Some Problems in General Algebra

by

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Abstract

In the first part of the thesis we construct a finitely based variety, whose equational theory is undecidable, yet whose word problems are recursively solvable, which solves a problem stated by G. McNulty. The construction produces a discriminator variety with the aforementioned properties, starting from a class of structures in some multisorted language (which may include relations), axiomatized by a finite set of universal sentences in the given multisorted signature.

This result also presents a common generalization of the earlier results obtained by B. Wells and A. Mekler, E. Nelson, and S. Shelah.

In the second part of the dissertation the classification of finite graph M-algebras which have finite equational bases is given in terms of omitted induced subgraphs. The result is related to an earlier result obtained for finite graph algebras by K. Baker, G. McNulty, and H. Werner.

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Ova teza je posvećena mojoj majci, Darinki.

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Introduction

In this body of work we investigate and solve two problems in general algebra.

The first problem concerns finitely axiomatizable varieties whose equational theory is undecidable yet all of whose finitely presented algebras have solvable word problems. This problem is intimately related to some earlier results obtained by A. Mekler, E. Nelson, and S. Shelah, ([39]), as well as those of B. Wells ([52]). We give an affirmative answer as to whether a variety with the properties listed above exists, thereby resolving a problem due to G. McNulty ([37]).

The second part of the thesis is dedicated to the problem of classifying finite graph M-algebras whose equational theory is finitely based. Our principal result is the characterization of such algebras in terms of the induced subgraphs of their underlying graphs. This result presents a natural extension of a similar theorem for finite graphs, obtained by K. Baker, G. McNulty, and H. Werner in [2].

The organization of this dissertation is as follows. Sections 1.1 and 1.2 present a basic introduction to universal algebra and first–order logic including some basic facts about first-order logic with several sorts of objects. The last section of Chapter 1 introduces the notion of a discriminator variety and several results are stated including the one due to R. McKenzie concerning the reduction of the universal theory of a discriminator variety to its equational theory.

In Chapter 2 we first introduce the notions of a finite presentation and uniform as well as non–uniform solvability of the word problem in a variety. In Section 1.3 we explore the relationship between the solvability of word problems for a variety and the decidability of the fragments of its elementary theory.

Chapters 3 and 4 contain the proof of the first main result of the dissertation. In Chapter 3 we show how to construct, given a universally pseudorecursive class of structures in a multisorted first–order language, a finitely axiomatized pseudorecursive equational class.

Chapter 4 gives the explicit axiomatization of such a multisorted class and investigates some relevant properties of its finitely generated members.

Section 5.1 presents a survey of the known finite basis results for varieties of algebras, while in Section 5.2 we turn our attention to a particular class of algebras having a height–one meet–semilattice as a reduct, and get acquainted with some of its basic features.

In Chapter 6, we include the second main result of the thesis, which gives the classification of finite graph M–algebras which are finitely based. This is accomplished by using shift automorphism methods, developed by K. Baker, G. McNulty, and H. Werner (Section 6.1) and the analysis of subdirectly irreducible members of a variety generated by a finite graph M–algebra which is not inherently finitely based (Section 6.2). To this end, we introduce the notion of definable ordered principal congruences, after which the dichotomy between finitely based and inherently nonfinitely based finite M-graph algebras is proven (Section 6.3).

Regarding the authorship of the material contained in this dissertation, we would like to mention the following. The material in Chapters 1, 2, and 5, comprises of known results due to other authors, while the results in Chapters 3, 4, and 6 are original in their entirety (to the best of our knowledge) and have not appeared previously in the literature. Chapters 3 and 4 essentially constitute the manuscript [12] and the material of Chapter 6 is contained in [11]

Chapter 1

Background

In this chapter we survey some of the basic notions in universal algebra and mathematical logic, to which we will refer freely in the rest of the dissertation.

Section 1.1. gives a brief introduction to universal algebra. The exposition in this section is taken mainly from [9] and [54]. Our primary reference for the subject is [9]. Another useful reference is [34]. Section 1.2. presents basic terminology of first-order logic, such as the notions of a language, structure, and satisfiability. There are several good references for this part, however, our preference is [17], accompanied by the Chapter V of [9]. In this section, we also introduce multisorted first-order logic, a natural extension of the first-order logic. Our exposition of multisorted logic is based on the Chapter 5 of [21].

Finally, in Section 1.3. we discuss the notion of a discriminator variety, state some of the basic structure results for this class of varieties, and cite a result of R. McKenzie, which relates the theory of the universal sentences of a discriminator variety to its equational theory.

1.1 A brief introduction to universal algebra

Given a nonempty set A, an **n-ary operation on A** is any function $f : A^n \to A$; n is said to be the **arity** of f. If $n = 0$, f is just an element of A. Thus, we can identify 0-ary operations with distinguished elements of A.

Definition 1.1 *A* ***language of algebras*** *$\mathcal{L}$ is a set of function symbols, so that each symbol $f \in \mathcal{L}$ is assigned a nonnegative integer n. This integer is called the* ***arity*** *of f, and f is said to be an n-ary function symbol.*

Definition 1.2 *Let $\mathcal{L}$ be a language of algebras. We define an* **algebra $\mathbf{A}$** *of language $\mathcal{L}$ (or, of* **type** *$\mathcal{L}$) to be an ordered pair $\langle A, F\rangle$ consisting of a nonempty set A and F, a family of operations on A which is indexed by $\mathcal{L}$, so that an n-ary operation $f^{\mathbf{A}}$ on $\mathbf{A}$ is corresponded to an n-ary function symbol $f \in \mathcal{L}$. We call A the* ***universe*** *of $\mathbf{A} = \langle A, F\rangle$, while the operations $f^{\mathbf{A}}$ are said to be the* ***fundamental operations*** *of $\mathbf{A}$.*

Among the algebras in the same language there are the natural notions (i.e. analogous to the notions for groups and rings) of *subalgebra, isomorphism, homomorphism, homomorphic image,* and *direct product.*

Definition 1.3 *Let $\mathbf{A}$ be an algebra. We say that a subset $B \subseteq A$ is a* ***subuniverse*** *of $\mathbf{A}$ if either B is the universe of some subalgebra $\mathbf{B} \leq \mathbf{A}$, or B is empty and the language contains no 0-ary function symbols. For each $X \subseteq A$, there is a smallest subuniverse of $\mathbf{A}$ containing X; it is called the subuniverse* ***generated*** *by X, and we denote it by $Sg^{\mathbf{A}}(X)$.*

If A is a nonempty set, the set of all equivalence relations on A, $Eqv(A)$, forms a lattice under inclusion. The operations of this lattice are the following: the *meet* of two relations is their intersection, while the *join* is the transitive closure of their union. The least and the greatest elements in this lattice are

$$\Delta_A = \{\langle a, a\rangle : a \in A\}$$

and

$$\nabla_A = A^2,$$

respectively.

Suppose $\theta \in Eqv(A)$ and $a, b \in A$. Then, we write $a \overset{\theta}{\equiv} b$ if $\langle a, b\rangle \in \theta$. The relation θ partitions the set A, and the θ-equivalence class containing a is denoted by a/θ. The set of all θ-equivalence classes of A is denoted by A/θ.

Definition 1.4 *Let* $\mathbf{A}$ *be an algebra and let* $\theta \in Eqv(A)$. *We say that* θ *is a* ***congruence*** *of* $\mathbf{A}$ *if, every fundamental operation* $f^{\mathbf{A}}$ *of arity* n *satisfies the following:*

$$a_1 \overset{\theta}{\equiv} b_1, \ldots, a_n \overset{\theta}{\equiv} b_n \textit{ implies } f^{\mathbf{A}}(a_1, \ldots, a_n) \overset{\theta}{\equiv} f^{\mathbf{A}}(b_1, \ldots, b_n).$$

In fact, it turns out that, as in the case of groups and rings, every congruence of an algebra is the kernel of some surjective homomorphism, and conversely.

The set of all congruences of a given algebra $\mathbf{A}$ forms a lattice under inclusion whose operations coincide with those for $Eqv(A)$. This lattice is called a *congruence lattice* of $\mathbf{A}$, and it will be denoted by $Con(\mathbf{A})$.

If θ is a congruence of $\mathbf{A}$, there is a natural way to define the *quotient algebra* $\mathbf{A}/\theta$: the universe of this algebra is A/θ, and for any function symbol f in the language of $\mathbf{A}$ and $a_1/\theta, \ldots, a_n/\theta \in A/\theta$,

$$f^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) = f(a_1, \ldots, a_n)/\theta.$$

Definition 1.5 *An algebra* $\mathbf{A}$ *is* ***congruence distributive*** *if its congruence lattice* $Con(\mathbf{A})$ *is a distributive lattice. Similarly,* $\mathbf{A}$ *is said to be* ***congruence modular*** *if its congruence lattice satisfies Dedekind's modular law:*

$$x \geq z \text{ implies } x \wedge (y \vee z) = (x \wedge y) \vee z.$$

We say that a class of algebras is congruence distributive (congruence modular) if every algebra in the class has the corresponding property.

We also mention another property of the congruence lattice of an algebra, which, strictly speaking, cannot be characterized via a lattice law.

Definition 1.6 *We say that an algebra* $\mathbf{A}$ *is* ***congruence permutable*** *if, for every pair of congruences* $\theta_1, \theta_2 \in Con(\mathbf{A})$

$$\theta_1 \vee \theta_2 = \theta_1 \circ \theta_2,$$

where $\circ$ *denotes the composition of binary relations.*

It is not difficult to show that every congruence permutable algebra is congruence modular as well.

1. We say that an algebra $\mathbf{A}$ is a **subdirect product** of the family of algebras $(\mathbf{A}_i : i \in I), I \neq \emptyset$, if

2. $\mathbf{A} \leq \Pi_{i \in I} \mathbf{A}_i$, and

3. For each coordinate map $\pi_i : \Pi_{i \in I} \mathbf{A}_i \to \mathbf{A}_i$, $\pi_i(\mathbf{A}) = \mathbf{A}_i$.

An embedding

$$\phi : \mathbf{A} \to \Pi_{i \in I} \mathbf{A}_i$$

is said to be *subdirect* if $\phi(\mathbf{A})$ is a subdirect product of $(\mathbf{A}_i : i \in I)$.

$\mathbf{A}$ is **subdirectly irreducible** if for every subdirect embedding

$$\alpha : \mathbf{A} \to \Pi_{i \in I} \mathbf{A}_i$$

there is an $i \in I$ such that

$$\pi_i \circ \alpha : \mathbf{A} \to \mathbf{A}_i$$

is an isomorphism. An equivalent condition is that, either $|A| = 1$, or the lattice $Con(\mathbf{A})$ contains a minimum congruence distinct from $\Delta_{\mathbf{A}}$.

It can be shown (see Thm. II.8.6 of [9]) that every algebra is a subdirect product of a collection of subdirectly irreducible algebras, which are homomorphic images of the original algebra.

If a variety V is such that every algebra in V is a subdirect product of finite subdirectly irreducible algebras, we say that V is **residually finite**.

An algebra $\mathbf{A}$ is said to be **simple** if

$$Con(\mathbf{A}) = \{\Delta_{\mathbf{A}}, \nabla_{\mathbf{A}}\}.$$

Clearly, every simple algebra is subdirectly irreducible.

Definition 1.7 *Let X be a set of distinct objects which will represent variables and $\mathcal{L}$ a language of algebras, whose set of 0-ary function symbols will be denoted by C. The set $T_{\mathcal{L}}(X)$ of **terms in the language $\mathcal{L}$ over X** is the smallest set such that*

1. $X \cup C \subseteq T_{\mathcal{L}}(X)$.
2. *If $t_1, \ldots, t_n \in T_{\mathcal{L}}(X)$ and $f \in \mathcal{L}$ is of arity n, then $f(t_1, \ldots, t_n) \in T_{\mathcal{L}}(X)$.*

The set $T_{\mathcal{L}}(X)$ can be turned into an algebra $\mathbf{T}_{\mathcal{L}}(X)$ in the language $\mathcal{L}$ in the following way: for every n-ary fundamental operation $f \in \mathcal{L}$ and $t_1, \ldots, t_n \in T_{\mathcal{L}}(X)$,

$$f^{\mathbf{T}_{\mathcal{L}}(X)}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n).$$

It is a standard practice to identify t with the pair $t(x_1, \ldots, x_n)$, where $(x_1, \ldots, x_n)$ is an ordered list of distinct elements from X. This list has to be such that it includes all the variables actually occurring in the term t. With this convention in mind, we see that a term can be identified with many different pairs of this kind, depending on the choice of variables in the list.

Every term $t(x_1, \ldots, x_n)$ in $\mathcal{L}$ defines an n-ary function $t^{\mathbf{A}}$ on the universe of any algebra $\mathbf{A}$ of type $\mathcal{L}$, as follows: let $a_1, \ldots, a_n \in A$;

1. If t is a variable x_i, then

$$t^{\mathbf{A}}(a_1, \ldots, a_n) = a_i.$$

2. If t is of the form $f(t_1(x_1, \ldots, x_n), \ldots, t_k(x_1, \ldots, x_n))$, where f is a k-ary function symbol in $\mathcal{L}$,

$$t^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{A}}\left(t_1^{\mathbf{A}}(a_1, \ldots, a_n), \ldots, t_k^{\mathbf{A}}(a_1, \ldots, a_n)\right).$$

Such a function $t^{\mathbf{A}}$ is called a *term operation* of $\mathbf{A}$.

It can be shown that if $\mathbf{A}$ is an algebra and $X \subseteq A$, then the subuniverse of $\mathbf{A}$ generated by X is

$$Sg^{\mathbf{A}}(X) = \{t^{\mathbf{A}}(a_1, \ldots, a_n) : t^{\mathbf{A}} \text{ is an } n\text{-ary term operation of } \mathbf{A} \text{ and}$$

$$\{a_1, \ldots, a_n\} \subseteq X\}.$$

If $t^{\mathbf{A}}$ is an $(n+k)$-ary term operation of $\mathbf{A}$ and $b_1, \ldots, b_k \in A$, the operation $p : A^n \to A$ defined by

$$p(a_1, \ldots, a_n) = t^{\mathbf{A}}(a_1, \ldots, a_n, b_1, \ldots, b_k)$$

is called a *polynomial operation* of $\mathbf{A}$. $Pol_1(\mathbf{A})$ will denote the set of all unary polynomials of $\mathbf{A}$.

If $\mathbf{A}$ is an algebra and $a, b \in A$, the intersection of all congruences of $\mathbf{A}$ which contain the pair $\langle a, b \rangle$ is itself a congruence, called a *principal congruence of* $\mathbf{A}$ *generated by* $\langle a, b \rangle$. We denote this congruence by $Cg^{\mathbf{A}}(a, b)$. In general, if $X \subseteq A^2$ then $Cg^{\mathbf{A}}(X)$ will denote the smallest congruence of $\mathbf{A}$ containing X. According to a theorem of Mal'cev, $Cg^{\mathbf{A}}(X)$ is the transitive closure of the following set

$$\{\langle p(z), p(u) \rangle : \ p \in Pol_1(\mathbf{A}) \text{ and } \langle z, u \rangle \in X \text{ or } \langle u, z \rangle \in X\},$$

if $X \neq \emptyset$, while for $X = \emptyset$,

$$Cg^{\mathbf{A}}(\emptyset) = \Delta_{\mathbf{A}}.$$

If K is a class of algebras in some algebraic language $\mathcal{L}$, $\mathbf{I}(K)$, $\mathbf{H}(K)$, $\mathbf{S}(K)$, and $\mathbf{P}(K)$ denote respectively the closure of K under isomorphisms, taking homomorphic images, taking subalgebras, and forming direct products of algebras in K. K is said to be a **variety** if it is nonempty and closed under the operators $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$. $\mathbf{V}(K)$ denotes the smallest variety containing K. If K is a finite set of finite algebras, $\mathbf{V}(K)$ is said to be *finitely generated.*

Definition 1.8 *We say that an algebra* $\mathbf{A}$ *is* ***locally finite*** *if for every finite* $X \subseteq A$ *the subuniverse* $Sg^{\mathbf{A}}(X)$ *is finite.*

A variety V *is locally finite if every member of* V *is locally finite.*

A celebrated theorem of G. Birkhoff states that varieties are precisely those classes of algebras which are defined by a set of *identities*,

Definition 1.9 *An **identity** in the language $\mathcal{L}$ over X is an expression of the form*

$$p = q,$$

where $p, q \in T_{\mathcal{L}}(X)$. An algebra $\mathbf{A}$ in $\mathcal{L}$ satisfies the identity $p = q$ if $p^{\mathbf{A}} = q^{\mathbf{A}}$. We write this as

$$\mathbf{A} \models p = q.$$

In other words,

$$\mathbf{A} \models p = q,$$

if and only if

$$p^{\mathbf{A}}(a_1, \ldots, a_n) = q^{\mathbf{A}}(a_1, \ldots, a_n),$$

for all $a_1, \ldots, a_n \in A$.

If Σ is a set of identities in $\mathcal{L}$ and $p = q$ an identity in $\mathcal{L}$, we write

$$\Sigma \models p = q$$

if, for every algebra $\mathbf{A}$ which satisfies every identity in Σ,

$$\mathbf{A} \models p = q.$$

An algebra $\mathbf{A}$ in a class of algebras K is said to satisfy the *universal mapping property* if there is a subset $X \subseteq A$ such that, for every $\mathbf{B} \in K$ and every map $\alpha : X \to B$, there is a unique homomorphism from $\mathbf{A}$ to $\mathbf{B}$ which extends α. Every variety contains algebras with the universal mapping property; these algebras are the *free algebras* of the variety.

If V is a nontrivial variety, there is a canonical way to produce the free algebra over a nonempty set of a designated cardinality $|X|$. This algebra will be a quotient of $\mathbf{T}_{\mathcal{L}}(X)$, and is usually denoted by $\mathbf{F}_V(\bar{X})$ (if $|X| = n$, where $n < \omega$, we also write $\mathbf{F}_V(n)$).

As mentioned before, a variety is a class of algebras defined by a set of identities. Given a variety V in $\mathcal{L}$, there is a syntactic way to describe those identities which are satisfied by every algebra $\mathbf{A} \in V$. This is the driving idea behind the notion of **equational logic**.

From this point on, until the end of this section, we assume that X is an infinite set of variables which is linearly ordered.

Definition 1.10 *Given a term $t \in T_{\mathcal{L}}(X)$, the subterms of t are defined by:*

1. *The term t is a subterm of itself.*

2. *If $f(t_1, \ldots, t_n)$ is a subterm of t and $f \in \mathcal{L}$ is n-ary then each t_i is a subterm of t.*

Definition 1.11 *A set of identities Σ over X is closed under replacement if, given $p = q$ in Σ and a term $r \in T_{\mathcal{L}}(X)$, if p occurs as a subterm of r, and s is the result of replacing that occurrence of p in r with q, then $r = s$ is also in Σ.*

Definition 1.12 *A set Σ of identities over X is closed under substitution, if for every $p = q$ in Σ and every $\alpha : X \to T_{\mathcal{L}}(X)$, if every occurrence of a fixed variable $x \in X$ in $p = q$ is replaced with $\alpha(x)$, the resulting identity is in Σ.*

Definition 1.13 *The* ***deductive closure*** *of a set Σ of identities is the smallest set $Ded(\Sigma)$ of identities in $\mathcal{L}$ which contains Σ such that*

1. $p = p \in Ded(\Sigma)$, *for every* $p \in T_{\mathcal{L}}(X)$.
2. *If* $p = q \in Ded(\Sigma)$ *then* $q = p \in Ded(\Sigma)$.
3. *If* $p = q, q = r \in Ded(\Sigma)$ *then* $p = r \in Ded(\Sigma)$.
4. $Ded(\Sigma)$ *is closed under replacement.*
5. $Ded(\Sigma)$ *is closed under substitution.*

If Σ is a set of identities over X, and $p = q$ is an identity over X, we write

$$\Sigma \vdash p = q$$

and say that $p = q$ is *provable* from Σ, if $p = q \in Ded(\Sigma)$.

The following theorem due to G. Birkhoff is the fundamental theorem of equational logic:

Theorem 1.14 (The Completeness Theorem for Equational Logic) *If Σ is a set of identities in $\mathcal{L}$ over X, and $p = q$ is an identity in $\mathcal{L}$ over X,*

$$\Sigma \models p = q \text{ if and only if } \Sigma \vdash p = q.$$

1.2 Structures, first-order logic, and multisorted first-order logic

Definition 1.15 *A* ***language*** *(or, a* ***signature****)* $\mathcal{L}$ *consists of a set* $\mathcal{R}$ *of relation symbols and a set* $\mathcal{F}$ *of function symbols. To each symbol of* $\mathcal{R} \cup \mathcal{F}$ *is assigned a nonnegative integer, the* ***arity*** *of the symbol. If* $\mathcal{R} = \emptyset$*, the language is said to be algebraic.*

Hence, the notion of a language for algebras as introduced in Section 1.1 is precisely a special case of a language.

The notion of a *structure in the language* $\mathcal{L}$ will be a generalization of that of an algebra:

Definition 1.16 *If* $\mathcal{L}$ *is a language, a* ***structure*** $\mathbf{A}$ ***in language*** $\mathcal{L}$ *is an ordered pair* $\mathbf{A} = \langle A, L \rangle$*, where* $A \neq \emptyset$ *and* $L = R \cup F$*, so that* R *is a family of fundamental relations* $r^{\mathbf{A}}$ *on* A *indexed by* $\mathcal{R}$*, and* F *is a family of fundamental operations* $f^{\mathbf{A}}$ *on* A *indexed by* $\mathcal{F}$*. The arity of* $r^{\mathbf{A}}$ *is equal to the one of the corresponding symbols* $r \in \mathcal{R}$ *and the same is true of every fundamental operation in* F*.*

The notions of isomorphism, homomorphic image, substructure, and direct product can be extended naturally to structures in the same language.

Given a language $\mathcal{L}$ and a set of variables X, the notion of a term is defined in the same way as for algebraic languages. We proceed to define the notion of a *first-order formula*:

Definition 1.17 *The* ***atomic formulas*** *in the language* $\mathcal{L}$ *are expressions of one of the following two forms:*

1. $p = q$, *where* p *and* q *are terms in* $\mathcal{L}$.

2. $r(t_1, \ldots, t_n)$, *where* r *is an* n*-ary relation symbol in* $\mathcal{R}$ *and* $t_1, \ldots, t_n$ *are terms in* $\mathcal{L}$.

Definition 1.18 *Let* X *be an infinite set of variables and* $\mathcal{L}$ *a language with equality. The set of* ***first-order formulas****, denoted by* $Form_{\mathcal{L}}(X)$*, is the smallest set of expressions which :*

1. *contains all atomic formulas in* $\mathcal{L}$;

2. *is closed under under negations* ($\neg$)*, conjunctions* ($\wedge$)*, disjunctions* ($\vee$)*, implications* ($\rightarrow$)*, and bi-implications* ($\leftrightarrow$)*;*

3. *is closed under universal* ($\forall x$) *and existential* ($\exists x$) *quantification for every variable* $x \in X$.

A **sentence** is a formula which has no *free* (i.e., unquantified) variables.

If $\mathbf{A}$ is a structure in the language $\mathcal{L}$, then any sentence ϕ in $\mathcal{L}$ is either true or false in $\mathbf{A}$; we write

$$\mathbf{A} \models \phi$$

to denote that ϕ is true in $\mathbf{A}$ (we also say that $\mathbf{A}$ satisfies ϕ). If ϕ is not necessarily a sentence, but a formula whose free variables are among $\{x_1, \ldots, x_n\}$, and $a_1, \ldots, a_n \in A$, then under the assignment $x_i \mapsto a_i$ the formula ϕ will be either true or false in $\mathbf{A}$. If under this assignment ϕ is true in $\mathbf{A}$, we write this as

$$\mathbf{A} \models \phi(a_1, \ldots, a_n).$$

A formula is said to be *universal* if it is of the form

$$\forall x_1 \forall x_2 \ldots \forall x_k \theta,$$

where θ contains no occurrences of quantifiers. The notion of an *existential* formula is defined analogously. Often, instead of $\forall x_1 \forall x_2 \ldots \forall x_k$ we write $\forall x_1 x_2 \ldots x_k$ or $\forall \bar{x}$, and similarly for blocks of existential quantifiers.

A *positive* formula is one that contains no occurrences of $\neg$, $\rightarrow$, or $\leftrightarrow$. A formula is said to be *positive primitive* if it is of the form

$$\exists x_1 \ldots x_k \; (\theta_1 \wedge \ldots \wedge \theta_m)$$

where each θ_i $(i = 1, \ldots m)$ is atomic.

Definition 1.19 *A* ***principal congruence formula*** *in the algebraic language $\mathcal{L}$ is a formula $\psi(x, y, z, u)$ of the form*

$$\exists w_1 \ldots w_k \left(\begin{array}{c} x = t_1 (v_1, w_1, \ldots, w_k) \wedge \\ \bigwedge_{1 \leq i < n} t_i (v_i', w_1, \ldots, w_k) = t_{i+1} (v_{i+1}, w_1, \ldots, w_k) \;\; \wedge \\ t_n (v_n', w_1, \ldots, w_k) = y \end{array} \right)$$

where $\{v_i, v_i'\} = \{z, u\}$ for $1 \leq i \leq n$, and t_j $(1 \leq j \leq n)$ are terms.

It is easy to show, using Malcev's characterization of congruence generation that, given an algebra $\mathbf{A}$ and $a, b, c, d \in A$,

$$\langle a, b \rangle \in Cg^{\mathbf{A}}(c, d) \text{ if and only if } \mathbf{A} \models \psi(a, b, c, d),$$

for some principal congruence formula ψ.

We say that a variety has **definable principal congruences** if there is a finite set Ψ of principal conguence formulas such that for every $\mathbf{A}$ in the variety and all $a, b, c, d \in A$,

$$\langle a, b\rangle \in Cg^{\mathbf{A}}(c, d) \text{ if and only if, for some } \psi \in \Psi,\ \mathbf{A} \models \psi(a, b, c, d).$$

A **universal Horn formula** is one of the form

$$\forall x_1 \ldots x_k\ (\phi_1 \wedge \ldots \wedge \phi_m)$$

where each ϕ_i $(i = 1, \ldots, m)$ is an atomic formula, or $\neg(\theta_1 \wedge \ldots \wedge \theta_r)$, or of the form $(\theta_1 \wedge \ldots \wedge \theta_r) \rightarrow \theta_{r+1}$, where every θ_i is atomic. It can be shown that the truth of universal Horn formulas is inherited by substructures and direct products of structures in the same language.

A special type of universal Horn formulas are **quasi-identities**. A quasi-identity is a universal Horn formula whose quantifier-free part is of the form

$$(\theta_1 \wedge \ldots \wedge \theta_r) \rightarrow \theta_{r+1},$$

where every θ_i is atomic, or, in the special case when the set $\{\theta_1, \ldots, \theta_r\}$ is empty, of the form θ_{r+1}, where θ_{r+1} is atomic.

For further information on first-order logic see [9] or [17].

The next concept we introduce is a generalization of a language. Namely, we allow objects represented by variables to be of different *sorts*:

Definition 1.20 *A* ***k*-sorted language** *$\mathcal{L}$ consists of:*

1. *k disjoint infinite sets $X_{\mathcal{L}}^{(1)}, \ldots, X_{\mathcal{L}}^{(k)}$, where $X_{\mathcal{L}}^{(i)}$ $(i = 1, \ldots, k)$ is the set of variables of the i-th sort,*

2. *the set $\mathcal{R}$ of relation symbols such that, to each n-ary $r \in \mathcal{R}$ an n-tuple $(i_1, \ldots, i_n)$ is associated so that $i_1, \ldots, i_n \in \{1, \ldots, k\}$,*

3. *the set $\mathcal{F}$ of operation symbols such that, to each n-ary $f \in \mathcal{F}$ an $(n+1)$-tuple $(i_1, \ldots, i_n, i)$ is associated so that $i_1, \ldots, i_n, i \in \{1, \ldots, k\}$.*

The notion of a term is defined as in the case of first-order logic.

Definition 1.21 *Let $t(x_1, \ldots, x_n)$ be a term in the k-sorted language $\mathcal{L}$.*

1. *If t is a variable $x_i \in X_{\mathcal{L}}^{(j)}$ $(1 \leq j \leq k)$, its sort is defined to be j.*

2. *If $t_1, \ldots .t_m$ are terms of sorts $i_1, \ldots, i_m$, respectively, and $f \in \mathcal{F}$ is an m-ary function symbol to which the tuple $(i_1, \ldots, i_m, i)$ is associated, then $f(t_1, \ldots, t_m)$ is defined to be of sort i.*

Since we treat function symbols of arity 0 as constants, this definition will still apply once a constant symbol is introduced as an abbreviation for a function which depends on no variables.

If $t(x_1, \ldots, x_n)$ is a term of sort i, along with the list of variables $x_1, \ldots, x_n$ which contains all the variables actually occurring in t, such that $x_j \in X_{\mathcal{L}}^{(i_j)}$ $(j = 1, \ldots, n)$, we define the *type* of $t(x_1, \ldots, x_n)$ to be

$$(i_1, \ldots, i_n, i).$$

Notice that the type of a term is not uniquely determined; it will depend on which variables are listed as the variables of t. However, in general, this will not present a problem since it will always be clear from the context what the variables occurring in t are.

The atomic formulas of $\mathcal{L}$ are of one of the following forms:

1. $r(t_1, \ldots, t_n)$, where $r \in \mathcal{R}$ is n-ary, of type $(i_1, \ldots, i_n)$ and each t_j is a term of the sort i_j.

2. $t_1 = t_2$, where t_1 and t_2 are two terms of the same sort.

The set of all formulas of $\mathcal{L}$ is now defined in the same way as for standard languages.

Definition 1.22 *Let $\mathcal{L}$ be a k-sorted language. A* ***k-sorted structure*** $\mathbf{A}$ *for $\mathcal{L}$ consists of:*

1. k nonempty sets $A^{(1)}, \ldots, A^{(k)}$; $A^{(i)}$ is said to be the universe of sort i of $\mathbf{A}$,

2. a family R of relations, indexed by $\mathcal{R}$, so that if $r \in \mathcal{R}$ is n-ary of type $(i_1, \ldots, i_n)$,
$$r^{\mathbf{A}} \subseteq A^{(i_1)} \times \cdots \times A^{(i_n)},$$

3. a family F of mappings, indexed by $\mathcal{F}$, so that if $f \in \mathcal{F}$ is n-ary of type $(i_1, \ldots, i_n, i)$,
$$f^{\mathbf{A}} : A^{(i_1)} \times \cdots \times A^{(i_n)} \to A^{(i)}.$$

The symbol $=$ is interpreted as the usual equality relation between elements on every $A^{(i)}$, $(i = 1, \ldots, k)$.

The notion of satisfiability of a sentence (formula) in a multisorted structure is defined as in the case of first-order logic.

For more on multisorted languages and structures, see e.g. [21].

1.3 Discriminator varieties

Definition 1.23 *The* ***discriminator function*** *on a set A is a ternary function $t : A^3 \to A$ defined by*

$$t(a,b,c) = \begin{cases} a, & \text{if } a \neq b, \\ c, & \text{if } a = b. \end{cases}$$

Given an algebra $\mathbf{A}$ in language $\mathcal{L}$, if there is a ternary term $t(x,y,z)$ in $\mathcal{L}$ which induces the discriminator function on the universe of $\mathbf{A}$, it is called a ***discriminator term*** *for $\mathbf{A}$.*

If K is a class of algebras with a common discriminator term $t(x,y,z)$, $\mathbf{V}(K)$ is said to be a discriminator variety. Such a variety is both congruence distributive and congruence permutable (see Thm IV.9.4 of [9]) and the class of its subdirectly irreducible members coincides with the class of simple algebras of $\mathbf{V}(K)$. Some examples of discriminator varieties include Boolean algebras, cylindric algebras of a given finite dimension, varieties of rings generated by finitely many finite fields, etc.

From this point on, we user the following notation: the class of subdirectly irreducible algebras in a variety V will be denoted by V_{SI}, while $V_{SI}(\mathbf{A})$ will denote the class of subdirectly irreducible members of the variety $V(\mathbf{A})$.

At this point, it is worth mentioning the following result which even though being more general, applies to discriminator varieties; it is the well-known theorem of Bjarni Jónsson which gives the description of the class of subdirectly irreducible members of a congruence distributive variety:

Theorem 1.24 *(B. Jónsson, [19]) Let $\mathbf{V}(K)$ be a congruence distributive variety. If $\mathbf{A}$ is a subdirectly irreducible algebra in $\mathbf{V}(K)$, then*

$$\mathbf{A} \in \mathbf{HSP}_U(K),$$

where $\mathbf{P}_U$ denotes the closure under taking ultraproducts of families of algebras in the class.

Using elementary properties of ultraproducts (see [17]), if K is a finite class of finite algebras, i.e. if $\mathbf{V}(K)$ is finitely generated, and $\mathbf{A}$ is a subdirectly irreducible algebra, it is easy to see that

$$\mathbf{A} \in \mathbf{HS}(K).$$

In fact, if $\mathbf{V}(K)$ is a discriminator variety, we can give a reasonably nice characterization of $\mathbf{V}_{SI}(K)$ using Jónsson's theorem ; namely, it can be checked that if $t(x,y,z)$ is a discriminator term for the class K, it will also be the discriminator term for $\mathbf{SP}_U(K)$. However, the presence of a discriminator term implies the simplicity of an algebra. Hence, every algebra in $\mathbf{SP}_U(K)$ will be a simple algebra, and the class $\mathbf{HSP}_U(K)$ will consist of $\mathbf{SP}_U(K)$ and the one-element algebra, if it is not already contained in $\mathbf{SP}_U(K)$.

Given a class of algebras K in the language $\mathcal{L}$, let $\mathcal{L}_t$ denote the language obtained by adding a new ternary function symbol t to $\mathcal{L}$. Also, given an algebra $\mathbf{A}$ in $\mathcal{L}$, we define $\mathbf{A}^t$ to be the algebra in $\mathcal{L}_t$, obtained by interpreting t as the discriminator function on A. The class K^t in $\mathcal{L}_t$ is defined to be

$$K^t = \left\{\mathbf{A}^t : \mathbf{A} \in K\right\}.$$

Clearly, the fact that t defines the discriminator function on every algebra in language $\mathcal{L}_t$ can be expressed by finitely many universal sentences in $\mathcal{L}_t$. Thus, if K is a class of algebras axiomatized by a set of universal sentences in $\mathcal{L}$, the class K^t will also be such. Therefore, $\mathbf{V}(K^t)$ will be a discriminator variety whose class of subdirectly irreducible algebras consists of K^t plus the one-element algebra. This provides us with a canonical way of generating discriminator varieties starting from classes of algebras axiomatized by a set of universal sentences.

Since quasi-identities are always true in the one-element algebra and their truth is preserved in subdirect products, the quasi-identities true in K^t and $\mathbf{V}(K^t)$ coincide. The reason for this is the fact that every algebra in a variety is either a one-element algebra or a subdirect product of nontrivial subdirectly irreducible members of the variety.

Next, we consider the connection between the universal theory and the equational theory of a discriminator variety more closely. It is straightforward to verify that, if $t(x, y, z)$ is a term in the algebraic language $\mathcal{L}$ which defines the discriminator function in every algebra of some class K, the following identities will be true in every algebra of K:

$$t(x,x,y)=y,\ t(x,y,x)=x,\ t(x,y,y)=x; \tag{1.1}$$

$$t(x,t(x,y,z),y)=y; \tag{1.2}$$

$$t(x,y,f(z_1,\ldots,z_n))=t(x,y,f(t(x,y,z_1),\ldots,t(x,y,z_n))) \tag{1.3}$$

where f is an n-ary fundamental operation in $\mathcal{L}$.

In [29], McKenzie proves the following theorem (Thm 1.3.):

Theorem 1.25 *If K is a class of algebras in some algebraic language $\mathcal{L}$, such that some ternary term $t(x,y,z)$ satisfies (1.1) , (1.2), (1.3) in K, the following are true:*

1. *A model $\mathbf{A}$ of (1.1) , (1.2), (1.3) is subdirectly irreducible if and only if $t^{\mathbf{A}}$ is the discriminator function on A.*

2. *Every finite model of (1.1) , (1.2), (1.3) is isomorphic to a direct product of subdirectly irreducible models of (1.1) , (1.2), (1.3).*

3. *Given any universal sentence ϕ in $\mathcal{L}$, one can effectively find an identity $p=q$ in $\mathcal{L}$, such that $\phi \leftrightarrow \forall\bar{x}(p(\bar{x})=q(\bar{x}))$ in every nontrivial subdirectly irreducible*

algebra in K.

Chapter 2

The word problem and its connections with decidability of fragments of elementary theory

In this chapter , we introduce the notion of the word problem for a finitely presented algebra in a variety. We give several examples of varieties of algebras which have solvable and unsolvable word problems with a discussion of several methods for proving that a variety has (un)solvable word problems. The notion of uniform solvability of word problems is introduced, and the result of Mekler, Nelson, and Shelah is quoted. The chapter ends with the discussion of the relationship between two different levels of solvability of word problems for a variety with the decidability of certain fragments of its elementary theory.

2.1 Word problems - definitions and examples

Let $\mathcal{L}$ be an algebraic language. We denote by G a set of new constant symbols such that $\mathcal{L} \cap G = \emptyset$, and let $\mathcal{L}_G$ denote the set $\mathcal{L} \cup G$.

In what follows, we abuse the notation slightly, and make no distinction between symbols in G and their interpretations in a structure.

Let $\mathbf{A}$ be an algebra in the language $\mathcal{L}$ and $G \subseteq A$. $\mathbf{A}_G$ will stand for the following expansion of $\mathbf{A}$:

$$\langle \mathbf{A}, \{x : x \in G\} \rangle .$$

We say that an identity in a language is **ground**, if it contains no variables.

Definition 2.1 *If R is a set of ground identities in $\mathcal{L}_G$, the ordered pair*

$$\langle G, R \rangle$$

is said to be a ***presentation*** *relative to $\mathcal{L}_G$.*

Definition 2.2 *Let Σ be a set of identities in the language $\mathcal{L}$, V the variety defined by Σ, and $\langle G, R \rangle$ a presentation relative to $\mathcal{L}_G$. If $\mathbf{A}$ is an algebra in $\mathcal{L}$, we say that it is* ***given by the presentation*** *(or,* ***presented by****) $\langle G, R \rangle$* ***relative to*** *V, if the following holds:*

1. *$\mathbf{A}$ is generated by G;*

2. $\mathbf{A} \models \Sigma \cup R$;

3. For every ground identity $s = t$ in $\mathcal{L}_G$,

$$\mathbf{A}_G \models s = t \textit{ if and only if } \Sigma \cup R \models s = t.$$

If there exist finite sets G and R such that $\mathbf{A}$ is given by $\langle G, R\rangle$ relative to V, $\mathbf{A}$ is said to be ***finitely presented*** *relative to V.*

Notice that, in light of the completeness theorem for equational logic (Theorem 1.14), if Σ axiomatizes V and $\mathbf{A}$ is finitely presented relative to V, the following holds for every ground identity $s = t$ in $\mathcal{L}_G$,

$$\mathbf{A}_G \models s = t \text{ if and only if } \Sigma \cup R \vdash s = t.$$

An immediate corollary of Definition 2.2 is that the algebra presented by $\langle G, R\rangle$ relative to V is unique, up to isomorphism.

Definition 2.3 *Let Σ be a recursive set of identities in the language $\mathcal{L}$, V the variety defined by Σ, and $\mathbf{A}$ the algebra given by a finite presentation $\langle G, R\rangle$ relative to V. The* ***word problem*** *for $\langle G, R\rangle$ relative to V asks whether there exists an algorithm which decides, given as an input a ground identity $s = t$ in $\mathcal{L}_G$, whether*

$$\mathbf{A}_G \models s = t$$

or not. If such an algorithm exists, we say that $\langle G, R\rangle$ has ***solvable word problem*** *(relative to V); otherwise, the word problem for $\mathbf{A}$ is said to be unsolvable.*

A nonobvious, though an easy consequence of this definition is the following:

Proposition 2.4 *If $\langle G_1, R_1\rangle$ and $\langle G_2, R_2\rangle$ are two finite presentations of the same algebra* $\mathbf{A}$ *relative to the variety V then the word problem for $\langle G_1, R_1\rangle$ has a solvable word problem if and only if*

$\langle G_2, R_2\rangle$ has a solvable word problem.

With this in view, we can relax the language and say that the word problem for an algebra is solvable when the word problem for one of its finite presentations is.

The study of word problems for classes of algebras was given impetus by the early works of Markov ([27]) and Post ([45]). They showed, independently from each other, that there exists a finite semigroup presentation with unsolvable word problem. Both constructions utilize Turing machine with undecidable Halting Problem into a finite semigroup presentation.

The following table gives a brief summary of some results showing the unsolvability of word problems in certain more familiar classes (varieties) of algebras:

Variety	Has an undecidable Word Problem
semigroups	Post ([45]), Markov ([27])
groups	Boone ([5]), Novikov ([41])
rings	follows from the results of Markov and Post ([27], [45])
modular lattices	Hutchinson ([18])
relation algebras	Tarski ([49])

Table 2.1: Varieties with unsolvable word problems

Each of the papers in the table provides a specific finite presentation relative to the class in question, for which the word problem is unsolvable.

The celebrated result of Boone and Higman states that a finitely presented group has solvable word problem if and only if it can be embedded in a finitely generated simple group. The following analogue of their result was obtained by Evans in [14]:

Theorem 2.5 *Let V be a recursively axiomatized variety in a finite language defined by a recursive set of identities in that language. A finitely presented algebra* $\mathbf{A}$ *in V has a solvable word problem if and only if it can be embedded in a finitely generated simple algebra.*

Definition 2.6 *Let V be a variety in the language $\mathcal{L}$. V is said to have* ***solvable word problems*** *if every finitely presented algebra in V has solvable word problem.*

If there exists a single algorithm which for every finite presentation $\langle G, R\rangle$, relative to V, and every ground identity $s = t$ in the language $\mathcal{L}_G$ decides whether

$$\mathbf{A}_G \models s = t,$$

we say that V has ***uniformly solvable word problems.***

Obviously, if a variety does not have solvable word problems, the word problems of the variety cannot be uniformly solvable.

On the other hand, if a variety has solvable word problems, it is usually so because the word problems are uniformly solvable. For example in the cases of Abelian groups and commutative semigroups, the question of the existence of an algorithm which solves the word problem for every finitely presented algebra in the variety reduces to the question of the existence of an algorithm for finding solutions to finite systems of linear equations over $\mathbb{Z}$.

The first intricate result concerning uniform solvability of word problems was obtained for the variety of lattices. This result was implicitly stated in [36], while the rudiments of the proof can be traced back to [48].

Theorem 2.7 *(Skolem [48], McKinsey [36]) The variety of lattices has uniformly solvable word problems.*

The essential ingredient underlying the proof of this theorem is the following fact, isolated and stated for the first time in a paper by A. Mal'cev ([26]); it also appears in Evans ([13]).

Theorem 2.8 *Let V be a finitely axiomatized variety in a finite language. If every finitely presented algebra of V is residually finite, then V has uniformly solvable word problems.*

In particular, every finitely based variety in a finite language which is residually finite has uniformly solvable word problems.

This theorem applies to a variety of cases. In particular, it applies to every finitely generated congruence distributive variety (see [19]):

Corollary 2.9 *Every finitely generated congruence distributive variety in a finite language has uniformly solvable word problems. In particular, the varieties of Boolean algebras and distributive lattices as well as every finitely generated discriminator variety have uniformly solvable word problems.*

In his paper [13], Evans gives certain criteria for the uniform solvability of word problems in a variety. One of them is, for instance, the *finite embeddability property* for partial algebras relative to the variety. This notion is not related to the remainder of our investigation, and the relevant notions will not be defined here. An interested reader is referred to the original article of Evans or [8] for more information. In fact, it turns out that the finite embeddability property for a variety is equivalent to every finite algebra in the variety being residually finite.

As consequences of the results mentioned in the preceding paragraph, Evans derives the uniform solvability of word problems for the varieties of lattices, loops, quasigroups, groupoids, as well as every variety of groups generated by a finite nilpotent group.

We still need to address the question whether every variety with solvable word problems is such that its word problems are uniformly solvable. It might appear that this is indeed true, in light of the results mentioned so far. However, the negative answer to this question was provided by a deep paper due to A. Mekler, E. Nelson,and S. Shelah ([39]). In this paper, the authors use an intricate way of interpreting a version of a Turing machine with an undecidable Halting Problem into an equational theory in a finite language, to prove the following:

Theorem 2.10 *There exists a finitely axiomatized equational theory in a finite language, such that its word problems are solvable, yet there is no uniform algorithm for the word problems of the variety.*

2.2 Interdependency between solvability of word problems and decidability of fragments of the elementary theory

In this section, we investigate the connections between the (uniform) solvability of word problems of a variety and the decidability of certain fragments of the elementary theory of the variety.

The particular relationships we will be mostly interested in are those that exist between the solvability of word problems and the decidability of equational and quasi-equational theories.

Definition 2.11 *Let V be a variety of algebras in the language $\mathcal{L}$. We denote with $Th_Q(V)$ the set of all quasi-identities true in every algebra of V.*

The following theorem is stated (without a proof) in [26]:

Theorem 2.12 *Let Σ be a set of identities in $\mathcal{L}$, and V the variety defined by Σ. Then, V has uniformly solvable word problems if and only if $Th_Q(V)$ is decidable.*

PROOF. Suppose $\mathbf{A} \in V$ is given by a finite presentation $\langle G, R\rangle$ and $s = t$ is a ground identity in $\mathcal{L}_G$. Then,

$$\Sigma \cup R \vdash s = t \text{ if and only if } \Sigma \vdash \wedge R \rightarrow s = t.$$

By the Lemma on constants for deductions in first-order theories (see Lemma 2.3.2 in [17]), after replacing every element of $G = \{g_1, \dots, g_n\}$ with x_i $(1 \leq i \leq n)$, we have that the second entailment displayed above is equivalent to

$$\Sigma \vdash \forall x_1 \ldots x_n \left(\wedge R(\bar{x}) \rightarrow s(\bar{x}) = t(\bar{x}) \right).$$

Thus, the existence of an algorithm which decides whether a given quasi-identity is in $Th_Q(V)$ or not, is equivalent to the existence of an algorithm which solves the word problem for every finite presentation $\langle G, R \rangle$ relative to V. $\square$

Definition 2.13 *Let V be a variety of algebras in the language $\mathcal{L}$. We denote with $Th_{Q,n}(V)$ the set of all quasi-identities with at most n variables which are true in every algebra of V.*

From the proof of Theorem 2.12, it is easy to extract the following corollary:

Corollary 2.14 *If V is a variety in a finite language, then, if $Th_{Q,n}(V)$ is decidable for every $n < \omega$, V has solvable word problems.*

Finally, we would like to state a few facts about the relationship between word problems and the decidability of the equational theory of a variety.

Definition 2.15 *Let V be a variety in the language $\mathcal{L}$. The equational theory of V, $Th_{Eq}(V)$, is the set of all identities $s = t$ in $\mathcal{L}$, which are true in every algebra from V.*

With $Th_{Eq,n}(V)$, we denote the set of all identities in at most n variables true in V.

Clearly, the recursiveness of $Th_{Eq}(V)$ will imply the recursiveness of $Th_{Eq,n}(V)$, for every $n < \omega$. The converse, however, fails. The first known example of such a variety was given by a student of Tarski's, B. Wells, in his doctoral dissertation [52].

Theorem 2.16 *(Wells [52]) There exists a finitely axiomatized equational theory in a finite language such that the equational theory of V is undecidable, yet, for every $n < \omega$, $Th_{Eq,n}(V)$ is decidable.*

Following the terminology proposed by Tarski, we call the varieties with the properties of Theorem 2.16 **pseudorecursive**.

Since $\mathbf{F}_V(n)$ is given by the finite presentation of the form $\langle \{x_1, \ldots, x_n\}, \emptyset \rangle$, where all x_i are pairwise distinct, the solvability of word problems of V will imply the decidability of $Th_{Eq,n}(V)$, for every $n < \omega$.

We can convince ourselves that the converse is false on the example of the variety of all groups. It has an undecidable word problem, while its equational theory is decidable (Dehn [10]).

In [37], G. McNulty stated the following problem: Is there a finitely axiomatized equational theory in a finite language whose word problems are solvable and whose equational theory is undecidable?

The existence of such a variety would imply Theorems 2.10 and 2.16; the first theorem would follow since the undecidability of the equational theory will yield the undecidability of the universal Horn theory, after which Theorem 2.12 can be used; the result of Wells would follow by the remarks following the statement of Theorem 2.16.

In Chapters 3 and 4, we provide the affirmative answer to this problem, by constructing a finitely axiomatized variety in a finite language with required properties, starting from a class of structures in a multisorted language.

Chapter 3

From multisorted structures to pseudorecursive varieties

In Section 2.2, we defined a variety V to be pseudorecursive if its equational theory is undecidable, yet, for every $n < \omega$, the theory consisting of equations involving at most n variables which are true in V is decidable.

The existence of finitely based pseudorecursive varieties was first established in [52], using a suitable encoding of computations of a Turing machine with an undecidable Halting problem in the equational theory of a variety.

In the following two chapters we provide a positive answer to the problem stated at the end of the previous chapter, i.e., we construct a finitely based variety V whose equational theory is undecidable, yet whose word problem is solvable. This will subsume the main results of [39] and [52], since:

1° The theory of universal Horn sentences of V will be undecidable, which is, according to a result of Mal'cev ([26]), equivalent to the non-existence of a uniform algorithm which would solve the word problem for any finite presentation relative to V. Hence, the uniform word problem for V will be unsolvable, and the main result of [39] follows.

2° On the other hand, since in V, for every $n < \omega$, the n- generated free algebra in V is finitely presented and the solvability of the word problem for this n-generated free algebra in V is equivalent to the decidability of the n-variable equational theory of V, V will be pseudorecursive.

Instead of trying to encode the undecidability of the Halting problem directly into the equations defining V, our approach will be to use a class of structures, defined in some multisorted signature (including relations), and then to translate different (un)decidability properties of that class into the corresponding properties of a variety in a 1-sorted language without relation symbols.

This construction will be driven by the following version of the Halting Problem: there is a Turing machine T such that there is no uniform algorithm to decide for which initial configurations T eventually halts; on the other hand, for each n, the set of initial configurations of length at most n from which T halts is decidable. This nonuniformity for the Halting Problem is essentially what makes things work out nicely. For more on the theory of computability and recursive functions, see [6].

One of the main tools throughout the following few chapters will be multisorted logic (with the identity) and we shall freely make use of standard notions and results of it. An introduction to the basic model theory of multisorted structures was given in Section 1.2 and the reader is invited to consult it for the sake of a further reference.

In order to make the exposition more readable, we have decided to deviate from the standard practice of designating different sets of letters for the variables of different sorts. Instead, we typically use the same letters (x,y,z, ...) for variables of all sorts. Since we are almost exclusively interested in sentences, we indicate the sort of a variable by restricting the range of the quantifier in question. For example,

$$(\forall x \in S_i)\phi(x)$$

means: "for every x of sort S_i, $\phi(x)$", while

$$(\exists x \in S_i)\phi(x)$$

should be interpreted as: "there exists x of sort S_i, such that $\phi(x)$ ".

Where a multisorted formula is not a sentence, we indicate explicitly the sort of a variable, as necessary.

We abuse notation by writing

$$\rho \subseteq S_{i_1} \times \ldots \times S_{i_m}$$

to mean that ρ is a relation symbol whose associated tuple is $(i_1, \ldots, i_m)$. Similarly, we write

$$f : S_{i_1} \times \ldots \times S_{i_k} \to S_j$$

to mean that f is a function symbol whose associated tuple is $(i_1, \ldots, i_k, j)$.

A brief discussion on the theory of multisorted varieties can be found in either [50] or [35]. Actually, the techniques developed in Section 4 of this chapter were inspired by the first part of Chapter 11 of [35], where the authors describe the categorical equivalence between the varieties of multisorted algebras and the varieties in 1–sorted algebraic signatures.

The most important tool in our proof is McKenzie's reduction of first order logic to equational logic using discriminator varieties, which was mentioned in Theorem 1.25.

The most exhaustive reference on algorithmic problems for varieties of algebras is [20], where further information can be found.

Next, we adopt several notational conventions and give some definitions that will be used in what follows.

If a is an n-tuple from, say, $A_1 \times A_2 \times \cdots \times A_n$, where $A_1, \ldots, A_n$ are any sets, the i-th component of a will sometimes be referred to as

$$a^i \in A_i,$$

or as

$$a[i] \in A_i,$$

depending on which form will be more convenient in a given context.

By the universal theory of a class $\mathcal{K}$ in a (multisorted) signature $\mathcal{L}$, we mean the theory of all universal sentences, in prenex form, true in every structure in $\mathcal{K}$, and it will be denoted by

$$Th_\forall(\mathcal{K}).$$

If $\mathcal{K}$ is a (multisorted) class of structures,

$$Th_{\forall,n}(\mathcal{K})$$

will denote the fragment of the universal theory of $\mathcal{K}$, consisting of the sentences from $Th_\forall(\mathcal{K})$ containing at most n variables of each sort.

Given a (multisorted) class $\mathcal{K}$ of algebras,

$$Th_{Eq,n}(\mathcal{K})$$

will stand for the set of equations in the signature of $\mathcal{K}$, having at most n variables of each sort which are true in every structure in $\mathcal{K}$.

We say that the (multisorted) class $\mathcal{K}$ is **universally pseudorecursive**, if $Th_\forall(\mathcal{K})$ is undecidable, while $Th_{\forall,n}(\mathcal{K})$ is decidable, for every $n < \omega$.

Sections 3.1 to 3.5 constitute the proof of the following theorem.

Theorem 3.1 *Given a universally pseudorecursive class $\mathcal{K}$ of multisorted structures axiomatized by a finite set Φ of universal sentences in a multisorted language $\mathcal{L}$, one can construct a finitely based pseudorecursive discriminator variety $\mathcal{V}$ whose word problems are solvable.*

3.1 From a multisorted universal class to a pseudorecursive variety

The idea of the proof of Theorem 3.1 is the following:

(1) *Eliminate the relation symbols.* Given a universally pseudorecursive class $\mathcal{K}$ of multisorted structures in a language $\mathcal{L}$, axiomatized by a finite set Φ of universal sentences, construct a universally pseudorecursive multisorted class $\mathcal{K}^*$, in a language $\mathcal{L}^*$, which does not contain relation symbols, and which is axiomatized by a finite set of universal sentences Ψ in $\mathcal{L}^*$.

(2) *Eliminate the constant symbols.* Given a universally pseudorecursive class $\mathcal{K}$ of multisorted structures in a language $\mathcal{L}$ which does not contain relation symbols, and which is axiomatized by a finite set Φ of universal sentences in $\mathcal{L}$, construct a universally pseudorecursive multisorted class $\mathcal{K}^*$ in a language $\mathcal{L}^*$, which does not contain either relation or constant symbols, and which is axiomatized by a finite set Ψ of universal sentences in $\mathcal{L}^*$.

(3) *Reduce to a single sort.* Given a universally pseudorecursive class $\mathcal{K}$ of multisorted structures in a language $\mathcal{L}$ not containing any relation or constant symbols and axiomatized by a finite set Ψ of universal sentences in $\mathcal{L}$, construct a universally pseudorecursive class of structures $\mathcal{K}^*$ in an ordinary 1–sorted algebraic language $\mathcal{L}^*$, such that $\mathcal{K}^*$ is axiomatized by a finite set of universal sentences.

(4) *Reduce to an equational class.* Given a universally pseudorecursive class of algebras $\mathcal{K}$ axiomatized by a finite set of universal sentences in $\mathcal{L}$, the associated discriminator variety denoted $\mathbf{V}(\mathcal{K}^t)$ is a finitely based pseudorecursive variety in the language $\mathcal{L}^t$, whose word problems are solvable.

The general method that will be used in all the steps (1)–(3) will depend on the following definition:

Definition 3.2 *Let $\mathcal{L}$ and $\mathcal{L}^*$ be two multisorted languages, and Σ a finite set of universal sentences in $\mathcal{L}^*$. The system of transformations $\langle *, \Sigma \rangle$ is called a Σ-* ***equivalence system*** *if it satisfies the following conditions:*

1. *For any structure $\mathbf{A}$ of signature $\mathcal{L}$, $\mathbf{A}^*$ is a structure of signature $\mathcal{L}^*$, which satisfies Σ. Conversely, for any structure $\mathbf{B}$ of signature $\mathcal{L}^*$, satisfying Σ, $\mathbf{B}_*$ is a structure of signature $\mathcal{L}$, so that*

$$(\mathbf{A}^*)_* \cong \mathbf{A},$$

and

$$(\mathbf{B}_*)^* \cong \mathbf{B}.$$

2. *For any sentence ϕ in $\mathcal{L}$, ϕ^* is a sentence in $\mathcal{L}^*$ constructed effectively from ϕ, and, conversely, for any sentence ψ in $\mathcal{L}^*$, ψ_* is a sentence constructed effectively from ψ in $\mathcal{L}$, so that, for any structure $\mathbf{A}$ of signature $\mathcal{L}$, any structure $\mathbf{B}$ of signature $\mathcal{L}^*$ satisfying Σ, and ϕ, ψ sentences in $\mathcal{L}, \mathcal{L}^*$, respectively, we*

have the following:

$$\mathbf{A} \models \phi \qquad \textit{if and only if} \qquad \mathbf{A}^* \models \phi^*,$$

and

$$\mathbf{B} \models \psi \qquad \textit{if and only if} \qquad \mathbf{B}_* \models \psi_*.$$

Moreover, these two transformations are such that they carry universal sentences into universal sentences.

Then, $\mathcal{K}^*$ is defined to be the class

$$\mathcal{K}^* = \mathbf{I}\{\mathbf{A}^* : \mathbf{A} \in \mathcal{K}\},$$

where $\mathbf{I}$ denotes the closure of the class under isomorphisms.

Definition 3.3 *Suppose that the language $\mathcal{L}$ is k–sorted, with the sorts $S_1, \ldots, S_k$. The transformation $\psi \to \psi_*$, which, given a universal sentence ψ in $\mathcal{L}^*$, associates to it a universal sentence ψ_* in $\mathcal{L}$ is said to be* **variable bounded** *if and only if there are functions $\theta_1, \ldots, \theta_k$ on the natural numbers such that for every formula ψ, if ψ has at most n variables of each sort, then ψ_* has at most $\theta_i(n)$ variables of sort S_i, for each $i = 1, \ldots, k$.*

If a Σ-equivalence system $\langle *, \Sigma \rangle$ is such that its transformation function on the sentences from $\mathcal{L}$ into $\mathcal{L}^*$ is variable-bounded, than $\langle *, \Sigma \rangle$ will be said to be variable-bounded, as well.

In that case, the following theorem will be true:

Theorem 3.4 *If $\mathcal{K}$ is a universally pseudorecursive class axiomatized by a set Φ of universal sentences, and $\langle *, \Sigma \rangle$ is a variable-bounded Σ-equivalence system, then $\mathcal{K}^*$ is a universally pseudorecursive class axiomatized by $\Phi^* \cup \Sigma$, where*

$$\Phi^* = \{\phi^* : \phi \in \Phi\}.$$

PROOF. It is quite a straightforward exercise to show that $\mathcal{K}^*$ is axiomatized by $\Phi^* \cup \Sigma$, and that $Th_\forall(\mathcal{K}^*)$ is undecidable.

Let ψ be a universal sentence in $\mathcal{L}^*$, which contains at most n variables of each sort, and let $\theta_1, \ldots, \theta_k$ be the functions which witness variable-boundedness. Consider

$$m = \max\{\theta_1(n), \theta_2(n), \ldots, \theta_k(n)\}.$$

Then,

$$\psi \in Th_{\forall,n}(\mathcal{K}^*) \qquad \text{if and only if} \qquad \psi_* \in Th_{\forall,m}(\mathcal{K}).$$

Since $Th_{\forall,m}(\mathcal{K})$ is decidable, so is $Th_{\forall,n}(\mathcal{K}^*)$. □

This concludes our preliminary discussion on how the steps (1)–(3) will be carried out. In the sequel, we shall refer to these results and confine ourselves to defining $\mathcal{L}^*$, Σ and the corresponding transformations for each particular case.

3.2 Getting rid of relations

Suppose $\mathcal{L}$ is a k–sorted language with sorts $S_1, \ldots, S_k$. We define $\mathcal{L}^*$ to be a $(k+1)$–sorted language whose sorts are those of $\mathcal{L}$, plus a new sort S_{k+1}, which is the sort of two new constant symbols 0 and 1.

Also, $\mathcal{L}^*$ retains all function and constant symbols from $\mathcal{L}$, but the relation symbols of $\mathcal{L}$ are replaced by function symbols intended to denote their characteristic functions, in the following way: if ρ is an m-ary relation symbol of $\mathcal{L}$,

$$\rho \subseteq S_{i_1} \times \cdots \times S_{i_m},$$

then ρ is replaced by an m-ary function symbol R in $\mathcal{L}^*$, so that

$$R : S_{i_1} \times \cdots \times S_{i_m} \mapsto S_{k+1}.$$

Let Σ consist of the following universal sentences:

$$(\forall x \in S_{k+1})(x = 0 \vee x = 1),$$

$$0 \neq 1.$$

Now, given a structure $\mathbf{A}$ of signature $\mathcal{L}$, where

$$\mathbf{A} = (S_1^{\mathbf{A}}, \ldots, S_k^{\mathbf{A}}),$$

define $\mathbf{A}^*$ to be the structure of signature $\mathcal{L}^*$, whose universe is

$$A^* = (S_1^A, \ldots, S_k^A, \{0, 1\}),$$

and if ρ is an m–ary relation symbol in $\mathcal{L}$, such that

$$\rho \subseteq S_{i_1} \times \cdots \times S_{i_m},$$

define

$$R^{\mathbf{A}^*} : S^A_{i_1} \times \cdots \times S^A_{i_m} \mapsto \{0,1\}$$

in the following way: if $a_j \in S^A_{i_j}$, $(1 \leqslant j \leqslant m)$,

$$R^{\mathbf{A}^*}(a_1,\dots,a_m) = 1 \qquad \text{if and only if} \qquad \rho^{\mathbf{A}}(a_1,\dots,a_m).$$

Conversely, if $\mathbf{B}$, where

$$\mathbf{B} = (S^{\mathbf{B}}_1,\dots,S^{\mathbf{B}}_k,\{0^{\mathbf{B}},1^{\mathbf{B}}\}),$$

is any structure of signature $\mathcal{L}^*$, which satisfies Σ, define $\mathbf{B}_*$ to be the structure whose universe is

$$B = (S^B_1,\dots,S^B_k),$$

and, if

$$R^{\mathbf{B}} : S_{i_1} \times \cdots \times S_{i_m} \mapsto \{0,1\},$$

then, for the corresponding ρ in $\mathcal{L}$, define, for $b_j \in S^{\mathbf{B}}_{i_j}$ $(1 \leqslant j \leqslant m)$,

$$\rho^{\mathbf{B}_*}(b_1,\dots,b_m) \qquad \text{if and only if} \qquad R^{\mathbf{B}}(b_1,\dots,b_m) = 1^{\mathbf{B}}.$$

It is easily seen that, given any structure $\mathbf{A}$ of signature $\mathcal{L}$, and any structure

$\mathbf{B}$ of signature $\mathcal{L}^*$, which satisfies Σ,

$$\begin{aligned} \mathbf{A} &\cong (\mathbf{A}^*)_*, \\ \mathbf{B} &\cong (\mathbf{B}_*)^*. \end{aligned}$$

Next, we define the corresponding transformations on the sentences in $\mathcal{L}$ and $\mathcal{L}^*$.

Let ϕ be a sentence in $\mathcal{L}$. Replace every atomic subformula of ϕ of the form

$$\rho(t_1(\bar{x}), \dots , t_m(\bar{x})),$$

where

$$\rho \subseteq S_{i_1} \times \cdots \times S_{i_m},$$

is a relation and $t_j(\bar{x})$ is a term of the sort S_{i_j}, for $1 \leqslant j \leqslant m$, by

$$R(t_1(\bar{x}), \dots , t_m(\bar{x})) = 1,$$

where R is the corresponding function symbol in $\mathcal{L}^*$. Denote the formula in $\mathcal{L}^*$, obtained from ϕ in this way, by ϕ^*.

Proposition 3.5 *Let* $\mathbf{A}$ *be a structure of signature* $\mathcal{L}$ *and* ϕ *a sentence in* $\mathcal{L}$. *Then,*

$$\mathbf{A} \models \phi \qquad \textit{if and only if} \qquad \mathbf{A}^* \models \phi^*.$$

PROOF. If $\psi(\bar{x})$ is an atomic formula in $\mathcal{L}$, and $\bar{a}$ is a tuple of elements in A of the appropriate sorts, it is easily seen that

$$\mathbf{A} \models \psi(\bar{a}) \qquad \text{if and only if} \qquad \mathbf{A}^* \models \phi^*(\bar{a}).$$

The proof can now be extended using induction on the complexity of ϕ. □

Now, suppose ψ is a sentence in $\mathcal{L}^*$, which can be assumed to be in prenex normal form. If ψ is of the form

$$Q\bar{x}(\forall z \in S_{k+1})\vartheta(\bar{x}, z),$$

where $Q\bar{x}$ is a quantifier prefix in ψ, this formula will be equivalent, modulo Σ, to

$$Q\bar{x}(\vartheta(\bar{x}, 0) \wedge \vartheta(\bar{x}, 1)); \tag{3.1}$$

while, if ψ is of the form

$$Q\bar{x}(\exists z \in S_{k+1})\vartheta(\bar{x}, z),$$

it will be equivalent to

$$Q\bar{x}(\vartheta(\bar{x}, 0) \vee \vartheta(\bar{x}, 1)). \tag{3.2}$$

Hence, after putting each of (3.1), (3.2) into prenex normal form, we may assume that ψ does not contain variables of sort S_{k+1}.

Thus, the atomic subformulas of ψ which involve terms of the sort S_{k+1} are of one of the following forms;

(1) $R(t_1(\bar{x}), \dots, t_m(\bar{x})) = 1$,

where

$$R : S_{i_1} \times \cdots \times S_{i_m} \to S_{k+1},$$

and every $t_j(\bar{x})$ is a term of the sort S_{i_j}.

(2) $R(t_1(\bar{x}), \dots, t_m(\bar{x})) = 0$,

where

$$R : S_{i_1} \times \cdots \times S_{i_m} \to S_{k+1},$$

and every $t_j(\bar{x})$ is a term of the sort S_{i_j}.

(3) $R_1(t_1(\bar{x}), \dots, t_m(\bar{x})) = R_2(u_1(\bar{x}), \dots, u_n(\bar{x}))$.

(4) $0 = 0$.

(5) $1 = 1$.

(6) $0 = 1$.

If an atomic subformula of ψ is of one of the forms (1) or (2), we replace it by

$$\rho(t_1(\bar{x}), \dots, t_m(\bar{x})),$$

or

$$\neg\rho(t_1(\bar{x}), \dots, t_m(\bar{x})),$$

respectively, where ρ is the relation symbol in $\mathcal{L}$ corresponding to R.

When an atomic subformula of ψ is of the form (3), we first replace it by

$$(R_1(t_1(\bar{x}),\dots,t_m(\bar{x}))=0 \ \wedge \ R_2(u_1(\bar{x}),\dots,u_n(\bar{x}))=0)$$
$$\vee(R_1(t_1(\bar{x}),\dots,t_m(\bar{x}))=1 \ \wedge \ R_2(u_1(\bar{x}),\dots,u_n(\bar{x}))=1),$$

and then handle this formula as in the previous case.

Now, if the sentence ψ contains at least one variable, say x of sort S_j, where j can be assumed not to be equal to $k+1$, according to our previous remark, and ψ contains an atomic subformula of the form $0=0$ or $1=1$, replace that subformula with $x=x$. If ψ contains an atomic subformula $0=1$, replace it with $x\neq x$. Finally, put the quantifier $(\forall x \in S_j)$ in front of the formula.

If ψ is a variable–free sentence, choose any variable x, not of the sort S_{k+1}, and replace $0=0$, $1=1$, and $0=1$, with $x=x$ and $x\neq x$, as in the preceding paragraph, and put the quantifier $(\forall x \in S_j)$ in front of the sentence.

What happens if ψ is universal? If ψ contains some variables of sort S_{k+1}, then, after eliminating these variables from ψ, ψ can be assumed to be of the form

$$\psi_1 \wedge \cdots \wedge \psi_m,$$

where ψ_i are universal sentences having the same quantifier prefixes, and containing no variables of sort S_{k+1}.

Now, after an application of the transformation described above, and after putting the sentence obtained in this way in the prenex normal form, we obtain a universal sentence ψ_*, having at most one variable more than the original sentence

ψ.

Let ψ_* be the formula in $\mathcal{L}$ obtained from ψ in the manner described above.

Proposition 3.6 *If* $\mathbf{B}$ *is a structure of signature* $\mathcal{L}^*$ *which satisfies* Σ *and* ψ *is a sentence in* $\mathcal{L}^*$, *then*

$$\mathbf{B} \models \psi \qquad \textit{if and only if} \qquad \mathbf{B}_* \models \psi_*.$$

PROOF. For each atomic formula $\vartheta(\bar{x})$, having no variables of sort S_{k+1}, in $\mathcal{L}^*$, let $\vartheta_*(\bar{x})$ be the formula obtained from $\vartheta(\bar{x})$ as above, and let $\bar{b}$ be a tuple of elements of $\mathbf{B}$ of the appropriate sorts. Then,

$$\mathbf{B} \models \vartheta(\bar{b}) \qquad \text{if and only if} \qquad \mathbf{B}_* \models \vartheta_*(\bar{b}).$$

The argument can now be easily extended to all the sentences in $\mathcal{L}^*$ having no variables of sort S_{k+1}, using induction on the complexity of a formula in $\mathcal{L}^*$. □

Finally, observe that the effective transformation, which given a sentence ψ in $\mathcal{L}^*$, produces a sentence ψ_* in $\mathcal{L}$, increases the number of variables by at most 1.

3.3 Elimination of constants

Throughout this section, we assume that the language $\mathcal{L}$ does not include relation symbols. Let $S_1, \ldots, S_k$ be the sorts of $\mathcal{L}$.

Let $\mathcal{L}^*$ be the language with the same sorts as $\mathcal{L}$, obtained from $\mathcal{L}$ by replacing all the constant symbols $c \in \mathcal{L}$ of sort S_i by the corresponding unary function symbols

$$f_c : S_i \to S_i.$$

Let Σ be the set of the following universal sentences in $\mathcal{L}^*$:

$$(\forall x \in S_i)(\forall y \in S_i)(f_c(x) = f_c(y)), \qquad c \in \mathcal{L} \setminus \mathcal{L}^*.$$

If $\mathbf{A}$ is a structure of signature $\mathcal{L}$, let $\mathbf{A}^*$ be the structure with the same universe as $\mathbf{A}$, such that, for every function symbol $g \in \mathcal{L} \cap \mathcal{L}^*$,

$$g^{\mathbf{A}} = g^{\mathbf{A}^*},$$

and,for $f_c \in \mathcal{L}^* \setminus \mathcal{L}$, where c is of the sort S_i, let

$$f_c : S_i \to S_i$$

be such that $f_c^{\mathbf{A}^*}$ has the constant value $c^{\mathbf{A}}$.

On the other hand, if $\mathbf{B}$ is any structure of signature $\mathcal{L}^*$ which satisfies Σ, define $\mathbf{B}_*$ to be the structure of signature $\mathcal{L}$ whose universe is that of $\mathbf{B}$ and, for $g \in \mathcal{L} \cap \mathcal{L}^*$,

$$g^{\mathbf{B}_*} = g^{\mathbf{B}},$$

while, for $f_c \in \mathcal{L}^* \setminus \mathcal{L}$, the corresponding $c \in \mathcal{L}$ will be interpreted in $\mathbf{B}_*$ as the constant value of $f_c^{\mathbf{B}}$.

It is straightforward to check that, for any structure $\mathbf{A}$ of signature $\mathcal{L}$, and any

B of signature $\mathcal{L}^*$ which satisfies the universal sentences in Σ,

$$(\mathbf{A}^*)_* \cong \mathbf{A}$$

and

$$(\mathbf{B}_*)^* \cong \mathbf{B}.$$

Given a formula $\phi(\bar{x})$ in $\mathcal{L}$, let $\phi'(\bar{x}, \bar{u})$ be a formula obtained from $\phi(\bar{x})$ by replacing each occurrence of $c \in \mathcal{L}$ of the sort S_i with $f_c(u)$, where u is a variable of the same sort which does not occur in $\phi(\bar{x})$. We assume that a systematic effective way to choose such variables is set up in advance. Finally, let $\phi^*(\bar{x})$ be $\forall \bar{u} \phi'(\bar{x}, \bar{u})$.

Proposition 3.7 *Let* $\mathbf{A}$ *be a structure of signature* $\mathcal{L}$, $\phi(\bar{x})$ *a formula in* $\mathcal{L}$, *and* $\bar{a}$ *a tuple of elements of* A *of the appropriate sorts. Then, for all* $b_1 \in S_1^{\mathbf{A}}, \ldots, b_k \in S_k^{\mathbf{A}}$,

$$\mathbf{A} \models \phi(\bar{a}) \qquad \textit{if and only if} \qquad \mathbf{A}^* \models \phi'(\bar{a}, \bar{b}).$$

Corollary 3.8 *If* $\mathbf{A}$ *is a structure of signature* $\mathcal{L}$, *and* ϕ *a sentence in* $\mathcal{L}$, *then*

$$\mathbf{A} \models \phi \qquad \textit{if and only if} \qquad \mathbf{A}^* \models \phi^*.$$

If $t(\bar{x})$ is a term in $\mathcal{L}^*$ which contains an occurrence of some $f_c \in \mathcal{L}^* \setminus \mathcal{L}$, and if $f_c(u(\bar{x}))$ is a subterm of $t(\bar{x})$, which is not nested inside any other subterm of $t(\bar{x})$ of the form $f_d(v(\bar{x}))$, where $f_d \in \mathcal{L}^* \setminus \mathcal{L}$, then replace $f_c(u(\bar{x}))$ by the corresponding $c \in \mathcal{L} \setminus \mathcal{L}^*$.

Now, given a formula $\psi(\bar{x})$ in $\mathcal{L}^*$, define $\psi_*(\bar{x})$ to be the formula in $\mathcal{L}$ which is obtained from $\psi(\bar{x})$ by transforming each term $t(\bar{x})$ occurring in $\psi(\bar{x})$ in the above described manner, plus deleting all the quantifiers that refer to the variables eliminated in this process.

Hence, we have the following proposition, whose proof is straightforward:

Proposition 3.9 *For a structure* $\mathbf{B}$ *of signature* $\mathcal{L}^*$ *which satisfies the universal sentences in* Σ, *a formula* $\psi(\bar{x})$ *in* $\mathcal{L}^*$, *and a tuple* $\bar{b}$ *of elements of* $\mathbf{B}$ *of the appropriate sorts,*

$$\mathbf{B} \models \psi(\bar{b}) \qquad \textit{if and only if} \qquad \mathbf{B}_* \models \psi_*(\bar{b}).$$

This transformation does not increase the number of the variables of any sort in ψ.

3.4 From the multisorted to a 1–sorted class

In this section, we make the transition from multisorted to one-sorted structures. The standard approach of taking the disjoint union of the sorts (an early reference is [46]) will not be used here; instead we opt for the technique developed in [35] for classes of unary algebras, and put it in a more general setting.

Let $\mathcal{L}$ be a language not containing any relation or constant symbols, whose sorts will be denoted by $S_1, S_2, \ldots, S_n$.

We define the signature $\mathcal{L}^*$ in the following way: If

$$f : S_{i_1} \times \cdots \times S_{i_k} \to S_j$$

is a k-ary operation symbol in $\mathcal{L}$, we define f^* to be a k-ary operation symbol in $\mathcal{L}^*$. Besides the operation symbols f^* corresponding to $f \in \mathcal{L}$, the language $\mathcal{L}^*$ will contain an n-ary function symbol d.

Let Σ consist of the following identities in $\mathcal{L}^*$:

$$d(x, x, \dots, x) = x \tag{3.3}$$

$$d(d(\bar{x_1}), d(\bar{x_2}), \dots, d(\bar{x_n})) = d(x_1^1, x_2^2, \dots, x_n^n) \tag{3.4}$$

where $\bar{x_i} = \langle x_1^i, \dots, x_n^i \rangle$ is an n-tuple of variables in $\mathcal{L}^*$, and

$$f^*(d(\bar{x_1}), \dots, d(\bar{x_k})) = d(x_1^1, \dots, x_{j-1}^1, f^*(x_{i_1}^1, \dots, x_{i_k}^k), x_{j+1}^1, \dots, x_n^1) \tag{3.5}$$

for $f \in \mathcal{L}$ k-ary of type $\langle S_{i_1}, \dots, S_{i_k}, S_j \rangle$.

Given a multisorted algebra

$$\mathbf{A} = (S_1^{\mathbf{A}}, \dots, S_n^{\mathbf{A}})$$

of signature $\mathcal{L}$, we define an algebra $\mathbf{A}^*$ of signature $\mathcal{L}^*$ in the following way:

- the universe of $\mathbf{A}^*$ is
$$S_1^{\mathbf{A}} \times \dots \times S_n^{\mathbf{A}};$$
- if $f \in \mathcal{L}$ is k-ary of the type $\langle S_{i_1}, \dots, S_{i_k}, S_j \rangle$, and $a_1, \dots, a_k \in A^*$ such that
$$a_i = \langle a_1^i, \dots, a_n^i \rangle, \qquad a_r^i \in S_{i_r} (1 \leqslant r \leqslant n),$$

then

$$(f^*)^{\mathbf{A}^*} : (A^*)^k \to A^*$$

is defined by

$$(f^*)^{\mathbf{A}^*}(a_1, \dots, a_k) = \langle a_1^1, a_2^1, \dots, a_{j-1}^1, f^{\mathbf{A}}(a_{i_1}^1, \dots, a_{i_k}^k), a_{j+1}^1, \dots, a_n^1 \rangle;$$

- $d^{\mathbf{A}^*}$ is the natural n–ary decomposition operation on $\mathbf{A}^*$, given by

$$d^{\mathbf{A}^*}(a_1, \dots, a_n) = \langle a_1^1, a_2^2, \dots, a_n^n \rangle.$$

It is a rather straightforward exercise to verify that, in any $\mathbf{A}^*$ defined in this way, all the identities in Σ are satisfied.

Now, given any algebra $\mathbf{B}$ of signature $\mathcal{L}^*$, which satisfies Σ, we show how to construct the corresponding $\mathbf{B}_*$.

Since $\mathbf{B}$ satisfies (3.3),(3.4), $d^{\mathbf{B}}$ will be the natural n–ary decomposition operation on B (see [35]). In other words, there exist $B_1, \dots, B_n$ such that B is in a bijective correspondence with $B_1 \times \cdots \times B_n$ via the bijection

$$\iota : B \to B_1 \times \cdots \times B_n;$$

and, then,

$$d^{\mathbf{B}}(b_1, \dots, b_n) = \iota^{-1}(\iota(b_1)[1], \dots, \iota(b_n)[n]).$$

For the sake of simplicity, we shall identify B with $B_1 \times \cdots \times B_n$, i.e., we assume that

$$B = B_1 \times \cdots \times B_n,$$

and

$$d^{\mathbf{B}}(b_1, \ldots, b_n) = (b_1[1], \ldots, b_n[n]),$$

for all $b_1, \ldots, b_n \in B$.

Fix some $c_1 \in B_1, \ldots, c_n \in B_n$. The n–sorted algebra $\mathbf{B}_*$ will have the universe $(B_1, \ldots, B_n)$.

The interpretation of the k–ary function symbol $f \in \mathcal{L}$ of the type $\langle S_{i_1}, \ldots, S_{i_k}, S_j \rangle$ is defined in the following way: if $b_1 \in B_{i_1}, \ldots, b_k \in B_{i_k}$, $f^{\mathbf{B}_*}(b_1, \ldots, b_k)$ is defined to be the j–th component of

$$(f^*)^{\mathbf{B}}(\langle c_1, \ldots, c_{i_1-1}, b_1, c_{i_1+1}, \ldots, c_n \rangle, \langle c_1, \ldots, c_{i_2-1}, b_2, c_{i_2+1}, \ldots, c_n \rangle, \ldots,$$

$$\langle c_1, \ldots, c_{i_k-1}, b_k, c_{i_k+1}, \ldots, c_n \rangle).$$

It is readily seen that, for any algebra $\mathbf{A}$ of signature $\mathcal{L}$ and any algebra $\mathbf{B}$ of signature $\mathcal{L}^*$, satisfying Σ,

$$(\mathbf{A}^*)_* \cong \mathbf{A},$$

and

$$(\mathbf{B}_*)^* \cong \mathbf{B}.$$

For each term $t(x_1, \ldots, x_k)$ in $\mathcal{L}$, we define the term

$$t^*(x_1, \ldots, x_k)$$

in $\mathcal{L}^*$, as follows:

1° If $t(x_1, \ldots, x_k)$ is a variable x_i, then $t^*(x_1, \ldots, x_k)$ is an (unsorted) variable x_i.

2° If $t(x_1, \ldots, x_k)$ is of the form

$$f(s_1(x_1, \ldots, x_k), \ldots, s_l(x_1, \ldots, x_k)),$$

where $f \in \mathcal{L}$ is l-ary of type $\langle S_{i_1}, \ldots, S_{i_l}, S_j \rangle$, and $s_{i_m}(x_1, \ldots, x_k)$ is of the sort S_{i_m} $(1 \leqslant m \leqslant l)$, we define $t^*(x_1, \ldots, x_k)$ to be

$$f^*(s_1^*(x_1, \ldots, x_k), \ldots, s_l^*(x_1, \ldots, x_k)),$$

and all the variables $x_1, \ldots, x_k$ are unsorted.

If ψ is an equation in $\mathcal{L}$ of the form

$$t_1(x_1, \ldots, x_k) = t_2(x_1, \ldots, x_k),$$

where t_1, t_2 are terms, which are both of the sort S_i, define ψ^* to be the following equation in $\mathcal{L}^*$:

$$d(x_1, \ldots, x_1, t_1^*(x_1, \ldots, x_k), x_1, \ldots, x_1) =$$

$$d(x_1, \ldots, x_1, t_2^*(x_1, \ldots, x_k), x_1, \ldots, x_1),$$

where t_1^* and t_2^* occur as the i-th entries in the list of arguments of d.

Given a formula ϕ in $\mathcal{L}$, we define ϕ^* to be a formula in $\mathcal{L}^*$ obtained by replacing every atomic subformula ψ of ϕ by ψ^* and making all variables unsorted.

Lemma 3.10 *Let* $\mathbf{A}$ *be an algebra of signature* $\mathcal{L}$, *and let* $a_1, \ldots, a_k \in A^*$, *where*

$$a_i = \langle a_1^i, \ldots, a_n^i \rangle \in S_1^{\mathbf{A}} \times \cdots \times S_n^{\mathbf{A}}, \qquad (1 \leqslant i \leqslant k).$$

If $t(x_1, \ldots, x_k)$ *is a term of the type* $\langle S_{i_1}, \ldots, S_{i_k}, S_j \rangle$ *in* $\mathcal{L}$, *then, in* $\mathbf{A}^*$, *the* j*-th component of*

$$(t^*)^{\mathbf{A}^*}(a_1, \ldots, a_k)$$

is

$$t^{\mathbf{A}}(a_{i_1}^1, \ldots, a_{i_k}^k).$$

Proposition 3.11 *Let* $\mathbf{A}$ *be an algebra of signature* $\mathcal{L}$ *and* $\phi(x_1, \ldots, x_k)$ *a formula in* $\mathcal{L}$, *whose variables are of the sorts* $S_{i_1}, \ldots, S_{i_k}$, *respectively. Let* $a_1 \in S_{i_1}^{\mathbf{A}}, \ldots, a_k \in S_{i_k}^{\mathbf{A}}$, *and let* $b_1, \ldots, b_k \in A^*$ *be such that*

$$b_1[i_1] = a_1, \ldots, b_k[i_k] = a_k.$$

Then,

$$\mathbf{A} \models \phi(a_1, \ldots, a_k) \qquad \textit{if and only if} \qquad \mathbf{A}^* \models \phi^*(b_1, \ldots, b_k).$$

PROOF. The proof is by induction on the complexity of ϕ.

1° ϕ is an equation of the form

$$t_1(x_1, \ldots, x_k) = t_2(x_1, \ldots, x_k).$$

Then,

$$\mathbf{A} \models t_1^{\mathbf{A}}(a_1, \ldots, a_k) = t_2^{\mathbf{A}}(a_1, \ldots, a_k)$$

if and only if

$$\mathbf{A}^* \models d^{\mathbf{A}^*}(b_1, \ldots, b_1, (t_1^*)^{\mathbf{A}^*}(b_1, \ldots, b_k), b_1, \ldots, b_1) =$$

$$d^{\mathbf{A}^*}(b_1, \ldots, b_1, (t_2^*)^{\mathbf{A}^*}(b_1, \ldots, b_k), b_1, \ldots, b_1),$$

where t_1^*, t_2^* occur as the i–th entries in the list of the arguments of d. (Here, we have made use of the Lemma 3.10.) This is, in turn, equivalent to

$$\mathbf{A}^* \models \phi^*(b_1, \ldots, b_k).$$

2° If $\phi(x_1, \ldots, x_k)$ is either $\neg\psi(x_1, \ldots, x_k)$ or $\psi(x_1, \ldots, x_k) \wedge \theta(x_1, \ldots, x_k)$, for some formulas $\psi(x_1, \ldots, x_k), \theta(x_1, \ldots, x_k)$, the claim follows immediately.

3° Assume that $\phi(x_1, \ldots, x_k)$ is of the form

$$(\exists y \in S_i)\psi(y, x_1, \ldots, x_k).$$

Then,

$$\mathbf{A} \models (\exists y \in S_i)\psi(y, a_1, \ldots, a_k)$$

if and only if there exists $a \in S_i^{\mathbf{A}}$ such that

$$\mathbf{A} \models \psi(a, a_1, \ldots, a_k).$$

Choosing $b \in A^*$ to be an arbitrary n-tuple whose i-th coordinate is a, and applying the induction hypothesis, we get

$$\mathbf{A} \models \psi(a, a_1, \ldots, a_k) \quad \text{if and only if} \quad \mathbf{A}^* \models \psi^*(b, b_1, \ldots, b_k).$$

Since ϕ^* is

$$\psi^*(y, x_1, \ldots, x_k),$$

this yields

$$\mathbf{A} \models (\exists y \in S_i)\psi(y, a_1, \ldots, a_k)$$

if and only if

$$\mathbf{A}^* \models (\exists y)\psi^*(y, b_1, \ldots, b_k). \qquad \square$$

Corollary 3.12 *If* $\mathbf{A}$ *is any algebra of signature* $\mathcal{L}$, *and* ϕ *a sentence in* $\mathcal{L}$, *then*

$$\mathbf{A} \models \phi \quad \textit{if and only if} \quad \mathbf{A}^* \models \phi^*.$$

Our next task is to define the corresponding effective transformation on sentences in $\mathcal{L}^*$. We first establish certain properties of terms in $\mathcal{L}^*$.

Lemma 3.13 *Let* $\mathbf{B}$ *be an algebra of signature* $\mathcal{L}^*$, *satisfying* Σ. *If* $t(x_1, \ldots, x_k)$ *is a term in* $\mathcal{L}$ *of the type* $\langle S_{i_1}, \ldots, S_{i_k}, S_j\rangle$, *and* $b_1, \ldots, b_k \in B$, *then any coordinate of*

$$(t^*)^{\mathbf{B}}(b_1, \ldots, b_k)$$

is equal to

$$s^{\mathbf{B}_\bullet}(b_1[i_1], \ldots, b_k[i_k]),$$

for some subterm $s(x_1, \ldots, x_k)$ *of* t.

PROOF. The proof is by induction on the complexity of t.

1° $t(x_1, \ldots, x_k)$ is a variable x_m $(1 \leqslant m \leqslant k)$, of sort S_{i_m}. In this case, the claim is trivial.

2° Suppose $t(x_1, \ldots, x_k)$ is of the form

$$f(s_1(x_1, \ldots, x_k), \ldots, s_m(x_1, \ldots, x_k)),$$

where each $s_r(x_1, \ldots, x_k)$ is of the type $\langle S_{i_1}, \ldots, S_{i_k}, S_{j_r}\rangle$ $(1 \leqslant r \leqslant m)$, while f is of the type $\langle S_{j_1}, \ldots, S_{j_r}, S_j\rangle$. Then,

$$(f^*)^{\mathbf{B}}((s_1^*)^{\mathbf{B}}(\bar{b}), \ldots, (s_m^*)^{\mathbf{B}}(\bar{b})) =$$
$$\langle (s_1^*)^{\mathbf{B}}(\bar{b})[1], \ldots, (s_1^*)^{\mathbf{B}}(\bar{b})[j-1], f^{\mathbf{B}_\bullet}((s_1^*)^{\mathbf{B}}(\bar{b})[j_1], \ldots, (s_m^*)^{\mathbf{B}}(\bar{b})[j_r]),$$
$$\ldots, (s_1^*)^{\mathbf{B}}(\bar{b})[j+1], \ldots, (s_1^*)^{\mathbf{B}}(\bar{b})[n]\rangle,$$

and the claim follows from the induction hypothesis applied to the elements $s_1^*(\bar{b})$, $\ldots, s_m^*(\bar{b})$. □

Moreover, observe that, from the proof of the lemma, we have the following: for each $t(x_1, \ldots, x_k)$ and $b_1, \ldots, b_k \in B$, we can effectively recover the subterms $s_1, \ldots, s_n$ of t, which determine the coordinates of $t^*(b_1, \ldots, b_k)$ in a unique way.

Lemma 3.14 *Modulo Σ, every term $t^*(x_1, \ldots, x_k)$ in $\mathcal{L}^*$ is equal to a term of the form*

$$d(t_1^*(x_1, \ldots, x_k), \ldots, t_n^*(x_1, \ldots, x_k)),$$

where d does not occur in any of $t_1^*, \ldots, t_n^*$.

PROOF. The proof is by induction on the complexity of t^*.

1° If $t^*(x_1, \ldots, x_k)$ is $f^*(x_1, \ldots, x_k)$, then

$$t^*(x_1, \ldots, x_k) = d(f^*(x_1, \ldots, x_k), \ldots, f^*(x_1, \ldots, x_k)).$$

If $t^*(x_1, \ldots, x_k)$ is $d(x_{i_1}, \ldots, x_{i_n})$, where $\{x_{i_1}, \ldots, x_{i_n}\} \subseteq \{x_1, \ldots, x_k\}$, then t^* itself will do.

2° If $t^*(x_1, \ldots, x_k)$ is of the form

$$d(s_1^*(x_1, \ldots, x_k), \ldots, s_n^*(x_1, \ldots, x_k)),$$

let

$$s_i^*(x_1, \ldots, x_k) = d(t_{i,1}^*(x_1, \ldots, x_k), \ldots, t_{i,n}^*(x_1, \ldots, x_k)),$$

where none of $t_{i,1}^*, \ldots, t_{n,i}^*$ contains d, for $1 \leqslant i \leqslant n$. Then,

$$\begin{aligned} t^*(x_1, \ldots, x_k) &= d(d(t^*_{1,1}(\bar{x}), \ldots, t^*_{1,n}(\bar{x})), \ldots, d(t^*_{n,1}(\bar{x}), \ldots, t^*_{n,n}(\bar{x}))) \\ &= d(t^*_{1,1}(\bar{x}), t^*_{2,2}(\bar{x}), \ldots, t^*_{n,n}(\bar{x})). \end{aligned}$$

If $t^*(x_1, \ldots, x_k)$ is of the form

$$f^*(s_1(\bar{x}), \ldots, s_k(\bar{x})),$$

where f^* is k–ary, using the induction hypothesis, we get

$$s^*_i(\bar{x}) = d(t^*_{i,1}(\bar{x}), \ldots, t^*_{i,n}(\bar{x})),$$

where none of $t^*_{i,j}$ contains d. Then,

$$\begin{aligned} t^*(\bar{x}) &= f^*(d(t^*_{1,1}(\bar{x}), \ldots, t^*_{1,n}(\bar{x})), \ldots, d(t^*_{k,1}(\bar{x}), \ldots, t^*_{k,n}(\bar{x}))) \\ &= d(t^*_{1,1}(\bar{x}), \ldots, t^*_{1,j-1}(\bar{x}), f(t^*_{1,i}(\bar{x}), \ldots, t^*_{k,i}(\bar{x})), t^*_{1,j+1}(\bar{x}), \ldots, t^*_{1,n}(\bar{x}))). \end{aligned}$$

This concludes the proof of the lemma. □

Hence, we can effectively transform any equation ψ in $\mathcal{L}^*$ into one both sides of which are of the form from the statement of Lemma 3.14. So, by Lemma 3.14, without any loss of generality, we may assume that any equation in $\mathcal{L}^*$ is of the form

$$d(t^*_1(x_1, \ldots, x_k), \ldots, t^*_n(x_1, \ldots, x_k))$$

$$= d(u_1^*(x_1, \dots, x_k), \dots, u_n^*(x_1, \dots, x_k)), \tag{3.6}$$

where the terms t_i^*, u_j^* contain no occurrences of d.

Given an equation ψ in $\mathcal{L}^*$ of the form (3.6), define ψ_* as follows:

$$\psi_* = \bigwedge_{i=1}^{n} \theta_i,$$

where θ_i is constructed in the following way: let p_i be the subterm of t_i, such that, for every $b_1, \dots, b_k \in B$,

$$t_i^*(b_1, \dots, b_k)[i] = p_i(b_1[i_1], \dots, b_k[i_k]),$$

where t_i is of the type $\langle S_{i_1}, \dots, S_{i_k}, S_j \rangle$, and let q_i be the subterm of u_i with the same property, namely

$$u_i^*(b_1, \dots, b_k)[i] = q_i(b_1[j_1], \dots, b_k[j_k]).$$

Set θ_i to be equal to

$$p_i(x_{i_1}^1, \dots, x_{i_k}^k) = q_i(x_{j_1}^1, \dots, x_{j_k}^k).$$

Now, if $\phi(x_1, \dots, x_k)$ is a formula in $\mathcal{L}^*$, let

$$\phi_*(x_1^1, \dots, x_n^1, x_1^2, \dots, x_n^2, \dots, x_1^k, \dots, x_n^k)$$

be the formula in $\mathcal{L}$, obtained by replacing all equations (atomic subformulas) ψ of ϕ by ψ_* as described above. Also, each quantifier Qx_i in ϕ will be replaced by

$$Qx_1^i \cdots x_n^i.$$

Clearly, if ϕ contains m variables, ϕ_* will contain $m \cdot n$ variables.

Proposition 3.15 *Suppose*

$$\mathbf{B} \models \Sigma.$$

For

$$b_1 = \langle b_1^1, \ldots, b_n^1 \rangle, \ldots, b_k = \langle b_1^k, \ldots, b_n^k \rangle$$

in B, and a formula $\phi(x_1, \ldots, x_k)$ in $\mathcal{L}^*$, we have

$$\mathbf{B} \models \phi(b_1, \ldots, b_k) \quad \textit{if and only if} \quad \mathbf{B}_* \models \phi_*(b_1^1, \ldots, b_n^1, \ldots, b_1^k, \ldots, b_n^k).$$

PROOF. If ϕ is an equation in $\mathcal{L}^*$, the validity of the claim follows from the preceding discussion. The only remaining case which is nontrivial is when $\phi(x_1, \ldots, x_k)$ is of the form

$$(\exists x)\psi(x, x_1, \ldots, x_k).$$

Then,

$$\mathbf{B} \models (\exists x)\psi(x, b_1, \ldots, b_k)$$

if and only if there exists

$$b_0 = \langle b_1^0, \ldots, b_n^0 \rangle$$

in B such that

$$\mathbf{B} \models \psi(b, b_1, \dots, b_k).$$

Then, by the induction hypothesis,

$$\mathbf{B} \models \psi(b, b_1, \dots, b_k)$$

if and only if

$$\mathbf{B}_* \models \psi_*(b_1^0, \dots, b_n^0, b_1^1, \dots, b_n^1, \dots, b_1^k, \dots, b_n^k),$$

but observe that

$$\phi_*(x_1^1, \dots, x_n^1, \dots, x_1^k, \dots, x_n^k)$$

is actually

$$(\exists x_1 \cdots x_n)\psi_*(x_1, \dots, x_n, x_1^1, \dots, x_n^1, \dots, x_1^k, \dots, x_n^k). \qquad \square$$

An immediate corollary of Proposition 3.15 is:

Corollary 3.16 *For any algebra* $\mathbf{B}$ *of signature* $\mathcal{L}^*$ *and any sentence* ϕ *in* $\mathcal{L}^*$,

$$\mathbf{B} \models \phi \qquad \textit{if and only if} \qquad \mathbf{B}_* \models \phi_*.$$

3.5 Adding the discriminator

Let $\mathcal{K}$ be a class of algebras of a 1–sorted signature $\mathcal{L}$, axiomatized by a finite set of universal sentences Φ in $\mathcal{L}$, which is universally pseudorecursive.

Let $\mathcal{K}^t$ be the class of algebras of signature $\mathcal{L} \cup \{t\}$, where t is a ternary function symbol not occurring in $\mathcal{L}$, and which is axiomatized by $\Phi \cup \{\delta_t\}$, where δ_t is a universal sentence asserting that t is interpreted by the discriminator function in every algebra in $\mathcal{K}^t$. Also, for any algebra $\mathbf{A} \in \mathcal{K}$, let $\mathbf{A}^t$ stand for the expansion of $\mathbf{A}$ by the discriminator function on A.

Clearly, we have the following:

Proposition 3.17 *$\mathcal{K}^t$ has an undecidable universal theory.*

Now, given a universal sentence ϕ in $\mathcal{L}^t = \mathcal{L} \cup \{t\}$, one can effectively construct a universal sentence ϕ_* in $\mathcal{L}$ with the same number of variables, so that

$$\mathbf{A} \models \phi_* \qquad \text{if and only if} \qquad \mathbf{A}^t \models \phi,$$

where $\mathbf{A}$ is any algebra in $\mathcal{K}$.

Here is how the construction proceeds. If ϕ contains an atomic subformula of the form

$$t(s_1, s_2, s_3) = s,$$

where s_1, s_2, s_3, s are terms in $\mathcal{L}^t$, replace it by:

$$(s_1 = s_2 \wedge s_3 = s) \vee (s_1 \neq s_2 \wedge s_1 = s). \tag{3.7}$$

Since $t(s_1, s_2, s_3) = s$ is equivalent to (3.7) in any model of $\Phi \cup \{\delta_t\}$, eliminating t

successively in ϕ, in the manner described above, one obtains ϕ_* such that

$$\mathbf{A}^t \models \phi \qquad \text{if and only if} \qquad \mathbf{A} \models \phi_*.$$

Now, it is easy to prove the following:

Proposition 3.18 *$Th_{\forall,n}(\mathcal{K}^t)$ is decidable, for every $n < \omega$.*

Consider the variety $\mathbf{V}(\mathcal{K}^t)$. Theorem 1.25 shows that, for each universal sentence ϕ in $\mathcal{L}^t$, one can effectively construct an identity $\hat{\phi}$ in $\mathcal{L}^t$, such that in every nontrivial subdirectly irreducible algebra $\mathbf{A}$ in $\mathbf{V}(\mathcal{K}^t)$,

$$\mathbf{A} \models \phi \qquad \text{if and only if} \qquad \mathbf{A} \models \hat{\phi}.$$

Let $(\mathcal{K}^t)^+$ denote the class of nontrivial members of $\mathcal{K}^t$. It is easy to see that the set $Th_\forall(\mathcal{K}^t)$ is recursive with respect to $Th_\forall((\mathcal{K}^t)^+)$ and the undecidability of $Th_\forall(\mathcal{K}^t)$ will therefore imply the undecidability of $Th_\forall((\mathcal{K}^t)^+)$. However, Theorem 1.25 shows that the undecidability of the latter is equivalent to the undecidability of the equational theory of $\mathbf{V}(\mathcal{K}^t)$. The reason for this lies in the following fact:$\mathcal{K}^t$ consists of the subdirectly irreducible algebras in $\mathbf{V}(\mathcal{K}^t)$ plus the one-element algebras (in case $\mathcal{K}$ has any one-element algebras). Therefore, an equation is true in $\mathbf{V}(\mathcal{K}^t)$ if and only if it is true in $\mathcal{K}^t$. Hence, the undecidability of the universal theory of $\mathcal{K}^t$ implies the undecidability of the equational theory of $\mathbf{V}(\mathcal{K}^t)$.

Theorem 3.19 *$\mathbf{V}(\mathcal{K}^t)$ has the undecidable equational theory.*

Theorem 3.20 *$\mathbf{V}(\mathcal{K}^t)$ is finitely based.*

PROOF. As explained in Section 1.3, the class of subdirectly irreducible members of $\mathbf{V}(\mathcal{K}^t)$ consists of $\mathcal{K}^t$ plus the one-element algebra, if not already included in $\mathcal{K}^t$.

For each $\phi \in \Phi \cup \{\delta_t\}$, let ϕ' be an identity such that

$$\mathbf{A} \models \phi \leftrightarrow \phi'$$

for every nontrivial subdirectly irreducible algebra $\mathbf{A}$ which satisfies the identities (1.1), (1.2), and (1.3) from Section 1.3. The existence of such an identity ϕ' is guaranteed by Theorem 1.25.

Now, Φ' will be the set of identities consisting of $\{\phi' : \phi \in \Phi\}$ along with (1.1), (1.2), and (1.3).

Let W be the variety defined by the set of identities Φ'. We need to show that

$$W_{SI} = \mathbf{V}_{SI}(\mathcal{K}^t).$$

If $\mathbf{A} \in \mathbf{V}_{SI}(\mathcal{K}^t)$ is a nontrivial algebra, $\mathbf{A} \in \mathcal{K}^t$ and

$$\mathbf{A} \models \Phi \cup \{\delta_t\}.$$

Since $t^{\mathbf{A}}$ will be a discriminator function on A, $\mathbf{A}$ will satisfy the identities (1.1)–(1.3), and this implies

$$\mathbf{A} \models \Phi'.$$

Conversely, if $\mathbf{A} \in W_{SI}$, and $\mathbf{A}$ is nontrivial, $\mathbf{A}$ satisfies (1.1)–(1.3), and since $\mathbf{A}$ is subdirectly irreducible, Theorem 1.25 yields that $t^{\mathbf{A}}$ will be the discriminator function on A. Thus,

$$\mathbf{A} \models \Phi$$

and we are done. □

Since $Th_{\forall,n}(\mathcal{K}^t)$ is decidable for every $n < \omega$, $Th_{\forall H,n}(\mathcal{K}^t)$, the theory of universal Horn sentences of $\mathcal{K}^t$ with at most n variables, is decidable.

We have already seen in Section 1.3 that the $Th_Q(\mathcal{K}^t)$ is precisely $Th_Q(\mathbf{V}(\mathcal{K}^t))$, whence $Th_{Q,n}(\mathbf{V}(\mathcal{K}^t))$ is decidable, for every $n < \omega$.

Corollary 2.14 states that the decidability of $Th_{Q,n}(\mathbf{V}(\mathcal{K}^t))$ implies the existence of a uniform algorithm which solves the word problem for any finitely presented algebra in $\mathbf{V}(\mathcal{K}^t)$ with at most n generators in its finite presentation. This yields the following theorem:

Theorem 3.21 *Every finitely presented algebra in* $\mathbf{V}(\mathcal{K}^t)$ *has a solvable word problem.*

By combining all the steps so far, we complete the proof of Theorem 3.1.

Chapter 4

Definition of the class $\mathcal{K}$

In this chapter, our goal is to construct a class $\mathcal{K}$ of structures in some multisorted language $\mathcal{L}$, axiomatized by a finite set of universal sentences in $\mathcal{L}$, which will have the desired properties; namely, its universal theory will be undecidable, while the theory consisting of the universal sentences true in $\mathcal{K}$ involving at most n variables of each sort will be decidable, for every $n \in \omega$.

To ensure that the universal theory of $\mathcal{K}$ is undecidable, we design $\mathcal{K}$ so as to reflect the computations of a Turing machine with an undecidable halting problem.

On the other hand, the decidability of every $(n, n, \ldots, n)$-universal theory will follow from the fact that, given a universal sentence ϕ in $\mathcal{L}$ containing at most n variables of each sort, we can effectively check whether ϕ is false in some structure in $\mathcal{K}$ by listing effectively all pairs$(\mathbf{A}, \bar{a})$ where $\mathbf{A}$ is any structure in $\mathcal{K}$ generated by at most n elements of each sort and $\bar{a}$ is any (n, n, n, n)–tuple of elements belonging to $\mathbf{A}$.

In particular, if a structure is generated by at most n elements of every sort, there will be only finitely many finite components (which will be defined later) in such a structure, and that number will be bounded by some computable function of n. These finite components, together with their location in the structure and at most n configurations of our Turing machine will, speaking informally, completely determine the structure.

The version of a Turing machine that we use is such that the tape of T is semi-infinite (to the right), and, if it happens that the machine head is scanning the leftmost cell of the tape and the appropriate instruction directs it to the left, the machine halts in *no* state.

Throughout the rest of the exposition, the machine T in question is assumed to be such that its halting problem in state q_0 is undecidable. The version of the Halting Problem that we have in mind, as mentioned in the introduction, is the following: there is a machine T for which the set of the initial configurations which lead to the halting state is undecidable.

By a **Turing machine** we mean a quintuple

$$T = \langle Q, S, \delta, q_0, q_1 \rangle,$$

where

- $S = \{s_0, s_1, \dots, s_m, s_{m+1}\}$ is the set of the *tape symbols* of T, where $B = s_{m+1}$ stands for the blank symbol.
- $Q = \{q_0, q_1, \dots, q_n\}$ is the set of the *states* of T.
- $q_0 \in Q$ is the unique *halting state* of T.

- $q_1 \in Q$ is the *initial state* of T.
- $\delta : S \times Q \to S \times Q \times \{L, N, R\}$ is a partial mapping (the *transition function* of T) and the intended interpretation of

$$\delta(s, q) = (s', q', \Gamma), \qquad \Gamma \in \{L, N, R\}$$

is: T, upon reading s in the scanned cell with its head in the state q, prints s' in place of s, moves in the direction Γ (L-left, N-no move, R-right), and changes the state of its head to q'.

Also, the transition function δ of T is assumed to satisfy the following: for every $s \in S$

$$\delta(s, q_0) = (s, q_0, N).$$

I.e., once the halting state has been reached there is no further action of T.

By a **configuration** we mean a finite initial portion of the tape, together with the current position of the head and its current state (we also allow for the case where the head is off the tape); i.e. a configuration is a finite sequence from $S \cup (S \times Q)$, having at most one coordinate from $S \times Q$. The set of all configurations of T will be denoted by *Config*.

T determines a unary function T : *Config* $\to$ *Config* via its transition function δ in the following way:

1° If $C \in Config$ is of the form

$$C = s_{i_1} \dots s_{i_{j-1}} \left(s_{i_j}, q\right) s_{i_{j+1}} \dots s_{i_n},$$

where $s_{i_k} \in S$, $(k = 1, \ldots, n)$, $q \in Q$ and

$$\delta\left(s_{i_j}, q\right) = (s', q', L),$$

then, $T(C) = C'$, where

$$C' = s_{i_1} \ldots \left(s_{i_{j-1}}, q'\right) s' s_{i_{j+1}} \ldots s_{i_n} B.$$

2° If $C \in Config$ is of the form

$$C = s_{i_1} \ldots s_{i_{j-1}} \left(s_{i_j}, q\right) s_{i_{j+1}} \ldots s_{i_n},$$

where $s_{i_k} \in S$, $(k = 1, \ldots, n)$, $q \in Q$ and

$$\delta\left(s_{i_j}, q\right) = (s', q', R),$$

then, $T(C) = C'$, where

$$C' = s_{i_1} \ldots s_{i_{j-1}} s'(s_{i_{j+1}}, q') \ldots s_{i_n} B.$$

3° If $C \in Config$ is of the form

$$C = s_{i_1} \ldots s_{i_{j-1}} \left(s_{i_j}, q\right) s_{i_{j+1}} \ldots s_{i_n},$$

where $s_{i_k} \in S$, $(k = 1, \ldots, n)$, $q \in Q$ and

$$\delta\left(s_{i_j}, q\right) = (s', q', N),$$

then, $T(C) = C'$, where

$$C' = s_{i_1} \ldots s_{i_{j-1}} (s', q') s_{i_{j+1}} \ldots s_{i_n} B.$$

4° If $C \in Config$ is of the form

$$C = s_{i_1} \ldots s_{i_{j-1}} s_{i_j} s_{i_{j+1}} \ldots s_{i_n},$$

where $s_{i_k} \in S$, $(k = 1, \ldots, n)$, then, $T(C) = C'$, where

$$C' = s_{i_1} \ldots s_{i_{j-1}} s_{i_j} s_{i_{j+1}} \ldots s_{i_n} B.$$

If $C \in$ *Config* is of the form

$$C = (s_{i_1}, q) s_{i_2} \cdots s_{i_n},$$

and

$$\delta(s_{i_1}, q) = (s', q', L),$$

then

$$T(C) = s' s_{i_2} \cdots s_{i_n} B.$$

In particular, the action of T on a particular configuration always increases the length of the configuration by one.

Before proceeding to the axiomatization of $\mathcal{K}$, we sketch the idea of a computation structure. These structures will be in $\mathcal{K}$. They are intended to fully capture computations of T. Our axiomatization for $\mathcal{K}$ results from assembling finitely many universal sentences which are true in these structures and which describe how these structures capture computations.

Given a configuration $\mathcal{C}$ of T, we can associate to it the computation structure $\mathbf{S}_{\mathcal{C}}$: Each horizontal line captures a particular configuration from the computation

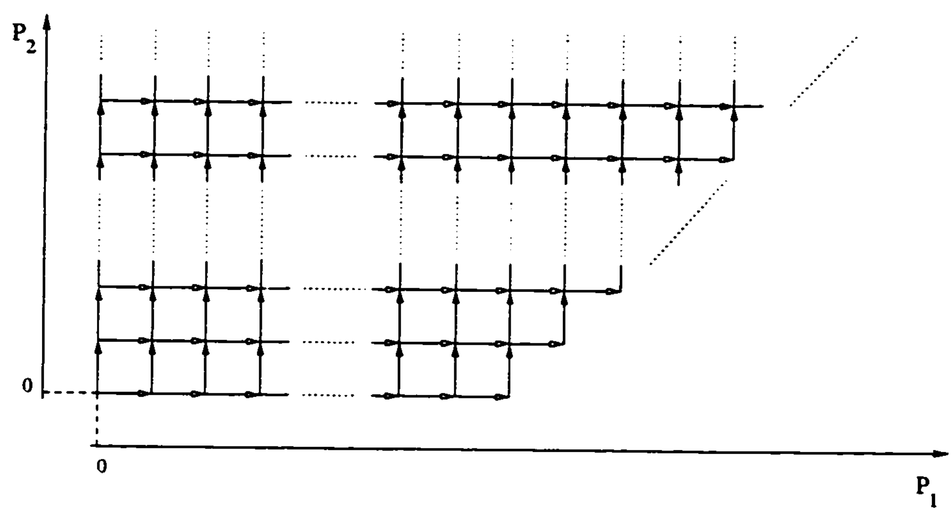

Figure 4.1: Computation structure

of T when started on $\mathcal{C}$. The computation proceeds upwards, one line at a time, in the following manner: assume that, given a configuration

$$\mathcal{C}_1 : \qquad s_{i_1} s_{i_2} \cdots (s_{i_j}, q) \cdots s_{i_n},$$

$T(C_1) = C_2$, where C_2 is, say,

$$C_2 : \qquad s_{i_1} s_{i_2} \cdots (s_{i_{j-1}}, q') s'_{i_j} \cdots s_{i_n} B.$$

C_2 will constitute the next "level" of our structure, being immediately "above" C_1. In other words, we introduce a new relation $\twoheadrightarrow$ which will connect the encodings of C_1 and C_2 in the following way: The intention of the arrows in the figure is

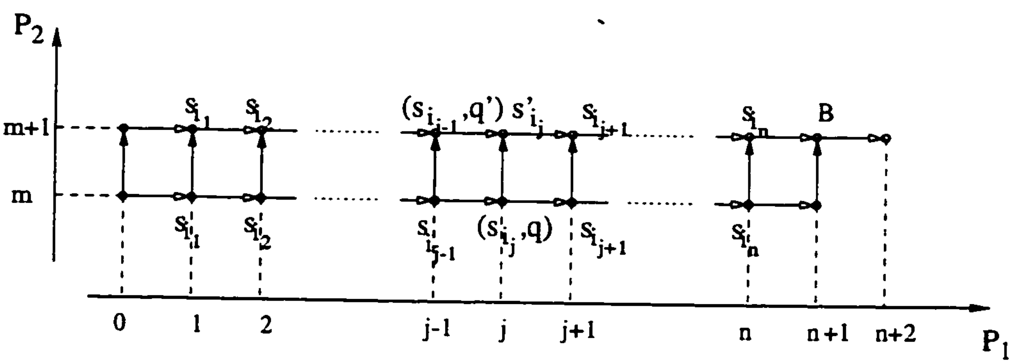

Figure 4.2: Two consecutive lines of a computation structure

to represent the binary relations $\rightarrow$ and $\twoheadrightarrow$ on the universe of the computation structure. The horizontal arrows in the diagram correspond to the relation $\rightarrow$ while the vertical ones represent $\twoheadrightarrow$.

The language $\mathcal{L}$ in which configuration structures, as well as the rest of the class $\mathcal{K}$, will be defined has four sorts that will be denoted by A, P_1, P_2, and C. The symbols of $\mathcal{L}$ will be the following:

- Function symbols

$diam : A^2 \to A$	$bord : A^2 \to A$
$h : A^3 \to A$	$\pi_1 : A \to P_1$
$f : A^4 \to A$	$\pi_2 : A \to P_2$
$g : A^2 \to A$	$\chi : A \to C$

- Relation symbols

$\to \subseteq A \times A$	$\leqslant_1 \subseteq P_1 \times P_1$
$\twoheadrightarrow \subseteq A \times A$	$\leqslant_2 \subseteq P_2 \times P_2$

- Constant symbols

∞ in A	$s \in S$ in C
∞_1 in P_1	$(s, q) \in S \times Q$ in C
∞_2 in P_2	c in C
∞' in C	d in C
$\diamondsuit$ in C	$\heartsuit$ in C

The computation structure is intended to correspond to a two-dimensional pattern in the grid that is coordinatized by two linear orderings, $(P_1, \leqslant_1)$ and $(P_2, \leqslant_2)$. π_1 and π_2 represent the projections of the "main structure part" A on the coordinate axes P_1 and P_2. The role of $f, g, h, diam$ and $bord$ is to build A according to the sequence of computations of T, started on C. If, at some point, a pattern is detected which cannot be a part of a genuine content of the tape of T, the appropriate structure-building operation has the default value ∞. ∞_1 and ∞_2 are the projections of ∞ onto P_1 and P_2, respectively. Finally, the role of the constant symbols c and d is to detect whether T eventually halts: if at some point, the halting state q_0 is reached, the axioms force $c = d$ in $\mathbf{S}_C$.

The role of χ is to assign a "label" from $S \cup (S \times Q)$ to each element of A, so that this label indicates the tape/state symbol of T at the corresponding position on the tape.

Next, we proceed to the definition of a **computation structure** corresponding to a particular configuration $\mathcal{C}$.

Let $\mathcal{C} \in Config$ be of the form

$$s_{i_1} s_{i_2} \ldots \left(s_{i_j}, q\right) \ldots s_{i_n},$$

or

$$s_{i_1} s_{i_2} \ldots s_{i_j} \ldots s_{i_n},$$

where $s_{i_k} \in S$ $(k = 1, \ldots, n)$ and $q \in Q$. In order to make the notation less cumbersome, we will adopt the following way of representing $\mathcal{C}$:

$$\mathcal{C} = c_1 c_2 \ldots c_j \ldots c_n,$$

where $c_k \in S \cup (S \times Q)$, $(k = 1, \ldots, n)$.

Given $\mathcal{C} \in Config$ and $i \geq 0$, define $\mathcal{C}^{(i)} \in Config$ inductively by: $\mathcal{C}^{(0)} = \mathcal{C}$, and for $i \geqslant 0$, $\mathcal{C}^{(i+1)} = T(\mathcal{C}^{(i)})$.

Suppose

$$\mathcal{C}^{(0)} = c_1^{(0)} c_2^{(0)} \ldots c_n^{(0)}.$$

Since every action of T increases the length of a configuration by one, we can write

$$\mathcal{C}^{(i)} = c_1^{(i)} c_2^{(i)} \dots c_{n+i}^{(i)}.$$

Now, we are ready to define the universes of a computation structure for $\mathcal{C}$, one for each of the sorts A, P_1, P_2, and C.

The universe for the sort of constants C will consist of the elements $\infty', \Diamond, \heartsuit, c$, and d, along with the elements of $S \cup (S \times Q)$. The way in which these elements will interpret the symbols in C should be transparent. We also assume that all the elements of this sort are pairwise distinct, except for, possibly, c and d.

The universe for the sort P_i $(i = 1, 2)$ will be the ordinal $\omega + 1$, whose maximal element will be denoted by ∞_i.

Finally, the universe for the sort A will be the following set:

$$A = \left\{ a_j^{(i)} : \ i < \omega, \ 0 \leq j \leq (n+1) + i \right\} \cup \{\infty\},$$

so that the binary relations $\rightarrow$ and $\twoheadrightarrow$ on A are defined in the following way:

$$x \twoheadrightarrow y \text{ if and only if } x = a_j^{(i)}, y = a_j^{(i+1)} \text{ for } i < \omega, \ 0 \leq j \leq (n+1) + i,$$

$$x \rightarrow y \text{ if and only if } x = a_j^{(i)}, y = a_{j+1}^{(i)} \text{ for } i < \omega, \ 0 \leq j \leq (n+1) + i.$$

The operations π_i $(i = 1, 2)$ and χ are defined so that

$$\pi_1(a_j^{(i)}) = j, \ \pi_2\left(a_j^{(i)}\right) = i, \ \pi_1(\infty) = \infty_1, \ \pi_2(\infty) = \infty_2,$$

$$\chi\left(a_0^{(i)}\right) = \diamondsuit,\ \chi\left(a_{(n+1)+i}^{(i)}\right) = \heartsuit,\ \chi(\infty) = \infty',\ \chi\left(a_j^{(i)}\right) = c_j^{(i)},$$

for $i < \omega$, $1 \leq j \leq n+i$.

The operations $f, g, h, diam$, and $bord$ are defined as follows:

$$f(x,y,z,u) = \begin{cases} a_{j+2}^{(i+1)}, & \text{if } x = a_j^{(i)}, y = a_{j+1}^{(i)}, z = a_{j+2}^{(i)}, u = a_{j+3}^{(i)}, \\ \infty, & \text{otherwise} \end{cases}$$

$$g(x,y) = \begin{cases} a_{(n+1)+i}^{(i+1)}, & \text{if } x = a_{(n+1)+(i-1)}^{(i)}, y = a_{(n+1)+i}^{(i)}, \\ \infty, & \text{otherwise} \end{cases}$$

$$h(x,y,z) = \begin{cases} a_1^{(i+1)}, & \text{if } x = a_0^{(i)}, y = a_1^{(i)}, z = a_2^{(i)} \\ \infty, & \text{otherwise} \end{cases}$$

$$diam(x,y) = \begin{cases} a_0^{(i+1)}, & \text{if } x = a_0^{(i)}, y = a_1^{(i)}, \\ \infty, & \text{otherwise} \end{cases}$$

$$bord(x,y) = \begin{cases} a_{(n+1)+(i+1)}^{(i+1)}, & \text{if } x = a_{(n+1)+(i-1)}^{(i)}, y = a_{(n+1)+i}^{(i)}, \\ \infty, & \text{otherwise} \end{cases}$$

This concludes the definition of a computation structure for a given configuration C. In fact, C will determine the structure completely, except for whether $c = d$ or not. Hence, for a given $C \in Config$, there will exist precisely two computation structures, $\mathbf{S}_C^{=}$, in which $c = d$, and $\mathbf{S}_C^{\neq}$, in which $c \neq d$.

We adopt the following convention: given a structure $\mathbf{S}$ in the language $\mathcal{K}$, the sets of elements of S of corresponding sorts, will be denoted by $A^{\mathbf{S}}, P_1^{\mathbf{S}}, P_2^{\mathbf{S}}, C^{\mathbf{S}}$, and

$$\mathbf{S} = \langle A^{\mathbf{S}}, P_1^{\mathbf{S}}, P_2^{\mathbf{S}}, C^{\mathbf{S}} \rangle.$$

The class $\mathcal{K}$ will be axiomatized by a finite list of universal sentences Φ in the language $\mathcal{L}$, whose explicit statements will be deferred to the next section of this chapter. These axioms ensure that all members of $\mathcal{K}$ share certain simple properties with the computation structures. For example, some of the asserted properties of $\mathcal{K}$ are the following:

1. $\leqslant_i$ linearly orders P_i, with the largest element ∞_i.

2. π_1 and π_2 coordinatize A via P_1 and P_2. By a (*horizontal*) *line* we understand the set of the elements of A with the same P_2 coordinate.

3. Every element of A is labelled, via χ, by some element of $S \cup (S \times Q) \cup \{\diamondsuit, \heartsuit, \infty'\}$, so that the first element of each line is labelled by $\diamondsuit$, and the last one by $\heartsuit$.

4. The structure-building operations f, g, and h reflect the computations of T.

5. *diam* is used to put $\diamondsuit$ to the first position of a line representing a configuration, while *bord* marks the right end of a configuration by putting $\heartsuit$.

6. $\twoheadrightarrow$ connects vertically the lines arising from two successive configurations in the computation of T, so that the elements labelled with $\diamondsuit$ are vertically aligned, while there is no $x \in A$ which is $\twoheadrightarrow$-connected to an element labelled by $\heartsuit$.

7. ∞ is isolated: i.e, there is no $x \in A$ such that either $x \rightarrow \infty$ or $x \twoheadrightarrow \infty$.

8. For every $s \in S$, there is an axiom which detects whether the final state q_0

has been reached:

$$(\forall x \in A)(\chi(x) = (s, q_0) \Rightarrow c = d).$$

4.1 The list of axioms for $\mathcal{K}$

1. Axioms which assert that $\leqslant_i$ is a linear ordering with the maximal element ∞_i:

   - $(\forall x \in P_i)(x \leq_i x)$
   - $(\forall x \in P_i)(\forall y \in P_i)(x \leq_i y \wedge y \leq_i x \Rightarrow x = y)$
   - $(\forall x \in P_i)(\forall y \in P_i)(\forall z \in P_i)(x \leq_i y \wedge y \leq_i z \Rightarrow x \leq_i z)$
   - $(\forall x \in P_i)(\forall y \in P_i)(x \leq_i y \vee y \leq_i x)$
   - $(\forall x \in P_i)(x \leq_i \infty_i)$

2. Axioms which describe π_i and their relation to $\rightarrow$ and $\twoheadrightarrow$:

   - $(\forall x \in A)(\forall y \in P_1)(\chi(x) = \diamondsuit \Rightarrow \pi_1(x) \leq_1 y)$
   - $(\forall x \in A)(\forall y \in A)(\chi(x) = \heartsuit \wedge \pi_2(x) = \pi_2(y) \Rightarrow \pi_1(y) \leq_1 \pi_1(x))$
   - $(\forall x \in A)(\forall y \in A)(\pi_1(x) = \pi_1(y) \wedge \pi_2(x) = \pi_2(y) \Rightarrow x = y)$
   - $(\forall x \in A)(\forall y \in A)(\chi(x) = \diamondsuit \wedge \pi_1(x) = \pi_1(y) \Rightarrow \chi(y) = \diamondsuit)$
   - $(\forall x \in A)(\pi_i(x) = \infty_i \Leftrightarrow x = \infty)$
   - $(\forall x \in A)(\forall y \in A)(\forall z \in P_1)(x \rightarrow y \Rightarrow \pi_1(x) <_1 \pi_1(y) \wedge \neg(\pi_1(x) <_1 z <_1 \pi_1(y)) \wedge \pi_2(x) = \pi_2(y))$

- $(\forall x \in A)(\forall y \in A)(\forall z \in P_2)(x \twoheadrightarrow y \Rightarrow \pi_2(x) <_2 \pi_2(y) \wedge \neg(\pi_2(x) <_2 z <_2 \pi_2(y)) \wedge \pi_1(x) = \pi_1(y))$
- $(\forall x \in A)(\forall y \in A)(\forall z \in A)(\forall u \in A)(x \rightarrow y \wedge x \twoheadrightarrow z \wedge y \twoheadrightarrow u \Rightarrow z \rightarrow u)$
- $(\forall x \in A)(\forall y \in A)(\forall z \in A)(\forall u \in A)(x \rightarrow y \wedge x \twoheadrightarrow z \wedge z \rightarrow u \Rightarrow y \twoheadrightarrow u)$
- $(\forall x \in A)(\forall y \in A)(\forall z \in A)(\forall u \in A)(x \rightarrow y \wedge z \rightarrow u \wedge y \twoheadrightarrow u \Rightarrow x \twoheadrightarrow z)$
- $(\forall x \in A)(\forall y \in A)(\forall z \in A)(\forall u \in A)(x \twoheadrightarrow z \wedge z \rightarrow u \wedge y \twoheadrightarrow u \Rightarrow x \rightarrow y)$

3. Axioms which describe the labelling by χ and the detection of the halting state:

   - For each $s \in S$
   $$(\forall x \in A)(\chi(x) = (s, q_0) \Rightarrow c = d)$$
   - $(\forall x \in C)(\bigvee\{x = s : s \in S \cup (S \times Q) \cup \{c, d, \infty'\}\} \wedge \bigwedge\{c_1 \neq c_2 : c_1, c_2 \in C, \text{ except for } \{c_1, c_2\} = \{c, d\}\})$
   - $(\forall x \in A)(\bigvee\{\chi(x) = c : c \in S \cup (S \times Q) \cup \{\diamondsuit, \heartsuit, \infty'\}\})$
   - $(\forall x \in A)(\forall y \in A)(\pi_2(x) = \pi_2(y) \wedge \chi(x) = \chi(y) = \heartsuit \Rightarrow x = y)$
   - $(\forall x \in A)(\chi(x) = \infty' \Leftrightarrow x = \infty)$
   - For $(s, q), (s', q') \in S \times Q$:

     $(\forall x \in A)(\forall y \in A)(\pi_2(x) = \pi_2(y) \wedge \chi(x) = (s, q) \wedge \chi(y) = (s', q') \Rightarrow x = y)$,
   - $(\forall x \in A)\neg(x \rightarrow \infty \vee x \twoheadrightarrow \infty)$
   - $(\forall x \in A)(\forall y \in A)\neg(x \twoheadrightarrow y \wedge \chi(y) = \heartsuit)$

- $(\forall x \in A)(\forall y \in A)\neg(\chi(x) = \Diamond \wedge \chi(y) = \heartsuit \wedge x \to y)$

4. Axioms which describe *bord* operation:

   - $(\forall x \in A)(\forall y \in A)(x \to y \wedge \chi(y) = \heartsuit \Rightarrow g(x,y) \to bord(x,y) \wedge \chi(bord(x,y)) = \heartsuit)$
   - $(\forall x \in A)(\forall y \in A)(\neg x \to y \vee \chi(y) \neq \heartsuit \Rightarrow g(x,y) = \infty)$

5. Axioms which describe operation g:

   - $(\forall x \in A)(\forall y \in A)(x \to y \wedge \chi(y) = \heartsuit \wedge \chi(x) = s \ (s \in S) \Rightarrow y \twoheadrightarrow g(x,y) \wedge \chi(g(x,y)) = B)$
   - If $T(s,q) = (s', q', R)$:

     $(\forall x \in A)(\forall y \in A)(x \to y \wedge \chi(y) = \heartsuit \wedge \chi(x) = (s,q) \Rightarrow y \twoheadrightarrow g(x,y) \wedge \chi(g(x,y)) = (B,q'))$
   - If $T(s,q) = (s', q', L)$ or $T(s,q) = (s', q', N)$:

     $(\forall x \in A)(\forall y \in A)(x \to y \wedge \chi(y) = \heartsuit \wedge \chi(x) = (s,q) \Rightarrow y \twoheadrightarrow g(x,y) \wedge \chi(g(x,y)) = B)$.

6. Axioms which describe *diam* operation:

   - $(\forall x \in A)(\forall y \in A)(x \to y \wedge \chi(x) = \Diamond \Rightarrow x \twoheadrightarrow diam(x,y) \wedge \chi(diam(x,y)) = \Diamond)$
   - $(\forall x \in A)(\forall y \in A)(\neg x \to y \vee \chi(x) \neq \Diamond \Rightarrow diam(x,y) = \infty)$

7. Axioms which describe the operation f: for all $s, s', s_{i_1}, s_{i_2}, s_{i_3}, s_{i_4} \in S$:

- $(\forall x \in A)(\forall y \in A)(\forall z \in A)(\forall w \in A)(x \to y \to z \to w \wedge ((\chi(x) = s_{i_1}) \vee (\chi(x) = \diamondsuit)) \wedge \chi(y) = s_{i_2} \wedge \chi(z) = s_{i_3} \wedge ((\chi(w) = s_{i_4}) \vee (\chi(w) = \heartsuit)) \Rightarrow z \twoheadrightarrow f(x,y,z,w) \wedge \chi(f(x,y,z,w)) = s_{i_3})$.
- If $T(s_{i_4}, q) = (s', q', N)$, or $T(s_{i_4}, q) = (s', q', R)$:

  $(\forall x \in A)(\forall y \in A)(\forall z \in A)(\forall w \in A)(x \to y \to z \to w \wedge ((\chi(x) = s_{i_1}) \vee (\chi(x) = \diamondsuit)) \wedge \chi(y) = s_{i_2} \wedge \chi(z) = s_{i_3} \wedge \chi(w) = (s_{i_4}, q) \Rightarrow z \twoheadrightarrow f(x,y,z,w) \wedge \chi(f(x,y,z,w)) = s_{i_3})$.
- If $T(s_{i_4}, q) = (s', q', L)$:

  $(\forall x \in A)(\forall y \in A)(\forall z \in A)(\forall w \in A)(x \to y \to z \to w \wedge ((\chi(x) = s_{i_1}) \vee (\chi(x) = \diamondsuit)) \wedge \chi(y) = s_{i_2} \wedge \chi(z) = s_{i_3} \wedge \chi(w) = (s_{i_4}, q) \Rightarrow z \twoheadrightarrow f(x,y,z,w) \wedge \chi(f(x,y,z,w)) = (s_{i_3}, q'))$.
- If $T(s_{i_3}, q) = (s', q', N)$:

  $(\forall x \in A)(\forall y \in A)(\forall z \in A)(\forall w \in A)(x \to y \to z \to w \wedge ((\chi(x) = s_{i_1}) \vee (\chi(x) = \diamondsuit)) \wedge \chi(y) = s_{i_2} \wedge \chi(z) = (s_{i_3}, q) \wedge ((\chi(w) = s_{i_4}) \vee (\chi(w) = \heartsuit)) \Rightarrow z \twoheadrightarrow f(x,y,z,w) \wedge \chi(f(x,y,z,w)) = (s', q'))$.
- If $T(s_{i_3}, q) = (s', q', L)$ or $T(s_{i_3}, q) = (s', q', R)$

  $(\forall x \in A)(\forall y \in A)(\forall z \in A)(\forall w \in A)(x \to y \to z \to w \wedge ((\chi(x) = s_{i_1}) \vee (\chi(x) = \diamondsuit)) \wedge \chi(y) = s_{i_2} \wedge \chi(z) = (s_{i_3}, q) \wedge ((\chi(w) = s_{i_4}) \vee (\chi(w) = \heartsuit)) \Rightarrow z \twoheadrightarrow f(x,y,z,w) \wedge \chi(f(x,y,z,w)) = s')$.
- If $T(s_{i_2}, q) = (s', q', L)$ or $T(s_{i_2}, q) = (s', q', N)$

  $(\forall x \in A)(\forall y \in A)(\forall z \in A)(\forall w \in A)(x \to y \to z \to w \wedge ((\chi(x) = s_{i_1}) \vee (\chi(x) = \diamondsuit)) \wedge \chi(y) = (s_{i_2}, q) \wedge \chi(z) = s_{i_3} \wedge ((\chi(w) = s_{i_4}) \vee (\chi(w) = \heartsuit)) \Rightarrow z \twoheadrightarrow f(x,y,z,w) \wedge \chi(f(x,y,z,w)) = s_{i_3})$.

- If $T(s_{i_2}, q) = (s', q', R)$:

  $(\forall x \in A)(\forall y \in A)(\forall z \in A)(\forall w \in A)(x \to y \to z \to w \wedge ((\chi(x) = s_{i_1}) \vee (\chi(x) = \diamondsuit)) \wedge \chi(y) = (s_{i_2}, q) \wedge \chi(z) = s_{i_3} \wedge ((\chi(w) = s_{i_4}) \vee (\chi(w) = \heartsuit)) \Rightarrow z \twoheadrightarrow f(x,y,z,w) \wedge \chi(f(x,y,z,w)) = (s_{i_3}, q')).$

- $(\forall x \in A)(\forall y \in A)(\forall z \in A)(\forall w \in A)(x \to y \to z \to w \wedge (\chi(x) = (s_{i_1}, q) \wedge \chi(y) = s_{i_2} \wedge \chi(z) = s_{i_3} \wedge ((\chi(w) = s_{i_4}) \vee (\chi(w) = \heartsuit)) \Rightarrow z \twoheadrightarrow f(x,y,z,w) \wedge \chi(f(x,y,z,w)) = s_{i_3})$

- $(\forall x \in A)(\forall y \in A)(\forall z \in A)(\forall w \in A)($all previous antecedents in this group false $\Rightarrow f(x,y,z,w) = \infty)$

8. Axioms which describe operation h: for all $s, s', s_{i_1}, s_{i_2}, s_{i_3} \in S$:

- $(\forall x \in A)(\forall y \in A)(\forall z \in A)(x \to y \to z \wedge \chi(x) = \diamondsuit \wedge \chi(y) = s_{i_2} \wedge ((\chi(z) = s_{i_3}) \vee (\chi(z) = \heartsuit)) \Rightarrow y \twoheadrightarrow h(x,y,z) \wedge \chi(h(x,y,z)) = s_{i_2})$

- If $T(s_{i_3}, q) = (s', q', N)$ or (s', q', R):

  $(\forall x \in A)(\forall y \in A)(\forall z \in A)(x \to y \to z \wedge \chi(x) = \diamondsuit \wedge \chi(y) = s_{i_2} \wedge \chi(z) = (s_{i_3}, q) \Rightarrow y \twoheadrightarrow h(x,y,z) \wedge \chi(h(x,y,z)) = s_{i_2}).$

- If $T(s_{i_3}, q) = (s', q', L)$:

  $(\forall x \in A)(\forall y \in A)(\forall z \in A)(x \to y \to z \wedge \chi(x) = \diamondsuit \wedge \chi(y) = s_{i_2} \wedge \chi(z) = (s_{i_3}, q) \Rightarrow y \twoheadrightarrow h(x,y,z) \wedge \chi(h(x,y,z)) = (s_{i_2}, q'))$

- If $T(s_{i_2}, q) = (s', q', N)$:

  $(\forall x \in A)(\forall y \in A)(\forall z \in A)(x \to y \to z \wedge \chi(x) = \diamondsuit \wedge \chi(y) = (s_{i_2}, q) \wedge ((\chi(z) = s_{i_3}) \vee (\chi(z) = \heartsuit)) \Rightarrow y \twoheadrightarrow h(x,y,z) \wedge \chi(h(x,y,z)) = (s', q'))$

- If $T(s_{i_2}, q) = (s', q', L)$ or (s', q', R):
  $(\forall x \in A)(\forall y \in A)(\forall z \in A)(x \to y \to z \wedge \chi(x) = \diamond \wedge \chi(y) = (s_{i_2}, q) \wedge ((\chi(z) = s_{i_3} \vee (\chi(z) = \heartsuit)) \Rightarrow y \twoheadrightarrow h(x,y,z) \wedge \chi(h(x,y,z)) = s')$
- $(\forall x \in A)(\forall y \in A)(\forall z \in A)$(all previous previous antecedents in this group are false $\Rightarrow h(x,y,z) = \infty$)

Definition 4.1 *$\mathcal{K}$ is the class of structures in the multisorted language $\mathcal{L}$ axiomatized by the axioms listed above.*

Clearly, $\mathcal{K}$ is a universal class which is finitely axiomatizable. For the sake of further refernce, we denote this set of axioms Φ.

4.2 $Th_\forall(\mathcal{K})$ is undecidable

In this section we prove that $Th_\forall(\mathcal{K})$ is undecidable, where $\mathcal{K}$ is the class of multisorted structures defined in Section 4.1.

Lemma 4.2 *Let $\mathcal{K}$ be the class of multisorted structures defined in Section 4.1.*

Given $C \in Config$, if the Turing machine T does not halt when started on C, then $\mathbf{S}^{=}_{C}, \mathbf{S}^{\neq}_{C} \in \mathcal{K}$.

If $C \in Config$ is such that T halts when started on C, then $\mathbf{S}^{=}_{C} \in \mathcal{K}$, while $\mathbf{S}^{\neq}_{C} \notin \mathcal{K}$.

Proof. By inspection, for every $C \in Config$, both $\mathbf{S}^{=}_{C}$ and $\mathbf{S}^{\neq}_{C}$ satisfy all axioms from Section 4.1 , except for the following ones

$$(\forall x \in A)\,(\chi\,(x) = (s, q_0) \Rightarrow c = d)\,. \tag{4.1}$$

If T halts when started on $\mathcal{C}$, then, for some $i, j < \omega$,

$$\chi\left(a_j^{(i)}\right) = (s, q_0),$$

so $\mathbf{S}_{\mathcal{C}}^{=}$ will satisfy the axiom (4.1), while this will not be the case with $\mathbf{S}_{\mathcal{C}}^{\neq}$.

Otherwise, if T does not halt when started on $\mathcal{C}$, we will have

$$\chi\left(a_j^{(i)}\right) \neq (s, q_0),$$

for all $i, j < \omega$, since none of $C^{(i)}$ will contain (s, q_0). Thus, both $\mathbf{S}_{\mathcal{C}}^{=}$ and $\mathbf{S}_{\mathcal{C}}^{\neq}$ will satisfy (4.1). □

The following lemma is obvious:

Lemma 4.3 *In any* $\mathbf{S} \in \mathcal{K}$,

1. $(P_i^{\mathbf{S}}, \leqslant_i^{\mathbf{S}})$ *is a linear ordering,* $i = 1, 2$;
2. $A^{\mathbf{S}}$ *is coordinatized by* $(P_1^{\mathbf{S}}, \leqslant_1^{\mathbf{S}}) \times (P_2^{\mathbf{S}}, \leqslant_2^{\mathbf{S}})$, *in the sense that if* $a, b \in A^{\mathbf{S}}$ *and*

$$(\pi_1^{\mathbf{S}}(a), \pi_2^{\mathbf{S}}(a)) = (\pi_1^{\mathbf{S}}(b), \pi_2^{\mathbf{S}}(b)),$$

then $a = b$.

Lemma 4.4 *If* $\mathbf{S} \in \mathcal{K}$ *and* $X \subseteq A^{\mathbf{S}}$ *are such that* $X = \{a_0, \dots, a_m\}$, *where*

$$a_0 \to a_1 \to \dots \to a_{m-1} \to a_m,$$

$$\chi^{\mathbf{S}}(a_0) = \diamondsuit, \chi^{\mathbf{S}}(a_m) = \heartsuit,$$

$$\chi^{\mathbf{S}}(a_i) \in S \cup (S \times Q), 1 \leqslant i \leqslant m-1,$$

so that for at most one i such that $1 \leqslant i \leqslant m-1$,

$$\chi^{\mathbf{S}}(a_i) \in S \times Q,$$

and if $\mathbf{S}_1$ is the substructure of $\mathbf{S}$ generated by X, then $\mathbf{S}_1$ is isomorphic to a computation structure.

PROOF. Let C be the following configuration

$$\chi(a_1)\dots\chi(a_{m-1}).$$

We claim that $\mathbf{S}_1$ is one of the two computation structures corresponding to C. First, we define inductively, for $i \geq 0$, the set $A^{(i)}$: let $A^{(0)} = X$, and for $i > 0$,

$$A^{(i)} = \left\{x \in A^{\mathbf{S}} : \ x = F(a_1, \dots, a_n), \text{ where } F : A^n \to A \text{ is in } \mathcal{L} \text{ and } a_j \in A^{(i-1)}\right\}.$$

First, observe the following: if $x, y \in A^{\mathbf{S}}$ are such that $x \to y$, then there is no element $z \in P_1^{\mathbf{S}}$ such that

$$\pi_1^{\mathbf{S}}(x) <_1 z <_2 \pi_1^{\mathbf{S}}(y),$$

and similarly for P_2 and π_2.

Using induction on i and the axioms for $\mathcal{K}$, it is easy to see that if $x, y \in A^{(i)}$, for some $i < \omega$, then

$$\pi_2(x) = \pi_2(y).$$

The axioms for the operations of the sort A show that these operations output ∞ unless all arguments belong to the same $A^{(i)}$, for some $i \geq 0$. Hence,

$$A^{\mathbf{S}_1} = \bigcup_{i<\omega} A^{(i)} \cup \{\infty^{\mathbf{S}}\}.$$

This, along with the axioms for $\twoheadrightarrow$, implies that $P_2^{\mathbf{S}_1}$ is a well-ordered set of type $\omega + 1$.

By inspecting the axioms for the operations $f, g, h, diam$,and $bord$, we can see that if $x \in A^{(i)}$, then either there is $y \in A^{(i-1)}$ such that $y \twoheadrightarrow x$, or $\chi(x) = \heartsuit$ and for some $x' \in A^{(i)}$, $x' \rightarrow x$.

Thus, $\pi_1^{\mathbf{S}}(A^{(1)})$ forms a finite convex subset of $(P_1^{\mathbf{S}}, \leqslant_1^{\mathbf{S}})$. Another induction on i shows that $\pi_1^{\mathbf{S}}(A^{(i)})$ will be a finite linear ordering, which will be an initial segment of $\pi_1^{\mathbf{S}}(A^{(i+1)})$, for every $i < \omega$.

Hence, $\bigcup_{i<\omega} \pi_1^{\mathbf{S}}(A^{(i)}) \cup \{\infty^{\mathbf{S}_1}\}$ will be a well-ordering isomorphic to $(\omega + 1, \leqslant)$.

In light of the previous remarks, we can fix two isomorphisms

$$\phi_1 : (P_1^{\mathbf{S}_1}, \leqslant_1^{\mathbf{S}_1}) \rightarrow (\omega + 1, \leqslant)$$
$$\phi_2 : (P_2^{\mathbf{S}_1}, \leqslant_2^{\mathbf{S}_1}) \rightarrow (\omega + 1, \leqslant)$$

Next, we proceed to the construction of an isomorphism between $\mathbf{S}_{\mathcal{C}}^{=}$ (or $\mathbf{S}_{\mathcal{C}}^{\neq}$) and $\mathbf{S}_1$. The choice between $\mathbf{S}_{\mathcal{C}}^{=}$ and $\mathbf{S}_{\mathcal{C}}^{\neq}$ will be determined by whether $c = d$ in $\mathbf{S}_1$ or not. For the sake of simplicity, we assume that $c = d$ in $\mathbf{S}_1$, the proof for the other case being analogous.

Clearly, care only needs to be taken as to how this isomorphism should be defined on the sorts A, P_1, and P_2.

We fix the notation from Section 4.1, and assume that the elements of the A-part of $\mathbf{S}_{\mathcal{C}}^{\mp}$ are

$$\{a_j^{(i)} : 0 \leqslant i < \omega, 0 \leqslant j \leqslant m + i\} \cup \{\infty\}.$$

We define the mapping

$$\psi : A^{\mathbf{S}_{\mathcal{C}}^{\mp}} \to A^{\mathbf{S}_1},$$

in the following way: $\psi(a_j^{(i)}) = x$, where $\phi_1(\pi_1^{\mathbf{S}_1}(x)) = j$ and $\phi_2(\pi_2^{\mathbf{S}_1}(x)) = i$. Also, $\psi(\infty) = \infty^{\mathbf{S}_1}$.

Clearly, ψ will be defined for every element $a_j^{(i)}$ since the corresponding x will be the element in $A^{(i)}$ such that $\psi_1(\pi_1^{\mathbf{S}_1}(x)) = j$. Also, it is immediate to verify that ψ is well-defined and injective.

Now, we show that ψ, together with ϕ_1^{-1} and ϕ_2^{-1}, defines the required isomorphism.

The axioms for $\mathcal{K}$ will now imply the following:

$$\psi(a_j^{(i)}) \to \psi(a_{j+1}^{(i)}), \text{ for } j = 0, \dots, m + i - 1$$

and

$$\psi(a_j^{(i)}) \twoheadrightarrow \psi(a_j^{(i+1)}), \text{ for } j = 0, \dots, m + i.$$

The first axiom in the sixth group shows that

$$\chi\left(\psi(a_0^{(i)})\right) = \diamond,$$

for all $i < \omega$. Similarly, the first axiom in the fourth group yields

$$\chi\left(\psi(a^{(i)}_{m+i})\right) = \heartsuit,$$

for every $i < \omega$.

By analysing various cases for C', C'', and δ, and using the axioms from the groups 4-8, one can show that if $C' \in Config$ is of the form

$$\chi\left(\psi(a^{(i)}_1)\right) \ldots \chi\left(\psi(a^{(i)}_{(m-1)+i})\right),$$

and $T(C') = C''$, then

$$C'' = \chi\left(\psi(a^{(i+1)}_1)\right) \ldots \chi\left(\psi(a^{(i+1)}_{(m-1)+i+1})\right).$$

The second axiom in the third group will ensure that all pairs of elements from the constant sort other than c, d are distinct.

Thus, $\mathbf{S}_1$ will be isomorphic to a computation structure for the configuration C. □

Definition 4.5 *Let*

$$C = (s_{i_1}, q_1)s_{i_2} \ldots s_{i_n}$$

be an initial configuration of the Turing machine T. We define θ_C, the halting sentence *corresponding to C to be:*

$$(\forall x_0 x_1 \cdots x_{n+1} \in A)(x_0 \to x_1 \to \cdots \to x_n \to x_{n+1} \wedge$$

$$\chi(x_0) = \Diamond \wedge \chi(x_1) = (s_{i_1}, q_1) \wedge \cdots \wedge \chi(x_n) = s_{i_n} \wedge \chi(x_{n+1}) = \heartsuit \Rightarrow c = d)$$

Proposition 4.6 *For any initial configuration $\mathcal{C}$ of T, $\theta_{\mathcal{C}} \in Th_{\forall}(\mathcal{K})$ if and only if T halts when started on the configuration $\mathcal{C}$.*

PROOF. Suppose the halting sentence is in the theory. Let $\mathbf{S}$ be a computation structure for $\mathcal{C}$. Clearly, the elements corresponding to $\mathcal{C}$ in $\mathbf{S}$ will fulfill the hypotheses of $\theta_{\mathcal{C}}$. Thus,

$$\mathbf{S} \models c = d.$$

In other words, $\mathbf{S}_{\mathcal{C}}^{=} \notin \mathcal{K}$. Thus, by Lemma 4.2, T halts when started on $\mathcal{C}$.

Conversely, suppose $\theta_{\mathcal{C}} \notin Th_{\forall}(\mathcal{K})$. Then, there exists a structure $\mathbf{S} \in \mathcal{K}$ and elements in S which witness the failure of $\theta_{\mathcal{C}}$ in $\mathbf{S}$. These elements will fulfill all the hypotheses of $\theta_{\mathcal{C}}$, yet $c \neq d$ in $\mathbf{S}$. The selected elements will constitute a line encoding a configuration, and according to Lemma 4.2, the substructure $\mathbf{S}_1$ of $\mathbf{S}$ generated by this line will be a configuration structure, in fact one which corresponds to $\mathcal{C}$. Since $c \neq d$ in $\mathbf{S}_1$, $\mathbf{S}_1 = \mathbf{S}_{\mathcal{C}}^{=}$. As $\mathbf{S}_1 \in \mathcal{K}$, it follows by Lemma 4.2 that T does not halt when started on $\mathcal{C}$. □

As an immediate consequence of Proposition 4.6, we obtain

Theorem 4.7 *$Th_{\forall}(\mathcal{K})$ is undecidable.*

4.3 The structure of (n, n, n, n)–generated structures of $\mathcal{K}$

Let $n \geq 1$ and let $\mathbf{S} = \langle A^{\mathbf{S}}, P_1^{\mathbf{S}}, P_2^{\mathbf{S}}, C^{\mathbf{S}} \rangle$ be an (n, n, n, n)–generated member of $\mathcal{K}$, where the value of n will be fixed throughout this section.

Assume $\mathbf{S} = Sg(X, Y_1, Y_2, Z)$, where

$$|X|, |Y_1|, |Y_2|, |Z| \leqslant n.$$

We define $\preccurlyeq$ to be the reflexive-transitive closure of $\rightarrow^{\mathbf{S}} \cup \twoheadrightarrow^{\mathbf{S}}$ in $A^{\mathbf{S}}$.

Lemma 4.8 *For every $a \in A^{\mathbf{S}} \setminus \{\infty^{\mathbf{S}}\}$, there exists $x \in X$ such that $x \preccurlyeq a$.*

PROOF. For an arbitrary subset $B \subseteq A^{\mathbf{S}}$, define

$$E(B) = (\{y \in A^{\mathbf{S}} : y = F^{\mathbf{S}}(x_1, \ldots, x_m), \text{ for some } x_1, \ldots, x_m \in B, \text{ and some } F : A^m \to A \text{ in } \mathcal{L}\} \cup B) \setminus \{\infty^{\mathbf{S}}\}.$$

Now, let

$$E^{(0)}(X) = X,$$

$$E^{(i+1)}(X) = E(E^{(i)}(X)), \qquad \text{for } i < \omega.$$

It is easily seen that, for $i \leqslant j$,

$$E^{(i)}(X) \subseteq E^{(j)}(X),$$

and

$$A^{\mathbf{S}} \setminus \{\infty^{\mathbf{S}}\} = \bigcup_{i<\omega} E^{(i)}(X).$$

We prove the lemma by induction on i such that $a \in E^{(i)}(X)$.

If $i = 0$, then $a \in E^{(0)}(X) = X$, so $x = a$ would do.

Suppose that the lemma is true for every $i < k$ and let

$$a \in E^{(k)}(X).$$

Then, either $a \in E^{(k-1)}(X)$ and we are done, or $a = F^{\mathbf{S}}(b_1, \ldots, b_m)$, for some $F : A^m \to A$ and $b_1, \ldots, b_m \in E^{(k-1)}$, so that, for some i, $b_i \twoheadrightarrow a$, or, in case when $a = bord(b_1, b_2)$, for some $b_1, b_2 \in E^{(k-1)}(X)$,

$$b_2 \twoheadrightarrow g(b_1, b_2) \to a.$$

In any case, the induction hypothesis yields

$$x \preccurlyeq a$$

for some $x \in X$. □

Lemma 4.9 *For every $b \in P_i^{\mathbf{S}}, (i = 1, 2)$, either $b \in Y_i$ or $b = \pi_i(x)$, for some $x \in A^{\mathbf{S}}$.*

Proof. Immediate, from the fact that the only operation which involves elements from $P_i^{\mathbf{S}}$ is $\pi_i^{\mathbf{S}} : A^{\mathbf{S}} \to P_i^{\mathbf{S}}$. □

Let $X = \{x_1, \dots, x_n\}$. For $1 \leqslant j \leqslant m$ we define

$$B_j = \{a \in A^{\mathsf{S}} : x_j \preccurlyeq a\}.$$

Then, from Lemma 4.8, we conclude that

$$A^{\mathsf{S}} \setminus \{\infty^{\mathsf{S}}\} = B_1 \cup \dots \cup B_n.$$

Lemma 4.10 *For every $j \in \{1, \dots, n\}$,*

$$\pi_i^{\mathsf{S}}(B_j) = \{\pi_i^{\mathsf{S}}(a) : a \in B_j\}$$

is well-ordered by $\leqslant_i$, for $i = 1, 2$.

PROOF. If $x, y \in P_i^{\mathsf{S}}$,$x \neq y$, $x \leqslant_i y$ and there is no $z \in P_i^{\mathsf{S}}$, $z \neq x, z \neq y$ such that

$$x \leqslant_i z \leqslant_i y,$$

we write

$$x \prec_i y.$$

We prove the lemma for the case $i = 1$, the proof for $i = 2$ being similar.

Let $y \in \pi_1^{\mathsf{S}}(B_j)$ be arbitrary. Then,

$$y = \pi_1^{\mathsf{S}}(a)$$

for some $a \in A^{\mathsf{S}}$ such that $x_j \preccurlyeq a$. Let

$$x_j = b_0 \hookrightarrow_1 b_1 \hookrightarrow_2 \cdots \hookrightarrow_{m-1} b_{m-1} \hookrightarrow_m b_m = a,$$

where $\hookrightarrow_1, \dots, \hookrightarrow_m \in \{\to^{\mathsf{S}} \twoheadrightarrow^{\mathsf{S}}\}$, $b_1, \dots, b_{m-1} \in A^{\mathsf{S}}$. We can also assume that $b_0, b_1, \dots, b_m$ are all distinct.

Let $\{c_1, \dots, c_{l-1}\} = \{\pi_1^{\mathsf{S}}(b_i) : 1 \leqslant i \leqslant l-1\}$, so that

$$c_1 \leqslant_1 c_2 \leqslant_1 \dots \leqslant_1 c_{l-1}.$$

If $\hookrightarrow_k = \to^{\mathsf{S}}$, then

$$\pi_1^{\mathsf{S}}(b_{k-1}) \prec_1 \pi_1^{\mathsf{S}}(b_k).$$

Claim: If $c_i \leqslant_1 c_{i+1}$ then there exists $k \leqslant l$ such that $c_i = \pi_1^{\mathsf{S}}(b_{k-1})$ and $c_{i+1} = \pi_1^{\mathsf{S}}(b_k)$.

Proof of the claim: Let b_p and b_q $(1 \leqslant p, q \leqslant l-1)$ be such that

$$c_i = \pi_1^{\mathsf{S}}(b_p), c_{i+1} = \pi_1^{\mathsf{S}}(b_q).$$

It is now easy to see that, in this case, $p < q$. We have already seen that the axioms for $\mathcal{K}$ imply the following: if $x \to^{\mathsf{S}} y$ then

$$\pi_1^{\mathsf{S}}(x) <_1 \pi_1^{\mathsf{S}}(y),$$

and there is no $z \in P_1^{\mathbf{S}}$ such that

$$\pi_1^{\mathbf{S}} <_1 z <_1 \pi_1^{\mathbf{S}}(y).$$

This shows that $p < q$ and, since

$$b_p \hookrightarrow_{p+1} b_{p+1} \hookrightarrow_{p+2} \dots \hookrightarrow_{q-1} b_{q-1} \hookrightarrow_q b_q,$$

where $\hookrightarrow_k \in \{\rightarrow^{\mathbf{S}}, \twoheadrightarrow^{\mathbf{S}}\}$ $(p+1 \leqslant k \leqslant q)$, precisely one of $\hookrightarrow_k$ can be $\rightarrow^{\mathbf{S}}$. Let r be such that $\hookrightarrow_r = \rightarrow^{\mathbf{S}}$. Then,

$$\pi_1^{\mathbf{S}}(b_p) = \pi_1^{\mathbf{S}}(b_r),\ \pi_1^{\mathbf{S}}(b_{r+1}) = \pi_1^{\mathbf{S}}(b_q)$$

and the claim is proved.

Hence,

$$\pi_1^{\mathbf{S}}(x_j) \prec_1 c_1 \prec_1 \cdots \prec_1 c_{l-1} \prec_1 y.$$

Also, $\pi_1^{\mathbf{S}}(B_j)$ is at most countable, since $A^{\mathbf{S}}$ is such. Thus, $\pi_1^{\mathbf{S}}(B_j)$ is either a finite well–ordering, or a countable one isomorphic to $(\omega, \leqslant)$. □

Proposition 4.11 *$(P_i^{\mathbf{S}}, \leqslant_i)$ is a well–ordering, for $i = 1, 2$.*

Proof. If $\mathbf{S} = Sg(X, Y_1, Y_2, Z)$, due to Lemma 4.9,

$$P_i^{\mathbf{S}} = Y_i \cup \pi_i^{\mathbf{S}}(B_1) \cup \cdots \cup \pi_i^{\mathbf{S}}(B_n) \cup \{\infty_i^{\mathbf{S}}\}.$$

Clearly, each of the sets in this finite union is well–ordered, with respect to the restriction of $\leqslant_i$, Y_i and $\{\infty_i^{\mathsf{S}}\}$ being finite, and $\pi_i^{\mathsf{S}}(B_j)$ by Lemma 4.10.

Since every linearly ordered set that is a finite union of well–ordered sets is itself well–ordered, the proposition is proved. □

Thus,

$$(P_i^{\mathsf{S}}, \leqslant_i) \cong (\lambda_i + 1, \leqslant),$$

where λ_i is some at most countable ordinal which is yet to be determined.

From now on, we shall assume that

$$P_i^{\mathsf{S}} = \lambda_i + 1, \qquad i = 1, 2.$$

A **(horizontal) line** in A^{S} is the set of all elements with the same P_2–coordinate; i. e. if $m \in \lambda_2$, the m-th horizontal line in A^{S} is

$$l_m = \{x \in A^{\mathsf{S}} : \pi_2^{\mathsf{S}}(x) = m\}.$$

Definition 4.12 *If l is a horizontal line in A^{S}, then its* index *is an ordinal $m < \lambda_2$ such that*

$$l = l_m = \{x \in A^{\mathsf{S}} : \pi_2(x) = m\}.$$

Lemma 4.13 *Every horizontal line in A^{S} is finite.*

PROOF. We use transfinite induction on the index of a line.

1° If the index m of a line is a limit ordinal or 0, then $l_m \subseteq X$, since there is neither an $a \in A^{\mathsf{S}}$ such that

$$a \twoheadrightarrow x,$$

nor $b, c \in A^{\mathsf{S}}$ such that

$$b \twoheadrightarrow c \rightarrow x.$$

Since X is finite, then so is l_0.

2° Now, assume that μ is a successor ordinal in λ_2, $\mu = \nu + 1$, and consider l_μ. Every $x \in l_\mu$ is either a generator in X or $F^{\mathsf{S}}(a_1, \ldots, a_k)$, for some $F : A^k \rightarrow A$, and $a_1, \ldots, a_k \in l_\nu$. Since l_ν is finite (by the induction hypothesis) and $\mathcal{L}$ is such, l_μ is finite as well. □

Definition 4.14 *A* ***configuration line*** *in A^{S} is a line consisting of $a_0, a_1, \ldots, a_m$, $a_{m+1} \in A^{\mathsf{S}}$, such that*

$$a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_m \rightarrow a_{m+1},$$

$$\chi^{\mathsf{S}}(a_0) = \diamondsuit, \chi^{\mathsf{S}}(a_{m+1}) = \heartsuit,$$

$$\chi^{\mathsf{S}}(a_i) \in S \cup (S \times Q), \qquad 1 \leqslant i \leqslant m,$$

so that for at most one $1 \leqslant i \leqslant m$,

$$\chi^{\mathsf{S}}(a_i) \in S \times Q.$$

Definition 4.15 *A* ***non–configuration segment*** *in A^{S} is a sequence $a_0, a_1, \ldots$, $a_k \in A^{\mathsf{S}}$ such that*

$$a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_k,$$

and either

(a) $\chi^{\mathsf{S}}(a_i) \in S \cup (S \times Q)$, for $0 \leqslant i \leqslant k$, or

(b) $\chi^{\mathbf{S}}(a_0) = \diamondsuit, \chi^{\mathbf{S}}(a_i) \in S \cup (S \times Q)$, $1 \leqslant i \leqslant k$, or

(c) $\chi^{\mathbf{S}}(a_k) = \heartsuit, \chi^{\mathbf{S}}(a_i) \in S \cup (S \times Q)$, $0 \leqslant i \leqslant k-1$.

Note that the axioms in Φ imply that, in every line, at most one element has a label in $(S \times Q)$.

Lemma 4.16 *Non-configuration segments generate finite substructures of* $\mathbf{S}$ *whose A-parts, omitting* $\infty^{\mathbf{S}}$*, have one of the three forms in the Figure 4.3*

PROOF. The proof follows from the axioms which determine the behaviour of *diam*,*f*,*g*,*h* and *bord*. □

Lemma 4.17 *Each line in* $A^{\mathbf{S}}$ *is either:*

1. *A configuration line, or*

2. *one or more* $\rightarrow$ *-connected components, each of which is a nonconfiguration segment.*

PROOF. By Lemma 4.13, the ninth and the tenth axiom from group 3, and the first six axioms of group 2. □

We define the binary relation $\sim$ on $A^{\mathbf{S}}$ to be the equivalence relation on $A^{\mathbf{S}}$ generated by $\rightarrow^{\mathbf{S}} \cup \twoheadrightarrow^{\mathbf{S}}$; i.e. $\sim$ is the smallest equivalence relation on $A^{\mathbf{S}}$ containing $\rightarrow^{\mathbf{S}} \cup \twoheadrightarrow^{\mathbf{S}}$.

Definition 4.18 *A* $(\rightarrow \cup \twoheadrightarrow)$*-**connected component** (or, simply,* $(\rightarrow \cup \twoheadrightarrow)$*-component) in* $A^{\mathbf{S}}$ *is an equivalence class of the binary relation* $\sim$*, other than* $\{\infty^{\mathbf{S}}\}$.

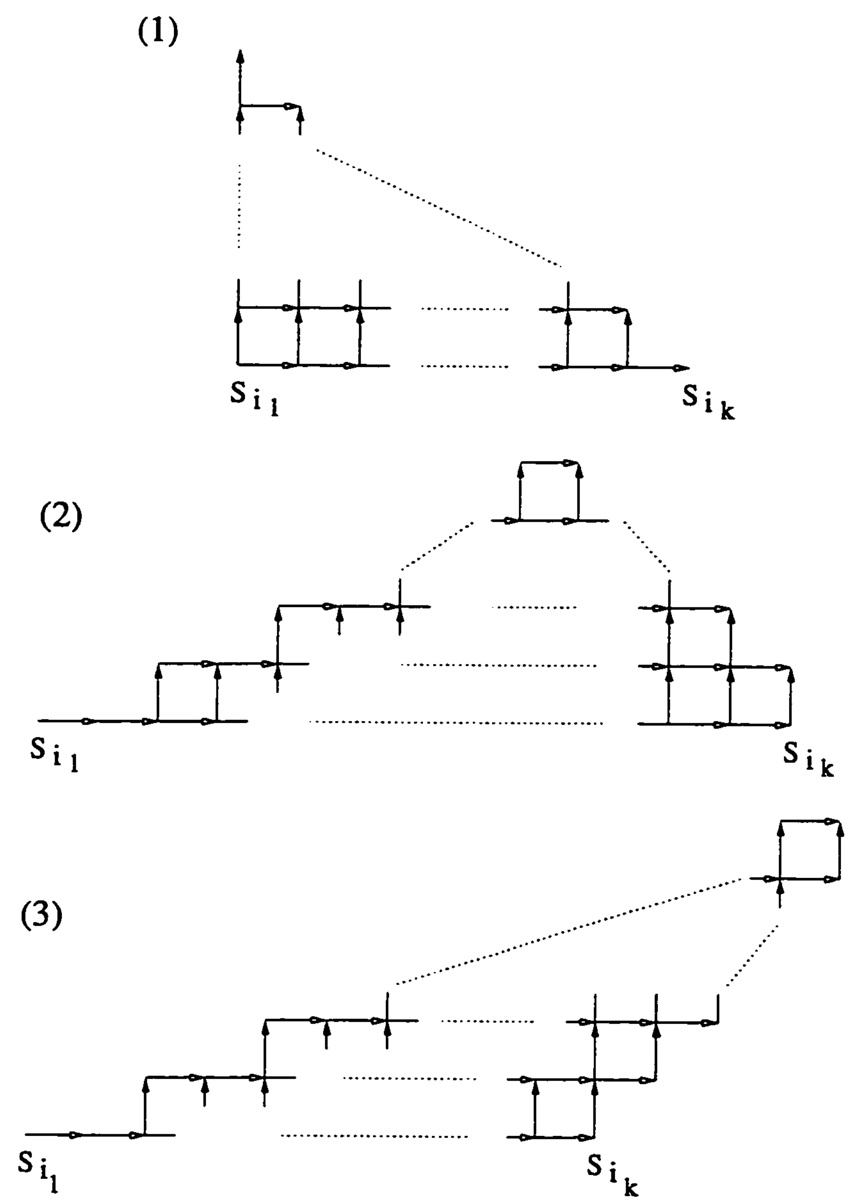

Figure 4.3: Structures generated by non-configuration segments

Let B be an arbitrary $(\rightarrow \cup \twoheadrightarrow)$–component in A^{S}, and let

$$X_B = X \cap B.$$

Let $l_{\mu_1}, \ldots, l_{\mu_k}$, where $\mu_1 < \mu_2 < \cdots < \mu_k$, be the lines in A^{S} in which the elements of X_B occur. Clearly, since

$$|X_B| \leqslant n,$$

where n is the number of elements in X, we have

$$k \leqslant n.$$

Also, define

$$X_i = X_B \cap l_{\mu_i}, \qquad 1 \leqslant i \leqslant k.$$

Then,

$$X_B = X_1 \cup \cdots \cup X_k.$$

Suppose that

$$|X_i| = m_i, \qquad 1 \leqslant i \leqslant k,$$

which yields

$$n \geqslant |X_B| = m_1 + \cdots + m_k.$$

Lemma 4.19 *Let B be an arbitrary $(\rightarrow \cup \twoheadrightarrow)$–component. Let $\mu_1, \ldots, \mu_k$, $m_1, \ldots,$ m_k be as defined above.*

If none of

$$B \cap l_{\mu_1}, B \cap l_{\mu_2}, \ldots, B \cap l_{\mu_k}$$

are configuration lines, then

1. *$B \cap l_j$ is not a configuration line for any $j < \lambda_2$.*

2. *$B \cap l_j = \emptyset$ for $j < \mu_1$.*

3. *$|B \cap l_j| \leqslant \max(0, (\sum_{i:\mu_i \leqslant j} m_i) - (j - \mu_1))$, for all $j < \lambda_2$ such that $j \geqslant \mu_1$.*

PROOF. (1) If $j \leqslant \mu_k$, $B \cap l_j$ cannot be a configuration line, for, otherwise, l_{μ_k} would be contained in a computation structure generated by $B \cap l_j$, which is impossible.

If $j > \mu_k$ and $B \cap l_j$ is a configuration line for which j is minimal, we get an immediate contradiction, since the axioms would imply that $l_{j'}$, where $j = j' + 1$, is a configuration line as well.

(2) If $x \in B \cap l_j$, for some $j < \mu_1$ then, either $x \in X_B$ or $x = t(a_1, \ldots, a_m)$ for some m-ary term t and $a_1, \ldots, a_m \in X$. Clearly, $x \notin X_B$ since $j < \mu_1$. Hence, for some a_j, we must have $a_j \preccurlyeq x$ and a_j would be in the same component as x, which is B. Thus, $a_j \in X_B$, but $\pi_2^{\mathsf{S}}(a_j) \leqslant j < \mu_1$, which is impossible. Therefore, $B \cap l_j = \emptyset$.

(3) The proof is by induction on $j \geqslant \mu_1$.

If $j = \mu_1$, the inequality is obvious, for

$$|B \cap l_j| = |B \cap l_{\mu_1}| = |X_B \cap l_{\mu_1}| = m_1.$$

Suppose j is a successor ordinal and $j = j' + 1$, for some $\mu_1 \leqslant j' \leqslant \lambda_2$. If

$B \cap l_{j'} = \emptyset$, then since B is connected, and $j > j' \geqslant \mu_1$, we must have $B \cap l_j = \emptyset$.

Next, assume that $B \cap l_{j'} \neq \emptyset$. If $j \notin \{\mu_1, \dots, \mu_k\}$ then, by the induction hypothesis,

$$\sum_{i:\mu_i \leqslant j'} m_i > (j' - \mu_1).$$

Thus,

$$\sum_{i:\mu_i \leqslant j} m_i \geqslant (j' - \mu_1) + 1.$$

As $B \cap l_{j'}$ is not a configuration line, and $j \notin \{\mu_1, \dots, \mu_k\}$, then, by Lemma 4.16,

$$\begin{aligned} |B \cap l_j| &\leqslant |B \cap l_{j'}| - 1 \\ &\leqslant (\sum_{i:\mu_i \leqslant j'} m_i) - (j' - \mu_1) - 1 \\ &\leqslant (\sum_{i:\mu_i \leqslant j'+1} m_i) - (j - \mu_1). \end{aligned}$$

On the other hand, if $j = \mu_p$, for some $1 \leqslant p \leqslant k$, then

$$\begin{aligned} |B \cap l_p| &\leqslant |B \cap l_{j'}| - 1 + m_p \\ &\leqslant (\sum_{i:\mu_i \leqslant j'} m_i) - (j' - \mu_1) - 1 + m_p \\ &= (\sum_{i:\mu_i \leqslant j} m_i) - (j' - \mu_1) - 1. \end{aligned}$$

Finally, if j is a limit ordinal $\mu_1 < j < \lambda_2$ then, $B \cap l_j = \emptyset$. For, since B is connected, then, for some $x \in l_j$ and some $y \in A^{\mathrm{S}} \cap B$, $y \twoheadrightarrow x$, which would contradict the fact that j is not a successor ordinal. □

Definition 4.20 *Let $X \subseteq A^{\mathsf{S}}$. We define the height and the width of X in the following way:*

$$height(X) = |\pi_2^{\mathsf{S}}(X)|$$

$$width(X) = |\pi_1^{\mathsf{S}}(X)|.$$

Corollary 4.21 *1. If a $(\rightarrow \cup \twoheadrightarrow)$-component B contains no configuration lines, then B is finite, and*

$$height(B) \leqslant n, \qquad width(B) \leqslant n^2.$$

Moreover, $\pi_1(B)$ and $\pi_2(B)$ are intervals in λ_1 and λ_2 respectively (of lengths $\leqslant n^2$ and $\leqslant n$, respectively).

2. If B does contain a configuration line, and j is the line number of the first configuration line occurring in B, then

$$j < \mu_1 + n.$$

Furthermore, if

$$B_0 = \bigcup_{i \leqslant j} B \cap l_i,$$

and

$$B_1 = \bigcup \{B \cap l_i : i \geqslant j\}.$$

Then,

(a) $\pi_1^{\mathsf{S}}(B_0) = [0, r]$ for some $r < n^3$ (an interval of length $< n^3$), while

$\pi_2^{\mathbf{S}}(B_0) = [\mu_1, j]$ *(an interval of length $\leqslant n$);*

(b) $B_1 \cup \{\infty^{\mathbf{S}}\}$ is identical to the A-part of the substructure generated by $B \cap l_j$.

PROOF. Lemma 4.19 tells us that, in the case when B contains no configuration lines, if $j \geqslant \mu_1$, then

$$|B \cap l_j| \leqslant \max(0, n - (j - \mu_1)).$$

So, if $B \cap l_j \neq \emptyset$,

$$n - (j - \mu_1) > 0$$

and

$$j < \mu_1 + n.$$

Since μ_1 is the line number of the first line in B,

$$height(B) < n.$$

Also, $\pi_2^{\mathbf{S}}(B)$ is an interval in λ_2, for B is connected, and its length is equal to $height(B)$.

Now,

$$height(B) \leqslant n,$$

and

$$\begin{aligned} width(B) &\leqslant \sum_i |B \cap l_i| \\ &\leqslant height(B) \cdot \max_i |B \cap l_i| \\ &\leqslant n^2. \end{aligned}$$

Again, since B is connected, $\pi_1^S(B)$ is an interval in λ_1, and the length of this interval is $width(B)$.

2. Let j be the index of the first configuration line in B. Clearly, l_μ is a configuration line, for every μ, $j < \mu < j + \omega$.

If l_j is not preceded by any other line in B, then, in the notation of Lemma 4.19, $\mu_1 = j$ and

$$j < \mu_1 + n$$

will be trivially true.

Suppose $j = j' + 1$ (notice that this index j cannot be a limit ordinal, unless $j = \mu_1$). Then, $j = \mu_k$, for otherwise there would be no configuration lines occurring in B.

Thus,

$$|B \cap l_{j'}| \leqslant \max(0, m_1 + \ldots + m_{k-1} - (j' - m_1)),$$

and, since $B \cap l_{j'} \neq \emptyset$,

$$j' - \mu_1 < m_1 + \ldots + m_{k-1}.$$

Therefore,

$$j \leqslant \mu_1 + (m_1 + \ldots + m_{k-1}) < \mu_1 + n.$$

Next, we proceed to the proof of (a). So, assume

$$B_0 = \bigcup_{i \leqslant j} B \cap l_i.$$

As we have already seen, $\pi_1^{\mathsf{S}}(B_0)$ is an interval in λ_2. Since there exists $x \in B \cap l_j$ such that $\chi^{\mathsf{S}}(x) = \Diamond$ and $\pi_1^{\mathsf{S}}(x) = 0$, this interval is of the form $[0, r]$ for some $r < \lambda_2$. Now,

$$|B \cap l_j| \leqslant n.$$

For every $x \in B \cap l_j$, define

$$C_x = \{y \in B \cap l_i : \mu_1 \leqslant i \leqslant j, y \preccurlyeq x\} \cup \{x\}$$

and consider $|\pi_1^{\mathsf{S}}(C_x)|$: clearly,

$$|\pi_1^{\mathsf{S}}(C_x)| \leqslant n^2,$$

by the same argument that was used to prove the bound for $width(B)$ in the first part of the proof.

Thus,

$$|\pi_1^{\mathsf{S}}(B_0)| \leqslant \sum_{x \in B \cap l_j} |\pi_1^{\mathsf{S}}(C_x)| \leqslant n^3.$$

Since l_{μ_1} is the first line in B,

$$\pi_2^{\mathbf{S}}(B_0) = [\pi_2^{\mathbf{S}}(l_{\mu_1}), \pi_2^{\mathbf{S}}(l_j)] = [\mu_1, j],$$

and the length of this interval is less than n.

For (b), notice that if $B \cap l_j$ is a configuration line, so is $B \cap l_k$, for every $k > j$. Consider the substructure $\mathbf{S}_1 \leqslant \mathbf{S}$ generated by $B \cap l_j$. As was proven in Section 4.1, $\mathbf{S}_1$ will be a computation structure, and for every k $(j \leqslant k < j + \omega)$, $A^{\mathbf{S}_1}$ intersects the k–th line. However, the axioms imply that if a line contains in it a configuration line, then it cannot contain anything else. Thus, $B = A^{\mathbf{S}_1}$. □

Definition 4.22 *An infinite $(\rightarrow \cup \twoheadrightarrow)$-component will be called* **standard**, *if its first line (with respect to the ordering $\leqslant_2^{\mathbf{S}}$) is a configuration line. Otherwise, it will be called a* **nonstandard** *component.*

An immediate corollary of the definition above and Lemma 4.19 is the following proposition.

Proposition 4.23 *Every infinite $(\rightarrow \cup \twoheadrightarrow)$-component is either standard or is a copy of a standard component preceded by finitely many lines.*

Lemma 4.24 *If B is a connected $(\rightarrow \cup \twoheadrightarrow)$-component, then $\pi_1^{\mathbf{S}}(B)$ and $\pi_2^{\mathbf{S}}(B)$ are subintervals of λ_1 and λ_2, respectively.*

PROOF. The proof follows easily from the following two facts: if $x \rightarrow y$ then

$$\pi_1^{\mathbf{S}}(y) = \pi_1^{\mathbf{S}}(x) + 1 \text{ and } \pi_2^{\mathbf{S}}(x) = \pi_2^{\mathbf{S}}(y),$$

and if $x \twoheadrightarrow y$, then

$$\pi_1^{\mathbf{S}}(x) = \pi_1^{\mathbf{S}}(y) \text{ and } \pi_2^{\mathbf{S}}(y) = \pi_2^{\mathbf{S}}(x) + 1 \qquad \square$$

Definition 4.25 *If $I \subseteq \lambda_1 \times \lambda_2$, and if $B \subseteq A^{\mathbf{S}} \setminus \{\infty^{\mathbf{S}}\}$, then we say that B is **contained in** I if*

$$\{\langle \pi_1^{\mathbf{S}}(b), \pi_2^{\mathbf{S}}(b)\rangle : b \in B\} \subseteq I.$$

Our knowledge of (n, n, n, n)-generated structures in $\mathcal{K}$ can now be summarized in the form of the following theorem:

Theorem 4.26 *Suppose $\mathbf{S}$ is an (n, n, n, n)-generated structure in $\mathcal{K}$. Let k be the number of infinite $(\rightarrow \cup \twoheadrightarrow)$-connected components in $A^{\mathbf{S}}$, and let t be the number of finite $(\rightarrow \cup \twoheadrightarrow)$-connected components. Then $k + t \leqslant n$, and*

1. *If $k = 0$, then $\mathbf{S}$ is finite and, in fact,*

$$|P_1^{\mathbf{S}}| \leqslant t \cdot n^2 + n + 1$$

$$|P_2^{\mathbf{S}}| \leqslant t \cdot n + n + 1$$

$$|A^{\mathbf{S}}| \leqslant t \cdot n^3 + 1$$

$$|C^{\mathbf{S}}| \leqslant |S \cup (S \times Q)| + 5$$

2. *If $k \neq 0$, then (replacing $\mathbf{S}$ with a structure isomorphic to it)*

(i) there exist $m_1 \leqslant n^3$ *and* $m_2 \leqslant n^2$ *such that*

$$\begin{aligned} (P_1^{\mathsf{S}}, \leqslant_1^{\mathsf{S}}) &= (\omega + m_1 + n + 1, \leqslant) \\ (P_2^{\mathsf{S}}, \leqslant_2^{\mathsf{S}}) &= (k\omega + m_2 + n + 1, \leqslant); \end{aligned}$$

(ii) there exist $\mu_i \leqslant j_i < \omega$, $i = 1, \ldots, k$, $j_i - \mu_i < n$, *such that for some enumeration* $B_0, B_1, \ldots, B_{k-1}$ *of the infinite* $(\rightarrow \cup \twoheadrightarrow)$*-components, the first configuration line in* B_i *is on line* $i\omega + j_i$ *and* B_i *is contained in*

$$([0, \omega) \times [i\omega + j_i, (i+1)\omega)) \cup ([0, n^3 - 1] \times [\mu_i, j_i]);$$

(iii) Each finite $(\rightarrow \cup \twoheadrightarrow)$*-component is contained in*

$$[x_i, x_i + r_i] \times [\nu_i\omega + s_i, \nu_i\omega + t_i],$$

for some $x_i, r_i, \nu_i, s_i, t_i$ *such that*

$$\begin{aligned} &x_i < \omega + m_1 + n \\ &r_i < n^2 \\ &t_i < s_i + n \\ &\nu_i \leqslant k \end{aligned}$$

and

$$\text{if } \nu_i < k \text{ then } t_i < j_{\nu_i}.$$

Hence, a typical finitely generated structure in $\mathcal{K}$ can be visualized as:

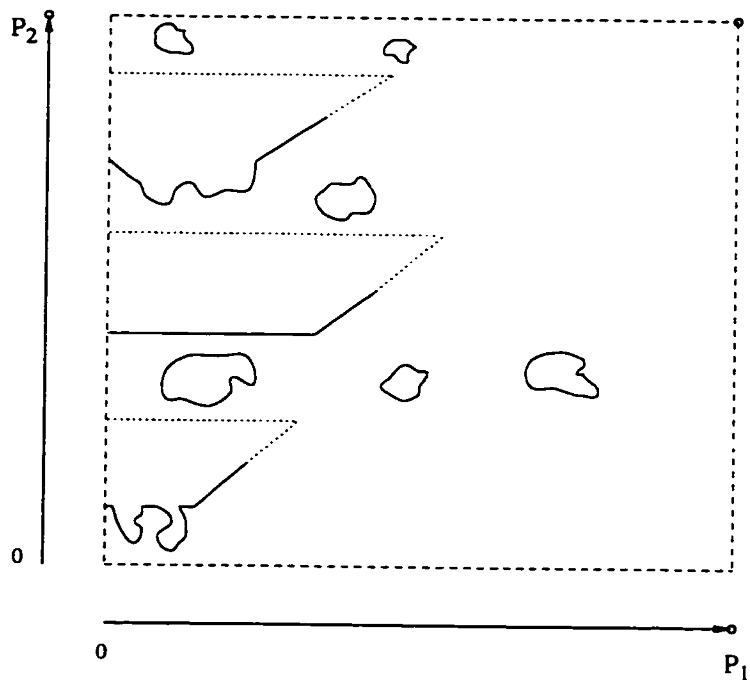

Figure 4.4: Finitely generated structure in $\mathcal{K}$

4.4 $Th_{\forall,n}(\mathcal{K})$ is decidable, for every $n < \omega$

In this section, we construct an algorithm which, given a universal sentence ψ in $\mathcal{L}$ which contains at most n variables of each sort, decides whether $\psi \in Th_{\forall,n}(\mathcal{K})$ or not. In Section 4.3, we have examined some of the most relevant features of the class $\mathcal{K}$ and gotten a good grasp of those structures in $\mathcal{K}$ which are generated by at most n elements of every sort. This will now be used to give a complete description of such structures using some "finite amount of information" about the structure, which will depend on n. Such descriptions, along with the specification of at most n elements of each sort in the structure given by the description, will witness the failure of ψ to be in $Th_{\forall,n}(\mathcal{K})$

We have already seen that all finite $(\rightarrow \cup \twoheadrightarrow)$–components of a structure in $\mathcal{K}$ generated by at most (n, n, n, n)–elements are bounded in height and width by n and n^2, respectively. The same is true of the "non-computation parts" of infinite components, where the bounds are given by n and n^3, respectively.

Throughout this section we assume that n is a fixed nonnegative integer.

Let $\mathcal{M}$ be the set of all $(n^2 \times n)$-matrices $M = (m_{ij})$, $(i = 0, \dots, n^2 - 1, j = 0, \dots, n-1)$ whose entries are elements of $S \cup (S \times Q) \cup \{\diamondsuit, \heartsuit, \clubsuit\}$, equipped with two binary relations $\rightarrow_M$ and $\twoheadrightarrow_M$ defined on the entries of M which are distinct from $\clubsuit$ so that, if

$$m_{ij} \rightarrow_M m_{i'j'},$$

then

$$i' = i + 1, j' = j,$$

and if

$$m_{ij} \twoheadrightarrow_M m_{i'j'},$$

then

$$i' = i, j' = j + 1.$$

$\tilde{\mathcal{M}}$ is defined to be the set of all $(n^3 \times n)$–matrices M, whose entries come from $S \cup (S \times Q) \cup \{\diamondsuit, \heartsuit, \clubsuit\}$, and the binary relations $\rightarrow_M$ and $\twoheadrightarrow_M$ are defined in the same way as for $\mathcal{M}$.

Let $\mathcal{L}'$ be a two-sorted language, whose sorts are A and C, and which is defined in the following way:

$$\mathcal{L}' = \mathcal{L} \setminus \{\leqslant_1, \leqslant_2, \pi_1, \pi_2, \infty_1, \infty_2, c, d\}.$$

Definition 4.27 *Let $M \in \mathcal{M} \cup \tilde{\mathcal{M}}$. We define a partial structure $\mathbf{S}_M$ in the language $\mathcal{L}'$ in the following way:*

1. The universe of the A-part of the structure is the set

$$\{(i,j) : 0 \leqslant i \leqslant N-1, 0 \leqslant j \leqslant n-1, m_{ij} \neq \clubsuit\} \cup \{\infty\},$$

where $N = n^2$ or n^3, respectively.

2. The operation χ is defined as:

$$\chi((i,j)) = m_{ij},$$

while $\chi(\infty) = \infty'$.

3. The two binary relations $\rightarrow$ and $\twoheadrightarrow$ are defined as on M:

$$(i,j) \rightarrow (i',j') \text{ if and only if } m_{ij} \rightarrow_M m_{i'j'},$$

and similarly for $\twoheadrightarrow$.

4. Let F be an m-ary fundamental operation in $\mathcal{L}'$ other than χ, and

$$(i_1, j_1), \ldots, (i_m, j_m) \in A^{\mathbf{S}_M}.$$

To show how the operation $F^{\mathbf{S}_M}$ is to be defined, we will demonstrate it on the particular example of $F = g$; the definitions for other function symbols are obtained in a similar fashion, guided by the axioms from groups 4–8.

- *If it is not the case that*

$$(i_1, j_1) \to (i_2, j_2),$$

we define

$$g^{\mathbf{S}_M}((i_1, j_1), (i_2, j_2)) = \infty.$$

- *Otherwise, if*

$$(i_1, j_1) \twoheadrightarrow (i_2, j_2),$$

and $\chi((i_2, j_2)) \neq \heartsuit$,

$$g^{\mathbf{S}_M}((i_1, j_1), (i_2, j_2)) = \infty.$$

On the other hand, if $\chi((i_2, j_2)) = \heartsuit$, and $(i_2, j_2 + 1) \in A^{\mathbf{S}_M}$,

$$g^{\mathbf{S}_M}((i_1, j_1), (i_2, j_2)) = (i_2, j_2 + 1),$$

while, if $(i_2, j_2 + 1) \notin A^{\mathbf{S}_M}$, $g^{\mathbf{S}_M}((i_1, j_1), (i_2, j_2))$ is undefined.

- *If $\infty \in \{a_1, \ldots, a_m\} \subseteq A^{\mathbf{S}_M}$, then $g^{\mathbf{S}_M}(a_1, \ldots, a_m) = \infty$.*

We say that $M \in \mathcal{M} \cup \tilde{\mathcal{M}}$ is **connected** if $A^{\mathbf{S}_M} \setminus \{\infty\}$ is $(\to \cup \twoheadrightarrow)$-connected (or empty).

Definition 4.28 *Let $\mathcal{M}'$ consist of those matrices $M \in \mathcal{M}$ which are connected and such that $\mathbf{S}_M$ is a total $\mathcal{L}'$-structure. (We allow for the empty matrix M; i.e.*

such that $A^{\mathbf{S}_M} = \{\infty\}$.) $\mathcal{M}''$ will consist of those connected $M \in \tilde{\mathcal{M}}$ such that $\mathbf{S}_M$ is not a total structure, and

(1) for some p, $0 \leqslant p \leqslant n-1$,

$$\chi((0,n-1)) = \diamondsuit, \chi((p+1,n-1)) = \heartsuit,$$

$$\chi((1,n-1)),\ldots,\chi((p,n-1)) \in S \cup (S \times Q)),$$

so that at most one of $\chi((1,n-1)),\ldots,\chi((p,n-1))$ is in $(S \times Q)$, and

$$(0,n-1) \to (1,n-1) \to \ldots \to (p,n-1) \to (p+1,n-1).$$

(2) If F is an m-ary operation symbol and

$$(i_1,j_1),\ldots,(i_m,j_m) \in A^{\mathbf{S}_M},$$

so that $0 \leqslant i_1,\ldots,i_m < n-1$, then $F^{\mathbf{S}_M}((i_1,j_1),\ldots,(i_m,j_m))$ is defined.

Intuitively, we want the matrices from $\mathcal{M}'$ to represent finite components in a structure, while $\mathcal{M}''$ is intended to represent non-computation parts of infinite components, including the first configuration line occurring in the component.

Now, suppose for the moment that $\mathbf{S}$ is an (n,n,n,n)-generated member of $\mathcal{K}$ such that $(P_1^{\mathbf{S}}, \leqslant_1^{\mathbf{S}})$ and $(P_2^{\mathbf{S}}, \leqslant_2^{\mathbf{S}})$ are ordinals. If B_1 and B_2 are two infinite

$(\rightarrow \cup \twoheadrightarrow)$–components of A^{S} such that, for all $x \in B_1$ and $y \in B_2$,

$$k\omega + l \leqslant \pi_2^{\mathsf{S}}(x) < (k+1)\omega,$$

$$(k+1)\omega + m \leqslant \pi_2^{\mathsf{S}}(y) < (k+2)\omega,$$

where $k, l, m < \omega$, then m satisfies

$$0 \leqslant m < n^2 + n,$$

for there are at most n finite $(\rightarrow \cup \twoheadrightarrow)$– components lying "between" B_1 and B_2, each of the height at most n, plus at most n generators of P_2^{S}. For our purposes, the bound on m can be given by $2n^2$, which will make it easier to work with.

Definition 4.29 *A (**potential**) **description** of an $\mathcal{L}$-structure consists of*

(1) an integer k $(0 \leqslant k \leqslant n)$;

(2) k configurations $C_1, \ldots, C_k$, such that every C_i is of the form

$$\diamondsuit c_1^{(i)} c_2^{(i)} \ldots c_{r_i}^{(i)} \heartsuit,$$

where $r_i + 2 \leqslant n$.

(3) k matrices $\tilde{M}_1, \ldots, \tilde{M}_k \in \mathcal{M}''$;

(4) $n - k$ matrices $M_1, \ldots, M_k \in \mathcal{M}'$;

(5) $n-k$ ordered pairs

$$\langle x_1, y_1\rangle, \ldots, \langle x_{n-k}, y_{n-k}\rangle \in [0, a\omega + n^3 + n + 1) \times [0, k\omega + n^2 + n + 1),$$

where $a = 0$ if $k = 0$, and $a = 1$ otherwise;

(6) the symbol $* \in \{=, \neq\}$.

We write this description as

$$I = \langle k; C_1, \ldots, C_k; \tilde{M}_1, \ldots, \tilde{M}_k; M_1, \ldots, M_{n-k}; \langle x_1, y_1\rangle, \ldots, \langle x_{n-k}, y_{n-k}\rangle; *\rangle.$$

The intended meaning of I is to represent a structure with the following features:

- The structure contains at most n connected components, of which k are infinite.
- The computation structure in the i-th infinite component is generated by a configuration line corresponding to C_i, while the finite non-computation part of that component is given by $\mathbf{S}_{\tilde{M}_i}$.
- The finite components of the structure are given by $\mathbf{S}_{M_j}$ $(1 \leqslant j \leqslant n-k)$ and the coordinates of their "left-bottom" entries are determined by the pair of coordinates $\langle x_j, y_j\rangle$.

There are two problems: (1) there may be no structure having these features, or (2) there may be such a structure, but it may not be in $\mathcal{K}$.

Next, we define the notion of an *allowable description*. The idea behind this notion is to fully address item (1) and partially address item (2).

Definition 4.30 *A description*

$$I = \langle k; C_1, \ldots, C_k; \tilde{M}_1, \ldots, \tilde{M}_k; M_1, \ldots, M_{n-k}; \langle x_1, y_1 \rangle, \ldots, \langle x_{n-k}, y_{n-k} \rangle; * \rangle$$

*of an $\mathcal{L}$-structure is said to be $\mathcal{K}$-**allowable** if the following requirements on I are met:*

(1) For C_i and $\tilde{M}_i$, if $C_i = \diamondsuit c_1^{(i)} c_2^{(i)} \ldots c_{r_i}^{(i)} \heartsuit$ and $\tilde{M}_i = (m_{ij})$, then

$$m_{0,n-1} = \diamondsuit, m_{1,n-1} = c_1^{(i)}, \ldots, m_{r_i,n-1} = c_{r_i}^{(i)}, m_{r_i+1,n-1} = \heartsuit,$$

and

$$m_{r_i+2,n-1} = m_{r_i+3,n-1} = \ldots = m_{n^3-1,n-1} = \clubsuit.$$

(2) Given $\tilde{M}_i$ and M_j, if

$$(i-1)\omega + 2n^2 - n + l = y_j + q$$

and

$$k = x_j + p$$

for some

$$0 \leqslant l, q \leqslant n-1$$

$$0 \leqslant k \leqslant n^3 - 1$$

$$0 \leqslant p \leqslant n^2 - 1$$

then at least one of the (k,l)-th entry of $\tilde{M}_i$ and the (p,q)-th entry of M_j is $\clubsuit$.

(3) For M_i and M_j $(i \neq j)$, if

$$\begin{aligned} y_i + l &= y_j + q \\ x_i + k &= x_j + p, \end{aligned}$$

for some

$$\begin{aligned} &0 \leqslant l, q \leqslant n-1 \\ &0 \leqslant k, p \leqslant n^2 - 1, \end{aligned}$$

then at least one of the (k,l)-th entry of M_i and the (p,q)-th entry of M_j is $\clubsuit$.

(4) For $i = 1, \ldots, n-k$,

$$y_i \in [0, 2n^2 - n] \cup [\omega, \omega + 2n^2 - n] \cup \ldots \cup [k\omega, k\omega + n^2 + n + 1].$$

To each $\mathcal{K}$-allowable description one can assign a structure in $\mathcal{L}$:

Definition 4.31 *If*

$$I = \langle k; \mathcal{C}_1, \ldots, \mathcal{C}_k; \tilde{M}_1, \ldots, \tilde{M}_k; M_1, \ldots, M_{n-k}; \langle x_1, y_1 \rangle, \ldots, \langle x_{n-k}, y_{n-k} \rangle; * \rangle$$

is a $\mathcal{K}$-allowable description then $\mathbf{S}_I$ is the structure defined in the following way:

(1) *Let* $a = 0$ *if* $k = 0$, *and* $a = 1$ *if* $k \geqslant 1$. *Then,*

$$(P_1^{\mathbf{S}_I}, \leqslant_1^{\mathbf{S}_I}) = (a\omega + n^3 + n + 1, \leqslant)$$
$$(P_2^{\mathbf{S}_I}, \leqslant_2^{\mathbf{S}_I}) = (k\omega + n^2 + n + 1, \leqslant)$$

Also, define

$$\infty_1 = a\omega + n^3 + n \in P_1^{\mathbf{S}_I}$$

and

$$\infty_1 = k\omega + n^2 + n \in P_2^{\mathbf{S}_I}.$$

(2) $C^{\mathbf{S}_I}$ *consists of* $S \cup (S \times Q) \cup \{c, d, \diamondsuit, \heartsuit, \infty'\}$ *so that all of the elements are pairwise distinct, with the possible exception of* c *and* d. *Moreover,*

$$c = d \text{ if and only if } * \text{ is } = .$$

The constants of sort C *have the obvious interpretation.*

(3) *Define*

$$d_j = (j-1)\omega + 2n^2$$
$$e_j = (\text{ length of } C_j) + 2,$$

for $j = 1, \ldots, k$. *Now,* $A^{\mathbf{S}_I}$ *will consist of the following ordered pairs in*

$P_1 \times P_2$: $A_1 \cup A_2 \cup A_3 \cup \{\infty\}$, *where*

$$A_1 = \bigcup_{i=1}^{n-k} \{(x_i + p, y_i + q) : p \leqslant n^2 - 1, q \leqslant n - 1, (p, q) \in S_{M_i}\},$$

$$A_2 = \bigcup_{j=1}^{k} \{(d_j - (n-1) + r, s) : r \leqslant n^3 - 1, s \leqslant n - 1, (r, s) \in S_{\tilde{M}_j}\},$$

$$A_3 = \bigcup_{j=1}^{k} \{\langle x, y\rangle : d_j \leqslant y < d_j + \omega \text{ and } 0 \leqslant x \leqslant e_j + (y - d_j)\}.$$

(4) $\rightarrow$ and $\twoheadrightarrow$ are now defined on A^{S_I} as follows:

- *If $(p, q), (r, s) \in A_1$ and, for some i:*

$$(p - x_i, q - y_i), (r - x_i, s - y_i) \in S_{M_i},$$

then

$$(p, q) \rightarrow (r, s)$$

if and only if

$$(p - x_i, q - y_i) \rightarrow_{M_i} (r - x_i, s - y_i),$$

while

$$(p, q) \twoheadrightarrow (r, s)$$

if and only if

$$(p - x_i, q - y_i) \twoheadrightarrow_{M_i} (r - x_i, s - y_i).$$

- *If $(p,q),(r,s) \in A_2$ and, for some j:*

$$(p - d_j + (n-1), q) \in S_{\bar{M}_j} \text{ and } (r - d_j + (n-1), s) \in S_{\bar{M}_j},$$

then

$$(p,q) \to (r,s)$$

if and only if

$$(p - d_j + (n-1), q) \to_{\bar{M}_j} (r - d_j + (n-1), s),$$

while

$$(p,q) \twoheadrightarrow (r,s)$$

if and only if

$$(p - d_j + (n-1), q) \twoheadrightarrow_{\bar{M}_j} (r - d_j + (n-1), s).$$

- *Finally, if $(p,q),(r,s) \in A_3$, and for some j*

$$d_j \leqslant q, s < d_j + \omega,$$

$$0 \leqslant p \leqslant e_j + (q - d_j),$$

$$0 \leqslant r \leqslant e_j + (s - d_j),$$

then $(p,q) \to (r,s)$ *if and only if*

$$r = p + 1 \text{ and } q = s,$$

while $(p,q) \twoheadrightarrow (r,s)$ *will be true if and only if*

$$r = p \text{ and } s = q + 1.$$

- *In all other cases the relations do not hold.*

(5) Given $(x,y) \in A^{\mathbf{S}_I}$, *we define*

$$\pi_1^{\mathbf{S}_I}((x,y)) = x \text{ and } \pi_2^{\mathbf{S}_I}((x,y)) = y.$$

Also,

$$\pi_1^{\mathbf{S}_I}(\infty) = \infty_1,\ \pi_2^{\mathbf{S}_I}(\infty) = \infty_2.$$

(6) If $(x,y) \in A_1$ *and, for some* i*:*

$$(p - x_i, q - y_i), (r - x_i, s - y_i) \in S_{M_i},$$

then

$$\chi^{\mathbf{S}_I}((x,y)) = \chi^{\mathbf{S}_{M_i}}((x,y))$$

If $(x, y) \in A_2$ *and, for some* j*:*

$$(p - d_j + (n-1), q) \in S_{\bar{M}_j} \text{ and } (r - d_j + (n-1), s) \in S_{\bar{M}_j},$$

then

$$\chi^{\mathbf{S}_I}((x, y)) = \chi^{\mathbf{S}_{\bar{M}_j}}((x, y)).$$

If $(x, y) \in A_3$, *then* $\chi^{\mathbf{S}_I}((x, y))$ *will be defined below (item (8)).*

Finally, $\chi^{\mathbf{S}_I}(\infty) = \infty'$.

(7) Given an m*-ary function symbol* F*, which is one of* $f, g, h, diam$ *or* $bord$*, and*

$$(a_1, b_1), \ldots, (a_m, b_m) \in A^{\mathbf{S}_I},$$

$F^{\mathbf{S}_I}((a_1, b_1), \ldots, (a_m, b_m))$ *is defined as follows:*

- *If* $(a_1, b_1), \ldots, (a_m, b_m) \in A_1$*, and for some* i*,*

$$(p - x_i, q - y_i), (r - x_i, s - y_i) \in S_{M_i},$$

then

$$F^{\mathbf{S}_I}((a_1, b_1), \ldots, (a_m, b_m)) = F^{\mathbf{S}_{M_i}}((a_1 - x_i, b_1 - y_i), \ldots, (a_m - x_i, b_m - y_i)).$$

- *If* $(a_1, b_1), \ldots, (a_m, b_m) \in A_2$ *and, for some* j*:*

$$(p - d_j + (n-1), q) \in S_{\bar{M}_j} \text{ and } (r - d_j + (n-1), s) \in S_{\bar{M}_j},$$

then, if

$$b_1 = \ldots = b_m = d_j - 1,$$

$F^{\mathbf{S}_I}((a_1, b_1), \ldots, (a_m, b_m))$ *will be defined below (item (8)).*

Otherwise, $F^{\mathbf{S}_I}((a_1, b_1), \ldots, (a_m, b_m))$ *is defined to be*

$$F^{\mathbf{S}_{\tilde{M}_j}}((a_1, b_1 - d_j + (n-1)), \ldots, (a_m, b_m - d_j + (n-1))).$$

- *If* $(a_1, b_1), \ldots, (a_m, b_m) \in A_3$, *then* $F^{\mathbf{S}_I}((a_1, b_1), \ldots, (a_m, b_m))$ *will be defined below (item (8)).*
- *In all other cases, the value of* $F^{\mathbf{S}_I}((a_1, b_1), \ldots, (a_m, b_m))$ *is defined to be* ∞.

(8) *The definitions of* χ, f, g, h, *diam and bord are completed by requiring that, for each* $j = 1, \ldots, k$, *the map*

$$A^{\mathbf{S}^*_{\mathcal{C}_j}} \setminus \{\infty\} \to A^{\mathbf{S}_I},$$

where $* \in \{=, \neq\}$ *is chosen as in* $\mathbf{S}_I$, *given by*

$$a_x^{(y)} \mapsto (x, d_j + y)$$

extends to an embedding of $\mathcal{L}$*-structures*

$$\mathbf{S}^*_{\mathcal{C}_j} \to \mathbf{S}_I.$$

Lemma 4.32 *Recall that n is fixed. Given a $\mathcal{K}$–allowable description*

$$I = \langle k; \mathcal{C}_1, \dots, \mathcal{C}_k; \tilde{M}_1, \dots, \tilde{M}_k; M_1, \dots, M_{n-k}; \langle x_1, y_1\rangle, \dots, \langle x_{n-k}, y_{n-k}\rangle; *\rangle$$

it is decidable whether $\mathbf{S}_I \in \mathcal{K}$.

PROOF. All the axioms in groups 1–3 will be trivially satisfied in $\mathbf{S}_I$, with the only possible exception being the last four axioms in group 2. However, these can be checked for the given choice of $\mathbf{S}_{M_i}$ and $\mathbf{S}_{\tilde{M}_j}$ in at most n^4 steps.

- Given $*$ in the description I, we need to verify whether $c * d$ is consistent with the values of χ on $A^{\mathbf{S}_I}$.

  If there is an element $(x, y) \in A^{\mathbf{S}_I}$ such that $(x, y) \in A_1 \cup A_2$ and it is χ–labelled by (s, q_0), where q_0 is the halting state, and $*$ is $\neq$, the algorithm outputs the negative answer and comes to a halt.

  If the Turing machine T started on any of the inputs $\mathcal{C}_1, \dots, \mathcal{C}_k$ eventually halts in state q_0, and $*$ is $\neq$, the algorithm outputs the negative answer and halts. (This step is fully determined, since the set

  $$\{\mathcal{C}_i : \text{ length of } \mathcal{C}_i \text{ is } \leqslant n-2, \text{ and } T \text{ started on } \mathcal{C}_i \text{ halts in state } q_0\}$$

  is finite, hence recursive.)

  In all other cases, the answer is positive, and the value of $*$ and the labelling by χ are consistent.

- Finally, we need an algorithm which would check whether the axioms from groups 4–8 hold in $\mathbf{S}_I$.

  Let Φ' denote the subset of Φ consisting of the axioms from groups 4–8. First, we remark the following:all of the axioms from Φ' mention only the elements that come from two successive lines in the coordinatization of the A-part. Hence, if an axiom $\phi \in \Phi'$ fails to be true in some $\mathcal{L}$-structure $\mathbf{S}$, then there are elements $a_1, \ldots, a_n \in A^{\mathbf{S}}$ so that

$$\mathbf{S} \models \neg\phi(a_1, \ldots, a_n)$$

and

$$\pi_2(a_i) \in \{j_0, j_0 + 1\},$$

for some $j_0 \in P_2^{\mathbf{S}}$.

Thus, an axiom $\phi \in \Phi$ from one of the groups 4–8 can fail to be to hold in $\mathbf{S}_I$ for one of the following two reasons:

(1) For some $a_1, \ldots, a_m \in A_1$ such that, for some i_0,

$$\pi_2^{\mathbf{S}_I}(\{a_1, \ldots, a_m\}) \subseteq \{i_0, i_0 + 1\},$$

we have

$$\mathbf{S}_I \models \neg\phi(a_1, \ldots, a_m);$$

or (2) For some $a_1, \ldots, a_m \in A_2$ such that, for some i_0,

$$\pi_2^{\mathbf{S}_I}(\{a_1, \ldots, a_m\}) \subseteq \{i_0, i_0 + 1\},$$

where $i_0 \notin \{2n^2, \omega + 2n^2, \ldots, (k-1)\omega + 2n^2\}$ we have

$$\mathbf{S}_I \models \neg\phi(a_1, \ldots, a_m).$$

The existence of an algorithm which checks whether $\mathbf{S}_I \in \mathcal{K}$ now follows easily from all the facts listed in the course of the proof so far. □

Definition 4.33 *Let I be a $\mathcal{K}$-allowable description such that $\mathbf{S}_I \in \mathcal{K}$, and let*

$$\bar{e} = \langle a_1, \ldots, a_n; b_1, \ldots, b_n; c_1, \ldots, c_n; d_1, \ldots, d_n \rangle$$

be an (n, n, n, n)-tuple such that

1. $a_1, \ldots, a_n \in A^{\mathbf{S}_I}$;
2. $b_1, \ldots, b_n \in [0, a\omega + n^3 + n + 1]$;
3. $c_1, \ldots, c_n \in [0, k\omega + n^2 + n + 1]$;
4. $d_1, \ldots, d_n \in S \cup (S \times Q) \cup \{\diamondsuit, \heartsuit, \infty', c, d\}$

then $\langle I, \bar{e} \rangle$ is said to be a description of a ***good pair****.*

Intuitively, $\bar{e}$ represents the $4n$-tuple of elements of $\mathbf{S}_I$ which generates some substructure of $\mathbf{S}_I$, where $\bar{b}$ belong to the sort $P_1^{\mathbf{S}_I}$, $\bar{c}$ belong to the sort $P_2^{\mathbf{S}_I}$, and $\bar{d}$ belong to the constant sort $C^{\mathbf{S}_I}$.

At this point, we remark that the existence of an algorithm which lists all descriptions of good pairs $\langle I, \bar{e}\rangle$ now follows easily from Lemma 4.32 and the fact that the ranges of the components of a $\mathcal{K}$–allowable description I, as well as those of the components of $\bar{e}$, are recursively enumerable sets.

Now we have the following proposition:

Proposition 4.34 *(1) If $\langle I, \bar{e}\rangle$ is the description of a good pair, then $\mathbf{S}_I \in \mathcal{K}$ and $\bar{e}$ is an (n, n, n, n)–tuple of elements of S_I.*

(2) Given any (n, n, n, n)–generated structure $\mathbf{S} \in \mathcal{K}$ and any (n, n, n, n)–tuple $\bar{e}'$ in S, there is a description $\langle I, \bar{e}\rangle$ of a good pair and an embedding

$$\alpha : (\mathbf{S}, \bar{e}') \to (\mathbf{S}_I, \bar{e}).$$

The final step towards the construction of the algorithm that decides the universal (n, n, n, n)–theory of $\mathcal{K}$ will be accomplished by the following proposition:

Proposition 4.35 *There is an algorithm which, given a description of a good pair $\langle I, \bar{e}\rangle$ and a quantifier–free formula $\phi(\bar{x})$, determines whether*

$$\mathbf{S}_I \models \phi(\bar{e}).$$

PROOF. The description I for $\mathbf{S}_I$ provides us with the specification of $\mathbf{S}_I$, with one possible difficulty: the code for $\mathbf{S}_I$ specifies *only* the initial line of the standard part of an infinite $(\to \cup \twoheadrightarrow)$–component in $A^{\mathbf{S}_I}$. In order to check the nonvalidity of a given quantifier–free formula ϕ in $\mathbf{S}_I$ when the variables of ϕ are substituted with the corresponding entries from $\bar{e}$, we must be able to recover a "sufficiently large" standard part of any infinite $(\rightarrow \cup \twoheadrightarrow)$-component.

Let $\mathcal{C}_i$ be the i-th configuration in the $\mathcal{K}$-allowable description I defining $\mathbf{S}_I$. Suppose that

$$\mathcal{C}_i = \Diamond c_1 \dots c_k \heartsuit,$$

where $c_j \in S \cup (S \times Q)$. Consider the elements $p_0, p_1, \dots, p_{k+1} \in A_3$ such that

$$p_0 \rightarrow p_1 \rightarrow \cdots \rightarrow p_k \rightarrow p_{k+1},$$
$$\chi^{\mathbf{S}_I}(p_0) = \Diamond, \chi^{\mathbf{S}_I}(p_j) = c_j (1 \leqslant j \leqslant k)\ \chi^{\mathbf{S}_I}(p_{k+1}) = \heartsuit,$$
$$\pi_1^{\mathbf{S}_I}(p_0) = 0, \dots, \pi_1^{\mathbf{S}_I}(p_{k+1}) = k+1,$$
$$\pi_2^{\mathbf{S}_I}(p_0) = \cdots = \pi_2^{\mathbf{S}_I}(p_{k+1}) = (i-1)\omega + 2n^2.$$

Set

$$B^{(0)} = \langle A^{(0)}, \rightarrow^{(0)}, \twoheadrightarrow^{(0)} \rangle,$$

where

$$A^{(0)} = \{\langle x_i, \chi^{\mathbf{S}_I}(p_i), \pi_1^{\mathbf{S}_I}(p_i), \pi_2^{\mathbf{S}_I}(p_i)\rangle : 0 \leqslant i \leqslant k+1\},$$
$$\rightarrow^{(0)} = \{\langle p_j, p_{j+1}\rangle : 0 \leqslant j \leqslant k\}$$
$$\twoheadrightarrow^{(0)} = \emptyset.$$

Given $B^{(n)} = \langle A^{(n)}, \rightarrow^{(n)}, \twoheadrightarrow^{(n)}\rangle$, we define $B^{(n+1)}$ in the following way

$$B^{(n+1)} = \langle A^{(n+1)}, \rightarrow^{(n+1)}, \twoheadrightarrow^{(n+1)}\rangle,$$

where

$$
\begin{aligned}
A^{(n+1)} &= A^{(n)} \cup \{\langle F^{\mathbf{S}_I}(z_1, \ldots, z_m), \chi^{\mathbf{S}_I}(F^{\mathbf{S}_I}(z_1, \ldots, z_m)), \pi_1^{\mathbf{S}_I}(z_F) \text{ (or } \pi_1^{\mathbf{S}_I}(z_F) + 1 \\
&\quad \text{when } F = bord), \ \pi_2^{\mathbf{S}_I}(z_F) + 1\rangle \mid F : A^m \to A, \ z_1, \ldots, z_m \in A^{(n)}\},
\end{aligned}
$$

$$
z_F = \begin{cases} z_1, & F = diam, \\ z_2, & F = g, h \text{ or } bord, \\ z_3, & F = f; \end{cases}
$$

and

$$
\begin{aligned}
\to^{(n+1)} &= \to^{(n)} \cup \{\langle a, b\rangle \in (A^{(n+1)} \setminus A^{(n)})^2 : \pi_1^{\mathbf{S}}(b) = \pi_1^{\mathbf{S}}(a) + 1\}, \\
\twoheadrightarrow^{(n+1)} &= \twoheadrightarrow^{(n)} \cup \{\langle a, b\rangle \in A^{(n)} \times A^{(n+1)} : \pi_1^{\mathbf{S}}(a) = \pi_1^{\mathbf{S}}(b)\}.
\end{aligned}
$$

Then,

$$
Sg^{\mathbf{S}_I}(p_0, \ldots, p_k + 1) = \bigcup_{n<\omega} B^{(n)},
$$

and a "sufficiently large" part of B can be effectively recovered by constructing $B^{(n)}$, $n < \omega$. Let d be the maximal number of occurrences of function symbols $f, g, h, diam$, or $bord$ appearing in a single term in ϕ. Let $y^{(0)}$ be the number defined in the following way: if

$$
b_i = (k_i\omega + 2n^2) + b_i',
$$

where $0 \leqslant k_i \leqslant n$, $0 \leqslant b_i' \leqslant \omega$, let

$$y^{(0)} = \max(b_i' : 1 \leqslant i \leqslant n).$$

(Here, b_i are the coordinates, which are included in the description of $\bar{e}$.)

Then in order to be able to test whether the given elements satisfy the quantifier-free part of an axiom from Φ, we need to reconstruct the first $y^{(0)} + d + 1$ lines in each of the computation substructures of the structure. Hence, only a finite portion of the structure, whose size can be bounded by a recursive function of n, needs to be inspected. $\square$

Hence, we have the following theorem:

Theorem 4.36 *$Th_{\forall,n}(\mathcal{K})$ is decidable, for every $n < \omega$.*

PROOF. First, we prove the following: given a quantifier-free formula $\phi(\bar{x})$ with at most n variables of each sort, if, for some (n, n, n, n)-generated structure $\mathbf{S} \in \mathcal{K}$ and an (n, n, n, n)-tuple $\bar{e}'$ in $\mathbf{S}$,

$$\mathbf{S} \models \neg\phi(\bar{e}')$$

then, for some description $\langle I, \bar{e}\rangle$ of a good pair,

$$\mathbf{S}_I \models \neg\phi(\bar{e}).$$

We use Proposition 12(2); namely, if such a structure $\mathbf{S} \in \mathcal{K}$ and (n, n, n, n)-tuple

$\bar{e}'$ exist, then there is a description $\langle I, \bar{e}\rangle$ of a good pair and an embedding

$$\alpha : (\mathbf{S}, \bar{e}') \rightarrow (\mathbf{S}_I, \bar{e}).$$

Since α preserves the non–validity of quantifier-free formulas in a language extended by constants, the claim has been proved.

Next, we focus our attention on the construction of an algorithm which decides $Th_{\forall,n}(\mathcal{K})$. The algorithm will be the following one:

1. Procedure $\mathcal{P}_1$ lists all universal sentences in $\mathcal{L}$ which contain at most n variables of each sort and which are in the universal theory of $\mathcal{K}$. Such a procedure must exist since $Th_{\forall}(\mathcal{K})$ is finitely axiomatizable and, hence, recursively enumerable.

2. Procedure $\mathcal{P}_2$ lists all good pairs $\langle I, \bar{e}\rangle$ and, for a given $\forall\bar{x}\phi(\bar{x})$, checks whether

$$\mathbf{S}_I \models \neg\phi(\bar{e}),$$

and if this happens to be the case lists the code for $\langle I, \bar{e}\rangle$. The existence of $\mathcal{P}_2$ follows from the following facts: by Lemma 15 and the remark following Definition 46, there exists a procedure which lists all descriptions of good pairs. Now, after such a pair has been generated, use the algorithm whose existence is asserted by Proposition 13 to check whether

$$\mathbf{S}_I \models \neg\phi(\bar{e}).$$

Now, the algorithm proceeds in the following manner: list alternately outputs of $\mathcal{P}_1$ and $\mathcal{P}_2$; if, at some point, $\forall\bar{x}\phi(\bar{x})$ appears in the list produced by $\mathcal{P}_1$, the algorithm outputs

$$\text{“}\forall\bar{x}\phi(\bar{x}) \in Th_{\forall,n}(\mathcal{K})\text{”}$$

and halts. If the second procedure $\mathcal{P}_2$ at some point outputs any description $\langle I, e\rangle$ of a good pair, the output of the algorithm will be

$$\text{“}\forall\bar{x}\phi(\bar{x}) \notin Th_{\forall,n}(\mathcal{K})\text{”}$$

and the algorithm will halt.

We need to prove that the algorithm constructed in this way is correct. If

$$\forall\bar{x}\phi(\bar{x}) \in Th_{\forall,n}(\mathcal{K}),$$

$\mathcal{P}_1$ will eventually list $\forall\bar{x}\phi(\bar{x})$. On the other hand, if

$$\forall\bar{x}\phi(\bar{x}) \notin Th_{\forall,n}(\mathcal{K}),$$

then, for some (n, n, n, n)–generated $\mathbf{S} \in \mathcal{K}$ and its generators $\bar{e}'$,

$$\mathbf{S} \models \neg\phi(\bar{e}').$$

However, by the claim made at the beginning of the proof, there is a description of a good pair $\langle I, \bar{e}\rangle$ such that

$$\mathbf{S}_I \models \neg\phi(\bar{e}).$$

Thus, $\langle I, \bar{e}\rangle$ will appear in the list generated by $\mathcal{P}_2$. Thus, the algorithm always terminates and gives the correct answer.

Therefore, $Th_{\forall,n}(\mathcal{K})$ is decidable, for every $n < \omega$. $\square$

Chapter 5

Some finite basis results and M-algebras

In this chapter, we introduce the notion of a finitely based variety and survey some of the most relevant results in the literature. The notion of an inherently nonfinitely based algebra is introduced, along with the definition of a graph algebra. We also quote the result of Baker, McNulty, and Werner which characterizes the dichotomy between finitely based and inherently nonfinitely based graph algebras in terms of the induced subgraphs of the corresponding underlying graph.

In the second part of the chapter we introduce the notion of an M-algebra. The basic properties of M-algebras are listed and the description of subdirectly irreducibles in a variety generated by an M-algebra is given.

5.1 Finitely based varieties and inherently non-finitely based algebras

Definition 5.1 *Let V be a variety. We say that V is* ***finitely based****, if there exists a finite subset Σ of $Th_{Eq}(V)$ such that*

$$\Sigma \models Th_{Eq}(V).$$

If $V = \mathbf{V}(\mathbf{A})$, and V is finitely based, the algebra $\mathbf{A}$ is said to be finitely based.

The notion of finite bases for equationally defined classes of algebras was first studied by G. Birkhoff ([4]). The first remarkable result in this area was proved in 1951 by R.Lyndon.

Theorem 5.2 *(Lyndon [24]) Every two-element algebra in a finite language is finitely based.*

As will be mentioned later, two is the best possible bound for this type of result. Namely, as we will see later, there are three-element algebras in a finite language which fail to be finitely based in a very strong sense.

Further attempts were made to establish finite basis results for other relatively well-understood classes of algebras. Here we list some of them:

Theorem 5.3 *1. (Oates, Powell [42]) Every finite group is finitely based.*

2. (Perkins [43]) Every commutative semigroup is finitely based.

3. (McKenzie [28]) Every finite lattice is finitely based.

4. (Kruse [22], Lvov [23]) Every finite ring is finitely based.

Apart from these sporadic results it seemed natural to ask whether a similar result could be established for broader classes of varieties, in particular, within the classes of congruence permutable or congruence distributive varieties. The first significant step in that direction which put the question of finite bases for varieties in a broader perspective was the well-known theorem of K. Baker:

Theorem 5.4 *(Baker [1]) Every finite algebra belonging to a congruence distributive variety in a finite language is finitely based.*

The theorem of Baker clearly implies Theorem 5.3 (3), as well as analogous results for relation algebras, cylindric algebras, Heyting algebras, etc.

An immediate corollary of Jónsson's theorem (Theorem 1.24) is that every finitely generated congruence distributive variety contains only finitely many subdirectly irreducibles, all of which are finite. It points to the direction of investigation of the possible relationship between the property of a variety being finitely based and the diversity of the class of its subdirectly irreducibles.

Definition 5.5 *Let V be a variety. V is said to be* ***residually less than*** *κ, where κ is a cardinal, if, for every subdirectly irreducible $\mathbf{A} \in V$,*

$$|A| < \kappa.$$

If such κ does not exist, we say that V is ***residually large****. Otherwise, V is* ***residually small****. If all subdirectly irreducibles in V are finite, V is* ***residually finite****.*

Another breakthrough was accomplished by a deep result of R. McKenzie. McKenzie proved a generalization of Baker's theorem for congruence modular varieties, under the additional assumption that the variety is residually small.

Theorem 5.6 *(McKenzie [31]) Every finite algebra in a finite language which belongs to a residually small congruence modular variety is finitely based.*

The proof of Theorem 5.6 makes extensive use of the commutator theory for congruence modular varieties developed by R. Freese and McKenzie in [16].

It is worth mentioning at this point that the assumption of residual smallness in the statement of Theorem 5.6 cannot be dropped. Polin ([44]) gives an example of a nonfinitely based finite bilinear algebra. Other examples were given by M. Vaughan-Lee ([51]), R. Bryant ([7]), et al.

Recently, R. Willard proved the following theorem which generalizes Baker's result for finitely generated congruence–distributive varieties:

Theorem 5.7 *([53]) Every congruence meet-semidistributive variety in a finite language which is residually less than* n*, for some* $n < \omega$*, is finitely based.*

The results of Baker and McKenzie prompt the following question:

Problem 1 *Is every residually finite finitely generated variety in a finite language finitely based?*

This question is usually attributed in the literature to B. Jónsson, although there is evidence that it was first stated in the doctoral dissertation of R. Park.

On the other side of the spectrum of finite algebras from those that are finitely based, are the algebras which are nonfinitely based in a rather strong manner. These are the algebras which do not belong to any locally finite finitely based variety.

Definition 5.8 *A finite algebra* **A** *with finitely many fundamental operations is said to be* ***inherently nonfinitely based*** *if* **A** *does not belong to any locally finite finitely based variety.*

Clearly, any finite algebra which is inherently nonfinitely based is nonfinitely based.

The first known example of an inherently nonfinitely based finite algebra was given by Murskiĭ in [40]. He constructed a three-element algebra in the language of a single binary operation which was inherently nonfinitely based.

Example 5.9 *(Murskiĭ's groupoid)*

	0	a	b
0	0	0	0
a	0	0	a
b	0	b	b

Table 5.1: Murskiĭ's groupoid

In [31], McKenzie proves that a residually small congruence modular variety does not contain an inherently nonfinitely based algebra.

A particularly abundant source of inherently nonfinitely based algebras is the class of *graph algebras*.

Definition 5.10 *Let* $\mathbf{G} = \langle V, E\rangle$ *be a graph, where* V *is the set of vertices of* $\mathbf{G}$ *and* E *is the set of undirected edges of* $\mathbf{G}$. *Let* $A = V \cup \{0\}$, *where* $0 \notin V$, *and define the binary operation* $\circ$ *on* A *by*

$$a \circ b = \begin{cases} a, & \text{if } \langle a, b\rangle \in E \\ 0, & \text{otherwise} \end{cases}.$$

The algebra $\mathbf{A} = \langle A, \circ\rangle$ *is called the* graph algebra *of* $\mathbf{G}$.

Most often the difference between $\mathbf{G}$ and $\mathbf{A}$ will be blurred, and we will identify a graph algebra with its underlying graph. We also assume that graphs are allowed to contain loops.

The notion of a graph algebra was first introduced and studied by C. Shallon in [47].

It turns out that the groupoid of Murskiĭ is a graph algebra; its underlying graph is

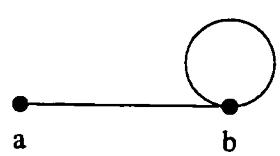

Figure 5.1: Underlying graph of Murskiĭ 's groupoid

McNulty and Shallon in [38] study inherently nonfinitely based algebras, and, in particular, those that carry the structure of a graph algebra.

The graph algebras which are finitely based were classified by Baker, McNulty, and Werner in [2]. The classification is given in terms of induced subgraphs of the underlying graph of the algebra.

Theorem 5.11 *(Baker, McNulty, Werner [3]) A graph* **G** *has a finitely based graph algebra if and only if* **G** *has no induced subgraph isomorphic to one of the four graphs listed below.*

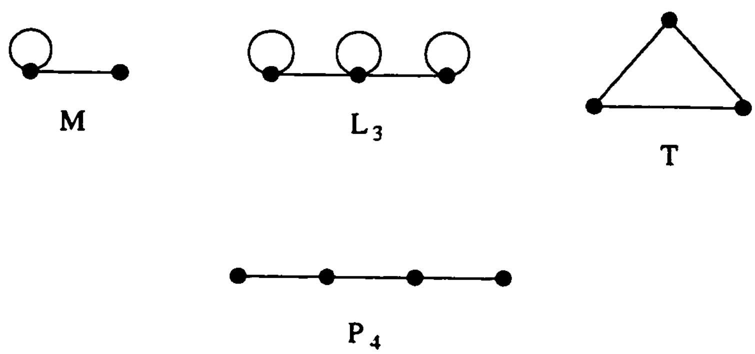

Figure 5.2: Prohibited subgraphs

Moreover, every nonfinitely based graph algebra is inherently nonfinitely based.

Theorem 5.11 will be used as a template upon which the results of Chapter 6 will be based. We shall show that the analogous graph-theoretic characterization holds for graph M-algebras.

Finally, in closing this section, we would like to mention a result which relates the finite basis property to the one of having definable principal congruences.

Theorem 5.12 *(McKenzie [30]) If V is a locally finite variety in a finite language which is residually less than n, for some $n < \omega$, and which has definable principal* *congruences, then it is finitely based.*

In the following chapter, we introduce a weakening of the concept of definability of principal congruences and use it to develop an analogue of Theorem 5.12 for graph M-algebras.

5.2 M-algebras: basic properties and subdirectly irreducible M-algebras

Definition 5.13 *An* ***M-algebra*** *is an algebra* $\mathbf{A}$ *whose type includes a binary meet-semilattice operation* $\wedge$ *and a constant 0, such that*

1. $\langle A, \wedge \rangle$ *is a height-1 meet-semilattice with least element 0.*

2. *0 is an absorbing element; that is, if* f *is an* n*-ary fundamental operation of* $\mathbf{A}$ *and* $0 \in \{a_1, \ldots, a_n\}$, *then* $f(a_1, \ldots, a_n) = 0$.

In recent years M-algebras have drawn considerable interest on the part of general algebraists. The reason for this lies in the fact that expansions of these algebras were cleverly used by R. McKenzie to refute some of long-standing conjectures in universal algebra, such as the Quackenbush's conjecture, as well as answer Tarski's Finite Basis Problem. (see e.g. [32], [33]).

The varieties generated by M-algebras display a rather agreeable behaviour: the class $V_{SI}(\mathbf{A})$, for an M-algebra $\mathbf{A}$, can be described in a uniform way.

Our exposition will follow closely the one given in [55], which is also our main reference for this section.

Definition 5.14 *Let* $\mathbf{A}$ *be an M-algebra, and let* $U = A \setminus \{0\}$. *We define the binary relation* $\gg$ *on* U *in the following way:* $a \gg b$ *if and only if* $f(a_1, \ldots, a_n) = b$, *for some n-ary fundamental operation* f *of* $\mathbf{A}$ *and* $a_1, \ldots, a_n \in U$, *such that* $a \in \{a_1, \ldots, a_n\}$.

Clearly, the relation $\gg$ is reflexive on U. The transitive closure of $\gg$ will be denoted by $\ggg$. Then, $\ggg$ is a preorder on U.

The following proposition characterizes subdirectly irreducible M-algebras in terms of the relation $\ggg$:

Proposition 5.15 *Let* $\mathbf{A}$ *be an M-algebra with* $U = A \setminus \{0\}$.

1. $\mathbf{A}$ *is subdirectly irreducible if and only if there exists* $b \in U$ *such that every* $a \in U$ *satisfies* $a \ggg b$. *If* $\mathbf{A}$ *is subdirectly irreducible, the monolith of* $\mathbf{A}$ *is the following congruence*

$$\mu = 0_{\mathbf{A}} \cup (X \cup \{0\})^2,$$

where $X = \{a \in U : b \ggg a\}$.

2. $\mathbf{A}$ *is simple if and only if* $\mathbf{A}$ *is subdirectly irreducible and the relation* $\ggg$ *is symmetric.*

For the proof, see [55], Lemma 1.1.

In what follows, we fix the notation from the statement of Proposition 5.15 and the paragraphs preceding it.

Consider an arbitrary power $\mathbf{A}^I$ of $\mathbf{A}$, where $I \neq \emptyset$. The reduct $\langle A^I, \wedge \rangle$ of $\mathbf{A}^I$ to the language $\{\wedge\}$ is a meet-semilattice whose smallest element is the constant I-sequence $\hat{0}$.

It is easy to see now that the I-sequences in U^I are precisely the maximal elements of $\langle A^I, \wedge \rangle$. If $\mathbf{B} \leq \mathbf{A}^I$, we can define the binary relation $\gg$ on $B \setminus \{\hat{0}\}$ in the same way as it was done for $\mathbf{A}$. The definition of $\ggg$ extends now naturally to $B \setminus \{\hat{0}\}$.

Let $B(U) = B \cap U^I$. Then, if $g \in B(U)$ and $f \ggg g$, it implies $f \in B(U)$, as well.

If p is an arbitrary element of $B(U)$, we define

$$B_p = \{f \in B(U) : f \ggg p\}.$$

$\mathbf{B}(p)$ is now defined to be the algebra in the language of $\mathbf{A}$, whose universe is the set $B_p \cup \{\hat{0}\}$, and whose fundamental operations are given by:

$$f^{\mathbf{B}(p)}(h_1, \dots, h_n) = \begin{cases} f^{\mathbf{B}}(h_1, \dots, h_n) & \text{if } f^{\mathbf{B}}(h_1, \dots, h_n) \in B_p \\ \hat{0} & \text{otherwise} \end{cases}$$

The $\wedge$-reduct of the algebra $\mathbf{B}(p)$ is a meet-semilattice of height one, and $\hat{0}$ is an absorbing element for every fundamental operation of $\mathbf{B}(p)$. Thus, $\mathbf{B}(p)$ is an M-algebra. Also, $\gg$ and $\ggg$ will be the restrictions to B_p of the corresponding relations defined on $B \setminus \{\hat{0}\}$.

The following theorem gives an explicit description of subdirectly irreducible members of a variety generated by an M-algebra.

Theorem 5.16 *(Willard [55]) Let* $\mathbf{A}$ *be an M-algebra.*

1. *If* $\mathbf{B} \leq \mathbf{A}^I$ $(I \neq \emptyset)$ *and* $p \in B(U)$, *then* $\mathbf{B}(p)$ *is in* $\mathbf{V}_{SI}(\mathbf{A})$.
2. *Conversely, if* $\mathbf{A}$ *is finite, then every member of* $\mathbf{V}_{SI}(\mathbf{A})$ *is isomorphic to* $\mathbf{B}(p)$ *for some such* $\mathbf{B}$ *and* p.

We finish this section by giving a brief discussion of the relationship between residual smallness for a variety generated by a finite M-algebra and its being finitely based.

Definition 5.17 *Let* $\mathbf{A}$ *be an M-algebra. We say that* $\mathbf{A}$ ***commutes*** *with* $\wedge$ *if*

$$f(a_1, \ldots, a_n) \wedge f(b_1, \ldots, b_n) = f(a_1 \wedge b_1, \ldots, a_n \wedge b_n)$$

for every n*-ary fundamental operation of* $\mathbf{A}$ $(n > 0)$ *and all* $a_i, b_i \in A$ $(i = 1, \ldots, n)$.

The following two results are taken from [56].

Theorem 5.18 *Let* $\mathbf{A}$ *be a finite M-algebra. Then,* $\mathbf{V}(\mathbf{A})$ *is residually small if and only if* $\mathbf{A}$ *commutes with* $\wedge$.

Theorem 5.19 *If* $\mathbf{A}$ *is a finite M-algebra such that* $\mathbf{V}(\mathbf{A})$ *is residually small, then* $\mathbf{A}$ *is finitely based.*

The converse of Theorem 5.19 is false:

Example 5.20 *Let* $\mathbf{A}$ *be the algebra in the language* $\{\wedge, *, 0\}$ *whose universe is* $A = \{0, a, b\}$, *and* $\wedge, *$ *are binary operations. The* $\wedge$*-reduct of* $\mathbf{A}$ *is a height-one semilattice with 0 as the smallest element, while* $*$ *is given by:*

$*$	0	a	b
0	0	0	0
a	0	a	a
b	0	a	b

Table 5.2: The *-reduct of a residually large finitely based M-algebra

It is easily seen that $*$ does not commute with $\wedge$, since

$$a = (a * a) \wedge (a * b) \neq (a \wedge a) * (a \wedge b) = 0,$$

and $\mathbf{V}(\mathbf{A})$ is residually large. However, one can show that the identities of $\mathbf{A}$ are finitely based.

Chapter 6

Finitely based graph M-algebras

In this chapter the notion of a graph M-algebra is introduced. Next, we characterize those graph M-algebras whose equational theory is inherently nonfinitely based in terms of induced subgraphs of their underlying graphs. It turns out that this classification is analogous to the one given in [3] for graph algebras. We prove that every graph M-algebra which omits certain finite graphs as induced subgraphs is finitely based. To this end, the notion of definability of ordered principal congruences is introduced and it is shown that this property in conjunction with the definability of the class of subdirectly irreducibles in the variety implies finite axiomatizability of the variety.

The following definition plays the crucial role in our further exposition:

Definition 6.1 *Suppose* $\mathbf{G} = \langle G, \circ, 0 \rangle$ *is a graph algebra. We define the* ***graph M-algebra*** $\mathbf{G}^{\wedge}$*, corresponding to* $\mathbf{G}$*, to be the M-algebra in the language* $\{\circ, \wedge, 0\}$ *which is an expansion of* $\mathbf{G}$ *and whose smallest element under the semilattice ordering is* 0*, the absorbing element for the multiplication in* $\mathbf{G}$*.*

6.1 Varieties which are inherently nonfinitely based

In this section we prove that if a finite graph contains a subgraph isomorphic to one of the four graphs which will be introduced below, its corresponding graph M-algebra will generate an inherently nonfinitely based variety. In fact, as we shall see later in this chapter, these are precisely those varieties generated by a finite graph M-algebra which are nonfinitely based.

The main result of [3] (Theorem 5.11) states the following:

A graph **G** has a finitely based graph algebra if and only if **G** has no induced subgraph isomorphic to one of the four graphs listed below.

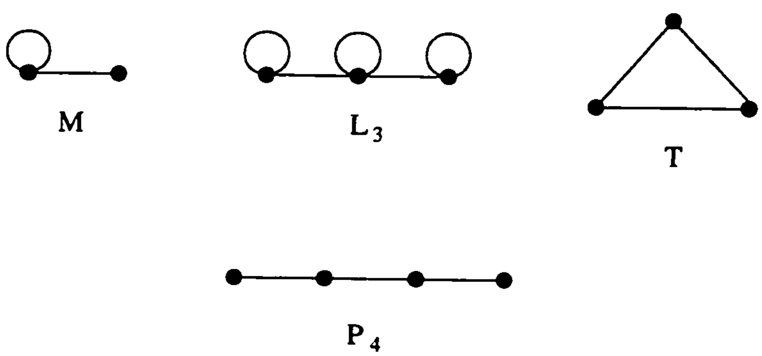

Figure 6.1: Prohibited subgraphs

Moreover, every nonfinitely based graph algebra is inherently nonfinitely based.

Now we give some definitions which are necessary in order to state one of the main tools used in the proof of Theorem 5.11.

Definition 6.2 *An element* ∞ *of an algebra* **A** *is* absorbing *if every fundamental operation applied to an n-tuple of elements of A containing* ∞ *has the value* ∞.

We say that an element of A is proper *if it is distinct from ∞.*

If $\alpha \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$, we denote by $\alpha^{(i)}$ the $\mathbb{Z}$-sequence obtained from α in the following way:

$$\alpha^{(i)}(n) = \alpha(n+i).$$

Now we are ready to state the theorem which is the crux of the proof of Theorem 5.11.

Theorem 6.3 *(Baker, McNulty, Werner [2]) Let $\mathbf{A}$ be a finite algebra of finite type, with an absorbing element ∞. Suppose that a $\mathbb{Z}$-sequence α of proper elements of $\mathbf{A}$ can be found with the following properties:*

(a) in $\mathbf{A}^{\mathbb{Z}}$, any fundamental operation f applied to translates of α is either a translate of α or a sequence containing ∞;

(b) there are only finitely many equations $f(\alpha^{(i_1)}, \ldots, \alpha^{(i_n)}) = \alpha^{(j)}$, where f is an n-ary fundamental operation and some argument is α itself;

(c) there is at least one equation $f(\alpha^{(i_1)}, \ldots, \alpha^{(i_n)}) = \alpha^{(1)}$, where some argument is α itself, in a variable on which f depends.

Then $\mathbf{A}$ is inherently nonfinitely based.

As mentioned at the beginning of the section, it turns out that the situation is the same for the finitely generated varieties of graph M-algebras. The main result of this section can be stated as follows:

Theorem 6.4 *If* $\mathbf{G}^\wedge$ *is a graph M-algebra whose underlying graph* $\mathbf{G}$ *contains one of the four graphs listed above as a subgraph,* $\mathbf{G}^\wedge$ *is inherently nonfinitely based.*

The following three lemmas comprise the proof of our theorem:

Lemma 6.5 *If the underlying graph of a graph M-algebra* $\mathbf{A}$ *contains a connected component which is neither a complete looped graph nor a complete bipartite graph, then it contains one of the four graphs* $\mathbf{M}, \mathbf{L}_3, \mathbf{T}, \mathbf{P}_4$ *as an induced subgraph.*

PROOF. Let C be a connected component of the underlying graph of $\mathbf{A}$, which is neither a complete looped graph nor a complete bipartite graph.

If C is a bipartite graph which is not complete, then it is immediate that C contains $\mathbf{P}_4$ as an induced subgraph.

If C is not a bipartite graph and contains no loops then C must contain an odd length cycle. From this, it is easy to conclude that either $\mathbf{T}$ or $\mathbf{P}_4$ will be contained in C.

Finally, suppose C contains loops and C is not complete. If there is a vertex without a loop, $\mathbf{M}$ must be an induced subgraph of C; otherwise, all the vertices are looped and there is a pair of vertices with no edge connecting them. However, these two vertices must be connected by some path of length n. Choose this pair of vertices so that n is minimal. Now, it is an easy exercise to verify that $\mathbf{L}_3$ will be an induced subgraph of the underlying graph of $\mathbf{A}$. □

Lemma 6.6 *If* $\mathbf{H}$ *is an induced subgraph of a graph* $\mathbf{G}$, *then* $\mathbf{H}^\wedge$ *is a subalgebra of* $\mathbf{G}^\wedge$.

PROOF. Trivial. □

Lemma 6.7 $\mathbf{M}^\wedge, \mathbf{L}_3^\wedge, \mathbf{T}^\wedge, \mathbf{P}_4^\wedge$ *generate inherently nonfinitely based varieties.*

PROOF. It is proved in the examples 4.2-4.6 of [2] that the graph algebras corresponding to these four graphs are inherently nonfinitely based. The authors exhibit a specific $\mathbb{Z}$-sequence α for each of them which witnesses the conditions of Theorem 6.3. Let $\mathbf{G}^\wedge$ be the graph M-algebra corresponding to $\mathbf{G}$, where $\mathbf{G} \in \{\mathbf{M}, \mathbf{L_3}, \mathbf{T}, \mathbf{P_4}\}$, and let α be the corresponding $\mathbb{Z}$-sequence for $\mathbf{G}$. Now, we need to check that the conditions of Theorem 6.3 are met after augmenting the language with $\wedge$, with the same α witnessing them. However, this follows almost immediately from the fact that $\wedge$ induces the height-1 semilattice ordering of $\mathbf{G}$. □

6.2 The structure of $V(\mathbf{A})$ when $\mathbf{A}$ is not inherently nonfinitely based

From this point on, we are primarily interested in those finite graph M-algebras which omit every one of the four graphs $\mathbf{M}, \mathbf{L_3}, \mathbf{T}$,and $\mathbf{P_4}$ as induced subgraphs. We prove that, in this case, the class of subdirectly irreducibles is first-order definable.

Definition 6.8 *Let $\kappa, \lambda \geq 1$ be any cardinals. K_κ^0 will denote the complete looped graph on κ vertices, while $K_{\kappa,\lambda}$ denotes the complete bipartite graph with no loops whose blocks are of size κ and λ, respectively.*

Let $\mathbf{A}$ be a finite graph M-algebra with the aforementioned property.

Proposition 6.9 *In $V(\mathbf{A})$, every subdirectly irreducible algebra is a simple graph M-algebra whose underlying graph is connected.*

PROOF. Let $\mathbf{B} \in V_{SI}(\mathbf{A})$. According to the description of subdirectly irreducibles in varieties generated by a single M-algebra (see Theorem 5.16), it is easy to see that $\mathbf{M}$ is a graph M-algebra and the underlying graph of $\mathbf{B}$ has to be connected. In fact, this is proved in [55].

The fact that $\mathbf{B}$ is simple follows from Lemma 1.1 (2) of [55]. □

Proposition 6.10 *If $\mathbf{A}$ omits every one of $\mathbf{M}, \mathbf{L_3}, \mathbf{T}$, and $\mathbf{P_4}$ as an induced subgraph, then the same is true of every $\mathbf{B} \in V_{SI}(\mathbf{A})$.*

PROOF. We use the description of subdirectly irreducibles in a variety generated by a finite M-algebra, as given in Theorem 5.16.

Assume $\mathbf{B} \in V_{SI}(\mathbf{A})$. Let A^+ and B^+ denote the underlying graphs of nonzero elements of A and B, respectively. We claim that for some set I,

$$B^+ \leq (A^+)^I.$$

as graphs.

Let $\mathbf{B} = \mathbf{B}_1(p)$ as in the statement of Theorem 5.16, where $\mathbf{B}_1 \leqslant \mathbf{A}^I$, for some I and some $p \in \mathbf{B}_1(U)$. The choice of I is now obvious, and

$$B^+ = B_1(p) \setminus \{0\}.$$

We need to show that, if $b, c \in B_1(p) \setminus \{0\}$ are such that $b \circ c \neq 0$ in $\mathbf{A}$, then $(b \circ c)(i)$ is in the same connected component with $p(i)$, for every $i \in I$. However, this follows immediately from the fact that $B_1(p)$ is $\ggg$–connected and $(b \circ c)(i) \neq 0$, for all $i \in I$.

If **B** contains an induced subgraph isomorphic to **M**, it is easily seen that the same is true of **A**.

Now, if **B** contains an induced subgraph isomorphic to $\mathbf{L_3}$, where $a, b, c \in (A^+)^I$,

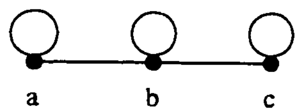

Figure 6.2: L_3

pick $i_0 \in I$ such that

$$a(i_0) \circ c(i_0) = 0.$$

Then, $b(i_0) \notin \{a(i_0), c(i_0)\}$, and $a(i_0), b(i_0), c(i_0)$ induce a subgraph in A^+ isomorphic to $\mathbf{L_3}$.

Suppose B^+ contains a copy of **T**, where $a, b, c \in (A^+)^I$. Now choose $i_0 \in I$ so

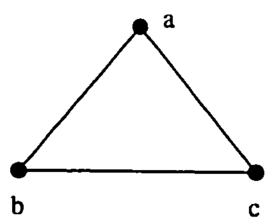

Figure 6.3: T

that

$$a(i_0) \circ a(i_0) = 0$$

in A^+. By analysing different cases we arrive at the conclusion that either a copy of $\mathbf{T}$ or one of $\mathbf{M}$ will be present in A^+.

Finally, if B^+ contains a copy of $\mathbf{P}_4$, where $a, b, c, d \in (A^+)^I$, pick a coordinate

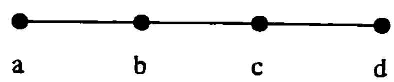

Figure 6.4: P_4

$i_0 \in I$ for which

$$a(i_0) \circ d(i_0) = 0.$$

Then, the possibilities that $a(i_0) = c(i_0)$ or that $b(i_0) = d(i_0)$ can be ruled out immediately. If $a(i_0) = b(i_0)$ or $c(i_0) = d(i_0)$ or $b(i_0) = c(i_0)$, A^+ will contain $\mathbf{M}$ as an induced subgraph. In the case when all four of $a(i_0), b(i_0), c(i_0)$, and $d(i_0)$ are distinct, A^+ will contain $\mathbf{P}_4$ as an induced subgraph. □

Thus, if $\mathbf{B}$ is a subdirectly irreducible member of $V(\mathbf{A})$, $\mathbf{B}$ can belong only to one of the following classes of graph M-algebras:

1. $(K_\kappa^0)^\wedge$, where $\kappa \geq 1$.

2. $(K_{\kappa,\lambda})^\wedge$, where $\kappa, \lambda \geq 1$.

3. a graph M-algebra whose underlying graph consists of a single vertex without a loop,

4. a trivial algebra whose only element is 0.

However, the presence of **B** of type (1) or (2) in $V_{SI}(\mathbf{A})$ where $\kappa, \lambda \geq 2$ will be sufficient to describe "almost all" algebras in $V_{SI}(\mathbf{A})$. Namely, we have the following proposition.

Proposition 6.11 *(a) If $\kappa \geq 2$, $V((K_\kappa^0)^\wedge)$ contains every $\left(K_\mu^0\right)^\wedge$, where $\mu \geq 1$.*

(b) If $\kappa \geq 2$, $V((K_{\kappa,1})^\wedge)$ contains every $(K_{\mu,1})^\wedge$, where $\mu \geq 1$.

(c) If $\kappa, \lambda \geq 2$, $V((K_{\kappa,\lambda})^\wedge)$ contains every $(K_{\mu,\nu})^\wedge$, where $\mu, \nu \geq 1$.

PROOF. (a) Let a, b be two distinct elements of $(K_\kappa^0)^\wedge$. Now, define $c^{(i)}$ $(0 \leq i < \mu)$ to be the μ-sequence of elements from $(K_\kappa^0)^\wedge$, so that

$$c^{(i)}(j) = \begin{cases} a, & j \leq i \\ b, & i < j \end{cases}.$$

Let **A** be the subalgebra of $((K_\kappa^0)^\wedge)^\mu$ generated by $c^{(i)}$ $(0 \leq i < \mu)$, and let θ be the smallest congruence of **A** which identifies all μ-sequences in **A** which contain a 0-entry. The reader can check that $\mathbf{A}/\theta$ will be a graph M-algebra isomorphic to $(K_\mu^0)^\wedge$.

The proof of (b) is a slight modification of a more general proof for (c), and is therefore omitted.

(c) Let a, b be two distinct elements of one of the two bipartite blocks of $K_{\kappa,\lambda}$, and c, d two distinct elements of the other block. Let $\xi = \max(\mu, \nu)$. Define $e^{(i)}$

$(0 \le i < \mu)$, a ξ-sequence of elements from $K_{\kappa,\lambda}$, and $f^{(i)}$ $(0 \le i < \nu)$, a ξ-sequence of elements from $K_{\kappa,\lambda}$ as follows:

$$e^{(i)}(j) = \begin{cases} a, & j \le i \\ b, & i < j \end{cases},$$

$$f^{(i)}(j) = \begin{cases} c, & j \le i \\ d, & i < j \end{cases}.$$

Let $\mathbf{A}$ be the subalgebra of $((K_{\kappa,\lambda})^{\wedge})^{\xi}$ generated by $e^{(i)}$ $(0 \le i < \mu)$ and $f^{(i)}$ $(0 \le i < \nu)$. Again, let θ be the congruence of $\mathbf{A}$ obtained by identifying all ξ-sequences containing 0. Then, $\mathbf{A}/\theta$ will be a graph M-algebra isomorphic to $(K_{\mu,\nu})^{\wedge}$. □

As an immediate consequence of the proposition, we get:

Corollary 6.12 *If* $\mathbf{A}$ *is a finite graph M-algebra which omits* $\mathbf{M}, \mathbf{L_3}, \mathbf{T}$, *and* $\mathbf{P_4}$, $V_{SI}(\mathbf{A})$ *can be defined by a single first-order sentence.*

6.3 Definable ordered principal congruences and finite basis

In this section we proceed with the proof of the finite basis theorem for the finitely generated varieties of graph M-algebras in which the generating algebra omits as induced subgraphs $\mathbf{M}, \mathbf{L_3}, \mathbf{T}$, and $\mathbf{P_4}$. We have already established the fact that in such finitely generated varieties, the class of subdirectly irreducibles is first-order definable. The next step would be to prove the DPC property for the variety. Instead, we prove a seemingly weaker property, which will turn out to be sufficient for the case of graph M-algebras. Moreover, this property, *DOPC*, which is to be defined, will first be established for the class of subdirectly irreducibles in the variety; after that, the property will be lifted to every member of the variety (as it will turn out to be expressible by a collection of quasi-identities).

Definition 6.13 *Let V be a variety such that there is a binary term operation in the language of V which induces a meet-semilattice operation in every algebra of V. We say that V has* DOPC *(or* ***definable ordered principal congruences****) if there are finitely many principal congruence formulas which define the congruences of the form $Cg^{\mathbf{A}}(a,b)$, where $b \leq a$, in every $\mathbf{A} \in V$.*

Obviously, DPC implies implies DOPC.

Proposition 6.14 *Let V be a variety which has a binary term operation $\wedge$ which is a semilattice operation in every member of V. Suppose V_{SI} is first-order definable and V has DOPC. Then, V is finitely based.*

Proof. Note, first, that if W is any variety in the language of V in which $\wedge$ is a meet-semilattice operation and $\mathbf{A} \in W$, any nontrivial principal congruence of $\mathbf{A}$ contains a pair $\langle a, b\rangle$ such that $b < a$. Next, if the congruences of the form $Cg^{\mathbf{A}}(a,b)$, where $b \leq a$ are definable in V by a disjunction of principal congruence formulas Ψ, we can write down a sentence Φ_1 such that, for every such variety W,

$$W \models \Phi_1$$

if and only if

$$\forall zu \left(u \leq z \to \text{“}\{\langle x, y\rangle : \Psi(x, y, z, u)\} \text{ is } Cg^{\mathbf{A}}(z, u)\text{”}\right)$$

holds in every $\mathbf{A} \in W$. Since the class of subdirectly irreducibles of V is first-order definable, there is a sentence Θ such that, if $\mathbf{A}$ is any algebra in the language of V,

$$\mathbf{A} \models \Theta \text{ if and only if } \mathbf{A} \in V_{SI}.$$

Let Φ_2 be a sentence asserting that the $\wedge$-reduct of an algebra is a semilattice. Now, define Φ_3 to be the sentence

$$\Phi_1 \ \wedge \Phi_2 \ \wedge \ [\exists xy(x \neq y \wedge \forall zu(z < u \to \Psi(x, y, z, u))) \to \Theta].$$

Let Σ be the set of identities defining V. By the note at the beginning of the proof,

$$\Sigma \models \Phi_3,$$

by compactness, there exists a finite subset Σ' of Σ such that

$$\Sigma' \models \Phi_3.$$

If $\mathbf{B}$ is any subdirectly irreducible algebra in the language of Vwhich satisfies Σ', it will also satisfy Φ_3, and thus, it will be isomorphic to some subdirectly irreducible member of V. Hence, Σ' will be a finite equational basis for V. □

In fact, Keith Kearnes proved that if a variety has DOPC then it has DPC as well. The following argument is due to Kearnes, and we include it here with his kind permission:

Proposition 6.15 *Let V be a variety with a binary term meet-semilattice operation $\wedge$. If V has DOPC then V has DPC.*

PROOF. Let Φ be the collection of principal congruence formulas which witnesses the fact that V has DOPC. Let $\mathbf{A} \in V$ and $a, b, c, d \in A$ so that

$$\langle c, d\rangle \in Cg^{\mathbf{A}}(a, b).$$

Define $\theta = Cg^{\mathbf{A}}(a \wedge b, b)$ and let

$$\mathbf{B} = \mathbf{A}/\theta \in V.$$

Clearly,

$$\langle c/\theta, d/\theta\rangle \in Cg^{\mathbf{B}}(a/\theta, b/\theta),$$

since, in $\mathbf{A}$, $\theta \subseteq Cg^{\mathbf{A}}(a, b)$.

Since V has DOPC, there exists a principal congruence formula $\phi \in \Phi$ such that

$$\mathbf{B} \models \phi(c/\theta, d/\theta, a/\theta, b/\theta).$$

Then, for some $(m+1)$-ary terms $t_1(x,\bar{y}),\dots,t_n(x,\bar{y})$, where $\bar{y}$ is an m-tuple of variables,

$$\mathbf{B} \models \exists \bar{y} \left[c/\theta = t_1(e_1/\theta,\bar{y}) \ \wedge \bigwedge_{1\le i<n} t_i(f_i/\theta,\bar{y}) = t_{i+1}(e_{i+1}/\theta,\bar{y}) \ \wedge \ t_n(f_n/\theta,\bar{y}) = d/\theta \right]$$

where

$$\{e_i,f_i\} = \{a,b\}$$

for $1 \le i \le n$. Thus,

$$\mathbf{A} \models \exists \bar{y} \left[\begin{array}{c} \text{“}\langle c, t_1(e_1,\bar{y})\rangle \in \theta\text{”} \wedge \bigwedge_{1\le i<n} \text{“}\langle t_i(f_i,\bar{y}), t_{i+1}(e_{i+1},\bar{y})\rangle \in \theta\text{”} \\ \wedge \ \text{“}\langle t_n(f_n,\bar{y}), d\rangle \in \theta\text{”} \end{array} \right] \tag{6.1}$$

Hence, there are some principal congruence formulas $\psi_0,\dots,\psi_n \in \Phi$ so that

$$\mathbf{A} \models \exists \bar{y} \left[\begin{array}{c} \psi_0\,(c, t_1(e_1,\bar{y}), a\wedge b, b\wedge b) \\ \wedge \bigwedge_{1\le i<n} \psi_i\,(t_i(f_i,\bar{y}), t_{i+1}(e_{i+1},\bar{y}), a\wedge b, b\wedge b) \\ \wedge \ \psi_n\,(t_n(f_n,\bar{y}), d, a\wedge b, b\wedge b) \end{array} \right].$$

Therefore,

$$\mathbf{A} \models \exists \bar{y} z \left[\begin{array}{c} \psi_0\,(c, t_1(e_1,\bar{y}), a\wedge z, b\wedge z) \wedge \\ \bigwedge_{1\le i<n} \psi_i\,(t_i(f_i,\bar{y}), t_{i+1}(e_{i+1},\bar{y}), a\wedge z, b\wedge z) \\ \wedge \ \psi_n\,(t_n(f_n,\bar{y}), d, a\wedge z, b\wedge z) \end{array} \right], \tag{6.2}$$

and clearly, this last formula is equivalent to a principal congruence formula. Hence,

$$\langle c, d\rangle \in Cg^{\mathbf{A}}(a, b)$$

can be witnessed by a principal congruence formula of the form (6.2). From the course of the proof, it is obvious that there are only finitely many such formulas, which entails the definability of principal congruences of V. □

Thus, V has DOPC if and only if it has DPC. Thus Proposition 6.15 is implied by the known one that if a variety has definable principal congruences and its class of subdirectly irreducibles is first-order definable, then its equational theory is finitely axiomatizable.

However, in our case trying to prove that the variety has definable ordered principal congruences can considerably simplify the task of being able to list principal congruence formulas witnessing the property, since, as can be inferred from the proof of the proposition, in general, the number of principal congruence formulas witnessing DPC will be an exponential function of the number of principal congruence formulas witnessing DOPC.

Lemma 6.16 *Let V be a variety generated by a finite M-algebra and let $s(x, \bar{v}), t(x, \bar{w})$ be two terms in which x occurs explicitly. Then,*

$$V \models \forall zu\bar{v}\bar{w}(u \leq z \rightarrow (\{s(z, \bar{v}), s(u, \bar{v})\} \cap \{t(z, \bar{w}), t(u, \bar{w})\} \neq \emptyset \rightarrow s(u, \bar{v}) = t(u, \bar{w}))).$$

PROOF. First of all, every subdirectly irreducible member of a variety generated by a finite M-algebra is itself an M-algebra (see e.g. [55]). Also, if $p(x)$ is a unary polynomial of $\mathbf{B} \in V$, built from a term in which x occurs explicitly, then $p(0) = 0$, since 0 is an absorbing element for any algebra of V. Using these two facts it is straightforward to prove the validity of the sentence in every subdirectly irreducible algebra of V.

The condition

$$\{s(z,\bar{v}), s(u,\bar{v})\} \cap \{t(z,\bar{w}), t(u,\bar{w})\} \neq \emptyset$$

is equivalent to the following formula

$$s(z,\bar{v}) = s(z,\bar{w}) \ \vee \ s(z,\bar{v}) = t(u,\bar{w}) \ \vee s(u,\bar{v}) = t(z,\bar{w}) \ \vee \ s(u,\bar{v}) = t(u,\bar{w}).$$

Hence, the original formula is equivalent to a conjunction of four universal Horn sentences, and, since it is true in every member of V_{SI}, it will be true in every algebra of V. □

Fix a finite M-algebra $\mathbf{A}$ and consider the quantifier-free formulas of the forms

(0) $x = y$

(1) $x = s(x,z) \wedge y = s(x,u)$

(2) $x = s(y,u) \wedge y = s(y,z)$

(3) $x = s(x,z) \wedge y = t(y,z) \wedge s(x,u) = t(y,u)$

where s and t are terms which may contain additional parameters and in which z (resp. u) occurs explicitly. We also allow for the possibility that x (resp. y) does not occur in s (resp. t).

Let $\Phi(x,y,z,u) = \{\phi_i(x,y,z,u) : i < \omega\}$ be the collection of all principal congruence formulas whose quantifier-free parts are of the forms (1)-(3). Also, $\sigma_i(x,y,z,u,\bar{v})$ will denote the quantifier-free part of $\phi_i(x,y,z,u)$.

Lemma 6.17 *Let* $\mathbf{A}$ *be a finite M-algebra and* $\sigma_i(x,y,z,u,\bar{v})$ *and* $\sigma_j(x,y,z,u,\bar{w})$ *be such that*

$$\phi_i(x,y,z,u) = \exists\bar{v}\sigma_i(x,y,z,u,\bar{v}),$$
$$\phi_j(x,y,z,u) = \exists\bar{w}\sigma_i(x,y,z,u,\bar{w})$$

are both in $\Phi(x,y,z,u)$. *Then there is* $\sigma_k(x,y,z,u,\bar{v},\bar{w})$ *such that, for every* $\mathbf{B} \in V_{SI}(\mathbf{A})$, *and all* $a,b,c,d,e \in B$,

$$\mathbf{B} \models b \leq a \to \forall\bar{v}\bar{w}\,(\sigma_i(c,d,a,b,\bar{v}) \ \wedge\ \sigma_j(d,e,a,b,\bar{w}) \to \sigma_k(c,e,a,b,\bar{v},\bar{w}))$$

PROOF. The proof breaks into 16 cases, depending on the types of σ_i and σ_j. If the type of either formula is (0), the choice for σ_k is obvious.

Now, suppose that both σ_i and σ_j are of type (1); that is, we have

$$\begin{aligned} c &= s(c,a,\bar{f}),\ d = s(c,b,\bar{f}), \\ d &= s'(d,a,\bar{f}'),\ e = s'(d,b,\bar{f}') \end{aligned}$$

where $\mathbf{B} \in V_{SI}(\mathbf{A})$, $a, b, c, d, e \in B$, $b \leq a$, $\bar{f}$ and $\bar{f}'$ are tuples of elements of B, while s and s' are terms. Using Lemma 6.16, we get

$$e = s'(d, b, \bar{f}') = s(c, b, \bar{f}),$$

so σ_k can be chosen to be:

$$x = s(x, z) \ \wedge \ y = s(x, u).$$

Next, suppose σ_i is of the type (1) while σ_j is of the type (2). Again, suppose $\mathbf{B} \in V_{SI}(\mathbf{A})$, $a, b, c, d, e \in B$, $b \leq a$, $\bar{f}$ and $\bar{f}'$ are tuples of elements of B, while s and s' are terms. Then,

$$\begin{aligned} c &= s(c, a, \bar{f}), \ d = s(c, b, \bar{f}), \\ d &= s'(e, b, \bar{f}'), \ e = s'(e, a, \bar{f}'). \end{aligned}$$

Thus, we have σ_k of type (3):

$$x = s(x, z, \bar{v}) \ \wedge \ y = s'(y, z, \bar{w}) \ \wedge \ s(x, u, \bar{v}) = s'(y, u, \bar{w}).$$

In the next case, suppose σ_i is of type (1), and σ_j is of type (3). As before, we assume that $\mathbf{B} \in V_{SI}(\mathbf{A})$, $a, b, c, d, e \in B$, $b \leq a$, $\bar{f}$ and $\bar{f}'$ are tuples of elements of

B, while s, s' , and s'' are terms, so that

$$\begin{aligned} c &= s(c, a, \bar{f}),\ d = s(c, b, \bar{f}), \\ d &= s'(d, a, \bar{f}'),\ e = s''(e, a, \bar{f}'),\ s'(d, b, \bar{f}') = s''(e, b, \bar{f}'). \end{aligned}$$

Using Lemma 6.16, we get

$$d = s(c, b, \bar{f}) = s'(d, b, \bar{f}') = s''(e, b, \bar{f}').$$

Hence, we have σ_k of type (3):

$$x = s(x, z, \bar{v}) \ \wedge \ y = s''(y, z, \bar{w}) \ \wedge \ s(x, u, \bar{v}) = s(y, u, \bar{w}).$$

The other six cases can be handled in a similar fashion. The following table shows how σ_k depends on σ_i and σ_j:

	(1)	(2)	(3)
(1)	(1)	(3)	(3)
(2)	(0)	(2)	(2)
(3)	(1)	(3)	(3)

Table 6.1: Dependence of σ_k on σ_i and σ_j

□

We are in the position to prove the following theorem.

Theorem 6.18 *The collection $\Phi(x,y,z,u)$ of principal congruence formulas defines ordered principal congruences in $V(\mathbf{A})$, for every finite M-algebra $\mathbf{A}$.*

PROOF. First, we claim that $\Phi(x,y,z,u)$ defines ordered principal congruences in $V_{SI}(\mathbf{A})$. Let $\mathbf{B} \in V_{SI}(\mathbf{A})$, $a,b \in B$, such that $b \leq a$, and let $\rho(a,b)$ be the binary relation on B defined by:

$$\langle c,d\rangle \in \rho(a,b) \text{ if and only if } \mathbf{B} \models \phi_i(c,d,a,b),$$

for some $\phi_i \in \Phi$.

Obviously, $\rho(a,b)$ is reflexive, symmetric, and compatible with all fundamental operations of $\mathbf{B}$. By the preceding lemma, it is transitive. Obviously, $Cg^{\mathbf{B}}(a,b)$ is contained in $\rho(a,b)$, since if $p(x)$ is a unary polynomial of $\mathbf{B}$, $\langle p(a),p(b)\rangle \in \rho(a,b)$, and the latter equivalence relation is transitive. Conversely, $\rho(a,b) \subseteq Cg^{\mathbf{B}}(a,b)$, for if $\mathbf{B} \models \phi_i(c,d,a,b)$, where $\phi_i \in \Phi$, then, either there is $p(x) \in Pol_1(\mathbf{B})$ such that

$$\{c,d\} = \{p(a),p(b)\},$$

or there are two unary polynomials p and q, and $e \in B$, so that

$$\{c,e\} = \{p(a),p(b)\} \text{ and } \{e,d\} = \{q(a),q(b)\}.$$

Thus, Φ defines ordered principal congruences in V_{SI}.

To see that the same holds for the whole variety V, observe that, in light of Lemma 6.17, it is possible to write down a collection of universal Horn formulas which express transitivity of the binary relation $\rho(a,b)$ defined as above, for every algebra $\mathbf{B} \in V$, and all $a, b \in V$ such that $b \leq a$. All other properties (reflexivity, symmetry, and compatibility) will lift automatically from the class of subdirectly irreducibles to the whole of V. □

In order to show that every variety generated by a finite graph M-algebra which omits the four graphs must be finitely based, we show that the number of the parameters required in the definition of quantifier-free formulas $\sigma_i(x, y, z, u, \bar{v})$ can be bounded. The number of such formulas will then be essentially finite, since the language in question is finite, and Φ can be reduced to a finite subset $\Phi' \subseteq \Phi$, which will define ordered principal congruences in V.

Let L' be the restriction of the language of graph M-algebra language $L = \{\circ, \wedge, 0\}$ to the language of graph algebras.

We show first that, for every finite graph M-algebra $\mathbf{A}$ which omits $\mathbf{M}$, $\mathbf{L_3}$, $\mathbf{T}$, and $\mathbf{P_4}$, the unary L'-polynomials depend on at most two parameters from A.

Lemma 6.19 *Let* $\mathbf{A}$ *be a graph M-algebra which omits* $\mathbf{M}$, $\mathbf{L_3}$, $\mathbf{T}$, *and* $\mathbf{P_4}$, *and let* $V = V(\mathbf{A})$. *Then, for every* $(n+1)$*-ary term* $s(x, \bar{u})$ *in* L', *where* $n \geq 2$, *there exists a ternary* L'*-term* $p_s(x, y, z)$ *and two* $(n+1)$*-ary terms* $t_1(x, \bar{u})$ *and* $t_2(x, \bar{u})$ *such that*

$$V \models s(x, \bar{u}) = p_s(x, t_1(x, \bar{u}), t_2(x, \bar{u})).$$

PROOF. At this point, we are using the remarks made immediately after the proof of Proposition 6.10. The following identities are true in every subdirectly irreducible algebra in V (and, thus, in V):

$$\begin{aligned}(x \circ y) \circ z &= (x \circ z) \circ y \\ (x \circ (y \circ z)) \circ u &= (x \circ u) \circ (y \circ z) \\ ((x \circ y) \circ z) \circ u &= ((x \circ u) \circ y) \circ z \\ ((x \circ y) \circ z) \circ u &= (x \circ y) \circ (u \circ (x \circ z))\end{aligned}$$

Using these identities, it follows, by induction on the number of occurrences of the operation symbol $\circ$ in a $(n+1)-$term $s(x, \bar{u})$, where $n \geq 3$, that the term $p_s(x, y, z)$ can be chosen to be one of these:

$$x, y,\ x \circ y,\ y \circ x,\ z \circ (y \circ x),\ z \circ (x \circ y),\ (y \circ x) \circ z,\ (x \circ y) \circ z,$$

while the two $(n+1)$-ary terms $t_1(x, \bar{u})$ and $t_2(x, \bar{u})$ will depend on $s(x, \bar{u})$. □

Lemma 6.20 *Let $t(x, \bar{u})$ be an $(n+1)$-ary term in the language of graph M-algebras in which x occurs explicitly. Then, there is an $(m+1)$-ary term $t'(x, \bar{v})$ in L' in which x occurs explicitly, where $m \leq n$ and $\bar{v}$ is a sub-tuple of $\bar{u}$, such that*

$$V \models t(x, \bar{u}) \leq t'(x, \bar{v})$$

The proof of this lemma is by induction on the complexity of the term $t(x, \bar{u})$, and will be left to the reader.

Now that we have a bound on the number of parameters for L'-polynomials, we can establish a similar bound for all L-polynomials:

Lemma 6.21 *Let* $\mathbf{A}$ *be a graph M-algebra, which omits* $\mathbf{M}$, $\mathbf{L}_3$, $\mathbf{T}$, *and* $\mathbf{P}_4$, *and let* $V = V(\mathbf{A})$. *Then, for every* $(n+1)$*-ary term* $s(x,\bar{u})$ *in* L, *where* $n \geq 3$, *there exists a 4-ary* L*-term* $p_s(x,y,z,w)$ *and three* $(n+1)$*-ary terms* $t_1(x,\bar{u})$, $t_2(x,\bar{u})$ *and* $t_3(x,\bar{u})$ *such that*

$$V \models s(x,\bar{u}) = p_s(x, t_1(x,\bar{u}), t_2(x,\bar{u}), t_3(x,\bar{u})).$$

Proof. Let $s(x,\bar{u})$ be an $(n+1)$-ary term in L, where $n \geq 3$. By the Lemma 6.19, there is an L'-term s', which can be assumed to contain the same variables as s, such that

$$s(x,\bar{u}) = s(x,\bar{u}) \wedge s'(x,\bar{u}).$$

By Lemma 6.19, one can find a ternary term $p_{s'}(x,y,z)$, and two $(n+1)$-ary terms $t_1'(x,\bar{u})$ and $t_2'(x,\bar{u})$, so that

$$V \models s'(x,\bar{u}) = p_s(x, t_1'(x,\bar{u}), t_2'(x,\bar{u})).$$

Then, for the 4-ary term $p_s(x,y,z,w)$ we can choose

$$w \wedge p_{s'}(x,y,z),$$

and for the terms $t_1(x,\bar{u}), t_2(x,\bar{u})$ and $t_3(x,\bar{u})$,

$$\begin{aligned} t_1(x,\bar{u}) &= t_1'(x,\bar{u}), \\ t_2(x,\bar{u}) &= t_2'(x,\bar{u}), \\ t_3(x,\bar{u}) &= s(x,\bar{u}). \end{aligned}$$

In that case,

$$V \models s(x,\bar{u}) = s(x,\bar{u}) \wedge s'(x,\bar{u}) = s(x,\bar{u}) \wedge p_{s'}(x, t_1'(x,\bar{u}), t_2'(x,\bar{u})).$$

□

As an immediate corollary of Lemma 6.21, we deduce the following proposition.

Proposition 6.22 *There is a finite subset $\Phi'(x,y,z,u) \subseteq \Phi(x,y,z,u)$, so that for every finite graph M-algebra $\mathbf{A}$, which omits $\mathbf{M}$, $\mathbf{L}_3$, $\mathbf{T}$, and $\mathbf{P}_4$,*

$$V(\mathbf{A}) \models \forall xyzu \left(\bigvee \Phi'(x,y,z,u) \longleftrightarrow \bigvee \Phi(x,y,z,u) \right).$$

In particular, $V(\mathbf{A})$ has DOPC.

This completes the proof of the main result of this section:

Theorem 6.23 *If $\mathbf{A}$ is a finite graph M-algebra, which omits $\mathbf{M}$, $\mathbf{L}_3$, $\mathbf{T}$, and $\mathbf{P}_4$, then $\mathbf{A}$ has a finitely axiomatizable equational theory.*

Open problems

In conclusion, we state the following list of some problems arising from the thesis.

1. Does there exist a variety of unary algebras in a finite language whose equational theory is finitely axiomatizable and undecidable, yet whose word problems are unsolvable?

2. Do there exist finitely axiomatizable equational theories of groups or rings which are undecidable, yet whose word problems are solvable?

3. Does there exist a finitely axiomatizable decidable equational theory such that the theory of quasi-identities based on it is pseudorecursive?

4. Which finite M-algebras have finitely based equational theory? Does there exist a characterization of such algebras in terms of prohibited "configurations" of a combinatorial nature, which distinguish between finitely based and nonfinitely based ones?

5. Is every nonfinitely based finite M-algebra inherently nonfinitely based?

6. Which (finite) M-algebras have DPC?

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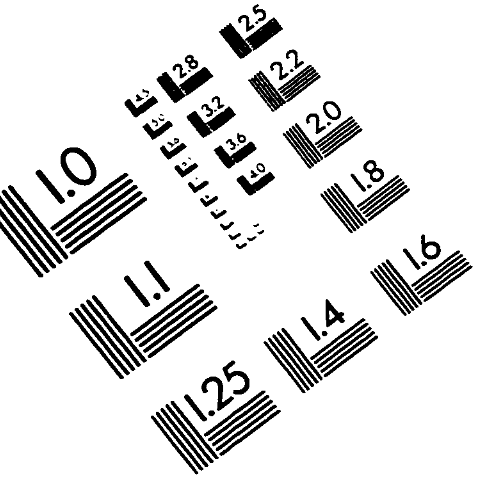
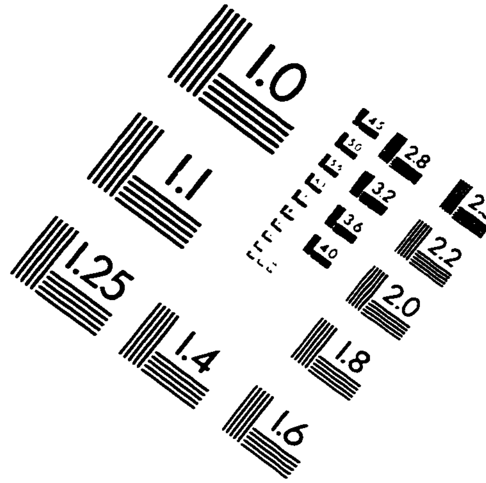
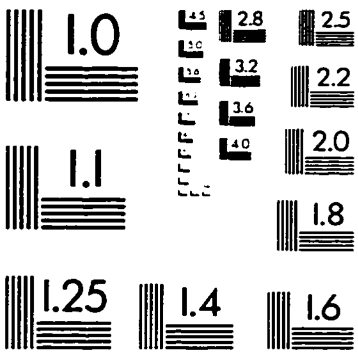
IMAGE EVALUATION TEST TARGET (QA-3)







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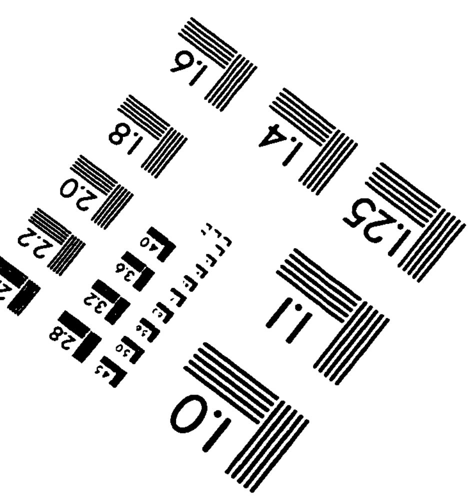

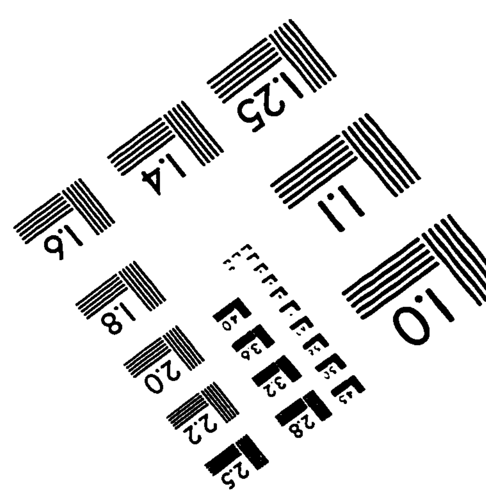

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