction of Type IIB on a torus and after performing a consistent truncation is given by [52]:

$$\begin{aligned}\mathcal{L}_{6B} &= eR - \frac{1}{2}e(\partial\varphi_1)^2 - \frac{1}{2}e(\partial\varphi_2)^2 - \frac{1}{2}ee^{2\varphi_1}(\partial\chi_1)^2 - \frac{1}{2}ee^{2\varphi_2}(\partial\chi_2)^2 \\ &\quad -\frac{1}{12}ee^{-\varphi_1-\varphi_2}(F^{NS}_{(3)})^2 - \frac{1}{12}ee^{\varphi_1-\varphi_2}(F^{RR}_{(3)})^2 + \chi_2 dA^{NS}_{(2)} \wedge dA^{RR}_{(2)} \qquad \text{(C.1)}\end{aligned}$$

where $F^{NS}_{(3)} = dA^{NS}_{(2)}$ and $F^{RR}_{(3)} = dA^{RR}_{(2)} + \chi_1 dA^{NS}_{(2)}$. This Lagrangian is related by T-duality in $D = 5$ to a different six-dimensional theory obtained by making a consistent truncation of Type IIA compactified on a four-dimensional torus. The corresponding Lagrangian is given by:

$$\begin{aligned}\mathcal{L}_{6A} &= eR - \frac{1}{2}e(\partial\varphi_1)^2 - \frac{1}{2}e(\partial\varphi_2)^2 - \frac{1}{48}ee^{\frac{\varphi_1}{2}-\frac{3\varphi_2}{2}}(F_{(4)})^2 - \frac{1}{12}ee^{-\varphi_1-\varphi_2}(F_{(3)})^2 \\ &\quad -\frac{1}{4}ee^{\frac{3\varphi_1}{2}-\frac{\varphi_2}{2}}(F_{(2)})^2 \qquad \text{(C.2)}\end{aligned}$$

where $F_{(4)} = dA_{(3)} - dA_{(2)} \wedge A_{(1)}$, $F_{(3)} = dA_{(2)}$ corresponds to the NS-NS 3-form $F_{(3)1}$ and $F_{(2)} = dA_{(1)}$ is the RR 2-form $\mathcal{F}^1_{(2)}$, with the index '1' denoting here the first reduction step from $D = 11$ to $D = 10$.

Let us focus on Type IIA theory first. Under a dimensional reduction using the formulae from the previous appendix we have:

$$ds_6^2 = e^{\frac{\varphi}{\sqrt{6}}} ds_5^2 + e^{\frac{-3\varphi}{\sqrt{6}}} (dz + \mathcal{A}_{(1)})^2 \tag{C.3}$$

and we obtain the following 5-dimensional Lagrangian:

$$\begin{aligned}\mathcal{L}_{5A} = {} & eR - \frac{1}{2}e(\partial\varphi_1)^2 - \frac{1}{2}e(\partial\varphi_2)^2 - \frac{1}{2}e(\partial\varphi)^2 - \frac{1}{48}ee^{-\frac{3\varphi}{\sqrt{6}}+\frac{\varphi_1}{2}-\frac{3\varphi_2}{2}}(F'_{(4)})^2 - \frac{1}{12}ee^{\frac{\varphi}{\sqrt{6}}+\frac{\varphi_1}{2}-\frac{3\varphi_2}{2}}(F_{(3)1})^2 \\ & -\frac{1}{12}ee^{-\frac{2\varphi}{\sqrt{6}}-\varphi_1-\varphi_2}(F'_{(3)})^2 - \frac{1}{2}ee^{-\frac{4\varphi}{\sqrt{6}}}\mathcal{F}_{(2)}^2 \\ & -\frac{1}{4}ee^{-\frac{\varphi}{\sqrt{6}}+\frac{3\varphi_1}{2}-\frac{\varphi_2}{2}}(F'_{(2)})^2 - \frac{1}{4}ee^{\frac{2\varphi}{\sqrt{6}}-\varphi_1-\varphi_2}(F_{(2)1})^2 - \frac{1}{2}ee^{\frac{3\varphi}{\sqrt{6}}+\frac{3\varphi_1}{2}-\frac{\varphi_2}{2}}(dA_{(0)1})^2\end{aligned} \tag{C.4}$$

where the field strengths are defined as follows:

$$\begin{aligned} F'_{(2)} &= dA_{(1)} - dA_{(0)1}\wedge\mathcal{A}_{(1)}, \qquad F'_{(3)} = dA_{(2)} - dA_{(1)}\wedge\mathcal{A}_{(1)} \\ F_{(3)1} &= dA_{(2)1} + dA_{(1)}\wedge A_{(1)} - dA_{(2)}\wedge A_{(0)1}, \qquad F'_{(4)} = dA_{(3)} - dA_{(2)}\wedge A_{(1)} - F_{(3)1}\wedge\mathcal{A}_{(1)}\end{aligned}$$

while $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$ and $F_{(2)1} = dA_{(1)1}$. Upon dualising $F_{(4)}$ to a 1-form field strength $d\chi'$ its kinetic term in the above Lagrangian will be replaced by:

$$-\frac{1}{2}ee^{\frac{3\varphi}{\sqrt{6}}-\frac{\varphi_1}{2}+\frac{3\varphi_2}{2}}(d\chi')^2 + \chi' F'_{(3)}\wedge F'_{(2)} + \chi' F_{(3)1}\wedge\mathcal{F}_{(2)} \tag{C.5}$$

If we perform the field redefinitions:

$$A'_{(1)} = A_{(1)} - A_{(0)1}\wedge\mathcal{A}_{(1)}, \; A'_{(2)} = A_{(2)} - A_{(1)1}\wedge\mathcal{A}_{(1)}, \;\; A'_{(2)1} = A_{(2)1} + A_{(1)1}\wedge A'_{(1)}$$

we find:

$$\begin{aligned} F'_{(2)} = dA'_{(1)} + A_{(0)1}\wedge\mathcal{F}_{(2)}, \qquad F'_{(3)} = dA'_{(2)} - A_{(1)1}\wedge\mathcal{F}_{(2)} \\ F_{(3)1} = dA'_{(2)1} + dA'_{(1)}\wedge A_{(1)1} - A_{(0)1}(dA'_{(2)} - A_{(1)1}\wedge\mathcal{F}_{(2)}) \\ \chi' F'_{(3)}\wedge F'_{(2)} + \chi' F_{(3)1}\wedge\mathcal{F}_{(2)} = \chi'(dA'_{(2)}\wedge dA'_{(1)} + dA'_{(2)1}\wedge\mathcal{F}_{(2)})\end{aligned} \tag{C.6}$$

Similarly, for the dimensional reduction of Type IIB Lagrangian we obtain:

$$\begin{aligned}\mathcal{L}_{5B} = {} & eR - \frac{1}{2}e(\partial\varphi_1)^2 - \frac{1}{2}e(\partial\varphi_2)^2 - \frac{1}{2}e(\partial\varphi)^2 - \frac{1}{2}ee^{2\varphi_1}(\partial\chi_1)^2 - \frac{1}{2}ee^{2\varphi_2}(\partial\chi_2)^2 \\ & -\frac{1}{12}ee^{-\frac{2\varphi}{\sqrt{6}}+\varphi_1-\varphi_2}(F'^{RR}_{(3)})^2 - \frac{1}{12}ee^{-\frac{2\varphi}{\sqrt{6}}-\varphi_1-\varphi_2}(F'^{NS}_{(3)})^2 - \frac{1}{2}ee^{-\frac{4\varphi}{\sqrt{6}}}\mathcal{F}_{(2)}^2 - \frac{1}{4}ee^{\frac{2\varphi}{\sqrt{6}}+\varphi_1-\varphi_2}(F^{RR}_{(2)1})^2 \\ & -\frac{1}{4}ee^{\frac{2\varphi}{\sqrt{6}}-\varphi_1-\varphi_2}(F^{NS}_{(2)1})^2 - \chi_2 dA^{RR}_{(2)}\wedge dA^{NS}_{(1)1} + \chi_2 dA^{NS}_{(2)}\wedge dA^{RR}_{(1)1}\end{aligned} \tag{C.7}$$

where $F^{NS}_{(2)1} = dA^{NS}_{(1)1}$, $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$ and:

$$F'^{NS}_{(3)} = dA^{NS}_{(2)} - dA^{NS}_{(1)1} \wedge \mathcal{A}_{(1)}, \qquad F^{RR}_{(2)1} = dA^{RR}_{(1)1} + \chi_1 dA^{NS}_{(1)1}$$
$$F'^{RR}_{(3)} = dA^{RR}_{(2)} - dA^{RR}_{(1)1} \wedge \mathcal{A}_{(1)} + \chi_1 dA^{NS}_{(2)} \wedge \mathcal{A}_{(1)} \tag{C.8}$$

As shown in [52], the T-duality rules relating the two truncated theories (C.4) and (C.7) are:

$$A_{(0)1} \to \chi_1, \quad A_{(1)1} \to \mathcal{A}_{(1)}, \quad \mathcal{A}_{(1)} \to A^{NS}_{(1)1}, \quad \chi' \to \chi_2,$$
$$A'_{(1)} \to A^{RR}_{(1)1}, \quad A'_{(2)} \to A^{NS}_{(2)}, \quad A'_{(2)1} \to -A^{RR}_{(2)}, \tag{C.9}$$

together with a rotation of the scalars:

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi \end{pmatrix}_{IIA,B} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{\sqrt{6}}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{\sqrt{6}}{4} \\ -\frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi \end{pmatrix}_{IIB,A} \tag{C.10}$$

which takes care of the dilaton couplings of the field strengths.

We are also interested in performing a timelike T-duality. As it is known this duality will relate Type IIA (respectively IIB) to Type IIB* (respectively IIA*). We wish to see if at the level of our truncated theories the timelike T-duality rules are still valid.

Consider first the Type IIA theory. Upon a timelike dimensional reduction the Lagrangian of the reduced theory will have a form similar with (C.4); however the kinetic terms for $F_{(3)1}$, $F_{(2)1}$, $\mathcal{F}_{(2)}$ and $dA_{(0)1}$ will have the reversed sign [43]. The five-dimensional metric is now of Euclidean signature and when we dualize the 4-form $F'_{(4)}$ to a scalar field strength χ' we obtain a positive kinetic term for this scalar. We expect to be able to relate this theory to a timelike reduction of a truncated six-dimensional Type IIB* theory by applying the T-duality rules given above. Now, it is known that the action of Type IIB* in ten dimensions is obtained from the usual Type IIB action after we reverse the signs of the RR kinetic terms. As the sign of such kinetic terms was irrelevant when discussing the truncation to six dimensions we see that a consistent truncation of Type IIB* in six dimensions will be given by the Lagrangian (C.1) in which we must reverse the sign on the kinetic terms for the RR fields, i.e. we must reverse the sign of the kinetic terms for $F^{RR}_{(3)}$ and also for χ_2 (which appears from the dualisation of the RR field $B_{(4)}$). When performing a timelike dimensional reduction the final Type IIB* Lagrangian will be similar with (C.7) with reverted signs for the kinetic terms of χ_2, $F^{RR}_{(3)}$, $F^{NS}_{(2)1}$ respectively $\mathcal{F}_{(2)}$. It is then straightforward to see that the T-duality will relate our truncated Type IIA theory with the truncated Type IIB*. It is easy to extend the above considerations to show that a timelike T-duality will relate Type IIB with Type IIA* at the level of our truncated theories. Also Type IIA* and Type IIB* are related by a usual T-duality along a spacelike direction.

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