Entire Solutions to Dirichlet Type Problems

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any final revisions, as accepted by my examiners. I understand that my thesis may be made electronically available to the public.

Scott Sitar

Abstract

In this thesis, we examined some Dirichlet type problems of the form:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^n \\ u &= f \text{ on } \psi = 0, \end{aligned}$$

and we were particularly interested in finding entire solutions when entire data was prescribed. This is an extension of the work of D. Siegel, M. Mouratidis, and M. Chamberland, who were interested in finding polynomial solutions when polynomial data was prescribed. In the cases where they found that polynomial solutions always existed for any polynomial data, we tried to show that entire solutions always existed given any entire data. For half space problems we were successful, but when we compared this to the heat equation, we found that we needed to impose restrictions on the type of data allowed. For problems where data is prescribed on a pair of intersecting lines in the plane, we found a surprising dependence between the existence of an entire solution and the number theoretic properties of the angle between the lines. We were able to show that for numbers α with ω_1 finite according to Mahler's classification of transcendental numbers, there will always be an entire solution given entire data for the angle $2\alpha\pi$ between the lines. We were also able to construct an uncountable, dense set of angles of measure 0, much in the spirit of Liouville's number, for which there will not always be an entire solution for all entire data. Finally, we investigated a problem where data is given on the boundary of an infinite strip in the plane. We were unable to settle this problem, but we were able to reduce it to other *a priori* more tractable problems.

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Contents

Chapter 1

Introduction

1.1 Origins of the Problem

According to Whittaker and Watson in chapter 18 of [23], Laplace first introduced the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

in a 1787 memoir on the motion of Saturn's rings. This was republished later in [21], Laplace's five volume work on celestial mechanics, in which he showed that many of the potential functions used in mathematical physics will always satisfy this equation. Therefore, being able to find explicit solutions to this equation and to understand the properties shared by all solutions would lead to advances in virtually all of the physical sciences. This equation would also be the starting point for other areas of both pure and applied mathematics such as potential theory and harmonic analysis.

It was not until later that a systematic study of solutions to Laplace's equation began. A stereotypical problem which was investigated was the Dirichlet problem. For a region D with boundary ∂D, it was asked whether or not, for a given function f, we could find a function u which satisfied:

$$\begin{aligned} \triangle u &= 0 \text{ in } D \\ u &= f \text{ on } \partial D. \end{aligned}$$

Some general results on the solvability of this system were found, such as existence and uniqueness of the solution u for a bounded domain D with smooth boundary ∂D and continuous boundary data f. For a discussion of the history and more modern development of these types of problems, see [2], specifically section 1.3 and chapter 6.

Perhaps the first and most detailed account of solutions to Laplace's equation was the 1879 work [20], which is of particular interest to us. Here, the preliminary section on kinematics deals with the spherical harmonics, given in spherical coordinates by:

$$r^n P_n^m(\cos\phi)\cos n\theta \quad \text{and} \quad r^n P_n^m(\cos\phi)\sin n\theta.$$

Using the surprising fact that these are in fact polynomials in the usual Cartesian coordinates x, y, and z, the tools are developed to show that if S denotes the unit ball in three dimensions, then the boundary value problem

$$\begin{aligned} \triangle u &= 0 \text{ in } S \\ u &= f \text{ on } \partial S \end{aligned}$$

will always possess a polynomial solution for any polynomial data f, where the solution u will have the same degree as the data f. Although this result is not stated explicitly in [20], it was certainly well within the reach of the authors.

1.2 Extensions of Boundary Value Problems

The previous boundary value problem may also be written in this form:

$$\begin{aligned} \triangle u &= 0 \text{ for } x^2 + y^2 + z^2 < 1 \\ u &= f \text{ on } x^2 + y^2 + z^2 - 1 = 0. \end{aligned}$$

However, since we were interested in the existence of polynomial solutions, the restriction on the domain was unnecessary. Therefore, the problem may be rephrased as:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^3 \\ u &= f \text{ on } x^2+y^2+z^2-1=0. \end{aligned}$$

This means that we are not dealing with a strict boundary value problem anymore. We are rather taking a function defined on a certain set and asking whether or not we can find a harmonic extension of this function to the entire space. With this in mind, it is then possible to choose different ways to prescribe our data. It is convenient to still refer to the set of points where we are prescribing data as the “boundary surface”, although this will not always technically be a surface or the boundary of a connected region. This is not a cause for concern, however, since we will be interested in solutions defined on the whole space. For example, the problem we shall investigate in chapter 5 is:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= f \text{ on } y \pm mx = 0. \end{aligned}$$

Here, we shall still refer to the pair of lines $y \pm mx = 0$ as a boundary surface, although they are, strictly speaking, just curves which do not divide the plane into an “inside” and an “outside”. In general, our notion of a boundary surface will be the zero set of some multivariate polynomial.

1.3 Polynomial Problem on Quadric Surfaces

In 1954, Brelot and Choquet were able to generalize the result for the polynomial problem on the unit sphere to an arbitrary ellipsoid in n dimensions by proving the remarkable fact that any polynomial divisor of a harmonic polynomial cannot have a constant sign. This result can be found as lemma 3 in [6], and its proof only makes use of the orthogonality of spherical harmonics over the unit ball, using the standard L^2 inner product.

The next step in the evolution of this problem was the beginning of the classification of surfaces. We can write the boundary surface for Dirichlet's problem in the form $\psi = 0$, for some function ψ. For example, the sphere would be represented by $\psi = x_1^2 + \cdots + x_n^2 - 1$. A surface represented by ψ would then be labelled as either "good" or "bad" according to whether the problem:

$$\begin{aligned} \triangle u &= 0 \text{ in some domain} \\ u &= f \text{ on } \psi = 0 \end{aligned}$$

would always possess a polynomial solution given any polynomial data f. Specifically, a "good" surface would always possess such a polynomial solution, whereas a "bad" surface would not.

This classification began in [9], where Siegel and Chamberland classified the quadric curves in the plane. Perhaps the most interesting results were the cases of hyperbolas and intersecting lines, given by

$$\psi = \frac{x^2}{a^2} - \frac{y^2}{b^2} \pm 1 \text{ and } \psi = \frac{x^2}{a^2} - \frac{y^2}{b^2},$$

respectively, for some choice of constants a and b. The determining factor as to whether these surfaces were good or bad turned out to be the angle between the asymptotes of the hyperbola and the angle between the intersecting lines. If these angles were rational multiples of π, then the corresponding surface would be bad. Otherwise, the surface would be good. In chapter 5, we will consider an extension to this problem which requires even more number theory to settle, and which represents the most interesting result from our work.

This classification of surfaces was continued in three dimensions in [16], where it was discovered that it was sometimes necessary for the polynomial solution u to have a larger degree than the polynomial data f. More examples of this phenomenon were found in [17], where the classification of quadric surfaces in three dimensions was nearly completed. The obstruction to completing this work stemmed from the intricate arguments involving the linear algebra of polynomial spaces, where determining whether a surface was good or bad reduced to a question about the properties of roots of orthogonal polynomials. In particular, knowledge about whether two Legendre polynomials can share any non-zero roots would be helpful, and this will be briefly explored in chapter 7.

1.4 Extensions of the Polynomial Problem

A natural step to take is to ask the following question. Suppose we have a boundary surface given by $\psi = 0$ in $\mathbb{R}^n$, and suppose that the problem

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^n \\ u &= f \text{ on } \psi = 0 \end{aligned}$$

always has a polynomial solution u for any polynomial data f. Is it then true that this problem will always possess an entire solution u given any entire data f? Initially, one might simply hope to be able to use the linearity of the problem to break the entire data f into homogeneous polynomials, solve each of these problems, and then add the corresponding solutions back together. In practice, however, it turns out to be very difficult to show that these formal solutions actually converge. The only boundary where this approach has been successful is the ellipsoid in n dimensions, which was proved by Khavinson and Shapiro in [13]. The proof depended in an essential way on the fact that the ellipsoid plus its interior is a compact set, and so for other boundaries in which we may be interested, we will need to develop some new tools.

Another extension to the polynomial problem is [5], where the authors looked at prescribing rational data on curves in the plane to see when the solution would also be a rational function. They proved that the disk was the only bounded domain with this property.

There are also other algebraic approaches to the problem that may be taken. For example, E. Fischer introduced an inner product on the space of polynomials under which the operations of partial differentiation and multiplication by the indeterminates are adjoint. An intuitive development of this inner product can be found in [18], and it can be seen in use in the latter part of [13]. These methods lead to a quick proof that the Dirichlet problem on the ellipsoid will always have a polynomial solution given polynomial data, but they have the added advantage of being able to generalize to differential operators other than the Laplacian.

1.5 Outline of Main Results

Our work will focus on the first extension to the polynomial problem mentioned above. Suppose for a surface given by $\psi = 0$, the problem

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^n \\ u &= f \text{ on } \psi = 0 \end{aligned}$$

always has a polynomial solution u given any polynomial data f. Then, we wish to determine whether this problem will always possess an entire solution u given any entire data f. The boundaries which we shall deal with will all be unbounded.

In chapter 3, we will start with a half space problem in n dimensions. We will prescribe data on the boundary $x_n = 0$ and see if this can be extended to an entire harmonic function in all of $\mathbb{R}^n$. Our strategy will be a direct approach, and we will construct a formal series solution and try to demonstrate convergence. This problem was investigated in [12], where the Neumann problem was settled with some advanced machinery. We have taken a more elementary approach and have solved the Dirichlet, Neumann, and Cauchy problems in theorem 3.4.4, theorem 3.4.5, and theorem 3.4.6, respectively. While the solution to the Neumann problem is not new, and the solutions to the Dirichlet and Cauchy problems were well within the reach of these techniques as well, we have fleshed out the elementary arguments alluded to in [12].

In chapter 4, we will provide a nice contrast to the results from chapter 3. It will again be a half space problem, but this time in only two dimensions. We will also be considering the heat equation rather than Laplace's equation, as this will lead to complications not found in the work in chapter 3. The new results from this chapter are theorem 4.4.1 and theorem 4.4.2, which describe the polynomial problem and the corresponding problem for entire functions.

Chapter 5 contains the most interesting results. We will look at the case of two intersecting lines in the plane, where the angle between them is an irrational multiple of π. The problem is known to always have a polynomial solution given any polynomial data, but trying to extend this search to entire functions reveals a surprising dependence on the number theoretic properties of the angle between the lines. In particular, the positive results in proposition 5.5.2 and theorem 5.6.2 are new.

In chapter 6, we will investigate an infinite strip in two dimensions, but will unfortunately be unable to settle the problem for entire functions. The work is included because it turned up many interesting questions and techniques.

Finally, chapter 7 will provide a summary of the open problems which this work has created. While we were unable to solve these problems, there were some promising ideas which may still prove useful.

Before getting to these results, we will first need to review some basic facts about entire functions, and about entire harmonic functions in particular. This will be the focus of the next chapter.

Chapter 2

Entire Functions

We begin our search for entire solutions to Laplace's equation by gathering some information regarding entire functions themselves. Starting with functions of a single variable, we will move to functions of many variables, and record the results which we shall need in later chapters. We will then gather some facts about entire harmonic functions in particular.

2.1 Basic Properties

We will begin with functions of a single variable. For simplicity, all power series will be given about the origin. Also, we will restrict ourselves to working with real variables unless otherwise noted, although the definitions and properties of entire functions which we shall need are the same whether we are working with real or complex variables.

If we have a function f which is analytic at the origin, we can write it as a power series with a positive radius of convergence:

$$f(x) = \sum_{n=0}^{\infty} f_n x^n,$$

where the f_n are constants. The function f will be called *entire* if, in fact, it has an infinite radius of convergence. This can be turned into a statement about the coefficients f_n by using Hadamard's formula, for example, as found in [15]:

$$\text{radius of convergence } = \left(\limsup_{n\to\infty} |f_n|^{1/n}\right)^{-1}.$$

Therefore, f will be entire if and only if

$$\limsup_{n\to\infty} |f_n|^{1/n} = 0.$$

This can be turned into a more convenient form for use in inequalities by the following basic lemma.

Lemma 2.1.1 *Let $\{a_n\}$ be a sequence of real numbers for $n \geq 0$. Then*

$$\limsup_{n\to\infty} |a_n|^{1/n} = 0$$

if and only if for every $R > 0$, we can find a constant $C_R > 0$ such that

$$|a_n| \leq \frac{C_R}{R^n}$$

for each $n \geq 0$.

Proof: First, suppose that

$$\limsup_{n\to\infty} |a_n|^{1/n} = 0.$$

Pick any $R > 0$. Then, we can find an $N > 0$ such that for all $n > N$, we have:

$$|a_n|^{1/n} \leq \frac{1}{R} \implies |a_n| \leq \frac{1}{R^n}.$$

Next, we define

$$C_R = \max\{R^0|a_0|, R^1|a_1|, \ldots, R^N|a_N|, 1\}.$$

Then, with this choice of C_R, we see that

$$|a_n| \leq \frac{C_R}{R^n}$$

for all $n \geq 0$.

Next, suppose that for every $R > 0$, we can find a constant $C_R > 0$ such that

$$|a_n| \leq \frac{C_R}{R^n}$$

for all $n \geq 0$. Pick any $R > 0$. Then, by taking the n^{th} root of the above inequality, we get

$$|a_n|^{1/n} \leq \frac{(C_R)^{1/n}}{R}.$$

Therefore:

$$\limsup_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} \frac{(C_R)^{1/n}}{R} = \frac{1}{R}.$$

However, since this will be true for any $R > 0$, we must have that

$$\limsup_{n\to\infty} |a_n|^{1/n} = 0.$$

□

Suppose that we have two entire functions, f and g. Writing each as a power series, it is straightforward to show that the sum $f + g$ and product fg will also be entire functions. Also, in the proof found in [15] that the composition of analytic functions is analytic, the estimates derived are also sufficient to show that the composition of entire functions is entire. Finally, since convergent power series can be differentiated term by term, we have the following formula for the k^{th} derivative of an entire function. It can be proved by repeatedly differentiating the series and re-indexing. If $f(x)$ is given by:

$$f(x) = \sum_{n=0}^{\infty} f_n x^n,$$

then the k^{th} derivative of f is given by:

$$f^{(k)}(x) = \sum_{n=0}^{\infty} (n+1)_k f_{n+k} x^n, \tag{2.1}$$

where $(n+1)_k$ denotes the Pochhammer symbol, or rising factorial:

$$(n+1)_k = (n+1)(n+2)\cdots(n+k).$$

If $k = 0$, the empty product is assumed to have the value of 1.

Next, in order to look at entire functions of many variables, we will need to review the very useful multi-index notation. A multi-index α in n dimensions is an ordered n-tuple of non-negative integers:

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n).$$

Addition and subtraction of multi-indices are performed componentwise. Scalar multiplication by non-negative integers is defined in the natural way:

$$c\alpha = (c\alpha_1, c\alpha_2, \dots, c\alpha_n).$$

Next, if we let

$$x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n,$$

then we also have the following notational conveniences:

$$\begin{aligned} |\alpha| &= \alpha_1 + \cdots + \alpha_n, \\ \alpha! &= (\alpha_1!)(\alpha_2!)\cdots(\alpha_n!), \text{ and} \\ x^\alpha &= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}. \end{aligned}$$

Also, for two multi-indices α and β, we will say that $\alpha \geq \beta$ if and only if $\alpha_i \geq \beta_i$ for each $1 \leq i \leq n$. Finally, if f is a function of n variables, then we can denote partial derivatives as follows:

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

Since we will always be dealing with entire functions, we are guaranteed that the order of the derivatives is unimportant.

Suppose that f is an analytic function of n variables. Then, using this multi-index notation, we can very compactly write down a power series about the origin:

$$f(x) = \sum_\alpha f_\alpha x^\alpha,$$

where we now have a sequence of coefficients indexed by multi-indices. As with functions of a single variable, this function will be called *entire* if this series converges everywhere. We also have similar bounds for the coefficients of entire functions of n variables. The following proposition will establish these bounds by making use of the fact that $\binom{n}{k} \leq 2^n$ for any positive integer n and any integer k such that $0 \leq k \leq n$.

Proposition 2.1.2 *Suppose we have a function f given by the following series:*

$$f(x) = \sum_{\alpha} f_\alpha x^\alpha.$$

Then f is entire if and only if for each $R > 0$, we can find a constant $C_R > 0$ such that:

$$|f_\alpha| \leq \frac{C_R}{R^{|\alpha|}}$$

for each multi-index α.

Proof: First, suppose that f is entire. In particular, for each $R > 0$, this means that the above series will converge at the point $x_R = (R, R, \ldots, R)$. Since the terms of a convergent series must be bounded, let us denote this bound by C_R. For each multi-index α, we then have:

$$|f_\alpha x_R^\alpha| = |f_\alpha R^{\alpha_1} R^{\alpha_2} \cdots R^{\alpha_n}| \leq C_R \implies |f_\alpha| \leq \frac{C_R}{R^{|\alpha|}}.$$

Next, let us suppose that for each $R > 0$, we can find a constant $C_R > 0$ such that

$$|f_\alpha| \leq \frac{C_R}{R^{|\alpha|}}$$

for each multi-index α. In particular, we can certainly find a constant $C_{3R} > 0$ such that

$$|f_\alpha| \leq \frac{C_{3R}}{(3R)^{|\alpha|}}$$

for each multi-index α. For each $x \in \mathbb{R}^n$ such that $|x| \leq R$, we then have the following estimate:

$$\begin{aligned}\sum_\alpha |f_\alpha||x^\alpha| &\leq \sum_\alpha \frac{C_{3R}}{(3R)^{|\alpha|}} R^{|\alpha|} \\ &= C_{3R} \sum_\alpha \frac{1}{3^{|\alpha|}}.\end{aligned}$$

Finally, by lemma 3.7.3 in [7], there will be $\binom{n+k-1}{k}$ multi-indices α over $\mathbb{R}^n$ such that $|\alpha| = k$, so we can finish our estimation as follows:

$$\begin{aligned}\sum_\alpha |f_\alpha||x^\alpha| &\leq C_{3R} \sum_\alpha \frac{1}{3^{|\alpha|}} \\ &= C_{3R} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{1}{3^k} \\ &= C_{3R} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{1}{3^k} \\ &\leq C_{3R} 2^{n-1} \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k,\end{aligned}$$

which is finite. Therefore, since this was true for all $|x| \leq R$ and R can be chosen as large as we wish, the series:

$$f(x) = \sum_\alpha f_\alpha x^\alpha$$

is absolutely convergent for all $x \in \mathbb{R}^n$. Since absolute convergence implies convergence, the series for f must converge for all $x \in \mathbb{R}^n$, and so f is an entire function. □

Finally, as with entire functions of a single variable, entire functions of n variables may be differentiated as many times as we please to yield other entire functions. If we let

$$f(x) = \sum_\alpha f_\alpha x^\alpha,$$

then by repeated differentiation and re-indexing, we get the following formula:

$$D^\beta f = \sum_\alpha (\alpha+1)_\beta f_{\alpha+\beta} x^\alpha, \tag{2.2}$$

where we have used a multi-index shorthand for a product of Pochhammer symbols:

$$(\alpha+1)_\beta = (\alpha_1+1)_{\beta_1}(\alpha_2+1)_{\beta_2}\cdots(\alpha_n+1)_{\beta_n}.$$

2.2 Entire Harmonic Functions

We will now focus our attention on entire harmonic functions, with a specific interest in entire harmonic functions in the plane, or $\mathbb{R}^2$.

One standard way of generating harmonic functions in the plane is to take either the real or imaginary part of a complex analytic function. In a sense, this is the only way to generate them since a harmonic function in a simply connected planar domain will be the real part of a complex analytic function in that domain. For a proof, see theorem 1.1.3 in [2]. The following proposition shows us that this behaves as expected with respect to entire functions.

Proposition 2.2.1 *Let $f(z)$ be an entire function of the complex variable z. Then, if we let $z = x + iy$ and write the real and imaginary parts of f as*

$$f(z) = v(x, y) + iw(x, y),$$

then both v and w are entire harmonic functions in the plane.

Proof: Since f is entire, we know that we can write:

$$f(z) = \sum_{n=0}^{\infty} f_n z^n,$$

where for each $R > 0$, we can find a constant $C_{2R} > 0$ such that

$$|f_n| \le \frac{C_{2R}}{(2R)^n}$$

for all $n \geq 0$. To get the real and imaginary parts, we substitute $z = x + iy$ and expand using the Binomial Theorem:

$$\begin{aligned} f(x+iy) &= \sum_{n=0}^{\infty} f_n (x+iy)^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_n \binom{n}{k} x^{n-k} (iy)^k. \end{aligned}$$

Therefore, the coefficients of v and w will be binomial coefficients multiplied by the coefficients of f, and we can bound the coefficient of $x^{n-k} y^k$ by:

$$\left| \binom{n}{k} f_n \right| \leq 2^n |f_n| \leq 2^n \frac{C_{2R}}{(2R)^n} = \frac{C_{2R}}{R^n},$$

which is exactly the power of R we need since $(n-k) + k = n$. Since this is true for all $R > 0$, this ensures that both v and w will be entire. That they are both harmonic follows from the Cauchy-Riemann equations. $\square$

Next, it is well known that if a harmonic function u in the plane is analytic at the origin, then it has a local expansion in polar coordinates there:

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta \right). \tag{2.3}$$

Proofs of this statement can be found in books like [2] and [3]. However, we would like to know more detailed information about the coefficients a_n and b_n if we also know that u is an entire harmonic function. With a little work, we will in fact be able to show that these coefficient sequences satisfy similar bounds to the coefficient sequences considered in the previous section.

First, we will suppose that u is entire and show that both a_n and b_n can be bounded in a very familiar way. We begin with a small lemma.

Lemma 2.2.2 *Let α_1 and α_2 be non-negative integers with $\alpha_1 + \alpha_2 = n$. Then:*

$$(\cos\theta)^{\alpha_1} (\sin\theta)^{\alpha_2} = \sum_{i=0}^{n} a_i \cos i\theta + \sum_{i=1}^{n} b_i \sin i\theta,$$

for some real numbers a_i and b_i.

Proof: We will proceed by induction on n. If $n = 0$, then $\alpha_1 = \alpha_2 = 0$ and the statement is trivial. So, suppose that the statement is true for all $\alpha_1 + \alpha_2 = n$. Then, if $\alpha_1 + \alpha_2 = n + 1$, at least one of α_1 or α_2 will be greater than 0. First, suppose that it is α_1 which is greater than 0:

$$(\cos\theta)^{\alpha_1}(\sin\theta)^{\alpha_2} = (\cos\theta)^{\alpha_1 - 1}(\sin\theta)^{\alpha_2}\cos\theta = \left(\sum_{i=0}^{n} a_i \cos i\theta + \sum_{i=1}^{n} b_i \sin i\theta\right)\cos\theta,$$

by the induction hypothesis. Then, we can continue:

$$\begin{aligned}
(\cos\theta)^{\alpha_1}(\sin\theta)^{\alpha_2} &= \sum_{i=0}^{n} a_i \cos i\theta\cos\theta + \sum_{i=1}^{n} b_i \sin i\theta\cos\theta \\
&= \sum_{i=1}^{n} \frac{a_i}{2}\left(\cos(i+1)\theta + \cos(i-1)\theta\right) \\
&\quad + \sum_{i=1}^{n} \frac{b_i}{2}\left(\sin(i+1)\theta + \sin(i-1)\theta\right).
\end{aligned}$$

The result now follows from re-indexing the above sums and taking care of the negative indices. The argument is virtually identical if we instead assume that it is α_2 which is greater than 0, and so we are done. □

Using this lemma, we can show that the coefficients of u in equation (2.3) satisfy normal looking bounds for entire functions.

Proposition 2.2.3 *Let u be an entire harmonic function in the plane. Then, we can express u in polar coordinates as:*

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n\left(a_n \cos n\theta + b_n \sin n\theta\right),$$

where for every $R > 0$, we can find a constant $C_R > 0$ such that:

$$|a_n| \leq \frac{C_R}{R^n} \text{ and } |b_n| \leq \frac{C_R}{R^n},$$

for each $n \geq 0$, and where $b_0 = 0$ by convention.

Proof: Since u is entire, we know we can write

$$u = \sum_{\alpha} c_\alpha x^{\alpha_1} y^{\alpha_2},$$

where the sum is taken over all multi-indices $\alpha = (\alpha_1, \alpha_2)$, and where we know that for any $R > 0$, we can find a $C_R > 0$ such that

$$|c_\alpha| \leq \frac{C_R}{R^{|\alpha|}}$$

for each multi-index α. Also, this sum will be absolutely and uniformly convergent on compact sets. By letting $x = r\cos\theta$ and $y = r\sin\theta$, we have:

$$u = \sum_{\alpha} c_\alpha (r\cos\theta)^{\alpha_1}(r\sin\theta)^{\alpha_2} = \sum_{\alpha} c_\alpha r^{|\alpha|}(\cos\theta)^{\alpha_1}(\sin\theta)^{\alpha_2}.$$

Since u converges absolutely and uniformly on all compact sets, this will certainly be true on the unit circle. Therefore, the sum:

$$w(\theta) = u(1,\theta) = \sum_{\alpha} c_\alpha (\cos\theta)^{\alpha_1}(\sin\theta)^{\alpha_2}$$

will be absolutely and uniformly convergent for all $\theta \in [-\pi, \pi]$, and will define a continuous, 2π-periodic function there.

Now, since u is an entire harmonic function, it will certainly be analytic in a neighbourhood of the origin. Therefore, we know that it can be written in the form:

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right).$$

Evaluating this series along the unit circle and comparing it with what we had before gives:

$$\sum_{\alpha} c_\alpha (\cos\theta)^{\alpha_1}(\sin\theta)^{\alpha_2} = w(\theta) = u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right).$$

Since w is smooth and 2π-periodic, it will have a pointwise convergent Fourier series. Therefore, we can get our desired estimates on the coefficients of u by estimating the

Fourier coefficients of w. First, we know:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} w(\theta)\cos n\theta d\theta \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{\alpha} c_\alpha(\cos\theta)^{\alpha_1}(\sin\theta)^{\alpha_2}\cos n\theta d\theta. \end{aligned}$$

However, since the sum for w is uniformly convergent, we may interchange the order of summation and integration to write:

$$a_n = \frac{1}{\pi}\sum_{\alpha} c_\alpha \int_{-\pi}^{\pi}(\cos\theta)^{\alpha_1}(\sin\theta)^{\alpha_2}\cos n\theta d\theta.$$

However, by lemma 2.2.2, we know that these integrals will be 0 by the orthogonality of the sine and cosine functions unless we have $|\alpha| \geq n$. This leads to:

$$a_n = \frac{1}{\pi}\sum_{|\alpha|\geq n} c_\alpha \int_{-\pi}^{\pi}(\cos\theta)^{\alpha_1}(\sin\theta)^{\alpha_2}\cos n\theta d\theta.$$

Now, we can use some simple bounds to proceed:

$$\begin{aligned} |a_n| &\leq \frac{1}{\pi}\sum_{|\alpha|\geq n} |c_\alpha| \int_{-\pi}^{\pi}\left|(\cos\theta)^{\alpha_1}(\sin\theta)^{\alpha_2}\cos n\theta\right| d\theta \\ &\leq \frac{1}{\pi}\sum_{|\alpha|\geq n} |c_\alpha| \int_{-\pi}^{\pi} d\theta \\ &\leq 2\sum_{|\alpha|\geq n} |c_\alpha|. \end{aligned}$$

Next, since u is an entire function, for any $R > 1$, we can find a constant $C_{2R} > 0$ such that:

$$|c_\alpha| \leq \frac{C_{2R}}{(2R)^{|\alpha|}},$$

for all multi-indices α. Using this, we can keep working away at our bound for a_n:

$$\begin{aligned} |a_n| &\leq 2\sum_{|\alpha|\geq n} |c_\alpha| \\ &= 2\sum_{k=n}^{\infty}\sum_{|\alpha|=k} |c_\alpha| \\ &\leq 2C_{2R}\sum_{k=n}^{\infty}\sum_{|\alpha|=k} \frac{1}{(2R)^k} \\ &= 2C_{2R}\sum_{k=n}^{\infty} \frac{k+1}{(2R)^k} \\ &\leq 4C_{2R}\sum_{k=n}^{\infty} \left(\frac{1}{R}\right)^k \\ &= 4C_{2R}\left(\frac{1}{R}\right)^n \frac{R}{R-1}, \end{aligned}$$

where we know that the geometric series converges since we chose $R > 1$. Therefore, for any $R > 1$, we can find a constant $C'_R > 0$ depending on R but independent of n such that

$$|a_n| \leq \frac{C'_R}{R^n}$$

for all $n \geq 0$. This can be trivially extended to be true for any $R > 0$, and we can perform a similar calculation for b_n. Therefore, if u is harmonic and entire, we can write

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right),$$

where for any $R > 0$, we can find a constant $C_R > 0$ such that:

$$|a_n| \leq \frac{C_R}{R^n} \text{ and } |b_n| \leq \frac{C_R}{R^n}$$

for all $n \geq 0$, with the convention that $b_0 = 0$. $\square$

Next, we will show that if the coefficient bounds are satisfied that we in fact have an entire harmonic function.

Proposition 2.2.4 *Let u be harmonic, and suppose that we can write:*

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right),$$

where for any $R > 0$, we can find a constant $C_R > 0$ such that

$$|a_n| \leq \frac{C_R}{R^n} \text{ and } |b_n| \leq \frac{C_R}{R^n}$$

for all $n \geq 0$, with the convention that $b_0 = 0$. Then, u is an entire function.

Proof: Our strategy will be to make use of the facts that $r^n \cos n\theta = \Re\left((x+iy)^n\right)$ and $r^n \sin n\theta = \Im\left((x+iy)^n\right)$. Because of the bounds we have on a_n and b_n, the sum for u will be absolutely and uniformly convergent on compact sets. So, we can rearrange terms to write:

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \Re\left((x+iy)^n\right) + \sum_{n=1}^{\infty} b_n \Im\left((x+iy)^n\right).$$

Now, we can use the Binomial Theorem to see:

$$(x+iy)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} (iy)^k = \sum_{k=0}^{n} i^k \binom{n}{k} x^{n-k} y^k.$$

So, the coefficients in the polynomials $\Re\left((x+iy)^n\right)$ and $\Im\left((x+iy)^n\right)$ are all binomial coefficients, and so we can bound them all in absolute value by 2^n. Note also that these polynomials will have no monomials in common. So, if we look at the expansion

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \Re\left((x+iy)^n\right) + \sum_{n=1}^{\infty} b_n \Im\left((x+iy)^n\right),$$

we see that the coefficient of each monomial of degree n can be bounded in absolute value by either $2^n|a_n|$ or $2^n|b_n|$. Now, since for each $R > 0$ we can find a constant $C_R > 0$ such that

$$|a_n| \leq \frac{C_R}{R^n} \text{ and } |b_n| \leq \frac{C_R}{R^n},$$

each of $2^n|a_n|$ and $2^n|b_n|$ can also be bounded in a similar manner by simply choosing R to be twice as large as the desired bound. Therefore, we have:

$$u = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha x^{\alpha_1} y^{\alpha_2} = \sum_{\alpha} c_\alpha x^{\alpha_1} y^{\alpha_2},$$

where for each $R > 0$, we can find a constant $C_R > 0$ such that

$$|c_\alpha| \leq \frac{C_R}{R^{|\alpha|}}$$

for each multi-index α. Therefore, u is an entire function. □

Putting together the results of the last two propositions, we have the following result.

Theorem 2.2.5 *Let u be a harmonic function in the plane. Then it will be entire if and only if we can write:*

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right),$$

where for each $R > 0$, we can find a constant $C_R > 0$ such that

$$|a_n| \leq \frac{C_R}{R^n} \text{ and } |b_n| \leq \frac{C_R}{R^n}$$

for all $n \geq 0$, with the convention that $b_0 = 0$.

Proof: This follows directly from propositions 2.2.3 and 2.2.4. □

Finally, we state the following result, which is a useful companion to the previous theorem.

Theorem 2.2.6 *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers for $n \geq 0$, with the assumption that $b_0 = 0$. If for any $R > 0$, we can find a constant $C_R > 0$ such that:*

$$|a_n| \leq \frac{C_R}{R^n} \text{ and } |b_n| \leq \frac{C_R}{R^n}$$

for all $n \geq 0$, then the function u given by:

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right)$$

is an entire harmonic function.

Proof: First, we will show that the sum is uniformly convergent on all compact sets. Pick any $R > 0$. Then, we can find a constant $C_{2R} > 0$ such that

$$|a_n| \leq \frac{C_{2R}}{(2R)^n} \text{ and } |b_n| \leq \frac{C_{2R}}{(2R)^n}$$

for all $n \geq 0$. Therefore, for each n, we have the following bound in the disk of radius R:

$$|r^n \left(a_n \cos n\theta + b_n \sin n\theta\right)| \leq 2R^n \frac{C_{2R}}{(2R)^n} = \frac{C_{2R}}{2^{n-1}}.$$

Since we know

$$\sum_{n=1}^{\infty} \frac{C_{2R}}{2^{n-1}} < \infty,$$

the defining sum for u converges uniformly in every disk, and hence in every compact set.

Therefore, u will be the uniform limit of harmonic functions everywhere, and will therefore be a harmonic function in the whole plane by theorem 1.5.1 of [2]. Finally, by theorem 2.2.5 and the coefficient bounds we are given, we see that it will be an entire harmonic function in the plane. $\square$

Chapter 3

The Half Space

The first problem on which we will focus our attention is modelled after the classical Dirichlet problem in the half space. The goal there is to find a harmonic function in the upper half space ($x_n > 0$) which is equal to a given data function on the boundary of the half space ($x_n = 0$). However, since we are interested in the pursuit of entire solutions, restricting the domain of our solution to the upper half space is no longer necessary. This means that the concrete problem of this chapter is the following: given an entire function $f = f(x_1, \ldots, x_{n-1})$, can we always find an entire function u which satisfies:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^n \\ u &= f \text{ when } x_n = 0? \end{aligned}$$

3.1 Review of the Polynomial Problem

Before turning to the task at hand, we will recall the corresponding problem for polynomial data: given a polynomial $f = f(x_1, \ldots, x_{n-1})$, can we always find a polynomial

u which satisfies:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^n \\ u &= f \text{ when } x_n = 0? \end{aligned}$$

This problem has an affirmative answer, and there are many ways to show it. One is as follows. First, consider the problem in the plane, or $\mathbb{R}^2$. Then, for any polynomial $f(x)$, we can use a simple formula involving complex variables to construct our solution, namely, let $u = \Re(f(x+iy))$. Clearly, $u(x,0) = f(x)$. Also, since f is a polynomial, $f(x+iy)$ will be a holomorphic function of the complex variable $z = x+iy$. Therefore, it will have a harmonic real part which can be seen to be a polynomial by expanding the powers of $x+iy$ using the Binomial Theorem. Finally, we can extend this affirmative answer in the plane to all higher dimensions using a theorem from [17], which states that if a boundary surface always admits a polynomial solution for any polynomial data in $\mathbb{R}^n$, then this will be the case in all higher dimensions as well.

3.2 Motivation for the Main Formula

Knowing that the polynomial half space problem will always possess a polynomial solution given any polynomial data leads us to suspect that the corresponding problem for entire data will always possess an entire solution. Unlike the polynomial problem, however, we will now need to deal with issues of convergence. To motivate our technical work in the next section, we will for the moment throw away these considerations to get a feel for the kinds of estimates we may need.

Our main strategy will be to take a harmonic function in n variables, expand it as a power series in the last variable, and then try to make use of any structure that may result. We begin by writing our harmonic function u as:

$$u(x_1, \ldots, x_n) = \sum_{k=0}^{\infty} f_k(x_1, \ldots, x_{n-1}) x_n^k.$$

Then, we may formally take the Laplacian of u, which will vanish since u was assumed to be harmonic. Dropping the notational dependence on the x_i for clarity, we have:

$$\begin{aligned} 0 &= \triangle u = \triangle \left(\sum_{k=0}^{\infty} f_k x_n^k \right) \\ &= \sum_{k=0}^{\infty} \triangle f_k \cdot x_n^k + \sum_{k=2}^{\infty} f_k \cdot k(k-1) x_n^{k-2} \\ &= \sum_{k=0}^{\infty} \triangle f_k \cdot x_n^k + \sum_{k=0}^{\infty} f_{k+2} \cdot (k+2)(k+1) x_n^k \\ &= \sum_{k=0}^{\infty} \left(\triangle f_k + (k+2)(k+1) f_{k+2} \right) x_n^k. \end{aligned}$$

Since each coefficient must identically vanish, we get the following recurrence relation between the coefficients of u:

$$f_{k+2} = -\frac{1}{(k+2)(k+1)} \triangle f_k.$$

Therefore, if we make a choice for f_0 and f_1, the rest of the coefficients will be completely determined by:

$$\begin{aligned} f_{2k} &= \frac{(-1)^k \triangle^k f_0}{(2k)!} \quad \text{and} \\ f_{2k+1} &= \frac{(-1)^k \triangle^k f_1}{(2k+1)!}. \end{aligned}$$

Substituting this back into our original series for u and grouping the even and odd terms together, we obtain the following form for our harmonic function u:

$$u = \sum_{k=0}^{\infty} \frac{(-1)^k \triangle^k f_0}{(2k)!} x_n^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k \triangle^k f_1}{(2k+1)!} x_n^{2k+1}. \tag{3.1}$$

Note also that on setting $x_n = 0$ we recover f_0, which still remains an arbitrary function of $x_1, \ldots, x_{n-1}$. So, if we wish to use this formula to show that the half space problem will always have an entire solution given any entire data, we need to build up some estimates on the repeated application of the Laplacian operator on entire functions. This formula can also be found in section 9.1 of [12], where it is mentioned that problems of this nature can be solved directly by estimating derivatives, which is the strategy we shall adopt.

3.3 Bounds on Repeated Laplacians

Throughout this section, we will be working over $\mathbb{R}^n$. We first need some inequalities dealing with multi-indices.

Lemma 3.3.1 *Let β be a multi-index. Then:*

$$\frac{(2\beta)!}{\beta!} \leq 2^{2|\beta|}\beta!.$$

Proof: By expanding out the factorials, we have:

$$\begin{aligned}
\frac{(2\beta)!}{\beta!} &= \frac{(2\beta_1)!}{\beta_1!}\cdots\frac{(2\beta_n)!}{\beta_n!} \\
&= \frac{(2\beta_1)!}{\beta_1!\beta_1!}\cdots\frac{(2\beta_n)!}{\beta_n!\beta_n!}\beta_1!\cdots\beta_n! \\
&= \binom{2\beta_1}{\beta_1}\cdots\binom{2\beta_n}{\beta_n}\beta! \\
&\leq 2^{2\beta_1}\cdots 2^{2\beta_n}\beta! = 2^{2|\beta|}\beta!
\end{aligned}$$

□

Lemma 3.3.2 *Let α and β be multi-indices. Then $(\alpha+1)_{2\beta} \leq 2^{|\alpha+2\beta|}(2\beta)!$, where:*

$$(\alpha+1)_{2\beta} = (\alpha_1+1)_{2\beta_1}\cdots(\alpha_n+1)_{2\beta_n}.$$

Proof: Here, we convert the Pochhammer symbols into the corresponding factorials, and then bound using binomial coefficients as before:

$$
\begin{aligned}
(\alpha+1)_{2\beta} &= (\alpha_1+1)_{2\beta_1}\cdots(\alpha_n+1)_{2\beta_n} \\
&= \frac{(\alpha_1+2\beta_1)!}{\alpha_1!}\cdots\frac{(\alpha_n+2\beta_n)!}{\alpha_n!} \\
&= \frac{(\alpha_1+2\beta_1)!}{\alpha_1!(2\beta_1)!}\cdots\frac{(\alpha_n+2\beta_n)!}{\alpha_n!(2\beta_n)!}(2\beta_1)!\cdots(2\beta_n)! \\
&= \binom{\alpha_1+2\beta_1}{\alpha_1}\cdots\binom{\alpha_n+2\beta_n}{\alpha_n}(2\beta)! \\
&\leq 2^{\alpha_1+2\beta_1}\cdots 2^{\alpha_n+2\beta_n}(2\beta)! \\
&= 2^{|\alpha+2\beta|}(2\beta)!
\end{aligned}
$$

□

Lemma 3.3.3 *Let α and β be multi-indices with $|\beta| = k \geq 0$. Then:*

$$\frac{k!}{\beta!}(\alpha+1)_{2\beta} \leq (k!)^2 \cdot 2^{|\alpha|+4k}.$$

Proof: We can combine the results from lemmas 3.3.1 and 3.3.2:

$$
\begin{aligned}
\frac{k!}{\beta!}(\alpha+1)_{2\beta} &\leq \frac{k!}{\beta!}2^{|\alpha+2\beta|}(2\beta)! \\
&= k!\cdot 2^{|\alpha|+2k}\cdot\frac{(2\beta)!}{\beta!} \\
&\leq k!\cdot 2^{|\alpha|+2k}\cdot 2^{2k}\beta! \\
&\leq (k!)^2\cdot 2^{|\alpha|+4k},
\end{aligned}
$$

since $|\beta| = k \implies \beta! \leq k!$. □

Lemma 3.3.4 *Let α and β be multi-indices with $|\beta| = k \geq 0$. Suppose that $R > 1$ and that $|x_i| \leq R$ for each $i = 1,\ldots,n$. Also, let $\{f_\gamma\}$ be a sequence of real numbers indexed by multi-indices, and suppose for each $S > 0$ that we can find a constant $C_S > 0$ such that:*

$$|f_\gamma| \leq \frac{C_S}{S^{|\gamma|}}$$

for each multi-index γ. Then for each $S > 0$:

$$\left|\frac{k!}{\beta!}(\alpha+1)_{2\beta} f_{\alpha+2\beta} x^\alpha\right| \leq (k!)^2 \cdot C_S \cdot \left(\frac{4R}{S}\right)^{|\alpha|+2k}.$$

Proof: We will make use of lemma 3.3.3 and the given bounds on f_γ and x_i:

$$\begin{aligned}
\left|\frac{k!}{\beta!}(\alpha+1)_{2\beta} f_{\alpha+2\beta} x^\alpha\right| &= \frac{k!}{\beta!}(\alpha+1)_{2\beta} \left|f_{\alpha+2\beta}\right| \left|x^\alpha\right| \\
&\leq (k!)^2 \cdot 2^{|\alpha|+4k} \cdot \frac{C_S}{S^{|\alpha|+2k}} \cdot R^{|\alpha|} \\
&\leq (k!)^2 \cdot C_S \cdot \frac{4^{|\alpha|+2k} \cdot R^{|\alpha|+2k}}{S^{|\alpha|+2k}} \\
&= (k!)^2 \cdot C_S \cdot \left(\frac{4R}{S}\right)^{|\alpha|+2k}
\end{aligned}$$

□

Lemma 3.3.5 *Let $0 < r < \frac{1}{2}$. Then the sum:*

$$\sum_{|\beta|=k} \sum_{\alpha} r^{|\alpha+2\beta|}$$

converges, where the sum runs over all multi-indices α and β such that $|\beta| = k \geq 0$.

Proof: Note that for any $k \geq 0$, there are $\binom{n+k-1}{k}$ multi-indices β such that $|\beta| = k$ by lemma 3.7.3 of [7]. Then, for any $N > 0$, we can make the following estimate:

$$\begin{aligned}
\sum_{|\beta|=k} \sum_{|\alpha| \leq N} r^{|\alpha+2\beta|} &= \sum_{|\beta|=k} r^{2k} \cdot \sum_{|\alpha|\leq N} r^{|\alpha|} \\
&= \binom{n+k-1}{k} r^{2k} \sum_{s=0}^{N} \sum_{|\alpha|=s} r^s \\
&= \binom{n+k-1}{k} r^{2k} \sum_{s=0}^{N} \binom{n+s-1}{s} r^s \\
&\leq \binom{n+k-1}{k} r^{2k} \sum_{s=0}^{N} 2^{n-1}(2r)^s \\
&\leq \binom{n+k-1}{k} 2^{n-1} r^{2k} \sum_{s=0}^{\infty} (2r)^s \\
&= \binom{n+k-1}{k} 2^{n-1} r^{2k} \frac{1}{1-2r},
\end{aligned}$$

where the last sum converges since $0 < 2r < 1$. Since this bound is independent of N and we are summing positive terms, the sum

$$\sum_{|\beta|=k} \sum_{\alpha} r^{|\alpha+2\beta|}$$

must converge. $\square$

Now, we are in a position to prove our estimate regarding the repeated application of the Laplacian operator to an entire function in $\mathbb{R}^n$.

Proposition 3.3.6 *Let f be an entire function given by the series:*

$$f = \sum_{\alpha} f_\alpha x^\alpha.$$

Then, for any $S > 0$, we can find a constant $C_S > 0$ such that if we write:

$$\triangle^k f = \sum_{\alpha} f_{\alpha,k} x^\alpha,$$

we have the following estimate for all $k \geq 1$ and all multi-indices α:

$$|f_{\alpha,k}| \leq C_S 2^{n-1} (k!)^2 \left(\frac{8}{S}\right)^{|\alpha|+2k}.$$

Proof: Let $\partial_1, \ldots, \partial_n$ be the usual partial derivative operators on $\mathbb{R}^n$, and note that since f is entire, they will all commute with each other. Therefore, we may use the Multinomial Theorem to write:

$$\triangle^k = \left(\partial_1^2 + \cdots + \partial_n^2\right)^k = \sum_{|\beta|=k} \frac{k!}{\beta!} \partial_1^{2\beta_1} \cdots \partial_n^{2\beta_n} = \sum_{|\beta|=k} \frac{k!}{\beta!} D^{2\beta}.$$

Then, given the series for f, we can use equation (2.2) to write:

$$D^{2\beta} f = \sum_{\alpha} (\alpha+1)_{2\beta} f_{\alpha+2\beta} x^\alpha$$

for any multi-index β. Therefore, we have:

$$\triangle^k f = \sum_{|\beta|=k} \frac{k!}{\beta!} D^{2\beta} f = \sum_{|\beta|=k} \sum_{\alpha} \frac{k!}{\beta!} (\alpha+1)_{2\beta} f_{\alpha+2\beta} x^\alpha.$$

We will now show that this series is absolutely convergent on compact subsets which will permit us to rearrange the terms in the series. Suppose that each $|x_i| \leq R$ for some $R > 1$ for each $i = 1, \ldots, n$. Also, since f is entire, for any $S > 0$, we can find a constant $C_S > 0$ such that

$$|f_\alpha| \leq \frac{C_S}{S^{|\alpha|}}$$

for each multi-index α. Therefore, by lemma 3.3.4, we have

$$\left| \frac{k!}{\beta!} (\alpha + 1)_{2\beta} f_{\alpha+2\beta} x^\alpha \right| \leq (k!)^2 \cdot C_S \left(\frac{4R}{S} \right)^{|\alpha+2\beta|}.$$

Since we can pick S large enough so that $\frac{4R}{S} < \frac{1}{2}$, we see that:

$$\sum_{|\beta|=k} \sum_{\alpha} \left| \frac{k!}{\beta!} (\alpha + 1)_{2\beta} f_{\alpha+2\beta} x^\alpha \right| \leq (k!)^2 \cdot C_S \cdot \sum_{|\beta|=k} \sum_{\alpha} \left(\frac{4R}{S} \right)^{|\alpha+2\beta|},$$

where we know by lemma 3.3.5 that this sum converges. Therefore, the series for $\triangle^k f$ converges absolutely on compact subsets, and we may rearrange the series to get:

$$\triangle^k f = \sum_{\alpha} \left(\sum_{|\beta|=k} \frac{k!}{\beta!} (\alpha + 1)_{2\beta} f_{\alpha+2\beta} \right) x^\alpha.$$

So, we can make the identification:

$$f_{\alpha,k} = \sum_{|\beta|=k} \frac{k!}{\beta!} (\alpha + 1)_{2\beta} f_{\alpha+2\beta}.$$

We can now use lemma 3.3.3 and the fact that f is entire to complete our estimate:

$$
\begin{aligned}
|f_{\alpha,k}| &\leq \sum_{|\beta|=k} \left| \frac{k!}{\beta!}(\alpha+1)_{2\beta} f_{\alpha+2\beta} \right| \\
&= \sum_{|\beta|=k} \frac{k!}{\beta!}(\alpha+1)_{2\beta} \left| f_{\alpha+2\beta} \right| \\
&\leq \sum_{|\beta|=k} (k!)^2 \cdot 2^{|\alpha|+4k} \cdot \frac{C_S}{S^{|\alpha|+2k}} \\
&= \binom{n+k-1}{k} (k!)^2 \cdot C_S \cdot \frac{2^{|\alpha|+4k}}{S^{|\alpha|+2k}} \\
&\leq 2^{n-1}(k!)^2 \cdot C_S \cdot \frac{2^k \cdot 2^{|\alpha|+4k}}{S^{|\alpha|+2k}} \\
&\leq C_S 2^{n-1}(k!)^2 \frac{2^{|\alpha|+2k} \cdot 4^{|\alpha|+2k}}{S^{|\alpha|+2k}} \\
&= C_S 2^{n-1}(k!)^2 \left(\frac{8}{S}\right)^{|\alpha|+2k}.
\end{aligned}
$$

□

3.4 Main Results

Using our estimates from proposition 3.3.6, we can prove the two crucial facts that we need in order to solve the half-space problem.

Proposition 3.4.1 *Let f be an entire function in $x_1, \ldots, x_n$, and define:*

$$u = \sum_{k=0}^{\infty} \frac{(-1)^k \triangle^k f}{(2k)!} y^{2k}.$$

Then, u is an entire function in $x_1, \ldots, x_n, y$ which satisfies $u = f$ when $y = 0$ and $\frac{\partial u}{\partial y} = 0$ when $y = 0$.

Proof: By proposition 3.3.6, we can write

$$\triangle^k f = \sum_{\alpha} f_{\alpha,k} x^{\alpha}$$

where for any $S > 0$, we can find a constant $C_S > 0$ such that

$$|f_{\alpha,k}| \leq C_S 2^{n-1}(k!)^2 \left(\frac{8}{S}\right)^{|\alpha|+2k}$$

for all $k \geq 1$ and all multi-indices α. Therefore, we can formally expand u as a power series in its variables:

$$u = \sum_{k=0}^{\infty} \sum_{\alpha} \frac{(-1)^k f_{\alpha,k}}{(2k)!} x^\alpha y^{2k}.$$

Now, using our estimate for $f_{\alpha,k}$, we can estimate the coefficient of $x^\alpha y^{2k}$:

$$\left|\frac{(-1)^k f_{\alpha,k}}{(2k)!}\right| \leq C_S 2^{n-1} \frac{(k!)^2}{(2k)!} \left(\frac{8}{S}\right)^{|\alpha|+2k} \leq C_S 2^{n-1} \left(\frac{8}{S}\right)^{|\alpha|+2k},$$

since $\frac{(k!)^2}{(2k)!} \leq 1$. Since S can be chosen to be arbitrarily large and

$$|\alpha| + 2k = |(\alpha_1, \ldots, \alpha_n, 2k)|$$

is the norm of the multi-index of the monomial under consideration, we see that the series will converge, so our expansion was justified. Also, by these estimates, we see that u is an entire function. Finally, we can see directly from the series that $u = f$ when $y = 0$ and that $\frac{\partial u}{\partial y} = 0$ when $y = 0$. $\square$

Proposition 3.4.2 *Let f be an entire function in $x_1, \ldots, x_n$, and define:*

$$u = \sum_{k=0}^{\infty} \frac{(-1)^k \triangle^k f}{(2k+1)!} y^{2k+1}.$$

Then, u is an entire function in $x_1, \ldots, x_n, y$ which satisfies $u = 0$ when $y = 0$ and $\frac{\partial u}{\partial y} = f$ when $y = 0$.

Proof: By proposition 3.3.6, we can write

$$\triangle^k f = \sum_{\alpha} f_{\alpha,k} x^\alpha$$

where for any $S > 0$, we can find a constant $C_S > 0$ such that

$$|f_{\alpha,k}| \leq C_S 2^{n-1}(k!)^2 \left(\frac{8}{S}\right)^{|\alpha|+2k}$$

for all $k \geq 1$ and all multi-indices α. Therefore, we can formally expand u as a power series in its variables:

$$u = \sum_{k=0}^{\infty} \sum_{\alpha} \frac{(-1)^k f_{\alpha,k}}{(2k+1)!} x^\alpha y^{2k+1}.$$

As before, we will estimate the coefficient of $x^\alpha y^{2k+1}$ using our estimate for $f_{\alpha,k}$:

$$\begin{aligned}
\left| \frac{(-1)^k f_{\alpha,k}}{(2k+1)!} \right| &\leq C_S 2^{n-1} \frac{(k!)^2}{(2k+1)!} \left(\frac{8}{S} \right)^{|\alpha|+2k} \\
&\leq C_S 2^{n-1} \left(\frac{8}{S} \right)^{|\alpha|+2k} \\
&= \left(\frac{C_S 2^{n-1} S}{8} \right) \left(\frac{8}{S} \right)^{|\alpha|+2k+1},
\end{aligned}$$

since $\frac{(k!)^2}{(2k+1)!} \leq 1$. Since S can be chosen to be arbitrarily large and $|\alpha| + 2k + 1 = |(\alpha_1, \ldots, \alpha_n, 2k+1)|$ is the norm of the multi-index of the monomial under consideration, we see that the series will converge, so our expansion was justified. Also by these estimates, we see that u will again be an entire function. Finally, we see from the series that $u = 0$ when $y = 0$ and $\frac{\partial u}{\partial y} = f$ when $y = 0$. □

Finally, we note the following fact, which was the motivation behind the above constructions.

Proposition 3.4.3 *Let f be an entire function in $x_1, \ldots, x_n$, and define:*

$$\begin{aligned}
u &= \sum_{k=0}^{\infty} \frac{(-1)^k \triangle^k f}{(2k)!} y^{2k} \quad \text{and} \\
w &= \sum_{k=0}^{\infty} \frac{(-1)^k \triangle^k f}{(2k+1)!} y^{2k+1}.
\end{aligned}$$

Then, both u and w are harmonic.

Proof: First, consider u. By proposition 3.4.1, we know that it will be an entire function, and so term by term differentiation is justified. Remembering that f is not a function of y, we can split the action of the Laplacian into the variables $x_1, \ldots, x_n$ and

y to see:

$$\begin{aligned}\triangle u &= \sum_{k=0}^{\infty}\frac{(-1)^k\triangle^{k+1}f}{(2k)!}y^{2k}+\sum_{k=1}^{\infty}\frac{(-1)^k\triangle^{k}f}{(2k-2)!}y^{2k-2}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k\triangle^{k+1}f}{(2k)!}y^{2k}+\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\triangle^{k+1}f}{(2k)!}y^{2k}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k\triangle^{k+1}f}{(2k)!}y^{2k}-\sum_{k=0}^{\infty}\frac{(-1)^k\triangle^{k+1}f}{(2k)!}y^{2k}=0.\end{aligned}$$

Next, consider w. By proposition 3.4.2, we know that it too will be an entire function, and so we can perform a similar calculation:

$$\begin{aligned}\triangle w &= \sum_{k=0}^{\infty}\frac{(-1)^k\triangle^{k+1}f}{(2k+1)!}y^{2k+1}+\sum_{k=1}^{\infty}\frac{(-1)^k\triangle^{k}f}{(2k-1)!}y^{2k-1}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k\triangle^{k+1}f}{(2k+1)!}y^{2k+1}+\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\triangle^{k+1}f}{(2k+1)!}y^{2k+1}\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k\triangle^{k+1}f}{(2k+1)!}y^{2k+1}-\sum_{k=0}^{\infty}\frac{(-1)^k\triangle^{k+1}f}{(2k+1)!}y^{2k+1}=0.\end{aligned}$$

$\square$

We are now in a position to easily prove our main results for half space problems. To pin down terminology, we will let the coordinates of $\mathbb{R}^n$ be $x_1,\ldots,x_n$, where this will be the domain of our boundary data, and we will let the coordinates of $\mathbb{R}^{n+1}$ be $x_1,\ldots,x_n,y$, where this will be the domain of our solutions. We then have three main results.

Theorem 3.4.4 *Let f be an entire function on $\mathbb{R}^n$. Then the Dirichlet type problem*

$$\begin{aligned}\triangle u &= 0 \text{ in } \mathbb{R}^{n+1}\\ u &= f \text{ on } y=0\end{aligned}$$

has an entire solution u.

Proof: We define

$$u=\sum_{k=0}^{\infty}\frac{(-1)^k\triangle^k f}{(2k)!}y^{2k}.$$

By proposition 3.4.3, we know that u is harmonic. By proposition 3.4.1, we know that u is entire and $u = f$ when $y = 0$. So, u is our desired entire solution. $\square$

Theorem 3.4.5 *Let g be an entire function on $\mathbb{R}^n$. Then the Neumann type problem*

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^{n+1} \\ \frac{\partial u}{\partial y} &= g \text{ on } y = 0 \end{aligned}$$

has an entire solution u.

Proof: We define

$$u = \sum_{k=0}^{\infty} \frac{(-1)^k \triangle^k g}{(2k+1)!} y^{2k+1}.$$

By proposition 3.4.3, we know that u is harmonic. By proposition 3.4.2, we know that u is entire and satisfies the boundary conditions. Therefore, it is our desired solution. $\square$

Theorem 3.4.6 *Let f and g be entire functions on $\mathbb{R}^n$. Then the Cauchy type problem*

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^{n+1} \\ u &= f \text{ on } y = 0 \\ \frac{\partial u}{\partial y} &= g \text{ on } y = 0 \end{aligned}$$

has a unique, entire solution u.

Proof: We define

$$u = \sum_{k=0}^{\infty} \frac{(-1)^k \triangle^k f}{(2k)!} y^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k \triangle^k g}{(2k+1)!} y^{2k+1}.$$

By proposition 3.4.3, we see that u is the sum of two harmonic functions and is thus harmonic. By propositions 3.4.1 and 3.4.2, we see that u is the sum of two entire functions, and so u is entire. Also by propositions 3.4.1 and 3.4.2, we see that u

satisfies the boundary conditions. Therefore, it is an entire solution to our problem. To see that it is unique, suppose that we have two solutions, u_1 and u_2, to our problem. Then, the difference $w = u_1 - u_2$ satisfies:

$$\begin{aligned} \triangle w &= 0 \text{ in } \mathbb{R}^{n+1} \\ w &= 0 \text{ on } y = 0 \\ \frac{\partial w}{\partial y} &= 0 \text{ on } y = 0. \end{aligned}$$

However, w is entire, and so we may write it as a series in y with coefficients which are functions of $x_1, \ldots, x_n$. Since we may differentiate entire functions term by term, we may follow the derivation in section 3.2 to write w in the form of equation (3.1):

$$w = \sum_{k=0}^{\infty} \frac{(-1)^k \triangle^k w_0}{(2k)!} y^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k \triangle^k w_1}{(2k+1)!} y^{2k+1},$$

for some functions w_0 and w_1. However, by applying our boundary conditions, we see that $w_0 \equiv w_1 \equiv 0$, and so $w = 0$. This means that our two solutions u_1 and u_2 were in fact equal, and so our half space problem has a unique, entire solution. $\square$

Chapter 4

The Heat Equation

This chapter deviates slightly from our main topic, but it serves as a nice comparison to the half space results of the previous chapter. The source of the deviation comes from considering the heat equation rather than Laplace's equation, but the main thrust of exploration will be the same: the search for entire solutions on a half space given entire data. Concretely, we will be examining whether for any entire function f, we can find an entire function u which satisfies the following one dimensional heat problem:

$$\begin{aligned} u_t &= u_{xx} \text{ in } \mathbb{R}^2 \\ u(x,0) &= f(x). \end{aligned}$$

4.1 Conditions for Entire Solutions

In the paper [14], Kovalevskaya made a thorough study of analytic solutions to the heat equation given analytic initial conditions. In particular, she constructed the following entire function where the solution converges nowhere:

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{\frac{1}{3}}}.$$

So, unlike Laplace's equation, the heat equation will not have an entire solution for all entire functions f given as data. However, we can impose conditions on our entire data to ensure that we have an entire solution. Suppose that our entire data is given to us as a power series:

$$f(x) = \sum_{k=0}^{\infty} f_k x^k.$$

Then, we will show in the remainder of this chapter that the following conditions are both necessary and sufficient for the existence of an entire solution to our one dimensional heat problem:

$$\lim_{k\to\infty} \left| \frac{(2k)!}{k!} f_{2k} \right|^{\frac{1}{k}} = \lim_{k\to\infty} \left| \frac{(2k+1)!}{k!} f_{2k+1} \right|^{\frac{1}{k+1}} = 0.$$

Therefore, the existence of an entire solution depends only upon the rate of decay of the Taylor coefficients of the data.

4.2 Necessity of the Conditions

Proposition 4.2.1 *Suppose that we have an entire function $u(x,t)$ which satisfies the following one dimensional heat problem:*

$$\begin{aligned} u_t &= u_{xx} \text{ in } \mathbb{R}^2 \\ u(x,0) &= f(x), \end{aligned}$$

where $f(x)$ is also an entire function. Then:

$$\lim_{k\to\infty} \left| \frac{(2k)!}{k!} f_{2k} \right|^{\frac{1}{k}} = \lim_{k\to\infty} \left| \frac{(2k+1)!}{k!} f_{2k+1} \right|^{\frac{1}{k+1}} = 0.$$

Proof: Since u is entire, this permits us to write it as a power series in t:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x) t^k,$$

where the functions $u_k(x)$ need to be determined. We begin by calculating some derivatives of our solution:

$$\begin{aligned} u_t &= \sum_{k=1}^{\infty} u_k(x) k t^{k-1} = \sum_{k=0}^{\infty} (k+1) u_{k+1}(x) t^k \\ u_{xx} &= \sum_{k=0}^{\infty} u_k''(x) t^k \end{aligned}$$

Since we know $u_t = u_{xx}$, we equate coefficients of t^k to obtain the relation

$$(k+1)u_{k+1}(x) = u_k''(x).$$

This gives us $u_1(x) = u_0''(x)$, $u_2(x) = \frac{1}{2!}u_0^{(4)}(x)$, and in general, we have:

$$u_k(x) = \frac{1}{k!} u_0^{(2k)}(x).$$

Finally, we can use our initial condition to completely determine our solution. Since we know $u(x,0) = f(x)$, we see that:

$$f(x) = u(x,0) = u_0(x).$$

Therefore, if we start by assuming that we have an entire solution to our heat problem, then it must have the form:

$$u(x,t) = \sum_{k=0}^{\infty} f^{(2k)}(x) \frac{t^k}{k!}. \tag{4.1}$$

The next step will be to obtain a series solution about the origin for our solution, using the above expansion as a starting point. Since f is an entire function, we can write it as a series:

$$f(x) = \sum_{n=0}^{\infty} f_n x^n,$$

for some coefficients f_n. We may also differentiate it $2k$ times using equation (2.1):

$$f^{(2k)}(x) = \sum_{n=0}^{\infty} (n+1)_{2k} f_{n+2k} x^n,$$

where $(n+1)_{2k}$ denotes the usual Pochhammer symbol, or rising factorial. Substituting this back into our entire solution gives:

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n+1)_{2k}}{k!} f_{n+2k} x^n t^k = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n+2k)!}{n!k!} f_{n+2k} x^n t^k.$$

We will now impose the condition that u is entire. For any $R > 0$, we can find a constant $C_R > 0$ such that:

$$\left| \frac{(n+2k)!}{n!k!} f_{n+2k} \right| \leq \frac{C_R}{R^{n+k}}$$

for all n, $k \geq 0$. In particular, we can set $n = 0$ and $n = 1$ to obtain the following necessary conditions:

$$\left| \frac{(2k)!}{k!} f_{2k} \right| \leq \frac{C_R}{R^k} \text{ and } \left| \frac{(2k+1)!}{k!} f_{2k+1} \right| \leq \frac{C_R}{R^{k+1}}.$$

Taking the k^{th} and $(k+1)^{\text{st}}$ roots of the above inequalities, we get:

$$\begin{aligned} \limsup_{k\to\infty} \left| \frac{(2k)!}{k!} f_{2k} \right|^{\frac{1}{k}} &\leq \limsup_{k\to\infty} \frac{(C_R)^{\frac{1}{k}}}{R} = \frac{1}{R} \text{ and} \\ \limsup_{k\to\infty} \left| \frac{(2k+1)!}{k!} f_{2k+1} \right|^{\frac{1}{k+1}} &\leq \limsup_{k\to\infty} \frac{(C_R)^{\frac{1}{k+1}}}{R} = \frac{1}{R}. \end{aligned}$$

However, since this is true for all $R > 0$, our conditions can be stated equivalently as:

$$\limsup_{k\to\infty} \left| \frac{(2k)!}{k!} f_{2k} \right|^{\frac{1}{k}} = \limsup_{k\to\infty} \left| \frac{(2k+1)!}{k!} f_{2k+1} \right|^{\frac{1}{k+1}} = 0.$$

However, since both of these expressions are non-negative, this is the same as saying:

$$\lim_{k\to\infty} \left| \frac{(2k)!}{k!} f_{2k} \right|^{\frac{1}{k}} = \lim_{k\to\infty} \left| \frac{(2k+1)!}{k!} f_{2k+1} \right|^{\frac{1}{k+1}} = 0.$$

So, assuming that we have an entire solution to our heat problem leads to the above necessary conditions. □

The proof of the above proposition also leads to the following corollary.

Corollary 4.2.2 *The heat problem:*

$$\begin{aligned} u_t &= u_{xx} \text{ in } \mathbb{R}^2 \\ u(x,0) &= f(x) \end{aligned}$$

has at most one entire solution.

Proof: Suppose that we can find two entire solutions, u_1 and u_2, to our heat problem. Then, the function $w = u_1 - u_2$ will be entire and will satisfy:

$$\begin{aligned} w_t &= w_{xx} \text{ in } \mathbb{R}^2 \\ w(x,0) &= 0. \end{aligned}$$

So, following the proof of proposition 4.2.1, we can write w in the form of equation (4.1):

$$w(x,t) = \sum_{k=0}^{\infty} v^{(2k)}(x) \frac{t^k}{k!},$$

for some function v. However, by applying our initial condition, we see that $v \equiv 0$, and so $w = u_1 - u_2 \equiv 0$. Therefore, $u_1 = u_2$, and so our heat problem can have at most one entire solution. □

4.3 Sufficiency of the Conditions

In order to demonstrate the sufficiency of the conditions, we will first need a technical lemma.

Lemma 4.3.1 *Let $R > 0$, and let k and m be integers such that $k \geq 1$ and $0 \leq m \leq k$. Then:*

$$\frac{R^m (k-m+1) \cdots k}{(2m)!} \leq e^R e^k,$$

where for $m = 0$, the empty product in the numerator is understood to be 1.

Proof: We will only need the fact that for $m \geq 0$, we have:

$$m!m! \leq (2m)! \implies \frac{1}{(2m)!} \leq \frac{1}{m!m!}.$$

This gives us the following estimate:

$$\begin{aligned}
\frac{R^m(k-m+1)\cdots k}{(2m)!} &\leq \frac{R^m k^m}{(2m)!} \\
&\leq \frac{R^m}{m!}\frac{k^m}{m!} \\
&\leq \left(\sum_{m=0}^{\infty}\frac{R^m}{m!}\right)\left(\sum_{m=0}^{\infty}\frac{k^m}{m!}\right) \\
&= e^R e^k
\end{aligned}$$

□

We now present a pair of similar lemmas dealing with the slight difference between the conditions on even and odd coefficients.

Lemma 4.3.2 *Suppose that for all $R > 0$, we can find a constant $C_R > 0$ such that:*

$$|f_{2k}| \leq \frac{C_R}{R^k}\frac{k!}{(2k)!}$$

for all $k \geq 0$. Then, for all $R > 0$, we can find a constant $D_R > 0$ such that:

$$|f_{2k}| \leq \frac{D_R}{R^{k+m}}\frac{(k-m)!(2m)!}{(2k)!}$$

for all $k \geq 0$ and for all integers m such that $0 \leq m \leq k$.

Proof: We know that for any $R > 0$, we can find a constant $C_R > 0$ such that

$$|f_{2k}| \leq \frac{C_R}{R^k}\frac{k!}{(2k)!}$$

for all $k \geq 0$. In particular, for any $R > 0$, we can find a constant $C_{eR} > 0$ such that

$$|f_{2k}| \leq \frac{C_{eR}}{(eR)^k}\frac{k!}{(2k)!}$$

for all $k \geq 0$. We can then make use of lemma 4.3.1. For any $k \geq 1$ and $0 \leq m \leq k$ we have:

$$\begin{aligned}
|f_{2k}| &\leq \frac{C_{eR}}{(eR)^k}\frac{k!}{(2k)!} \\
&= \frac{C_{eR}}{(eR)^{k+m}} \cdot \frac{(k-m)!(2m)!}{(2k)!} \cdot \frac{(eR)^m (k-m+1)\cdots k}{(2m)!} \\
&= \frac{C_{eR}}{(eR)^{k+m}} \cdot \frac{(k-m)!(2m)!}{(2k)!} \cdot \frac{e^m R^m (k-m+1)\cdots k}{(2m)!} \\
&\leq \frac{C_{eR}}{(eR)^{k+m}} \cdot \frac{(k-m)!(2m)!}{(2k)!} \cdot e^m e^R e^k \\
&= \frac{C_{eR}e^R}{R^{k+m}}\frac{(k-m)!(2m)!}{(2k)!} \\
&= \frac{C'_R}{R^{k+m}}\frac{(k-m)!(2m)!}{(2k)!},
\end{aligned}$$

where we have collected the R dependence into C'_R. To finish the proof, we must only include the case where $k = 0$ and $m = 0$, which requires the choice:

$$D_R = \max\{C'_R, |f_0|\}.$$

With this choice, we have that

$$|f_{2k}| \leq \frac{D_R}{R^{k+m}}\frac{(k-m)!(2m)!}{(2k)!}$$

for all $k \geq 0$ and $0 \leq m \leq k$. □

Lemma 4.3.3 *Suppose that for all $R > 0$, we can find a constant $C_R > 0$ such that:*

$$|f_{2k+1}| \leq \frac{C_R}{R^{k+1}}\frac{k!}{(2k+1)!}$$

for all $k \geq 0$. Then, for all $R > 0$, we can find a constant $D_R > 0$ such that:

$$|f_{2k+1}| \leq \frac{D_R}{R^{k+m+1}}\frac{(k-m)!(2m+1)!}{(2k+1)!}$$

for all $k \geq 0$ and for all integers m such that $0 \leq m \leq k$.

Proof: We proceed exactly as in the proof of the previous lemma, using lemma 4.3.1 along the way. For all $R > 0$, we can find a constant $C_{eR} > 0$ such that:

$$|f_{2k+1}| \leq \frac{C_{eR}}{(eR)^{k+1}} \frac{k!}{(2k+1)!}$$

for all $k \geq 0$. Then, for any integers $k \geq 1$ and $0 \leq m \leq k$, we can estimate as follows:

$$\begin{aligned}
|f_{2k+1}| &\leq \frac{C_{eR}}{(eR)^{k+1}} \frac{k!}{(2k+1)!} \\
&= \frac{C_{eR}}{(eR)^{k+m+1}} \cdot \frac{(k-m)!(2m+1)!}{(2k+1)!} \cdot \frac{(eR)^m (k-m+1)\cdots k}{(2m+1)!} \\
&\leq \frac{C_{eR}}{(eR)^{k+m+1}} \cdot \frac{(k-m)!(2m+1)!}{(2k+1)!} \cdot \frac{e^m R^m (k-m+1)\cdots k}{(2m)!} \\
&\leq \frac{C_{eR}}{(eR)^{k+m+1}} \cdot \frac{(k-m)!(2m+1)!}{(2k+1)!} \cdot e^m e^R e^k \\
&= \frac{C_{eR}}{(eR)^{k+m+1}} \cdot \frac{(k-m)!(2m+1)!}{(2k+1)!} \cdot \frac{e^{k+m+1} e^R}{e} \\
&= \frac{C_{eR} e^R}{e} \frac{1}{R^{k+m+1}} \cdot \frac{(k-m)!(2m+1)!}{(2k+1)!} \\
&= \frac{C'_R}{R^{k+m+1}} \frac{(k-m)!(2m+1)!}{(2k+1)!},
\end{aligned}$$

where we have collected the R dependence into C'_R. Again, to complete the proof, we must only include the case where $k = 0$ and $m = 0$, so we make the choice:

$$D_R = \max\{C'_R, |f_1|R\}.$$

With this choice, we have that

$$|f_{2k+1}| \leq \frac{D_R}{R^{k+m+1}} \frac{(k-m)!(2m+1)!}{(2k+1)!}$$

for all $k \geq 0$ and $0 \leq m \leq k$. □

We are now in a position to perform our main estimation. In the following proposition, note that for each particular coefficient f_m, there will be multiple choices of n and k such that $n + 2k = m$. So, we will be getting multiple bounds on each coefficient. While this means that some of the estimates may not be as sharp as possible, this is the form that we will need in our final proof of the sufficiency of the conditions on the coefficients of f to guarantee the existence of an entire solution.

Proposition 4.3.4 *Suppose that the following limits hold:*

$$\lim_{k\to\infty}\left|\frac{(2k)!}{k!}f_{2k}\right|^{\frac{1}{k}} = \lim_{k\to\infty}\left|\frac{(2k+1)!}{k!}f_{2k+1}\right|^{\frac{1}{k+1}} = 0.$$

Then, for all $R > 0$, we can find a constant $C_R > 0$ such that:

$$\left|\frac{(n+2k)!}{n!k!}f_{n+2k}\right| \leq \frac{C_R}{R^{n+k}}$$

for all $n \geq 0$ and $k \geq 0$.

Proof: First, since we have

$$\lim_{k\to\infty}\left|\frac{(2k)!}{k!}f_{2k}\right|^{\frac{1}{k}} = 0,$$

we can use lemma 2.1.1 to see that for any $R > 0$, we can find a constant $A_R > 0$ such that:

$$\left|\frac{(2k)!}{k!}f_{2k}\right| \leq \frac{A_R}{R^k} \implies |f_{2k}| \leq \frac{A_R}{R^k}\frac{k!}{(2k)!}$$

for all $k \geq 0$. Therefore, by lemma 4.3.2, for any $R > 0$ we know that we can find a constant $A'_R > 0$ such that:

$$|f_{2k}| \leq \frac{A'_R}{R^{k+m}}\frac{(k-m)!(2m)!}{(2k)!}$$

for all $k \geq 0$ and $0 \leq m \leq k$. Similarly, since

$$\lim_{k\to\infty}\left|\frac{(2k+1)!}{k!}f_{2k+1}\right|^{\frac{1}{k+1}} = 0,$$

lemma 2.1.1 tells us that for any $R > 0$, we can find a constant $B_R > 0$ such that

$$\left|\frac{(2k+1)!}{k!}f_{2k+1}\right| \leq \frac{B_R}{R^{k+1}} \implies |f_{2k+1}| \leq \frac{B_R}{R^{k+1}}\frac{k!}{(2k+1)!}$$

for all $k \geq 0$. Therefore, by lemma 4.3.3, for each $R > 0$ we can find a constant $B'_R > 0$ such that

$$|f_{2k+1}| \leq \frac{B'_R}{R^{k+m+1}}\frac{(k-m)!(2m+1)!}{(2k+1)!}$$

for all $k \geq 0$ and $0 \leq m \leq k$. Let $C_R = \max\{A'_R, B'_R\}$. Then, we have:

$$\begin{aligned}
|f_{2k}| &\leq \frac{C_R}{R^{k+m}} \frac{(k-m)!(2m)!}{(2k)!} \quad \text{and} \\
|f_{2k+1}| &\leq \frac{C_R}{R^{k+m+1}} \frac{(k-m)!(2m+1)!}{(2k+1)!}
\end{aligned}$$

for all $k \geq 0$ and $0 \leq m \leq k$. To complete the argument, we make the change of variables $k' = k - m$ and $n = 2m$ in the first inequality, and $k' = k - m$ and $n = 2m+1$ in the second inequality. Then, both inequalities get transformed to:

$$|f_{n+2k'}| \leq \frac{C_R}{R^{n+k'}} \frac{(k')!n!}{(n+2k')!},$$

and the conditions that $k \geq 0$ and $0 \leq m \leq k$ get transformed into $k' \geq 0$ and $n \geq 0$. Removing the primes for clarity, this means that for any $R > 0$, we have found a constant $C_R > 0$ such that

$$\left| \frac{(n+2k)!}{n!k!} f_{n+2k} \right| \leq \frac{C_R}{R^{n+k}}$$

for all $k \geq 0$ and $n \geq 0$. □

We can now demonstrate the sufficiency of our conditions for ensuring that our one dimensional heat problem has an entire solution.

Proposition 4.3.5 *Let f be an entire function, and consider the problem:*

$$\begin{aligned}
u_t &= u_{xx} \text{ in } \mathbb{R}^2 \\
u(x,0) &= f(x) = \sum_{n=0}^{\infty} f_n x^n.
\end{aligned}$$

If the following conditions on f are satisfied:

$$\lim_{k\to\infty} \left| \frac{(2k)!}{k!} f_{2k} \right|^{\frac{1}{k}} = \lim_{k\to\infty} \left| \frac{(2k+1)!}{k!} f_{2k+1} \right|^{\frac{1}{k+1}} = 0,$$

then this problem will have a unique, entire solution u.

Proof: By proposition 4.3.4, for any $R > 0$, we can find a constant $C_R > 0$ such that:

$$\left|\frac{(n+2k)!}{n!k!} f_{n+2k}\right| \leq \frac{C_R}{R^{n+k}}$$

for all $n \geq 0$ and $k \geq 0$. Therefore, if we construct the double series:

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n+2k)!}{n!k!} f_{n+2k} x^n t^k,$$

then this will define an entire function. It remains to show that this function solves our heat problem.

Since we are dealing with an entire power series, we can differentiate term by term to get:

$$\begin{aligned}
u_t &= \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} \frac{(n+2k)!}{n!(k-1)!} f_{n+2k} x^n t^{k-1} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n+2k+2)!}{n!k!} f_{n+2k+2} x^n t^k \\
u_x &= \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \frac{(n+2k)!}{(n-1)!k!} f_{n+2k} x^{n-1} t^k = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n+2k+1)!}{n!k!} f_{n+2k+1} x^n t^k \\
u_{xx} &= \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \frac{(n+2k+1)!}{(n-1)!k!} f_{n+2k+1} x^{n-1} t^k = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(n+2k+2)!}{n!k!} f_{n+2k+2} x^n t^k
\end{aligned}$$

So, our function satisfies $u_t = u_{xx}$. Finally, we have:

$$u(x,0) = \sum_{n=0}^{\infty} \frac{n!}{n!} f_n x^n = \sum_{n=0}^{\infty} f_n x^n = f(x).$$

Therefore, we have constructed an entire solution to our heat problem, thereby showing the sufficiency of the conditions on the coefficients of f. The uniqueness of this solution follows directly from corollary 4.2.2. □

4.4 Comparison with Laplace's Equation

We can now state our results and compare them to those of the previous chapter. First, we show that the heat equation will always have a polynomial solution given polynomial initial conditions. More information about these *heat polynomials* can be found in [8].

Theorem 4.4.1 *The one dimensional heat problem:*

$$\begin{aligned} u_t &= u_{xx} \text{ in } \mathbb{R}^2 \\ u(x,0) &= f(x) \end{aligned}$$

will have a unique polynomial solution u for any polynomial data f.

Proof: Since the power series for f will have only finitely many terms, the conditions:

$$\lim_{k\to\infty}\left|\frac{(2k)!}{k!}f_{2k}\right|^{\frac{1}{k}} = \lim_{k\to\infty}\left|\frac{(2k+1)!}{k!}f_{2k+1}\right|^{\frac{1}{k+1}} = 0$$

will be satisfied. Therefore, following the proof of proposition 4.3.5, we see that the function:

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n+2k)!}{n!k!}f_{n+2k}x^n t^k$$

is well defined and will be a solution to our heat problem. However, because the coefficients of f are eventually all 0, this will be a finite sum and will give us our desired polynomial solution. Since polynomials are entire functions, we can apply corollary 4.2.2 to see that this solution must be unique. □

Theorem 4.4.2 *Let f be an entire function. Then the one dimensional heat problem:*

$$\begin{aligned} u_t &= u_{xx} \text{ in } \mathbb{R}^2 \\ u(x,0) &= f(x) \end{aligned}$$

will have an entire solution u if and only if:

$$\lim_{k\to\infty}\left|\frac{(2k)!}{k!}f_{2k}\right|^{\frac{1}{k}} = \lim_{k\to\infty}\left|\frac{(2k+1)!}{k!}f_{2k+1}\right|^{\frac{1}{k+1}} = 0.$$

When this solution exists, it is unique.

Proof: The necessity of the conditions was proved in proposition 4.2.1, and the sufficiency was shown in proposition 4.3.5. Once again, if a solution exists, uniqueness follows from corollary 4.2.2. □

We have now observed a phenomenon which did not occur in the previous chapter. The half space heat problem in two dimensions always possessed a polynomial solution given polynomial data, but the existence of an entire solution given entire data depended on the decay rate of the Taylor coefficients of the data. As we shall see in the next chapter where showing convergence of our solution turns into a very delicate matter, even Laplace's equation is not always free from these types of complications.

4.5 Unification of the Conditions

As it stands, we have separate conditions to ensure entire solutions for even and odd coefficients of our data. While in practice this poses no difficulties, it is aesthetically less than optimal. In this final section, we will show that these two conditions can in fact be unified into a single condition on our data, namely:

$$\lim_{n\to\infty}\left|\frac{n!}{\left\lfloor\frac{n}{2}\right\rfloor!}f_n\right|^{\frac{2}{n}}=0.$$

On substituting $n = 2k$ into this limit, we immediately obtain our condition for even coefficients. On substituting $n = 2k + 1$, we note that $\left\lfloor\frac{n}{2}\right\rfloor = k$, and:

$$\lim_{k\to\infty}\left|\frac{(2k+1)!}{k!}f_{2k+1}\right|^{\frac{1}{k+1/2}}=0.$$

This suggests that we need the following straightforward lemma.

Lemma 4.5.1 *Let $g(k)$ be a function defined on the non-negative integers such that $g(k) \geq 0$ for all $k \geq 0$. Then:*

$$\lim_{k\to\infty}g(k)^{\frac{2k+2}{2k+1}}=0\Leftrightarrow\lim_{k\to\infty}g(k)=0.$$

Proof: First, suppose that

$$\lim_{k\to\infty}g(k)=0.$$

Then, we can find a constant $M \geq 0$ such that $g(k) \leq M$ for all $k \geq 0$. Therefore:

$$0 \leq g(k)^{\frac{2k+2}{2k+1}} = g(k) \cdot g(k)^{\frac{1}{2k+1}} \leq g(k) \cdot M^{\frac{1}{2k+1}}.$$

However, by letting k tend to infinity, this means

$$\lim_{k\to\infty} g(k)^{\frac{2k+2}{2k+1}} = 0.$$

Next, suppose that

$$\lim_{k\to\infty} g(k)^{\frac{2k+2}{2k+1}} = 0.$$

This means that we must have

$$\limsup_{k\to\infty} g(k) = c < \infty.$$

For the sake of contradiction, suppose that $c > 0$. Now, we will pick

$$\epsilon = \min\left(\frac{c^2}{4}, 1\right) > 0.$$

Therefore, there is a constant $K > 0$ such that

$$g(k)^{\frac{2k+2}{2k+1}} \leq \epsilon$$

for all $k \geq K$. However, this means that

$$g(k) \leq \epsilon^{\frac{2k+1}{2k+2}} \leq \sqrt{\epsilon} \leq \frac{c}{2}$$

for all $k \geq K$. This cannot be true since

$$\limsup_{k\to\infty} g(k) = c > 0,$$

and so we must have that

$$\limsup_{k\to\infty} g(k) = 0 \implies \lim_{k\to\infty} g(k) = 0$$

since $g(k) \geq 0$ for all $k \geq 0$, establishing our equivalence. $\square$

We can now show that our two conditions can be unified into a single statement.

Proposition 4.5.2 *Let $\{f_n\}$ be a sequence of real numbers for $n \geq 0$. Then:*

$$\lim_{k\to\infty} \left|\frac{(2k)!}{k!} f_{2k}\right|^{\frac{1}{k}} = \lim_{k\to\infty} \left|\frac{(2k+1)!}{k!} f_{2k+1}\right|^{\frac{1}{k+1}} = 0$$

if and only if:

$$\lim_{n\to\infty} \left|\frac{n!}{\lfloor\frac{n}{2}\rfloor!} f_n\right|^{\frac{2}{n}} = 0.$$

Proof: First, suppose that:

$$\lim_{k\to\infty} \left|\frac{(2k)!}{k!} f_{2k}\right|^{\frac{1}{k}} = \lim_{k\to\infty} \left|\frac{(2k+1)!}{k!} f_{2k+1}\right|^{\frac{1}{k+1}} = 0,$$

and consider

$$\left|\frac{n!}{\lfloor\frac{n}{2}\rfloor!} f_n\right|^{\frac{2}{n}}.$$

Then, if we let $n = 2k$ and look at the subsequence of even coefficients, we have:

$$\lim_{n\to\infty, n \text{ even}} \left|\frac{n!}{\lfloor\frac{n}{2}\rfloor!} f_n\right|^{\frac{2}{n}} = \lim_{k\to\infty} \left|\frac{(2k)!}{k!} f_{2k}\right|^{\frac{1}{k}} = 0.$$

If we let $n = 2k+1$ and look at the subsequence of odd coefficients, we have:

$$\begin{aligned}
\lim_{n\to\infty, n \text{ odd}} \left|\frac{n!}{\lfloor\frac{n}{2}\rfloor!} f_n\right|^{\frac{2}{n}} &= \lim_{k\to\infty} \left|\frac{(2k+1)!}{k!} f_{2k+1}\right|^{\frac{1}{k+1/2}} \\
&= \lim_{k\to\infty} \left(\left|\frac{(2k+1)!}{k!} f_{2k+1}\right|^{\frac{1}{k+1}}\right)^{\frac{2k+2}{2k+1}} \\
&= 0
\end{aligned}$$

by lemma 4.5.1. Therefore, since both of the even and odd subsequences of

$$\left|\frac{n!}{\lfloor\frac{n}{2}\rfloor!} f_n\right|^{\frac{2}{n}}$$

tend to 0 as $n \to \infty$, we must have that:

$$\lim_{n\to\infty} \left|\frac{n!}{\lfloor\frac{n}{2}\rfloor!} f_n\right|^{\frac{2}{n}} = 0.$$

Next, suppose that

$$\lim_{n\to\infty}\left|\frac{n!}{\lfloor\frac{n}{2}\rfloor!}f_n\right|^{\frac{2}{n}}=0.$$

Then, both of the even and odd subsequences must also tend to 0 as $n\to\infty$. Setting $n=2k$ gives us:

$$0=\lim_{n\to\infty,n\text{ even}}\left|\frac{n!}{\lfloor\frac{n}{2}\rfloor!}f_n\right|^{\frac{2}{n}}=\lim_{k\to\infty}\left|\frac{(2k)!}{k!}f_{2k}\right|^{\frac{1}{k}}.$$

Setting $n=2k+1$ gives us:

$$\begin{aligned}0 &= \lim_{n\to\infty,n\text{ odd}}\left|\frac{n!}{\lfloor\frac{n}{2}\rfloor!}f_n\right|^{\frac{2}{n}}\\ &= \lim_{k\to\infty}\left|\frac{(2k+1)!}{k!}f_{2k+1}\right|^{\frac{1}{k+1/2}}\\ &= \lim_{k\to\infty}\left(\left|\frac{(2k+1)!}{k!}f_{2k+1}\right|^{\frac{1}{k+1}}\right)^{\frac{2k+2}{2k+1}}.\end{aligned}$$

Therefore, by lemma 4.5.1, this means that

$$\lim_{k\to\infty}\left|\frac{(2k+1)!}{k!}f_{2k+1}\right|^{\frac{1}{k+1}}=0,$$

establishing the equivalence between our two conditions. □

Note that there are actually many different ways that we could have unified our conditions on even and odd coefficients. For example, if we desired the growth condition to be a smooth function of n, we could have just as well chosen:

$$\lim_{n\to\infty}\left|\frac{\Gamma(1+n)}{\Gamma\left(1+\frac{n}{2}\right)}f_n\right|^{\frac{2}{n}}=0.$$

However, demonstrating that this is an equivalent condition requires a bit more technical work and does not really offer any tangible improvements over what we have done.

Chapter 5

Intersecting Lines in Two Dimensions

The next problem on which we will focus is a Dirichlet type problem in the plane. We are again searching for entire harmonic functions, but now we are prescribing values on a pair of intersecting lines through the origin given by $y = \pm mx$, or by $\theta = \pm\theta_0$ in polar coordinates. Specifically, we are asking whether we can find an entire function u in the plane which satisfies:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= f \text{ on } \theta = \pm\theta_0, \end{aligned}$$

where f is a given entire function. The existence of such a function u will turn out to depend on the nature of the angle θ_0 in a rather unexpected way. Angles for which there will always be an entire solution for any entire data will be called "good", while angles for which we can find data such that the problem has no entire solution will be called "bad".

5.1 Review of the Polynomial Problem

First, let us recall the corresponding polynomial problem. This asks when we can find a polynomial u such that:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= f \text{ when } \theta = \pm\theta_0, \end{aligned}$$

where f is our given polynomial data. In [9], it is shown that this problem will always possess a polynomial solution for any polynomial data if θ_0 is not a rational multiple of π. If θ_0 is a rational multiple of π, then it is possible to construct polynomial data for which the problem has no polynomial solution.

5.2 Potential Series Solution

In this section, we will develop a series expansion for a solution to our problem which will help us determine when such a solution exists. We note first the following fact.

Lemma 5.2.1 *Let f be an entire function in the plane. If r and θ are the standard polar coordinates in the plane, then evaluating f along $\theta = \pm\theta_0$ will yield entire functions of r.*

Proof: Since f is entire, we can write

$$f = \sum_{\alpha} c_\alpha x^{\alpha_1} y^{\alpha_2},$$

where the sum is taken over all multi-indices $\alpha = (\alpha_1, \alpha_2)$, and where for any $R > 0$, we can find a constant $C_R > 0$ such that

$$|c_\alpha| \leq \frac{C_R}{R^{|\alpha|}}$$

for all multi-indices α. This means that for any $R > 0$, we can certainly find a constant $C_{2R} > 0$ such that

$$|c_\alpha| \leq \frac{C_{2R}}{(2R)^{|\alpha|}}$$

for all multi-indices α. Now, along $\theta = \theta_0$ we have that $x = r\cos\theta_0$ and $y = r\sin\theta_0$. Making these substitutions into our expression for f gives:

$$\begin{aligned} f\Big|_{\theta=\theta_0} &= \sum_\alpha c_\alpha (r\cos\theta_0)^{\alpha_1}(r\sin\theta_0)^{\alpha_2} = \sum_\alpha c_\alpha (\cos\theta_0)^{\alpha_1}(\sin\theta_0)^{\alpha_2} r^{|\alpha|} \\ &= \sum_{n=0}^{\infty}\left(\sum_{|\alpha|=n} c_\alpha (\cos\theta_0)^{\alpha_1}(\sin\theta_0)^{\alpha_2}\right) r^n. \end{aligned}$$

Next, by using our knowledge about the c_α, we can bound the coefficient of r^n as follows:

$$\left|\sum_{|\alpha|=n} c_\alpha (\cos\theta_0)^{\alpha_1}(\sin\theta_0)^{\alpha_2}\right| \leq C_{2R}\sum_{|\alpha|=n}\frac{1}{(2R)^n} = \frac{C_{2R}(n+1)}{(2R)^n} \leq 2C_{2R}\left(\frac{1}{R}\right)^n.$$

Since this is true for all $R > 0$, we see that f evaluated on $\theta = \theta_0$ will in fact be an entire function of r. Similarly, we have that f evaluated on $\theta = -\theta_0$ will be an entire function of r. □

In light of the previous lemma, let us make the following definitions regarding our data in polar coordinates along the two rays $\theta = \pm\theta_0$:

$$f(r,\theta_0) = \sum_{n=0}^{\infty} f_n^{(+)} r^n \quad \text{and} \tag{5.1}$$

$$f(r,-\theta_0) = \sum_{n=0}^{\infty} f_n^{(-)} r^n, \tag{5.2}$$

where we must have that $f_0^{(+)} = f_0^{(-)}$, since f only has a single value at the origin. Also, for any $R > 0$, we can find a constant $C_R > 0$ such that

$$\left|f_n^{(+)}\right| \leq \frac{C_R}{R^n} \text{ and } \left|f_n^{(-)}\right| \leq \frac{C_R}{R^n}$$

for each $n \geq 0$. We can now prove a necessary condition for the existence of an entire solution to our intersecting lines problem.

Proposition 5.2.2 *Suppose that we can find an entire function u which satisfies:*

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= f \text{ on } \theta = \pm\theta_0, \end{aligned}$$

where f is a given entire function. Then, we can write:

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right),$$

where we have the following equations for the coefficients:

$$\begin{aligned} a_0 &= 2f_0^{(+)}, \\ a_n \cos n\theta_0 &= \frac{f_n^{(+)} + f_n^{(-)}}{2}, \quad \text{and} \\ b_n \sin n\theta_0 &= \frac{f_n^{(+)} - f_n^{(-)}}{2}. \end{aligned}$$

Proof: Suppose that our intersecting lines problem has an entire solution u. By theorem 2.2.5, we know that we can write:

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right),$$

where for any $R > 0$, we can find a constant $C_R > 0$ such that

$$|a_n| \leq \frac{C_R}{R^n} \text{ and } |b_n| \leq \frac{C_R}{R^n}.$$

By evaluating u on $\theta = \pm\theta_0$, we obtain the following two equalities:

$$\begin{aligned} f(r,\theta_0) = u(r,\theta_0) \quad &\Longrightarrow \quad \sum_{n=0}^{\infty} f_n^{(+)} r^n = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta_0 + b_n \sin n\theta_0\right) \\ f(r,-\theta_0) = u(r,-\theta_0) \quad &\Longrightarrow \quad \sum_{n=0}^{\infty} f_n^{(-)} r^n = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta_0 - b_n \sin n\theta_0\right). \end{aligned}$$

By adding and subtracting these equations, we get:

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{f_n^{(+)} + f_n^{(-)}}{2}\right) r^n &= \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n a_n \cos n\theta_0 \\ \sum_{n=0}^{\infty} \left(\frac{f_n^{(+)} - f_n^{(-)}}{2}\right) r^n &= \sum_{n=1}^{\infty} r^n b_n \sin n\theta_0 \end{aligned}$$

So, by matching coefficients of the powers of r, we see that

$$\begin{aligned}
a_0 &= 2f_0^{(+)} \\
a_n \cos n\theta_0 &= \frac{f_n^{(+)} + f_n^{(-)}}{2} \\
b_n \sin n\theta_0 &= \frac{f_n^{(+)} - f_n^{(-)}}{2}
\end{aligned}$$

for $n \geq 1$, which was our goal. □

Already, we can see some problems on the horizon for angles θ_0 which are rational multiples of π. If either $\cos n\theta_0$ or $\sin n\theta_0$ vanish for a particular value of n, then it might not be possible for the coefficient equations of proposition 5.2.2 to be true. This will be dealt with in the next section.

Assuming for the moment that $\cos n\theta_0$ and $\sin n\theta_0$ do not vanish for any value of $n \geq 1$, we can rearrange the coefficient equations to get:

$$a_n = \frac{f_n^{(+)} + f_n^{(-)}}{2\cos n\theta_0} \text{ and } b_n = \frac{f_n^{(+)} - f_n^{(-)}}{2\sin n\theta_0}$$

for $n \geq 1$. So, assuming an entire solution u exists, it must have the above coefficients, and so we must have a unique solution. Whether or not these coefficients decay quickly enough will determine whether or not entire solutions actually exist, and this will depend on our ability to bound the denominators of the coefficients.

5.3 Rational Multiples of π

We will first dispatch the case where θ_0 is a rational multiple of π by giving examples of entire data for which the intersecting lines problem has no entire solution. Our strategy will be to assume the existence of such an entire solution, and then obtain a contradiction from proposition 5.2.2. In fact, we will be able to succeed in this endeavour using only polynomial data.

Proposition 5.3.1 *Let θ_0 be a rational multiple of π such that $0 < \theta_0 < \frac{\pi}{2}$. Then, there exists an entire function f such that there is no entire function u which satisfies the intersecting lines problem:*

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= f \text{ on } \theta = \pm\theta_0. \end{aligned}$$

Proof: Note that since $0 < \theta_0 < \frac{\pi}{2}$, both $\cos\theta_0$ and $\sin\theta_0$ will be non-zero. Also, since θ_0 is a rational multiple of π, we will fix notation so that we can write $\theta_0 = \frac{p}{q}\pi$, where p and q are relatively prime integers. The proof consists of taking cases based on whether q is even or odd.

First, suppose that q is odd and let:

$$f = r^q \sin\theta = r^{q-1} r \sin\theta = \left(x^2 + y^2\right)^{\frac{q-1}{2}} y.$$

Then, f is a polynomial, and so is certainly entire. Next, we notice that:

$$f\Big|_{\theta=\pm\theta_0} = \pm r^q \sin\theta_0 \implies f_q^{(\pm)} = \pm\sin\theta_0.$$

Therefore, assuming that we can find an entire solution u to the intersecting lines problem with this data, proposition 5.2.2 tells us the form of the series that it must have. Examining the coefficient b_q tells us:

$$b_q \sin q\theta_0 = \frac{f_q^{(+)} - f_q^{(-)}}{2} = \sin\theta_0.$$

However, this means that

$$0 \neq \sin\theta_0 = b_q \sin q\theta_0 = b_q \sin p\pi = 0,$$

a contradiction. Therefore, the intersecting lines problem has no entire solutions for this particular choice of data.

Next, suppose that q is even and write $q = 2q'$. Then, for the fraction $\frac{p}{q}$ to be in lowest terms, p must be odd, or $p = 2p' + 1$. However, we then have:

$$\cos q'\theta_0 = \cos q' \frac{2p'+1}{2q'}\pi = \cos\left(p'\pi + \frac{\pi}{2}\right) = 0.$$

This will be the basis for our contradictions. We now need two further cases depending on whether q' is even or odd. First, suppose that q' is even, and let:

$$f = r^{q'} = \left(x^2 + y^2\right)^{\frac{q'}{2}},$$

which is a polynomial and hence an entire function. Since there is no θ dependence, note that $f_{q'}^{(\pm)} = 1$. Assuming that the intersecting lines problem has an entire solution, proposition 5.2.2 tells us that the coefficient $a_{q'}$ must satisfy:

$$a_{q'} \cos q'\theta_0 = \frac{f_{q'}^{(+)} + f_{q'}^{(-)}}{2} = 1.$$

However, this means that

$$1 = a_{q'} \cos q'\theta_0 = 0,$$

a contradiction. Next, suppose that q' is odd and let:

$$f = r^{q'} \cos\theta = r^{q'-1} r \cos\theta = \left(x^2 + y^2\right)^{\frac{q'-1}{2}} x,$$

which is a polynomial and so an entire function. Now, evaluating f on our two lines gives us:

$$f\Big|_{\theta=\pm\theta_0} = r^{q'} \cos\theta_0 \implies f_{q'}^{(\pm)} = \cos\theta_0.$$

Assuming that the intersecting lines problem has an entire solution, proposition 5.2.2 tells us that the coefficient $a_{q'}$ must satisfy:

$$a_{q'} \cos q'\theta_0 = \frac{f_{q'}^{(+)} + f_{q'}^{(-)}}{2} = \cos\theta_0.$$

However, this means that

$$0 \neq \cos\theta_0 = a_{q'} \cos q'\theta_0 = 0,$$

a contradiction.

So, in all cases, if θ_0 is a rational multiple of π, we can construct entire data for which the intersecting lines problem has no entire solutions. $\square$

5.4 Preliminary Bounds on the Denominator

With the results for θ_0 being a rational multiple of π settled in the last section, we can now turn to the more interesting case of when θ_0 is not a rational multiple of π. For this section, we will fix the notation of $\theta_0 = \alpha\pi$, where α is irrational. Then, we know that both $\sin n\theta_0$ and $\cos n\theta_0$ will never vanish for any $n \geq 1$. However, since $n\theta_0 = n\alpha\pi$, both of these quantities will get arbitrarily close to 0 infinitely often since α can be approximated arbitrarily well by rational numbers. Using this as a starting point, we can derive the following two estimates, given here as propositions 5.4.1 and 5.4.2.

Proposition 5.4.1 *Let α be an irrational number such that*

$$0 < \theta_0 = \alpha\pi < \frac{\pi}{2}.$$

For $n \geq 1$, if we define

$$q(n) = \min_{m \in \mathbb{Z}} \left| \alpha - \frac{m}{n} \right|,$$

then we have the following inequalities:

$$\frac{1}{\pi} \frac{1}{nq(n)} \leq \frac{1}{|\sin n\alpha\pi|} \leq \frac{1}{2} \frac{1}{nq(n)}$$

for all $n \geq 1$.

Proof: We know that the value of $q(n)$ will be achieved for a particular value of m, so we can define the function $p(n)$ by:

$$q(n) = \left| \alpha - \frac{p(n)}{n} \right|.$$

In particular, note that $q(n) < \frac{1}{2n}$. Therefore, we can write:

$$nq(n)\pi = |n\alpha\pi - p(n)\pi|,$$

where $0 < nq(n)\pi < \frac{\pi}{2}$. Taking the sine of both sides gives:

$$\sin nq(n)\pi = \sin |n\alpha\pi - p(n)\pi|.$$

Next, on the interval $(-\pi/2, \pi/2)$, we know that $\sin |x| = |\sin x|$, so we have:

$$\sin nq(n)\pi = \sin |n\alpha\pi - p(n)\pi| = |\sin(n\alpha\pi - p(n)\pi)| = |\sin n\alpha\pi|,$$

since $p(n)$ is an integer. Finally, on the interval $(0, \pi/2)$, we know by the concavity of the sine function that

$$\frac{2}{\pi}x \leq \sin x \leq x,$$

so we can write:

$$\frac{2}{\pi}nq(n)\pi \leq \sin nq(n)\pi \leq nq(n)\pi.$$

Therefore, since $\sin nq(n)\pi = |\sin n\alpha\pi|$, we have arrived at our upper and lower bounds:

$$\frac{1}{\pi}\frac{1}{nq(n)} \leq \frac{1}{|\sin n\alpha\pi|} \leq \frac{1}{2}\frac{1}{nq(n)}.$$

□

Proposition 5.4.2 *Let α be an irrational number such that*

$$0 < \theta_0 = \alpha\pi < \frac{\pi}{2}.$$

For $n \geq 1$, if we define

$$q_e(n) = \min_{m \in \mathbb{Z}} \left| \alpha - \frac{2m+1}{2n} \right|,$$

then we have the following inequalities:

$$\frac{1}{\pi}\frac{1}{nq_e(n)} \leq \frac{1}{|\cos n\alpha\pi|} \leq \frac{1}{2}\frac{1}{nq_e(n)}$$

for all $n \geq 1$.

Proof: Apart from the slightly different form of our rational approximation to α, the proof follows the same course as the previous proposition. We know that the value of $q_e(n)$ will be achieved at a particular value of m, so we can define the function $p_e(n)$ by:

$$q_e(n) = \left|\alpha - \frac{2p_e(n)+1}{2n}\right|.$$

Note that we again have $0 < q_e(n) < \frac{1}{2n}$. Therefore, we have

$$nq_e(n)\pi = \left|n\alpha\pi - \frac{2p_e(n)+1}{2}\pi\right| = \left|n\alpha\pi - p_e(n)\pi - \frac{\pi}{2}\right|,$$

where $0 < nq_e(n)\pi < \frac{\pi}{2}$. Taking the sine of both sides and making use of the fact that $\sin|x| = |\sin x|$ on the interval $(-\pi/2, \pi/2)$, we have:

$$\begin{aligned}
\sin nq_e(n)\pi &= \sin\left|n\alpha\pi - p_e(n)\pi - \frac{\pi}{2}\right| \\
&= \left|\sin\left(n\alpha\pi - p_e(n)\pi - \frac{\pi}{2}\right)\right| \\
&= \left|\sin\left(n\alpha\pi - \frac{\pi}{2}\right)\right| \\
&= |\cos n\alpha\pi|,
\end{aligned}$$

since $p_e(n)$ is an integer. Finally, by the concavity of $\sin x$ on the interval $(0, \pi/2)$, we can write

$$\frac{2}{\pi}x \le \sin x \le x$$

for all x in this interval. Therefore:

$$2nq_e(n) \le \sin nq_e(n)\pi \le nq_e(n)\pi.$$

Since we know that $\sin nq_e(n)\pi = |\cos n\alpha\pi|$, we can rearrange the above inequalities to get our final result:

$$\frac{1}{\pi}\frac{1}{nq_e(n)} \le \frac{1}{|\cos n\alpha\pi|} \le \frac{1}{2}\frac{1}{nq_e(n)}$$

for all $n \ge 1$. □

Returning to the equations of proposition 5.2.2, we see that in order to get an upper bound on our coefficients by making use of the above propositions, we will need to be able to bound

$$q(n) = \min_{m\in\mathbb{Z}} \left|\alpha - \frac{m}{n}\right| \text{ and } q_e(n) = \min_{m\in\mathbb{Z}} \left|\alpha - \frac{2m+1}{2n}\right|$$

away from 0 in terms of n.

5.5 Algebraic Multiples of π

In order to produce an initial infinite set of angles θ_0 which make our intersecting lines problem always have an entire solution given any entire data, we need to borrow a few more ideas from number theory. Recall that a number α is called *algebraic* if it is the root of a polynomial with integer coefficients, or if $p(\alpha) = 0$, where:

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

and each a_i is an integer. The set of real algebraic numbers is infinite, countable, and has Lebesgue measure 0. However, one interesting property of algebraic numbers is the following approximation result.

Theorem 5.5.1 *Let α be a real algebraic number. Then, there exists an integer $d > 0$ and a constant $K > 0$ such that:*

$$\left|\alpha - \frac{m}{n}\right| \geq \frac{K}{n^d}$$

for all integers m and n with $n > 0$.

This result is standard, and the proof can be found in [11]. Being able to bound rational approximations away from 0 in this manner turns out to be one way to ensure that our intersecting lines problem always has an entire solution given entire data, as the next proposition shows.

Proposition 5.5.2 *Let $\theta_0 = \alpha\pi$ be an angle between 0 and $\frac{\pi}{2}$, where α is irrational. Suppose that we can find constants K and d such that*

$$\left|\alpha - \frac{m}{n}\right| \geq \frac{K}{n^d}$$

for all integers m and n with $n > 0$. Then given any entire function f, we can find a unique, entire function u which satisfies:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= f \text{ on } \theta = \pm\theta_0. \end{aligned}$$

Proof: Using our lower bound on rational approximations to α, we can estimate:

$$q(n) = \min_{m\in\mathbb{Z}} \left|\alpha - \frac{m}{n}\right| \geq \frac{K}{n^d}$$

for all $n \geq 1$. Also, we have:

$$q_e(n) = \min_{m\in\mathbb{Z}} \left|\alpha - \frac{2m+1}{2n}\right| \geq \frac{K}{(2n)^d} = \frac{K}{2^d}\frac{1}{n^d}$$

for all $n \geq 1$. Putting these estimates into the upper bounds in propositions 5.4.1 and 5.4.2 gives us:

$$\begin{aligned} \frac{1}{|\sin n\alpha\pi|} &\leq \frac{1}{2}\frac{1}{nq(n)} \leq \frac{1}{2}\frac{n^d}{nK} = \frac{1}{2K}n^{d-1} \quad \text{and} \\ \frac{1}{|\cos n\alpha\pi|} &\leq \frac{1}{2}\frac{1}{nq_e(n)} \leq \frac{1}{2}\frac{2^d n^d}{nK} = \frac{2^{d-1}}{K}n^{d-1}. \end{aligned}$$

Following equations (5.1) and (5.2), we let

$$f(r, \pm\theta_0) = \sum_{n=0}^{\infty} f_n^{(\pm)} r^n,$$

where for any $R > 0$, we can find a constant $C_R > 0$ such that:

$$|f_n^{(\pm)}| \leq \frac{C_R}{R^n}$$

for all $n \geq 0$. Then, using the coefficient formulas in proposition 5.2.2, we can make the following estimates:

$$\begin{aligned}
|a_n| &= \left| \frac{f_n^{(+)} + f_n^{(-)}}{2\cos n\alpha\pi} \right| \\
&\leq \frac{1}{2}\left(|f_n^{(+)}| + |f_n^{(-)}|\right) \frac{1}{|\cos n\alpha\pi|} \\
&\leq \frac{C_R}{R^n} \frac{2^{d-1}}{K} n^{d-1} \\
&\leq \frac{2^{d-1} C_R}{K} \left(\frac{2^{d-1}}{R} \right)^n
\end{aligned}$$

and

$$\begin{aligned}
|b_n| &= \left| \frac{f_n^{(+)} - f_n^{(-)}}{2\sin n\alpha\pi} \right| \\
&\leq \frac{1}{2}\left(|f_n^{(+)}| + |f_n^{(-)}|\right) \frac{1}{|\sin n\alpha\pi|} \\
&\leq \frac{C_R}{R^n} \frac{1}{2K} n^{d-1} \\
&\leq \frac{C_R}{2K} \left(\frac{2^{d-1}}{R} \right)^n .
\end{aligned}$$

Since K and d are constants which depend only upon α, and since we may choose R to be arbitrarily large, we see that the conditions of theorem 2.2.6 are satisfied for our sequence of coefficients. Therefore, the function defined by:

$$u = f_0^{(+)} + \sum_{n=1}^{\infty} r^n \left(\frac{f_n^{(+)} + f_n^{(-)}}{2\cos n\theta_0} \cos n\theta + \frac{f_n^{(+)} - f_n^{(-)}}{2\sin n\theta_0} \sin n\theta \right)$$

is an entire harmonic function in the plane. To see that it satisfies our intersecting lines problem, we must now only show that it takes on the right values on the lines $\theta = \pm\theta_0$. First, we have:

$$\begin{aligned}
u(r,\theta_0) &= f_0^{(+)} + \sum_{n=1}^{\infty} r^n \left(\frac{f_n^{(+)} + f_n^{(-)}}{2} + \frac{f_n^{(+)} - f_n^{(-)}}{2} \right) \\
&= f_0^{(+)} + \sum_{n=1}^{\infty} f_n^{(+)} r^n \\
&= f(r,\theta_0).
\end{aligned}$$

Next, using the fact that $f_0^{(+)} = f_0^{(-)}$ since f has only a single value at the origin, we have:

$$\begin{aligned}
u(r, -\theta_0) &= f_0^{(+)} + \sum_{n=1}^{\infty} r^n \left(\frac{f_n^{(+)} + f_n^{(-)}}{2} - \frac{f_n^{(+)} - f_n^{(-)}}{2} \right) \\
&= f_0^{(-)} + \sum_{n=1}^{\infty} f_n^{(-)} r^n \\
&= f(r, -\theta_0).
\end{aligned}$$

Therefore, we have constructed an entire harmonic function which satisfies our intersecting lines problem. Finally, since the coefficients were completely determined by the data, this solution must be unique. $\square$

In particular, note that the combination of theorem 5.5.1 and proposition 5.5.2 tells us that angles of the form $\theta_0 = \alpha\pi$ where α is algebraic will all guarantee that the intersecting lines problem will always have a unique, entire solution given any entire data. While this is encouraging, we must bear in mind that the set of real algebraic numbers has Lebesgue measure 0. Therefore, while we have shown that there are infinitely many angles θ_0 which will guarantee an entire solution given any entire data, the set of such angles is still small, in some sense. We remedy this situation in the next section, however, with the introduction of a little machinery from transcendental number theory.

5.6 Extension of the Positive Results

In 1932, Mahler introduced a classification of all complex numbers into four disjoint sets, which are labelled A, S, T, and U. The idea was based on a generalization of the concept of approximation of a number by rationals. In this exposition, we will follow the notation of Baker, as given in [4].

Given a polynomial with integer coefficients,

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

we define the *height* of $p(x)$ to be:

$$\max\{|a_0|, |a_1|, \ldots, |a_n|\}.$$

In particular, note that for fixed values of n and h, there are only finitely many such polynomials of degree at most n with height at most h. So, given any complex number α, there will be a polynomial $p(x)$ with degree at most n and height at most h such that $|p(\alpha)|$ is minimized. We then define the quantity $\omega(n, h)$ by the following relation:

$$|p(\alpha)| = \frac{1}{h^{n\omega(n,h)}},$$

with the convention that $\omega(n, h) = \infty$ if $p(\alpha) = 0$. Of course, the most interesting situations happen when α is transcendental, or when we are guaranteed that $p(\alpha)$ will never vanish. Further to this, we define the quantities ω_n by:

$$\omega_n = \limsup_{h\to\infty} \omega(n, h),$$

so we are allowing polynomials with larger and larger heights. Next, we define:

$$\omega = \limsup_{n\to\infty} \omega_n.$$

Finally, we let $\nu = k$, where k is the smallest integer such that $\omega_k = \infty$. If ω_k is finite for all k, then we let $\nu = \infty$. A number α will then belong to one of the sets A, S, T, or U according to the behaviour of these quantities:

$$\begin{aligned}
\alpha \in A \quad &\Leftrightarrow \quad \omega = 0,\ \nu = \infty,\\
\alpha \in S \quad &\Leftrightarrow \quad 0 < \omega < \infty,\ \nu = \infty,\\
\alpha \in T \quad &\Leftrightarrow \quad \omega = \infty,\ \nu = \infty, \quad \text{and}\\
\alpha \in U \quad &\Leftrightarrow \quad \omega = \infty,\ \nu < \infty.
\end{aligned}$$

Roughly speaking, for an irrational number α, a larger value of ω means that α can be better approximated by rationals.

In [4], it is then shown that the set A is just the usual set of algebraic numbers. Therefore, A has Lebesgue measure 0 since the set of algebraic numbers is countable. It is also shown that both T and U have Lebesgue measure 0. So, in fact, most numbers are S numbers. We will now use that fact to show that in the same sense, most angles $\theta_0 = \alpha\pi$ will guarantee that the intersecting lines problem has an entire solution given any entire data.

Proposition 5.6.1 *Let α be an irrational number such that $0 < \alpha < \frac{1}{2}$, i.e. $0 < \theta_0 = \alpha\pi < \frac{\pi}{2}$. Also, suppose that for this α, we have that $\nu > 1$, or that ω_1 is finite. Then, we can find constants K and d such that:*

$$\left|\alpha - \frac{m}{n}\right| \geq \frac{K}{n^d}$$

for all integers m and n with $n > 0$.

Proof: Since we know that

$$\omega_1 = \limsup_{h\to\infty} \omega(1,h) < \infty,$$

we can find a constant $H > 0$ such that

$$\sup\{\omega(1,h) : h \geq H\} \leq \omega_1 + 1.$$

Bearing this in mind, we define:

$$M = \max\{1, \omega(1,1), \ldots, \omega(1,H-1), \omega_1 + 1\},$$

so we have $\omega(1,h) \leq M$ for all $h \geq 1$, and $M \geq 1$. In particular:

$$h^{\omega(1,h)} \leq h^M \implies \frac{1}{h^M} \leq \frac{1}{h^{\omega(1,h)}}$$

for all $h \geq 1$. Now, let $p(x) = hx - k$, where h and k are integers with $h > 0$ and $0 \leq k \leq h$. Then, the height of $p(x)$ is h, and so by the definition of $\omega(1,h)$, we have:

$$|p(\alpha)| \geq \frac{1}{h^{1\cdot\omega(1,h)}} \geq \frac{1}{h^M}.$$

Substituting in our choice of $p(x)$ and rearranging, we are left with:

$$|h\alpha - k| \geq \frac{1}{h^M} \implies \left|\alpha - \frac{k}{h}\right| \geq \frac{1}{h^{M+1}}$$

for all $h \geq 1$ and $0 \leq k \leq h$. Finally, we need to extend this inequality to be true for all $h \geq 1$ and all integers k. To see that this is true, note that since $0 < \alpha < \frac{1}{2}$,

$$\left|\alpha - \frac{k}{h}\right| \geq \frac{1}{h}$$

if $k < 0$ or $k > h$. Also, since $h \geq 1$ and $M \geq 1$, we know

$$h^M \geq 1 \implies h^{M+1} \geq h \implies \frac{1}{h} \geq \frac{1}{h^{M+1}}.$$

Putting these two facts together, we have that

$$\left|\alpha - \frac{k}{h}\right| \geq \frac{1}{h^{M+1}}$$

for all $h \geq 1$ and for all integers k. Switching notation to that of proposition 5.5.2, we have found constants $K = 1$ and $d = M + 1$ such that

$$\left|\alpha - \frac{m}{n}\right| \geq \frac{K}{n^d}$$

for all integers m and n with $n > 0$. $\square$

In the previous section, we proved that angles which are algebraic multiples of π will always guarantee that the intersecting lines problem will have an entire solution given any entire data. We can now extend this result to a much larger set, in terms of Lebesgue measure.

Theorem 5.6.2 *Consider the intersecting lines problem:*

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= f \text{ on } \theta = \pm\theta_0, \end{aligned}$$

where $0 < \theta_0 < \frac{\pi}{2}$*. Then, the set of angles* θ_0 *which guarantee that we can find an entire solution* u *given any entire data* f *has full measure, i.e. has measure* $\frac{\pi}{2}$*.*

Proof: Let α be an irrational number such that $0 < \alpha < \frac{1}{2}$, or $0 < \theta_0 = \alpha\pi < \frac{\pi}{2}$. Also, suppose that for this α, we know $\nu > 1$, or that ω_1 is finite. Then, by proposition 5.6.1, we know that we can find constants K and d such that

$$\left|\alpha - \frac{m}{n}\right| \geq \frac{K}{n^d}$$

for all integers m and n such that $n > 0$. Therefore, by proposition 5.5.2, we know that for this particular $\theta_0 = \alpha\pi$, the intersecting lines problem will always have a unique, entire solution given any entire data. To prove our theorem, note that the only angles which remain are those of the form $\theta_0 = \alpha\pi$ where α is irrational and $\nu \leq 1$, or $\theta_0 = \alpha\pi$ where α is rational. However, we know that if $\nu \leq 1$ then $\alpha \in U$ which has measure 0, and we know that the rational multiples of π also have measure 0. So, all of the angles which may not guarantee entire solutions given entire data live inside a set of measure 0. Therefore, the set of angles which do guarantee entire solutions given entire data must have full measure, or measure $\frac{\pi}{2}$. $\square$

5.7 An Example with No Entire Solution

Up until this point, the only examples of angles where we constructed entire data for which the intersecting lines problem did not have an entire solution have been the rational multiples of π. In theorem 5.6.2, the possibility arose that there may be more such angles, namely certain transcendental multiples of π. In this section, we will construct uncountably many of them.

Referring back to propositions 5.4.1 and 5.4.2, we see that the success of this construction will depend on our ability to find numbers which can be approximately extremely well by rational numbers. To this end, we prove the following proposition, which was inspired by Liouville's number, which is defined as:

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}.$$

The proof that α is transcendental found in [4] is the motivation behind the construction in the next proposition.

Proposition 5.7.1 *Let h be a function such that for any $C > 0$, we know that $h(n) > Cn$ for sufficiently large integers n. Then, we can construct uncountably many irrational numbers α such that the inequality*

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{h(b)}$$

has infinitely many coprime integer solutions (a, b).

Proof: First, we will demonstrate how to construct one of these numbers. We will use the same form as Liouville's number, but instead of using $n!$ in the exponent in the denominator, we will replace it by an initially unknown, increasing function from the natural numbers to the natural numbers:

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{10^{g(k)}}.$$

Note that the function $g(n)$ determines the decimal expansion of the number α. Now, given a function h, we wish to construct the function g and hence the number α such that the following inequality:

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{h(b)}$$

has infinitely many integer solutions (a, b) where a and b are relatively prime. To this end, we define the following sequences of integers:

$$a_n = 10^{g(n)} \sum_{k=1}^{n} \frac{1}{10^{g(k)}} \text{ and } b_n = 10^{g(n)}.$$

Note that since we will construct g to be increasing, each a_n is congruent to 1 modulo 10. Therefore, each pair (a_n, b_n) is relatively prime. Next, note that we have the following approximations:

$$\begin{aligned}
\left|\alpha - \frac{a_n}{b_n}\right| &= \left|\alpha - \sum_{k=1}^{n} \frac{1}{10^{g(k)}}\right| \\
&= \sum_{k=n+1}^{\infty} \frac{1}{10^{g(k)}} \\
&\leq \frac{1}{10^{g(n+1)}} \left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right) \\
&= \frac{10}{9} \frac{1}{10^{g(n+1)}}.
\end{aligned}$$

We can guarantee that these sequences of integers will be our desired solutions by imposing the condition that:

$$\left|\alpha - \frac{a_n}{b_n}\right| \leq \frac{10}{9} \frac{1}{10^{g(n+1)}} < \frac{1}{h(b_n)} = \frac{1}{h\left(10^{g(n)}\right)}. \tag{5.3}$$

This condition will give us a recursive definition for g which can be found by rearranging:

$$\frac{10}{9} \frac{1}{10^{g(n+1)}} < \frac{1}{h\left(10^{g(n)}\right)} \Leftrightarrow g(n+1) > \frac{\log\left(\frac{10}{9} h\left(10^{g(n)}\right)\right)}{\log 10}.$$

Thus, we can take $g(1)$ to be any positive integer, and then let:

$$g(n+1) = \max\left(\left\lceil \frac{\log\left(\frac{10}{9} h\left(10^{g(n)}\right)\right)}{\log 10} \right\rceil + 1, g(n) + 1\right).$$

This insures that g is increasing and that inequality (5.3) is satisfied. Since $g(1)$ is arbitrary, given the function h, we can construct at least countably many numbers α such that the inequality:

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{h(b)}$$

has infinitely many coprime integer solutions (a, b). Next, we will show that each such α which we have constructed is irrational.

Let α be any number constructed by the above process, and let $g(n)$ be the corresponding function used in the construction. We will now make use of the fact that a number is rational if and only if its decimal expansion is eventually repeating. Since we know that for any constant C, $h(n) > Cn$ for sufficiently large n, we can make the particular choice of

$$h\left(10^{g(n)}\right) > 9 \cdot 10^{N-1} \cdot 10^{g(n)}$$

for each $N \geq 1$, for sufficiently large n. Multiplying through by $\frac{10}{9}$ gives:

$$\frac{10}{9} h\left(10^{g(n)}\right) > 10^{g(n)+N}.$$

We can then take the logarithm of both sides and then add 1 to obtain:

$$\frac{\log\left(\frac{10}{9} h\left(10^{g(n)}\right)\right)}{\log 10} + 1 > g(n) + N + 1.$$

However, by the definition of $g(n)$, we know that

$$g(n+1) \geq \frac{\log\left(\frac{10}{9} h\left(10^{g(n)}\right)\right)}{\log 10} + 1 > g(n) + N + 1.$$

Since this is true for each $N \geq 1$ for n sufficiently large, we see that

$$\sup_n (g(n+1) - g(n)) = \infty.$$

Therefore, α cannot possibly have a repeating decimal expansion, and so it must be irrational.

Finally, we will show that this construction technique will produce uncountably many such numbers α. Since the choice of $g(1)$ was arbitrary, we know that there are at least countably many such numbers α, so suppose that these are all of them, and label them α_1, α_2, $\cdots$. We will now proceed to construct an α in the same manner as above for which we can find infinitely many coprime integer solutions to:

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{h(b)}$$

and which is not in our list. For each i, let k_i be the position of the i^{th} 1 in α_i. Then, define the function $g(n)$ recursively by $g(1) = k_1 + 1$, and:

$$g(n+1) = \max\left(\left\lceil \frac{\log\left(\frac{10}{9}h\left(10^{g(n)}\right)\right)}{\log 10}\right\rceil + 1, g(n)+1, k_{n+1}+1\right).$$

This means that $g(n)$ is increasing, and if we let:

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{10^{g(n)}},$$

then as before, α will be irrational, and we will be able to find infinitely many coprime integer solutions to:

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{h(b)}.$$

However, for each $n \geq 1$, the position of the n^{th} 1 in α is greater than or equal to $k_n + 1$, which is strictly greater than k_n, or the position of the n^{th} 1 in α_n. Therefore, $\alpha \neq \alpha_n$ for any n, and so there must be uncountably many such α. □

Finally, we will give an example of entire data f for which we can construct uncountably many angles θ_0 such that the intersecting lines problem has no entire solutions. As our data, we will choose $f = ye^x$, and we will leave our angle $\theta_0 = \alpha\pi$ undetermined for the moment. Evaluating f on $\theta = \theta_0$ gives:

$$f\Big|_{\theta=\theta_0} = r\sin\theta_0 \sum_{n=0}^{\infty} \frac{(r\cos\theta_0)^n}{n!} = \sum_{n=1}^{\infty} \frac{(\cos\theta_0)^{n-1}\sin\theta_0}{(n-1)!} r^n.$$

Similarly, we have:

$$f\Big|_{\theta=-\theta_0} = -\sum_{n=1}^{\infty} \frac{(\cos\theta_0)^{n-1}\sin\theta_0}{(n-1)!} r^n.$$

This means that

$$b_n = \frac{f_n^{(+)} - f_n^{(-)}}{2\sin n\theta_0} = \frac{1}{\sin n\theta_0} \frac{(\cos\theta_0)^{n-1}\sin\theta_0}{(n-1)!}.$$

Now, since we know that

$$\frac{1}{|\sin n\theta_0|} = \frac{1}{|\sin n\alpha\pi|} \geq \frac{1}{\pi}\frac{1}{nq(n)}$$

where

$$q(n) = \min_{m \in \mathbb{Z}} \left| \alpha - \frac{m}{n} \right|,$$

we arrive at the following lower bound:

$$|b_n| = \frac{|\cos\theta_0|^{n-1}|\sin\theta_0|}{(n-1)!} \frac{1}{|\sin n\theta_0|} \geq \frac{|\cos\theta_0|^{n-1}|\sin\theta_0|}{(n-1)!} \frac{1}{\pi n q(n)} = \frac{|\cos\theta_0|^{n-1}|\sin\theta_0|}{\pi n! q(n)}.$$

Now, suppose that u is an entire solution to our intersecting lines problem with the data as chosen above. We can write:

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta \right).$$

Then, pick any $R > 0$. We should be able to find a constant $C_R > 0$ such that

$$|b_n| \leq \frac{C_R}{R^n}$$

for all $n \geq 1$. However, this means that:

$$\frac{|\cos\theta_0|^{n-1}|\sin\theta_0|}{\pi n! q(n)} \leq |b_n| \leq \frac{C_R}{R^n}$$

for all $n \geq 1$. We can rearrange this to see that

$$q(n) \geq \frac{|\cos\theta_0|^{n-1}|\sin\theta_0| R^n}{C_R \pi n!}$$

for all $n \geq 1$. We will now construct a number α and hence our angle θ_0 so that this statement is not true. To this end, define the function $h(n)$ by:

$$h(n) = \pi n!.$$

Clearly, for any constant $K > 0$, we have $h(n) > Kn$ for all sufficiently large n. Therefore, by proposition 5.7.1, we can construct uncountably many irrational numbers α such that the inequality

$$\left| \alpha - \frac{m}{n} \right| < \frac{1}{h(n)}$$

has infinitely many solutions in coprime integers. Pick any such α, let the sequence of the denominators of these solutions be given by $n_1, n_2, n_3, \ldots$, and let $\theta_0 = \alpha\pi$. This means that

$$q(n_k) \le \frac{1}{h(n_k)}$$

for all $k \ge 1$, and so:

$$\frac{|\cos\theta_0|^{n_k-1}|\sin\theta_0|R^{n_k}}{C_R \pi n_k!} \le q(n_k) \le \frac{1}{h(n_k)} = \frac{1}{\pi n_k!}$$

for each $k \ge 1$. Now, this can be rearranged to get:

$$|R\cos\theta_0|^{n_k}\tan\theta_0 \le C_R$$

for each $k \ge 1$. However, since this must be true for any $R > 0$, we can certainly choose an R such that $|R\cos\theta_0| > 1$. Then, the above inequality will not be true for k sufficiently large, and so it will not be the case that for any $R > 0$, we can find a constant $C_R > 0$ such that

$$|b_n| \le \frac{C_R}{R^n}$$

for each $n \ge 1$. Therefore, for any of our constructed angles and the given data, the intersecting lines problem does not have an entire solution.

5.8 Density of Bad Angles

To summarize our work on the intersecting lines problem thus far, we have found that the existence of entire solutions given any entire data depends on the angle between the lines and the x-axis. We found a set of angles of full measure for which the problem will always possess an entire solution given entire data. For angles which are rational multiples of π, we found that there will not always be entire solutions. However, the corresponding problem will not always possess a polynomial solution given any polynomial data, and so this is not really a case of interest. What we wish to investigate now are the rest of the bad angles, of which we constructed uncountably many in the previous section.

We know that the set of angles which will not always guarantee us an entire solution will be uncountable and have Lebesgue measure 0. We will now show that this set of angles is also dense in $[0, \pi/2]$.

Proposition 5.8.1 *The set of angles $\theta_0 = \alpha\pi$ with α irrational for which the problem*

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= f \text{ on } \theta = \pm\theta_0 \end{aligned}$$

does not have an entire solution u for all entire data f is dense in $[0, \pi/2]$.

Proof: We will make extensive use of our example in the previous section, where for the data $f = ye^x$, we constructed uncountably many angles for which the intersecting lines problem did not have an entire solution. For this proof, fix $\theta_0 = \alpha\pi$ to be any one of those angles constructed in the manner of proposition 5.7.1. Our strategy will then be to show that for all rational numbers of the form:

$$s = \frac{p}{10^q}$$

where p is an integer and q is a positive integer, the intersecting lines problem with the angle $(s + \alpha)\pi$ will fail to have an entire solution for the same data, $f = ye^x$. Since rational numbers of this form are dense in the real numbers, this will show that the set of angles which do not always guarantee us an entire solution will have a dense subset, and hence be dense itself.

Let

$$s = \frac{p}{10^q},$$

where p is any integer and q is any positive integer. Choose any $q' > q$ and let $n = 10^{q'}$. Then, note that ns is an integer, and:

$$\begin{aligned} \left|\sin\left(n\left(s+\alpha\right)\pi\right)\right| &= \left|\sin ns\pi \cdot \cos n\alpha\pi + \cos ns\pi \cdot \sin n\alpha\pi\right| \\ &= \left|\cos ns\pi \cdot \sin n\alpha\pi\right| \\ &= \left|\sin n\alpha\pi\right|. \end{aligned}$$

Now, let $\theta_0 = (s+\alpha)\pi$, and suppose that the problem

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= ye^x \text{ on } \theta = \pm\theta_0 \end{aligned}$$

has an entire solution u. Since it is entire, we know that we are able to write

$$u = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right),$$

where for every $R > 0$, we can find a constant $C_R > 0$ such that

$$|a_n| \leq \frac{C_R}{R^n} \text{ and } |b_n| \leq \frac{C_R}{R^n}$$

for all $n \geq 0$. As before, we have an explicit formula for the coefficients, and our particular interest lies with the b_n:

$$b_n = \frac{1}{\sin n\theta_0} \frac{(\cos\theta_0)^{n-1}\sin\theta_0}{(n-1)!}.$$

Also as before, we can find infinitely many coprime integer solutions to the inequality

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{\pi n!}.$$

Let the sequence of denominators of these solutions be $n_1, n_2, \ldots$, and note by the construction technique of proposition 5.7.1 that these integers will all be powers of 10.

Therefore, for sufficiently large k, $n_k s$ will be an integer, and we will have

$$\begin{aligned}
|b_{n_k}| &= \frac{1}{|\sin n_k\theta_0|}\frac{|\cos\theta_0|^{n_k-1}|\sin\theta_0|}{(n_k-1)!} \\
&= \frac{1}{|\sin\left(n_k\left(r+\alpha\right)\pi\right)|}\frac{|\cos\theta_0|^{n_k-1}|\sin\theta_0|}{(n_k-1)!} \\
&= \frac{1}{|\sin n_k\alpha\pi|}\frac{|\cos\theta_0|^{n_k-1}|\sin\theta_0|}{(n_k-1)!} \\
&\geq \frac{|\cos\theta_0|^{n_k-1}|\sin\theta_0|}{\pi n_k! q(n_k)} \\
&\geq |\cos\theta_0|^{n_k-1}|\sin\theta_0|,
\end{aligned}$$

since

$$q(n_k) = \min_{m\in\mathbb{Z}}\left|\alpha - \frac{m}{n_k}\right| \leq \frac{1}{\pi n_k!}.$$

Now, pick any R such that $|R\cos\theta_0| > 1$. Then, we should be able to find a constant $C_R > 0$ such that

$$|b_n| \leq \frac{C_R}{R^n}$$

for all $n \geq 0$. Using our above bound, this means:

$$|\cos\theta_0|^{n_k-1}|\sin\theta_0| \leq \frac{C_R}{R^{n_k}} \implies |R\cos\theta_0|^{n_k}|\tan\theta_0| \leq C_R,$$

which will be false for sufficiently large k. Therefore, the problem

$$\begin{aligned}
\triangle u &= 0 \text{ in } \mathbb{R}^2 \\
u &= ye^x \text{ on } \theta = \pm\theta_0
\end{aligned}$$

will not have an entire solution u, and so the angle

$$\theta_0 = \left(\frac{p}{10^q} + \alpha\right)\pi$$

will be bad. Since p and q were arbitrary, we have shown the existence of a dense subset of bad angles, and so the set of bad angles itself must also be dense. $\square$

Chapter 6

The Infinite Strip in Two Dimensions

This chapter is somewhat different from the previous chapters in that we were unable to determine whether or not our problem always possessed an entire solution given entire data. However, the search led to many interesting ideas and related questions, and we will discuss them here. The specific question that we were interested in answering was: given any entire functions f and g, can we construct an entire function u which satisfies

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= f \text{ on } y = 0 \\ u &= g \text{ on } y = 1? \end{aligned}$$

Note that we can pose the question using only one function as our entire data. If we let:

$$h(x, y) = (1 - y)f(x) + yg(x),$$

then $h(x,0) = f(x)$ and $h(x,1) = g(x)$, and our infinite strip problem becomes:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= h \text{ on } y(y-1) = 0. \end{aligned}$$

6.1 Review of the Polynomial Problem

Suppose that we restrict ourselves to looking at polynomial data. Then, we can show that the infinite strip problem will always have a polynomial solution. Our boundary surface is given by the polynomial $y(y-1) = y^2 - y = 0$, and this has a leading order term of y^2. Since this does not change sign, we can use lemma 3 of [6] by Brelot and Choquet to say that our boundary polynomial is not a harmonic divisor. Therefore, using the operator method from [16] and [17], we see that the polynomial problem in the infinite strip will always possess a polynomial solution.

6.2 An Initial Simplification

Our first step in searching for entire solutions will be to make a slight simplification of the problem. The initial formulation of the infinite strip problem involved prescribing data on both of the lines $y = 0$ and $y = 1$. However, we will show that being able to prescribe data on just one of these lines is enough.

Proposition 6.2.1 *Suppose that for any entire function f, we can find an entire function u which satisfies:*

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= f \text{ on } y = 0 \\ u &= 0 \text{ on } y = 1. \end{aligned}$$

Then, for any entire functions f and g, we can find an entire function u which satisfies:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= f \text{ on } y = 0 \\ u &= g \text{ on } y = 1. \end{aligned}$$

Proof: Given an entire function f, we know that we can find an entire function u which satisfies:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= f \text{ on } y = 0 \\ u &= 0 \text{ on } y = 1. \end{aligned}$$

Then, the function $v(x, y) = u(x, 1 - y)$ will also be entire, and it will satisfy:

$$\begin{aligned} \triangle v &= 0 \text{ in } \mathbb{R}^2 \\ v &= 0 \text{ on } y = 0 \\ v &= f \text{ on } y = 1. \end{aligned}$$

We can then construct a solution to the full infinite strip problem by finding an entire function v which satisfies:

$$\begin{aligned} \triangle v &= 0 \text{ in } \mathbb{R}^2 \\ v &= f \text{ on } y = 0 \\ v &= 0 \text{ on } y = 1, \end{aligned}$$

and an entire function w which satisfies:

$$\begin{aligned} \triangle w &= 0 \text{ in } \mathbb{R}^2 \\ w &= g \text{ on } y = 0 \\ w &= 0 \text{ on } y = 1. \end{aligned}$$

Then, the function $u(x, y) = v(x, y) + w(x, 1 - y)$ will be an entire function which satisfies:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= f \text{ on } y = 0 \\ u &= g \text{ on } y = 1. \end{aligned}$$

$\square$

Note that there is nothing special about us choosing the line $y = 0$ as the line where we prescribe data. With the simple transformation $y \mapsto 1 - y$, we could have just as easily chosen the line $y = 1$. Also note that we can never guarantee uniqueness for these problems, as the entire function $u = e^{\pi x} \sin \pi y$ satisfies:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= 0 \text{ on } y = 0 \\ u &= 0 \text{ on } y = 1. \end{aligned}$$

6.3 Reformulation as an Infinite Differential Operator

Our first attempt will be to consider the infinite strip problem where we prescribe data on the line $y = 1$:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= 0 \text{ on } y = 0 \\ u &= f \text{ on } y = 1. \end{aligned}$$

Following our success with the half space problems, we will try to write our solution u as a series in y with coefficients which are functions of x. Again, assuming u is entire leads to the following series:

$$u = \sum_{k=0}^{\infty} \frac{(-1)^k g_0^{(2k)}(x)}{(2k)!} y^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k g_1^{(2k)}(x)}{(2k+1)!} y^{2k+1}.$$

However, since we know that $u = 0$ when $y = 0$, this means that $g_0 \equiv 0$, or

$$u = \sum_{k=0}^{\infty} \frac{(-1)^k g_1^{(2k)}(x)}{(2k+1)!} y^{2k+1}.$$

By evaluating this on $y = 1$ and using our other boundary condition, we arrive at:

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k g_1^{(2k)}(x)}{(2k+1)!}. \tag{6.1}$$

This is a differential equation which depends upon all of the derivatives of the unknown function, and it can be written in a slightly more compact form. We know that the power series for the sinc function is:

$$\operatorname{sinc}(x) = \frac{\sin x}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!}.$$

So, if we replace the variable x by the derivative operator D, we obtain a differential operator:

$$\operatorname{sinc}(D) = \sum_{k=0}^{\infty} \frac{(-1)^k D^{2k}}{(2k+1)!},$$

which acts on the space of entire functions in the natural way. Therefore, equation (6.1) can be written as:

$$\operatorname{sinc}(D) g_1 = f. \tag{6.2}$$

The homogeneous version of these types of problems were considered in [19] by Ritt. For this non-homogeneous problem, we are guaranteed that the left hand side of equation (6.2) is well-defined by the following proposition.

Proposition 6.3.1 *Let g be an entire function. Then* $\operatorname{sinc}(D)g$ *is an entire function.*

Proof: Since g is entire, we can write it as

$$g(x) = \sum_{n=0}^{\infty} g_n x^n.$$

We can also differentiate it as many times as we please by using equation (2.1) to get the following series:

$$g^{(2k)}(x) = \sum_{n=0}^{\infty} (n+1)_{2k} g_{n+2k} x^n.$$

Formally substituting this into the series for $\mathrm{sinc}(D)g$, we have:

$$\begin{aligned}
\mathrm{sinc}(D)g &= \sum_{k=0}^{\infty} \frac{(-1)^k g^{(2k)}(x)}{(2k+1)!} \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(n+1)_{2k} g_{n+2k} x^n \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^k}{2k+1}\frac{(n+2k)!}{n!(2k)!} g_{n+2k} x^n \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^k}{2k+1}\binom{n+2k}{n} g_{n+2k} x^n.
\end{aligned}$$

To finish the proof, we will show that this series is absolutely convergent on compact subsets, rearrange the series to get expressions for the coefficients of $\mathrm{sinc}(D)g$, and then estimate them to show that in fact this will define an entire function.

First, pick any $R > 1$. Since g is entire, we can find a constant $C_{3R} > 0$ such that

$$|g_n| \leq \frac{C_{3R}}{(3R)^n}$$

for all $n \geq 0$. Then, for $|x| \leq R$, we have the following estimate:

$$\begin{aligned}
\left| \frac{(-1)^k}{2k+1}\binom{n+2k}{n} g_{n+2k} x^n \right| &= \frac{1}{2k+1}\binom{n+2k}{n} |g_{n+2k}||x|^n \\
&\leq \frac{1}{2k+1} 2^{n+2k} \frac{C_{3R}}{(3R)^{n+2k}} R^n \\
&\leq \frac{1}{2k+1} 2^{n+2k} \frac{C_{3R}}{(3R)^{n+2k}} R^{n+2k} \\
&= \frac{C_{3R}}{2k+1} \left(\frac{2}{3}\right)^{n+2k} \\
&\leq C_{3R} \left(\frac{2}{3}\right)^{n+2k}.
\end{aligned}$$

However, this means that:

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left|\frac{(-1)^k}{2k+1}\binom{n+2k}{n}g_{n+2k}x^n\right| &\leq \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}C_{3R}\left(\frac{2}{3}\right)^{n+2k} \\
&= C_{3R}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{2k}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n},
\end{aligned}$$

which is finite. Therefore, the formal sum for $\operatorname{sinc}(D)g$ is absolutely convergent for all x on compact subsets. This permits us to rearrange the series to obtain:

$$\begin{aligned}
\operatorname{sinc}(D)g &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^k}{2k+1}\binom{n+2k}{n}g_{n+2k}x^n \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\binom{n+2k}{n}g_{n+2k}\right)x^n.
\end{aligned}$$

Finally, we will estimate these coefficients to show that $\operatorname{sinc}(D)g$ is entire.

For any $R > 1$, we can find a constant $C_{2R} > 0$ such that

$$|g_n| \leq \frac{C_{2R}}{(2R)^n}$$

for all $n \geq 0$, since g is entire. This allows us to bound the coefficient as follows:

$$\begin{aligned}
\left|\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\binom{n+2k}{n}g_{n+2k}\right| &\leq \sum_{k=0}^{\infty}\frac{1}{2k+1}\binom{n+2k}{n}|g_{n+2k}| \\
&\leq \sum_{k=0}^{\infty}2^{n+2k}\frac{C_{2R}}{(2R)^{n+2k}} \\
&= \sum_{k=0}^{\infty}\frac{C_{2R}}{R^{n+2k}} \\
&= \frac{C_{2R}}{R^n}\sum_{k=0}^{\infty}\frac{1}{R^{2k}} \\
&= \frac{C_{2R}}{R^n}\cdot\frac{R^2}{R^2-1} \\
&= \frac{C'_R}{R^n},
\end{aligned}$$

where we know that this geometric series converges since $R > 1$, and where we have collected simple R dependence into C'_R. Therefore, for every $R > 1$, we have found a constant $C_R' > 0$ such that

$$\left|\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \binom{n+2k}{n} g_{n+2k}\right| \leq \frac{C_R'}{R^n}$$

for all $n \geq 0$. Since this can be trivially extended to be true for all $R > 0$, we see that $\text{sinc}(D)g$ will be an entire function. □

Based on this proposition, we see that the operator $\text{sinc}(D)$ maps entire functions to entire functions. The question now becomes whether or not this map is surjective. The importance of this question is shown in the next theorem.

Theorem 6.3.2 *Consider the infinite strip problem:*

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= 0 \text{ on } y = 0 \\ u &= f \text{ on } y = 1, \end{aligned}$$

where f is an entire function. Then, we will always be able to find an entire solution u given any entire data f if and only if the operator $\text{sinc}(D)$ *is a surjective map from the space of entire functions to itself.*

Proof: First, suppose that for every entire function f, we can find an entire function u which satisfies the infinite strip problem with f as data. This can be written as a series in y as:

$$u(x, y) = \sum_{k=0}^{\infty} \frac{(-1)^k g^{(2k)}(x)}{(2k+1)!} y^{2k+1}.$$

Note that we can recover the function g:

$$g(x) = \left.\frac{\partial u}{\partial y}\right|_{y=0}.$$

Therefore, g will be an entire function, and by evaluating u on $y = 1$, we see that $\text{sinc}(D)g = f$. Since f was chosen arbitrarily, the map $\text{sinc}(D)$ is surjective.

Next, suppose that the map $\operatorname{sinc}(D)$ is surjective. Let f be any entire function, and let g be such that $\operatorname{sinc}(D)g = f$. Then, consider the function

$$u = \sum_{k=0}^{\infty} \frac{(-1)^k g^{(2k)}(x)}{(2k+1)!} y^{2k+1}.$$

By proposition 3.4.2, we know this this will be an entire function, and by proposition 3.4.3 we know that it will be harmonic. Evaluating along $y = 0$ clearly gives 0, and evaluating along $y = 1$ gives:

$$u(x,1) = \sum_{k=0}^{\infty} \frac{(-1)^k g^{(2k)}(x)}{(2k+1)!} = \operatorname{sinc}(D)g = f.$$

Therefore, u is an entire function which satisfies the infinite strip problem with f given as data. Since f was arbitrarily chosen, the infinite strip problem will have an entire solution for any entire data. $\square$

Unfortunately, we were unable to determine whether or not the operator $\operatorname{sinc}(D)$ was surjective. Other unanswered questions related to infinite differential operators will be discussed in chapter 7.

6.4 Reformulation as a Difference Equation

It may be possible to view the question of the surjectivity of the $\operatorname{sinc}(D)$ operator in a different light. Through a series of transformations which preserve entire functions, we can arrive at a difference equation instead of a differential equation, although the sacrifice to be made is that we must now allow complex variables. However, the proof of proposition 6.3.1 still works if the real variable x is replaced by the complex variable z, and so $\operatorname{sinc}(D)$ will map complex entire functions to complex entire functions. Let us start with the following equation, where f and g are complex entire functions:

$$\operatorname{sinc}(D)g = f \implies \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} g^{(2k)}(z) = f(z).$$

Differentiating with respect to z gives us:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} g^{(2k+1)}(z) = f'(z) \implies \sin(D)g(z) = f'(z).$$

Next, we can convert the $\sin(D)$ into an exponential form:

$$\frac{e^{iD} - e^{-iD}}{2i} g(z) = f'(z) \implies e^{iD}g(z) - e^{-iD}g(z) = 2if'(z).$$

We can now make use of a form of Taylor's Theorem which states that $e^{aD}g(z) = g(z+a)$:

$$g(z+i) - g(z-i) = 2if'(z).$$

Finally, we will perform a series of simple scale changes and relabellings to bring this into a nice form. First, we let $g_2(z) = g(2iz)$ and $2if'(z) = f_2(z)$. Our equation then becomes:

$$g_2\left(\frac{z}{2i} + \frac{1}{2}\right) - g_2\left(\frac{z}{2i} - \frac{1}{2}\right) = f_2(z).$$

Then, we make the change of variable $w = \frac{z}{2i}$, which brings us to:

$$g_2\left(w + \frac{1}{2}\right) - g_2\left(w - \frac{1}{2}\right) = f_2(2iw).$$

Finally, we let $g_3(w) = g_2\left(w - \frac{1}{2}\right)$, which gives us:

$$g_3(w+1) - g_3(w) = f_2(2iw).$$

Reclaiming our original letters, being able to find a complex entire function g which satisfies

$$g(z+1) - g(z) = f(z) \tag{6.3}$$

for any complex entire function f will mean that the $\mathrm{sinc}(D)$ operator is surjective, since we can just invert through our series of transformations.

This problem has been studied before, and in particular, it is at the heart of algorithms for indefinite summation in modern computer algebra systems, which can be found in [22]. We will show how to solve equation (6.3) for polynomials f and g, but we were unable to extend these results to entire functions. A discussion of the difficulties can be found in chapter 7.

If f is a polynomial, the most direct way to solve equation (6.3) is to make use of a different basis of the space of polynomials. First, we will define some notation for a falling factorial:

$$(x)^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+1),$$

with the convention that $(x)^{\underline{0}} = 1$. For each k, $(x)^{\underline{k}}$ will have degree k, and so we can use these as a basis for polynomials. The coefficients for this change of basis are the well known Stirling numbers of the second kind, denoted by:

$$\left\{ {n \atop k} \right\},$$

where n and k are non-negative integers with $k \leq n$. A description of their properties can be found in [10]. In particular, the change of basis is given by:

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)^{\underline{k}}.$$

The reason for choosing this basis is the following formula, which is a discrete analog of the derivative:

$$\begin{aligned}
(x+1)^{\underline{k}} - (x)^{\underline{k}} &= (x+1)x\cdots(x-k+2) - x(x-1)\cdots(x-k+1) \\
&= x(x-1)\cdots(x-k+2)(x+1-(x-k+1)) \\
&= kx(x-1)\cdots(x-k+2) \\
&= k(x)^{\underline{k-1}}.
\end{aligned}$$

Therefore, suppose that we are given a polynomial $f(x)$, and we wish to solve equation (6.3). Then, if we write $f(x)$ as:

$$f(x) = \sum_{n=0}^{N} f_n x^n,$$

we can perform a change of basis to obtain

$$\begin{aligned} f(x) &= \sum_{n=0}^{N} f_n x^n \\ &= \sum_{n=0}^{N} \sum_{k=0}^{n} f_n \left\{ {n \atop k} \right\} (x)^{\underline{k}} \\ &= \sum_{k=0}^{N} \left(\sum_{n=k}^{N} f_n \left\{ {n \atop k} \right\} \right) (x)^{\underline{k}}. \end{aligned}$$

So, using the discrete derivative property of the falling factorial, we can write down a solution directly:

$$g(x) = \sum_{k=0}^{N} \frac{1}{k+1} \left(\sum_{n=k}^{N} f_n \left\{ {n \atop k} \right\} \right) (x)^{\underline{k+1}}.$$

Note that this solution will not be unique, since we may add an arbitrary constant, just like with indefinite integration. In fact, the processes of applying a difference operator to a function and trying to solve equation (6.3) share many algebraic properties with the usual operations of differentiation and integration, and an explanation of some of these similarities can be found in [22].

As straightforward as solving equation (6.3) was for polynomials, extending the solution to entire functions proved to be difficult. This will be discussed in chapter 7.

6.5 A Formula with Complex Variables

Our lack of success in the previous sections might lead us to think that the infinite strip problem might not always possess an entire solution given any entire data. After our experience with the heat equation, we might suspect that the infinite strip problem will only possess an entire solution if the given data satisfies some sort of growth condition. In this section, we will give some compelling evidence which suggests that this is not the case by finding explicit solutions for data which seems to grow as quickly as we please.

Suppose we have an entire complex function which satisfies the following conditions:

$$f(z+2i) = f(z) \text{ for all } z \tag{6.4}$$

$$f(z) = \overline{f(\overline{z})} \text{ for all } z. \tag{6.5}$$

Note that this second condition is the same as that used to extend a complex analytic function across the real axis when it is real valued there. Then, if we consider the problem:

$$\begin{aligned} \triangle u &= 0 \text{ in } \mathbb{R}^2 \\ u &= f \text{ on } y = 0 \\ u &= 0 \text{ on } y = 1, \end{aligned}$$

we can write down a solution in a very compact way:

$$u(x,y) = \Re\left((1 + i(x+iy))f(x+iy)\right). \tag{6.6}$$

Certainly, this will be an entire function by proposition 2.2.1. Also, it will be harmonic. To see that it satisfies the boundary conditions, we first note two facts. First, on $y = 0$, we have:

$$f(x+iy) = \overline{f(x-iy)} \implies f(x) = \overline{f(x)} \implies f(x) \in \mathbb{R}.$$

Next, on $y = 1$, we can perform the same calculation, also making use of the periodicity of f to see:

$$f(x+i) = \overline{f(x-i)} = \overline{f(x+i)} \implies f(x+i) \in \mathbb{R}.$$

Therefore, on $y = 0$ we have

$$u(x,0) = \Re\left((1+ix)f(x)\right) = f(x),$$

and on $y = 1$ we have

$$u(x,1) = \Re\left((1 + i(x+i))f(x+i)\right) = f(x+i) - f(x+i) = 0.$$

So, equation (6.6) works as claimed.

As particular examples of functions which satisfy conditions (6.4) and (6.5), we will consider towers of exponentials. Denoting e^x by $\exp(x)$ for clarity, look at the functions:

$$f(x) = \exp(\pi x),\ f(x) = \exp(\exp(\pi x)),\ f(x) = \exp(\exp(\exp(\pi x))), \ldots$$

These will all satisfy the periodicity condition (6.4) due to the innermost $\exp(\pi x)$, and they will all satisfy condition (6.5) because they are complex analytic functions which assume real values on the real axis. However, by using equation (6.6), we see that we can find solutions to the infinite strip problem for data which can grow seemingly as quickly as we please.

So, we should not be discouraged by our lack of success in the previous sections, as it does not appear to be the case that a growth condition on the data for the infinite strip problem will determine the existence of an entire solution.

Chapter 7

Summary of Open Problems

We will conclude our exposition with some unanswered questions which have arisen as a result of our work. Although it is somewhat unsatisfying to have so many loose ends, each of the following problems has generated interesting methods which may yet produce results.

7.1 Regarding Legendre Polynomials

In chapter 1, we remarked that the difficulties in completing the classification of quadric surfaces in three dimensions for the polynomial problem were related to questions about the roots of orthogonal polynomials. Problems of this sort were mentioned briefly in [1], where the authors voiced the opinion that settling these questions would very likely be non-trivial.

The problem which we attempted was determining whether or not two Legendre polynomials could have a non-zero root in common. Trying to answer this question led to the discovery of the following formula. Suppose $\alpha \neq 0$ is a root of $P_n(x)$, where $P_n(x)$ is the Legendre polynomial of degree n. Then, for $n \geq 2$, we have the following factorization:

$$P_n(x) = \frac{1}{nP_{n-1}(\alpha)}(x-\alpha)\sum_{k=0}^{n-1}(2k+1)P_k(\alpha)P_k(x).$$

Note that the factor $P_{n-1}(\alpha)$ in the denominator is not a cause for concern because the roots of consecutive orthogonal polynomials are always interleaved. Therefore, since α is a root of $P_n(x)$, it cannot be a root of $P_{n-1}(x)$. This factorization formula can be verified by expanding the right hand side using the three term recurrence relation for Legendre polynomials:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0.$$

However, the argument which was used to discover this factorization uses an interesting technique which may still prove useful in showing that Legendre polynomials cannot share non-zero roots. We will illustrate it here.

Computation has shown that if two Legendre polynomials share a non-zero root, then one of the polynomials must have degree at least 500. So, it is reasonable to conjecture that in fact two Legendre polynomials cannot share a non-zero root. Since they are polynomials with rational coefficients, it is natural to ask about their irreducibility over the rational numbers, or in particular whether they have any non-zero rational roots at all. This led us to the following argument.

Suppose $\alpha \neq 0$ is a rational root of $P_n(x)$. Then, we can write $P_n(x) = (x-\alpha)f(x)$, where f is a polynomial of degree $n-1$. Since there is a unique Legendre polynomial of each degree, we can use them as a basis of the space of polynomials. This allows us to write f as:

$$f(x) = \sum_{k=0}^{n-1} f_k P_k(x),$$

for some constants f_k. Next, we can take the three term recurrence relation for the Legendre polynomials and rearrange it into the following form for $n \geq 1$:

$$xP_n(x) = \frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x).$$

Using this and our factorization of $P_n(x)$, we can perform a calculation:

$$\begin{aligned}
P_n(x) &= (x-\alpha)f(x) \\
&= (x-\alpha)\sum_{k=0}^{n-1} f_k P_k(x) \\
&= \sum_{k=0}^{n-1} f_k x P_k(x) - \alpha \sum_{k=0}^{n-1} f_k P_k(x) \\
&= f_0 x P_0(x) + \sum_{k=1}^{n-1} f_k \left(\frac{k+1}{2k+1} P_{k+1}(x) + \frac{k}{2k+1} P_{k-1}(x) \right) - \alpha \sum_{k=0}^{n-1} f_k P_k(x) \\
&= f_0 P_1(x) + \sum_{k=1}^{n-1} \frac{k+1}{2k+1} f_k P_{k+1}(x) + \sum_{k=1}^{n-1} \frac{k}{2k+1} f_k P_{k-1}(x) - \alpha \sum_{k=0}^{n-1} f_k P_k(x) \\
&= \sum_{k=0}^{n-1} \frac{k+1}{2k+1} f_k P_{k+1}(x) + \sum_{k=1}^{n-1} \frac{k}{2k+1} f_k P_{k-1}(x) - \alpha \sum_{k=0}^{n-1} f_k P_k(x) \\
&= \sum_{k=1}^{n} \frac{k}{2k-1} f_{k-1} P_k(x) + \sum_{k=0}^{n-2} \frac{k+1}{2k+3} f_{k+1} P_k(x) - \alpha \sum_{k=0}^{n-1} f_k P_k(x)
\end{aligned}$$

However, we know that the Legendre polynomials are linearly independent, so we can equate the coefficients of $P_k(x)$ on both sides of the equation to obtain a system of equations for the coefficients f_k. The coefficient of $P_0(x)$ gives us:

$$0 = \frac{1}{3} f_1 - \alpha f_0 \implies f_1 = 3\alpha f_0.$$

Next, for $1 \le k \le n-2$, the coefficient of $P_k(x)$ gives us:

$$0 = \frac{k}{2k-1} f_{k-1} - \alpha f_k + \frac{k+1}{2k+3} f_{k+1}.$$

The coefficient for $P_{n-1}(x)$ yields:

$$0 = \frac{n-1}{2n-3} f_{n-2} - \alpha f_{n-1} \implies f_{n-2} = \alpha \frac{2n-3}{n-1} f_{n-1}.$$

Finally, the coefficient of $P_n(x)$ gives us:

$$1 = \frac{n}{2n-1} f_{n-1} \implies f_{n-1} = \frac{2n-1}{n}.$$

Note that this gives us a system of $n+1$ equations for our n unknown coefficients f_k. To solve the system, we will take the approach of trying to solve the three term recurrence relation obtained from the coefficient of $P_k(x)$ for $1 \leq k \leq n-2$. By a little manipulation:

$$\begin{aligned}
& \frac{k}{2k-1} f_{k-1} - \alpha f_k + \frac{k+1}{2k+3} f_{k+1} = 0 \\
\implies \quad & \frac{k}{k-\frac{1}{2}} f_{k-1} - 2\alpha f_k + \frac{k+1}{k+\frac{3}{2}} f_{k+1} = 0 \\
\implies \quad & \frac{k}{k-\frac{1}{2}} f_{k-1} - 2\alpha \frac{k+\frac{1}{2}}{k+\frac{1}{2}} f_k + \frac{k+1}{k+\frac{3}{2}} f_{k+1} = 0 \\
\implies \quad & k r_{k-1} - \alpha(2k+1) r_k + (k+1) r_{k+1} = 0,
\end{aligned}$$

where we have set $r_k = \frac{f_k}{k+\frac{1}{2}}$. However, this is the three term recurrence relation for Legendre polynomials, and so we can write the general solution as:

$$r_k = AP_k(\alpha) + BQ_k(\alpha),$$

where $Q_k(x)$ is the Legendre function of the second kind. For a description of these functions, see [23]. Reverting back to our constants f_k gives:

$$f_k = \left(k + \frac{1}{2}\right) \left(AP_k(\alpha) + BQ_k(\alpha)\right).$$

To determine the constants A and B, we will suppose that f_k will have this form for $0 \leq k \leq n-1$, and make use of the other equations in our system for the f_k. For $k=0$, this gives us:

$$f_0 = \frac{1}{2} \left(AP_0(\alpha) + BQ_0(\alpha)\right) = \frac{A}{2} + \frac{B\beta}{2},$$

where we have defined

$$\beta = \log\left(\frac{1+\alpha}{1-\alpha}\right)$$

for convenience. On isolating A, we get:

$$A = 2f_0 - B\beta.$$

Next, for $k = 1$ we can see:

$$\begin{aligned} 3\alpha f_0 &= f_1 = \frac{3}{2}\left(AP_1(\alpha) + BQ_1(\alpha)\right) \\ &= \frac{3}{2}A\alpha + \frac{3}{2}B\left(\frac{\alpha\beta}{2} - 1\right) \\ &= \frac{3}{2}\alpha\left(2f_0 - B\beta\right) + \frac{3}{4}B\alpha\beta - \frac{3}{2}B \\ &= 3\alpha f_0 - \frac{3}{4}\alpha\beta B - \frac{3}{2}B. \end{aligned}$$

After cancelling the $3\alpha f_0$ from both sides, we are left with:

$$\frac{3}{4}\alpha\beta B + \frac{3}{2}B = 0.$$

Now, suppose that $B \neq 0$. Then, we can cancel it out to obtain $\alpha\beta = -2$. Putting in the value of β, this means that:

$$\log\left(\frac{1+\alpha}{1-\alpha}\right) = -\frac{2}{\alpha} \implies \frac{1+\alpha}{1-\alpha} = e^{-\frac{2}{\alpha}}.$$

However, we have now arrived at a contradiction. Since we originally assumed that α was rational, we see that the left hand side will be rational. However, the right hand side will be a rational power of e, which is transcendental. Therefore, we must have $B = 0$, and:

$$f_k = A\left(k + \frac{1}{2}\right)P_k(\alpha).$$

Finally, we will make use of our last equation:

$$\frac{2n-1}{n} = f_{n-1} = A\left(n - \frac{1}{2}\right)P_{n-1}(\alpha).$$

On solving for A, we obtain:

$$A = \frac{2}{nP_{n-1}(\alpha)}.$$

As was already noted, the factor $P_{n-1}(\alpha)$ is okay to appear in the denominator since consecutive Legendre polynomials cannot share roots. Putting this into our expression for f_k gives:

$$f_k = \frac{2}{nP_{n-1}(\alpha)}\left(k + \frac{1}{2}\right)P_k(\alpha) = \frac{1}{nP_{n-1}(\alpha)}(2k+1)P_k(\alpha).$$

Putting this back into our factorization of $P_n(x)$ gives us our formula:

$$P_n(x) = (x-\alpha)f(x) = \frac{1}{nP_{n-1}(\alpha)}(x-\alpha)\sum_{k=0}^{n-1}(2k+1)P_k(\alpha)P_k(x).$$

As was mentioned at the start of this section, this formula can be verified directly using the three term recurrence relation for the Legendre polynomials for any root α of $P_n(x)$, without needing the assumption that α is rational. Therefore, our quest to show that Legendre polynomials do not have non-zero rational roots failed, but we uncovered an interesting formula along the way.

As an example of how this formula might be used, suppose that α is a non-zero root of $P_n(x)$. Then, since $P_n(x)$ is either an even or odd function depending on whether n is even or odd, we must have that $-\alpha$ is also a root of $P_n(x)$. Putting this into our formula gives:

$$0 = P_n(-\alpha) = \frac{1}{nP_{n-1}(\alpha)}(-2\alpha)\sum_{k=0}^{n-1}(2k+1)P_k(\alpha)P_k(-\alpha).$$

Due to the fact that $P_k(x)$ will either be even or odd as k is even or odd, we can simplify this to:

$$\sum_{k=0}^{n-1}(-1)^k(2k+1)P_k(\alpha)^2 = 0,$$

for any non-zero root α of $P_n(x)$. This has not been thoroughly pursued.

7.2 Regarding Infinite Differential Operators

In chapter 6, we came across the following equation when trying to solve the infinite strip problem:

$$\mathrm{sinc}(D)g = f,$$

where $\mathrm{sinc}(D)$ is the differential operator:

$$\mathrm{sinc}(D) = \sum_{k=0}^{\infty}\frac{(-1)^k D^{2k}}{(2k+1)!}.$$

One approach to try to solve this equation for an entire function g when f is an entire function is to formally invert the operator. Since $\operatorname{sinc}(0) \neq 0$, we can find a power series for its reciprocal:

$$\frac{1}{\operatorname{sinc}(x)} = \sum_{k=0}^{\infty} c_k x^k,$$

for some constants c_k. Therefore, if we define the differential operator:

$$\frac{1}{\operatorname{sinc}(D)} = \sum_{k=0}^{\infty} c_k D^k,$$

then by a formal manipulation of power series, we see that

$$\operatorname{sinc}(D)g = f \implies g = \frac{1}{\operatorname{sinc}(D)} f.$$

However, the power series for the reciprocal of $\operatorname{sinc}(x)$ will only have a finite radius of convergence, and so the corresponding differential operator will not always map entire functions to entire functions due to failure of convergence of the series. It will correctly provide solutions for polynomials, though, since convergence will not be an issue.

We can reasonably ask many questions at this point, such as the exact domain of the $\frac{1}{\operatorname{sinc}(D)}$ operator. Also, if the decay rate of coefficients of g is known, then it might be possible to determine estimates for the decay rate of the coefficients of $\operatorname{sinc}(D)g$. Finally, we can generalize our problem to other differential operators. If we have an entire function

$$p(x) = \sum_{k=0}^{\infty} p_k x^k$$

which does not vanish at the origin, then we may pose the same questions for the operators $p(D)$ and $\frac{1}{p(D)}$.

In particular, it would be interesting to know conditions on $p(x)$ which would ensure that $p(D)$ maps entire functions to entire functions. For example, with the given p, suppose we apply this operator to a function

$$g(x) = \sum_{k=0}^{\infty} g_k x^k,$$

and evaluate the resulting function at the origin. Then, we know that:

$$g^{(k)}(0) = k! g_k,$$

and so:

$$\begin{aligned} p(D)g(0) &= \sum_{k=0}^{\infty} p_k D^k g(0) \\ &= \sum_{k=0}^{\infty} k! p_k g_k. \end{aligned}$$

Surprisingly, this is the Fischer inner product of entire functions as described in [18]! Since this inner product is only defined on a restricted class of entire functions, it is perhaps not surprising that the operators $p(D)$ will not always map entire functions to entire functions. As a particular example, we may choose:

$$p(x) = g(x) = \sum_{k=2}^{\infty} \frac{x^k}{(\log k)^k}.$$

The root test shows that this is an entire function, but $p(D)g(0)$ will clearly not converge.

7.3 Regarding Difference Equations

Also in chapter 6, we came across the following difference equation:

$$g(z+1) - g(z) = f(z),$$

where we wished to determine if, given an entire function f, we could find an entire function g as to make the above true. We showed how to solve this problem for polynomials, but mentioned that we were unable to extend these results to entire functions. To demonstrate the difficulty, suppose that we are given an entire function f:

$$f(z) = \sum_{n=0}^{\infty} f_n z^n.$$

Then, disregarding questions of convergence for the sake of exploration, we can formally change our basis to the falling factorials by making use of the Stirling numbers of the second kind:

$$f(z) = \sum_{n=0}^{\infty} f_n z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_n \left\{ {n \atop k} \right\} (z)^{\underline{k}}.$$

Proceeding formally again, we can interchange the order of summation to obtain:

$$f(z) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} f_n \left\{ {n \atop k} \right\} \right) (z)^{\underline{k}}. \tag{7.1}$$

It is then a simple matter to write down a solution of the difference equation by making use of the discrete derivative property of falling factorials:

$$g(z) = \sum_{k=0}^{\infty} \frac{1}{k+1} \left(\sum_{n=k}^{\infty} f_n \left\{ {n \atop k} \right\} \right) (z)^{\underline{k+1}}.$$

We must always keep in mind, however, that we have not proven anything yet.

We cannot avoid the question of convergence forever, and so we will begin with equation (7.1). To see that the coefficients of $(z)^{\underline{k}}$ actually make sense, we must borrow some facts from combinatorics. In particular, we can make use of the following generating series for Stirling numbers of the second kind, as found in [10]:

$$\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} z^n = \frac{z^k}{(1-z)(1-2z)\cdots(1-kz)}, \tag{7.2}$$

where we see by examining the poles of the right hand side that this series will converge for all $|z| < \frac{1}{k}$. This already poses us problems, since as $k \to \infty$, the radius of convergence of the series will tend to 0, but it is good enough to make some progress.

Fix a $k \geq 1$, and choose an $R > k$, so that:

$$\frac{1}{R} < \frac{1}{k}.$$

Then, since f is an entire function, we can find a constant $C_R > 0$ such that:

$$|f_n| \leq \frac{C_R}{R^n}$$

for all $n \geq 0$. Therefore, we can estimate:

$$\sum_{n=k}^{\infty} |f_n| \left\{ n \atop k \right\} \leq C_R \sum_{n=k}^{\infty} \left\{ n \atop k \right\} \left(\frac{1}{R} \right)^n ,$$

which is equation (7.2) with $z = \frac{1}{R} < \frac{1}{k}$, and so this will be finite. Since absolute convergence implies convergence, we see that the coefficients in equation (7.1) will exist.

To make further progress, we will need some estimates on these coefficients. The best that we are able to do is the following proposition, which replaces the infinite sum with a finite sum in terms of function values. We will first need two small lemmas.

Lemma 7.3.1 *Let $g(x)$ be a polynomial such that $(x-1)^k \mid g(x)$ for some $k \geq 1$. Then $(x-1)^{k-1} \mid g'(x)$.*

Proof: This is a simple consequence of the product rule for derivatives. Since $(x-1)^k \mid g(x)$, we may write $g(x) = (x-1)^k h(x)$ for some polynomial $h(x)$. We then have:

$$\begin{aligned} g'(x) &= k(x-1)^{k-1} h(x) + (x-1)^k h'(x) \\ &= (x-1)^{k-1} \left(k h(x) + (x-1) h'(x) \right), \end{aligned}$$

so $(x-1)^{k-1} \mid g'(x)$ as stated. □

Lemma 7.3.2 *If k is an integer such that $k \geq 1$, then:*

$$\sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n = \begin{cases} (-1)^{k+1}, & n = 0 \\ 0, & n = 1,\ 2,\ \ldots,\ k-1 \end{cases}$$

Proof: Consider the function $g(x) = (x-1)^k - (-1)^k$. This has a power series expansion of:

$$\begin{aligned} g(x) &= \sum_{j=0}^{k} \binom{k}{j} x^j (-1)^{k-j} - (-1)^k \\ &= \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} x^j. \end{aligned}$$

Denoting the derivative operator by D, we can differentiate this identity and multiply by x to obtain:

$$\begin{aligned}(xD)g(x) &= x\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}jx^{j-1}\\ &= \sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}jx^{j}.\end{aligned}$$

Applying this sequence of operators repeatedly will then give us:

$$(xD)^n g(x) = \sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^n x^j$$

for all $n \geq 0$. Therefore, we see that

$$(xD)^n g(1) = \sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

In particular, for $n = 0$, we have:

$$\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j} = g(1) = (-1)^{k+1}.$$

For $n > 0$, note that $g'(x) = k(x-1)^{k-1}$, or $(x-1)^{k-1} \mid g'(x)$. This also means that $(x-1)^{k-1} \mid (xD)g(x)$. Continuing this process and using the previous lemma, we know $(x-1)^{k-2} \mid D(xD)g(x)$, so $(x-1)^{k-2} \mid (xD)^2 g(x)$, and in general we will have

$$(x-1)^{k-n} \mid (xD)^n g(x)$$

for each $n = 1, 2, \ldots, k-1$. Therefore, $(xD)^n g(1) = 0$ for these n, and this gives us:

$$\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^n = (xD)^n g(1) = 0,$$

completing our proof. □

We now can transform the coefficients in (7.1) from an infinite sum to a finite sum.

Proposition 7.3.3 *Given an entire function*

$$f(z) = \sum_{n=0}^{\infty} f_n z^n,$$

we have the following identity for all $k \geq 1$:

$$\sum_{n=k}^{\infty} f_n \left\{ n \atop k \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(j).$$

Proof: There are many explicit formulas for Stirling numbers of the second kind, but the one which will prove useful to us here can be found in [10]:

$$\left\{ n \atop k \right\} = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

This gives us:

$$\sum_{n=k}^{\infty} f_n \left\{ n \atop k \right\} = \frac{1}{k!} \sum_{n=k}^{\infty} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} f_n j^n.$$

Pick any $R > k$. Since f is entire, we can find a constant $C_R > 0$ such that

$$|f_n| \leq \frac{C_R}{R^n}$$

for all $n \geq 0$. Therefore, we see that:

$$\begin{aligned} \sum_{j=1}^{k} \left| (-1)^{k-j} \binom{k}{j} f_n j^n \right| &\leq \sum_{j=1}^{k} 2^k \frac{C_R}{R^n} j^n \\ &\leq k 2^k C_R \left(\frac{k}{R} \right)^n. \end{aligned}$$

Since $R > k$, the sum

$$\frac{1}{k!} \sum_{n=k}^{\infty} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} f_n j^n$$

will be absolutely convergent, and so we may rearrange terms as we please. This allows us to make the following calculation:

$$\begin{aligned}
\sum_{n=k}^{\infty} f_n \left\{ {n \atop k} \right\} &= \frac{1}{k!} \sum_{n=k}^{\infty} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} f_n j^n \\
&= \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \sum_{n=k}^{\infty} f_n j^n \\
&= \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \left(f(j) - \sum_{n=0}^{k-1} f_n j^n \right) \\
&= \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} f(j) - \frac{1}{k!} \sum_{j=1}^{k} \sum_{n=0}^{k-1} (-1)^{k-j} \binom{k}{j} f_n j^n \\
&= \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} f(j) - \frac{1}{k!} \sum_{n=0}^{k-1} f_n \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n \\
&= \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} f(j) - \frac{1}{k!} f_0 (-1)^{k+1} \\
&= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(j),
\end{aligned}$$

where we have used the previous lemma and the fact that $f_0 = f(0)$. □

Even with this slight simplification of the coefficients in (7.1), we were not able to justify our rearrangement of the sum to obtain (7.1) or demonstrate convergence on disks. We are now also in the possession of seemingly contradictory pieces of evidence. The above sum seems to suggest that the growth of our function $f(z)$ will have an impact on the convergence of our formal solution, but in section 6.5 of the previous chapter, we were able to solve the infinite strip problem for data which could grow extremely rapidly.

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