Topics in Delayed Renewal Risk Models

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Actuarial Science

Waterloo, Ontario, Canada, 2007

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Main focus is to extend the analysis of the ruin related quantities, such as the surplus immediately prior to ruin, the deficit at ruin or the ruin probability, to the delayed renewal risk models.

First, the background for the delayed renewal risk model is introduced and two important equations that are used as frameworks are derived. These equations are extended from the ordinary renewal risk model to the delayed renewal risk model. The first equation is obtained by conditioning on the first drop below the initial surplus level, and the second equation by conditioning on the amount and the time of the first claim.

Then, we consider the deficit at ruin in particular among many random variables associated with ruin and six main results are derived. We also explore how the Gerber-Shiu expected discounted penalty function can be expressed in closed form when distributional assumptions are given for claim sizes or the time until the first claim.

Lastly, we consider a model that has premium rate reduced when the surplus level is above a certain threshold value until it falls below the threshold value. The amount of the reduction in the premium rate can also be viewed as a dividend rate paid out from the original premium rate when the surplus level is above some threshold value. The constant barrier model is considered as a special case where the premium rate is reduced to 0 when the surplus level reaches a certain threshold value. The dividend amount paid out during the life of the surplus process until ruin, discounted to the beginning of the process, is also considered.

Acknowledgements

First of all, I would like to express my greatest appreciation to my supervisor Professor Gordon E. Willmot for his continuous encouragement and support for my research and Ph.D. thesis. I am so grateful for all his shared ideas and valuable time spent with me.

Also, I wish to thank all members of the examining committee, Professor Steve Drekic, Professor Jun Cai, and Professor Etienne Marceau, for their careful reading and comments.

Finally, I wish to express my gratitude to my parents, In-Soo Kim and Dukkee Kwon, and my husband, Chang-Kee Lee, for their sacrifices, encouragement, and love throughout my life and especially these academic years.

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Chapter 1

Background

1.1 Introduction

Much active research has been going on in the area of ruin theory, more specifically renewal risk processes, since the introduction of the expected discounted penalty function suggested by Gerber and Shiu in their paper in NAAJ in 1998, which marked an epoch in the area. It started in the framework of the classical Poisson model where the inter-claim times have exponential distributions. This model is attractive in the sense that the memoryless property of the exponential distribution makes calculations easy. Then the research was extended to ordinary Sparre-Andersen renewal risk models where the interclaim times have other distributions than the exponential distribution. Dickson and Hipp (1998, 2001) considered the Erlang-2 distribution, Li and Garrido (2004a) the Erlang-n distribution, Gerber and Shiu (2005) the generalized Erlang-n distribution (a sum of n independent exponential distributions with different scale parameters) and Li and Garrido (2005) looked into the Coxian class distributions. One difficulty with these models is that we have to assume that a claim occurs at time 0, which is not the case in usual settings.

The delayed or modified renewal risk model solves this problem by assuming that the time until the first claim has a different distribution than the rest of the inter-claim times. Not much research has been done for this model at this stage. Among the first works was Willmot (2004) where a mixture of a "generalized equilibrium" distribution and an exponential distribution is considered for the distribution of the time until the first claim. Special cases of the model include the stationary renewal risk model and the delayed renewal risk model with the time until the first claim exponentially distributed.

The stationary or equilibrium renewal risk model is a special case of the delayed renewal risk model where the time until the first claim has an equilibrium distribution of the other inter-claim times' distribution. The motivation for choosing this distribution is that it is the limiting distribution of the time until the next claim occurs, i.e. the forward recurrence time, in an ordinary renewal process. See Karlin and Taylor (1975) for details. Willmot and Dickson (2003) have looked into the Gerber-Shiu discounted penalty function in general and Willmot et al. (2004) into the deficit at ruin for this model in particular.

Another special case of the delayed renewal risk model where the time until the first claim is exponentially distributed is of much interest. This is the simplest delayed renewal risk model that we consider yet with an important property. Because of the memoryless property of the exponential distribution, we do not need to know the time of the last claim before time 0. This model will be explored in chapter 5 of this paper.

Our main focus is to extend the analysis of the ruin related quantities such as the surplus immediately prior to ruin, the deficit at ruin or the ruin probability to the delayed renewal risk model. The background for the delayed renewal risk model is introduced in section 1.2. In chapter 2, we derive two important equations that will be used as a framework for later chapters. These equations are extended from the ordinary renewal risk model to the delayed renewal risk model. The first equation is obtained by conditioning on the first drop below the initial surplus level in section 2.1, and the second equation by conditioning on the amount and the time of the first claim in section 2.3. Chapter 3 considers the deficit at ruin in particular among many random variables associated with ruin and six main results are derived. Chapter 4 and 5 show how the Gerber-Shiu expected discounted penalty function can be expressed in closed form when distributional assumptions are given for claim sizes and the first interclaim times, respectively. In chapter 6, we consider a model that has premium rate reduced when the surplus level is above a certain threshold value until it falls below the threshold value. The amount of the reduction in the premium rate can also be viewed as a dividend rate paid out from the original premium rate when the surplus level is above some threshold value. The constant barrier model is considered as a special case where the premium rate is reduced to 0 when the surplus level reaches a certain threshold value. The dividend amount paid out during the life of the surplus process until ruin, discounted to the beginning of the process, is also considered in Chapter 6.

1.2 The Delayed Renewal Risk Model

In the delayed renewal risk model, the number of claims process $\{N(t); t \geq 0\}$ is assumed to be a delayed renewal process, with V_1 the time until the first claim occurs, and V_i the time between the $(i-1)$th and the ith claim for $i = 2, 3, 4, ...$ It is assumed that $\{V_2, V_3, ...\}$ is a sequence of independent and identically distributed (IID) positive random variables with common distribution function (DF) $K(t) = 1-\bar{K}(t) = Pr(V \leq$

t), probability density function (PDF) $k(t)$ and the mean $E(V) = \int_0^\infty v dK(v) < \infty$, where V is an arbitrary V_i for $i = 2, 3, 4, ...$ The random variable V_1 is also assumed to be positive and independent of $\{V_2, V_3, ...\}$ but with a (possibly) different DF $K_1(t)$ and PDF $k_1(t)$.

If $K_1(t) = K(t)$, then the above model becomes the ordinary (or equivalently the Sparre-Andersen) renewal risk model. Also a special case of the delayed model, the equilibrium or stationary renewal risk model can be defined if the PDF of the time until the first claim is $k_1(t) = k_e(t) = \bar{K}(t)/E(V)$.

Individual claim sizes $\{Y_1, Y_2, ...\}$, independent of $N(t)$ and $\{V_1, V_2, ...\}$, are positive IID random variables with DF $P(y) = 1 - \bar{P}(y) = Pr(Y \leq y)$, PDF $p(y)$ and moments $E(Y^j) = \int_0^\infty y^j dP(y) < \infty$, where Y is an arbitrary Y_i, and Y_i is the size of the ith claim. The associated equilibrium DF is defined as $P_1(y) = 1 - \bar{P}_1(y) = \int_0^y \bar{P}(t)dt/E(Y)$.

The surplus of the insurer at time t is defined as

$$U_t = u + ct - \sum_{i=1}^{N(t)} Y_i, \ t \geq 0, \tag{1.1}$$

where $u = U_0 \geq 0$ is the initial surplus, $c = (1+\theta)E(Y)/E(V)$ is the constant premium rate per unit time received continuously, and $\theta > 0$ is the relative security loading.

Let $T_d = inf\{t : U_t < 0\}$ be the time of ruin, where $T_d = \infty$ if $U_t \geq 0$ for all $t \geq 0$. Two important non-negative random variables in connection with the time of ruin are the deficit at ruin $|U_{T_d}|$ and the surplus immediately prior to ruin $U_{T_d^-}$, where T_d^- is the left limit of T_d. The sum of the two random variables, $\{U_{T_d^-} + |U_{T_d}|\}$, is the amount of the claim causing ruin. These random variables are depicted in the figure below with a sample path of a surplus process.

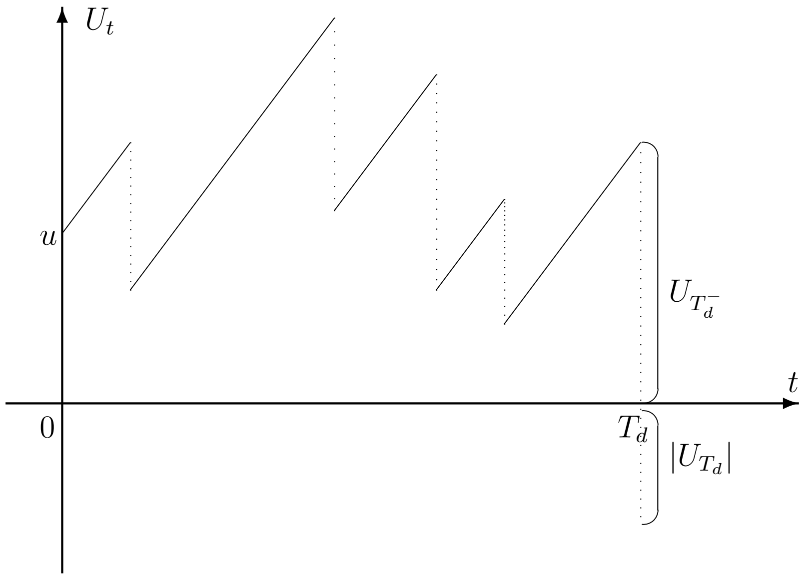

Figure 1.1: Graphical representation of the surplus process U_t

For the delayed renewal risk process, the widely known Gerber-Shiu expected discounted penalty function (Gerber and Shiu, 1998) is given by

$$m_\delta^d(u) = E\{e^{-\delta T_d} w(U_{T_d^-}, |U_{T_d}|) I(T_d < \infty)|\ U_0 = u\} \tag{1.2}$$

where $w(x_1, x_2)$ is a non-negative function for $x_1 > 0$ and $x_2 > 0$, and $I(A) = 1$ if A is true and $I(A) = 0$ otherwise. The parameter δ may be viewed as a Laplace transform argument for the time of ruin or as a discount factor. The ruin probability $\psi^d(u)$ can be obtained from the above function by letting $w(x_1, x_2) = 1$ and $\delta = 0$:

$$\psi^d(u) = E\{I(T_d < \infty)|\ U_0 = u\} = Pr(T_d < \infty|\ U_0 = u),\ u \geq 0. \tag{1.3}$$

We also need to introduce the Gerber-Shiu function of the ordinary renewal risk process since the Gerber-Shiu function in the delayed renewal risk process is expressed in terms of it. This is natural considering that the delayed renewal risk process follows an ordinary renewal risk process after its first claim. The Gerber-Shiu function in the stationary renewal risk process is also introduced because the process is the most widely used special case of the delayed renewal risk process. The corresponding Gerber-Shiu expected discounted penalty function for the ordinary renewal risk process and the stationary renewal risk process, respectively, are as follows:

$$m_\delta(u) = E\{e^{-\delta T} w(U_{T^-}, |U_T|) I(T < \infty)|\ U_0 = u\}, \tag{1.4}$$

$$m_\delta^e(u) = E\{e^{-\delta T_e} w(U_{T_e^-}, |U_{T_e}|) I(T_e < \infty)|\ U_0 = u\}. \tag{1.5}$$

where T and T_e are random variables of the time of ruin for the ordinary and the stationary renewal risk process, respectively.

Some special cases of the Gerber-Shiu function obtained by specifying the penalty function are widely used in later chapters, particularly when we have information on the distribution function for the claim sizes. These special cases of the Gerber-Shiu function allow us to obtain explicit forms in some situations where it is difficult to analyze the general Gerber-Shiu function $m_\delta^d(u)$. The first special function we consider is $m_{\delta,s}^d(u)$ with the penalty function $w(x, y)$ having the form $w(x, y) = e^{-sx} w_2(y)$. In the delayed renewal risk process it is defined as

$$m^d_{\delta,s}(u) = E\{e^{-\delta T_d - sU_{T_d^-}} w_2(|U_{T_d}|)I(T_d < \infty)|\ U_0 = u\}. \tag{1.6}$$

The corresponding functions in the ordinary renewal risk process and the stationary renewal risk process are

$$m_{\delta,s}(u) = E\{e^{-\delta T - sU_{T-}} w_2(|U_T|)I(T < \infty)|\ U_0 = u\}, \tag{1.7}$$

$$m^e_{\delta,s}(u) = E\{e^{-\delta T_e - sU_{T_e^-}} w_2(|U_{T_e}|)I(T_e < \infty)|\ U_0 = u\}, \tag{1.8}$$

respectively.

The second special function we consider can be seen as a special function of the first special function with $s = 0$. The penalty function is a function of the deficit only, with no information regarding the surplus prior to ruin. The Gerber-Shiu function in the delayed renewal risk process reduces to

$$m^d_{\delta,0}(u) = E\{e^{-\delta T_d} w_2(|U_{T_d}|)I(T_d < \infty)|\ U_0 = u\} \tag{1.9}$$

and the corresponding functions in the ordinary renewal risk process and the stationary renewal risk process are

$$m_{\delta,0}(u) = E\{e^{-\delta T} w_2(|U_T|)I(T < \infty)|\ U_0 = u\}, \tag{1.10}$$

$$m^e_{\delta,0}(u) = E\{e^{-\delta T_e} w_2(|U_{T_e}|)I(T_e < \infty)|\ U_0 = u\}, \tag{1.11}$$

respectively.

1.3 Preliminaries

(1)Dickson-Hipp Transform

We will define the Dickson-Hipp transform (Li and Garrido, 2004) which is used often in later chapters. Let $h(x)$ be a real valued continuous function with Laplace transform (LT)

$$\tilde{h}(s) = \int_0^\infty e^{-sx} h(x)dx. \tag{1.12}$$

Then a Dickson-Hipp function of $h(t)$ is defined as

$$h_r(x) = T_r h(x) = e^{rx} \int_x^\infty e^{-rt} h(t)dt, \; x \geq 0 \tag{1.13}$$

for r that satisfies $|\tilde{h}(r)| < \infty$.

The LT of a Dickson-Hipp function $h_r(x)$ is

$$\tilde{h}_r(s) = \int_0^\infty e^{-sx} h_r(x)dx = \frac{\tilde{h}(r) - \tilde{h}(s)}{s - r} \tag{1.14}$$

by integration by parts.

We can generalize the definition of Dickson-Hipp transform to functions where they are not continuous. If $H(x) = 1 - \bar{H}(x)$ is a distribution function then a Dickson-Hipp Stieltjes transform can be defined as

$$h_r(x) = T_r h(x) = e^{rx} \int_x^\infty e^{-rt} dH(t), \ x \geq 0, \tag{1.15}$$

analogous to defining a Laplace Stieltjes transform (LST) as

$$\tilde{h}(s) = \int_0^\infty e^{-sx} dH(x). \tag{1.16}$$

(2)Lundberg Adjustment Coefficient

Let $-R_\delta$ be a negative root of the generalized Lundberg's fundamental equation (defined in the ordinary renewal risk model), i.e. R_δ satisfies

$$\tilde{k}(\delta + cR_\delta)\tilde{p}(-R_\delta) = 1 \tag{1.17}$$

or

$$\tilde{b}_\delta(-R_\delta) = \frac{1}{\phi_\delta} \tag{1.18}$$

where $b_\delta(u)$ is the PDF of the discounted ladder-height random variable and $\phi_\delta = \bar{G}_\delta(0) = E\{e^{-\delta T} I(T < \infty) |\, U_0 = 0\}$, in the ordinary renewal risk model.

It is shown (Rolski et al., 1999, pp.255-9) that the two equations are equivalent when $\delta = 0$. In particular, when $\delta = 0$ let $\kappa = R_0$. Landriault and Willmot (2007) show the equivalence of the two equations for any non-negative value of δ in section

3.2 of their paper.

(3)Compound Geometric Convolution

Let function $\bar{G}(x) = 1 - G(x) = \sum_{n=1}^{\infty}(1-\phi)\phi^n \bar{F}^{*n}(x),\ x \geq 0$, be a compound geometric tail and function $A(x) = 1 - \bar{A}(x)$, $x \geq 0$, be a distribution function with $A(0) = 0$. Then the tail of a compound geometric convolution function $\bar{W}(x)$ is defined as

$$\bar{W}(x) = \int_x^\infty dA * G(y) = \bar{A}(x) + \int_0^x \bar{G}(x-y)dA(y),\ x \geq 0. \tag{1.19}$$

Of course, the role of $\bar{A}$ and $\bar{G}$ can be interchanged in the above convolution. This type of convolution is frequently used in chapter 3, when we talk about the deficit at ruin in particular.

Willmot and Lin (2001, Section 9.3) show that $\bar{W}(x)$ satisfies a defective renewal equation

$$\bar{W}(x) = \phi \int_0^x \bar{W}(x-y)dF(y) + \phi\bar{F}(x) + (1-\phi)\bar{A}(x). \tag{1.20}$$

Also, Willmot(2002b) shows that the equilibrium distribution of $W(y)$, denoted by $W_1(y)$, is also a convolution DF with Laplace-Stieltjes transform

$$\tilde{w}_1(s) = \int_0^\infty e^{-sy}dW_1(y) = \tilde{g}(s)\{\theta\tilde{f}_1(s) + (1-\theta)\tilde{a}_1(s)\} \tag{1.21}$$

where

$$\theta = \frac{\phi\int_0^\infty \bar{F}(t)dt}{\phi\int_0^\infty \bar{F}(t)dt + (1-\phi)\int_0^\infty \bar{A}(t)dt}, \tag{1.22}$$

and $\tilde{g}(s) = \int_0^\infty e^{-sy} dG(y)$, $\tilde{a}_1(s) = \int_0^\infty e^{-sy} dA_1(y) = \{1-\tilde{a}(s)\}/\{s \int_0^\infty \bar{A}(t)dt\}$, $\tilde{f}_1(s) = \int_0^\infty e^{-sy} dF_1(y) = \{1 - \tilde{f}(s)\}/\{s \int_0^\infty \bar{F}(t)dt\}$.

1.4 Laplace Transform of the Time of Ruin in the Ordinary Model

This section expands the ideas of Willmot and Woo (2007) and Willmot (2007). The derivation in section 2.1 of Willmot and Woo (2007) can be extended to the case where some of the weights, p_{ik}'s, are negative as long as all the weights sum up to 1 since none of the arguments necessitate the weights being positive. As a result, the Coxian class distribution can be expressed in a form of combination of Erlangs. Also, the expression of the Laplace transform of the time of ruin in Example 3.2 of Willmot (2007) that is derived for mixture of Erlangs holds for combination of Erlangs as well.

Assume that claims are of countable scale and shape mixture of Erlangs, i.e.,

$$p(y) = \sum_{i=1}^{n} \sum_{k=1}^{\infty} p_{ik} \frac{\beta_i^k y^{k-1} e^{-\beta_i y}}{(k-1)!}, \ y > 0 \tag{1.23}$$

where $n \in \{2, 3, \ldots\}$, $p_{ik} \in \Re$, $\sum_{i=1}^{n} \sum_{k=1}^{\infty} p_{ik} = 1$, and (1.23) is a proper distribution.

Then following the same idea and procedure as in section 2.1 of Willmot and Woo (2007), since the derivation does not require p_{ik}'s to be positive, $p(y)$ can be expressed in the exact same form;

$$p(y) = \sum_{j=1}^{\infty} q_j \frac{\beta_n^j y^{j-1} e^{-\beta_n y}}{(j-1)!}, \ y > 0, \tag{1.24}$$

where we can assume $\beta_n > \beta_i$ for $i = 1, 2, \ldots, n-1$ without loss of generality and

$$q_j = \sum_{i=1}^{n} \sum_{k=1}^{j} p_{ik} \binom{j-1}{k-1} (\frac{\beta_i}{\beta_n})^k (1 - \frac{\beta_i}{\beta_n})^{j-k};\ j = 1, 2, \ldots. \tag{1.25}$$

Note that q_j's could be negative but $\sum_{j=1}^{\infty} q_j = 1$ and thus (1.24) is a combination of Erlangs.

Now, we follow the idea of Example 3.2 in Willmot (2007). Define the Erlang-j PDF

$$\tau_j(y) = \frac{\beta(\beta y)^{j-1} e^{-\beta y}}{(j-1)!},\ j = 1, 2, \ldots, \tag{1.26}$$

and from (1.24)

$$\begin{aligned} p(x+y) &= \sum_{l=1}^{\infty} q_l \beta^l \frac{(x+y)^{l-1}}{(l-1)!} e^{-\beta(x+y)} \\ &= \beta^{-1} \sum_{j=1}^{\infty} \tau_j(y) \sum_{l=j}^{\infty} q_l \tau_{l-j+1}(x) \\ &= \beta^{-1} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} q_{j+k-1} \tau_k(x) \tau_j(y) \end{aligned}$$

where the last equality is obtained by replacing l by $k = l - j + 1$.
Thus $p(x+y)$ can be expressed as

$$p(x+y) = \sum_{j=1}^{\infty} \eta_j(x) \tau_j(y),\ x \geq 0, y \geq 0 \tag{1.27}$$

where

$$\eta_j(x) = \beta^{-1} \sum_{k=1}^{\infty} q_{j+k-1} \tau_k(x). \tag{1.28}$$

Willmot (2007) shows that when $p(x+y)$ admits this factorization the discounted ladder height PDF $b_\delta(y)$ satisfies

$$b_\delta(y) = \sum_{j=1}^{\infty} \tilde{\eta}_{j,\delta} \tau_j(y) \tag{1.29}$$

where

$$\tilde{\eta}_{j,\delta} = \frac{1}{\phi_\delta} \int_0^\infty \eta_j(x) \frac{f_\delta(x|0)}{\bar{P}(x)} dx \tag{1.30}$$

and $f_\delta(x|0)$ is the discounted defective marginal density of the surplus prior to ruin (x), given $U_0 = 0$.

He also shows that the Laplace transform of the time of ruin

$$\bar{G}_\delta(u) = E\{e^{-\delta T} I(T < \infty) |\, U_0 = u\} \tag{1.31}$$

satisfies

$$\bar{G}_\delta(u) = e^{-\beta u} \sum_{m=0}^{\infty} \bar{C}_{m,\delta} \frac{(\beta u)^m}{m!} \tag{1.32}$$

where $\bar{C}_{m,\delta} = \sum_{k=m+1}^{\infty} c_{k,\delta}$ and $\{c_{k,\delta}; k = 0, 1, 2, \ldots\}$ has compound geometric generating function

$$\sum_{k=0}^{\infty} c_{k,\delta} z^k = \frac{1 - \phi_\delta}{1 - \phi_\delta \sum_{j=1}^{\infty} \tilde{\eta}_{j,\delta} z^j}. \tag{1.33}$$

Note that $\eta_j(x)$ and $\tilde{\eta}_{j,\delta}$ can be negative and thus the discounted ladder height PDF $b_\delta(y)$ is a combination of Erlangs and $\{c_{k,\delta}; k = 0, 1, 2, \ldots\}$ has compound geometric generating function, instead of compound geometric probability generating function. All the arguments we used also hold for these negative values and the results we derived are extended to combination of Erlangs from mixture of Erlangs. Thus, the Laplace transform of the time of ruin when claims are of Coxian-class type, can be expressed in the form of equation (1.32).

Chapter 2

Two Framework Equations

In this section, we derive two equations that are used as a framework for solving specific problems. The first equation is derived by conditioning on the first drop below its initial surplus level (section 2.1) and the second equation by conditioning on the first claim (section 2.3).

Li and Garrido (2005, Section 6.) show how the defective renewal equation approach of Gerber and Shiu (1998) can be extended to the ordinary renewal risk process. In section 2.1, we are going to see how the general form of the defective renewal equation is modified for the delayed renewal risk process. In section 2.2, we derive another expression for the distribution function of the ladder-height random variable, in addition to the one obtained in section 2.1. The former is expressed in terms of the Laplace transform of the time of ruin and the DF of the ladder-height random variable in the ordinary renewal process, whereas the latter in terms of the defective joint PDF of the surplus prior to ruin, the deficit at ruin and the time of ruin, with the distribution of the claim size.

2.1 Adapted General Defective Renewal Equation

The motivation of the derivation in this section comes from the corresponding derivation in the classical Poisson model in Gerber and Shiu (1998).

Lemma 2.1 *The general form of the modified defective renewal equation in the delayed renewal risk process is*

$$m_\delta{}^d(u) = \phi_\delta{}^d \int_0^u m_\delta(u-y) b_\delta{}^d(y) dy + \int_u^\infty \int_0^\infty w(x+u, y-u) p_x(y) f_\delta{}^d(x|\ 0) dx dy. \quad (2.1)$$

In (2.1),

$$\phi_\delta{}^d = \int_0^\infty f_\delta{}^d(x|\ 0) dx \quad (2.2)$$

and

$$b_\delta{}^d(y) = \int_0^\infty p_x(y) \{ \frac{f_\delta{}^d(x|\ 0)}{\phi_\delta{}^d} \} dx \quad (2.3)$$

with

$$f_\delta{}^d(x|\ 0) = \int_0^\infty \int_0^\infty e^{-\delta t} f^d(x, y, t|\ 0) dt dy \quad (2.4)$$

where $f^d(x, y, t|0)$ *is the defective joint PDF of the surplus prior to ruin* (x)*, the deficit at ruin* (y) *and the time of ruin* (t)*, given initial surplus* 0*.*

Proof:

By conditioning on the first drop in surplus below its initial level u, we obtain the following equation. The first term in the equation explains the case where ruin does not occur in the first drop whereas the second term explains the case where ruin does occur due to the first drop. If the first drop is less than u, then ruin does not occur and the process starts again with an ordinary renewal process since the first claim should have already occurred by this time, with reduced new initial surplus $u-y$ where y is the amount of the first drop. If the first drop is greater than u, ruin occurs with the deficit $y-u$ and the surplus prior to ruin $u+x$ where x is the surplus gained above the initial level u before the first drop.

$$\begin{aligned}
m_\delta^d(u) &= \int_0^u \int_0^\infty \int_0^\infty e^{-\delta t} m_\delta(u-y) f^d(x,y,t|\ 0)dtdxdy \\
&\quad + \int_u^\infty \int_0^\infty \int_0^\infty e^{-\delta t} w(x+u,y-u) f^d(x,y,t|\ 0)dtdxdy \\
&= \int_0^u \int_0^\infty m_\delta(u-y) f_\delta^d(x,y|\ 0)dxdy \\
&\quad + \int_u^\infty \int_0^\infty w(x+u,y-u) f_\delta^d(x,y|\ 0)dxdy.
\end{aligned}$$

where

$$f_\delta^d(x,y|\ 0) = \int_0^\infty e^{-\delta t} f^d(x,y,t|\ 0)dt. \tag{2.5}$$

Thus, we can rewrite $m_\delta^d(u)$ as

$$m_\delta^d(u) = \phi_\delta^d \int_0^u m_\delta(u-y) b_\delta^d(y)dy + \int_u^\infty \int_0^\infty w(x+u,y-u) f_\delta^d(x,y|\ 0)dxdy \tag{2.6}$$

where

$$\phi_\delta^d = \int_0^\infty \int_0^\infty f_\delta^d(x,y|\ 0)dxdy \tag{2.7}$$

and

$$b_\delta^d(y) = \frac{1}{\phi_\delta^d} \int_0^\infty f_\delta^d(x,y|\ 0)dx. \tag{2.8}$$

This choice of ϕ_δ^d makes $b_\delta^d(y)$ a probability density function, i.e., $\int_0^\infty b_\delta^d(y)dy = 1$.

Note that ϕ_δ^d and $b_\delta^d(y)$ can be expressed in a different form using the argument of Gerber and Shiu (1998, p.53). Using the conditional probability formula of Bayes' rule,

$$P(A \bigcap B) = P(A)\ P(B|A), \tag{2.9}$$

the joint PDF of $U(T_d-)$, $|U(T_d)|$ and T_d at the point (x, y, t) can be written as the joint PDF of $U(T_d-)$ and T_d at the point (x, t) multiplied by the conditional PDF of $|U(T_d)|$ at y, given $U(T_d-) = x$ and $T_d = t$. The deficit at ruin, $|U(T_d)|$, depends on the claim sizes and the time of ruin, $T_d = t$, depends on the interclaim times but since the claim size random variables are independent of the interclaim time random variables, $|U(T_d)|$ does not depend on $T_d = t$. And the conditional PDF of $|U(T_d)|$ at y, given $U(T_d-) = x$ is

$$\frac{p(x+y)}{\int_0^\infty p(x+y)dy} = \frac{p(x+y)}{\bar{P}(x)}. \tag{2.10}$$

Thus,

$$f^d(x, y, t|\ 0) = {f_1}^d(x, t|\ 0)\ p_x(y) \tag{2.11}$$

where ${f_1}^d(x, t|\ 0)$ is the joint PDF of the surplus prior to ruin and the time at ruin, and $p_x(y) = p(x+y)/\bar{P}(x)$.

The discounted joint PDF of the surplus and the deficit becomes

$$f_\delta^d(x,y|\ 0) = \int_0^\infty e^{-\delta t} f^d(x,y,t|\ 0)dt = p_x(y){f_\delta}^d(x|\ 0) \tag{2.12}$$

where

$$f_\delta^d(x|\ 0) = \int_0^\infty e^{-\delta t} {f_1}^d(x,t|\ 0)dt \tag{2.13}$$

is the discounted density of the surplus, given zero initial surplus.

Then,

$${\phi_\delta}^d = \int_0^\infty \int_0^\infty p_x(y){f_\delta}^d(x|\ 0)dxdy = \int_0^\infty {f_\delta}^d(x|\ 0) \int_0^\infty p_x(y)dydx = \int_0^\infty {f_\delta}^d(x|\ 0)dx, \tag{2.14}$$

$${b_\delta}^d(y) = \frac{\int_0^\infty p_x(y){f_\delta}^d(x|\ 0)dx}{{\phi_\delta}^d} = \int_0^\infty p_x(y)\{\frac{{f_\delta}^d(x|\ 0)}{{\phi_\delta}^d}\}dx \tag{2.15}$$

and the expression for ${m_\delta}^d(u)$ in equation (2.6) can be reexpressed as

$${m_\delta}^d(u) = {\phi_\delta}^d \int_0^u m_\delta(u-y){b_\delta}^d(y)dy + \int_u^\infty \int_0^\infty w(x+u, y-u)p_x(y){f_\delta}^d(x|\ 0)dxdy \tag{2.16}$$

Q.E.D.

Note that ${\phi_\delta}^d \leq {\phi_0}^d = \psi^d(0)$ and ${b_\delta}^d(y)$ is a mixture PDF over x of $p_x(y)$. Also, ${b_\delta}^d(y)$ has the same mixture form as in the ordinary renewal risk model (Willmot, 2007) but with different mixing weights. The mixing weights in $b_\delta(y)$ shown in Willmot (2007) are discounted density of the surplus in the ordinary renewal process, i.e.,

$b_\delta(y) = \int_0^\infty p_x(y)\{\frac{f_\delta(x|\ 0)}{\phi_\delta}\}dx.$

Equation (2.1) is not a renewal equation since ${m_\delta}^d(u)$ is expressed in terms of $m_\delta(u)$ but we name it "general defective renewal equation adapted to the delayed renewal risk process" because the logic and the form of the equation are similar to those of the corresponding equation in the ordinary renewal risk process.

The general form of ${b_\delta}^d(y)$ we just derived is very useful in obtaining the explicit PDF of ${b_\delta}^d(y)$ in some special cases, as we can see from the following examples.

Example 2.1.1 Exponential claim size distribution

In the special case where $p(y) = \beta e^{-\beta y}$, it is easy to see that

$$p_x(y) = \frac{p(x+y)}{\bar{P}(x)} = \frac{\beta e^{-\beta(x+y)}}{e^{-\beta x}} = \beta e^{-\beta y} \tag{2.17}$$

and

$${b_\delta}^d(y) = \int_0^\infty p_x(y)\{\frac{{f_\delta}^d(x|\ 0)}{{\phi_\delta}^d}\}dx = \beta e^{-\beta y}\int_0^\infty \{\frac{{f_\delta}^d(x|\ 0)}{{\phi_\delta}^d}\}dx = \beta e^{-\beta y}. \tag{2.18}$$

Thus, $p_x(y) = {b_\delta}^d(y) = p(y)$. ♣

Example 2.1.2 Combination of Erlangs with countable scale and shape parameters for the claim size distribution

More generally, let's assume the claim sizes have a distribution with PDF of the form

$$p(y) = \sum_{i=1}^{k}\sum_{j=1}^{\infty} a_{ij}\frac{\beta_i(\beta_i y)^{j-1}e^{-\beta_i y}}{(j-1)!}$$

where a_{ij}'s are weights that could be negative or positive but $\sum_{i=1}^{k}\sum_{j=1}^{\infty} a_{ij} = 1$. This includes a combination of exponentials when $a_{ij} = 0$ for $j \geq 2$ and also an Erlang mixture with single scale parameter when $k = 1$.

Then

$$b_\delta{}^d(y) = \sum_{i=1}^{k}\sum_{j=1}^{\infty} a_{ij}{}^d(\delta)\frac{\beta_i(\beta_i y)^{j-1}e^{-\beta_i y}}{(j-1)!}$$

where

$$a_{ij}{}^d(\delta) = \frac{1}{\phi_\delta{}^d}\sum_{m=j}^{\infty}\frac{a_{im}}{(m-j)!}\int_0^\infty \frac{(\beta_i x)^{m-j} f_\delta{}^d(x|\ 0)}{e^{\beta_i x}\bar{P}(x)}dx.$$

To show this, we begin with $p(x+y)$ to derive $p_x(y)$ and use it to obtain the form of $b_\delta^d(y)$.

$$\begin{aligned}
p(x+y) &= \sum_{i=1}^{k}\sum_{m=1}^{\infty} a_{im}\frac{\beta_i{}^m (x+y)^{m-1}e^{-\beta_i(x+y)}}{(m-1)!} \\
&= \sum_{i=1}^{k}\sum_{m=1}^{\infty} a_{im}\frac{\beta_i{}^m e^{-\beta_i(x+y)}}{(m-1)!}\sum_{j=0}^{m-1}\binom{m-1}{j}x^{m-1-j}y^j \\
&= \sum_{i=1}^{k}\sum_{m=1}^{\infty} a_{im}\frac{\beta_i{}^m e^{-\beta_i(x+y)}}{(m-1)!}\sum_{j=1}^{m}\binom{m-1}{j-1}x^{m-j}y^{j-1} \\
&= \sum_{i=1}^{k}\sum_{j=1}^{\infty}\frac{\beta_i{}^j y^{j-1}e^{-\beta_i y}}{(j-1)!}\sum_{m=j}^{\infty} a_{im}\frac{(\beta_i x)^{m-j}}{(m-j)!}e^{-\beta_i x} \\
&= \sum_{i=1}^{k}\sum_{j=1}^{\infty}\{e^{-\beta_i x}\sum_{m=j}^{\infty} a_{im}\frac{(\beta_i x)^{m-j}}{(m-j)!}\}\frac{\beta_i(\beta_i y)^{j-1}e^{-\beta_i y}}{(j-1)!}
\end{aligned}$$

and this leads to

$$p_x(y) = \frac{p(x+y)}{\bar{P}(x)} = \sum_{i=1}^{k}\sum_{j=1}^{\infty} a_{ij}{}^*(x)\frac{\beta_i(\beta_i y)^{j-1}e^{-\beta_i y}}{(j-1)!}$$

where

$$a_{ij}{}^*(x) = \frac{e^{-\beta_i x}}{\bar{P}(x)} \sum_{m=j}^{\infty} a_{im} \frac{(\beta_i x)^{m-j}}{(m-j)!}.$$

Thus,

$$\begin{aligned} b_\delta{}^d(y) &= \int_0^\infty \{\sum_{i=1}^{k} \sum_{j=1}^{\infty} a_{ij}{}^*(x) \frac{\beta_i(\beta_i y)^{j-1} e^{-\beta_i y}}{(j-1)!}\}\{\frac{f_\delta{}^d(x|\ 0)}{\phi_\delta{}^d}\}dx \\ &= \sum_{i=1}^{k} \sum_{j=1}^{\infty} a_{ij}{}^d(\delta) \frac{\beta_i(\beta_i y)^{j-1} e^{-\beta_i y}}{(j-1)!} \end{aligned}$$

where

$$\begin{aligned} a_{ij}{}^d(\delta) &= \frac{1}{\phi_\delta{}^d} \int_0^\infty a_{ij}{}^*(x) f_\delta{}^d(x|\ 0)dx = \frac{1}{\phi_\delta{}^d} \int_0^\infty f_\delta{}^d(x|\ 0)\{\frac{e^{-\beta_i x}}{\bar{P}(x)} \sum_{m=j}^{\infty} a_{im} \frac{(\beta_i x)^{m-j}}{(m-j)!}\}dx \\ &= \frac{1}{\phi_\delta{}^d} \sum_{m=j}^{\infty} \frac{a_{im}}{(m-j)!} \int_0^\infty \frac{(\beta_i x)^{m-j} f_\delta{}^d(x|\ 0)}{e^{\beta_i x} \bar{P}(x)}\ dx \end{aligned}$$

Notice that $p_x(y)$ is a PDF of the same form as $p(y)$ but with a_{ij} replaced by $a^*_{ij}(x)$, and that $b^d_\delta(y)$ is also of the same form but with a_{ij} replaced by $a^d_{ij}(\delta)$. ♣

As shown in the examples, the form of $b^d_\delta(y)$ in (2.3) makes $b^d_\delta(y)$ to have the same form of distribution as the original claim size distribution. This mixing representation of $b^d_\delta(y)$ also preserves the DFR property. If $p(y)$ is DFR, the residual lifetime distribution $p_x(y)$ is also DFR. Noting that $b^d_\delta(y)$ is a mixture of $p_x(y)$ over x, it preserves the DFR property (p.10, Willmot and Lin, 2001).

2.2 Another Expression for $b_\delta^d(y)$

Let $\bar{G}_\delta^d(u)$ be the Laplace transform of the time of ruin in the delayed renewal process, i.e.

$$\bar{G}_\delta^d(u) = E\{e^{-\delta T_d} I(T_d < \infty) |\, U_0 = u\}. \tag{2.19}$$

Using equation (2.1) with the argument $w(x, y) = 1$, we obtain an equation for $\bar{G}_\delta^d(u)$, expressed in terms of $\bar{G}_\delta(u)$, the Laplace transform of the time of ruin in the ordinary renewal process and $B_\delta^d(y)$, the DF of the ladder-height random variable in the delayed renewal process.

$$\bar{G}_\delta^d(u) = \phi_\delta^d \int_0^u \bar{G}_\delta(u-t) b_\delta^d(t) dt + \phi_\delta^d \bar{B}_\delta^d(u). \tag{2.20}$$

Out of this equation we get an expression for $\bar{B}_\delta^d(u)$ that is simpler than the one we obtained in the previous section.

Lemma 2.2 $\bar{B}_\delta^d(u)$ *satisfies*

$$\bar{B}_\delta^d(u) = \frac{\phi_\delta \bar{B}_\delta(u) - \frac{\bar{G}_\delta^d(u)}{\phi_\delta^d} + \phi_\delta \int_0^u \frac{\bar{G}_\delta^d(u-t)}{\phi_\delta^d} dB_\delta(t)}{\phi_\delta - 1} \tag{2.21}$$

Proof:

Taking Laplace transforms on both sides of (2.20), we obtain

$$\int_0^\infty e^{-su} \bar{G}_\delta^d(u) du = \phi_\delta^d \tilde{b}_\delta^d(s) \int_0^\infty e^{-su} \bar{G}_\delta(u) du + \phi_\delta^d \{\frac{1 - \tilde{b}_\delta^d(s)}{s}\} \tag{2.22}$$

and isolating $\tilde{b}_\delta^d(s)$, we get

$$\tilde{b}_\delta^d(s) = \frac{s\frac{\int_0^\infty e^{-su}\bar{G}_\delta^d(u)du}{\phi_\delta^d} - 1}{s\int_0^\infty e^{-su}\bar{G}_\delta(u)du - 1}. \tag{2.23}$$

It is convenient to work in terms of the tail, so we rearrange (2.23) to get

$$\frac{1-\tilde{b}_\delta^d(s)}{s} = \frac{\int_0^\infty e^{-su}\bar{G}_\delta(u)du - \frac{\int_0^\infty e^{-su}\bar{G}_\delta^d(u)du}{\phi_\delta^d}}{s\int_0^\infty e^{-su}\bar{G}_\delta(u)du - 1}. \tag{2.24}$$

Noting that

$$\bar{G}_\delta(u) = \sum_{n=1}^{\infty} {\phi_\delta}^n(1-\phi_\delta){\bar{B}_\delta}^{*n}(u) \tag{2.25}$$

where $B_\delta(u)$ is a distribution function of the ladder-height random variable in the ordinary renewal process (Willmot and Lin, 2001), $\int_0^\infty e^{-su}\bar{G}_\delta(u)du$ can be expressed as

$$\int_0^\infty e^{-su}\bar{G}_\delta(u)du = \frac{1}{s}\{1 - \frac{1-\phi_\delta}{1-\phi_\delta\tilde{b}_\delta(s)}\} \tag{2.26}$$

and again

$$s\int_0^\infty e^{-su}\bar{G}_\delta(u)du - 1 = \frac{\phi_\delta - 1}{1-\phi_\delta\tilde{b}_\delta(s)}. \tag{2.27}$$

Substituting (2.27) into (2.24) gives

$$\begin{aligned}\frac{1-\tilde{b}_\delta^d(s)}{s} &= \frac{\int_0^\infty e^{-su}\bar{G}_\delta(u)du - \frac{\int_0^\infty e^{-su}\bar{G}_\delta^d(u)du}{\phi_\delta^d}}{\frac{\phi_\delta - 1}{1-\phi_\delta\tilde{b}_\delta(s)}} \\ &= \frac{(\phi_\delta^d\int_0^\infty e^{-su}\bar{G}_\delta(u)du - \int_0^\infty e^{-su}\bar{G}_\delta^d(u)du)(1-\phi_\delta\tilde{b}_\delta(s))}{\phi_\delta^d(\phi_\delta - 1)}\end{aligned}$$

Expanding the numerator of the right hand side and inverting the Laplace-Stieljes transform we get

$$\begin{aligned}
\bar{B}_\delta^d(u) &= \frac{[\phi_\delta^d \bar{G}_\delta(u) - \bar{G}_\delta^d(u) - \phi_\delta \phi_\delta^d \int_0^u \bar{G}_\delta(u-t) b_\delta(t) dt + \phi_\delta \int_0^u \bar{G}_\delta^d(u-t) b_\delta(t) dt]}{\phi_\delta^d(\phi_\delta - 1)} \\
&= \frac{\bar{G}_\delta(u) - \phi_\delta \int_0^u \bar{G}_\delta(u-t) b_\delta(t) dt}{\phi_\delta - 1} - \frac{\bar{G}_\delta^d(u) - \phi_\delta \int_0^u \bar{G}_\delta^d(u-t) b_\delta(t) dt}{\phi_\delta^d(\phi_\delta - 1)} \\
&= \frac{\phi_\delta \bar{B}_\delta(u)}{\phi_\delta - 1} - \frac{\bar{G}_\delta^d(u) - \phi_\delta \int_0^u \bar{G}_\delta^d(u-t) b_\delta(t) dt}{\phi_\delta^d(\phi_\delta - 1)}
\end{aligned}$$

Q.E.D.

Note that rearrangement of (2.21) will give the defective renewal equation for $\bar{G}_\delta^d(u)$. This will be shown and discussed in detail in section 3.1.

In particular, when $\delta = 0$, (2.21) simplifies to

$$\bar{B}_0^d(u) = \frac{\psi(0)\bar{B}_0(u) - \frac{\psi^d(u)}{\psi^d(0)} + \psi(0) \int_0^u \frac{\psi^d(u-t)}{\psi^d(0)} b_0(t) dt}{\psi(0) - 1} \tag{2.28}$$

as

$$\bar{G}_0^d(u) = E\{I(T_d < \infty) | U_0 = u\} = \psi^d(u) \tag{2.29}$$

and

$$\phi_0^d = \int_0^\infty \int_0^\infty \int_0^\infty f^d(x, y, t|0) dx dy dt = \psi^d(0). \tag{2.30}$$

$\bar{B}_0^d(u)$ is not in a nice simple form and involves the ruin probabilities of both the ordinary and the delayed renewal risk process but is a useful form in solving other problems such as obtaining the asymptotic formula for the proper distribution of the deficit as we will see in Chapter 3. But when $\psi^d(u)$ can be expressed in terms of $\psi(u)$, the expression of $\bar{B}_0^d(u)$ will only involve $\psi(u)$ and we can obtain an explicit expression for $\bar{B}_0^d(u)$ if $\psi(u)$ has explicit form, as can be seen in the next example.

Example 2.2.1 $k_1(t)$ **specified as in Willmot (2004)**

Let

$$k_1(t) = q\frac{e^{-\alpha t}\int_t^\infty e^{\alpha y}k(y)dy}{\int_0^\infty e^{\alpha y}\bar{K}(y)dy} + (1-q)\alpha e^{-\alpha t}, t \geq 0 \tag{2.31}$$

where $\alpha > 0$ if $0 \leq q < 1$, and $-\infty < \alpha < \infty$ if $q = 1$. Motivation for this choice of $k_1(t)$ can be found in Willmot (2004).

It is shown that the ruin probability in this case satisfies

$$\psi^d(u) = \frac{k_1(0)E(V)}{1+\theta}\int_0^u \psi(u-y)p_1(y)dy + \int_u^\infty e^{-\alpha(t-u)/c}\rho(t)dt \tag{2.32}$$

where

$$\rho(t) = \frac{\alpha}{c}\psi(t) + \frac{k_1(0)}{c}\bar{P}(t) - \frac{\alpha k_1(0)E(V)}{c(1+\theta)}\int_0^t \psi(t-y)p_1(y)dy. \tag{2.33}$$

Then

$$\psi^d(0) = \int_0^\infty e^{-\alpha t/c}\rho(t)dt = T_{\frac{\alpha}{c}}\rho(0) \tag{2.34}$$

and from (2.28)

$$\bar{B}_0^d(u) = \frac{1}{\psi(0)-1}[\psi(0)\bar{B}_0(u) - \beta(u) + \psi(0)\int_0^u \beta(u-t)b_0(t)dt] \tag{2.35}$$

where

$$\beta(u) = \frac{k_1(0)E(V)}{(1+\theta)T_{\frac{\alpha}{c}}\rho(0)}\int_0^u \psi(u-y)p_1(y)dy + \frac{T_{\frac{\alpha}{c}}\rho(u)}{T_{\frac{\alpha}{c}}\rho(0)}. \tag{2.36}$$

We can see that our interest now narrows down to obtaining the ruin probability for the ordinary process. It is entirely expressed in terms of the quantities from the ordinary process.

Suppose that the claim amount distribution is a hyperexponential (or a mixture of exponentials) with a tail of the form

$$\bar{P}(y) = qe^{-\lambda y} + (1-q)e^{-\beta y}, \; y \geq 0,$$

where $0 < q < 1$, and without loss of generality we can assume $\lambda < \beta$. With this assumption, Willmot (2002a) shows that the ruin probability in the ordinary renewal process is explicitly expressed as

$$\psi(u) = C_1 e^{-r_1 u} + C_2 e^{-r_2 u}, \; u \geq 0, \tag{2.37}$$

where r_1 and r_2 are the two distinct positive roots of the Lundberg equation $\tilde{k}(cr)\tilde{p}(-r) = 1$ which satisfies $0 < r_1 < \lambda < r_2 < \beta$, and

$$C_1 = \frac{r_2(\lambda - r_1)(\beta - r_1)}{\lambda\beta(r_2 - r_1)}, \; C_2 = \frac{r_1(r_2 - \lambda)(\beta - r_2)}{\lambda\beta(r_2 - r_1)}. \tag{2.38}$$

After some tedious calculation,

$$\begin{aligned}
&\frac{k_1(0)E(V)}{1+\theta}\int_0^t \psi(t-y)p_1(y)dy \\
&= \frac{k_1(0)}{c}\int_0^t \{C_1e^{-r_1(t-y)} + C_2e^{-r_2(t-y)}\}\{qe^{-\lambda y} + (1-q)e^{-\beta y}\}dy \\
&= \frac{k_1(0)}{c}\{A_1e^{-r_1t} + A_2e^{-r_2t} + B_1e^{-\lambda t} + B_2e^{-\beta t}\} \qquad (2.39)
\end{aligned}$$

where

$$A_1 = \frac{C_1\{q\beta + (1-q)\lambda - r_1\}}{(\lambda - r_1)(\beta - r_1)} = \frac{r_2\{q\beta + (1-q)\lambda - r_1\}}{\lambda\beta(r_2 - r_1)}, \qquad (2.40)$$

$$A_2 = \frac{C_2\{q\beta + (1-q)\lambda - r_2\}}{(\lambda - r_2)(\beta - r_2)} = \frac{-r_1\{q\beta + (1-q)\lambda - r_2\}}{\lambda\beta(r_2 - r_1)}, \qquad (2.41)$$

$$B_1 = q\frac{C_1r_2 + C_2r_1 - (C_1+C_2)\lambda}{(\lambda - r_1)(\lambda - r_2)} = -\frac{q}{\lambda}, \qquad (2.42)$$

and

$$B_2 = (1-q)\frac{C_1r_2 + C_2r_1 - (C_1+C_2)\beta}{(\beta - r_1)(\beta - r_2)} = -\frac{(1-q)}{\beta}. \qquad (2.43)$$

Using (2.39),

$$\begin{aligned}
\rho(t) &= (\frac{\alpha}{c}C_1 - \frac{\alpha k_1(0)}{c^2}A_1)e^{-r_1t} + (\frac{\alpha}{c}C_2 - \frac{\alpha k_1(0)}{c^2}A_2)e^{-r_2t} \\
&\quad +(\frac{k_1(0)}{c}q - \frac{\alpha k_1(0)}{c^2}B_1)e^{-\lambda t} + (\frac{k_1(0)}{c}(1-q) - \frac{\alpha k_1(0)}{c^2}B_2)e^{-\beta t}\}, \quad (2.44)
\end{aligned}$$

and

$$\begin{aligned}
T_{\frac{\alpha}{c}}\rho(u) &= e^{\frac{\alpha u}{c}}\int_u^\infty e^{-\frac{\alpha t}{c}}\rho(t)dt \\
&= (\frac{\alpha C_1 - \alpha\frac{k_1(0)}{c}A_1}{\alpha + cr_1})e^{-r_1u} + (\frac{\alpha C_2 - \alpha\frac{k_1(0)}{c}A_2}{\alpha + cr_2})e^{-r_2u} \\
&\quad +(\frac{k_1(0)q - \alpha\frac{k_1(0)}{c}B_1}{\alpha + c\lambda})e^{-\lambda u} + (\frac{k_1(0)(1-q) - \alpha\frac{k_1(0)}{c}B_2}{\alpha + c\beta})e^{-\beta u}. \quad (2.45)
\end{aligned}$$

Thus, from (2.32), the ruin probability is

$$
\begin{aligned}
\psi^d(u) &= (\frac{\alpha C_1 + r_1 k_1(0) A_1}{\alpha + c r_1})e^{-r_1 u} + (\frac{\alpha C_2 + r_2 k_1(0) A_2}{\alpha + c r_2})e^{-r_2 u} \\
&\quad + (\frac{k_1(0) q + \lambda k_1(0) B_1}{\alpha + c\lambda})e^{-\lambda u} + (\frac{k_1(0)(1-q) + \beta k_1(0) B_2}{\alpha + c\beta})e^{-\beta u}.
\end{aligned}
$$

Willmot (2002a) also shows that the ladder-height distribution has tail

$$\bar{B}_0(u) = q_1 e^{-\lambda u} + (1 - q_1) e^{-\beta u} \tag{2.46}$$

where

$$q_1 = \frac{\beta(\lambda - r_1)(r_2 - \lambda)}{(\beta - \lambda)(\lambda\beta - r_1 r_2)}. \tag{2.47}$$

Since we have explicit expressions for $\psi(u)$, $\psi^d(u)$, and $b_0(u)$, (2.28) can be expressed in explicit form in the case where claims are of hyperexponential. We can see that $\bar{B}_0^d(u)$ is expressed as sum of exponential terms. ♣

2.3 Equation Conditioning on the First Claim

There is another expression for $m_\delta^d(u)$ that is used often as a framework for solving more specified problems. By conditioning on the time(t) and the amount(y) of the first claim, we can write $m_\delta^d(u)$ as follows. Assume the time of the first claim is t. Then the accumulated surplus up to that time point is $u + ct$. Thus ruin occurs if the amount of the first claim is greater than $u + ct$ with the surplus prior to ruin being $u + ct$, the deficit at ruin $y - u - ct$, and thus the Gerber-Shiu discounted function becomes $e^{-\delta t} w(u + ct, y - u - ct)$. Otherwise ruin does not occur and the process would start again as an ordinary renewal process with new reduced initial surplus $u + ct - y$ and thus the Gerber-Shiu discounted function in this case is $e^{-\delta t} m_\delta(u + ct - y)$. Therefore,

$$\begin{aligned} m_\delta^d(u) &= \int_0^\infty e^{-\delta t}\{\int_0^{u+ct} m_\delta(u+ct-y)p(y)dy \\ &\quad + \int_{u+ct}^\infty w(u+ct, y-u-ct)p(y)dy\}k_1(t)dt, \end{aligned}$$

i.e.

$$m_\delta^d(u) = \int_0^\infty e^{-\delta t}\sigma_\delta(u+ct)k_1(t)dt \tag{2.48}$$

where

$$\sigma_\delta(t) = \int_0^t m_\delta(t-y)p(y)dy + \int_t^\infty w(t, y-t)p(y)dy. \tag{2.49}$$

Note that $\sigma_\delta(t)$ is the same as in the ordinary renewal risk process since it does not involve $k_1(t)$.

We can also breakdown $\sigma_\delta(t)$ into

$$\sigma_\delta(t) = \int_0^t m_\delta(t-y)p(y)dy + \alpha(t) \tag{2.50}$$

where

$$\alpha(t) = \int_t^\infty w(t, y-t)p(y)dy. \tag{2.51}$$

Then $\alpha(t)$ is a function which does not depend on δ.

With a change of variable from t to $r = u + ct$, $m_\delta^d(u)$ can be rewritten as

$$\begin{aligned} m_\delta^d(u) &= \int_u^\infty e^{-\delta(\frac{r-u}{c})}\sigma_\delta(r)k_1(\frac{r-u}{c})\frac{dr}{c} \\ &= \frac{1}{c}\int_u^\infty e^{-\delta(\frac{t-u}{c})}\sigma_\delta(t)k_1(\frac{t-u}{c})dt. \end{aligned} \tag{2.52}$$

Usually (2.52) is used more often than (2.48) as the form of the function $k_1(t)$ is assumed and we can substitute $\frac{t-u}{c}$ in the function.

In the stationary renewal risk model, using equation (2.52), Willmot and Dickson (2003) show that the Gerber-Shiu discounted penalty function m_δ^e may be expressed as

$$m_\delta^e(u) = \frac{1}{1+\theta}\int_0^u m_\delta(u-t)dP_1(t) + q(u) \tag{2.53}$$

where

$$q(u) = e^{(\delta/c)u}\int_u^\infty e^{-(\delta/c)t}\{\tau(t) - \frac{\delta}{c(1+\theta)}\int_0^t m_\delta(t-y)dP_1(y)\}dt$$

and

$$\tau(t) = \frac{1}{(1+\theta)E(Y)}\int_t^\infty w(t, y-t)p(y)dy.$$

When $\delta = 0$, the Gerber-Shiu function simplifies further to

$$m_0^e(u) = \frac{1}{1+\theta}\int_0^u m_0(u-t)dP_1(t) + \frac{1}{(1+\theta)E(Y)}\int_u^\infty\int_t^\infty w(t, y-t)p(y)dydt. \tag{2.54}$$

Chapter 3

The Deficit at Ruin in the General Delayed Model

In section 1.2, we have introduced several quantities that are associated with ruin. The deficit at ruin, in particular, is our focus for this chapter. Willmot (2007) has studied the discounted moments of the deficit in the ordinary Sparre Andersen model and Willmot et al. (2004) have studied the proper distribution of the deficit, stochastic decomposition of the residual lifetime of L_e (the maximal aggregate loss in the stationary model) involving the deficit, and asymptotic distribution of the proper deficit in the case of the stationary renewal risk model. These results are extended to the delayed renewal risk model.

If $w(x, y) = w_2(y)$, a function of deficit only, ${m_\delta}^d(u)$ in equation (2.6) simplifies to

$$\begin{aligned} {m_\delta}^d(u) &= {\phi_\delta}^d \int_0^u m_\delta(u-y) b_\delta^d(y) dy + \int_u^\infty w_2(y-u) \int_0^\infty f_\delta^d(x, y|\ 0) dx dy \\ &= {\phi_\delta}^d \int_0^u m_\delta(u-y) b_\delta^d(y) dy + {\phi_\delta}^d \int_u^\infty w_2(y-u) b_\delta^d(y) dy. \end{aligned} \quad (3.1)$$

The analysis of the deficit at ruin starts with this nice simpler equation.

3.1 Laplace Transform of the Time of Ruin

As already introduced in section 2.2, if $w_2(x_2) = 1$, then

$$m_\delta{}^d(u) = \bar{G}_\delta^d(u) = E\{e^{-\delta T_d} I(T_d < \infty)|\ U_0 = u\}$$

satisfies

$$\bar{G}_\delta^d(u) = \phi_\delta{}^d \int_0^u \bar{G}_\delta(u-y) b_\delta^d(y) dy + \phi_\delta{}^d \bar{B}_\delta^d(u). \tag{3.2}$$

Equation (3.2) can be rewritten as

$$\bar{G}_\delta^d(u) = \phi_\delta{}^d \bar{\Lambda}_\delta(u) \tag{3.3}$$

where

$$\bar{\Lambda}_\delta(u) = \int_0^u \bar{G}_\delta(u-y) b_\delta^d(y) dy + \bar{B}_\delta^d(u) \tag{3.4}$$

is a tail of a compound geometric convolution.

Note that this can also be written as

$$\bar{\Lambda}_\delta(u) = Pr\{L_\delta + X_\delta^d > u\} \tag{3.5}$$

where X_δ^d has DF $B_\delta^d(x) = Pr\{X_\delta^d \leq x\}$ and is independent of L_δ, where L_δ satisfies $Pr\{L_\delta > x\} = \bar{G}_\delta(x)$.

Thus, the Laplace transform representation, in PDF form, of (3.4) is

$$\tilde{\lambda}_\delta(s) = \tilde{b}_\delta^d(s) E(e^{-sL_\delta}). \tag{3.6}$$

When $\delta = 0$, (3.4) simplifies to

$$\bar{\Lambda}_0(u) = \int_0^u \psi(u-t) b_0^d(t) dt + \bar{B}_0^d(u) = Pr\{L + X^d > u\} \tag{3.7}$$

where X^d has DF $B_0^d(x) = Pr\{X^d \leq x\}$ and is independent of maximal aggregate loss L in the ordinary renewal risk process.

Willmot and Lin (2001) show that a tail of a compound geometric convolution satisfies a defective renewal equation, as discussed in section 1.3 of this paper. This, in turn, implies that $\bar{G}_\delta^d(u)$ satisfies a defective renewal equation since it is just a matter of multiplying by the constant ${\phi_\delta}^d = \bar{G}_\delta^d(0)$ which is less than 1.

The explicit form of the defective renewal equation can be obtained as follows. Take Laplace transforms on both sides of (3.2) to obtain

$$\int_0^\infty e^{-su} \bar{G}_\delta^d(u) du = \phi_\delta^d \{ \frac{1 - \frac{1-\phi_\delta}{1-\phi_\delta \tilde{b}_\delta(s)}}{s} \} \tilde{b}_\delta^d(s) + \phi_\delta^d \frac{1 - \tilde{b}_\delta^d(s)}{s}.$$

Now we multiply by $\{1 - \phi_\delta \tilde{b}_\delta(s)\}$ to get

$$\begin{aligned} \{1 - \phi_\delta \tilde{b}_\delta(s)\} \int_0^\infty e^{-su} \bar{G}_\delta^d(u) du &= \phi_\delta^d \frac{1 - \phi_\delta \tilde{b}_\delta(s) - \tilde{b}_\delta^d(s) + \phi_\delta \tilde{b}_\delta^d(s)}{s} \\ &= \phi_\delta^d \{ \phi_\delta \frac{1 - \tilde{b}_\delta(s)}{s} + (1 - \phi_\delta) \frac{1 - \tilde{b}_\delta^d(s)}{s} \}, \end{aligned}$$

and thus

$$\int_0^\infty e^{-su}\bar{G}_\delta^d(u)du = \phi_\delta \tilde{b}_\delta(s)\int_0^\infty e^{-su}\bar{G}_\delta^d(u)du + \phi_\delta^d\{\phi_\delta \frac{1-\tilde{b}_\delta(s)}{s} + (1-\phi_\delta)\frac{1-\tilde{b}_\delta^d(s)}{s}\}. \tag{3.8}$$

Inverting the Laplace transform gives us

$$\bar{G}_\delta^d(u) = \phi_\delta \int_0^u \bar{G}_\delta^d(u-y)b_\delta(y)dy + \phi_\delta^d\{\phi_\delta \bar{B}_\delta(u) + (1-\phi_\delta)\bar{B}_\delta^d(u)\}. \tag{3.9}$$

We already know in the ordinary renewal risk process that $0 < \phi_\delta < 1$ and $b_\delta(y)$ is a PDF (see Willmot, 2007). This implies that (3.9) is a defective renewal equation for $\bar{G}_\delta^d(u)$.

For the solution of the renewal equation, asymptotic estimate and bounds can be obtained using the results in Willmot, Cai and Lin (2001). If $\kappa > 0$ satisfies the Lundberg condition

$$\tilde{b}_\delta(-\kappa) = \frac{1}{\phi_\delta}, \tag{3.10}$$

then the Cramer-Lundberg asymptotic formula yields

$$\bar{G}_\delta^d(u) \sim \frac{\phi_\delta^d \int_0^\infty e^{\kappa y}\{\bar{B}_\delta(y) + (1/\phi_\delta - 1)\bar{B}_\delta^d(y)\}dy}{\int_0^\infty y e^{\kappa y} b_\delta(y)dy} e^{-\kappa u},\ u \to \infty, \tag{3.11}$$

and the bounds are

$$\sigma_L(u)\psi_L(u)e^{-\kappa u} \le \bar{G}_\delta^d(u) \le \sigma_U(u)\psi_U(u)e^{-\kappa u} \tag{3.12}$$

where

$$\begin{aligned}\psi_U(u) &= \phi_\delta^d\{1+(1/\phi_\delta-1)\sup_{0\le z\le u,\bar{B}_\delta(z)>0}\frac{\bar{B}_\delta^d(z)}{\bar{B}_\delta(z)}\},\ u\ge 0,\\ \psi_L(u) &= \phi_\delta^d\{1+(1/\phi_\delta-1)\inf_{0\le z\le u,\bar{B}_\delta(z)>0}\frac{\bar{B}_\delta^d(z)}{\bar{B}_\delta(z)}\},\ u\ge 0,\\ \sigma_U(u) &= \sup_{0\le z\le u,\bar{B}_\delta(z)>0}\frac{e^{\kappa z}\bar{B}_\delta(z)}{\int_z^\infty e^{\kappa y}b_\delta(y)dy},\ u\ge 0,\\ \sigma_L(u) &= \inf_{0\le z\le u,\bar{B}_\delta(z)>0}\frac{e^{\kappa z}\bar{B}_\delta(z)}{\int_z^\infty e^{\kappa y}b_\delta(y)dy},\ u\ge 0.\end{aligned}$$

Lin (1996) has analyzed $\sigma_U(u)$ and $\sigma_L(u)$ for many reliability classes.

The asymptotic estimate is helpful in observing the behavior of large u's, whereas the bounds give insight into the behavior of small u's.

3.2 Discounted kth Moment of the Deficit

If $w_2(x_2)=x_2^k$, then $m_\delta{}^d(u)=r_{k,\delta}^d(u)=E\{e^{-\delta T_d}|U_{T_d}|^k I(T_d<\infty)|\ U_0=u\}$ satisfies

$$r_{k,\delta}^d(u)=\phi_\delta{}^d\int_0^u r_{k,\delta}(u-t)b_\delta^d(t)dt+\phi_\delta{}^d\int_u^\infty (y-u)^k b_\delta^d(y)dy \tag{3.13}$$

Now, let $\mu_{k,\delta}=\int_0^\infty y^k dB_\delta(y)$ where $B_\delta(y)$ is the ladder-height DF in the ordinary renewal risk process. Also define the kth order equilibrium DF $B_{k,\delta}^d(y)=1-\bar{B}_{k,\delta}^d(y)$ recursively by

$$B_{k,\delta}^d(y)=\frac{\int_0^y \bar{B}_{k-1,\delta}^d(t)dt}{\int_0^\infty \bar{B}_{k-1,\delta}^d(t)dt},\ k=1,2,...,\tag{3.14}$$

where $B^d_{0,\delta}(y) = B^d_\delta(y)$, as long as $\int_0^\infty \bar{B}^d_{k-1,\delta}(t)dt < \infty$. Then Hesselager et al. (1998) show that the representation in (3.14) could be rewritten as

$$\bar{B}^d_{k,\delta}(y) = \frac{\int_y^\infty (t-y)^k b^d_\delta(t)dt}{\int_0^\infty t^k b^d_\delta(t)dt}, \ k = 1, 2, ... \tag{3.15}$$

If we define the DF of the compound geometric convolution $H_{k,\delta}(u) = 1 - \bar{H}_{k,\delta}(u)$ by

$$\bar{H}_{k,\delta}(u) = \int_0^u \bar{G}_\delta(u-y)dB_{k,\delta}(y) + \bar{B}_{k,\delta}(u) \tag{3.16}$$

where $B_{k,\delta}(y)$ is the kth order equilibrium DF of the ladder-height DF $B_\delta(y)$ in the ordinary renewal risk process, Willmot (2007) shows that

$$r_{k,\delta}(u) = \frac{\phi_\delta \mu_{k+1,\delta}}{(1-\phi_\delta)(k+1)} h_{k+1,\delta}(u) \tag{3.17}$$

where $h_{k+1,\delta}(u)$ is the density of $H_{k+1,\delta}(u)$.

Using equation (3.17) and (3.15), (3.13) results in

$$r^d_{k,\delta}(u) = \frac{{\phi_\delta}^d \phi_\delta \mu_{k+1,\delta}}{(1-\phi_\delta)(k+1)} \int_0^u h_{k+1,\delta}(u-t)b^d_\delta(t)dt + {\phi_\delta}^d \{\int_0^\infty y^k b^d_\delta(y)dy\} \bar{B}^d_{k,\delta}(u). \tag{3.18}$$

From Theorem 4.1 of Willmot (2007), the above equation can be expressed in terms of stop loss moments. Hence,

$$\begin{aligned} r^d_{k,\delta}(u) &= \phi^d_\delta \int_0^u r_{k,\delta}(u-y) b^d_\delta(y) dy + \phi^d_\delta \int_u^\infty (y-u)^k b^d_\delta(y) dy \\ &= \frac{\phi^d_\delta}{1-\phi_\delta} v_k(u) - \frac{\phi_\delta \phi^d_\delta}{1-\phi_\delta} \sum_{j=0}^{k} \binom{k}{j} \mu_{k-j,\delta} v_j(u) \\ &\quad + \phi^d_\delta \int_u^\infty (y-u)^k b^d_\delta(y) dy \qquad (3.19) \end{aligned}$$

where

$$\begin{aligned} v_j(u) &= \int_0^u \int_{u-y}^\infty (t-(u-y))^j dG_\delta(t) b^d_\delta(y) dy \qquad (3.20) \\ &= \int_0^u [\bar{G}_{j,\delta}(u-y) \int_0^\infty t^j dG_\delta(t)] b^d_\delta(y) dy \qquad (3.21) \end{aligned}$$

since the tail of the kth order equilibrium DF of $G_\delta(y)$, $\bar{G}_{k,\delta}(y)$ satisfies

$$\bar{G}_{k,\delta}(y) = \frac{\int_y^\infty (t-y)^k dG_\delta(t)}{\int_0^\infty t^k dG_\delta(t)} \qquad (3.22)$$

and thus

$$v_j(u) = \int_0^\infty t^j dG_\delta(t) \cdot \int_0^u \bar{G}_{j,\delta}(u-y) b^d_\delta(y) dy. \qquad (3.23)$$

The Laplace transform of $v_j(u)$ can be obtained as

$$\tilde{v}_j(s) = \int_0^\infty t^j dG_\delta(t) \tilde{b}^d_\delta(s) \sum_{k=1}^{j} p_k(j) \{ \frac{1-\tilde{g}_\delta(s)\tilde{b}_{k,\delta}(s)}{s} \} \qquad (3.24)$$

using corollary 4.1. of Willmot (2002b), i.e.

$$\tilde{g}_{j,\delta}(s) = \tilde{g}_{\delta}(s) \sum_{k=1}^{j} p_k(j) \tilde{b}_{k,\delta}(s) \tag{3.25}$$

where $\{p_1(j), p_2(j), \ldots, p_j(j)\}$ is a discrete probability measure and $\tilde{b}_{k,\delta}(s)$ is the Laplace transform of the kth equilibrium density of $b_\delta(y)$.

Example 3.2.1 Erlang mixture claim size distribution

When claim sizes are of mixed Erlang type, i.e.

$$p(y) = \sum_{j=1}^{\infty} a_j \frac{\beta(\beta y)^{j-1} e^{-\beta y}}{(j-1)!}, \tag{3.26}$$

Willmot (2007) shows that the stop-loss moments of the compound geometric distribution satisfy

$$\int_u^\infty (t-u)^j dG_\delta(t) = e^{-\beta u} \sum_{m=0}^{\infty} \gamma_{m,j} \frac{(\beta u)^m}{m!} \tag{3.27}$$

where $\gamma_{m,j} = \beta^{-j} \sum_{n=1}^{\infty} c_{m+n,\delta} \Gamma(j+n)/\Gamma(n)$ and again $c_{n,\delta}$ is defined by (1.33) in section 1.4, except that $\{c_{n,\delta}; n = 0, 1, 2, ...\}$ now has compound geometric probability generating function, instead of compound geometric generating function, since the mixing weights in the ladder-height DF are all positive and sum to 1, i.e.,

$$C(z) = \sum_{n=0}^{\infty} c_{n,\delta} z^n = P\{Q(z)\}, \tag{3.28}$$

where P(z) is the probability generating function of the geometric distribution and Q(z) is the probability generating function of the mixing weights for Erlangs in the ladder-height distribution.

Also, $b_\delta^d(y)$ is known in this case from example 2.1.2 with $k = 1$.

$$b_\delta^d(y) = \sum_{l=1}^{\infty} a_l^d(\delta) \frac{\beta(\beta y)^{l-1} e^{-\beta y}}{(l-1)!} \tag{3.29}$$

where $\{a_1^d(\delta), a_2^d(\delta), \ldots\}$ are constants depending on δ.
Thus, substitution of (3.27) and (3.29) into equation (3.21) gives

$$\begin{aligned} v_j(u) &= \int_0^u e^{-\beta(u-y)} \sum_{m=0}^{\infty} \gamma_{m,j} \frac{\beta^m (u-y)^m}{m!} \sum_{l=1}^{\infty} a_l^d(\delta) \frac{\beta^l y^{l-1} e^{-\beta y}}{(l-1)!} dy \\ &= e^{-\beta u} \sum_{m=0}^{\infty} \gamma_{m,j} \sum_{l=1}^{\infty} a_l^d(\delta) \frac{\beta^{m+l}}{m!(l-1)!} \int_0^u (u-y)^m y^{l-1} dy. \end{aligned}$$

Using Beta type integration, $\int_0^u (u-y)^m y^{l-1} dy = u^{m+l}\Gamma(m+1)\Gamma(l)/\Gamma(m+l+1)$, $v_j(u)$ becomes

$$\begin{aligned} v_j(u) &= e^{-\beta u} \sum_{m=0}^{\infty} \gamma_{m,j} \sum_{l=1}^{\infty} a_l^d(\delta) \frac{(\beta u)^{m+l}}{(m+l)!} \\ &= e^{-\beta u} \sum_{m=0}^{\infty} \gamma_{m,j} \sum_{r=m+1}^{\infty} a_{r-m}^d(\delta) \frac{(\beta u)^r}{r!} \\ &= e^{-\beta u} \sum_{r=1}^{\infty} \sum_{m=0}^{r-1} \gamma_{m,j} a_{r-m}^d(\delta) \frac{(\beta u)^r}{r!} \\ &= e^{-\beta u} \sum_{r=1}^{\infty} B_{j,r}(\delta) \frac{(\beta u)^r}{r!} \end{aligned} \tag{3.30}$$

where $B_{j,r}(\delta) = \sum_{m=0}^{r-1} \gamma_{m,j} a_{r-m}^d(\delta)$. The second equality is obtained from changing variables by letting $r = m + l$ and the third equality from changing order of summations.

Also the last term in equation (3.19) can be simplified in the case of Erlang mixture claim sizes.

$$\begin{aligned}\int_u^\infty (y-u)^k b_\delta^d(y)dy &= \int_0^\infty x^k b_\delta^d(x+u)dx \\ &= \int_0^\infty x^k \sum_{l=1}^\infty a_l^d(\delta)\frac{\beta^l (x+u)^{l-1}e^{-\beta(x+u)}}{(l-1)!}dx \\ &= \sum_{l=1}^\infty a_l^d(\delta)e^{-\beta u}\frac{\beta^l}{(l-1)!}\int_0^\infty x^k(x+u)^{l-1}e^{-\beta x}dx.\end{aligned}$$

The integral $\int_0^\infty x^k(x+u)^{l-1}e^{-\beta x}dx$ can be evaluated as

$$\begin{aligned}\int_0^\infty x^k(x+u)^{l-1}e^{-\beta x}dx &= \sum_{i=0}^{l-1}\binom{l-1}{i}u^{l-1-i}\int_0^\infty x^{k+i}e^{-\beta x}dx \\ &= \sum_{i=0}^{l-1}\binom{l-1}{i}u^{l-1-i}\frac{(k+i)!}{\beta^{k+i+1}}.\end{aligned}$$

And thus,

$$\begin{aligned}\int_u^\infty (y-u)^k b_\delta^d(y)dy &= \sum_{l=1}^\infty a_l^d(\delta)e^{-\beta u}\sum_{i=0}^{l-1}\frac{(k+i)!\beta^{l-k-i-1}u^{l-1-i}}{i!(l-1-i)!} \\ &= e^{-\beta u}\sum_{l=1}^\infty a_l^d(\delta)\sum_{i=0}^{l-1}\frac{(k+i)!}{\beta^k i!}\frac{(\beta u)^{l-i-1}}{(l-i-1)!} \\ &= e^{-\beta u}\sum_{l=1}^\infty a_l^d(\delta)\sum_{r=0}^{l-1}\frac{(k+l-r-1)!}{\beta^k(l-r-1)!}\frac{(\beta u)^r}{r!} \\ &= e^{-\beta u}\sum_{r=0}^\infty\sum_{l=r+1}^\infty a_l^d(\delta)\frac{(k+l-r-1)!}{\beta^k(l-r-1)!}\frac{(\beta u)^r}{r!} \\ &= e^{-\beta u}\sum_{r=0}^\infty A_{k,r}(\delta)\frac{(\beta u)^r}{r!}\end{aligned}$$

where $A_{k,r}(\delta) = \sum_{l=r+1}^{\infty} a_l^d(\delta)(k+l-r-1)!/(\beta^k(l-r-1)!)$.

Therefore,

$$\begin{aligned}
r_{k,\delta}^d(u) &= \frac{\phi_\delta^d}{1-\phi_\delta} e^{-\beta u} \sum_{r=1}^{\infty} B_{k,r}(\delta) \frac{(\beta u)^r}{r!} \\
&\quad - \frac{\phi_\delta^d \phi_\delta}{1-\phi_\delta} \sum_{j=0}^{k} \binom{k}{j} \mu_{k-j,\delta} e^{-\beta u} \sum_{r=1}^{\infty} B_{j,r}(\delta) \frac{(\beta u)^r}{r!} \\
&\quad + \phi_\delta^d e^{-\beta u} \sum_{r=0}^{\infty} A_{k,r}(\delta) \frac{(\beta u)^r}{r!} \\
&= e^{-\beta u} \sum_{r=1}^{\infty} [\frac{\phi_\delta^d}{1-\phi_\delta} B_{k,r}(\delta) - \frac{\phi_\delta^d \phi_\delta}{1-\phi_\delta} \sum_{j=0}^{k} \binom{k}{j} \mu_{k-j,\delta} B_{j,r}(\delta)] \frac{(\beta u)^r}{r!} \\
&\quad + e^{-\beta u} \sum_{r=0}^{\infty} [\phi_\delta^d A_{k,r}(\delta)] \frac{(\beta u)^r}{r!} \\
&= e^{-\beta u} \sum_{r=0}^{\infty} D_{k,r}(\delta) \frac{(\beta u)^r}{r!}
\end{aligned}$$

where

$$D_{k,r}(\delta) = \begin{cases} \phi_\delta^d A_{k,0}(\delta) & r = 0 \\ \frac{\phi_\delta^d}{1-\phi_\delta} B_{k,r}(\delta) - \frac{\phi_\delta^d \phi_\delta}{1-\phi_\delta} \sum_{j=0}^{k} \binom{k}{j} \mu_{k-j,\delta} B_{j,r}(\delta) + \phi_\delta^d A_{k,r}(\delta) & r = 1, 2, ... \end{cases}, \tag{3.31}$$

which is a damped exponential series.♣

3.3 Discounted Distribution Function of the Proper Deficit

When $w_2(x_2) = I(x_2 > y)$, the Gerber-Shiu expected discounted penalty function becomes the discounted defective survival function of the deficit.
Then, ${m_\delta}^d(u) = \bar{G}^d_\delta(u, y) = E\{e^{-\delta T_d} I(|U_{T_d}| > y) I(T_d < \infty)|\ U_0 = u\}$ satisfies

$$\begin{aligned}\bar{G}^d_\delta(u, y) &= {\phi_\delta}^d \int_0^u \bar{G}_\delta(u-t, y) b^d_\delta(t)dt + {\phi_\delta}^d \int_u^\infty I(t-u > y) b^d_\delta(t)dt \\ &= {\phi_\delta}^d \int_0^u \bar{G}_\delta(u-t, y) b^d_\delta(t)dt + {\phi_\delta}^d \int_{u+y}^\infty b^d_\delta(t)dt\end{aligned}$$

i.e.

$$\bar{G}^d_\delta(u, y) = {\phi_\delta}^d \int_0^u \bar{G}_\delta(u-t, y) b^d_\delta(t)dt + {\phi_\delta}^d \bar{B}^d_\delta(u+y). \tag{3.32}$$

Further, if we let $\delta = 0$, then

$$ {m_0}^d(u) = \bar{G}^d(u, y) = E\{I(|U_{T_d}| > y) I(T_d < \infty)|\ U_0 = u\} \tag{3.33}$$

satisfies

$$\bar{G}^d(u, y) = \psi^d(0) \int_0^u \bar{G}(u-t, y) b^d_0(t)dt + \psi^d(0) \bar{B}^d_0(u+y) \tag{3.34}$$

where $\psi^d(0) = \phi^d_0$ is the ruin probability in the delayed renewal risk process with initial surplus 0.

Equation (3.34) can also be argued probabilistically. When the first drop below its initial surplus level occurs it may cause ruin. Otherwise ruin does not occur and the process starts again in the ordinary renewal risk process with initial surplus $u - t$ where $t < u$. If ruin occurs, for the deficit to be greater than y when the initial surplus is u, the drop amount should be greater than $u + y$. The first term in (3.34) explains the case where ruin does not occur on the first drop and the second term explains the case where ruin occurs. Since the argument conditions on the first drop below initial surplus level, the probability for the occurrence of it is multiplied.

Theorem 3.1 *The discounted proper distribution function of the deficit, $\bar{G}^d_{\delta,u}(y)$, is expressed as*

$$\begin{aligned}
\bar{G}^d_{\delta,u}(y) &= \frac{\bar{G}^d_\delta(u,y)}{\bar{G}^d_\delta(u)} \\
&= \frac{\bar{G}_\delta(0)\int_0^u \bar{B}_{\delta,u-t}(y)\bar{B}_\delta(u-t)d\Lambda_\delta(t) + (1-\bar{G}_\delta(0))\bar{B}^d_{\delta,u}(y)\bar{B}^d_\delta(u)}{\bar{G}_\delta(0)\int_0^u \bar{B}_\delta(u-t)d\Lambda_\delta(t) + (1-\bar{G}_\delta(0))\bar{B}^d_\delta(u)}, \quad (3.35)
\end{aligned}$$

where $B_{\delta,t}(x) = 1 - \bar{B}_{\delta,t}(x)$ is the residual lifetime DF of $B_\delta(x)$, i.e., $\bar{B}_{\delta,t}(x) = \bar{B}_\delta(t+x)/\bar{B}_\delta(t)$.

When $\delta = 0$, the proper distribution function of the deficit, $\bar{G}^d_u(y)$, is

$$\begin{aligned}
\bar{G}^d_u(y) &= \frac{\bar{G}^d(u,y)}{\psi^d(u)} \\
&= \frac{\psi(0)\int_0^u \bar{B}_{0,u-t}(y)\bar{B}_0(u-t)d\Lambda_0(t) + (1-\psi(0))\bar{B}^d_{0,u}(y)\bar{B}^d_0(u)}{\psi(0)\int_0^u \bar{B}_0(u-t)d\Lambda_0(t) + (1-\psi(0))\bar{B}^d_0(u)}. \quad (3.36)
\end{aligned}$$

Proof:

The Laplace transform of $\bar{G}_\delta(u)$, in PDF form, can be written as

$$\tilde{g}_\delta(s) = E(e^{-sL_\delta}) = \frac{1-\phi_\delta}{1-\phi_\delta \tilde{b}_\delta(s)} \tag{3.37}$$

where $\phi_\delta = \bar{G}_\delta(0)$ and using Proposition 2.1. of Willmot (2002a) we obtain

$$\bar{G}_\delta(u, y) = \frac{\phi_\delta}{1-\phi_\delta} \int_{0^-}^{u} \bar{B}_\delta(u+y-t) dG_\delta(t) \tag{3.38}$$

where $G_\delta(t) = 1 - \bar{G}_\delta(t) = P(L_\delta \leq t)$.

Using equation (3.38), we want to rewrite (3.32) in terms of simpler functions $B_\delta(x)$, $B_\delta^d(x)$ and $G_\delta(x)$.

The LT of $\int_0^u \bar{G}_\delta(u-t, y) b_\delta^d(t) dt$ in equation (3.32) is

$$\int_0^\infty e^{-su} \int_0^u \bar{G}_\delta(u-t, y) b_\delta^d(t) dt du = \tilde{b}_\delta^d(s) \int_0^\infty e^{-su} \bar{G}_\delta(u, y) du. \tag{3.39}$$

Using (3.38),

$$\begin{aligned} \int_0^\infty e^{-su} \bar{G}_\delta(u, y) du &= \frac{\phi_\delta}{1-\phi_\delta} \int_0^\infty e^{-su} \bar{B}_\delta(u+y) du \int_0^\infty e^{-su} dG_\delta(u) \\ &= \frac{\phi_\delta}{1-\phi_\delta} E(e^{-sL_\delta}) \int_0^\infty e^{-su} \bar{B}_\delta(u+y) du. \end{aligned}$$

Thus, equation (3.39) becomes

$$\begin{aligned}\int_0^\infty e^{-su}\int_0^u \bar{G}_\delta(u-t,y)b_\delta^d(t)dtdu &= \tilde{b}_\delta^d(s)\frac{\phi_\delta}{1-\phi_\delta}E(e^{-sL_\delta})\int_0^\infty e^{-su}\bar{B}_\delta(u+y)du \\ &= \tilde{\lambda}_\delta(s)\frac{\phi_\delta}{1-\phi_\delta}\int_0^\infty e^{-su}\bar{B}_\delta(u+y)du.\end{aligned}$$

By inversion of the LT, we have

$$\int_0^u \bar{G}_\delta(u-t,y)b_\delta^d(t)dtdu = \frac{\phi_\delta}{1-\phi_\delta}\int_0^u \bar{B}_\delta(u+y-t)d\Lambda_\delta(t) \tag{3.40}$$

and thus equation (3.32) becomes

$$\bar{G}_\delta^d(u,y) = \phi_\delta^d\frac{\phi_\delta}{1-\phi_\delta}\int_0^u \bar{B}_\delta(u+y-t)d\Lambda_\delta(t) + \phi_\delta^d\bar{B}_\delta^d(u+y). \tag{3.41}$$

The discounted ruin probability can be obtained by letting $y=0$,

$$\bar{G}_\delta^d(u) = \bar{G}_\delta^d(u,0) = \phi_\delta^d\frac{\phi_\delta}{1-\phi_\delta}\int_0^u \bar{B}_\delta(u-t)d\Lambda_\delta(t) + \phi_\delta^d\bar{B}_\delta^d(u). \tag{3.42}$$

If we denote the discounted proper distribution function of the deficit by $\bar{G}_{\delta,u}^d(y)$, it is expressed as

$$\begin{aligned}\bar{G}^d_{\delta,u}(y) &= \frac{\bar{G}^d_\delta(u,y)}{\bar{G}^d_\delta(u)} \\ &= \frac{\phi_\delta \int_0^u \bar{B}_\delta(u+y-t)d\Lambda_\delta(t) + (1-\phi_\delta)\bar{B}^d_\delta(u+y)}{\phi_\delta \int_0^u \bar{B}_\delta(u-t)d\Lambda_\delta(t) + (1-\phi_\delta)\bar{B}^d_\delta(u)} \\ &= \frac{\phi_\delta \int_0^u \bar{B}_{\delta,u-t}(y)\bar{B}_\delta(u-t)d\Lambda_\delta(t) + (1-\phi_\delta)\bar{B}^d_{\delta,u}(y)\bar{B}^d_\delta(u)}{\phi_\delta \int_0^u \bar{B}_\delta(u-t)d\Lambda_\delta(t) + (1-\phi_\delta)\bar{B}^d_\delta(u)}. \qquad (3.43)\end{aligned}$$

Q.E.D.

Note that the discounted proper distribution function of the deficit, $G^d_{\delta,u}(y)$ is a mixture of the residual lifetime DF's $B_{\delta,t}(y)$ for $0 < t < u$ and of $B^d_{\delta,u}(y)$. The proper distribution function of the deficit in the delayed renewal risk model has the same form as in the stationary renewal risk model (see Willmot et al., 2004, eq.(2.5), p.245), with the change of the ladder-height DF of the stationary model, which is the equilibrium DF of the claim sizes ($P_1(y)$), replaced by that of the delayed model.

The associated probability density function can be obtained by taking the derivative of $\bar{G}^d_{\delta,u}(y)$ with respect to y.

Corollary 3.2 *The discounted probability density function of the proper deficit is*

$$g^d_{\delta,u}(y) = \frac{\phi_\delta \int_0^u b_{\delta,u-t}(y)\bar{B}_\delta(u-t)d\Lambda_\delta(t) + (1-\phi_\delta)b^d_{\delta,u}(y)\bar{B}^d_\delta(u)}{\phi_\delta \int_0^u \bar{B}_\delta(u-t)d\Lambda_\delta(t) + (1-\phi_\delta)\bar{B}^d_\delta(u)}, \qquad (3.44)$$

which is a mixture of residual lifetime probability densities $b_{\delta,u-t}(y)$ *and* $b^d_{\delta,u}(y)$.

We can identify the proper distribution function of the deficit easily using the mixture representations above in some special cases.

Example 3.3.1 Exponential claim size distribution

When claim sizes are exponentially distributed, i.e., $p(y) = \beta e^{-\beta y}$, we know from Example 2.1.1 that $b_\delta^d(y)$ also has the same exponential distribution and it is easy to see that

$$\begin{aligned}\bar{G}_{\delta,u}^d(y) &= \frac{\phi_\delta \int_0^u \bar{B}_\delta(u+y-t)d\Lambda_\delta(t) + (1-\phi_\delta)\bar{B}_\delta^d(u+y)}{\phi_\delta \int_0^u \bar{B}_\delta(u-t)d\Lambda_\delta(t) + (1-\phi_\delta)\bar{B}_\delta^d(u)} \\ &= \frac{\phi_\delta \int_0^u e^{-\beta(u+y-t)}d\Lambda_\delta(t) + (1-\phi_\delta)e^{-\beta(u+y)}}{\phi_\delta \int_0^u e^{-\beta(u-t)}d\Lambda_\delta(t) + (1-\phi_\delta)e^{-\beta u}} \\ &= e^{-\beta y}.\end{aligned}$$

Thus, the discounted proper distribution function of the deficit in this example has the same exponential distribution as the claim size distribution function, regardless of the first interclaim time distribution function. ♣

Example 3.3.2 Coxian - Class claim size distribution

More generally, let's assume that the claim sizes have a distribution function of Coxian - class with PDF of the form as in Example 2.1.2. Then we also know from the same Example that

$$b_0^d(y) = \sum_{i=1}^{k}\sum_{j=1}^{r} {a_{ij}}^d(0)\frac{\beta_i(\beta_i y)^{j-1}e^{-\beta_i y}}{(j-1)!}$$

where

$$a_{ij}{}^d(0) = \frac{1}{\phi_0{}^d} \sum_{m=j}^{r} \frac{a_{im}}{(m-j)!} \int_0^\infty \frac{(\beta_i x)^{m-j} f_0{}^d(x|\ 0)}{e^{\beta_i x} \bar{P}(x)} dx.$$

Then $b_0^d(x+y)$ satisfies

$$\begin{aligned}
b_0^d(x+y) &= \sum_{i=1}^{k} \sum_{j=1}^{r} a_{ij}{}^d(0) \frac{\beta_i^j (x+y)^{j-1} e^{-\beta_i(x+y)}}{(j-1)!} \\
&= \sum_{i=1}^{k} \sum_{j=1}^{r} a_{ij}{}^d(0) \frac{\beta_i^j}{(j-1)!} e^{-\beta_i(x+y)} \sum_{m=0}^{j-1} \frac{(j-1)!}{m!(j-m-1)!} x^{j-m-1} y^m \\
&= \sum_{i=1}^{k} \sum_{j=1}^{r} a_{ij}{}^d(0) e^{-\beta_i x} \sum_{m=0}^{j-1} \frac{(\beta_i x)^{j-m-1}}{(j-m-1)!} \frac{\beta_i (\beta_i y)^m e^{-\beta_i y}}{m!} \\
&= \sum_{i=1}^{k} \sum_{j=1}^{r} a_{ij}{}^d(0) e^{-\beta_i x} \sum_{m=1}^{j} \frac{(\beta_i x)^{j-m}}{(j-m)!} \frac{\beta_i (\beta_i y)^{m-1} e^{-\beta_i y}}{(m-1)!}
\end{aligned}$$

and changing the order of summation between j and m gives

$$\begin{aligned}
b_0^d(x+y) &= \sum_{i=1}^{k} \sum_{m=1}^{r} \sum_{j=m}^{r} a_{ij}{}^d(0) e^{-\beta_i x} \frac{(\beta_i x)^{j-m}}{(j-m)!} \frac{\beta_i (\beta_i y)^{m-1} e^{-\beta_i y}}{(m-1)!} \\
&= \sum_{i=1}^{k} \sum_{j=1}^{r} \sum_{m=j}^{r} a_{im}{}^d(0) e^{-\beta_i x} \frac{(\beta_i x)^{m-j}}{(m-j)!} \left(\frac{\beta_i (\beta_i y)^{j-1} e^{-\beta_i y}}{(j-1)!} \right),
\end{aligned}$$

i.e.

$$b_0^d(x+y) = \sum_{i=1}^{k} \sum_{j=1}^{r} \alpha_{ij}^d(x) \left(\frac{\beta_i (\beta_i y)^{j-1} e^{-\beta_i y}}{(j-1)!} \right) \tag{3.45}$$

where

$$\alpha_{ij}^{d}(x) = e^{-\beta_i x} \sum_{m=j}^{r} a_{im}{}^{d}(0) \frac{(\beta_i x)^{m-j}}{(m-j)!}. \tag{3.46}$$

And by the analogy between $b_0^d(y)$ and $b_0(y)$,

$$b_0(x+y) = \sum_{i=1}^{k} \sum_{j=1}^{r} \alpha_{ij}(x) \left(\frac{\beta_i (\beta_i y)^{j-1} e^{-\beta_i y}}{(j-1)!}\right) \tag{3.47}$$

where

$$\alpha_{ij}(x) = e^{-\beta_i x} \sum_{m=j}^{r} a_{im}(0) \frac{(\beta_i x)^{m-j}}{(m-j)!}. \tag{3.48}$$

From equation (3.44),

$$g_u^d(y) = \frac{\psi(0) \int_0^u b_0(u-t+y) d\Lambda_0(t) + (1-\psi(0)) b_0^d(u+y)}{\psi(0) \int_0^u \bar{B}_0(u-t) d\Lambda_0(t) + (1-\psi(0)) \bar{B}_0^d(u)}, \tag{3.49}$$

and substitution of (3.45) and (3.47) into (3.49) leads to

$$\begin{aligned}
& g_u^d(y) \\
&= \frac{\psi(0) \int_0^u \sum_{i=1}^{k} \sum_{j=1}^{r} \alpha_{ij}(u-t) \tau_{ij}(y) d\Lambda_0(t) + (1-\psi(0)) \sum_{i=1}^{k} \sum_{j=1}^{r} \alpha_{ij}^d(u) \tau_{ij}(y)}{\psi(0) \int_0^u \bar{B}_0(u-t) d\Lambda_0(t) + (1-\psi(0)) \bar{B}_0^d(u)} \\
&= \sum_{i=1}^{k} \sum_{j=1}^{r} \left\{ \frac{\psi(0) \int_0^u \alpha_{ij}(u-t) d\Lambda_0(t) + (1-\psi(0)) \alpha_{ij}^d(u)}{\psi(0) \int_0^u \bar{B}_0(u-t) d\Lambda_0(t) + (1-\psi(0)) \bar{B}_0^d(u)} \right\} \tau_{ij}(y)
\end{aligned}$$

where

$$\tau_{ij}(y) = \frac{\beta_i(\beta_i y)^{j-1}e^{-\beta_i y}}{(j-1)!},$$

i.e.

$$g_u^d(y) = \sum_{i=1}^{k}\sum_{j=1}^{r} C_{ij}(u)\frac{\beta_i(\beta_i y)^{j-1}e^{-\beta_i y}}{(j-1)!} \tag{3.50}$$

where

$$C_{ij}(u) = \frac{\psi(0)\int_0^u \alpha_{ij}(u-t)d\Lambda_0(t) + (1-\psi(0))\alpha_{ij}^d(u)}{\psi(0)\int_0^u \bar{B}_0(u-t)d\Lambda_0(t) + (1-\psi(0))\bar{B}_0^d(u)}. \tag{3.51}$$

Therefore, $g_u^d(y)$ is of the same form as the claim size distribution, which is Coxian - class, with weights changed. ♣

As already shown in section 2.1 for $b_\delta^d(y)$, the mixture representation of $g_{\delta,u}^d(y)$ in (3.44) also makes $g_{\delta,u}^d(y)$ to have the same form of distribution as the discounted ladder-height distributions $b_\delta^d(y)$ and $b_\delta^d(y)$, which in turn makes $g_{\delta,u}^d(y)$ to have the same form of distribution as the original claim size distribution. Also with the same logic already mentioned at the end of section 2.1 for $b_\delta^d(y)$, mixing representation of $g_{\delta,u}^d(y)$ also preserves the DFR property if $b_\delta^d(y)$ and $b_\delta^d(y)$ are DFR (p.10, Willmot and Lin, 2001). Thus, if the claim size distribution $p(y)$ has DFR property, the discounted ladder-height distributions $b_\delta^d(y)$ and $b_\delta^d(y)$ do as well (section 2.1) and so does the discounted proper distribution of the deficit $g_{\delta,u}^d(y)$.

3.4 Asymptotic Distribution of the Proper Deficit

In this section we will examine what will happen when the DF of the proper deficit is asymptotic in the initial surplus u. We can guess that it will be a mixture representation as in Section 3.3 but with simpler mixing weights, since the weights in Section 3.3 are functions of u and these should converge for an asymptotic formula to exist. It is not straightforward to take the limit of u from the proper distribution derived in Section 3.3, so we follow the idea of Theorem 2.3 in Willmot, Dickson, Drekic and Stanford (2004).

Theorem 3.3 *The asymptotic distribution of the proper deficit is*

$$\lim_{u\to\infty} \bar{G}_u^d(y) = \lim_{u\to\infty} \bar{G}_u(y) = \frac{\int_0^\infty e^{\kappa t}\bar{B}_0(y+t)dt}{\int_0^\infty e^{\kappa t}\bar{B}_0(t)dt} = \frac{\int_0^\infty e^{\kappa t}\bar{B}_{0,t}(y)\bar{B}_0(t)dt}{\int_0^\infty e^{\kappa t}\bar{B}_0(t)dt}. \tag{3.52}$$

Proof:

Multiplying by $e^{\kappa u}$ where κ is the adjustment coefficient defined as in Section 1.3 and taking the limit as $u \to \infty$ on both sides of the equation (3.34), we obtain

$$\lim_{u\to\infty} e^{\kappa u}\bar{G}^d(u,y) = \psi^d(0)\lim_{u\to\infty} e^{\kappa u}\int_0^u \bar{G}(u-t,y)dB_0^d(t) + \psi^d(0)\lim_{u\to\infty} e^{\kappa u}\bar{B}_0^d(u+y). \tag{3.53}$$

By Lundberg's inequality in the ordinary renewal risk model,

$$e^{\kappa u}\bar{G}(u,y) \le e^{\kappa u}\psi(u) \le 1, \tag{3.54}$$

and by dominated convergence, equation (3.53) can be rewritten as

$$\lim_{u\to\infty} e^{\kappa u}\bar{G}^d(u,y) = \psi^d(0)\int_0^\infty \{\lim_{u\to\infty} e^{\kappa u}\bar{G}(u,y)\}e^{\kappa t}dB_0^d(t) + \psi^d(0)\lim_{u\to\infty} e^{\kappa u}\bar{B}_0^d(u+y). \tag{3.55}$$

By equation (2.28),

$$\lim_{u\to\infty} e^{\kappa u}\bar{B}_0^d(u) = \lim_{u\to\infty} e^{\kappa u}\frac{\psi(0)\bar{B}_0(u) - \frac{\psi^d(u)}{\psi^d(0)} + \psi(0)\int_0^u \frac{\psi^d(u-t)}{\psi^d(0)}dB_0(t)}{\psi(0)-1} \tag{3.56}$$

and since (1.18) holds,

$$\lim_{u\to\infty} e^{\kappa u}\bar{B}_0(u) \le \lim_{u\to\infty}\int_u^\infty e^{\kappa y}dB_0(y) = 0, \tag{3.57}$$

the first term in equation (3.56) disappears and the equation reduces to

$$\begin{aligned}\lim_{u\to\infty} e^{\kappa u}\bar{B}_0^d(u) &= \frac{\psi(0)}{\{\psi(0)-1\}\psi^d(0)}\int_0^\infty \{\lim_{u\to\infty} e^{\kappa u}\psi^d(u)\}e^{\kappa t}dB_0(t)\\ &\quad -\frac{1}{\{\psi(0)-1\}\psi^d(0)}\lim_{u\to\infty} e^{\kappa u}\psi^d(u).\end{aligned}$$

The Lundberg's adjustment coefficient, κ satisfies $\tilde{b}_0(-\kappa) = \frac{1}{\psi(0)}$ and $\lim_{u\to\infty} e^{\kappa u}\psi^d(u)$ is a constant (Willmot and Lin (2001), Theorem 11.4.3). These lead to

$$\begin{aligned}&\lim_{u\to\infty} e^{\kappa u}\bar{B}_0^d(u)\\ &= \frac{\psi(0)}{\{\psi(0)-1\}\psi^d(0)}\frac{1}{\psi(0)}\lim_{u\to\infty} e^{\kappa u}\psi^d(u) - \frac{1}{\{\psi(0)-1\}\psi^d(0)}\lim_{u\to\infty} e^{\kappa u}\psi^d(u)\\ &= 0.\end{aligned}$$

Also,

$$0 \leq \lim_{u\to\infty} e^{\kappa u}\bar{B}_0^d(u+y) \leq \lim_{u\to\infty} e^{\kappa u}\bar{B}_0^d(u) = 0 \tag{3.58}$$

implies that

$$\lim_{u\to\infty} e^{\kappa u}\bar{B}_0^d(u+y) = 0. \tag{3.59}$$

Now, equation (3.55) becomes

$$\lim_{u\to\infty} e^{\kappa u}\bar{G}^d(u,y) = \psi^d(0)\tilde{b}_0^d(-\kappa)\{\lim_{u\to\infty} e^{\kappa u}\bar{G}(u,y)\} \tag{3.60}$$

and since

$$\bar{G}_u^d(y) = \frac{\bar{G}^d(u,y)}{\bar{G}^d(u,0)}, \tag{3.61}$$

$$\lim_{u\to\infty} \bar{G}_u^d(y) = \frac{\lim_{u\to\infty} e^{\kappa u}\bar{G}^d(u,y)}{\lim_{u\to\infty} e^{\kappa u}\bar{G}^d(u,0)} = \frac{\lim_{u\to\infty} e^{\kappa u}\bar{G}(u,y)}{\lim_{u\to\infty} e^{\kappa u}\bar{G}(u,0)} = \lim_{u\to\infty} \bar{G}_u(y). \tag{3.62}$$

Q.E.D.

Theorem 3.3 shows that the asymptotic distribution of the proper deficit as the initial surplus u goes to ∞ in the delayed renewal risk model is still of the same mixture form as in the ordinary or the stationary renewal risk model, independently of the distribution of the first interclaim time. This is because large u implies large t, and as the initial surplus u gets large, the effect of the assumed distribution for the time until the first claim becomes insignificant. Also note that the asymptotic DF of the proper deficit is a mixture of the residual lifetime DF's $B_{0,t}(y)$ for $t \geq 0$ as in the previous section but with much simpler mixing weights.

3.5 Stochastic Decomposition of the Residual Lifetime of L_δ^d

Let L^d be the maximal aggregate loss in the delayed renewal risk model and $\varphi^d(u) = 1 - \psi^d(u)$ be the distribution function of L^d. Let's extend this to the case where $\delta > 0$ and define L_δ^d to be a random variable with DF $G_\delta^d(u) = 1 - \bar{G}_\delta^d(u)$

Theorem 3.4 *The conditional survival function of L_δ^d satisfies*

$$P(L_\delta^d > u + y | L_\delta^d > u) = P(L_\delta + X_{\delta,u}^d > y) \tag{3.63}$$

for $y \geq 0$, where $X_{\delta,u}^d$ is a random variable statistically independent of L_δ with distribution function $G_{\delta,u}^d(y)$.

Proof:

We will first give probabilistic proof in the case when $\delta = 0$ for better understanding of the equation, where the interpretation is clear. In much the same way as in Theorem 2.2 of Willmot et al. (2004) we can argue as follows. Suppose the initial surplus level is $u + y$. The event $\{L^d \leq u + y\}$ can be divided into two mutually exclusive and exhaustive events $\{L^d \leq u\}$ and $\{u \leq L^d \leq u + y\}$. The probability of the first event is $\varphi^d(u)$ and the surplus always remains above y. For the second event to happen, the surplus level should fall below level y at some point in time to a level $y - t \geq 0$ with probability $dG^d(u, t)$ but then ruin should not occur after that with probability

$\varphi(y-t)$. Hence,

$$\varphi^d(u+y) = \varphi^d(u) + \int_0^y \varphi(y-t)dG^d(u,t), \tag{3.64}$$

and equivalently,

$$1-\psi^d(u+y) = 1-\psi^d(u) + \int_0^y (1-\psi(y-t))dG^d(u,t), \tag{3.65}$$

and

$$\psi^d(u+y) = \psi^d(u) - \int_0^y dG^d(u,t) + \int_0^y \psi(y-t)dG^d(u,t). \tag{3.66}$$

Divide both sides with $\psi^d(u)$ to get

$$\begin{aligned} \frac{\psi^d(u+y)}{\psi^d(u)} &= 1 - \int_0^y dG_u^d(t) + \int_0^y \psi(y-t)dG_u^d(t) && (3.67)\\ &= \bar{G}_u^d(y) + \int_0^y \psi(y-t)dG_u^d(t), && (3.68)\end{aligned}$$

which is (3.63) when $\delta = 0$.

Now, we will give an analytic proof which can be applied for all values of non-negative δ, by following the idea of Theorem 2.1. in Willmot and Cai (2004).

Divide both sides of (3.41) by ϕ_δ^d to obtain

$$\begin{aligned} \frac{\bar{G}_\delta^d(u,y)}{\phi_\delta^d} &= \frac{\phi_\delta}{1-\phi_\delta}\int_0^u \bar{B}_\delta(u+y-t)d\Lambda_\delta(t) + \bar{B}_\delta^d(u+y)\\ &= \bar{B}_\delta^d(u+y) + \frac{\phi_\delta}{1-\phi_\delta}\bar{B}_\delta(y)\{\int_0^u \bar{B}_{\delta,y}(u-t)d\Lambda_\delta(t)\}\end{aligned}$$

where $\bar{B}_{\delta,y}(t)$ is the residual lifetime tail of $\bar{B}_\delta(t)$ defined by

$$\bar{B}_{\delta,y}(t) = \frac{\bar{B}_\delta(y+t)}{\bar{B}_\delta(y)}. \tag{3.69}$$

And interchanging the role of $B_{\delta,y}$ and Λ_δ in the convolution,

$$\begin{aligned}
\frac{\bar{G}^d_\delta(u,y)}{\phi^d_\delta} &= \bar{B}^d_\delta(u+y) + \frac{\phi_\delta}{1-\phi_\delta}\bar{B}_\delta(y)\{\int_0^u \bar{\Lambda}_\delta(u-t)dB_{\delta,y}(t) + \bar{B}_{\delta,y}(u) - \bar{\Lambda}_\delta(u)\} \\
&= \bar{B}^d_\delta(u+y) + \frac{\phi_\delta}{1-\phi_\delta}\{\int_0^u \bar{\Lambda}_\delta(u-t)d_tB_\delta(y+t) + \bar{B}_\delta(u+y) - \bar{\Lambda}_\delta(u)\bar{B}_\delta(y)\} \\
&= \bar{B}^d_\delta(u+y) + \frac{\phi_\delta}{1-\phi_\delta}\{\int_y^{u+y} \bar{\Lambda}_\delta(u+y-t)dB_\delta(t) + \bar{B}_\delta(u+y) - \bar{\Lambda}_\delta(u)\bar{B}_\delta(y)\}.
\end{aligned}$$

Since (3.3) holds, equation (3.9) is equivalent to

$$\bar{\Lambda}_\delta(u) = \phi_\delta \int_0^u \bar{\Lambda}_\delta(u-y)b_\delta(y)dy + \phi_\delta\bar{B}_\delta(u) + (1-\phi_\delta)\bar{B}^d_\delta(u). \tag{3.70}$$

and using (3.70) with u replaced by $u+y$ leads to

$$\begin{aligned}
&\phi_\delta\{\int_y^{u+y} \bar{\Lambda}_\delta(u+y-t)dB_\delta(t) + \bar{B}_\delta(u+y)\} \\
&= \phi_\delta\{\int_0^{u+y} \bar{\Lambda}_\delta(u+y-t)dB_\delta(t) + \bar{B}_\delta(u+y)\} - \phi_\delta\int_0^y \bar{\Lambda}_\delta(u+y-t)dB_\delta(t) \\
&= \bar{\Lambda}_\delta(u+y) - (1-\phi_\delta)\bar{B}^d_\delta(u+y) - \phi_\delta\int_0^y \bar{\Lambda}_\delta(u+y-t)dB_\delta(t).
\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\bar{G}^d_\delta(u,y)}{\phi^d_\delta} &= \bar{B}^d_\delta(u+y)+\frac{1}{1-\phi_\delta}\{\bar{\Lambda}_\delta(u+y)-(1-\phi_\delta)\bar{B}^d_\delta(u+y)\} \quad (3.71)\\ &-\frac{\phi_\delta}{1-\phi_\delta}\{\int_0^y \bar{\Lambda}_\delta(u+y-t)dB_\delta(t)+\bar{\Lambda}_\delta(u)\bar{B}_\delta(y)\}\\ &= \frac{1}{1-\phi_\delta}\bar{\Lambda}_\delta(u+y)\\ &-\frac{\phi_\delta}{1-\phi_\delta}\{\int_0^y \bar{\Lambda}_\delta(u+y-t)dB_\delta(t)+\bar{\Lambda}_\delta(u)\bar{B}_\delta(y)\} \quad (3.72)\end{aligned}$$

Divide both sides of (3.72) by $\bar{\Lambda}_\delta(u)$ and this results in

$$\bar{G}^d_{\delta,u}(y)=\frac{1}{1-\phi_\delta}\frac{\bar{G}^d_\delta(u+y)}{\bar{G}^d_\delta(u)}-\frac{\phi_\delta}{1-\phi_\delta}\{\int_0^y\frac{\bar{G}^d_\delta(u+y-t)}{\bar{G}^d_\delta(u)}dB_\delta(t)+\bar{B}_\delta(y)\}. \quad (3.73)$$

It follows by taking LT that

$$\frac{1-E(e^{-sX^d_{\delta,u}})}{s}=\{\frac{1}{1-\phi_\delta}-\frac{\phi_\delta}{1-\phi_\delta}\tilde{b}_\delta(s)\}\int_0^\infty e^{-sy}\frac{\bar{G}^d_\delta(u+y)}{\bar{G}^d_\delta(u)}dy-\frac{\phi_\delta}{1-\phi_\delta}\frac{1-\tilde{b}_\delta(s)}{s}, \quad (3.74)$$

and rearranging that

$$\begin{aligned}\int_0^\infty e^{-sy}\frac{\bar{G}^d_\delta(u+y)}{\bar{G}^d_\delta(u)}dy &= \frac{\frac{1-E(e^{-sX^d_{\delta,u}})}{s}+\frac{\phi_\delta}{1-\phi_\delta}\frac{1-\tilde{b}_\delta(s)}{s}}{\frac{1-\phi_\delta\tilde{b}_\delta(s)}{1-\phi_\delta}}\\ &= \frac{1-\phi_\delta-(1-\phi_\delta)E(e^{-sX^d_{\delta,u}})+\phi_\delta-\phi_\delta\tilde{b}_\delta(s)}{s(1-\phi_\delta\tilde{b}_\delta(s))}\\ &= \frac{1-\tilde{g}_\delta(s)E(e^{-sX^d_{\delta,u}})}{s}.\end{aligned}$$

By the uniqueness of the LT, the theorem is proved.

Q.E.D.

As a result of equation (3.63), the mean and the moments of the discounted (proper) deficit can be calculated as in the following corollary.

Corollary 3.5 *The mean of the discounted proper deficit is*

$$E(X^d_{\delta,u}) = E(L_\delta + X^d_{\delta,u}) - E(L_\delta) = \int_0^\infty (\frac{\bar{G}^d_\delta(u+y)}{\bar{G}^d_\delta(u)} - \bar{G}_\delta(y))dy. \tag{3.75}$$

When $\delta = 0$, this simplifies to

$$E(X^d_u) = E(L + X^d_u) - E(L) = \int_0^\infty (\frac{\psi^d(u+y)}{\psi^d(u)} - \psi(y))dy, \tag{3.76}$$

the same form as in the ordinary or the stationary renewal risk model.

The second and higher moments can be calculated recursively, i.e.

$$E\{(X^d_u)^n\} = E\{(L + X^d_u)^n\} - \sum_{k=0}^{n-1} \binom{n}{k} E\{(X^d_u)^k\} E(L^{n-k}), \tag{3.77}$$

since $E\{(X^d_u)^k L^{n-k}\} = E\{(X^d_u)^k\}E(L^{n-k})$ by independence of X^d_u and L.

3.6 Joint Distribution of the Surplus and the Deficit

Let δ be non-negative and $w(x, y) = e^{-sx-zy}$. Then the Gerber-Shiu expected discounted penalty function, $m^d_\delta(u)$ in this case, can be expressed in terms of the discounted joint distribution of the surplus and the deficit, i.e.

$$m_\delta^d(u) = E\{e^{-\delta T_d - sU_{T_d^-} - z|U_{T_d}|} I(T_d < \infty)|U_0 = u\} = \int_0^\infty e^{-sx} \int_0^\infty e^{-zy} f_\delta^d(x, y|u) dy dx \tag{3.78}$$

where $f_\delta^d(x, y|u)$ is the discounted joint distribution of the surplus and the deficit in the delayed renewal risk process.

On the other hand, we can obtain alternative expression for $m_\delta^d(u)$ in this case using the equations in Section 2.3. Using equation (2.49),

$$\begin{aligned}
\sigma_\delta(u + ct) &= \int_0^{u+ct} m_\delta(u + ct - y)p(y)dy + \int_{u+ct}^\infty w(u + ct, y - u - ct)p(y)dy \\
&= \int_0^{u+ct} m_\delta(v)p(u + ct - v)dv + \int_{u+ct}^\infty e^{-s(u+ct)} e^{-z(y-u-ct)} p(y)dy \\
&= \int_0^{u+ct} \int_0^\infty e^{-sx} \int_0^\infty e^{-zy} f_\delta(x, y|v) dy dx p(u + ct - v)dv \\
&\quad + e^{-s(u+ct)} \int_0^\infty e^{-zy} p(y + u + ct)dy \\
&= \int_0^\infty e^{-sx} \int_0^\infty e^{-zy} \{\int_0^{u+ct} f_\delta(x, y|v)p(u + ct - v)dv\} dy dx \\
&\quad + e^{-s(u+ct)} \int_0^\infty e^{-zy} p(y + u + ct)dy
\end{aligned}$$

and thus from equation (2.48),

$$\begin{aligned}
m_\delta^d(u) &= \int_0^\infty e^{-\delta t}\sigma_\delta(u+ct)k_1(t)dt \\
&= \int_0^\infty e^{-\delta t}\int_0^\infty e^{-sx}\int_0^\infty e^{-zy}\{\int_0^{u+ct} f_\delta(x,y|v)p(u+ct-v)dv\}dydxk_1(t)dt \\
&\quad +e^{-su}\int_0^\infty e^{-\delta t}e^{-sct}\int_0^\infty e^{-zy}p(y+u+ct)dyk_1(t)dt \\
&= \int_0^\infty e^{-sx}\int_0^\infty e^{-zy}\{\int_0^\infty e^{-\delta t}\int_0^{u+ct} f_\delta(x,y|v)p(u+ct-v)dvk_1(t)dt\}dydx \\
&\quad +e^{-su}\int_0^\infty e^{-\delta\frac{x}{c}}e^{-sx}\int_0^\infty e^{-zy}p(y+u+x)k_1(\frac{x}{c})\frac{1}{c}dydx \\
&= \int_0^\infty e^{-sx}\int_0^\infty e^{-zy}\{\int_0^\infty e^{-\delta t}\int_0^{u+ct} f_\delta(x,y|v)p(u+ct-v)dvk_1(t)dt\}dydx \\
&\quad +\int_u^\infty e^{-sx}\int_0^\infty e^{-zy}p(x+y)k_1(\frac{x-u}{c})e^{-\delta(\frac{x-u}{c})}\frac{1}{c}dydx. \qquad (3.79)
\end{aligned}$$

Comparing (3.78) with (3.79), we obtain

$$f_\delta^d(x,y|u) = \begin{cases} \int_0^\infty\int_0^{u+ct} f_\delta(x,y|v)p(u+ct-v)dvk_1(t)e^{-\delta t}dt \\ \text{for } 0\le x<u \\ \int_0^\infty\int_0^{u+ct} f_\delta(x,y|v)p(u+ct-v)dvk_1(t)e^{-\delta t}dt + \frac{1}{c}p(x+y)e^{-\delta(\frac{x-u}{c})}k_1(\frac{x-u}{c}) \\ \text{for } x\ge u \end{cases}. \qquad (3.80)$$

Once we have the discounted joint density of the surplus and the deficit in the ordinary renewal risk process, we can solve for the discounted joint density of the surplus and the deficit in the delayed renewal risk process.

The discounted marginal defective density of the surplus can be easily obtained from the above equation by integrating with respect to the deficit y, which is

$$f_\delta^d(x|u) = \int_0^\infty f_\delta^d(x,y|u)dy$$
$$= \begin{cases} \int_0^\infty \int_0^{u+ct} f_\delta(x|v)p(u+ct-v)dve^{-\delta t}k_1(t)dt \\ \text{for } 0 \le x < u \\ \int_0^\infty \int_0^{u+ct} f_\delta(x|v)p(u+ct-v)dve^{-\delta t}k_1(t)dt + \frac{1}{c}\bar{P}(x)e^{-\delta(\frac{x-u}{c})}k_1(\frac{x-u}{c}) \\ \text{for } x \ge u \end{cases}.$$

From Gerber and Shiu (1998, p.53) we know that the following holds in the ordinary renewal risk process;

$$f_\delta(x,y|v) = f_\delta(x|v)\frac{p(x+y)}{\bar{P}(x)} \tag{3.81}$$

and thus

$$f_\delta^d(x|u) = \begin{cases} \int_0^\infty \int_0^{u+ct} f_\delta(x,y|v)\frac{\bar{P}(x)}{p(x+y)}p(u+ct-v)dve^{-\delta t}k_1(t)dt \\ \text{for } 0 \le x < u \\ \int_0^\infty \int_0^{u+ct} f_\delta(x,y|v)\frac{\bar{P}(x)}{p(x+y)}p(u+ct-v)dve^{-\delta t}k_1(t)dt + \frac{1}{c}\bar{P}(x)e^{-\delta(\frac{x-u}{c})}k_1(\frac{x-u}{c}) \\ \text{for } x \ge u \end{cases},$$

i.e.

$$f_\delta^d(x|u) = f_\delta^d(x,y|u)\frac{\bar{P}(x)}{p(x+y)} \quad \text{or} \quad f_\delta^d(x,y|u) = f_\delta^d(x|u)\frac{p(x+y)}{\bar{P}(x)}. \tag{3.82}$$

The same relationship between the discounted joint density of the surplus and the deficit and the discounted marginal density of the surplus holds in the delayed renewal risk process as in the ordinary renewal risk process.

Example 3.6.1 Exponential claim sizes and Coxian class interclaim times in stationary model

In the stationary renewal risk model with $\delta = 0$, from equation (2.54), $f^e(x, y|u)$ can be written as

$$f^e(x, y|u) = \frac{1}{1+\theta}\int_0^u f(x, y|u-z)dP_1(z) + \int_u^\infty \frac{1}{(1+\theta)E(Y)}\int_t^\infty w(t, z-t)p(z)dzdt \tag{3.83}$$

Using the expression of $f(x, y|u)$ when interclaim times have a Coxian distribution in section 7. of Li and Garrido (2005) with the assumption of exponential claim sizes ($p(y) = \beta e^{-\beta y}$), we can show

$$f(x, y|u) = \begin{cases} \beta e^{-\beta(x+y)} e^{-R_\delta u} \sum_{j=1}^n b_j \frac{\beta - R_\delta}{R_\delta + \rho_j}(e^{R_\delta x} - e^{-\rho_j x}), & 0 \le x < u \\ \beta e^{-\beta(x+y)} \sum_{j=1}^n b_j e^{-\rho_j x}(\frac{\beta+\rho_j}{R_\delta+\rho_j}e^{\rho_j u} - \frac{\beta - R_\delta}{R_\delta+\rho_j}e^{-R_\delta u}), & x \ge u \end{cases}. \tag{3.84}$$

Thus after substitution and simplification

$$f^e(x, y|u) = \begin{cases} \frac{1}{(1+\theta)}\beta e^{-\beta(x+y)} \sum_{j=1}^n b_j \frac{\beta}{R_0+\rho_j}(e^{R_0 x} - e^{-\rho_j x})e^{-R_0 u}, & 0 \le x < u \\ \frac{1}{(1+\theta)}\beta e^{-\beta(x+y)}\{\sum_{j=1}^n b_j e^{-\rho_j x}\frac{\beta}{R_0+\rho_j}(e^{\rho_j u} - e^{-R_0 u}) + \beta\}, & x \ge u \end{cases}. \tag{3.85}$$

Note that the deficit at ruin is also exponential and is independent of the surplus immediately prior to ruin, for both ordinary and stationary renewal risk models, when claims follow exponential distribution. This also holds for the delayed renewal risk model as well as with penalty function $w(x, y) = e^{-sx}w_2(y)$ where $w_2(y)$ is an arbitrary function of deficit at ruin. This is shown in Section 4.1.

The marginal densities are

$$\begin{aligned} f^e(x|u) &= \int_0^\infty f^e(x, y|u)dy \\ &= \begin{cases} \frac{1}{(1+\theta)}e^{-\beta x}\sum_{j=1}^n b_j \frac{\beta}{R_0+\rho_j}(e^{R_0 x} - e^{-\rho_j x})e^{-R_0 u}, & 0 \le x < u \\ \frac{1}{(1+\theta)}e^{-\beta x}\{\sum_{j=1}^n b_j e^{-\rho_j x}\frac{\beta}{R_0+\rho_j}(e^{\rho_j u} - e^{-R_0 u}) + \beta\}, & x \ge u \end{cases} \end{aligned}$$

and

$$\begin{aligned} g^e(y|u) &= \int_0^\infty f^e(x, y|u)dx \\ &= \frac{1}{(1+\theta)}\beta e^{-\beta y}e^{-R_0 u}. \end{aligned}$$

Also when the initial surplus is 0,

$$f^e(x, y|0) = \frac{1}{(1+\theta)}\beta e^{-\beta(x+y)}\{\sum_{j=1}^n b_j e^{-\rho_j x}\frac{\beta}{R_0+\rho_j} + \beta\}. \tag{3.86}$$

♣

Chapter 4

Distributional Assumptions for Claim Sizes

We have made assumptions about the distributions of the claim sizes and the interclaim times in the examples of the previous chapters for specific expressions of interest. This and the next chapter deal with more general penalty functions and Gerber-Shiu functions and provides comprehensive and unified results. In Chapter 4, assumptions are made for DF's of the claim sizes with arbitrary DF's of the interclaim times, and vice versa in Chapter 5. These chapters also provide insight into the nice properties of assumed distributions used for derivation.

4.1 Exponential Claim Sizes

4.1.1 Delayed Renewal Risk Process

When claim amounts follow an exponential distribution, ${b_\delta}^d(y) = p(y) = \beta e^{-\beta y}$ for $y > 0$ as shown in Example 2.1.1.

Also when $w(x,y) = w_2(y)$, using equation (3.1), ${m_\delta}^d(u)$ satisfies

$$\begin{aligned} {m_{\delta,0}}^d(u) &= {\phi_\delta}^d \int_0^u m_{\delta,0}(u-y)(\beta e^{-\beta y})dy + {\phi_\delta}^d \int_u^\infty w_2(y-u)(\beta e^{-\beta y})dy \\ &= {\phi_\delta}^d \int_0^u m_{\delta,0}(u-y)(\beta e^{-\beta y})dy + {\phi_\delta}^d \int_0^\infty w_2(t)\beta e^{-\beta(t+u)}dt \\ &= {\phi_\delta}^d \int_0^u m_{\delta,0}(u-y)(\beta e^{-\beta y})dy + {\phi_\delta}^d e^{-\beta u} E\{w_2(Y)\} \end{aligned}$$

where $E\{w_2(Y)\} = \int_0^\infty w_2(y)p(y)dy$.

In the ordinary renewal risk process, Willmot (2007) shows that

$$m_{\delta,0}(u) = \phi_\delta E\{w_2(Y)\} e^{-\beta(1-\phi_\delta)u} \tag{4.1}$$

or

$$m_{\delta,0}(u) = (1 - \frac{R_\delta}{\beta}) E\{w_2(Y)\} e^{-R_\delta u} \tag{4.2}$$

where $-R_\delta$ is a negative root of the generalized Lundberg's fundamental equation defined as in section 1.3, i.e. R_δ satisfies

$$\tilde{k}(\delta + cR_\delta) = \frac{1}{\tilde{p}(-R_\delta)} = 1 - \frac{R_\delta}{\beta}, \tag{4.3}$$

and $R_\delta = \beta(1-\phi_\delta)$.

Thus, ${m_{\delta,0}}^d(u)$ becomes

$$\begin{aligned} m_{\delta,0}^{d}(u) &= \phi_{\delta}^{d}(1-\frac{R_{\delta}}{\beta})E\{w_2(Y)\}\int_0^u e^{-R_{\delta}(u-y)}\beta e^{-\beta y}dy + {\phi_{\delta}}^{d}e^{-\beta u}E\{w_2(Y)\} \\ &= {\phi_{\delta}}^{d}e^{-R_{\delta}u}E\{w_2(Y)\}\int_0^u (\beta-R_{\delta})e^{-(\beta-R_{\delta})y}dy + {\phi_{\delta}}^{d}e^{-\beta u}E\{w_2(Y)\} \\ &= {\phi_{\delta}}^{d}e^{-R_{\delta}u}E\{w_2(Y)\}[1-e^{-(\beta-R_{\delta})u}] + {\phi_{\delta}}^{d}e^{-\beta u}E\{w_2(Y)\} \end{aligned}$$

i.e.

$$m_{\delta,0}^{d}(u) = {\phi_{\delta}}^{d}E\{w_2(Y)\}e^{-R_{\delta}u}. \tag{4.4}$$

But conditioning on the amount and the time of the first claim, we get

$$m_{\delta,0}^{d}(u) = \int_0^{\infty} e^{-\delta t}\sigma_{\delta}(u+ct)k_1(t)dt \tag{4.5}$$

where

$$\sigma_{\delta}(t) = \int_0^t m_{\delta,0}(t-y)dP(y) + \int_t^{\infty} w_2(y-t)dP(y) = E\{w_2(Y)\}e^{-R_{\delta}t} \tag{4.6}$$

after some simple algebra.

Now, substitute equation (4.4) and (4.6) into (4.5) to get

$$\phi_{\delta}^{d}E\{w_2(Y)\}e^{-R_{\delta}u} = \int_0^{\infty} e^{-\delta t}E\{w_2(Y)\}e^{-R_{\delta}(u+ct)}k_1(t)dt \tag{4.7}$$

and this leads to the result that

$$\phi_\delta{}^d = \int_0^\infty e^{-(\delta+cR_\delta)t} k_1(t)dt = \tilde{k_1}(\delta + cR_\delta) \tag{4.8}$$

i.e.

$$m_{\delta,0}{}^d(u) = \tilde{k_1}(\delta + cR_\delta)E\{w_2(Y)\}e^{-R_\delta u}. \tag{4.9}$$

Note that when claims are exponentially distributed,

$$m_{\delta,0}(u) = \phi_\delta E\{w_2(Y)\}e^{-R_\delta u} \tag{4.10}$$

in the ordinary renewal risk process and

$$m_{\delta,0}{}^d(u) = \phi_\delta{}^d E\{w_2(Y)\}e^{-R_\delta u} \tag{4.11}$$

in the delayed renewal risk process and thus the ratio of the two functions are not a function of u but just a constant;

$$\frac{m_{\delta,0}{}^d(u)}{m_{\delta,0}(u)} = \frac{\phi_\delta{}^d}{\phi_\delta} = \frac{\tilde{k_1}(\delta + cR_\delta)}{\tilde{k}(\delta + cR_\delta)}. \tag{4.12}$$

When $w_2(Y) = 1$, (4.9) becomes

$$\bar{G}_\delta^d(u) = \tilde{k_1}(\delta + cR_\delta)e^{-R_\delta u}. \tag{4.13}$$

Note that (4.13) also satisfies (3.2) and equating Laplace transforms of (4.13) and (3.2) yields

$$\frac{\phi_\delta^d}{s + R_\delta} = \frac{\phi_\delta^d \phi_\delta}{s + R_\delta}\tilde{b}_\delta^d(s) + \phi_\delta^d \frac{1 - \tilde{b}_\delta^d(s)}{s} \tag{4.14}$$

from which it follows that

$$\tilde{b}^d_\delta(s)(\frac{1}{s} - \frac{\phi_\delta}{s+R_\delta}) = \frac{1}{s} - \frac{1}{s+R_\delta} \tag{4.15}$$

and using $\phi_\delta = 1 - \frac{R_\delta}{\beta}$, $\tilde{b}^d_\delta(s)$ reduces to

$$\tilde{b}^d_\delta(s) = \frac{\beta}{s+\beta}, \tag{4.16}$$

which immediately yields that

$$b^d_\delta(y) = \beta e^{-\beta y}. \tag{4.17}$$

Define

$$G^d(u,y) = Pr(|U_{T_d}| \le y, T_d < \infty |\ U_0 = u), \tag{4.18}$$

then with $w_2(x) = I(x \le y)$,

$$E\{w_2(Y)\} = \int_0^\infty I(x \le y) dP(x) = P(y) \tag{4.19}$$

and with $\delta = 0$, $m^d_{0,0}(u) = G^d(u,y)$ is obtained from (4.9), i.e.

$$G^d(u,y) = P(y)\psi^d(u) \tag{4.20}$$

where

$$\psi^d(u) = \bar{G}^d_0(u) = \tilde{k}_1(cR_0)e^{-R_0 u}. \tag{4.21}$$

Also note that the proper distribution function of the deficit is also exponential, i.e.

$$\frac{G^d(u,y)}{\psi^d(u)} = P(y). \tag{4.22}$$

Now, let's consider the function $w(x,y)$ has the form of $w(x,y) = e^{-sx}w_2(y)$, extended from $w(x,y) = w_2(y)$. Then the Gerber-Shiu expected discounted penalty function is defined as

$$m^d_{\delta,s}(u) = E\{e^{-\delta T_d - sU_{T_d^-}} w_2(|U_{T_d}|)I(T_d < \infty)|\ U_0 = u\} \tag{4.23}$$

and the solution to this can be found in the following way.

Conditioning on the time (t) and the amount (y) of the first claim, we obtain

$$m^d_{\delta,s}(u) = \int_0^\infty e^{-\delta t}\sigma_{\delta,s}(u+ct)k_1(t)dt \tag{4.24}$$

where

$$\sigma_{\delta,s}(t) = \int_t^\infty w(t, y-t)p(y)dy + \int_0^t m_{\delta,s}(t-y)p(y)dy. \tag{4.25}$$

Noting that $\sigma_{\delta,s}(t)$ is same for ordinary and delayed renewal processes, we can use the result already derived in Willmot (2007),

$$\sigma_{\delta,s}(t) = (E\{w_2(Y)\} - \beta\tilde{\rho}_\delta(s))e^{-(\beta+s)t} + \frac{\beta}{\phi_\delta}\tilde{\rho}_\delta(s)\bar{G}_\delta(t) \tag{4.26}$$

where

$$\tilde{\rho}_\delta(s) = \frac{E\{w_2(Y)\}\tilde{k}\{\delta + c(\beta+s)\}}{s + \beta\tilde{k}\{\delta + c(\beta+s)\}}. \tag{4.27}$$

Substitution of (4.26) into (4.24) yields,

$$\begin{aligned}
m^d_{\delta,s}(u) &= (E\{w_2(Y)\} - \beta\tilde{\rho}_\delta(s))\int_0^\infty e^{-\delta t}e^{-(\beta+s)(u+ct)}dK_1(t) \\
&\quad + \frac{\beta}{\phi_\delta}\tilde{\rho}_\delta(s)\int_0^\infty e^{-\delta t}\bar{G}_\delta(u+ct)dK_1(t) \\
&= (E\{w_2(Y)\} - \beta\tilde{\rho}_\delta(s))e^{-(\beta+s)u}\tilde{k}_1\{\delta + c(\beta+s)\} \\
&\quad + \beta\tilde{\rho}_\delta(s)\tilde{k}_1(\delta + cR_\delta)e^{-R_\delta u} \\
&= \frac{E\{w_2(Y)\}s\tilde{k}_1\{\delta + c(\beta+s)\}}{s + \beta\tilde{k}\{\delta + c(\beta+s)\}}e^{-(\beta+s)u} \\
&\quad + \frac{E\{w_2(Y)\}\beta\tilde{k}\{\delta + c(\beta+s)\}}{s + \beta\tilde{k}\{\delta + c(\beta+s)\}}\tilde{k}_1(\delta + cR_\delta)e^{-R_\delta u},
\end{aligned}$$

i.e.

$$\begin{aligned}
m^d_{\delta,s}(u) &= E\{w_2(Y)\}[a(\delta,s)\tilde{k}_1\{\delta + c(\beta+s)\}e^{-(\beta+s)u} \\
&\quad + (1 - a(\delta,s))\tilde{k}_1\{\delta + c\beta(1-\phi_\delta)\}e^{-\beta(1-\phi_\delta)u}]
\end{aligned} \tag{4.28}$$

where

$$a(\delta,s) = \frac{s}{s + \beta\tilde{k}\{\delta + c(\beta+s)\}}. \tag{4.29}$$

$m^d_{\delta,s}(u)$ can be viewed as an weighted average of $E\{w_2(Y)\}\tilde{k}_1\{\delta + c(\beta + s)\}e^{-(\beta+s)u}$ and $E\{w_2(Y)\}\tilde{k}_1\{\delta + c\beta(1 - \phi_\delta)\}e^{-\beta(1-\phi_\delta)u}$.

Example 4.1.1 Assumption of exponential distributions for interclaim times

In addition to the assumption of exponential claim sizes, assume that the interclaim times also follow exponential distributions; $k_1(t) = \lambda_1 e^{-\lambda_1 t}$ and $k(t) = \lambda e^{-\lambda t}$. Then using (4.28) and (4.29), it is easy to see that

$$\begin{aligned} m^d_{\delta,s}(u) &= \lambda_1 E\{w_2(Y)\}[\frac{\beta\lambda}{cs^2 + (\lambda + \delta + c\beta)s + \beta\lambda} \cdot \frac{e^{-\beta(1-\phi_\delta)u}}{\lambda_1 + \delta + c\beta(1 - \phi_\delta)} \\ &\quad + \frac{cs^2 + (\lambda + \delta + c\beta)s}{cs^2 + (\lambda + \delta + c\beta)s + \beta\lambda} \cdot \frac{e^{-(\beta+s)u}}{\lambda_1 + \delta + c(\beta + s)}]. \end{aligned}$$

When $\lambda_1 = \lambda$, $m^d_{\delta,s}(u)$ further reduces to

$$m^d_{\delta,s}(u) = \frac{\lambda E\{w_2(Y)\}}{cs^2 + (\lambda + \delta + c\beta)s + \beta\lambda}[(\beta - R_\delta)e^{-R_\delta u} + se^{-(\beta+s)u}] \tag{4.30}$$

and this coincides with the formula derived in Example 3.1 in Willmot (2007).

We know that the deficit and the surplus at ruin are independent of each other when claims are from exponential distribution. This implies that the LT of the discounted surplus at ruin is

$$E_\delta\{e^{-sU_{T_d^-}} | U_0 = u\} = \frac{\lambda}{cs^2 + (\lambda + \delta + c\beta)s + \beta\lambda}[(\beta - R_\delta)e^{-R_\delta u} + se^{-(\beta+s)u}] \tag{4.31}$$

where the expectation is taken with respect to the defective PDF of the discounted surplus when the initial surplus is u, $f_\delta^d(x|u)$. This is shown when $\delta = 0$ in the following. From Willmot (2005, p.24), the marginal defective density of the surplus is

$$f(x|u) = \frac{\lambda(1+\theta)}{c\theta}e^{-\beta x}\begin{cases}\{\psi(u-x)-\psi(u)\}, & x < u\\ \{1-\psi(u)\}, & x > u\end{cases}, \tag{4.32}$$

and from Klugman et al. (2004), the exact ruin probability can be obtained with exponential claims in classical Poisson model, which is

$$\psi(u) = \frac{1}{1+\theta}e^{-\frac{\theta}{1+\theta}\beta u}. \tag{4.33}$$

Then,

$$\begin{aligned}
E\{e^{-sU_{T_d^-}}|U_0 = u\} &= \int_0^\infty e^{-sx}f(x|u)dx\\
&= \frac{\lambda}{c\theta}\{\int_0^u e^{-(s+\beta)x}(e^{-\frac{\theta}{1+\theta}\beta(u-x)} - e^{-\frac{\theta}{1+\theta}\beta u})dx\\
&\quad + \int_u^\infty e^{-(s+\beta)x}(1+\theta - e^{-\frac{\theta}{1+\theta}\beta u})dx\}\\
&= \frac{\lambda}{c\theta}\{(\frac{1+\theta}{(1+\theta)s+\beta} - \frac{1}{s+\beta})e^{-\frac{\theta}{1+\theta}\beta u}\\
&\quad + (\frac{1+\theta}{s+\beta} - \frac{1+\theta}{(1+\theta)s+\beta})e^{-(s+\beta)u}\}\\
&= \frac{\lambda}{c}\{\frac{\beta}{(1+\theta)s^2+(2+\theta)\beta s+\beta^2}e^{-\frac{\theta}{1+\theta}\beta u}\\
&\quad + \frac{(1+\theta)}{(1+\theta)s^2+(2+\theta)\beta s+\beta^2}e^{-(s+\beta)u}\}.
\end{aligned} \tag{4.34}$$

Using $c = \frac{(1+\theta)\lambda}{\beta}$ and $R_0 = \frac{\theta}{1+\theta}\beta$, (4.34) becomes

$$E\{e^{-sU_{T_d^-}}|U_0 = u\} = \frac{\lambda}{cs^2+(\lambda+c\beta)s+\beta\lambda}[(\beta - R_0)e^{-R_0 u} + se^{-(\beta+s)u}], \tag{4.35}$$

which is (4.31) when $\delta = 0$. ♣

Alternatively, $m^d_{\delta,s}(u)$ can be rewritten as

$$m^d_{\delta,s}(u) = \tilde{\rho}_\delta(s)\frac{\tilde{k}_1(\delta + c(\beta + s))}{\tilde{k}(\delta + c(\beta + s))}se^{-(\beta+s)u} + \tilde{\rho}_\delta(s)\beta\tilde{k}_1(\delta + cR_\delta)e^{-R_\delta u}$$

i.e., using (4.13),

$$m^d_{\delta,s}(u) = \tilde{\rho}_\delta(s)\{\frac{\tilde{k}_1(\delta + c(\beta + s))}{\tilde{k}(\delta + c(\beta + s))}se^{-(\beta+s)u} + \beta\bar{G}^d_\delta(u)\}. \tag{4.36}$$

4.1.2 Special Case : Stationary Renewal Risk Process

Using the generalized Lundberg equation (4.3), the Laplace transform of the equilibrium inter-claim time distribution may be expressed as

$$\tilde{k}_e(\delta + cR_\delta) = \frac{1 - \tilde{k}(\delta + cR_\delta)}{(\delta + cR_\delta)E(V)} = \frac{R_\delta}{\beta E(V)(\delta + cR_\delta)}, \tag{4.37}$$

and with $\beta E(V) = (1 + \theta)/c$, (4.37) may be reexpressed as

$$\tilde{k}_e(\delta + cR_\delta) = \frac{1}{1 + \theta}\frac{cR_\delta}{\delta + cR_\delta}. \tag{4.38}$$

For the stationary process, (4.9) thus becomes

$$m^e_{\delta,0}(u) = \frac{E\{w_2(Y)\}}{1 + \theta}\frac{cR_\delta}{\delta + cR_\delta}e^{-R_\delta u}. \tag{4.39}$$

When $\delta = 0$, (4.39) becomes

$$m_{0,0}^{e}(u) = \frac{E\{w_2(Y)\}}{1+\theta} e^{-R_0 u}, \tag{4.40}$$

whereas when $w_2 = 1$, (4.39) becomes

$$\bar{G}_{\delta}^{e}(u) = \frac{1}{1+\theta} \frac{cR_\delta}{\delta + cR_\delta} e^{-R_\delta u}. \tag{4.41}$$

The stationary ruin probability is

$$\psi^{e}(u) = \bar{G}_{0}^{e}(u) = \frac{1}{1+\theta} e^{-R_0 u}, \tag{4.42}$$

which agrees with a known result (See Willmot and Lin (2001), p.230) and also with the exact ruin probability with exponential claims in classical Poisson model.

As explored in this section, assumption of exponential claim sizes gives great simplification to many functions derived from the Gerber-Shiu expected discounted penalty function, even in the delayed renewal risk model. We can easily obtain exact values for these functions, when assumptions for the time until the first claim and for the interclaim times are made, not worrying about the bounds and asymptotics.

4.2 Infinite Mixture of Erlangs with Single Scale Parameter

Assume that the distribution of the claim sizes is of the form

$$p(y) = \sum_{j=1}^{\infty} q_j \frac{\beta(\beta y)^{j-1} e^{-\beta y}}{(j-1)!}, \; y \geq 0 \tag{4.43}$$

where $\sum_{j=1}^{\infty} q_j = 1$.

Finite mixtures of Erlangs are special cases of the Coxian-class, but infinite mixtures of the above form do not belong to Coxian-class. One example of this is the sum of two Chi-square distributions, where it can be expressed as an infinite mixture of Erlang but not in a form of Coxian-class distribution.

4.2.1 Delayed Renewal Risk Process

From Example 2.1.2 we know that $p_x(y) = p(x+y)/\bar{P}(x)$ is of the form

$$p_x(y) = \sum_{j=1}^{\infty} \eta_j^*(x) \frac{\beta(\beta y)^{j-1} e^{-\beta y}}{(j-1)!} = \sum_{j=1}^{\infty} \eta_j^*(x) \tau_j(y) \tag{4.44}$$

where

$$\eta_j^*(x) = \frac{e^{-\beta x}}{\bar{P}(x)} \sum_{m=j}^{\infty} q_m \frac{\beta x^{m-j}}{(m-j)!} \tag{4.45}$$

and

$$\tau_j(y) = \frac{\beta(\beta y)^{j-1} e^{-\beta y}}{(j-1)!}. \tag{4.46}$$

Also, since $p(x+y) = \bar{P}(x)p_x(y) = \sum_{j=1}^{\infty} \bar{P}(x)\eta_j^*(x)\tau_j(y)$,

$$p(x+y) = \sum_{j=1}^{\infty} \eta_j(x)\tau_j(y) \tag{4.47}$$

where $\eta_j(x) = \bar{P}(x)\eta_j^*(x)$.

As in the previous section, assume that the penalty function is of the form $w(x,y) = e^{-sx}w_2(y)$. Willmot (2007) has shown that, in the ordinary renewal risk process, $m_{\delta,s}(u)$ satisfies

$$m_{\delta,s}(u) = e^{-su}\sum_{j=1}^{\infty} \tilde{b}_{j,\delta}(s)\{\tau_j(u) + \sum_{k=0}^{\infty}(\frac{s}{\beta})^k \sum_{n=1}^{\infty} \frac{c_{n,\delta}}{1-\phi_\delta}\binom{n+k-1}{k}\tau_{n+k+j}(u)\} \tag{4.48}$$

where

$$\tilde{b}_{j,\delta}(s) = \int_0^\infty e^{-sx}\frac{f_\delta(x|0)}{\bar{P}(x)}\int_0^\infty w_2(y)\bar{P}(x+y)\eta_j^*(x+y)dydx, \tag{4.49}$$

and a probability distribution $\{c_{0,\delta}, c_{1,\delta}, c_{2,\delta}, ...\}$ is defined in terms of the compound geometric probability generating function

$$\sum_{n=0}^{\infty} c_{n,\delta}z^n = \frac{1-\phi_\delta}{1-\phi_\delta\sum_{j=1}^{\infty}\tilde{\eta}_{j,\delta}z^j} \tag{4.50}$$

with $\tilde{\eta}_{j,\delta} = \int_0^\infty \eta_j^*(x)\frac{f_\delta(x|0)}{\phi_\delta}dx$.

Using equation (2.1) with $w(x,y) = e^{-sx}w_2(y)$,

$$m_{\delta,s}^d(u) = \phi_\delta^d\int_0^u m_{\delta,s}(u-y)b_\delta^d(y)dy + v_{\delta,s}^d(u) \tag{4.51}$$

where

$$\begin{aligned} b^d_\delta(y) &= \int_0^\infty p_x(y)\{\frac{f^d_\delta(x|0)}{\phi^d_\delta}\}dx = \int_0^\infty \{\sum_{j=1}^\infty \eta^*_j(x)\tau_j(y)\}\{\frac{f^d_\delta(x|0)}{\phi^d_\delta}\}dx \\ &= \sum_{j=1}^\infty \{\int_0^\infty \eta^*_j(x)\frac{f^d_\delta(x|0)}{\phi^d_\delta}dx\}\tau_j(y) = \sum_{j=1}^\infty \tilde{\eta}^d_{j,\delta}\tau_j(y) \qquad (4.52)\end{aligned}$$

with $\tilde{\eta}^d_{j,\delta} = \int_0^\infty \eta^*_j(x)\frac{f^d_\delta(x|0)}{\phi^d_\delta}dx$

and

$$\begin{aligned} v^d_{\delta,s}(u) &= e^{-su}\int_0^\infty e^{-sx}f^d_\delta(x|0)\int_u^\infty w_2(y-u)p_x(y)dydx \\ &= e^{-su}\int_0^\infty e^{-sx}f^d_\delta(x|0)\int_0^\infty w_2(y)\frac{p(x+y+u)}{\bar{P}(x)}dydx \\ &= e^{-su}\sum_{j=1}^\infty \tilde{b}^d_{j,\delta}(s)\tau_j(u) \qquad (4.53)\end{aligned}$$

with $\tilde{b}^d_{j,\delta}(s) = \int_0^\infty e^{-sx}\frac{f^d_\delta(x|0)}{\bar{P}(x)}\int_0^\infty w_2(y)\eta_j(x+y)dydx$.

Note that $b^d_\delta(y)$ and $v^d_{\delta,s}(u)$ are sums of damped exponential series.

Now, using (4.48) and (4.52),

$$\begin{aligned}\int_0^u m_{\delta,s}(u-y)b_\delta^d(y)dy &= \int_0^u e^{-s(u-y)}\sum_{j=1}^{\infty}\tilde{b}_{j,\delta}(s)\{\tau_j(u-y)\\&+\sum_{k=0}^{\infty}(\frac{s}{\beta})^k\sum_{n=1}^{\infty}\frac{c_{n,\delta}}{1-\phi_\delta}\binom{n+k-1}{k}\tau_{n+k+j}(u-y)\}\\&\times\sum_{m=1}^{\infty}\tilde{\eta}_{m,\delta}^d\tau_m(y)dy\\&= e^{-su}\sum_{j=1}^{\infty}\tilde{b}_{j,\delta}(s)\sum_{m=1}^{\infty}\tilde{\eta}_{m,\delta}^d\{\int_0^u e^{sy}\tau_j(u-y)\tau_m(y)dy\\&+\sum_{k=0}^{\infty}(\frac{s}{\beta})^k\sum_{n=1}^{\infty}\frac{c_{n,\delta}}{1-\phi_\delta}\binom{n+k-1}{k}\\&\int_0^u e^{sy}\tau_{n+k+j}(u-y)\tau_m(y)dy\}\end{aligned}\tag{4.54}$$

Using the beta-type integral,

$$\begin{aligned}\int_0^u e^{sy}\tau_j(u-y)\tau_m(y)dy &= \frac{\beta^{j+m}e^{-\beta u}}{(j-1)!(m-1)!}\int_0^u e^{sy}(u-y)^{j-1}y^{m-1}dy\\&= \frac{\beta^{j+m}e^{-\beta u}}{(j-1)!(m-1)!}\sum_{i=0}^{\infty}\frac{s^i}{i!}\int_0^u(u-y)^{j-1}y^{m+i-1}dy\\&= \frac{\beta^{j+m}e^{-\beta u}}{(j-1)!(m-1)!}\sum_{i=0}^{\infty}\frac{s^i}{i!}\frac{(j-1)!(m+i-1)!}{(j+m+i-1)!}u^{j+m+i-1}\\&= \sum_{i=0}^{\infty}(\frac{s}{\beta})^i\binom{m+i-1}{i}\tau_{j+m+i}(u),\end{aligned}\tag{4.55}$$

and thus

$$\begin{aligned}\int_0^u m_{\delta,s}(u-y)b_\delta^d(y)dy &= e^{-su}\sum_{j=1}^{\infty}\tilde{b}_{j,\delta}(s)\sum_{m=1}^{\infty}\tilde{\eta}_{m,\delta}^d\sum_{i=0}^{\infty}(\frac{s}{\beta})^i\binom{m+i-1}{i} \\ &\quad \times\{\tau_{j+m+i}(u)+\sum_{k=0}^{\infty}(\frac{s}{\beta})^k\sum_{n=1}^{\infty}\frac{c_{n,\delta}}{1-\phi_\delta} \\ &\quad \binom{n+k-1}{k}\tau_{m+i+n+k+j}(u)\}. \end{aligned} \tag{4.56}$$

Finally,

$$\begin{aligned} m_{\delta,s}^d(u) &= \phi_\delta^d e^{-su}\sum_{j=1}^{\infty}\tilde{b}_{j,\delta}(s)\sum_{m=1}^{\infty}\tilde{\eta}_{m,\delta}^d\sum_{i=0}^{\infty}(\frac{s}{\beta})^i\binom{m+i-1}{i} \\ &\quad \times\{\tau_{j+m+i}(u)+\sum_{k=0}^{\infty}(\frac{s}{\beta})^k\sum_{n=1}^{\infty}\frac{c_{n,\delta}}{1-\phi_\delta}\binom{n+k-1}{k}\tau_{m+i+n+k+j}(u)\} \\ &\quad +e^{-su}\sum_{j=1}^{\infty}\tilde{b}_{j,\delta}^d(s)\tau_j(u), \end{aligned} \tag{4.57}$$

which is in a form of damped exponential series.

When $s=0$, the penalty function becomes $w(x,y)=w_2(y)$ and

$$\begin{aligned} m_{\delta,0}^d(u) &= \phi_\delta^d\sum_{j=1}^{\infty}\tilde{b}_{j,\delta}(0)\sum_{m=1}^{\infty}\tilde{\eta}_{m,\delta}^d\{\tau_{j+m}(u)+\sum_{n=1}^{\infty}\frac{c_{n,\delta}}{1-\phi_\delta}\tau_{j+m+n}(u)\}+\sum_{j=1}^{\infty}\tilde{b}_{j,\delta}^d(0)\tau_j(u) \\ &= \phi_\delta^d\sum_{j=1}^{\infty}\tilde{b}_{j,\delta}(0)\sum_{m=1}^{\infty}\tilde{\eta}_{m,\delta}^d\sum_{n=0}^{\infty}\frac{c_{n,\delta}}{1-\phi_\delta}\tau_{j+m+n}(u)+\sum_{j=1}^{\infty}\tilde{b}_{j,\delta}^d(0)\tau_j(u) \end{aligned} \tag{4.58}$$

the second line follows since $c_{0,\delta}=1-\phi_\delta$ from (4.50).

Let $l = m + n$ and interchange the order of summation to get,

$$\begin{aligned}\sum_{m=1}^{\infty}\tilde{\eta}^{d}_{m,\delta}\sum_{n=0}^{\infty}\frac{c_{n,\delta}}{1-\phi_\delta}\tau_{j+m+n}(u) &= \sum_{m=1}^{\infty}\tilde{\eta}^{d}_{m,\delta}\sum_{l=m}^{\infty}\frac{c_{l-m,\delta}}{1-\phi_\delta}\tau_{j+l}(u)\\ &= \sum_{l=1}^{\infty}\sum_{m=1}^{l}\tilde{\eta}^{d}_{m,\delta}\frac{c_{l-m,\delta}}{1-\phi_\delta}\tau_{j+l}(u). \qquad (4.59)\end{aligned}$$

Substituting this into the following expression and letting $n = l + j$,

$$\begin{aligned}&\sum_{j=1}^{\infty}\tilde{b}_{j,\delta}(0)\sum_{m=1}^{\infty}\tilde{\eta}^{d}_{m,\delta}\sum_{n=0}^{\infty}\frac{c_{n,\delta}}{1-\phi_\delta}\tau_{j+m+n}(u)\\ =& \sum_{j=1}^{\infty}\tilde{b}_{j,\delta}(0)\sum_{l=1}^{\infty}\sum_{m=1}^{l}\tilde{\eta}^{d}_{m,\delta}\frac{c_{l-m,\delta}}{1-\phi_\delta}\tau_{j+l}(u)\\ =& \sum_{j=1}^{\infty}\sum_{n=j+1}^{\infty}\sum_{m=1}^{n-j}\tilde{b}_{j,\delta}(0)\tilde{\eta}^{d}_{m,\delta}\frac{c_{n-m-j,\delta}}{1-\phi_\delta}\tau_{n}(u)\\ =& \sum_{n=2}^{\infty}\sum_{j=1}^{n-1}\sum_{m=1}^{n-j}\tilde{b}_{j,\delta}(0)\tilde{\eta}^{d}_{m,\delta}\frac{c_{n-m-j,\delta}}{1-\phi_\delta}\tau_{n}(u). \qquad (4.60)\end{aligned}$$

Thus,

$$\begin{aligned}m^{d}_{\delta,0}(u) &= \sum_{n=2}^{\infty}\{\sum_{j=1}^{n-1}\sum_{m=1}^{n-j}\tilde{b}_{j,\delta}(0)\phi^{d}_{\delta}\tilde{\eta}^{d}_{m,\delta}\frac{c_{n-m-j,\delta}}{1-\phi_\delta}\}\tau_{n}(u)+\sum_{j=1}^{\infty}\tilde{b}^{d}_{j,\delta}(0)\tau_{j}(u)\\ &= e^{-\beta u}\sum_{i=0}^{\infty}R^{d}_{i,\delta}\frac{(\beta u)^{i}}{i!} \qquad (4.61)\end{aligned}$$

where

$$R_{i,\delta}^d = \begin{cases} \beta \tilde{b}_{1,\delta}^d(0) & \text{for} \quad i = 0 \\ \beta[\tilde{b}_{i+1,\delta}^d(0) + \sum_{j=1}^{i} \sum_{m=1}^{i+1-j} \tilde{b}_{j,\delta}(0) \phi_\delta^d \tilde{\eta}_{m,\delta}^d \frac{c_{i+1-m-j,\delta}}{1-\phi_\delta}] & \text{for} \quad i = 1, 2, 3, ... \end{cases} \tag{4.62}$$

Thus, $m_{\delta,0}^d(u)$ is a damped exponential series, as in the ordinary renewal process.

4.2.2 Special Case : Stationary Renewal Risk Process

Let $\delta = 0$ in the Gerber-Shiu function in the stationary renewal risk process; $m_0^e(u) = E\{w(U_{T_e^-}, |U_{T_e}|)I(T_e < \infty)|U_0 = u\}$. In this case, from (2.54) we know that

$$m_0^e(u) = \frac{1}{1+\theta} \int_0^u m_0(u-y)p_1(y)dy + \frac{1}{(1+\theta)E(Y)} \int_u^\infty \int_t^\infty w(t, y-t)p(y)dydt. \tag{4.63}$$

Also let $w(x, y) = e^{-sx} w_2(y)$ as before. Then the double integration in the second term is

$$\begin{aligned} \int_u^\infty \int_t^\infty w(t, y-t)p(y)dydt &= \int_u^\infty \int_t^\infty e^{-st} w_2(y-t)p(y)dydt \\ &= \int_u^\infty e^{-st} \int_0^\infty w_2(y)p(y+t)dydt. \end{aligned}$$

Using (4.47), a property of Erlang mixtures, we obtain

$$\begin{aligned}
\int_u^\infty \int_t^\infty w(t, y-t)p(y)dydt &= \sum_{j=1}^\infty \int_0^\infty w_2(y)\tau_j(y)dy \int_u^\infty e^{-st}\eta_j(t)dt \\
&= \sum_{j=1}^\infty E_j[w_2(Y)] \int_u^\infty e^{-st-\beta t} \sum_{m=j}^\infty q_m \frac{(\beta t)^{m-j}}{(m-j)!}dt \\
&= \sum_{j=1}^\infty E_j[w_2(Y)] \sum_{m=j}^\infty \frac{q_m \beta^{m-j}}{(\beta+s)^{m-j+1}} \times \\
&\quad \int_u^\infty \frac{(\beta+s)^{m-j+1}t^{m-j}e^{-(\beta+s)t}}{(m-j)!}dt \\
&= \sum_{j=1}^\infty E_j[w_2(Y)] \sum_{m=j}^\infty q_m \beta^{m-j} e^{-(\beta+s)u} \times \\
&\quad \sum_{k=0}^{m-j} \frac{(\beta+s)^{k-m+j-1}u^k}{k!} \\
&= e^{-su} \sum_{j=1}^\infty E_j[w_2(Y)] \sum_{k=0}^\infty \sum_{q=0}^k \binom{k}{q} \frac{\beta^q s^{k-q} u^k}{k!} e^{-\beta u} \times \\
&\quad \sum_{m=k+j}^\infty q_m \beta^{m-j}(\beta+s)^{j-m-1} \\
&= e^{-su} \sum_{j=1}^\infty E_j[w_2(Y)] \sum_{k=0}^\infty \sum_{q=0}^k \frac{(su)^{k-q}}{(k-q)!} \times \\
&\quad \sum_{m=k+j}^\infty q_m \beta^{m-j}(\beta+s)^{j-m-1}\tau_{q+1}(u) \qquad (4.64)
\end{aligned}$$

Now, using the form of $m_{0,s}$ in Willmot (2007) when claims are of mixed Erlang type, the integral in the first term of (4.63) is

$$
\begin{aligned}
&\int_0^u m_{0,s}(u-y)\bar{P}(y)dy \\
&= \int_0^u e^{-s(u-y)} \sum_{i=1}^{\infty} \tilde{b}_{i,0}(s)\{\tau_i(u-y) \\
&+ \sum_{l=0}^{\infty} (\frac{s}{\beta})^l \sum_{n=1}^{\infty} \frac{c_{n,0}}{1-\phi_0} \binom{n+l-1}{l} \tau_{n+l+i}(u-y)\} \sum_{j=1}^{\infty} q_j \sum_{k=0}^{j-1} \frac{\beta^k y^k}{k!} e^{-\beta y} dy \\
&= e^{-su} \sum_{j=1}^{\infty} q_j \sum_{k=0}^{j-1} \sum_{i=1}^{\infty} \tilde{b}_{i,0}(s)\{\int_0^u e^{sy} \tau_i(u-y) \frac{\beta^k y^k}{k!} e^{-\beta y} dy \\
&+ \sum_{l=0}^{\infty} (\frac{s}{\beta})^l \sum_{n=1}^{\infty} \frac{c_{n,0}}{1-\phi_0} \binom{n+l-1}{l} \int_0^u e^{sy} \tau_{n+l+i}(u-y) \frac{\beta^k y^k}{k!} e^{-\beta y} dy
\end{aligned}
$$

Using beta-type integral (4.55), we obtain

$$
\int_0^u e^{sy} \tau_i(u-y) \frac{\beta^k y^k}{k!} e^{-\beta y} dy = \frac{1}{\beta} \sum_{h=0}^{\infty} (\frac{s}{\beta})^h \binom{k+h}{h} \tau_{i+k+h+1}(u).
$$

Thus,

$$
\begin{aligned}
&\int_0^u m_{0,s}(u-y)\bar{P}(y)dy \\
&= \frac{1}{\beta} e^{-su} \sum_{j=1}^{\infty} q_j \sum_{k=0}^{j-1} \sum_{i=1}^{\infty} \tilde{b}_{i,0}(s) \sum_{h=0}^{\infty} (\frac{s}{\beta})^h \binom{k+h}{h} \{\tau_{i+k+h+1}(u) \\
&+ \sum_{l=0}^{\infty} (\frac{s}{\beta})^l \sum_{n=1}^{\infty} \frac{c_{n,0}}{1-\phi_0} \binom{n+l-1}{l} \tau_{n+l+i+k+h+1}(u)\} \qquad (4.65)
\end{aligned}
$$

Therefore, using (4.65) and (4.64) we obtain

$$\begin{aligned}
&m^e_{0,s}(u)\\
&= \frac{1}{(1+\theta)E(Y)}\{\int_0^u m_{0,s}(u-y)\bar{P}(y)dy + \int_u^\infty \int_t^\infty e^{-st}w_2(y-t)p(y)dydt\\
&= \frac{e^{-su}}{\beta(1+\theta)E(Y)}[\sum_{j=1}^\infty q_j \sum_{k=0}^{j-1}\sum_{i=1}^\infty \tilde{b}_{i,0}(s)\sum_{h=0}^\infty (\frac{s}{\beta})^h \binom{k+h}{h} \times\\
&\{\tau_{i+k+h+1}(u) + \sum_{l=0}^\infty (\frac{s}{\beta})^l \sum_{n=1}^\infty \frac{c_{n,0}}{1-\phi_0}\binom{n+l-1}{l}\tau_{n+l+i+k+h+1}(u)\} +\\
&\sum_{j=1}^\infty E_j[w_2(Y)]\sum_{k=0}^\infty\sum_{q=0}^k \frac{(su)^{k-q}}{(k-q)!}\sum_{m=k+j}^\infty q_m\beta^{m-j}(\beta+s)^{j-m-1}\tau_{q+1}(u)
\end{aligned} \tag{4.66}$$

When $s=0$, the penalty function $w(x,y)=w_2(y)$ is a function of the deficit only, and the Gerber-Shiu function simplifies to

$$\begin{aligned}
m^e_{0,0}(u) &= \frac{e^{-\beta u}}{(1+\theta)E(Y)}\sum_{j=1}^\infty q_j[\sum_{k=0}^{j-1}\sum_{i=1}^\infty \tilde{b}_{i,0}(0)\sum_{n=0}^\infty \frac{c_{n,0}}{1-\phi_0}\frac{(\beta u)^{n+i+k}}{(n+i+k)!}\\
&\quad + \sum_{m=1}^j E_m[w_2(Y)]\sum_{k=0}^{j-m}\frac{\beta^{k-1}u^k}{k!}].
\end{aligned} \tag{4.67}$$

Further simplification by letting $w_2(y)=1$ leads us to the ruin probability

$$\psi^e(u) = \frac{e^{-\beta u}}{(1+\theta)E(Y)}\sum_{j=1}^\infty q_j \sum_{k=0}^{j-1}[\sum_{i=1}^\infty \tilde{b}_{i,0}(0)\sum_{n=0}^\infty \frac{c_{n,0}}{1-\phi_0}\frac{(\beta u)^{n+i+k}}{(n+i+k)!} + (j-k)\frac{\beta^{k-1}u^k}{k!}]. \tag{4.68}$$

Note that $\psi^e(u)$ is a compound geometric convolution, which is already known from Willmot and Dickson (2003) by

$$\psi^e(u) = \frac{1}{1+\theta}\int_0^u \psi(u-y)dP_e(y) + \frac{1}{1+\theta}\bar{P}_e(u), \tag{4.69}$$

and that $\tilde{b}_{i,0}(0)$ can be evaluated easily when interclaim times are of Coxian-type, as mentioned in Example 3.2 of Willmot (2007).

4.3 Coxian Class

In this section, assume that the claim size distribution $p(y)$ belongs to the Coxian class, i.e. the Laplace transform $\tilde{p}$ is of the form

$$\tilde{p}(s) = \frac{\lambda^* + s\beta(s)}{\prod_{i=1}^{n}(s+\lambda_i)} \tag{4.70}$$

with $\lambda^* = \prod_{i=1}^{n} \lambda_i$ for $\lambda_i > 0$ and $\beta(s)$ is a polynomial of degree $n-2$ or less.

Li and Garrido (2005) show that if $\lambda_1, \lambda_2, ..., \lambda_n$ are distinct then, using partial fractions,

$$\tilde{p}(s) = \frac{\lambda^* + s\beta(s)}{\prod_{i=1}^{n}(s+\lambda_i)} = \sum_{i=1}^{n} \frac{a_i}{s+\lambda_i}, \tag{4.71}$$

where $a_i = (\lambda^* - \lambda_i\beta(-\lambda_i))/\prod_{j=1,j\neq i}^{n}(\lambda_j - \lambda_i)$.

Also, if some of the λ_i are not distinct, then (4.70) can be rewritten as

$$\tilde{p}(s) = \frac{\lambda^* + s\beta(s)}{\prod_{i=1}^{k}(s+\lambda_i)^{n_i}} \tag{4.72}$$

where $\lambda_1, \lambda_2, ..., \lambda_k$ are distinct, $\lambda^* = \prod_{i=1}^{k} {\lambda_i}^{n_i}$, and $n = n_1 + n_2 + ... + n_k$. Using partial fractions,

$$\tilde{p}(s) = \frac{\lambda^* + s\beta(s)}{\prod_{i=1}^{k}(s+\lambda_i)^{n_i}} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} a_{i,j}(\frac{\lambda_i}{s+\lambda_i})^j \tag{4.73}$$

where

$$a_{i,j} = \frac{1}{{\lambda_i}^j (n_i - j)!} \frac{d^{n_i-j}}{ds^{n_i-j}} \prod_{m=1,m\neq i}^{k} \frac{\lambda^* + s\beta(s)}{(s+\lambda_m)^{n_m}}|_{s=-\lambda_i}, \tag{4.74}$$

and, hence,

$$p(y) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{i,j} \frac{\lambda_i{}^j y^{j-1} e^{-\lambda_i y}}{(j-1)!}. \tag{4.75}$$

We will assume that $\tilde{p}(s)$ is of the form as in (4.72) with $\lambda_1, \lambda_2, ..., \lambda_k$ distinct, since it actually includes (4.71) when all n_i's equal to 1.

Landriault and Willmot (2007) have considered an ordinary renewal risk process with arbitrary interclaim times and Coxian-distributed claim sizes. When the penalty function of the Gerber-Shiu is a function of the deficit only, they have shown that the Gerber-Shiu expected discounted penalty function in this case,

$$m_{\delta,0}(u) = E[e^{-\delta T} w_2(|U(T)|) I(T < \infty) | U_0 = u], \tag{4.76}$$

is

$$m_{\delta,0}(u) = \sum_{l=1}^{n} \eta_l e^{R_l u} \tag{4.77}$$

where $\{\eta_l\}_{l=1}^{n}$ satisfy the system of linear equations

$$\sum_{l=1}^{n} \eta_l \left(\frac{\lambda_i}{\lambda_i + R_l}\right)^j = \int_0^\infty w_2(x) \frac{\lambda_i{}^j x^{j-1} e^{-\lambda_i x}}{(j-1)!} dx \tag{4.78}$$

for $i = 1, ..., k$ and $j = 1, ..., n_i$, and $R_l = R_l(\delta)$ for $l = 1, ..., n$ are all the roots on the left-half complex plane (i.e. $Re(R_l(\delta)) < 0$ for $l = 1, 2, ..., n$) of the Lundberg's generalized fundamental equation, $\tilde{k}(\delta - cs)\tilde{p}(s) = 1$.

In the delayed renewal risk process, $m^d_{\delta,0}(u)$ satisfies

$$m^d_{\delta,0}(u) = \phi^d_\delta \int_0^u m_{\delta,0}(u-y) b^d_\delta(y) dy + \phi^d_\delta \int_u^\infty w_2(y-u) b^d_\delta(y) dy \tag{4.79}$$

where $b^d_\delta(y) = \int_0^\infty p_x(y) \{\frac{f^d_\delta(x|0)}{\phi^d_\delta}\} dx$.

From Landriault and Willmot (2007), when claim sizes are Coxian-distributed, we know that $p_x(y)$ is of the form

$$p_x(y) = \sum_{i=1}^{k} \sum_{j=0}^{n_i - 1} b_{i,j}(x) \frac{\lambda_i^{j+1} y^j e^{-\lambda_i y}}{j!} \tag{4.80}$$

where $b_{i,j}(x) = \sum_{m=0}^{n_i-j-1} \frac{a_{i,m+j}}{\bar{P}(x)} x^m \frac{\lambda_i^m e^{-\lambda_i x}}{j!}$, and hence,

$$b^d_\delta(y) = \sum_{i=1}^{k} \sum_{j=0}^{n_i - 1} \tilde{b}^d_{i,j,\delta}(0) \frac{\lambda_i^{j+1} y^j e^{-\lambda_i y}}{j!} \tag{4.81}$$

where $\tilde{b}^d_{i,j,\delta}(s) = \int_0^\infty e^{-sx} b_{i,j}(x) \frac{f^d_\delta(x|0)}{\phi^d_\delta} dx$.

Using equations (4.77) and (4.81), equation (4.79) becomes

$$
\begin{aligned}
m^d_{\delta,0}(u) &= \phi^d_\delta \int_0^u \sum_{l=1}^{n} \eta_l e^{R_l(u-y)} \sum_{i=1}^{k}\sum_{j=0}^{n_i-1} \tilde{b}^d_{i,j,\delta}(0) \frac{\lambda_i^{j+1} y^j e^{-\lambda_i y}}{j!} dy \\
&+\phi^d_\delta \int_u^\infty w_2(y-u) \sum_{i=1}^{k}\sum_{j=0}^{n_i-1} \tilde{b}^d_{i,j,\delta}(0) \frac{\lambda_i^{j+1} y^j e^{-\lambda_i y}}{j!} dy \\
&= \phi^d_\delta \sum_{l=1}^{n} \eta_l e^{R_l u} \sum_{i=1}^{k}\sum_{j=0}^{n_i-1} \tilde{b}^d_{i,j,\delta}(0) (\frac{\lambda_i}{\lambda_i+R_l})^{j+1} \int_0^u \frac{(\lambda_i+R_l)^{j+1} y^j e^{-(\lambda_i+R_l)y}}{j!} dy \\
&+\phi^d_\delta \sum_{i=1}^{k}\sum_{j=0}^{n_i-1} \tilde{b}^d_{i,j,\delta}(0) \int_u^\infty w_2(y-u) \frac{\lambda_i^{j+1} y^j e^{-\lambda_i y}}{j!} dy \\
&= \phi^d_\delta \sum_{i=1}^{k}\sum_{j=0}^{n_i-1} \tilde{b}^d_{i,j,\delta}(0) \sum_{l=1}^{n} \eta_l e^{R_l u} (\frac{\lambda_i}{\lambda_i+R_l})^{j+1} [1 - \sum_{h=0}^{j} \frac{((\lambda_i+R_l)u)^h e^{-(\lambda_i+R_l)u}}{h!}] \\
&+\phi^d_\delta \sum_{i=1}^{k}\sum_{j=0}^{n_i-1} \tilde{b}^d_{i,j,\delta}(0) \sum_{h=0}^{j} u^h \lambda_i^h e^{-\lambda_i u} \\
&\times \int_u^\infty w_2(y-u) \binom{j}{h} \frac{\lambda_i^{j-h+1} (y-u)^{j-h} e^{-\lambda_i (y-u)}}{j!} dy \\
&= \phi^d_\delta \sum_{i=1}^{k}\sum_{j=0}^{n_i-1} \tilde{b}^d_{i,j,\delta}(0) \sum_{l=1}^{n} \eta_l e^{R_l u} (\frac{\lambda_i}{\lambda_i+R_l})^{j+1} [1 - \sum_{h=0}^{j} \frac{((\lambda_i+R_l)u)^h e^{-(\lambda_i+R_l)u}}{h!}] \\
&+\phi^d_\delta \sum_{i=1}^{k}\sum_{j=0}^{n_i-1} \tilde{b}^d_{i,j,\delta}(0) \sum_{h=0}^{j} \frac{(\lambda_i u)^h e^{-\lambda_i u}}{h!} \gamma_{i,j-h+1} \\
&= \phi^d_\delta \sum_{i=1}^{k}\sum_{j=0}^{n_i-1} \tilde{b}^d_{i,j,\delta}(0) \sum_{l=1}^{n} \eta_l e^{R_l u} (\frac{\lambda_i}{\lambda_i+R_l})^{j+1} \\
&+\phi^d_\delta \sum_{i=1}^{k}\sum_{j=0}^{n_i-1} \tilde{b}^d_{i,j,\delta}(0) \sum_{h=0}^{j} \frac{(\lambda_i u)^h e^{-\lambda_i u}}{h!} [\gamma_{i,j-h+1} - \sum_{l=1}^{n} \eta_l (\frac{\lambda_i}{\lambda_i+R_l})^{j-h+1}].
\end{aligned}
$$

By (4.78), the second term on the right hand side disappears and thus

$$
\begin{aligned}
m_{\delta,0}^{d}(u) &= \phi_{\delta}^{d}\sum_{i=1}^{k}\sum_{j=0}^{n_i-1}\tilde{b}_{i,j,\delta}^{d}(0)\sum_{l=1}^{n}\eta_l e^{R_l u}(\frac{\lambda_i}{\lambda_i+R_l})^{j+1} \\
&= \sum_{l=1}^{n}[\eta_l\phi_{\delta}^{d}\sum_{i=1}^{k}\sum_{j=0}^{n_i-1}\tilde{b}_{i,j,\delta}^{d}(0)(\frac{\lambda_i}{\lambda_i+R_l})^{j+1}]e^{R_l u} \\
&= \sum_{l=1}^{n}\eta_l^{d}e^{R_l u} \qquad (4.82)
\end{aligned}
$$

where $\eta_l^d = \eta_l\phi_\delta^d\sum_{i=1}^{k}\sum_{j=0}^{n_i-1}\tilde{b}_{i,j,\delta}^{d}(0)(\frac{\lambda_i}{\lambda_i+R_l})^{j+1}$.

Note that $m_{\delta,0}^{d}(u)$ has the same form of linear combination of $e^{R_l u}$'s, as in the ordinary renewal risk model.

Chapter 5

Distributional Assumptions for the Time until the First Claim

In this chapter, we are going to solve $m_\delta^d(u)$ directly from the equation conditioning on the time and the amount of the first claim, for different distributions of time until the first claim. We expand the argument from a simple distribution (exponential) to a more complicated distribution (Coxian class).

5.1 Exponential

Let's look at the simplest cases, the exponential distribution. Assuming that the time until the first claim has a exponential distribution in the delayed renewal risk model seems to be useful and convenient. This is because the memoryless property of the exponential distribution takes care of the difficulty in observing the last claim before time 0. When the classical Poisson renewal risk process was extended to the ordinary Sparre-Andersen renewal risk process this useful property was lost.

The distribution of the time until the first claim, $k_1(t)$, is

$$k_1(t) = \lambda e^{-\lambda t} \tag{5.1}$$

with a LT of

$$\tilde{k}_1(s) = \frac{\lambda}{\lambda + s}. \tag{5.2}$$

As introduced in section 2.3, $m_\delta^d(u)$ can be expressed as

$$\begin{aligned} m_\delta^d(u) &= \int_0^\infty e^{-\delta t}\sigma_\delta(u+ct)k_1(t)dt \\ &= \int_0^\infty e^{-\delta t}\sigma_\delta(u+ct)\lambda e^{-\lambda t}dt \\ &= \lambda \int_0^\infty e^{-(\lambda+\delta)t}\sigma_\delta(u+ct)dt \end{aligned}$$

where

$$\sigma_\delta(t) = \int_0^t m_\delta(t-y)dP(y) + \int_t^\infty w(t, y-t)dP(y). \tag{5.3}$$

With a change in the variable $x = u + ct$, $m_\delta^d(u)$ results in

$$m_\delta^d(u) = \frac{\lambda}{c} e^{\frac{\lambda+\delta}{c}u} \int_u^\infty e^{-\frac{\lambda+\delta}{c}t}\sigma_\delta(t)dt = \frac{\lambda}{c} T_{\frac{\lambda+\delta}{c}}\sigma_\delta(u). \tag{5.4}$$

That is, $m_\delta^d(u)$ is a Dickson-Hipp transform of $\sigma_\delta(t)$. In later section it is necessary to use the LT of $m_\delta^d(u)$ and invert back to obtain an analytic solution for $m_\delta^d(u)$, but in the exponential case it is straightforward to obtain an explicit form out of the equation derived in section 2.3.

5.2 Combination of Exponentials

The second simplest distribution in this chapter is a combination of exponentials. It loses the nice memoryless property that the exponential distribution has, but the explicit form of the Gerber-Shiu expected discounted penalty function can still be obtained without using LT.

This PDF is of the form

$$k_1(t) = \sum_{i=1}^{r} p_i \lambda_i e^{-\lambda_i t}. \tag{5.5}$$

where p_i's are real numbers and $\sum_{i=1}^{r} p_i = 1$.

Following the same approach as in the previous section we get an explicit solution expressed as a combination of Dickson-Hipp transforms, i.e.

$$m_\delta^d(u) = \sum_{i=1}^{r} p_i(\frac{\lambda_i}{c}) e^{\frac{\lambda_i+\delta}{c}u} \int_u^\infty e^{-\frac{\lambda_i+\delta}{c}t} \sigma_\delta(t) dt = \sum_{i=1}^{r} p_i(\frac{\lambda_i}{c}) T_{\frac{\lambda_i+\delta}{c}} \sigma_\delta(u). \tag{5.6}$$

Both (5.4) and (5.6) are expressed in nice simple forms, a Dickson-Hipp transform and a combination of Dickson-Hipp transforms. If we can obtain an explicit expression for $\sigma_\delta(t)$, also can we for $m_\delta^d(u)$ in these cases.

5.3 Coxian Class

Let's assume that the distribution of the time until the first claim has a PDF of the form

$$k_1(y) = \sum_{i=1}^{r} \sum_{j=1}^{n_i} p_{ij} \frac{\lambda_i^j y^{j-1} e^{-\lambda_i y}}{(j-1)!} \tag{5.7}$$

with a LT of

$$\tilde{k}_1(s) = \sum_{i=1}^{r} \sum_{j=1}^{n_i} p_{ij} (\frac{\lambda_i}{\lambda_i + s})^j . \tag{5.8}$$

Now, we are going to solve $m_\delta^d(u)$ directly from the equation conditioning on the time and amount of the first claim. This method was used in Li and Garrido (2005) for the ordinary renewal risk model.

Once again using (2.48) and (2.49) introduced in section 2.3,

$$m_\delta^d(u) = \int_0^\infty e^{-\delta t} \sigma_\delta(u + ct) k_1(t) dt \tag{5.9}$$

where

$$\sigma_\delta(t) = \int_0^t m_\delta(t - y) dP(y) + \int_t^\infty w(t, y - t) dP(y).$$

Then, the LT of $m_\delta^d(u)$ is

$$\begin{aligned}
\tilde{m}_\delta^d(s) &= \int_0^\infty e^{-su} \int_0^\infty e^{-\delta t} \sigma_\delta(u+ct) k_1(t) dt du \\
&= \int_0^\infty e^{-\delta t} k_1(t) \int_0^\infty e^{-su} \sigma_\delta(u+ct) du dt \\
&= \int_0^\infty e^{-\delta t} k_1(t) \int_{ct}^\infty e^{-s(y-ct)} \sigma_\delta(y) dy dt \\
&= \int_0^\infty e^{-(\delta-cs)t} k_1(t) \int_{ct}^\infty e^{-sy} \sigma_\delta(y) dy dt \\
&= \int_0^\infty e^{-sy} \sigma_\delta(y) \int_0^{\frac{y}{c}} e^{-(\delta-cs)t} k_1(t) dt dy \\
&= \int_0^\infty e^{-st} \sigma_\delta(t) \int_0^{\frac{t}{c}} e^{-(\delta-cs)y} k_1(y) dy dt.
\end{aligned}$$

Note that

$$
\begin{aligned}
\int_0^{\frac{t}{c}} e^{-(\delta-cs)y} k_1(y)dy &= \sum_{i=1}^{r}\sum_{j=1}^{n_i} p_{ij}\frac{\lambda_i{}^j}{(j-1)!}\int_0^{\frac{t}{c}} y^{j-1}e^{-(\lambda_i+\delta-cs)y}dy \\
&= \sum_{i=1}^{r}\sum_{j=1}^{n_i}\{p_{ij}(\frac{\lambda_i}{\lambda_i+\delta-cs})^j \times \\
&\quad \int_0^{\frac{t}{c}} \frac{(\lambda_i+\delta-cs)^j y^{j-1}e^{-(\lambda_i+\delta-cs)y}}{(j-1)!}dy\} \\
&= \sum_{i=1}^{r}\sum_{j=1}^{n_i}[p_{ij}(\frac{\lambda_i}{\lambda_i+\delta-cs})^j \times \\
&\quad \{1-\sum_{m=0}^{j-1}\frac{(\lambda_i+\delta-cs)^m(\frac{t}{c})^m e^{-(\lambda_i+\delta-cs)(\frac{t}{c})}}{m!}\}] \\
&= \sum_{i=1}^{r}\sum_{j=1}^{n_i} p_{ij}(\frac{\lambda_i}{\lambda_i+\delta-cs})^j \\
&\quad -e^{st}\sum_{i=1}^{r} e^{-\frac{\lambda_i+\delta}{c}t}\sum_{j=1}^{n_i} p_{ij}\lambda_i{}^j\sum_{m=0}^{j-1}\frac{(\lambda_i+\delta-cs)^{m-j}(\frac{t}{c})^m}{m!} \\
&= \tilde{k}_1(\delta-cs) \\
&\quad -e^{st}\sum_{i=1}^{r} e^{-\frac{\lambda_i+\delta}{c}t}\sum_{j=1}^{n_i} p_{ij}\lambda_i{}^j\sum_{m=0}^{j-1}\frac{(\lambda_i+\delta-cs)^{m-j}(\frac{t}{c})^m}{m!}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\tilde{m}_\delta^d(s) &= \int_0^\infty e^{-st}\sigma_\delta(t)\tilde{k_1}(\delta - cs)dt \\
&\quad - \int_0^\infty \sigma_\delta(t)\sum_{i=1}^{r} e^{-\frac{\lambda_i+\delta}{c}t}\sum_{j=1}^{n_i} p_{ij}\lambda_i{}^j \sum_{m=0}^{j-1}\frac{(\lambda_i+\delta-cs)^{m-j}(\frac{t}{c})^m}{m!}dt \\
&= \tilde{\sigma_\delta}(s)\tilde{k_1}(\delta-cs) \\
&\quad - \sum_{i=1}^{r}\sum_{j=1}^{n_i} p_{ij}\lambda_i{}^j \sum_{m=0}^{j-1}\frac{(\lambda_i+\delta-cs)^{m-j}}{m!c^m}\int_0^\infty t^m e^{-\frac{\lambda_i+\delta}{c}t}\sigma_\delta(t)dt \\
&= \tilde{\sigma_\delta}(s)\tilde{k_1}(\delta-cs) - \sum_{i=1}^{r}\sum_{j=1}^{n_i} p_{ij}\lambda_i{}^j \sum_{m=0}^{j-1}\frac{(\lambda_i+\delta-cs)^{m-j}(-1)^m\tilde{\sigma_\delta}^{(m)}(\frac{\lambda_i+\delta}{c})}{m!c^m} \\
&= \tilde{\sigma_\delta}(s)\tilde{k_1}(\delta-cs) \\
&\quad - \sum_{i=1}^{r}\sum_{j=1}^{n_i} p_{ij}(\frac{\lambda_i}{\lambda_i+\delta-cs})^j \sum_{m=0}^{j-1}(-\frac{\lambda_i+\delta-cs}{c})^m\frac{\tilde{\sigma_\delta}^{(m)}(\frac{\lambda_i+\delta}{c})}{m!},
\end{aligned}
$$

i.e.

$$
\tilde{m}_\delta^d(s) = \tilde{\sigma_\delta}(s)\tilde{k_1}(\delta-cs) - \sum_{i=1}^{r}\sum_{j=1}^{n_i} p_{ij}\sum_{m=0}^{j-1}(-\frac{\lambda_i}{c})^m(\frac{\lambda_i}{\lambda_i+\delta-cs})^{j-m}\frac{\tilde{\sigma_\delta}^{(m)}(\frac{\lambda_i+\delta}{c})}{m!}. \tag{5.10}
$$

Note

$$
(\frac{\lambda_i}{\lambda_i+\delta-cs})^{j-m} = \frac{(-\lambda_i/c)^{j-m}}{(j-m-1)!}\frac{(j-m-1)!}{(s-(\lambda_i+\delta)/c)^{j-m}} \tag{5.11}
$$

and inverting this gives

$$\frac{(-\lambda_i/c)^{j-m}}{(j-m-1)!}u^{j-m-1}e^{\frac{\lambda_i+\delta}{c}u}. \tag{5.12}$$

Thus, inverting equation (5.10) gives

$$\begin{aligned} m_\delta^d(u) &= \int_0^u \sigma_\delta(y)\sum_{i=1}^{r}\sum_{j=1}^{n_i} p_{ij}\frac{(-\lambda_i/c)^j}{(j-1)!}(u-y)^{j-1}e^{\frac{\lambda_i+\delta}{c}(u-y)}dy \\ &\quad -\sum_{i=1}^{r}\sum_{j=1}^{n_i} p_{ij}\sum_{m=0}^{j-1}(\frac{-\lambda_i}{c})^m \frac{\tilde{\sigma_\delta}^{(m)}(\frac{\lambda_i+\delta}{c})}{m!}\frac{(-\lambda_i/c)^{j-m}}{(j-m-1)!}u^{j-m-1}e^{\frac{\lambda_i+\delta}{c}u} \\ &= \sum_{i=1}^{r}\sum_{j=1}^{n_i} p_{ij}\frac{(-\lambda_i/c)^j}{(j-1)!}e^{\frac{\lambda_i+\delta}{c}u}\int_0^u \sigma_\delta(y)(u-y)^{j-1}e^{-\frac{\lambda_i+\delta}{c}y}dy \\ &\quad -\sum_{i=1}^{r}\sum_{j=1}^{n_i} p_{ij}(\frac{-\lambda_i}{c})^j e^{\frac{\lambda_i+\delta}{c}u}\sum_{m=0}^{j-1}\frac{\tilde{\sigma_\delta}^{(m)}(\frac{\lambda_i+\delta}{c})}{m!(j-m-1)!}u^{j-m-1}, \end{aligned}$$

i.e.

$$m_\delta^d(u) = \sum_{i=1}^{r}\sum_{j=1}^{n_i} p_{ij}\frac{(-\lambda_i/c)^j}{(j-1)!}e^{\frac{\lambda_i+\delta}{c}u}d_{ij}(u) \tag{5.13}$$

where

$$d_{ij}(u) = \int_0^u e^{-\frac{\lambda_i+\delta}{c}y}\sigma_\delta(y)(u-y)^{j-1}dy - \sum_{m=0}^{j-1}\binom{j-1}{m}\tilde{\sigma_\delta}^{(m)}(\frac{\lambda_i+\delta}{c})u^{j-m-1}. \tag{5.14}$$

When $r = 1$ and $n_1 = 1$, $k_1(x)$ is an exponential distribution and equation (5.13) becomes

$$m_\delta^d(u) = \frac{\lambda}{c} e^{\frac{\lambda+\delta}{c}u} \int_u^\infty e^{-\frac{\lambda+\delta}{c}t} \sigma_\delta(t)dt = \frac{\lambda}{c} T_{\frac{\lambda+\delta}{c}} \sigma_\delta(u), \tag{5.15}$$

which is the same equation as we derived in section 5.1.

When $n_i = 1$ for all i's, $k_1(x)$ is a combination of exponential distributions and equation (5.13) reduces to

$$m_\delta^d(u) = \sum_{i=1}^r p_i(\frac{\lambda_i}{c}) e^{\frac{\lambda_i+\delta}{c}u} \int_u^\infty e^{-\frac{\lambda_i+\delta}{c}t} \sigma_\delta(t)dt = \sum_{i=1}^r p_i(\frac{\lambda_i}{c}) T_{\frac{\lambda_i+\delta}{c}} \sigma_\delta(u), \tag{5.16}$$

which also coincides with the equation derived in section 5.2.

5.4 Impact of the Dist. of the Time until the First Claim

In this section, we explore how different distributions of the time until the first claim have impacts on the ruin probabilities. In particular, we consider short-tail distribution and long-tail distribution along with the exponential distribution, for the time until the first claim. The examples we use for the short-tail distribution, which has coefficient of variation (CV) less than 1 and increasing failure rate (IFR), are sum of exponentials and Gamma distribution with shape parameter equal to or greater than 1. For examples of the long-tail distribution, which has CV greater than 1 and decreasing failure rate (DFR), we use mixture of exponentials and Inverse Gaussian distribution. For definitions, see Willmot and Lin (2001) and references therein.

For simplification, we assume that the claims and the interclaim times are of exponential, i.e., $p(y) = \beta e^{-\beta y}$ and $k(t) = \lambda e^{\lambda t}$. Then from (4.9), by letting $\delta = 0$ and $w_2(y) = 1$, we know

$$\psi^d(u) = \tilde{k}_1(c\kappa)e^{-\kappa u}. \tag{5.17}$$

And with the assumption that $k(t) = \lambda e^{\lambda t}$, κ can be solved as

$$\kappa = \frac{\beta\theta}{1+\theta}$$

and the ruin probability in (5.17) is

$$\psi^d(u) = \tilde{k}_1(\lambda\theta)e^{-\frac{\beta\theta}{1+\theta}u}. \tag{5.18}$$

The first case we consider is $k_1(t) = \lambda e^{\lambda t}$, where the process reduces to the classical Poisson process. Note that the exponential distribution has both DFR and IFR and has a CV of 1. In this case, the ruin probability is

$$\psi^d(u) = \frac{1}{1+\theta}e^{-\frac{\beta\theta}{1+\theta}u}. \tag{5.19}$$

The second case we consider is the sum of two exponentials, where the LT of $k_1(t)$ has the form

$$\tilde{k}_1(s) = (\frac{\lambda_1}{\lambda_1 + s})(\frac{\lambda_2}{\lambda_2 + s}), \tag{5.20}$$

and thus

$$\psi^d(u) = (\frac{\lambda_1}{\lambda_1 + \lambda\theta})(\frac{\lambda_2}{\lambda_2 + \lambda\theta})e^{-\frac{\beta\theta}{1+\theta}u}. \tag{5.21}$$

The last case we consider is the mixture of two exponentials, where $k_1(t) = q\lambda_1 e^{-\lambda_1 t} + (1-q)\lambda_2 e^{-\lambda_2 t}$ and the ruin probability in this case is

$$\psi^d(u) = \{q\frac{\lambda_1}{\lambda_1 + \lambda\theta} + (1-q)\frac{\lambda_2}{\lambda_2 + \lambda\theta}\}e^{-\frac{\beta\theta}{1+\theta}u}. \tag{5.22}$$

To further simplify the calculation, let $\beta = \lambda = 1$ and $\theta = 0.2$. The parameters λ_1 and λ_2 are also chosen so that the mean of the time until the first claim is 1.

For the sum of two exponentials, $\lambda_1 = \lambda_2 = 2$. Note that this is Erlang-2 and has the smallest CV among all the possible combinations of λ's. The CV is 0.707.

For the mixture of two exponentials, $\lambda_1 = 0.2, \lambda_2 = 2$ and $q = 1/9$. The CV in this case is 2.236.

It can be seen from table 5.1 that the short-tail distribution brings lower ruin probability and the long-tail distribution higher ruin probability, when the mean of these distributions are the same. As pointed out in Landriault and Willmot (2007), this is due to the greater variance of the increment $cW_1 - Y_1$ of the surplus process for long-tail distribution. The greater the variance of this increment, the higher the

	$k_1(t)$		
u	Exponential	Sum of Exponentials	Mixture of Exponential
0	0.833333333	0.826446281	0.863636364
0.1	0.819559545	0.812786325	0.849361710
0.5	0.766703679	0.760367285	0.794583813
1	0.705401437	0.699571673	0.731052399
5	0.362165174	0.359172073	0.375334816
10	0.157396336	0.156095540	0.163119839
20	0.029728328	0.029482639	0.030809358
30	0.005614956	0.005568551	0.005819136
50	0.000200308	0.000198652	0.000207592

Table 5.1: Calculation of the ruin probability, $\psi^d(u)$, for Different Distributions of Time until the First Claim

probability of ruin. Also the higher ruin probability associated with long-tail distribution can be explained with more probability being shifted to the left for long-tail distribution. When a long-tail distribution has the same mean as a short-tail distribution, the probability on small values of the long-tail distribution should be more than that of the short-tail distribution to relax the effect of the large values of the long-tail distribution. This can be shown by comparing the values of the cumulative distribution function (CDF). In our case, the CDF of value 1 for mixture of exponentials (long-tail) is 0.788732 whereas the CDF of value 1 for sum of exponentials (short-tail) is 0.593994. This translates into the likelihood of earlier claim occurrence for long-tail distribution and thus higher ruin probability.

The same results can be observed from table 5.2, where Gamma (5, 5) is used for short-tail distribution and Inverse Gaussian (1, 0.2) for long-tail distribution, for the time until the first claim. Gamma (5, 5) has mean of 1, CV of 0.447 and ruin probability

$$\psi^d(u) = (\frac{5}{5+\lambda\theta})^5 e^{-\frac{\beta\theta}{1+\theta}u}, \tag{5.23}$$

and Inverse Gaussian (1, 0.2) has mean of 1, CV of 2.236 and ruin probability

$$\psi^d(u) = e^{[0.2(1-(1+2\lambda\theta/0.2)^{0.5})]} e^{-\frac{\beta\theta}{1+\theta}u}. \tag{5.24}$$

	$k_1(t)$		
u	Exponential	Gamma(5,5)	Inverse Gaussian(1,0.2)
0	0.833333333	0.821927107	0.863803332
0.1	0.819559545	0.808341847	0.849525918
0.5	0.766703679	0.756209444	0.794737431
1	0.705401437	0.695746275	0.731193734
5	0.362165174	0.357208048	0.375407380
10	0.157396336	0.155241978	0.163151375
20	0.029728328	0.029321422	0.030815314
30	0.005614956	0.005538101	0.005820261
50	0.000200308	0.000197566	0.000207632

Table 5.2: Calculation of the ruin probability, $\psi^d(u)$, for Different Distributions of Time until the First Claim

Chapter 6

Two-Step Premium and Discounted Dividends

In this chapter, we employ the dividend strategy to the models we have studied. De Finetti (1957) first proposed this strategy into the insurance risk models. There is an active on-going research in this topic these days. Lin et al.(2003) have studied the constant barrier strategy where all the premium is paid out as a dividend as soon as the surplus level reaches the barrier level, in a classical compound Poisson model. Lin and Pavlova (2006) have extended this idea to the threshold dividend strategy model where only part of the premium is paid out as a dividend, instead of the whole premium being paid out. Other authors have extended the constant barrier problem from the classical Poisson model to the Sparre Andersen model. Li and Garrido (2004b) have extended to the model where the interclaim times have Generalized Erlang (n) distribution and Landriault (2007) studied the case where the interclaim times follow Coxian distribution. But for the threshold dividend strategy model not much research has been done outside of the classical Poisson model.

6.1 Two-Step Premium and Barrier

Our goal is to see how the delayed renewal risk process can be expressed in term of the ordinary renewal risk process that follow after the first claim, with the presence of a dividend barrier.

Define $T_{d,b}$ as the time of ruin when the threshold dividend strategy is present with threshold level b, in the delayed renewal risk process and T_b the corresponding random variable in the ordinary renewal risk model. Then the Gerber-Shiu expected discounted penalty function in each case, $m_\delta^d(u;b)$ and $m_\delta(u;b)$, can be written as

$$m_\delta^d(u;b) = E\{e^{-\delta T_{d,b}} w(U_b(T_{d,b}^-), |U_b(T_{d,b})|) I(T_{d,b} < \infty) | U(0) = u\}, \tag{6.1}$$

and

$$m_\delta(u;b) = E\{e^{-\delta T_b} w(U_b(T_b^-), |U_b(T_b)|) I(T_b < \infty) | U(0) = u\}, \tag{6.2}$$

respectively.

We assume that the premium rate is c_1 when the surplus level is below the threshold level b, and the premium rate changes to c_2 as soon as the surplus level is above b. Thus the dividend is paid out in the amount of $c_1 - c_2$ when the surplus level is above the threshold level and the surplus process under the threshold dividend strategy, $U_b(t)$, can be described as

$$dU_b(t) = \begin{cases} c_1 dt - dS(t) & \text{for} \quad U_b(t) \le b \\ c_2 dt - dS(t) & \text{for} \quad U_b(t) > b \end{cases}. \tag{6.3}$$

Figure 6.1 depicts a sample path of the surplus process with a threshold dividend strategy.

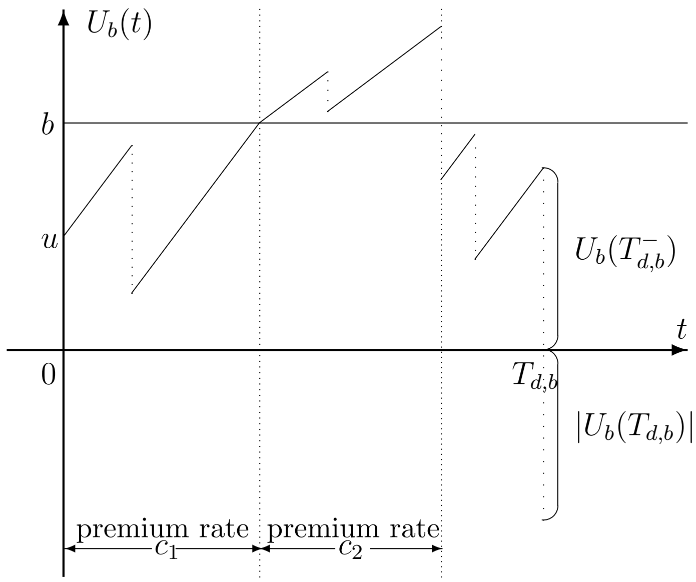

Figure 6.1: Graphical representation of the surplus process $U_b(t)$

6.1.1 Threshold Dividend Strategy Model

Let's now derive an equation for $m_\delta^d(u;b)$ conditioning on the time and the amount of the first claim. $m_\delta^d(u;b)$ behaves differently depending on whether the initial surplus level u is above or below b. Thus $m_\delta^d(u;b)$ should be considered in two disjoint cases,

$$m_\delta^d(u;b) = \begin{cases} m_1^d(u) & \text{for} \quad 0 \le u \le b \\ m_2^d(u) & \text{for} \quad u > b \end{cases} . \tag{6.4}$$

The Gerber-Shiu function in the ordinary renewal risk process with threshold dividend strategy will also take the same form as above.

$$m_\delta(u;b) = \begin{cases} m_1(u) & \text{for} \quad 0 \le u \le b \\ m_2(u) & \text{for} \quad u > b \end{cases} . \tag{6.5}$$

When we derive an equation for the threshold dividend strategy conditioning on the first claim, there is one more step to consider compared to the corresponding equation without a threshold. Since the premium level changes depending on where the surplus level is at, we have to consider whether the first claim will occur before the surplus level reaches the threshold level b or after exceeding it.

Let's first consider the case $0 \leq u \leq b$. When u is below the threshold level b, the premium rate is c_1 and if the first claim occurs before the surplus level reaching b, $u + c_1 t \leq b$ and thus t should be in the range of $0 \leq t \leq \frac{b-u}{c_1}$. Otherwise, i.e. when $t > \frac{b-u}{c_1}$, the surplus level will exceed b. Once the surplus level exceeds b, the premium rate changes to c_2, and the accumulated surplus at time t for $t > \frac{b-u}{c_1}$ will be $b+c_2(t-\frac{b-u}{c_1})$ where $t-\frac{b-u}{c_1}$ is the time elapsed right after the surplus level exceeds b.

Thus, when $0 \leq u \leq b$, $m_\delta^d(u;b)$ can be written as

$$\begin{aligned}
m_\delta^d(u;b) &= m_1^d(u) \\
&= \int_0^{\frac{b-u}{c_1}} e^{-\delta t}[\int_0^{u+c_1 t} m_\delta(u+c_1 t-y;b)p(y)dy \\
&\quad + \int_{u+c_1 t}^{\infty} w(u+c_1 t, y-u-c_1 t)p(y)dy]k_1(t)dt \\
&\quad + \int_{\frac{b-u}{c_1}}^{\infty} e^{-\delta t}[\int_0^{b+c_2(t-\frac{b-u}{c_1})} m_\delta(b+c_2(t-\frac{b-u}{c_1})-y;b)p(y)dy \\
&\quad + \int_{b+c_2(t-\frac{b-u}{c_1})}^{\infty} w(b+c_2(t-\frac{b-u}{c_1}), y-b-c_2(t-\frac{b-u}{c_1}))p(y)dy]k_1(t)dt \\
&= \int_0^{\frac{b-u}{c_1}} e^{-\delta t}\sigma_\delta(u+c_1 t;b)k_1(t)dt \\
&\quad + \int_{\frac{b-u}{c_1}}^{\infty} e^{-\delta t}\sigma_\delta(b+c_2(t-\frac{b-u}{c_1});b)k_1(t)dt \qquad (6.6)
\end{aligned}$$

where

$$\sigma_\delta(t;b) = \int_0^t m_\delta(t-y;b)p(y)dy + \int_t^\infty w(t,y-t)p(y)dy. \tag{6.7}$$

Change of variables, $x = u + c_1 t$ in the first term of equation (6.6) and $x = b + c_2(t - \frac{b-u}{c_1})$ in the second term leads to

$$\begin{aligned} m_1^d(u) &= \frac{1}{c_1}\int_u^b e^{-\delta(\frac{x-u}{c_1})}\sigma_\delta(x;b)k_1(\frac{x-u}{c_1})dx \\ &\quad + \frac{1}{c_2}\int_b^\infty e^{-\delta(\frac{x-b}{c_2}+\frac{b-u}{c_1})}\sigma_\delta(x;b)k_1(\frac{x-b}{c_2}+\frac{b-u}{c_1})dx \end{aligned} \tag{6.8}$$

When $u \geq b$, the surplus level will always stay above the threshold level b until the first claim occurs. Thus,

$$m_\delta^d(u;b) = m_2^d(u) = \int_0^\infty e^{-\delta t}\sigma_\delta(u + c_2 t;b)k_1(t)dt. \tag{6.9}$$

Or by change of variable, $x = u + c_2 t$

$$m_2^d(u) = \frac{1}{c_2}\int_u^\infty e^{-\delta(\frac{x-u}{c_2})}\sigma_\delta(x;b)k_1(\frac{x-u}{c_2})dx. \tag{6.10}$$

Example 6.1.1 Exponentially distributed first interclaim time

If the first interclaim time is exponentially distributed, i.e. $k_1(t) = \lambda e^{-\lambda t}$, then for

$0 \le u \le b$,

$$\begin{aligned} m_1^d(u) &= \frac{\lambda}{c_1} e^{(\lambda+\delta)\frac{u}{c_1}} \int_u^b e^{-(\lambda+\delta)\frac{x}{c_1}} \sigma_\delta(x;b)dx \\ &\quad + \frac{\lambda}{c_2} e^{\{(\lambda+\delta)\frac{b}{c_2}+(\lambda+\delta)\frac{u-b}{c_1}\}} \int_b^\infty e^{-(\lambda+\delta)\frac{x}{c_2}} \sigma_\delta(x;b)dx \end{aligned}$$

and for $u > b$,

$$m_2^d(u) = \frac{\lambda}{c_2} e^{(\lambda+\delta)\frac{u}{c_2}} \int_u^\infty e^{-(\lambda+\delta)\frac{x}{c_2}} \sigma_\delta(x;b)dx.♣$$

Ruin Probability

In this section, the probability of ultimate ruin is studied. Let $\delta = 0$ and the penalty function $w(x, y) = 1$ to obtain the ruin probability from equation (6.8) and (6.10). Then the ruin probability $\psi^d(u; b)$ satisfies

$$\psi^d(u;b) = \begin{cases} \psi_1^d(u) & \text{for} \quad 0 \le u \le b \\ \psi_2^d(u) & \text{for} \quad u > b \end{cases}, \tag{6.11}$$

where

$$\begin{aligned} \psi_1^d(u) &= \frac{1}{c_1} \int_u^b \{\int_0^x \psi(x-y;b)p(y)dy + \bar{P}(x)\} k_1(\frac{x-u}{c_1})dx \\ &\quad + \frac{1}{c_2} \int_b^\infty \{\int_0^x \psi(x-y;b)p(y)dy + \bar{P}(x)\} k_1(\frac{x-b}{c_2} + \frac{b-u}{c_1})dx \end{aligned} \tag{6.12}$$

and

$$\psi_2^d(u) = \frac{1}{c_2} \int_u^\infty \{\int_0^x \psi(x-y;b)p(y)dy + \bar{P}(x)\} k_1(\frac{x-u}{c_2})dx. \tag{6.13}$$

Example 6.1.2 Erlang mixture for the time until the first claim, Exponential interclaim times, and Exponential claim sizes

Let $k_1(t) = \sum_{j=1}^{\infty} q_j \frac{\lambda_1(\lambda_1 t)^{j-1}e^{-\lambda_1 t}}{(j-1)!}$, $k(t) = \lambda e^{-\lambda t}$, and $p(y) = \beta e^{-\beta y}$.
Then from section 6. of Lin and Pavlova (2006), we know that

$$\psi(u;b) = \begin{cases} \psi_1(u) = 1 - q(b) + \frac{q(b)}{(1+\theta_1)}e^{-\rho_1 u} & \text{for} \quad 0 \le u \le b \\ \psi_2(u) = \frac{1}{(1+\theta_2)}[1 - q(b) + q(b)e^{-\rho_1 b}]e^{-\rho_2(u-b)} & \text{for} \quad u > b \end{cases}. \tag{6.14}$$

where $\rho_i = \frac{\theta_i}{1+\theta_i}\beta$ for $i = 1, 2$ and $q(b) = \frac{(1+\theta_1)\theta_2}{(\theta_1-\theta_2)e^{-\rho_1 b}+(1+\theta_1)\theta_2}$.

Equation (6.12) and (6.13) require the evaluation of $\int_0^x \psi(x-y;b)p(y)dy$. Since $\psi(u;b)$ is a piecewise function, the form of it changes depending on the range of the variable and thus,

$$\int_0^x \psi(x-y;b)p(y)dy = \int_0^{x-b} \psi_2(x-y)p(y)dy + \int_{x-b}^x \psi_1(x-y)p(y)dy. \tag{6.15}$$

The first term on the right hand side of (6.15) is evaluated as

$$\begin{aligned} &\int_0^{x-b} \psi_2(x-y)p(y)dy \\ &= \frac{\beta}{1+\theta_2}[1 - q(b) + q(b)e^{-\rho_1 b}]e^{-\rho_2(x-b)} \int_0^{x-b} e^{-(\beta-\rho_2)y}dy \\ &= \frac{\beta}{\beta-\rho_2}\frac{1}{1+\theta_2}[1 - q(b) + q(b)e^{-\rho_1 b}]e^{-\rho_2(x-b)}[1 - e^{-(\beta-\rho_2)(x-b)}] \\ &= [1 - q(b) + q(b)e^{-\rho_1 b}](e^{-\rho_2(x-b)} - e^{-\beta(x-b)}). \end{aligned} \tag{6.16}$$

And the second term is evaluated as

$$\begin{aligned} &\int_{x-b}^x \psi_1(x-y)p(y)dy \\ &= [1-q(b)]\int_{x-b}^x \beta e^{-\beta y}dy + \frac{q(b)}{1+\theta_1}e^{-\rho_1 x}\beta \int_{x-b}^x e^{-(\beta-\rho_1)y}dy \\ &= [1-q(b)]e^{-\beta x}\{e^{\beta b} - 1\} + q(b)e^{-\beta x}\{e^{(\beta-\rho_1)b} - 1\} \end{aligned} \tag{6.17}$$

From (6.16) and (6.17),

$$\begin{aligned}\int_0^x \psi(x-y;b)p(y)dy &= \{1-q(b)\}\{e^{-\rho_2(x-b)} - e^{-\beta x}\} + q(b)\{e^{-\rho_2 x-(\rho_1-\rho_2)b} - e^{-\beta x}\} \\ &= \{1-q(b)\}e^{-\rho_2(x-b)} + q(b)e^{-\rho_2 x-(\rho_1-\rho_2)b} - e^{-\beta x} \\ &= e^{-\rho_2(x-b)}\{1-q(b)+q(b)e^{-\rho_1 b}\} - e^{-\beta x} \\ &= e^{-\rho_2(x-b)}\{\frac{(\theta_1-\theta_2)e^{-\rho_1 b}+(1+\theta_1)\theta_2 e^{-\rho_1 b}}{(\theta_1-\theta_2)e^{-\rho_1 b}+(1+\theta_1)\theta_2}\} - e^{-\beta x} \\ &= e^{-\rho_2(x-b)-\rho_1 b}\{\frac{\theta_1(1+\theta_2)}{(\theta_1-\theta_2)e^{-\rho_1 b}+(1+\theta_1)\theta_2}\} - e^{-\beta x} \\ &= q(b)\frac{\rho_1}{\rho_2}e^{-\rho_2(x-b)-\rho_1 b} - e^{-\beta x}\end{aligned}$$

and

$$\{\int_0^x \psi(x-y;b)p(y)dy + \bar{P}(x)\} = q(b)\frac{\rho_1}{\rho_2}e^{-(\rho_1-\rho_2)b}e^{-\rho_2 x} \tag{6.18}$$

Now, substituting (6.18) into equation (6.12), we obtain

$$\psi_1^d(u) = q(b)\frac{\rho_1}{\rho_2}e^{-(\rho_1-\rho_2)b}\{\frac{1}{c_1}\int_u^b e^{-\rho_2 x}k_1(\frac{x-u}{c_1})dx + \frac{1}{c_2}\int_b^\infty e^{-\rho_2 x}k_1(\frac{x-b}{c_2}+\frac{b-u}{c_1})dx$$

The first integration, with the change of variable from x to $z = \frac{x-u}{c_1}$, is

$$\begin{aligned}
&\int_u^b e^{-\rho_2 x} k_1(\frac{x-u}{c_1})dx \\
&= c_1 e^{-\rho_2 u} \int_0^{\frac{b-u}{c_1}} e^{-c_1\rho_2 z} k_1(z)dz \\
&= c_1 e^{-\rho_2 u} \sum_{j=1}^{\infty} q_j \frac{\lambda_1^j}{(\lambda_1 + c_1\rho_2)^j} \int_0^{\frac{b-u}{c_1}} \frac{(\lambda_1 + c_1\rho_2)^j z^{j-1} e^{-(\lambda_1+c_1\rho_2)z}}{(j-1)!} dz \\
&= c_1 e^{-\rho_2 u} \sum_{j=1}^{\infty} q_j \frac{\lambda_1^j}{(\lambda_1 + c_1\rho_2)^j} \sum_{k=j}^{\infty} \frac{(\lambda_1 + c_1\rho_2)^k (\frac{b-u}{c_1})^k}{k!} e^{-(\lambda_1+c_1\rho_2)(\frac{b-u}{c_1})} \\
&= c_1 \sum_{j=1}^{\infty} q_j \frac{\lambda_1^j}{(\lambda_1 + c_1\rho_2)^j} \sum_{k=j}^{\infty} \frac{(\frac{\lambda_1}{c_1} + \rho_2)^k (b-u)^k}{k!} e^{-(\frac{\lambda_1}{c_1}+\rho_2)b + \frac{\lambda_1}{c_1}u}. \qquad (6.19)
\end{aligned}$$

And the second integration, with the change of variable from x to $z = \frac{x-b}{c_2} + \frac{b-u}{c_1}$, is

$$\int_b^\infty e^{-\rho_2 x} k_1(\frac{x-b}{c_2}+\frac{b-u}{c_1})dx$$
$$= c_2 e^{-\rho_2(b-\frac{c_2}{c_1}(b-u))}\int_{\frac{b-u}{c_1}}^\infty e^{-c_2\rho_2 z}k_1(z)dz$$
$$= c_2 e^{-\rho_2(b-\frac{c_2}{c_1}(b-u))}\sum_{j=1}^\infty q_j\frac{\lambda_1^j}{(\lambda_1+c_2\rho_2)^j}$$
$$\times\int_{\frac{b-u}{c_1}}^\infty \frac{(\lambda_1+c_2\rho_2)^j z^{j-1}e^{-(\lambda_1+c_2\rho_2)z}}{(j-1)!}dz$$
$$= c_2 e^{-\rho_2(b-\frac{c_2}{c_1}(b-u))}\sum_{j=1}^\infty q_j\frac{\lambda_1^j}{(\lambda_1+c_2\rho_2)^j}$$
$$\times\sum_{k=0}^{j-1}\frac{(\lambda_1+c_2\rho_2)^k(\frac{b-u}{c_1})^k}{k!}e^{-(\lambda_1+c_2\rho_2)(\frac{b-u}{c_1})}$$
$$= c_2\sum_{j=1}^\infty q_j\frac{\lambda_1^j}{(\lambda_1+c_2\rho_2)^j}\sum_{k=0}^{j-1}\frac{(\lambda_1+c_2\rho_2)^k(\frac{b-u}{c_1})^k}{k!}$$
$$\times e^{-(\frac{\lambda_1}{c_1}+\rho_2)b+\frac{\lambda_1}{c_1}u} \tag{6.20}$$

Finally,

$$\psi_1^d(u) = q(b)\frac{\rho_1}{\rho_2}\sum_{j=1}^\infty q_j\lambda_1^j\{\sum_{k=j}^\infty\frac{(\lambda_1+c_1\rho_2)^{k-j}(\frac{b-u}{c_1})^k}{k!}$$
$$+\sum_{k=0}^{j-1}\frac{(\lambda_1+c_2\rho_2)^{k-j}(\frac{b-u}{c_1})^k}{k!}\}e^{-(\rho_1+\frac{\lambda_1}{c_1})b+\frac{\lambda_1}{c_1}u}. \tag{6.21}$$

Also, substituting (6.18) into equation (6.13) and changing the variable from x to

$z = \frac{x-u}{c_2}$, we obtain

$$\begin{aligned} \psi_2^d(u) &= \frac{1}{c_2} q(b) \frac{\rho_1}{\rho_2} e^{-(\rho_1-\rho_2)b} \int_u^\infty e^{-\rho_2 x} k_1(\frac{x-u}{c_2}) dx \\ &= q(b) \frac{\rho_1}{\rho_2} e^{-(\rho_1-\rho_2)b} \int_0^\infty e^{-\rho_2(c_2 z+u)} k_1(z) dz \\ &= q(b) \frac{\rho_1}{\rho_2} e^{-(\rho_1-\rho_2)b} \sum_{j=1}^\infty q_j \frac{\lambda_1^j}{(\lambda_1 + c_2\rho_2)^j} e^{-\rho_2 u}. \clubsuit \end{aligned} \tag{6.22}$$

6.1.2 Constant Barrier Model

Constant Barrier model is a special case of the threshold dividend strategy model when $c_2 = 0$ and thus the surplus level can never go above b. Dividends are paid continuously at rate c_1 once the surplus level reaches barrier b, until a new claim occurs.

In the threshold dividend strategy model, only a proportion of the premium received is paid out, whereas in the constant barrier model, the whole amount of the premium received is paid out as a dividend when the surplus level reaches b. As pointed out by Lin et al. (2003), the time of ruin is finite and thus the ultimate ruin probability is 1 in constant barrier model. This is intuitive in a sense that the total surplus level is always limited at b whereas the total claim amount to be paid increases to infinity as time goes on. Since there is only one premium rate for this model, we will use premium rate c instead of c_1.

The surplus process, $U_b(t)$, reduces to

$$dU_b(t) = \begin{cases} cdt - dS(t) & \text{for} \quad U_b(t) < b \\ -dS(t) & \text{for} \quad U_b(t) = b \end{cases} \tag{6.23}$$

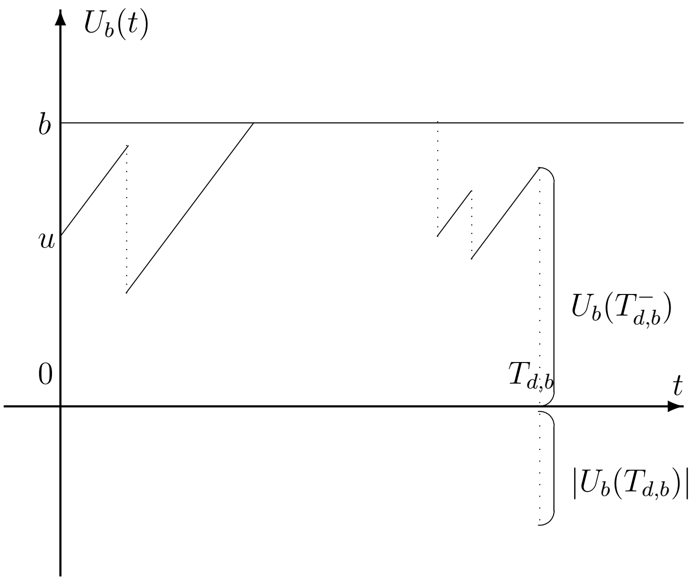

Figure 6.2: Graphical representation of the surplus process $U_b(t)$ in constant barrier model

and the Gerber-Shiu function, $m^d_\delta(u;b)$ also reduces from (6.6) to

$$\begin{aligned} m^d_\delta(u;b) &= m^d_1(u) \\ &= \int_0^{\frac{b-u}{c}} e^{-\delta t}\sigma_\delta(u+ct;b)k_1(t)dt \\ &\quad + \int_{\frac{b-u}{c}}^{\infty} e^{-\delta t}\sigma_\delta(b;b)k_1(t)dt \\ &= \frac{1}{c}\int_u^b e^{-\delta(\frac{x-u}{c})}\sigma_\delta(x;b)k_1(\frac{x-u}{c})dx \\ &\quad +\sigma_\delta(b;b)\int_{\frac{b-u}{c}}^{\infty} e^{-\delta t}k_1(t)dt \end{aligned} \tag{6.24}$$

for $0 \le u \le b$.

$m^d_2(u)$ does not exist in the constant barrier model since u cannot go above b.

When $u = b$,

$$m_\delta^d(b;b) = \sigma_\delta(b;b)\int_0^\infty e^{-\delta t}k_1(t)dt, \tag{6.25}$$

and thus

$$\sigma_\delta(b;b) = \frac{m_\delta^d(b;b)}{\tilde{k}_1(\delta)}. \tag{6.26}$$

Example 6.1.3 Generalized Erlang (n) distribution for interclaim times after the first claim

Lin et al.(2003) show that the Gerber-Shiu expected discounted penalty function with a constant barrier in a classical Poisson model satisfies

$$m_\delta(u;b) = m_\delta(u;\infty) - \frac{m_\delta'(b;\infty)}{v'(b)}v(u),\ \ 0 \le u \le b \tag{6.27}$$

where $m_\delta(u;\infty)$ is the corresponding Gerber-Shiu function without a barrier and $v(u)$ is a function that is a solution to the homogeneous integro-differential equation, which is part of the nonhomogeneous integro-differential equation satisfied by $m_\delta(u;b)$,

$$v'(u) = -\frac{\lambda}{c}\int_0^u v(u-y)p(y)dy. \tag{6.28}$$

Li and Garrido (2004b) extends the argument to the case where the interclaim times follow generalized Erlang(n) distribution and show $m_\delta(u;b)$ can be expressed as

$$m_\delta(u;b) = m_\delta(u;\infty) + \sum_{i=1}^{n} \eta_i v_i(u) \tag{6.29}$$

where $\{v_i(u)\}_{i=1}^{n}$ are n linearly independent solutions to the homogeneous integro-differential equation

$$\prod_{j=1}^{n} [(1 + \frac{\delta}{\lambda_j})I - \frac{c}{\lambda_j}D]v(u) = \int_0^u v(u-x)p(x)dx \tag{6.30}$$

if we define I, D to be the identity operator and the differentiation operator, respectively, and η_i's are chosen to satisfy the boundary conditions,

$$m_\delta^{(k)}(b;b) = 0, k = 1, 2, ...n. \tag{6.31}$$

Especially when the claim size Y is rationally distributed, i.e.

$$\tilde{p}(s) = \frac{Q_{m-1}(s)}{Q_m(s)},\ Re(s) \in (h_Y, \infty) \tag{6.32}$$

where $m \in N^+$, $h_Y := inf\{s \in \Re : E[e^{-sY}] < \infty\}$, Q_m is a polynomial of degree m with leading coefficient 1, Q_{m-1} is a polynomial of degree $m-1$ or less, and Q_m and Q_{m-1} do not have any common zeros, $v_i(u)$'s satisfy

$$v_i(u) = \sum_{j=1}^{n} \alpha_{ij} e^{\rho_j u} + \sum_{k=1}^{m} \beta_{ik} e^{-R_k u} \tag{6.33}$$

where $\rho_1, \rho_2, ..., \rho_n$ are all the roots with positive real parts to the generalized Lundberg's equation

$$\prod_{j=1}^{n} [(1 + \frac{\delta}{\lambda_j}) - \frac{c}{\lambda_j}s] = \tilde{p}(s) \tag{6.34}$$

and $-R_1, -R_2, ..., -R_m$ are the roots with negative real parts to the equation. Also,

$$\alpha_{ij} = \frac{\lambda_1\lambda_2\cdots\lambda_n}{c^n}\frac{-d_i(\rho_j)Q_m(\rho_j)}{\prod_{l=1}^{m}(R_l+\rho_j)\prod_{l=1,l\neq j}^{n}(\rho_l-\rho_j)}, \tag{6.35}$$

$$\beta_{ik} = \frac{\lambda_1\lambda_2\cdots\lambda_n}{c^n}\frac{d_i(-R_k)Q_m(-R_k)}{\prod_{l=1}^{n}(R_k+\rho_l)\prod_{l=1,l\neq k}^{m}(R_l-R_k)}, \tag{6.36}$$

and

$$d_i(s) = \sum_{m=0}^{n-i} B_{m+i}s^m,\ \ i = 1, 2, ...n \tag{6.37}$$

where $\sum_{k=0}^{n} B_k s^k = \prod_{j=1}^{n}[(1+(\delta/\lambda_j)) - (c/\lambda_j)s]$.

Thus,

$$\begin{aligned}
\sigma_\delta(x;b) &= \int_0^x m_\delta(x-y;b)p(y)dy + \int_x^\infty w(x,y-x)p(y)dy \\
&= \int_0^x \{m_\delta(x-y;\infty) + \sum_{i=1}^{n}\eta_i v_i(x-y)\}p(y)dy + \int_x^\infty w(x,y-x)p(y)dy \\
&= \sigma_\delta(x) + \sum_{i=1}^{n}\eta_i\int_0^x v_i(x-y)p(y)dy \\
&= \sigma_\delta(x) + \sum_{i=1}^{n}\eta_i\int_0^x\{\sum_{j=1}^{n}\alpha_{ij}e^{\rho_j(x-y)} + \sum_{k=1}^{m}\beta_{ik}e^{-R_k(x-y)}\}p(y)dy \\
&= \sigma_\delta(x) + \sum_{i=1}^{n}\eta_i\{\sum_{j=1}^{n}\alpha_{ij}\int_0^x e^{\rho_j(x-y)}p(y)dy + \sum_{k=1}^{m}\beta_{ik}\int_0^x e^{-R_k(x-y)}p(y)dy\}
\end{aligned}$$

If $p(y) = \beta e^{-\beta y}$, then the claim size Y is rationally distributed with $m = 1$ and $\sigma_\delta(x;b)$

reduces to

$$\sigma_\delta(x;b) = \sigma_\delta(x) + \sum_{i=1}^{n}\sum_{j=1}^{n}(\eta_i\alpha_{ij}\frac{\beta}{\beta+\rho_j})\{e^{\rho_j x} - e^{-\beta x}\} + \sum_{i=1}^{n}(\eta_i\beta_{i1}\frac{\beta}{\beta-R_1})\{e^{-R_1 x} - e^{-\beta x}\}. \tag{6.38}$$

When the time until the first claim also follows an exponential distribution, i.e. $k_1(t) = \lambda e^{-\lambda t}$, $m_\delta^d(u;b)$ can be written as

$$\begin{aligned} m_\delta^d(u;b) &= \frac{\lambda}{c}\int_u^b e^{-(\lambda+\delta)(\frac{x-u}{c})}\sigma_\delta(x;b)dx + \lambda\sigma_\delta(b;b)\int_{\frac{b-u}{c}}^{\infty} e^{-(\lambda+\delta)t}dt \\ &= \frac{\lambda}{c}e^{(\frac{\lambda+\delta}{c})u}\int_u^b e^{-(\frac{\lambda+\delta}{c})x}\sigma_\delta(x;b)dx + \frac{\lambda}{\lambda+\delta}\sigma_\delta(b;b)e^{-(\lambda+\delta)(\frac{b-u}{c})} \\ &= \frac{\lambda}{c}e^{(\frac{\lambda+\delta}{c})u}\int_u^b e^{-(\frac{\lambda+\delta}{c})x}\sigma_\delta(x;b)dx + m_\delta^d(b;b)e^{-(\lambda+\delta)(\frac{b-u}{c})}. \end{aligned}$$

The integration in the first term, with the substitution using equation (6.38), leads to

$$\begin{aligned} \int_u^b e^{-(\frac{\lambda+\delta}{c})x}\sigma_\delta(x;b)dx &= \int_u^b e^{-(\frac{\lambda+\delta}{c})x}\sigma_\delta(x)dx \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}[\frac{1}{\frac{\lambda+\delta}{c}-\rho_j}\{e^{-(\frac{\lambda+\delta}{c}-\rho_j)u} - e^{-(\frac{\lambda+\delta}{c}-\rho_j)b}\} \\ &\quad - \frac{1}{\frac{\lambda+\delta}{c}+\beta}\{e^{-(\frac{\lambda+\delta}{c}+\beta)u} - e^{-(\frac{\lambda+\delta}{c}+\beta)b}\}] \\ &\quad + \sum_{i=1}^{n} b_{i1}[\frac{1}{\frac{\lambda+\delta}{c}+R_1}\{e^{-(\frac{\lambda+\delta}{c}+R_1)u} - e^{-(\frac{\lambda+\delta}{c}+R_1)b}\} \\ &\quad - \frac{1}{\frac{\lambda+\delta}{c}+\beta}\{e^{-(\frac{\lambda+\delta}{c}+\beta)u} - e^{-(\frac{\lambda+\delta}{c}+\beta)b}\}] \end{aligned}$$

where we define a_{ij} and b_{i1} to be $a_{ij} = \eta_i \alpha_{ij} \frac{\beta}{\beta+\rho_j}$ and $b_{i1} = \eta_i \beta_{i1} \frac{\beta}{\beta-R_1}$.

And finally the Gerber-Shiu function $m_\delta^d(u;b)$ becomes

$$\begin{aligned} m_\delta^d(u;b) &= m_\delta^d(b;b) e^{-(\frac{\lambda+\delta}{c})(b-u)} + \frac{\lambda}{c} e^{(\frac{\lambda+\delta}{c})u} \int_u^b e^{-(\frac{\lambda+\delta}{c})x} \sigma_\delta(x) dx \\ &+ \sum_{i=1}^n \sum_{j=1}^n \frac{\lambda a_{ij}}{\lambda+\delta-c\rho_j} \{ e^{\rho_j u} - e^{-(\frac{\lambda+\delta}{c})(b-u)+\rho_j b} \} \\ &+ \sum_{i=1}^n \frac{\lambda b_{i1}}{\lambda+\delta+cR_1} \{ e^{-R_1 u} - e^{-(\frac{\lambda+\delta}{c})(b-u)+R_1 b} \} \\ &- \sum_{i=1}^n \frac{\lambda(\sum_{j=1}^n a_{ij} + b_{i1})}{\lambda+\delta+c\beta} \{ e^{-\beta u} - e^{-(\frac{\lambda+\delta}{c})(b-u)-\beta b} \}. \end{aligned}$$

Other than the second term, all the rest are in exponential form with respect to u. The second term needs to be evaluated from the function that comes from the ordinary process without a barrier, which can be found.♣

6.1.3 Stationary Renewal Risk Model with Threshold Dividend Strategy

When the time until the first claim has an equilibrium distribution, i.e. $k_1(t) = \frac{\bar{K}(t)}{E(V)}$, the delayed renewal risk model with threshold dividend strategy reduces to the stationary renewal risk model with threshold dividend strategy. Let $T_{e,b}$ be the time of ruin when the threshold dividend strategy is present with threshold level b, in the stationary renewal risk process. Also the Gerber-Shiu expected discount penalty function will be defined as

$$m_\delta^e(u;b) = E\{e^{-\delta T_{e,b}} w(U_b(T_{e,b}^-), |U_b(T_{e,b})|) I(T_{e,b} < \infty) | U(0) = u\}, \tag{6.39}$$

and this will be composed of two functions

$$m_\delta^e(u;b) = \begin{cases} m_1^e(u) & \text{for} \quad 0 \le u \le b \\ m_2^e(u) & \text{for} \quad u > b \end{cases} . \tag{6.40}$$

Theorem 6.1 *The two functions, $m_1^e(u)$ and $m_2^e(u)$, in a threshold dividend strategy model can be solved as*

$$m_1^e(u) = e^{-\frac{\delta}{c_1}(b-u)} m_1^e(b) + \frac{1}{c_1 E(V)} \int_u^b e^{-\frac{\delta}{c_1}(t-u)} \{\sigma_\delta(t;b) - m_1(t)\} dt \tag{6.41}$$

where

$$m_1^e(b) = \frac{1}{E(V)} \int_0^\infty e^{-\delta t} \sigma_\delta(b + c_2 t; b) \bar{K}(t) dt \tag{6.42}$$

and

$$m_2^e(u) = e^{-\frac{\delta}{c_2}(b-u)} m_2^e(b) + \frac{1}{c_2 E(V)} \int_b^u e^{-\frac{\delta}{c_2}(t-u)} \{m_2(t) - \sigma_\delta(t;b)\} dt \tag{6.43}$$

where $m_2^e(b) = m_1^e(b)$.

Proof:

For $0 \le u \le b$, from equation (6.8), we know that

$$
\begin{aligned}
m_1^e(u) &= \frac{1}{c_1 E(V)} \int_u^b e^{-\delta(\frac{x-u}{c_1})} \sigma_\delta(x;b) \bar{K}(\frac{x-u}{c_1}) dx \\
&+ \frac{1}{c_2 E(V)} \int_b^\infty e^{-\delta(\frac{x-b}{c_2}+\frac{b-u}{c_1})} \sigma_\delta(x;b) \bar{K}(\frac{x-b}{c_2}+\frac{b-u}{c_1}) dx. \qquad (6.44)
\end{aligned}
$$

We usually would use Laplace Transform argument to obtain a solution but in the threshold dividend strategy problem this is not possible for $m_1^e(u)$ or $m_2^e(u)$ since these are only defined with a range that is not $(0, \infty)$. In section 7. of Lin et al. (2003) they show a different way of obtaining a solution in a constant barrier model that works within the stationary renewal risk settings. Following their idea, we can solve for $m_1^e(u)$ and $m_2^e(u)$ in a threshold dividend strategy model.

First let's differentiate equation (6.44) with respect to u to get

$$
\begin{aligned}
m_1^{e\prime}(u) &= \frac{\delta}{c_1} m_1^e(u) - \frac{1}{c_1 E(V)} \sigma_\delta(u;b) \bar{K}(0) \\
&+ \frac{1}{c_1^2 E(V)} \int_u^b e^{-\delta(\frac{x-u}{c_1})} \sigma_\delta(x;b) k(\frac{x-u}{c_1}) dx \\
&+ \frac{1}{c_1 c_2 E(V)} \int_b^\infty e^{-\delta(\frac{x-b}{c_2}+\frac{b-u}{c_1})} \sigma_\delta(x;b) k(\frac{x-b}{c_2}+\frac{b-u}{c_1}) dx \\
&= \frac{\delta}{c_1} m_1^e(u) - \frac{1}{c_1 E(V)} \sigma_\delta(u;b) + \frac{1}{c_1 E(V)} m_1(u). \qquad (6.45)
\end{aligned}
$$

Changing the variable u to t and multiplying $e^{-\frac{\delta}{c_1}t}$ on both sides of the equation lead us to

$$
e^{-\frac{\delta}{c_1}t} m_1^{e\prime}(t) - \frac{\delta}{c_1} e^{-\frac{\delta}{c_1}t} m_1^e(t) = \frac{e^{-\frac{\delta}{c_1}t}}{c_1 E(V)} \{m_1(t) - \sigma_\delta(t;b)\}. \qquad (6.46)
$$

Notice that the left hand side is equal to $\frac{d}{dt}[e^{-\frac{\delta}{c_1}t}m_1^e(t)]$. Integrating both sides from u to b, we get

$$e^{-\frac{\delta}{c_1}b}m_1^e(b) - e^{-\frac{\delta}{c_1}u}m_1^e(u) = \frac{1}{c_1E(V)}\int_u^b e^{-\frac{\delta}{c_1}t}\{m_1(t) - \sigma_\delta(t;b)\}dt, \tag{6.47}$$

and thus the solution to $m_1^e(u)$ is

$$m_1^e(u) = e^{-\frac{\delta}{c_1}(b-u)}m_1^e(b) + \frac{1}{c_1E(V)}\int_u^b e^{-\frac{\delta}{c_1}(t-u)}\{\sigma_\delta(t;b) - m_1(t)\}dt \tag{6.48}$$

where

$$\begin{aligned} m_1^e(b) &= \frac{1}{c_2E(V)}\int_b^\infty e^{-\delta(\frac{x-b}{c_2})}\sigma_\delta(x;b)\bar{K}(\frac{x-b}{c_2})dx \\ &= \frac{1}{E(V)}\int_0^\infty e^{-\delta t}\sigma_\delta(b+c_2t;b)\bar{K}(t)dt \end{aligned}$$

Now, for $u > b$, from equation (6.10), we know that

$$m_2^e(u) = \frac{1}{c_2E(V)}\int_u^\infty e^{-\delta(\frac{x-u}{c_2})}\sigma_\delta(x;b)\bar{K}(\frac{x-u}{c_2})dx. \tag{6.49}$$

Applying the same idea as before, differentiate with respect to u to get

$$m_2^{e\prime}(u) = \frac{\delta}{c_2}m_2^e(u) - \frac{1}{c_2E(V)}\sigma_\delta(u;b) + \frac{1}{c_2E(V)}m_2(u) \tag{6.50}$$

Changing the variable u to t, multiplying $e^{-\frac{\delta}{c_2}t}$ on both sides of the equation and moving the first term on the right hand side to the left yields

$$\frac{d}{dt}\{e^{-\frac{\delta}{c_2}t}m_2^e(t)\} = \frac{e^{-\frac{\delta}{c_2}t}}{c_2E(V)}\{m_2(t) - \sigma_\delta(t;b)\}. \tag{6.51}$$

Integrate both sides with respect to t from b to u, rearrange to isolate for $m_2^e(u)$ and we obtain the solution

$$m_2^e(u) = e^{-\frac{\delta}{c_2}(b-u)}m_2^e(b) + \frac{1}{c_2E(V)}\int_b^u e^{-\frac{\delta}{c_2}(t-u)}\{m_2(t) - \sigma_\delta(t;b)\}dt \tag{6.52}$$

where $m_2^e(b) = m_1^e(b)$.

Q.E.D.

6.2 Discounted Dividends

One interest that arises in connection with the threshold dividend strategy model is the value of the dividend payments. For $0 \leq u \leq b$, dividends will be paid as soon as the surplus reaches b and for $u \geq b$ dividends are paid from the beginning at rate $\hat{c}$ until the surplus level falls below the threshold level b. Let $V(u,b)$ be the expected present value of dividend payments at force of interest δ, until ruin occurs in the ordinary renewal risk process and $V^d(u,b)$ the corresponding quantity in the delayed renewal risk process. Then as in the previous section, we can see $V(u,b)$ will consist of two functions, i.e.

$$V(u;b) = \begin{cases} V_1(u,b) & \text{for} \quad 0 \le u \le b \\ V_2(u,b) & \text{for} \quad u > b \end{cases}. \tag{6.53}$$

Also, in the delayed renewal risk process,

$$V^d(u;b) = \begin{cases} V_1^d(u,b) & \text{for} \quad 0 \le u \le b \\ V_2^d(u,b) & \text{for} \quad u > b \end{cases}. \tag{6.54}$$

Since our focus is on the dividend and not the premium rate change above the threshold level, we will use different notations than the ones used in the previous section. We denote c as premium rate when the surplus level is below the threshold level and $\hat{c}$ as dividend rate that is paid out when the surplus level is above the threshold level. Then, comparing with the rates used in the previous section, $c = c_1$ and $\hat{c} = c_1 - c_2$.

The Gerber-Shiu function which provides unified approach in analyzing quantities related to ruin and is used throughout the paper cannot be used for the analysis in this section since dividends are paid during the survival of the process but the Gerber-Shiu function conditions on the occurrence of ruin. Thus we need to derive an equation that is suitable for $V^d(u,b)$ and that relates $V^d(u,b)$ to $V(u,b)$.

Dickson and Drekic (2006) study $V(u,b)$ in a classical compound Poisson model with threshold dividend strategy and show that for $0 \le u \le b$, $V(u,b)$ satisfies

$$V_1(u,b) = V(b,b)E[exp\{-\delta T_{u,b}\}] \tag{6.55}$$

and for $u \ge b$,

$$V_2(u,b) = \frac{\hat{c}}{\delta}(1 - E[e^{-\delta \hat{T}_{u-b}}]) + V(b,b)\int_0^\infty e^{-\delta t}\int_0^b \hat{w}(u-b,y,t)E[exp\{-\delta T_{b-y,b}\}]dydt \tag{6.56}$$

where $\hat{T}_u$ is the time to ruin with initial surplus u and premium rate $c - \hat{c}$, $T_{u,b}$ is the time of the first upcrossing of the surplus process through b from u without ruin occurring, and $\hat{w}(u,y,t)$ is the defective density of the time (t) and severity (y) of ruin, from initial surplus u, when premium rate is $c - \hat{c}$.

Landriault (2007) studies $V(u,b)$ in a Sparre Andersen model with the assumption that interclaim times follow Coxian distribution when constant barrier is present. The approach in this paper is different from the approach used in Dickson and Drekic (2006). He derives an equation for $V(u,b)$ conditioning on the first claim. We will follow this idea and apply it to the threshold dividend strategy model in deriving a relation for $V^d(u,b)$ with $V(u,b)$.

6.2.1 Threshold Dividend Strategy Model

Conditioning on the time and the amount of the first claim we obtain the following equation. Let's first consider the case $0 \le u \le b$. When u is below the threshold level b, the surplus will accumulate at a rate of c as long as the surplus level does not reach the threshold level b. If the first claim occurs before the surplus level reaching b, it means that t should be in the range of $0 \le t \le \frac{b-u}{c}$. When the first claim occurs within the range, no dividends are paid out and the process will start over again with an ordinary renewal risk process with new initial surplus $u + ct - y$ where y is the amount of the first claim. Otherwise, i.e. when $t > \frac{b-u}{c}$, the surplus level will exceed b and dividends are paid out at rate $\hat{c}$ beginning at time $\frac{b-u}{c}$. When the first claim occurs, the process will start again with the ordinary renewal process with the new initial surplus $b + (c - \hat{c})(t - \frac{b-u}{c}) - y$. The ordinary renewal risk process starts as long as the first claim amount does not bring ruin, i.e. y is less than the accumulated surplus level at the time of first claim occurrence. Otherwise, when ruin occurs, there is no more dividends in the later process. Thus, for $0 \leq u \leq b$,

$$\begin{aligned} V^d(u,b) &= V_1^d(u,b) = \int_0^{\frac{b-u}{c}} e^{-\delta t} \int_0^{u+ct} V_1(u+ct-y,b)p(y)dyk_1(t)dt \\ &+ \int_{\frac{b-u}{c}}^{\infty} \{e^{-\delta(\frac{b-u}{c})}\hat{c}\bar{a}_{\overline{t-\frac{b-u}{c}}|} + e^{-\delta t} \int_0^{b+(c-\hat{c})(t-\frac{b-u}{c})} \\ & V(b+(c-\hat{c})(t-\frac{b-u}{c})-y,b)p(y)dy\}k_1(t)dt \end{aligned} \tag{6.57}$$

where

$$\bar{a}_{\overline{t}|} = \int_0^t e^{-\delta x}dx = \frac{1-e^{-\delta t}}{\delta}$$

and the second term on the right hand side can be rewritten as

$$\begin{aligned} \int_{\frac{b-u}{c}}^{\infty} e^{-\delta(\frac{b-u}{c})}\hat{c}\bar{a}_{\overline{t-\frac{b-u}{c}}|}k_1(t)dt &= \int_{\frac{b-u}{c}}^{\infty} e^{-\delta(\frac{b-u}{c})}\hat{c}\frac{1-e^{-\delta(t-\frac{b-u}{c})}}{\delta}k_1(t)dt \\ &= \frac{\hat{c}}{\delta}e^{-\delta(\frac{b-u}{c})}\bar{K}_1(\frac{b-u}{c}) - \frac{\hat{c}}{\delta}\int_{\frac{b-u}{c}}^{\infty} e^{-\delta t}k_1(t)dt. \end{aligned}$$

The third term on the right can also be elaborated as

$$\begin{aligned}
&\int_{\frac{b-u}{c}}^{\infty} e^{-\delta t} \int_0^{b+(c-\hat{c})(t-\frac{b-u}{c})} V(b+(c-\hat{c})(t-\frac{b-u}{c})-y,b)p(y)dyk_1(t)dt \\
&= \frac{1}{c}\int_b^{\infty} e^{-\delta(\frac{x-u}{c})} \int_0^{b+(c-\hat{c})(\frac{x-b}{c})} V(b+(c-\hat{c})(\frac{x-b}{c})-y,b)p(y)dyk_1(\frac{x-u}{c})dx \\
&= \frac{1}{c}\int_b^{\infty} e^{-\delta(\frac{x-u}{c})} \int_{(c-\hat{c})(\frac{x-b}{c})}^{b+(c-\hat{c})(\frac{x-b}{c})} V_1(b+(c-\hat{c})(\frac{x-b}{c})-y,b)p(y)dyk_1(\frac{x-u}{c})dx \\
&\quad + \frac{1}{c}\int_b^{\infty} e^{-\delta(\frac{x-u}{c})} \int_0^{(c-\hat{c})(\frac{x-b}{c})} V_2(b+(c-\hat{c})(\frac{x-b}{c})-y,b)p(y)dyk_1(\frac{x-u}{c})dx
\end{aligned}$$

Let's now consider the case where $u \geq b$. When the initial surplus level is above b, dividends are paid from the beginning of the process and when the first claim occurs, the ordinary renewal process will start. In this situation we don't have to divide up the range of the time of the first claim occurrence as before since the surplus process will always stay above b until we get the first claim. Thus, for $u \geq b$,

$$\begin{aligned}
V^d(u,b) &= V_2^d(u,b) \\
&= \int_0^{\infty} \hat{c}\bar{a}_{\overline{t}|}k_1(t)dt + \int_0^{\infty} e^{-\delta t}\int_0^{u+(c-\hat{c})t} V(u+(c-\hat{c})t-y,b)p(y)dyk_1(t)dt \\
&= \frac{\hat{c}}{\delta}\int_0^{\infty}(1-e^{-\delta t})k_1(t)dt \\
&\quad + \frac{1}{c-\hat{c}}\int_u^{\infty} e^{-\delta(\frac{x-u}{c-\hat{c}})}\int_0^x V(x-y,b)p(y)dyk_1(\frac{x-u}{c-\hat{c}})dx \\
&= \frac{\hat{c}}{\delta}(1-\tilde{k}_1(\delta)) + \frac{1}{c-\hat{c}}\int_u^{\infty} e^{-\delta(\frac{x-u}{c-\hat{c}})}\int_{x-b}^x V_1(x-y,b)p(y)dyk_1(\frac{x-u}{c-\hat{c}})dx \\
&\quad + \frac{1}{c-\hat{c}}\int_u^{\infty} e^{-\delta(\frac{x-u}{c-\hat{c}})}\int_0^{x-b} V_2(x-y,b)p(y)dyk_1(\frac{x-u}{c-\hat{c}})dx. \qquad (6.58)
\end{aligned}$$

6.2.2 Constant Barrier Model

We can reduce the threshold dividend strategy model to the constant barrier model by letting $c = \hat{c}$. Then the initial surplus u can only take values in the range of $[0, b]$ and the expected discounted dividend $V^d(u, b)$ is

$$\begin{aligned} V^d(u,b) &= \frac{1}{c}\int_u^b e^{-\delta(\frac{x-u}{c})}\int_0^x V_1(x-y,b)p(y)dyk_1(\frac{x-u}{c})dx \\ &+\frac{1}{c}\int_b^\infty e^{-\delta(\frac{x-u}{c})}\int_0^b V(b-y,b)p(y)dyk_1(\frac{x-u}{c})dx \\ &+\frac{c}{\delta}e^{-\delta(\frac{b-u}{c})}\bar{K}_1(\frac{b-u}{c}) - \frac{1}{\delta}\int_b^\infty e^{-\delta(\frac{x-u}{c})}k_1(\frac{x-u}{c})dx \end{aligned} \tag{6.59}$$

for $0 \le u \le b$.

6.2.3 Stationary Model with Threshold Dividend Strategy

If we reduce the delayed renewal risk process to the stationary renewal risk process, then our expected discounted dividends in the threshold dividend strategy model can be written as, for $0 \le u \le b$,

$$\begin{aligned} V^e(u,b) &= V_1^e(u,b) = \frac{1}{cE(V)}\int_u^b e^{-\delta(\frac{x-u}{c})}\int_0^x V_1(x-y,b)p(y)dy\bar{K}(\frac{x-u}{c})dx \\ &+\frac{1}{cE(V)}\int_b^\infty e^{-\delta(\frac{x-u}{c})}\int_0^{b+(c-\hat{c})(\frac{x-b}{c})} V(b+(c-\hat{c})(\frac{x-b}{c})-y,b) \\ &p(y)dy\bar{K}(\frac{x-u}{c})dx \\ &+\frac{\hat{c}}{\delta E(V)}e^{-\delta(\frac{b-u}{c})}\int_{\frac{b-u}{c}}^\infty \bar{K}(t)dt - \frac{\hat{c}}{c\delta E(V)}\int_b^\infty e^{-\delta(\frac{x-u}{c})}\bar{K}(\frac{x-u}{c})dx, \end{aligned} \tag{6.60}$$

and for $u \geq b$,

$$\begin{aligned} V^e(u,b) &= V_2^e(u,b) \\ &= \frac{\hat{c}}{\delta}(1 - \frac{1-\tilde{k}(\delta)}{\delta E(V)}) \\ &\quad + \frac{1}{(c-\hat{c})E(V)} \int_u^\infty e^{-\delta(\frac{x-u}{c-\hat{c}})} \int_0^x V(x-y,b)p(y)dy\bar{K}(\frac{x-u}{c-\hat{c}})dx. \end{aligned} \tag{6.61}$$

We can obtain a solution in the case of the stationary renewal risk model by differentiating the equations given above since the differentiation of $\bar{K}(t)$ gives $k(t)$, which makes it possible to express some terms in ordinary renewal risk process.

Theorem 6.2 *The solutions to $V_1^e(u,b)$ and $V_2^e(u,b)$ are*

$$V_1^e(u,b) = e^{-\frac{\delta}{c}(b-u)}V_1^e(b,b) + \frac{1}{cE(V)} \int_u^b e^{-\frac{\delta}{c}(t-u)}\{\int_0^t V_1(t-y,b)p(y)dy - V_1(t,b)\}dt \tag{6.62}$$

for $0 \leq u \leq b$, where

$$V_1^e(b,b) = \frac{\hat{c}}{\delta}(1-\frac{1-\tilde{k}(\delta)}{\delta E(V)})+\frac{1}{(c-\hat{c})E(V)} \int_b^\infty e^{-\delta(\frac{x-b}{c-\hat{c}})} \int_0^x V(x-y,b)p(y)dy\bar{K}(\frac{x-b}{c-\hat{c}})dx. \tag{6.63}$$

and

$$\begin{aligned} V_2^e(u,b) &= \frac{\hat{c}}{\delta}(1-e^{-\frac{\delta}{c-\hat{c}}(b-u)}) + e^{-\frac{\delta}{c-\hat{c}}(b-u)}V_2^e(b,b) \\ &+ \frac{1}{(c-\hat{c})E(V)}\int_b^u e^{-\frac{\delta}{c-\hat{c}}(t-u)}\{V_2(t,b) - \int_0^t V(t-y,b)p(y)dy\}dt \end{aligned} \tag{6.64}$$

for $u \geq b$*, where* $V_2^e(b,b) = V_1^e(b,b)$.

Proof:

Differentiate equation (6.60) with respect to u to get

$$\frac{d}{du}V_1^e(u,b) = \frac{\delta}{c}V_1^e(u,b) + \frac{1}{cE(V)}V_1(u,b) - \frac{1}{cE(V)}\int_0^u V_1(u-y,b)p(y)dy \tag{6.65}$$

for $0 \leq u \leq b$.

Applying the same method used in the previous section, we change the variable u to t and multiply $e^{-\frac{\delta}{c}t}$ on both sides, we obtain

$$e^{-\frac{\delta}{c}t}\frac{d}{dt}V_1^e(t,b) - \frac{\delta}{c}e^{-\frac{\delta}{c}t}V_1^e(t,b) = \frac{e^{-\frac{\delta}{c}t}}{cE(V)}V_1(t,b) - \frac{e^{-\frac{\delta}{c}t}}{cE(V)}\int_0^t V_1(t-y,b)p(y)dy. \tag{6.66}$$

Noticing that $\frac{d}{dt}[e^{-\frac{\delta}{c}t}V_1^e(t,b)] = e^{-\frac{\delta}{c}t}\frac{d}{dt}V_1^e(t,b) - \frac{\delta}{c}e^{-\frac{\delta}{c}t}V_1^e(t,b)$ and integrating with respect to t from u to b will lead us to

$$e^{\frac{-\delta}{c}b}V_1^e(b,b) - e^{-\frac{\delta}{c}u}V_1^e(u,b) = \int_u^b \frac{e^{-\frac{\delta}{c}t}}{cE(V)}V_1(t,b)dt - \int_u^b \frac{e^{-\frac{\delta}{c}t}}{cE(V)}\int_0^t V_1(t-y,b)p(y)dydt. \tag{6.67}$$

Thus, isolating for $V_1^e(u,b)$ gives

$$V_1^e(u,b) = e^{-\frac{\delta}{c}(b-u)}V_1^e(b,b) + \frac{1}{cE(V)}\int_u^b e^{-\frac{\delta}{c}(t-u)}\{\int_0^t V_1(t-y,b)p(y)dy - V_1(t,b)\}dt \tag{6.68}$$

for $0 \leq u \leq b$, where

$$\begin{aligned} V_1^e(b,b) &= \frac{\hat{c}}{\delta}(1 - \frac{1-\tilde{k}(\delta)}{\delta E(V)}) \\ &+ \frac{1}{(c-\hat{c})E(V)}\int_b^\infty e^{-\delta(\frac{x-b}{c-\hat{c}})}\int_0^x V(x-y,b)p(y)dy\bar{K}(\frac{x-b}{c-\hat{c}})dx. \end{aligned} \tag{6.69}$$

For $u \geq b$, differentiate equation (6.61) with respect to u, and we get

$$
\begin{aligned}
\frac{d}{du}V_2^e(u,b) &= -\frac{1}{(c-\hat{c})E(V)}\int_0^u V(u-y,b)p(y)dy \\
&+\frac{\delta}{(c-\hat{c})^2E(V)}\int_u^\infty e^{-\delta(\frac{x-u}{c-\hat{c}})}\int_0^x V(x-y,b)p(y)dy\bar{K}(\frac{x-u}{c-\hat{c}})dx \\
&+\frac{1}{(c-\hat{c})^2E(V)}\int_u^\infty e^{-\delta(\frac{x-u}{c-\hat{c}})}\int_0^x V(x-y,b)p(y)dyk(\frac{x-u}{c-\hat{c}})dx \\
&= -\frac{1}{(c-\hat{c})E(V)}\int_0^u V(u-y,b)p(y)dy \\
&+\frac{\delta}{c-\hat{c}}\{V_2^e(u,b)-\frac{\hat{c}}{\delta}(1-\frac{1-\tilde{k}(\delta)}{\delta E(V)})\} \\
&+\frac{1}{(c-\hat{c})E(V)}\{V_2(u,b)-\frac{\hat{c}}{\delta}(1-\tilde{k}(\delta))\} \\
&= \frac{\delta}{c-\hat{c}}V_2^e(u,b)+\frac{1}{(c-\hat{c})E(V)}V_2(u,b)-\frac{\hat{c}}{c-\hat{c}} \\
&-\frac{1}{(c-\hat{c})E(V)}\int_0^u V(u-y,b)p(y)dy. \qquad (6.70)
\end{aligned}
$$

Changing the variable u to t and multiply $e^{-\frac{\delta}{c-\hat{c}}t}$ on both sides,

$$
\begin{aligned}
e^{-\frac{\delta}{c-\hat{c}}t}\frac{d}{dt}V_2^e(t,b)-\frac{\delta}{c-\hat{c}}e^{-\frac{\delta}{c-\hat{c}}t}V_2^e(t,b) &= \frac{e^{-\frac{\delta}{c-\hat{c}}t}}{(c-\hat{c})E(V)}V_2(t,b)-\frac{\hat{c}}{c-\hat{c}}e^{-\frac{\delta}{c-\hat{c}}t} \\
&-\frac{e^{-\frac{\delta}{c-\hat{c}}t}}{(c-\hat{c})E(V)}\int_0^t V(t-y,b)p(y)dy.
\end{aligned}
$$

Integrating t from b to u and rearranging the terms we get a solution for $V_2^e(u,b)$,

$$
\begin{aligned}
V_2^e(u,b) &= \frac{\hat{c}}{\delta}(1-e^{-\frac{\delta}{c-\hat{c}}(b-u)})+e^{-\frac{\delta}{c-\hat{c}}(b-u)}V_2^e(b,b) \\
&+\frac{1}{(c-\hat{c})E(V)}\int_b^u e^{-\frac{\delta}{c-\hat{c}}(t-u)}\{V_2(t,b)-\int_0^t V(t-y,b)p(y)dy\}dt
\end{aligned}
\qquad (6.71)
$$

for $u \geq b$, where $V_2^e(b,b) = V_1^e(b,b)$.

Q.E.D.

Chapter 7

Summary and Highlights

In Chapter 2, we derived two framework equations (2.1) and (2.48) that are used throughout the later chapters. The former is derived conditioning on the first drop in surplus below its initial level u, whereas the latter is derived conditioning on the first claim that occurs in the process. Both of the equation uses the property that the process starts over again with the ordinary renewal risk process in those points in time. And thus, these equations show relationship between the delayed and ordinary renewal risk processes. The widely used stationary renewal risk process is a special case of the delayed renewal risk process and the results derived in this thesis can all be applied to the stationary renewal risk process. Section 2.2 expresses the ladder height distribution $B_\delta^d(u)$ in terms of $G_\delta^d(u)$ and $B_\delta(u)$, where the expression of it is different from section 2.1. This expression may not be of much interest by itself but was a useful tool in proving the theorem in section 3.4.

Chapter 3 provides a valuable overview of many quantities and distributions related to the deficit at ruin in detail; the discounted kth moment of the deficit, the discounted distribution function of the deficit, the asymptotic distribution of the proper deficit, stochastic decomposition of the residual lifetime of L^d_δ which is possible using the random variable from the discounted DF of the discounted proper deficit, and the discounted joint distribution of the surplus and the deficit. We also discuss about the Laplace transform of the time of ruin at the beginning of Chapter 3. It is included in Chapter 3, since not only can it be obtained by simplifying equation (3.1) which is a starting point for the chapter but also let us define $\bar{\Lambda}_\delta(u)$ which is essential function used in later sections where we present the DF of the deficit at ruin (section 3.3) and the stochastic decomposition of the residual lifetime of L^d_δ (section 3.5). The discounted kth moment of the deficit at ruin in delayed renewal risk model (section 3.2) is an extension of the result in section 4 of Willmot (2007) in the ordinary renewal risk model. Section 3.3, 3.4 and 3.5 extend the ideas of Willmot et al. (2004) for DF of the proper deficit, from the stationary renewal risk model to the delayed renewal risk model. In section 3.3 and 3.5, we further extend the argument to the case where $\delta > 0$. The discounted joint PDF of the surplus and the deficit at ruin is derived in the delayed renewal risk process, in section 3.6, and the important relationship between the joint density of the surplus and the deficit and the marginal density of the surplus, (3.82), is proved.

We assume particular distributions for claim sizes and derive the general form of the Gerber-Shiu expected discounted penalty function in Chapter 4. The most general penalty function that we work with is $w(x, y) = e^{-sx}w_2(y)$, which has a useful property in a sense that the function of the surplus and the function of the deficit is separated. This is a nice extension of the widely used penalty function $w(x, y) = w_2(y)$, since it contains more information than the penalty function of the deficit only. When claims are from exponential distribution (section 4.1), $m^d_{\delta,s}(u)$ is expressed a weighted average of two exponential terms. When claims are from Erlang mixture of single scale parameter (section 4.2), $m^d_{\delta,s}(u)$ is of damped exponential series form.

In Chapter 5, the distributional assumption is made for the time until the first claim but the distribution of the claim sizes are left to be arbitrary. When the time until the first claim is of exponential or combination of exponentials, the derivations of the Gerber-Shiu function $m_\delta^d(u)$ are straightforward and have simple Dickson-Hipp transform of $\sigma_\delta(t)$, or combination of Dickson-Hipp transforms of $\sigma_\delta(t)$, respectively. Coxian class distributions are also considered in section 5.3.

Chapter 6 is concerned with the threshold dividend strategy in the delayed renewal risk process, where part of the premium is paid out as a dividend when the surplus level reaches the barrier level. Unlike in other chapters, now the Gerber-Shiu function behaves differently depending on the surplus level. The general form of the Gerber-Shiu function in the delayed renewal risk process, in terms of that of the ordinary renewal risk process, is derived in section 6.1.1. Section 6.1.2 contains the simplified result for constant barrier model which is a special case of the threshold dividend strategy model. The stationary renewal risk process with the threshold dividend strategy is also considered in section 6.1.3. Finally, the values of the dividend payments under all three models considered in section 6.1 are studied in section 6.2.

One of the difficulties in dealing with the delayed renewal risk model is that the functions in this model are expressed in terms of the functions from the ordinary renewal risk process and thus we can not obtain a renewal equation which is a convenient tool. But we were able to obtain equation (2.1), where the logic and the form of it is similar to the defective renewal equation in the ordinary renewal risk process. From this equation, many subsequent results were derived.

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